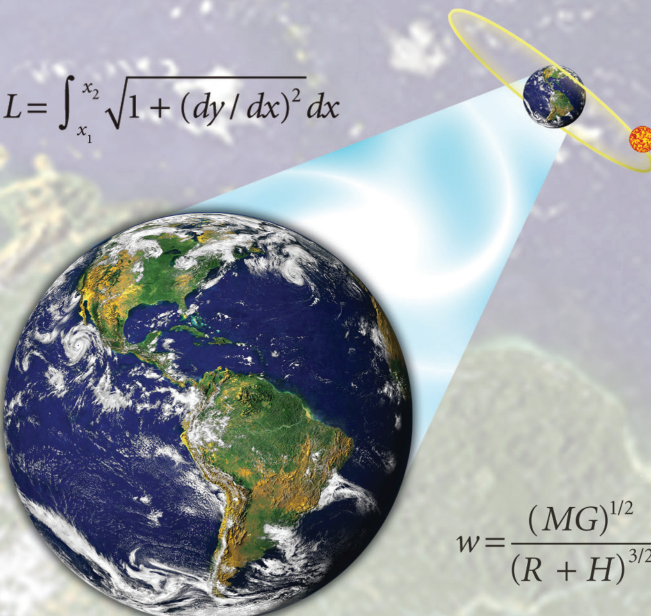
Advances in Applied Mathematics

Pocket Book of Integrals and Mathematical Formulas

5th Edition

Ronald J. Tallarida

$$L = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\, dx$$



$$w = \frac{(MG)^{1/2}}{(R + H)^{3/2}}$$

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Pocket Book of Integrals and Mathematical Formulas

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Advances in Applied Mathematics

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Pocket Book of Integrals and Mathematical Formulas, 5th Edition
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Advances in Applied Mathematics

Pocket Book of Integrals and Mathematical Formulas

5th Edition

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Preface to the Fifth Edition

Pocket Book of Integrals and Mathematical Formulas, Fifth Edition, a revision of a very successful pocket book, provides a handy reference source for students, engineers, scientists, and others seeking essential mathematical formulas, concepts, and definitions. Topics range from precalculus to vector analysis and from Fourier series to statistics. The previous editions added material on business and financial mathematics that has been well received since it provided information on progressions, especially geometric progressions, which form the basis for many formulas related to annuities, growth of funds, and interest payments. That material has been retained. The fourth edition also retained topics in statistics, nonlinear regression, and an expanded discussion in the differential equations section by adding a treatment of Runge Kutta methods and a new application to drug kinetics. This edition includes several classic calculus applications. These gems of calculus illustrate its power and practical use. Readers of the previous editions have enjoyed special topics that included the derivation leading to the geostationary satellite orbit, a timely topic, as well as an interesting set of topics in number theory whose inclusion was motivated by the recent proof of Fermat's last theorem. An interesting Fermat offshoot, namely, "near misses," is included, thereby extending the range of interest of this popular book. The table of integrals, which contains the most useful forms, has been reformatted and has been rechecked for accuracy. Although we strive to keep the book size small, we have enlarged the type slightly without sacrificing special topics. These include Fourier series, Laplace and Z-transforms, vector analysis, complex numbers, orthogonal polynomials and infinite series. Many other handbooks go too far in their attempts, essentially trying to mimic larger comprehensive texts. The result is a reference less detailed than the full texts and too big to be conveniently portable so that users would not carry them. Through a careful selection of topics and detail, *Pocket Book of Integrals and Mathematical Formulas* truly meets the needs of students and professionals in being a convenient, compact, and usable resource that also provides worked examples where most necessary. The book is portable, comprehensive, and easy to use.

Ronald J. Tallarida
Philadelphia, Pennsylvania

Preface to the Fourth Edition

As in the previous works, this new edition preserves the content, size, and convenience of this portable reference source for students and workers who use mathematics, while introducing much new material. New in this fourth edition is an expanded chapter on series that now includes many fascinating properties of the natural numbers that follow from number theory, a field that has attracted much new interest since the recent proof of Fermat's last theorem. While the proofs of many of these theorems are deep, and in some cases still lacking, all the number theory topics included here are easy to describe and form a bridge between arithmetic and higher mathematics. The fourth edition also includes new applications such as the geostationary satellite orbit, drug kinetics (as an application of differential equations), and an expanded statistics section that now discusses the normal approximation of the binomial distribution as well as a treatment of nonlinear regression. The widespread use of computers now makes the latter topic amenable to all students, and thus all users of the *Pocket Book of Integrals* can benefit from the concise summary of this topic. The chapter on financial mathematics, introduced in the third edition, has proved successful and is retained without change in this edition, whereas the Table of Integrals has been reformatted for easier usage. This change in format also allowed the inclusion of all the new topics without the necessity of increasing the physical size of the book, thereby keeping its wide appeal as a true, handy pocket book that students and professionals will find useful in their mathematical pursuits.

Ronald J. Tallarida
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Preface to the Third Edition

This new edition has been enlarged to contain all the material in the second edition, an expanded chapter on statistics that now includes sample size estimations for means and proportions, and a totally new chapter on financial mathematics. In adding this new chapter we have also included a number of tables that aid in performing the calculations on annuities, true interest, amortization schedules, compound interest, systematic withdrawals from interest accounts, etc. The treatment and style of this material reflect the rest of the book, i.e., clear explanations of concepts, relevant formulas, and worked examples. The new financial material includes analyses not readily found in other sources, such as the effect of lump sum payments on amortization schedules and a novel "in-out formula" that calculates current regular deposits to savings in order to allow the start of systematic withdrawals of a specified amount at a later date. While many engineers, mathematicians, and scientists have found much use for this handy pocket book, this new edition extends its usage to them and to the many business persons and individuals who make financial calculations.

Ronald J. Tallarida
Philadelphia, Pennsylvania

Preface to the Second Edition

This second edition has been enlarged by the addition of several new topics while preserving its convenient pocket size. New in this edition are the following topics: z-transforms, orthogonal polynomials, Bessel functions, probability and Bayes' rule, a summary of the most common probability distributions (binomial, Poisson, normal, t, Chi square, and F), the error function, and several topics in multivariable calculus that include surface area and volume, the ideal gas laws, and a table of centroids of common plane shapes. A list of physical constants has also been added to this edition.

I am grateful for many valuable suggestions from users of the first edition, especially Lt. Col. W. E. Skeith and his colleagues at the U.S. Air Force Academy.

Ronald J. Tallarida
Philadelphia, Pennsylvania

Preface to the First Edition

The material of this book has been compiled so that it may serve the needs of students and teachers as well as professional workers who use mathematics. The contents and size make it especially convenient and portable. The widespread availability and low price of scientific calculators have greatly reduced the need for many numerical tables (e.g., logarithms, trigonometric functions, powers, etc.) that make most handbooks bulky. However, most calculators do not give integrals, derivatives, series, and other mathematical formulas and figures that are often needed. Accordingly, this book contains that information in addition to a comprehensive table of integrals. A section on statistics and the accompanying tables, also not readily provided by calculators, have also been included.

The size of the book is comparable to that of many calculators, and it is really very much a companion to the calculator and the computer as a source of information for writing one's own programs. To facilitate such use, the author and the publisher have worked together to make the format attractive and clear. Yet, an important requirement in a book of this kind is accuracy.

Toward that end we have checked each item against at least two independent sources.

Students and professionals alike will find this book a valuable supplement to standard textbooks, a source for review, and a handy reference for many years.

Ronald J. Tallarida
Philadelphia, Pennsylvania

Author

Ronald J. Tallarida holds BS and MS degrees in physics/mathematics and a PhD in pharmacology. His primary appointment is professor of pharmacology at Temple University School of Medicine, Philadelphia, Pennsylvania. For more than 30 years, he also served as an adjunct professor of Biomedical Engineering at Drexel University in Philadelphia where he received the Lindback Award for Distinguished Teaching of mathematics. As an author and researcher, he has published more than 290 works that include eight books, has been the recipient of research grants from NIH, and has served as a consultant to both industry and government agencies. His main research interests are in the areas of mathematical modeling of biological systems, feedback control, and the action of drugs and drug combinations.

Greek Letters

α	Α	Alpha
β	Β	Beta
γ	Γ	Gamma
δ	Δ	Delta
ε	Ε	Epsilon
ζ	Ζ	Zeta
η	Η	Eta
θ	Θ	Theta
ι	Ι	Iota
κ	Κ	Kappa
λ	Λ	Lambda
μ	Μ	Mu
ν	Ν	Nu
ξ	Ξ	Xi
ο	Ο	Omicron
π	Π	Pi
ρ	Ρ	Rho
σ	Σ	Sigma
τ	Τ	Tau
υ	ϒ	Upsilon
ϕ	Φ	Phi
χ	Χ	Chi
ψ	Ψ	Psi
ω	Ω	Omega

The Numbers π and e

π	=	3.14159	26535	89793
e	=	2.71828	18284	59045
$\log_{10} e$	=	0.43429	44819	03252
$\log_e 10$	=	2.30258	50929	94046

Prime Numbers

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
…				…					…

Important Numbers in Science (Physical Constants)

Avogadro constant (N_A)	6.02×10^{26} kmole^{-1}
Boltzmann constant (k)	1.38×10^{-23} J·°K^{-1}
Electron charge (e)	1.602×10^{-19} C
Electron, charge/mass (e/m_e)	1.760×10^{11} C · kg^{-1}
Electron rest mass (m_e)	9.11×10^{-31} kg (0.511 MeV)
Faraday constant (F)	9.65×10^{4} C · mole^{-1}
Gas constant (R)	8.31×10^{3} J·°K^{-1} kmole^{-1}
Gas (ideal) normal volume (V_o)	22.4 m^3 · kmole^{-1}
Gravitational constant (G)	6.67×10^{-11} N · m^2 · kg^{-2}
Hydrogen atom (rest mass) (m_H)	1.673×10^{-27} kg (938.8 MeV)
Neutron (rest mass) (m_n)	1.675×10^{-27} kg (939.6 MeV)
Planck constant (h)	6.63×10^{-34} J · s
Proton (rest mass) (m_p)	1.673×10^{-27} kg (938.3 MeV)
Speed of light (c)	3.00×10^{8} m · s^{-1}

1

Elementary Algebra and Geometry

1.1 Fundamental Properties (Real Numbers)

$a + b = b + a$	Commutative Law for Addition
$(a + b) + c = a + (b + c)$	Associative Law for Addition
$a + 0 = 0 + a$	Identity Law for Addition
$a + (-a) = (-a) + a = 0$	Inverse Law for Addition
$a(bc) = (ab)c$	Associative Law for Multiplication
$a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1, \quad a \neq 0$	Inverse Law for Multiplication
$(a)(1) = (1)(a) = a$	Identity Law for Multiplication

$ab = ba$	Commutative Law for Multiplication
$a(b + c) = ab + ac$	Distributive Law

Division by zero is not defined.

1.2 Exponents

For integers m and n,

$$a^n a^m = a^{n+m}$$

$$a^n / a^m = a^{n-m}$$

$$(a^n)^m = a^{nm}$$

$$(ab)^m = a^m b^m$$

$$(a/b)^m = a^m / b^m$$

1.3 Fractional Exponents

$$a^{p/q} = (a^{1/q})^p$$

where $a^{1/q}$ is the positive qth root of a if $a > 0$ and the negative qth root of a if a is negative and q is odd. Accordingly, the five rules of exponents given above (for integers) are also valid if m and n are fractions, provided a and b are positive.

1.4 Irrational Exponents

If an exponent is irrational, e.g., $\sqrt{2}$, the quantity, such as $a^{\sqrt{2}}$, is the limit of the sequence $a^{1.4}$, $a^{1.41}$, $a^{1.414}$, ….

- *Operations with Zero*

$$0^m = 0; \quad a^0 = 1$$

1.5 Logarithms

If x, y, and b are positive and $b \neq 1$,

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b x^p = p\log_b x$$

$$\log_b(1/x) = -\log_b x$$

$$\log_b b = 1$$

$$\log_b 1 = 0 \quad \textit{Note}: b^{\log_b x} = x.$$

- *Change of Base* $(a \neq 1)$

$$\log_b x = \log_a x \log_b a$$

1.6 Factorials

The factorial of a positive integer n is the product of all the positive integers less than or equal to the integer n and is denoted $n!$ Thus,

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n.$$

Factorial 0 is defined 0! = 1.

- *Stirling's Approximation*

$$\lim_{n \to \infty} (n/e)^n \sqrt{2\pi n} = n!$$

(See also Section 9.2.)

1.7 Binomial Theorem

For positive integer n,

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n.$$

1.8 Factors and Expansion

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a^2 - b^2) = (a-b)(a+b)$$

$$(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$$

$$(a^3 + b^3) = (a+b)(a^2 - ab + b^2)$$

1.9 Progression

An *arithmetic progression* is a sequence in which the difference between any term and the preceding term is a constant (d):

$$a, a+d, a+2d, \ldots, a+(n-1)d.$$

If the last term is denoted $l[= a + (n-1)d]$, then the sum is

$$s = \frac{n}{2}(a+l).$$

A *geometric progression* is a sequence in which the ratio of any term to the preceding terms is a constant r. Thus, for n terms,

$$a, ar, ar^2, \ldots, ar^{n-1}$$

The sum is

$$S = \frac{a - ar^n}{1 - r}$$

1.10 Complex Numbers

A complex number is an ordered pair of real numbers (a, b).

Equality: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Addition: $(a, b) + (c, d) = (a + c, b + d)$

Multiplication: $(a, b)(c, d) = (ac - bd, ad + bc)$

The first element of (a, b) is called the *real* part; the second, the *imaginary* part. An alternate notation for (a, b) is $a + bi$, where $i^2 = (-1, 0)$, and $i(0, 1)$ or $0 + 1i$ is written for this complex number as a convenience. With this understanding, i behaves as a number, i.e., $(2 - 3i)(4 + i) = 8 - 12i + 2i - 3i^2 = 11 - 10i$. The conjugate of $a + bi$ is $a - bi$, and the product of a complex number and its conjugate is $a^2 + b^2$.

Thus, *quotients* are computed by multiplying numerator and denominator by the conjugate of the denominator, as illustrated below:

$$\frac{2+3i}{4+2i} = \frac{(4-2i)(2+3i)}{(4-2i)(4+2i)} = \frac{14+8i}{20} = \frac{7+4i}{10}$$

1.11 Polar Form

The complex number $x + iy$ may be represented by a plane vector with components x and y:

$$x + iy = r\left(\cos\theta + i\sin\theta\right)$$

(see Figure 1.1). Then, given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, the product and quotient are:

Product: $z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Quotient: $z_1/z_2 = (r_1/r_2)[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Powers:
$$z^n = \left[r\left(\cos\theta + i\sin\theta\right)\right]^n = r^n\left[\cos n\theta + i\sin n\theta\right]$$

Roots:
$$z^{1/n} = \left[r\left(\cos\theta + i\sin\theta\right)\right]^{1/n} = r^{1/n}\left[\cos\frac{\theta + k.360}{n} + i\sin\frac{\theta + k.360}{n}\right],$$
$$k = 0, 1, 2, \ldots, n-1$$

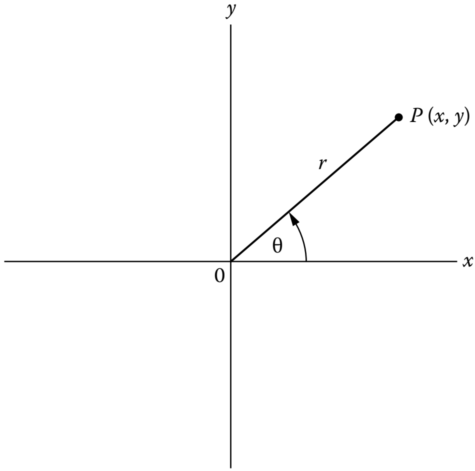


FIGURE 1.1
Polar form of complex number.

1.12 Permutations

A permutation is an ordered arrangement (sequence) of all or part of a set of objects. The number of permutations of *n* objects taken *r* at a time is

$$p(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

A permutation of positive integers is even or odd if the total number of inversions is an even integer or an odd integer, respectively. Inversions are counted relative to each integer j in the permutation by counting the number of integers that follow j and are less than j. These are summed to give the total number of inversions. For example, the permutation 4132 has four inversions: three relative to 4 and one relative to 3. This permutation is therefore even.

1.13 Combinations

A combination is a selection of one or more objects from among a set of objects regardless of order. The number of combinations of n different objects taken r at a time is

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

1.14 Algebraic Equations

- *Quadratic*
  If $ax^2 + bx + c = 0$, and $a \neq 0$, then roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- *Cubic*
  To solve $x^3 + bx^2 + cx + d = 0$, let $x = y - b/3$. Then the *reduced cubic* is obtained:

$$y^3 + py + q = 0$$

  where $p = c - (1/3)b^2$ and $q = d - (1/3)bc + (2/27)b^3$. Solutions of the original cubic are then in terms of the reduced cubic roots y_1, y_2, y_3:

$$x_1 = y_1 - (1/3)b \quad x_2 = y_2 - (1/3)b$$

$$x_3 = y_3 - (1/3)b$$

The three roots of the reduced cubic are

$$y_1 = (A)^{1/3} + (B)^{1/3}$$

$$y_2 = W(A)^{1/3} + W^2(B)^{1/3}$$

$$y_3 = W^2(A)^{1/3} + W(B)^{1/3}$$

where

$$A = -\frac{1}{2}q + \sqrt{(1/27)p^3 + \frac{1}{4}q^2},$$

$$B = -\frac{1}{2}q - \sqrt{(1/27)p^3 + \frac{1}{4}q^2},$$

$$W = \frac{-1 + i\sqrt{3}}{2}, \quad W^2 = \frac{-1 - i\sqrt{3}}{2}.$$

When $(1/27)p^3 + (1/4)p^2$ is negative, A is complex; in this case, A should be expressed in trigonometric form: $A = r(\cos\theta + i\sin\theta)$, where θ is a first or second quadrant angle, as q is negative or positive. The three roots of the reduced cubic are

$$y_1 = 2(r)^{1/3}\cos(\theta/3)$$

$$y_2 = 2(r)^{1/3}\cos\left(\frac{\theta}{3} + 120°\right)$$

$$y = 2(r)^{1/3}\cos\left(\frac{\theta}{3} + 240°\right)$$

1.15 Geometry

Figures 1.2 through 1.12 are a collection of common geometric figures. Area (A), volume (V), and other measurable features are indicated.

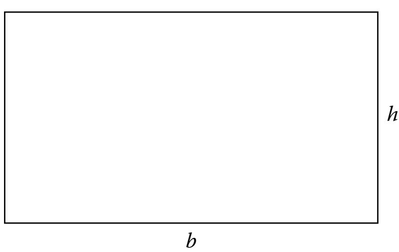


FIGURE 1.2
Rectangle. $A = bh$.

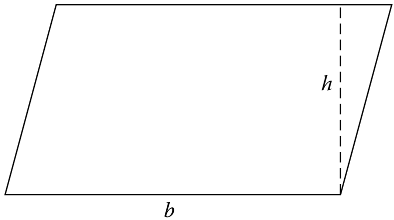


FIGURE 1.3
Parallelogram. $A = bh$.

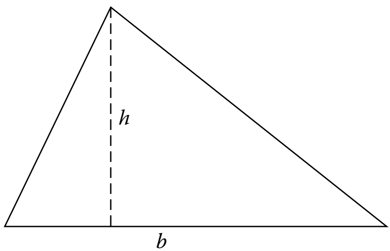


FIGURE 1.4
Triangle. $A = \frac{1}{2}bh$.

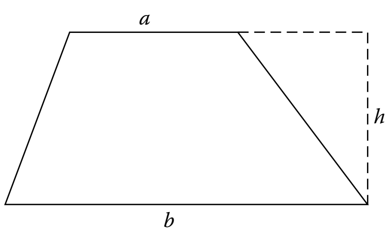


FIGURE 1.5
Trapezoid. $A = \frac{1}{2}(a + b)h$.

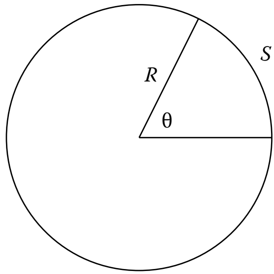


FIGURE 1.6
Circle. $A = \pi R^2$; circumference = $2\pi R$; arc length $S = R\theta$ (θ in radians).

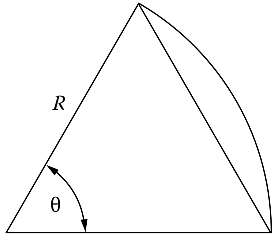


FIGURE 1.7
Sector of circle. $A_{\text{sector}} = \frac{1}{2}R^2\theta$; $A_{\text{segment}} = \frac{1}{2}R^2(\theta - \sin\theta)$.

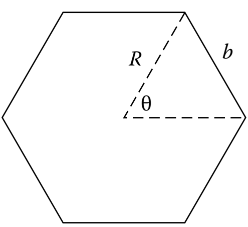


FIGURE 1.8
Regular polygon of n sides. $A = \frac{n}{4}b^2 \operatorname{ctn}\frac{\pi}{n}$; $R = \frac{b}{2}\csc\frac{\pi}{n}$.

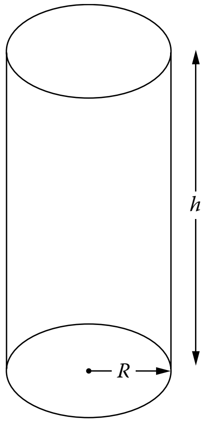


FIGURE 1.9
Right circular cylinder. $V = \pi R^2 h$; lateral surface area = $2\pi Rh$.

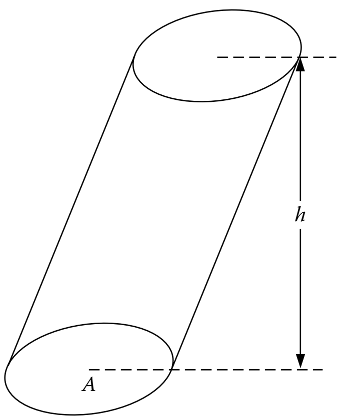


FIGURE 1.10
Cylinder (or prism) with parallel bases. $V = Ah$.

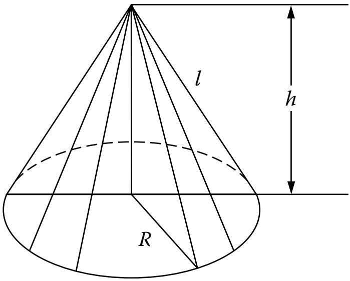


FIGURE 1.11
Right circular cone. $V = \frac{1}{3}\pi R^2 h$; lateral surface area = $\pi R l = \pi R\sqrt{R^2 + h^2}$.

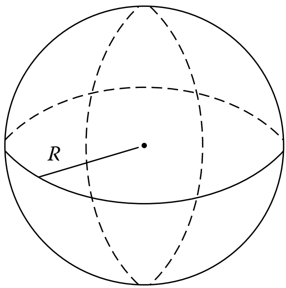


FIGURE 1.12
Sphere. $V = \frac{4}{3}\pi R^3$; surface area = $4\pi R^2$.

1.16 Pythagorean Theorem

For any right triangle with perpendicular sides a and b, the hypotenuse c is related by the formula

$$c^2 = a^2 + b^2$$

This famous result is central to many geometric relations, e.g., see Section 4.2.

2

Determinants, Matrices, and Linear Systems of Equations

2.1 Determinants

Definition. The square array (matrix) A, with n rows and n columns, has associated with it the determinant

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},$$

a number equal to

$$\sum (\pm) a_{1i} a_{2j} a_{3k} \ldots a_{nl}$$

where $i, j, k, \ldots, l$ is a permutation of the n integers $1, 2, 3, \ldots, n$ in some order. The sign is plus if the permutation is *even* and is minus if the permutation is *odd* (see Section 1.12). The 2×2 determinant

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

has the value $a_{11}a_{22}-a_{12}a_{21}$ since the permutation (1, 2) is even and (2, 1) is odd. For 3×3 determinants, permutations are as follows:

1,	2,	3	even
1,	3,	2	odd
2,	1,	3	odd
2,	3,	1	even
3,	1,	2	even
3,	2,	1	odd

Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{Bmatrix} +a_{11} & \cdot & a_{22} & \cdot & a_{33} \\ -a_{11} & \cdot & a_{23} & \cdot & a_{32} \\ -a_{12} & \cdot & a_{21} & \cdot & a_{33} \\ +a_{12} & \cdot & a_{23} & \cdot & a_{31} \\ +a_{13} & \cdot & a_{21} & \cdot & a_{32} \\ -a_{13} & \cdot & a_{22} & \cdot & a_{31} \end{Bmatrix}$$

A determinant of order n is seen to be the sum of $n!$ signed products.

2.2 Evaluation by Cofactors

Each element a_{ij} has a determinant of order $(n - 1)$ called a *minor* (M_{ij}) obtained by suppressing all elements in row i and column j. For example, the minor of element a_{22} in the 3×3 determinant above is

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

The cofactor of element a_{ij}, denoted A_{ij}, is defined as $\pm M_{ij}$, where the sign is determined from i and j:

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

The value of the $n \times n$ determinant equals the sum of products of elements of any row (or column) and their respective cofactors. Thus, for the 3×3 determinant,

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \text{ (first row)}$$

or

$$= a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \text{ (first column)}$$

etc.

2.3 Properties of Determinants

a. If the corresponding columns and rows of A are interchanged, det A is unchanged.

b. If any two rows (or columns) are interchanged, the sign of det A changes.

c. If any two rows (or columns) are identical, det $A = 0$.

d. If A is triangular (all elements above the main diagonal equal to zero), $A = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$:

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

e. If to each element of a row or column there is added C times the corresponding element in another row (or column), the value of the determinant is unchanged.

2.4 Matrices

Definition. A matrix is a rectangular array of numbers and is represented by a symbol A or $[a_{ij}]$:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

The numbers a_{ij} are termed *elements* of the matrix; subscripts i and j identify the element, as the number is row i and column j. The order of the matrix is $m \times n$ ("m by n"). When $m = n$, the matrix is square and is said to be of order n. For a square matrix of order n the elements $a_{11}, a_{22}, \ldots, a_{nn}$ constitute the main diagonal.

2.5 Operations

Addition: Matrices A and B of the same order may be added by adding corresponding elements, i.e., $A + B = [(a_{ij} + b_{ij})]$.

Scalar multiplication: If $A = [a_{ij}]$ and c is a constant (scalar), then $cA = [ca_{ij}]$, that is, every element of A is multiplied by c. In particular, $(-1)A = -A = [-a_{ij}]$ and $A + (-A) = 0$, a matrix with all elements equal to zero.

Multiplication of matrices: Matrices A and B may be multiplied only when they are conformable, which means that the number of columns of A equals the number of rows of B. Thus, if A is $m \times k$ and B is $k \times n$, then the product $C = AB$ exists as an $m \times n$ matrix with elements c_{ij} equal to the sum of products of elements in row i of A and corresponding elements of column j of B:

$$c_{ij} = \sum_{l=1}^{k} a_{il} b_{lj}$$

For example, if

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & \cdots & a_{mk} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{k1} & b_{k2} & \cdots & b_{kn} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

then element c_{21} is the sum of products $a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2k}b_{k1}$.

2.6 Properties

$$A + B = B + A$$

$$A + (B + C) = (A + B) + C$$

$$(c_1 + c_2)A = c_1A + c_2A$$

$$c(A + B) = cA + cB$$

$$c_1(c_2A) = (c_1c_2)A$$

$$(AB)(C) = A(BC)$$

$$(A + B)(C) = AC + BC$$

$$AB \neq BA \text{(in general)}$$

2.7 Transpose

If A is an $n \times m$ matrix, the matrix of order $m \times n$ obtained by interchanging the rows and columns of A is called the *transpose* and is denoted A^T. The following are properties of A, B, and their respective transposes:

$$(A^T)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(cA)^T = cA^T$$

$$(AB)^T = B^T A^T$$

A *symmetric* matrix is a square matrix A with the property $A = A^T$.

2.8 Identity Matrix

A square matrix in which each element of the main diagonal is the same constant a and all other elements zero is called a *scalar* matrix.

$$\begin{bmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \cdots & a \end{bmatrix}$$

When a scalar matrix multiplies a conformable second matrix A, the product is aA, that is, the same as multiplying A by a scalar a. A scalar matrix with diagonal elements 1 is called the *identity,* or *unit* matrix, and is denoted I. Thus, for any nth-order matrix A, the identity matrix of order n has the property

$$AI = IA = A$$

2.9 Adjoint

If A is an n-order square matrix and A_{ij} the cofactor of element a_{ij}, the transpose of $[A_{ij}]$ is called the *adjoint* of A:

$$adj\, A = [A_{ij}]^T$$

2.10 Inverse Matrix

Given a square matrix A of order n, if there exists a matrix B such that $AB = BA = I$, then B is called the *inverse* of A. The inverse is denoted A^{-1}. A necessary and sufficient condition that the square matrix A have an inverse is $\det A \neq 0$.

Such a matrix is called *nonsingular*; its inverse is unique and is given by

$$A^{-1} = \frac{adj\,A}{\det A}$$

Thus, to form the inverse of the nonsingular matrix, A, form the adjoint of A and divide each element of the adjoint by det A. For example,

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 4 & 5 & 6 \end{bmatrix} \text{ has matrix of cofactors}$$

$$\begin{bmatrix} -11 & -14 & 19 \\ 10 & -2 & -5 \\ 2 & 5 & -1 \end{bmatrix},$$

$$\text{adjoint} = \begin{bmatrix} -11 & 10 & 2 \\ -14 & -2 & 5 \\ 19 & -5 & -1 \end{bmatrix} \text{ and determinant } 27.$$

Therefore,

$$A^{-1} = \begin{bmatrix} \frac{-11}{27} & \frac{10}{27} & \frac{2}{27} \\ \frac{-14}{27} & \frac{-2}{27} & \frac{5}{27} \\ \frac{19}{27} & \frac{-5}{27} & \frac{-1}{27} \end{bmatrix}.$$

2.11 Systems of Linear Equations

Given the system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & +\cdots+ & a_{nn}x_n & = & b_n \end{array}$$

a unique solution exists if det $A \neq 0$, where A is the $n \times n$ matrix of coefficients $[a_{ij}]$.

- *Solution by Determinants (Cramer's Rule)*

$$x_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & & \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & & a_{nn} \end{vmatrix} \div \det A$$

$$x_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & & \cdots \\ \vdots & \vdots & & & \\ a_{n1} & b_n & a_{n3} & & a_{nn} \end{vmatrix} \div \det A$$

$\vdots$

$$x_k = \frac{\det A_k}{\det A},$$

where A_k is the matrix obtained from A by replacing the kth column of A by the column of b's.

2.12 Matrix Solution

The linear system may be written in matrix form $AX = B$ where A is the matrix of coefficients $[a_{ij}]$ and X and B are

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If a unique solution exists, $\det A \neq 0$; hence, A^{-1} exists and

$$X = A^{-1}B.$$

3

Trigonometry

3.1 Triangles

In any triangle (in a plane) with sides a, b, and c and corresponding opposite angles A, B, C,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad \text{(Law of Sines)}$$

$$a^2 = b^2 + c^2 - 2cb\cos A. \quad \text{(Law of Cosines)}$$

$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}. \quad \text{(Law of Tangents)}$$

$$\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \text{where } s = \frac{1}{2}(a+b+c).$$

$$\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$$

$$\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$Area = \frac{1}{2}bc\sin A$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

If the vertices have coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), the area is the *absolute value* of the expression

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

3.2 Trigonometric Functions of an Angle

With reference to Figure 3.1, $P(x, y)$ is a point in either one of the four quadrants and A is an angle whose initial side is coincident with the positive x-axis and whose terminal side contains the point $P(x, y)$. The distance from the origin $P(x, y)$

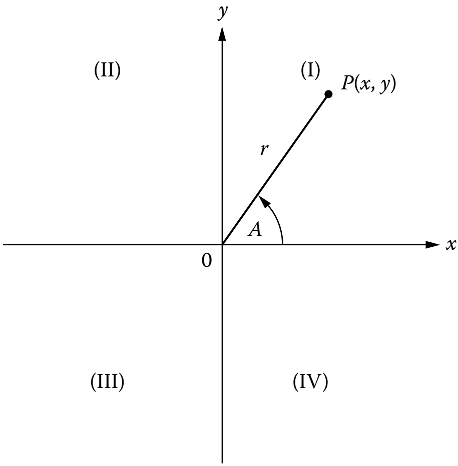


FIGURE 3.1
The trigonometric point. Angle *A* is taken to be positive when the rotation is counterclockwise and negative when the rotation is clockwise. The plane is divided into quadrants as shown.

is denoted by *r* and is positive. The trigonometric functions of the angle *A* are defined as

$$\sin A = \text{sine } A = y/r$$
$$\cos A = \text{cosine } A = x/r$$
$$\tan A = \text{tangent } A = y/x$$
$$\text{ctn } A = \text{cotangent } A = x/y$$
$$\sec A = \text{secant } A = r/x$$
$$\csc A = \text{cosecant } A = r/y$$

Angles are measured in degrees or radians: 180° = π radians; 1 radian = 180°/π degrees.

The trigonometric functions of 0°, 30°, 45°, and integer multiples of these are directly computed:

	0°	30°	45°	60°	90°	120°	135°	150°	180°
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	–1
tan	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	–1	$-\frac{\sqrt{3}}{3}$	0
ctn	∞	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	–1	$-\sqrt{3}$	∞
sec	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞	–2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	–1
csc	∞	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	∞

3.3 Trigonometric Identities

$$\sin A = \frac{1}{\csc A}$$

$$\cos A = \frac{1}{\sec A}$$

$$\tan A = \frac{1}{\operatorname{ctn} A} = \frac{\sin A}{\cos A}$$

$$\csc A = \frac{1}{\sin A}$$

$$\sec A = \frac{1}{\cos A}$$

$$\operatorname{ctn} A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}$$

$$\sin^2 A + \cos^2 A = 1$$
$$1 + \tan^2 A = \sec^2 A$$
$$1 + \operatorname{ctn}^2 A = \csc^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\sin nA = 2 \sin(n-1)A \cos A - \sin(n-2)A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\cos 3A = 4\cos^3 A - 3 \cos A$$

$$\cos nA = 2 \cos(n-1)A \cos A - \cos(n-2)A$$

$$\sin A + \sin B = 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\sin A - \sin B = 2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\cos A + \cos B = 2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)$$

$$\cos A - \cos B = -2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)$$

$$\tan A \pm \tan B = \frac{\sin(A \pm B)}{\sin A \cos B}$$

$$\text{ctn } A \pm \text{ctn } B = \pm\frac{\sin(A \pm B)}{\sin A \sin B}$$

$$\sin A \sin B = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)$$

$$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

$$\sin A \cos B = \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B)$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan\frac{A}{2} = \frac{1-\cos A}{\sin A} = \frac{\sin A}{1+\cos A} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\sin^2 A = \frac{1}{2}(1-\cos 2A)$$

$$\cos^2 A = \frac{1}{2}(1+\cos 2A)$$

$$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$$

$$\cos^3 A = \frac{1}{4}(\cos 3A + 3\cos A)$$

$$\sin ix = \frac{1}{2}i(e^x - e^{-x}) = i\sinh x$$

$$\cos ix = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

$$\tan ix = \frac{i(e^x - e^{-x})}{e^x + e^{-x}} = i\tanh x$$

$$e^{x+iy} = e^x(\cos y + i\sin y)$$

$$(\cos x \pm i\sin x)^n = \cos nx \pm i\sin nx$$

3.4 Inverse Trigonometric Functions

The inverse trigonometric functions are multiple valued, and this should be taken into account in the use of the following formulas:

$$\sin^{-1} x = \cos^{-1} \sqrt{1-x^2}$$

$$= \tan^{-1} \frac{x}{\sqrt{1-x^2}} = \text{ctn}^{-1} \frac{\sqrt{1-x^2}}{x}$$

$$= \sec^{-1} \frac{1}{\sqrt{1-x^2}} = \csc^{-1} \frac{1}{x}$$

$$= -\sin^{-1}(-x)$$

$$\cos^{-1} x = \sin^{-1} \sqrt{1-x^2}$$

$$= \tan^{-1} \frac{\sqrt{1-x^2}}{x} = \text{ctn}^{-1} \frac{x}{\sqrt{1-x^2}}$$

$$= \sec^{-1} \frac{1}{x} = \csc^{-1} \frac{1}{\sqrt{1-x^2}}$$

$$= \pi - \cos^{-1}(-x)$$

$$\tan^{-1} x = \text{ctn}^{-1} \frac{1}{x}$$

$$= \sin^{-1} \frac{x}{\sqrt{1+x^2}} = \cos^{-1} \frac{1}{\sqrt{1-x^2}}$$

$$= \sec^{-1} \sqrt{1+x^2} = \csc^{-1} \frac{\sqrt{1+x^2}}{x}$$

$$= -\tan^{-1}(-x)$$

4

Analytic Geometry

4.1 Rectangular Coordinates

The points in a plane may be placed in one-to-one correspondence with pairs of real numbers. A common method is to use perpendicular lines that are horizontal and vertical and intersect at a point called the *origin*. These two lines constitute the coordinate axes; the horizontal line is the x-axis and the vertical line is the y-axis. The positive direction of the x-axis is to the right, whereas the positive direction of the y-axis is up. If P is a point in the plane, one may draw lines through it that are perpendicular to the x- and y-axes (such as the broken lines of Figure 4.1). The lines intersect the x-axis at a point with coordinate x_1 and the y-axis at a point with coordinate y_1. We call x_1 the x-coordinate, or *abscissa*, and y_1 is termed the y-coordinate, or *ordinate*, of the point P.

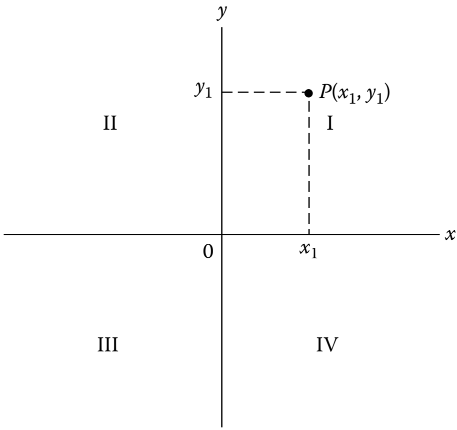


FIGURE 4.1
Rectangular coordinates.

Thus, point P is associated with the pair of real numbers (x_1, y_1) and is denoted $P(x_1, y_1)$. The coordinate axes divide the plane into quadrants I, II, III, and IV.

4.2 Distance between Two Points: Slope

The distance d between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In the special case when P_1 and P_2 are both on one of the coordinate axes, for instance, the x-axis,

$$d = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|,$$

or on the y-axis,

$$d = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|.$$

The midpoint of the line segment P_1P_2 is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The slope of the line segment P_1P_2, provided it is not vertical, is denoted by m and is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is related to the angle of inclination α (Figure 4.2) by

$$m = \tan \alpha$$

Two lines (or line segments) with slopes m_1 and m_2 are perpendicular if

$$m_1 = -1/m_2$$

and are parallel if

$$m_1 = m_2.$$

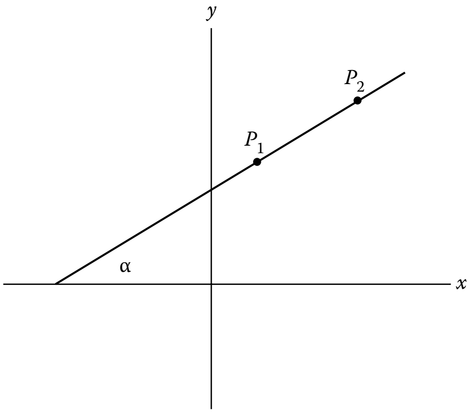


FIGURE 4.2
The angle of inclination is the smallest angle measured counterclockwise from the positive x-axis to the line that contains P_1P_2.

4.3 Equations of Straight Lines

A *vertical* line has an equation of the form

$$x = c$$

where $(c, 0)$ is its intersection with the x-axis. A line of slope m through point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

Thus, a *horizontal line* (slope = 0) through point (x_1, y_1) is given by

$$y = y_1.$$

A nonvertical line through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by either

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

or

$$y - y_2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_2).$$

A line with x-intercept a and y-intercept b is given by

$$\frac{x}{a} + \frac{y}{b} = 1 \; (a \neq 0, b \neq 0).$$

The *general equation* of a line is

$$Ax + By + C = 0$$

The *normal form* of the straight line equation is

$$x \cos\theta + y \sin\theta = p$$

where p is the distance along the normal from the origin and θ is the angle that the normal makes with the x-axis (Figure 4.3).

The general equation of the line $Ax + By + C = 0$ may be written in normal form by dividing by

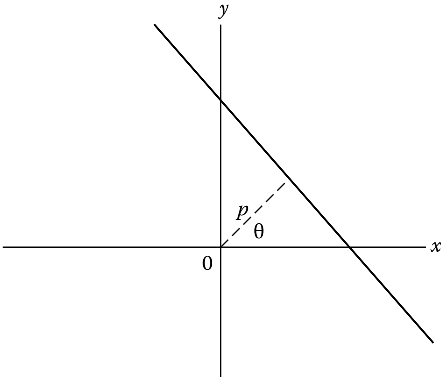


FIGURE 4.3
Construction for normal form of straight line equation.

$\pm\sqrt{A^2 + B^2}$, where the plus sign is used when C is negative and the minus sign is used when C is positive:

$$\frac{Ax + By + C}{\pm\sqrt{A^2 + B^2}} = 0,$$

so that

$$\cos\theta = \frac{A}{\pm\sqrt{A^2 + B^2}}, \quad \sin\theta = \frac{B}{\pm\sqrt{A^2 + B^2}}$$

and

$$p = \frac{|C|}{\sqrt{A^2 + B^2}}.$$

4.4 Distance from a Point to a Line

The perpendicular distance from a point $P(x_1, y_1)$ to the line $Ax + By + C = 0$ is given by d:

$$d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}.$$

4.5 Circle

The general equation of a circle of radius r and center at $P(x_1, y_1)$ is

$$(x - x_1)^2 + (y - y_1)^2 = r^2.$$

4.6 Parabola

A parabola is the set of all points (x, y) in the plane that are equidistant from a given line called the *directrix* and a given point called the *focus.* The parabola is symmetric about a line that contains the focus and is perpendicular to the directrix. The line of symmetry intersects the parabola at its *vertex* (Figure 4.4). The eccentricity $e = 1$.

The distance between the focus and the vertex, or vertex and directrix, is denoted by p (> 0)

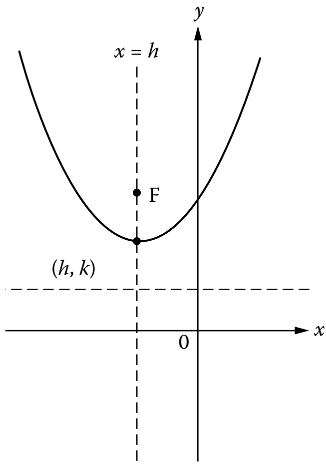


FIGURE 4.4
Parabola with vertex at (h, k). F identifies the focus.

and leads to one of the following equations of a parabola with vertex at the origin (Figures 4.5 and 4.6):

$$y = \frac{x^2}{4p} \quad \text{(opens upward)}$$

$$y = -\frac{x^2}{4p} \quad \text{(opens downward)}$$

$$x = \frac{y^2}{4p} \quad \text{(opens to right)}$$

$$x = -\frac{y^2}{4p} \quad \text{(opens to left)}$$

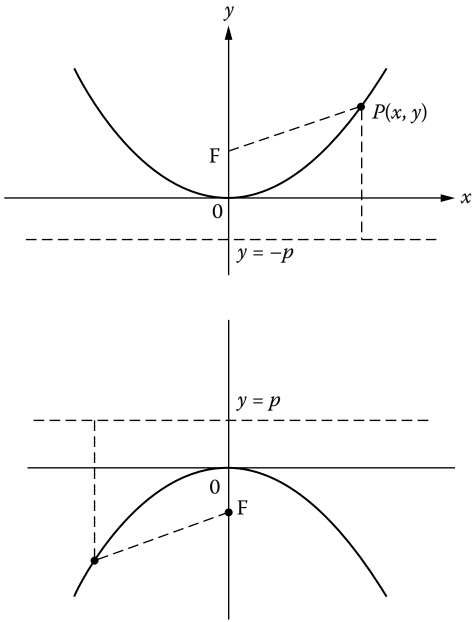


FIGURE 4.5
Parabolas with y-axis as the axis of symmetry and vertex at the origin. Upper, $y = \frac{x^2}{4p}$; lower, $Y = -\frac{x^2}{4p}$.

For each of the four orientations shown in Figures 4.5 and 4.6, the corresponding parabola with vertex (h, k) is obtained by replacing x by $x - h$ and y by $y - k$. Thus, the parabola in Figure 4.7 has the equation

$$x - h = -\frac{(y - k)^2}{4p}.$$

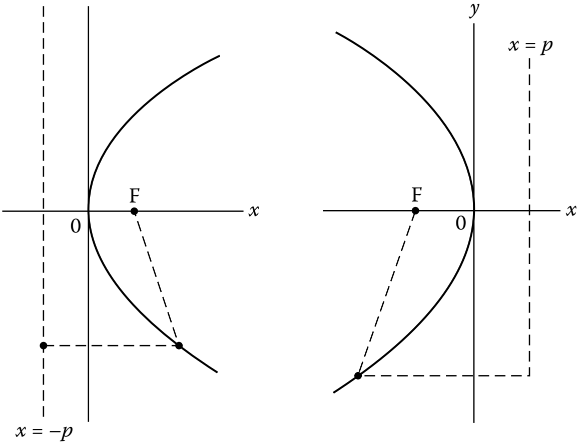


FIGURE 4.6
Parabolas with x-axis as the axis of symmetry and vertex at the origin. Left, $x = \frac{y^2}{4p}$; right, $x = -\frac{y^2}{4p}$.

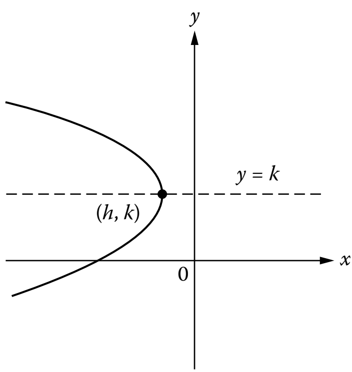


FIGURE 4.7
Parabola with vertex at (h, k) and axis parallel to the x-axis.

4.7 Ellipse

An ellipse is the set of all points in the plane such that the sum of their distances from two fixed points, called *foci*, is a given constant $2a$. The distance between the foci is denoted $2c$; the length of the major axis is $2a$, whereas the length of the minor axis is $2b$ (Figure 4.8), and

$$a = \sqrt{b^2 + c^2}.$$

The eccentricity of an ellipse, e, is <1. An ellipse with center at point (h, k) and major axis *parallel to the x-axis* (Figure 4.9) is given by the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

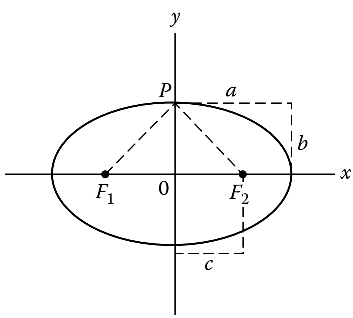


FIGURE 4.8
Ellipse; since point P is equidistant from foci F_1 and F_2, the segments F_1P and $F_2P = a$; hence, $a = \sqrt{b^2 + c^2}$.

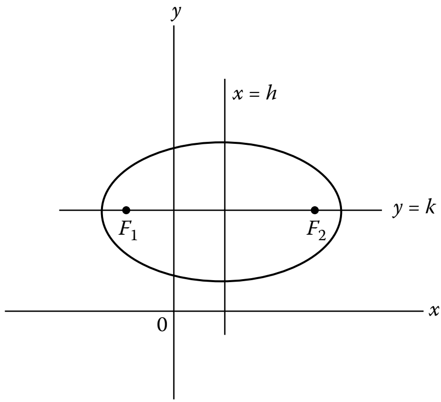


FIGURE 4.9
Ellipse with major axis parallel to the x-axis. F_1 and F_2 are the foci, each a distance c from center (h, k).

An ellipse with center at (h, k) and major axis parallel to the y-axis is given by the equation (Figure 4.10)

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1.$$

4.8 Hyperbola ($e > 1$)

A hyperbola is the set of all points in the plane such that the difference of its distances from two fixed points (foci) is a given positive constant denoted $2a$. The distance between the two

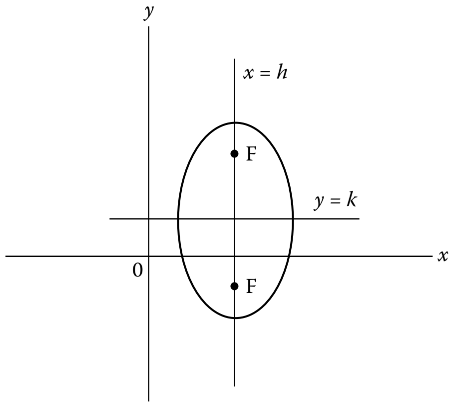


FIGURE 4.10
Ellipse with major axis parallel to the y-axis. Each focus is a distance c from center (h, k).

foci is $2c$, and that between the two vertices is $2a$. The quantity b is defined by the equation

$$b = \sqrt{c^2 - a^2}$$

and is illustrated in Figure 4.11, which shows the construction of a hyperbola given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

When the focal axis is parallel to the y-axis, the equation of the hyperbola with center (h, k) (Figures 4.12 and 4.13) is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

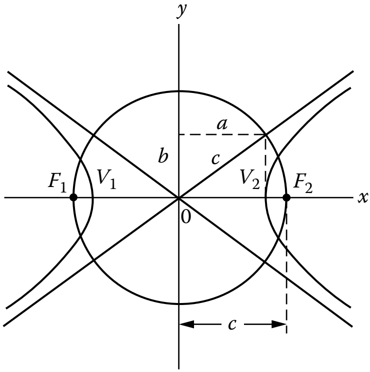


FIGURE 4.11
Hyperbola. V_1, V_2 = vertices; F_1, F_2 = foci. A circle at center 0 with radius c contains the vertices and illustrates the relations among a, b, and c. Asymptotes have slopes b/a and $-b/a$ for the orientation shown.

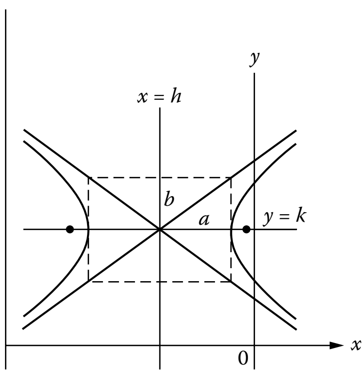


FIGURE 4.12
Hyperbola with center at (h, k): $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$; slope of asymptotes, $\pm\, b/a$.

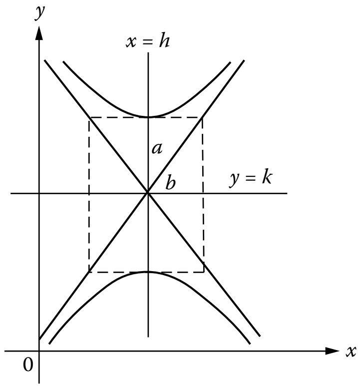


FIGURE 4.13

Hyperbola with center at (*h*, *k*): $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$; slopes of asymptotes, $\pm a/b$.

If the focal axis is parallel to the x-axis and center (*h*, *k*), then

$$\frac{(x-h)^2}{a^2} - \frac{(y+k)^2}{b^2} = 1$$

4.9 Change of Axes

A change in the position of the coordinate axes will generally change the coordinates of the points in the plane. The equation of a particular curve will also generally change.

- Translation
  When the new axes remain parallel to the original, the transformation is called

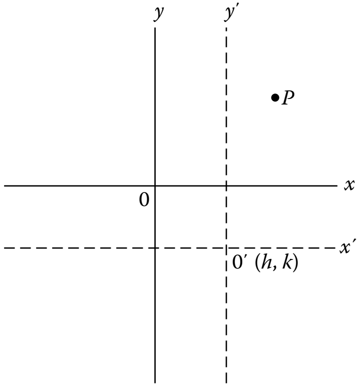


FIGURE 4.14
Translation of axes.

a *translation* (Figure 4.14). The new axes, denoted x' and y', have origin $0'$ at (h, k) with reference to the x- and y-axes.

A point P with coordinates (x, y) with respect to the original has coordinates (x', y') with respect to the new axes. These are related by

$$x = x' + h$$
$$x = y' + k$$

For example, the ellipse of Figure 4.10 has the following simpler equation with respect to axes x' and y' with the center at (h, k):

$$\frac{y'^2}{a^2} + \frac{x'^2}{b^2} = 1.$$

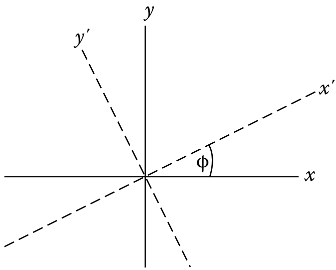


FIGURE 4.15
Rotation of axes.

- Rotation
  When the new axes are drawn through the same origin, remaining mutually perpendicular, but tilted with respect to the original, the transformation is one of rotation. For angle of rotation ϕ (Figure 4.15), the coordinates (x, y) and (x', y') of a point P are related by

$$x = x'\cos\phi - y'\sin\phi$$

$$y = x'\sin\phi + y'\cos\phi$$

4.10 General Equation of Degree 2

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Every equation of the above form defines a conic section or one of the limiting forms of a conic.

By rotating the axes through a particular angle ϕ, the xy term vanishes, yielding

$$A'x^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

with respect to the axes x' and y'. The required angle ϕ (see Figure 4.15) is calculated from

$$\tan 2\phi = \frac{B}{A-C}, \quad (\phi < 90°).$$

4.11 Polar Coordinates (Figure 4.16)

The fixed point O is the origin or *pole*, and a line OA drawn through it is the polar axis. A point P in the plane is determined from its distance r,

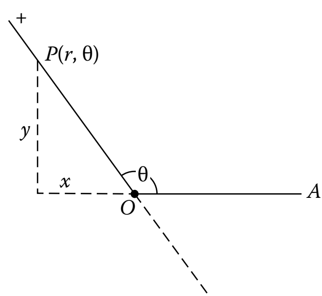


FIGURE 4.16
Polar coordinates.

measured from O, and the angle θ between OP and OA. Distances measured on the terminal line of θ from the pole are positive, whereas those measured in the opposite direction are negative.

Rectangular coordinates (x, y) and polar coordinates (r, θ) are related according to

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

Several well-known polar curves are shown in Figures 4.17 through 4.21.

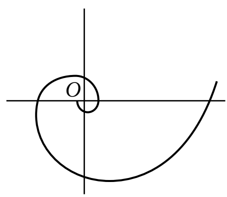


FIGURE 4.17
Polar curve $r = e^{a\theta}$.

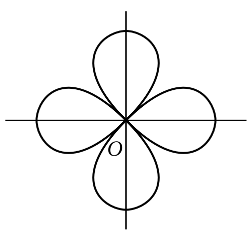


FIGURE 4.18
Polar curve $r = a\cos 2\theta$.

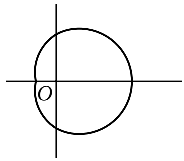


FIGURE 4.19
Polar curve $r = 2a\cos\theta + b$.

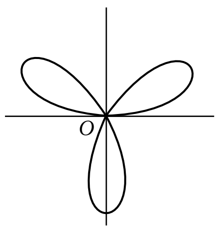


FIGURE 4.20
Polar curve $r = a\sin 3\theta$.

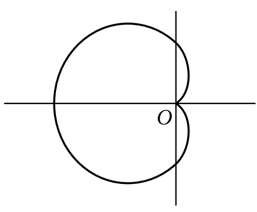


FIGURE 4.21
Polar curve $r = a(1 - \cos\theta)$.

The polar equation of a conic section with focus at the pole and distance $2p$ from directrix to focus is either

$$r = \frac{2ep}{1 - e\cos\theta} \quad \text{(directrix to left of pole)}$$

or

$$r = \frac{2ep}{1 + e\cos\theta} \quad \text{(directrix to right of pole)}$$

The corresponding equations for the directrix below or above the pole are as above, except that sin θ appears instead of cos θ.

4.12 Curves and Equations (Figures 4.22 through 4.35)

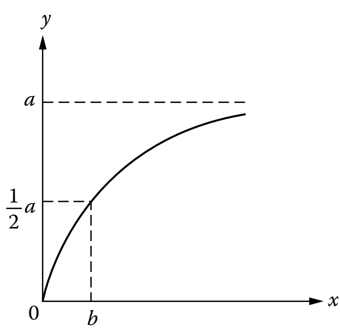


FIGURE 4.22
$y = \dfrac{ax}{x + b}.$

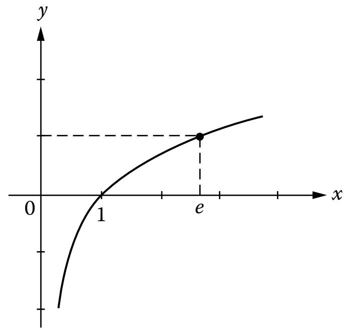


FIGURE 4.23
$y = \log x$.

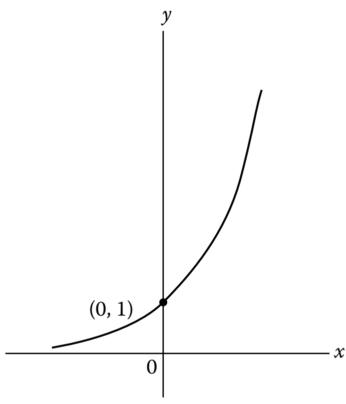


FIGURE 4.24
$y = e^x$.

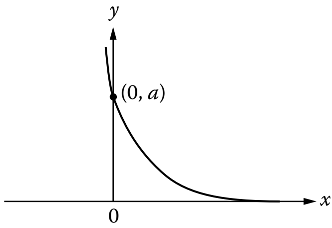


FIGURE 4.25
$y = ae^{-x}$.

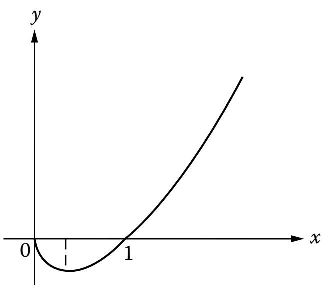


FIGURE 4.26
$y = x \log x$.

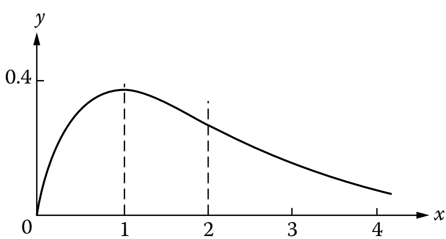


FIGURE 4.27
$y = xe^{-x}$.

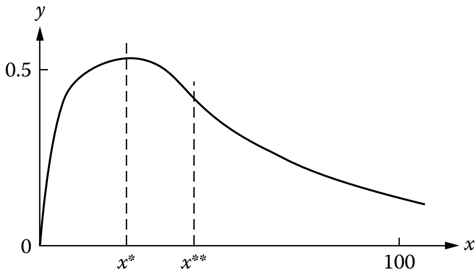


FIGURE 4.28
$y = e^{-ax} - e^{-bx}$, $0 < a < b$ (drawn for $a = 0.02$, $b = 0.1$, and showing maximum and inflection).

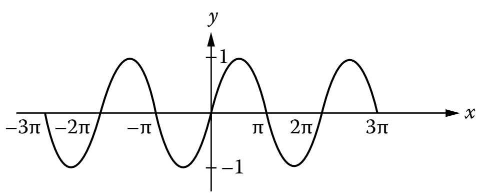


FIGURE 4.29
$y = \sin x$.

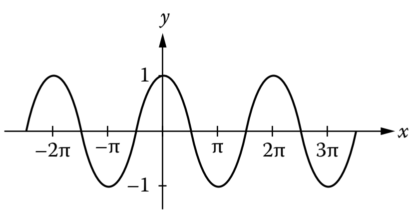


FIGURE 4.30
$y = \cos x$.

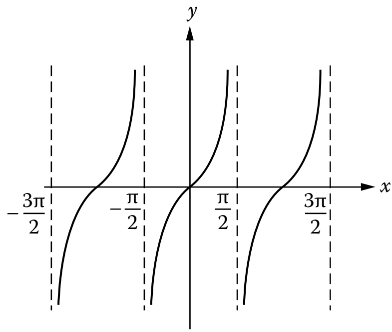


FIGURE 4.31
$y = \tan x$.

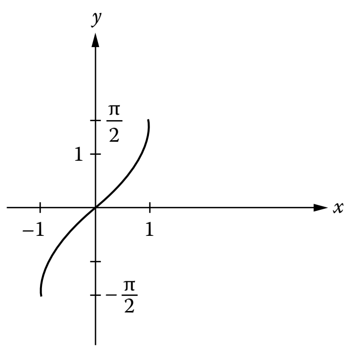


FIGURE 4.32
$y = \arcsin x$.

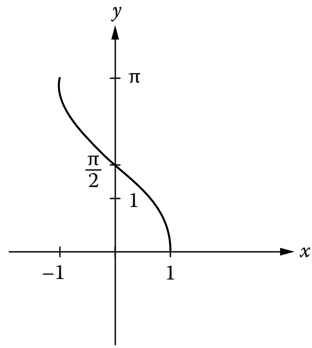


FIGURE 4.33
$y = \arccos x$.

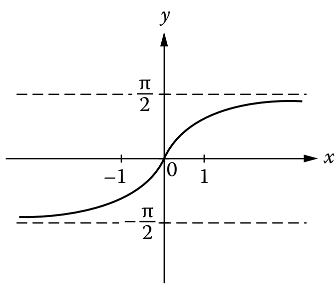


FIGURE 4.34
$y = \arctan x$.

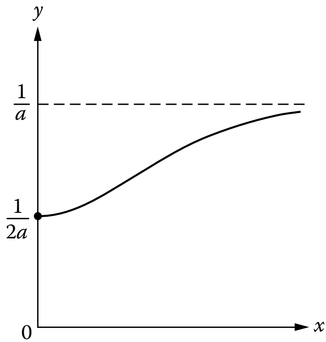


FIGURE 4.35
$y = e^{bx}/a\ \ (1 + e^{bx}),\ \ x \geq 0$ (logistic equation).

4.13 Exponential Function (Half-Life)

The function given by $y = e^x$ is the well-known exponential function (e = base of natural logarithms; see Figures 4.24 and 4.25). In many applications, e.g., radioactive decay, pharmacokinetics, growth models, etc., one encounters this function with time (t) as the independent variable, i.e., $y = Ae^{kt}$, for constants A and k. For positive k, the function increases and doubles in time $\ln(2)/k$. When k is negative, the function decreases and is often characterized by the *half-life*, which is the time to decrease to $A/2$. Half-life is therefore $-\ln(2)/k$.

5

Series, Number Facts, and Theory

5.1 Bernoulli and Euler Numbers

A set of numbers, $B_1, B_3, \ldots, B_{2n-1}$ (Bernoulli numbers) and B_2, $B_4, \ldots$, B_{2n} (Euler numbers), appears in the series expansions of many functions. A partial listing follows; these are computed from the following equations:

$$B_{2n} - \frac{2n(2n-1)}{2!} B_{2n-2} + \frac{2n(2n-1)(2n-2)(2n-3)}{4!} B_{2n-4} - \cdots + (-1)^n = 0,$$

and

$$\frac{2^{2n}(2^{2n}-1)}{2n}B_{2n-1} = (2n-1)B_{2n-2} - \frac{(2n-1)(2n-2)(2n-3)}{3!}B_{2n-4} + \cdots + (-1)^{n-1}.$$

$B_1 = 1/6$	$B_2 = 1$
$B_3 = 1/30$	$B_4 = 5$
$B_5 = 1/42$	$B_6 = 61$
$B_7 = 1/30$	$B_8 = 1385$
$B_9 = 5/66$	$B_{10} = 50521$
$B_{11} = 691/2730$	$B_{12} = 2702765$
$B_{13} = 7/6$	$B_{14} = 199360981$
$\vdots$	$\vdots$

5.2 Series of Functions

In the following, the interval of convergence is indicated; otherwise, it is all x. Logarithms are to the base e. Bernoulli and Euler numbers (B_{2n-1} and B_{2n}) appear in certain expressions.

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2$$

$$+\frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots$$

$$+\frac{n!}{(n-j)!j!}a^{n-j}x^j + \cdots \left[x^2 < a^2\right]$$

$$(a-bx)^{-1} = \frac{1}{a}\left[1 + \frac{bx}{a} + \frac{b^2x^2}{a^2} + \frac{b^3x^3}{a^3} + \cdots\right]\left[b^2x^2 < a^2\right]$$

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!}x^2$$

$$\pm\frac{n(n-1)(n-2)x^3}{3!} + \cdots\left[x^2 < 1\right]$$

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!}x^2$$

$$\mp\frac{n(n+1)(n+2)}{3!}x^3 + \cdots\left[x^2 < 1\right]$$

$$(1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1}{2\cdot 4}x^2 \pm \frac{1\cdot 3}{2\cdot 4\cdot 6}x^3$$

$$-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^4 \pm \cdots [x^2 < 1]$$

$$(1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1\cdot 3}{2\cdot 4}x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3$$

$$+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^4 \mp \cdots [x^2 < 1]$$

$$(1 \pm x^2)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x^2 - \frac{x^4}{2\cdot 4} \pm \frac{1\cdot 3}{2\cdot 4\cdot 6}x^6 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8}x^8 \pm \cdots [x^2 < 1]$$

$$(1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \cdots [x^2 < 1]$$

$$(1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp x^5 + \cdots [x^2 < 1]$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots$$

$$a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + \frac{(x\log a)^3}{3!} + \cdots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots [0 < x < 2]$$

$$\log x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \cdots \left[x > \frac{1}{2}\right]$$

$$\log x = 2\left[\left(\frac{x-1}{x+1}\right) + \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 + \cdots\right] \quad [x > 0]$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \left[x^2 < 1\right]$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots\right]\left[x^2 < 1\right]$$

$$\log\left(\frac{x+1}{x-1}\right) = 2\left[\frac{1}{x} + \frac{1}{3}\left(\frac{1}{x}\right)^3 + \frac{1}{5}\left(\frac{1}{x}\right)^5 + \cdots\right]\left[x^2 < 1\right]$$

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \frac{x^7}{7\,!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \frac{x^6}{6\,!} + \cdots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315}$$

$$+ \cdots + \frac{2^{2n}(2^{2n}-1)B_{2n-1}x^{2n-1}}{(2n)!} \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945}$$

$$- \cdots - \frac{B_{2n-1}(2x)^{2n}}{(2n)!\,x} - \cdots \left[x^2 < \pi^2\right]$$

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots$$

$$+ \frac{B_{2n}x^{2n}}{(2n)\,!} + \cdots \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\csc x = \frac{1}{x} + \frac{x}{3!} + \frac{7x^3}{3\cdot 5\,!} + \frac{31x^5}{3\cdot 7!}$$

$$+\cdots + \frac{2(2^{2n+1}-1)}{(2n+2)\,!} B_{2n+1} x^{2n+1} + \cdots \left[x^2 < \pi^2\right]$$

$$\sin^{-1} x = x + \frac{x^3}{6} + \frac{(1\cdot 3)x^5}{(2\cdot 4)5} + \frac{(1\cdot 3\cdot 5)x^7}{(2\cdot 4\cdot 6)7} + \cdots \left[x^2 < 1\right]$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots [x^2 < 1]$$

$$\sec^{-1} x = \frac{\pi}{2} - \frac{1}{x} - \frac{1}{6x^3}$$

$$- \frac{1\cdot 3}{(2\cdot 4)5x^2} - \frac{1\cdot 3\cdot 5}{(2\cdot 4\cdot 6)7x^7} - \cdots \left[x^2 > 1\right]$$

$$\sinh x = x + \frac{x^3}{3\,!} + \frac{x^5}{5\,!} + \frac{x^7}{7!} + \cdots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \cdots$$

$$\tanh x = (2^2 - 1)2^2 B_1 \frac{x}{2!} - (2^4 - 1)2^4 B_3 \frac{x^3}{4!}$$

$$+\left(2^6 - 1\right)2^6 B_5 \frac{x^5}{6!} - \cdots \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{ctnh} x = \frac{1}{x}\left(1 + \frac{2^2 B_1 x^2}{2!} - \frac{2^4 B_3 x^4}{4!} + \frac{2^6 B_2 x^6}{6!} - \cdots\right)$$

$$\left[x^2 < \pi^2\right]$$

$$\operatorname{sech} x = 1 - \frac{B_2 x^2}{2!} + \frac{B_4 x^4}{4!} - \frac{B_6 x^6}{6!} + \cdots \left[x^2 < \frac{\pi^2}{4}\right]$$

$$\operatorname{csch} x = \frac{1}{x} - (2-1)2\ B_1 \frac{x}{2!}$$

$$+ (2^3 - 1)\ 2B_3 \frac{x^3}{4!} - \cdots \dagger \left[x^2 < \pi^2\right]$$

$$\sinh^{-1} x = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

$$\dagger \qquad \left[x^2 < 1\right]$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots \left[x^2 < 1\right]$$

$$\operatorname{ctnh}^{-1} x = \frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots \left[x^2 > 1\right]$$

$$\operatorname{csch}^{-1} x = \frac{1}{x} - \frac{1}{2\cdot 3x^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5x^5}$$

$$- \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7x^7} + \cdots \left[x^2 > 1\right]$$

$$\int_0^x e^{-t^2}\,dt = x - \frac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots$$

5.3 Error Function

The following function, known as the error function, erf x, arises frequently in applications:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

The integral cannot be represented in terms of a finite number of elementary functions; therefore, values of erf x have been compiled in tables. The following is the series for erf x:

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots \right]$$

There is a close relation between this function and the area under the standard normal curve (Table A.1). For evaluation, it is convenient to use z instead of x; then erf z may be evaluated from the area $F(z)$ given in Table A.1 by use of the relation

$$\operatorname{erf} z = 2F(\sqrt{2}z)$$

Example

$$\text{erf}(0.5) = 2F\,[(1.414)(0.5)] = 2F(0.707)$$

By interpolation from Table A.1, $F\,(0.707) = 0.260$; thus, erf $(0.5) = 0.520$.

5.4 Fermat's Little Theorem

This theorem provides a condition that a prime number must satisfy.

Theorem

If p is a prime, then for any integer a, $(a^P - a)$ is divisible by p.

Examples

$2^8 - 2 = 254$ is not divisible by 8; thus, 8 cannot be prime.
$3^7 - 3 = 2184$ is divisible by 7, because 7 is prime.

5.5 Fermat's Last Theorem

If n is an integer greater than 2, then $a^n + b^n = c^n$ has no solutions in nonzero integers a, b, and c. For example, there are no integers a, b, and c such that $a^3 + b^3 = c^3$. This author has generated "near misses," i.e., $a^3 + b^3 = c^3 \pm 1$, as shown below, and shown further that if $(a + b)$ is odd, c is even, whereas if $(a + b)$ is even, then c is odd.

"Near Misses" in the Cubic Form of Fermat's Last Theorem[a]
Near misses for integers a and b between 2 and 1,000 … and beyond

a	b	c	$a^3 + b^3$	c^3
6	8	9	728	729
9	10	12	1729	1728
64	94	103	1092728	1092727
71	138	144	2985983	2985984
73	144	150	3375001	3375000
135	138	172	5088447	5088448
135	235	249	15438250	15438249
242	720	729	387420488	387420489
244	729	738	401947273	401947272
334	438	495	121287376	121287375
372	426	505	128787624	128787625
426	486	577	192100032	192100033
566	823	904	738763263	738763264
791	812	1010	1030300999	1030301000
…	…	…	…	…
2304	577	2316	12422690497	12422690496
…	…	…	…	…
11161	11468	14258	2898516861513	2898516861512
…	…	…	…	…

[a] Table derived from a computer program written by this author.

5.6 Beatty's Theorem

If a and b are positive and irrational with the property that $\frac{1}{a} + \frac{1}{b} = 1$, then for positive integers n, the integer parts of na and nb constitute a partition of the set of positive integers, i.e., the two sequences

$$\lfloor a \rfloor, \lfloor 2a \rfloor, \lfloor 3a \rfloor, \ldots$$

$$\lfloor b \rfloor, \lfloor 2a \rfloor, \lfloor 3b \rfloor, \ldots$$

where $\lfloor x \rfloor$ is the greatest integer function, containing all positive integers but having no common terms.

An interesting example occurs if $a = \sqrt{2}$, which yields the two sequences

$\{S_1\}$ 1, 2, 4, 5, 7, 8, 9, 11, 12, …

$\{S_2\}$ 3, 6, 10, 13, 17, 20, 23, 27, 30 …

which partition the integers and also have the property that the difference between successive terms $\{S_{2i} - S_{1i}\}$ is the sequence

2, 4, 6, 8, 10, 12, 14, …

5.7 An Interesting Prime

73939133 is a prime number as is each number obtained by deleting the right-most digit; each of the following is a prime number:

7393913, 739391, 73939, 7393, 739, 73, 7

5.8 Goldbach Conjecture

Every even number greater than or equal to 4 can be expressed as the sum of two prime numbers.

Examples

$6 = 3 + 3$

$12 = 5 + 7$

$18 = 5 + 13$

$20 = 3 + 17 = 7 + 13$

5.9 Twin Primes

Twin primes are pairs of primes that differ by 2, e.g., {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}, …, {137, 139}, etc. It is believed, but not proved, that there are infinitely many twin primes.

5.10 Collatz Conjecture

Consider a sequence that begins with any positive integer and applies the following rule for successive terms: if it is odd, multiply by 3 and add 1; if it is even, divide it by 2. All such sequences terminate with 4, 2, 1. (This conjecture is still unproven.)

Example

Start with **23** to give 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

6

Differential Calculus

6.1 Notation

For the following equations, the symbols $f(x)$, $g(x)$, etc., represent functions of x. The value of a function $f(x)$ at $x = a$ is denoted $f(a)$. For the function $y = f(x)$ the derivative of y with respect to x is denoted by one of the following:

$$\frac{dy}{dx},\ f'(x),\ D_x y,\ y'.$$

Higher derivatives are as follows:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x) = f''(x)$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx} f''(x) = f'''(x),\ \text{etc.}$$

and values of these at $x = a$ are denoted $f''(a), f'''(a)$, etc. (see Table of Derivatives).

6.2 Slope of a Curve

The tangent line at a point $P(x, y)$ of the curve $y = f(x)$ has a slope $f'(x)$ provided that $f'(x)$ exists at P. The slope at P is defined to be that of the tangent line at P. The tangent line at $P(x_1, y_1)$ is given by

$$y - y_1 = f'(x_1)(x - x_1).$$

The *normal line* to the curve at $P(x_1, y_1)$ has slope $-1/f'(x_1)$ and thus obeys the equation

$$y - y_1 = [-1/f'(x_1)](x - x_1)$$

(The slope of a vertical line is not defined.)

6.3 Angle of Intersection of Two Curves

Two curves $y = f_1(x)$ and $y = f_2(x)$, that intersect at a point $P(X, Y)$ where derivatives $f'_1(X), f'_2(X)$ exist, have an angle (α) of intersection given by

$$\tan\alpha = \frac{f_2'(X) - f_1'(X)}{1 + f_2'(X)\cdot f_1'(X)}.$$

If $\tan\alpha > 0$, then α is the acute angle; if $\tan\alpha < 0$, then α is the obtuse angle.

6.4 Radius of Curvature

The radius of curvature R of the curve $y = f(x)$ at point $P(x, y)$ is

$$R = \frac{\left\{1 + [f'(x)]^2\right\}^{3/2}}{f''(x)}$$

In polar coordinates (θ, r) the corresponding formula is

$$R = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}$$

The *curvature* K is $1/R$.

6.5 Relative Maxima and Minima

The function f has a relative maximum at $x = a$ if $f(a) \geq f(a + c)$ for all values of c (positive or negative) that are sufficiently near zero. The function f has a relative minimum at $x = b$ if $f(b) \leq f(b + c)$ for all values of c that are sufficiently close to zero. If the function f is defined on the closed interval $x_1 \leq x \leq x_2$, and has a relative maximum or minimum at $x = a$, where $x_1 < a < x_2$, and if the derivative $f'(x)$ exists at $x = a$, then $f'(a) = 0$. It is noteworthy that a relative maximum or minimum may occur at a point where the derivative does not exist. Further, the derivative may vanish at a point that is neither a maximum nor a minimum for the function. Values of x for which $f'(x) = 0$ are called critical values. To determine whether a critical value of x, say, x_c, is a relative maximum or minimum for the function at x_c, one may use the second derivative test:

1. If $f''(x_c)$ is positive, $f(x_c)$ is a minimum.
2. If $f''(x_c)$ is negative, $f(x_c)$ is a maximum.
3. If $f''(x_c)$ is zero, no conclusion may be made.

The sign of the derivatives as x advances through x_c may also be used as a test. If $f'(x)$ changes from positive to zero to negative, then a maximum occurs at x_c, whereas a change in $f'(x)$ from negative to zero to positive indicates a minimum. If $f'(x)$ does not change sign as x advances through x_c, then the point is neither a maximum nor a minimum.

6.6 Points of Inflection of a Curve

The sign of the second derivative of f indicates whether the graph of $y = f(x)$ is concave upward or concave downward:

$$f''(x) > 0 : \text{concave upward}$$

$$f''(x) < 0 : \text{concave downward}$$

A point of the curve at which the direction of concavity changes is called a point of inflection

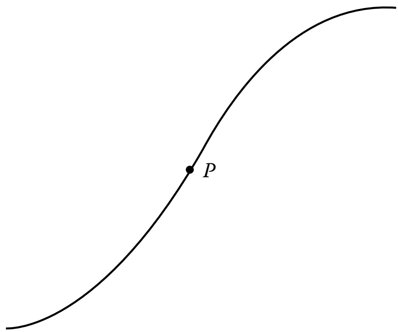


FIGURE 6.1
Point of inflection.

(Figure 6.1). Such a point may occur where $f''(x) = 0$ or where $f''(x)$ becomes infinite. More precisely, if the function $y = f(x)$ and its first derivative $y' = f'(x)$ are continuous in the interval $a \leq x \leq b$, and if $y'' = f''(x)$ exists in $a < x < b$, then the graph of $y = f(x)$ for $a < x < b$ is concave upward if $f''(x)$ is positive and concave downward if $f''(x)$ is negative.

6.7 Taylor's Formula

If f is a function that is continuous on an interval that contains a and x, and if its first $(n + 1)$ derivatives are continuous on this interval, then

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R,$$

where R is called the *remainder.* There are various common forms of the remainder.

- *Lagrange's Form*

$$R = f^{(n+1)}(\beta)\cdot\frac{(x-a)^{n+1}}{(n+1)!};\quad \beta \text{ between } a \text{ and } x.$$

- *Cauchy's Form*

$$R = f^{(n+1)}(\beta)\cdot\frac{(x-\beta)^{n}(x-a)}{n!};\quad \beta \text{ between } a \text{ and } x.$$

- *Integral Form*

$$R = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

6.8 Indeterminant Forms

If $f(x)$ and $g(x)$ are continuous in an interval that includes $x = a$, and if $f(a) = 0$ and $g(a) = 0$, the limit $\lim_{x\to a}(f(x)/g(x))$ takes the form "0/0", called an *indeterminant form. L'Hôpital's rule* is

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Similarly, it may be shown that if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

(The above holds for $x \to \infty$.)

Examples

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to a}\frac{\cos x}{1} = 1$$

$$\lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0$$

6.9 Numerical Methods

a. *Newton's method* for approximating roots of the equation $f(x) = 0$: A first estimate x_1 of the root is made; then provided that $f'(x_1) \neq 0$, a better approximation is x_2:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

The process may be repeated to yield a third approximation, x_3, to the root:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

provided $f'(x_2)$ exists. The process may be repeated. (In certain rare cases the process will not converge.)

b. *Trapezoidal rule for areas* (Figure 6.2): For the function $y = f(x)$ defined on the interval (a, b) and positive there, take n equal subintervals of width $\Delta x = (b - a)/n$. The area bounded by the curve between $x = a$ and $x = b$ (or definite integral of $f(x)$) is approximately the sum of trapezoidal areas, or

$$A \sim \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right)(\Delta x)$$

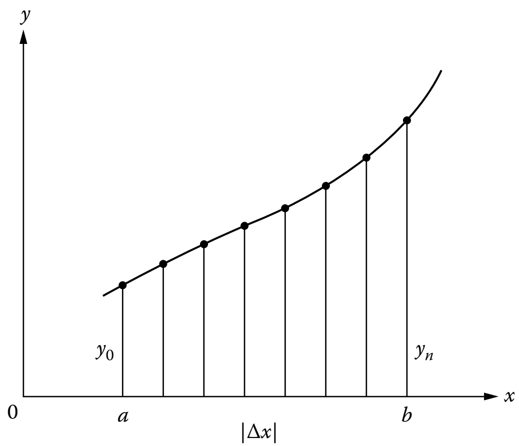


FIGURE 6.2
Trapezoidal rule for area.

Estimation of the error (E) is possible if the second derivative can be obtained:

$$E = \frac{b-a}{12} f''(c)(\Delta x)^2,$$

where c is some number between a and b.

6.10 Functions of Two Variables

For the function of two variables, denoted $z = f(x, y)$, if y is held constant, say, at $y = y_1$, then the resulting function is a function of x only. Similarly, x may be held constant at x_1, to give the resulting function of y.

- *The Gas Laws*
  A familiar example is afforded by the ideal gas law that relates the pressure p, the volume V, and the absolute temperature T of an ideal gas:

  $$pV = nRT$$

  where n is the number of moles and R is the gas constant per mole, 8.31 ($J \cdot {}^\circ K^{-1} \cdot mole^{-1}$). By rearrangement, any one of the three variables may be expressed as a function of the other two. Further, either one of these two may be held constant. If T is held constant, then we get the form known as Boyle's law:

  $$p = kV^{-1} \quad \text{(Boyle's law)}$$

where we have denoted nRT by the constant k and, of course, $V > 0$. If the pressure remains constant, we have Charles' law:

$$V = bT \quad \text{(Charles' law)}$$

where the constant b denotes nR/p. Similarly, volume may be kept constant:

$$p = aT$$

where now the constant, denoted a, is nR/V.

6.11 Partial Derivatives

The physical example afforded by the ideal gas law permits clear interpretations of processes in which one of the variables is held constant. More generally, we may consider a function $z = f(x, y)$ defined over some region of the x-y-plane in which we hold one of the two coordinates, say, y, constant. If the resulting function of x is differentiable at a point (x, y), we denote this derivative by one of the following notations:

$$f_x, \ \delta f / \delta x, \ \delta z / \delta x$$

called the *partial derivative with respect to x*. Similarly, if x is held constant and the resulting function of y is differentiable, we get the *partial derivative with respect to y*, denoted by one of the following:

$$f_y,\ \delta f/\delta y,\ \delta z/\delta y$$

Example

Given $z = x^4y^3 - y\sin x + 4y$, then

$$\delta z/\delta x = 4(xy)^3 - y\cos x;$$
$$\delta z/\delta y = 3x^4y^2 - \sin x + 4.$$

6.12 Application of Derivatives

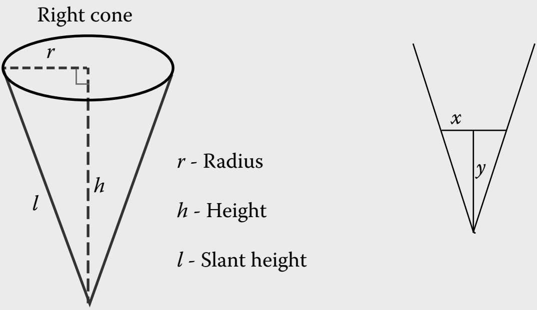
6.12.1 Related Rate Problems

Two time-varying quantities that are related will have rates of change that are also related. The chain rule (p. 235) is applied.

Example

Liquid enters a conical container at a constant rate of $dV/dt = 2$ cu ft/min. As the container fills, the rate of change of liquid height y varies with the height even though the volume input is constant. Our aim is to obtain that rate of change of height. At all times, of course, the liquid radius

x and height y are in the proportion of the cone's dimensions, that is, $y/x = h/r$.

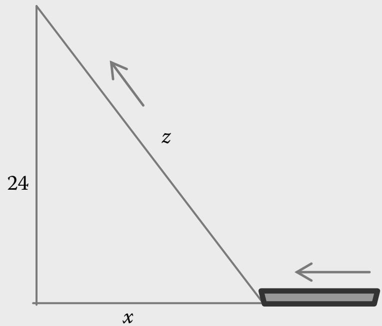


If the cone's dimensions are $r = 5$ ft and $h = 20$ ft, then the liquid height y corresponds to liquid radius x in such a way that $y/x = h/\mathbf{r}$; thus $x = ry/h$. The volume of the cone's content is therefore $V = (1/3) \times \pi x^2 y = (1/3)\, \pi\, (ry/h)^2\, y$, or $V = (1/3)\, \pi r^2 y^3/h^2$. Since $r^2/h^2 = 1/16$, we have $V = (1/48)\, \pi y^3$. The application of the chain rule yields $dV/dt = (1/16)\, \pi y^2\, dy/dt$, an expression that connects the rates dV/dt (= 2) and dy/dt for all times $t > 0$. *For example, at an instant at which $y = 5$ ft, we have*

$2 = (\pi/16)\, 5^2\, dy/dt$ so that $dy/dt = 32/(25\pi)$ ft./min.

Example

A person standing on a wharf 24 feet above the water is pulling in a rope attached to a boat at the rate of 5 feet/second. How fast is the boat moving toward the wharf when the amount of rope out is 25 feet?



Here we are relating dx/dt to dz/dt, and we wish to determine dx/dt. From the right triangle we have

$$x^2 + 24^2 = z^2$$

and therefore $2x\frac{dx}{dt} = 2z\frac{dz}{dt}$ or $\frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}$.

When $z = 25$, $x = 7$ and, since $dz/dt = 5$ ft./sec., $dx/dt = 125/7$ ft./sec.

Example

A large spherical balloon is leaking gas at the rate of 2 ft^3/hr. How fast is the radius decreasing when the radius = 20 feet?

We start with the volume of the sphere, $V = \frac{4}{3}\pi r^3$, and relate the rates dV/dt and dr/dt, that is, $\frac{dV}{dt} = 4{\neq}r^2 \frac{dr}{dt}$. Since $dV/dt = -2$, it follows that $dr/dt = \frac{-1}{800\,\pi}$ ft./hr. is the rate of decrease of radius.

6.12.2 Rectilinear Motion

Objects that move along a straight line may be described by an equation of motion. For example, if the movement is horizontal, an equation of the form $x = f(t)$ would give the position (x) at any time t.

Example

$x = \frac{1}{4}t^4 + t^3 - 2t + 1$. *Velocity and acceleration follow from differentiation.*

The velocity $v = dx/dt = t^3 + 3t^2 - 2$

and the acceleration $a = dv/dt = 3t^2 + 6t$.

Substituting a specific value of t gives the velocity and acceleration values at that instant.

Example

A particle undergoes oscillatory motion along a horizontal plane with position given by the equation $x = A\cos wt + B\sin wt$. This describes a type of straight line motion called *simple harmonic motion*. It is oscillatory with period

$\frac{2\neq}{w}$ and amplitude $\sqrt{A^2 + B^2}$. The velocity v and acceleration a are obtained from the time derivatives

$$v = \frac{dx}{dt} = -Aw \sin wt + Bw \cos wt$$

$$a = -Aw^2 \cos wt - Bw^2 \sin wt$$
$$= -w^2 x$$

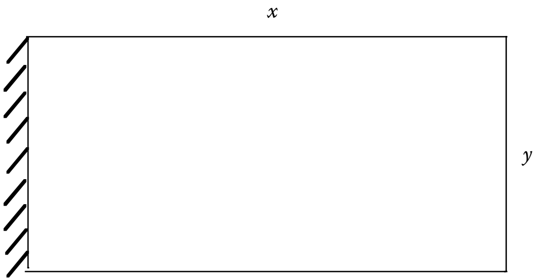
The last expression shows that the acceleration is proportional to x, the displacement. This is a situation that can be viewed as a spring attached to the particle that is stretched and set in motion on a smooth surface. The force, and thus the acceleration, is proportional to the displacement x.

Example

A particle fired vertically upward from the surface of the earth attains a height y that is given by the time-varying equation $y = v_0 t - \frac{1}{2} g t^2$, where v_0 is the initial velocity and g = the gravitational constant (32 ft/s^2 or 9.8 m/s^2).

Its velocity v at any time during the flight is the derivative $\frac{dy}{dt} = v_0 - gt$ and the acceleration is, of course, $\frac{dv}{dt} = -g$.

6.12.3 Applied Problem in Maximum and Minimum



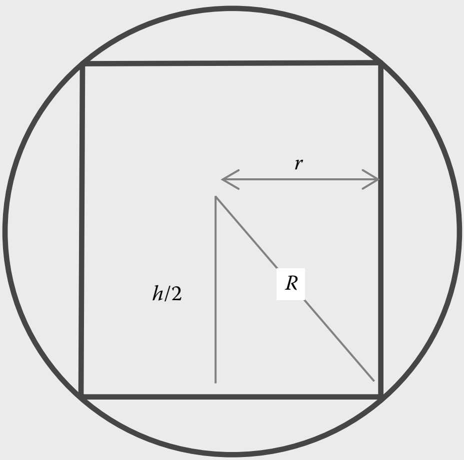
Example

A rectangular plot of 1500 square feet is to be fenced on three sides as shown, and we wish to determine the dimensions that minimize the amount of fencing. This is a problem in which we are given a rectangular area and we seek the minimum of the perimeter. The area $A = xy = 1500$ and the fence length $P = 2x + y$. Thus, $P = 2x + 1500/x$, and this will be a minimum when $dP/dx = 0$. Therefore, $dP/dx = 2 - 1500/x^2 = 0$, which gives $x^2 = 750$ so that $x = \sqrt{750} = 5\sqrt{30}$ and $y = 1500/x = 10\sqrt{30}$.

Example

A right circular cylinder with radius r and height h is to be inscribed within a sphere of radius R. We wish to determine the dimensions of the largest cylinder that will fit. We begin with the expression for the volume V of the cylinder

$$V = \pi r^2 h$$



and note the constraint that relates the spherical and cylindrical dimensions using the Pythagorean theorem

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2 .$$

Therefore the volume V is expressed in terms of the variable h from which dV/dh easily follows, as shown in the following:

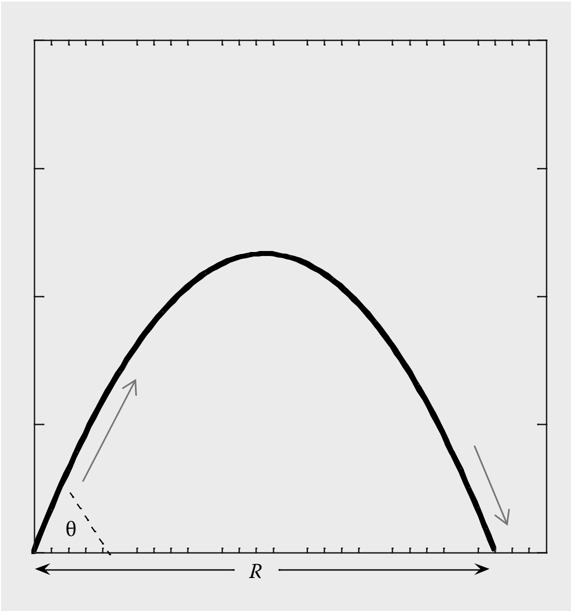
$$V = \pi h\left(R^2 - \frac{h^2}{4}\right)$$

$$\frac{dV}{dh} = \pi R^2 - \frac{3}{4}\pi h^2$$

Equating this derivative to zero gives the cylinder height $h = 2\frac{\sqrt{3}}{3}R$.

Example

A projectile fired with initial velocity v from a gun at an angle θ with the horizontal will follow a path as shown below when air resistance is neglected. The range R is given by $\frac{v^2}{g}\sin 2\theta$ where v is the initial velocity and g is acceleration due to gravity. To find the maximum range, we take the derivative $dR/d\theta$ and equate it to zero, yielding $\frac{2v^2}{g}\cos 2\theta = 0$, which leads to $\theta = 45°$. Therefore the gun should be fired at an angle of 45° with the horizontal.


θ
R

7

Integral Calculus

7.1 Indefinite Integral

If $F(x)$ is differentiable for all values of x in the interval (a, b) and satisfies the equation $dy/dx = f(x)$, then $F(x)$ is an integral of $f(x)$ with respect to x. The notation is $F(x) = \int f(x)dx$ or, in differential form, $dF(x) = f(x)dx$.

For any function $F(x)$ that is an integral of $f(x)$ it follows that $F(x) + C$ is also an integral. We thus write

$$\int f(x)dx = F(x) + C.$$

(See Table of Integrals)

7.2 Definite Integral

Let $f(x)$ be defined on the interval $[a, b]$, which is partitioned by points $x_1, x_2, \ldots, x_j, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. The jth interval has length $\Delta x_j = x_j - x_{j-1}$, which may vary with j. The sum $\Sigma_{j=1}^{n} f(v_j)\Delta x_j$, where v_j is arbitrarily chosen in the jth subinterval, depends on the numbers $x_0, \ldots, x_n$ and the choice of the v as well as f, but if such sums approach a common value as all Δx approach zero, then this value is the definite integral of f over the interval (a, b) and is denoted $\int_a^b f(x)dx$. The *fundamental theorem of integral calculus* states that

$$\int_a^b f(x)dx = F(b) - F(a),$$

where F is any continuous indefinite integral of f in the interval (a, b).

7.3 Properties

$$\int_a^b [f_1(x) + f_2(x) + \cdots + f_j(x)]dx$$

$$= \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \cdots + \int_a^b f_j(x)dx.$$

$$\int_a^b c\, f(x)dx = c\int_a^b f(x)dx,\text{ if } c \text{ is a constant.}$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

7.4 Common Applications of the Definite Integral

- *Area* (*Rectangular Coordinates*)

  Given the function $y = f(x)$ such that $y > 0$ for all x between a and b, the area bounded by the curve $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b f(x)dx.$$

- *Length of Arc (Rectangular Coordinates)*

  Given the smooth curve $f(x, y) = 0$ from point (x_1, y_1) to point (x_2, y_2), the length between these points is

$$L = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2}\, dx,$$

$$L = \int_{x_1}^{x_2} \sqrt{1 + (dx/dy)^2}\, dy.$$

- *Mean Value of a Function*

  The mean value of a function $f(x)$ continuous on $[a, b]$ is

$$\frac{1}{(b-a)} \int_a^b f(x)dx.$$

- *Area (Polar Coordinates)*

  Given the curve $r = f(\theta)$, continuous and nonnegative for $\theta_1 \le \theta \le \theta_2$, the area enclosed by this curve and the radial lines $\theta = \theta_1$ and $\theta = \theta_2$ is given by

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} [f(\theta)]^2 d\theta.$$

- *Length of Arc (Polar Coordinates)*

  Given the curve $r = f(\theta)$ with continuous derivative $f'(\theta)$ on $\theta_1 \le \theta \le \theta_2$, the length of arc from $\theta = \theta_1$ to $\theta = \theta_2$ is

$$L = \int_{\theta_1}^{\theta_2} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta.$$

- *Volume of Revolution*

  Given a function $y = f(x)$ continuous and nonnegative on the interval (a, b), when the region bounded by $f(x)$ between a and b is revolved about the x-axis, the volume of revolution is

$$V = \pi \int_a^b [f(x)]^2\, dx.$$

- *Surface Area of Revolution (Revolution about the x-axis, between a and b)*

  If the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x-axis, the area A of the surface generated is given by the following:

$$A = \int_a^b 2\pi f(x)\{1 + [f'(x)]^2\}^{1/2}\, dx$$

- *Work*

  If a variable force $f(x)$ is applied to an object in the direction of motion along the x-axis between $x = a$ and $x = b$, the work done is

$$W = \int_a^b f(x)dx.$$

7.5 Cylindrical and Spherical Coordinates

a. Cylindrical coordinates (Figure 7.1):

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Element of volume, $dV = rdrd\theta dz$.

b. Spherical coordinates (Figure 7.2):

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

Element of volume, $dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$.

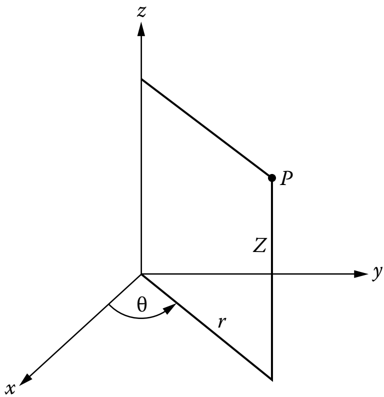


FIGURE 7.1
Cylindrical coordinates.

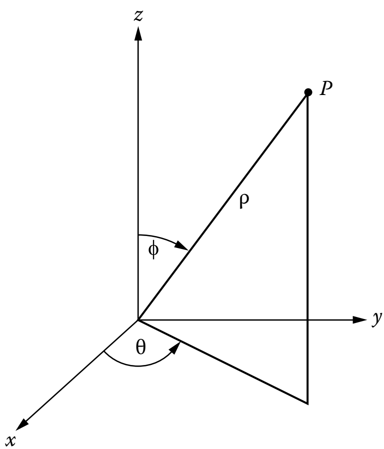


FIGURE 7.2
Spherical coordinates.

7.6 Double Integration

The evaluation of a double integral of $f(x, y)$ over a plane region R,

$$\iint_R f(x,y)dA$$

is practically accomplished by iterated (repeated) integration. For example, suppose that a vertical straight line meets the boundary of R in at most two points so that there is an upper boundary, $y = y_2(x)$, and a lower boundary, $y = y_1(x)$. Also, it is assumed that these functions are continuous from a to b (see Figure 7.3). Then

$$\iint_R f(x,y)dA = \int_a^b \left(\int_{y_1(x)}^{y_2(x)} f(x,y)dy \right) dx$$

If R has a left-hand boundary, $x = x_1(y)$, and a right-hand boundary, $x = x_2(y)$, which are continuous from c to d (the extreme values of y in R), then

$$\iint_R f(x,y)dA = \int_c^d \left(\int_{x_1(y)}^{x_2(y)} f(x,y)dx \right) dy$$

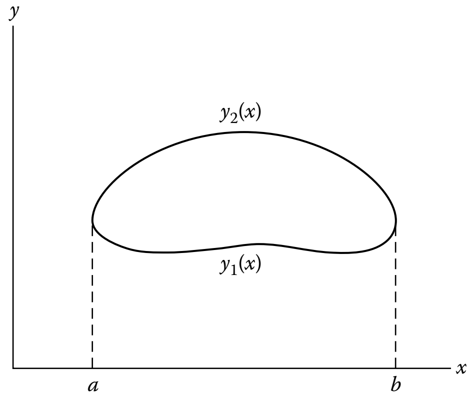


FIGURE 7.3
Region R bounded by $y_2(x)$ and $y_1(x)$.

Such integrations are sometimes more convenient in polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$; $dA = r dr\, d\theta$.

7.7 Surface Area and Volume by Double Integration

For the surface given $z = f(x, y)$, which projects onto the closed region R of the x-y-plane, one may calculate the volume V bounded above by the surface and below by R, and the surface area S by the following:

$$V = \iint_R z\,dA = \iint_R f(x,y)\,dx\,dy$$

$$S = \iint_R [1 + (\delta z/\delta x)^2 + (\delta z/\delta y)^2]^{1/2}\,dx\,dy$$

[In polar coordinates, (r, θ), we replace dA by $r\,dr\,d\theta$.]

7.8 Centroid

The centroid of a region R of the x-y-plane is a point (x', y') where

$$x' = \frac{1}{A}\iint_R x\,dA; \quad y' = \frac{1}{A}\iint_R y\,dA$$

and A is the area of the region.

Example

For the circular sector of angle 2α and radius R, the area A is αR^2; the integral needed for x', expressed in polar coordinates, is

$$\iint x\,dA = \int_{-\alpha}^{\alpha}\int_0^R (r\cos\theta)\,r\,dr\,d\theta$$

$$= \left[\frac{R^3}{3}\sin\theta\right]_{-\alpha}^{+\alpha} = \frac{2}{3}R^3\sin\alpha$$

and thus,

$$x' = \frac{\frac{2}{3}R^3 \sin\alpha}{\alpha R^2} = \frac{2}{3} R \frac{\sin\alpha}{\alpha}.$$

Centroids of some common regions are shown in Figure 7.4.

Centroids of some common regions are shown below:

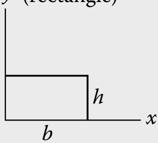
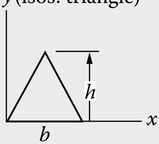
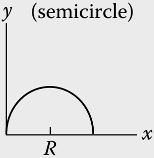
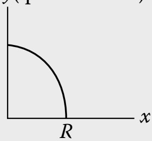
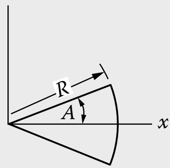
		Centroids	
	Area	x'	y'
y (rectangle) 	bh	$b/2$	$h/2$
y (isos. triangle)* 	$bh/2$	$b/2$	$h/3$
y (semicircle) 	$\pi R^2/2$	R	$4R/3\pi$

FIGURE 7.4
Centroids of common regions. (*Continued*)

y(quarter circle) 	$\pi R^2/4$	$4R/3\pi$	$4R/3\pi$
y(circular sector) 	R^2A	$2R \sin A/3A$	0

*$y' = h/3$ for any triangle of altitude h.

FIGURE 7.4 (*Continued*)
Centroids of common regions.

7.9 Applications of Integration

7.9.1 Indefinite Integral

Example: Growth through Regular Saving

The indefinite integral is used in numerous applications in diverse fields. Illustrated here is a situation involving regular savings and growth through interest that leads to calculated amounts at any time. We start with an interest rate (per month) of k and a fixed dollar amount (P) that is saved each month, that is, we start with no money on hand but deposit P at the end of each month. At time t the amount is denoted by x and is given by the following expression (a differential equation) that expresses the rate dx/dt from deposits plus growth:

$$\frac{dx}{dt} = P + kx$$

After separating the terms, we get the integral expression

$$\int \frac{dx}{P + kx} = \int dt$$

which integrates to

$$ln\left\{\frac{P + kx}{C}\right\} = kt, \text{ or } P + kx = C\,e^{kt}.$$

Using $x = 0$ at $t = 0$ gives $C = P$ and, thus,

$$x = \frac{P}{k}\left(e^{kt} - 1\right).$$

Suppose that P = 600/month and that the growth rate is 4% per year (0.003333…/month). After 10 years (120 months), the value of x grows to (600/.003333)($e^{0.4000}$ –1) = \$ 88,537.30. This calculation, done in this way, assumes that the interest and input payment are continuous, whereas the formula given in Section 12.18 for the same data uses discrete monthly payments and interest and gets a slightly different result, that is, $88,348.02

Example: Falling Bodies

Bodies that fall in the earth's gravitational field are subject to a constant acceleration (g) and, if air resistance is negligible, the body gains velocity during the fall. If, however, there is appreciable air resistance, then the situation is different. Suppose that the air exerts a force proportional to the velocity, that is, a force = kv and opposite to the velocity vector. The equation governing the motion is then

$$mg - kv = m\frac{dv}{dt}$$

This is a separable expression and is integrated as follows:

$$m\frac{dv}{mg - kv} = dt$$

$$m\int \frac{dv}{mg - kv} = \int dt$$

$$\frac{-m}{k} \; ln\left\{mg - kv\right\} = t + constant$$

This expression simplifies to

$$mg - kv = Ce^{\frac{-kt}{m}}$$

If the body falls from rest, then $v = 0$ at $t = 0$ so that $C = mg$ and, using that in this equation and rearranging, we get the velocity as a function of time

$$v = \frac{mg}{k}\left(1 - e^{\frac{-kt}{m}}\right)$$

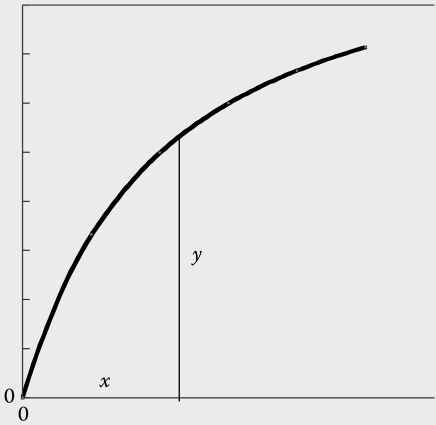
This result shows that the velocity approaches a limit, that is, it becomes approximately *constant at the value* $\frac{mg}{k}$. This is termed "limiting velocity" (or terminal velocity).

7.9.2 The Definite Integral

Example: Volumes of Revolution

A curve in rectangular coordinates that is given by $y = f(x)$ has a height y at any value of x that is defined by the equation. If the portion of that curve (between values $x = 0$ and $x = a$) is rotated about the x-axis, the result is a solid of revolution whose volume is determinable from the integral (Section 7.4). The size of the resulting "egg-shaped" solid is based on the value $x = a$, and the volume is given by

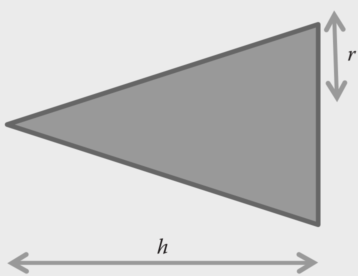
$$V = (\pi)\int_0^a y^2\,dx = (\pi)\int_0^a f(x)^2\,dx$$



Example

If the function is a straight line of slope *r*/*h*, the resulting volume will be a right circular cone of radius *r* and height *h*, whose volume is given by

$$V = (\pi)\int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi r^2 h/3$$

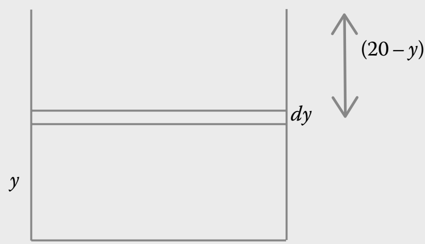


Example: Work

The definite integral is used to calculate the work in situations in which the force or distance through which the force acts varies. An illustrative problem is that of a cylindrical tank, radius = 10 feet and height = 20 feet, that contains water. To calculate the work in pumping this out, we consider a slab of thickness *dy* at height *y*. The density of water (62.4 lbs/ft^3) is denoted here by *w*, and therefore the weight of this circular slab is $w\pi(10)^2 dy$ and, for a small *dy*, should be moved a vertical distance = $(20 - y)$. Thus, the integral below is evaluated

$$W = 100\pi w \int_0^{20} (20 - y)\, dy$$

$$= 100\pi w(200) = 1{,}248{,}000\pi \text{ ft/lbs}$$



Example: Average Value

The average value of a function is determined from the definite integral. A continuous function defined on the x-axis between a and b has the average value given by

$$\frac{1}{(b-a)}\int_a^b f(x)\,dx$$

the

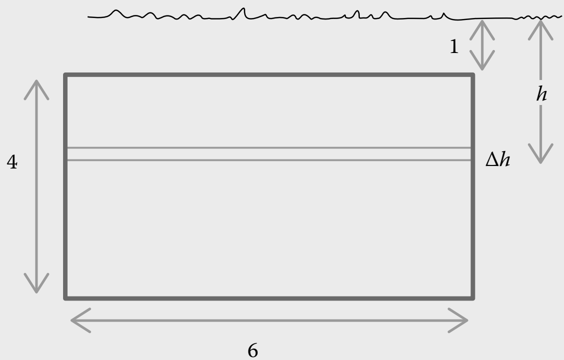
For example, one arch of a sine curve has average value $= \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{2}{\pi}$

Example: Liquid Pressure

The pressure at any point of a resting liquid is the same and is proportional to the depth below the surface of the liquid. If ω is the weight density and h is the depth, then the pressure $p = \omega h$. A flat surface submerged (as shown below) will experience a force F given by

$$F = \int_1^5 l\omega h\,dh = \int_1^5 6\omega h\,dh = 72\,\omega$$

where l is the surface width (in this case 6) at depth h. In the following diagram, a face representing one end of a rectangular tank is 6 ft wide, 4 feet deep, and is submerged **one foot below the water surface**. Thus, the integration limits are 1 to 5.



If the liquid is water, then ω = 62.5 pounds per cubic foot and the force from above F = 4500 pounds.

8

Vector Analysis

8.1 Vectors

Given the set of mutually perpendicular unit vectors **i**, **j**, and **k** (Figure 8.1), any vector in the space may be represented as $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, where a, b, and c are *components.*

- Magnitude of ***F***

$$|\mathbf{F}| = (a^2 + b^2 + c^2)^{\frac{1}{2}}$$

- *Product by Scalar **p***

$$p\mathbf{F} = pa\mathbf{i} + pb\mathbf{j} + pc\mathbf{k}.$$

- *Sum of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$*

$$\mathbf{F}_1 + \mathbf{F}_2 = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$$

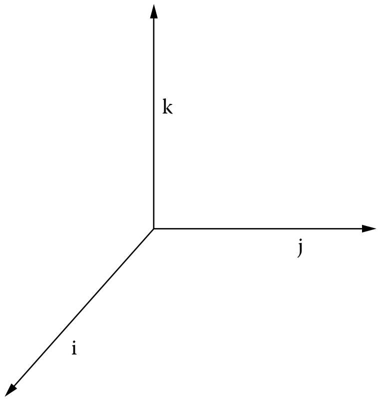


FIGURE 8.1
The unit vectors **i**, **j**, and **k**.

- *Scalar Product*

$$\mathbf{F}_1 \bullet \mathbf{F}_2 = a_1a_2 + b_1b_2 + c_1c_2$$

(Thus, $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$.)

Also

$$\mathbf{F}_1 \bullet \mathbf{F}_2 = \mathbf{F}_2 \bullet \mathbf{F}_1$$

$$(\mathbf{F}_1 + \mathbf{F}_2) \bullet \mathbf{F}_3 = \mathbf{F}_1 \bullet \mathbf{F}_3 + \mathbf{F}_2 \bullet \mathbf{F}_3$$

- *Vector Product*

$$\mathbf{F}_1 \times \mathbf{F}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

(Thus, $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$, $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Also,

$$\mathbf{F}_1 \times \mathbf{F}_2 = -\mathbf{F}_2 \times \mathbf{F}_1$$

$$(\mathbf{F}_1 + \mathbf{F}_2) \times \mathbf{F}_3 = \mathbf{F}_1 \times \mathbf{F}_3 + \mathbf{F}_2 \times \mathbf{F}_3$$

$$\mathbf{F}_1 \times (\mathbf{F}_2 + \mathbf{F}_3) = \mathbf{F}_1 \times \mathbf{F}_2 + \mathbf{F}_1 \times \mathbf{F}_3$$

$$\mathbf{F}_1 \times (\mathbf{F}_2 + \mathbf{F}_3) = (\mathbf{F}_1 \bullet \mathbf{F}_3)\mathbf{F}_2 - (\mathbf{F}_1 \bullet \mathbf{F}_2)\mathbf{F}_3$$

$$\mathbf{F}_1 \bullet (\mathbf{F}_2 \times \mathbf{F}_3) = (\mathbf{F}_1 \times \mathbf{F}_2) \bullet \mathbf{F}_3$$

8.2 Vector Differentiation

If $\mathbf{V}$ is a vector function of a scalar variable t, then

$$\mathbf{V} = a(t)\mathbf{i} + b(t)\mathbf{j} + c(t)\mathbf{k}$$

and

$$\frac{d\mathbf{V}}{dt} = \frac{da}{dt}\mathbf{i} + \frac{db}{dt}\mathbf{j} + \frac{dc}{dt}\mathbf{k}.$$

For several vector functions $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n$

$$\frac{d}{dt}\left(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n\right) = \frac{d\mathbf{V}_1}{dt} + \frac{d\mathbf{V}_2}{dt} + \cdots + \frac{d\mathbf{V}_n}{dt},$$

$$\frac{d}{dt}\left(\mathbf{V}_1 \bullet \mathbf{V}_2\right) = \frac{d\mathbf{V}_1}{dt} \bullet \mathbf{V}_2 + \mathbf{V}_1 \bullet \frac{d\mathbf{V}_2}{dt},$$

$$\frac{d}{dt}\left(\mathbf{V}_1 \times \mathbf{V}_2\right) = \frac{d\mathbf{V}_1}{dt} \times \mathbf{V}_2 + \mathbf{V}_1 \times \frac{d\mathbf{V}_2}{dt}.$$

For a scalar-valued function $g(x, y, z)$,

$$(\textbf{gradient})\quad \text{grad}\, g = \boldsymbol{\nabla} g = \frac{\delta g}{\delta x}\mathbf{i} + \frac{\delta g}{\delta y}\mathbf{j} + \frac{\delta g}{\delta z}\mathbf{k}.$$

For a vector-valued function $\mathbf{V}(a, b, c)$, where a, b, and c are each a function of x, y, and z, respectively,

$$(\textbf{divergence})\ \text{div}\, \mathbf{V} = \nabla \bullet \mathbf{V} = \frac{\delta a}{\delta x} + \frac{\delta b}{\delta y} + \frac{\delta c}{\delta z}$$

$$(\textbf{curl})\quad \text{curl}\, \mathbf{V} = \nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ a & b & c \end{vmatrix}$$

Also,

$$\text{div}\,\text{grad}\, g = \nabla^2 g = \frac{\delta^2 g}{\delta x^2} + \frac{\delta^2 g}{\delta y^2} + \frac{\delta^2 g}{\delta z^2}$$

and

$$\text{curl}\,\text{grad}\, g = \mathbf{0};\quad \text{div}\,\text{curl}\,\mathbf{V} = 0;$$

$$\text{curl}\,\text{curl}\,\mathbf{V} = \text{grad}\,\text{div}\,\mathbf{V} - (\mathbf{i}\boldsymbol{\nabla}^2 a + \mathbf{j}\boldsymbol{\nabla}^2 b + \mathbf{k}\boldsymbol{\nabla}^2 c).$$

8.3 Divergence Theorem (Gauss)

Given a vector function ***F*** with continuous partial derivatives in a region ***R*** bounded by a closed surface ***S***,

$$\iiint_R \operatorname{div} \mathbf{F}\, d\mathbf{V} = \iint_S \mathbf{n} \bullet \mathbf{F}\, dS,$$

where **n** is the (sectionally continuous) unit normal to ***S***.

8.4 Stokes' Theorem

Given a vector function with continuous gradient over a surface ***S*** that consists of portions that are piecewise smooth and bounded by regular closed curves such as ***C***,

$$\iint_S \mathbf{n} \bullet \operatorname{curl} \mathbf{F}\, dS = \oint_C \mathbf{F} \bullet d\mathbf{r}$$

8.5 Planar Motion in Polar Coordinates

Motion in a plane may be expressed with regard to polar coordinates (r, θ). Denoting the position vector by $\mathbf{r}$ and its magnitude by r, we have $\mathbf{r} = r\mathbf{R}(\theta)$, where **R** is the unit vector. Also, $d\mathbf{R}/d\theta = \mathbf{P}$, a unit vector perpendicular to **R**. The velocity and acceleration are then

$$\mathbf{V} = \frac{dr}{dt}\mathbf{R} + r\frac{d\theta}{dt}\mathbf{P};$$

$$\mathbf{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{R} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{P}.$$

Note that the component of acceleration in the **P** direction (transverse component) may also be written

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

so that in purely radial motion it is zero and

$$r^2\frac{d\theta}{dt} = C(\text{constant})$$

which means that the position vector sweeps out area at a constant rate (see "Area (Polar Coordinates)," Section 7.4).

8.6 Geostationary Satellite Orbit

A satellite in circular orbit with velocity v around the equator at height h has a central acceleration, $\frac{v^2}{R+h}$, where R is the radius of the earth. From

Newton's second law this acceleration equals $\frac{MG}{(R+h)^2}$, where M is the mass of the earth and G is the gravitational constant, thereby giving orbital velocity $\left[\frac{MG}{R+h}\right]^{1/2}$ and angular velocity $\omega = \frac{(MG)^{1/2}}{(R+h)^{3/2}}$. Inserting constants $M = 5.98 \times 10^{24}$ kg, $R = 6.37 \times 10^6$ m, $G = 6.67 \times 10^{-11}$ N·m²/kg², and earth's angular velocity $\omega = 7.27 \times 10^{-5}$/s, one finds $h \approx 35{,}790$ km. Thus, a satellite orbiting around the equator at this height above the earth's surface appears stationary.

9

Special Functions

9.1 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \operatorname{ctnh} x = \frac{1}{\tanh x}$$

$$\sinh(-x) = -\sinh x \qquad \operatorname{ctnh}(-x) = -\operatorname{ctnh} x$$

$$\cosh(-x) = \cosh x \qquad \operatorname{sech}(-x) = \operatorname{sech} x$$

$$\tanh(-x) = -\tanh x \qquad \operatorname{csch}(-x) = -\operatorname{csch} x$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \operatorname{ctnh} x = \frac{\cosh x}{\sinh x}$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad \cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$$

$$\sinh^2 x = \frac{1}{2}(\cos 2x - 1) \qquad \text{ctnh}^2\, x - \text{csch}^2\, x = 1$$

$$\text{csch}^2\, x - \text{sech}^2\, x \qquad \tanh^2 x + \text{sech}^2\, x = 1$$

$$= \text{csch}^2\, x \ \text{sech}^2\, x$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$$

$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$$

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$$

9.2 Gamma Function (Generalized Factorial Function)

The gamma function, denoted $\Gamma(x)$, is defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$$

- *Properties*

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0$$

$$\Gamma(1) = 1$$

$$\Gamma(n+1) = n\Gamma(n) = n! \quad (n = 1, 2, 3, \ldots)$$

$$\Gamma(x)\Gamma(1-x) = \pi / \sin\pi x$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$2^{2x-1}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2x)$$

9.3 Laplace Transforms

The Laplace transform of the function $f(t)$, denoted by $F(s)$ or $L\{f(t)\}$, is defined

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

provided that the integration may be validly performed. A sufficient condition for the existence of $F(s)$ is that $f(t)$ be of exponential order as $t \to \infty$ and that it is sectionally continuous over every finite interval in the range $t \geq 0$. The Laplace transform of $g(t)$ is denoted by $L\{g(t)\}$ or $G(s)$.

- *Operations*

$f(t)$	$F(s)=\int_0^\infty f(t)e^{-st}dt$
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)$ $-s^{n-2}f'(0)$ $-\cdots-f^{(n-1)}(0)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$e^{at}f(t)$	$F(s-a)$
$\int_0^t f(t-\beta)\cdot g(\beta)d\beta$	$F(s)\cdot G(s)$
$f(t-a)$	$e^{-as}F(s)$
$f\left(\frac{t}{a}\right)$	$aF(as)$
$\int_0^t g(\beta)d\beta$	$\frac{1}{s}G(s)$

$f(t-c)\delta(t-c)$	$e^{-cs}F(s), c > 0$

where

$$\delta(t-c) = 0 \quad \text{if } 0 \le t < c$$
$$= 1 \quad \text{if } t \ge c$$

$f(t) = f(t+\omega)$ (periodic)	$\dfrac{\int_0^{\omega} e^{-s\tau} f(\tau)d\tau}{1-e^{-s\omega}}$

- *Table of Laplace Transforms*

$f(t)$	$F(s)$
1	$1/s$
t	$1/s^2$
$\dfrac{t^{n-1}}{(n-1)!}$	$1/s^n \quad (n = 1, 2, 3\ldots)$
$\sqrt{t}$	$\dfrac{1}{2s}\sqrt{\dfrac{\neq}{s}}$
$\dfrac{1}{\sqrt{t}}$	$\sqrt{\dfrac{\neq}{s}}$
e^{at}	$\dfrac{1}{s-a}$
te^{at}	$\dfrac{1}{(s-a)^2}$
$\dfrac{t^{n-1}e^{at}}{(n-1)!}$	$\dfrac{1}{(s-a)^n} \quad (n = 1, 2, 3\ldots)$

$\frac{t^x}{\Gamma(x+1)}$	$\frac{1}{s^{x+1}}, \quad x > -1$
$\sin at$	$\frac{a}{s^2 + a^2}$
$\cos at$	$\frac{s}{s^2 + a^2}$
$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cosh at$	$\frac{s}{s^2 - a^2}$
$e^{at} - e^{bt}$	$\frac{a-b}{(s-a)(s-b)}, \quad (a \neq b)$
$ae^{at} - be^{bt}$	$\frac{s(a-b)}{(s-a)(s-b)}, \quad (a \neq b)$
$t \sin at$	$\frac{2as}{(s^2 + a^2)^2}$
$t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$
$\frac{\sin at}{t}$	$Arc\ tan \frac{a}{s}$
$\frac{\sinh at}{t}$	$\frac{1}{2} \log_e \left(\frac{s+a}{s-a} \right)$

9.4 *z*-Transform

For the real-valued sequence $\{f(k)\}$ and complex variable z, the z-transform, $F(z) = Z\{f(k)\}$, is defined by

$$Z\{f(k)\} = F(\mathrm{z}) = \sum_{k=0}^{\infty} f(k)z^{-k}$$

For example, the sequence $f(k) = 1, k = 0, 1, 2, \ldots$, has the z-transform

$$F(z) = 1 + z^{-1} + z^{-2} + z^{-3} \cdots + z^{-k} + \cdots$$

- *z-Transform and the Laplace Transform*
  For function $U(t)$, the output of the ideal sampler $U^*(t)$ is a set of values $U(kT)$, $k = 0, 1, 2, \ldots$, that is,

  $$U^*(t) = \sum_{k=0}^{\infty} U(t)\delta(t - kT)$$

  The Laplace transform of the output is

  $$\{U^*(t)\} = \int_0^{\infty} e^{-st} U^*(t)dt$$

  $$= \int_0^{\infty} e^{-st} \sum_{k=0}^{\infty} U(t)\delta(t - kT)dt$$

  $$= \sum_{k=0}^{\infty} e^{-sKT} U(kT)$$

Defining $z = e^{sT}$ gives

$$Z\{U^*(t)\} = \sum_{k=0}^{\infty} U(kT)z^{-k}$$

which is the z-transform of the sampled signal $U(kT)$.

- *Properties*

Linearity: $Z\{af_1(k) + bf_2(k)\}$

$$= aZ\{f_1(k)\} + bZ\{f_2(k)\}$$

$$= aF_1(z) + bF_2(z)$$

Right-shifting property: $Z\{f(k-n)\} = z^{-n}F(z)$

Left-shifting property: $Z\{f(k+n)\} = z^{n}F(z)$

$$- \sum_{k=0}^{n-1} f(k)z^{n-k}$$

Time scaling: $Z\{a^k f(k)\} = F(z/a)$

Multiplication by k: $Z\{kf(k)\} = -zdF(z)/dz$

Initial value: $f(0) = \lim_{z\to\infty}(1-z^{-1})F(z) = F(\infty)$

Final value: $\lim_{k\to\infty} f(k) = \lim_{z\to 1}(1-z^{-1})F(z)$

Convolution: $Z\{f_1(k) * f_2(k)\} = F_1(z)F_2(z)$

- *z-Transforms of Sampled Functions*

$f(k)$	$Z\{f(kT)\} = F(z)$
1 at k; else 0	z^{-k}
1	$\frac{z}{z-1}$
kT	$\frac{Tz}{(z-1)^2}$
$(kT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$
$\sin \omega kT$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$
$\cos \omega T$	$\frac{z(z - \cos \omega T)}{z^2 - 2z\cos \omega T + 1}$
e^{-akT}	$\frac{z}{z - e^{-aT}}$
kTe^{-akT}	$\frac{zTe^{-aT}}{(z - e^{-aT})^2}$
$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT} z(z + e^{-aT})}{(z - e^{-aT})^3}$
$e^{-akT} \sin \omega kT$	$\frac{ze^{-aT} \sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$
$e^{-akT} \cos \omega kT$	$\frac{z(z - e^{-aT}\cos \omega T)}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$

$a^k \sin \omega kT$	$\dfrac{az \sin \omega T}{z^2 - 2az \cos \omega T + a^2}$
$a^k \cos \omega kT$	$\dfrac{z(z - a \cos \omega T)}{z^2 - 2az \cos \omega T + a^2}$

9.5 Fourier Series

The periodic function $f(t)$, with period 2π, may be represented by the trigonometric series

$$a_0 + \sum_{1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where the coefficients are determined from

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \quad (n = 1, 2, 3, \ldots)$$

Such a trigonometric series is called the Fourier series corresponding to $f(t)$, and the coefficients are termed Fourier coefficients of $f(t)$. If the function is piecewise continuous in the interval

$-\pi \le t \le \pi$, and has left- and right-hand derivatives at each point in that interval, then the series is convergent with sum *f*(*t*) except at points t_i at which *f*(*t*) is discontinuous. At such points of discontinuity, the sum of the series is the arithmetic mean of the right- and left-hand limits of *f*(*t*) at t_i. The integrals in the formulas for the Fourier coefficients can have limits of integration that span a length of 2π, for example, 0 to 2π (because of the periodicity of the integrands).

9.6 Functions with Period Other than 2π

If *f*(*t*) has period *P*, the Fourier series is

$$f(t) \sim a_0 + \sum_{1}^{\infty}\left(a_n \cos\frac{2\pi n}{P}t + b_n \sin\frac{2\pi n}{P}t\right),$$

where

$$a_0 = \frac{1}{P}\int_{-P/2}^{P/2} f(t)dt$$

$$a_n = \frac{2}{P}\int_{-P/2}^{P/2} f(t)\cos\frac{2\pi n}{P}tdt$$

$$b_n = \frac{2}{P}\int_{-P/2}^{P/2} f(t)\sin\frac{2\pi n}{P}tdt$$

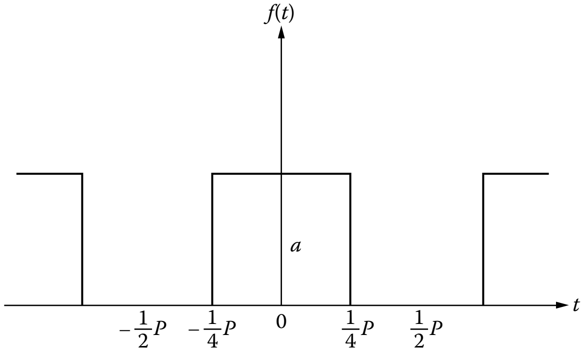


FIGURE 9.1
Square wave: $f(t) \sim \frac{a}{2} + \frac{2a}{\pi}\left(\cos\frac{2\pi t}{P} - \frac{1}{3}\cos\frac{6\pi t}{P} + \frac{1}{5}\cos\frac{10\pi t}{P} + \cdots\right)$.

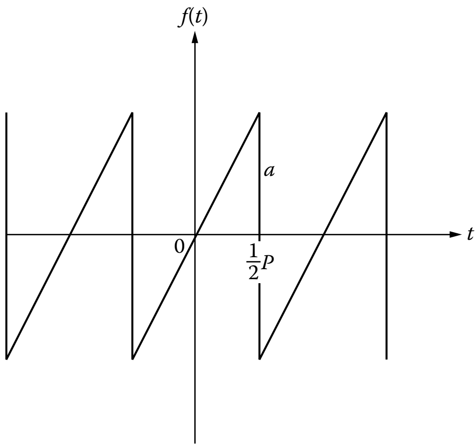


FIGURE 9.2
Sawtooth wave: $f(t) \sim \frac{2a}{\pi}\left(\sin\frac{2\pi t}{P} - \frac{1}{2}\sin\frac{4\pi t}{P} + \frac{1}{3}\sin\frac{6\pi t}{P} - \cdots\right)$.

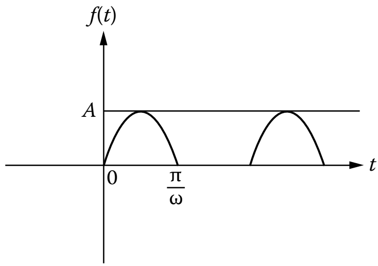


FIGURE 9.3
Half-wave rectifier: $f(t) \sim \frac{A}{\pi} + \frac{A}{2}\sin\omega t - \frac{2A}{\pi}\left(\frac{1}{(1)(3)}\cos 2\omega t + \frac{1}{(3)(5)}\cos 4\omega t + \cdots\right).$

Again, the interval of integration in these formulas may be replaced by an interval of length P, for example, 0 to P.

9.7 Bessel Functions

Bessel functions, also called cylindrical functions, arise in many physical problems as solutions of the differential equation

$$x^2 y'' + xy' + (x^2 - n^2)y = 0$$

which is known as Bessel's equation. Certain solutions of the above, known as *Bessel functions of the first kind of order n*, are given by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{n+2k}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{-n+2k}$$

In the above it is noteworthy that the gamma function must be defined for the negative argument q: $\Gamma(q) = \Gamma(q+1)/q$, provided that q is not a negative integer. When q is a negative integer, $1/\Gamma(q)$ is defined to be zero. The functions $J_{-n}(x)$ and $J_n(x)$ are solutions of Bessel's equation for all real n. It is seen, for $n = 1, 2, 3, \ldots$ that

$$J_{-n}(x) = (-1)^n J_n(x)$$

and therefore, these are not independent; hence, a linear combination of these is not a general solution. When, however, n is not a positive integer, negative integer, nor zero, the linear combination with arbitrary constants c_1 and c_2

$$y = c_1 J_n(x) + c_2 J_{-n}(x)$$

is the general solution of the Bessel differential equation.

The zero-order function is especially important as it arises in the solution of the heat equation (for a "long" cylinder):

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \cdots$$

while the following relations show a connection to the trigonometric functions:

$$J_{\frac{1}{2}}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \sin x$$

$$J_{-\frac{1}{2}}(x) = \left[\frac{2}{\pi x}\right]^{1/2} \cos x$$

The following recursion formula gives $J_{n+1}(x)$ for any order in terms of lower-order functions:

$$\frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

9.8 Legendre Polynomials

The Legendre equation arises in the solution of certain physical problems that are analyzed in spherical coordinates. The basic equation is as follows:

$$\frac{d}{dx}\left[\left(1 - x^2\right)\frac{d}{dx}Y\right] + n\left(n+1\right)Y = 0$$

If n is an integer, then the solutions are polynomials; hence the equation is often written as

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{d}{dx}P_n\left(x\right)\right]+n\left(n+1\right)P_n\left(x\right)=0.$$

First few solutions are listed as follows:

$$P_0\left(x\right)=1$$

$$P_1\left(x\right)=x$$

$$P_2\left(x\right)=\left(\frac{1}{2}\right)(3x^2-1)$$

$$P_3\left(x\right)=\left(\frac{1}{2}\right)(5x^3-3x)$$

$$P_4\left(x\right)=\left(\frac{1}{8}\right)(35x^4-30x^2+3)$$

$$P_5\left(x\right)=\left(\frac{1}{8}\right)(63x^5-70x^3+15x)$$

Additional Legendre polynomials may be determined from the *Rodrigues formula*:

$$P_n\left(x\right)=\left(\frac{1}{2^n n!}\right)\frac{d^n}{dx^n}(x^2-1)^n$$

9.9 Laguerre Polynomials

Laguerre polynomials, denoted $L_n(x)$, are solutions of the differential equation

$$xy^n + (1 - x)y' + ny = 0$$

and are given by

$$L_n(x) = \sum_{j=0}^{n} \frac{(-1)^j}{j!} C_{(n,j)} x^j \quad (n = 0, 1, 2, \ldots)$$

Thus,

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$

$$L_2(x) = 1 - 2x + \frac{1}{2}x^2$$

$$L_3(x) = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3$$

Additional Laguerre polynomials may be obtained from the recursion formula

$$(n + 1)L_{n+1}(x) - (2n + 1 - x)L_n(x) + nL_{n-1}(x) = 0$$

9.10 Hermite Polynomials

The Hermite polynomials, denoted $H_n(x)$, are given by

$$H_0 = 1, H_n(x) = (-1)^n e^{x^2} \frac{d^n e^{-x^2}}{dx^n}, \quad (n = 1, 2, \ldots)$$

and are solutions of the differential equation

$$y^n - 2xy' + 2ny = 0 \quad (n = 0, 1, 2, \ldots)$$

The first few Hermite polynomials are

$$H_0 = 1 \qquad H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2 \qquad H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

Additional Hermite polynomials may be obtained from the relation

$$H_{n+1}(x) = 2xH_n(x) - H_n'(x),$$

where prime denotes differentiation with respect to x.

9.11 Orthogonality

A set of functions $\{f_n(x)\}(n = 1, 2, \ldots)$ is orthogonal in an interval (a, b) with respect to a given weight function $w(x)$ if

$$\int_a^b w(x) f_m(x) \bar{f}_n(x) dx = 0 \quad \text{when } m \neq n$$

The following polynomials are orthogonal on the given interval for the given $w(x)$:

$$\text{Legendre polynomials}: P_n(x) \quad w(x) = 1$$

$$a = -1, b = 1$$

$$\text{Laguerre polynomials}: L_n(x) \quad w(x) = \exp(-x)$$

$$a = 0, b = \infty$$

$$\text{Hermite polynomials}: H_n(x) \quad w(x) = \exp(-x^2)$$

$$a = -\infty, b = \infty$$

The Bessel functions of order n, $J_n(\lambda_1 x)$, $J_n(\lambda_2 x), \ldots,$ are orthogonal with respect to $w(x) = x$ over the interval $(0, c)$ provided that the λ_i are the positive roots of $J_n(\lambda c) = 0$:

$$\int_0^c x J_n(\lambda_j x) J_n(\lambda_k x) dx = 0 \quad (j \neq k)$$

where n is fixed and $n \geq 0$.

10

Differential Equations

10.1 First-Order, First-Degree Equations

$$M(x, y)dx + N(x, y)dy = 0$$

a. If the equation can be put in the form $A(x)dx + B(y)dy = 0$, it is *separable* and the solution follows by integration: $\int A(x)\,dx + \int B(y)\,dy = C$; thus, $x(1 + y^2)dx + ydy = 0$ is separable since it is equivalent to $xdx + ydy/(1 + y^2) = 0$, and integration yields $x^2/2 + \frac{1}{2}\log(1 + y^2) + C = 0$.

b. If $M(x, y)$ and $N(x, y)$ are *homogeneous* and of the *same degree* in x and y, then substitution of vx for y (thus, $dy = vdx + xdv$) will yield a separable equation in the variables x and y. [A function such as $M(x, y)$ is homogeneous of degree n in x and y

if $M(cx, cy) = c^n M(x, y)$.] For example, $(y - 2x)dx + (2y + x)dy$ has M and N each homogenous and of degree 1 so that substitution of $y = vx$ yields the separable equation

$$\frac{2}{x}dx + \frac{2v+1}{v^2+v-1}dv = 0.$$

c. If $M(x, y)dx + N(x, y)dy$ is the differential of some function $F(x, y)$, then the given equation is said to be *exact.* A necessary and sufficient condition for exactness is $\partial M/\partial y = \partial N/\partial x$. When the equation is exact, F is found from the relations $\partial F/\partial x = M$ and $\partial F/\partial y = N$, and the solution is $F(x, y) = C$ (constant). For example, $(x^2 + y)dy + (2xy - 3x^2)dx$ is exact since $\partial M/\partial y = 2x$ and $\partial N/\partial x = 2x$. F is found from $\partial F/\partial x = 2xy - 3x^2$ and $\partial F/\partial y = x^2 + y$. From the first of these, $F = x^2y - x^3 + \phi(y)$; from the second, $F = x^2y + y^2/2 + \Psi(x)$. It follows that $F = x^2y - x^3 + y^2/2$, and $F = C$ is the solution.

d. Linear, order 1 in y: Such an equation has the form $dy + P(x)ydx = Q(x)dx$. Multiplication by exp $[P(x)dx]$ yields

$$d\left[y\exp\left(\int P\,dx\right)\right] = Q(x)\exp\left(\int P dx\right)dx.$$

For example, $dy + (2/x)ydx = x^2dx$ is linear in y. $P(x) = 2/x$, so $\int Pdx = 2\ln x = \ln x^2$, and $\exp\left(\int Pdx\right) = x^2$. Multiplication by x^2 yields $d(x^2y) = x^4dx$, and integration gives the solution $x^2y = x^5/5 + C$.

- *Application of Linear-Order 1 Differential Equations: Drug Kinetics*

  A substance (e.g., a drug) placed in one compartment is eliminated from that compartment at a rate proportional to the quantity it contains, and this elimination moves it to a second compartment (such as blood) that originally does not contain the substance. The second compartment also eliminates the substance to an external sink and does so at a rate proportional to the quantity it contains. If D denotes the initial amount in the first compartment, and the elimination rate constants from each compartment are denoted k_1 and k_2, respectively, then the quantities in compartment 1 (denoted X) and compartment 2 (denoted Y) at any time t are described by

$$\frac{dX}{dt} = -k_1X \quad X(0) = D \quad \text{(compartment 1)}$$

$$\frac{dY}{dt} = k_1X - k_2Y \quad Y(0) = D \quad \text{(compartment 2)}$$

from which

$$X = De^{-k_1 t}$$

so that

$$\frac{dY}{dt} + k_2 Y = k_1 D e^{-k_1 t}, \text{ a linear order 1 equation}$$

with solution

$$Y = \left(\frac{k_1 D}{k_2 - k_1}\right)(e^{-k_1 t} - e^{-k_2 t})$$

This illustrates a model that is commonly used to describe the movement of a drug from some entry site into and out of the blood.

10.2 Second-Order Linear Equations (with Constant Coefficients)

$$(b_0 D^2 + b_1 D + b_2)y = f(x), \quad D = \frac{d}{dx}.$$

a. Right-hand side = 0 (homogeneous case)

$$(b_0 D^2 + b_1 D + b_2)y = 0.$$

The *auxiliary equation* associated with the above is

$$b_0m^2 + b_1m + b_2 = 0.$$

If the roots of the auxiliary equation are *real and distinct*, say, m_1 and m_2, then the solution is

$$y = C_1e^{m_1x} + C_2e^{m_2x}$$

where the C's are arbitrary constants.

If the roots of the auxiliary equation are *real and repeated*, say, $m_1 = m_2 = p$, then the solution is

$$y = C_1e^{px} + C_2xe^{px}.$$

If the roots of the auxiliary equation are *complex* $a + ib$ and $a - ib$, then the solution is

$$y = C_1e^{ax}\cos bx + C_2e^{ax}\sin bx.$$

b. Right-hand side $\neq 0$ (nonhomogeneous case)

$$(b_0D^2 + b_1D + b_2)y = f(x)$$

The general solution is $y = C_1y_1(x) + C_2y_2(x) + y_p(x)$, where y_1 and y_2 are solutions of the corresponding homogeneous equation and y_p is a solution of the given nonhomogeneous differential equation. y_p has the form $y_p(x) = A(x)y_1(x) + B(x)y_2(x)$, and A and B are found from simultaneous solution of $A'y_1 + B'y_2 = 0$ and $A'y'_1 + B'y'_2 = f(x)/b_0$. A solution exists if the determinant

$$\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$$

does not equal zero. The simultaneous equations yield A' and B' from which A and B follow by integration. For example,

$$(D^2 + D - 2)y = e^{-3x}.$$

The auxiliary equation has the distinct roots 1 and –2; hence, $y_1 = e^x$ and $y_2 = e^{-2x}$, so that $y_p = Ae^x + Be^{-2x}$. The simultaneous equations are

$$A'e^x - 2B'e^{-2x} = e^{-3x}$$

$$A'e^x + B'e^{-2x} = 0$$

and give $A' = (1/3)e^{-4x}$ and $B' = (-1/3)e^{-x}$. Thus, $A = (-1/12)e^{-4x}$ and $B = (1/3)e^{-x}$, so that

$$y_p = (-1/12)e^{-3x} + (1/3)e^{-3x}$$

$$= \frac{1}{4}e^{-3x}.$$

$$\therefore y = C_1e^x + C_2e^{-2x} + \frac{1}{4}e^{-3x}.$$

10.3 Runge Kutta Method (of Order 4)

The solution of differential equations may be approximated by numerical methods as described here for the differential equation $dy/dx = f(x, y)$, with $y = y_0$ at $x = x_0$. Step size h is chosen and the solution is approximated over the interval $[x_0, x_n]$, where $x_n = nh$. The approximation follows from the recursion formula

$$y_{n+1} = y_n + (1/6)\,(K_1 + 2K_2 + 2K_3 + K_4)$$

where

$K_1 = hf(x_n, y_n)$
$K_2 = hf(x_n + h/2, y_n + K_1/2)$
$K_3 = hf(x_n + h/2, y_n + K_2/2)$
$K_4 = hf(x_n + h, y_n + K_3)$

11

Statistics

11.1 Arithmetic Mean

$$\mu = \frac{\Sigma X_i}{N},$$

where X_i is a measurement in the population and N is the total number of X_i in the population. For a *sample* of size n, the sample mean, denoted $\bar{X}$, is

$$\bar{X} = \frac{\sum X_i}{n}.$$

11.2 Median

The median is the middle measurement when an odd number (n) of measurement are arranged in order; if n is even, it is the midpoint between the two middle measurements.

11.3 Mode

It is the most frequently occurring measurement in a set.

11.4 Geometric Mean

$$\textit{geometric mean} = \sqrt[n]{X_i X_2 \ldots X_n}$$

11.5 Harmonic Mean

The harmonic mean H of n numbers $X_1, X_2, \ldots,$ X_n, is

$$H = \frac{n}{\sum (1/X_i)}$$

11.6 Variance

The mean of the sum of squares of deviations from the means (μ) is the population variance, denoted σ^2:

$$\sigma^2 = \sum (X_i - \mu)^2 / N.$$

The sample variance, s^2, for sample size n is

$$s^2 = \sum (X_i - \bar{X})^2 / (n-1).$$

A simpler computational form is

$$s^2 = \frac{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}}{n-1}$$

11.7 Standard Deviation

The positive square root of the population variance is the standard deviation. For a population,

$$\sigma = \left[\frac{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{N}}{N} \right]^{1/2};$$

for a sample,

$$s = \left[\frac{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}}{n-1} \right]^{1/2}$$

11.8 Coefficient of Variation

$$V = s/\bar{X}.$$

11.9 Probability

For the sample space U, with subsets A of U (called events), we consider the probability measure of an event A to be a real-valued function p defined over all subsets of U such that

$$0 \le p(A) \le 1$$

$$p(U) = 1 \text{ and } p(\Phi) = 0$$

If A_1 and A_2 are subsets of U

$$p(A_1 \cup A_2) = p(A_1) + p(A_2) - p(A_1 \cap A_2)$$

Two events A_1 and A_2 are called mutually exclusive if and only if $A_1 \cap A_2 = \phi$ (null set). These events are said to be independent if and only if $p(A_1 \cap A_2) = p(A_1)p(A_2)$.

- *Conditional Probability and Bayes' Rule*
  The probability of an event A, given that an event B has occurred, is called the conditional probability and is denoted $p(A/B)$. Further,

$$p(A/B) = \frac{p(A \cap B)}{p(B)}$$

Bayes' rule permits a calculation of an *a posteriori* probability from given *a priori* probabilities and is stated below:

If $A_1, A_2, \ldots, A_n$ are n mutually exclusive events, and $p(A_1) + p(A_2) + \cdots + p(A_n) = 1$, and B is any event such that $p(B)$ is not 0, then the conditional probability $p(A_i/B)$ for any one of the events A_i, *given that B has occurred*, is

$$p(A_i/B) = \frac{p(A_i)p(B/A_i)}{p(A_1)p(B/A_1) + p(A_2)p(B/A_2) + \cdots + p(A_n)p(B/A_n)}$$

Example

Among five different laboratory tests for detecting a certain disease, one is effective with probability 0.75, whereas each of the others is effective with probability 0.40. A medical student, unfamiliar with the advantage of the best test, selects one of them and is successful in detecting the disease in a patient. What is the probability that the most effective test was used?

Let B denote (the event) of detecting the disease, A_1 the selection of the best test, and A_2 the selection of one of the other four tests; thus, $p(A_1) = 1/5$, $p(A_2) = 4/5$, $p(B/A_1) = 0.75$ and $p(B/A_2) = 0.40$. Therefore,

$$p(A_1/B) = \frac{\frac{1}{5}(0.75)}{\frac{1}{5}(0.75) + \frac{4}{5}(0.40)} = 0.319$$

Note, the *a priori* probability is 0.20; the outcome raises this probability to 0.319.

- *Expected Value*
  For the random variable X that assumes n finite values x_1, x_2, …, x_n, with corresponding probabilities $P(x_i)$ such that $\sum_1^n P(x_i) = 1$, the expected value (also called the mean) is given by $E(x) = \sum x_i P(x_i)$. For a continuous random variable with $a \le x \le b$, $E(x) = \int_a^b xP(x)$.

11.10 Binomial Distribution

In an experiment consisting of n independent trials in which an event has probability p in a single trial, the probability p_X of obtaining X successes is given by

$$P_X = C_{(n,X)} p^X q^{(n-X)}$$

where

$$q = (1 - p) \text{ and } C_{(n,X)} = \frac{n!}{X!(n-X)!}$$

The probability of between a and b successes (both a and b included) is $P_a + P_{a+1} + \cdots + P_b$, so if $a = 0$ and $b = n$, this sum is

$$\sum_{X=0}^{n} C_{(n,X)} p^X q^{(n-X)} = q^n + C_{(n,1)} q^{n-1} p + C_{(n,2)} q^{n-2} p^2 + \cdots + p^n$$

$$= (q + p)^n = 1.$$

11.11 Mean of Binomially Distributed Variable

The mean number of successes in n independent trials is $m = np$ with standard deviation $\sigma = \sqrt{npq}$.

11.12 Normal Distribution

In the binomial distribution, as n increases the histogram of heights is approximated by the bell-shaped curve (normal curve),

$$Y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

where m = the mean of the binomial distribution = np, and $\sigma = \sqrt{npq}$ is the standard deviation. For any normally distributed random variable X with mean m and standard deviation σ, the probability function (density) is given by the above.

The *standard* normal probability curve is given by

$$y = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

and has mean = 0 and standard deviation = 1. The total area under the standard normal curve is 1. Any normal variable X can be put into standard form by defining $Z = (X - m)/\sigma$; thus, the probability of X between a given X_1 and X_2 is the area under the standard normal curve between the corresponding Z_1 and Z_2 (Table A.1).

- *Normal Approximation to the Binomial Distribution*

  The standard normal curve is often used instead of the binomial distribution in experiments with discrete outcomes. For example, to determine the probability of obtaining 60 to 70 heads in a toss of 100 coins, we take X = 59.5 to X = 70.5 and compute corresponding values of Z from mean np = $100 \frac{1}{2} = 50$, and the standard deviation $\sigma = \sqrt{(100)(1/2)(1/2)} = 5$. Thus, Z = (59.5 − 50)/5 = 1.9 and Z = (70.5 − 50)/5 = 4.1. From Table A.1, the area between Z = 0 and Z = 4.1 is 0.5000, and between Z = 0 and Z = 1.9

is 0.4713; hence, the desired probability is 0.0287. The binomial distribution requires a more lengthy computation:

$$C_{(100,60)}(1/2)^{60}(1/2)^{40} + C_{(100,61)}(1/2)^{61}(1/2)^{39} + \cdots + C_{(100,70)}(1/2)^{70}(1/2)^{30}.$$

Note that the normal curve is symmetric, whereas the histogram of the binomial distribution is symmetric only if $p = q = 1/2$. Accordingly, when p (hence q) differs appreciably from 1/2, the difference between probabilities computed by each increases. It is usually recommended that the normal approximation not be used if p (or q) is so small that np (or nq) is less than 5.

11.13 Poisson Distribution

$$P = \frac{e^{-m} m^r}{r!}$$

is an approximation to the binomial probability for r successes in n trials when $m = np$ is small (<5) and the normal curve is not recommended to approximate binomial probabilities (Table A.2). The variance σ^2 in the Poisson distribution is np, the same value as the mean.

Example

A school's expulsion rate is 5 students per 1,000. If class size is 400, what is the probability that 3 or more will be expelled? Since p = 0.005 and n = 400, $m = np = 2$ and $r = 3$. From Table A.2 we obtain for $m = 2$ and r (= x) = 3 the probability $p = 0.323$.

11.14 Empirical Distributions

A distribution that is skewed to the right (positive skewness) has a median to the right of the mode and a mean to the right of the median. One that is negatively skewed has a median to the left of the mode and a mean to the left of the median. An approximate relationship among the three parameters is given by

$$\text{Median} \doteqdot 2/3(\text{mean}) + 1/3(\text{mode})$$

Skewness may be measured by either of the following formulas:

$$\text{Skewness} = (\text{mean} - \text{mode})/s$$

$$\text{Skewness} = 3(\text{mean} - \text{median})/s$$

11.15 Estimation

Conclusions about a population parameter such as mean μ may be expressed in an interval estimation containing the sample estimate in such a way that the interval includes the unknown μ with probability (1 − α). A value Z_α is obtained from the table for the normal distribution. For example, $Z_\alpha = 1.96$ for $\alpha = 0.05$. Sample values $X_1, X_2, \ldots, X_n$ permit computation of the variance s^2, which is an estimate of σ^2. A confidence interval for μ is

$$\left(\bar{X} - Z_\alpha s/\sqrt{n}, \bar{X} + Z_\alpha s/\sqrt{n}\right)$$

For $a = 0.05$ this interval is

$$\left(\bar{X} - 1.96s/\sqrt{n}, \bar{X} + 1.96s/\sqrt{n}\right)$$

The ratio $s/\sqrt{n}$ is the *standard error of the mean* (see Section 11.17).

11.16 Hypotheses Testing

Two groups may have different sample means and it is desired to know if the apparent difference arises from random or significant deviation in the items of the samples. The *null hypothesis* (H_0) is that both samples belong to the same population, i.e., the differences are random. The alternate hypothesis (H_1) is that these are two different populations. Test procedures are designed so one may accept or reject the null hypothesis. The decision to accept is made with probability α of error. The value of α is usually 0.05, 0.01, or 0.001. If the null hypothesis is rejected, though correct, the error is called an *error of the first kind*. The error of acceptance of the null hypothesis, when false, is an *error of the second kind*.

11.17 *t*-Distribution

In many situations, μ and σ are unknown and must be estimated from $\bar{X}$ and s in a sample of small size n, so use of the normal distribution is not recommended. In such situations the Student's t-distribution is used and is given by the probability density function

$$y = A(1 + t^2/f)^{-(f+1)/2}$$

where f stands for degrees of freedom and A is a constant

$$= \Gamma(f/2 + 1/2)/\Gamma(f/2)\sqrt{f\pi}$$

so that the total area (probability) under the curve of y vs. t is 1. In a normally distributed population with mean μ, if all possible samples of size n and mean $\bar{X}$ are taken, the quantity $(\bar{X} - \mu)\sqrt{n}/s$ satisfies the t-distribution with

$$f = n - 1$$

or

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}}.$$

Thus, confidence limits for μ are

$$\left(\bar{X} - t \cdot s/\sqrt{n}, \bar{X} + t \cdot s/\sqrt{n}\right)$$

where t is obtained from Table A.3 for (n − 1) degrees of freedom and confidence level (1 − α).

11.18 Hypothesis Testing with *t*- and Normal Distributions

When two normal, independent populations with means μ_X and μ_Y and standard deviations σ_X and σ_Y are considered, and all possible pairs of samples are taken, the distribution of the difference between sample means $\bar{X} - \bar{Y}$ is also normally distributed. This distribution has mean $\mu_X - \mu_Y$ and standard deviation

$$\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}$$

where n_1 is the sample size of X_i variates and n_2 is the sample size of Y_i variates. The quantity Z computed as

$$Z = \frac{\left(\bar{X} - \bar{Y}\right) - \left(\mu_X - \mu_Y\right)}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}}$$

satisfies a standard normal probability curve (Section 11.12).

Accordingly, to test whether two sample means differ significantly, i.e., whether they are drawn from the same or different populations, the null hypothesis (H_0) is $\mu_X - \mu_Y = 0$, and

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_X^2}{n_1} + \frac{\sigma_Y^2}{n_2}}}$$

is computed. For sufficiently large samples ($n_1 >$ 30 and $n_2 >$ 30), sample standard deviations s_X and s_Y are used as estimates of σ_X and σ_Y, respectively. The difference is significant if the value of Z indicates a small probability, say, <0.05 (or $|Z| >$ 1.96; Table A.1).

For *small samples* where the standard deviation of the population is unknown and estimated from the sample, the t-distribution is used instead of the standard normal curve.

$$t = \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{s^2}{n_1} + \frac{s^2}{n_2}}},$$

where s is the "pooled estimate of the standard deviation" computed from

$$s^2 = \frac{(n_1 - 1)s_X^2 + (n_2 - 1)s_Y^2}{n_1 + n_2 - 2}$$

The computed t is compared to the tabular value (Table A.3) for degrees of freedom $f = n_1 + n_2 - 2$ at the appropriate confidence level (such as $\alpha = 0.05$ or 0.01). When the computed t exceeds in magnitude the value from the table, the null hypothesis is rejected and the difference is said to be significant. In cases that involve *pairing* of the variates, such as heart rate before and after exercise, the difference $D = X - Y$ is analyzed. The mean (sample) difference $\bar{D}$ is computed and the null hypothesis is tested from

Example

Mean exam scores for two groups of students on a standard exam were 75 and 68, with other pertinent values:

$$\bar{X} = 75 \qquad \bar{Y} = 68$$

$$s_x = 4 \qquad s_y = 3$$

$$n_1 = 20 \qquad n_2 = 18$$

Thus,

$$s^2 = \frac{(19)(4)^2 + (17)(3)^2}{36} = 12.7,$$

and

$$t = \frac{75 - 68}{\sqrt{\frac{12.7}{20} + \frac{12.7}{18}}} = 6.05.$$

From Table A.3, $t_{0.01}$, for 36 degrees of freedom, is between 2.756 and 2.576; hence, these means are significantly different at the 0.01 level.

$$t = \frac{\bar{D}}{s_D / \sqrt{n}},$$

where s_D is the standard deviation of the set of differences:

$$s_D = \left[\sum \left(D - \bar{D}\right)^2 / \left(n - 1\right)\right]^{1/2}$$

In this case, $f = n - 1$.

11.19 Chi-Square Distribution

In an experiment with two outcomes (e.g., heads or tails), the observed frequencies can be compared to the expected frequencies by applying the normal distribution. For more than two outcomes, say, n, the observed frequencies $O_1, O_2, \ldots, O_n$ and the expected frequencies, $e_1, e_2, \ldots, e_n$, are compared with the chi-square statistic (χ^2):

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - e_i)^2}{e_i}.$$

The χ^2 is well approximated by a theoretical distribution expressed in Table A.4. The probability that χ^2 is between two numbers χ_1^2 and χ_2^2 is the area under the curve between χ_1^2 and χ_2^2 for degrees of freedom f. The probability density function is

$$y = \frac{1}{2^{f/2}\Gamma(f/2)} e^{-\frac{1}{2}\chi^2} (\chi^2)^{(f-2)/2}, \quad (0 \le \chi^2 \le \infty).$$

In a *contingency table* of j rows and k columns, $f = (j - 1)(k - 1)$. In such a matrix arrangement the observed and expected frequencies are determined for each of the $j \times k = n$ cells or positions and entered in the above equation.

When $f = 1$, the adjusted χ^2 formula (Yates' correction) is recommended:

$$\chi^2_{\text{adj}} = \sum_{i=1}^{n} \frac{(|O_i - e_i| - 1/2)^2}{e_i}.$$

Example: Contingency Table

Men and women were sampled for preference of three different brands of breakfast cereal. The number of each gender that liked the brand is shown in the contingency table. The expected number for each cell is given in parentheses and is calculated as row total × column total/grand total. Degrees of freedom = (2 – 1) × (3 – 1) = 2 and X^2 is calculated as:

$$\chi^2 = \frac{(50-59.7)^2}{59.7} + \cdots + \frac{(60-75.7)^2}{75.7} = 11.4$$

Brands

	A	**B**	**C**	**Totals**
Men	50 (59.7)	40 (45.9)	80 (64.3)	170
Women	80 (70.3)	60 (54.1)	60 (75.7)	200
Totals	130	100	140	370

Since the tabular value at the 5% level for $f = 2$ is 5.99, the result is significant for a relationship between gender and brand preference.

X^2 is frequently used to determine whether a population satisfies a normal distribution. A large sample of the population is taken and divided into C classes, in each of which the observed frequency is noted and the expected frequency calculated. The latter is calculated from the assumption of a normal distribution. The class intervals should contain an expected frequency of 5 or more. Thus, for the interval (X_i, X_{i+1}), calculations of $Z_i = (X_i - \bar{X})/s$ and $Z_{i+1} = (X_{i+1} - \bar{X})/s$ are made and the probability is determined from the area under the standard normal curve. This probability, $(P_i) \times N$, gives the expected frequency for the class interval. Degrees of freedom = $C - 3$ in this application of the X^2 test.

11.20 Least Squares Regression

A set of n values (X_i, Y_i) that display a linear trend is described by the linear equation $\hat{Y}_i = \alpha + \beta X_i$. Variables α and β are constants (population parameters) and are the intercept and slope, respectively. The rule for determining the line is minimizing the sum of the squared deviations,

$$\sum_{i=1}^{n} \left(Y_i - \hat{Y} \right)_i^2$$

and with this *criterion* the parameters α and β are best estimated from a and b, calculated as

$$b = \frac{\sum X_i Y_i - \frac{\left(\sum X_i\right)\left(\sum Y_i\right)}{n}}{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{n}}$$

and

$$a = \bar{Y} - b\bar{X},$$

where $\bar{X}$ and $\bar{Y}$ are mean values, assuming that for any value of X the distribution of Y values is normal with variances that are equal for all X, and the latter (X) are obtained with negligible error. The null hypothesis, H_0:β = 0, is tested with analysis of variance:

Source	**SS**	**DF**	**MS**
Total $\left(Y_i - \bar{Y}\right)$	$\Sigma\left(Y_i - \bar{Y}\right)^2$	$n - 1$	
Regression $\left(\hat{Y}_i - \bar{Y}\right)$	$\Sigma\left(\hat{Y}_i - \bar{Y}\right)^2$	1	
Residual $(Y_i - \hat{Y}_i)$	$\Sigma(Y_i - \hat{Y}_i)^2$	$n - 2$	$\dfrac{SS_{\text{resid}}}{(n-2)} = S^2_{Y \cdot X}$

Computing forms for SS terms are

$$SS_{\text{total}} = \sum\left(Y_i - \bar{Y}\right)^2 = \sum Y_i^2 - \left(\sum Y_i\right)^2 / n$$

$$SS_{\text{regr.}} = \sum\left(\hat{Y}_i - \bar{Y}\right)^2 = \frac{\left[\sum X_i Y_i - \left(\sum X_i\right)\left(\sum Y_i\right)/n\right]^2}{\sum X_i^2 - \left(\sum X_i\right)^2 / n}$$

$F = MS_{\text{regr.}} / MS_{\text{resid.}}$ is calculated and compared with the critical value of F for the desired confidence level for degrees of freedom 1 and $n - 2$ (Table A.5). The coefficient of determination, denoted r^2, is

$$r^2 = SS_{\text{regr.}} / SS_{\text{total}}$$

Example

Given points: (0, 1), (2, 3), (4, 9), (5, 16). Analysis proceeds with the following calculations:

	SS	DF	MS	
Total	136.7	3		$F = \frac{121}{7.85} = 15.4$ (significant)[a]
Regression	121	1	121	
Residual	15.7	2	$7.85 = S^2_{Y\cdot X}$	$r^2 = 0.885$ $s_b = 0.73$

[a] See *F*-distribution, Section 11.21.

r is the *correlation coefficient*. The *standard error of estimate* is $\sqrt{s^2_{Y\cdot X}}$ and is used to calculate confidence intervals for α and β. For the confidence limits of β and α,

$$b \pm t s_{Y\cdot X}\sqrt{\frac{1}{\sum\left(X_i - \bar{X}\right)^2}}$$

$$a \pm t s_{Y\cdot X}\sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum\left(X_i - \bar{X}\right)^2}}$$

where t has $n - 2$ degrees of freedom and is obtained from Table A.3 for the required probability.

The null hypothesis, H_0:β = 0, can also be tested with the t-statistic:

$$t = \frac{b}{s_b}$$

where s_b is the standard error of b:

$$s_b = \frac{s_{Y \cdot X}}{\left[\sum \left(X_i - \bar{X}\right)^2\right]^{1/2}}$$

- *Standard Error of* $\hat{Y}$
  An estimate of the mean value of Y for a given value of X, say, X_0, is given by the regression equation

  $$\hat{Y}_0 = a + bX_0.$$

  The standard error of this predicted value is given by

  $$S_{\hat{Y}_0} = S_{Y \cdot X}\left[\frac{1}{n} + \frac{(X_0 - \bar{X})^2}{\sum \left(X_i - \bar{X}\right)^2}\right]^{\frac{1}{2}}$$

  and is a minimum when $X_0 = \bar{X}$ and increases as X_0 moves away from $\bar{X}$ in either direction.

11.21 Nonlinear Regression Analysis

Given a data set (x_i, y_i), $i = 1, \ldots, N$, it is desired to fit these to a nonlinear equation.

The basis of nonlinear curve fitting is as follows. A function Y of x contains, say, two parameters denoted here by α and β, that is, $Y = f(x, \alpha, \beta)$.

We seek here a representation in which α and β are estimated by *a* and *b*. These estimates are initially a_0 and b_0. A Taylor series representation is made about these initial estimates a_0 and b_0:

$$Y \approx f\left(a_0, b_0, x\right) + \left(\partial f/\partial\alpha\right)\left(\alpha - a_0\right) + \left(\partial f/\partial\beta\right)\left(\beta - b_0\right)$$

$$Y - f\left(a_0, b_0, x\right) \approx \left(\partial f/\partial\alpha\right)\left(\alpha - a_0\right) + \left(\partial f/\partial\beta\right)\left(\beta - b_0\right)$$

For this choice of a_0 for α and b_0 for β, each value x_i gives the left-hand side of the above equation, $Y_i - f(a_0, b_0, z_i)$, denoted here by $\hat{Y}$. The partial derivative $\partial f/\partial a$ uses the a_0 and b_0 values and also has a value for each x_i value, denoted here by X_{1i}. Similarly, the partial derivative $\partial f/\partial\beta$ has a value at this x_i, which we denote by X_{2i}. Thus, we get a set of values of a dependent variable $\hat{Y} = cX_1 + dX_2$ that is linearly related to the independent variables X_1 and X_2. A multiple linear regression (described below) yields the two regression coefficients *c* and *d*.

There are *N* data points (x_i, y_i). Using estimates (a_0, b_0) of parameters, the data are transformed into three different sets, denoted by $\hat{Y}$, X_1, and X_2, defined as follows:

$$\hat{Y}_i = y_i - f(a_0, b_0, x_i)$$

$$X_{1i} = (\partial f/\partial\alpha)$$

$$X_{2i} = (\partial f/\partial\beta)$$

where the partial derivatives are evaluated with a_0, b_0 at each x_i value.

Thus, the original data set gives rise to three data columns of length N:

$\hat{Y}$	X_1	X_2
…	…	…
…	…	…

The values of $\hat{Y}$, X_1, and X_2 in the table are entered into a linear multiple regression procedure to yield

$$\hat{Y} = cX_1 + dX_2$$

The coefficients c and d are determined (with standard errors) from equations given below; these allow improved estimates of parameters a and b by taking a new set of estimates $a_1 = c + a_0$ and $b_1 = d + b_0$.

The new set of estimates, a_1 and b_1, are then used to calculate $\hat{Y}$, X_1, and X_2, and the process is repeated to yield new parameters, a_2 and b_2 (with standard errors). A stopping criterion is applied, e.g., if the difference between two iterates is less than some specified value. This last set is retained and the last set's standard errors are retained as the standard errors of the final estimate.

- *Multiple Regression (Equations)*
  In the discussion on nonlinear curve fitting above, we saw the need for iterative use of the two-parameter linear regression given by

$$\hat{Y} = cX_1 + dX_2.$$

At every step of the iterative process a set of X_1, X_2 and corresponding $\hat{Y}$ values is calculated, and at that step we wish to calculate the coefficients c and d. The procedure for doing this is a special case of the general multiple regression algorithm based on $Y = b_0 + b_1 X_1 + b_2 X_2 + \cdots + b_N X_N$, which estimates all the coefficients. In our application (two-parameter nonlinear analysis) there is no b_0 term and $N = 2$. The data array is that shown above. Our model equation is given by

$$\hat{Y} = cX_1 + dX_2$$

Using a least squares procedure we calculate the following by first getting the determinant D:

$$D = \begin{vmatrix} \sum X_1^2 & \sum X_1X_2 \\ \sum X_1X_2 & \sum X_2^2 \end{vmatrix}$$

$$= \left(\sum X_1^2\right)\left(\sum X_2^2\right) - \left(\sum X_1X_2\right)^2$$

The coefficients c and d are calculated:

$$c = \begin{vmatrix} \sum YX_1 & \sum X_1X_2 \\ \sum YX_2 & \sum X_2^2 \end{vmatrix} \div D$$

$$d = \begin{vmatrix} \sum X_1^2 & \sum YX_1 \\ \sum X_1X_2 & \sum YX_2 \end{vmatrix} \div D$$

The following *Gaussian coefficients* are needed in the error estimates and these are given by

$$c_{11} = \frac{\sum X_2^2}{D} \quad c_{22} = \frac{\sum X_1^2}{D} \quad c_{12} = \frac{-\sum X_1 X_2}{D}$$

The squared differences between the observed and estimated $\hat{Y}$ values are summed to give $SS_{res} = \sum(\hat{Y}_{obs} - \hat{Y}_{est})^2$. From SS_{res} we get the variance

$$s^2 = \frac{SS_{res}}{N-2}$$

which is used to obtain the needed variances and standard errors from the following:

$$V(c) = c_{11}s^2 \quad V(d) = c_{22}s^2$$

$$SE(c) = \sqrt{V(c)} \quad SE(d) = \sqrt{V(d)}$$

It is seen that the procedure for nonlinear curve fitting requires extensive computation that is almost always done on a computer. The iteration stops when the changes in coefficients *c* and *d* become sufficiently small. At that point in the process the standard errors are those given above at this last turn of the cycle.

11.22 The *F*-Distribution (Analysis of Variance)

Given a normally distributed population from which two independent samples are drawn, these provide estimates, s_1^2 and s_2^2, of the variance, σ^2. Quotient $F = s_1^2 / s_2^2$ has this probability density function for f_1 and f_2 degrees of freedom of s_1 and s_2:

$$y = \frac{\Gamma\left(\dfrac{f_1 + f_2}{2}\right)}{\Gamma\left(\dfrac{f_1}{2}\right)\Gamma\left(\dfrac{f_2}{2}\right)} \cdot f_1^{\frac{f_1}{2}} f_2^{\frac{f_2}{2}} \cdot \frac{F^{\frac{f_1 - 2}{2}}}{\left(f_2 + f_1 F\right)^{\frac{f_1 + f_2}{2}}},$$

$$(0 \le f < \infty)$$

In testing among k groups (with sample size n) and sample means $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_k$, the F-distribution tests the null hypothesis, $\mu_1 = \mu_2 = \cdots = \mu_k$, for the means of populations from which the sample is drawn. Individual values from the jth sample (j = 1 to k) are denoted $A_{ij} (i = 1 \text{ to } n)$. The "between means" sums of squares (S.S.T.) is computed

$$\text{S.S.T} = n\left(\bar{A}_1 - \bar{A}\right)^2 + n\left(\bar{A}_2 - \bar{A}\right)^2 + \cdots + n\left(\bar{A}_k - \bar{A}\right)^2,$$

where $\bar{A}$ is the means of all group means, as well as the "within samples" sum of squares (S.S.E.), where

$$\text{S.S.E.} = \sum_{i=1}^{n}\left(A_{i1} - \bar{A}_1\right)^2 + \sum_{i=1}^{n}\left(A_{i2} - \bar{A}_2\right)^2 + \cdots + \sum_{i=1}^{n}\left(A_{ik} - \bar{A}_k\right)^2$$

Then

$$s_1^2 = \frac{\text{S.S.T.}}{k-1}$$

and

$$s_2^2 = \frac{\text{S.S.E.}}{k(n-1)}$$

are calculated and the ratio F is obtained:

$$F = \frac{s_1^2}{s_2^2},$$

with numerator degrees of freedom $k - 1$ and denominator degrees of freedom $k(n - 1)$. If the calculated F exceeds the tabular value of F at the desired probability (say, 0.05), we *reject* the null hypothesis that the samples came from populations with equal means (see Table A.5 and gamma function, Section 9.2).

11.23 Summary of Probability Distributions

- *Continuous Distributions*
  Normal

$$y = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-(x-m)^2/2\sigma^2\right]$$

Mean = m

Variance = σ^2

Standard Normal

$$y = \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right)$$

Mean = 0

Variance = 1

F-Distribution

$$y = A \frac{F^{\frac{f_1-2}{2}}}{\left(f_2 + f_1 F\right)^{\frac{f_1+f_2}{2}}};$$

$$\text{where } A = \frac{\Gamma\left(\frac{f_1+f_2}{2}\right)}{\Gamma\left(\frac{f_1}{2}\right)\Gamma\left(\frac{f_2}{2}\right)} f_1^{\frac{f_1}{2}} f_2^{\frac{f_2}{2}}.$$

$$\text{Mean} = \frac{f_2}{f_2 - 2}$$

$$\text{Variance} = \frac{2f_2^2\left(f_1 + f_2 - 2\right)}{f_1\left(f_2 - 2\right)^2\left(f_2 - 4\right)}$$

Chi-Square

$$y = \frac{1}{2^{f/2}\Gamma(f/2)}\exp\left(-\frac{1}{2}x^2\right)\left(x^2\right)^{\frac{f-2}{2}}$$

$$\text{Mean} = f$$

$$\text{Variance} = 2f$$

Students t

$$y = A(1 + t^2/f)^{-(f+1)/2}$$

where $A = \dfrac{\Gamma(f/2 + 1/2)}{\sqrt{f\pi}\,\Gamma\left(f/2\right)}$.

$$\text{Mean} = 0$$

$$\text{Variance} = \frac{f}{f-2} \quad (\text{for } f > 2)$$

- *Discrete Distributions*
  Binomial Distribution

$$y = C_{(n,x)}p^x(1-p)^{n-x}$$

Mean = np

Variance = $np\,(1 - p)$

Poisson Distribution

$$y = \frac{e^{-m} m^x}{x!}$$

Mean = m

Variance = m

11.24 Sample Size Determinations

- *Single Proportion*
  The sample size required to detect a difference between a test proportion, p_1, and a standard proportion value, p_0, is calculated from

$$n = \left\{ \frac{z_\alpha \sqrt{p_0(1-p_0)} - z_\beta \sqrt{p_1(1-p_1)}}{p_1 - p_0} \right\}^2$$

where z_α is the two-tailed z-value from the standard normal curve for the desired level of significance and z_β is the lower one-tailed z-value selected for the power required (probability of rejecting the null hypothesis when it should be rejected). For $\alpha < 0.05$, z_α is 1.96, while z_β is one of the following: –1.28 (90% power), –0.84 (80% power), or –0.525 (70% power).

Example

It is well established that 30% of the residents of a certain community experience allergy symptoms each year. It is desired to show that newly developed preventive inoculations can reduce this proportion to 10%. We have $p_0 = 0.30$ and $p_1 = 0.10$, and thus, at the 5% level of significance and power 80%, n is given by

$$n = \left\{1.96\sqrt{(0.3)(0.7)} + 0.84\sqrt{(0.1)(0.9)}\right\}^2 /(0.10 - 0.30)^2$$

$$= 33.07$$

meaning that 34 patients should be tested.

- *Two Proportions*
  When control and treatment groups are sampled, and the respective proportions expected are p_c and p_t, the needed sample size of *each group* to show a difference between these is calculated from

$$n = \left\{\frac{z_\alpha\sqrt{2p_c(1-p_c)} - z_\beta\sqrt{p_t(1-p_t)+p_c(1-p_c)}}{p_c - p_t}\right\}^2$$

Example

Suppose shock is known to occur in 15% of the patients who get a certain infection and we wish to show that a new preventive treatment can reduce this proportion to 5%; thus, $p_c = 0.15$ and $p_t = 0.05$. Using $z_\alpha = 1.96$ and $z_\beta = -0.84$ (for 80% power), the sample size needed in *each group* is calculated from

$$n = \left\{ \frac{1.96\sqrt{2(0.15)(0.85)} + 0.84\sqrt{(0.05)(0.95) + (0.15)(0.85)}}{(0.15 - 0.05)^2} \right\}^2$$

$$= 179.9$$

Thus, 180 patients are needed in each group.

- *Sample Mean*
  When the mean of a sample (μ_1) is to be compared to a standard value (μ_0), the number to be sampled in order to show a significant difference is calculated from

$$n = \left\{ \frac{(z_\alpha - z_\beta)\sigma}{\mu_1 - \mu_0} \right\}^2$$

  where σ is an estimate of the population standard deviation.

Example

A certain kind of light bulb is known to have a mean lifetime of 1,000 hours, with standard deviation = 100 hours. A new manufacturing process is installed by the manufacturer and it is desired to know whether the mean lifetime changes by, say, 25 hours; thus, $\mu_1 - \mu_0 = 25$. The sample size required for testing the new bulbs, based on the 0.05 level of significance and 90% power, is calculated from

$$n = \{(1.96 + 1.28)(100)/25\}^2 = 167.96$$

so that 168 bulbs should be tested.

- *Two Means*
  When two groups are sampled with the aim of detecting a difference in their means, $\mu_1 - \mu_2$, the sample size of *each group* is calculated from

$$n = 2\left\{\frac{(z_\alpha - z_\beta)\sigma}{\mu_1 - \mu_2}\right\}^2$$

Example

Examination scores of students from two different school districts are being compared in certain standardized examinations (scale, 200–800, where the standard deviation is 100). A difference in mean scores of 20 would be regarded as important. Using the 5% level of significance and 80% power, the number of student scores from each school district that should be included is

$$n = 2\{(1.96 + 0.84)(100)/20\}^2 = 392$$

12

Financial Mathematics

12.1 Simple Interest

An item or service costs an amount C and is to be paid off over time in equal installment payments. The difference between the cost C and the total amount paid in installments is the interest I. The interest rate r is the amount of interest divided by the cost and the time of the loan T (usually expressed in years):

$$r = I/CT$$

Example

An item purchased and costing \$4,000 is to be paid off in 18 equal monthly payments of \$249.

The total amount paid is $18 \times \$249 = \$4{,}482$, so that $I = \$482$. The time of the loan is 1.5 years; hence, the rate is $r = 482/(4000 \times 1.5) = 0.0803$ or 8.03%.

Note: While the above computation is correct, the computed rate, 8.03%, is misleading. This would be the true rate only if the $4,482 were repaid in one payment at the end of 18 months. But since you are reducing the unpaid balance with each payment, you are paying a rate higher than 8.03%. True interest rates are figured on the unpaid balance. The monthly payment based on the true rate is discussed below.

12.2 True Interest Formula (Loan Payments)

The interest rate is usually expressed per year; thus, the monthly rate r is 1/12th of the annual interest rate. The monthly payment P is computed from the amount borrowed, A, and the number of monthly payments, n, according to the formula

$$P = Ar\frac{(1+r)^n}{(1+r)^n - 1} \quad (r > 0)$$

Example

A mortgage of $80,000 ($A$) is to be paid over 20 years (240 months) at a rate of 9% per year. The monthly payment is computed from the above formula with n = 240 months and r = 0.09/12 = 0.0075 per month.

It is necessary to calculate $(1 + 0.0075)^{240}$ for use in the formula. This is accomplished with the calculator key [y^x]; that is, enter 1.0075, press the [y^x] key, then 240 = to give 6.00915. The above formula yields

$$P = 80000 \times 0.0075 \times 6.00915/(6.00915 - 1)$$
$$= \$719.78$$

Example

An automobile costing $20,000 is to be financed at the annual rate of 8% and paid in equal monthly payments over 60 months. Thus, $n = 60$, $A = 20000$, and $r = 0.08/12 = 0.006667$.

First compute $(1 + 0.006667)^{60}$ (by entering 1.006667 then pressing the key [y^x], followed by 60) = 1.48987. Thus, the monthly payment is

$$P = 20000 \times .006667 \times 1.48987/(1.48987 - 1)$$
$$= \$405.53$$

Table A.6 gives the monthly payment for each $1,000 of the loan at several different interest rates.

Example

Use Table A.6 to get the monthly payment for the previous example.

Note that the table entry for 8% and 5 years is $20.28 per thousand. Since the loan is $20,000, you must multiply $20.28 by 20, which gives $405.60. (This differs by a few cents from the above due to rounding in the tables.)

12.3 Loan Payment Schedules

Once the monthly loan payment is determined, it usually remains constant throughout the duration of the loan. The amount that goes to interest and principal changes with each payment as illustrated below.

Example

Show the payment schedule for a loan of $10,000 at the annual interest rate of 12%, which is to be paid in equal monthly payments over 5 months.

The monthly payment P is computed using the monthly interest rate $r = 0.12/12 = 0.01$ and the formula in Section 12.2:

$$P = 10000 \times \frac{(0.01) \times (1.01)^5}{(1.01)^5 - 1}$$

The value $(1.01)^5$ is calculated by entering 1.01 then pressing [$\mathbf{y}^x$] followed by 5 to give 1.0510101, so that the above becomes

$$P = 10000 \times \frac{0.01 \times 1.0510101}{1.0510101 - 1} = 2060.40$$

Thus, monthly payments are $2,060.40. The first month's interest is 1% of $10,000, or $100. Since the monthly payment is constant, the following table shows the application of the monthly payment to both principal and interest as well as the balance.

Payment Schedule

Payment	To Interest	To Principal	Balance
1	100	1960.40	8039.60
2	80.40	1980.00	6059.60
3	60.60	1999.80	4059.80
4	40.60	2019.80	2040
5	20.40	2040.00	–

12.4 Loan Balance Calculation

The balance after some number of payments, illustrated in Section 12.3 above, may be calculated directly from a formula that is given below. In this calculation it is assumed that the monthly payments in amount P are made every month. The amount of these payments was determined from the original amount of the loan, denoted A, the number of months of the loan (e.g., 120 months for a 10-year loan), and the monthly interest rate r as given in Section 12.3. We now wish to determine what the balance is after a specific number of payments, denoted by k, have been made. The balance is given by

$$Bal_k = (1+r)^k\left(A - \frac{P}{r}\right) + \frac{P}{r} \quad (r > 0)$$

Example

A 15-year loan of $100,000 at 7% annual interest rate was made and requires a monthly payment of $899. This monthly payment was determined from the formula in Section 12.3. It is desired to know what the balance is after 5 years (60 payments).

The calculation requires the use of r at the monthly rate; thus, $r = 0.07/12 = 0.0058333$, and substitution yields

$$\begin{aligned} Bal_{60} &= (1+0.0058333)^{60}\left\{100000 - \frac{899}{0.0058333}\right\} \\ &\quad + \frac{899}{0.0058333} \\ &= (1.41762)[100000 - 154115.17] + 154115.17 \\ &= \$77,400.43 \end{aligned}$$

12.5 Accelerated Loan Payment

The monthly payment P on a loan depends on the amount borrowed, A, the monthly interest rate, r, and the number of payments, n (the length of the loan). If the monthly payment is increased to a new amount, P', then the number of monthly payments will be reduced to some lesser number, n', which is calculated as follows:

First, calculate *term* 1 from the formula

$$term\ 1 = \frac{P'}{P' - Ar}$$

and *term* 2:

$$term\ 2 = (1 + r)$$

From *term* 1 and *term* 2 the number of months n' is calculated as

$$n' = \frac{\log(term\ 1)}{\log(term\ 2)}$$

Example

A mortgage of $50,000 for 30 years (360 months) at an annual rate of 8% requires monthly payments of $7.34 per thousand; thus, 50 thousand requires a monthly payment of 50 × $7.34 = $367 (see Table A.6). If the borrower decides to pay more than the required monthly payment, say $450, how long would it take to pay off the loan?

The monthly interest rate is 0.08/12 and is used in the calculations of *term* 1 and *term* 2:

$$term\ 1 = \frac{450}{450 - (50000)(0.08/12)} = 3.8571$$

$$term\ 2 = (1 + 0.08/12) = 1.00667$$

Thus,

$$n' = \frac{\log(3.8571)}{\log(1.00667)} = \frac{0.5863}{0.002887} = 203.1 \text{ months}$$

The loan time is reduced to 203.1 months (16.9 years).

12.6 Lump Sum Payment

A way to reduce the length of a loan is to make a lump payment that immediately reduces the amount owed to some lower amount, which we denote by *Bal*. The original monthly payment remains at the amount *P*, which was previously determined from the original terms of the loan, but now the number of future payments *M* will be fewer because of the reduction in the amount owed. This number *M* is calculated from quantities *X* and *Y*, defined as follows:

$$X = \frac{P}{P - (Bal)(r)}$$

$$Y = 1 + r \quad (r > 0)$$

and

$$M = \frac{\log(X)}{\log(Y)}$$

Example

In a previous example (Section 12.4) we considered a situation at the end of 5 years of a loan of \$100,000 for 15 years at the annual interest rate of 7% (0.0058333/month). The balance after 5 years was \$77,400.43 and the monthly payment is \$899.00 and scheduled to remain at that amount for the remaining 120 months. Suppose a lump payment of \$20,000 is made, thereby reducing the amount owed to \$57,400.43, denoted here by *Bal*. The monthly payments remain at $899. The number of future payments *M* is calculated from the above formulas:

$$X = \frac{899}{899 - (57400.43)(0.0058333)} = 1.59350$$

$$Y = (1 + 0.0058333) = 1.0058333$$

The quantity *M* is then calculated:

$$M = \frac{\log(1.59350)}{\log(1.0058333)} = 80.1 \text{ months}$$

12.7 Compound Interest

An amount of money (A) deposited in an interest-bearing account will earn interest that is added to the deposited amount at specified time intervals. Rates are usually quoted on an annual basis, as a percent. The interest is added at some fixed time interval or interest period such as a year, a month, or a day. The annual rate is divided by this interval for the purpose of calculation; e.g., if the annual rate is 9% and the interest period is 1 month, then the periodic rate r is 0.09/12 = 0.0075; if the period is 3 months (quarter of a year), then r = 0.09/4 = 0.0225. After n time intervals (compounding periods) the money grows to an amount S given by

$$S = A(1 + r)^n$$

where
A is the original amount
n is the number of interest periods
r is the rate per period

Example

$500 is deposited with an annual interest rate of 10% compounded quarterly. What is the amount after 2 years?

$A = \$500$

$r = 0.10/4 = 0.025$ (the periodic rate = 12-month rate/4)

$n = 2/(1/4) = 8$ (no. of interest periods)

and

$$S = 500 \times (1.025)^8$$
$$S = 500 \times 1.2184 = \$609.20$$

If this annual rate were compounded monthly, then $r = 0.10/12 = 0.008333$ and $n = 2/(1/12) = 24$, so that S becomes

$$S = 500 \times (1.008333)^{24}$$
$$= 500 \times 1.22038 = \$610.19$$

- *Effective Rate of Interest*
  When annual interest of, say, 8% is compounded at an interval such as four times

per year (quarterly), the effective yield is calculated by using the annual rate divided by 4, thus 2% or 0.02, and the number of compounding periods, in this case 4. Thus,

$$(1.02)^4 = 1.0824$$

and the effective annual rate is 0.0824, or 8.24%. In contrast, 8% is the nominal rate. Table A.7 shows the growth of $1 for different effective annual interest rates and numbers of years.

12.8 Time to Double (Your Money)

The time (in years) to double an amount deposited depends on the annual interest rate (r) and is calculated from the following formula:

$$Time\,(years) = \frac{\log 2}{\log(1+r)} = \frac{0.3010}{\log(1+r)}$$

Example

For interest rate 6% ($r = 0.06$), the time in years is

$$\frac{0.3010}{\log(1.06)} = \frac{0.3010}{0.2531} = 11.89\text{+ years}$$

Table A.8 gives the doubling time for various annual interest rates.

12.9 Present Value of a Single Future Payment

If one is to receive a specified amount (A) of money at some future time, say, n years from now, this benefit has a present value (V) that is based on the current interest rate (r) and calculated according to the formula

$$V = \frac{A}{(1+r)^n}$$

Example

You are to receive $1,000 ten years from now and the current annual interest rate is 8% (r = 0.08) and constant. The present value of this benefit is

$$V = 1000/(1.08)^{10} = 1000/(2.1589) = \$463.20$$

12.10 Regular Saving to Accumulate a Specified Amount

- *Payments at the Beginning of the Year*
  We wish to determine an amount P that should be saved each year in order to accumulate S dollars in n years, given that the annual interest rate is r. The payment P,

calculated from the formula below, is made on a certain date and on that same date each year, so that after n years (and n payments) the desired amount S is available.

$$P = \frac{rS}{(1+r)^{n+1} - (1+r)} \quad (r > 0)$$

To make this schedule more clear, say that the payment is at the beginning of the year, then at the beginning of the next year, and so on for 10 payments, the last being made at the beginning of the 10th year. At the end of this 10th year (and no further payments) we have the amount S. The payment amounts P are computed from the above formula.

Example

It is desired to accumulate $20,000 for college expenses needed 10 years hence in a savings account that pays the constant rate of 6% annually.

$$S = 20000,\ r = 0.06, \text{ and } n = 10.$$

The quantity $(1.06)^{11} = 1.8983$. Thus,

$$P = \frac{0.06 \times 20000}{1.8983 - 1.06} = 1431.47$$

so that $1,431.47 must be saved each year.

- *Payments at the End of the Year*
  Payments of amount P are deposited in an interest-bearing account at the end of each year for n years so that n such payments are made. The annual interest is r. It is desired to have S dollars immediately after the last payment. The annual payment P to attain this is given by the formula

$$P = \frac{rS}{(1+r)^n - 1} \quad (r > 1)$$

Example

It is desired to accumulate \$100,000 by making annual deposits in amount P at the end of each year for 40 years (say, from age 25 to 65 in a retirement plan) on the assumption that the interest rate is 10% per year and remains constant over the entire period. P is then

$$P = \frac{0.10 \times 100000}{(1.10)^{40} - 1} = \$225.94$$

Example

It is desired to accumulate \$100,000 in 10 years by making semiannual payments in an account paying 4% annually, but compounded semiannually, i.e., at the end of each 6-month period, for 20 periods. In this case we use the interest rate 0.04/2 = 0.02 for the compounding period, and insert $n = 20$ into the above formula.

$$P = \frac{0.02 \times 100000}{(1.02)^{20} - 1} = 4,116$$

so that deposits of $4,116 are required every 6 months. (Result rounded to nearest dollar.)

12.11 Monthly Payments to Achieve a Specified Amount

It is convenient to have tables of monthly payments for several different annual interest rates and compounding periods, and these are given in Tables A.9 and A.10.

12.12 Periodic Withdrawals from an Interest-Bearing Account

- *Balance Calculation*
  An account with an initial amount A is earning interest at the rate r. If a fixed amount P is withdrawn at regular intervals, then the balance B after n withdrawals is given by

$$B = A(1+r)^n - P\left[\frac{(1+r)^n - 1}{r}\right] \quad (r > 0)$$

In a common application the withdrawals are made monthly so that the annual interest rate r used in the formula is the annual rate divided by 12 (with monthly compounding). In this application the withdrawal is made at the end of the month. (*Note*: Balance decreases only if $P > Ar$.)

Example

An account earning interest at 10% per year and compounded monthly contains $25,000, and monthly withdrawals of $300 are made at the end of each month. How much remains after 6 withdrawals? After 12 withdrawals?

Since the rate is 10% and withdrawals are monthly, we use the rate $r = 0.10/12 = 0.008333$, with $A = 25{,}000$ and $P = 300$. First, for $n = 6$:

$$B = 25000 \times (1.008333)^6 - 300 \times \left[\frac{(1.008333)^6 - 1}{0.008333}\right]$$

Note: $(1.008333)^6 = 1.05105$. Thus,

$$\begin{aligned} B &= 25000 \times 1.05105 - 300 \times \left[\frac{1.05105 - 1}{0.008333}\right] \\ &= \$24,438 \text{ (rounded)} \end{aligned}$$

After 12 withdrawals,

$$B = 25000 \times (1.008333)^{12} - 300 \times \left[\frac{(1.008333)^{12} - 1}{0.008333}\right]$$

$$B = \$23,848 \text{ (rounded)}$$

Figure 12.1 shows the result of depositing \$10,000 at 8% annually (0.6667% monthly) and

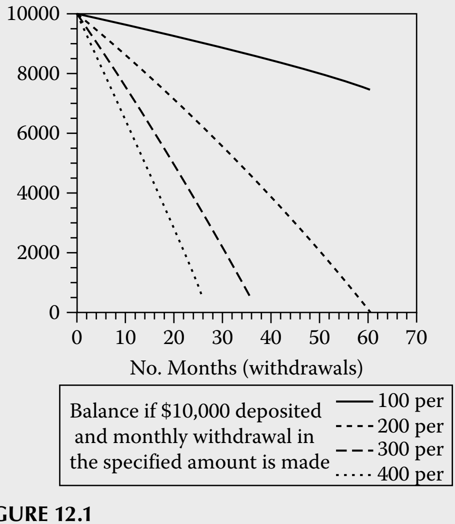


FIGURE 12.1
Balance of \$10,000 for specified monthly withdrawal. Interest rate is 8% per year.

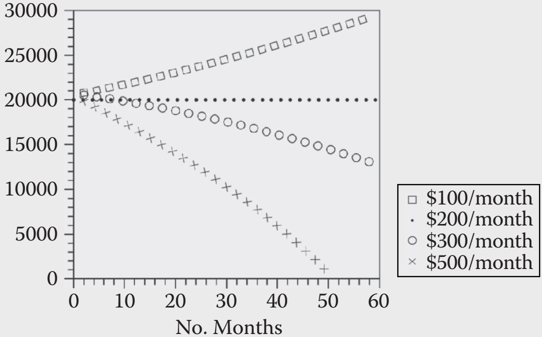


FIGURE 12.2
Balance of $20,000 with specified withdrawals in an account that earns 12% per year. (*Note:* Withdrawals up to $200/month do not decrease the balance.)

withdrawing a specified amount each month, while Figure 12.2 gives the results for $20,000 and annual interest 12%.

- *Amount on Deposit*
  The amount of money A, earning annual interest r, that must be on deposit in order to withdraw amount P at the end of each year for n years is given by

$$A = \frac{P}{r}\left[1 - \frac{1}{(1+r)^n}\right] \quad (r > 0)$$

Example

For an annual interest rate of 6%, withdrawals of $1,000 at the end of each of 20 years require an amount A on deposit that is calculated as

$$\frac{\$1000}{0.06}\left[1-\frac{1}{1.06^{20}}\right]=\$11,469.92$$

Note: If the withdrawals are monthly, then the interest rate is $r/12$ (assumed monthly compounding) and n is the number of months.

12.13 Periodic Withdrawals That Maintain the Principal

The amount of monthly withdrawals that will neither increase nor decrease the principal, called the *critical amount*, is given by

$$P = rA$$

where A is the principal and r is the interest rate.

Example

Suppose an amount A = \$25,000 is deposited and r = 0.0083333 (monthly); then

$$P = 0.008333 \times 25000$$
$$= \$208.32$$

so that \$208.32 may be withdrawn monthly while maintaining the original \$25,000.

Figure 12.2 shows the change in principal (\$20,000) following a number of withdrawals for several different monthly amounts in an account earning 12% per year and compounded monthly (r = 0.01). It is noteworthy that withdrawing less than \$200 per month (critical amount) does not decrease, but actually increases the principal.

12.14 Time to Deplete an Interest-Bearing Account with Periodic Withdrawals

If withdrawals at regular time intervals are in amounts greater than the critical amount (see Section 12.13), the balance decreases. The number of withdrawals to depletion may be calculated as follows:

$$n = \frac{\log\left[\dfrac{-P/r}{A-P/r}\right]}{\log(1+r)} \quad (P > Ar)$$

where

P = monthly amount
A = amount of the principal
r = interest rate
n = number of withdrawals to depletion

Example

An account with principal $10,000 is earning interest at the annual rate of 10% and monthly withdrawals of $200 are made.

To determine the number of withdrawals to depletion we use the monthly interest rate, r = 0.1/12 = 0.008333, with P = 200 and A = $10,000. The bracketed quantity is

$$[(-200/0.008333)/(10000 - 200/0.008333)]$$
$$= 1.7142$$

and its logarithm is 0.23406. The quantity in parentheses is 1.008333 and its logarithm is 0.003604; hence,

$$n = \frac{0.23406}{0.003604} = 64.94$$

Effectively this means 65 payments (months).

12.15 Amounts to Withdraw for a Specified Number of Withdrawals I: Payments at the End of Each Year

Suppose an amount A has accumulated in a savings account or pension plan and continues to earn annual interest at the rate r. How much can one withdraw each year, *at the end of each year*, for n years? We denote the annual withdrawal amount by P and it is computed from the formula below:

$$P = \frac{Ar}{1 - \dfrac{1}{(1+r)^n}} \qquad (r > 0)$$

Example

The amount in savings is \$100,000 and regular payments are desired for 20 years over which it assumed that the annual rate of interest is 6% and payable once a year. Using $r = 0.06$, $n = 20$, and $A = 100{,}000$ in the above gives

$$P = \frac{100000 \times 0.06}{1 - \dfrac{1}{(1+0.06)^{20}}}$$

Note that $(1.06)^{20} = 3.20713$ and its reciprocal is 0.31180.

Thus, $P = 6000/(1 - 0.31180) = \$8{,}718.40$.

Payments of \$8,718.40 per year at the end of each year for 20 years are possible from this

\$100,000. Of course, if 10 times this, or \$1,000,000, were on hand, then 10 times this, or \$87,184 would be paid for 20 years.

Example

If the same amounts above earn 8% annually instead of 6%, the calculation is

$$P = \frac{100000 \times 0.08}{1 - \dfrac{1}{(1 + 0.08)^{20}}}$$

Note that $(1.08)^{20} = 4.66096$ and its reciprocal is 0.214548. Thus,

$$P = 8000/(1 - 0.214548) = \$10,185.22$$

Payments of \$10,185.22 are possible for 20 years from the \$100,000 fund; from a \$1,000,000 fund the annual payments are 10 times this, or \$101,852.20.

12.16 Amounts to Withdraw for a Specified Number of Withdrawals II: Payments at the Beginning of Each Year

An amount A has accumulated in a savings account or pension plan and continues to earn annual interest at the annual rate r and is payable yearly. How much can you withdraw each year, at the beginning of each of n years? We denote the annual withdrawal amount by P, and it is computed from the formula below:

$$P = \frac{Ar}{(1+r) - \frac{1}{(1+r)^{n-1}}} \quad (r > 0)$$

Example

There is $100,000 in an account that earns 8% annually. It is desired to determine how much can be withdrawn (P), at the beginning of each year, for 25 years. In this application, $r = 0.08$, $n = 25$ years, and $A = 100{,}000$. Thus, P is given by

$$P = \frac{100000 \times 0.08}{1.08 - \frac{1}{(1.08)^{24}}}$$

Note that $1.08^{24} = 6.34118$ and the reciprocal of this is 0.15770, so that P is given by

$$P = \frac{8000}{1.08 - 0.15770}$$

which is $8,673.97.

Example

Suppose that there is \$100,000 in an account earning 8% annually and you desire to withdraw it at the beginning of each year for only 10 years. The amount per year P is now computed as

$$P = \frac{100000 \times 0.08}{1.08 - \dfrac{1}{(1.08)^9}}$$

We calculate that $1.08^9 = 1.9990$ and its reciprocal is 0.50025, so that P is given by

$$P = \frac{8000}{1.08 - 0.50025} = 13{,}799.05$$

Since the original amount is \$100,000, this annual withdrawal amount is 13.799% of the original. It is convenient to have a table of the percent that may be withdrawn for a specified number of years at various interest rates, and this is given in Table A.11.

Example

Find the percent of a portfolio that may be withdrawn at the beginning of each year for 15 years if the annual average rate of interest is 12%.

From Table A.11, in the 12% column, the entry at 15 years is 13.109%. Thus, a portfolio of \$100,000 allows annual withdrawals of \$13,109.

12.17 Present Value of Regular Payments

Suppose you are to receive yearly payments of a certain amount over a number of years. This occurs, for example, when one wins a state lottery. The current value of this stream of payments depends on the number of years (n), the interest rate (r) that money earns (assumed constant), and the amount (P) of the yearly payment. The current value (V) is computed from the formula

$$V = \frac{P}{r}\left[1 - \frac{1}{(1+r)^n}\right] \quad (r > 0)$$

Example

The current interest rate is 7% and annual payments of \$100 are to be paid for 25 years. The current value of these payments is

$$V = \frac{100}{0.07}\left[1 - \frac{1}{(1.07)^{25}}\right]$$

Note: $(1.07)^{25} = 5.42743$; using this in the above formula we compute

$$V = \$1165.36$$

12.18 Annuities

- *Deposits at the End of the Year*
  The same amount, denoted by P, is deposited in an interest-bearing account at the *end* of each year. The annual interest rate is r. At the end of n years these deposits grow to an amount S given by

  $$S = P\left[\frac{(1+r)^n - 1}{r}\right] \quad (r > 0)$$

  If the deposits are made every month, the above formula holds for the accumulated amount after n months. In this case, the interest rate, r, is the annual rate divided by 12 and compounded monthly.

Example

The sum of $500 is deposited at the end of every year in an account that earns 6% annually. What is the total at the end of 12 years?

$$P = 500,\ r = 0.06,\ \text{and}\ n = 12$$

Thus,

$$S = 500 \times \frac{(1+0.06)^{12} - 1}{0.06}$$

We must calculate $(1.06)^{12}$, which equals 2.012196. Thus, the above becomes

$$S = 500 \times 1.012196/0.06 = 500 \times 16.8699$$

$$= \$8434.97.$$

Example

Monthly payments of $500 are made into a retirement plan that has an average annual interest rate of 12% with monthly compounding. How much does this grow to in 25 years?

Because payments are made monthly, the rate r and the value of n must be based on monthly payments. Thus, the rate r is (0.12/12 = 0.01), and $n = 25 \times 12 = 300$ months. Thus, the value of S is

$$S = 500 \times \frac{(1+0.01)^{300} - 1}{0.01}$$

Note: $(1.01)^{300} = 19.7885$; thus,

$$S = 500 \times 18.7885/.01 = \$939,425$$

Table A.12 shows the result of depositing $1,000 at the end of each year in an account that earns annual interest at several different rates (payable yearly).

- *Deposits at the Beginning of the Year*
  Amount P is deposited each time and the annual interest rate is r; after n years the accumulated amount is S given by

$$S = \frac{P}{r}\left[(1+r)^{n+1} - (1+r)\right]$$

Example

$1,000 is deposited at the beginning of each year in a savings account that yields 8% annually and paid annually. At the end of 15 years the amount is S given by

$$S = \frac{1000}{0.08} \times \left[(1.08)^{16} - 1.08\right]$$

$$S = (12500) \times [3.426 - 1.080] = 12500 \times 2.346$$
$$= 29325$$

Thus, the amount grows to $29,325. Table A.13 illustrates the accumulation of funds when $1,000 is deposited at the beginning of each year in an account that earns a specified annual rate. *Note: If interest is paid more often than once a year, then the effective annual interest should be used in the application of these annuity formulas.*

12.19 The In-Out Formula

We wish to determine the amount of money (A) to be saved each month for a specified number of months (M) in order that withdrawals of

$1,000 monthly for another specified time (N) may begin. It is assumed that the interest rate (r) remains constant throughout the saving and collecting periods and that compounding occurs monthly. Thus, the interest rate, r, is the annual interest rate divided by 12, and N and M are in months. The monthly amount, A, which must be saved is given by the formula

$$A = 1000\left[\frac{(1+r)^N - 1}{(1+r)^N} \cdot \frac{1}{(1+r)^M - 1}\right] \quad (r > 0)$$

Example

The amount to be saved monthly for 15 years ($M = 15 \times 12 = 180$ months) is to be determined in order that one can receive $1,000 per month for the next 10 years ($N = 10 \times 12 = 120$ months). The annual interest rate is 6%; thus, $r = 0.06/12 = 0.005$ per month. From the above formula,

$$A = 1000\left[\frac{(1.005)^{120} - 1}{(1.005)^{120}} \cdot \frac{1}{(1.005)^{180} - 1}\right]$$

$$A = (1000)[(0.450367) \cdot (0.6877139)]$$

$$A = 309.72$$

Thus, $309.72 must be saved each month for 15 years in order to receive $1,000 per month for the next 10 years.

Table A.15, for annual interest 6%, gives the results of this calculation by reading down to

15 years and across to 10 years, as well as a number of different combinations of savings years and collection years. Tables A.14 to A.17 apply to annual interest rates of 4, 6, 8, and 10%. The use of these tables is illustrated in the next example.

Example

For an annual interest rate of 4%, how much should be saved monthly for 25 years in order to collect $1,000 monthly for the next 20 years?

From Table A.14, reading down to 25 years and across to 20 years, the table shows $320.97. Thus, $320.97 must be saved for each $1,000 monthly collected for 20 years. If, say, $3,000 per month is to be collected, we multiply $320.97 by 3 to give $962.91 as the amount to be saved each month for 25 years.

12.20 Stocks and Stock Quotations

The stocks of various corporations require familiarity with the terms used and the underlying calculations. Besides the *high*, *low*, and *closing price*, and the change from the previous trading day, the stock quotations, as listed in newspapers, contain additional terms that are calculated.

Yield: The dividend or other cash distribution that is paid on the security and usually expressed as a percentage of the closing price. The dollar amount of the distribution divided by the closing price, when multiplied by 100, gives the yield. Thus, a dividend of \$3.50 for a stock selling for \$40.75 has a yield of

$$100 \times (3.50/40.75) = 8.6\% \quad \text{(rounded)}$$

Price-earnings ratio (P/E): The closing price divided by the earnings per share (for the most recent four quarters); for example, if annual earnings = \$2.25 for the above stock, priced at \$40.75, then P/E = 40.75/2.25 = 18.1.

Volume: The volume traded, usually on a daily basis, is quoted in units of 100. For example, a volume figure of 190 means 190 × 100 = 19,000 shares traded.

A listing might look as follows:

Stock	**Div**	**Yield**	**Vol**	**Hi**	**Lo**	**Close**	**Change**
XYZ	3.50	8.6	190	42¼	40⅛	40¾	+½

which means that this stock attained daily highs and lows of 42¼ and 40⅛, respectively, and closed at ½ above the previous day's closing price of 40¼.

12.21 Bonds

Bonds are issued by many corporations (and governments), usually with a par value or face value of \$1,000, and mature at a specified time that is part of the quotation information found in newspapers. The corporation (or government) thus promises to pay the face value of $1,000 at maturity and also pays interest to the bond holder. The quotation also includes this annual interest expressed in percent. Although the face value of the bond may be $1,000, the price that purchases it is based on units of $100; for example, the quoted purchase price, such as $95, means that the bond costs 10 times this, or $950, whereas a purchase listing of $110 would mean that it costs $1,100. Thus, XYZ corporation bonds that pay interest at 8.5% and mature in 1998 would be listed as

XYZ	8½	98

If the purchase price is $110, then the cost (without commission) is 10 × $110 = $1,100 but pays interest of 8.5% of the face value of $1,000, or $85. This is the amount paid annually regardless of the purchase price. Thus, the effective yield is computed from this earned interest and the purchase price:

$$100 \times (85/1100) = 7.7\%$$

The listing, as published in newspapers, might look as follows:

Bond	Current Yield	Close	Net Change
XYZ 8½ 98	7.7	110	+½

The last column, "Net Change," means that the closing price on the previous trading day was 109½. The quotation might also include the sales volume (usually in units of $1,000) as well as the high and low prices of the bond during the trading day.

- *Bond Value*
  The value of a bond is determined from the number of years to maturity and the amount of the annual coupon payments paid each year until the bond matures. The face value (par value) of most bonds is $1,000.00. The current value uses the current interest rate, e.g., 7%, to compute the current value of $1,000 at 7% for the number of years to maturity, such as 30. This is given by $1000/(1 + 0.07)^{30}$ = $131.37. This is the first part of the computation. The next part uses the amount of the coupon payment, e.g., $70 per year for 30 years. This is calculated from the product of $70 and the *factor* $[1 - 1/(1.07)^{30}]/0.07$. This factor is 12.4090 and when multiplied by $70 gives $868.63. This is the second part of the calculation. When these parts are added, $131.37 + $863.63, the sum is $1,000. Accordingly, this bond is presently worth $1,000, i.e., a bond with face value of $1,000 that pays $70 per year for 30 years should have a current selling price of $1,000 (assuming safety) based on the current interest rate of 7%.

  The two parts of the calculation are based on the formulas below, in which *r* is the annual interest rate and N is the number of years:

$$Z = (\textit{face value}) / (1 + r)^N$$

The second part uses these values and the annual payment C:

$$T = \frac{C}{r}\left[1 - \frac{1}{(1+r)^N}\right]$$

Example

The previously illustrated 30-year bond pays \$70 per year, but the current interest rate is now only 6%. For this calculation we need $(1.06)^{30}$, which is 5.7435. Thus, Z = \$1,000/5.7435 and T = (70/0.06) × (1 − 1/5.7435) = \$963.54. Adding the two parts, \$174.11 + \$963.54, gives \$1,137.65. *Note*: The bond value has increased as a result of this interest drop.

12.22 Tax-Free Yield

Certain securities such as municipal bonds may be purchased tax-free. The relationship between the tax-free yield (F) and the tax-equivalent yield (T) depends on one's tax rate (R) according to the formula

$$F = T(1 - R)$$

Example

If one is in the 28% tax bracket, i.e., $R = 0.28$, then the tax-free equivalent of a corporate bond paying 6.5% is

$$F = 0.065 \times (1 - 0.28) = 0.0468, \text{ or } 4.68\%$$

(The tax rate is taken to be the total of the federal and effective state rates.)

12.23 Stock Options (Puts and Calls)

Various stock exchanges permit the purchase of stock options such as "puts" and "calls." Each of these has an exercise price and an expiration date. The call option is the right to buy shares at the exercise price at any time on or before the expiration date. The put option is the right to sell shares at the exercise price. Thus, if the stock of XYZ corporation is currently trading at 52½ ($52.50) and the exercise price is $50 with an expiration date 3 weeks hence, the call provides a guarantee of $2.50 if sold now (less commissions). Thus, the call has a value of at least that amount and would sell for even more since the stock price might increase even further. The price of the call might thus be $3.25. In contrast, the put, if exercised now, would lose $2.50, a negative value. But because the exercise date is still weeks away, the put still has worth since the stock price could fall below $50 (the exercise price), giving the option some value, such as ⅜ (37½ cents). As the time of expiration gets nearer, this value would dwindle to zero. The listing of these options (in early March 1997) would appear as follows:

XYZ	**C**	**521/2**	
Date	Strike	Call	Put
March 97	50	3¼	⅜
(C is a code for the exchange.)			

If the expiration is a month later, April 1997, the call and put prices would be greater, say 4 and 1⅛, respectively, because of the time to expiration (the third Friday of the month).

12.24 Market Averages

The simple average of a set of n numbers, also called the arithmetic mean, is computed by summing the numbers and dividing by n. The closing prices of groups of stocks, such as the stocks of 30 large companies that comprise the well-known *Dow Jones Industrials*, provide an average. Because corporations often split shares, thereby changing their price per share, and because some of the corporations on the list of 30 may change over time, the simple formula for getting these averages is modified. For example, in the summer of 1997 the total of the 30 prices was divided by 0.26908277 to get the average (or average change). For example, if each gained 1 point, the sum 30 divided by 0.26908277 is $111.49, a gain in the average. Thus, even over several years, with stock splits (and even some different corporations), a change in the average is a useful indicator of performance.

Other popular averages such as Standard & Poor's and the New York Stock Exchange are comprised of different groups of stocks in segments such as transportation, utilities, etc., as well as broad, composite averages. Each group has its own divisor.

12.25 Mutual and Quotations

Mutual funds are usually listed in newspapers with values of the net asset value (*NAV*) of a share, the buy price of a share, and the change in net asset value from the previous day's closing price. The net asset value is computed as the total of securities and cash in the fund divided by the number of shares. When the buy price is greater than the *NAV*, the difference is known as the *load* or cost (commission) of buying the fund.

The percent as commission is computed as 100 × *load*/*NAV*.

Example

The XYZ fund is listed as follows:

Fund	**NAV**	**Buy**	**Change**
XYZ	18.40	19.52	–0.03

The load is 100 × (19.52 – 18.40)/(18.40) = 6.087%.

The listing also indicates that on the previous trading day the *NAV* was $18.43. If the fund is sold without a load, the symbol "NL" (no load) appears in the buy column. Total return may be computed from the difference between your cost (buy price) and the *NAV* when you sell and will also include dividend and distributions that the fund may pay.

Example

The fund above, which was purchased at $19.52 per share, attains a net asset value of $22 eight months later. It also declares a dividend (*D*) of 25 cents and a capital gain distribution (*CG*) of 40 cents during that time. These are added to the difference between the net asset value and the buy price, and this quantity is divided by the buy price to give the proportional return (*PR*); percent return is 100 × *PR*:

$$PR = \frac{D + CG + (NAV - Buy)}{Buy}$$

$$PR = \frac{0.25 + 0.40 + (22 - 19.52)}{19.52} = \frac{3.13}{19.52} = 0.1603$$

Thus, the percent return is 16.03%. Because this was attained in only 8 months, it is equivalent to a 12-month return obtained by multiplying by $^{12}/_{8}$, or 1.5. Thus, the *annual percent return* is 1.5 × 16.03%, or 24.04%.

12.26 Dollar Cost Averaging

The share price of a stock or mutual fund varies so that regular investment of a fixed amount of money will buy more shares when the price is low and fewer shares when the price is high. The table below illustrates the results of investing $100 each month for 9 months in a stock whose price is initially $15.00 and which fluctuates over the 9-month period but returns to $15.00 per share. The same $100 divided by the share price gives the number of shares purchased each month. The total number of shares accumulated is 62.742 and has a price of $15 at the end of 9 months so that the total is worth $941.13. This is a gain of $41.13, even though the share price is the same at the beginning and end of the time period.

Month	Price/Share	No. of Shares
1	15.00	6.6667
2	14.50	6.8966
3	14.00	7.1429
4	14.00	7.1429
5	13.50	7.4074
6	14.00	7.1429
7	14.50	6.8966
8	14.75	6.7797
9	15.00	6.6667
Total shares		62.7424
Value = \$15.00 × 62.7424 = \$941.14		

12.27 Moving Average

Stocks, bonds, mutual funds, and other instruments whose prices change are sometimes plotted along with their *moving average* over some specified time interval. For example, suppose the closing prices of a mutual fund for a sequence of days were as shown below:

14.00, 14.25, 14.35, 15.02, 14.76, 14.81, 14.92, 14.99, 15.32, 15.45, 15.32, 15.05, …, 17.45

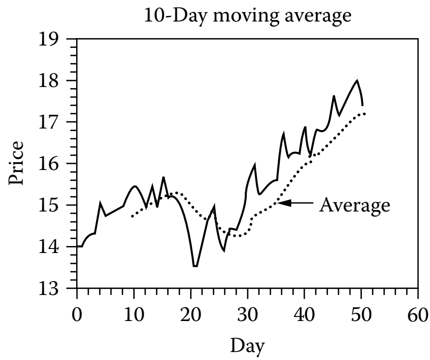


FIGURE 12.3
The moving average.

Illustrated here is the 10-day moving average. The average of the first 10 prices is the sum (14.00 + 14.25 + ⋯ + 15.45) divided by 10, which is 14.79. The next average is obtained from day 2 to day 11, that is, drop 14.00, which is day 1's price, and average by summing to day 11 (14.25 + 14.35 + ⋯ + 15.32) and dividing by 10, which gives 14.92. These numbers, computed on days 10, 11, etc., are the 10-day moving average values. They are plotted, along with the daily prices, in the graph in Figure 12.3.

While the daily prices fluctuate considerably, the moving average has much lower fluctuation, as seen by the smoother curve. The usefulness of a moving average is that it indicates the main trend in prices. Whereas this example uses the 10-day moving average, other time intervals may be used, such as 30-day, 200-day, etc. Some mutual funds use a 39-week moving average.

Table of Derivatives

In the following table, a and n are constants, e is the base of the natural logarithms, and u and v denote functions of x.

1. $\frac{d}{dx}(a) = 0$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(au) = a\frac{du}{dx}$

4. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

5. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

6. $\frac{d}{dx}(u/v) = \frac{v\frac{dv}{dx} - u\frac{dv}{dx}}{v^2}$

7. $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$

8. $\frac{d}{dx}e^u = e^u\frac{du}{dx}$

9. $\frac{d}{dx} a^u = (\log_e a) a^u \frac{du}{dx}$

10. $\frac{d}{dx} \log_e u = (1/u) \frac{du}{dx}$

11. $\frac{d}{dx} \log_a u = (\log_a e)(1/u) \frac{du}{dx}$

12. $\frac{d}{dx} u^v = v u^{v-1} \frac{du}{dx} + u^v (\log_e u) \frac{dv}{dx}$

13. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$

14. $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$

15. $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$

16. $\frac{d}{dx} \operatorname{ctn} u = -\csc^2 u \frac{du}{dx}$

17. $\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$

18. $\frac{d}{dx} \csc u = -\text{csu} \operatorname{ctn} u \frac{du}{dx}$

19. $\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx},$

$$\left(-\frac{1}{2}\pi \leq sin^{-1} u \leq \frac{1}{2}\pi\right)$$

20. $$\frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}, (0 \le \cos^{-1}u \le \pi)$$

21. $$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$$

22. $$\frac{d}{dx}\text{ctn}^{-1}u = \frac{-1}{1+u^2}\frac{du}{dx}$$

23. $$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx},$$

$$\left(-\pi \le \sec^{-1}u < -\frac{1}{2}\pi; \quad 0 \le \sec^{-1}u < \frac{1}{2}\pi\right)$$

24. $$\frac{d}{dx}\csc^{-1}u = \frac{-1}{u\sqrt{u^2-1}}\frac{du}{dx},\dagger$$

$$\left(-\pi < \csc^{-1}u \le -\frac{1}{2}\pi; \quad 0 < \csc^{-1}u \le \frac{1}{2}\pi\right)$$

25. $$\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}$$

26. $$\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}$$

27. $$\frac{d}{dx}\tanh u = \text{sech}^2 u\frac{du}{dx}$$

28. $$\frac{d}{dx}\text{ctn}\, u = -\text{csch}^2 u\frac{du}{dx}$$

29. $$\frac{d}{dx}\text{sech}\, u = -\text{sech}\, u \tanh u\frac{du}{dx}$$

30. $\frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \operatorname{ctnh} u \frac{du}{dx}$

31. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

32. $\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$

33. $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}$

34. $\frac{d}{dx}\operatorname{ctnh}^{-1} u = \frac{-1}{u^2-1}\frac{du}{dx}$

35. $\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$

36. $\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{u\sqrt{u^2+1}}\frac{du}{dx}$

Additional Relations with Derivatives

$$\frac{d}{dt}\int_a^t f(x)dx = f(t)$$

$$\frac{d}{dt}\int_t^a f(x)dx = -f(t)$$

If $x = f(y)$, then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \text{ (chain rule)}$$

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)},$$

and

$$\frac{d^2y}{dx^2} = \frac{f'(t)\,g''(t) - g'(t)\,f''(t)}{[f'(t)]^3}$$

(*Note*: exponent in denominator is 3.)

Table of Integrals: Indefinite and Definite Integrals

Table of Indefinite Integrals

Basic Forms (all logarithms are to base *e*)

1. $\int dx = x + C$

2. $\int x^n dx = \dfrac{x^{n+1}}{n+1} + C, \ (n \neq 1)$

3. $\int \dfrac{dx}{x} = \log x + C$

4. $\int e^x dx = e^x + C$

5. $\int a^x dx = \dfrac{a^x}{\log a} + C$

6. $\int \sin x \, dx = -\cos x + C$

7. $\int \cos x \, dx = \sin x + C$

8. $\int \tan x \, dx = -\log \cos x + C$

9. $\int \sec^2 x \, dx = \tan x + C$

10. $\int \csc^2 x \, dx = -\text{ctn}\, x + C$

11. $\int \sec x \tan x \, dx = \sec x + c$

12. $\int \sin^2 x \, dx = \frac{1}{2} x - \frac{1}{2} \sin x \cos x + C$

13. $\int \cos^2 x \, dx = \frac{1}{2} x + \frac{1}{2} \sin x \cos x + C$

14. $\int \log x \, dx = x \log x - x + C$

15. $\int a^x \log a \, dx = a^x + C, \ (a > 0)$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan \frac{x}{a} + C$

17. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x-a}{x+a} + C, \ (x^2 > a^2)$

$$= \frac{1}{2a} \log \frac{a-x}{a+x} + C, \ (x^2 < a^2)$$

18. $$\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)+C$$

19. $$\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left(x+\sqrt{x^2-a^2}\right)+C$$

20. $$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}+C$$

21. $$\int \sqrt{a^2-x^2}\,dx$$

$$=1/2\left\{x\sqrt{a^2-x^2}+a^2\arcsin\frac{x}{a}\right\}+C$$

22. $$\int \sqrt{a^2+x^2}\,dx$$

$$=1/2\left\{x\sqrt{a^2+x^2}+a^2\log(x+\sqrt{a^2+x^2})\right\}+C$$

23. $$\int \sqrt{x^2-a^2}\,dx$$

$$=1/2\left\{x\sqrt{x^2-a^2}-a^2\log(x+\sqrt{x^2-a^2})\right\}+C$$

Form $ax+b$

In the following list, a constant of integration C should be added to the result of each integration.

24. $$\int (ax+b)^m\,dx = \frac{(ax+b)^{m+1}}{a(m+1)} \quad (m \neq 1)$$

25. $$\int x(ax+b)^m\,dx = \frac{(ax+b)^{m+2}}{a^2(m+2)} - \frac{b(ax+b)^{m+1}}{a^2(m+1)},$$

$$(m \neq -1, -2)$$

26. $$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b)$$

27. $$\int \frac{dx}{(ax+b)^2} = -\frac{1}{a(ax+b)}$$

28. $$\int \frac{dx}{(ax+b)^3} = -\frac{1}{2a(ax+b)^2}$$

29. $$\int \frac{xdx}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\log(ax+b)$$

30. $$\int \frac{xdx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\log(ax+b)$$

31. $$\int \frac{x\,dx}{(ax+b)^3} = \frac{b}{2a^2(ax+b)^2} - \frac{1}{a^2(ax+b)}$$

32. $$\int x^2(ax+b)^m\,dx$$

$$= \frac{1}{a^3}\left[\frac{(ax+b)^{m+3}}{m+3} - \frac{2b(ax+b)^{m+2}}{m+2} + \frac{b^2(ax+b)^{m+1}}{m+1}\right]$$

$$(m \neq -1, -2, -3)$$

33. $$\int \frac{x^2 dx}{ax+b}$$

$$= \frac{1}{a^3}\left[\frac{1}{2}(ax+b)^2 - 2b(ax+b) + b^2 \log(ax+b)\right]$$

34. $$\int \frac{x^2 dx}{(ax+b)^2}$$

$$= \frac{1}{a^3}\left[(ax+b) - \frac{b^2}{ax+b} - 2b\log(ax+b)\right]$$

35. $$\int \frac{x^2 dx}{(ax+b)^3}$$

$$= \frac{1}{a^3}\left[\log(ax+b) + \frac{2b}{ax+b} - \frac{b^2}{2(ax+b)^2}\right]$$

36. $$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\log\left(\frac{x}{ax+b}\right)$$

37. $$\int \frac{dx}{x^3(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\log\left(\frac{ax+b}{x}\right)$$

38. $$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2}\log\left(\frac{ax+b}{x}\right)$$

39. $$\int \frac{dx}{x^2(ax+b)^2}$$

$$= -\frac{2ax+b}{b^2x(ax+b)} + \frac{2a}{b^3}\log\left(\frac{ax+b}{x}\right)$$

40. $$\int x^m (ax+b)^n \, dxv$$

$$= \frac{1}{a(m+n+1)} \left[x^m (ax+b)^{n+1} - mb \int x^{m-1} (ax+b)^n \, dx \right]$$

$$= \frac{1}{m+n+1} \left[x^{m+1} (ax+b)^n + nb \int x^m (ax+b)^{n-1} \, dx \right]$$

$$(m > 0, m+n+1 \neq 0)$$

Forms *ax* + *b* and *cx* + *d*

41. $$\int \frac{dx}{(ax+b)(cx+d)} = \frac{1}{bc-ad} \log\left(\frac{cx+x}{ax+b}\right)$$

42. $$\int \frac{x\,dx}{(ax+b)(cx+d)}$$

$$= \frac{1}{bc-ad} \left[\frac{b}{a} \log(ax+b) - \frac{d}{c} \log(cx+d) \right]$$

43. $$\int \frac{dx}{(ax+b)^2(cx+d)}$$

$$= \frac{1}{bc-ad} \left[\frac{1}{ax+b} + \frac{c}{bc-ad} \log\left(\frac{cx+d}{ax+b}\right) \right]$$

44. $$\int \frac{x\,dx}{(ax+b)^2(cx+d)}$$

$$= \frac{1}{bc-ad}\left[-\frac{b}{a(ax+b)} - \frac{d}{bc-ad}\log\left(\frac{cx+d}{ax+b}\right)\right]$$

Forms with $ax+b, cx+d,$ and $\sqrt{ax+b}$

45. $$\int \frac{x^2\,dx}{(ax+b)^2(cx+d)} = \frac{b^2}{a^2(bc-ad)(ax+b)}$$

$$+ \frac{1}{(bc-ad)^2}$$

$$\times\left[\frac{d^2}{c}\log|cx+d| + \frac{b(bc-2ad)}{a^2}\log(ax+b)\right]$$

46. $$\int \frac{ax+b}{cx+d}\,dx = \frac{ax}{c} + \frac{bc-ad}{c^2}\log(cx+d)$$

47. $$\int (ax+b)^m(cx+d)^n\,dx$$

$$= \frac{1}{a(m+n+1)}\Big[(ax+b)^{m+1}(cx+d)^n$$

$$-n(bc-ad)\int (ax+b)^m(cx+d)^{n-1}\,dx\Big]$$

Forms with $\sqrt{ax+b}$

48. $$\int \sqrt{ax+b}\;dx = \frac{2}{3a}\sqrt{(ax+b)^3}$$

49. $$\int x\sqrt{ax+b}\;dx = \frac{2(3ax-2b)}{15a^2}\sqrt{(ax+b)^3}$$

50. $$\int x^2\sqrt{ax+b}\,dx$$

$$= \frac{2(15a^2x^2-12abx+8b^2)}{105a^3}\sqrt{(ax+b)^3}$$

51. $$\int x^m\sqrt{ax+b}\;dx$$

$$= \frac{2}{a(2m+3)}$$

$$\times\left[x^m\sqrt{(ax+b)^3} - mb\int x^{m-1}\sqrt{ax+b}\;dx\right]$$

52. $$\int \frac{(ax+b)^{\frac{m}{2}}\,dx}{x}$$

$$= a\int (ax+b)^{\frac{m-2}{2}}\,dx + b\int \frac{(ax+b)^{\frac{m-2}{2}}\,dx}{x}$$

53. $$\int \frac{dx}{x(ax+b)^{\frac{m}{2}}} = \frac{1}{b}\int \frac{dx}{x(ax+b)^{\frac{m-2}{2}}} - \frac{a}{b}\int \frac{dx}{(ax+b)^{\frac{m}{2}}}$$

54. $$\int \frac{\sqrt{ax+b}\,dx}{cx+d} = \frac{2\sqrt{ax+b}}{c}$$

$$+\frac{1}{c}\sqrt{\frac{bc-ad}{c}}\log\left|\frac{\sqrt{c(ax+b} - \sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}\right|$$

$$(c>0,\ bc>ad)$$

55. $$\int \frac{\sqrt{ax+b}\,dx}{cx+d} = \frac{2\sqrt{ax+b}}{c} - \frac{2}{c}\sqrt{\frac{ad-bc}{c}}\arctan\sqrt{\frac{c(ax+b)}{ad-bc}}$$

$$(c > 0, bc < ad)$$

56. $$\int \frac{(cx+d)dx}{\sqrt{ax+b}} = \frac{2}{3a^2}(3ad - 2bc + acx)\sqrt{ax+b}$$

57. $$\int \frac{dx}{(cx+d)\sqrt{ax+b}}$$

$$= \frac{2}{\sqrt{c}\sqrt{ad-bc}}\arctan\sqrt{\frac{c(ax+b}{ad-bc}}$$

$$(c > 0, bc < ad)$$

58. $$\int \frac{dx}{(cx+d)\sqrt{ax+b}}$$

$$= \frac{1}{\sqrt{c}\sqrt{bc-ad}}\log\left|\frac{\sqrt{c(ax+b)}-\sqrt{bc-ad}}{\sqrt{c(ax+b)}+\sqrt{bc-ad}}\right|$$

$$(c > 0, bc > ad)$$

59. $$\int \sqrt{ax+b}\sqrt{cx+d}\ dx$$

$$= \int \sqrt{acx^2 + (ad+bc)x + bd}\,dx$$

(see 154)

60. $$\int \frac{\sqrt{ax+b}\;dx}{x} =$$

$$2\sqrt{ax+b} + \sqrt{b}\log\left(\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right)$$

$$(b > 0)$$

61. $$\int \frac{\sqrt{ax+b}\,dx}{x} = 2\sqrt{ax+b}$$

$$-2\sqrt{-b}\ \arctan\left(\frac{\sqrt{ax+b}}{\sqrt{-b}}\right)$$

$$(b < 0)$$

62. $$\int \frac{\sqrt{ax+b}}{x^2}\,dx = \frac{\sqrt{ax+b}}{x} + \frac{a}{2}\int \frac{dx}{x\sqrt{ax+b}}$$

63. $$\int \frac{\sqrt{ax+b}\,dx}{x^m}$$

$$= -\frac{1}{(m-1)b}$$

$$\times\left[\frac{\sqrt{(ax+b)^3}}{x^{m-1}} + \frac{(2m-5)a}{2}\int \frac{\sqrt{ax+b}\;dx}{x^{m-1}}\right]$$

$$(m \neq 1)$$

64. $$\int \frac{dx}{\sqrt{ax+b}} = \frac{2\sqrt{ax+b}}{a}$$

65. $$\int \frac{x\,dx}{\sqrt{ax+b}} = \frac{2(ax-2b)\sqrt{ax+b}}{3a^2}$$

66. $$\int \frac{x^2dx}{\sqrt{ax+b}} = \frac{2(3a^2x^2-4abx+8b^2)\sqrt{ax+b}}{15a^3}$$

67. $$\int \frac{x^m dx}{\sqrt{ax+b}}$$

$$= \frac{2}{a(2m+1)}\left[x^m\sqrt{ax+b} - mb\int \frac{x^{m-1}dx}{\sqrt{ax+b}}\right]$$

$$\left(m \neq -\frac{1}{2}\right)$$

68. $$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}}\log\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| \qquad (b>0)$$

69. $$\int \frac{dx}{x\sqrt{ax+b}} = \frac{2}{\sqrt{-b}}\arctan\sqrt{\frac{ax+b}{-b}} \qquad (b<0)$$

Forms with $\sqrt{ax+b}$ and $ax^2 + c$

70. $$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}$$

71. $$\int \frac{dx}{x^m\sqrt{ax+b}}$$

$$= -\frac{\sqrt{ax+b}}{(m-1)bx^{m-1}}$$

$$-\frac{(2m-3)a}{(2m-2)b}\int \frac{dx}{x^{m-1}\sqrt{ax+b}} \qquad (m \neq 1)$$

72. $$\int (ax+b)^{\pm\frac{m}{2}}\,dx = \frac{2(ax+b)^{\frac{2\pm m}{2}}}{a(2\pm m)}$$

73. $$\int x(ax+b)^{\pm\frac{m}{2}}\,dx$$

$$= \frac{2}{a^2}\left[\frac{(ax+b)^{\frac{4\pm m}{2}}}{4\pm m} - \frac{b(ax+b)^{\frac{2\pm m}{2}}}{2\pm m}\right]$$

Form $ax^2 + c$

74. $$\int \frac{dx}{ax^2+c} = \frac{1}{\sqrt{ac}}\arctan\left(x\sqrt{\frac{a}{c}}\right) \quad (a>0, c>0)$$

75. $$\int \frac{dx}{ax^2+c}$$

$$= \frac{1}{2\sqrt{-ac}}\log\left(\frac{x\sqrt{a}-\sqrt{-c}}{x\sqrt{a}+\sqrt{-c}}\right) \quad (a>0, c<0)$$

76. $$\int \frac{dx}{ax^2+c}$$

$$= \frac{1}{2\sqrt{-ac}}\log\left(\frac{\sqrt{c}+x\sqrt{-a}}{\sqrt{c}-x\sqrt{-a}}\right) \quad (a<0, c>0)$$

77. $$\int \frac{x\,dx}{ax^2+c} = \frac{1}{2a}\log(ax^2+c)$$

78. $$\int \frac{x^2 dx}{ax^2 + c} = \frac{x}{a} - \frac{c}{a} \int \frac{dx}{ax^2 + c}$$

79. $$\int \frac{x^m dx}{ax^2 + c} = \frac{x^{m-1}}{a(m-1)} - \frac{c}{a} \int \frac{x^{m-2} dx}{ax^2 + c} \quad (m \neq 1)$$

80. $$\int \frac{dx}{x(ax^2 + c)} = \frac{1}{2c} \log\left(\frac{ax^2}{ax^2 + c} \right)$$

81. $$\int \frac{dx}{x^2(ax^2 + c)} = -\frac{1}{cx} - \frac{a}{c} \int \frac{dx}{ax^2 + c}$$

82. $$\int \frac{dx}{x^m(ax^2 + c)}$$

$$= -\frac{1}{c(m-1)x^{m-1}} - \frac{a}{c} \int \frac{dx}{x^{m-2}(ax^2 + c)} \quad (m \neq 1)$$

83. $$\int \frac{dx}{(ax^2 + c)^m} = \frac{1}{2(m-1)c} \cdot \frac{x}{(ax^2 + c)^{m-1}} + \frac{2m-3}{2(m-1)c} \int \frac{dx}{(ax^2 + c)^{m-1}} \quad (m \neq 1)$$

Forms $ax^2 + c$ and $ax^2 + bx + c$

84. $$\int \frac{x\, dx}{(ax^2 + c)^m}$$

$$= -\frac{1}{2a(m-1)(ax^2 + c)^{m-1}} \quad (m \neq 1)$$

85. $$\int \frac{x^2 dx}{(ax^2+c)^m} = -\frac{x}{2a(m-1)(ax^2+c)^{m-1}} + \frac{1}{2a(m-1)}\int \frac{dx}{(ax^2+c)^{m-1}} \quad (m \neq 1)$$

86. $$\int \frac{dx}{x(ax^2+c)^m} = \frac{1}{2c(m-1)(ax^2+c)^{m-1}} + \frac{1}{c}\int \frac{dx}{x(ax^2+c)^{m-1}} \quad (m \neq 1)$$

87. $$\int \frac{dx}{x^2(ax^2+c)^m} = \frac{1}{c}\int \frac{dx}{x^2(ax^2+c)^{m-1}} - \frac{a}{c}\int \frac{dx}{(ax^2+c)^m}$$

(see 82 and 83)

Form $ax^2 + bx + c$

88. $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \log \frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}} \quad (b^2 > 4ac)$$

89. $$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{4ac-b^2}} \tan^{-1} \frac{2ax+b}{\sqrt{4ac-b^2}} \quad (b^2 < 4ac)$$

90. $$\int \frac{dx}{ax^2+bx+c} = -\frac{2}{2ax+b}, \quad (b^2 = 4ac)$$

91. $$\int \frac{dx}{(ax^2+bx+c)^{n+1}}$$
$$= \frac{2ax+b}{n(4ac-b^2)(ax^2+bx+c)^n}$$
$$+ \frac{2(2n-1)a}{n(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^n}$$

92. $$\int \frac{xdx}{ax^2+bx+c}$$
$$= \frac{1}{2a}\log(ax^2+bx+c) - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c}$$

93. $$\int \frac{x^2dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\log(ax^2+bx+c)$$
$$+ \frac{b^2-2ac}{2a^2}\int \frac{dx}{ax^2+bx+c}$$

94. $$\int \frac{x^n dx}{ax^2+bx+c} = \frac{x^{n-1}}{(n-1)a} - \frac{c}{a}\int \frac{x^{n-2}dx}{ax^2+bx+c}$$
$$- \frac{b}{a}\int \frac{x^{n-1}dx}{ax^2+bx+c}$$

Forms with ax^2+bx+c and $\sqrt{2ax-x^2}$

95. $$\int \frac{xdx}{(ax^2+bx+c)^{n+1}} = \frac{-(2c+bx)}{n(4ac-b^2)(ax^2+bx+c)^n}$$
$$- \frac{b(2n-1)}{n(4ac-b^2)}\int \frac{dx}{(ax^2+bx+c)^n}$$

96. $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\log\left(\frac{x^2}{ax^2+bx+c}\right) - \frac{b}{2c}\int \frac{dx}{(ax^2+bx+c)}$$

97. $$\int \frac{dx}{x^2(ax^2+bx+c)} = \frac{b}{2c^2}\log\left(\frac{ax^2+bx+c}{x^2}\right) - \frac{1}{cx} + \left(\frac{b^2}{2c^2} - \frac{a}{c}\right)\int \frac{dx}{(ax^2+bx+c)}$$

Forms with $\sqrt{2ax - x^2}$

98. $$\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x-a}{a}\right)$$

99. $$\int x\sqrt{2ax-x^2}\,dx = -\frac{3a^2+ax-2x^2}{6}\sqrt{2ax-x^2} + \frac{a^3}{2}\arcsin\left(\frac{x-a}{a}\right)$$

100. $$\int x^m\sqrt{2ax-x^2}\,dx = -\frac{x^{m-1}\sqrt{(2ax-x^2)^3}}{m+2} + \frac{a(2m+1)}{m+2}\int x^{m-1}\sqrt{2ax-x^2}\,dx$$

101. $$\int \frac{\sqrt{2ax - x^2}\,dx}{x} = \sqrt{2ax - x^2} + a \arcsin\left(\frac{x-a}{a}\right)$$

102. $$\int \frac{\sqrt{2ax - x^2}\,dx}{x^m} = -\frac{\sqrt{(2ax - x^2)^3}}{a(2m-3)x^m} + \frac{m-3}{a(2m-3)}\int \frac{\sqrt{2ax - x^2}\,dx}{x^{m-1}}$$

103. $$\int \frac{dx}{\sqrt{2ax - x^2}} = \arcsin\left(\frac{x-a}{a}\right)$$

104. $$\int \frac{x\,dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \arcsin\left(\frac{x-a}{a}\right)$$

105. $$\int \frac{x^m dx}{\sqrt{2ax - x^2}} = -\frac{x^{m-1}\sqrt{2ax - x^2}}{m} + \frac{a(2m-1)}{m}\int \frac{x^{m-1}dx}{\sqrt{2ax - x^2}}$$

Forms with $\sqrt{2ax - x^2}$ and Forms $\sqrt{a^2 - x^2}$

106. $$\int \frac{dx}{2\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{ax}$$

107. $$\int \frac{dx}{x^m\sqrt{2ax - x^2}} = -\frac{\sqrt{2ax - x^2}}{a(2m-1)x^m} + \frac{m-1}{a(2m-1)}\int \frac{dx}{x^{m-1}\sqrt{2ax - x^2}}$$

Forms with $\sqrt{a^2 - x^2}$

108. $$\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right)$$

109. $$\int x\sqrt{a^2 - x^2}\,dx = -\frac{1}{3}\sqrt{(a^2 - x^2)^3}$$

110. $$\int x^2\sqrt{a^2 - x^2}\,dx = -\frac{x}{4}\sqrt{(a^2 - x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right)$$

111. $$\int x^3\sqrt{a^2 - x^2}\,dx = \left(-\frac{1}{5}x^2 - \frac{2}{15}a^2\right)\sqrt{(a^2 - x^2)^3}$$

112. $$\int \frac{\sqrt{a^2 - x^2}\,dx}{x} = \sqrt{a^2 - x^2} - a\log\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|$$

113. $$\int \frac{\sqrt{a^2 - x^2}\,dx}{x^2} = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin\frac{x}{a}$$

114. $$\int \frac{\sqrt{a^2 - x^2}\,dx}{x^3} = -\frac{\sqrt{a^2 - x^2}}{2x^2} + \frac{1}{2a}\log\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|$$

115. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a}$$

116. $$\int \frac{x\,dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2}$$

117. $$\int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = -\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}$$

118. $$\int \frac{x^3 dx}{\sqrt{a^2 - x^2}} = \tfrac{1}{3}\sqrt{(a^2 - x^2)^3} - a^2\sqrt{a^2 - x^2}$$

119. $$\int \frac{dx}{x\sqrt{a^2 - x^2}} = -\frac{1}{a}\log\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|$$

120. $$\int \frac{dx}{x^2\sqrt{a^2 - x^2}} = -\sqrt{\frac{a^2 - x^2}{a^2 x}}$$

121. $$\int \frac{dx}{x^3\sqrt{a^2 - x^2}}$$

$$= -\frac{\sqrt{a^2 - x^2}}{2a^2x^2} - \frac{1}{2a^3}\log\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|$$

Forms with $\sqrt{x^2 - a^2}$

122. $$\int \sqrt{x^2 - a^2}\,dx$$

$$= \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2 - a^2}\right|$$

123. $$\int x\sqrt{x^2 - a^2}\,dx = \tfrac{1}{3}\sqrt{(x^2 - a^2)^3}$$

124. $$\int x^2\sqrt{x^2-a^2}\,dx = \frac{x}{4}\sqrt{(x^2-a^2)^3} + \frac{a^2x}{8}\sqrt{x^2-a^2}$$
$$-\frac{a^4}{8}\log\left|x+\sqrt{x^3-a^2}\right|$$

125. $$\int x^3\sqrt{x^2-a^2}\,dx$$
$$= \frac{1}{5}\sqrt{(x^2-a^2)^5} + \frac{a^2}{3}\sqrt{(x^2-a^2)^3}$$

126. $$\int\frac{\sqrt{x^2-a^2}\,dx}{x} = \sqrt{x^2-a^2} - a\arccos\frac{a}{x}$$

127. $$\int\frac{\sqrt{x^2-a^2}}{x^2}\,dx$$
$$= \frac{-1}{x}\sqrt{x^2-a^2} + \log\left|x+\sqrt{x^2-a^2}\right|$$

128. $$\int\frac{\sqrt{x^2-a^2}\,dx}{x^3} = -\frac{\sqrt{x^2-a^2}}{2x^2} + \frac{1}{2a}\arccos\frac{a}{x}$$

129. $$\int\frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right|$$

130. $$\int\frac{x\,dx}{\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}$$

131. $$\int\frac{x^2dx}{\sqrt{x^2-a^2}}$$
$$= \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|$$

132. $$\int \frac{x^3 dx}{\sqrt{x^2 - a^2}} = \frac{1}{3}\sqrt{(x^2 - a^2)^3} + a^2\sqrt{x^2 - a^2}$$

Forms $\sqrt{x^2 - a^2}$ and $\sqrt{a^2 + x^2}$

133. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\arccos\frac{a}{x}$$

134. $$\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2 x}$$

135. $$\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{2a^2 x^2} + \frac{1}{2a^2}\arccos\frac{a}{x}$$

Forms with $\sqrt{a^2 + x^2}$

136. $$\int \sqrt{a^2 + x^2}\,dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\log\left(x + \sqrt{a^2 + x^2}\right)$$

137. $$\int x\sqrt{a^2 + x^2}\,dx = \frac{1}{3}\sqrt{(a^2 + x^2)^3}$$

138. $$\int x^2\sqrt{a^2 + x^2}\,dx = \frac{x}{4}\sqrt{(a^2 + x^2)^3} - \frac{a^2 x}{8}\sqrt{a^2 + x^2} - \frac{a^4}{8}\log(x + \sqrt{a^2 + x^2})$$

139. $$\int x^3\sqrt{a^2 + x^2}\,dx = \left(\tfrac{1}{5}x^2 - \tfrac{2}{15}a^2\right)\sqrt{(a^2 + x^2)^3}$$

140. $$\int \frac{\sqrt{a^2+x^2}\,dx}{x} = \sqrt{a^2+x^2} - a\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

141. $$\int \frac{\sqrt{a^2+x^2}\,dx}{x^2} = -\frac{\sqrt{a^2+x^2}}{x} + \log\left(x+\sqrt{a^2+x^2}\right)$$

142. $$\int \frac{\sqrt{a^2+x^2}\,dx}{x^3} = -\frac{\sqrt{a^2+x^2}}{2x^2} - \frac{1}{2a}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

143. $$\int \frac{dx}{\sqrt{a^2+x^2}} = \log\left(x+\sqrt{a^2+x^2}\right)$$

144. $$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}$$

145. $$\int \frac{x^2dx}{\sqrt{a^2+x^2}} = \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$$

Forms $\sqrt{a^2+x^2}$ and $\sqrt{ax^2+bx+c}$

146. $$\int \frac{x^3dx}{\sqrt{a^2+x^2}} = \frac{1}{3}\sqrt{(a^2+x^2)^3} - a^2\sqrt{a^2+x^2}$$

147. $$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

148. $$\int \frac{dx}{x^2\sqrt{a^2+x^2}} = -\frac{\sqrt{a^2+x^2}}{a^2x}$$

149. $$\int \frac{dx}{x^3\sqrt{a^2+x^2}}$$

$$= -\frac{\sqrt{a^2+x^2}}{2a^2x^2} + \frac{1}{2a^3}\log\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|$$

Forms with $\sqrt{ax^2 + bx^2 + c}$

150. $$\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

$$= \frac{1}{\sqrt{a}}\log\left(2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right),\ a>0$$

151. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{-a}}\sin^{-1}\frac{-2ax-b}{\sqrt{b^2-4ac}},\ a<0$$

152. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a}$$

$$-\frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

153. $$\int \frac{x^n dx}{\sqrt{ax^2+bx+c}} = \frac{x^{n-1}}{an}\sqrt{ax^2+bx+c} - \frac{b(2n-1)}{2an}\int \frac{x^{n-1}dx}{\sqrt{ax^2+bx+c}} - \frac{c(n-1)}{an}\int \frac{x^{n-2}dx}{\sqrt{ax^2+bx+c}}$$

154. $$\int \sqrt{ax^2+bx+c}\ dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ac-b^2}{8a}\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

155. $$\int x\sqrt{ax^2+bx+c}\ dx = \frac{(ax^2+bx+c)^{\frac{3}{2}}}{3a} - \frac{b}{2a}\int \sqrt{ax^2+bx+c}\ dx$$

156. $$\int x^2\sqrt{ax^2+bx+c}\ dx$$

$$= \left(x - \frac{5b}{6a}\right)\frac{(ax^2+bx+c)^{\frac{3}{2}}}{4a} + \frac{(5b^2-4ac)}{16a^2}\int \sqrt{ax^2+bx+c}\ dx$$

Form $\sqrt{ax^2 + bx + c}$

157. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

$$= -\frac{1}{\sqrt{c}} \log\left(\frac{\sqrt{ax^2+bx+c}+\sqrt{c}}{x} + \frac{b}{2\sqrt{c}} \right),$$

$$c > 0$$

158. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}}$$

$$= \frac{1}{\sqrt{-c}} \sin^{-1} \frac{bx+2c}{x\sqrt{b^2-4ac}}, \quad c < 0$$

159. $$\int \frac{dx}{x\sqrt{ax^2+bx}} = -\frac{2}{bx}\sqrt{ax^2+bx}, \quad c = 0$$

160. $$\int \frac{dx}{x^n\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{c(n-1)x^{n-1}}$$

$$+ \frac{b(3-2n)}{2c(n-1)} \int \frac{dx}{x^{n-1}\sqrt{ax^2+bx+c}}$$

$$+ \frac{a(2-n)}{c(n-1)} \int \frac{dx}{x^{n-2}\sqrt{ax^2+bx+c}}$$

161. $$\int \frac{dx}{(ax^2+bx+c)^{\frac{3}{2}}}$$

$$= -\frac{2(2ax+b)}{(b^2-4ac)\sqrt{ax^2+bx+c}}, \quad b^2 \neq 4ac$$

162. $$\int \frac{dx}{(ax^2+bx+c)^{\frac{3}{2}}} = -\frac{1}{2\sqrt{a^3}\,(x+b/2a)^2}, \quad b^2 = 4ac$$

Miscellaneous Algebraic Forms

163. $$\int \sqrt{\frac{a+x}{b+x}}\,dx = \sqrt{(a+x)(b+x)} + (a-b)\log\left(\sqrt{a+x}+\sqrt{b+x}\right)$$
$$(a+x>0 \text{ and } b+x>0)$$

164. $$\int \sqrt{\frac{a+x}{b-x}}\,dx = -\sqrt{(a+x)(b-x)} - (a+b)\arcsin\sqrt{\frac{b-x}{a+b}}$$

165. $$\int \sqrt{\frac{a-x}{b+x}}\,dx = \sqrt{(a-x)(b+x)} + (a+b)\arcsin\sqrt{\frac{b+x}{a+b}}$$

166. $$\int \sqrt{\frac{1+x}{1-x}}\,dx = -\sqrt{1-x^2} + \arcsin x$$

167. $$\int \frac{dx}{\sqrt{(x-a)(b-x)}} = 2\arcsin\sqrt{\frac{x-a}{b-a}}$$

168. $$\int \frac{dx}{ax^3+b} = \frac{k}{3b}\left[\sqrt{3}\arctan\frac{2x-k}{k\sqrt{3}} + \log\left|\frac{k+x}{\sqrt{x^2-kx+k^2}}\right|\right] \quad \left(b \neq 0, k = \sqrt[3]{\frac{b}{a}}\right)$$

Form $\sqrt{ax^2+bx+c}$ and Miscellaneous Algebraic Forms

169. $$\int \frac{x\,dx}{ax^3+b} = \frac{1}{3ak}\left[\sqrt{3}\arctan\frac{2x-k}{k\sqrt{3}} - \log\left|\frac{k+x}{\sqrt{x^2-kx+k^2}}\right|\right] \quad \left(b \neq 0, k = \sqrt[3]{\frac{a}{b}}\right)$$

170. $$\int \frac{dx}{x(ax^m+b)} = \frac{1}{bm}\log\left|\frac{x^m}{ax^m+b}\right| \qquad (b \neq 0)$$

171. $$\int \frac{dx}{\sqrt{(2ax-x^2)^3}} = \frac{x-a}{a^2\sqrt{2ax-x^2}}$$

172. $$\int \frac{x\,dx}{(2ax-x^2)^3} = \frac{x}{a\sqrt{(2ax-x^2)}}$$

173. $$\int \frac{dx}{\sqrt{2ax+x^2}} = \log\left|x+a+\sqrt{2ax+x^2}\right|$$

174. $$\int \sqrt{\frac{cx+d}{ax+b}}\,dx = \frac{\sqrt{ax+b}\cdot\sqrt{cx+d}}{a} + \frac{(ad-bc)}{2a}\int \frac{dx}{\sqrt{ax+b}\cdot\sqrt{cx+d}}$$

Trigonometric Forms

175. $$\int (\sin ax)dx = -\frac{1}{a}\cos ax$$

176. $$\int (\sin^2 ax)dx = -\frac{1}{2a}\cos ax \sin ax + \frac{1}{2}x = \frac{1}{2}x - \frac{1}{4a}\sin 2ax$$

177. $$\int (\sin^3 ax)dx = -\frac{1}{3a}(\cos ax)(\sin^2 ax + 2)$$

178. $$\int (\sin^4 ax)dx = \frac{3x}{8} - \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

179. $$\int (\sin^n ax)dx = \frac{\sin^{n-1} ax \cos ax}{na} + \frac{n-1}{n}\int (\sin^{n-2} ax)dx$$

180. $$\int \frac{dx}{\sin^2 ax} = \int (\csc^2 ax)dx = -\frac{1}{a}\cot ax$$

181. $$\int \frac{dx}{\sin^m ax} = \int (\csc^m ax)dx$$

$$= -\frac{1}{(m-1)a} \cdot \frac{\cos ax}{\sin^{m-1} ax}$$

$$+ \frac{m-2}{m-1} \int \frac{dx}{\sin^{m-2} ax}$$

182. $$\int \sin(a+bx)dx = -\frac{1}{b}\cos(a+bx)$$

183. $$\int \frac{dx}{1 \pm \sin ax} = \mp \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

184. $$\int \frac{\sin ax}{1 \pm \sin ax} dx = \pm x + \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right)$$

185. $$\int \frac{dx}{(\sin ax)(1 \pm \sin ax)}$$

$$= \frac{1}{a} \tan\left(\frac{\pi}{4} \mp \frac{ax}{2}\right) + \frac{1}{a} \log \tan \frac{ax}{2}$$

186. $$\int \frac{dx}{(1+\sin ax)^2} = -\frac{1}{2a} \tan\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

$$-\frac{1}{6a} \tan^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

187. $$\int \frac{dx}{(1-\sin ax)^2} = \frac{1}{2a} \cot\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

$$+ \frac{1}{6a} \cot^3\left(\frac{\pi}{4} - \frac{ax}{2}\right)$$

188. $$\int \frac{\sin ax}{(1+\sin ax)^2}\,dx$$

$$= -\frac{1}{2a}\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{1}{6a}\tan^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

189. $$\int \frac{\sin ax}{(1-\sin ax)^2}\,dx$$

$$= -\frac{1}{2a}\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right)+\frac{1}{6a}\cot^3\left(\frac{\pi}{4}-\frac{ax}{2}\right)$$

190. $$\int \frac{\sin x\,dx}{a+b\sin x} = \frac{x}{b}-\frac{a}{b}\int \frac{dx}{a+b\sin x}$$

191. $$\int \frac{dx}{(\sin x)(a+b\sin x)}$$

$$= \frac{1}{a}\log\tan\frac{x}{2}-\frac{b}{a}\int \frac{dx}{a+b\sin x}$$

192. $$\int \frac{dx}{(a+b\sin x)^2}$$

$$= \frac{b\cos x}{(a^2-b^2)(a+b\sin x)}+\frac{a}{a^2-b^2}\int \frac{dx}{a+b\sin x}$$

193. $$\int \frac{\sin x\,dx}{(a+b\sin x)^2}$$

$$= \frac{a\cos x}{(b^2-a^2)(a+b\sin x)}+\frac{b}{b^2-a^2}\int \frac{dx}{a+b\sin x}$$

194. $$\int \sqrt{1+\sin x}\, dx = \pm 2\left(\sin\frac{x}{2} - \cos\frac{x}{2}\right)$$

$$\left[\text{use} + \text{if} (8k-1)\frac{\pi}{2} < x \le (8k+3)\frac{\pi}{2},\right.$$

$$\left.\text{otherwise} -; \quad k \text{ an integer}\right]$$

195. $$\int \sqrt{1-\sin x}\, dx = \pm 2\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)$$

$$\left[\text{use} + \text{if} (8k-3)\frac{\pi}{2} < x \le (8k+1)\frac{\pi}{2},\right.$$

$$\left.\text{otherwise} -; \quad k \text{ an integer}\right]$$

196. $$\int (\cos ax) dx = \frac{1}{a}\sin ax$$

197. $$\int (\cos^2 ax) dx = \frac{1}{2a}\sin ax \cos ax + \frac{1}{2}x$$

$$= \frac{1}{2}x + \frac{1}{4a}\sin 2ax$$

198. $$\int (\cos^3 ax) dx = \frac{1}{3a}(\sin ax)(\cos^2 ax + 2)$$

199. $$\int (\cos^4 ax) dx = \frac{3x}{8} + \frac{\sin 2ax}{4a} + \frac{\sin 4ax}{32a}$$

200. $$\int (\cos^n ax) dx$$

$$= \frac{1}{na}\cos^{n-1} ax \sin ax + \frac{n-1}{n}\int (\cos^{n-2} ax) dx$$

201. $\int(\cos^{2m}ax)dx$

$$= \frac{\sin ax}{a}\sum_{r=0}^{m-1}\frac{(2m)!(r!)^2}{2^{2m-2r}(2r+1)!(m!)^2}\cos^{2r+1}ax$$

$$+ \frac{(2m)!}{2^{2m}(m!)^2}x$$

202. $\int(\cos^{2m+1}ax)dx$

$$= \frac{\sin ax}{a}\sum_{r=0}^{m}\frac{2^{2m-2r}(m!)^2(2r)!}{(2m+1)!(r!)^2}\cos^{2r}ax$$

203. $\int\frac{dx}{\cos^2 ax} = \int(\sec^2 ax)dx = \frac{1}{a}\tan ax$

204. $\int\frac{dx}{\cos^n ax} = \int(\sec^n ax)dx$

$$= \frac{1}{(n-1)a}\cdot\frac{\sin ax}{\cos^{n-1}ax}$$

$$+ \frac{n-2}{n-1}\int\frac{dx}{\cos^{n-2}ax}$$

205. $\int\cos(a+bx)dx = \frac{1}{b}\sin(a+bx)$

206. $\int\frac{dx}{1+\cos ax} = \frac{1}{a}\tan\frac{ax}{2}$

207. $\int\frac{dx}{1-\cos ax} = -\frac{1}{a}\cot\frac{ax}{2}$

208. $$\int \frac{dx}{a + b\cos x} = \begin{cases} \dfrac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \dfrac{\sqrt{a^2 - b^2}\tan\dfrac{x}{2}}{a + b} \\ \text{or} \\ \dfrac{1}{\sqrt{b^2 - a^2}} \log\left(\dfrac{\sqrt{b^2 - a^2}\tan\dfrac{x}{2} + a + b}{\sqrt{b^2 - a^2}\tan\dfrac{x}{2} - a - b}\right) \end{cases}$$

209. $$\int \frac{\cos ax}{1 + \cos ax}\,dx = x - \frac{1}{a}\tan\frac{ax}{2}$$

210. $$\int \frac{\cos ax}{1 - \cos ax}\,dx = -x - \frac{1}{a}\cot\frac{ax}{2}$$

211. $$\int \frac{dx}{(\cos ax)(1 + \cos ax)}$$
$$= \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a}\tan\frac{ax}{2}$$

212. $$\int \frac{dx}{(\cos ax)(1 - \cos ax)}$$
$$= \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right) - \frac{1}{a}\cot\frac{ax}{2}$$

213. $$\int \frac{dx}{(1 + \cos ax)^2} = \frac{1}{2a}\tan\frac{ax}{2} + \frac{1}{6a}\tan^3\frac{ax}{2}$$

214. $$\int \frac{dx}{(1 - \cos ax)^2} = -\frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

215. $$\int \frac{\cos ax}{(1+\cos ax)^2}\,dx = \frac{1}{2a}\tan\frac{ax}{2} - \frac{1}{6a}\tan^3\frac{ax}{2}$$

216. $$\int \frac{\cos ax}{(1-\cos ax)^2}\,dx = \frac{1}{2a}\cot\frac{ax}{2} - \frac{1}{6a}\cot^3\frac{ax}{2}$$

217. $$\int \frac{\cos x dx}{a+b\cos x} = \frac{x}{b} - \frac{a}{b}\int \frac{dx}{a+b\cos x}$$

218. $$\int \frac{dx}{(\cos x)(a+b\cos x)}$$

$$= \frac{1}{a}\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) - \frac{b}{a}\int \frac{dx}{a+b\cos x}$$

219. $$\int \frac{dx}{(a+b\cos x)^2}$$

$$= \frac{b\sin x}{(b^2-a^2)(a+b\cos x)}$$

$$- \frac{a}{b^2-a^2}\int \frac{dx}{a+b\cos x}$$

220. $$\int \frac{\cos x}{(a+b\cos x)^2}\,dx$$

$$= \frac{a\sin x}{(a^2-b^2)(a+b\cos x)}$$

$$- \frac{a}{a^2-b^2}\int \frac{dx}{a+b\cos x}$$

221. $\int \sqrt{1-\cos ax}\, dx$

$$= -\frac{2\sin ax}{a\sqrt{1-\cos ax}} = -\frac{2\sqrt{2}}{a}\cos\left(\frac{ax}{2}\right)$$

222. $\int \sqrt{1+\cos ax}\, dx$

$$= \frac{2\sin ax}{a\sqrt{1+\cos ax}} = \frac{2\sqrt{2}}{a}\sin\left(\frac{ax}{2}\right)$$

223. $\int \frac{dx}{\sqrt{1-\cos ax}} = \pm\sqrt{2}\log\tan\frac{x}{4},$

[use + if $4k\pi < x < (4k+2)\pi$,

otherwise −; k an integer]

224. $\int \frac{dx}{\sqrt{1+\cos ax}} = \pm\sqrt{2}\log\tan\left(\frac{x+\pi}{4}\right),$

[use + if $(4k-1)\pi < x < (4k+1)\pi$,

otherwise −; k an integer]

225. $\int (\sin mx)(\sin nx) dx$

$$= \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

226. $\int (\cos mx)(\cos nx) dx$

$$= \frac{\sin(m-n)x}{2(m-n)} + \frac{\sin(m+n)x}{2(m+n)}, \quad (m^2 \neq n^2)$$

227. $$\int (\sin ax)(\cos ax)dx = \frac{1}{2a}\sin^2 ax$$

228. $$\int (\sin mx)(\cos nx)dx$$

$$= -\frac{\cos(m-n)x}{2(m-n)} - \frac{\cos(m+n)x}{2(m+n)},$$

$$(m^2 \neq n^2)$$

229. $$\int (\sin^2 ax)(\cos^2 ax)dx = -\frac{1}{32a}\sin 4ax + \frac{x}{8}$$

230. $$\int (\sin ax)(\cos^m ax)dx = -\frac{\cos^{m+1} ax}{(m+1)a}$$

231. $$\int (\sin^m ax)(\cos ax)dx = \frac{\sin^{m+1} ax}{(m+1)a}$$

232. $$\int \frac{\sin ax}{\cos^2 ax}dx = \frac{1}{a\cos ax} = \frac{\sec ax}{a}$$

233. $$\int \frac{\sin^2 ax}{\cos ax}dx = -\frac{1}{a}\sin ax + \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

234. $$\int \frac{\cos ax}{\sin^2 ax}dx = -\frac{1}{a\sin ax} = -\frac{\csc ax}{a}$$

235. $$\int \frac{dx}{(\sin ax)(\cos ax)} = \frac{1}{a}\log\tan ax$$

236. $$\int \frac{dx}{(\sin ax)(\cos^2 ax)} = \frac{1}{a}\left(\sec ax + \log\tan\frac{ax}{2}\right)$$

237. $$\int \frac{dx}{(\sin ax)(\cos^n ax)} = \frac{1}{a(n-1)\cos^{n-1} ax} + \int \frac{dx}{(\sin ax)(\cos^{n-2} ax)}$$

238. $$\int \frac{dx}{(\sin^2 ax)(\cos ax)} = -\frac{1}{a}\csc ax + \frac{1}{a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

239. $$\int \frac{dx}{(\sin^2 ax)(\cos^2 ax)} = -\frac{2}{a}\cot 2ax$$

240. $$\int \frac{\sin ax}{1 \pm \cos ax}\,dx = \mp\frac{1}{a}\log(1 \pm \cos ax)$$

241. $$\int \frac{\cos ax}{1 \pm \sin ax}\,dx = \pm\frac{1}{a}\log(1 \pm \sin ax)$$

242. $$\int \frac{dx}{(\sin ax)(1 \pm \cos ax)} = \pm\frac{1}{2a(1 \pm \cos ax)} + \frac{1}{2a}\log\tan\frac{ax}{2}$$

243. $$\int \frac{dx}{(\cos ax)(1 \pm \sin ax)} = \mp\frac{1}{2a(1 \pm \sin ax)} + \frac{1}{2a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

244. $$\int \frac{\sin ax}{(\cos ax)(1 \pm \cos ax)}\,dx = \frac{1}{a}\log(\sec ax \pm 1)$$

245. $$\int \frac{\cos ax}{(\sin ax)(1 \pm \sin ax)}\,dx = -\frac{1}{a}\log(\csc ax \pm 1)$$

246. $$\int \frac{\sin ax}{(\cos ax)(1 \pm \sin ax)}\,dx$$

$$= \frac{1}{2a(1 \pm \sin ax)} \pm \frac{1}{2a}\log\tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

247. $$\int \frac{\cos ax}{(\sin ax)(1 \pm \cos ax)}\,dx$$

$$= -\frac{1}{2a(1 \pm \cos ax)} \pm \frac{1}{2a}\log\tan\frac{ax}{2}$$

248. $$\int \frac{dx}{\sin ax \pm \cos ax} = \frac{1}{a\sqrt{2}}\log\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)$$

249. $$\int \frac{dx}{(\sin ax \pm \cos ax)^2} = \frac{1}{2a}\tan\left(ax \mp \frac{\pi}{4}\right)$$

250. $$\int \frac{dx}{1 + \cos ax \pm \sin ax} = \pm\frac{1}{a}\log\left(1 \pm \tan\frac{ax}{2}\right)$$

251. $$\int \frac{dx}{a^2\cos^2 cx - b^2\sin^2 cx}$$

$$= \frac{1}{2abc}\log\frac{b\tan cx + a}{b\tan cx - a}$$

252. $$\int \frac{\cos ax}{\sqrt{1+b^2\sin^2 ax}}\,dx$$

$$= \frac{1}{ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right)$$

253. $$\int \frac{\cos ax}{\sqrt{1-b^2\sin^2 ax}}\,dx = \frac{1}{ab}\sin^{-1}(b\sin ax)$$

254. $$\int (\cos ax)\sqrt{1+b^2\sin^2 ax}\,dx$$

$$= \frac{\sin ax}{2a}\sqrt{1+b^2\sin^2 ax}$$

$$+ \frac{1}{2ab}\log\left(b\sin ax + \sqrt{1+b^2\sin^2 ax}\right)$$

255. $$\int (\cos ax)\sqrt{1-b^2\sin^2 ax}\,dx$$

$$= \frac{\sin ax}{2a}\sqrt{1-b^2\sin^2 ax}$$

$$+ \frac{1}{2ab}\sin^{-1}(b\sin ax)$$

256. $$\int (\tan ax)dx = -\frac{1}{a}\log\cos ax = \frac{1}{a}\log\sec ax$$

257. $$\int (\cot ax)dx = \frac{1}{a}\log\sin ax = -\frac{1}{a}\log\csc ax$$

258. $$\int (\sec ax)dx = \frac{1}{a}\log(\sec ax + \tan ax)$$
$$= \frac{1}{a}\log \tan\left(\frac{\pi}{4} + \frac{ax}{2}\right)$$

259. $$\int (\csc ax)dx = \frac{1}{a}\log(\csc ax - \cot ax)$$
$$= \frac{1}{a}\log \tan\frac{ax}{2}$$

260. $$\int (\tan^2 ax)dx = \frac{1}{a}\tan ax - x$$

261. $$\int (\tan^3 ax)dx = \frac{1}{2a}\tan^2 ax + \frac{1}{a}\log\cos ax$$

262. $$\int (\tan^4 ax)dx = \frac{\tan^3 ax}{3a} - \frac{1}{a}\tan x + x$$

263. $$\int (\tan^n ax)dx = \frac{\tan^{n-1} ax}{a(n-1)} - \int (\tan^{n-2} ax)dx$$

Forms with Trigonometric Functions and Inverse Trigonometric Functions

264. $$\int (\cot^2 ax)dx = -\frac{1}{a}\cot ax - x$$

265. $$\int (\cot^3 ax)dx = -\frac{1}{2a}\cot^2 ax - \frac{1}{a}\log\sin ax$$

266. $$\int (\cot^4 ax)dx = -\frac{1}{3a}\cot^3 ax + \frac{1}{a}\cot ax + x$$

267. $$\int (\cot^n ax)dx = -\frac{\cot^{n-1} ax}{a(n-1)} - \int (\cot^{n-2} ax)dx$$

Forms with Inverse Trigonometric Functions

268. $$\int (\sin^{-1} ax)dx = x\sin^{-1} ax + \frac{\sqrt{1-a^2x^2}}{a}$$

269. $$\int (\cos^{-1} ax)dx = x\cos^{-1} ax - \frac{\sqrt{1-a^2x^2}}{a}$$

270. $$\int (\tan^{-1} ax)dx = x\tan^{-1} ax - \frac{1}{2a}\log(1+a^2x^2)$$

271. $$\int (\cot^{-1} ax)dx = x\cot^{-1} ax + \frac{1}{2a}\log(1+a^2x^2)$$

272. $$\int (\sec^{-1} ax)dx = x\sec^{-1} ax - \frac{1}{a}\log\left(ax + \sqrt{a^2x^2-1}\right)$$

273. $$\int (\csc^{-1} ax)dx = x\csc^{-1} ax + \frac{1}{a}\log\left(ax + \sqrt{a^2x^2-1}\right)$$

274. $$\int x[\sin^{-1}(ax)]dx = \frac{1}{4a^2}\left[(2a^2x^2-1)\sin^{-1}(ax) + ax\sqrt{1-a^2x^2}\right]$$

275. $$\int x[\cos^{-1}(ax)]dx$$

$$= \frac{1}{4a^2}[(2a^2x^2 - 1)\cos^{-1}(ax) - ax\sqrt{1 - a^2x^2}]$$

Mixed Algebraic and Trigonometric Forms

276. $$\int x(\sin ax)dx = \frac{1}{a^2}\sin ax - \frac{x}{a}\cos ax$$

277. $$\int x^2(\sin ax)dx = \frac{2x}{a^2}\sin ax - \frac{a^2x^2 - 2}{a^3}\cos ax$$

278. $$\int x^3(\sin ax)dx$$

$$= \frac{3a^2x^2 - 6}{a^4}\sin ax - \frac{a^2x^3 - 6x}{a^3}\cos ax$$

279. $$\int x(\cos ax)dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax$$

280. $$\int x^2(\cos ax)dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax$$

281. $$\int x^3(\cos ax)dx$$

$$= \frac{3a^2x^2 - 6}{a^4}\cos ax + \frac{a^2x^3 - 6x}{a^3}\sin ax$$

282. $$\int x(\sin^2 ax)dx = \frac{x^2}{4} - \frac{x\sin 2ax}{4a} - \frac{\cos 2ax}{8a^2}$$

283. $\int x^2(\sin^2 ax)dx$

$$= \frac{x^3}{6} - \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax - \frac{x\cos 2ax}{4a^2}$$

284. $\int x(\sin^3 ax)dx$

$$= \frac{x\cos 3ax}{12a} - \frac{\sin 3ax}{36a^2}$$

$$- \frac{3x\cos ax}{4a} + \frac{3\sin ax}{4a^2}$$

285. $\int x(\cos^2 ax)dx = \frac{x^2}{4} + \frac{x\sin 2ax}{4a} + \frac{\cos 2ax}{8a^2}$

286. $\int x^2(\cos^2 ax)dx$

$$= \frac{x^3}{6} + \left(\frac{x^2}{4a} - \frac{1}{8a^3}\right)\sin 2ax + \frac{x\cos 2ax}{4a^2}$$

287. $\int x(\cos^3 ax)dx = \frac{x\sin 3ax}{12a} + \frac{\cos 3ax}{36a^2}$

$$+ \frac{3x\sin ax}{4a} + \frac{3\cos ax}{4a^2}$$

288. $\int \frac{\sin ax}{x^m}dx$

$$= -\frac{\sin ax}{(m-1)x^{m-1}} + \frac{a}{m-1}\int \frac{\cos ax}{x^{m-1}}dx$$

289. $$\int \frac{\cos ax}{x^m}\,dx = -\frac{\cos ax}{(m-1)x^{m-1}} - \frac{a}{m-1}\int \frac{\sin ax}{x^{m-1}}\,dx$$

290. $$\int \frac{x}{1 \pm \sin ax}\,dx = \mp \frac{x \cos ax}{a(1 \pm \sin ax)} + \frac{1}{a^2}\log(1 \pm \sin ax)$$

291. $$\int \frac{x}{1+\cos ax}\,dx = \frac{x}{a}\tan\frac{ax}{2} + \frac{2}{a^2}\log\cos\frac{ax}{2}$$

292. $$\int \frac{x}{1-\cos ax}\,dx = -\frac{x}{a}\cot\frac{ax}{2} + \frac{2}{a^2}\log\sin\frac{ax}{2}$$

293. $$\int \frac{x+\sin x}{1+\cos x}\,dx = x\tan\frac{x}{2}$$

294. $$\int \frac{x-\sin x}{1-\cos x}\,dx = -x\cot\frac{x}{2}$$

295. $$\int \frac{x}{\sin^2 ax}\,dx = \int x(\csc^2 ax)dx = -\frac{x \cot ax}{a} + \frac{1}{a^2}\log\sin ax$$

296. $$\int \frac{x}{\sin^n ax}\,dx = \int x(\csc^n ax)\,dx$$
$$= -\frac{x\cos ax}{a(n-1)\sin^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\sin^{n-2} ax} + \frac{(n-2)}{(n-1)}\int \frac{x}{\sin^{n-2} ax}\,dx$$

297. $$\int \frac{x}{\cos^2 ax}\,dx = \int x(\sec^2 ax)\,dx$$
$$= \frac{1}{a} x \tan ax + \frac{1}{a^2} \log \cos ax$$

298. $$\int \frac{x}{\cos^n ax}\,dx = \int x(\sec^n ax)\,dx$$
$$= \frac{x \sin ax}{a(n-1)\cos^{n-1} ax} - \frac{1}{a^2(n-1)(n-2)\cos^{n-2} ax} + \frac{n-2}{n-1}\int \frac{x}{\cos^{n-2} ax}\,dx$$

Logarithmic Forms

299. $$\int (\log x)\,dx = x \log x - x$$

300. $\int x(\log x)dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$

Mixed Algebraic and Trigonometric Forms and Logarithmic Forms

301. $\int x^2(\log x)dx = \frac{x^3}{3}\log x - \frac{x^3}{9}$

302. $\int x^n(\log ax)dx = \frac{x^{n+1}}{n+1}\log ax - \frac{x^{n+1}}{(n+1)^2}$

303. $\int (\log x)^2 dx = x(\log x)^2 - 2x\log x + 2x$

304. $\int \frac{(\log x)^n}{x}dx = \frac{1}{n+1}(\log x)^{n+1}$

305. $$\int \frac{dx}{\log x} = \log(\log x) + \log x + \frac{(\log x)^2}{2\cdot 2!} + \frac{(\log x)^3}{3\cdot 3!} + \cdots$$

306. $\int \frac{dx}{x\log x} = \log(\log x)$

307. $\int \frac{dx}{x(\log x)^n} = -\frac{1}{(n-1)(\log x)^{n-1}}$

308. $\int [\log(ax+b)]dx = \frac{ax+b}{a}\log(ax+b) - x$

309. $$\int \frac{\log(ax+b)}{x^2}\,dx = \frac{a}{b}\log x - \frac{ax+b}{bx}\log(ax+b)$$

310. $$\int \left[\log\frac{x+a}{x-a}\right] dx$$

$$= (x+a)\log(x+a) - (x-a)\log(x-a)$$

311. $$\int x^n (\log X)\,dx$$

$$= \frac{x^{n+1}}{n+1}\log X - \frac{2c}{n+1}\int \frac{x^{n+2}}{X}\,dx$$

$$- \frac{b}{n+1}\int \frac{x^{n+1}}{X}\,dx$$

where $X = a + bx + cx^2$

312. $$\int [\log(x^2+a^2)]\,dx$$

$$= x\log(x^2+a^2) - 2x + 2a\tan^{-1}\frac{x}{a}$$

313. $$\int [\log(x^2-a^2)]\,dx$$

$$= x\log(x^2-a^2) - 2x + a\log\frac{x+a}{x-a}$$

314. $$\int x[\log(x^2 \pm a^2)]\,dx$$

$$= \frac{1}{2}(x^2 \pm a^2)\log(x^2 \pm a^2) - \frac{1}{2}x^2$$

315. $$\int \left[\log\left(x+\sqrt{x^2 \pm a^2}\right)\right] dx$$

$$= x\log\left(x+\sqrt{x^2 \pm a^2}\right) - \sqrt{x^2 \pm a^2}$$

316. $$\int x\left[\log\left(x+\sqrt{x^2 \pm a^2}\right)\right] dx$$

$$= \left(\frac{x^2}{2} \pm \frac{a^2}{4}\right)\log\left(x+\sqrt{x^2 \pm a^2}\right) - \frac{x\sqrt{x^2 \pm a^2}}{4}$$

317. $$\int x^m\left[\log\left(x+\sqrt{x^2 \pm a^2}\right)\right] dx$$

$$= \frac{x^{m+1}}{m+1}\log\left(x+\sqrt{x^2 \pm a^2}\right)$$

$$- \frac{1}{m+1}\int \frac{x^{m+1}}{\sqrt{x^2 \pm a^2}}\, dx$$

318. $$\int \frac{\log\left(x+\sqrt{x^2 + a^2}\right)}{x^2} dx$$

$$= -\frac{\log\left(x+\sqrt{x^2 + a^2}\right)}{x} - \frac{1}{a}\log\frac{a+\sqrt{x^2 + a^2}}{x}$$

319. $$\int \frac{\log\left(x+\sqrt{x^2 - a^2}\right)}{x^2} dx$$

$$= -\frac{\log\left(x+\sqrt{x^2 - a^2}\right)}{x} + \frac{1}{|a|}\sec^{-1}\frac{x}{a}$$

Exponential Forms

320. $\int e^x dx = e^x$

321. $\int e^{-x} dx = -e^{-x}$

322. $\int e^{ax} dx = \frac{e^{ax}}{a}$

323. $\int xe^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$

324. $\int \frac{e^{\alpha x}}{x^m} dx = -\frac{1}{m-1}\frac{e^{ax}}{x^{m-1}} + \frac{a}{m-1}\int \frac{e^{ax}}{x^{m-1}} dx$

325. $\int e^{ax} \log x\, dx = \frac{e^{ax} \log x}{a} - \frac{1}{a}\int \frac{e^{ax}}{x} dx$

326. $\int \frac{dx}{1+e^x} = x - \log(1+e^x) = \log \frac{e^x}{1+e^x}$

327. $\int \frac{dx}{a+be^{px}} = \frac{x}{a} - \frac{1}{ap}\log(a+be^{px})$

328. $\int \frac{dx}{ae^{mx}+be^{-mx}} = \frac{1}{m\sqrt{ab}} \tan^{-1}\left(e^{mx}\sqrt{\frac{a}{b}}\right),$

$(a > 0, b > 0)$

329. $\int (a^x - a^{-x})dx = \frac{a^x + a^{-x}}{\log a}$

330. $$\int \frac{e^{ax}}{b + ce^{ax}}\,dx = \frac{1}{ac}\log(b + ce^{ax})$$

331. $$\int \frac{xe^{ax}}{(1 + ax)^2}\,dx = \frac{e^{ax}}{a^2(1 + ax)}$$

332. $$\int xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}$$

333. $$\int e^{ax}[\sin(bx)]dx = \frac{e^{ax}[a\sin(bx) - b\cos(bx)]}{a^2 + b^2}$$

334. $$\int e^{ax}\left[\sin(bx)\right]\left[\sin(cx)\right]dx$$

$$= \frac{e^{ax}\left[(b - c)\sin(b - c)x + a\cos(b - c)x\right]}{2\left[a^2 + (b - c)^2\right]}$$

$$- \frac{e^{ax}\left[(b + c)\sin(b + c)x + a\cos(b + c)x\right]}{2\left[a^2 + (b + c)^2\right]}$$

335. $$\int e^{ax}[\cos(bx)]dx = \frac{e^{ax}}{a^2 + b^2}[a\cos(bx) + b\sin(bx)]$$

336. $$\int e^{ax}\left[\cos(bx)\right]\left[\cos(cx)\right]dx$$

$$= \frac{e^{ax}\left[(b - c)\sin(b - c)x + a\cos(b - c)x\right]}{2\left[a^2 + (b - c)^2\right]}$$

$$+ \frac{e^{ax}\left[(b + c)\sin(b + c)x + a\cos(b + c)x\right]}{2\left[a^2 + (b + c)^2\right]}$$

337. $\int e^{ax}\left[\sin^n bx\right]dx$

$$= \frac{1}{a^2+n^2b^2}\Big[(a\sin bx - nb\cos bx)e^{ax}\sin^{n-1}bx$$

$$+ n(n-1)b^2\int e^{ax}\left[\sin^{n-2}bx\right]dx\Big]$$

Hyperbolic Forms

338. $\int e^{ax}\left[\cos^n bx\right]dx$

$$= \frac{1}{a^2+n^2b^2}\Big[\left(a\cos bx + nb\sin bx\right)e^{ax}\cos^{n-1}bx$$

$$+ n(n-1)b^2\int e^{ax}\left[cos^{n-2}bx\right]dx\Big]$$

339. $\int xe^{ax}[\sin bx]dx$

$$= \frac{xe^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$- \frac{e^{ax}}{(a^2+b^2)^2}\left[\left(a^2-b^2\right)\sin bx - 2ab\cos bx\right]$$

340. $\int xe^{ax}[\cos bx]dx$

$$= \frac{xe^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

$$- \frac{e^{ax}}{\left(a^2+b^2\right)^2}\left[\left(a^2-b^2\right)\cos bx + 2ab\sin bx\right]$$

341. $\int (\sinh x)dx = \cosh x$

342. $\int (\cosh x)dx = \sinh x$

343. $\int (\tanh x)dx = \log \cosh x$

344. $\int (\coth x)dx = \log \sinh x$

345. $\int (\operatorname{sech} x)dx = \tan^{-1}(\sinh x)$

346. $\int \operatorname{csch} x\, dx = \log \tanh\left(\frac{x}{2}\right)$

347. $\int x(\sinh x)dx = x \cosh x - \sinh x$

348. $\int x^n(\sinh x)dx = x^n \cosh x$

$$- n\int x^{n-1}(\cosh x)dx$$

349. $\int x(\cosh x)dx = x \sinh x - \cosh x$

350. $\int x^n(\cosh x)dx = x^n \sinh x - n\int x^{n-1}(\sinh x)dx$

351. $\int (\operatorname{sech} x)(\tanh x)dx = -\operatorname{sech} x$

352. $\int (\operatorname{csch} x)(\coth x)dx = -\operatorname{csch} x$

353. $\int (\sinh^2 x)dx = \frac{\sinh 2x}{4} - \frac{x}{2}$

354. $\int (\tanh^2 x)dx = x - \tanh x$

355. $\int \left(\tanh^n x\right)dx = -\frac{\tanh^{n-1}}{n-1} + \int (\tanh^{n-2} x)dx,$

$(n \neq 1)$

356. $\int (\operatorname{sech}^2 x)dx = \tanh x$

357. $\int (\cosh^2 x)dx = \frac{\sinh 2x}{4} + \frac{x}{2}$

358. $\int (\coth^2 x)dx = x - \coth x$

359. $\int (\coth^n x)dx = -\frac{\coth^{n-1} x}{n-1} + \int \coth^{n-2} x dx,$

$(n \neq 1)$

Table of Definite Integrals

360. $\int_1^\infty \frac{dx}{x^m} = \frac{1}{m-1}, \quad [m > 1]$

361. $$\int_0^\infty \frac{dx}{(1+x)x^p} = \pi \csc p\pi, \quad [p<1]$$

362. $$\int_0^\infty \frac{dx}{(1-x)x^p} = -\pi \cot p\pi, \quad [p<1]$$

363. $$\int_0^\infty \frac{x^{p-1}dx}{1+x} = \frac{\pi}{\sin p\pi} = B(p, 1-p) = \Gamma(p)\Gamma(1-p), \quad [0<p<1]$$

364. $$\int_0^\infty \frac{x^{m-1}dx}{1+x^n} = \frac{\pi}{n\sin\frac{m\pi}{n}}, \quad [0<m<n]$$

365. $$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}} = \pi$$

366. $$\int_0^\infty \frac{a\,dx}{a^2+x^2} = \frac{\pi}{2}, \text{ if } a>0; \quad 0, \text{ if } a=0; \quad -\frac{\pi}{2}, \text{ if } a<0$$

367. $$\int_0^\infty e^{-ax}dx = \frac{1}{a}, \quad (a>0)$$

368. $$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}dx = \log\frac{b}{a}, \quad (a,\, b>0)$$

369. $$\int_0^\infty x^n e^{-ax}dx = \begin{cases} \dfrac{\Gamma(n+1)}{a^{n+1}}, & (n > -1, a > 0) \\ & \text{or} \\ \dfrac{n!}{a^{n+1}}, & (a > 0, n \text{ positive integer}) \end{cases}$$

370. $$\int_0^\infty x^n \exp(-ax^p)dx = \frac{\Gamma(k)}{pa^k},$$

$$\left(n > -1,\ p > 0,\ a > 0,\ k = \frac{n+1}{p}\right)$$

371. $$\int_0^\infty e^{-a^2x^2}dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma\left(\frac{1}{2}\right), \quad (a > 0)$$

372. $$\int_0^\infty xe^{-x^2}dx = \frac{1}{2}$$

373. $$\int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$$

374. $$\int_0^\infty x^{2n}e^{-ax^2}dx = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

375. $$\int_0^\infty x^{2n+1}e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}}, \quad (a > 0)$$

376. $$\int_0^1 x^m e^{-ax}\,dx = \frac{m!}{a^{m+1}}\left[1 - e^{-a}\sum_{r=0}^{m}\frac{a^r}{r!}\right]$$

377. $$\int_0^\infty e^{\left(-x^2 - \frac{a^2}{x^2}\right)}dx = \frac{e^{-2a}\sqrt{\pi}}{2}, \quad (a \geq 0)$$

378. $$\int_0^\infty e^{-nx}\sqrt{x}\,dx = \frac{1}{2n}\sqrt{\frac{\pi}{n}}$$

379. $$\int_0^\infty \frac{e^{-nx}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{n}}$$

380. $$\int_0^\infty e^{-ax}(\cos mx)dx = \frac{a}{a^2 + m^2}, \quad (a > 0)$$

381. $$\int_0^\infty e^{-ax}(\sin mx)dx = \frac{m}{a^2 + m^2}, \quad (a > 0)$$

382. $$\int_0^\infty xe^{-ax}[\sin(bx)]dx = \frac{2ab}{(a^2 + \mathrm{b}^2)^2}, \quad (a > 0)$$

383. $$\int_0^\infty xe^{-ax}[\cos(bx)]dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}, \quad (a > 0)$$

384. $\int_0^\infty x^n e^{-ax}\left[\sin(bx)\right]dx$

$$= \frac{n!\left[(a+ib)^{n+1}-(a-ib)^{n+1}\right]}{2i\left(a^2+b^2\right)^{n+1}}, \quad \left(i^2=-1,\ a>0\right)$$

385. $\int_0^\infty x^n e^{-ax}\left[\cos(bx)\right]dx$

$$= \frac{n![(a-ib)^{n+1}+(a+ib)^{n+1}]}{2(a^2+b^2)^{n+1}}, \quad (i^2=-1,\ a>0)$$

386. $\int_0^\infty \frac{e^{-ax}\sin x}{x}dx = \cot^{-1} a, \quad (a>0)$

387. $\int_0^\infty e^{-a^2x^2}\cos bx\,dx = \frac{\sqrt{\pi}}{2a}\exp\left(-\frac{b^2}{4a^2}\right), \quad (ab \neq 0)$

388. $\int_0^\infty e^{-t\cos\phi}t^{b-1}\left[\sin(t\sin\phi)\right]dt = \left[\Gamma(b)\right]\sin(b\phi),$

$$\left(b>0, -\frac{\pi}{2}<\phi<\frac{\pi}{2}\right)$$

389. $\int_0^\infty e^{-t\cos\phi}t^{b-1}[\cos(t\sin\phi)]\,dt = [\Gamma(b)]\cos(b\phi),$

$$\left(b>0, -\frac{\pi}{2}<\phi<\frac{\pi}{2}\right)$$

390. $$\int_0^\infty t^{b-1}\cos t\,dt = [\Gamma(b)]\cos\left(\frac{b\pi}{2}\right),$$

$$(0 < b < 1)$$

391. $$\int_0^\infty t^{b-1}(\sin t)\,dt = [\Gamma(b)]\sin\left(\frac{b\pi}{2}\right),$$

$$(0 < b < 1)$$

392. $$\int_0^1 (\log x)^n\,dx = (-1)^n \cdot n!$$

393. $$\int_0^1 \left(\log\frac{1}{x}\right)^{\frac{1}{2}} dx = \frac{\sqrt{\pi}}{2}$$

394. $$\int_0^1 \left(\log\frac{1}{x}\right)^{-\frac{1}{2}} dx = \sqrt{\pi}$$

395. $$\int_0^1 \left(\log\frac{1}{x}\right)^{n} dx = n!$$

396. $$\int_0^1 x\log(1-x)\,dx = -\frac{3}{4}$$

397. $$\int_0^1 x\log(1+x)\,dx = \frac{1}{4}$$

398. $$\int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n n!}{(m+1)^{n+1}},$$

$$m > -1,\ n = 0, 1, 2, \ldots$$

If $n \neq 0, 1, 2, \ldots$, replace $n!$ by $\Gamma(n + 1)$

399. $$\int_0^\infty \frac{\sin x}{x^p} \, dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \quad 0 < p < 1$$

400. $$\int_0^\infty \frac{\cos x}{x^p} \, dx = \frac{\pi}{2\Gamma(p)\sin(p\pi/2)}, \quad 0 < p < 1$$

401. $$\int_0^\infty \frac{1 - \cos px}{x^2} \, dx = \frac{\pi p}{2}$$

402. $$\int_0^\infty \frac{\sin px \cos qx}{x} \, dx$$

$$= \left\{ 0,\ q > p > 0;\ \frac{\pi}{2},\ p > q > 0;\ \frac{\pi}{4},\ p = q > 0 \right\}$$

403. $$\int_0^\infty \frac{\cos(mx)}{x^2 + a^2} \, dx = \frac{\pi}{2|a|} e^{-|ma|}$$

404. $$\int_0^\infty \cos(x^2) \, dx = \int_0^\infty \sin(x^2) \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

405. $$\int_0^\infty \sin ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\sin\frac{\pi}{2n}, \quad n > 1$$

406. $$\int_0^\infty \cos ax^n\,dx = \frac{1}{na^{1/n}}\Gamma(1/n)\cos\frac{\pi}{2n}, \quad n > 1$$

407. $$\int_0^\infty \frac{\sin x}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

408. (a) $$\int_0^\infty \frac{\sin^3 x}{x}\,dx = \frac{\pi}{4}$$ (b) $$\int_0^\infty \frac{\sin^3 x}{x^2}\,dx = \frac{3}{4}\log 3$$

409. $$\int_0^\infty \frac{\sin^3 x}{x^3}\,dx = \frac{3\pi}{8}$$

410. $$\int_0^\infty \frac{\sin^4 x}{x^4}\,dx = \frac{\pi}{3}$$

411. $$\int_0^{\pi/2} \frac{dx}{1 + a\cos x} = \frac{\cos^{-1} a}{\sqrt{1 - a^2}}, \quad (a < 1)$$

412. $$\int_0^{\pi} \frac{dx}{a + b\cos x} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad (a > b \geq 0)$$

413. $$\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1 - a^2}}, \quad (a^2 < 1)$$

414. $$\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx = \log\frac{b}{a}$$

415. $$\int_0^{\pi/2} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{\pi}{2ab}$$

416. $$\int_0^{\pi/2} (\sin^n x)dx = \begin{cases} \int_0^{\pi/2} (\cos^n x)dx \\ \text{or} \\ \dfrac{1\cdot 3\cdot 5\cdot 7\ldots(n-1)}{2\cdot 4\cdot 6\cdot 8\ldots(n)}\dfrac{\pi}{2}, \\ (n \text{ an even integer, } n \neq 0) \\ \text{or} \\ \dfrac{2\cdot 4\cdot 6\cdot 8\ldots(n-1)}{1\cdot 3\cdot 5\cdot 7\ldots(n)}, \\ (n \text{ an odd integer, } n \neq 1) \\ \text{or} \\ \dfrac{\sqrt{\pi}}{2}\dfrac{\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\dfrac{n}{2}+1\right)}, \quad (n > -1) \end{cases}$$

417. $$\int_0^\infty \frac{\sin mx\,dx}{x} = \frac{\pi}{2}, \quad \text{if } m > 0;\ 0, \quad \text{if } m = 0;$$

$$-\frac{\pi}{2}, \quad \text{if } m < 0$$

418. $$\int_0^\infty \frac{\cos x \, dx}{x} = \infty$$

419. $$\int_0^\infty \frac{\tan x \, dx}{x} = \frac{\pi}{2}$$

420. $$\int_0^\pi \sin ax \cdot \sin bx \, dx = \int_0^\pi \cos ax \cdot \cos bx \, dx = 0,$$

$$(a \neq b; \ a, b \text{ integers})$$

421. $$\int_0^{\pi/a} [\sin(ax)][\cos(ax)] \, dx$$

$$= \int_0^\pi [\sin(ax)][\cos(ax)] \, dx = 0$$

422. $$\int_0^\pi [\sin(ax)][\cos(bx)] \, dx$$

$$= \frac{2a}{a^2 - b^2}, \text{ if } a - b \text{ is odd, or } 0 \text{ if } a - b \text{ is even}$$

423. $$\int_0^\infty \frac{\sin x \cos mx \, dx}{x} = 0, \quad \text{if } m < -1 \text{ or } m > 1;$$

$$\frac{\pi}{4}, \quad \text{if } m = \pm 1; \quad \frac{\pi}{2}, \quad \text{if } m^2 < 1$$

424. $$\int_0^\infty \frac{\sin ax \sin bx}{x^2}\,dx = \frac{\pi a}{2}, \quad (a \le b)$$

425. $$\int_0^\pi \sin^2 mx\,dx = \int_0^\pi \cos^2 mx\,dx = \frac{\pi}{2}$$

426. $$\int_0^\infty \frac{\sin^2(px)}{x^2}\,dx = \frac{\pi p}{2}$$

427. $$\int_0^1 \frac{\log x}{1+x}\,dx = -\frac{\pi^2}{12}$$

428. $$\int_0^1 \frac{\log x}{1-x}\,dx = -\frac{\pi^2}{6}$$

429. $$\int_0^1 \frac{\log(1+x)}{x}\,dx = \frac{\pi^2}{12}$$

430. $$\int_0^1 \frac{\log(1-x)}{x}\,dx = -\frac{\pi^2}{6}$$

431. $$\int_0^1 (\log x)[\log(1+x)]\,dx = 2 - 2\log 2 - \frac{\pi^2}{12}$$

432. $$\int_0^1 (\log x)[\log(1-x)]\,dx = 2 - \frac{\pi^2}{6}$$

433. $$\int_0^1 \frac{\log x}{1-x^2}\,dx = -\frac{\pi^2}{8}$$

434. $$\int_0^1 \log\left(\frac{1+x}{1-x}\right)\cdot\frac{dx}{x} = \frac{\pi^2}{4}$$

435. $$\int_0^1 \frac{\log x\,dx}{\sqrt{1-x^2}} = -\frac{\pi}{2}\log 2$$

436. $$\int_0^1 x^m\left[\log\left(\frac{1}{x}\right)\right]^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}},$$

$$\text{if } m+1>0,\ n+1>0$$

437. $$\int_0^1 \frac{(x^p - x^q)\,dx}{\log x} = \log\left(\frac{p+1}{q+1}\right),$$

$$(p+1>0,\ q+1>0)$$

438. $$\int_0^1 \frac{dx}{\sqrt{\log\left(\frac{1}{x}\right)}} = \sqrt{\pi}$$

439. $$\int_0^\pi \log\left(\frac{e^x+1}{e^x-1}\right)dx = \frac{\pi^2}{4}$$

440. $$\int_0^{\pi/2} (\log\sin x)\,dx = \int_0^{\pi/2} \log\cos x\ dx = -\frac{\pi}{2}\log 2$$

441. $$\int_0^{\pi/2} (\log \sec x)dx = \int_0^{\pi/2} \log \csc x \ \ dx = \frac{\pi}{2}\log 2$$

442. $$\int_0^{\pi} x(\log \sin x)dx = -\frac{\pi}{2}\log 2$$

443. $$\int_0^{\pi/2} (\sin x)(\log \sin x)dx = \log 2 - 1$$

444. $$\int_0^{\pi/2} (\log \tan x)dx = 0$$

445. $$\int_0^{\pi} \log(a \pm b \cos x)dx$$

$$= \pi \log\left(\frac{a + \sqrt{a^2 - b^2}}{2}\right), \quad (a \geq b)$$

446. $$\int_0^{\pi} \log\left(a^2 - 2ab \cos x + b^2\right)dx$$

$$= \begin{cases} 2\pi \log a, & a \geq b > 0 \\ 2\pi \log b, & b \geq a > 0 \end{cases}$$

447. $$\int_0^{\infty} \frac{\sin ax}{\sinh bx}dx = \frac{\pi}{2b}\tanh\frac{a\pi}{2b}$$

448. $$\int_0^{\infty} \frac{\cos ax}{\cosh bx}dx = \frac{\pi}{2b}\operatorname{sech}\frac{\alpha\pi}{2b}$$

449. $$\int_0^\infty \frac{dx}{\cosh ax} = \frac{\pi}{2a}$$

450. $$\int_0^\infty \frac{x\ dx}{\sinh ax} = \frac{\pi^2}{4a^2}$$

451. $$\int_0^\infty e^{-ax}(\cosh bx)dx = \frac{a}{a^2 - b^2}, \quad (0 \le |b| < a)$$

452. $$\int_0^\infty e^{-ax}(\sinh bx)dx = \frac{b}{a^2 - b^2}, \quad (0 \le |b| < a)$$

453. $$\int_0^\infty \frac{\sinh ax}{e^{bx} + 1} dx = \frac{\pi}{2b} \csc \frac{a\pi}{b} - \frac{1}{2a}$$

454. $$\int_0^\infty \frac{\sinh ax}{e^{bx} - 1} dx = \frac{1}{2a} - \frac{\pi}{2b} \cot \frac{a\pi}{b}$$

455. $$\int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

$$= \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 5}\right)^2 k^6 + \cdots\right],$$

if $k^2 < 1$

456. $$\int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 x}\, dx$$

$$= \frac{\pi}{2}\left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{k^6}{5} - \cdots\right],$$

if $k^2 < 1$

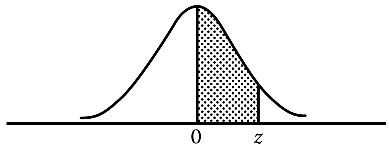
457. $$\int_0^{\infty} e^{-x} \log x\, dx = -\gamma = -0.5772157\ldots$$

458. $$\int_0^{\infty} e^{-x^2} \log x\, dx = -\frac{\sqrt{\pi}}{4}\left(\gamma + 2\log 2\right)$$

Appendix

TABLE A.1

Areas under the Standard Normal Curve



z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852

(Continued)

TABLE A.1 (*Continued*)

Areas under the Standard Normal Curve

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767

(*Continued*)

TABLE A.1 (*Continued*)
Areas under the Standard Normal Curve

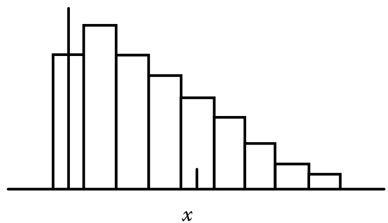
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990

Source: With kind permission from Springer+Business Media: Manual of Pharmacologic Calculations with Computer Programs, 2nd ed., 1987, R.J. Tallarida and R.B. Murray.

TABLE A.2

Poisson Distribution

Each number in this table represents the probability of obtaining at least *X* successes, or the area under the histogram to the right of and including the rectangle whose center is at *X*.



m	*X* = 0	*X* = 1	*X* = 2	*X* = 3	*X* = 4	*X* = 5	*X* = 6	*X* = 7	*X* = 8	*X* = 9	*X* = 10	*X* = 11	*X* = 12	*X* = 13	*X* = 14
.10	1.000	.095	.005												
.20	1.000	.181	.018	.001											
.30	1.000	.259	.037	.004											
.40	1.000	.330	.062	.008	.001										

(Continued)

TABLE A.2 (*Continued*)

Poisson Distribution

m	X = 0	X = 1	X = 2	X = 3	X = 4	X = 5	X = 6	X = 7	X = 8	X = 9	X = 10	X = 11	X = 12	X = 13	X = 14
.50	1.000	.393	.090	.014	.002										
.60	1.000	.451	.122	.023	.003										
.70	1.000	.503	.156	.034	.006	.001									
.80	1.000	.551	.191	.047	.009	.001									
.90	1.000	.593	.228	.063	.063	.013	.002								
1.00	1.000	.632	.264	.080	.019	.004	.001								
1.1	1.000	.667	.301	.100	.026	.005	.001								
1.2	1.000	.699	.337	.120	.034	.008	.002								
1.3	1.000	.727	.373	.143	.043	.011	.002								
1.4	1.000	.753	.408	.167	.054	.014	.003	.001							
1.5	1.000	.777	.442	.191	.066	.019	.004	.001							
1.6	1.000	.798	.475	.217	.079	.024	.006	.001							

(*Continued*)

TABLE A.2 (*Continued*)

Poisson Distribution

m	*X* = 0	*X* = 1	*X* = 2	*X* = 3	*X* = 4	*X* = 5	*X* = 6	*X* = 7	*X* = 8	*X* = 9	*X* = 10	*X* = 11	*X* = 12	*X* = 13	*X* = 14
1.7	1.000	.817	.517	.243	.093	.030	.008	.002							
1.8	1.000	.835	.537	.269	.109	.036	.010	.003	.001						
1.9	1.000	.850	.566	.296	.125	.044	.013	.003	.001						
2.0	1.000	.865	.594	.323	.143	.053	.017	.005	.001						
2.2	1.000	.889	.645	.377	.181	.072	.025	.007	.002						
2.4	1.000	.909	.692	.430	.221	.096	.036	.012	.003	.001					
2.6	1.000	.926	.733	.482	.264	.123	.049	.017	.005	.001					
2.8	1.000	.939	.769	.531	.308	.152	.065	.024	.008	.002	.001				
3.0	1.000	.950	.801	.577	.353	.185	.084	.034	.012	.004	.001				
3.2	1.000	.959	.829	.620	.397	.219	.105	.045	.017	.006	.002				
3.4	1.000	.967	.853	.660	.442	.256	.129	.058	.023	.008	.003	.001			
3.6	1.000	.973	.874	.697	.485	.294	.156	.073	.031	.012	.004	.001			
3.8	1.000	.978	.893	.731	.527	.332	.184	.091	.040	.016	.006	.002			

(*Continued*)

TABLE A.2 (*Continued*)

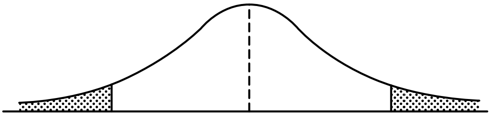
Poisson Distribution

m	*X* = 0	*X* = 1	*X* = 2	*X* = 3	*X* = 4	*X* = 5	*X* = 6	*X* = 7	*X* = 8	*X* = 9	*X* = 10	*X* = 11	*X* = 12	*X* = 13	*X* = 14
4.0	1.000	.982	.908	.762	.567	.371	.215	.111	.051	.021	.008	.003	.001		
4.2	1.000	.985	.922	.790	.605	.410	.247	.133	.064	.028	.011	.004	.001		
4.4	1.000	.988	.934	.815	.641	.449	.280	.156	.079	.036	.015	.006	.002	.001	
4.6	1.000	.990	.944	.837	.674	.487	.314	.182	.095	.045	.020	.008	.003	.001	
4.8	1.000	.992	.952	.857	.706	.524	.349	.209	.113	.056	.025	.010	.004	.001	
5.0	1.000	.993	.960	.875	.735	.560	.384	.238	.133	.068	.032	.014	.005	.002	.001

Source: Alder, H.L. and Roessler, E.B., *Introduction to Probability and Statistics*, 6th ed., 1977. With permission of W.H. Freeman, New York.

TABLE A.3

t-Distribution



Deg. Freedom, f	90% (P = 0.1)	95% (P = 0.05)	99% (P = 0.01)
1	6.314	12.706	63.657
2	2.920	4.303	9.925
3	2.353	3.182	5.841
4	2.132	2.776	4.604
5	2.015	2.571	4.032
6	1.943	2.447	3.707
7	1.895	2.365	3.499
8	1.860	2.306	3.355
9	1.833	2.262	3.250
10	1.812	2.228	3.169
11	1.796	2.201	3.106
12	1.782	2.179	3.055
13	1.771	2.160	3.012
14	1.761	2.145	2.977
15	1.753	2.131	2.947
16	1.746	2.120	2.921
17	1.740	2.110	2.898
18	1.734	2.101	2.878
19	1.729	2.093	2.861
20	1.725	2.086	2.845
21	1.721	2.080	2.831
22	1.717	2.074	2.819

(*Continued*)

TABLE A.3 (*Continued*)

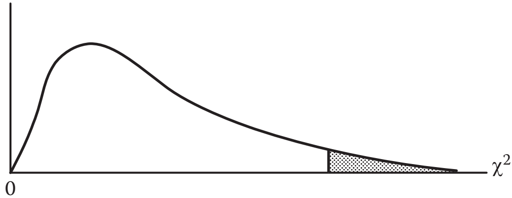
t-Distribution

Deg. Freedom, *f*	90% (P = 0.1)	95% (P = 0.05)	99% (P = 0.01)
23	1.714	2.069	2.807
24	1.711	2.064	2.797
25	1.708	2.060	2.787
26	1.706	2.056	2.779
27	1.703	2.052	2.771
28	1.701	2.048	2.763
29	1.699	2.045	2.756
inf.	1.645	1.960	2.576

Source: With kind permission from Springer+Business Media: Manual of Pharmacologic Calculations with Computer Programs, 2nd ed., 1987, R.J. Tallarida and R.B. Murray.

TABLE A.4

χ^2-Distribution



v	**0.05**	**0.025**	**0.01**	**0.005**
1	3.841	5.024	6.635	7.879
2	5.991	7.378	9.210	10.597
3	7.815	9.348	11.345	12.838
4	9.488	11.143	13.277	14.860
5	11.070	12.832	15.086	16.750
6	12.592	14.449	16.812	18.548
7	14.067	16.013	18.475	20.278
8	15.507	17.535	20.090	21.955
9	16.919	19.023	21.666	23.589
10	18.307	20.483	23.209	25.188
11	19.675	21.920	24.725	26.757
12	21.026	23.337	26.217	28.300
13	22.362	24.736	27.688	29.819
14	23.685	26.119	29.141	31.319
15	24.996	27.488	30.578	32.801
16	26.296	28.845	32.000	34.267
17	27.587	30.191	33.409	35.718
18	28.869	31.526	34.805	37.156
19	30.144	32.852	36.191	38.582
20	31.410	34.170	37.566	39.997
21	32.671	35.479	38.932	41.401

(*Continued*)

TABLE A.4 (*Continued*)

χ^2-Distribution

v	**0.05**	**0.025**	**0.01**	**0.005**
22	33.924	36.781	40.289	42.796
23	35.172	38.076	41.638	44.181
24	36.415	39.364	42.980	45.558
25	37.652	40.646	44.314	46.928
26	38.885	41.923	45.642	48.290
27	40.113	43.194	46.963	49.645
28	41.337	44.461	48.278	50.993
29	42.557	45.722	49.588	52.336
30	43.773	46.979	50.892	53.672

Source: Freund, J.E. and Williams, F.J., *Elementary Business Statistics: The Modern Approach*, 2nd ed., 1972. With permission of Prentice Hall, Englewood Cliffs, NJ.

TABLE A.5

Variance Ratio

	F(95%)									
	n_1									
n_3	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	**∞**
1	161.4	199.5	215.7	224.6	230.2	234.0	238.9	243.9	249.0	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.37	19.41	19.45	19.30
3	10.13	9.55	9.28	9.12	9.01	8.94	8.84	8.74	8.64	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.91	5.77	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.68	4.53	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.00	3.84	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.57	3.41	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.28	3.12	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.07	2.90	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.91	2.74	2.54

(Continued)

TABLE A.5 (*Continued*)

Variance Ratio

	F(95%)									
	n_1									
n_2	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	**∞**
11	4.84	3.98	3.59	3.36	3.20	3.09	2.95	2.79	2.61	2.40
12	4.75	3.88	3.49	3.26	3.11	3.00	2.85	2.69	2.50	2.30
13	4.67	3.80	3.41	3.18	3.02	2.92	2.77	2.60	2.42	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.53	2.35	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.64	2.48	2.29	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.42	2.24	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.55	2.38	2.19	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.34	2.15	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.48	2.31	2.11	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.38	2.08	1.84

(*Continued*)

TABLE A.5 (*Continued*)

Variance Ratio

	F(95%)									
	n_1									
n_3	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	∞
21	4.32	3.47	3.07	2.84	2.68	2.57	2.42	2.25	2.05	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.40	2.23	2.03	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.38	2.20	2.00	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.36	2.18	1.98	1.73
25	4.24	3.38	2.99	2.76	2.60	2.49	2.34	2.16	1.96	1.71
26	4.22	3.37	2.98	2.74	2.59	2.47	2.32	2.15	1.95	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.30	2.13	1.93	1.67
28	4.20	3.34	2.95	2.71	2.56	2.44	2.29	2.12	1.91	1.65
29	4.18	3.33	2.93	2.70	2.54	2.43	2.28	2.10	1.90	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.09	1.89	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.00	1.79	1.51

(*Continued*)

TABLE A.5 (*Continued*)

Variance Ratio

	F(95%)									
	n_1									
n_2	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	**∞**
60	4.00	3.15	2.76	2.52	2.37	2.25	2.10	1.92	1.70	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.02	1.83	1.61	1.25
∞	3.84	2.99	2.60	2.37	2.21	2.10	1.94	1.75	1.52	1.00

	F(99%)									
	n_1									
n_2	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	**∞**
1	4.052	4.999	5.403	5.625	5.764	5.859	5.982	6.106	6.234	6.366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.37	99.42	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.49	27.05	26.60	26.12
4	21.20	18.00	16.69	15.98	15.52	15.21	14.80	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.29	9.89	9.47	9.02

(*Continued*)

TABLE A.5 (*Continued*)

Variance Ratio

	F(99%)									
	n_1									
n_2	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	**∞**
6	13.74	10.92	9.71	9.15	8.75	8.47	8.10	7.72	7.31	6.88
7	12.25	9.55	1.45	7.85	7.46	7.19	6.84	6.47	6.07	5.65
8	11.26	8.65	7.59	7.01	6.63	6.37	6.03	5.67	5.28	4.86
9	10.56	8.02	6.99	6.42	6.06	5.80	5.47	5.11	4.73	4.31
10	10.04	7.56	6.55	5.99	5.64	5.39	5.06	4.71	4.33	3.91
11	9.65	7.20	6.22	5.67	5.32	5.07	4.74	4.40	4.02	3.60
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.16	3.78	3.36
13	9.07	6.70	5.74	5.20	4.86	4.62	4.30	3.96	3.59	3.16
14	8.86	6.51	5.56	5.03	4.69	4.46	4.14	3.80	3.43	3.00
15	8.68	6.36	5.42	4.89	4.56	4.32	4.00	3.67	3.29	2.87

(*Continued*)

TABLE A.5 (*Continued*)

Variance Ratio

	F(99%)									
	n_1									
n_2	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	∞
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.55	3.18	2.75
17	8.40	6.11	5.18	4.67	4.34	4.10	3.79	3.45	3.08	2.65
18	8.28	6.01	5.09	4.58	4.25	4.01	3.71	3.37	3.00	2.57
19	8.18	5.93	5.01	4.50	4.17	3.94	3.63	3.30	2.92	2.49
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.23	2.86	2.42
21	8.02	5.78	4.87	4.37	4.04	3.81	3.51	3.17	2.80	2.36
22	7.94	5.72	4.82	4.31	3.99	3.76	3.45	3.12	2.75	2.31
23	7.88	5.66	4.76	4.26	3.94	3.71	3.41	3.07	2.70	2.26
24	7.82	5.61	4.72	4.22	3.90	3.67	3.36	3.03	2.66	2.21
25	7.77	5.57	4.68	4.18	3.86	3.63	3.32	2.99	2.62	2.17

(Continued)

TABLE A.5 (*Continued*)

Variance Ratio

	F(99%)									
	n_1									
n_2	**1**	**2**	**3**	**4**	**5**	**6**	**8**	**12**	**24**	∞
26	7.72	5.53	4.64	4.14	3.82	3.59	3.29	2.96	2.58	2.13
27	7.68	5.49	4.60	4.11	3.78	3.56	3.26	2.93	2.55	2.10
28	7.64	5.45	4.57	4.07	3.75	3.53	3.23	2.90	2.52	2.06
29	7.60	5.42	4.54	4.04	3.73	3.50	3.20	2.87	2.49	2.03
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.84	2.47	2.01
40	7.31	5.18	4.31	3.83	3.51	3.29	2.99	2.66	2.29	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.50	2.12	1.60
120	6.85	4.79	3.95	3.48	3.17	2.96	2.66	2.34	1.95	1.38
∞	6.64	4.60	3.78	3.32	3.02	2.80	2.51	2.18	1.79	1.00

Source: Fisher, R.A. and Yates, F., *Statistical Tables for Biological Agricultural and Medical Research*, The Longman Group Ltd., London, U.K., With permission.

TABLE A.6

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
	3-Year Loan		
4.00	29.52	9.500	32.03
4.25	29.64	9.750	32.15
4.50	29.75	10.00	32.27
4.75	29.86	10.25	32.38
5.00	29.97	10.50	32.50
5.25	30.08	10.75	32.62
5.50	30.20	11.00	32.74
5.75	30.31	11.25	32.86
6.00	30.42	11.50	32.98
6.25	30.54	11.75	33.10
6.50	30.65	12.00	33.21
6.75	30.76	12.25	33.33
7.00	30.88	12.50	33.45
7.25	30.99	12.75	33.57
7.50	31.11	13.00	33.69
7.75	31.22	13.25	33.81
8.00	31.34	13.50	33.94
8.25	31.45	13.75	34.06
8.50	31.57	14.00	34.18
8.75	31.68	14.25	34.30
9.00	31.80	14.50	34.42
9.25	31.92	14.75	34.54
	5-Year Loan		
4.00	18.42	9.500	21.00
4.25	18.53	9.750	21.12

(*Continued*)

TABLE A.6 (*Continued*)

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
4.50	18.64	10.00	21.25
4.75	18.76	10.25	21.37
5.00	18.87	10.50	21.49
5.25	18.99	10.75	21.62
5.50	19.10	11.00	21.74
5.75	19.22	11.25	21.87
6.00	19.33	11.50	21.99
6.25	19.45	11.75	22.12
6.50	19.57	12.00	22.24
6.75	19.68	12.25	22.37
7.00	19.80	12.50	22.50
7.25	19.92	12.75	22.63
7.50	20.04	13.00	22.75
7.75	20.16	13.25	22.88
8.00	20.28	13.50	23.01
8.25	20.40	13.75	23.14
8.50	20.52	14.00	23.27
8.75	20.64	14.25	23.40
9.00	20.76	14.50	23.53
9.25	20.88	14.75	23.66
	10-Year Loan		
4.00	10.12	9.500	12.94
4.25	10.24	9.750	13.08
4.50	10.36	10.00	13.22
4.75	10.48	10.25	13.35

(*Continued*)

TABLE A.6 (*Continued*)

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
5.00	10.61	10.50	13.49
5.25	10.73	10.75	13.63
5.50	10.85	11.00	13.78
5.75	10.98	11.25	13.92
6.00	11.10	11.50	14.06
6.25	11.23	11.75	14.20
6.50	11.35	12.00	14.35
6.75	11.48	12.25	14.49
7.00	11.61	12.50	14.64
7.25	11.74	12.75	14.78
7.50	11.87	13.00	14.93
7.75	12.00	13.25	15.08
8.00	12.13	13.50	15.23
8.25	12.27	13.75	15.38
8.50	12.40	14.00	15.53
8.75	12.53	14.25	15.68
9.00	12.67	14.50	15.83
9.25	12.80	14.75	15.98
15-Year Loan			
4.00	7.39	9.500	10.44
4.25	7.52	9.750	10.59
4.50	7.65	10.00	10.75
4.75	7.78	10.25	10.90
5.00	7.91	10.50	11.05
5.25	8.04	10.75	11.21

(*Continued*)

TABLE A.6 (*Continued*)

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
5.50	8.17	11.00	11.37
5.75	8.30	11.25	11.52
6.00	8.44	11.50	11.68
6.25	8.57	11.75	11.84
6.50	8.71	12.00	12.00
6.75	8.85	12.25	12.16
7.00	8.99	12.75	12.49
7.50	9.27	13.00	12.65
7.75	9.41	13.25	12.82
8.00	9.56	13.50	12.98
8.25	9.70	13.75	13.15
8.50	9.85	14.00	14.32
8.75	9.99	14.25	13.49
9.00	10.14	14.50	13.66
9.25	10.29	14.75	13.83
	20-Year Loan		
4.00	6.06	9.50	9.32
4.25	6.19	9.75	9.49
4.50	6.33	10.00	9.65
4.75	6.46	10.25	9.82
5.00	6.60	10.50	9.98
5.25	6.74	10.75	10.15
5.50	6.88	11.00	10.32
5.75	7.02	11.25	10.49
6.00	7.16	11.50	10.66

(*Continued*)

TABLE A.6 (*Continued*)

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
6.25	7.31	11.75	10.84
6.50	7.46	12.00	11.01
6.75	7.60	12.25	11.19
7.00	7.75	12.50	11.36
7.25	7.90	12.75	11.54
7.50	8.06	13.00	11.72
7.75	8.21	13.50	12.07
8.25	8.52	13.75	12.25
8.50	8.68	14.00	12.44
8.75	8.84	14.25	12.62
9.00	9.00	14.50	12.80
9.25	9.16	14.75	12.98
	25-Year Loan		
4.00	5.28	9.500	8.74
4.25	5.42	9.750	8.91
4.50	5.56	10.00	9.09
4.75	5.70	10.25	9.26
5.00	5.85	10.50	9.44
5.25	5.99	10.75	9.62
5.50	6.14	11.00	9.80
5.75	6.29	11.00	9.80
5.75	6.29	11.25	9.98
6.00	6.44	11.50	10.16
6.25	6.60	11.75	10.35
6.50	6.75	12.00	10.53

(*Continued*)

TABLE A.6 (*Continued*)

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
6.75	6.91	12.25	10.72
7.00	7.07	12.50	10.90
7.25	7.23	12.75	11.09
7.50	7.39	13.00	11.28
7.75	7.55	13.25	11.47
8.00	7.72	13.50	11.66
8.25	7.88	13.75	11.85
8.50	8.05	14.00	12.04
8.75	8.22	14.25	12.23
9.00	8.39	14.50	12.42
9.25	8.56	14.75	12.61
	30-Year Loan		
4.00	4.77	9.500	8.41
4.25	4.92	9.750	8.59
4.50	5.07	10.00	8.78
4.75	5.22	10.25	8.96
5.00	5.37	10.50	9.15
5.25	5.52	10.75	9.34
5.50	5.68	11.00	9.52
5.75	5.84	11.25	9.71
6.00	6.00	11.50	9.90
6.25	6.16	11.75	10.09
6.50	6.32	12.00	10.29
6.75	6.49	12.25	10.48
7.00	6.65	12.50	10.67
7.25	6.82	12.75	10.87

(*Continued*)

TABLE A.6 (*Continued*)

Monthly Payments per $1,000 of Loan Value

Annual Rate (%)	Payment ($) Monthly	Annual Rate (%)	Payment ($) Monthly
7.50	6.99	13.00	11.06
7.75	7.16	13.25	11.26
8.00	7.34	13.50	11.45
8.25	7.51	13.75	11.65
8.50	7.69	14.00	11.85
8.75	7.87	14.25	12.05
9.00	8.05	14.50	12.25
9.25	8.23	14.75	12.44

Note: The number of thousands borrowed is multiplied by the listed monthly payment for the indicated annual interest rate. The product is the total monthly payment. Due to rounding, this may be off by a few cents from the actual.

TABLE A.7

The Growth of $1 at Various Annual Interest Rates and Specified Number of Years

Years	3%	4%	5%	6%	7%
1	1.0300	1.0400	1.0500	1.0600	1.0700
2	1.0609	1.0816	1.1025	1.1236	1.1449
3	1.0927	1.1249	1.1576	1.1910	1.2250
4	1.1255	1.1699	1.2155	1.2625	1.3108
5	1.1593	1.2167	1.2763	1.3382	1.4026
6	1.1941	1.2653	1.3401	1.4185	1.5007
7	1.2299	1.3159	1.4071	1.5036	1.6058
8	1.2668	1.3686	1.4775	1.5938	1.7182
9	1.3048	1.4233	1.5513	1.6895	1.8385
10	1.3439	1.4802	1.6289	1.7908	1.9672
11	1.3842	1.5395	1.7103	1.8983	2.1049
12	1.4258	1.6010	1.7959	2.0122	2.2522
13	1.4685	1.6651	1.8856	2.1329	2.4098
14	1.5126	1.7317	1.9799	2.2609	2.5785
15	1.5580	1.8009	2.0789	2.3966	2.7590
20	1.8061	2.1911	2.6533	3.2071	3.8697
25	2.0938	2.6658	3.3864	4.2919	5.4274
30	2.4273	3.2434	4.3219	5.7435	7.6123
35	2.8139	3.9461	5.5160	7.861	10.677
40	3.2620	4.8010	7.0400	10.286	14.974
45	3.7816	5.8412	8.9850	13.765	21.002
50	4.3839	7.1067	11.467	18.420	29.457

(*Continued*)

TABLE A.7 (*Continued*)

The Growth of $1 at Various Annual Interest Rates and Specified Number of Years

Years	8%	9%	10%	11%	12%
1	1.0800	1.0900	1.1000	1.1100	1.1200
2	1.1664	1.1881	1.2100	1.2321	1.2544
3	1.2597	1.2950	1.3310	1.3676	1.4049
4	1.3605	1.4116	1.4641	1.5181	1.5735
5	1.4693	1.5386	1.6105	1.6851	1.7623
6	1.5869	1.6771	1.7716	1.8704	1.9738
7	1.7138	1.8280	1.9487	2.0762	2.2107
8	1.8509	1.9926	2.1436	2.3045	2.4760
9	1.9990	2.1719	2.3579	2.5580	2.7731
10	2.1589	2.3674	2.5937	2.8394	3.1058
11	2.3316	2.5804	2.8531	3.1518	3.4785
12	2.5182	2.8127	3.1384	3.4985	3.8960
13	2.7196	3.0658	3.4523	3.8833	4.3635
14	2.9372	3.3417	3.7975	4.3104	4.8871
15	3.1722	3.6425	4.1772	4.7846	5.4736
20	4.6610	5.6044	6.7275	8.0623	9.6463
25	6.8485	8.6231	10.835	13.585	17.000
30	10.063	13.268	17.449	22.892	29.960
35	14.785	20.414	28.102	38.575	52.800
40	21.725	31.409	45.259	65.001	93.051
45	31.920	48.327	72.890	109.53	163.99
50	46.902	74.358	117.39	184.56	289.00

TABLE A.8

Doubling Time for Various Annual Interest Rates

Rate (%)	Years
1	69.7
2	35.0
3	23.4
4	17.7
5	14.2
6	11.9
7	10.2
8	9.01
9	8.04
10	7.27
11	6.64
12	6.12
13	5.67
14	5.29
15	4.96

TABLE A.9

Monthly Savings to Produce $1,000 in the Specified Number of Years at the Given Annual Interest Rate (Compounded Monthly)

Years	3%	4%	5%	6%	7%
1	82.19	81.82	81.44	81.07	80.69
2	40.48	40.09	39.70	39.32	38.94
3	26.58	26.19	25.80	25.42	25.04
4	19.63	19.25	18.86	18.49	18.11
5	15.47	15.08	14.70	14.33	13.97
6	12.69	12.31	11.94	11.57	11.22
7	10.71	10.34	9.97	9.61	9.26
8	9.23	8.85	8.49	8.14	7.80
9	8.08	7.71	7.35	7.01	6.67
10	7.16	6.79	6.44	6.10	5.78
15	4.41	4.06	3.74	3.44	3.16
20	3.05	2.73	2.43	2.16	1.92
25	2.24	1.94	1.68	1.44	1.23
30	1.72	1.44	1.20	0.99	0.82
35	1.35	1.09	0.88	0.71	0.56
40	1.08	0.85	0.66	0.50	0.38
Years	**8%**	**9%**	**10%**	**11%**	**12%**
1	80.32	79.95	79.58	79.21	78.85
2	38.56	38.18	37.81	37.44	37.07
3	24.67	24.30	23.93	23.57	23.21
4	17.75	17.39	17.03	16.68	16.33
5	13.61	13.26	12.91	12.58	12.24
6	10.87	10.53	10.19	9.87	9.55
7	8.92	8.59	8.27	7.96	7.65
8	7.47	7.15	6.84	6.54	6.25

(*Continued*)

TABLE A.9 (*Continued*)

Monthly Savings to Produce $1,000 in the Specified Number of Years at the Given Annual Interest Rate (Compounded Monthly)

Years	8%	9%	10%	11%	12%
9	6.35	6.04	5.74	5.46	5.18
10	5.47	5.17	4.88	4.61	4.35
15	2.89	2.64	2.41	2.20	2.00
20	1.70	1.50	1.32	1.16	1.01
25	1.05	0.89	0.75	0.63	0.53
30	0.67	0.55	0.44	0.36	0.29
35	0.44	0.34	0.26	0.20	0.16
40	0.29	0.21	0.16	0.12	0.08

TABLE A.10

Monthly Savings to Produce $1,000 in Specified Number of Years at the Given Annual Interest Rate (Compounded Annually)

Years	3%	4%	5%	6%	7%
1	83.33	83.33	83.33	83.33	83.33
2	41.05	40.85	40.65	40.45	40.26
3	26.96	26.70	26.43	26.18	25.92
4	19.92	19.62	19.33	19.05	18.77
5	15.70	15.39	15.08	14.78	14.49
6	12.88	12.56	12.25	11.95	11.65
7	10.88	10.55	10.223	9.93	9.63
8	9.37	9.04	8.73	8.42	8.12
9	8.20	7.87	7.56	7.25	6.96
10	7.27	6.94	6.62	6.32	6.03
15	4.48	4.16	3.86	3.58	3.32
20	3.10	2.80	2.52	2.26	2.03
25	2.29	2.00	1.75	1.52	1.32
30	1.75	1.49	1.25	1.05	0.88
35	1.38	1.13	0.92	0.75	0.60
40	1.10	0.88	0.69	0.54	0.42

Years	8%	9%	10%	11%	12%
1	83.33	83.33	83.33	83.33	83.33
2	40.06	39.87	39.68	39.49	39.31
3	25.67	25.42	25.18	24.93	24.70
4	18.49	18.22	17.96	17.69	17.44
5	14.20	13.92	13.65	13.38	13.12
6	11.36	11.08	10.80	10.53	10.27
7	9.34	9.06	8.78	8.52	8.26
8	7.83	7.56	7.29	7.03	6.78

(*Continued*)

TABLE A.10 (*Continued*)

Monthly Savings to Produce $1,000 in Specified Number of Years at the Given Annual Interest Rate (Compounded Annually)

Years	8%	9%	10%	11%	12%
9	6.67	6.40	6.14	5.88	5.64
10	5.75	5.48	5.23	4.98	4.75
15	3.07	2.84	2.62	2.42	2.23
20	1.82	1.63	1.45	1.30	1.16
25	1.14	0.98	0.88	0.73	0.63
30	0.74	0.61	0.51	0.42	0.35
35	0.48	0.39	0.31	0.24	0.19
40	0.32	0.25	0.19	0.14	0.11

TABLE A.11

Percentage of Funds That May Be Withdrawn Each Year at the Beginning of the Year at Different Annual Interest Rates

Years	4%	5%	6%	7%	8%
1	100.000	100.000	100.000	100.000	100.000
2	50.980	51.220	51.456	51.691	51.923
3	34.649	34.972	35.293	35.612	35.929
4	26.489	26.858	27.226	27.591	27.956
5	21.599	21.998	22.396	22.794	23.190
6	18.343	18.764	19.185	19.607	20.029
7	16.020	16.459	16.900	17.341	17.784
8	14.282	14.735	15.192	15.651	16.112
9	12.932	13.399	13.870	14.345	14.822
10	11.855	12.334	12.818	13.306	13.799
15	8.6482	9.1755	9.7135	10.261	10.818
20	7.0752	7.6422	8.2250	8.8218	9.4308
25	6.1550	6.7574	7.3799	8.0197	8.6740
30	5.5606	6.1954	6.8537	7.5314	8.2248
35	5.1517	5.8164	6.5070	7.2181	7.9447
40	4.8580	5.5503	6.2700	7.0102	7.7648
45	4.6406	5.3583	6.1038	6.8691	7.6470
50	4.4760	5.2168	5.9853	6.7719	7.5688
Years	**9%**	**10%**	**11%**	**12%**	**13%**
1	100.000	100.000	100.000	100.000	100.000
2	52.153	52.381	52.607	52.830	53.052
3	36.244	36.556	36.866	37.174	37.480
4	28.318	28.679	229.038	29.396	29.752
5	23.586	23.982	24.376	24.769	25.161
6	20.451	20.873	21.295	21.717	22.137

(*Continued*)

TABLE A.11 (*Continued*)

Percentage of Funds That May Be Withdrawn Each Year at the Beginning of the Year at Different Annual Interest Rates

Years	9%	10%	11%	12%	13%
7	18.228	18.673	19.118	19.564	20.010
8	16.576	17.040	17.506	17.973	18.441
9	15.303	15.786	16.270	16.757	17.245
10	14.295	14.795	15.297	15.802	16.309
15	11.382	11.952	12.528	13.109	13.694
20	10.050	10.678	11.313	11.953	12.598
25	9.3400	10.015	10.697	11.384	12.073
30	8.9299	9.6436	10.363	11.084	11.806
35	8.6822	9.4263	10.174	10.921	11.666
40	8.5284	9.2963	10.065	10.831	11.592
45	8.4313	9.2174	10.001	10.780	11.552
50	8.3694	9.1690	9.9639	10.751	11.530

TABLE A.12

Growth of Annual Deposits of $1,000 at the End of the Year at Specified Annual Interest Rates

Years	6%	8%	10%
1	1000	1000	1000
2	2060	2080	2100
3	3183.60	3246.4	3310
4	4374.62	4506.11	4641
5	5637.09	5866.60	6105.11
6	6975.32	7335.93	7715.61
7	8393.84	8922.80	9487.17
8	9897.47	10636.63	11435.89
9	11491.32	12487.56	13579.48
10	13180.79	14486.56	15937.42
11	14971.64	16645.49	18531.17
12	16869.94	18977.13	21384.28
13	18882.14	21495.30	24522.71
14	21015.07	24214.92	27974.98
15	23275.97	27152.11	31772.48
20	36785.59	45761.96	57275.00
25	54864.51	73105.94	98347.06
30	79058.19	113283.21	164494.02
35	111434.78	172316.8	271024.38
40	154761.97	259056.52	442592.56

TABLE A.13

Growth of Annual Deposits of $1,000 at the Beginning of the Year at Specified Annual Interest Rates

Years	6%	8%	10%
1	1060.00	1080.00	1100.00
2	2183.60	2246.40	2310.00
3	3374.62	3506.11	3641.00
4	4637.09	4866.60	5105.10
5	5975.32	6335.93	6715.61
6	7393.84	7922.80	8487.17
7	8897.47	9636.63	10435.89
8	10491.32	11487.56	12579.48
9	12180.79	13486.56	14937.42
10	13971.64	15645.49	17531.17
11	15869.94	17977.13	20384.28
12	17882.14	20495.30	23522.71
13	20015.07	23214.92	26974.98
14	22275.97	26152.11	30772.48
15	24672.53	29324.28	34949.73
20	38992.73	49422.92	63002.50
25	58156.38	78954.41	108181.77
30	83801.68	122345.87	180943.42
35	118120.87	186102.14	298126.81
40	164047.69	279781.03	486851.81

TABLE A.14

Monthly Amount That Must Be Saved for the Years Indicated (Down) in Order to Collect $1,000 per Month Thereafter (Across) at 4% Annual Interest Compounded Monthly

Years Saving	Years Collecting				
	5	10	15	20	25
5	819.00	1489.80	2039.10	2489.10	2857.50
10	368.75	670.77	918.11	1120.69	1286.61
15	220.65	401.36	549.36	670.57	769.85
20	148.04	269.29	368.60	449.93	516.54
25	105.61	192.11	262.95	320.97	368.49
30	78.24	142.31	194.79	237.77	272.97
35	59.43	108.10	147.96	180.60	207.34
40	45.94	83.56	114.38	139.62	160.29

TABLE A.15

Monthly Amount That Must Be Saved for the Years Indicated (Down) in Order to Collect $1,000 per Month Thereafter (Across) at 6% Annual Interest Compounded Monthly

Years Saving	Years Collecting				
	5	10	15	20	25
5	714.37	1291.00	1698.50	2000.60	2224.55
10	315.63	549.63	723.11	851.73	947.08
15	177.86	309.72	407.48	479.96	533.69
20	111.95	194.95	256.48	302.10	335.92
25	74.64	129.98	171.00	201.42	223.97
30	51.49	89.67	117.97	138.95	154.51
35	36.31	63.22	83.18	97.97	108.94
40	25.97	45.23	59.50	70.09	77.94

TABLE A.16

Monthly Amount That Must Be Saved for the Years Indicated (Down) in Order to Collect $1,000 per Month Thereafter (Across) at 8% Annual Interest Compounded Monthly

Years Saving	Years Collecting 5	10	15	20	25
5	671.21	1121.73	1424.13	1627.10	1763.34
10	269.58	450.52	571.98	653.49	708.21
15	142.52	238.19	302.40	345.49	374.42
20	83.73	139.93	177.65	202.97	219.97
25	51.86	86.67	110.03	125.71	136.24
30	33.09	55.30	70.21	80.22	86.94
35	21.50	35.93	45.62	52.12	56.48
40	14.13	23.61	29.97	34.25	37.11

TABLE A.17

Monthly Amount That Must Be Saved for the Years Indicated (Down) in Order to Collect $1,000 per Month Thereafter (Across) at 10% Annual Interest Compounded Monthly

Years Saving	Years Collecting 5	10	15	20	25
5	607.79	977.20	1201.72	1338.18	1421.12
10	229.76	369.41	454.28	505.87	537.22
15	113.56	182.57	224.52	250.02	265.51
20	61.98	99.65	122.55	136.46	144.92
25	35.47	57.03	70.13	78.10	82.94
30	20.82	33.48	41.17	45.84	48.68
35	12.40	19.93	24.51	27.29	28.99
40	7.44	11.97	14.71	6.39	17.40

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