The Pigeonhole Principle

1 Pigeonhole Principle: Simple form

Theorem 1.1. *If $n+1$ objects are put into n boxes, then at least one box contains two or more objects.*

Proof. Trivial. $\square$

Example 1.1. Among 13 people there are two who have their birthdays in the same month.

Example 1.2. There are n married couples. How many of the $2n$ people must be selected in order to guarantee that one has selected a married couple?

Other principles related to the pigeonhole principle:

- If n objects are put into n boxes and no box is empty, then each box contains exactly one object.
- If n objects are put into n boxes and no box gets more than one object, then each box has an object.

The abstract formulation of the three principles: Let X and Y be finite sets and let $f : X \longrightarrow Y$ be a function.

- If X has more elements than Y, then f is not one-to-one.
- If X and Y have the same number of elements and f is onto, then f is one-to-one.
- If X and Y have the same number of elements and f is one-to-one, then f is onto.

Example 1.3. In any group of n people there are at least two persons having the same number friends. (It is assumed that if a person x is a friend of y then y is also a friend of x.)

Proof. The number of friends of a person x is an integer k with $0 \le k \le n-1$. If there is a person y whose number of friends is $n-1$, then everyone is a friend of y, that is, no one has 0 friend. This means that 0 and $n-1$ can not be simultaneously the numbers of friends of some people in the group. The pigeonhole principle tells us that there are at least two people having the same number of friends. $\square$

Example 1.4. Given n integers $a_1, a_2, \ldots, a_n$, not necessarily distinct, there exist integers k and l with $0 \le k < l \le n$ such that the sum $a_{k+1} + a_{k+2} + \cdots + a_l$ is a multiple of n.

Proof. Consider the n integers

$$a_1, \quad a_1 + a_2, \quad a_1 + a_2 + a_3, \quad \ldots, \quad a_1 + a_2 + \cdots + a_n.$$

Dividing these integers by n, we have

$$a_1 + a_2 + \cdots + a_i = q_i n + r_i, \quad 0 \le r_i \le n-1, \quad i = 1, 2, \ldots, n.$$

If one of the remainders $r_1, r_2, \ldots, r_n$ is zero, say, $r_k = 0$, then $a_1 + a_2 + \cdots + a_k$ is a multiple of n. If none of $r_1, r_2, \ldots, r_n$ is zero, then two of them must the same (since $1 \le r_i \le n-1$ for all i), say, $r_k = r_l$ with $k < l$. This means that the two integers $a_1 + a_2 + \cdots + a_k$ and $a_1 + a_2 + \cdots + a_l$ have the same remainder. Thus $a_{k+1} + a_{k+2} + \cdots + a_l$ is a multiple of n. $\square$

Example 1.5. A chess master who has 11 weeks to prepare for a tournament decides to play at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calendar week. Show that there exists a succession of consecutive days during which the chess master will have played exactly 21 games.

Proof. Let a_1 be the number of games played on the first day, a_2 the total number of games played on the first and second days, a_3 the total number games played on the first, second, and third days, and son. Since at least one game is played each day, the sequence of numbers $a_1, a_2, \ldots, a_{77}$ is strictly increasing, that is, $a_1 < a_2 < \cdots < a_{77}$. Moreover, $a_1 \geq 1$; and since at most 12 games are played during any one week, $a_{77} \leq 12 \times 11 = 132$. Thus

$$1 \leq a_1 < a_2 < \cdots < a_{77} \leq 132.$$

Note that the sequence $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ is also strictly increasing, and

$$22 \leq a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \leq 132 + 21 = 153.$$

Now consider the 154 numbers

$$a_1,\ a_2,\ \ldots,\ a_{77},\ a_1 + 21,\ a_2 + 21,\ \ldots,\ a_{77} + 21;$$

each of them is between 1 and 153. It follows that two of them must be equal. Since $a_1, a_2, \ldots, a_{77}$ are distinct and $a_1 + 21, a_2 + 21, \ldots, a_{77} + 21$ are also distinct, then the two equal numbers must be of the forms a_i and $a_j + 21$. Since the number games played up to the ith day is $a_i = a_j + 21$, we conclude that on the days $j + 1$, $j + 2$, ..., i the chess master played a total of 21 games. □

Example 1.6. Given 101 integers from $1, 2, \ldots, 200$, there are at least two integers such that one of them is divisible by the other.

Proof. By factoring out as many 2's as possible, we see that any integer can be written in the form $2^k \cdot a$, where $k \geq 0$ and a is odd. The number a can be one of the 100 numbers $1, 3, 5, \ldots, 199$. Thus among the 101 integers chosen, two of them must have the same a's when they are written in the form, say, $2^r \cdot a$ and $2^s \cdot a$ with $r \neq s$. if $r < s$, then the first one divides the second. If $r > s$, then the second one divides the first. □

Example 1.7 (Chines Remainder Theorem). Let m and n be relatively prime positive integers. Then the system

$$\begin{cases} x \equiv a \pmod{m} \\ x \equiv b \pmod{n} \end{cases}$$

has a solution.

Proof. We may assume that $0 \leq a < m$ and $0 \leq b < n$. Let us consider the n integers

$$a, \quad m + a, \quad 2m + a, \quad \ldots, \quad (n-1)m + a.$$

Each of these integers has remainder a when divided by m. Suppose that two of them had the same remainder r when divided by n. Let the two numbers be $im + a$ and $jm + a$, where $0 \leq i < j \leq n - 1$. Then there are integers q_i and q_j such that

$$im + a = q_i n + r \quad \text{and} \quad jm + a = q_j n + r.$$

Subtracting the first equation from the second, we have

$$(j - i)m = (q_j - q_i)n.$$

Since $\gcd(m, n) = 1$, we conclude that $n | (j - i)$. Note that $0 < j - i \leq n - 1$. This is a contradiction. Thus the n integers $a, m + a, 2m + a, \ldots, (n-1)m + a$ have distinct remainders when divided by n. That is, each of the n numbers $0, 1, 2, \ldots, n - 1$ occur as a remainder. In particular, the number b does. Let p be the integer with $0 \leq p \leq n - 1$ such that the number $x = pm + a$ has remainder b when divided by n. Then for some integer q, $x = qn + b$. So

$$x = pm + a = qn + b,$$

and x has the required property. □

2 Pigeonhole Principle: Strong Form

Theorem 2.1. *Let $q_1, q_2, \ldots, q_n$ be positive integers. If*

$$q_1 + q_2 + \cdots + q_n - n + 1$$

objects are put into n boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, …, the nth box contains at least q_n objects.

Proof. Suppose it is not true, that is, the ith box contains at most $q_i - 1$ objects, $i = 1, 2, \ldots, n$. Then the total number of objects contained in the n boxes can be at most

$$(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n,$$

which is one less than the number of objects distributed. This is a contradiction. □

The simple form of the pigeonhole principle is obtained from the strong form by taking $q_1 = q_2 = \cdots = q_n = 2$. Then

$$q_1 + q_2 + \cdots + q_n - n + 1 = 2n - n + 1 = n + 1.$$

In elementary mathematics the strong form of the pigeonhole principle is most often applied in the special case when $q_1 = q_2 = \cdots = q_n = r$. In this case the principle becomes:

- If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes contains r or more of the objects. Equivalently,
- If the average of n nonnegative integers $a_1, a_2, \ldots, a_n$ is greater than $r - 1$, i.e.,

  $$\frac{a_1 + a_2 + \cdots + a_n}{n} > r - 1,$$

  then at leats one of the integers is greater than or equal to r.

Example 2.1. A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

Answer: $8 + 6 + 9 - 3 + 1 = 21$.

Example 2.2. Given two disks, one smaller than the other. Each disk is divided into 200 congruent sectors. In the larger disk 100 sectors are chosen arbitrarily and painted red; the other 100 sectors are painted blue. In the smaller disk each sector is painted either red or blue with no stipulation on the number of red and blue sectors. The smaller disk is placed on the larger disk so that the centers and sectors coincide. Show that it is possible to align the two disks so that the number of sectors of the smaller disk whose color matches the corresponding sector of the larger disk is at least 100.

Proof. We fix the larger disk first, then place the smaller disk on the top of the larger disk so that the centers and sectors coincide. There 200 ways to place the smaller disk in such a manner. For each such alignment, some sectors of the two disks may have the same color. Since each sector of the smaller disk will match the same color sector of the larger disk 100 times among all the 200 ways and there are 200 sectors in the smaller disk, the total number of matched color sectors among the 200 ways is $100 \times 200 = 20,000$. Note that there are only 200 ways. Then there is at least one way that the number of matched color sectors is $\frac{20,000}{200} = 100$ or more. □

Example 2.3. Show that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of $n^2 + 1$ real numbers contains either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

Proof. Suppose there is no increasing subsequence of length $n+1$. We suffices to show that there must be a decreasing subsequence of length $n+1$.

Let ℓ_k be the length of the longest increasing subsequence which begins with a_k, $1 \le k \le n^2+1$. Since it is assumed that there is no increasing subsequence of length $n+1$, we have $1 \le \ell_k \le n$ for all k. By the strong form of the pigeonhole principle, $n+1$ of the n^2+1 integers $\ell_1, \ell_2, \ldots, \ell_{n^2+1}$ must be equal, say,

$$\ell_{k_1} = \ell_{k_2} = \cdots = \ell_{k_{n+1}},$$

where $1 \le k_1 < k_2 < \cdots < k_{n+1} \le n^2+1$. If there is one k_i $(1 \le i \le n)$ such that $a_{k_i} < a_{k_{i+1}}$, then any increasing subsequence of length $\ell_{k_{i+1}}$ beginning with $a_{k_{i+1}}$ will result a subsequence of length $\ell_{k_{i+1}}+1$ beginning with a_{k_i} by adding a_{k_i} in the front; so $\ell_{k_i} > \ell_{k_{i+1}}$, which is contradictory to $\ell_{k_i} = \ell_{k_{i+1}}$. Thus we must have

$$a_{k_1} \ge a_{k_{2+1}} \ge \cdots \ge a_{k_{n+1}},$$

which is a decreasing subsequence of length $n+1$. □

3 Ramsey Theory

Theorem 3.1 (Ramsey Theorem). *Let S be a finite set with n elements. Let $P_r(S)$ be the collection of all r-subsets of S with $r \ge 1$, i.e.,*

$$P_r(S) = \{X \subseteq S : |X| = r\}.$$

Then for any integers $p, q \ge r$ there exists a smallest integer $R(p,q;r)$ such that, if $n \ge R(p,q;r)$ and $P_r(S)$ is 2-colored with two color classes $\mathcal{C}_1$ and $\mathcal{C}_2$, then there is either a p-subset $S_1 \subseteq S$ such that $P_r(S_1) \subseteq \mathcal{C}_1$, or a q-subset $S_2 \subseteq S$ such that $P_r(S_2) \subseteq \mathcal{C}_2$.

Proof. We proceed by induction on p, q, and r. For $r=1$, we have $R(p,q;1) = p+q-1$. Note that every element of $P_1(S)$ is a singleton set and $|P_1(S)| = |S|$. For an n-set S with $n \ge p+q-1$, if $|\mathcal{C}_1| \ge p$, we take any p-subset $S_1 \subseteq \bigcup_{X \in \mathcal{C}_1} X$, then obviously $P_1(S_1) \subseteq \mathcal{C}_1$. If $|\mathcal{C}_1| < p$, then $|\mathcal{C}_2| \ge q$; we take any q-subset $S_2 \subseteq \bigcup_{X \in \mathcal{C}_2} X$ and obviously have $P_1(S_2) \subseteq \mathcal{C}_2$. Thus $R(p,q;1) \le p+q-1$. For $n = p+q-2$, let $\mathcal{C}_1$ be the set of $p-1$ singleton sets and $\mathcal{C}_2$ the set of the other $q-1$ singleton sets. Then it is impossible to have a p-subset $S_1 \subseteq S$ such that $P_1(S_1) \subseteq \mathcal{C}_1$ or a q-subset S_2 such that $P_1(S_2) \subseteq \mathcal{C}_2$. Thus $R(p,q;1) \ge p+q-1$.

Moreover, for any integer $r \ge 1$ it can be easily verified that

$$R(r,q;r) = q, \quad R(p,r;r) = p.$$

In fact, for $p = r$, let S be a q-set. For a 2-coloring $\{\mathcal{C}_1, \mathcal{C}_2\}$ of $P_r(S)$, if $\mathcal{C}_1 = \emptyset$, then $P_r(S) = \mathcal{C}_2$ and obviously $P_r(S_2) \subseteq \mathcal{C}_2$ for $S_2 = S$. If $\mathcal{C}_1 \ne \emptyset$, take an r-subset $A \in \mathcal{C}_1$; obviously, $P_r(A) = \{A\} \subseteq \mathcal{C}_1$. Thus $R(k,q;k) \le q$. Let $|S| \le q-1$. If $\mathcal{C}_1 = \emptyset$, then $\mathcal{C}_2 = P_r(S)$. It is clear that there is neither an r-subset $A \subseteq S$ such that $P_r(A) \subseteq \mathcal{C}_1$ nor a q-subset $B \subseteq S$ such that $P_r(B) \subseteq \mathcal{C}_2$. Thus $R(r,q;r) \ge q$. It is similar for the case $R(p,r;r) = p$.

Next we establish a recurrence relation about $R(p,q;r)$ for $r \ge 2$ as follows:

$$R(p,q;r) \le R(p_1,q_1;r-1)+1, \quad p_1 = R(p-1,q;r), \quad q_1 = R(p,q-1;r).$$

Let $n \ge R(p_1,q_1;r-1)+1$ and $|S| = n$. Take an element $x \in S$ and let $S_1 = S - \{x\}$. Then $|S_1| = n-1$ and $|S_1| \ge R(p_1,q_1;r-1)$. Let $\{\mathcal{C}, \mathcal{D}\}$ be a 2-coloring of $P_r(S)$ and let

$$\mathcal{C}_1 = \{A \in \mathcal{C} : x \notin A\}, \quad \mathcal{D}_1 = \{A \in \mathcal{D} : x \notin A\}.$$

Obviously, $\{\mathcal{C}_1, \mathcal{D}_1\}$ is a 2-coloring of $P_r(S_1)$. Let

$$\mathcal{C}_x = \{A \in P_{r-1}(S_1) : A \cup \{x\} \in \mathcal{C}\}, \quad \mathcal{D}_x = \{A \in P_{r-1}(S_1) : A \cup \{x\} \in \mathcal{D}\}.$$

For any $A \in P_{r-1}(S_1)$, it is obvious that either $A \cup \{x\} \in \mathcal{C}$ or $A \cup \{x\} \in \mathcal{D}$; then either $A \in \mathcal{C}_x$ or $A \in \mathcal{D}_x$. Thus $\{\mathcal{C}_x, \mathcal{D}_x\}$ is a 2-coloring of $P_{r-1}(S_1)$. Since $|S_1| \ge R(p_1,q_1;r-1)$ and by the induction hypothesis on k, we have (I) there exists a p_1-subset $X \subseteq S_1$ such that $P_{r-1}(X) \subseteq \mathcal{C}_x$, or (II) there exists a q_1-subset $Y \subseteq S_1$ such that $P_{r-1}(Y) \subseteq \mathcal{D}_x$.

Case (I): Since $p_1 = R(p-1, q; r)$ and $\{\mathcal{C}_1, \mathcal{D}_1\}$ is a 2-coloring of $P_r(S_1)$, by induction hypothesis on p (when r is fixed) there exists either a $(p-1)$-subset $X_1 \subseteq X$ such that $P_r(X_1) \subseteq \mathcal{C}_1 \subset \mathcal{C}$ or a q-subset $Y_1 \subseteq X$ such that $P_r(Y_1) \subseteq \mathcal{D}_1 \subset \mathcal{D}$. In the former case, consider the p-subset $X' = X_1 \cup \{x\} \subseteq S$. For any r-subset $A \subset X'$, if $x \notin A$, obviously $A \subset X_1$, so $A \in \mathcal{C}$; if $x \in A$, obviously $A - \{x\}$ is an $(r-1)$-subset of X, so $A - \{x\} \in \mathcal{C}_x$, then $A = (A - \{x\}) \cup \{x\} \in \mathcal{C}$. This means that X' is an r-subset of S and $P_r(X') \subseteq \mathcal{C}$. In the latter case, we already have a q-subset $Y_1 \subseteq S$ such that $P_r(Y_1) \subseteq \mathcal{D}$.

Case (II): Since $q_1 = R(p, q-1; r)$ and $\{\mathcal{C}_1, \mathcal{D}_1\}$ is a partition of $P_r(S_1)$, then by induction hypothesis on q (when r is fixed) there exists either a p-subset $X_1 \subseteq X$ such that $P_r(X_1) \subseteq \mathcal{C}_1 \subset \mathcal{C}$ or a $(q-1)$-subset $Y_1 \subseteq X$ such that $P_r(Y_1) \subseteq \mathcal{D}_1 \subset \mathcal{D}$. In the former case, we already have a p-subset $X_1 \subseteq S$ such that $P_r(X_1) \subseteq \mathcal{C}$. In the latter case, we have a q-subset $Y' = Y_1 \cup \{x\} \subset S$ and $P_r(Y') \subseteq \mathcal{D}$.

Now we have obtained a recurrence relation:

$$R(p, q; r) \leq R\Big(R(p-1, q; r), R(p, q-1; r); r-1\Big) + 1.$$

□

Theorem 3.2. *Let S be an n-set. Let $q_1, q_2, \ldots, q_k, r$ be positive integers such that $q_1, q_2, \ldots, q_k \geq r$. Then there exists a smallest integer $R(q_1, q_2, \ldots, q_k; r)$ such that, if $n \geq R(q_1, q_2, \ldots, q_k; r)$ and for any k-coloring $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_k\}$ of $P_r(S)$ there is at least one i $(1 \leq i \leq k)$ and a q_i-subset $S_i \subseteq S$ such that $P_r(S_i) \subseteq \mathcal{C}_i$.*

Proof. We proceed induction on k. For $k = 1$, then $P_r(S)$ is a 1-coloring of $P_r(S)$ itself; the theorem is obviously true. For $k = 2$, it is Theorem 3.1. For $k \geq 3$, let $\mathcal{D}_i = \mathcal{C}_i$ for $1 \leq i \leq k-2$ and $\mathcal{D}_{k-1} = \mathcal{C}_{k-1} \cup \mathcal{C}_k$. By the induction hypothesis there exists the integer $q'_{k-1} = R(q_{k-1}, q_k; r)$ and subsequently the integer $R(q_1, \ldots, q_{k-2}, q'_{k-1}; r)$.

Now for $|S| = n \geq R(q_1, \ldots, q_{k-2}, q'_{k-1}; r)$, since $\{\mathcal{D}_1, \ldots, \mathcal{D}_{k-1}\}$ is a $(k-1)$-coloring of $P_r(S)$, there exists either at least one q_i-subset $S_i \subseteq S$ such that $P_r(S_i) \subseteq \mathcal{D}_i = \mathcal{C}_i$ with $1 \leq i \leq k-2$ or a q'_{k-1}-subset $S'_{k-1} \subseteq S$ such that $P_r(S'_{k-1}) \subseteq \mathcal{D}_{k-1}$. In the former case, nothing is to be proved. In the latter case, let $\mathcal{D}'_{k-1} = \{A \in P_r(S'_{k-1}) : A \in \mathcal{C}_{k-1}\}$ and $\mathcal{D}'_k = \{A \in P_r(S'_{k-1}) : A \in \mathcal{C}_k\}$, then $\{\mathcal{D}'_{k-1}, \mathcal{D}'_k\}$ is a 2-coloring of $P_r(S'_{k-1})$. Since $|S'_{k-1}| = q'_{k-1} = R(q_{k-1}, q_k; r)$, there exists either a q_{k-1}-subset $S_{k-1} \subseteq S'_{k-1}$ such that $P_r(S_{k-1}) \subseteq \mathcal{D}'_{k-1}$ or a q_k-subset $S_k \subseteq S$ such that $P_r(S_k) \subseteq \mathcal{D}'_k$. Note that $\mathcal{D}'_{k-1} \subseteq \mathcal{P}_{k-1}$ and $\mathcal{D}'_k \subseteq \mathcal{P}_k$. Then there exists either a q_{k-1}-subset $S_{k-1} \subseteq S$ such that $P_r(S_{k-1}) \subseteq \mathcal{C}_{k-1}$ or a q_k-subset $S_k \subseteq S$ such that $P_r(S_k) \subseteq \mathcal{C}_k$.

Summarizing the above argument we obtain the recurrence relation:

$$R(q_1, q_2, \ldots, q_k; r) \leq R(q_1, q_2, \ldots, q_{k-2}, q'_{k-1}; r).$$

□

For positive integers $q_1, q_2, \ldots, q_k, r$ such that $q_1, q_2, \ldots, q_k \geq r$, $R(q_1, q_2, \ldots, q_k; r)$ are called *Ramsey numbers.* For any permutation $(\sigma_1, \sigma_2, \ldots, \sigma_k)$ of $(1, 2, \ldots, k)$, we have

$$R(q_{\sigma_1}, q_{\sigma_2}, \ldots, q_{\sigma_k}; r) = R(q_1, q_2, \ldots, q_k; r).$$

Proposition 3.3 (Pigeonhole Principle: The Strong Form). *If $r = 1$, then the Ramsey number $R(q_1, q_2, \ldots, q_k; 1)$ is the smallest integer n such that if the elements of an n-set are colored with k colors $c_1, c_2, \ldots, c_k$, then either there are q_1 elements of color c_1, or q_2 elements of color c_2, ..., or q_k elements of color c_k. Moreover,*

$$R(q_1, q_2, \ldots, q_k; 1) = q_1 + q_2 + \cdots + q_k - k + 1.$$

4 Applications of the Ramsey Theorem

Theorem 4.1. *For positive integers $q_1, \ldots, q_k$ there exists a smallest positive integer $R(q_1, \ldots, q_k; 2)$ such that, if $n \geq R(q_1, \ldots, q_k; 2)$ and for any edge coloring of the complete graph K_n with k colors $c_1, \ldots, c_k$, there is at least one i $(1 \leq i \leq k)$ such that K_n has a complete subgraph K_{q_i} of the color c_i.*

Proof. Each edge of K_n can be considered as a 2-subset of its vertices. □

In the book of Richard Brualdi, the Ramsey numbers $R(q_1, \ldots, q_k; 2)$ are denoted by $r(q_1, \ldots, q_k)$.

Theorem 4.2 (Erdös-Szekeres). *For any integer $k \geq 3$ there exists a smallest integer $N(k)$ such that, if $n \geq N(k)$ and for any n points on a plane having no three points through a line, then there is a convex k-gon whose vertices are among the given n points.*

Before proving the theorem we prove the following two lemmas first.

Lemma 4.3. *Among any 5 points on a plane, no three points through a line, 4 of them must form a convex quadrangle.*

Proof. Join every pair of two points by a segment to have a configuration of 10 segments. The circumference of the configuration forms a convex polygon. If the convex polygon is a pentagon or a quadrangle, the problem is solved. Otherwise the polygon must be a triangle, and the other two points must be located inside the triangle. Draw a straight line through the two points; two of the three vertices must be located in one side of the straight line. The two vertices on the same side and the two points inside the triangle form a quadrangle. □

Lemma 4.4. *Given $k \geq 4$ points on a plane, no 3 points through a line. If any 4 points are vertices of a convex quadrangle, then the k points are actually the vertices of a convex k-gon.*

Proof. Join every pair of two points by a segment to have a configuration of $k(k-1)/2$ segments. The circumference of the configuration forms a convex l-polygon. If $l = k$, the problem is solved. If $l < k$, there must be at least one point inside the l-polygon. Let $v_1, v_2, \ldots, v_l$ be the vertices of the convex l-polygon, and draw segments between v_1 and $v_3, v_4, \ldots, v_{l-1}$ respectively. The point inside the convex l-polygon must be located in one of the triangles $\triangle v_1v_2v_3, \triangle v_1v_3v_4, \ldots, \triangle v_1v_{l-1}v_l$. Obviously, the three vertices of the triangle with a given point inside together with the point do not form a convex quadrangle. This is a contradiction. □

Proof of Theorem 4.2. We apply the Ramsey theorem to prove Theorem 4.2. For $k = 3$, it is obviously true. Now for $k \geq 4$, if $n \geq R(k, 5; 4)$, we divide the 4-subsets of the n points into a class $\mathcal{C}$ of 4-subsets whose points are vertices of a convex quadrangle, and another class $\mathcal{D}$ of 4-subsets whose points are not vertices of any convex quadrangle. By the Ramsey theorem, there is either k points whose any 4-subset belongs to $\mathcal{C}$, or 5 points whose any 4-subset belongs to $\mathcal{D}$. In the formal case, the problem is solved by Lemma 4.4. In the latter case, it is impossible by Lemma 4.3. □

Theorem 4.5 (Schur). *For any positive integer k there exists a smallest integer N_k such that, if $n \geq N_k$ and for any k-coloring of $[1, n]$, there is a monochromatic sequence $x_1, x_2, \ldots, x_l$ ($l \geq 2$) such that $x_l = \sum_{i=1}^{l-1} x_i$.*

Proof. Let $n \geq R(l, \ldots, l; 2)$ and let $\{A_1, \ldots, A_k\}$ be a k-coloring of $[1, n]$. Let $\{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ be a k-coloring of $P_2([1, n])$ defined by

$$\{a, b\} \in \mathcal{C}_i \quad \text{if and only if} \quad |a - b| \in A_i, \quad \text{where} \quad 1 \leq i \leq k.$$

By the Ramsey theorem, there is one r ($1 \leq r \leq k$) and an l-subset $A = \{a_1, a_2, \ldots, a_l\} \subset [1, n]$ such that $P_2(A) \subseteq \mathcal{C}_r$. We may assume $a_1 < a_2 < \cdots < a_l$. Then

$$\{a_i, a_j\} \in \mathcal{C}_r \quad \text{and} \quad a_j - a_i \in A_r \quad \text{for all } i < j.$$

Let $x_i = a_{i+1} - a_i$ for $1 \leq i \leq l - 1$ and $x_l = a_l - a_1$. Then $x_i \in A_r$ for all $1 \leq i \leq l$ and $x_l = \sum_{i=1}^{l-1} x_i$. □

5 Van der Waerden Theorem

The Van der Waerden theorem states that for any k-coloring of the set $\mathbb{Z}$ of integers there always exists a monochromatic progression series.

Let $[a, b]$ denote the set of integers x such that $a \leq x \leq b$. Two tuples $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ of $[1, l]^m$ are said *l-equivalent*, written $(x_1, \ldots, x_m) \sim (y_1, \ldots, y_m)$, if all entries before the last l in each tuple are the same. For instance, for $l = 5$, $m = 4$,

$$(3, 5, 2, 5) \sim (3, 5, 2, 5), \quad (2, 4, 5, 2) \sim (2, 4, 5, 4), \quad (4, 3, 1, 4) \sim (2, 3, 2, 1), \quad (3, 5, 5, 1) \not\sim (3, 5, 2, 4).$$

Obviously, l-equivalence is an equivalence relation on $[0, l]^m$. All tuples of $[0, l-1]^m$ are l-equivalent.

Definition 5.1. For integers $l, m \geq 1$, let $A(l, m)$ denote the statement: For any integer $k \geq 1$ there exists a smallest integer $N(l, m, k)$ such that, if $n \geq N(l, m, k)$ and $[1, n]$ is k-colored, then there are integers $a, d_1, d_2, \ldots, d_m \geq 1$ such that $a + l\sum_{i=1}^{m} d_i \leq n$ and for each l-equivalence class E of $[0, l]^m$,

$$\left\{ a + \sum_{i=1}^{m} x_i d_i : (x_1, \ldots, x_m) \in E \right\}$$

is monochromatic (having the same color).

When $m = 1$, there are only two l-equivalence classes for $[0, l]^m$, i.e.,

$$\{0, 1, 2, \ldots, l-1\} \quad \text{and} \quad \{l\}.$$

The statement $A(l, 1)$ means that for any integer $k \geq 1$ there exists a smallest integer $N(l, 1, k)$ such that, if $n \geq N(l, 1, k)$ and $[1, n]$ is k-colored, then there are integers $a, d \geq 1$ such that $a + ld \leq n$ and the sequence, $a, a + d, a + 2d, \ldots, a + (l-1)d$, is monochromatic.

Theorem 5.2 (Graham-Rothschild). *The statement $A(l, m)$ is true for all integers $l, m \geq 1$.*

Proof. We proceed induction on l and m. For $l = m = 1$, the 1-equivalence classes of $[0, 1]$ are $\{0\}$ and $\{1\}$; the statement $A(1, 1)$ states that for any integer $k \geq 1$ there exists a smallest integer $N(1, 1; k)$ such that, if $n \geq N(1, 1; k)$ and $[1, n]$ is k-colored, then there are integers $a, d \geq 1$ such that $a + d \leq n$, and both $\{a\}$ and $\{a + d\}$ are monochromatic. This is obviously true and $N(1, 1; k) = 2$. We divide the induction argument into two statements:

(I) *If $A(l, m)$ is true for some $m \geq 1$ then $A(l, m + 1)$ is true.*

(II) *If $A(l, m)$ is true for all $m \geq 1$ then $A(l + 1, 1)$ is true.*

The induction goes as follows: The truth of $A(1, 1)$ implies the truth of $A(1, m)$ for all $m \geq 1$ by (I). Then by (II) the statement $A(2, 1)$ is true. Again by (I) the statement $A(2, m)$ is true for all $m \geq 1$. Continuing this procedure we obtain that $A(l, m)$ is true for all $l, m \geq 1$.

Proof of (I): Let the integer $k \geq 1$ be fixed. Since $A(l, m)$ is true, the integer $N(l, m, k)$ exists and set $p = N(l, m, k)$. Since $A(l, 1)$ is true, the integer $N(l, 1, k^p)$ exists and we set $q = N(l, 1, k^p)$, $N = pq$. Let $\phi : [1, N] \longrightarrow [1, k]$ be a k-coloring of $[1, N]$. Let $\psi : [1, q] \longrightarrow [1, k]^p$ be a k^p-coloring of $[1, q]$ defined by

$$\psi(i) = \Big(\phi((i-1)p + 1),\ \phi((i-1)p + 2),\ \ldots,\ \phi((i-1)p + p)\Big), \quad 1 \leq i \leq q. \tag{1}$$

Since $A(l, 1)$ is true, then for the k^p-coloring ψ of $[1, q]$ there are integers $a, d \geq 1$ such that

$$a + ld \leq q$$

and

$$\{a + xd : x = 0, 1, 2, \ldots, l-1\}$$

is monochromatic, i.e.,

$$\psi(a + xd) = \text{constant}, \quad x = 0, 1, 2, \ldots, l-1. \tag{2}$$

Note that $[(a-1)p + 1, ap] \subseteq [1, pq]$ because $a \leq q$. Since $A(l, m)$ is true, then when ϕ is restricted to the p-set $[(a-1)p + 1, ap]$ there are integers $b, d_1, d_2, \ldots, d_m \geq 1$ such that

$$(a-1)p + 1 \leq b, \quad b + l\sum_{i=1}^{m} d_i \leq ap,$$

and for each l-equivalence class E of $[0, l]^m$,

$$\left\{ b + \sum_{i=1}^{m} x_i d_i : (x_1, \ldots, x_m) \in E \right\}$$

is monochromatic, i.e.,

$$\phi\left(b+\sum_{i=1}^{m} x_i d_i\right) = \text{constant}, \quad (x_1,\dots,x_m)\in E. \tag{3}$$

Recall that our job is to prove that $A(l,m+1)$ is true. For the k-coloring ϕ of $[1,N]$, we have had the integers

$$b, d_1, d_2, \dots, d_{m+1} \geq 1, \quad \text{where} \quad d_{m+1} = dp.$$

Since $b + l\sum_{i=1}^{m} d_i \leq ap$ and $a + ld \leq q$, we have

$$b + l\sum_{i=1}^{m+1} d_i \leq ap + ldp = (a+dl)p \leq pq = N.$$

Now for any two l-equivalent tuples $(x_1,\dots,x_{m+1})$ and $(y_1,\dots,y_{m+1})$ of $[0,l]^{m+1}$, consider the numbers

$$\alpha = b + \textstyle\sum_{i=1}^{m+1} x_i d_i, \quad \beta = b + \sum_{i=1}^{m+1} y_i d_i,$$

$$\alpha_0 = b + \textstyle\sum_{i=1}^{m} x_i d_i, \quad \beta_0 = b + \sum_{i=1}^{m} y_i d_i.$$

Notice that our job is to show that α and β have the same color, i.e., $\phi(\alpha) = \phi(\beta)$. We divide the job into three cases:

Case 1. $x_{m+1} = y_{m+1} = l$. Then $x_i = y_i$ for all $1 \leq i \leq m$. Thus $\alpha = \beta$, and obviously, $\phi(\alpha) = \phi(\beta)$.

Case 2. $x_{m+1} = l$ and $y_{m+1} \leq l-1$, or, $x_{m+1} \leq l-1$ and $y_{m+1} = l$. This implies that $(x_1,\dots,x_{m+1})$ and $(y_1,\dots,y_{m+1})$ are not l-equivalent. This is a contradiction.

Case 3. $x_{m+1}, y_{m+1} \in [0,l-1]$. Then $(x_1,\dots,x_m)$ and $(y_1,\dots,y_m)$ are l-equivalent. It follows from (2) that $\psi(a) = \psi(a+x_{m+1}d)$, and by definition (1) of ψ, the corresponding coordinates of $\psi(a)$ and $\psi(a+x_{m+1}d)$ are equal, i.e.,

$$\phi\Big((a-1)p+i\Big) = \phi\Big((a+x_{m+1}d-1)p+i\Big), \quad i = 1,2,\dots,p.$$

Since $(a-1)p+1 \leq b \leq \alpha_0 \leq b + l\sum_{i=1}^{m} d_i \leq ap = (a-1)p+p$, there exists $j \in [1,p]$ such that $\alpha_0 = (a-1)p+j$. We then have

$$\alpha = \alpha_0 + x_{m+1}dp = (a-1)p + j + x_{m+1}dp = (a+x_{m+1}d-1)p + j.$$

Thus

$$\phi(\alpha) = \phi\Big((a+x_{m+1}d-1)p+j\Big) = \phi\Big((a-1)p+j\Big) = \phi(\alpha_0).$$

Similarly, $\phi(\beta) = \phi(\beta_0)$. Since $(x_1,\dots,x_m)$ and $(y_1,\dots,y_m)$ are l-equivalent, it follows from (3) that $\phi(\alpha_0) = \phi(\beta_0)$. Therefore $\phi(\alpha) = \phi(\beta)$. This means that $A(l,m+1)$ is true and

$$N(l,m+1,k) \leq N(l,m,k)\cdot N\left(l,1,k^{N(l,m,k)}\right).$$

Proof of (II): Fix an integer $k \geq 1$. Since $A(l,m)$ is true for all $m \geq 1$, the statement $A(l,k)$ is true and $N(l,k,k)$ exists. Let $N = 2N(l,k,k)$ and let ϕ be a k-coloring of $[1,N]$. Notice that the restriction of ϕ on $[1,N(l,k,k)]$ is a k-coloring. Then there are integers $a, d_1, d_2, \dots, d_k \geq 1$ such that

$$a + l\sum_{i=1}^{k} d_i \leq N(l,k,k),$$

and for l-equivalent tuples $(x_1,\dots,x_k), (y_1,\dots,y_k) \in [0,l]^k$,

$$\phi\left(a+\sum_{i=1}^{k} x_i d_i\right) = \phi\left(a+\sum_{i=1}^{k} y_i d_i\right).$$

Consider the $k+1$ tuples (none of them are l-equivalent)

$$(0,0,\dots,0),\ (l,0,\dots,0),\ (l,l,\dots,0),\ \dots,\ (l,l,\dots,l)$$

of $[0,l]^k$ to have $k+1$ distinct integers

$$a,\ a+ld_1,\ a+l(d_1+d_2),\ \ldots,\ a+l(d_1+d_2+\cdots+d_k).$$

At least two of them, say $a+l(d_1+\cdots+d_\lambda)$ and $a+l(d_1+\cdots+d_\mu)$ with $\lambda<\mu$, must have the same color, i.e.,

$$\phi\left(a+l\sum_{i=1}^{\lambda}d_i\right)=\phi\left(a+l\sum_{i=1}^{\mu}d_i\right). \tag{4}$$

For any $x\in[0,l-1]$, the two tuples

$$(\underbrace{l,\ldots,l}_{\lambda},\ \underbrace{x,\ldots,x}_{\mu-\lambda},\ 0,\ldots,0) \quad \text{and} \quad (\underbrace{l,\ldots,l}_{\lambda},\ \underbrace{0,\ldots,0}_{\mu-\lambda},\ 0,\ldots,0)$$

of $[0,l]^k$ are l-equivalent. Thus the numbers $a+l\sum_{i=1}^{\lambda}d_i+x\sum_{i=\lambda+1}^{\mu}d_i$ for $x\in[0,l-1]$ have the same color by ϕ, i.e.,

$$\phi\left(a+l\sum_{i=1}^{\lambda}d_i+x\sum_{i=\lambda+1}^{\mu}d_i\right)=\text{constant},\quad x=0,1,2,\ldots,l-1.$$

Combining this with (4) we have

$$\phi\left(a+l\sum_{i=1}^{\lambda}d_i+x\sum_{i=\lambda+1}^{\mu}d_i\right)=\text{constant},\quad x=0,1,2,\ldots,l-1,l.$$

Recall that our job is to prove the truth of $A(l+1,1)$. Let $b=a+l\sum_{i=1}^{\lambda}d_i$ and $d=\sum_{i=\lambda+1}^{\mu}d_i$. Then we have had the integers $b,d\geq 1$ such that

$$b+(l+1)d=a+l\sum_{i=1}^{\lambda}d_i+(l+1)\sum_{i=\lambda+1}^{\mu}d_i=a+l\sum_{i=1}^{\mu}d_i+\sum_{i=\lambda+1}^{\mu}d_i\leq N(l,k,k)+N(l,k,k)=N$$

and for the k-coloring ϕ of $[1,N]$, the l-equivalence class $\{0,1,2,\ldots,l\}$ of $[0,l+1]^1$ have the same color, i.e.,

$$\phi(b+xd)=\text{constant},\quad x=0,1,2,\ldots,l.$$

This means that the statement $A(l+1,1)$ is true. □

The truth of $A(l,m)$ for $m=1$ is called the Van der Waerden theorem.

Corollary 5.3 (Van der Waerden Theorem). *For any positive integers k and l there exists a smallest integer $N(l,k)$ such that, if $n\geq N(l,k)$ and $[1,n]$ is k-colored, then there is a monochromatic arithmetic sequence of length l in $[1,n]$.*

Supplementary Exercises

1. For the game of Nim, let us restrict that each player can move one or two coins. Find the winning strategy for each player.

2. Let n be a positive integer. In the game of Nim let us restrict that each player can move only $i\in\{1,2,\ldots,n\}$ coins each time from one heap. Find the winning strategy for each player.

3. Given $m(m-1)^2+1$ integral points on a plane, where m is odd. Show that there exists m points whose center is also an integral point.