$1.75

S153

INFINITE SEQUENCES AND SERIES

BY

PROFESSOR

KONRAD KNOPP

TRANSLATED BY

F. BAGEMIHL

poznanski

FIRST PUBLICATION IN ANY LANGUAGE!

This is the first appearance of a new work by Konrad Knopp, who is renowned as one of the best expositors in modern mathematics. Concentrated upon two topics of modern mathematics, it presents in detail the theory of infinite sequences and series. It is an introductory presentation, designed to give the student sufficient background to penetrate into more advanced topics by himself. Foundations are laid with special care; all definitions are clearly stated; all theorems proved with enough detail to make them entirely clear.

Partial Contents: Sequences and sets. Real & Complex numbers. Functions of a real and of a a complex variable. Sequences and series. The Main Tests for Infinite Series. Operating with Convergent Series. Power Series. Development of a Theory of Convergence. Expansion of the Elementary Functions. Numerical and Closed Evaluation of Series.

Translated by Frederick Bagemihl. Bibliography. v +186pp. 5-3/8 x 8.

INFINITE SEQUENCES AND SERIES

By

DR. KONRAD KNOPP
Emeritus Professor of Mathematics at the University of Tübingen

TRANSLATED BY

FREDERICK BAGEMIHL
The Institute for Advanced Study

NEW YORK
DOVER PUBLICATIONS, INC.

Manufactured in the United States of America

CONTENTS

FOREWORD

The purpose of this little book in the Dover Series is to develop the theory of infinite sequences and series from its beginnings—the construction of the system of real and of complex numbers—so far, that the reader will be in a position to penetrate into the more advanced parts of the theory by himself. The foundations are therefore presented carefully, but the development has been carried out only to the extent permitted by this purpose and the narrow compass of the book. Thus, important topics had to be omitted, which could perhaps be presented in a second small volume for advanced students. The table of contents indicates in detail the subjects treated.

Mr. Frederick Bagemihl has carried out the translation from my German manuscript with the same care and understanding as have already distinguished his translations of my previous little function-theoretical volumes. I take this opportunity to thank him heartily for all his trouble.

KONRAD KNOPP

Chapter 1

INTRODUCTION AND PREREQUISITES

1.1. Preliminary remarks concerning sequences and series

In manifold investigations in pure and applied mathematics, it often happens that a result is not obtained all at once, as in $7 \cdot 13 = 91$, but that one seeks to approximate the result in a definite way by steps. This is the case, for example, in calculating the area of a circle of radius 1, for which we obtain first perhaps 3, then 22/7, say, then 3.1415, *etc.*, as *approximate values* or *approximations*. This also occurs in the elementary method of calculating the square root of 2, where we get first 1, then 1.4, then 1.41, *etc.* We thus secure a sequence of values which lead, in a sense to be described later in more detail, to the value π, $\sqrt{2}$, respectively. If we compute in such a manner, corresponding to each *natural number* 0, 1, 2, ...,[1] a number $s_0, s_1, s_2, \ldots$, or if these are given or defined in some other way, we say that we have before us an *infinite sequence of numbers*, or briefly a *sequence*, with the *terms* s_ν, and we denote it by

$$(1) \qquad \{s_0, s_1, s_2, \ldots\}, \text{ or } \{s_0, s_1, \ldots, s_\nu, \ldots\}, \text{ or merely } \{s_\nu\}.^{2}$$

Such sequences may be given (generated, defined) in the most varied ways. Numerous examples will appear in the sequel. The following method occurs especially often: If a certain term of the sequence is already known, then the next term is given by means of the amount by which it differs from the former. *E.g.*, if s_3 is already known, then s_4 is determined by indicating the difference $s_4 - s_3 = a_4$, so that $s_4 = s_3 + a_4$. Thus, a_4 represents the amount that has to be added to

[1] The number 0 is often not counted as a natural number; here, however, it is more convenient to do so.

[2] Or by $\{s_n\}$, for it is of course immaterial which letter we choose as index. We prefer ν and n.

s_3 in order to obtain the next term s_4.[1] If, for the sake of uniformity, we set the initial term s_0 equal to a_0, and then, in general, write $s_\nu - s_{\nu-1} = a_\nu$, we have, for $n = 0, 1, 2, \ldots$,

$$(2) \qquad s_n = a_0 + a_1 + a_2 + \ldots + a_n.$$

The n^{th} term of the sequence $\{s_\nu\}$ is obtained by "adding up" the terms of another (infinite) sequence $\{a_\nu\}$. To indicate this continued summation process, the sequence thus obtained is denoted by

$$(3) \qquad a_0 + a_1 + \ldots + a_\nu + \ldots$$

and is called an *(infinite) series*. The following is a simple example: If we divide out the fraction $\frac{1}{1-a}$ according to the elementary rules, we get a sequence beginning with $s_0 = 1$, $s_1 = 1 + a$, $s_2 = 1 + a + a^2$ and yielding $s_{n-1} = 1 + a + \ldots + a^{n-1}$. The corresponding *remainder* $\frac{a^n}{1-a}$ now yields, at the next step, a^n, which must be added to s_{n-1} to produce s_n. Since this continues without end, we obtain the infinite series

$$(4) \qquad 1 + a + a^2 + \ldots + a^\nu + \ldots.$$

Whether or not, and in what sense, this infinite series is the same as the fraction $\frac{1}{1-a}$ which we started with, still requires, naturally, precise elucidation.

An (infinite) series is a means, employed particularly often in what follows, of defining an (infinite) sequence: A certain sequence $\{a_\nu\}$ is directly computed, defined—in short: given. It is, however, not itself the main object of the investigation; a new sequence $\{s_n\}$ is derived from it, whose terms are formed according to the specification (2), and this sequence $\{s_n\}$ is the one which furnishes the real subject of the investigation. Thus, in the series

$$(5) \qquad 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^\nu} + \ldots,$$

[1] a_4 (and, later on, a_ν), of course, need not be positive, but may denote an arbitrary number.

we are not so much interested in the individual terms, $\frac{1}{2^\nu}$, of this series, as in what we get if we sum them up without end, *i.e.*, form the sums

$$(6) \qquad s_n = 1 + \frac{1}{2} + \dots + \frac{1}{2^n} = 2 - \frac{1}{2^n}, \qquad (n = 0, 1, 2, \dots),$$

which approach the number 2 as the "value" of the infinite series. (For a precise definition, see 2.1.)

Of the two concepts *sequence* and *series*, the former is the simpler and more primitive one. In the first place, a series can only be defined if one already possesses the notion of a sequence; for to be able to write down the series (3), one must know the sequence of its terms. Furthermore, to form the series (3) requires the operation of addition, which does not enter at all into the concept of the sequence (1). In a sequence, the individual terms are not connected, but are merely ordered in a definite way by their indices. Historically it was just the opposite. Infinite series appeared, especially during the 17^{th} century, quite naturally, as in the example that led to (4), as well as in the computation of the values of the elementary functions (logarithms, *etc.*). It was more than a century later, however, before the meaning and significance of such expansions were clarified satisfactorily. Even in modern literature, sufficient distinction between the two concepts is, unfortunately, not always made. In the present exposition we shall, for the reasons which we have put forth, place the sequences, as the decisive new concept, in the forefront, and, as is proper, derive the series from them by annexing the operation of addition.

For sums of the kind appearing in formulas (2) to (6) and in similar cases, where summands of the same kind are to be added, it is customary to use an abbreviated notation: For $b_1 + b_2 + b_3 + b_4$ we write more briefly $\sum\limits_{\nu=1}^{4} b_\nu$ (read: the sum of b_ν as ν runs from 1 to 4), and analogously for the sums in formulas (2) – (6),

$$\sum_{\nu=0}^{n} a_\nu, \quad \sum_{\nu=0}^{\infty} a_\nu \text{ (or still shorter } \sum_{\nu} a_\nu, \text{ or simply } \sum a_\nu), \quad \sum_{\nu=0}^{\infty} a^\nu,$$

$$\sum_{\nu=0}^{\infty} \frac{1}{2^\nu}, \quad \sum_{\nu=0}^{n} \frac{1}{2^\nu}.$$

The corresponding expression for the familiar binomial expansion of $(a + b)^n$ (where $n \geqq 1$ is an integer) is

$$(7) \qquad (a + b)^n = \sum_{\nu=0}^{\infty} \binom{n}{\nu} a^\nu b^{n-\nu}.$$

In all these cases, ν is called the *index of summation*. It *runs* from 0 to n or from 0 to ∞, where the latter means that ν runs through all the natural numbers from 0 on without end.[1] The index of summation ν may of course be replaced by any other letter. We shall often use $n, \mu, \lambda, \rho, \ldots$.

We merely mention the fact that we arrive at *infinite products*, as they are called, if we make use of multiplication instead of addition: From a sequence $\{a_\nu\}$, which we suppose to be given directly, we derive the sequence $\{p_n\}$ of products which are formed in accordance with

$$(8) \qquad p_n = a_0 \cdot a_1 \cdot a_2 \cdot \ldots \cdot a_n, \qquad (n = 0, 1, 2, \ldots);$$

this sequence is then denoted, in analogy with (3), by

$$(9) \qquad a_0 \cdot a_1 \cdot \ldots \cdot a_\nu \ldots \quad \text{or} \quad \prod_{\nu=0}^{\infty} a_\nu.$$

These *infinite products* will be investigated, but only briefly, in 3.7.

1.2. Real and complex numbers

We must assume that the reader is familiar with the construction of the system of real numbers and the system of complex numbers, and also, of course, with their use. Because of its fundamental importance, however, for all that follows, we shall nevertheless explain the most essential idea which is employed in that construction.

Starting from the *natural numbers* (see 1.1), one introduces in the well-known manner first the *negative integers*, which together with the natural numbers are called briefly the *integers*, and then the fractions, which together with the integers are designated as the *rational numbers*.

---

[1] Whether or not, and under what conditions, a symbol of the form Σa_ν or $\sum\limits_{\nu=0}^{\infty} a_\nu$ represents a definite number will be discussed in detail in 2.6.

The latter may be written in the form $\frac{p}{q}$, where p is an arbitrary integer and q is a natural number *different from* 0.

In the domain of these rational numbers, the *four fundamental operations* of addition, subtraction, multiplication, and division—the latter with the sole exception of division by 0—can always be carried out and yield a unique result, which is again a rational number. In this sense the totality of rational numbers forms a closed domain. Such a domain is called a *field* if the operations of addition and multiplication defined in it obey the *associative*, *commutative*, and *distributive* laws, as is the case for the rational numbers. The field of rational numbers is, moreover, ordered, *i.e.*, between two rational numbers a and b, precisely one of the three relations

$$a < b, \quad a = b, \quad a > b$$

holds.

This order obeys the familiar simple fundamental laws of order. The numbers > 0 are called *positive*, those < 0 are called *negative*. If $a = 0$, we also say that *a vanishes*.

We also assume that the reader is familiar with the fact that the rational numbers can be made to correspond to certain points of a straight line, the *number axis*, on which two points, 0 and 1, are fixed arbitrarily, as well as with how this is accomplished. We usually imagine this axis to be drawn horizontally, and choose 1 to the right of 0. The image points of the rational numbers are called for brevity the *rational points* of the number axis. These rational points are dense on the line, *i.e.*, if a and b are any two such points, then there is at least another one, *e.g.* the point $c = \frac{1}{2}(a + b)$, lying between them.

All further rules of operation, some of which we shall list at the end of this section, can be derived rigorously in a purely formal manner (*i.e.*, without having to consider the meaning of the symbols) from the foregoing fundamental laws. The fact that it is unnecessary in this derivation to make use of the meaning of the symbols has the following important consequence: If a, b, ... are any other entities whatsoever besides the rational numbers, but which obey the same fundamental laws, then it is possible to operate with these entities according to exactly the same rules as with the rational numbers. One is therefore justified in calling any system of such entities a *number system*, and the entities themselves, *numbers*.

Such other entities, now, which obey all the fundamental laws valid for the rational numbers, are—and this leads us to the intimated single essentially new fundamental idea in the construction of the number systems—the *real numbers*, which we arrive at through the following consideration:

The system of rational numbers is incomplete in the sense that it is incapable of satisfying some very simple demands. Thus, as is well known, there is no rational number whose square is equal to 2. On the contrary, the square of every (positive) rational number is either <2 or >2, and in both cases there exists such a number whose square is arbitrarily close to 2 and less than it or greater than it, respectively. This, as well as the graphical representation on the number axis, leads us to divide all rational numbers into two classes: a class $\mathfrak{A}$, into which we put every positive rational number whose square is less than 2, as well as 0 and the negative rational numbers, and a class $\mathfrak{A}'$ containing every positive rational number whose square is greater than 2. The question arises, whether this classification, which we shall denote by $(\mathfrak{A}|\mathfrak{A}')$ and which is said to be a *Dedekind cut* in the domain of rational numbers, can be regarded as a substitute for the (still lacking) number whose square is equal to 2, and, in particular, whether it can be regarded as a number with which one can operate as with a rational number.

This question can be answered in no other way than the following: One considers *the totality of all conceivable Dedekind cuts* $(\mathfrak{A}|\mathfrak{A}')$ in the domain of rational numbers, *i.e.*, of all imaginable divisions of all rational numbers into two nonempty classes $\mathfrak{A}$ and $\mathfrak{A}'$ which satisfy (as above) the sole requirement that every number of the class $\mathfrak{A}$ be less than every number of the class $\mathfrak{A}'$. Then one shows that these cuts are such "other entities" which, under suitable agreements regarding their order $(<, =, >)$ as well as their addition $(+)$ and multiplication $(\cdot)$, obey all the fundamental laws valid for the rational numbers. How these agreements are to be made—the way to proceed is obvious when the matter is viewed on the number axis—will of course not be considered here, but will be regarded as familiar to the reader. If, however, one now denotes such a cut by a small Roman letter, setting, say, $(\mathfrak{A}|\mathfrak{A}') = a$, and calls these cuts numbers, then, with these stipulations, they obey without exception all the fundamental laws valid for the rational numbers. The entities obtained in this manner are therefore numbers (see above), and in their totality constitute *the system* or *the field of real numbers*. A part of the real numbers turns out to be equivalent (in the sense of the definition of the symbol $=$) to the hitherto existing rational numbers: The system of real numbers is a (proper) extension of the system of rational numbers. Those real numbers which are not rational are called *irrational*.

Thus at the moment—we set this down as the result of the foregoing discussion—a real number is regarded as defined or given, only if it is either rational, and hence can be represented in the form p/q (see above), or if it is realized by some cut in the domain of rational numbers.

With the construction of the system of real numbers, a certain closure is attained. It can be shown that no different system (distinct, in any essential respect, from the acquired system of real numbers) and no more extensive system of entities exists, which satisfies *all* the forenamed fundamental laws—no matter how order, addition, and multiplication be defined. (*Uniqueness theorem* and *completeness theorem* for the system of real numbers.)

A renewed classification in the domain of real numbers leads to nothing new: If all real numbers are divided into two (nonempty) classes $\mathfrak{A}$ and $\mathfrak{A}'$ in such a manner that every number a in $\mathfrak{A}$ is less than every number a' in $\mathfrak{A}'$, then there is the following *theorem of continuity*, also called the *fundamental theorem of Dedekind*, for the real numbers:

Theorem. *A Dedekind cut in the domain of real numbers invariably defines (determines, strikes, realizes) one, and only one, real number s, the cut-number, such that every $a \leqq s$, every $a' \geqq s$. The cut-number, s, itself may belong to $\mathfrak{A}$ or to $\mathfrak{A}'$, depending on the classificatory viewpoint. Every number less than s belongs to $\mathfrak{A}$ (it "is an a"), every number greater than s belongs to $\mathfrak{A}'$ (it "is an a'").*

These real numbers can now be put, in the familiar way, into one-to-one correspondence with the totality of points of the number axis. Operating with numbers has a graphical analogue in operating with the points of the number axis.

The step from the real to the complex numbers is in principle of a much simpler nature than that from the rational to the real numbers just considered. In contrast to the fact, which was taken above as point of departure, that the quadratic equation $x^2 - 2 = 0$ has no solution in the domain of rational numbers (but which is now solved by the cut-number of the cut which was presented as an example) is the new fact that the equation $x^2 + 2 = 0$, *e.g.*, (and many similar ones) has no solution even in the system of real numbers, and that there is even no real number which "nearly" satisfies the equation. It was noticed early, however, that one could operate formally with numbers of the form $\alpha + \alpha' i$, where α and α' denote arbitrary real numbers and i is a symbol which satisfies the (at the moment unrealizable) condition $i^2 = -1$, in almost the same way as with real numbers, and that quadratic equations of the kind indicated then possess a solution at least formally. If we leave aside the symbol i which at first appears to be meaningless, then in the aforesaid operations we are dealing with operations with number pairs (α, α'), which we immediately think of as being represented graphically in the usual manner as points $a = (\alpha, \alpha')$ of a plane provided with a set of rectangular coordinate axes (α-axis horizontal and directed toward the right, α'-axis vertical and directed upward, the same unit of length on both). The historical development alluded to has made it almost compulsory to call two number pairs $a = (\alpha, \alpha')$ and $b = (\beta, \beta')$ "equal" $(a = b)$, if, and only if, the points representing them coincide, *i.e.*, $\alpha = \beta$ and *at the same time* $\alpha' = \beta'$; otherwise they are said to be unequal $(a \neq b)$. It has also led to regarding the number pairs $(\alpha + \beta,\ \alpha' + \beta')$ and $(\alpha\beta - \alpha'\beta',\ \alpha\beta' + \alpha'\beta)$ as the sum $a + b$ and product $a \cdot b$, respectively. With these stipulations it is now an easy matter to show (*cf. Elem.*[1], § 4–§ 15) that the very same laws do indeed hold for operating with the number pairs $a = (\alpha, \alpha')$ as in the field of rational or of real num-

[1] This is an abbreviation for the author's *Elements of the Theory of Functions* listed in the Bibliography.

bers,[1] except for the laws of order: It is not possible to define for the number pairs an order corresponding to the symbols $<$ and $>$ for which the customary theorems ("laws of order") for operating with real numbers are valid.[2] Whereas invariably one of the three relations $a = b, a < b$, $a > b$ holds between two real numbers a and b, one has to be satisfied, in the case of two number pairs, with the alternative $a = b$ and its negation $a \neq b$.

With this exception, operations with the number pairs (α, α'), if they are denoted for brevity by a letter a, proceed formally the same as operations with real numbers. The number pairs are therefore likewise regarded as numbers, which are now designated as *complex numbers* to distinguish them from the hitherto existing real numbers.

Finally it is easy to verify that operating with the number pairs $(\alpha, 0)$ according to the rules set down for number pairs proceeds formally in just the same way as if one were operating with the real numbers α themselves: Equality, sum, and product of two such pairs, $(\alpha, 0)$, $(\beta, 0)$, goes over, if we abbreviate these pairs to α and β, into equality, sum, and product of α and β. We say: The subsystem of all number pairs $(\alpha, 0)$, relative to their equality and their combination by means of addition and multiplication, is *isomorphic* to the system of real numbers. We may therefore actually set $(\alpha, 0)$ equal to α, *i.e.*, regard the pair $(\alpha, 0)$ as merely another symbol for the real number α.

But then we may first put $(\alpha, 0) + (0, \alpha') = \alpha + (0, \alpha')$ and further $(0, \alpha') = (\alpha', 0)\ (0, 1)$, so that finally the arbitrary number pair (α, α') may be represented in the form

$$(\alpha, \alpha') = \alpha + \alpha' \cdot (0, 1). \tag{1}$$

Thus, all number pairs may, with the exclusive use of a single number pair $(0, 1)$, be written in the form (1). If, with *Euler*, we now introduce the letter i as an abbreviation for this number pair, then we may set

$$(\alpha, \alpha') = \alpha + \alpha' i, \tag{2}$$

[1] The number pair (0, 0) now takes the place of the number 0 which is excluded from being a denominator in division.

[2] We cannot enter here into the reasons for this impossibility.

where

$$(3) \qquad i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1.$$

With this the connection with the naïve way of using complex numbers mentioned in the beginning is established: Operations with complex numbers may be regarded as operations with sums of the form $\alpha + \alpha' i$, in which α and α' denote arbitrary real numbers and i is a symbol (number pair) for which $i^2 = -1$.

All further simple facts and customary agreements concerning the system of complex numbers will be regarded, as in the case of the real numbers, as known. These include, *e.g.*, the following: The plane in which we have represented the number pairs $(\alpha, \alpha') = a$ as points is called the *plane of complex numbers* or, briefly, the *complex plane;* $\alpha = \Re(a)$ is designated as the *real part* of a, $\alpha' = \Im(a)$ as the *imaginary part* of a; the α-axis is known as the *real axis*, the α'-axis as the *imaginary axis; etc.* If we introduce polar coordinates into this plane (so that $\alpha = \rho \cos \varphi$, $\alpha' = \rho \sin \varphi$), then we obtain the *trigonometric representation* or *polar form* of a:

$$(4) \qquad a = \rho (\cos \varphi + i \sin \varphi).$$

Here the nonnegative number ρ is called the *absolute value* or *modulus* of a and is denoted by $|a|$, φ is called the (or an) *amplitude* or *argument* of a (am a or arg a). The latter is infinitely multiple-valued,[1] for if φ is an amplitude of a, then so is $\varphi + 2k\pi$ (k an arbitrary integer). For every $a \neq 0$, the uniquely determined amplitude which satisfies the auxiliary condition $-\pi < \varphi \leqq +\pi$ is called the *principal value* of am a.

In conclusion, for the subsequent use of real and complex numbers we list without detailed comment a number of simple

1.2.1. Conventions and formulas

1. "Numbers", in the sequel, are always arbitrary complex numbers; our investigations take place "in the complex domain". Only if it follows unambiguously from the context shall the numbers be real,

[1] It is completely indeterminate (may be chosen arbitrarily) for $a = 0$.

the investigations take place "in the real domain". All assertions, however, concerning arbitrary complex numbers are also correct for real numbers, for these are a subset of the set of complex numbers.[1]

In particular, we are dealing with real numbers

a) if we speak of natural, positive, or negative numbers;

b) if two numbers are connected by the symbol $<$ or $>$, or if, in an assertion involving numbers, either explicit or implicit use is made of their order (see above);

c) if, to note a special case of b), we speak of an increasing or a decreasing sequence or function (definition in 2.1).

2. In addition to the fundamental laws, all those rules (the so-called derived rules) which are familiar to us from operating with rational or real numbers are valid for operating with complex numbers, in which, and in whose proofs, no use is made of (linear) order, *i.e.*, of the symbols $<$ and $>$. In short, one may operate formally with complex numbers in the same way as with real numbers, except that operations with inequalities have to be altered in the indicated manner.

Of these derived rules, we call particular attention to the rules of parentheses (formation of the product of two sums of several terms), the definition of the powers a^n with a positive integral exponent n, as well as the binomial theorem (see 1.1, (7)).

3. $|a|$ is equal to the *distance* of the point a from the point 0, $|a-b| = |b-a|$ is the distance between the points a and b. We have

$$|ab| = |a|\cdot|b|, \qquad \text{am}\, ab = \text{am}\, a + \text{am}\, b,^{2}$$

$$\left|\frac{b}{a}\right| = \frac{|b|}{|a|}, \qquad \text{am}\, \frac{b}{a} = \text{am}\, b - \text{am}\, a,^{2}$$

in particular,

$$\left|\frac{1}{a}\right| = \frac{1}{|a|}, \qquad \text{am}\, \frac{1}{a} = -\text{am}\, a,^{2} \qquad \text{provided that } a \neq 0.$$

[1] In other words, a real number is at the same time also a complex number (whose imaginary part equals 0). The converse, however, is naturally not true.

[2] More precisely: Every amplitude on the left is equal to one of the am-values on the right, and conversely.

4. *Inequalities*

Invariably $|a| \geqq 0$, and $|a| = 0$ if, and only if, $a = 0$. We have invariably

$$|a + b| \leqq |a| + |b| \quad \text{and} \quad \geqq \big||a| - |b|\big|,$$

and invariably

$$|a_1 + a_2 + \dots + a_p| \leqq |a_1| + |a_2| + \dots + |a_p|,$$

and likewise invariably

$$|\Re(a)| \leqq |a|, \quad |\Im(a)| \leqq |a|.$$

5. The totality of numbers (points) z for which

$$|z-a| = \rho, \quad \leqq \rho, \quad \geqq \rho, \quad < \rho, \quad > \rho$$

for fixed, given a and given $\rho > 0$, fill out easily recognizable circumferences or their interior or exterior regions with or without *boundary.*

The totality of numbers z for which (a arbitrary, $\varepsilon > 0$, fixed) $|z-a| < \varepsilon$, is called the (open) *ε-neighborhood* of the point a.

1.3. Sets of numbers

We also regard as known the fundamental concepts concerning sets of numbers (points), and we merely list briefly several definitions and facts:

If a finite or an infinite number of complex numbers are selected from the totality of complex numbers according to any rule, they constitute a *set of numbers*, and the corresponding points constitute a *set of points.* Such a set $\mathfrak{M}$ is regarded as given or defined, if the rule of selection is so formulated that for every number it is definite whether it belongs to the set or not, and if only the one or the other is possible. A particular number shall belong at most once to the set. (In this connection, *cf.* 2.2.) The individual numbers z of the set are called its *elements.* We write $z \in \mathfrak{M}$ to express the fact that z is an element of $\mathfrak{M}$. The rule of selection is permitted to be such that no number of the kind in question exists—we then speak of the *empty set*—or that all numbers belong to the set. A set which contains infinitely many (distinct) elements is designated expressly as an *infinite* set. Numerous examples will appear in the sequel.

If every point of $\mathfrak{M}'$ is also a point of $\mathfrak{M}$, then $\mathfrak{M}'$ is called a *subset* of

$\mathfrak{M}$, in symbols: $\mathfrak{M}' \subseteq \mathfrak{M}$. We say that $\mathfrak{M}'$ is a *proper subset* of $\mathfrak{M}$, if a $z \in \mathfrak{M}$ exists which does not belong to $\mathfrak{M}'$.

A set is said to be *bounded*, if there exists a positive number K such that "for all z of the set" (*i.e.*, for every $z \in \mathfrak{M}$) the inequality $|z| \leqq K$ holds. Such a number K is then called a *bound* for the (absolute values of the numbers of the) set. In the contrary case, $\mathfrak{M}$ is said to be *unbounded*. The totality of points (complex numbers) which do not belong to $\mathfrak{M}$ is called the *complement* of $\mathfrak{M}$.

If a point ζ of the plane possesses the property that in *every* ε-neighborhood of ζ there are infinitely many points of a given set $\mathfrak{M}$, then ζ is said to be a *limit point* of $\mathfrak{M}$.

A point belonging to $\mathfrak{M}$ is called *isolated*, if some ε-neighborhood of the point contains no other point of $\mathfrak{M}$. It is called an *interior point* of $\mathfrak{M}$, if some ε-neighborhood of the point belongs entirely to $\mathfrak{M}$.

A point ζ of the plane (ζ may or may not belong to $\mathfrak{M}$) is called a *boundary point* of $\mathfrak{M}$, if every ε-neighborhood of ζ contains at least one point which belongs to $\mathfrak{M}$ and at least one point which does not belong to $\mathfrak{M}$. A set is said to be *closed*, if it contains all its limit points; it is said to be *open*, if it consists solely of interior points.

The foregoing considerations of sets referred to arbitrary complex numbers, took place "in the complex domain". If we restrict ourselves to real numbers, we arrive at the concept of a *set of real numbers*. It must, however, be observed in this connection, that "in the real domain" the complement of $\mathfrak{M}$ is usually understood to be only the set of *real* numbers which do not belong to $\mathfrak{M}$, and that the ε-neighborhood of a real number ξ (ξ arbitrary, real; $\varepsilon > 0$) likewise consists of only the *real* numbers x for which $|x - \xi| < \varepsilon$; they constitute the open interval[1] $\xi - \varepsilon < x < \xi + \varepsilon$. Otherwise all definitions remain the same. Nevertheless, several new details (refinements) arise, due to the fact that the real numbers form an ordered set:

---

[1] An interval denotes the set of all real numbers which lie between two definite real numbers, say a and b. It is called *closed* or *open*, according as the end points are regarded as belonging to it or not. If $a < b$, then the closed interval $a \leqq x \leqq b$ is denoted by $< a, b >$ and the open interval $a < x < b$, by (a, b). The intervals $a \leqq x < b$ and $a < x \leqq b$ are called *semiopen*.

A real set is said to be *bounded on the left* (*right*), or *from below* (*above*), if there exists a number K_1 (K_2) such that for all x of the set, we have $x \geqq K_1$ ($\leqq K_2$). K_1 is called a *lower bound*, K_2 an *upper bound*, of the set. The former may be replaced by any smaller (but generally not by a larger) number, the latter by any larger (but generally not by a smaller) number. It is now a fact of fundamental importance, that of all lower bounds there is always a greatest, and of all upper bounds there is always a least. Thus, if the real set $\mathfrak{M}$ is bounded on the left (but is not empty), then invariably there exists precisely one real number γ with the following two properties:

(a) To the left of γ, there is no point of the set; briefly: *there is*

$$\textit{no} \quad x < \gamma.$$

(b) To the left of every number which is $> \gamma$, there is at least one point of the set; in other words: *for every* $\varepsilon > 0$ *there is*

$$\textit{at least one} \quad x < \gamma + \varepsilon.$$

This number γ which is uniquely determined by $\mathfrak{M}$ is called the *greatest lower bound* (in Latin: *finis inferior*) of $\mathfrak{M}$, and is denoted by

$$\text{g.l.b. } \mathfrak{M}, \quad \text{fin inf } \mathfrak{M}, \quad \underline{\text{fin}}\ \mathfrak{M}, \quad \text{or} \quad \inf \mathfrak{M},$$

and the analogue holds for every (nonempty) real set bounded on the right, whose *least upper bound* γ' is then denoted by

$$\text{l.u.b. } \mathfrak{M}, \quad \text{fin sup } \mathfrak{M}, \quad \overline{\text{fin}}\ \mathfrak{M}, \quad \text{or} \quad \sup \mathfrak{M}.$$

We shall prove this important

Theorem 1. *Every real set $\mathfrak{M}$ which is not empty and is bounded on the left (right) possesses a well-determined greatest lower (least upper) bound.*

The very simple proof is based on the fundamental definition of real numbers by means of the Dedekind cut. We divide the totality of real numbers into two classes, $\mathfrak{A}$, $\mathfrak{A}'$. Into the class $\mathfrak{A}$ we put all real numbers a for which no $x < a$. Into the class $\mathfrak{A}'$, however, we put every real number a' for which at least one $x < a'$.[1] By hypothesis,

[1] "No x", of course, means: no $x \in \mathfrak{M}$; likewise, "at least one x".

neither one of the two classes is empty. We have invariably $a < a'$, because otherwise there would exist an $x < a$. If γ is the real number which is realized by this cut, then γ possesses the two properties (a) and (b), and is therefore the greatest lower bound of $\mathfrak{M}$. For if $\xi < \gamma$, then ξ is also less than every number which lies between ξ and γ. Such a number, however, being $< \gamma$, is a number a in $\mathfrak{A}$. Since $\xi < a$, ξ cannot be an element of $\mathfrak{M}$. Hence, there is no $x < \gamma$. On the other hand, if $\varepsilon > 0$, then $\gamma + \varepsilon$ belongs to $\mathfrak{A}'$, and consequently there is at least one $x < \gamma + \varepsilon$, Q.E.D.

The corresponding proof for the least upper bound is left to the reader.

Greatest lower and least upper bounds may be regarded as an extension of the concepts of *minimum* and *maximum* from finite to infinite sets: Among finitely many real numbers, $a_1, a_2, \ldots, a_p$, there is always a least and a greatest value, which is denoted by

$$\min(a_1, a_2, \ldots, a_p), \quad \max(a_1, a_2, \ldots, a_p),$$

respectively. For infinite sets, this need not be the case. The set of positive numbers, *e.g.*, possesses no least element. We do, however, invariably have Theorem 1 which was just proved. We call explicit attention to the following: The greatest lower and least upper bounds of a set need not themselves be points of the set.

If a set is unbounded on the left or on the right, then we also say, respectively, that its greatest lower bound is equal to $-\infty$, its least upper bound is equal to $+\infty$.

A proof quite analogcus to that of Theorem 1 yields the equally fundamental *Bolzano-Weierstrass* theorem:

Theorem 2. *Every real, bounded, infinite set possesses at least one limit point.*[1]

PROOF. We again divide the totality of all real numbers into two classes, $\mathfrak{A}$, $\mathfrak{A}'$. Into the class $\mathfrak{A}$ we put every real number a, to the left of which lie no, or at most finitely many, points of the set:

$$\textit{at most finitely many} \quad x < a, \qquad x \in \mathfrak{M}.$$

[1] In 2.5 this theorem will also be proved for sets of complex numbers.

Into the class $\mathfrak{A}'$ we put every real number a', to the left of which lie infinitely many points of the set:

$$\textit{infinitely many } x < a', \qquad x \in \mathfrak{M}.$$

By virtue of the assumptions, we again have before us a Dedekind cut. Let it determine the real number λ. Then, if $\varepsilon > 0$ is chosen arbitrarily, $\lambda-\varepsilon$ belongs to $\mathfrak{A}$, $\lambda+\varepsilon$ to $\mathfrak{A}'$. Hence, there are at most finitely many $x<\lambda-\varepsilon$, but infinitely many $<\lambda+\varepsilon$. Thus, there are infinitely many x in the ε-neighborhood of λ, and consequently λ is a limit point. This already completes the proof of Theorem 2. The proof, however, shows even more: Since at most a finite number of points of the set lie to the left of $\lambda-\varepsilon$, there is certainly no further limit point there, *i.e.*, λ is the smallest limit point of the set. It is therefore called the *lower limit* (in Latin: *limes inferior*) and denoted by

$$\liminf \mathfrak{M} \qquad \text{or} \qquad \underline{\lim}\ \mathfrak{M}.$$

The upper limit, λ', of $\mathfrak{M}$ ($\limsup \mathfrak{M}$, $\overline{\lim}\ \mathfrak{M}$) is defined analogously.

If a set is not bounded on the left (right), we designate $-\infty$ $(+\infty)$ as its lower (upper) limit. Finally, if a set is bounded on the right but not on the left, and if it has no finite limit point whatsoever (as, say, the set of negative integers), then it is reasonable to call $-\infty$ its lim sup, and in the "mirror image" of this case, $+\infty$ its lim inf.

Henceforth, in addition to the Dedekind cut, we also have at our disposal for the realization (definition, determination) of real numbers the formation of the greatest lower and least upper bound, as well as the formation of the upper and lower limit, of sets.

1.4. Functions of a real and of a complex variable

In the sequel, familiarity with the *concept of a function* of a real variable as well as with that of a function of a complex variable, and with their fundamental (*i.e.*, simplest) properties, must be assumed (*cf. Elem.*, § 31 ff.).

If an investigation takes place in the real domain, we shall ordinarily denote the variable by x, the functional value by $y=f(x)$. If it takes place in the complex domain, we write z and $w=f(z)$.

In either case, there is an underlying point set $\mathfrak{M}$ which serves as the domain of definition, and with every point (x or z) of $\mathfrak{M}$ there is associated in an arbitrary but well-determined manner a new number, y or w, as the functional value. The totality of numbers y or w constitutes the *domain of values* of the function.

In particular, we regard the so-called *elementary functions* as known. More precise statements concerning them will be made in chapter 6.

We also assume that the reader is familiar with the concepts of *continuity* and *differentiability* (in the real and complex domains) as well as their manipulation, in other words, with the rudiments of differential calculus and function theory, the former roughly as far as Taylor's theorem, of the latter, only a part of that which is contained in the author's little volume: *Elements of the Theory of Functions*, New York, 1952.

To enter into further details would lead us beyond the limits of the present volume.

Chapter 2

SEQUENCES AND SERIES[1]

2.1. Arbitrary sequences. Null sequences

Definition 1. *The natural numbers* $0, 1, 2, \ldots$ *ordered according to magnitude form a sequence, the sequence of natural numbers.*[2] *If with every one of these numbers* ν *there is associated in any manner a single definite (complex) number* z_ν, *then these numbers* $z_0, z_1, \ldots, z_\nu, \ldots$ *form an infinite sequence of numbers, briefly, a sequence.* (*See* 1.1,(1) *for notation.*)

2.1.1. Remarks

1. Sometimes it is more convenient to let a sequence begin with z_1 or with z_m, where m may denote a fixed integer $\gtreqless 0$, instead of with the term z_0. The sequence then is $\{z_m, z_{m+1}, \ldots, z_\nu, \ldots\}$. Nevertheless, we shall always designate as the ν^{th} term that one which bears the index ν. Thus, the initial term need not be the "first" term, which does not appear at all in case $m > 1$.

2. Numerous examples will come up in the sequel. Here we mention only the sequence $\{c, c, c, \ldots\}$, all of whose terms have the same value c.

3. By calling the "sequence" infinite, we mean to indicate merely that every term is followed by another one.

4. If all numbers of the sequence are real, we speak of a *real sequence*, and then usually denote it by $\{x_\nu\}$; otherwise, we speak of a *complex sequence* or an *arbitrary sequence*. The points corresponding to

---

[1] We remind the reader expressly of the agreement made in 1.2.1,1.

[2] We usually denote an arbitrary one of them by ν or n—any other letter, however, is admissible.

the terms of the sequence form a *real* or a *complex point set.* We use these point sets to visualize sequences of numbers.

Definition 2. *A sequence* $\{z_\nu\}$ *is said to be* bounded, *if a number* $K > 0$ *exists such that invariably*[1]

$$(1) \qquad |z_\nu| \leqq K.$$

In the real domain we have (*cf.* 1.3) also the following finer distinctions: The real sequence $\{x_\nu\}$ is said to be *bounded on the left* (*right*) if a constant K_1 (K_2) exists such that, for all ν,

$$(2) \qquad x_\nu \geqq K_1, \qquad x_\nu \leqq K_2,$$

respectively. The following definition likewise pertains only to real sequences:

Definition 3. *A real sequence* $\{x_\nu\}$ *is said to be* monotonically increasing (*or simply* increasing, *or* nondecreasing), *if invariably*

$$(3) \qquad x_\nu \leqq x_{\nu+1}.$$

In symbols: $x_\nu \nearrow$. *If invariably* $x_\nu \geqq x_{\nu+1}$, $\{x_\nu\}$ *is said to be* monotonically decreasing (*or simply* decreasing, *or* nonincreasing), *in symbols:* $x_\nu \searrow$. *If we require the sharper relation* $x_\nu < x_{\nu+1}$ ($x_\nu > x_{\nu+1}$) *to hold, then the sequence is said to be* strictly increasing (strictly decreasing).

The most important concept for all that follows is that of a *null sequence:*

Definition 4. *An arbitrary complex sequence* $\{z_\nu\}$ *is called a* null sequence, *if it possesses the following property: If the positive* (*small*) *number* ε *is chosen arbitrarily, it is always possible to associate with it a number* $\mu > 0$ *such that*

$$(4) \qquad |z_\nu| < \varepsilon \quad \text{for all} \quad \nu > \mu.$$

[1] *I.e.*, for every index ν.

2.1.2. Remarks

1. Since $\varepsilon > 0$ may be chosen arbitrarily, and, in particular, to be very small, the essential content of the definition may also be expressed in the following looser but more intuitive form: For all sufficiently high indices, the terms are very small in absolute value, namely $< \varepsilon$, as soon as $\nu > \mu = \mu(\varepsilon)$.

2. The number μ does not have to be an integer. We may, however, assume it to be an integer, if it is advantageous to do so. For in (4), μ may be replaced by any larger number. We shall designate this number μ as the *stage* beyond which $|z_\nu| < \varepsilon$. Definition 4 then reads: Having chosen $\varepsilon > 0$, $|z_\nu| < \varepsilon$ *from a certain stage on.*

3. It is easy to verify that an equivalent definition of a null sequence is obtained if $< \varepsilon$ in (4) is replaced by $\leqq \varepsilon$, or $> \mu$ by $\geqq \mu$, or both.

4. The arbitrarily chosen (small) positive number is usually denoted by ε. Sometimes it is convenient to denote it by $\frac{\varepsilon}{2}$ or ε^2, $\frac{\varepsilon}{K}$ $(K > 0)$, *etc.*

5. The examples of null sequences nearest at hand are the sequences $\{0, 0, \ldots, 0, \ldots\}$, $\left\{1, \frac{1}{2}, \ldots, \frac{1}{\nu}, \ldots\right\}$ and $\{1, a, a^2, \ldots, a^\nu, \ldots\}$,[1] the latter provided that $|a| < 1$. For we have $\left|\frac{1}{\nu}\right| < \varepsilon$ as soon as $\nu > \frac{1}{\varepsilon}$. That also $|a^\nu| < \varepsilon$ after a certain stage, if $|a| < 1$, is not quite so self-evident. It is shown as follows: From $|a| < 1$ we get $|1/a| > 1$. We set $|1/a| = 1 + p$, so that $p > 0$. Then, for $\nu \geqq 1$, $|a^\nu| = \frac{1}{(1+p)^\nu} < \frac{1}{\nu p}$, where the inequality follows from the fact that, according to the binomial theorem, $(1+p)^\nu > \nu p$. Hence, $|a^\nu| < \varepsilon$ as soon as $\nu > 1/(\varepsilon p)$.

6. The examples in **5** show that, in order to prove that $\{z_\nu\}$ is a null sequence, it is necessary to indicate, for an arbitrarily chosen $\varepsilon > 0$, how to obtain the stage $\mu = \mu(\varepsilon)$ beyond which $|z_\nu| < \varepsilon$. Conversely, if a sequence $\{z_\nu\}$ is *assumed* to be a null sequence, this is to

---

[1] In the second example the numbering begins, naturally, with 1, in the third, with 0.

assume that, for every $\varepsilon > 0$, the corresponding stage μ of the kind required in the definition may be regarded as known.

7. To assert that $\{z_\nu\}$ is a null sequence means, if we visualize it according to 1.2, that an arbitrarily chosen ε-neighborhood of the origin contains all points z_ν of the sequence with at most a finite number of exceptions—namely all points whose index ν is greater than a suitable $\mu = \mu(\varepsilon)$.

A large part of all the following proofs will amount to showing that a given sequence, or one appearing in the course of an investigation, is a null sequence. Very often this will be accomplished, as stressed in **6**, by actually specifying the $\mu = \mu(\varepsilon)$ which corresponds to the chosen $\varepsilon > 0$. Very often, however, it will be accomplished by comparing the sequence to be investigated with a known null sequence, or by setting up a suitable relation between the two. The following simple theorems serve as a basis for this.

2.1.3. Theorems

1. *Every null sequence is a bounded sequence.* For—choose $\varepsilon = 1$—we have $|z_\nu| < 1$ for $\nu > \mu$, and hence $|z_\nu| \leqq K = \max(1, |z_0|, \ldots, |z_\mu|)$.

2. *Let $\{z_\nu\}$ be a null sequence. Suppose that for a fixed K the terms of a sequence $\{z'_\nu\}$ under investigation satisfy the condition that, for all ν after a certain stage μ',*

$$|z'_\nu| \leqq K|z_\nu|.$$

Then $\{z'_\nu\}$ is also a null sequence. For we have $|z'_\nu| < \varepsilon$ as soon as $|z_\nu| < \varepsilon/K$.[1]

3. *Let $\{z_\nu\}$ be a null sequence and $\{b_\nu\}$ be a bounded sequence. Then the sequence $\{z'_\nu\}$ with the terms $z'_\nu = b_\nu z_\nu$ is also a null sequence.* Proof according to **2**, where we take K to be a bound for the $|b_\nu|$.

4. *Let $\{x_\nu\}$ be a null sequence with positive (real) terms, and α be an arbitrary positive real number. Then $\{x_\nu^\alpha\}$ is also a null sequence.* For we have $x_\nu^\alpha < \varepsilon$ as soon as $x_\nu < \varepsilon^{1/\alpha}$.

---

[1] This form of the proof, which is generally employed in the sequel, reads in more detail: Let $\varepsilon > 0$ be chosen. By hypothesis, there exists a μ such that $|z_\nu| < \frac{\varepsilon}{K}$ for $\nu > \mu$. We may take this $\mu > \mu'$. Then for these ν we also have $|z'_\nu| \leqq K|z_\nu| < \varepsilon$, *i.e.*, $\{z'_\nu\}$ is a null sequence.

If $\nu_0 < \nu_1 < \ldots < \nu_n < \ldots$ is an arbitrary sequence of natural numbers, and if we set $z_{\nu_n} = z'_n$, then $\{z'_n\}$ is called a *subsequence* of the sequence $\{z_\nu\}$. Concerning subsequences, we have

5. *Let $\{z_\nu\}$ be a null sequence. Then every subsequence $\{z'_n\}$ of $\{z_\nu\}$ is also a null sequence.* For if $|z_\nu| < \varepsilon$ for $\nu > \mu$, then $\{z'_n\} = |z_{\nu_n}| < \varepsilon$ for $n > \mu$, because $\nu_n \geqq n$.

Let $\{z'_\nu\}$ be an (infinite) subsequence of $\{z_\nu\}$. Suppose that the z_ν which do not appear in $\{z'_\nu\}$ likewise form an infinite subsequence $\{z''_\nu\}$. Then we say that $\{z_\nu\}$ is decomposed into the two subsequences $\{z'_\nu\}$ and $\{z''_\nu\}$. Concerning such *decompositions*, we have

6. *If the sequence $\{z_\nu\}$ is decomposed into the sequences $\{z'_\nu\}$ and $\{z''_\nu\}$, and if these two sequences are null sequences, then $\{z_\nu\}$ is also a null sequence.* For if we choose $\varepsilon > 0$, there exist numbers μ' and μ'' such that $|z'_\nu| < \varepsilon$ for $\nu > \mu'$ and $|z''_\nu| < \varepsilon$ for $\nu > \mu''$. If μ is the largest index which the terms z'_ν with $\nu \leqq \mu'$ and the terms z''_ν with $\nu \leqq \mu''$ have in the original sequence, then $|z_\nu| < \varepsilon$ for $\nu > \mu$.

Corollary. *An analogous result holds for a decomposition of $\{z_\nu\}$ into a fixed number, say $p > 2$, of sequences.*

If $\{\nu_0, \nu_1, \ldots, \nu_n, \ldots\}$ is a sequence of natural numbers in which every natural number appears exactly once, then $\{\nu_n\}$ is called a *rearrangement* of the sequence of natural numbers, and, more generally, $\{z'_n\}$, with $z'_n = z_{\nu_n}$, is called a *rearrangement* of the sequence $\{z_\nu\}$. Concerning rearrangements, we have

7. *If $\{z_\nu\}$ is a null sequence, then every one of its rearrangements $\{z'_n\}$ is also a null sequence.* For if we choose $\varepsilon > 0$, there exists an integer μ such that $|z_\nu| < \varepsilon$ for $\nu > \mu$. Let μ' be the largest of the indices which the terms $z_0, z_1, \ldots, z_\mu$ bear in the sequence $\{z'_n\}$. Then $|z'_n| < \varepsilon$ for $n > \mu'$.

8. *Let $\{z'_\nu\}$ and $\{z''_\nu\}$ be two null sequences. Then the sequence $\{z_\nu\}$, with $z_\nu = z'_\nu + z''_\nu$, is also a null sequence.* ("Two null sequences may be added term by term.") For if we choose $\varepsilon > 0$, then $|z'_\nu| < \frac{\varepsilon}{2}$ for $\nu > \mu'$ and $|z''_\nu| < \frac{\varepsilon}{2}$ for $\nu > \mu''$. Hence,

$$|z_\nu| = |z'_\nu + z''_\nu| \leqq |z'_\nu| + |z''_\nu| < \varepsilon \quad \text{for} \quad \nu > \mu = \max(\mu', \mu'').$$

In conjunction with **2**, this yields

9. *Let* $\{z'_\nu\}$ *and* $\{z''_\nu\}$ *be two null sequences, and* c' *and* c'' *be two arbitrary fixed numbers. Then the sequence* $\{z_\nu\}$, *with* $z_\nu = c' z'_\nu + c'' z''_\nu$ *is also a null sequence.*

Corollary. *An analogous result holds for any fixed number, say* p, *of null sequences.* And in particular: *If* $\{z'_\nu\}$ *and* $\{z''_\nu\}$ *are null sequences, then so are* $\{z'_\nu - z''_\nu\}$ *and* $\{z'_\nu\, z''_\nu\}$—the latter according to **1** and **3**.

No general assertion of a similar nature can be made regarding the sequence $\{z''_\nu / z'_\nu\}$, but we add the following useful remark:

10. *If* $\{z_\nu\}$ *is an arbitrary sequence, and if there exists a* $\gamma > 0$ *such that all* $|z_\nu| \geqq \gamma$, *then the sequence* $\{1/z_\nu\}$ *is bounded.* For we have

$$|1/z_\nu| \leqq K = 1/\gamma.$$

Finally, we mention the following theorem for real sequences:

11. *Let* $\{x'_\nu\}$ *and* $\{x''_\nu\}$ *be two real null sequences. Suppose that for a real sequence* $\{x_\nu\}$ *under investigation we have* $x'_\nu \leqq x_\nu \leqq x''_\nu$ *after a certain stage. Then* $\{x_\nu\}$ *is also a null sequence.* For if we choose $\varepsilon > 0$, then, after a certain stage, $-\varepsilon < x'_\nu$ and $x''_\nu < \varepsilon$, and hence also $|x_\nu| < \varepsilon$. [1]

2.1.4. Special null sequences

1. $\left\{\dfrac{1}{\nu}\right\}$, *and* $\{a^\nu\}$ *for* $|a| < 1$. Proof above in 2.1.2,5.

2. *For* $|a| < 1$, $\{\nu a^\nu\}$ *is also a null sequence.* For, as in the proof of **1**, we may assume that $a \neq 0$, so that we may again set $|1/a| = 1 + p$ with $p > 0$. Then, for $\nu > 2$, we have

$$|\nu a^\nu| = \frac{\nu}{(1+p)^\nu} < \frac{2}{(\nu-1)\,p^2} < \frac{4}{\nu p^2},$$

where the last inequality follows from $\nu - 1 > \dfrac{\nu}{2}$. Hence, $|\nu a^\nu| < \varepsilon$ for $\nu > \mu = \max\left(2, \dfrac{4}{\varepsilon p^2}\right)$. [2]

---

[1] In somewhat more detail: If $\varepsilon > 0$ is chosen, then $-\varepsilon < x'_\nu$ for all $\nu > \mu'$, $x''_\nu < \varepsilon$ for all $\nu > \mu''$, hence $|x_\nu| < \varepsilon$ for all $\nu > \mu = \max(\mu', \mu'')$.

[2] Note that if $|a|$ lies close to 1, say $|a| = \frac{100}{101}$, then the terms νa^ν for $\nu = 1, 2, \ldots$ first grow rapidly and become small again only for very large ν. (For what values of ν in this case is $\nu a^\nu < 1/1000$?)

3. *For arbitrary* $\alpha > 0$, $\left\{\frac{1}{\nu^\alpha}\right\}$ *is a null sequence.* Proof according to Theorem 4 and Example 1.

4. *For* $|a| < 1$ *and arbitrary* $\alpha > 0$, $\{\nu^\alpha a^\nu\}$ *is a null sequence.* For if we set $|a|^{1/\alpha} = b$, then $b < 1$, and consequently $\{\nu b^\nu\}$ is a null sequence. Since $|\nu^\alpha a^\nu| = (\nu b^\nu)^\alpha$, the assertion now follows according to Theorem 4.

5. *The sequences* $\left\{\frac{\log \nu}{\nu}\right\}$, $\left\{\frac{\log \nu}{\nu^\alpha}\right\}$, $\left\{\frac{\log^\beta \nu}{\nu^\alpha}\right\}$ *are null sequences for arbitrary positive* α *and* β. The logarithm is supposed to be taken to a certain base $b > 1$.[1]

PROOF. It suffices to show that the second sequence is a null sequence. The choice $\alpha = 1$ then shows that the first is a null sequence, and that the third is, follows from Theorem 4 by writing $\frac{\log^\beta \nu}{\nu^\alpha}$ in the form $\left(\frac{\log \nu}{\nu^{\alpha/\beta}}\right)^\beta$. Now to every $\nu \geqq 2$ there corresponds a natural number k_ν such that $b^{k_\nu} \leqq \nu < b^{k_\nu + 1}$ and consequently

$$\frac{\log \nu}{\nu^\alpha} < \frac{k_\nu + 1}{b^{\alpha k_\nu}} = b^\alpha \frac{k_\nu + 1}{(b^\alpha)^{k_\nu + 1}}.$$

If we set $1/b^\alpha = a$ and $k_\nu + 1 = n$, then

$$\frac{\log \nu}{\nu^\alpha} < b^\alpha \cdot n a^n \qquad \text{and hence } < \varepsilon$$

as soon as $na^n < \varepsilon/b^\alpha$. This, however, is true for all $n > m = m(\varepsilon)$, because $0 < a < 1$ and hence $\{na^n\}$ is a null sequence. Thus, $\frac{\log \nu}{\nu^\alpha} < \varepsilon$ for all ν for which $k_\nu + 1 > m$. This is certainly the case for all $\nu > b^m$, because then $\log \nu > m$ and therefore *a fortiori* $k_\nu + 1 > m$.[2]

6. *Let* $a > 0$. *Then the numbers* $x_\nu = \sqrt[\nu]{a} - 1$ *form a null sequence.*

PROOF. This is trivial for $a = 1$. For $a > 1$, $x_\nu > 0$ and $(1 + x_\nu)^\nu = a$, the binomial expansion shows that $\nu x_\nu < a$, *i.e.*, $x_\nu < a/\nu$. Hence, ac-

---

[1] In this example as well as in some foregoing and in the following ones, some simple properties of the elementary functions are regarded as known.

[2] It is customary to interpret the fact that the last of our three sequences is a null sequence, as follows: Every (fixed and positive) power, however large, of $\log \nu$ increases more slowly than any power, however small (but fixed and positive), of ν itself.—Examples 2 and 4 may be interpreted analogously.

cording to 2.1.3,**2**, $\{x_\nu\}$ is a (positive) null sequence. For $0 < a < 1$ we have $1/a = b > 1$. Consequently $\{\sqrt[\nu]{b}-1\}$ is a null sequence. The sequence $\{\sqrt[\nu]{a}\}$, now, is bounded. Theorem 2.1.3,**3** therefore shows immediately that $\{1-\sqrt[\nu]{a}\}$ is also a positive null sequence, and hence $\{\sqrt[\nu]{a}-1\}$ is a negative null sequence.

7. *Let $\{x_\nu\}$ be a null sequence and $a > 0$. Then $\{a^{x_\nu}-1\}$ is also a null sequence.*

For, according to **6**, m can be chosen so that $a^{1/m}$ and $a^{-1/m}$ lie between $1-\varepsilon$ and $1+\varepsilon$. On the basis of the elementary properties of powers, a^{x_ν} lies between $a^{1/m}$ and $a^{-1/m}$ as soon as $|x_\nu| < 1/m$, which is the case for $\nu > \mu = \mu(m) = \mu(\varepsilon)$. Then, for $\nu > \mu$, a^{x_ν} lies between $1-\varepsilon$ and $1+\varepsilon$. This proves the assertion.

8. *The numbers $\sqrt[\nu]{\nu}-1 = y_\nu$ also form a null sequence.* For, by methods analogous to those employed in Examples 6 and 2, we find that, for $\nu > 2$, invariably $\dfrac{\nu(\nu-1)}{2}\, y_\nu^2 < \nu$, $y_\nu < \dfrac{2}{\nu^{1/2}}$.

9. *Let $\{x_\nu\}$ be a real null sequence whose terms are all > -1. Then $\{\log(1+x_\nu)\}$ is also a null sequence, no matter to what base $b > 1$ the logarithm is taken.*

Proof. Let $\varepsilon > 0$ be given. We set

$$b^\varepsilon - 1 = \varepsilon', \qquad 1 - b^{-\varepsilon} = \varepsilon'', \qquad \text{so that} \qquad \varepsilon' = b^\varepsilon \varepsilon'' > \varepsilon'' > 0.$$

We can therefore determine μ so that $|x_\nu| < \varepsilon''$ for $\nu > \mu$. For these ν we have *a fortiori*

$$-\varepsilon'' < x_\nu < \varepsilon' \qquad \text{or} \qquad b^{-\varepsilon} < 1 + x_\nu < b^\varepsilon, \qquad i.e., \qquad |\log(1+x_\nu)| < \varepsilon,$$

which proves the assertion. Similarly we find:

10. *The sequence $\{(1+x_\nu)^\alpha - 1\}$ (α an arbitrary real number) is a null sequence, if $\{x_\nu\}$ satisfies the same assumptions as in* **9**.

2.2. Sequences and sets of numbers

Sets and sequences of numbers have certain things in common, but also differences, which we should like to point out explicitly.

The numbers z_ν of a given sequence $\{z_\nu\}$ do not form a set of numbers in the sense of 1.3, because we demanded of a set of numbers that every number appear in this set at most once, which need not be true in the case of a sequence of numbers. We may therefore say only: The distinct numbers appearing in an infinite sequence $\{z_\nu\}$ form a set. This set may be infinite, but it may also be finite. For z_ν may have the same value for infinitely many ν. Consequently, instead of speaking of an element of the set, one must speak of a term of the sequence, that is to say, of a number ν, for which z_ν possesses this or that property. The terms of a sequence, moreover, are ordered in a definite way, every term z_ν is followed by a completely definite term $z_{\nu+1}$, whereas in general no order is specified for the numbers (points) of a set.

On the other hand, a set of numbers may contain very many more numbers, in a quite definite sense (see **6** below), than a sequence of numbers. These facts necessitate certain modifications of the definitions of limit point, greatest lower (least upper) bound, lower (upper) limit, *etc.* given in 1.3, which we shall discuss briefly.

1. Let $\{z_\nu\}$ be an arbitrary sequence of numbers. We say that ζ is a *limit point* or *limiting value* of this sequence, if, after choosing $\varepsilon > 0$, the inequality $|z_\nu - \zeta| < \varepsilon$ is satisfied for infinitely many indices. Thus, *e.g.*, the sequence $\{\zeta, \zeta, \zeta, \ldots\}$ has the value ζ (and only this one) as a limit point, whereas the point set consisting of the single point ζ has no limit point at all.

2. If $\{z_\nu\}$ is not bounded, we say that the terms z_ν cluster at "infinity". The totality of complex numbers z for which $|z| > K$ (where $K > 0$ is fixed) is said to be a neighborhood of the "point at infinity" or of the "point ∞". Of each individual complex number we often say expressly that it is "finite", in order to indicate that it is different from this so-called point "at infinity".

3. (*Greatest lower bound and least upper bound.*) If $\{x_\nu\}$ is a real sequence which is bounded on the left, then there always exists precisely one number γ possessing the following two properties:

(a) $x_\nu < \gamma$ for no ν,

(b) to every $\varepsilon > 0$ there corresponds at least one ν for which

$$x_\nu < \gamma + \varepsilon.$$

We write $\gamma = \text{g.l.b. } x_\nu$ or $\gamma = \underline{\text{fin}}\, x_\nu$, and define the least upper bound γ' in an analogous fashion. The proof is the same, except for minor changes, as that of Theorem 1 in 1.3.

4. *(Lower limit and upper limit.)* If $\{x_\nu\}$ is a real sequence which is bounded on the left, then there always exists one number λ with the following two properties: If $\varepsilon > 0$ is chosen arbitrarily, there are

(a) at most finitely many ν for which $x_\nu < \lambda - \varepsilon$, but

(b) infinitely many ν for which $x_\nu < \lambda + \varepsilon$.

We write $\lambda = \underline{\lim}\, x_\nu$. The value $\lambda' = \overline{\lim}\, x_\nu$ is defined analogously for real sequences bounded on the right.

5. We agree to set one of the numbers $\gamma, \gamma', \lambda, \lambda'$, in the definitions in **3** and **4**, equal to $-\infty$ or $+\infty$ in the cases which correspond exactly, in 1.3, to those for arbitrary sets.

6. Whereas to every sequence of numbers there corresponds a set, the converse does not hold. *E.g.*, there is no sequence which contains all the real (or indeed all the complex) numbers.[1] If, however, an (infinite) set possesses the property that its elements can be designated (enumerated, numbered, ordered) as $z_0, z_1, \ldots, z_\nu, \ldots$ in such a manner that every point of the set receives a number as its index, then the set is said to be *enumerable*, otherwise it is *nonenumerable*. We say that a nonenumerable set, *e.g.*, the set of all real or the set of all complex numbers, is of greater *power* than the set of natural numbers or any enumerable set. In this well-defined sense, enumerable sets are poorer in elements than nonenumerable ones.

7. The set of (real) rational numbers also appears at first glance to be very much richer in elements than the set or sequence of natural numbers $0, 1, 2, \ldots$. Nevertheless, the former too is enumerable. To see this, first write down successively the distinct positive rational numbers r in $0 < r < 1$ in order of increasing denominators: $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{5}$, $\frac{2}{5}$, ...; then, after every value r, insert the values $\frac{1}{r}$, $-r$, $-\frac{1}{r}$; and finally, put 0 and 1 at the beginning. The resulting sequence

$$0,\ 1,\ \tfrac{1}{2},\ 2,\ -\tfrac{1}{2},\ -2,\ \tfrac{1}{3},\ \ldots$$

---

[1] Since we shall have no occasion to make use of this result in the sequel, we omit the proof, which may be found in any work containing the rudiments of the theory of point sets, *e.g.*, E. Kamke, *Theory of Sets*, New York, 1950.

then contains every real rational number, and each precisely once: The set of rational numbers is enumerable.

8. The following very similar consideration is somewhat more general: For every $n=0, 1, 2, \ldots$, suppose that a sequence $\{a_{n0}, a_{n1}, \ldots, a_{n\nu}, \ldots\}$ is given. If we write down the elements of these sequences in rows, one below another, as in a determinant, we call the resulting configuration

$$\begin{pmatrix} a_{00}, & a_{01}, & \ldots, & a_{0\nu}, & \ldots \\ a_{10}, & a_{11}, & \ldots, & a_{1\nu}, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n0}, & a_{n1}, & \ldots, & a_{n\nu}, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

an infinite matrix or a double sequence, and denote it briefly by $(a_{n\nu})$ or $\{a_{n\nu}\}$, $(n, \nu = 0, 1, 2, \ldots)$. The totality of its elements is again enumerable; it is possible to "reorder" it into a simple sequence. There are many ways of doing this. We call special attention to the following:

a) *Arrangement by diagonals.* In this case we write down in succession for $k=0, 1, 2, \ldots$ the $k+1$ elements $a_{k0}, a_{k-1,1}, \ldots, a_{0k}$ which occupy the k^{th} diagonal. The resulting sequence $\{b_p\}$, which begins with

$$a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, a_{30}, \ldots \equiv b_0, b_1, \ldots, b_p, \ldots,$$

obviously contains every $a_{n\nu}$, and each precisely once.

b) *Arrangement by squares.* For $k=0, 1, 2, \ldots$, we write down in succession the $2k+1$ elements $a_{k0}, a_{k1}, \ldots, a_{kk}, a_{k-1,k}, \ldots, a_{0k}$ which are contained in the border of the k^{th} square in the upper left corner of the matrix. We again obtain a sequence $\{b'_p\}$ which contains every $a_{n\nu}$, and each precisely once.

Both methods show that the totality of the $a_{n\nu}$ is enumerable. Every rearrangement of one of the sequences accomplishes the same thing, and, conversely, any two arrangements of all the $a_{n\nu}$ in a simple sequence are merely rearrangements of one another.

2.3. Convergence and divergence

Definition 1. *If $\{z_\nu\}$ is a given sequence of numbers, and if it is related to a certain number z in such a way that $\{z_\nu - z\}$ is a null sequence, then we say* *that the sequence* $\{z_\nu\}$ converges *to* z, *that it is* convergent, *with the* limit z, *or that its terms* tend to *or* approach *the limit* z *as* $\nu \to \infty$, *and we write*

$$z_\nu \to z \quad \textit{as} \quad \nu \to \infty, \qquad \lim z_\nu = z, \qquad \lim_{\nu \to \infty} z_\nu = z.$$

According to the definition of a null sequence, $z_\nu \to z$ if, and only if, with every $\varepsilon > 0$ there can be associated a $\mu = \mu(\varepsilon)$ such that

$$|z_\nu - z| < \varepsilon \qquad \text{for every} \qquad \nu > \mu.$$

2.3.1. Remarks and examples

1. According to this definition, null sequences are sequences which converge to 0. Henceforth, therefore, we may express the fact that $\{z_\nu\}$ is a null sequence by writing

$$z_\nu \to 0 \ (\text{as } \nu \to \infty), \qquad \lim\ z_\nu = 0, \qquad \lim_{\nu \to \infty}\ z_\nu = 0.$$

2. Consequently, with suitable interpretation, the remarks made in connection with null sequences (2.1.2) remain valid for convergent sequences.

3. Examples 6 and 8 in 2.1.4 assert, respectively, that

$$\sqrt[\nu]{a} \to 1 \quad (\text{for fixed } a > 0), \qquad \sqrt[\nu]{\nu} \to 1.$$

4. The meaning of $z_\nu \to z$ for the corresponding sequence of points is the following: Given $\varepsilon > 0$, all points z_ν, with at most a finite number of exceptions, lie in the ε-neighborhood of z. In the real domain, $x_\nu \to x$ means that the points x_ν, with at most a finite number of exceptions, lie between $x - \varepsilon$ and $x + \varepsilon$.

5. With respect to the limit z approached by the sequence $\{z_\nu\}$, the term z_ν is regarded as the ν^{th} *approximation*, and the difference $z - z_\nu$, which has to be added to z_ν in order to obtain the limit itself, as the ν^{th} *error*. This value, which is also called the ν^{th} *remainder*, will usually be denoted by r_ν, so that $z_\nu + r_\nu = z$.

6. If $\{x_\nu\}$ is a real sequence which converges to x, and if the sequence is monotonic, then we write more expressively $x_\nu \nearrow x$ or $x_\nu \searrow x$ according as the sequence tends increasingly or decreasingly to x.

Definition 2. *Every sequence* $\{z_\nu\}$ *which does not tend to a definite limit in the sense of Definition* 1, *is said to be* divergent.

Sometimes it is desirable to make finer distinctions among divergent sequences:

a) *In the complex domain.* $\{z_\nu\}$ is called boundedly or unboundedly divergent, according as the divergent sequence $\{z_\nu\}$ is bounded or not.

b) *In the real domain.* We say that $x_\nu \to +\infty$, if, given an arbitrary (large) positive number G, it is possible to determine a $\mu = \mu(G)$ such that $x_\nu > G$ for $\nu > \mu$.[1] Similarly: $x_\nu \to -\infty$, if, given $G > 0$, we have invariably $x_\nu < -G$ for $\nu > \mu$. In these two cases, the sequence in question is called *definitely divergent.*[2] If, at the same time, the sequence is monotonic, then we write more clearly $x_\nu \nearrow +\infty$, $x_\nu \searrow -\infty$, respectively. In all other cases, $\{x_\nu\}$ is said to be *indefinitely divergent* or *oscillating.*

Definition 3. *Let* $\{z_\nu\}$ *and* $\{z'_\nu\}$ *be two sequences of numbers, and suppose that the terms of the first are different from* 0. *If these sequences are related so that*

(1) *the sequence* $\{z'_\nu/z_\nu\}$ *tends to* 0, *then we say that* $\{z'_\nu\}$ *is of lower order than* $\{z_\nu\}$, *and we write*

$$z'_\nu = o(z_\nu);$$

(2) *the sequence* $\{z'_\nu/z_\nu\}$ *is bounded, we say that* $\{z'_\nu\}$ *is (at most) of the same order as* $\{z_\nu\}$, *and we write*

$$z'_\nu = O(z_\nu);$$

(3) *the sequence* $\{z'_\nu/z_\nu\}$ *converges, say*

$$z'_\nu/z_\nu \to g,$$

and if $g \neq 0$,[3] *then we say that the sequence* $\{z'_\nu\}$ *is asymptotically proportional*

[1] "All terms x_ν with sufficiently large index ν are very large", and indeed, x_ν itself, not merely $|x_\nu|$.

[2] Definite divergence is still closely related to convergence. *E.g.*, if $x_\nu > 0$ and $x_\nu \to +\infty$, it follows that $1/x_\nu = x'_\nu \to 0$. For we have $x'_\nu < \varepsilon$ as soon as $x_\nu > 1/\varepsilon$, *i.e.*, for all $\nu > \mu = \mu(1/\varepsilon)$. We therefore say, in case $x_\nu \to +\infty$, that x_ν tends to the *improper limit* $+\infty$, and $x_\nu \to -\infty$ is described analogously.

[3] We assume, of course, that g is a finite number, *i.e.*, that g is different from the point at infinity (see above).

to the sequence $\{z_\nu\}$, *and we write*

$$z'_\nu \sim z_\nu.$$

In particular, if $g=1$, *we write*

$$z'_\nu \cong z_\nu,$$

and say that the two sequences are asymptotically equal.[1]

For the purpose of illustration, we mention, without further explanation,[2] the following

2.3.2. Examples

1) (*In the complex domain.*) $(1+i)^\nu = o(2^\nu)$ and $= O(2^{\nu/2})$. $(3+4i)^\nu = = O(5^\nu)$. $z_\nu = O(1)$ means that the sequence $\{z_\nu\}$ is bounded, $z_\nu = o(1)$ means that $\{z_\nu\}$ is a null sequence. Thus, according to 2.1.4,**1**, $\nu a^\nu = o(1)$ for every fixed a with $|a|<1$. Similarly $a+bz^\nu = O(z^\nu)$ for every fixed z with $|z| \geqq 1$.

2) (*In the real domain.*) $\sqrt{5\nu^2+8\nu} \sim \nu$, $\sqrt{\nu^2+1} \cong \nu$, $\log(5\nu^9+13) \sim \log\nu$ or $= O(\log\nu)$, $\sqrt{\nu+1} \cong \sqrt{\nu}$, $\sqrt{\nu+1} - \sqrt{\nu} \sim \dfrac{1}{\sqrt{\nu}}$ and, more precisely, $\cong \dfrac{1}{2\sqrt{\nu}}$.

2.3.3. The following **theorems** concerning convergent sequences can be read off easily and immediately from the corresponding theorems on null sequences (see 2.1):

1. *A convergent sequence determines its limit uniquely.* For if $z_\nu \to z$, and if $z' \neq z$, then a z_ν which lies in an ε-neighborhood of z cannot always lie at the same time in the ε-neighborhood of z'—certainly not, *e.g.*, if we take ε to be the positive number $\frac{1}{2}|z'-z|$.

2. *A convergent sequence is invariably bounded* (2.1.3,**1**); and if $z_\nu \to z$ and $|z_\nu| \leqq K$, then also $|z| \leqq K$.

3. $z_\nu \to z$ *implies* $|z_\nu| \to |z|$. For, according to 1.2.1,**4**, $\big||z_\nu|-|z|\big| \leqq |z_\nu - z|$.

4. *Let* $z_\nu \to z$. *For a fixed integer* $p \gtreqless 0$, *set* $z_{\nu+p} = z'_\nu$, $\nu = 0, 1, 2, \ldots$.

---

[1] In the literature, an express distinction is not always made between *asymptotically proportional* and *asymptotically equal.*

[2] The proofs follow very easily from the theorems in this and the next section.

Then we have also $z'_\nu \to z$.[1] For if $|z_\nu - z| < \varepsilon$ for $\nu > \mu$, then $|z'_\nu - z| = |z_{\nu+p} - z| < \varepsilon$ for $\nu + p > \mu$ or $\nu > \mu' = \mu - p$.

5. *If* $z_\nu \to z$, *then every subsequence* $\{z'_\nu\}$ *of* $\{z_\nu\}$ *also converges to* z (2.1.3,**5**).

6. *If* $z_\nu \to z$, *and if* $\{z'_\nu\}$ *is a rearrangement of* $\{z_\nu\}$, *then also* $z'_\nu \to z$ (2.1.3,**7**).

7. *Let* $z_\nu \to z$, *and let* $\{k_\nu\}$ *be any sequence of positive integers. Denote by* $\{z'_\nu\}$ *the sequence* $z_0, z_0, \ldots, z_0, z_1, \ldots, z_1, z_2, \ldots$, *where* z_0 *is taken* k_0 *times,* z_1 *is taken* k_1 *times,* $\ldots$, z_ν *is taken* k_ν *times,* $\ldots$ *in succession. Then also* $z'_\nu \to z$.

8. *If the sequence* $\{z_\nu\}$ *is decomposed into the sequences* $\{z'_\nu\}$ *and* $\{z''_\nu\}$, *and if both these sequences* $\to z$, *then also* $z_\nu \to z$. An analogous result holds for a decomposition into $p > 2$ sequences.

9. *Let* $\{x'_\nu\}$ *and* $\{x''_\nu\}$ *be real sequences which converge to the same limit* x. *Suppose that for a sequence* $\{x_\nu\}$ *under investigation we have* $x'_\nu \leqq x_\nu \leqq x''_\nu$ *after a certain stage. Then also* $x_\nu \to x$.

10. *Let* $z'_\nu \to z'$ *and* $z''_\nu \to z''$. *Then, for arbitrary fixed numbers* a *and* b, *we have*

$$az'_\nu + bz''_\nu \to az' + bz'', \qquad z'_\nu z''_\nu \to z'z'', \qquad \frac{z''_\nu}{z'_\nu} \to \frac{z''}{z'},$$

where the last relation holds provided that all $z'_\nu \neq 0$ *and also* $z' \neq 0$. The first relation follows immediately from 2.1.3,**9**. The second follows from

$$z'_\nu z''_\nu - z'z'' = (z'_\nu - z')z''_\nu + z'\,(z''_\nu - z'')$$

and the remark that on the right-hand side, two null sequences are multiplied by bounded factors and then added, which yields a null sequence. Since $\dfrac{z''_\nu}{z'_\nu} = z''_\nu \cdot \dfrac{1}{z'_\nu}$, the third relation follows from the second if we can show that, under our assumptions, $\dfrac{1}{z'_\nu} \to \dfrac{1}{z'}$. This, however, is indeed the case, because $\dfrac{1}{z'_\nu} - \dfrac{1}{z'} = \dfrac{z' - z'_\nu}{z' \cdot z'_\nu}$, and by our hypotheses and 2.1.3,**10**, $\left\{\dfrac{1}{z'\, z'_\nu}\right\}$ is a bounded sequence and $\{z' - z'_\nu\}$ is a null sequence.

11. *Let* $z_\nu \to z$, *and set* $z_\nu = x_\nu + iy_\nu$, $z = x + iy$ (x_ν, y_ν, x, y *real*).

[1] For $p > 0$, the theorem may be interpreted as asserting that finitely many terms may be disregarded in questions of convergence. — The numbers $z_{-1}, z_{-2}, \ldots, z_{-q}$ which appear for $p = -q < 0$ may be set equal to any value, say 0.

Then $x_\nu \to x$, $y_\nu \to y$, *and conversely—i.e., the last two relations imply* $z_\nu \to z$. For we have

$$\left.\begin{matrix}|x_\nu - x|\\|y_\nu - y|\end{matrix}\right\} \leqq |z_\nu - z| \leqq |x_\nu - x| + |y_\nu - y|.$$

By means of this theorem, the problem of the convergence of complex sequences is reduced completely to that of real sequences. Only seldom, however, is anything gained in practice by this reduction.

12. *If the real sequence* $\{x_\nu\}$ *converges to the limit* x, *and if* $a > 0$ *is fixed, then* $a^{x_\nu} \to a^x$ (see 2.1.4.7).

13. *If the real sequence* $\{x_\nu\}$ *converges to the limit* x, *and if all* x_ν *as well as* x *are positive, then* $\log x_\nu \to \log x$ *for any choice of the base* $b > 1$ *of the logarithm.*

2.4. Cauchy's limit theorem and its generalizations

The majority of the theorems in the last section not only affirm that a sequence under investigation is convergent, but also make assertions concerning its limit. Particularly numerous applications in this direction may be made of the following theorem, due to *Cauchy* (1821), and its generalizations:

Theorem 1. *If* $z_\nu \to z$, *then also the sequence of arithmetic means*

$$\frac{z_0 + z_1 + \ldots + z_n}{n+1} = z'_n \to z.$$

Proof. First let $z = 0$. If $\varepsilon > 0$ is given, there exists a μ such that $|z_\nu| < \frac{\varepsilon}{2}$ for $\nu > \mu$. Then, for $n > \mu$, we have $|z'_n| \leqq \frac{|z_0 + \ldots + z_\mu|}{n+1} + \frac{\varepsilon}{2}$. The numerator of the first fraction on the right-hand side is a fixed number, so that this fraction is $< \frac{\varepsilon}{2}$ for $n > m$ $(>\mu)$. But then $|z'_n| < \varepsilon$ for $n > m = m(\varepsilon)$, *i.e.*, $z'_n \to 0$.

For arbitrary z, $\{z_\nu - z\}$ is a null sequence, and therefore, by what we just proved, so is the sequence of numbers $\frac{(z_0 - z) + \ldots + (z_n - z)}{n+1} = z'_n - z$, which means that $z'_n \to z$.

We state without proof the following supplements and applications:

2.4.1. **1.** If $\{x_\nu\}$ is real and $x_\nu \to +\infty$, then also

$$\frac{x_0 + x_1 + \ldots + x_n}{n+1} \to +\infty.$$

2. For an arbitrary real sequence $\{x_\nu\}$,

$$\underline{\lim}\, x_\nu \leqq \overline{\underline{\lim}}\, \frac{x_0 + x_1 + \ldots + x_n}{n+1} \leqq \overline{\lim}\, x_\nu.$$ [1]

3. If $y_\nu > 0$ and $y_\nu \to \eta > 0$, then also the sequence of geometric means

$$\sqrt[n]{y_1\, y_2 \,\ldots\, y_n} \to \eta.$$

4. If $\{c_\nu\}$ is a sequence with positive terms, then

$$\underline{\lim}\, \frac{c_{\nu+1}}{c_\nu} \leqq \overline{\underline{\lim}}\, \sqrt[\nu]{c_\nu} \leqq \overline{\lim}\, \frac{c_{\nu+1}}{c_\nu};$$

in particular, if $\frac{c_{\nu+1}}{c_\nu} \to \gamma > 0$, then $\sqrt[\nu]{c_\nu} \to \gamma$.

Cauchy's limit theorem admits of the following far-reaching and important

2.4.2. Generalizations

1. If $z_\nu \to z$, and if $\{p_\nu\}$ is a sequence of positive numbers such that $p_0 + p_1 + \ldots + p_n = P_n \to +\infty$, then also

$$\frac{p_0 z_0 + p_1 z_1 + \ldots + p_n z_n}{p_0 + p_1 + \ldots + p_n} \to z.$$

(We obtain Theorem 1 if we take all $p_\nu = 1$.)

If we set $p_\nu z_\nu = w_\nu$, then **1** may also be formulated as follows:

2. $\frac{w_\nu}{p_\nu} \to z$ implies $\frac{w_0 + w_1 + \ldots + w_n}{p_0 + p_1 + \ldots + p_n} \to z.$

If we set the numerator of the last fraction $= W_n$ and the denominator $= P_n$, we may also state the theorem as follows:

---

[1] $\overline{\underline{\lim}}$ means that one may take either $\underline{\lim}$ or $\overline{\lim}$.

3. If $P_n \nearrow +\infty$, then

$$\frac{W_n}{P_n} \to z, \text{ provided that } \frac{W_n - W_{n-1}}{P_n - P_{n-1}} \to z.$$[1]

4. For example, if $k > 0$ is an integer, it follows that

$$\lim \frac{1^k + 2^k + \ldots + n^k}{n^{k+1}} = \lim \frac{n^k}{n^{k+1} - (n-1)^{k+1}},$$

provided that the last limit exists. Application of the binomial theorem shows, however, that this limit does exist and is equal to $\frac{1}{k+1}$.

5. The assertions **1, 2, 3** remain valid for complex p_ν provided that they satisfy the following condition: For a fixed $K > 0$ and every ν,

$$|p_0| + |p_1| + \cdots + |p_\nu| \leqq K |p_0 + p_1 + \cdots + p_\nu|,$$

and the sum on the left $\to \infty$ as $\nu \to \infty$.

And this theorem, which contains the preceding ones, is in turn merely a special case of the following one, which was discovered independently in 1911 by *Silverman* and *Toeplitz:*

Theorem 2. *Let* $(a_{n\nu})$ *be a row-finite matrix* (*see* 2.2,**8**), *i.e., for every* $n = 0, 1, 2, \ldots$, *let* $a_{n\nu} = 0$ *for* $\nu > \nu_n$, *where* $\{\nu_n\}$ *denotes an arbitrary sequence of natural numbers.*[2] *Suppose that this matrix satisfies the three conditions*

(N) $\displaystyle\sum_{\nu=0}^{\nu_n} |a_{n\nu}| \leqq M$ *for every* $n = 0, 1, 2, \ldots$,

(R) $\displaystyle A_n = \sum_{\nu=0}^{\nu_n} a_{n\nu} \to 1$ *as* $n \to \infty$,

(C) $a_{n\nu} \to 0$ *as* $n \to \infty$, *for every fixed* $\nu = 0, 1, 2, \ldots$.[3]

---

[1] In this formulation, the theorem bears a close relation to *l'Hospital's rule*, which is ordinarily proved in the differential calculus.

[2] In applications we have very often $\nu_n = n$, and the matrix is then called a *triangular matrix*. In 3.5 the above theorem will also be proved for matrices which are not row-finite.

[3] These three conditions (N), (R), (C) may be remembered as norm-, row-, and column-condition, respectively.

Then $z_\nu \to z$ *invariably implies that also*

$$\sum_{\nu=0}^{\nu_n} a_{n\nu} z_\nu = z'_n \to z.$$

The *proof* differs only slightly from that of Theorem 1. Again let us first assume that $z=0$. Then, given $\varepsilon>0$, there exists a μ such that $|z_\nu|<\varepsilon/(2M)$ for $\nu>\mu$. For $n>\mu$ we have

$$|z'_n| \leqq |a_{n0}z_0+\ldots+a_{n\mu}z_\mu| + \frac{\varepsilon}{2M}\sum_{\nu=\mu+1}^{\nu_n}|a_{n\nu}|.$$

Now because of (C), $a_{n\nu}z_\nu \to 0$ as $n\to\infty$, for fixed ν. Hence, according to 2.1.3,**8**, the first absolute value on the right also $\to 0$ as $n\to\infty$. Consequently, there exists an $m>\mu$ such that this absolute value is $<\frac{\varepsilon}{2}$ for $n>m$. For these n, then, we have $|z'_n|<\varepsilon$, so that $z'_n\to 0$. If $z\neq 0$, then

$$z'_n = \sum_{\nu=0}^{\nu_n} a_{n\nu}(z_\nu - z) + A_n z.$$

By what we have just proved, the value of the sum on the right $\to 0$ as $n\to\infty$. Since $A_n\to 1$, it follows that $z'_n\to z$.

Corollary. *If* (R)$A_n\to 1$ *is replaced by the condition* (R′)$A_n\to A$, *and tho other conditions remain the same, then* $z'_n\to Az$.[1]

Theorem 1 is contained as a special case in Theorem 2 for $\nu_n=n$, $a_{n\nu}=\dfrac{1}{n+1}$, $(0\leqq\nu\leqq n)$. Likewise 2.4.2,**1** for $\nu_n=n$ and

$$a_{n\nu}=\frac{p_\nu}{P_n},\quad (0\leqq\nu\leqq n).$$

Finally, we prove the following theorem, which, in a certain respect, is more general than Theorem 2:

Theorem 3. *Let* $z_\nu\to z$ *and* $w_\nu\to w$. *Suppose that* $(a_{n\nu})$ *is a triangular matrix, and set*

$$\sum_{\nu=0}^{n} a_{n\nu}z_\nu w_{n-\nu} = a_{n0}z_0w_n + a_{n1}z_1w_{n-1}+\ldots+a_{nn}z_nw_0 = z'_n,\quad (n=0,1,2,\ldots).$$

[1] *If* $z=0$, *then the respective conditions* (R), (R′) *may be omitted altogether.*

If the matrix satisfies, in addition to the three conditions (N), (R), (C) *of Theorem* 2 *(with* $\nu_n = n$*), the following condition:*

$$\text{(C}'\text{)}\ a_{n,\,n-\nu} \to 0 \quad \textit{as} \quad n \to \infty, \textit{ for every fixed } \nu = 0, 1, 2, \ldots,$$

then $z_n' \to zw$.

PROOF. If we set $a_{n\nu} w_{n-\nu} = b_{n\nu}$, then $z_n' = \sum_{\nu=0}^{n} b_{n\nu} z_\nu$. The matrix $(b_{n\nu})$ satisfies conditions of the form (N) and (C) of Theorem 2, and condition (R$'$) of the Corollary. For if K is an upper bound for all $|w_\nu|$, then $\sum_{\nu=0}^{n} |b_{n\nu}| \leq K \cdot M$ for all n; $b_{n\nu} \to 0$ as $n \to \infty$ because $a_{n\nu} \to 0$ and $w_{n-\nu}$ remains bounded (as $n \to \infty$, for fixed ν). Finally,

$$\sum_{\nu=0}^{n} b_{n\nu} = \sum_{\nu=0}^{n} a_{n\nu}\, w_{n-\nu} = \sum_{\nu=0}^{n} a_{n,\,n-\nu}\, w_\nu \to w$$

according to Theorem 2 itself, since the matrix $a_{n\nu}' = a_{n,\,n-\nu}$ fulfills the three conditions of that theorem. Hence, by the Corollary of Theorem 2, $z_n' \to wz$.

2.5. The main tests for sequences

The two principal problems in the sequel will be to decide whether or not a given sequence of numbers converges, and, in the first case, to assert something more precise about the limiting value, or to ascertain it. Both questions are very often answered as a consequence of the fact that the sequence under investigation is obtained in a way which enables one to recognize immediately that the sequence converges and what its limit is, *e.g.* on the basis of the rules of operation and the theorems in 2.3, or on the basis of Cauchy's limit theorem and its generalizations in 2.4, each of which asserts that the new sequence, which is the one in question, converges, and that it possesses a definite limit.

The situation, however, is not always so favorable. The questions of the convergence and of the limiting value of a given sequence must frequently be decided solely on the basis of the knowledge of its terms. Whereas there are hardly any general means available for answering the second question (*cf.* especially ch. 7), extensive aid in deciding the convergence question is afforded by the so-called convergence tests, only two of which, to begin with, will be treated here, which are nsually called, on account of their importance, the *main tests*. The first is concerned only with real, monotonic sequences, but is particularly important because of its simplicity and its extensive applications. The second is concerned with arbitrary sequences, and has therefore the greater theoretical significance. Both criteria provide necessary and sufficient conditions, and, accordingly, cannot be improved.

First main test for real, monotonic sequences

A real, monotonic sequence $\{x_\nu\}$ *is convergent if, and only if, it is bounded. If it is not bounded, then it* $\to +\infty$ *if it is increasing, it* $\to -\infty$ *if it is decreasing. Thus a monotonic sequence always behaves definitely* (2.3.1, Def. 2, b)).

PROOF. a) Let $\{x_\nu\}$ be decreasing and bounded, and denote the greatest lower bound by γ. Then if $\varepsilon > 0$, there exists a term, say x_μ, of the sequence, such that $x_\mu < \gamma + \varepsilon$. Consequently $x_\nu < \gamma + \varepsilon$ for all $\nu > \mu$. Since all terms of the sequence are $\geqq \gamma$, we have thus associated with the given $\varepsilon > 0$ a number μ such that $\gamma \leqq x_\nu < \gamma + \varepsilon$, in particular $|x_\nu - \gamma| < \varepsilon$, for $\nu > \mu$; *i.e.*, $x_\nu \to \gamma$, and invariably $x_\nu \geqq \gamma$. If $\{x_\nu\}$ is increasing and bounded, then it can be shown in an analogous manner that the sequence tends to its least upper bound γ', $x_\nu \to \gamma'$, and invariably $x_\nu \leqq \gamma'$.

b) Let $\{x_\nu\}$ be decreasing and unbounded. Then, since all $x_\nu \leqq x_0$, the sequence is unbounded on the left. It is therefore possible to associate with every $G > 0$ a μ such that $x_\mu < -G$ and, due to the fact that the sequence is monotonically decreasing, *a fortiori* $x_\nu < -G$ for all $\nu > \mu$. Hence, $x_\nu \to -\infty$. An analogous argument takes care of the case in which $\{x_\nu\}$ is increasing and unbounded.

This important criterion asserts, in particular, that every monotonic and bounded sequence determines or defines a definite real number —namely, it greatest lower or least upper bound. It therefore may and shall serve, in addition to the Dedekind cut as well as $\underline{\overline{\mathrm{fin}}}$ and $\underline{\overline{\lim}}$, as a new means for determining real numbers. It is especially handy in the following form:

Principle of nested intervals

Let $\{x_\nu\}$ *be a monotonically increasing, and* $\{x'_\nu\}$ *be a monotonically decreasing, (real) sequence. Suppose that* $x_\nu \leqq x'_\nu$ *for every* ν, *and that the differences* $d_\nu = x'_\nu - x_\nu \to 0$. *Then there always exists precisely one real number* x *which satisfies the condition* $x_\nu \leqq x \leqq x'_\nu$ *for every* ν. *Expressed graphically: If* $\{I_\nu\}$ *is a sequence of closed, nested intervals* $I_\nu = \langle x_\nu, x'_\nu \rangle$ *whose lengths decrease to* 0, *then there is always exactly one point* x *which belongs to every one of these intervals.*

The *proof* follows immediately from the first main test. For according to this test, $\{x_\nu\}$ is convergent. If $x_\nu \to x$, then also $x'_\nu = x_\nu + d_\nu \to x$, and according to the preceding proof we have invariably $x_\nu \leqq x \leqq x'_\nu$. If also invariably $x_\nu \leqq x' \leqq x'_\nu$, then, for every ν, $x'_\nu - x_\nu = d_\nu \geqq |x' - x|$. Since $\{d_\nu\}$ is a null sequence, we must have $x' = x$: there is only one point which belongs to all the intervals.

In applications, $I_{\nu+1}$ will usually be a certain one of the two halves of I_ν. In this case we speak of the *bisection method* of determining a real number.

The analogue in the complex domain is the *principle of nested squares*, which it will suffice to state and prove in graphical form: If $\{S_\nu\}$ is a sequence of nested, closed squares, whose sides we shall assume to be parallel to the coordinate axes in the plane, and if their diagonals $d_\nu \to 0$ as $\nu \to \infty$, then there exists precisely one point z which belongs to every S_ν.

PROOF. If we project the S_ν on the real axis, we obtain there a real nest of intervals which determines the point x, say. Similarly, by projecting the S_ν on the imaginary axis, we again obtain a real nest of intervals—let it determine the point y there. Then the point $z = x + iy$, and only this one, belongs to every one of the squares.

To the bisection method in the real domain corresponds the *quadrisection method* in the complex domain. By means of parallels to its sides, S_ν is divided into four congruent parts, and $S_{\nu+1}$ is a certain one of them.

Examples.

1. By expanding by the binomial theorem, it is easy to see (*cf.* 6.3,**8**, where the expansion is carried out) that the sequence of numbers $x_\nu = \left(1+\frac{1}{\nu}\right)^\nu$, $\nu = 1, 2, \ldots$, increases monotonically, but that invariably $x_\nu < 3$, so that $\lim \left(1+\frac{1}{\nu}\right)^\nu$ exists and is equal to a number which is > 2 but $\leqq 3$. This number is denoted by e. For further details, see 6.3.

2. The sequence of numbers $x'_\nu = \left(1+\frac{1}{\nu}\right)^{\nu+1}$ is monotonically decreasing. For $x'_\nu < x'_{\nu-1}$ is easily seen to be the same as $\left(1+\frac{1}{\nu^2-1}\right)^\nu >$ $> 1+\frac{1}{\nu}$; and this is true because the value on the left is, by the binomial theorem, even $> 1+\frac{\nu}{\nu^2-1}$. And as $\{x'_\nu\}$ is evidently bounded, it is convergent. It also $\to e$, because $x'_\nu = \left(1+\frac{1}{\nu}\right)x_\nu$. Hence, invariably $x_\nu < e < x'_\nu$.

3. The sequence of numbers $x_\nu = \left(1+\frac{1}{2^2}+\ldots+\frac{1}{\nu^2}\right)$ is obviously monotonically increasing. For $\nu > 1$, however, $x_\nu < 1+\frac{1}{1\cdot 2}+\ldots+$ $+\frac{1}{(\nu-1)\cdot\nu} = 1+\left(1-\frac{1}{2}\right)+\ldots+\left(\frac{1}{\nu-1}-\frac{1}{\nu}\right) = 2-\frac{1}{\nu}$. Thus the sequence $\{x_\nu\}$ is also bounded, and therefore convergent. Later on we shall find the value of its limit to be $\frac{\pi^2}{6}$.

4. The sequence of numbers $x_\nu = \left(\frac{1}{\nu+1}+\frac{1}{\nu+2}+\ldots+\frac{1}{2\nu}\right)$, $\nu = 1, 2, \ldots$, is also increasing and bounded—the latter because $x_\nu < \frac{\nu}{\nu} = 1$, the former because an easy calculation shows that $x_{\nu+1} - x_\nu > 0$. The sequence is therefore convergent. In **7** its limit will be shown to be the natural logarithm of 2.

5. The sums $\left(1+\frac{1}{2}+\ldots+\frac{1}{\nu}\right)$, $\nu = 1, 2, \ldots$, will be denoted (*cf.* 2.5.1,**3**) by h_ν. They obviously form a monotonically increasing sequence. An easy calculation, which is carried out in 2.5.1,**3**, shows that it is unbounded. It therefore $\to +\infty$.

6. If log denotes—as generally in the sequel—the natural logarithm, *i.e.*, the logarithm to the base e (see above), then Examples 1 and 2 assert that invariably

$$\frac{1}{\nu+1} < \log\left(1+\frac{1}{\nu}\right) < \frac{1}{\nu} \tag{1}$$

or

$$0 < \frac{1}{\nu} - \log\left(1+\frac{1}{\nu}\right) < \frac{1}{\nu} - \frac{1}{\nu+1}. \tag{2}$$

If we write this down for $\nu = 1, 2, \ldots, n-1$ and add, it follows that for $n > 1$ invariably

$$0 < h_{n-1} - \log n < 1 - \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < h_n - \log n < 1. \tag{3}$$

From this we see that

$$0 < \frac{h_n}{\log n} - 1 < \frac{1}{\log n}, \qquad (n > 1).$$

Hence, $h_n/\log n \to 1$, because $1/\log n \to 0$. Consequently,

$$h_n \cong \log n. \tag{4}$$

The (according to (3), positive) sequence of differences $d_n = h_n - \log n$, however, is monotonically decreasing, because $d_n - d_{n+1} = \log \frac{n+1}{n} - \frac{1}{n+1}$, which, by (1), is >0. Therefore

$$\lim_{n\to\infty} (h_n - \log n) = C \tag{5}$$

exists, and for the limit C we have $0 \leq C < 1$.[1] Later on we shall see that $C = 0.577\ldots$.

[1] Since the sequence of differences $d'_n = h_{n-1} - \log n$, as is equally easy to show, is monotonically increasing and of course also $\to C$, it follows, more precisely, that actually $0 < C < 1$.

7. If we write (5) in the form $h_n = \log n + C + o(1)$,[1] we see that the terms of the sequence considered in **4** are equal to

$$h_{2n} - h_n = \log(2n) + C + o(1) - \log n - C - o(1) = \log 2 + o(1),$$

which means that that sequence $\to \log 2$.

The principle of nested squares now enables us to prove the *Bolzano-Weierstrass* theorem (*cf.* 1.3) also for sequences of complex numbers:

Bolzano-Weierstrass Theorem. *Every bounded sequence* $\{z_\nu\}$ *possesses at least one limit point* z.

Proof. Since $\{z_\nu\}$ is bounded, it is possible to assign a square S_0 (whose sides are parallel to the coordinate axes) containing all the points of the sequence, and hence, in any case, infinitely many.[2] Consequently, of the four subsquares resulting from the quadrisection method, there must be at least one, which we shall call S_1, which in turn contains infinitely many of the z_ν. If there are several of the subsquares to choose from, call that one S_1 which, in the usual numbering of the four quadrants, is the first one of them (and similarly in the succeeding steps). Analogously, denote a definite one of the four subsquares of S_1 by S_2, *etc.* Then $\{S_\nu\}$ is obviously a sequence of nested squares, and each of the squares contains infinitely many terms of our sequence. If z is the innermost point of $\{S_\nu\}$, then z is a limit point of $\{z_\nu\}$. For let $\varepsilon > 0$ be chosen arbitrarily, and then determine p so that S_p has a diagonal $< \varepsilon$. The entire square S_p lies in the ε-neighborhood of z, and since infinitely many z_ν lie in S_p, the same is true in the ε-neighborhood of z, *i.e.*, z is a limit point of $\{z_\nu\}$, and the theorem is proved.

The Bolzano-Weierstrass theorem is the mainstay of the proof of the second main test, which was first formulated by *Cauchy* in 1821:

---

[1] According to 2.3.1, Definition 3, "$+o(1)$" means that we have to add here the terms of a null sequence (which is not known more precisely). Similarly, in the lines that follow, $+o(1)-o(1)$ means the difference of the terms of two null sequences that are not known more precisely, in any case, however, the term $o(1)$ of a null sequence.

[2] Here and in what follows, "infinitely many" means of course that there exist infinitely many ν for which z_ν lies in that square.

Second main test (*for arbitrary sequences*).

A sequence $\{z_\nu\}$ *is convergent if, and only if, with every* $\varepsilon > 0$ *it is possible to associate a* $\mu = \mu(\varepsilon)$ *such that*

(6) $|z_{\nu'} - z_\nu| < \varepsilon$ *for all pairs of indices* ν, ν' *which are* $> \mu$. [1]

PROOF. a) That (6) is necessary for the convergence of the sequence can be seen at once. For $z_\nu \to z$ implies that, having chosen $\varepsilon > 0$, there exists a number μ such that $|z_\nu - z| < \frac{\varepsilon}{2}$ for all $\nu > \mu$. Hence, if also $\nu' > \mu$, then also $|z_{\nu'} - z| < \frac{\varepsilon}{2}$, and consequently

$$|z_{\nu'} - z_\nu| \leqq |z_{\nu'} - z| + |z_\nu - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

b) In order to show that (6) is also sufficient for the convergence of $\{z_\nu\}$, we first prove that this sequence is bounded if (6) holds. According to (6), $|z_\nu - z_{\mu+1}| < \varepsilon$ and therefore $|z_\nu| < |z_{\mu+1}| + \varepsilon$ for $\nu > \mu$. Hence, for all ν,

$$|z_\nu| \leqq K = \max\,(|z_0|, \ldots, |z_\mu|, |z_{\mu+1}| + \varepsilon).$$

As a bounded sequence, however, $\{z_\nu\}$ possesses at least one limit point z.[2] It is now easy to see that our sequence converges to this value. For to an arbitrary $\varepsilon > 0$ there corresponds a μ such that for all ν, $\nu' > \mu$ we have invariably $|z_{\nu'} - z_\nu| < \frac{\varepsilon}{2}$. Since z is limit point, however, we may choose ν' so that also $|z_{\nu'} - z| < \frac{\varepsilon}{2}$ and hence $|z_\nu - z| \leqq$ $\leqq |z_{\nu'} - z| + |z_{\nu'} - z_\nu| < \varepsilon$, and this for all $\nu > \mu$. Thus $z_\nu \to z$, so that condition (6) is sufficient for the convergence of the sequence $\{z_\nu\}$.

Corollaries. 1. On the basis of what we have proved, the test may be formulated also as follows: A sequence $\{z_\nu\}$ is convergent if, and only if, it is bounded and possesses precisely one limit point, which is then the limit of the sequence.—Since, in the real domain, the

[1] In graphical terms, (6) asserts that all z_ν whose indices are sufficiently large lie very close to one another. For, the distance between z_ν and $z_{\nu'}$ is $< \varepsilon$ for all the pairs of indices mentioned above.

[2] In this step we thus appeal to the Bolzano-Weierstrass theorem, and hence to nested squares or intervals, and thereby to the creation of the system of real numbers.

extreme limit points of a sequence $\{x_\nu\}$ are furnished by $\underline{\lim}\, x_\nu$ and $\overline{\lim}\, x_\nu$, the test in this case can be given the following form: A real sequence $\{x_\nu\}$ is convergent if, and only if, $\underline{\lim}\, x_\nu = \overline{\lim}\, x_\nu$ (and both are finite).

2. Condition (6) in the second main test may obviously be expressed in the following equivalent forms:

(a) Having chosen $\varepsilon > 0$, there exists a μ such that

$$|z_{\nu+\lambda} - z_\nu| < \varepsilon \quad \text{for all } \nu > \mu \quad \text{and every } \lambda = 0, 1, 2, \ldots.$$

(b) If $\{\lambda_\nu\}$ is any sequence of natural numbers, then invariably

$$(z_{\nu+\lambda_\nu} - z_\nu) \to 0.$$

Example. Let z_0 and z_1 be any two points. For $\nu \geqq 2$, set $z_\nu = \frac{1}{2}(z_{\nu-1} + z_{\nu-2})$, *i.e.*, = the midpoint of the segment extending from $z_{\nu-2}$ to $z_{\nu-1}$. Then the sequence $\{z_\nu\}$ is convergent. For if $d = |z_1 - z_0|$ is the distance between the initial terms, then it is easy to verify by induction that the distance $|z_{\mu+1} - z_\mu| = d/2^\mu$, and that all z_ν, with $\nu > \mu + 1$, lie on the segment extending from z_μ to $z_{\mu+1}$. They are thus separated from one another by less than ε, if μ is chosen in accordance with $d/2^\mu < \varepsilon$. The limit of the sequence is easily found to be the value $\frac{1}{3}(z_0 + 2z_1)$. Additional examples occur frequently in the sequel.

2.6. Infinite series

As already emphasized introductorily in 1.1, sequences are very often given in the form 1.1 (3):

$$(1) \qquad a_0 + a_1 + \ldots + a_\nu + \ldots \quad \text{or} \quad \sum_{\nu=0}^{\infty} a_\nu.$$

If there is no possibility of uncertainty, this may be denoted more briefly by $\sum\limits_\nu a_\nu$ or $\sum a_\nu$. These expressions are thus merely other symbols for the sequence $\{s_\nu\}$ with $s_\nu = a_0 + a_1 + \ldots + a_\nu$. Every property for which we have introduced a special name in the case of sequences is carried over to series: A series is thus called convergent, divergent (definitely or indefinitely), *etc.*, according as the sequence $\{s_\nu\}$ of its partial sums possesses this property. If it converges to the limit s, we call s the *value* or the *sum* of the series and write

(2) $$\sum_{\nu=0}^{\infty} a_\nu = s.$$ [1]

The quite customary designation "sum" for the value s of a series is nevertheless unfortunate. For s is no sum, but rather the limit of a sequence of sums, namely, the sequence of partial sums of the series. It is also especially misleading because it engenders the belief that one may operate with infinite series exactly as with ordinary sums, *i.e.*, with sums having a definite finite number of terms, such as $a+b+c$ or $c_1+c_2+\ldots+c_p$, (p a fixed natural number >1). In 3.6, in connection with "operating with infinite series", we shall discover in greater detail that this is not the case; here we shall cite merely a particularly crude example: Consider the series

$$\sum_{\nu=0}^{\infty} (-1)^\nu \equiv 1-1+1-1+-\ldots.$$

If we were allowed to "insert parentheses" as in ordinary sums, then its sum would $=(1-1)+(1-1)+\ldots$, and hence certainly equal 0. It would, however, also $= 1-(1-1)-(1-1)-\ldots$, and hence certainly equal 1! The fallacy here will be explained in 3.6. Every case must be carefully tested to determine whether or to what extent the rules valid for operating with "ordinary" sums still hold for infinite series. For some rules this will be the case, for others not.

2.6.1. Examples and remarks

1. $\sum_{\nu=0}^{\infty} \frac{1}{2^\nu}$. Here we have (*cf.* 1.1,(6)) $s_\nu = 2-\frac{1}{2^\nu}$, $|s_\nu-2| = \frac{1}{2^\nu}$, and this, by 2.1.2,**5**, tends to 0, so that $s_\nu \to 2$. Thus, $\sum_{\nu=0}^{\infty} \frac{1}{2^\nu} = 2$.

---

[1] The notation (2) accordingly signifies two things: 1) The sequence of partial sums of (1) is convergent (or $\lim s_\nu$ exists), and 2) $\lim s_\nu = s$. In the case of convergence, the symbol on the left in (2) is frequently used actually as a symbol for the value s.

2. $\sum_{\nu=0}^{\infty} a^\nu \equiv 1+a+a^2+\ldots+a^\nu+\ldots$. We have $s_\nu = 1+a+\ldots+$ $+a^\nu = \frac{1-a^{\nu+1}}{1-a}$ provided that $a \neq 1$. Therefore $s_\nu - \frac{1}{1-a} = \frac{-a}{1-a}\cdot a^\nu$. According to 2.1.2,5, this tends to 0 if $|a|<1$. Hence,

$$\sum_{\nu=0}^{\infty} a^\nu = \frac{1}{1-a} \quad \text{for} \quad |a|<1. \tag{3}$$

We emphasize once more what this means: The sequence of partial sums $\{s_\nu\}$ of the series on the left, that is, the well-determined sequence of numbers $s_\nu = (1+a+\ldots+a^\nu)$, is convergent in the sense of 2.3, provided that $|a|<1$, and its limit, $\lim s_\nu$, has the value $1/(1-a)$. It is thus in this sense (in particular, under the restriction $|a|<1$) that the series 1.1,(4) has the same value as the fraction $\frac{1}{1-a}$ from which it was obtained in 1.1.

At the beginning the reader should clarify in the same way the meaning of every equality similar to (3) (say 4.2,(15)) until he is quite familiar with such assertions and their significance. If a is positive and less than 1, our calculation shows further that, for every $\nu = 0, 1, 2, \ldots$,

$$1+a+\ldots+a^\nu < \frac{1}{1-a}. \tag{4}$$

3. We express the fact that $\sum_{\nu=0}^{\infty} \frac{1}{\nu+1}$ denotes the series $1+\frac{1}{2}+$ $+\ldots+\frac{1}{\nu+1}+\ldots$, in the form

$$\sum_{\nu=0}^{\infty} \frac{1}{\nu+1} \equiv 1+\frac{1}{2}+\ldots+\frac{1}{\nu+1}+\ldots. \tag{5}$$ [1]

In such a case as this it is convenient to write the series also in the form $\sum_{\nu=1}^{\infty} \frac{1}{\nu}$. Exactly the same series is represented by $\sum_{\rho=1}^{\infty} \frac{1}{\rho}$, $1+\frac{1}{2}+\sum_{\lambda=3}^{\infty}\frac{1}{\lambda}$, $\sum_{\mu=3}^{\infty}\frac{1}{\mu-2}$, too. Likewise, in general,

$$\sum_{\nu=0}^{\infty} a_\nu \quad \text{and} \quad a_0+\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_0+a_1+\ldots+a_p+\sum_{q=p+1}^{\infty} a_q.$$

[1] The symbol "$\equiv$" thus means that the expressions on the left and on the right of it are merely different ways of writing the same thing.

The series (5) is definitely divergent and $\to +\infty$. For if $n > 2^k$, (k an integer $\geqq 0$), then

$$s_n = 1+\frac{1}{2}+\ldots+\frac{1}{n} > 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right) + \left(\frac{1}{5}+\ldots+\frac{1}{8}\right)+\ldots+ \\ +\left(\frac{1}{2^{k-1}+1}+\ldots+\frac{1}{2^k}\right).$$

If in every expression in parentheses we replace all the denominators by the greatest one of them, we see that the value of each of these expressions is $>\frac{1}{2}$. Therefore $s_n > \frac{1}{2}k$. Hence, if $G > 0$ is given, and if k is an integer $> 2G$, then $s_n > G$ for all $n > 2^k$. The series just considered is usually designated as the *harmonic series*. We shall therefore denote its partial sums by h_n:

$$1+\frac{1}{2}+\ldots+\frac{1}{n} = h_n. \tag{6}$$

4. $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)} \equiv \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{\nu(\nu+1)} + \ldots.$

In this case

$$s_n = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Hence, $s_n - 1 = -\dfrac{1}{n+1}$ and $\to 0$. Therefore $\displaystyle\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)} = 1.$

5. The partial sums of the series $\sum_{\nu=0}^{\infty} (-1)^\nu$ have alternately the values 1 and 0. The series is therefore indefinitely divergent. The partial sums of the series $\sum_{\nu=0}^{\infty} (-1)^\nu\,(2\nu+1)$ have the values 1, -2, $+3$, -4, ... The first sequence oscillates between finite, the second between infinite, bounds.[1]

We emphasized above that an infinite series is merely another

[1] A sequence $\{s_\nu\}$ oscillates between finite bounds if there exist two bounds, K_1 and K_2, such that invariably $K_1 \leqq s_\nu \leqq K_2$; it oscillates between infinite bounds if, no matter how $K_1 < K_2$ be chosen, there are infinitely many ν for which $s_\nu < K_1$ and infinitely many others for which $s_\nu > K_2$.

symbol for the sequence of its partial sums; conversely, every sequence may also be written in the form of an infinite series, namely, the sequence $\{s_\nu\}$ as the series

$$s_0 + \sum_{\nu=1}^{\infty} (s_\nu - s_{\nu-1}), \qquad \text{or} \qquad \sum_{\nu=0}^{\infty} (s_\nu - s_{\nu-1})$$

if we stipulate that the undefined term s_{-1} shall denote 0: $s_{-1} = 0$. For, the series written down obviously have as partial sums precisely the values s_ν, $(\nu = 0, 1, 2, \ldots)$. [1]

In chapters 3 and 5 we shall treat the convergence questions for infinite series systematically. To illustrate them, suffice it to say the following at this point: The series $\sum \frac{1}{2^\nu}$ proved to be convergent, the series $\sum \frac{1}{\nu}$, however, definitely divergent. Interpreted in schoolboy fashion: If I present someone first a dollar, then a half, then a quarter dollar, *etc.* with the denominators $8, 16, \ldots, 2^\nu, \ldots$, then the recipient will never acquire a full two dollars, no matter how long the presentation is continued. If, however, we present him first 1 dollar, then $\frac{1}{2}$, then $\frac{1}{3}$ of a dollar, *etc.* with the denominators $4, 5, \ldots, \nu, \ldots$, then the wealth of the recipient increases beyond all bounds if we merely continue the presentation long enough. What is the inner reason for this fundamental difference? What is the situation in the case, *e.g.*, of the presentations $1, \frac{1}{2^2}, \ldots, \frac{1}{\nu^2}, \ldots$? That is, is the series $\sum \frac{1}{\nu^2}$ convergent too or not? The purpose of the investigations of the following chapters is to put us in a position to decide for as many series $\sum a_\nu$ as possible whether they converge or not. At the same time, a larger stock of series whose convergence or divergence is known to us will be made available, and a feeling awakened for being able to

---

[1] Similarly, every sequence $\{s_\nu\}$ may be written as an infinite product, provided only that all $s_\nu \neq 0$. For

$$s_0 \cdot \prod_{\nu=1}^{\infty} \frac{s_\nu}{s_{\nu-1}}, \qquad \text{or} \qquad \prod_{\nu=0}^{\infty} \frac{s_\nu}{s_{\nu-1}}, \quad (s_{-1} = 1),$$

has as partial products precisely the numbers s_ν, $(\nu = 0, 1, \ldots)$.

tell whether a given series belongs to the one class or the other. In this direction we shall prove here only the following simple, but nevertheless important, theorems:

2.6.2. Theorem 1. *If Σa_ν is convergent, then the terms a_ν of the series form a null sequence.* In other words: *For the convergence of a series Σa_ν,*

$$a_\nu \to 0 \tag{1}$$

is a necessary condition.

PROOF. By hypothesis, $s_\nu \to s$, and hence, by 2.3.3,**4**, also $s_{\nu-1} \to s$. Therefore, by 2.3.3,**10**,

$$s_\nu - s_{\nu-1} \to s - s, \quad i.e., \quad a_\nu \to 0.$$

We add expressly that this condition $a_\nu \to 0$ for the convergence of Σa_ν is by no means sufficient. For, the series $\Sigma \frac{1}{\nu}$, *e.g.*, proved to be divergent, although $\frac{1}{\nu} \to 0$.[1]

Theorem 2. *The series $\sum\limits_{\nu=0}^{\infty} a_\nu$ and $\sum\limits_{\nu=0}^{\infty} a_{\nu+p}$, ($p \gtreqless 0$, integral, fixed), have the same convergence behavior. If the value of the first $= s$, then that of the second $= s - s_{p-1}$.*[2] For if s_ν are the partial sums of $a_0 + a_1 + \ldots + a_\nu + \ldots$, and s'_ν are those of $a_p + a_{p+1} + \ldots \equiv a'_0 + a'_1 + \ldots$, then

$$s'_\nu = s_{\nu+p} - s_{p-1} \quad \text{or} \quad s_\nu = s'_{\nu-p} + s_{p-1},\text{[3]}$$

from which, according to 2.3.3,**4**, the assertion follows as $\nu \to \infty$.

For $p = n+1$ this theorem asserts that with $\sum\limits_{\nu=0}^{\infty} a_\nu = s$,

$$\sum_{\nu=n+1}^{\infty} a_\nu \equiv a_{n+1} + a_{n+2} + \ldots + a_\nu + \ldots,$$

for every fixed $n = 0, 1, \ldots$, is also a convergent series, which has the

[1] As late as the eighteenth century this condition $a_\nu \to 0$ was rather generally regarded as sufficient for the convergence of Σa_ν.

[2] Again the terms a_ρ and s_ρ are to be set $= 0$ if $\rho < 0$.

[3] Bear in mind the agreement just made regarding a_ρ and s_ρ for negative ρ.

value $s-s_n$. It is called the n^{th} *remainder* of $\sum a_\nu$, and its value is denoted by r_n, so that

$$s-s_n = r_n, \qquad s_n+r_n = s.\,[1]$$

We shall also call r_n the "error" that accrues to the n^{th} partial sum s_n relative to the value of the whole series. Since $s-s_n \to s-s = 0$, we have, as a supplement to Theorem 1,

Theorem 3. *If $\sum a_\nu$ is convergent, then we have in addition to* (1), *that even the sequence of remainders* (*errors*)

$$(2) \qquad r_n \to 0.$$

The next theorem is almost only a special case of Theorem 2:

Theorem 4. *If a is an arbitrary number, then the two series*

$$a_0+a_1+\ldots+a_\nu+\ldots$$

and

$$a+a_0+a_1+\ldots+a_\nu+\ldots \equiv a_0'+a_1'+\ldots+a_\nu'+\ldots$$

are either both convergent or both divergent. In the case of convergence, if the value of the first $= s$, then that of the second $= a+s$. For if s_ν, s_ν' are the respective partial sums of the series, then $s_\nu' = a+s_{\nu-1}$ and conversely $s_\nu = s_{\nu+1}'-a$ for $\nu \geqq 0$. The proof now follows from 2.3.3, theorems 4 and 10.

Repeated application of Theorem 4 yields the following important "*Theorem on finitely many alterations*":

Theorem 5. *If we delete a finite number of terms from the series $\sum a_\nu$ which is assumed to converge to the value s, or if we insert finitely often a finite number of new terms between two successive terms, or if we place a finite number of new terms before the initial term, and denote the resulting altered series by $a_0'+a_1'+\ldots+a_\nu'+\ldots$, then this series is also convergent, and its value s'*

---

[1] Thus r_n is the value of the subseries commencing after the n^{th} term. To be consistent, the whole series $\sum_{\nu=0}^{\infty} a_\nu$ is then denoted by r_{-1}.

Note that one can only speak of a "remainder" in connection with a series whose convergence is assured. For divergent series, the concept of "remainder" is completely meaningless.

is obtained from s exactly as if we were dealing with an ordinary sum (*cf.* 2.6), *i.e.*, s' is obtained from s by subtracting from s the sum of all the deleted terms and adding to s the sum of all the added terms. For, the finitely many alterations described may be arrived at by deleting an existent initial term or prefixing a new initial term in accordance with Theorem 4, and carrying out this operation a finite number of times. [1]

We emphasize expressly that this theorem does not necessarily remain valid if in one place or another the words "finite number" are replaced by "infinite number". The following special case, however, is true:

Theorem 6. *If, in a convergent series* $\sum a_\nu$, *we delete in an arbitrary manner any terms* a_ν *which have the value* 0, *or if we insert finitely or infinitely often between successive terms in each case a finite number of terms every one of which has the value* 0,[2] *then the new series* $\sum a'_\nu$ *is also convergent and both series have the same value.*

We leave the details of the proof, which is based on 2.3.3, theorems 5 and 7, to the reader.

---

[1] We also say: In investigating convergence, finitely many terms do not matter; or: only the "late" or "distant" terms play a role.

[2] Make, *e.g.*, the series $a_0+a_1+a_2+\ldots$ into the series

$$0+0+a_0+0+a_1+0+0+0+a_2+a_3+0+\ldots \equiv a'_0+a'_1+a'_2+\ldots.$$

We then speak of a "dilution of the series".

Chapter 3

THE MAIN TESTS FOR INFINITE SERIES. OPERATING WITH CONVERGENT SERIES

3.1. Series of positive terms: The first main test and the comparison tests of the first and second kind

If the terms a_ν of a series $\sum a_\nu$ are nonnegative ($\geqq 0$), then we speak, for brevity, of a series of positive terms. Its partial sums s_ν form a monotonically increasing sequence. It is therefore convergent if, and only if, it is bounded (on the right):

First main test (for series of positive terms).

A series $\sum a_\nu$ of positive terms is convergent if, and only if, its partial sums are bounded. If, say, invariably $s_\nu \leqq K$ and $\sum a_\nu = s$, then also $s \leqq K$. If the partial sums are unbounded, then the series is definitely divergent to the (improper) value $+\infty$.

PROOF. Apply the first main test in 2.5, whose proof immediately establishes the validity of also the second part of our assertion.

3.1.1. Theorems. From this simple but especially important theorem, we obtain very easily the following theorems, in which, for greater clarity, we shall denote a series whose convergence is assumed, by $\sum c_\nu$, and likewise a series which is assumed to be divergent, by $\sum d_\nu$, but, in either instance, only in the case of positive terms. Similarly, series whose convergence or divergence, respectively, is assumed to be known will be denoted by $\sum c'_\nu$, $\sum d'_\nu$, ... The partial sums of such series will then be denoted by C_ν, D_ν, C'_ν, ..., and the values of $\sum c_\nu$, $\sum c'_\nu$, ... by C, C', ...

1. If $\sum c_\nu$ is convergent, and if the sequence $\{\gamma_\nu\}$ is positive[1] and bounded, then $\sum \gamma_\nu c_\nu$ is also convergent. For if $\bar{C}$ is a bound of the

[1] That is to say, of course, that all its terms are $\geqq 0$.

partial sums of $\sum c_\nu$, and γ a bound of the sequence $\{\gamma_\nu\}$, then obviously $\gamma \cdot \bar{C}$ is a bound for the partial sums of $\sum \gamma_\nu c_\nu$.

2. If $\sum d_\nu$ is divergent, and if the sequence $\{\delta_\nu\}$ has a positive lower bound δ: $\delta_\nu \geqq \delta > 0$, then $\sum \delta_\nu d_\nu$ is also divergent. For, its partial sums are $> G$ as soon as the partial sums of $\sum d_\nu$ are greater than G/δ.[1]

3. If $\sum c'_\nu$ is a "subseries" of $\sum c_\nu$, then $\sum c'_\nu$ is also convergent. Here $\sum c'_\nu$ is called a *subseries* of $\sum c_\nu$, if $\{c'_\nu\}$ is a subsequence of $\{c_\nu\}$. The proof is almost self-evident, because every bound of the C_ν is also a bound of the C'_ν.

The following theorem is a bit deeper:

4. If $\sum c'_\nu$ is a rearrangement of $\sum c_\nu$, then $\sum c'_\nu$ is also convergent and has the same value as $\sum c_\nu$. Here $\sum c'_\nu$ is called a *rearrangement* of $\sum c_\nu$, if $\{c'_\nu\}$ is a rearrangement of $\{c_\nu\}$.

This theorem is not so plausible as the preceding ones. For although the series $\sum c'_\nu$ may still be regarded in a certain sense as "the same series" as $\sum c_\nu$ (because the latter has "only" been rearranged), the sequences $\{C_\nu\}$ and $\{C'_\nu\}$ of partial sums of these series are completely different sequences. The theorem asserts, nevertheless, that the second sequence also converges and has the same limit as the first.

Proof. An arbitrary partial sum C'_ν is certainly $\leqq C_{\nu'}$, if we set ν' equal to the largest of the indices which the terms $c'_0, c'_1, \ldots, c'_\nu$ possessed in the series $\sum c_\nu$. Since the C_ρ increase toward their limit C, we have $C'_\nu \leqq C$, and this holds for every ν. Hence, $\sum c'_\nu$ is convergent, and the value C' of this series is at most $= C$: $C' \leqq C$. Since, however, conversely, $\sum c_\nu$ is a rearrangement of $\sum c'_\nu$, we have also $C \leqq C'$, and hence finally $C' = C$. Analogously, we have

5. If $\sum d'_\nu$ is a rearrangement of $\sum d_\nu$, then the first series is also divergent. This is proved either in a manner akin to the proof of **4**, or by immediately recognizing on the basis of **4** that the assumption that $\sum d'_\nu$ converges is false.

---

[1] This abbreviated mode of expression, which we shall frequently employ in similar cases in the sequel, means more precisely: Having chosen $G > 0$, there exists, by hypothesis, a μ such that $D_\nu > G/\delta$ for all $\nu > \mu$. Then obviously $D'_\nu > G$ for $\nu > \mu$, so that $D'_\nu \to +\infty$.

3.1.2. Examples

1. $\sum a^\nu$, $0<a<1$. For every ν we have (see 2.6.1,**2**)

$$1+a+\ldots+a^\nu<K=\frac{1}{1-a},$$

and hence the series is convergent. We already know this, but the proof here is simpler than that in 2.6.1,**2**. The convergence question for this so-called *geometric series* is now completely settled: For $|a|<1$ it converges and has the value $\frac{1}{1-a}$. For $|a|\geqq 1$ it diverges, because, on account of $|a^\nu|\geqq 1$, its terms do not decrease to 0 (see 2.6.2, Theorem 1).

2. $\sum\limits_{\nu=0}^{\infty}\frac{1}{\nu!}=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots$. In this case, for $\nu\geqq 1$, the ν^{th} partial sum

$$s_\nu\leqq 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{\nu-1}}=3-\frac{1}{2^{\nu-1}}<K=3.$$

Therefore the series is convergent, and its value, which is customarily denoted by e, is $\leqq 3$.[1]

3. The series $\sum\limits_{\nu=1}^{\infty}\frac{1}{\nu^\alpha}$ is designated as the *harmonic series* with the exponent α (*cf.* 2.6.1,**3**). For $\alpha<1$, its partial sums are greater than those of the series $\sum\frac{1}{\nu}$. Since these are not bounded (see 2.6.1,**3**), those of the present series are also unbounded: For $\alpha<1$ the series $\sum\frac{1}{\nu^\alpha}$ is divergent. If $\alpha>1$ and $2^k>n$, then the n^{th} partial sum of the series

$$=\sum_{\nu=1}^{n}\frac{1}{\nu^\alpha}<1+\left(\frac{1}{2^\alpha}+\frac{1}{3^\alpha}\right)+$$

$$+\left(\frac{1}{4^\alpha}+\ldots+\frac{1}{7^\alpha}\right)+\ldots+\left(\frac{1}{(2^k)^\alpha}+\ldots+\frac{1}{(2^{k+1}-1)^\alpha}\right).$$

---

[1] If we replace by 2 the factors of the denominator which are >2, only in the terms from $\frac{1}{4!}$ on, then we find actually that $e\leqq 2+\frac{11}{12}<3$, and this can easily be improved (see 7.2,**1**). The value of the series is the base e of the natural logarithms (see 6.4).

If we now replace (*cf.* 2.6.1,**3**) the natural numbers appearing in the denominators within each pair of parentheses, by the smallest one of them, it follows that our expression is

$$\leqq 1+\frac{1}{2^{\alpha-1}}+\frac{1}{(2^{\alpha-1})^2}+\ldots+\frac{1}{(2^{\alpha-1})^k}=1+a+\ldots+a^k,$$

where, for brevity, we have set $1/2^{\alpha-1}=a$. For $\alpha>1$, we have $0<a<1$. If we put $K=\dfrac{1}{1-a}$, then, according to 2.6.1,(4),

$$s_n \leqq K \qquad \text{for every } n,$$

the partial sums of our series are bounded, the series is convergent. Thus, to sum up: The harmonic series $\sum \frac{1}{\nu^\alpha}$ are divergent for $\alpha\leqq 1$, convergent for $\alpha>1$. One can make a satisfactory assertion regarding the sum of the series in the case of convergence only for integral even exponents α (see 7.3,**3**,(4)). For $\alpha=2$, *e.g.*, the value of the series $=\frac{\pi^2}{6}$ (see 7.3,**3**). For odd integral, and for nonintegral, exponents, no relation is known between the value of the series and any numbers arising in a different connection, such as π, e, or similar numbers.

Simpler and more convenient criteria can be derived from the first main test through the mediation of the following two comparison tests. In the case of these two tests, just as in the case of many that follow, the situation is this: A certain series $\sum a_\nu$ (invariably of positive terms in this section) is to be investigated as to its convergence or divergence. This takes place here by means of a suitable comparison with a series $\sum c_\nu$ (whose convergence thus is already known) or a series $\sum d_\nu$ (which is already known to be divergent), respectively. In the comparison test of the first kind, it is assumed very simply that for all ν, or at least for all ν which are not less than a certain natural number μ, the inequality

$$a_\nu \leqq c_\nu$$

holds.[1] In this case we say for brevity that this inequality is valid

[1] $\sum c_\nu$ is then called a *majorant* of the series $\sum a_\nu$.

"after a certain stage" or "for all sufficiently large ν". This immediately implies the convergence of the series $\sum a_\nu$. For, its partial sums s_ν, if $\mu = 0$, are not greater than the partial sums C_ν of $\sum c_\nu$, and are therefore, simultaneously with them, $\leqq C$. For $\mu \geqq 1$ we have correspondingly $s_\nu \leqq K = C + s_{\mu-1}$.[1]

If, however, we have (in the same sense) after a certain stage

$$a_\nu \geqq d_\nu,$$

then $\sum a_n$ is also divergent, because if this inequality is valid from the beginning on, the partial sums of our series are at least as large as those of $\sum d_\nu$, and hence are unbounded simultaneously with these. This theorem, which we have now explained in precise terms, will, for brevity, just as analogous cases in the sequel, be formulated as follows:

Comparison test of the first kind.

$$(1)\qquad \begin{cases} a_\nu \leqq c_\nu & : \ \mathrm{C} \\ a_\nu \geqq d_\nu & : \ \mathrm{D} \end{cases}$$

or, more generally,

$$(1')\qquad \begin{cases} a_\nu = O(c_\nu), \ \ i.e., \ \ a_\nu \leqq K c_\nu, & (K > 0, \text{ fixed}) : \mathrm{C} \\ a_\nu \geqq \ \delta d_\nu, & (\delta > 0, \text{ fixed}) : \mathrm{D}. \end{cases}$$

The first line of (1), to express it in words once more, means: If the terms a_ν of a series $\sum a_\nu$ under investigation stand in the relation $a_\nu \leqq c_\nu$ to the terms c_ν of an already known convergent series, for all ν from a certain index μ on, then $\sum a_\nu$ is also convergent. The other three lines are to be interpreted analogously.

We have already made *de facto* use of these very simple criteria in connection with the foregoing examples.

---

[1] Since finitely many alterations (see 2.6.2, Theorem 5) play no role in the question as to the convergence of Σa_ν, we may also imagine the terms a_0 to $a_{\mu-1}$ simply as being replaced by 0, or by $c_0, \ldots, c_{\mu-1}$, respectively. Due to this simple artifice, we may, without loss of generality, take $\mu = 0$ in the following proofs.

Comparison test of the second kind.

$$(2)\qquad \begin{cases} \dfrac{a_{\nu+1}}{a_\nu} \leqq \dfrac{c_{\nu+1}}{c_\nu} : \mathrm{C} \\[2ex] \dfrac{a_{\nu+1}}{a_\nu} \geqq \dfrac{d_{\nu+1}}{d_\nu} : \mathrm{D}.\text{[1]} \end{cases}$$

PROOF. We may assume (see footnote, p. 56) that the inequalities hold from $\nu=0$ on. If we write down the first of them for $\nu=0, 1, \ldots, n-1$, where n is understood to be a natural number >1, and multiply them together, we obtain

$$\frac{a_n}{a_0} \leqq \frac{c_n}{c_0} \quad \text{or} \quad a_n \leqq \frac{a_0}{c_0}\, c_n$$

for all $n = 0, 1, 2, \ldots$. Hence, according to the test of the first kind, $\sum a_\nu$ is convergent. Similarly, if the second inequality is fulfilled for all ν, we find that $a_n \geqq \dfrac{a_0}{d_0}\, d_n$ for all $n = 0, 1, 2, \ldots$. Consequently $\sum a_\nu$ is divergent.

From these comparison tests we shall now derive criteria that are more special, by substituting for $\sum c_\nu$ or $\sum d_\nu$ one of the series which we already know to be convergent or divergent, respectively.

3.2. The radical test and the ratio test

If we take the geometric series $\sum a^\nu$, $(0 \leqq a < 1)$, as the comparison series in 3.1,(1), it yields, in the brief formulation discussed above:

$$a_\nu \begin{cases} \leqq a^\nu, & (0 \leqq a < 1) \ : \mathrm{C} \\ \geqq a^\nu, & (a \geqq 1) \qquad : \mathrm{D}. \end{cases}$$

The divergence half of this theorem is trivial, because $a \geqq 1$ implies $a^\nu \geqq 1$, and the terms a_ν, if they are $\geqq a^\nu$, therefore do not form a null sequence. Its first half is equivalent to the following so-called *radical test* (*Cauchy* 1821):

$$(1)\qquad \sqrt[\nu]{a_\nu} \leqq a < 1 : \mathrm{C},$$

[1] Without saying it expressly, it must of course be assumed here that none of the terms a_ν, c_ν, $d_\nu = 0$ — at least "from a certain stage on".

whose more detailed formulation asserts: *If*, from a certain stage on, $\sqrt[\nu]{a_\nu}$ does not exceed a fixed positive number $a<1$, then $\sum a_\nu$ is convergent. We expressly emphasize that it is a *fixed* number a, which is *less* than 1, that is not to be exceeded, because beginners very often overlook this. [1]

Likewise, if we take $c_\nu = a^\nu$, 3.1,(2) yields:

$$\frac{a_{\nu+1}}{a_\nu}\begin{cases}\leqq a<1 : \mathrm{C}\\ \geqq a \geqq 1 : \mathrm{D}.\end{cases}$$

Here again the divergence half is trivial, for it asserts that the sequence $\{a_\nu\}$ of terms of the series increases monotonically, and therefore is certainly no null sequence. The hereby acquired convergence criterion

$$\frac{a_{\nu+1}}{a_\nu} \leqq a < 1 : \mathrm{C} \tag{2}$$

is commonly designated as the *ratio test* (*Cauchy* 1821).

Before applying these tests to given series, we shall make them handier by means of a couple of remarks:

1. It is often not at all easy to decide for a sequence such as $\sqrt[\nu]{a_\nu}$ or $\frac{a_{\nu+1}}{a_\nu}$ whether its terms exceed a fixed proper fraction [2] a from a certain stage on. It is usually easier, however, to ascertain the limit of such a sequence, or, if it has no limit, its principal limits. In terms of these we have: *If*

$$\overline{\lim}\ \sqrt[\nu]{a_\nu}<1, \textit{ in particular if } \lim \sqrt[\nu]{a_\nu} \textit{ exists and is } <1, \tag{3}$$

then $\sum a_\nu$ *is convergent.*

Indeed, if $\overline{\lim}\ \sqrt[\nu]{a_\nu} = \alpha<1$, and if we set, say, $\frac{1+\alpha}{2} = a$, then

[1] The convergence of Σa_ν need not follow from $\sqrt[\nu]{a_\nu}\leqq 1$ or even <1 (for all ν). This is already shown by $a_\nu = \frac{1}{\nu}$, $\nu = 1, 2, \ldots$ (we need not be troubled here by the fact that $\sqrt[\nu]{a_\nu}$ has no meaning for $\nu=0$).

[2] A number a for which $0 \leqq a < 1$ is often called a proper fraction—even if it is not rational.

$0 < a < 1$, and, from a certain stage on, $\sqrt[\nu]{a_\nu} < a$ (*cf.* 2.2,**4**), so that $\sum a_\nu$ is convergent. We call (3) the limit form of the radical test. Analogously we have: *If*

$$\text{(4)} \qquad \overline{\lim}\, \frac{a_{\nu+1}}{a_\nu} < 1, \text{ \textit{in particular if} } \lim \frac{a_{\nu+1}}{a_\nu} \text{ \textit{exists and is} } < 1,$$

then $\sum a_\nu$ *is convergent.* The matter is less simple for the corresponding divergence tests. We have:

$$\text{(5)} \qquad \overline{\lim}\, \sqrt[\nu]{a_\nu} > 1, \text{ (\textit{in particular}, } \lim \sqrt[\nu]{a_\nu} \text{ \textit{exists and is} } > 1) : \text{D}.$$

For this inequality means that infinitely often $\sqrt[\nu]{a_\nu} > 1$, and hence also $a_\nu > 1$, and therefore $\{a_\nu\}$ is not a null sequence. No decision, however, is afforded by $\overline{\lim}\, \sqrt[\nu]{a_\nu} = 1$, (or even $\lim \sqrt[\nu]{a_\nu}$ exists and $= 1$). For if we take $a_\nu = 1/\nu^\alpha$, then, no matter what value α may denote, $\sqrt[\nu]{a_\nu} \to 1$ (see 2.4,**4**), whereas $\sum a_\nu$ converges for $\alpha > 1$, diverges for $\alpha < 1$.

3.2.1. Examples. In the following series, let x be a positive number. It does not matter whether the summation begins with $\nu = 0$ or $\nu = 1$, because we are only interested in investigating the convergence of the series. In each example we denote the terms of the series under investigation by a_ν.

1. $\sum \nu^\alpha x^\nu$, (α arbitrary, real). $\dfrac{a_{\nu+1}}{a_\nu} = \left(\dfrac{\nu+1}{\nu}\right)^\alpha \cdot x \to x$. $x < 1$: C, $x > 1$: D. $x = 1$: harmonic series (see 3.1.2,**3**).

2. $\sum \dfrac{x^\nu}{\nu!}$. $\dfrac{a_{\nu+1}}{a_\nu} = \dfrac{x}{\nu+1} \to 0$ for every x. The series is everywhere convergent, *i.e.*, for every x ($\geqq 0$).

3. $\sum \dbinom{\nu+k}{\nu} x^\nu$, ($k \geqq 0$, arbitrary). $\dfrac{a_{\nu+1}}{a_\nu} = \dfrac{\nu+1+k}{\nu+1}\, x \to x$. For $x < 1$: C, for $x > 1$: D. For $x = 1$ we have invariably $a_\nu \geqq 1$:[1] D.

---

[1] The series $\sum\limits_{\nu=0}^{\infty} \dbinom{\nu+k}{\nu}$ obtained for $x = 1$ is still a series of positive terms for $k > -1$, but its convergence behavior is not so simple to determine (see 5.4, Theorem 1).

4. $\sum \frac{1}{\sqrt{\nu^2+1}}$: D, because $a_\nu \geqq \frac{1}{\sqrt{\nu^2+\nu^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\nu}$.

5. $\sum \frac{1}{\sqrt{\nu^3+1}}$: C, because $a_\nu \leqq \frac{1}{\nu^{3/2}}$.

6. $\sum \frac{1}{\sqrt{\nu(\nu^2-1)}}$:[1] C, because, for $\nu \geqq 2$, we have $a_\nu \leqq \frac{1}{\sqrt{\nu\left(\nu^2 - \frac{\nu^2}{2}\right)}} =$ $= \frac{\sqrt{2}}{\nu^{3/2}}$.

7. $\sum \left(\frac{1}{\log \nu}\right)^p$,[1] ($p > 0$, arbitrary): D. For, since $\frac{\log^p \nu}{\nu} \to 0$ (see 2.1.4,5), this value is <1, *i.e.*, $a_\nu > \frac{1}{\nu}$, after a certain stage.

8. $\sum \frac{1}{(\log \nu)^{\log \nu}}$:[1] C. For, since $(\log \nu)^{\log \nu} = (e^{\log \nu})^{\log \log \nu} = \nu^{\log \log \nu}$, we have $a_\nu < \frac{1}{\nu^2}$ for $\nu > \mu$.

9. $\sum \frac{1}{(2\nu+1)^\alpha}$: C for $\alpha > 1$, D for $\alpha \leqq 1$; because for $\alpha > 1$ we have $a_\nu < \frac{1}{2^\alpha} \cdot \frac{1}{\nu^\alpha}$, and for $\alpha \leqq 1$ (and $\nu \geqq 1$) we have $a_\nu > \frac{1}{(2\nu+\nu)^\alpha} = \frac{1}{3^\alpha} \cdot \frac{1}{\nu^\alpha}$.

10. In the sequence $\{t_\nu\}$, let t_0 denote any integer and t_ν, for $\nu \geqq 1$, a "digit", *i.e.*, one of the numbers 0, 1, 2, ..., 9. Then the series

$$\sum_{\nu=0}^{\infty} \frac{t_\nu}{10^\nu} \equiv t_0 + \frac{t_1}{10} + \frac{t_2}{10^2} + \ldots + \frac{t_\nu}{10^\nu} + \ldots$$

is convergent, because for $\nu \geqq 1$ we have $a_\nu < 10 \cdot \frac{1}{10^\nu}$, and $\sum 10^{-\nu}$ is convergent. In this sense every ordinary decimal fraction or—in readily understood notation—every expression of the form

$$t_0 + 0 . t_1 t_2 \ldots t_\nu \ldots$$

is to be regarded as a convergent infinite series. Its value, *i.e.*, the

[1] Here we let ν run from 2 on.

value s of this series, lies between t_0 and t_0+1. For every integer $p \geqq 1$, however, we also have

$$s_p \leqq s \leqq s_p + \frac{1}{10^p},$$

where s_p denotes the decimal fraction $t_0+0.t_1t_2\ldots t_p$ terminated after p places. The value of a decimal fraction is thus (for every $p \geqq 0$) not less than the decimal fraction terminated after p places, but not greater by more than 10^{-p}, *i.e.*, by more than "a unit in the last place".[1]

3.3. Series of positive, monotonically decreasing terms

The series of positive terms occurring in applications usually have the additional property that their terms decrease monotonically (in the wider sense): $a_\nu \searrow$. Partly simpler, partly more far-reaching theorems hold for the narrower class of these series. Thus we have here the following theorem, which goes beyond Theorem 1 in 2.6.2:

Theorem 1. *If $\sum a_\nu$ is a convergent series of positive, monotonically decreasing terms, then*

$$\nu a_\nu \to 0 \quad \text{as} \quad \nu \to \infty. \tag{1}$$

In other words: (1) *is a necessary condition for the convergence of such a series* $\sum a_\nu$.

Proof. Since $a_\nu \searrow 0$, we have, if n is a natural number,

$$na_{2n} \leqq a_{n+1}+\ldots+a_{2n} = s_{2n}-s_n.$$

Thus, as $n \to \infty$, $na_{2n} \to s-s = 0$, and hence also $2na_{2n} \to 0$. Likewise

$$(n+1)a_{2n+1} \leqq a_{n+1}+\ldots+a_{2n+1} = s_{2n+1}-s_n.$$

Thus also $(2n+1)a_{2n+1} \to 0$ as $n \to \infty$. Hence, according to theorem 6 in 2.1.3, we have (1). That this condition is not sufficient for convergence is shown by the series (4) considered below, with $\alpha = 1$, which series belongs to our class, diverges, and for which nevertheless $\nu a_\nu \to 0$.

---

[1] This easy "estimate of error" for decimal fractions is the main reason, next to the convenient comparison of the magnitudes of two decimal fractions, for their practical usefulness (*cf.* 7.1).

Especially impressive and capable of many applications is the following theorem, designated as *Cauchy's condensation test*, which asserts, *e.g.*, that the series $\sum a_n$ and $\sum 2^\nu a_{2^\nu}$ have the same convergence behavior (*i.e.*, either both converge or else both diverge). We prove, somewhat more generally,

Theorem 2. *Let $\sum a_\nu$ be a series with positive, monotonically decreasing terms. Suppose that $\{k_\nu\}$ is a sequence of natural numbers which is monotonically increasing in the narrower sense, and that there exists an $M>0$ such that, for $\nu=1, 2, \ldots,$*

$$k_{\nu+1}-k_\nu \leq M(k_\nu-k_{\nu-1}).\text{[1]} \tag{2}$$

Then the two series

$$\sum_n a_n \quad \textit{and} \quad \sum_\nu (k_{\nu+1}-k_\nu)a_{k_\nu} \tag{3}$$

have the same convergence behavior.

The *proof* will be preceded by several remarks and examples:

1. If we choose $k_\nu = 2^\nu, 3^\nu, \ldots$ or $=[k^\nu]$,[2] ($k>1$, arbitrary), or $=\nu^2, \nu^3, \ldots$, then condition (2) is fulfilled for a suitable M.

2. Accordingly, $\sum a_n$ and the series $\sum 2^\nu a_{2^\nu}$, $\sum(2\nu+1)a_{\nu^2}$, *etc.* therefore have the same convergence behavior, if $\{a_n\}$ is a positive, monotonically decreasing sequence.

3. As a very special case, $\sum \frac{1}{n^\alpha}$ and $\sum \frac{2^\nu}{(2^\nu)^\alpha} \equiv \sum \frac{1}{(2^{\alpha-1})^\nu}$, $(\alpha>0)$, have the same convergence behavior.[3] Since the last series is a geometric series, this furnishes a new proof of the convergence of $\sum \frac{1}{n^\alpha}$ for $\alpha>1$ and of the divergence of this series for $\alpha \leq 1$.

4. Likewise the series $\sum \frac{1}{n \log n}$ possesses the same convergence be-

[1] This means that the gaps between the k_ν do not increase too rapidly.

[2] If x is real, then $[x]$ denotes the greatest integer $g \leq x$, *i.e.*, the integer g satisfying $g \leq x < g+1$.

[3] In order that this series and likewise those mentioned in the following examples be meaningful, n or ν may run only from 1 or from a higher stage on; in the series (5), only from a number m on, for which $\log_p m$ exists and is >0.

havior as $\sum \frac{2^\nu}{2^\nu \log 2^\nu} \equiv \sum \frac{1}{(\log 2)\nu}$; it is therefore divergent. This holds *a fortiori* for the series

$$(4) \qquad \sum \frac{1}{n (\log n)^\alpha}$$

if $\alpha < 1$. If, however, $\alpha > 1$, then this series has the same convergence behavior as $\sum \frac{2 \cdot 3^\nu}{3^\nu (\log 3^\nu)^\alpha} \equiv \sum \frac{2}{(\log 3)^\alpha} \cdot \frac{1}{\nu^\alpha}$, and is therefore convergent. Proceeding in this manner, it is easy to verify that the series

$$(5) \qquad \sum_n \frac{1}{n \log n \log_2 n \ldots \log_{p-1} n \, (\log_p n)^\alpha}$$

also diverge for $\alpha \leqq 1$ and converge for $\alpha > 1$.[1] Here $\log_p n$ denotes the p-fold iterated (natural) logarithm of n ($p = 0, 1, \ldots$): $\log_0 n = n$, $\log_1 n = \log n$, $\log_p n = \log(\log_{p-1} n)$.

For fixed $\alpha > 1$, these series form, for $p = 0, 1, \ldots$, a scale of series which converge more and more weakly, and similarly, for fixed $\alpha < 1$, a scale of more and more weakly divergent series. This convergence behavior of the series (5) was discovered by *N. H. Abel.*

PROOF OF THEOREM 2. Let us denote the partial sums of the series (3) by s_n, t_ν, respectively. Then, for $n < k_\nu$, if we set $a_0 + \ldots + a_{k_0-1} = A$ (thus $A = 0$ for $k_0 = 0$),

$$\begin{aligned} s_n &\leqq s_{k_\nu} \leqq A + (a_{k_0} + \ldots + a_{k_1-1}) + \ldots + (a_{k_\nu} + \ldots + a_{k_{\nu+1}-1}) \\ &\leqq A + (k_1 - k_0) a_{k_0} + \ldots + (k_{\nu+1} - k_\nu) a_{k_\nu} \end{aligned}$$

$$(6) \qquad s_n \leqq A + t_\nu .$$

For $n > k_\nu$, however, we have

$$\begin{aligned} s_n &\geqq s_{k_\nu} \geqq (a_{k_0+1} + \ldots + a_{k_1}) + \ldots + (a_{k_{\nu-1}+1} + \ldots + a_{k_\nu}) \\ &\geqq (k_1 - k_0) a_{k_1} + \ldots + (k_\nu - k_{\nu-1}) a_{k_\nu}, \\ M s_n &\geqq (k_2 - k_1) a_{k_1} + \ldots + (k_{\nu+1} - k_\nu) a_{k_\nu}, \end{aligned}$$

$$(7) \qquad M s_n \geqq t_\nu - t_0 .$$

---

[1] In the proofs, which run exactly as for the series (4), it is only necessary to apply the fact that $2 < e < 3$.

Now (6) shows that if the sequence $\{t_\nu\}$ is bounded, then so is the sequence $\{s_n\}$, and (7) shows conversely that if $\{s_n\}$ is bounded then so is $\{t_\nu\}$. This, on the basis of the first main test, completes the proof of the theorem.

The following theorem, which is commonly designated as the *integral test* is particularly useful. It is based on the assumption that the terms of the series $\sum a_\nu$ under investigation are the values of a function $f(x)$ for $x=\nu$: $a_\nu = f(\nu)$.

Theorem 3. *Let $f(t)$ be defined for $t \geqq 1$ as a positive, monotonically decreasing function, and set*

$$f(\nu) = a_\nu$$

for $\nu = 1, 2, \ldots$. Then

$$\text{the series (8) } \sum_{\nu=1}^{\infty} a_\nu \quad \text{and the integral (9) } \int_1^\infty f(t)\,dt$$

have the same convergence behavior.[1] *Moreover, the partial sums of* (8) *and the partial integrals $\int_0^n f(t)\,dt = I_n$ are such that the sequence of differences*

$$s_n - I_n \tag{10}$$

approaches, in a monotonically decreasing fashion, a limit between 0 *and* a_1.

Proof. Since $f(t)\searrow$, we have, for $\nu = 2, 3, \ldots$,

$$\int_\nu^{\nu+1} f(t)\,dt \leqq a_\nu \leqq \int_{\nu-1}^{\nu} f(t)\,dt. \tag{11}$$

If we write down these inequalities for $\nu = 2, 3, \ldots, n$, $(n \geqq 2)$, and add, it follows that

$$\int_2^{n+1} f(t)\,dt \leqq s_n - a_1 \leqq \int_1^n f(t)\,dt,$$

$$I_{n+1} - I_2 \leqq s_n - a_1 \leqq I_n. \tag{12}$$

---

[1] Since $f(t) \geqq 0$, the partial integrals $\int_1^x f(t)\,dt = F(x)$ form a monotonically increasing function for $x \geqq 1$. They therefore have (according to the first main test for functions) a limit I, if, and only if, $F(x)$ is bounded. In this case the integral (9) is said to be convergent and to have the value I. Otherwise $F(x) \to +\infty$, and (9) is called divergent.

The second part of this double inequality shows that the s_n are bounded if the I_n are, the first part the converse. Hence, (8) and (9) have the same convergence behavior. Furthermore,

$$s_n - I_n - (s_{n+1} - I_{n+1}) = \int_n^{n+1} f(t)\,dt - a_{n+1} \geqq 0,$$

the latter according to (11). The differences (10) thus decrease monotonically and are therefore at most equal to the initial term a_1 and, according to the left half of (12), at least equal to $a_1 - I_2$,—and this with the exclusion of the stated bounds in case $f(t)$ decreases monotonically in the stricter sense in $1 < t < 2$.

A few examples will serve to illustrate the effectiveness of the test:

1. $f(t) = \frac{1}{t}$ shows that the series $\sum \frac{1}{\nu}$ has the same convergence behavior as $\int_1^\infty \frac{dt}{t}$. Hence, since $\int_1^x \frac{dt}{t} = \log x \to +\infty$, $\sum \frac{1}{\nu}$ is divergent, as we already know. We now learn further, however, that the sequence

$$\left\{1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right\} \searrow C, \tag{13}$$

where C is a number in $0 < C < 1$. This number C is called *Euler's constant.* Its value is equal to $0.577\ldots$.

2. $f(t) = \frac{1}{t^\alpha}$, $\alpha > 1$, shows again, since $\int_1^x \frac{dt}{t^\alpha} = \frac{1}{\alpha - 1}\left(1 - \frac{1}{x^{\alpha-1}}\right) < \frac{1}{\alpha - 1}$, that $\sum \frac{1}{\nu^\alpha}$ converges for $\alpha > 1$. From (11) it follows further, however, if we set $\nu = n+1,\ n+2, \ldots,\ n+p$ there and add, that

$$\int_{n+1}^{n+p+1} \frac{dt}{t^\alpha} \leqq s_{n+p+1} - s_n \leqq \int_n^{n+p} \frac{dt}{t^\alpha}.$$

If we evaluate the integrals and let $p \to \infty$, we obtain

$$\frac{1}{\alpha - 1}\,\frac{1}{(n+1)^{\alpha-1}} \leqq r_n \leqq \frac{1}{\alpha - 1}\,\frac{1}{n^{\alpha-1}} \tag{14}$$

as a very good estimate of the remainder or error for the series $\sum \frac{1}{\nu^\alpha}$, $(\alpha > 1)$, under consideration.

3. For $0 < \alpha < 1$ we find, as in **1**, that

$$(15) \qquad 1 + \frac{1}{2^\alpha} + \dots + \frac{1}{n^\alpha} - \frac{n^{1-\alpha}-1}{1-\alpha}$$

is a monotonically decreasing sequence which, as $n \to \infty$, tends to a limit lying between 0 and 1 (these bounds excluded). In particular,

$$(16) \qquad 1 + \frac{1}{2^\alpha} + \dots + \frac{1}{n^\alpha} \cong \frac{n^{1-\alpha}}{1-\alpha}, \qquad (0 < \alpha < 1),$$

and provides a measure of the rapidity with which the partial sums of the now divergent series $\sum \frac{1}{\nu^\alpha}$ tend to $+\infty$.

4. For $f(t) = \dfrac{1}{t \log t \dots \log_{p-1} t \, (\log_p t)^\alpha}$, the indefinite integral

$$\int f(t)dt = \begin{cases} \log_{p+1} t & \text{for } \alpha = 1 \\ -\dfrac{1}{\alpha - 1} \dfrac{1}{(\log_p t)^{\alpha-1}} & \text{for } \alpha \gtrless 1. \end{cases}$$

From this, as in **1, 2,** and **3,** we again read off the convergence behavior of the Abel series already considered in (5), with corresponding remainder estimates for $\alpha > 1$ and assertions concerning the strength of divergence for $0 < \alpha < 1$.

5. Now let $f(t)$ be a function which, for $t > x_0$, is positive, increases monotonically to $+\infty$, and has a derivative $f'(t)$ which decreases monotonically to 0 (and hence is also positive). Then, as in **4,**

$$\int \frac{f'(t)}{(f(t))^\alpha} dt = \begin{cases} \log f(t) & \text{for } \alpha = 1 \\ -\dfrac{1}{\alpha - 1} \dfrac{1}{(f(t))^{\alpha-1}} & \text{for } \alpha \lessgtr 1. \end{cases}$$

The definite integral taken from x_0 to x thus remains bounded for $\alpha > 1$ as $x \to \infty$, but tends to $+\infty$ for $\alpha \leqq 1$. Therefore the series

$$\sum_{\nu=p}^{\infty} \frac{f'(\nu)}{(f(\nu))^\alpha},$$

starting at a suitable stage $\nu = p$, is also convergent for $\alpha > 1$, divergent for $\alpha \leq 1$.

3.4. The second main test

In the preceding three sections we have considered only series of positive terms. We now turn once again to series Σa_ν of arbitrary (real or complex) terms a_ν. The partial sums of such a series form an arbitrary sequence $\{s_\nu\}$, for whose convergence behavior the second main test (2.5,(6)) is appropriate. If we carry it over to the present case, it asserts

Theorem 1 (*second main test for infinite series*). *The series Σa_ν is convergent if, and only if, having chosen $\varepsilon > 0$, a $\mu = \mu(\varepsilon)$ can be assigned such that for all pairs of indices ν and ν' with $\nu' > \nu > \mu$, we have*

(1) $$|a_{\nu+1} + a_{\nu+2} + \ldots + a_{\nu'}| < \varepsilon.$$

We shall leave it to the reader to convince himself that the following conditions, which are sometimes more convenient to use, are equivalent to (1):

(1^a) To every $\varepsilon > 0$ a μ can be assigned such that, for all $\nu > \mu$ and arbitrary natural ρ, we have

$$|a_{\nu+1} + \ldots + a_{\nu+\rho}| < \varepsilon.$$

(1^b) For every sequence $\{k_\nu\}$ of natural numbers, the "partial segments"

$$T_\nu = (a_{\nu+1} + \ldots + a_{\nu+k_\nu})$$

of the series form a null sequence.—Somewhat more generally:

(1^c) For every sequence $\{n_\nu\}$ which tends to $+\infty$, and every arbitrary sequence $\{k_\nu\}$ of natural numbers, the "partial segments"

$$T'_\nu = (a_{n_\nu+1} + \ldots + a_{n_\nu+k_\nu})$$

of the series form a null sequence.

In connection with sequences we were able to state the second main test in a form which was unprecise but which emphasized what was essential: A sequence is convergent if, and only if, its terms eventually all lie very close to one another. Here, in connection with series, we may say: A series Σa_ν is convergent if, and only if, from a certain stage on, the value of the sum obtained can be altered only very little by a further summing up of the terms of the series. Now, as then, it is merely necessary to make the "very little" precise by means of ε, and the "from a certain stage on" by means of the μ associated with ε.

We formulate explicitly the following (self-evident, according to Theorem 1)

Corollary. Σa_ν is divergent if a partial-segment sequence T_ν or T'_ν can be assigned which does **not** form a null sequence.

For the series $\Sigma \frac{1}{\nu}$, $T_\nu = \frac{1}{\nu+1} + \ldots + \frac{1}{2\nu}$ is such a partial-segment sequence, since $T_\nu > \frac{\nu}{2\nu} = \frac{1}{2}$ for all $\nu = 1, 2, \ldots$. The series is therefore divergent.[1]

For $\Sigma \frac{z^\nu}{\nu^2}$, if z is an arbitrary complex number with $|z| \leqq 1$, we have

$$\left|\frac{z^{\nu+1}}{(\nu+1)^2} + \cdots + \frac{z^{\nu+\rho}}{(\nu+\rho)^2}\right| \leqq \frac{1}{\nu(\nu+1)} + \cdots + \frac{1}{(\nu+\rho-1)(\nu+\rho)} =$$
$$= \frac{1}{\nu} - \frac{1}{\nu+\rho} < \frac{1}{\nu}$$

(*cf.* 2.6.1,4), and hence $< \varepsilon$ for $\nu > \mu$ if we take $\mu \geqq 1/\varepsilon$. The series is therefore convergent for the z in question.

The next example deals with alternating series, *i.e.*, real series Σa_ν whose terms have alternating signs, so that, if the initial term is positive, we may set $a_\nu = (-1)^\nu b_\nu$ with $b_\nu \geqq 0$.

Theorem 2. (*Leibniz's test.*) *A (real) alternating series of which the absolute values of the terms form a monotonic null sequence, is invariably*

[1] The reader should compare the various proofs which we have now given for the divergence of $\Sigma \frac{1}{\nu}$, and determine whether or to what extent they differ from one another.

convergent. If $\sum_{\nu=0}^{\infty}(-1)^\nu b_\nu$, *with* $b_\nu \searrow 0$, *is such a series, then its value lies between* b_0 *and* b_0-b_1, *more generally, between any two successive partial sums.*

PROOF. For arbitrary natural ν and ρ,

$$|(-1)^{\nu+1}b_{\nu+1}+\ldots+(-1)^{\nu+\rho}b_{\nu+\rho}| = |b_{\nu+1}-b_{\nu+2}+\ldots+(-1)^{\rho-1}b_{\nu+\rho}|.$$

The sum between the absolute-value signs can be written in the form

$$(b_{\nu+1}-b_{\nu+2}) + (b_{\nu+3}-b_{\nu+4})+\ldots+ \begin{cases}(b_{\nu+\rho-1}-b_{\nu+\rho}), & \text{if } \rho \text{ is even,}\\ b_{\nu+\rho}, & \text{if } \rho \text{ is odd.}\end{cases}$$

Since $\{b_\nu\}$ is decreasing, this shows that this sum is $\geqq 0$ and therefore the absolute-value signs on the right may be removed. If this sum is then written in the form

$$b_{\nu+1}-(b_{\nu+2}-b_{\nu+3})-\ \ldots\ -\begin{cases}b_{\nu+\rho}, & \text{if } \rho \text{ is even,}\\ (b_{\nu+\rho-1}-b_{\nu+\rho}), & \text{if } \rho \text{ is odd,}\end{cases}$$

then this shows further that the sum is $\leqq b_{\nu+1}$. Since $b_\nu \searrow 0$, this is $<\varepsilon$ for all $\nu>\mu$, if we choose μ so that $b_\mu<\varepsilon$.

Simple examples of this very useful Theorem 2, which is due to *G. W. Leibniz* (1705), are the series

$$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu^\alpha},\ (\alpha>0), \qquad \text{and} \qquad \sum_{\nu=2}^{\infty}\frac{(-1)^\nu}{(\log\nu)^\alpha},\ (\alpha>0).$$

Theorems 1 to 6 in 2.6.2 already pertain to arbitrary series. In 3.3 we were able to sharpen the first of these theorems for the case of series with positive, monotonically decreasing terms, to the theorem "$\nu a_\nu \to 0$". For arbitrary convergent series this need not be the case, as is shown by the last two given series. It is true, however, that the sequence $\{\nu a_\nu\}$ tends to 0 "in the mean"; more precisely, that

$$\frac{a_1+2a_2+\ldots+na_n}{n}\to 0. \tag{2}$$

We shall prove at the same time the somewhat more general

Theorem 3. *Let* $\sum a_\nu$ *be an arbitrary convergent series, and* $\{p_\nu\}$ *be an arbitrary sequence which tends monotonically to* $+\infty$, *or* (*more generally*) *a* *sequence of complex numbers satisfying the condition:* $|p_\nu| \to +\infty$ *and, for a suitable* $M > 0$ *and all* $\nu = 0, 1, \ldots,$

$$|p_0| + |p_1 - p_0| + \cdots + |p_\nu - p_{\nu-1}| \leqq M|p_\nu|.$$

Then also the sequence of quotients

$$\frac{p_0 a_0 + p_1 a_1 + \ldots + p_n a_n}{p_n} \to 0. \text{[1]} \tag{3}$$

Proof. If s_ν are the partial sums of Σa_ν, and if $s_\nu \to s$, then, according to 2.4.2,**5**, also

$$\frac{p_1 s_0 + (p_2 - p_1)s_1 + \ldots + (p_n - p_{n-1})s_{n-1}}{p_n} \to s.$$

Since $\dfrac{p_0 s_0}{p_n} \to 0$ and $s_n \to s$, we have

$$s_n - \frac{(p_1 - p_0)s_0 + \ldots + (p_n - p_{n-1})s_{n-1}}{p_n} \to 0.$$

This, however, is precisely the assertion (3), as one can verify by bringing the term s_n over the denominator p_n and then collecting the terms in the numerator involving $p_0, p_1, \ldots$. [2]

By means of the following theorem, which is almost self-evident because of 2.3.2,**11**, the treatment of complex series is completely reduced to that of real series,—a reduction which, to be sure, is only seldom of use in practice.

Theorem 4. *The series* Σa_ν *with* $a_\nu = \alpha_\nu + i\alpha'_\nu$, $(\alpha_\nu, \alpha'_\nu$ *real), is convergent if, and only if, the two real series* $\Sigma \alpha_\nu$ *and* $\Sigma \alpha'_\nu$ *both converge. If* s, σ, σ' *are the sums of the three series in the case of convergence, then* $s = \sigma + i\sigma'$.

3.5. Absolute convergence

Of the two series $\Sigma \dfrac{(-1)^{\nu-1}}{\nu}$ and $\Sigma \dfrac{1}{\nu}$, $(\nu = 1, 2, \ldots)$, the first proved to be convergent, the second divergent. The convergent series thus

[1] The p_ν are $\neq 0$ from a certain stage on. It suffices to consider the quotients (3) from this stage on.

[2] One thereby performs an "Abel partial summation"; *cf.* below, 5.3.

becomes a divergent series when in the former the negative terms are replaced by their absolute values. This is not the case for the series $\Sigma \frac{(-1)^{\nu-1}}{\nu^2}$. In the sequel it will usually make an essential difference whether a convergent series Σa_ν remains convergent if all its terms are replaced by their absolute values, or whether it thereby becomes divergent. Here we have, first of all, the following simple, but for applications especially important,

Theorem 1. *A series Σa_ν is certainly convergent, if the series $\Sigma |a_\nu|$, which is a series of positive terms, converges. If, in this case, $\Sigma a_\nu = s$ and $\Sigma |a_\nu| = S$, then moreover $|s| \leqq S$.*

Proof. Since (see 1.2.1,**4**)

$$|a_{\nu+1} + \ldots + a_{\nu+\rho}| \leqq |a_{\nu+1}| + \ldots + |a_{\nu+\rho}|,$$

the left side is $< \varepsilon$ if the right side is, from which the first assertion follows according to the second main test. [1] Since

$$|s_n| \leqq |a_0| + |a_1| + \ldots + |a_n| \leqq S,$$

the relation $|s| \leqq S$ now follows according to 2.3.2,**2**.

Convergent series Σa_ν thus fall into two classes, according as "even" the series $\Sigma |a_\nu|$ converges or not. We introduce

Definition 1. *A convergent series Σa_ν shall be called absolutely convergent, if the series $\Sigma |a_\nu|$ is also convergent. If this is not the case, then Σa_ν shall be called nonabsolutely convergent.* [2]

Convergent series of positive terms are automatically absolutely convergent.

---

[1] In greater detail: Having chosen $\varepsilon > 0$, a $\mu = \mu(\varepsilon)$ can be determined so that the right side of the inequality is $< \varepsilon$ for all $\nu > \mu$ and all $\rho > 0$, because condition (1^a) of the second main test is a necessary condition for convergence. Therefore the left side of the inequality is also $< \varepsilon$ for the same ν and ρ; and hence Σa_ν is convergent, because the condition in (1^a) is sufficient for the convergence of this series.

[2] We expressly emphasize: The designation "nonabsolutely convergent" shall be applied only to convergent series.

The geometric series Σz^ν is absolutely convergent for $|z|<1$ (*i.e.*, wherever it converges at all).

$\sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!}$ is convergent for every (complex) z; $\Sigma \binom{\nu+k}{\nu} z^\nu$, $(k \geqq 0)$, and $\sum_{\nu=1}^{\infty} \nu^\alpha z^\nu$, ($\alpha$ arbitrary, real), provided that $|z|<1$.

For the partial sums s_ν of a series Σa_ν, the latter's absolute convergence means that the series

$$s_0 + \sum_{\nu=1}^{\infty} |s_\nu - s_{\nu-1}|, \qquad \text{or} \qquad \sum_{\nu=0}^{\infty} |s_\nu - s_{\nu-1}|, \quad (s_{-1}=0),$$

converges. We introduce

Definition 2. *If a sequence $\{s_\nu\}$ has the property that (with $s_{-1}=0$)*

$$\sum_{\nu=0}^{\infty} |s_\nu - s_{\nu-1}| < +\infty, \text{[1]} \tag{1}$$

then we say that it is of bounded variation.

Such a sequence is invariably convergent, for with (1) the series $\Sigma(s_\nu - s_{\nu-1})$ also converges, and therefore, as $n \to \infty$,

$$s_n = \sum_{\nu=0}^{n} (s_\nu - s_{\nu-1})$$

tends to the value of this series. The value of the series (1) is also designated as the *total variation* of the sequence $\{s_\nu\}$.

Definition 3. *A sequence $\{s_\nu\}$ shall be called absolutely convergent with the limit s, if $s_\nu \to s$ and $\{s_\nu\}$ is of bounded variation.*

There are above all two reasons for the importance of absolute convergence: First of all, the series $\Sigma|a_\nu|$ is a series of positive terms, for which the numerous, and for the most part very simple, comparison tests in 3.1 are available. Thus, *e.g.*, we have immediately

Theorem 2. *If Σc_ν is a convergent series of positive terms, and if the*

---

[1] This convenient notation means simply that the series of positive terms written down converges.

terms a_ν of a series under investigation satisfy, from a certain stage on, the condition

$$|a_\nu| \leqq K c_\nu,^{1} \quad (K > 0, \ \textit{fixed}), \qquad \textit{or} \qquad \left|\frac{a_{\nu+1}}{a_\nu}\right| \leqq \frac{c_{\nu+1}}{c_\nu},$$

then $\sum a_\nu$ converges, and is actually absolutely convergent.

Corollary. *If $\sum a_\nu$ is absolutely convergent and $\{b_\nu\}$ is a bounded sequence, then $\sum a_\nu b_\nu$ is also absolutely convergent.*

For, $|a_\nu b_\nu| \leqq K|a_\nu|$, if K denotes a bound of the sequence $\{b_\nu\}$.

The second reason for the importance of the concept "absolute convergence" is the fact that one can operate with absolutely convergent series for the most part—the next section will show this in detail—as with ordinary sums.

Sometimes—although not often (*cf.* 3.4, Theorem 4)—it is convenient to decide the question of the absolute convergence of a series by separating the real and imaginary parts of its terms. In this connection we have

Theorem 3. *A series $\sum a_\nu$ of complex terms $a_\nu = \alpha_\nu + i\alpha'_\nu$ is absolutely convergent if, and only if, the two (real) series $\sum \alpha_\nu$ and $\sum \alpha'_\nu$ both converge absolutely.*

The proof can be read off immediately from the double inequality

$$\left.\begin{matrix} |\alpha_\nu| \\ |\alpha'_\nu| \end{matrix}\right\} \leqq |a_\nu| \leqq |\alpha_\nu| + |\alpha'_\nu|.$$

Finally, we can now prove the *Cauchy-Toeplitz theorem*, which we proved in 2.4.2 for row-finite matrices, also for matrices that are not row-finite.

Theorem 4. *Let $(a_{\kappa\nu})$ be an arbitrary matrix (see 2.2,8) satisfying the three conditions*

$$\text{(N)} \quad \sum_{\nu=0}^{\infty} |a_{\kappa\nu}| \leqq M \qquad \textit{for every } n = 0, 1, 2, \ldots,$$

$$\text{(R)} \quad \sum_{\nu=0}^{\infty} a_{\kappa\nu} = A_\kappa \to 1 \qquad \textit{as } n \to \infty,$$

$$\text{(C)} \quad a_{\kappa\nu} \to 0 \quad \textit{as} \quad n \to \infty \qquad \textit{for every fixed } \nu = 0, 1, 2, \ldots.$$

[1] Or: $a_\nu = O(c_\nu)$.

Then $z_\nu \to z$ *invariably implies that the series*

$$(2) \qquad \sum_{\nu=0}^{\infty} a_{n\nu} z_\nu = z'_n$$

converge, for every $n = 0, 1, 2, \ldots$, *and that also their values* $z'_n \to z$ *as* $n \to \infty$.

Indeed, (N) asserts that the series $\sum_\nu a_{n\nu}$ converges absolutely for every n. Hence, on account of the boundedness of the sequence $\{z_\nu\}$, the series (2) are also absolutely convergent. That $z'_n \to z$ is now proved word for word as in 2.4, Theorem 2, since in its proof no use at all was made of the fact that the matrix was supposed to be row-finite.

Corollary 1. If $z = 0$, so that $\{z_\nu\}$ is a null sequence, then $\{z'_n\}$ is also a null sequence. It is evident that in the proof of this special case of Theorem 4, no use is made of the condition (R); the fact that, on the basis of (N), the sequence $\{A_n\}$ of row-sums is bounded, is sufficient.

Corollary 2. Theorem 4 yields—as we have formulated it—sufficient conditions that, by means of (2), a convergent sequence $\{z_\nu\}$ be transformed into a sequence $\{z'_n\}$ which again converges, and, moreover, has the same limit. The importance of the theorem demonstrated goes beyond this fact: The established sufficient conditions are also necessary that *every* convergent sequence $\{z_\nu\}$ be transformed by means of (2) into a sequence $\{z'_n\}$ which is again convergent and has the same limit. [1] Extension of our considerations in this direction, however, would lead beyond the limits of the present little volume.

3.6. Operating with convergent series

We have already emphasized repeatedly, and shall immediately see

[1] The transformation (2) is then called *permanent*. If we waive the equality of the limiting values and require merely that the sequence $\{z'_n\}$ again converge (say to z'), then the transformation (2) is called *convergence-preserving*. It possesses this property if, and only if, $(a_{n\nu})$ satisfies, in addition to (N), the two conditions

(R′) $A_n \to a$ as $n \to \infty$, (C′) $a_{n\nu} \to \alpha_\nu$ as $n \to \infty$, for every $\nu = 0, 1, 2, \ldots$.

more precisely, that the same rules need not hold for operating with convergent series as for operating with ordinary sums. Indeed, in every case we must test to see whether and to what extent these rules can be carried over to operations with infinite series. Operation with ordinary sums is based on the following fundamental rules:

1. Associative law: $(a+b)+c = a+(b+c)$.
2. Commutative law: $a+b = b+a$.
3. Distributive law: $a(b+c) = ab+ac$.

They can be extended in the familiar way to sums with an arbitrary but fixed (and finite) number p of terms

$$a_1+a_2+\ldots+a_p, \tag{1}$$

and then assert:

1. The value of the sum (1) does not change if successive terms are united in an arbitrary way by means of parentheses to form single terms. Thus, if $1 < p_1 < p_2 < \ldots < p_k < p$, then

$$(a_1+\ldots+a_{p_1}) + (a_{p_1+1}+\ldots+a_{p_2})+\ldots+(a_{p_k+1}+\ldots+a_p) \tag{2}$$

has the same value as (1). Conversely, a sum of the form (2) retains its value if the parentheses are removed, *i.e.*, if (2) is changed back again into (1). (Insertion and removal of parentheses.)

2. The value of the sum (1) does not change if the terms are permuted in an arbitrary manner.

3. Two sums of the form (1), say $(a_1+\ldots+a_p)$ and $(b_1+\ldots+b_q)$, are multiplied together by multiplying every term of the first sum by every term of the second and adding these $p\cdot q$ products in an arbitrary order of succession.

If we imagine (1) to be replaced by an infinite series, then only half of the first of these rules still holds, and the other two are not at all valid any more in general. Specifically, we have the following theorems.

Theorem 1. *The insertion of parentheses in convergent series is permissible without restriction. More precisely: Let*

$$\sum_{\nu=0}^{\infty} a_\nu = s \tag{3}$$

be a convergent series. If $\{\nu_\lambda\}$, *with* $-1 = \nu_0 < \nu_1 < \nu_2 < \ldots$, *is an arbitrary sequence of integers, and if we set*

$$a_{\nu_\lambda+1} + \ldots + a_{\nu_{\lambda+1}} = A_\lambda,$$

then the series

$$\sum_{\lambda=0}^{\infty} A_\lambda \equiv \sum_{\lambda=0}^{\infty} (a_{\nu_\lambda+1} + \ldots + a_{\nu_{\lambda+1}}) \tag{4}$$

is also convergent and has the value s.

Proof. The sequence of partial sums S_λ of ΣA_λ is obviously the subsequence $s_{\nu_1}, s_{\nu_2}, \ldots$ of the sequence $\{s_\nu\}$, and therefore converges, as the latter, to s as its limit. That the removal of parentheses is not permissible in general is shown already by the crude example $(1-1)+(1-1)+\ldots$ in 2.6.

In particular cases, *i.e.*, under suitable restrictive conditions, it is, nevertheless, permissible. It is important to know such conditons. The following rule is trivial:

If (4) converges and has the value s, and if the series (3) obtained from it by the removal of parentheses also converges, then it too has the value s. For, according to what was proved, (4) is convergent along with (3), and both series have the same value. Consequently, on removing the parentheses in (4), it is merely necessary to secure the convergence of the resulting series (3). This is accomplished, for example, by the following

Theorem 2. *If* (4) *converges to the value* s, *and if the sequence of numbers*

$$A'_\lambda = |a_{\nu_\lambda+1}| + \ldots + |a_{\nu_{\lambda+1}}|$$

is a null sequence, then the series (3) *resulting from the removal of parentheses also converges to the same value* s.

Proof. To every integer $\nu > 0$ there corresponds a unique integer $\lambda \geqq 0$ such that

$$\nu_\lambda < \nu \leqq \nu_{\lambda+1}. \tag{5}$$

Think of every ν as having associated with it this λ. Then, with the notation of the preceding proof, we obviously have

$$|s_\nu - S_\lambda| \leqq A'_\lambda$$

and hence

(6) $$s_\nu - s = S_\lambda - s + \varepsilon_\lambda, \qquad \text{where } \varepsilon_\lambda \to 0 \text{ as } \lambda \to \infty.$$

From $S_\lambda - s \to 0$ and $\varepsilon_\lambda \to 0$ it follows that $(S_\lambda - s) + \varepsilon_\lambda \to 0$ as $\lambda \to \infty$. Having chosen $\varepsilon > 0$, one can therefore determine λ_0 so that in (6) the absolute value of the right side, and hence also that of the left side, is $< \varepsilon$ for $\lambda \geqq \lambda_0$. Thus, if we set $\nu_{\lambda_0} = \mu$, we have

$$|s_\nu - s| < \varepsilon \qquad \text{for all } \nu > \mu,$$

i.e., Σa_ν is convergent and $= s$.

An instructive example is afforded by the series

(7) $$\sum_{\lambda=0}^{\infty} A_\lambda \equiv \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \ldots + \\ + \left(\frac{1}{4\lambda+1} + \frac{1}{4\lambda+3} - \frac{1}{2\lambda+2}\right) + \ldots .$$

This series is convergent, because, as is immediately verified, its terms are positive, but, for $\lambda > 0$,

$$< \frac{1}{2}\,\frac{1}{\lambda(\lambda+1)} < \frac{1}{2}\,\frac{1}{\lambda^2},$$

so that $\Sigma \dfrac{1}{2\lambda^2}$ is a convergent majorant with positive terms. If we remove the parentheses, we obtain the series

(8) $$\Sigma a_\nu \equiv 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + + - \ldots ,$$

which, according to Theorem 2, also converges, because obviously (by 2.1) $A'_\lambda \leqq \dfrac{1}{4\lambda+1} + \dfrac{1}{4\lambda+3} + \dfrac{1}{2\lambda+2} \to 0$ as $\lambda \to \infty$. Its value is equal to that of (7), which we shall call S. Since (7) has positive terms, certainly $S > A_0 + A_1 > \frac{11}{12}$. This result is very remarkable. For, (8) is "only" a rearrangement, in the sense of 3.1.1,4, of the series

(9) $$\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \ldots ,$$

the value of which, since it is an alternating series, lies (see 3.4, Theorem 2) between $1-\frac{1}{2}=\frac{1}{2}$ and $1-\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, and hence in any case is $<\frac{10}{12}$. We shall investigate this state of affairs more precisely in just a moment, but first we shall prove the following two simple rules of operation:

Theorem 3. *If Σa_ν and Σb_ν are convergent series with the respective values s and t, then the series*

$$\Sigma(a_\nu+b_\nu) \quad \textit{and} \quad \Sigma(a_\nu-b_\nu) \tag{10}$$

are also convergent and have the respective values $s+t$ and $s-t$. (In short: Convergent series may be added and subtracted term by term.) *Likewise, the series*

$$a_0+b_0+a_1+b_1+\dots \qquad \textit{and} \qquad a_0-b_0+a_1-b_1+-\dots, \tag{11}$$

resulting from the removal of the parentheses, are convergent with the respective values $s+t$ and $s-t$. Finally, if c is an arbitrary number, then Σca_ν is also convergent and has the value cs. [1]

Proof. Let s_ν and t_ν be the partial sums of the given series. Then

$$(s_\nu+t_\nu), \qquad (s_\nu-t_\nu)$$

are the respective partial sums of the series (10), and (according to 2.3.3,**10**) they $\to s+t$, $s-t$, respectively, which proves the first assertion. By Theorem 2, however, the parentheses in these series may be removed, because $\{|a_\nu|+|b_\nu|\}$, according to 2.1.3,**2** and **8**, is a null sequence. This implies the truth of the assertions concerning the series (11). Finally, the partial sums of Σca_ν form the sequence $\{cs_\nu\}$, which, by 2.3.3,**10**, $\to cs$.

We shall illustrate this theorem by determining the relation between the values of the two series (8) and (9). We denoted the value of (7) and (8) by S; let that of (9) be s. Then, by Theorem 1, the series

$$\sum_{\lambda=0}^{\infty}\left(\frac{1}{2\lambda+1}-\frac{1}{2\lambda+2}\right) \quad \text{and} \quad \sum_{\lambda=0}^{\infty}\left(\frac{1}{4\lambda+1}-\frac{1}{4\lambda+2}+\frac{1}{4\lambda+3}-\frac{1}{4\lambda+4}\right)$$

[1] Thus, in this special sense, the distributive law is valid for arbitrary convergent series.

also converge to the value s. Multiply the first by $\frac{1}{2}$, getting

$$\sum_{\lambda=0}^{\infty}\left(\frac{1}{4\lambda+2}-\frac{1}{4\lambda+4}\right)=\frac{1}{2}s,$$

and add this (both by virtue of Theorem 3) to the second series to obtain

$$\sum_{\lambda=0}^{\infty}\left(\frac{1}{4\lambda+1}+\frac{1}{4\lambda+3}-\frac{1}{2\lambda+2}\right)=\frac{3}{2}s.$$

This, however, is the series (7). Hence, $S=\frac{3}{2}s$. The series (9) is transformed by rearrangement into the series (8) which, to be sure, also converges but has a different sum (for we saw that $s>0$):

The commutative law mentioned at the beginning in **2** *does not hold any more in general for arbitrary convergent series.* We shall now show—and this brings out the importance of absolute convergence especially clearly—that it remains valid for absolutely convergent series, and only for these.

Theorem 4. *If Σa_ν is an absolutely convergent series, and if $\Sigma a'_\nu$ is an arbitrary rearrangement of it* (*cf.* 3.1.1,**4**), *then this series is also convergent, and both series have the same value.*

Proof 1. Because of the convergence of $\Sigma|a_\nu|$ and the necessity of condition (1^a) in the second main test (3.4, Theorem 1), having chosen $\varepsilon>0$ it is possible to determine an index μ such that

$$(12)\qquad |a_{\mu+1}|+\dots+|a_{\mu+\rho}|<\varepsilon$$

for every integer $\rho\geqq 1$. Now if (*cf.* 3.1.1,**4**) $a'_\nu=a_{n_\nu}$, and m is so large that the numbers $0, 1, \dots, \mu$ all appear among the numbers $n_0, n_1, \dots, n_m$, then, for $n>m$, the terms $a_0, a_1, \dots, a_\mu$ evidently cancel in s'_n-s_n, since they appear in s'_n as well as in s_n. The difference s'_n-s_n is thus equal to the sum of finitely many of the terms $\pm a_{\mu+1}$, $\pm a_{\mu+2}, \dots$. According to (12), however, the sum of the absolute values of arbitrarily many of them remains $<\varepsilon$. Hence, for $n>m$, invariably $|s'_n-s_n|<\varepsilon$, *i.e.*, $\{s'_n-s_n\}$ is a null sequence. Thus, on account of $s'_n=s_n+(s'_n-s_n)$, $s'_n\to s$ with $s_n\to s$.

Proof 2. Because of Theorem 3 in 3.5, it suffices to prove the theorem for real series. If $\{a_\nu\}$ is real and $\Sigma|a_\nu|$ converges, then, by 3.6, Theorem 3, the series

$$\sum \frac{|a_\nu|+a_\nu}{2} \quad \text{and} \quad \sum \frac{|a_\nu|-a_\nu}{2}$$

are also convergent (*cf.* 3.5, Theorem 1). Both, however, are series of positive terms, which, by 3.1.1,**4**, are unaffected by a rearrangement. The first of these series is obtained from Σa_ν by replacing all negative terms by 0, the second by replacing all positive terms by 0 and multiplying the new series term by term by -1. If their respective sums are denoted by s^+ and s^-, then (by Theorem 3) $s = s^+ - s^-$. Again denoting rearrangement by an accent, the series

$$\Sigma \tfrac{1}{2}(|a'_\nu|+a'_\nu) \quad \text{and} \quad \Sigma \tfrac{1}{2}(|a'_\nu|-a'_\nu)$$

are thus also convergent with the respective values s^+ and s^-. By subtraction (once more according to Theorem 3) it follows finally that $\Sigma a'_\nu$ converges too and has the value $s' = s^+ - s^- = s$.

Before this Theorem 4, we presented a special series which was transformed, by a rearrangement, into another convergent series, but one with a different value. The importance of Theorem 4 is enhanced by the remark that the same is possible for *every* nonabsolutely convergent series. First we prove

Theorem 5. *If Σa_ν is a convergent, but not an absolutely convergent, series, then there are rearrangements, $\Sigma a'_\nu$, of it that diverge.*

Proof. Here again it is sufficient, on the basis of Theorem 3 in 3.5, to prove the assertion for real series. Let us then denote those a_ν which are $\geqq 0$, in the order of succession in which they appear[1] in Σa_ν, by $p_0, p_1, p_2, \ldots$; likewise those < 0 by $-q_0, -q_1, -q_2, \ldots$. Then Σp_ν and Σq_ν are series of positive terms, both of which contain infinitely many terms which are positive (> 0) in the narrower sense. If $\{P_\nu\}$ and $\{Q_\nu\}$ are the sequences of their partial sums, then at least one of these sequences must $\to +\infty$. For if both were bounded, then the sequence of partial sums of $\Sigma|a_\nu|$ would also remain bounded, and

[1] That is, skipping the negative terms.

hence this series would converge, contrary to our hypothesis.[1] If, say, $P_\nu \to +\infty$, then we construct a series of the form

$$(13) \qquad p_0+p_1+\dots+p_{\nu_0}-q_0+p_{\nu_0+1}+\dots+p_{\nu_1}-q_1+p_{\nu_1+1}+\dots+ \\ +p_{\nu_2}-q_2+p_{\nu_2+1}+\dots,$$

in which a group of positive terms is followed every time by a negative term. For a suitable choice of the indices $0 \leqq \nu_0 < \nu_1 < \nu_2 < \dots$, this series, which is obviously a rearrangement, $\Sigma a'_\nu$, of Σa_ν, is definitely divergent. For this purpose we need only choose ν_0 so large that $p_0+p_1+\dots+p_{\nu_0} > 1+q_0$, then $\nu_1 > \nu_0$ so that $p_0+\dots+p_{\nu_0}+\dots+p_{\nu_1} > > 2+q_0+q_1$, and in general $\nu_\lambda > \nu_{\lambda-1}$ so large that

$$p_0+p_1+\dots+p_{\nu_\lambda} > \lambda+1+q_0+q_1+\dots+q_\lambda,$$

$\lambda = 0, 1, 2, \dots$. Such a choice of ν_λ is always possible, because $P_\nu \to +\infty$. The series (13) is then obviously definitely divergent. For, that partial sum of this series whose last term is $-q_\lambda$ is $> \lambda$, and this holds all the more for the ones that follow. Hence, $s'_\nu \to +\infty$. We have thus proved

Theorem 6. *A convergent series remains convergent under every rearrangement if, and only if, it converges absolutely. It then also retains its value under every rearrangement.*[2]

Before Theorem 4, we saw that, in special cases, under a rearrangement, the convergence could be retained, but the value of the series could be altered. It is not difficult to show that this is possible for every real, nonabsolutely convergent series. This also holds for series of complex terms, but the proof is then essentially much more difficult.

It is customary to designate a convergent series which is unaffected by rearrangement as unconditionally convergent, one which is affected,[3]

---

[1] Actually both sequences $\to +\infty$. For if we had, say, $P_\nu \to +\infty$, $Q_\nu \to Q < \infty$, then the partial sums s_ν of the series Σa_ν would, as is easily seen, $\to +\infty$, contrary to the assumption that this series converges.

[2] We then say for brevity that it is *unaffected* by rearrangement. *The convergence of every nonabsolutely convergent series can be destroyed by a suitable rearrangement.*

[3] We also say that for such a series the order of the terms matters.

however, as (only) conditionally convergent. Theorem 6 then takes on the following form:

Theorem 6. *A series Σa_ν is unconditionally convergent if, and only if, it converges absolutely, and hence is conditionally convergent if, and only if, it is nonabsolutely convergent.*

What is essential in this theorem is that the classification of all convergent series on the one hand into those that converge absolutely and those that converge nonabsolutely, and on the other hand into those that converge unconditionally and those that converge conditionally, takes place according to two points of view which are inherently quite different. Nevertheless, the classes obtained in both cases are the same.

The considerations which we have carried out concerning rearrangements can be generalized in a very essential way. Let Σa_ν be an arbitrary absolutely convergent series. Denote its value by s. According to 2.6.2, Theorem 6, we may dilute this series in an arbitrary manner with zeros. For the sake of simplicity we shall denote the diluted series again by Σa_ν. It is also absolutely convergent and has the value s. If we now imagine $\{a_\nu\}$ to be decomposed, in accordance with 2.1.3,**6**, into two subsequences $\{a'_\nu\}$ and $\{a''_\nu\}$, then, by 3.1.1,**3**, the series $\Sigma a'_\nu$ and $\Sigma a''_\nu$ are also absolutely convergent, and if their respective values are denoted by s' and s'', then $s = s'+s''$ or $\Sigma a_\nu = \Sigma a'_\nu + \Sigma a''_\nu$, and the corresponding result holds if we decompose Σa_ν into $(r+1)$ subseries:

Theorem 7. *Let the absolutely convergent series Σa_ν have the value s, and let*

$$\sum_\nu a_\nu^{(0)} + \sum_\nu a_\nu^{(1)} + \ldots + \sum_\nu a_\nu^{(r)} \tag{14}$$

be a decomposition of Σa_ν into the $(r+1)$ subseries $\Sigma a_\nu^{(\rho)}$, ($\rho = 0, 1, \ldots, r$; $r > 0$ an integer and fixed). Then each of these subseries is absolutely convergent, and if their respective values are denoted by $\alpha_0, \alpha_1, \ldots, \alpha_r$, then

$$\alpha_0+\alpha_1+\ldots+\alpha_r = s \text{ and moreover } |\alpha_0|+|\alpha_1|+\ldots+|\alpha_r| \leqq \sum_\nu |a_\nu|. \tag{15}$$

PROOF. The absolute convergence of the subseries is guaranteed by 3.1.1, **3**. If we add these subseries term by term (removing the parentheses) according to 3.6, Theorem 3, we evidently obtain a rearrangement of Σa_ν, which thus again has the sum s. This proves the first of the relations (15). If we start with the series $\Sigma|a_\nu|$ and carry out the same steps, it follows quite analogously, if we set $\sum_\nu |a_\nu^{(\rho)}| = \beta_\rho$, $(\rho = 0, 1, \ldots, r)$, that $\beta_0+\beta_1+\ldots+\beta_r = \Sigma|a_\nu|$. Since $|\alpha_\rho| \leqq \beta_\rho$ (by 3.5, Theorem 1), this contains the second relation (15).

We shall show that the corresponding result holds actually for decompositions into infinitely many subseries. To this end, let $\sum_{\lambda=0}^{\infty} a_{0\lambda}$ be a first subseries of Σa_ν. Let it be chosen so that the remaining subseries retains infinitely many terms. From this remaining subseries, take a new subseries $\sum_{\lambda=0}^{\infty} a_{1\lambda}$, *etc.* We write down these series in rows, one under another:

$$(16)\qquad \begin{cases} a_{00}+a_{01}+\ldots+a_{0\lambda}+\ldots \\ a_{10}+a_{11}+\ldots+a_{1\lambda}+\ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{\varkappa 0}+a_{\varkappa 1}+\ldots+a_{\varkappa\lambda}+\ldots \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{cases}$$

By 3.1.1,**3**, every row is an absolutely convergent series. [1] We denote their values by $\alpha_0, \alpha_1, \ldots, \alpha_\varkappa, \ldots$:

$$(17)\qquad \sum_{\lambda=0}^{\infty} a_{\varkappa\lambda} = \alpha_\varkappa, \qquad (\varkappa = 0, 1, \ldots, \text{fixed}).$$

For corresponding reasons, the elements of every column are the elements of an absolutely convergent series. We set

$$(18)\qquad \sum_{\varkappa=0}^{\infty} a_{\varkappa\lambda} = \alpha'_\lambda, \qquad (\lambda = 0, 1, \ldots, \text{fixed}).$$

[1] The effect of the insertion of zeros introduced at the beginning is that for these row series the case is admitted that one or another of them contains only a finite number of terms (or none at all) of the original series, so that a row series may be only a finite subseries of the original.

We shall show that invariably

$$(19) \qquad s = \sum_{\varkappa=0}^{\infty} \alpha_\varkappa = \sum_{\lambda=0}^{\infty} \alpha'_\lambda,$$

or, in greater detail,

$$(20) \qquad \sum_{\nu=0}^{\infty} a_\nu = \sum_{\varkappa=0}^{\infty}\left(\sum_{\lambda=0}^{\infty} a_{\varkappa\lambda}\right) = \sum_{\lambda=0}^{\infty}\left(\sum_{\varkappa=0}^{\infty} a_{\varkappa\lambda}\right);$$

in other words, that the following *extended rearrangement theorem* holds:

Theorem 8. *Let the absolutely convergent series* $\sum a_\nu$ *have the value* s, *and let* (16) *be a decomposition of this series (which may be diluted beforehand in an arbitrary way) into a sequence of subseries in the manner described. Then every "row series"* $\sum\limits_{\lambda} a_{\varkappa\lambda}$ *and every "column series"* $\sum\limits_{\varkappa} a_{\varkappa\lambda}$ *in this schema* (16) *is absolutely convergent, and the values of all these series are connected by the relation* (19), *or, in greater detail,* (20), *where all the series that appear are again absolutely convergent.*

Proof. That the series (17) and (18) converge absolutely follows once more from 3.1.1, **3**. The proof of the first half of (19) is very similar to that of the preceding theorem; it is merely necessary to note that we now have infinitely many subseries. To this end, having chosen $\varepsilon > 0$, we first determine, according to 2.6.2, Theorem 3, a $\nu_0 = \nu_0(\varepsilon)$ so that the remainder

$$(21) \qquad |a_{\nu_0+1}| + |a_{\nu_0+2}| + \dots < \varepsilon,$$

and then choose a $\varkappa_0 = \varkappa_0(\varepsilon)$ so large that all the terms $a_0, a_1, \dots, a_{\nu_0}$ appear in the subseries $\sum\limits_{\lambda} a_{\varkappa\lambda}$ with $\varkappa = 0, 1, \dots, \varkappa_0$. Then, if $\mu = \mu(\varepsilon) = \max(\varkappa_0, \nu_0)$, and $\nu > \mu$, the series[1]

$$(22) \qquad (\alpha_0 + \alpha_1 + \dots + \alpha_\nu - s_\nu),$$

after all its terms that occur with a plus and with a minus sign have been cancelled (by virtue of 2.6.2, Theorems 5 and 6), contains only

[1] Here again the series with the sums $\alpha_0, \alpha_1, \dots, \alpha_\nu$ are to be added term by term and the parentheses removed in accordance with Theorem 3, and the finitely many terms $-a_0, -a_1, \dots, -a_\nu$ inserted, by virtue of 2.6.2, Theorem 5, anywhere in the resulting series.

such terms $\pm a_\rho$ for which $\rho > \nu_0$. Hence, by (21) and 3.5, Theorem 1,

$$|\alpha_0+\alpha_1+\ldots+\alpha_\nu - s_\nu| < \varepsilon \quad \text{for} \quad \nu > \mu.$$

The difference (22) thus $\to 0$ as $\nu \to \infty$, and consequently, since $\lim s_\nu$ exists and is equal to s, $\lim\limits_{\nu\to\infty} (\alpha_0+\alpha_1+\ldots+\alpha_\nu)$ also exists and equals s. Therefore $\sum\limits_{\varkappa} \alpha_\varkappa = s$. If we carry out exactly the same steps with $\Sigma|a_\nu|$ and set $\sum\limits_{\lambda}|a_{\varkappa\lambda}| = \beta_\varkappa$, it follows that $\sum\limits_{\varkappa}\beta_\varkappa$ converges. Since $|\alpha_\varkappa| \leqq \beta_\varkappa$ (by 3.5, Theorem 1), $\Sigma\alpha_\varkappa$ is also absolutely convergent. The corresponding assertions concerning the column series are obtained in an entirely similar manner.

The converse of this theorem (that is to say, that the absolute convergence of all the series (16) and of the series $\Sigma\alpha_\varkappa$ implies the convergence of the series Σa_ν and $\Sigma\alpha'_\lambda$ as well as the equality (19) or (20)) need not hold. This is shown already by the trivial example obtained by taking for all row series in (16) the series $1-1+0+0+0+\ldots$. Thus, in order to obtain such a converse, we must insert additional restrictive conditions. We prove

Theorem 9. *If we are given a sequence of absolutely convergent series* $\sum\limits_{\lambda} a_{\varkappa\lambda}$, $(\varkappa = 0, 1, \ldots)$, *written down one under another as in* (16), *and if, with the notation* $\sum\limits_{\lambda} a_{\varkappa\lambda} = \alpha_\varkappa$ *and* $\sum\limits_{\lambda}|a_{\varkappa\lambda}| = \beta_\varkappa$, *not only* $\Sigma\alpha_\varkappa$ *converges, but also* $\Sigma\beta_\varkappa$ *is convergent and has the value* β, *then all the series* (18), *the two series in* (19), *as well as the series* Σa_ν, *converge absolutely, and their sums are connected by the relation expressed in* (19) *and* (20).

The *proof* is extremely simple. According to 2.2,**8**, the totality of numbers $a_{\varkappa\lambda}$ can be arranged in a simple sequence $\{a_\nu\}$ in many ways, and the series Σa_ν formed with it. This series is absolutely convergent. For, a partial sum

$$(23) \qquad |a_0|+|a_1|+\cdots+|a_\nu|$$

of $\Sigma|a_\nu|$ is obviously $\leqq \beta_0+\beta_1+\ldots+\beta_\varkappa$ if we choose $\varkappa$ so large that the terms $a_0, a_1, \ldots, a_\nu$ appear in the series $\alpha_0, \alpha_1, \ldots, \alpha_\varkappa$. Hence, the partial sums (23) remain bounded, namely $\leqq \beta$, so that Σa_ν is absolutely convergent. Call its value s. Two different arrangements of the $a_{\varkappa\lambda}$ in a simple sequence obviously yield two series Σa_ν which are merely rearrangements of each other. All these series are therefore absolutely convergent and possess the same value s. If Σa_ν is a certain one of them, then (16), or better, the two iterated series in (20), are extended rearrangements of Σa_ν in the sense of Theorem 8. This theorem therefore immediately yields the further assertions of Theorem 9.

Corollary 1. The decisive auxiliary condition in Theorem 9 was that $\Sigma\beta_\varkappa$ also be convergent. It is easy to see that it is equivalent to the following: There exists a number $K>0$ such that the sum of the absolute values of finitely many terms of the schema (16) is invariably $\leqq K$. For if $\Sigma\beta_\varkappa$ is convergent and $=\beta$, then obviously $K=\beta$ does the trick. If, conversely, a $K>0$ of the kind described exists, then first of all $\Sigma|a_\nu|$ is convergent (because the partial sums are bounded), and Theorem 8 shows, provided that we apply it to $\Sigma|a_\nu|$, that all $\beta_\varkappa$ as well as $\Sigma\beta_\varkappa = \beta$ exist.—Likewise, the convergence of $\Sigma\beta'_\lambda$, $(\beta'_\lambda = \sum_\varkappa |a_{\varkappa\lambda}|)$, is equivalent to the two conditions just discussed.

Corollary 2. The series appearing in the second and third places of (20) are designated as iterated series, because summation is performed twice—first by rows and then over the row values, or first by columns and then over the column values. The entire schema (16), if we imagine a plus sign to be placed before each of the terms a_{10}, a_{20}, ..., is called a double series, and is also designated, for brevity, by $\Sigma a_{\varkappa\lambda}$, $(\varkappa, \lambda = 0, 1, \ldots)$. Under any one of the equivalent assumptions mentioned in Corollary 1, we regard the then well-determined number s as its value. (We shall consider double series only under one of these assumptions.)

Corollary 3. The theorems we have proved are frequently applied in the following way: An arbitrary series $\sum_\varkappa \alpha_\varkappa$ with the value s is given. Every one of its terms is represented in any manner as the value of an infinite series

(24) $$\alpha_\varkappa = a_{\varkappa 0}+a_{\varkappa 1}+\ldots+a_{\varkappa\lambda}+\ldots, \quad (\varkappa = 0, 1, \ldots),$$

these series being written down in rows one under another in the form of the schema (16). Then if these row series converge absolutely, and we set $\sum_\lambda |a_{\varkappa\lambda}| = \beta_\varkappa$, and if, finally, $\Sigma\beta_\varkappa$ converges too, then the column series are also all absolutely convergent,

$$\sum_{\varkappa=0}^{\infty} a_{\varkappa\lambda} = \alpha'_\lambda, \qquad (\lambda = 0, 1, \ldots),$$

the series $\sum_\lambda \alpha'_\lambda$ converges absolutely too, and the relations (19) and (20) hold.[1]

The convergence of the series $\Sigma\beta_\varkappa$ (in addition to the existence of the numbers $\beta_\varkappa$) proved to be sufficient for the validity of all these theorems. We conclude these considerations with the presentation of another condition concerning the interchangeability of the order of summation in (20), which is not only sufficient but is actually necessary, and which was given essentially by *A. A. Markoff.*[2]

For this purpose we imagine a situation similar to that just described in Corollary 3: A convergent series $\Sigma\alpha_\varkappa = s$ is given, and every one of its terms $\alpha_\varkappa$ is represented as the value of an infinite series (24). These series again are written down in rows one under another in the form of the schema (16). Instead, however, of assuming any absolute convergence, only the convergence of each individual column series $\sum_\varkappa a_{\varkappa\lambda} = \alpha'_\lambda$, $(\lambda = 0, 1, \ldots)$, is further assumed. Then automatically the series $\sum_\varkappa(\alpha_\varkappa - a_{\varkappa 0})$, $\sum_\varkappa(\alpha_\varkappa - a_{\varkappa 0} - a_{\varkappa 1})$, and, in general, the series

$$(25) \qquad \sum_\varkappa(\alpha_\varkappa - a_{\varkappa 0} - a_{\varkappa 1} - \ldots - a_{\varkappa\lambda}), \qquad (\lambda = 0, 1, \ldots, \text{fixed}),$$

are convergent. The general term of this series is obviously the remainder beginning after the λ^{th} term of the series (24). Let us denote this remainder by $\rho_{\varkappa\lambda}$. Consequently $\sum_\varkappa \rho_{\varkappa\lambda} = \rho_\lambda$ is convergent. With this notation we now have

Theorem 10. *In the situation just described, the series of column sums* $\sum_\lambda \alpha'_\lambda$ *is convergent and equal to* $\sum_\varkappa \alpha_\varkappa$ *(or, in other words, the transition from the*

[1] The second relation is also expressed by saying that in the iterated series in (20), the order of summation may be interchanged.

[2] *Cf.* in this connection: K. Knopp, *Einige Bemerkungen zur Kummerschen und Markoffschen Reihentransformation*, Sitzungsberichte der Berl. Math. Ges., vol. 19 (1921), pp. 4–17, and *Infinite Series* (see Bibliography), 2nd edition, p. 242.

first to the second of the iterated series (20) *is permissible*) *if, and only if,* $\rho_\lambda \to 0$ *as* $\lambda \to \infty$.

The *proof* is most simple. For, the value of the series (25) is, on the one hand, as already remarked, equal to $\sum\limits_{\varkappa} \rho_{\varkappa\lambda} = \rho_\lambda$, and, on the other hand, according its very formation, $= s - \alpha'_0 - \alpha'_1 - \ldots - \alpha'_\lambda$. Consequently

$$\alpha'_0 + \alpha'_1 + \ldots + \alpha'_\lambda = s - \rho_\lambda$$

and $\to s$ as $\lambda \to \infty$ (*i.e.*, equalities (19) and (20) hold) if, and only if, $\rho_\lambda \to 0$.[1]

The transition from the series $\sum\limits_{\varkappa} \alpha_\varkappa$ to the series $\sum\limits_{\lambda} \alpha'_\lambda$ through the mediation of (24) is designated as a *series transformation*—in this case, as *Markoff's transformation.*

The theorems acquired put us now in a position to make assertions concerning the further validity of the distributive law. Since $a \cdot \sum\limits_{\lambda} b_\lambda = \sum\limits_{\lambda} ab_\lambda$ (*i.e.*, if $\sum\limits_{\lambda} b_\lambda$ converges, then, for every a, $\sum\limits_{\lambda} (ab_\lambda)$ is also convergent, and the equality written down is valid), and in the corresponding sense, for a fixed number k of terms $a_\varkappa$, the equation

$$(26) \quad (a_1 + a_2 + \ldots + a_k) \cdot \sum_{\lambda} b_\lambda = \sum_{\lambda} (a_1 b_\lambda) + \sum_{\lambda} (a_2 b_\lambda) + \ldots + \sum_{\lambda} (a_k b_\lambda)$$

holds, the question here is merely whether or in what sense the product of two convergent series $\sum\limits_{\varkappa} a_\varkappa = A$ and $\sum\limits_{\lambda} b_\lambda = B$ can be formed in an analogous fashion. For example, is

$$(27) \quad \left(\sum_{\varkappa=0}^{\infty} a_\varkappa\right)\left(\sum_{\lambda=0}^{\infty} b_\lambda\right) = \sum_{\lambda} (a_0 b_\lambda) + \sum_{\lambda} (a_1 b_\lambda) + \ldots + \sum_{\lambda} (a_\varkappa b_\lambda) + \ldots,$$

i.e., is the series $\sum C_\varkappa$ with the terms $C_\varkappa = \sum\limits_{\lambda} (a_\varkappa b_\lambda)$ convergent and is

---

[1] Actually, it follows, more precisely, that the series $\sum\limits_{\lambda} \alpha'_\lambda$ of column sums converges if, and only if, the sequence $\{\rho_\lambda\}$ converges. If its limit is $= \rho$, then

$$\sum_{\lambda} \alpha'_\lambda = \sum_{\varkappa} \alpha_\varkappa + \rho.$$

The reader should construct an example in which all the assumptions used are fulfilled, but $\rho \neq 0$.

its value $C = AB$? This is evidently always the case, for, by what we said before, $C_\varkappa = a_\varkappa \cdot \Sigma b_\lambda = a_\varkappa \cdot B$, hence

$$C_0 + C_1 + \ldots + C_k = A_k \cdot B, \qquad (A_k = a_0 + a_1 + \ldots + a_k),$$

which $\to AB$. The answer is less simple if we adhere to the wording with which we formulated the distributive law at the beginning in 3.6: May one multiply two convergent series $\Sigma a_\varkappa = A$ and $\Sigma b_\lambda = B$ by multiplying every term $a_\varkappa$ of the first by every term b_λ of the second and forming a simple series Σp_ν from the products $a_\varkappa b_\lambda$ taken in an arbitrary order of succession—*i.e.*, is this series convergent and does it have the value AB? If, however, the equality

$$\Sigma a_\varkappa \cdot \Sigma b_\lambda = \Sigma p_\nu = AB \tag{28}$$

is to be valid for an arbitrary arrangement of the products (*cf.* 2.2,**8**) $a_\varkappa b_\lambda = p_\nu$, then Σp_ν must converge unconditionally, and hence absolutely. This must then also be the case for every subseries, *e.g.*, for the series of all products $a_\varkappa b_\lambda$ for which $\varkappa$ has a certain fixed value. Thus, for $\varkappa$ fixed, the series $\sum\limits_\lambda a_\varkappa b_\lambda$ or $a_\varkappa \cdot \sum\limits_\lambda b_\lambda$, hence finally Σb_λ, must converge absolutely. Similarly, the absolute convergence of $\Sigma a_\varkappa$ is a necessary condition for the validity of the equality (28) for an arbitrary arrangement of the products p_ν. We shall show that this is also sufficient.

Theorem 11. *Let $\Sigma a_\varkappa = A$ and $\Sigma b_\lambda = B$ be two convergent series, and $\{p_\nu\}$ be the totality of products $a_\varkappa b_\lambda$ arranged in a simple sequence. Then Σp_ν converges unconditionally (i.e., for every arrangement of the p_ν) if, and only if, $\Sigma a_\varkappa$ and Σb_λ are absolutely convergent; Σp_ν then has the "correct" value $P = AB$.*

Proof. It remains for us to show merely that the absolute convergence of $\Sigma a_\varkappa$ and Σb_λ is sufficient for the validity of (28). If we denote by K the product of the values of $\Sigma |a_\varkappa|$ and $\Sigma |b_\lambda|$, then obviously, for every r,

$$|p_0| + |p_1| + \cdots + |p_r| \leqq K. \tag{29}$$

For if μ is the greatest of the indices $\varkappa$, λ appearing in the products $a_\varkappa b_\lambda$ denoted by $p_0, p_1, \ldots, p_r$, then the sum on the left in (29) is

$$\leqq (|a_0|+\ldots+|a_\mu|)\,(|b_0|+\ldots+|b_\mu|) \leqq \sum_\varkappa |a_\varkappa|\cdot\sum_\lambda |b_\lambda| = K.$$

Thus, for every arrangement of the products $a_\varkappa b_\lambda$ in a sequence $\{p_\nu\}$, the series Σp_ν converges and has always the same value, call it P. If, however, in particular, we arrange the products "by squares" (*cf.* 2.2,8b)), then, provided that $a_n b_n$ occupies the lower right-hand corner of the square and we set $(n+1)^2-1 = m$,

$$p_0+p_1+\ldots+p_m = \left(\sum_{\varkappa=0}^{n} a_\varkappa\right)\left(\sum_{\lambda=0}^{n} b_\lambda\right).$$

As $n\to\infty$, the right side $\to AB$. On the left is a subsequence of the sequence of partial sums P_ν of Σp_ν; it, therefore, just as $\{P_\nu\}$ itself, $\to P$. Hence, $P=AB$.

In Theorem 11 we required Σp_ν to converge unconditionally (for every arrangement of the p_ν). It is conceivable—and it is in fact true—that for special arrangements of the products $a_\varkappa b_\lambda$, the series Σp_ν is convergent under weaker assumptions concerning the factor series $\Sigma a_\varkappa$, Σb_λ. For applications, the most important arrangement is that by diagonals (see 2.2,8a)), to which one is led by the elementary process of multiplying out two polynomials

$$\begin{aligned}(a_0+a_1z+\ldots+a_kz^k)\,(b_0+b_1z+\ldots+b_lz^l) &= \\ = a_0b_0+(a_0b_1+a_1b_0)z+\ldots+(a_0b_\nu+a_1b_{\nu-1}+\ldots+a_\nu b_0)z^\nu+\ldots.&\end{aligned}$$ [1]

The series

$$(30)\qquad a_0b_0+(a_0b_1+a_1b_0)+\ldots+(a_0b_\nu+a_1b_{\nu-1}+\ldots+a_\nu b_0)+\ldots$$

is to be regarded accordingly as the product series. It is designated as the *Cauchy product* of $\Sigma a_\varkappa$ and Σb_λ.[2]

According to Theorem 11, we immediately have the *theorem of Cauchy:*

---

[1] If we set $a_\varkappa=0$ for $\varkappa>k$ and $b_\lambda=0$ for $\lambda>l$, then the terms on the right, from a certain stage on, $=0$: the series "terminates".

[2] We emphasize expressly that the parentheses are to be left in.

Theorem 12. *If $\Sigma a_\varkappa = A$ and $\Sigma b_\lambda = B$ are two absolutely convergent series, then their Cauchy product*

$$\sum_{n=0}^{\infty} c_n \quad \text{with} \quad c_n = a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0 \tag{31}$$

is absolutely convergent and its value C is equal to AB. [1]

That the *Cauchy* product of two arbitrary convergent series $\Sigma a_\nu = A$ and $\Sigma b_\nu = B$ need not converge at all is shown by the following example, where the series $\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\sqrt{\nu+1}}$ is chosen for both factor series. Then we have

$$c_n = (-1)^n \left[\frac{1}{\sqrt{1\cdot(n+1)}} + \frac{1}{\sqrt{2\cdot n}} + \ldots + \frac{1}{\sqrt{(n+1)\cdot 1}} \right].$$

If we replace all the natural numbers (factors) under the radical sign by the largest of them, *viz.*, $(n+1)$, we see that

$$|c_n| \geqq \frac{n+1}{\sqrt{(n+1)\,(n+1)}} = 1$$

for all n; Σc_n is not convergent.

For $|z|<1$, Σz^ν is absolutely convergent and $= \frac{1}{1-z}$. The *Cauchy* product of this series by itself yields the representation

$$\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}, \qquad (|z|<1).$$

The series $\Sigma \frac{z^\nu}{\nu!}$ is (see 3.2.1,2) absolutely convergent for every z. We shall call the function furnished by it $f(z)$. If we choose any two numbers z_1 and z_2, then the *Cauchy* product

---

[1] Later on (5.7) we shall show that the absolute convergence of only *one* of the two factor series is already sufficient for the convergence of the *Cauchy* product (to the correct value).

$$f(z_1)\cdot f(z_2) = \sum_{\nu} \frac{z_1^{\nu}}{\nu!}\cdot\sum_{\nu} \frac{z_2^{\nu}}{\nu!} = \sum_{n=0}^{\infty}\left(\frac{z_2^n}{0!\,n!} + \frac{z_1 z_2^{n-1}}{1!\,(n-1)!} + \dots + \frac{z_1^n}{n!\,0!}\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left[\binom{n}{0} z_2^n + \binom{n}{1} z_2^{n-1} z_1 + \dots + \binom{n}{n} z_1^n\right]$$

$$= \sum_{n=0}^{\infty}\frac{(z_1+z_2)^n}{n!} = f(z_1+z_2)\,.$$

The function represented by our series is designated as the *exponential function* and denoted by $\exp z$ or e^z. It thus satisfies the important *functional equation* or the *addition theorem*

$$\exp\,(z_1+z_2) = \exp z_1\cdot \exp z_2 \quad \text{or} \quad e^{z_1+z_2} = e^{z_1}\cdot e^{z_2},$$

and more generally, of course, for p numbers $z_1, \dots, z_p$,

$$\exp\,(z_1+z_2+\dots+z_p) = \exp z_1\cdot \exp z_2\cdot \dots \cdot \exp z_p\,.$$

This function will be considered in greater detail in 6.3. At the moment we note merely that for $z = x \geqq 0$, $\exp x \geqq 1+x$, and hence, for any nonnegative numbers $a_0, a_1, \dots, a_n$, we have the inequality

(32) $$(1+a_0)\,(1+a_1)\dots(1+a_n) \leqq \exp\,(a_0+a_1+\dots+a_n).$$

3.7. Infinite products

Although infinite products (see 1.1, (9)) will not be treated in detail in this little volume, it is nevertheless useful to know a few simple facts about them, because they often serve as a good expedient for handling series.

If $\{u_\nu\}$ is an arbitrary sequence of numbers, then the symbol

(1) $$\prod_{\nu=0}^{\infty} u_\nu \equiv u_0\cdot u_1\cdot\dots\cdot u_\nu\cdot\dots$$

shall denote the sequence of partial products

(2) $$p_\nu = u_0\cdot u_1\cdot\dots\cdot u_\nu\,.$$

Every convergence property of the sequence (2) is then ascribed, under restrictions to be stated immediately, to the infinite product (1) itself.

These restrictions are called for by the exceptional role played by 0 in multiplication. If, *e.g.*, for a single factor, say u_n, of the product, we had $u_n = 0$, then we should have all $p_\nu = 0$ as soon as $\nu > n$. The sequence p_ν would thus be convergent with the limiting value 0, regardless of the nature of the factors u_ν. Likewise, every product (1) for which, for a fixed θ in $0 < \theta < 1$, $|u_\nu| \leqq \theta$ from a certain index on, would obviously be convergent and again have the value 0. To exclude these meaningless cases, it is customary to make the following definition:

Definition 1. *The infinite product* (1) *shall be called convergent* (*in the narrower sense*) *if, and only if, from a certain stage on, say for all* $\nu > \mu$,[1] *the factors* $u_\nu \neq 0$, *and the partial products*

$$p'_\nu = u_{\mu+1} \cdot u_{\mu+2} \cdot \ldots \cdot u_\nu, \qquad (\nu > \mu),$$

beginning after the μ^{th} *factor tend to a limit* P' *different from* 0. *The number*

$$P = u_0 \cdot u_1 \cdot \ldots \cdot u_\mu \cdot P'$$

is then regarded as the value of the product (1).[2]

According to this definition, we have first of all, as in the case of "ordinary products", *i.e.*, products with finitely many factors, the following

Theorem 1. *A convergent infinite product has the value* 0 *if, and only if, one of the factors* $= 0$.

Furthermore, in analogy with 2.6.2, Theorem 1, we have

Theorem 2. *In a convergent product* (1), *the sequence of factors* $u_\nu \to 1$.

For according to Definition 1, for $\nu > \mu + 1$ we have

$$u_\nu = \frac{p'_\nu}{p'_{\nu-1}} \quad \text{and thus} \quad \to \frac{P'}{P'} = 1.$$

[1] For μ we may thus take the index of the last factor u_ν having the value 0, or any larger number. If no such factor exists, set $\mu = -1$.

[2] P is independent of the choice of μ. For if μ is replaced by $\mu' > \mu$, then the partial products $p''_\nu = u_{\mu'+1} \ldots u_\nu$, $(\nu > \mu')$, obviously tend to the value $P'' = (u_{\mu+1} \ldots u_{\mu'})^{-1} \cdot P'$, and we have once more

$$u_0 \cdot u_1 \cdot \ldots \cdot u_{\mu'} \cdot P'' = u_0 \cdot u_1 \cdot \ldots \cdot u_\mu \cdot P' = P.$$

Because of this theorem, it is customary to write the factors u_ν in the form

$$u_\nu = 1+a_\nu, \tag{3}$$

and hence the infinite product (1) itself in the form

$$\prod_{\nu=0}^{\infty} (1+a_\nu), \tag{4}$$

for whose convergence $a_\nu \to 0$ is now a necessary (but by no means sufficient) condition. Whereas we called the u_ν the *factors* of the product (1), we shall call the a_ν in (4) its *terms*. On account of the prefatory remark, we only consider such products if there exists a μ such that $a_\nu \neq -1$ for $\nu > \mu$.

The products with positive terms are again especially simple to treat because for them the sequence of partial products $p_\nu = (1+a_0) \ldots (1+a_\nu)$ increases monotonically. The product is therefore convergent if, and only if, these p_ν form a bounded sequence. This leads to

Theorem 3. *A product* $\Pi(1+a_\nu)$ *with positive terms* a_ν *is convergent if, and only if,* Σa_ν *converges.*

Proof. I. If the product is convergent, that is, if the partial products are bounded, then, because of

$$a_0+a_1+\ldots+a_\nu \leqq (1+a_0)(1+a_1)\ldots(1+a_\nu),$$

the partial sums of Σa_ν are obviously also bounded, so that Σa_ν is convergent.

II. If Σa_ν is convergent, then, by 3.6,(32),

$$(1+a_0)(1+a_1)\ldots(1+a_\nu) \leqq \exp(a_0+a_1+\ldots+a_\nu),$$

and the boundedness of the partial sums of Σa_ν implies that of the partial products of $\Pi(1+a_\nu)$. Herewith everything is already proved —and beyond that, the inequality

$$P \leqq \exp s, \tag{5}$$

if P denotes the value of the product and s the value of the series.

For products (4) with arbitrary (real or complex) terms a_ν, the second main test for sequences is available. It furnishes, if we again set

$$(1+a_{\mu+1}) \dots (1+a_\nu) = p'_\nu$$

for $\nu > \mu$, a necessary and sufficient condition that p'_ν tend to a finite limit different from 0 as $\nu \to \infty$. We can give it the following form:

Theorem 4. *The infinite product* (4) *is convergent* (*in the narrower sense*) *if, and only if, having chosen an arbitrary number* $\varepsilon > 0$, *one can assign an index* ν_0 *such that, for all* $\nu > \nu_0$ *and all* $\rho \geqq 1$, *the inequality*

$$(6) \qquad |(1+a_{\nu+1}) \dots (1+a_{\nu+\rho}) - 1| < \varepsilon$$

is satisfied. [1]

Proof. I. If $\Pi(1+a_\nu)$ is convergent in the narrower sense, then there exists a μ such that $1+a_\nu \neq 0$ for $\nu > \mu$. The partial products p'_ν begun after this stage are thus $\neq 0$ and tend to a limit different from 0. Hence, there exists (*cf.* 2.3.1,4) a number $\gamma > 0$ such that $|p'_\nu| \geqq \gamma > 0$ for all $\nu > \mu$. According to the second main test for sequences, if $\varepsilon > 0$ is given, we can now determine a ν_0 such that, for all $\nu > \nu_0$ and all $\rho \geqq 1$,

$$|p'_{\nu+\rho} - p'_\nu| < \varepsilon\gamma \quad \text{or} \quad \left|\frac{p'_{\nu+\rho}}{p'_\nu} - 1\right| < \varepsilon.$$

The last, however, is precisely the relation (6) to be proved.

II. If, conversely, (6) is satisfiable to the extent stated in the theorem, then, by choosing, first of all, $\varepsilon = \frac{1}{2}$, (6) asserts that it is possible to determine μ so that, for $\nu > \mu$,

$$|(1+a_{\mu+1}) \dots (1+a_\nu) - 1| < \tfrac{1}{2}.$$

Therefore, in particular, $(1+a_\nu) \neq 0$ for $\nu > \mu$, and, moreover, $|p'_\nu - 1| < \frac{1}{2}$ or $\frac{1}{2} < |p'_\nu| < \frac{3}{2}$, if the p'_ν again denote the partial products begun after the μ^{th} factor ($\nu > \mu$). Thus the sequence $\{p'_\nu\}$, if it converges at all, has a limit different from 0. And that it does converge

[1] In the sense of the first paragraph on p. 68, this may be expressed as follows: The product is convergent if, and only if, the partial products (of arbitrary length) beginning after the index ν_0 lie close to 1 and hence do not noticeably alter the product of the preceding factors any more.

is shown once more by (6). For, (6) asserts that for arbitrary $\varepsilon > 0$ a ν_0 can be determined so that

$$\left| \frac{p'_{\nu+\rho}}{p'_\nu} - 1 \right| < \varepsilon \quad \text{or} \quad |p'_{\nu+\rho} - p'_\nu| < \varepsilon \cdot |p'_\nu| < 2\varepsilon$$

for all $\nu > \nu_0$ and all $\rho \geqq 1$. Hence, by the second main test, $\{p'_\nu\}$ is convergent.

From Theorem 4 we now obtain—*cf.* the corresponding Theorem 1 in 3.5 for series—

Theorem 5. *A product* $\Pi(1+a_\nu)$ *is certainly convergent if* $\Pi(1+|a_\nu|)$ *or (because of Theorem* 3) $\Sigma|a_\nu|$ *is convergent.*

For,

$$|(1+a_{\nu+1}) \ldots (1+a_{\nu+\rho}) - 1| \leqq (1+|a_{\nu+1}|) \ldots (1+|a_{\nu+\rho}|) - 1,$$

as is immediately seen by imagining the products on the left and on the right to be multiplied out, and cancelling $+1$ and -1 on both sides. Thus, if the right side is $< \varepsilon$, then so is the left.[1]

Products $\Pi(1+a_\nu)$ for which "even" $\Pi(1+|a_\nu|)$—which is a product with positive terms—converges shall be called *absolutely convergent*, in analogy with 3.5, Definition 1. According to Theorem 3, then, we have

Theorem 6. *A product* $\Pi(1+a_\nu)$ *is absolutely convergent if, and only if,* Σa_ν *converges absolutely.*

In analogy with Theorem 3, we can now easily prove

Theorem 7. *A product of the form* $\Pi(1-a_\nu)$, *with* $0 \leqq a_\nu < 1$, *is convergent if, and only if,* Σa_ν *converges.*

For if Σa_ν is convergent, then, since $|a_\nu| = |-a_\nu| = a_\nu$, $\Pi(1-a_\nu)$ is actually absolutely convergent. If, conversely, the product is convergent and its value $=P$, then, since $1+a \leqq \dfrac{1}{1-a}$ for $0 \leqq a < 1$, we have

$$(1+a_0)\ (1+a_1) \ldots (1+a_\nu) \leqq [(1-a_0)\ (1-a_1) \ldots (1-a_\nu)]^{-1} \leqq \frac{1}{P},$$

so that $\Pi(1+a_\nu)$, and hence Σa_ν, converges.

[1] *Cf.* in this connection footnote 1, p. 71.

For the purpose of illustration, we list the following examples[1] with brief explanations:

1. The products $\prod\limits_{\nu=1}^{\infty}\left(1+\frac{1}{\nu^\alpha}\right)$ are convergent for $\alpha>1$. For $\alpha=1$ it is divergent, because then the n^{th} partial product is $p_n=\frac{2}{1}\cdot\frac{3}{2}\cdots\cdot\frac{n+1}{n}=n+1$ and thus $\to+\infty$. Likewise, $\prod\limits_{\nu=2}^{\infty}\left(1-\frac{1}{\nu^\alpha}\right)$ is absolutely convergent for $\alpha>1$. For $\alpha=1$, the partial products $p_n=\frac{1}{2}\cdot\frac{2}{3}\cdots\cdot\frac{n-1}{n}=\frac{1}{n}\to 0$, and we say that the product diverges to 0. For $\alpha<1$ it diverges *a fortiori*, because then the partial products are positive but $<\frac{1}{n}$.

2. $\prod\limits_{\nu=2}^{\infty}\left(1-\frac{2}{\nu(\nu+1)}\right)$ is convergent and has the value $\frac{1}{3}$, as is immediately verified by forming the partial products.

3. $\prod\limits_{\nu=1}^{\infty}\left(1-\frac{(-1)^\nu}{\nu}\right)$ is convergent (but nonabsolutely). For, the n^{th} partial product

$$=\frac{2}{1}\cdot\frac{1}{2}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdots\left(1-\frac{(-1)^n}{n}\right),$$

which obviously $\to 1$.

4. $\prod\left(1+\frac{x}{\nu}\right)$ diverges to $+\infty$ for every real $x>0$, diverges to 0 for every real $x<0$. For, the factors are, from a certain stage on, of the form considered in Theorems 3 and 7, and $\sum\frac{x}{\nu}$ is divergent for $x\gtrless 0$.

5. $\prod\limits_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2}\right)$ is convergent[2] for every (complex) z.

6. $\prod\left(1-\frac{1}{\nu^2}\right)$ is (absolutely) convergent and $=\frac{1}{2}$. (Form the partial products.)

---

[1] In order that the following products be meaningful, ν in several of them must start from 1 or 2.

[2] Its value is $=\frac{\sin\pi z}{\pi z}$, as is shown, *e.g.*, in § 3 of the author's *Theory of Functions*, vol. II, listed in IV of the Bibliography.

7. $\Pi\,(1+z^\nu)$ is absolutely convergent for every z with $|z|<1$.

8. If Σa_ν is absolutely convergent, then $\Pi\,(1+a_\nu z)$ is also absolutely convergent for every z.

9. $\Pi\,(1+a_\nu z^\nu)$ converges absolutely for every z for which $\Sigma a_\nu z^\nu$ is absolutely convergent. (Concerning the convergence of such a "power series", see 4.1, Theorem 1.)

The convergence of nonabsolutely convergent products is somewhat more difficult to recognize, and will not be treated here.

Chapter 4

POWER SERIES

4.1. The circle of convergence

We have already encountered several times, series of the form $\Sigma a_\nu z^\nu$, where z has been permitted to be arbitrary to a certain extent. Such series, and, somewhat more generally, series of the form $\Sigma a_\nu (z-z_0)^\nu$, where z_0 is a fixed number, are called *power series.* In what follows, there is usually no loss of generality in considering only power series of the first form. For if we set $z-z_0 = z'$ for abbreviation, and then drop the accent, the second form goes over into the first.

Examples of such series were

$$\Sigma z^\nu, \quad \Sigma \binom{\nu+k}{\nu} z^\nu, \quad \Sigma \frac{z^\nu}{\nu!}, \ \ldots .$$

The first converges if, and only if, $|z| < 1$, *i.e.*, in the interior of the unit circle. The third converges for every z, *i.e.*, "in the entire plane". Finally, $\sum_{\nu=1}^{\infty} \nu^\nu z^\nu$ is an example of a power series that converges only for $z = 0$, because, for $z \neq 0$, $\nu^\nu z^\nu = (\nu z)^\nu$ obviously does not tend to 0.

We shall show, first of all, that every power series possesses an analogous convergence behavior, *i.e.*, that it converges either in the entire plane, or in a certain circle about 0 as center, or only for $z = 0$. Indeed, we have

Theorem 1. *Let $\Sigma a_\nu z^\nu$ be an arbitrary power series, and set $\overline{\lim}\ \sqrt[\nu]{|a_\nu|} = \alpha$. Then,*

a) *for $\alpha = 0$, the series is everywhere convergent,*

b) *for $\alpha = +\infty$, the series is divergent for every $z \neq 0$.*

c) *If, finally, $0 < \alpha < +\infty$, then the series is absolutely convergent for every z with $|z| < r = \dfrac{1}{\alpha}$, divergent for every z with $|z| > r$.* (The behavior of the series on the circumference $|z| = r$ can then be quite varied; see below.) *Thus we have in all three cases, with suitable interpretation,*

$$r = \frac{1}{\alpha} = \frac{1}{\overline{\lim}\ \sqrt[\nu]{|a_\nu|}}. \qquad \textit{(Cauchy-Hadamard formula.)}$$

PROOFS. a) $\alpha = 0$ means that $\sqrt[\nu]{|a_\nu|} \to 0$, because $\sqrt[\nu]{|a_\nu|} \geqq 0$. Hence, if z is an arbitrary number, then also $\sqrt[\nu]{|a_\nu z^\nu|} = |z| \cdot \sqrt[\nu]{|a_\nu|} \to 0$. The assertion now follows from the radical test.

b) Let $\alpha = +\infty$ and $z \neq 0$, so that $|z| > 0$. Then, according to 2.2,5, $\sqrt[\nu]{|a_\nu|} > \frac{1}{|z|}$ or $\sqrt[\nu]{|a_\nu z^\nu|} > 1$ infinitely often, and consequently $\Sigma a_\nu z^\nu$ is divergent.

c) In this case, let z be an arbitrary, but henceforth fixed, number, with $|z| < r = \frac{1}{\alpha}$. Choose a positive number ρ for which $|z| < \rho < \frac{1}{\alpha}$, and hence $\frac{1}{\rho} > \alpha$. Then, in accordance with the meaning of α, we have, for all ν from a certain stage on,

$$\sqrt[\nu]{|a_\nu|} < \frac{1}{\rho} \qquad \text{and therefore} \qquad \sqrt[\nu]{|a_\nu z^\nu|} < \frac{|z|}{\rho} = \theta < 1.$$

Thus, by the radical test, $\Sigma a_\nu z^\nu$ is convergent. If z' is a number with $|z'| > r = \frac{1}{\alpha}$, so that $\frac{1}{|z'|} < \alpha$, then $\sqrt[\nu]{|a_\nu|} > \frac{1}{|z'|}$ or $\sqrt[\nu]{|a_\nu z'^\nu|} > 1$ infinitely often, and consequently $\Sigma a_\nu z'^\nu$ is divergent.

In this (main) case c), then, $\Sigma a_\nu z^\nu$ is absolutely convergent at every interior point of the circle $|z| < r$, divergent at every point in the exterior of that circle. We therefore call this circle the *circle of convergence* of the power series, the number r the *radius of convergence*, and the points of the circumference $|z| = r$ the *boundary of the circle of convergence*. In case a) we set $r = +\infty$, in case b) $r = 0$. In this last case the circle of convergence thus degenerates to the origin and possesses no interior points.

Naturally, for the power series $\Sigma a_\nu (z - z_0)^\nu$, the circle $|z - z_0| < r$ is the circle of convergence. The behavior of power series on the boundary of the circle of convergence can be quite varied, as the following three examples show.

Σz^ν has $r=1$; the series is divergent at every boundary point (*i.e.*, for every z with $|z| = 1$).

$\Sigma \dfrac{z^\nu}{\nu^2}$ also has $r=1$ (why?); the series, however, is absolutely convergent at every boundary point.

$\Sigma \dfrac{z^\nu}{\nu}$ likewise has $r=1$. The series is divergent at $z=+1$, convergent at $z=-1$.

We shall sketch a second proof of the fundamental theorem concerning the existence of a definite circle of convergence for every power series, which, however, does not yield simultaneously a formula for the radius itself.

To this end we first prove: 1) If $\Sigma a_\nu z^\nu$ converges at $z = z_1 \neq 0$, then the series is absolutely convergent for every z for which $|z| < |z_1|$ (in other words, for every z lying in the interior of the circle that passes through z_1 and has 0 as center). For if z is a fixed point of this kind, then $a_\nu z^\nu = (a_\nu z_1^\nu)\left(\dfrac{z}{z_1}\right)^\nu$. If we denote by K an upper bound of the null sequence $\{|a_\nu z_1^\nu|\}$, and denote the proper fraction $\left|\dfrac{z}{z_1}\right|$ by θ, then $|a_\nu z^\nu| \leqq K\theta^\nu$, and consequently $\Sigma a_\nu z^\nu$ is absolutely convergent, as asserted.

Equivalent to this is: 2) If $\Sigma a_\nu z^\nu$ diverges at $z=z_1'$, then the series is divergent for every z for which $|z| > |z_1'|$, in other words, for every z whose distance from the origin is greater than that of z_1'.

If, now, $\Sigma a_\nu z^\nu$ converges neither everywhere nor nowhere (except at 0), then there exists a point of convergence $z_1 \neq 0$ and a point of divergence z_1'. According to the two preliminary remarks, it is therefore possible to assign a positive number r_0 $(<|z_1|)$ such that the series converges for $z=r_0$, and a positive number r_0' $(>|z_1'|)$ such that the series diverges for $z=r_0'$. We now apply the bisection method to the positive real interval $J_0 = (r_0, r_0')$. We designate its left or its right half as J_1, according as $\Sigma a_\nu z^\nu$ diverges or converges at the midpoint $z = \frac{1}{2}(r_0+r_0')$ of J_0. According to the same rule, we designate the left or the right half of J_1 as J_2, *etc.* The intervals J_ν then all have the property that our power series converges at the left endpoint r_ν of J_ν, but diverges at the right endpoint r'_ν. The (positive) number r determined by this nest of intervals is the radius of convergence of the series. For if $|z| < r$, then there exists an r_ν for which $|z| < r_\nu$ ($\leqq r$). Hence, by the first preliminary remark, $\Sigma a_\nu z^\nu$ is convergent, because z lies closer to the origin than r_ν does. If $|z'| > r$, then there exists an r'_ν with $|z'| > r'_\nu > r$, and since our series diverges at r'_ν, it also diverges at z'.

4.2. The functions represented by power series

Henceforth we shall consider only power series $\Sigma a_\nu (z-z_0)^\nu$ whose radius is not equal to 0. It is then absolutely convergent for every z in the interior of its circle of convergence—*i.e.*, for every z with $|z-z_0| < r$; in particular, for *every* z, if $r = +\infty$.[1] Its value is thus a function of z, which we shall denote by $f(z)$. We say that the power series represents the function $f(z)$ (in the interior of the circle of convergence), or conversely, the function $f(z)$ is expanded or developed in a power series there:

$$(1) \qquad f(z) = \sum_{\nu=0}^{\infty} a_\nu (z-z_0)^\nu, \qquad (r > 0).$$

It will be shown that such functions possess many desirable properties and are very important. They are called *regular* or *analytic functions.* These properties are established by means of the following theorems.

Theorem 1. *The function represented by a power series is continuous at the center z_0 of its circle of convergence.*

Proof. Let $0 < \rho < r$. Then $\sum_\nu |a_\nu| \rho^\nu$ is convergent, and, by 2.62, Theorem 2 and 3.1.1, Theorem 1, so is $\sum_{\nu=1}^{\infty} |a_\nu| \rho^{\nu-1}$. Call the value of the last series K. Then, for every z with $|z-z_0| \leqq \rho$,

$$|f(z)-f(z_0)| \leqq \sum_{\nu=1}^{\infty} |a_\nu (z-z_0)^\nu| \leqq |z-z_0| \sum_\nu |a_\nu|\, |z-z_0|^{\nu-1} \leqq K \cdot |z-z_0|.$$

[1] We leave aside, for the time being, the points on the boundary of the circle of convergence.

Having chosen $\varepsilon > 0$, a $\delta > 0$ can therefore be assigned (it suffices to take a δ which is $< \rho$ and $< \frac{\varepsilon}{K}$) so that

$$|f(z) - f(z_0)| < \varepsilon \qquad \text{for all} \qquad |z - z_0| < \delta.$$

Thus $f(z)$ is continuous at z_0.

Corollary. *Let* $f(z) = a_0 + a_1 z + \ldots + a_\nu z^\nu + \ldots$ *be convergent for* $|z| < r$. *Then, for every fixed* $p = 0, 1, \ldots,$ *as* $z \to 0$,

$$\frac{f(z) - (a_0 + a_1 z + \ldots + a_p z^p)}{z^{p+1}} \to a_{p+1}; \tag{2}$$

hence, in particular,

$$f(z) - (a_0 + a_1 z + \ldots + a_p z^p) = O(z^{p+1}), \qquad (z \to 0). \tag{3}$$

Proof. For $z \neq 0$, we have on the left in (2) the power series $f_p(z) = a_{p+1} + a_{p+2} z + \ldots$. If we apply Theorem 1 to this series, we obtain the assertion.

From Theorem 1 we get the following especially important

Theorem 2. (*Identity theorem for power series.*) *If each of the two power series* $\sum a_\nu z^\nu$ *and* $\sum b_\nu z^\nu$ [1] *has a radius* $\geqq \rho > 0$, *and if they have the same values in a neighborhood (no matter how small) of the origin, or if their values coincide only at the points* z_λ *of a certain sequence* $\{z_1, z_2, \ldots\}$ *with* $|z_\lambda| < \rho$, $z_\lambda \neq 0$, $z_\lambda \to 0$, *then they are completely identical, i.e.,* $a_\nu = b_\nu$ for $\nu = 0, 1, 2, \ldots$.

Proof. Let us denote by $f(z)$ and $g(z)$ the functions represented by the power series in $|z| < \rho$, so that $f(z_\lambda) = g(z_\lambda)$ for $\lambda = 0, 1, 2, \ldots$. The functions are continuous at the origin. Therefore, according to Theorem 1, as $\lambda \to \infty$ the limit of the left side $= f(0) = a_0$ and that of the right side $= g(0) = b_0$. Hence $a_0 = b_0$. We now consider the power series

$$a_1 + a_2 z + \ldots \qquad \text{and} \qquad b_1 + b_2 z + \ldots, \tag{4}$$

[1] To simplify the notation we make use here of the fact that there is no loss of generality in assuming that $z_0 = 0$.

which, for $z \neq 0$, represent, according to the rules for operating with convergent series, the respective functions

$$\frac{f(z)-a_0}{z}=f_1(z), \qquad \frac{g(z)-b_0}{z}=g_1(z). \tag{5}$$

Exactly the same assumptions hold for the series (4) and their values (5) as for the original power series and the functions $f(z)$ and $g(z)$ represented by them. The same reasoning shows that $a_1=b_1$. Suppose that the equality $a_\nu=b_\nu$ has already been proved for $\nu=0, 1, 2, \ldots, n$. Then consideration of the series

$$a_{n+1}+a_{n+2}z+\cdots \quad \text{and} \quad b_{n+1}+b_{n+2}z+\cdots \tag{4 }$$

which, for $z \neq 0$, represent the respective functions

$$\frac{f(z)-(a_0+a_1z+\ldots+a_nz^n)}{z^{n+1}}=f_{n+1}(z), \tag{5 }$$

$$\frac{g(z)-(b_0+b_1z+\ldots+b_nz^n)}{z^{n+1}}=g_{n+1}(z)$$

shows that, for $\nu=n+1$, likewise $a_\nu=b_\nu$. This equality thus holds for all $\nu=0, 1, 2, \ldots$.

Remarks. We shall explain the meaning of this important theorem by means of several *corollaries*.

1. The theorem asserts that if $f(z)$ can be represented in a neighborhood of the origin by a power series, then it is already completely determined by its values at the points z_λ of a null sequence $\{z_\lambda\}$ whose terms z_λ are $\neq 0$ and belong to the neighborhood in question.[1]

2. The corresponding theorem holds of course for power series of the more general form $\Sigma a_\nu(z-z_0)^\nu$. It can also be stated as follows: If it is at all possible to expand a function $f(z)$ in a power series for a neighborhood of a point z_0, then this can be done *in only one way.*

3. Thus, if we have arrived at power-series expansions

$$\Sigma a_\nu(z-z_0)^\nu \quad \text{and} \quad \Sigma b_\nu(z-z_0)^\nu,$$

[1] This is analogous to the fact that a quadratic polynomial $a+bz+cz^2$ is already fully determined by the knowledge of three functional values.

for one and the same function, in two different ways, then the theorem yields the infinitely many equations $a_\nu = b_\nu$, $(\nu = 0, 1, \ldots)$. This way of applying our theorem is called the *method of comparison of coefficients.* If, *e.g.*, a function $f(z)$ has the property that, for every z in $|z| < \rho$, invariably $f(-z) = f(z)$—such a function is called an *even function*—and if it is developable in a power series $\Sigma a_\nu z^\nu$ there, then the method in question shows that

$$a_0 - a_1 z + a_2 z^2 - + \ldots = a_0 + a_1 z + a_2 z^2 + \ldots,$$

and hence that $a_1 = a_3 = a_5 = \ldots = 0$: The power-series expansion of an even function contains only the even powers of z (more precisely, the coefficients of the odd powers are equal to 0). Analogously, for the *odd functions*, for which $f(-z) = -f(z)$, the coefficients of the even powers are all equal to 0.

Theorem 3. *Let $\Sigma a_\nu z^\nu$ be a power series with the positive radius r.*[1] *If z_1 is an interior point of its circle of convergence, then the function $f(z)$ represented by this series can also be expanded in a power series*

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_1)^n \tag{6}$$

in a neighborhood of z_1. Every coefficient b_n is represented by the absolutely convergent series

$$b_n = \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} a_{n+\nu} z_1^\nu, \qquad (n = 0, 1, \ldots), \tag{7}$$

which, regarded as a power series (drop the index 1 *in z_1), again has the exact radius r. Furthermore, the radius r_1* of (6) *is at least $= r - |z_1|$, i.e., at least equal to the distance of the point z_1 from the boundary of the circle of convergence of $\Sigma a_\nu z^\nu$.* (*Theorem on the transformation to a new center.*)

Proof. If $|z| < r$, then $f(z) = \Sigma a_\nu z^\nu$. In the sense of Theorem 9, Corollary 3 in 3.6, we now expand every term $a_\nu z^\nu = a_\nu (z_1 + (z - z_1))^\nu$ in the (only formally) infinite series

$$a_\nu z^\nu = a_\nu \left(z_1^\nu + \binom{\nu}{1} z_1^{\nu-1}(z - z_1) + \ldots + \binom{\nu}{n} z_1^{\nu-n}(z - z_1)^n + \ldots\right). \tag{8}$$

[1] It may also be $+\infty$.—We leave to the reader the formulation of the theorem for the case in which we start more generally with $\Sigma a_\nu (z - z_0)^\nu$.

It is trivially convergent, because for $n > \nu$ its terms are $=0$. It therefore remains convergent if we replace each of its terms by its absolute value. Its sum is then

$$= |a_\nu| \, (|z_1| + |z-z_1|)^\nu = \alpha_\nu.$$

After writing down the series (8) for $\nu = 0, 1, \ldots$ in rows one under another, we may therefore apply Theorem 9 in 3.6, provided that $\Sigma\alpha_n$ converges. That is certainly the case, however, if $|z_1|+|z-z_1| = = Z < r$, or what is the same, $|z-z_1| < r-|z_1|$. For, the series $\Sigma a_\nu Z^\nu$ is absolutely convergent, because Z then lies in the interior of the circle of convergence. The theorem cited now asserts, first of all, that the series appearing in the columns also converge absolutely. These, however, are the series

$$\left(\sum_{\nu=n}^{\infty} \binom{\nu}{n} a_\nu z_1^{\nu-n}\right)(z-z_1)^n.$$

But the series in parentheses is just another form of the series in (7), and the latter is therefore absolutely convergent. Since z_1 was arbitrary in $|z| < r$, this means that each of the power series (7)—we imagine the index 1 in z_1 to be removed for a moment—has at least the radius r. Nor can it be greater. For if $\Sigma \binom{n+\nu}{\nu} a_{n+\nu} z^\nu$ were convergent for a certain z with $|z| > r$, this series would also be *absolutely* convergent for a certain z with $|z| > r$. Then, by the first comparison test, $\sum\limits_\nu a_{n+\nu} z^\nu$, and consequently $\Sigma a_\nu z^\nu$, would converge absolutely. That is, however, not the case for $|z| > r$. Hence, the power series (7) —with z instead of z_1—have the exact radius r. If (now once more with z_1) we denote its value by b_n, then the n^{th} column-sum $= b_n(z-z_1)^n$. The theorem asserts further that the series whose terms are these column sums, *i.e.*, the series

$$\sum_{n=0}^{\infty} b_n \, (z-z_1)^n,$$

converges absolutely, and that its value is equal to $\Sigma a_\nu z^\nu = f(z)$. Herewith all assertions of the theorem are proved.

The following two theorems now follow easily from this particularly important theorem.

Theorem 4. *A function represented by a power series $\Sigma a_\nu z^\nu$ is continuous at every interior point z_1 of its circle of convergence.*

PROOF. Since $f(z)$ can also be represented by the power series (6), $f(z)$, according to Theorem 1, is continuous at z_1.

Theorem 5. *A function represented by a power series, say*

$$f(z) = \Sigma a_\nu z^\nu, \tag{9}$$

is differentiable arbitrarily often at every interior point of its circle of convergence $|z| < r$, and its derivatives may be obtained by term-by-term differentiation.

PROOF. It suffices to show that $f(z)$ can be differentiated once at every interior point z of the circle of convergence, and that the derivative $f'(z)$, as a function of the variable z, is represented by that power series which is obtained by a single term-by-term differentiation of $\Sigma a_\nu z^\nu$, *i.e.*, by the series

$$\sum_{\nu=1}^{\infty} \nu a_\nu z^{\nu-1} = \sum_{\nu=0}^{\infty} (\nu+1) a_{\nu+1} z^\nu, \tag{10}$$

which, by Theorem 3, again has the radius r. For if its value is $f'(z)$ for every z with $|z| < r$, then the same result can be applied once more to this representation of $f'(z)$, and we obtain the representation

$$f''(z) = \sum_{\nu=1}^{\infty} (\nu+1)\nu a_{\nu+1} z^{\nu-1},$$

which is valid again in $|z| < r$. This can be written more briefly in the following form:

$$\frac{1}{2!} f''(z) = \sum_{\nu=0}^{\infty} \binom{\nu+2}{2} a_{\nu+2} z^\nu, \qquad (|z| < r).$$

By applying this step k times in all, we arrive at the fact that the k^{th} derivative, too, exists at every point z with $|z| < r$, and that it is represented by the power series

$$\frac{1}{k!} f^{(k)}(z) = \sum_{\nu=0}^{\infty} \binom{\nu+k}{k} a_{\nu+k} z^\nu, \quad (k = 0, 1, 2, \ldots), \tag{11}$$

resulting from k-fold term-by-term differentiation of the series $\Sigma a_\nu z^\nu$. It again has the radius r, and is thus absolutely convergent for $|z| < r$.

In order to prove this now for $k = 1$, we represent the function $f(z)$, according to Theorem 3, in the form (6) for an arbitrary, but then fixed, z_1 with $|z_1| < r$, with the meaning (7) of b_n. If $z \neq z_1$ is an arbitrary interior point of the circle $|z - z_1| < r - |z_1|$, then

$$\frac{f(z) - f(z_1)}{z - z_1} = b_1 + b_2(z - z_1) + \ldots,$$

and the series on the right is absolutely convergent for the z in question. According to Theorem 1,

$$\lim_{z \to z_1} \frac{f(z) - f(z_1)}{z - z_1} = f'(z_1) = b_1 = \sum_{\nu=0}^{\infty} (\nu+1) a_{\nu+1} z_1^\nu.$$

Since z_1 was arbitrary in $|z| < r$, we see (just drop the index 1) that $f'(z)$ exists in $|z| < r$ and is represented by the absolutely convergent series (10). This proves the assertions pertaining to (11) for $k = 1$, and hence, according to the preliminary remarks, all the assertions of Theorem 5.

Corollary. From (11) we get, in particular,

$$\frac{1}{k!} f^{(k)}(0) = a_k. \tag{12}$$

If we substitute this in (9), this representation of $f(z)$ acquires the form

$$f(z) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(0)}{\nu!} z^\nu. \tag{13}$$

If we had started, in these investigations, with the more general power series $f(z) = \Sigma a_\nu (z - z_0)^\nu$, we should have arrived at the representation

$$f(z) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z_0)}{\nu!} (z - z_0)^\nu. \tag{14}$$

The series (14) is called the *Taylor series* or *Taylor expansion* of $f(z)$, and (13) the *Maclaurin form* of the same.

Examples and simple remarks

1. By differentiating the familiar representation $\frac{1}{1-z} = \Sigma z^\nu$ valid in $|z|<1$, we obtain immediately

$$\frac{1}{(1-z)^2} = \sum_\nu (\nu+1)z^\nu, \qquad \frac{1}{(1-z)^3} = \Sigma \binom{\nu+2}{2} z^\nu,$$

and in general, for $k = 0, 1, 2, \ldots$,

$$\frac{1}{(1-z)^{k+1}} = \sum_{\nu=0}^{\infty} \binom{\nu+k}{k} z^\nu, \qquad (|z|<1). \tag{15}$$

This is to be regarded as an extension, to negative integral exponents, of the binomial theorem $(1-z)^k = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} z^\nu$ which is valid for $k = 0, 1, \ldots$. In 6.5 we shall see that (15) is also correct for arbitrary nonintegral k.

2. The power series

$$\sum_{\nu=0}^{\infty} \frac{a_\nu}{\nu+1} z^{\nu+1}, \quad \Sigma \frac{a_\nu}{(\nu+1)(\nu+2)} z^{\nu+2}, \ldots$$

also have the same radius as $\Sigma a_\nu z^\nu$, because the latter is obtained from the former by a single or repeated term-by-term differentiation. Thus, $\Sigma \frac{a_\nu}{\nu+1} z^{\nu+1}$ is the definite integral

$$F(z) = \int_0^z f(t)dt. \tag{16}$$

Whereas the foregoing theorems have dealt only with properties of functions represented by power series, which they possess in the *interior* of the circle of convergence, the following theorem is concerned with the behavior of the series on the boundary of that circle:

Theorem 6. (*Abel's limit theorem.*) *Let the power series $\Sigma a_\nu z^\nu$ have the finite radius $r>0$, and suppose that it converges at a point z_0 on the boundary of the circle of convergence: $\Sigma a_\nu z_0^\nu = s$. Then the function $f(z)$ represented by the series for $|z|<r$ tends to the limit s as z tends radially from 0 to the point z_0: $\lim_{z\to z_0} f(z) = s$ for radial approach.*

PROOF. First of all, we see, as we often have before, that there is no loss of generality in assuming that $r=1$ and even $z_0=1$. For if we set $z=z_0z'$ and then $a_\nu z_0^\nu=a'_\nu$, the series goes over into $\Sigma a'_\nu z'^\nu$, which now converges at $z'=1$ and has the value s. It suffices, therefore, to prove the theorem in the following form: Let $\Sigma a_\nu z^\nu$ be convergent in $|z|<1$, and represent the function $f(z)$ there. If Σa_ν is convergent and equal to s, then $f(x)\to s$ as real $x\to 1-0$:[1]

$$\lim_{x\to 1-0} \Sigma a_\nu x^\nu = \lim_{x\to 1-0} f(x) = s = \Sigma a_\nu.$$

Now the *proof* is very simple and is closely related to *Cauchy's* limit theorem (2.4). For $0\leqq x<1$ we have $\frac{1}{1-x}\,\Sigma a_\nu x^\nu = \Sigma x^\nu\cdot\Sigma a_\nu x^\nu =$ $= \Sigma(a_0+a_1+\ldots+a_\nu)x^\nu = \Sigma s_\nu x^\nu$ (*cf.* the beginning of the next section, 4.3). Hence, if we set $s=s_\nu+r_\nu$, $f(x) = (1-x)\Sigma s_\nu x^\nu = s-(1-x)\,\Sigma r_\nu x^\nu$, or

$$|f(x)-s| \leqq (1-x)\,\Sigma\,|r_\nu|x^\nu.$$

Having chosen $\varepsilon>0$, we can find, since $r_\nu\to 0$, a μ such that $|r_\nu|<\frac{\varepsilon}{2}$ for $\nu>\mu$. Hence,

$$|f(x)-s| \leqq (1-x)\sum_{\nu=0}^{\mu}|r_\nu|x^\nu+\frac{\varepsilon}{2}.$$

Since μ is a fixed number, the first term on the right tends trivially to 0 as $x\to 1-0$. We can therefore determine such a small δ in $0<\delta<1$ that this term is $<\frac{\varepsilon}{2}$ for all x in $1-\delta<x<1$. For these x, then,

$$|f(x)-s|<\varepsilon,$$

which proves the assertion. Applications of this important theorem are given in 7.3,2.

4.3. Operating with power series. Expansion of composite functions

All rules for operating with (absolutely) convergent series hold of course for operating with power series, so long as we remain in the

[1] *I.e.*, for left-sided tending of x to $+1$.

interior of the circle of convergence. Thus we have, *e.g.*,

$$(1)\qquad \Sigma a_\nu z^\nu \pm \Sigma b_\nu z^\nu = \Sigma (a_\nu \pm b_\nu) z^\nu$$

and

$$(2)\qquad \Sigma a_\nu z^\nu \cdot \Sigma b_\nu z^\nu = \Sigma (a_0 b_\nu + a_1 b_{\nu-1} + \ldots + a_\nu b_0) z^\nu,$$

so long as z lies inside the circles of convergence of both series, in other words, for $|z| < \min(r, r')$, if r and r' are their two radii.[1] *E.g.*, if $\Sigma a_\nu z^\nu$ again has the radius r, then, for $|z| < \min(1, r)$,

$$(3)\qquad \begin{aligned} \frac{1}{1-z}\Sigma a_\nu z^\nu &= \Sigma z^\nu \cdot \Sigma a_\nu z^\nu = \Sigma (a_0 + a_1 + \ldots + a_\nu) z^\nu \\ &= \Sigma s_\nu z^\nu, \end{aligned}$$

or

$$(4)\qquad \Sigma a_\nu z^\nu = (1-z)\, \Sigma s_\nu z^\nu.$$

Here, as always, the s_ν denote the partial sums of Σa_ν. It is just as easy to see that, if $\Sigma a_\nu z^\nu = f(z)$ for $|z| < r$, then, for these z, the (positive integral) powers of $f(z)$ can be represented by power series. For we have, first of all, by (2):

$$(5)\quad (f(z))^2 = \sum_{\nu=0}^{\infty} (a_0 a_\nu + \ldots + a_\nu a_0) z^\nu, \text{ which we shall write as } \sum_{\nu=0}^{\infty} a_{2,\nu} z^\nu.$$

Again by (2) we then obtain

$$(6)\qquad (f(z))^3 = \sum_{\nu=0}^{\infty} (a_0 a_{2,\nu} + a_1 a_{2,\nu-1} + \ldots + a_\nu a_{2,0}) z^\nu,$$

and, in general, for $k = 0, 1, 2, \ldots$ and every $|z| < r$,

$$(7)\qquad (f(z))^k = \sum_{\nu=0}^{\infty} a_{k\nu} z^\nu \quad \text{with}^{2} \quad a_{k\nu} = a_0 a_{k-1,\nu} + \ldots + a_\nu a_{k-1,0}.$$

Whereas these facts are very simple, and therefore we do not formulate them as special theorems, the following problem is somewhat

[1] We emphasize expressly that $\min(r, r')$ does not have to be the true radius of convergence of the series (1) or (2). Thus, *e.g.*, $\Sigma(a_\nu - a_\nu)z^\nu$ obviously has the radius $+\infty$; likewise, $(1 + z + z^2 + \ldots)(1 - z + 0 + 0 + \ldots)$.

[2] For $k = 1$ we then have to set $a_{k\nu} = a_\nu$ (for $\nu = 0, 1, 2, \ldots$), and, for $k = 0$, $a_{00} = 1$ and $a_{0\nu} = 0$ $(\nu = 1, 2, \ldots)$.

deeper: Let $\Sigma b_\nu w^\nu$ be a power series in the variable w, with the radius r_1, and call the function that it represents $g(w)$, so that

$$(8) \qquad g(w) = \sum_{\rho=0}^{\infty} b_\rho w^\rho \qquad \text{for} \quad |w| < r_1.$$

Likewise let

$$(9) \qquad \varphi(z) = \Sigma\, a_\nu z^\nu \qquad \text{for} \quad |z| < r.$$

Let us set $\varphi(z) = w$. It is then also possible to represent the composite function $g(\varphi(z)) = f(z)$ by a power series, at least for all z in a certain neighborhood of 0, and this series is obtained, in particular,—as might appear most natural—by representing the powers $w^\rho = (\varphi(z))^\rho$ as power series $\sum_\nu a_{\rho\nu} z^\nu$ according to (7), substituting these in (8), and finally collecting all terms involving the same power z^n, $(n = 0, 1, \ldots)$, thus obtaining for the composite function $f(z) = g(\varphi(z))$ the power-series representation

$$(10) \qquad \sum_{n=0}^{\infty} c_n z^n \quad \text{with} \quad (11)\ c_n = \sum_{\rho=0}^{\infty} b_\rho a_{\rho n}, \qquad (n = 0, 1, \ldots).$$

If this is to be correct for a neighborhood of 0, it must hold, in particular, for $z = 0$ itself. We thus find $|a_0| < r_1$ [1] as a necessary condition for our problem to admit of the indicated answer. We shall prove that this condition is also sufficient:

Theorem 1. *Let* (8) and (9) *be two given power series with the respective radii* r_1, r; *let the functions represented by these series be* $g(w)$, $w = \varphi(z)$, *respectively; furthermore, in accordance with* (7), *set*

$$(12) \qquad w^\rho = (\varphi(z))^\rho = \sum_{\nu=0}^{\infty} a_{\rho\nu} z^\nu, \qquad (\rho = 0, 1, \ldots).$$

Then, under the sole assumption that $|a_0| < r_1$, *each of the series* (11) *is absolutely convergent. Moreover, for a suitable* R, *the power series* (10) *is convergent in* $|z| < R$ *and represents the composite function* $f(z) = g(\varphi(z))$ *there.*

[1] We have agreed (footnote, p. 102) to leave the boundary points out of consideration for the time being.

This holds at least for all those $|z| < r$ *for which* $\sum_{\nu=0}^{\infty} |a_\nu z^\nu|$ *remains* $< r_1$.[1]

PROOF. Theorem 9 in 3.6 is available for the proof. For, according to (12),

$$(13) \qquad b_\rho w^\rho = b_\rho a_{\rho 0} + b_\rho a_{\rho 1} z + \ldots + b_\rho a_{\rho n} z^n + \ldots, \qquad (\rho = 0, 1, 2, \ldots).$$

If the hypotheses of the theorem cited were satisfied, then all column series would be (absolutely) convergent, and the n^{th} one of them would have the value $c_n z^n$ with the meaning (11) of the c_n, $(n = 0, 1, 2, \ldots)$. Furthermore, the series $\sum c_n z^n$ constructed with the column sums would also be absolutely convergent, and its value would be equal to the value of the series built with the row sums, *i.e.*, equal to $\sum b_\rho w^\rho = g(\varphi(z))$, and everything would be proved.

The hypotheses of the theorem cited are certainly fulfilled, however, if also those series converge which arise from (13) on replacing each of the terms by its absolute value, and if, finally, the series whose terms are the values of the series thus altered converges too;—in short, if we can associate with the schema (13) another one, each of whose elements is $\geqq 0$ and $\geqq$ the absolute value of the corresponding element in (13), and which fulfills the conditions just mentioned. Such a schema can be obtained immediately in the following manner. If we set $|a_\nu| = \alpha_\nu$, $|b_\nu| = \beta_\nu$, then the power series $\omega = \sum \alpha_\nu z^\nu$ and $\sum \beta_\nu \omega^\nu$ also have the respective radii r, r_1.[2] Denote the value of the first by $\tilde{\varphi}(z)$. Let us now form the following schema corresponding to (13), substituting a positive number $\zeta < r$ for z:

$$(14) \qquad \beta_\rho \omega^\rho = \beta_\rho(\tilde{\varphi}(\zeta))^\rho = \beta_\rho \alpha_{\rho 0} + \beta_\rho \alpha_{\rho 1} \zeta + \ldots + \beta_\rho \alpha_{\rho n} \zeta^n + \ldots, \qquad (\rho = 0, 1, 2, \ldots).$$

Then

$$|b_\rho a_{\rho n} z^n| \leqq \beta_\rho |a_{\rho n}| \zeta^n \leqq \beta_\rho \alpha_{\rho n} \zeta^n.$$

---

[1] And, because of the continuity of $f(z)$ at 0 and the fact that $|a_0| < r_1$, this is certainly satisfied for all z in a certain circular neighborhood $|z| < R$ of the origin. For R we may take, *e.g.*, the certainly still positive least upper bound of the absolute values of those z with $|z| < r$ for which $\sum |a_\nu z^\nu| < r_1$.

[2] For the magnitude of the radius of a power series depends—as the *Cauchy-Hadamard* formula shows—only on the absolute values of the coefficients.

For certainly

$$|a_{\rho n}| \leqq \alpha_{\rho n} \quad \text{for} \quad \rho = 0 \text{ and } 1 \quad \text{and } n = 0, 1, 2, \ldots;$$

but then we conclude inductively from (5), (6), and (7), that this inequality is correct for all further $\rho = 2, 3, \ldots$. If, finally, the positive ζ is so small that $\omega = \tilde{\varphi}(\zeta) < r_1$—and this was supposed to be the case for all $\zeta < R$—then the series constructed with the row sums of (14), *i.e.*, the series $\Sigma\beta_\rho(\tilde{\varphi}(\zeta))^\rho = \Sigma\beta_\rho\omega^\rho$, is also convergent, since $\Sigma\beta_\rho\omega^\rho$, as emphasized, has the radius r_1. Thus, the hypotheses of Theorem 9 in 3.6 are satisfied. This therefore holds *a fortiori* for the schema (13), which, according to the remarks at the beginning, completes the proof of the theorem.

Remarks. 1. We find without difficulty that

$$c_0 = b_0 + b_1 a_0 + \ldots = g(a_0) = g(\varphi(0))$$

and

$$\begin{aligned} c_1 &= a_1(b_1 + 2b_2 a_0 + 3b_3 a_0^2 + \ldots) \\ &= g'(a_0) \cdot a_1 = g'(\varphi(0)) \cdot \varphi'(0). \end{aligned}$$

If we had started with power series in $(z-z_0)$ and $(w-w_0)$, we should have found analogously that

$$\frac{d}{dz} g(\varphi(z)) = g'(\varphi(z)) \cdot \varphi'(z)$$

for $z = z_0$,—hence, for every z at which the hypotheses of our theorem are satisfied, *i.e.*, for which $|\varphi(z)|$ is smaller than the radius of the outer power series.

2. The actual (numerical) derivation of the series (10) from the series (8) and (9) is possible only in simple cases. The theorem has more the character of an existence theorem: If its hypotheses are satisfied, then there exists a power series (10) with a positive radius, which represents the composite function $f(z) = g(\varphi(z))$. Since the series (10), according to the Identity Theorem 2 in 4.2, is uniquely determined, any other way may be used to find it. We shall encounter examples of such alternative ways in the sequel.

3. Nevertheless it is possible of course to determine several of the beginning coefficients c_n. If, *e.g.*, the composite function $\exp \dfrac{z}{1-z}$ is to be expanded for a certain neighborhood of 0, then the series (*cf.* 4.2,(15))

$$w^\nu = \left(\frac{z}{1-z}\right)^\nu = \sum_{n=0}^{\infty} \binom{n+\nu-1}{n} z^{\nu+n} = z^\nu + \binom{\nu}{1} z^{\nu+1} + \binom{\nu+1}{2} z^{\nu+2} + \dots$$

have to be substituted in the series $\exp w = 1+w+\frac{1}{2}w^2+\dots$, which yields

$$1+(z+z^2+z^3+\dots) + \frac{1}{2!}(z^2+2z^3+3z^4+\dots) + \frac{1}{3!}(z^3+3z^4+\dots) + \dots$$
$$= 1+z+\frac{3}{2}z^2+\frac{13}{6}z^3+\dots,$$

where it is not immediately possible, however, to perceive the formation of the later coefficients.

4. If the "outer" power series is everywhere convergent, as in the preceding example, then the hypotheses of the theorem are satisfied for all $|z|<r$. If the "inner" series is also everywhere convergent, then the hypotheses are satisfied for all z in the plane. Thus we find, *e.g.*, the expansion

$$\exp(e^z) = e^{e^z} = e\cdot e^{(e^z-1)} =$$
$$= e\left[1+\left(z+\frac{1}{2!}z^2+\dots\right) + \frac{1}{2!}\left(z+\frac{1}{2!}z^2+\dots\right)^2+\dots\right] =$$
$$= e\cdot\left(1+z+z^2+\frac{5}{6}z^3+\dots\right).$$

5. The foregoing examples show that it is advantageous if $\varphi(0) = 0$, so that the inner series begins with $a_1 z+\dots$.

6. The examples in **3** and **4** may also be obtained by means of the respective transformations $\exp(z+z^2+\dots) = e^z\cdot e^{z^2}\cdot \dots$,

$$e^{1+z+\frac{1}{2}z^2+\dots} = e\cdot e^z\cdot e^{\frac{1}{2}z^2}\dots .$$

7. Further examples will be given in chapter 6.

As an example of a more general character, we consider the expansion in a power series, of the reciprocal value of a power series.

Theorem. 2. *Let*

$$\Sigma a_\nu z^\nu = g(z) \tag{15}$$

be a power series with a positive radius. Then, under the sole assumption that $a_0 \neq 0$, $1/g(z)$ *can also be represented by a power series with a positive radius:*

$$1/g(z) = f(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{16}$$

PROOF. If $a_0 \neq 0$, then we may write the series for $g(z)$ in the form $a_0(1+a_1'z+\ldots)$. Thus we see that it suffices to prove the theorem for the special case $a_0 = 1$. The assertion then reads, if we make a change of sign: *If*

$$w = \varphi(z) = a_1 z + a_2 z^2 + \ldots \tag{17}$$

has a radius $r > 0$, *then there exists a power series* $\Sigma c_n z^n \equiv 1 + c_1 z + c_2 z^2 + \ldots$ *with a radius* $r' > 0$ *such that, in* $|z| < r'$,

$$\frac{1}{1-\varphi(z)} = 1 + c_1 z + c_2 z^2 + \ldots . \tag{18}$$

PROOF. For $|w| < 1$,

$$1/(1-w) = 1 + w + w^2 + \ldots .$$

In this expansion we may substitute for w, according to Theorem 1, the expansion (17), and we immediately obtain, in the sense of that theorem, the expansion (18). It is certainly valid for all those z for which

$$|a_1 z| + |a_2 z^2| + \ldots < 1,$$

which is certainly the case for all sufficiently small $|z|$ because of the continuity of the power series on the left side of the inequality sign. The expansion (18) therefore has a positive radius r', and this already completes the proof of the theorem.

If we wish, more generally, to derive the expansion (16) from (15) with $a_0 \neq 0$, then we have to expand

$$\frac{1}{g(z)} = \frac{1}{a_0} \frac{1}{1-\left(-\frac{a_1}{a_0} z - \frac{a_2}{a_0} z^2 - \ldots\right)} = \sum_{n=0}^{\infty} c_n z^n \tag{19}$$

in the sense of the proof just carried out.

In order to obtain this expansion, as already stressed in Remark 1 we shall not use the general method presented in the proof of Theorem 1. On the contrary, once the existence of the expansion (19) has been guaranteed, it suffices to set down the relation

$$(a_0+a_1z+a_2z^2+\ldots)\,(c_0+c_1z+c_2z^2+\ldots) = 1$$

with "undetermined coefficients" (*cf.* Remark 3 after Theorem 2 in 4.2). This yields, according to the identity theorem, the linear equations

$$\text{(20)}\qquad \begin{cases} a_0c_0 & = 1 \\ a_1c_0+a_0c_1 & = 0 \\ \cdots\cdots\cdots\cdots & \\ a_nc_0+a_{n-1}c_1+\ldots+a_0c_n & = 0 \\ \cdots\cdots\cdots\cdots & \end{cases}$$

The coefficients $c_0, c_1, c_2, \ldots$ can be calculated[1] from these equations successively and unambiguously, since, after the solution of the 0^{th} to the $(n-1)^{\text{st}}$ of these equations, the n^{th} equation contains only the single unknown c_n with the nonzero coefficient a_0. Whereas no synoptic formulas for the c_n are obtained in this way, the solution of the 0^{th} to the n^{th} equations by *Cramer*'s rule immediately yields, after a slight transformation, in addition to $c_0 = \dfrac{1}{a_0}$,

$$\text{(21)}\qquad c_n = \frac{(-1)^n}{a_0^{n+1}} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_{n-1} & \cdots & \cdots & a_1 \end{vmatrix}$$

for $n = 1, 2, \ldots$.

The following is a particularly important example of such a division: The problem is to develop

$$\text{(22)}\qquad \frac{z}{e^z-1} = \frac{1}{1+\dfrac{z}{2!}+\dfrac{z^2}{3!}+\ldots}$$

[1] We also call (20) *recursion formulas*, since they base the calculation of c_n on the previous calculation of $c_0, c_1, \ldots, c_{n-1}$.

in a power series. For historical reasons it is customary to denote it by

$$(23)\qquad B_0+\frac{B_1}{1!}z+\frac{B_2}{2!}z^2+\ldots=\sum_{n=0}^{\infty}\frac{B_n}{n!}z^n.$$

The recursion formulas (20) read in this case: $B_0 = 1$, and, for $n \geqq 1$,

$$\frac{1}{(n+1)!}\cdot\frac{B_0}{0!}+\frac{1}{n!}\cdot\frac{B_1}{1!}+\ldots+\frac{1}{1!}\cdot\frac{B_n}{n!}=0,$$

or, if we multiply by $(n+1)!$,

$$(24)\qquad \binom{n+1}{0}B_0+\binom{n+1}{1}B_1+\ldots+\binom{n+1}{n}B_n=0.$$

These equations are easier to remember if we replace the B_ν in them by B^ν. Then they assume the very short form

$$(25)\qquad (1+B)^{n+1}-B^{n+1}=0.$$

These equations, however, are to be taken only *symbolically*, *i.e.*, they go over into the true equations (24) only on the basis of the agreement that, after the binomial theorem has been applied to $(1+B)^{n+1}$, we again replace B^ν everywhere by B_ν. From these equations we find, by simple calculation, for the first few B_ν, the values

$$B_0=1,\quad B_1=-\tfrac{1}{2},\quad B_2=\tfrac{1}{6},\quad B_3=0,\quad B_4=-\tfrac{1}{30},\quad B_5=0,\ \ldots.$$

They are, as the calculation shows, well-determined rational numbers. They are called the *Bernoullian numbers*. Their calculation offers no difficulties in principle. They may therefore be regarded as "known", even though their values cannot be specified by means of a simple formula—except, say, by means of a determinant such as in (21)—and they do not exhibit any simple regularity at all. Since $\frac{z}{e^z-1}+\frac{z}{2}$ is, as is easily verified, an even function, we have $B_3 = B_5 = \ldots = 0$. For the next few B_ν with even ν, we find

$$B_6=\tfrac{1}{42},\quad B_8=-\tfrac{1}{30},\quad B_{10}=\tfrac{5}{66},\quad B_{12}=-\tfrac{691}{2730},\quad B_{14}=\tfrac{7}{6},\ \ldots.$$

The problem of expanding $1/\cos z$ in a power series can be handled quite similarly. Since the function is even, we may set down the relation with undetermined coefficients in the form

$$(26) \qquad \frac{1}{\cos z} = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{E_{2\nu}}{(2\nu)!} z^{2\nu},$$

which will converge for sufficiently small $|z|$. The numbers $E_{2\nu}$ obtained are called *Eulerian numbers.* From

$$\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - + \ldots\right) \left(E_0 - \frac{E_2}{2!} z^2 + \frac{E_4}{4!} z^4 - + \ldots\right) = 1,$$

we get $E_0 = 1$ and, for $\nu \geqq 1$, recursion formulas, which, after multiplication by $(2\nu)!$, may be written in the form

$$(27) \qquad E_{2\nu} + \binom{2\nu}{2} E_{2\nu-2} + \ldots + E_0 = 0,$$

or *symbolically* (see above) in the form

$$(E+1)^p + (E-1)^p = 0$$

which is valid for all $p = 1, 2, 3, \ldots$. Thus we have $E_1 = E_3 = E_5 = = \ldots = 0$ and

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad \ldots.$$

According to (27), these *Euler*ian numbers are rational and integral, and may easily be calculated and, accordingly, regarded as "known". Like the *Bernoulli*an numbers, however, they obey no simple law.

4.4. The inversion of a power series

As the last investigation concerning power series, we shall discuss the formation of the inverse power series of a given one. Once more let

$$(1) \qquad w = f(z) = w_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

be an arbitrarily given power series with the positive radius r.[1] To what extent then is z determined when w is given, that is to say, the

[1] It is useful, for what follows, to denote the constant term by w_0. It is the value of $w = f(z)$ for $z = z_0$.

equation $w = f(z)$ (uniquely) solvable for z, given w? We shall show that under the sole assumption that $a_1 \neq 0$, to every value w lying sufficiently close to w_0 there corresponds precisely one value z lying close to z_0 such that $f(z) = w$. This value z is then, under the indicated restriction of position of w and z, a single-valued function $z = g(w)$, so that for all these z and w,

$$(2) \qquad f(g(w)) = w \quad \text{or} \quad g(f(z)) = z.$$

The function $z = g(w)$, often also—interchanging the letters z and w —the function $w = g(z)$, is then called the *inverse function* or simply the *inverse* of $f(z)$. Thus, *e.g.*, the functions

$$w = \frac{2z-1}{z-1} \quad \text{and} \quad z = \frac{w-1}{w-2}, \quad (z \neq 1,\ w \neq 2),$$

or $\frac{2z-1}{z-1}$ and $\frac{z-1}{z-2}$, are inverses of each other. The elementary functions

$$e^z \text{ and } \log z,\ z^3 \text{ and } \sqrt{z},\ \sin z \text{ and } \arcsin z,$$

discussed in chapter 6, are likewise pairs of inverse functions. If the one is denoted by f and the other by g, then, with proper definition of these functions, equation (2) invariably holds for every z, w in certain regions of the z-, w-plane, respectively.—Following these preliminary remarks, we prove

Theorem 3. *Let* (1) *be an arbitrary power series with the radius* $r > 0$. *Then, under the sole assumption that* $a_1 \neq 0$, *there exists precisely one power series*

$$(3) \qquad z = z_0 + b_1(w-w_0) + b_2(w-w_0)^2 + \dots$$

with a positive radius r', *such that the functions represented by* (1) *and* (3), *with* z *and* w *restricted to sufficiently small neighborhoods of* z_0, w_0, *respectively, are inverses of each other. We have, moreover,*

$$b_1 = 1/a_1,\ \textit{i.e.},\ g'(w_0) = 1/f'(z_0).$$

Proof. Here again it is no restriction to assume that $z_0 = w_0 = 0$. Similarly, the assumption $a_1 = 1$ also involves no restriction. For if $a_1 \neq 0$, then $a_1 z + a_2 z^2 + \dots$ can be written in the form

$(a_1 z) + \frac{a_2}{a_1^2}(a_1 z)^2 + \ldots$. If we now set $a_1 z = z'$, $\frac{a_\nu}{a_1^\nu} = a'_\nu$, then the power series goes over into $z' + a'_2 z'^2 + \ldots$, which acquires the desired form if we drop the accent. Hence, let there be given the power series

$$z + a_2 z^2 + a_3 z^3 + \ldots \tag{4}$$

with the radius $r > 0$. Then we have to show that there exists precisely one power series

$$z = w + b_2 w^2 + b_3 w^3 + \ldots \tag{5}$$

with a positive radius r', such that the functions represented by them are inverses of each other, so that, in other words,

$$(w + b_2 w^2 + \ldots) + a_2 (w + b_2 w^2 + \ldots)^2 + \ldots \tag{6}$$

goes over into the power series $w + 0 + 0 + \ldots = w$ if (6) is ordered, by virtue of Theorem 1 in 4.3, according to increasing powers of w and, naturally, the hypotheses of that theorem are satisfied.[1] To prove this, we show first that if there exists at all a power series (5) with the required properties, then this power series is uniquely determined. Now the substitution of (5) in (4) in the sense of Theorem 1 is certainly permissible for all sufficiently small $|w|$. If we set, somewhat as in 4.3,(7),

$$(w + b_2 w^2 + \ldots)^\nu = w^\nu + b_{\nu,\nu+1} w^{\nu+1} + \ldots + b_{\nu n} w^n + \ldots \tag{7}$$

(so that, in particular, $b_{\nu\nu} = 1$, $b_{\nu\rho} = 0$ for $\rho < \nu$, as well as $b_{1\nu} = b_\nu$, $b_{11} = 1$), then, if we collect the terms in (6) involving w^n, $n \geqq 2$, 0 must appear as coefficient. This yields the equation

$$a_n b_{nn} + \ldots + a_\nu b_{\nu n} + \ldots + a_1 b_{1n} = 0, \qquad (n \geqq 2),$$

or, since $b_{nn} = 1$, $b_{1n} = b_n$, $a_1 = 1$,

$$a_n + a_{n-1} b_{n-1,n} + \ldots + a_\nu b_{\nu n} + \ldots + a_2 b_{2n} + b_n = 0. \tag{8}$$

These equations are again recursion formulas for calculating the b_n.

[1] Since we introduced z as an abbreviation of $a_1 z$, (5) has to be divided by a_1 to obtain the inverse of the series $a_1 z + a_2 z^2 + \ldots$ considered before. Finally, z and w have to be replaced by $(z - z_0)$, $(w - w_0)$, respectively, in order to get the pair (1) and (3) of mutually inverse power series, for which, then, $b_1 = 1/a_1$.

For we read off from the formulas 4.3,(7) for calculating the $b_{\nu n}$—we have only to replace a there by b—that, to determine

$$b_{2n} = b_{n-1} + b_2 b_{n-2} + \ldots + b_{n-2} b_2 + b_{n-1}, \quad (n = 3, 4, \ldots\ ^1),$$

it is only necessary to know $b_2, \ldots, b_{n-1}$, and also, that to calculate the $b_{\nu n}$, $(\nu > 2,\ n \geqq \nu)$, it is only necessary to know $b_2, \ldots, b_{n-1}$. Hence, if these are already known, then (8) immediately and unambiguously yields the value of b_n. We obtain

$$b_2 = -a_2, \quad b_3 = 2a_2^2 - a_3, \quad \ldots .^2$$

These calculations, however, can be carried out in every case (*i.e.*, without regard to the convergence behavior of (5)). We say: There exists precisely one power series (5) formally satisfying the conditions of the problem. The proof will be complete as soon as we have shown that this formally acquired series possesses a positive radius.

This can be done as follows: We choose any numbers $\alpha_\nu \geqq |a_\nu|$, $(\nu = 2, 3, \ldots)$, *e.g.*, the numbers m/ρ^ν, where $\rho > 0$ lies in the interior of the circle of convergence of (4), and m denotes the absolute value of the largest term of the null sequence $\{a_\nu \rho^\nu\}$. We then carry out exactly the same calculations as we made just now, starting merely with the series

$$(4') \qquad z - \alpha_2 z^2 - \alpha_3 z^3 - \ldots$$

instead of (4); let the series

$$(5') \qquad w + \beta_2 w^2 + \beta_3 w^3 + \ldots$$

be the series thus obtained instead of (5). The recursion formulas for calculating the β_ν read in this case: $\beta_2 = \alpha_2$, and, for $n \geqq 3$, corresponding to (8),

$$(8') \qquad \beta_n = \alpha_n + \alpha_{n-1}\, \beta_{n-1,n} + \ldots + \alpha_2\, \beta_{2n},$$

where the $\beta_{\nu n}$, in analogy with the $b_{\nu n}$, are defined as the coefficients

[1] We have $b_{22} = b_1 = 1$, — as can be read off immediately from (7).

[2] For small values of n, this calculation presents no difficulty and is recommended to the reader as an exercise. For larger n, it very soon becomes obscure. Satisfactory formulas expressing the b_n in terms of the a_ν are not known.

of the ν^{th} power of the series (5′), whose convergence for small $|w|$ will be shown in a moment. This shows (inductively) that the $\beta_{\nu n}$, and hence also the β_n, are >0, and the $|b_n|$ are $\leq \beta_n$, $(n = 2, 3, \ldots)$. With the choice $\alpha_\nu = m/\rho^\nu$ that we made, (4′) has the positive radius ρ, and the function represented by this series in $|z| < \rho$ is the function

$$w = z - m\frac{z^2}{\rho^2}\left(1+\frac{z}{\rho}+\ldots\right) = z - m\frac{z^2}{\rho(\rho-z)}.$$

The inverse of this function $w = f(z)$, however, can be obtained—here we anticipate several simple matters which will not be discussed until chapter 6—directly by solving the quadratic equation

$$(m+\rho)z^2 - \rho(\rho+w)\,z + \rho^2 w = 0$$

and we find that

$$z = \frac{\rho^2}{2(m+\rho)}\left[1+\frac{w}{\rho}-\sqrt{1-\frac{2(2m+\rho)}{\rho^2}\,w+\frac{w^2}{\rho^2}}\right],$$

where, for small $|w|$, that value of the square root is to be taken which lies close to $+1$. If we set the zeros of the radicand, *i.e.*, the values

$$(2m+\rho) \pm 2\sqrt{m(m+\rho)},$$

where the last root is understood to be the positive real value, $= w_1$, w_2, respectively, the radicand may be written in the form

$$\left(1-\frac{w}{w_1}\right)\left(1-\frac{w}{w_2}\right).$$

The values w_1 and w_2 are both positive and real. Thus we have

$$z = \frac{\rho^2}{2(m+\rho)}\left[1+\frac{w}{\rho}-\left(1-\frac{w}{w_1}\right)^{\frac{1}{2}}\left(1-\frac{w}{w_2}\right)^{\frac{1}{2}}\right].$$

Now in 6.5 it will be shown that the function $(1-\omega)^{\frac{1}{2}}$, taken to mean the square root of $(1-\omega)$ lying close to $+1$ for small $|\omega|$, can be expanded, for $|\omega| < 1$, in a power series beginning with

$$1-\frac{\omega}{2}-\frac{\omega^2}{8}-\ldots,$$

which, moreover, (except for the initial term 1) has only negative coefficients. Hence, as a little calculation shows, z can be expanded, in accordance with

$$z = \frac{\rho}{2(m+\rho)}\left[1+\frac{w}{\rho}-\left(1-\frac{w}{2w_1}-\ldots\right)\left(1-\frac{w}{2w_2}-\ldots\right)\right],$$

in a power series of the form

$$z = w+\beta_2 w^2+\beta_3 w^3+\ldots,$$

which, as already noted above, has exclusively positive (real) coefficients. Since it converges for $|w| < \min(w_1,\ w_2)$, and hence in any case has a positive radius, this completes the proof of Theorem 3.

Chapter 5

DEVELOPMENT OF THE THEORY OF CONVERGENCE

5.1. The theorems of Abel, Dini, and Pringsheim

In this section and the next, we again deal with series of positive terms (*cf.* 3.1–3.3). In 3.3 we were able to deduce the convergence behavior of $\Sigma \frac{1}{\nu(\log \nu)^\alpha}$ from the convergence behavior of $\Sigma \frac{1}{\nu}$. Something similar holds if we start with an arbitrary divergent series Σd_ν. In 1867, *U. Dini* proved the following theorem:

Theorem 1. *If Σd_ν is an arbitrary divergent series of positive terms, and if D_ν are its partial sums, then*

$$(1) \qquad \Sigma \frac{d_\nu}{D_\nu^\alpha} \textit{ is convergent for } \alpha > 1, \textit{ divergent for } \alpha \leqq 1.[1]$$

Proof. In case $\alpha = 1$,

$$\frac{d_{\nu+1}}{D_{\nu+1}} + \dots + \frac{d_{\nu+\rho}}{D_{\nu+\rho}} \geq \frac{d_{\nu+1} + \dots + d_{\nu+\rho}}{D_{\nu+\rho}} = 1 - \frac{D_\nu}{D_{\nu+\rho}}.$$

For every fixed ν, this is $> \frac{1}{2}$ for all sufficiently large ρ (because $D_n \to \infty$). Hence, according to the second main test, our series (1) is not convergent. For $\alpha < 1$ it is *a fortiori* divergent.

The convergence assertion of the theorem is a little more laborious to prove. It is contained in the following somewhat more general theorem which is due to *A. Pringsheim* (1890):

Theorem 2. *If d_ν and D_ν have the same meaning as in Theorem 1, then the series*

$$(2) \qquad \sum_{\nu=1}^{\infty} \frac{d_\nu}{D_\nu \cdot D_{\nu-1}^\delta}$$

is convergent for every $\delta > 0$.

---

[1] *Abel* had proved in 1828 that $\Sigma \frac{d_\nu}{D_{\nu-1}}$ diverges with Σd_ν. — We assume that $d_0 > 0$, so that also all $D_\nu > 0$.

PROOF. Choose a natural number p for which $\frac{1}{p} = \gamma < \delta$. Then it suffices to prove the theorem for the case in which the δ in the theorem is replaced by γ. Since, however, the series $\sum(D_{\nu-1}^{-\gamma} - D_\nu^{-\gamma})$, which again has positive terms, converges trivially—for, its n^{th} partial sum is $D_0^{-\gamma} - D_n^{-\gamma}$, which, since $D_n \to \infty$, tends to $D_0^{-\gamma}$—, it would, according to the comparison test of the first kind, suffice to show that

$$\frac{d_\nu}{D_\nu \cdot D_{\nu-1}^{\gamma}} \leqq \frac{1}{\gamma}(D_{\nu-1}^{-\gamma} - D_\nu^{-\gamma}) \tag{3}$$

or (since $d_\nu = D_\nu - D_{\nu-1}$) that

$$1 - \frac{D_{\nu-1}}{D_\nu} \leqq \frac{1}{\gamma}\left(1 - \frac{D_{\nu-1}^{\gamma}}{D_\nu^{\gamma}}\right). \tag{4}$$

If, for abbreviation, we set $(D_{\nu-1}/D_\nu)^\gamma = x$, so that $D_{\nu-1}/D_\nu = x^p$, then (4) is the same as

$$1-x^p \leqq p(1-x).$$

This, however, is certainly correct, because $0 < x \leqq 1$ and $1-x^p = (1-x)(1+x+\ldots+x^{p-1})$. Therefore (2) is convergent for $\delta > 0$.

The case $\alpha = 1$ in Theorem 1 admits of the following more precise assertion:

Theorem 3. (*Cesàro*) *If $\sum d_\nu$, $(d_\nu \geqq 0)$, is divergent, but if $d_\nu/D_\nu \to 0$, then, for the partial sums of $\sum d_\nu/D_\nu$, we have the asymptotic estimate*

$$\frac{d_0}{D_0} + \frac{d_1}{D_1} + \ldots + \frac{d_\nu}{D_\nu} \cong \log D_\nu. \tag{5}$$

PROOF. Since $\frac{d_\nu}{D_\nu} = d_\nu' \to 0$, we have, as is shown by formula (2) in 6.4 (whose proof is independent of the present investigations)

$$\frac{d_\nu'}{\log \frac{1}{1-d_\nu'}} = \frac{d_\nu/D_\nu}{\log (D_\nu/D_{\nu-1})} \to 1.$$ [1]

[1] These quotients may be set $=1$, if $d_\nu = 0$. Likewise, set $D_{-1} = 1$.

Hence, according to 2.4.2,**2**, we have also

$$\frac{d_0' + d_1' + \ldots + d_n'}{\log D_n} \to 1,$$

which proves the theorem.

Dini (1867) also proved a theorem corresponding to Theorem 1, starting from a convergent series:

Theorem 4. *If* $\sum c_\nu$ *is a convergent series of positive terms, and if* $r_{n-1} = c_n + c_{n+1} + \ldots$ *are its remainders, then*

$$(6) \qquad \sum \frac{c_n}{r_{n-1}^\alpha} = \sum \frac{c_n}{(c_n + c_{n+1} + \ldots)^\alpha} \quad \begin{cases} \textit{converges for } \alpha < 1, \\ \textit{diverges for } \alpha \geqq 1. \end{cases}$$

Proof. As in the proof of Theorem 1, we have, first of all, for $\alpha = 1$,

$$\frac{c_n}{r_{n-1}} + \ldots + \frac{c_{n+k}}{r_{n+k-1}} \geqq \frac{c_n + \ldots + c_{n+k}}{r_{n-1}} = 1 - \frac{r_{n+k}}{r_{n-1}}.$$

Since, for every (fixed) n, k can be chosen so large that $1-(r_{n+k}/r_{n-1}) > \frac{1}{2}$, our series is divergent for $\alpha = 1$. This holds *a fortiori* for $\alpha > 1$, because $r_n < 1$ from a certain stage on.

If $\alpha < 1$, then we choose the natural number p so that, with $\gamma = \dfrac{1}{p}$, we have $\alpha < 1-\gamma$. Since $r_n < 1$ from a certain stage on, it suffices, then, to prove the convergence of the series

$$\sum \frac{c_\nu}{(r_{\nu-1})^{1-\gamma}} = \sum \frac{r_{\nu-1} - r_\nu}{r_{\nu-1}} \cdot r_{\nu-1}^\gamma, \qquad \left(\gamma = \frac{1}{p}\right).$$

On account of $r_\nu \searrow 0$, $\sum (r_{\nu-1}^\gamma - r_\nu^\gamma)$ is trivially a convergent series of positive terms. It therefore suffices to show that, from a certain stage on,

$$\frac{r_{\nu-1} - r_\nu}{r_{\nu-1}} \cdot r_{\nu-1}^\gamma \leqq \frac{1}{\gamma} \, (r_{\nu-1}^\gamma - r_\nu^\gamma),$$

or, if we set $(r_\nu/r_{\nu-1})^\gamma = y$, that

$$(1-y^p) \leqq p(1-y).$$

This is certainly the case, however, because $0 < y \leqq 1$.

5.2. Scales of convergence tests

The theorems in the preceding section are of importance in various directions. We base our explanation of this on the following definition:

Definition 1. *A convergent series* Σc_ν *of positive terms is said to converge* faster or better *than another convergent series* $\Sigma c'_\nu$ *of positive terms* (*or the latter is said to converge* slower or worse *than the former*), *if—where, as usual,* r_n *and* r'_n *are understood to be the respective remainders of these series*— $r_n/r'_n \to 0$.[1]

Likewise, of two divergent series Σd_ν *and* $\Sigma d'_\nu$ *of positive terms, the first is said to diverge slower or more weakly than the second* (*the latter to diverge faster or more strongly than the former*), *if* $s_n/s'_n \to 0$*—where* s_ν *and* s'_ν *are understood to be the respective partial sums of the series.*

In this connection we have the following simple

Theorem 1. *If* $c_\nu/c'_\nu \to 0$, *then* Σc_ν *converges faster than* $\Sigma c'_\nu$. *If* $d_\nu/d'_\nu \to 0$, *then* Σd_ν *diverges slower than* $\Sigma d'_\nu$.

For if $c_\nu < \varepsilon c'_\nu$ for $\nu > \mu$, then, for every $n > \mu$,

$$\frac{r_n}{r'_n} = \frac{c_{n+1}+c_{n+2}+\cdots}{c'_{n+1}+c'_{n+2}+\cdots} < \varepsilon,$$

and hence $r_n/r'_n \to 0$. On the other hand, $d_\nu/d'_\nu \to 0$ immediately implies, according to 2.4.2,2, that also

$$\frac{d_0+d_1+\ldots+d_n}{d'_0+d'_1+\ldots+d'_n} = \frac{s_n}{s'_n} \to 0.$$

According to this, Theorem 1 in 5.1 asserts, in particular: For every divergent series Σd_ν, there exists a more weakly divergent series $\Sigma d'_\nu$, namely, the series with $d'_\nu = d_\nu/D_\nu$.

Likewise, Theorem 4 asserts: For every convergent series Σc_ν, one can find a more weakly convergent series $\Sigma c'_\nu$; *e.g.*, the series $\Sigma c'_\nu$ of the Theorem 4 just cited with an α in $0 < \alpha < 1$, for then $c_\nu/c'_\nu = (c_\nu + c_{\nu+1} + \ldots)^\alpha \to 0$.

[1] This limit need not exist; therefore, of two convergent (divergent) series, the one need not always converge (diverge) faster than the other.

With the aid of these theorems one can start with an arbitrary convergent or divergent series and form scales of series that are more and more weakly convergent or divergent, respectively. If we begin with $\Sigma 1 = 1+1+\ldots$, say, then we obtain, as in 3.3,(5), successively the divergent series

$$\Sigma \frac{1}{\nu}, \qquad \Sigma \frac{1}{\nu \log \nu}, \qquad \Sigma \frac{1}{\nu \log \nu \log_2 \nu}, \ldots,$$

and, for $\alpha > 1$, the convergent series

$$\Sigma \frac{1}{\nu^{\alpha}}, \qquad \Sigma \frac{1}{\nu (\log \nu)^{\alpha}}, \qquad \Sigma \frac{1}{\nu \log \nu (\log_2 \nu)^{\alpha}}, \ldots.$$

(Here the D_ν were replaced by asymptotically equal values, which has no effect on the convergence behavior.) Each of these series is more weakly divergent or convergent, respectively, than the preceding one. If we choose them as comparison series in the convergence tests of the first and second kind (3.1), we obtain increasingly finer divergence or convergence tests. Since, however, these series from the third on diverge or converge extraordinarily weakly, they have, on the whole, more of a theoretical than a practical value. We shall therefore not enter more closely into this, but shall only mention the form into which one can bring these criteria after an easy transformation. They then constitute an immediate generalization of the ratio test in 3.2: *If, for a series Σa_ν of positive terms and for a fixed integer $k \geqq 0$, we have, from a certain stage on,*

$$\left[\frac{a_{n+1}}{a_n} - 1 + \frac{1}{n} + \frac{1}{n \log n} + \ldots + \frac{1}{n \log n \ldots \log_k n}\right] \cdot n \log n \ldots \log_k n$$

$$\begin{cases} \leqq -\alpha < 0, & \textit{then } \Sigma a_\nu \textit{ is convergent,} \\ \geqq 0, & \textit{then } \Sigma a_\nu \textit{ is divergent.} \end{cases}$$

Of this scale, only the test for $k = 0$ and at most that for $k = 1$ have any practical importance. We shall derive these as well as several finer criteria pertaining to series of arbitrary complex terms, independently of the foregoing one, in the next section but one. These investigations will be preceded in 5.3 by some auxiliary considerations which will also be of importance later on.

5.3. Abel's partial summation. Lemmas

1. Let $\{a_\nu\}$ and $\{p_\nu\}$ be any two sequences (now once more with arbitrary complex terms). Let a be any number, and set

$$a+a_0+a_1+\ldots+a_\nu = s'_\nu, \qquad (\nu = 0, 1, 2, \ldots;\ s'_{-1} = a).$$

Then, for every integral $\mu \geqq -1$ and $\rho \geqq 1$,

$$(1) \qquad \sum_{\nu=\mu+1}^{\mu+\rho} a_\nu p_\nu = \sum_{\nu=\mu+1}^{\mu+\rho} s'_\nu(p_\nu - p_{\nu+1}) - s'_\mu p_{\mu+1} + s'_{\mu+\rho} p_{\mu+\rho+1}.$$

In particular ($\mu = -1$, $\mu+\rho = n$, $a = 0$, and, as usual, $s_\nu = a_0 + a_1 + \ldots + a_\nu$),

$$(2) \qquad \sum_{\nu=0}^{n} a_\nu p_\nu = \sum_{\nu=0}^{n} s_\nu(p_\nu - p_{\nu+1}) + s_n p_{n+1}.$$

It is customary to call the formulas (1) and (2) the *formulas for Abel's partial summation.* They correspond exactly to the formulas for integration by parts.

The *proof* is obtained immediately by setting $a_\nu = (s'_\nu - s'_{\nu-1})$, ($\nu = 0, 1, \ldots$), in the sum on the left, and then collecting the terms containing the same factor s'_ν.

Formula (2) yields at once

Theorem 1. *A series of the form $\Sigma a_\nu p_\nu$ is convergent, if, with $s_\nu = a_0+a_1+\ldots+a_\nu$,*

(3) *the series $\Sigma s_\nu(p_\nu - p_{\nu+1})$ converges and if at the same time*

(4) *the sequence $\{s_\nu p_{\nu+1}\}$ converges.*

For then the right side of (2), and with it the left side, tends to a limit as $n \to \infty$.

Lemma 1. Let Σa_ν and Σb_ν be any two series whose terms are different from 0. If the sequences $\{p_\nu\} \equiv \{a_\nu/b_\nu\}$ and $\{p_\nu^{-1}\} \equiv \{b_\nu/a_\nu\}$ are both of bounded variation (see 3.5, Definition 2 and the subsequent remark) and if they both have a limit different from 0, then either each one of the two series, or neither one of them, is (a) convergent, (b) absolutely convergent, (c) divergent with bounded partial sums, (d) divergent with unbounded partial sums.

PROOF. We denote the respective partial sums of Σa_ν and Σb_ν by s_ν, t_ν. Because of the symmetry in the assumptions and the conclusions, it suffices to argue from the series Σb_ν to the series Σa_ν. Suppose, then, that (*a*) Σb_ν is convergent. Then the sequence $\{t_\nu\}$ is also convergent, and therefore bounded. From the absolute convergence of the series $\Sigma(p_\nu - p_{\nu+1})$ and the resulting convergence of the sequence $\{p_\nu\}$, it now follows immediately that the right side of (2)[1], and with it, also the left side, tends to a limit as $n \to \infty$. (b) Since $\{|p_\nu|\}$ is of bounded variation if $\{p_\nu\}$ is[2], the correctness of (b) follows in the same way. If the sequence $\{t_\nu\}$ is divergent but bounded, than the first term in (2) tends, as before, to a limit as $n \to \infty$, the second, however, does not, but remains bounded. This proves (c). Finally, (d) is merely the negation of (a), (b), and (c).

Lemma 2. Let Σc_ν and $\Sigma c'_\nu$ be two absolutely convergent series (of complex terms) whose terms are all <1 in absolute value, and let c and c' be two arbitrary numbers different from 0. Then the sequences of numbers

$$p_n = c \cdot \prod_{\nu=0}^{n} (1 + c_\nu), \quad q_n = c' \cdot \prod_{\nu=0}^{n} (1 + c'_n)^{-1}, \quad \text{and } p_n q_n, \quad (n = 0, 1, \ldots),$$

are of bounded variation and their limits are different from 0.

PROOF. According to 3.7, Theorem 5, the sequences $\{p_\nu\}$ and $\{q_\nu\}$ are at any rate convergent. They are therefore also bounded. By 3.7, Theorem 1, their limits are different from 0. From

$$p_\nu - p_{\nu+1} = -p_\nu c_{\nu+1}, \qquad q_\nu - q_{\nu+1} = q_\nu \frac{c'_{\nu+1}}{1 + c'_{\nu+1}},$$

and the fact that $c'_\nu \to 0$, we can now read off, that the first two se-

[1] We have only to replace a_ν by b_ν and correspondingly s_ν by t_ν in (2).
[2] For we have $||p_\nu| - |p_{\nu+1}|| \leqq |p_\nu - p_{\nu+1}|$.

quences of the lemma are of bounded variation. That the third is then follows from

$$p_\nu q_\nu - p_{\nu+1}\, q_{\nu+1} = (p_\nu - p_{\nu+1}) q_\nu + p_{\nu+1}(q_\nu - q_{\nu+1}).$$

Lemma 3. Let Σa_ν and Σb_ν be two arbitrary series whose terms are different from 0. From a certain stage on, let

$$(5) \qquad \frac{a_{\nu+1}}{a_\nu} = 1 - \frac{\alpha}{\nu} + c_\nu, \qquad \frac{b_{\nu+1}}{b_\nu} = 1 - \frac{\alpha}{\nu} + c'_\nu,$$

where Σc_ν and $\Sigma c'_\nu$ are absolutely convergent series. Then the series Σa_ν and Σb_ν satisfy the hypotheses of Lemma 1, *i.e.*, the sequences $\{a_\nu/b_\nu\}$ and $\{b_\nu/a_\nu\}$ are of bounded variation and their limits are different from 0.

Proof. Choose m so large that for $\nu > m$ (5) invariably holds, that $|c_\nu| < 1$, $\left|\frac{\alpha}{\nu}\right| < 1$, and $\left|\frac{c_\nu}{1-\frac{\alpha}{\nu}}\right| < 1$, and that the corresponding relations hold for the c'_ν. Then, for $n \geqq m$,

$$a_{n+1} = a_m \cdot \prod_{\nu=m}^{n} \left(1 - \frac{\alpha}{\nu} + c_\nu\right) = a_m \cdot \prod_{\nu=m}^{n} \left(1 - \frac{\alpha}{\nu}\right) \prod_{\nu=m}^{n} \left(1 + \frac{c_\nu}{1-\frac{\alpha}{\nu}}\right)$$

and correspondingly

$$b_{n+1} = b_m \cdot \prod_{\nu=m}^{n} \left(1 - \frac{\alpha}{\nu}\right) \prod_{\nu=m}^{n} \left(1 + \frac{c'_\nu}{1-\frac{\alpha}{\nu}}\right).$$

Since the series $\Sigma c_\nu \Big/ \left(1-\frac{\alpha}{\nu}\right)$ and $\Sigma c'_\nu \Big/ \left(1-\frac{\alpha}{\nu}\right)$ converge absolutely, the correctness of the present assertion follows immediately from Lemma 2.

5.4. Special comparison tests of the second kind

We prove at once a very far-reaching test of the second kind which goes back essentially to *K. Weierstrass:* [1]

[1] Journ. f. d. reine u. angew. Math., vol. 51 (1856), p. 29; Werke, vol. I, p. 185. *Weierstrass* assumed instead of (1) that $a_{\nu+1}/a_\nu$ could be developed in a power series $1 - \frac{\alpha}{\nu} + \frac{\alpha_2}{\nu^2} + \ldots$ of ascending powers of $1/\nu$. The above form of the theorem and its quite considerably simplified proof compared to *Weierstrass*'s is due to *H. Jehle*, Math. Zeitschr., vol. 52 (1950). *Cf.*, in this connection, also *R. P. Agnew*, Pacific J. Math., vol. 1 (1951), pp. 1–3.

Theorem 1. *If the terms a_ν of a given series $\sum a_\nu$ of complex terms possess, for $\nu \geqq m$, a representation of the form*

$$(1) \qquad \frac{a_{\nu+1}}{a_\nu} = 1 - \frac{\alpha}{\nu} + c_\nu \qquad \text{with} \qquad \sum |c_\nu| < +\infty,$$

then $\sum a_\nu$ is convergent if, and only if, $\Re(\alpha) = \beta > 1$. If this is the case, then $\sum a_\nu$ is actually absolutely convergent. If $\Re(\alpha) = 1$ but $\alpha \neq +1$, then $\sum a_\nu$ is divergent but has bounded partial sums. If $\alpha = 1$ or $\Re(\alpha) < 1$, then $\sum a_\nu$ is divergent and the partial sums are unbounded.

Proof. We consider first the case $\alpha = +1$. For $b_\nu = 1/\nu$, $(\nu \geqq 1)$, we may set

$$(2) \qquad \frac{b_{\nu+1}}{b_\nu} = \frac{\nu}{\nu+1} = 1 - \frac{1}{\nu} + c'_\nu \qquad \text{with} \qquad \sum |c'_\nu| < \infty.$$ [1]

Thus, for $\alpha = +1$, $\sum a_\nu$ has, according to Lemma 3 and in the sense of Lemma 1 in 5.3, the same convergence behavior as $\sum 1/\nu$, *i.e.*, $\sum a_\nu$ is divergent and the partial sums are unbounded.

If, now, $\alpha \neq +1$, so that $\delta = 1 - \alpha \neq 0$, then we associate with the series $\sum a_\nu$ the series $\sum b_\nu$ for which

$$(3) \qquad b_\nu = e^{\delta h_\nu} - e^{\delta h_{\nu+1}}, \qquad \left(\nu = 1, 2, \ldots;\ h_\nu = 1 + \frac{1}{2} + \ldots + \frac{1}{\nu}\right).$$

An easy calculation shows that we may set

$$(4) \qquad \frac{b_{\nu+1}}{b_\nu} = 1 - \frac{\alpha}{\nu} + c'_\nu \qquad \text{with} \qquad \sum |c'_\nu| < \infty.$$ [2]

Therefore $\sum a_\nu$ has the same convergence behavior (in the sense of Lemma 1 in 5.3) as the series $\sum b_\nu$. The latter's partial sums, however, are

$$= \sum_{\nu=1}^{n} b_\nu = e^{1-\alpha} - e^{(1-\alpha) h_{n+1}}.$$

---

[1] We have $c'_\nu = 1/\nu(\nu+1)$.

[2] For we have

$$\frac{b_{\nu+1}}{b_\nu} = \frac{1 - e^{\delta/(\nu+2)}}{e^{-\delta/(\nu+1)} - 1} = \frac{\frac{1}{\nu+2} + \frac{\delta}{2(\nu+2)^2} + O\left(\frac{1}{\nu^3}\right)}{\frac{1}{\nu+1} - \frac{\delta}{2(\nu+1)^2} + O\left(\frac{1}{\nu^3}\right)} = 1 - \frac{\alpha}{\nu} + O\left(\frac{1}{\nu^2}\right) = 1 - \frac{\alpha}{\nu} + c'_\nu.$$

Thus, since $h_{n+1} \to +\infty$, Σb_ν, and hence also Σa_ν, is convergent if, and only if, $\Re(1-\alpha) < 0$ or $\Re(\alpha) > 1$. If $\Re(\alpha) = 1$ (but $\alpha \neq +1$), then $1-\alpha = i\gamma$ is a pure imaginary and $\{e^{i\gamma h_{n+1}}\}$ tends to no limit as $n \to \infty$, but remains bounded. If, finally, $\Re(\alpha) = \beta < 1$, then

$$\left|e^{(1-\alpha) h_{n+1}}\right| = e^{(1-\beta) h_{n+1}} \to +\infty,$$

the partial sums are unbounded. Consequently the same also holds for Σa_ν.

In order to prove, in conclusion, the absolute convergence of Σa_ν in the case $\Re(\alpha) = \beta > 1$, we associate with it the series Σb_ν with $b_\nu = e^{-\alpha h_\nu}$. It, too, as in (4), satisfies

$$(5) \qquad \frac{b_{\nu+1}}{b_\nu} = e^{-\frac{\alpha}{\nu+1}} = 1 - \frac{\alpha}{\nu} + c'_\nu, \qquad \text{with} \qquad \Sigma|c'_\nu| < \infty,$$

so that, by Lemma 3 in 5.3, either both Σa_ν and Σb_ν are, or both are not, absolutely convergent.

For the second series, however, $|b_\nu| = e^{-\beta h_\nu}$, and hence

$$\left|\frac{b_{\nu+1}}{b_\nu}\right| = 1 - \frac{\beta}{\nu} + c''_\nu, \qquad \text{with} \qquad \Sigma|c''_\nu| < \infty.$$

Since $\beta > 1$, $\Sigma|b_\nu|$ is therefore convergent by the part of Theorem 1 already proved, and consequently $\Sigma|a_\nu|$ is also convergent.

This completes the proof of Theorem 1. We shall supplement this theorem by means of several *remarks* and *corollaries*.

1. Under the assumption of Lemma 1 in 5.3, either both the series $\Sigma(a_\nu - a_{\nu+1})$ and $\Sigma(b_\nu - b_{\nu+1})$ are or both are not absolutely convergent.

PROOF. It suffices again to infer the absolute convergence of the first series from that of the second. That of the first, however, can be read off immediately from

$$a_\nu - a_{\nu+1} = b_\nu p_\nu - b_{\nu+1} p_{\nu+1} = p_\nu(b_\nu - b_{\nu+1}) + b_{\nu+1}(p_\nu - p_{\nu+1}),$$

since both terms on the right are terms of series which are absolutely convergent by the assumptions.[1]

[1] Since $\Sigma|b_\nu - b_{\nu+1}| < \infty$, $\lim b_\nu$ exists, therefore $\{b_\nu\}$ is bounded.

2. Under the assumptions of Theorem 1, $\Sigma(a_\nu - a_{\nu+1})$ is absolutely convergent if, and only if, $\Re(\alpha) = \beta > 0$.

PROOF. We again associate with the a_ν the $b_\nu = e^{-\alpha h_\nu}$ as in (5). Then, on account of (1) and (5), the assumptions of Lemma 1 in 5.3 are again (see Lemma 3 in 5.3) satisfied, so that, by the preceding corollary, $\Sigma(a_\nu - a_{\nu+1})$ is absolutely convergent if, and only if, this holds for $\Sigma(b_\nu - b_{\nu+1})$. Now we have, however,

$$b_\nu - b_{\nu+1} = b'_\nu = e^{-\alpha h_\nu} - e^{-\alpha h_{\nu+1}},$$

and the calculation in connection with (3)—δ there has to be replaced by $-\alpha$ and we have to write b'_ν instead of b_ν—has shown that $\Sigma b'_\nu$ converges, and then actually absolutely, if, and only if, $\Re(-\alpha) < 0$, *i.e.*, $\Re(\alpha) > 0$. The same therefore holds for $\Sigma(a_\nu - a_{\nu+1})$.

3. Under the assumptions of Theorem 1, $\lim a_\nu = 0$ if, and only if, $\Re(\alpha) > 0$.

PROOF. Once again we associate with the a_ν the $b_\nu = e^{-\alpha h_\nu}$. Then, on account of (1) and (5) and Lemma 3 in 5.3, $a_\nu = b_\nu p_\nu$, and $\lim p_\nu \neq 0$. Hence, $\lim a_\nu = 0$ if, and only if, $\lim b_\nu = 0$. This, however, is obviously the case if, and only if, $\Re(\alpha) > 0$.

4. Under the assumptions of Theorem 1, $\Sigma(-1)^\nu a_\nu$ is convergent if, and only if, $\Re(\alpha) > 0$. (*Cf.*, in this connection, 5.5, Example 5.)

PROOF. According to Corollary 3, $\Re(\alpha) > 0$ is at any rate necessary for the convergence of $\Sigma(-1)^\nu a_\nu$. If, however, $\Re(\alpha) > 0$, then, by Corollary 2, $\Sigma(a_\nu - a_{\nu+1})$, and therefore also $\sum_{\rho=0}^{\infty}(a_{2\rho} - a_{2\rho+1})$, is absolutely convergent. Since $a_\nu \to 0$, we have also $|a_{2\rho}| + |a_{2\rho+1}| \to 0$. By 3.6, Theorem 2, we may therefore remove the parentheses in the last series. This proves the assertion.

5. If, for all ν from a certain stage on, we have

$$\left|\frac{a_{\nu+1}}{a_\nu}\right| \leqq 1 - \frac{\alpha}{\nu} + c_\nu, \qquad \text{with } \alpha > 1 \qquad \text{and} \qquad \Sigma|c_\nu| < \infty,$$

then Σa_ν is absolutely convergent. If, however, from a certain stage on, we have

$$\left|\frac{a_{\nu+1}}{a_\nu}\right| \geqq 1 - \frac{1}{\nu} + c'_\nu, \qquad \text{with } |c'_\nu| < \infty,$$

then $\Sigma|a_\nu|$ is divergent. These are merely special cases of Theorem 1. A special case, in turn, of **5** is

6. *J. L. Raabe's test* (1832). If, from a certain stage on, we have

$$\left(\left|\frac{a_{\nu+1}}{a_\nu}\right|-1\right)\cdot\nu\begin{cases}\leqq-\alpha<-1, & \text{then } \Sigma|a_\nu| \text{ is convergent,}\\ \geqq-1, & \text{then } \Sigma|a_\nu| \text{ is divergent.}^{1}\end{cases}$$

For the assumptions assert that, from a certain stage on, $\left|\frac{a_{\nu+1}}{a_\nu}\right|\leqq 1-\frac{\alpha}{\nu}$ with $\alpha>1$, $>1-\frac{1}{\nu}$, respectively.[2]

7. Another special case of Theorem 1 is the test formulated by *C. F. Gauss* (although he gave it only for real series): If the quotient $a_{\nu+1}/a_\nu$ can be written in the form

$$\frac{a_{\nu+1}}{a_\nu}=\frac{\nu^k+b_1\nu^{k-1}+\ldots+b_k}{\nu^k+b_1'\nu^{k-1}+\ldots+b_k'},\qquad (k\geqq 1,\ \text{integral}),$$

then Σa_ν is absolutely convergent for $\Re(b_1'-b_1)>1$, divergent for $\Re(b_1'-b_1)\leqq 1$. For we have

$$\frac{a_{\nu+1}}{a_\nu}=1-\frac{b_1'-b_1}{\nu}+O\left(\frac{1}{\nu^2}\right).$$

5.5. Abel's and Dirichlet's tests and their generalizations

In 5.4 we associated with a series Σa_ν the series $\Sigma a_\nu p_\nu$ and derived assertions about $\Sigma a_\nu p_\nu$ from assumptions concerning Σa_ν and $\{p_\nu\}$. Since $\Sigma a_\nu p_\nu$, for a given Σa_ν, can be identical with any other series

---

[1] Basically, **5** and **6** are only tests for series of positive terms.

[2] A *direct proof* of *Raabe*'s test can be given as follows: We set $|a_\nu|=\alpha_\nu$. The convergence assumption then says that $\nu\alpha_{\nu+1}\leqq(\nu-1)\alpha_\nu-(\alpha-1)\alpha_\nu$ for $\nu>\mu$ and $\alpha>1$. According to this, the sequence $\{\nu\alpha_{\nu+1}\}$ is monotonically decreasing and therefore tends to a limit $\gamma\geqq 0$. Hence, $\Sigma\gamma_\nu$, with $\gamma_\nu=(\nu-1)\alpha_\nu-\nu\alpha_{\nu+1}$, is trivially convergent, and, since $\alpha_\nu\leqq\frac{\gamma_\nu}{\alpha-1}$, so is $\Sigma\alpha_\nu$. In a similar manner, the divergence assumption implies that $(\nu-1)\alpha_\nu-\nu\alpha_{\nu+1}<0$, so that $\{\nu\alpha_{\nu+1}\}$ increases monotonically, and therefore eventually remains greater than a fixed number $\gamma>0$. The divergence of $\Sigma\alpha_\nu$ now follows from $\alpha_{\nu+1}>\frac{\gamma}{\nu}$.

Σb_ν (we have merely to set $p_\nu = b_\nu/a_\nu$[1]), we may designate assertions of the kind just mentioned, as ***comparison tests in the extended sense.*** We shall derive a few such tests and give several applications of them.

Theorem 1. (*Abel*, 1826.) *$\Sigma a_\nu p_\nu$ is convergent if Σa_ν converges and $\{p_\nu\}$ is monotonic and bounded.*—We prove at once the somewhat more general.

Theorem 2. (*Dedekind*, 1863.) *$\Sigma a_\nu p_\nu$ is convergent if Σa_ν converges and $\{p_\nu\}$ is of bounded variation.*

The *proof* follows immediately from 5.3, Theorem 1 (*cf.* also the proof of Lemma 1, (a) in 5.3). For, the assumptions imply the (absolute) convergence of $\Sigma s_\nu(p_\nu - p_{\nu+1})$ and the convergence of the sequence $\{s_\nu p_{\nu+1}\}$.

An easy but quite essential modification is furnished by the following tests:

Theorem 3. (*Dirichlet*, 1863.) *$\Sigma a_\nu p_\nu$ is convergent if Σa_ν has bounded partial sums and $\{p_\nu\}$ is a monotonic null sequence.*—And somewhat more generally:

Theorem 4. (*Dedekind*, 1863.) *$\Sigma a_\nu p_\nu$ is convergent if Σa_ν has bounded partial sums and $\{p_\nu\}$ is a null sequence of bounded variation.*

The *proof* again follows directly from 5.3, Theorem 1. For, the assumptions once more imply the (absolute) convergence of $\Sigma s_\nu(p_\nu - p_{\nu+1})$ and the convergence of the sequence $\{s_\nu p_{\nu+1}\}$.[2]

Applications and examples

1. According to Theorem 1, the following series, *e.g.*, converge with Σa_ν:

$$\Sigma a_\nu x^\nu,\quad (0 \leq x \leq 1),\quad \Sigma \sqrt[\nu]{\nu}\cdot a_\nu,\quad \Sigma\left(1+\frac{1}{\nu}\right)^\nu a_\nu,\quad \Sigma\frac{a_\nu}{\nu},\quad \Sigma\frac{a_\nu}{\log\nu},\ \textit{etc.}$$

2. Σz^ν, $|z| = 1$, $z \neq 1$, has (see 2.6.1,**2**) bounded partial sums. Hence, for these z (*i.e.*, for all $z \neq +1$ on the boundary of the unit

---

[1] And assume that the terms $a_\nu \neq 0$.

[2] Note that in Theorems 3 and 4, less is assumed about Σa_ν, and therefore more is assumed about $\{p_\nu\}$, than in Theorems 1 and 2.

circle), $\Sigma a_\nu z^\nu$ is convergent if $\{a_\nu\}$ is a null sequence of bounded variation, in particular, a monotonic null sequence.

According to this, the series $\Sigma \frac{z^\nu}{\nu}$, *e.g.*, which has the radius 1 and, as will be shown in 6.4, represents $\log \frac{1}{1-z}$, is still convergent for all $z \neq 1$ on the boundary of the circle of convergence. The same holds, say, for the series $\Sigma \frac{z^\nu}{\log \nu}$, $\Sigma \frac{z^\nu}{\log \log \nu}$, *etc.* For $z = -1$ and $a_\nu \searrow 0$, this application furnishes a new proof of *Leibniz*'s test (3.4, Theorem 2). It shows also that a series of the form $\Sigma(-1)^\nu a_\nu$ is already convergent if the a_ν (real or complex) form a null sequence of bounded variation.

Since $\Sigma (-1)^\nu z^{2\nu}$ $\left(= \frac{1}{1+z^2}\right)$ has bounded partial sums for every $|z| \leqq 1$ except for $z = \pm i$, it follows analogously that the arc-tan series (see 6.6) $\Sigma(-1)^\nu \frac{z^{2\nu+1}}{2\nu+1}$ converges exactly for the z just mentioned, and hence, in particular, at all boundary points of the unit circle different from $\pm i$.

3. If in **2** we set $z = \cos x + i \sin x$ and separate real and imaginary parts, it follows that, for every null sequence $\{a_\nu\}$ of bounded variation, the series $\Sigma a_\nu \cos \nu x$ and $\Sigma a_\nu \sin \nu x$ are convergent for every real x—the first possibly with the exception of the values $x = 2k\pi$, $(k = 0, \pm 1, \pm 2, \ldots)$.

4. A series of the form $\sum\limits_{\nu=1}^{\infty} \frac{a_\nu}{\nu^z}$ is called an (ordinary) *Dirichlet series.* For such a series, the following holds: If the series converges for $z = z_0$, or if it has merely bounded partial sums for this value of z, then it is convergent for every z for which $\Re(z) > \Re(z_0)$.[1] Since $\Sigma \frac{a_\nu}{\nu^z} = \Sigma \frac{a_\nu}{\nu^{z_0}} \cdot \frac{1}{\nu^{z-z_0}}$, on the basis of Theorem 4 we have merely to show that the sequence $\left\{\frac{1}{\nu^{z-z_0}}\right\}$ is a null sequence of bounded variation if $z - z_0 = d$ has a positive real part $\Re(d) = \delta$. On account of $\left|\frac{1}{\nu^d}\right| =$

[1] Visualize this condition in the z-plane.

$=\frac{1}{\nu^\delta}\to 0$ for $\delta>0$ as $\nu\to\infty$, only the absolute convergence of $\sum\left(\frac{1}{\nu^d}-\frac{1}{(\nu+1)^d}\right)$ remains to be proved. We have

$$\left|\frac{1}{\nu^d}-\frac{1}{(\nu+1)^d}\right| = \frac{1}{\nu^\delta}\left|1-\left(1+\frac{1}{\nu}\right)^{-d}\right|,$$

and since, by 6.3 and 6.4,(1), $1-\left(1+\frac{1}{\nu}\right)^{-d}=\frac{d}{\nu}+O\left(\frac{1}{\nu^2}\right)$, it follows further that

$$\left|\frac{1}{\nu^d}-\frac{1}{(\nu+1)^d}\right| = \frac{|d|}{\nu^{1+\delta}}+O\left(\frac{1}{\nu^2}\right), \qquad (\delta>0).$$

The infinite series with these terms, however, is convergent.

5. In connection with the theorems in 5.4, we obtain the following far-reaching

Theorem. *If the coefficients of the power series* $\sum a_\nu z^\nu$ *have the property that it is possible to set*

$$\frac{a_{\nu+1}}{a_\nu}=1-\frac{\alpha}{\nu}+c_\nu,$$

with an absolutely convergent series $\sum c_\nu$, *then the power series* $\sum a_\nu z^\nu$ *has the radius* 1 *and, at the boundary points* $|z|=1$, *it is*

a) *absolutely convergent for* $\Re(\alpha)>1$,
b) *conditionally convergent for* $0<\Re(\alpha)\leqq 1$ *except at* $z=+1$,
c) *divergent for* $\Re(\alpha)\leqq 0$.

PROOF. Since $\left|\frac{a_{\nu+1}z^{\nu+1}}{a_\nu z^\nu}\right|\to|z|$, our series has the radius 1. The remaining assertions, however, follow easily from 5.4, Theorem 1 and its corollaries: a) If $\Re(\alpha)>1$, then according to those theorems $\sum a_\nu$ is absolutely convergent. If $|z|=1$, then the same holds for $\sum a_\nu z^\nu$. b) If $0<\Re(\alpha)\leq 1$, then, by Corollaries 2 and 3, $\{a_\nu\}$ is a null sequence of bounded variation. Since $\sum z^\nu$ has bounded partial sums for every $z\neq+1$ with $|z|=1$, $\sum a_\nu z^\nu$ converges for these z according to Theorem 4. For $z=+1$, however, we are dealing with the series $\sum a_\nu$, which, under the present assumption, diverges, according to 5.4, Theorem 1. c) If, finally, $\Re(\alpha)\leqq 0$, then, by Corollary 3 of that theorem, $\{a_\nu\}$ is not a null sequence. The same holds, then, for $\{a_\nu z^\nu\}$, if $|z| = 1$, so that $\Sigma a_\nu z^\nu$ does not converge.

6. By means of the preceding theorem, the behavior of the series $\Sigma\binom{\alpha}{\nu}z^\nu$, which we shall meet as the *binomial series* in 6.5, is clarified on the boundary of its circle of convergence $|z| < 1$. Here we set $(-1)^\nu\binom{\alpha}{\nu} = a_{\nu+1}$, so that, for $\nu \geqq 1$,

$$\frac{a_{\nu+1}}{a_\nu} = 1 - \frac{\alpha+1}{\nu}.$$

The preceding theorem therefore immediately yields: a) If $\Re(\alpha) > 0$, then the series is absolutely convergent at all boundary points $|z| = 1$. b) If $-1 < \Re(\alpha) \leqq 0$, then it is conditionally convergent at all these boundary points except at $z = -1$, where it diverges (with bounded partial sums). c) If, finally, $\Re(\alpha) \leqq -1$, then it is divergent at all boundary points.

5.6. Series transformations

In 3.6, Theorem 9 (Corollary 3) and Theorem 10, we represented each term of a series $\Sigma\alpha_n$ as the sum of a convergent series (*cf.* the series (24) there), wrote down these representations $\alpha_n = \sum_\nu a_{n\nu}$ in rows one under another, and then derived a new series by means of columnar summation. We shall now pursue this idea further in a somewhat different form.

To this end, we begin with an arbitrary series Σa_ν, introduce a matrix $B = (b_{n\nu})$, and set, somewhat as in 3.5, (2) and under the assumption of the convergence of the series that appear,

$$\text{(B)} \qquad \sum_{\nu=0}^{\infty} b_{n\nu}a_\nu = \alpha_n, \qquad (n = 0, 1, 2, \ldots).$$

In such a case we shall say, for brevity, that Σa_ν has been transformed into the series $\Sigma\alpha_n$ by means of the transformation (B).[1]

We seek conditions under which such a transformation is permanent,

[1] We have precisely the situation presented in 3.6, Theorem 9, Corollary 3, if we choose the $b_{n\nu}$ so that $b_{n\nu}a_\nu$ is equal to the $a_{n\nu}$ there. This is always possible, provided that $a_\nu \neq 0$.

i.e., changes a convergent series Σa_ν into a series $\Sigma \alpha_n$ that is again convergent and that also has the same value. In this connection, we have, in analogy with Theorem 4 in 3.5, the following

Theorem 1. *Let* $B = (b_{n\nu})$ *be a matrix which, if we set*

$$\sum_{\rho=0}^{n} b_{\rho\nu} = B_{n\nu}, \qquad (n, \nu = 0, 1, 2, \ldots), \tag{1}$$

satisfies the two conditions that for a suitable $M > 0$

$$\sum_{\nu=0}^{\infty} |B_{n\nu} - B_{n,\nu+1}| \leqq M \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{2}$$[1]

and that

$$\lim_{n\to\infty} B_{n\nu} = \sum_{n=0}^{\infty} b_{n\nu} = 1 \quad \text{for every } \nu = 0, 1, 2, \ldots. \tag{3}$$[2]

If Σa_ν, *with the partial sums* s_ν, *is an arbitrary convergent series, then, for every* $n = 0, 1, \ldots$,

$$\alpha_n = \sum_\nu b_{n\nu} a_\nu \tag{4}$$

exists, and $\Sigma \alpha_n$ *is again a convergent series. Moreover,* $\Sigma \alpha_n = \Sigma a_\nu$.

Proof. We show, first, that (2) and (3) imply that, for a suitable K and all $n, \nu = 0, 1, 2, \ldots$,

$$|B_{n\nu}| \leqq K. \tag{5}$$

Now, according to (2), $\sum_\nu (B_{n\nu} - B_{n,\nu+1}) = \lim_\nu (B_{n0} - B_{n\nu})$ exists and

$$|B_{n0} - B_{n\nu}| \leqq M \quad \text{for all} \quad n, \nu = 0, 1, 2, \ldots. \tag{6}$$

By (3), however, $\lim_n B_{n0} = 1$ exists, and hence $\{B_{n0}\}$ is bounded. Therefore, by (6), $|B_{n\nu}|$, for all $n, \nu = 0, 1, \ldots$, also lies below a fixed bound, *i.e.*, (5) holds.

---

[1] In words: For every fixed $n = 0, 1, \ldots$, $\{B_{n\nu}\}$ is a sequence of bounded variation, and the total variations of all these sequences lie below a bound M independent of n.

[2] In words: The column series of the matrix are all convergent, and all have the value 1.

Now

$$\sigma_n = \alpha_0+\alpha_1+\ldots+\alpha_n = \sum_\nu (b_{0\nu}+b_{1\nu}+\ldots+b_{n\nu})a_\nu = \sum_\nu B_{n\nu}a_\nu$$

or

$$\sigma_n = \sum_\nu B_{n\nu}(r_{\nu-1}-r_\nu), \tag{7}$$

if we denote by r_ν the remainders

$$r_\nu = a_{\nu+1}+a_{\nu+2}+\ldots \qquad \text{for} \qquad \nu = -1, 0, 1, \ldots,$$

so that, in particular, $r_{-1} = s = \Sigma a_\nu$. By the formula for *Abel*'s partial summation (5.3,(2)),

$$\sum_{\nu=0}^{\mu} B_{n\nu}(r_{\nu-1}-r_\nu) = -\sum_{\nu=0}^{\mu}(B_{n\nu}-B_{n,\nu+1})r_\nu+B_{n0}\cdot s - B_{n,\mu+1}\,r_\mu.$$

As $\mu\to\infty$, this yields, on account of (5) and $r_\mu\to 0$, in accordance with (7),

$$\sigma_n - B_{n0}\cdot s = -\sum_{\nu=0}^{\infty}(B_{n\nu}-B_{n,\nu+1})\,r_\nu. \tag{8}$$

The sequence $\{\sigma_n-B_{n0}\cdot s\}$ on the left thus arises from the null sequence $\{r_\nu\}$ by means of a transformation of the form discussed in 3.5, Theorem 4, with $a_{n\nu} = B_{n\nu}-B_{n,\nu+1}$. This transformation satisfies the conditions (N) and (C) there, because the first is identical with (2), and the second follows from (3), since $\lim\limits_n (B_{n\nu}-B_{n,\nu+1}) = 1-1 = 0$. According to (6), the row sums $A_n = \Sigma a_{n\nu} = \lim\limits_{\nu\to\infty}(B_{n0}-B_{n\nu})$ are not greater than M in absolute value. Hence, by Corollary 1 of Theorem 4 in 3.5, we have $(\sigma_n-B_{n0}\cdot s)\to 0$, *i.e.*, $\sigma_n\to s$, since $B_{n0}\to 1$.

Corollary 1. This theorem, just as Theorem 4 in 3.5, is a "best possible" theorem in the sense that conditions (2) and (3) are not only *sufficient* for its validity, but are also *necessary*. The proof of this, however, which follows easily from the corresponding fact regarding Theorem 4 in 3.5, will have to be omitted.

Corollary 2. Since, in the proof of Theorem 1, the quantities $b_{n\nu}$ did not appear at all any more, but only the quantities $B_{n\nu}$ were used, we have proved at the same time the following theorem:

Theorem 2. *Let* $(B_{n\nu})$ *be a matrix satisfying conditions* (2) *and* (3). *If* $\sum a_\nu$, *with the partial sums* s_ν *and the value* s, *is an arbitrary convergent series, then, for every* $n = 0, 1, 2, \ldots$,

$$\sigma_n = \sum_{\nu=0}^{\infty} B_{n\nu}\, a_\nu \tag{9}$$

exists and the sequence $\{\sigma_n\}$, *in turn, is convergent and has the limit* s.

Finally, one can convince oneself without difficulty that no use was made of the fact that n ranges precisely over the numbers $0, 1, 2, \ldots$, and it is seen that the following theorem therefore also holds:

Theorem 3. *Let* $\{B_\nu(x)\}$ *be a sequence of functions defined in a left-sided neighborhood,* U, *of* x_0 *(that is, in an interval of the form* $x_0 - \delta < x < x_0$*)*[1] *and satisfying the two conditions*

$$\sum_\nu |B_\nu(x) - B_{\nu+1}(x)| \leqq M \qquad \text{for all } x \text{ in } U, \tag{10}$$

$$\lim_{x\to x_0} B_\nu(x) = 1 \qquad \text{for every } \nu = 0, 1, \ldots. \tag{11}$$

If $\sum a_\nu$, *with the partial sums* s_ν *and the value* s, *is an arbitrary convergent series, then, for every* x,

$$\sigma(x) = \sum_\nu B_\nu(x) a_\nu \tag{12}$$

exists, and the function $\sigma(x)$ *converges to* s *as* $x \to x_0 - 0$.[2]

Applications

1. If $\sum a_\nu = s$, then also $\sum \alpha_n = s$, if we set $\alpha_0 = a_0$ and

$$\alpha_n = \frac{a_1 + 2a_2 + \ldots + na_n}{n(n+1)}$$

for $n \geqq 1$. Here

$b_{0\nu} = 1$ for $\nu = 0$ and $= 0$ for $\nu > 0$, and for $n \geqq 1$ we have

$$b_{n\nu} = \frac{\nu}{n(n+1)} \quad \text{or} \quad = 0$$

[1] Here we may have also $x_0 = +\infty$. By a left-sided neighborhood of x_0 we then mean an interval of the form $x_1 < x < +\infty$, where x_1 may denote an arbitrary number.

[2] *I.e.*, for left-sided approach of x to x_0, $+\infty$, respectively.

according as $0 \leqq \nu \leqq n$ or $\nu > n$. Consequently $B_{n\nu} = \nu\left(\frac{1}{\nu} - \frac{1}{n+1}\right) = 1 - \frac{\nu}{n+1}$ for $\nu \leqq n$, and $=0$ for $\nu > n$. According to this, conditions (2) and (3) of Theorem 1 are obviously satisfied, and the transformation is therefore permanent. It corresponds exactly to *Cauchy*'s limit theorem in 2.4.

2. If $\Sigma a_\nu = s$, then also $\Sigma \alpha_n = s$, if, for $n = 0, 1, 2, \ldots$, we set

$$\alpha_n = \frac{1}{2^{n+1}}\left[\binom{n}{0} a_0 + \binom{n}{1} a_1 + \ldots + \binom{n}{n} a_n\right]$$

(*Euler's transformation of series*). Here

$$b_{n\nu} = \frac{1}{2^{n+1}}\binom{n}{\nu} \quad \text{for} \quad 0 \leqq \nu \leqq n, \quad \text{and} \quad = 0 \quad \text{for} \quad \nu > n.$$

Consequently, $B_{n\nu} = \sum\limits_{\rho=\nu}^{n} \frac{1}{2^{\rho+1}}\binom{\rho}{\nu}$ for $0 \leqq \nu \leqq n$, and $=0$ for $\nu > n$. An elementary transformation shows that, for $0 \leqq \nu \leqq n$, also

$$B_{n\nu} = \frac{1}{2^{n+1}}\left[\binom{n+1}{\nu+1} + \binom{n+1}{\nu+2} + \ldots + \binom{n+1}{n+1}\right].$$

This representation shows that $B_{n\nu}$, for fixed n, decreases monotonically and tends to zero as $\nu \to \infty$, so that $\sum\limits_{\nu} |B_{n\nu} - B_{n,\nu+1}| = \frac{2^{n+1}-1}{2^{n+1}} \leqq 1$. Hence, (2) in Theorem 1 is satisfied. Further, for $n > \nu$,

$$B_{n\nu} = \frac{1}{2^{n+1}}\left[2^{n+1} - \binom{n+1}{0} - \ldots - \binom{n+1}{\nu}\right],$$

and thus tends, for fixed ν, to 1 as $n \to \infty$, so that (3) in Theorem 1 is also satisfied: *Euler's transformation is permanent.*

3. It is remarkable that even some divergent series are changed into convergent ones by means of the transformations **1** and **2.** If, *e.g.*, $\Sigma a_\nu = \Sigma(-1)^\nu$, then the transformation **1** yields the convergent series

$$1 - \frac{1}{2} + \frac{1}{2\cdot 3} - \frac{1}{2\cdot 3} + \frac{1}{2\cdot 5} - \frac{1}{2\cdot 5} + - \ldots = \frac{1}{2}$$

and the second yields the series

$$\tfrac{1}{2}+0+0+\ldots = \tfrac{1}{2}.$$

These remarks are the starting point of the extensive theory of methods for the summation of divergent series.

4. *Abel*'s limit theorem (see 4.2, Theorem 6) also follows immediately as an application of Theorem 3. We have only to set $B_\nu(x) = x^\nu$ $(0 \leqq x < 1,\ \ \nu = 0, 1, 2, \ldots)$ as well as $x_0 = +1$. For then conditions (10) and (11) of the latter theorem are obviously satisfied. Hence, if $\Sigma a_\nu = s$ is convergent, then $\sigma(x) = \sum\limits_\nu a_\nu x^\nu$ exists in $0 \leqq x < 1$ —which is self-evident here—and $\sigma(x) \to s = \Sigma a_\nu$ as $x \to 1-0$. This, however, is *Abel*'s limit theorem.

5.7. Multiplication of series

In 3.6 we considered the *Cauchy* product of two convergent series $\Sigma a_\nu = A$ and $\Sigma b_\nu = B$, and saw in Theorem 12 that this product Σc_ν, $(c_\nu = a_0 b_\nu + \ldots + a_\nu b_0)$, again converges and has the expected value $C = AB$ if the two factor series converge absolutely. An example showed that without this assumption the series Σc_ν need not converge at all. We now investigate the question as to whether the convergence of Σc_ν can be guaranteed under weaker assumptions than are made in *Cauchy*'s theorem, and we first prove the following theorem, which is due to *F. Mertens* (1875):

Theorem 1. *If at least* o n e *of the two convergent series $\Sigma a_\nu = A$ and $\Sigma b_\nu = B$ is absolutely convergent, then the Cauchy product series Σc_ν is also convergent, and has the value $C = AB$.*

Proof. Let us assume that Σb_ν converges absolutely. We denote the partial sums of our three series by A_ν, B_ν, C_ν, respectively. Then

$$\begin{aligned} C_n &= c_0+c_1+\ldots+c_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \ldots + (a_0 b_n + \ldots + a_n b_0) \\ &= A_0 b_n + A_1 b_{n-1} + \ldots + A_n b_0 \\ &= A \cdot B_n - (\alpha_0 b_n + \alpha_1 b_{n-1} + \ldots + \alpha_n b_0), \end{aligned}$$

if we set $A_\nu = A - \alpha_\nu$, so that $\alpha_\nu \to 0$. Since $A \cdot B_n \to AB$, there remains to be shown merely that $\alpha_0 b_n + \alpha_1 b_{n-1} + \ldots + \alpha_n b_0 = \gamma_n \to 0$ if $\Sigma|b_\nu|$ converges and $\{\alpha_\nu\}$ is a null sequence. The sequence $\{\gamma_n\}$, however, is obtained from $\{\alpha_\nu\}$ by means of a linear transformation in the sense of 2.4, Theorem 2 (and Corollary) with the matrix $a_{n\nu} = b_{n-\nu}$ for $n = 0, 1, \ldots, \; 0 \leqq \nu \leqq n$, and $= 0$ for $\nu > n$. The two conditions of that theorem read here

$$\sum_{n=\nu}^{\infty} |b_{n-\nu}| \leqq M, \quad i.e., \quad \sum_{\nu=0}^{\infty} |b_\nu| \leqq M, \quad \text{and} \quad \sum_{\nu=0}^{\infty} b_\nu = B,$$

both of which are satisfied because of the absolute convergence of Σb_ν. Hence $\gamma_n \to B \cdot 0 = 0$, and consequently $C \to AB$. This proves the theorem.

Corollary. As in the case of Theorem 4 in 3.5 and Theorem 1 in 5.6, the present theorem is also a best possible one in a certain sense. It can be shown (but the proof must be omitted here) that the absolute convergence of Σb_ν is also necessary in order that its *Cauchy* product with *every* convergent series Σa_ν turn out to be convergent. Its value is then automatically the correct one, *i.e.*, equal to AB. For we have

Theorem 2. *Let* $\Sigma a_\nu = A$ *and* $\Sigma b_\nu = B$ *be two convergent series. If their Cauchy product* Σc_ν *is also convergent, then it has the correct value* $C = AB$.

PROOF. We consider the three power series $f_a(x) = \Sigma a_\nu x^\nu$, $f_b(x) = \Sigma b_\nu x^\nu$, and $f_c(x) = \Sigma c_\nu x^\nu$, for $0 \leqq x < 1$. They are absolutely convergent in the interval $0 \leqq x < 1$. Therefore, by *Cauchy*'s multiplication theorem (3.6, Theorem 12), $f_a(x)\, f_b(x) = f_c(x)$. According to *Abel*'s limit theorem (Theorem 6 in 4.2), these functions tend to the respective limits A, B, C as $x \to 1-0$. Therefore $AB = C$.

There remains the (up to this day not satisfactorily settled) question, under what assumptions concerning the factor series Σa_ν and Σb_ν the product series Σc_ν turns out to be convergent. Of the numerous theorems which provide sufficient conditions for this to occur, we shall present only the following one, discovered by *G. H. Hardy*[1] in 1908:

---

[1] The proof that follows was given by *E. Landau* in 1920.

Theorem 3. *Let* $\Sigma a_\mu = A$ *and* $\Sigma b_\nu = B$ *be two convergent series. If* $a_\mu = O\left(\frac{1}{\mu}\right)$, $b_\nu = O\left(\frac{1}{\nu}\right)$, *then* Σc_ν, $(c_\nu = a_0 b_\nu + \ldots + a_\nu b_0)$, *is convergent and* $= AB$.

PROOF. We have

$$c_0+c_1+\ldots+c_n = \sum_{\mu+\nu\leq n} a_\mu b_\nu. \tag{1}$$

If we arrange the products $p_{\mu\nu} = a_\mu b_\nu$ as the elements of a matrix $(p_{\mu\nu})$, then (1) is the sum of all these elements as far as the n^{th} diagonal, which joins the elements p_{n0} and p_{0n}. Setting $n/2 = m$, we decompose the triangle determined by these elements and p_{00} into a square and two smaller triangles, in accordance with

$$\Sigma p_{\mu\nu} = \sum_{\mu\leq m,\ \nu\leq m} p_{\mu\nu} + \sum_{\nu>m} p_{\mu\nu} + \sum_{\mu>m} p_{\mu\nu}, \tag{2}$$

where all the sums are to be extended only over those μ and ν satisfying the additional condition $\mu+\nu \leq n$.

The first term on the right in (2) is equal to $\sum_{\mu\leq m} a_\mu \cdot \sum_{\nu\leq m} b_\nu$, and therefore $\to AB$. It suffices to show that the second and third $\to 0$. For reasons of symmetry, only one of these two sums, say the last, has to be treated.

To this end, we divide the triangle corresponding to this sum into two parts by means of the vertical line at $\nu = y = y_n$, sum in the left part by columns, in the right by rows, and thus, having chosen a $y = y_n$ in $0 \leq y \leq n$,[1] set

$$\sum_{\mu>m} p_{\mu\nu} = \sum_{\nu\leq y} b_\nu \cdot \sum_{m<\mu\leq n-\nu} a_\mu + \sum_{m<\mu\leq n-y} a_\mu \cdot \sum_{y<\nu\leq n-\mu} b_\nu. \tag{3}$$

Here we now choose $y = y_n$ as follows: Let

$$r_\lambda = a_{\lambda+1} + a_{\lambda+2} + \ldots \quad \text{and} \quad \rho_n = \overline{\text{fin}}_{\nu>n} |r_\nu|,$$

so that $\rho_n > 0$ (except if, from a certain stage on, all $a_\nu = 0$, and then there is nothing to prove) and $\searrow 0$ as $n \to \infty$. Then choose $y = y_n$ so that

$$\sum_{0\leq\lambda\leq y} |b_\lambda| \leq \frac{1}{\sqrt{\rho_n}},$$

[1] This representation can be made clear conveniently by means of a little sketch.

y_n remains invariably $\leqq n$, but $y_n \to \infty$, which is obviously possible. Then the first term on the right in (3) is

$$\leqq \sum_{0 \leqq \lambda \leqq y} |b_\lambda| \cdot 2\rho_n \leqq 2\sqrt{\rho_n}$$

and thus $\to 0$. The second term in (3) is, on the basis of the O-assumption, and because its second factor $\to 0$ on account of the convergence of Σb_ν, $= o\left(\sum_{m<\mu\leqq n} \frac{1}{\mu}\right) = o(1)$, because the sum in parentheses tends to log 2 (see 2.5, Example 4). Hence, the partial sum in (1) tends to AB, Q.E.D.

Chapter 6

EXPANSION OF THE ELEMENTARY FUNCTIONS

6.1. List of the elementary functions

Knowledge of the elementary functions and their power-series expansions, in the real as well as in the complex domain, will be presupposed here in the main. It is acquired in the real domain as an application of *Taylor*'s theorem in the differential calculus; in the complex domain it belongs to the rudiments of the theory of functions (*cf. Elem.*,[1] section V). This knowledge will not be deepened here; rather, only several fundamental definitions and the most important properties of these functions, in so far as they are of series-theoretical interest, will be listed, but proofs will only be indicated briefly.

First of all, only the two functions z and e^z need to be regarded as *elementary functions*. All functions, however, that can be obtained from these two functions and arbitrary complex numbers a, b, ... by performing the following operations a finite number of times, are also designated as elementary:

I. Linear combination $a \cdot f(z) + b \cdot g(z)$ as well as multiplication $f(z) \cdot g(z)$ and division $f(z)/g(z)$ of two already existing functions f and g.

II. Formation of the composite function $f(g(z))$ of two already existing functions.

III. Formation of the inverse of an already existing function.

Here z, of course, must be restricted to those points (regions) of the z-plane at which the operations mentioned are meaningful (can be carried out). Several of the functions thus obtained are given special names:

---

[1] This is an abbreviation of the title of the book referred to above at the end of chapter 1.

1. All functions that can be obtained from z by means of the rational operations I are designated as *rational functions* (*cf.* 6.2).

2. The function defined for all z by the series (see 3.2.1,**2**)

$$\sum_{\nu=0}^{\infty} \frac{z^\nu}{\nu!} \equiv 1+z+\frac{z^2}{2!}+\dots \tag{1}$$

is denoted by $\exp z$ or e^z (*cf.* 6.3).

3. The functions

$$\frac{e^{iz}+e^{-iz}}{2} = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-+\dots, \tag{2}$$

$$\frac{e^{iz}-e^{-iz}}{2i} = z-\frac{z^3}{3!}+\frac{z^5}{5!}-+\dots \tag{3}$$

are denoted by $\cos z$, $\sin z$, respectively. Further, we set $\frac{\sin z}{\cos z} = \tan z$, $\frac{\cos z}{\sin z} = \cot z$. These four functions are called *trigonometric* or *circular functions* (*cf.* 6.3). The functions

$$\frac{e^{z}+e^{-z}}{2} = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\dots, \tag{4}$$

$$\frac{e^{z}-e^{-z}}{2} = z+\frac{z^3}{3!}+\frac{z^5}{5!}+\dots \tag{5}$$

are denoted by $\cosh z$ and $\sinh z$, and we set $\frac{\sinh z}{\cosh z} = \tanh z$, $\frac{\cosh z}{\sinh z} = \coth z$. These four functions are designated as *hyperbolic functions*. The hyperbolic functions differ only a little (in the complex domain) from the circular functions, for we have $\cosh z = \cos(iz)$, $\sinh z = -i \sin(iz)$. We shall therefore not treat these functions and their inverses any further.

4. The inverses of the functions e^z, $\sin z$, $\cos z$, $\tan z$, and $\cot z$ are denoted respectively by $\log z$, $\operatorname{arc\,sin} z$, $\operatorname{arc\,cos} z$, $\operatorname{arc\,tan} z$, and $\operatorname{arc\,cot} z$ (*cf.* 6.4 and 6.6).

5. The composite function $\exp(a \log z)$, (a fixed, arbitrary, complex), is denoted by z^a and designated as the *general power*, and the composite function $\exp(z \log a)$, ($a \neq 0$, arbitrary, complex) is denoted by a^z and designated as the *general exponential function*; for further details, see 6.5 and 6.3.

6.2. The rational functions

For an integral $k \geqq 0$ and $|z| < 1$ we found the expansion (see 4.2,(15))

$$(1) \qquad \frac{1}{(1-z)^{k+1}} = \sum_{\nu=0}^{\infty} \binom{\nu+k}{\nu} z^{\nu}.$$

If we set $(k+1) = -p$ and replace z by $-z$, then (1) shows, since $\binom{\nu+a}{\nu} = (-1)^{\nu}\binom{-a-1}{\nu}$ (for arbitrary a), that

$$(2) \qquad (1+z)^p = \sum_{\nu=0}^{\infty} \binom{p}{\nu} z^{\nu}$$

for fixed $p = -1, -2, \ldots$ and $|z| < 1$. Thus, (1) or (2) may be regarded as an extension of the binomial theorem to negative integral exponents p. For positive integral p the series (2) is only formally infinite; it is then valid for all z. For $p = 0$ it yields, likewise for all z (including $z = -1$), the value 1.[1] For further details concerning the expansion (2), see 6.5.

More generally, for arbitrary a and $z_0 \neq a$ we have

$$(3) \qquad \frac{1}{(z-a)^{k+1}} = \frac{1}{(z_0-a)^{k+1}} \cdot \frac{1}{\left(1 - \dfrac{z-z_0}{a-z_0}\right)^{k+1}},$$

and hence

$$(4) \quad \frac{c}{(z-a)^{k+1}} = \frac{c}{(z_0-a)^{k+1}} \cdot \sum_{\nu=0}^{\infty} \binom{\nu+k}{\nu} \left(\frac{z-z_0}{a-z_0}\right)^{\nu}, \quad (|z-z_0| < |a-z_0|).$$

An arbitrary rational function, however, may be represented as the sum of an entire rational and a proper fractional rational function.[2] Since every proper fractional rational function may be represented as the sum of finitely many partial fractions, *i.e.*, fractions of the form (4),

---

[1] This is in agreement with the definition in footnote 1, p. 159.

[2] Either one of the two parts here may $= 0$.

this points the way to the representation of a given rational function by means of a power series.

6.3. The exponential function and the circular functions

All further elementary functions can be derived from the rational functions and the exponential function. We shall list here briefly the properties of $\exp z$, $\cos z$, and $\sin z$ that are the most important from a series-theoretical standpoint.

1. The series 6.1,(1), (2), (3) defining these functions are everywhere convergent. Since these functions, accordingly, are defined for all z of the entire z-plane, they are called *entire functions*.

2. It is customary to supplement the definition of e^z by the following: Let a be fixed and *positive*. By a^z we mean the entire function uniquely defined by

$$a^z = \exp(z \log a) = \sum_{\nu=0}^{\infty} \frac{(\log a)^\nu}{\nu!} z^\nu, \tag{1}$$

where $\log a$ denotes the (real) natural logarithm of a.

3. For $\exp z$ we have (see 3.6, following Theorem 12) the addition theorem

$$\exp(z_1+z_2) = \exp z_1 \cdot \exp z_2,$$

and an analogue for p summands. Similarly,

$$a^{z_1+z_2} = a^{z_1} \cdot a^{z_2}.$$

4. Since the functions mentioned in **1** are represented by everywhere convergent power series, they are continuous, and differentiable arbitrarily often, for every z, and

$$(e^z)' = e^z, \quad (a^z)' = a^z \cdot \log a, \quad (\cos z)' = -\sin z, \quad (\sin z)' = \cos z.$$

5. It follows from 6.1,(2) and (3), that

$$e^{iz} = \cos z + i \sin z$$

for every z. Hence, in particular, for a real y,

$$e^{iy} = \cos y + i \sin y,$$

so that e^{iy}, and for $z = x+iy$ also

$$e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y),$$

can easily be calculated with the help of ordinary logarithmic and trigonometric tables.

6. From the second formula in **5** we get—the values of $\cos y$ and $\sin y$ for $y = 2\pi$, π, and $\pi/2$ will be regarded as known—, in particular, the important fact that

$$e^{2\pi i} = 1 \quad \text{and hence also} \quad e^{2k\pi i} = 1 \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots .$$

Therefore

$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z,$$

so that the function e^z has the *period* $2\pi i$. We have the more precise result, however, that $e^z = 1$ if, and only if, $z = 2k\pi i$, with $k = 0$, ± 1, $\pm 2, \ldots$. For if we set $z = x+iy$, then, by **5** (and by 1.2,(4)), we must have $e^x = 1$ and at the same time $\cos y + i \sin y = 1$. This is the case for real x, y only for $x = 0$ and $y = 2k\pi$.

7. The addition theorem for e^z leads, without difficulty, with the aid of 6.1,(2) and (3), to the corresponding theorems for $\cos z$ and $\sin z$:

$$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$
$$\sin(z_1+z_2) = \cos z_1 \sin z_2 + \sin z_1 \cos z_2.$$

From these theorems follow, as is well known, all formulas of goniometry, as it is called; in other words, the formulas for

$$\cos 2z, \quad \sin 2z, \quad \cos\left(z+\frac{\pi}{2}\right), \quad \cos(z+\pi), \quad \cos(z+2\pi), \quad \textit{etc.}$$

The formulas of real goniometry, therefore, also hold unchanged in the complex domain.

8. For every (fixed) z, $\left(1+\dfrac{z}{\nu}\right)^\nu = z_\nu \to e^z$.

Proof. Expanding z_ν, for $\nu > 2$, by the binomial theorem, we may write:

$$z_\nu = 1 + z + \frac{1}{2!}\left(1-\frac{1}{\nu}\right)z^2 + \ldots + \frac{1}{k!}\left[\left(1-\frac{1}{\nu}\right)\left(1-\frac{2}{\nu}\right)\cdots\left(1-\frac{k-1}{\nu}\right)\right]z^k + \ldots .$$

The series is only formally infinite, because, for $k > \nu$, its terms are $=0$. The coefficient of z^k is $\geqq 0$ but $\leqq \frac{1}{k!}$. The same is then true of the coefficient of z^k in the difference

$$e^z - z_\nu = \sum_{k=2}^{\infty} \frac{1}{k!}\left[1-\left(1-\frac{1}{\nu}\right)\cdots\left(1-\frac{k-1}{\nu}\right)\right]z^k.$$

For fixed z and a given $\varepsilon > 0$, we now choose a p so large, that the remainder

$$\sum_{k=p+1}^{\infty} \frac{|z|^k}{k!} < \frac{\varepsilon}{2}.$$

Then, for $\nu > p$,

$$|e^z - z_\nu| < \sum_{k=2}^{p} \frac{1}{k!}\left[1-\left(1-\frac{1}{\nu}\right)\ \cdots\ \left(1-\frac{k-1}{\nu}\right)\right]\cdot|z|^k + \frac{\varepsilon}{2}.$$

Now the sum on the right has a fixed number, $p-1$, of terms, each of which $\to 0$ as $\nu \to \infty$. Hence, for a suitable μ, this sum is $< \varepsilon/2$ for $\nu > \mu$, and therefore

$$|e^z - z_\nu| < \varepsilon \quad \text{for} \quad \nu > \mu,$$

which proves the assertion.

9. The important question as to the domain of values of the function $w = e^z$ is completely answered by the following theorem: For an arbitrarily given $w \neq 0$, there exists precisely one z whose imaginary part lies between $-\pi$ (excl.) and $+\pi$ (incl.), for which $e^z = w$. The value $w = 0$, however, is assumed for no z.

The latter assertion follows already from the equation $e^z \cdot e^{-z} = e^0 = 1$, according to which no factor on the left can have the value 0. The first assertion is verified as follows: If we set $z = x+iy$, $w = R(\cos\psi + {}$ ${}+ i\sin\psi)$, then we are supposed to have

$$e^x = R,\ e^{iy} = e^{i\psi}.$$

The first of these equations is satisfied for precisely one real x, because e^x increases monotonically from 0 to ∞ (both excl.) as x ranges over the real numbers from $-\infty$ to $+\infty$ (both excl.). The second implies that $e^{i(y-\psi)} = 1$, and hence, according to **6**, $y = \psi + 2k\pi$, $(k = 0, \pm 1, \ldots)$.

If $\psi = \text{am } w$ is fixed, then there exists precisely one integer k such that $y = \psi + 2k\pi$ lies between $-\pi$ (excl.) and $+\pi$ (incl.). Hence, $x = \log R$, $y = \text{am } w$, provided that the principal value of this amplitude is taken.

10. The question as to the domain of values of the function $\sin z$ can be answered analogously. In the period-strip $-\pi < \Re(z) \leqq +\pi$, $\sin z$ assumes every value w different from ± 1 at precisely two distinct points, whereas each of the values ± 1 is assumed at exactly one point (namely at $\pi/2$ and $-\pi/2$). Precisely one solution of $\sin z = w$ lies in the region $-\pi/2 \leqq \Re(z) \leqq +\pi/2$, provided that the part of the boundary lying below the axis of reals (that is to say, the set of points $z = x + iy$ with $x = \pm\pi/2$, $y < 0$) is deleted from this strip (*cf. Elem.*, § 46).

11. We arrive at the power-series expansions of $\tan z$ and $\cot z$ with the aid of the expansion (23) in 4.3. According to 6,1,(2) and (3), we have, for every $z \neq k\pi$, $(k = 0, \pm 1, \pm 2, \ldots)$,

$$z \cot z = iz \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} = iz + \frac{2iz}{e^{2iz} - 1}.$$

Hence, according to (23) in 4.3, if we bear in mind that $B_1 = -\frac{1}{2}$ and that $B_3 = B_5 = \ldots = 0$, we have

$$z \cot z = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{2^{2\nu} B_{2\nu}}{(2\nu)!} z^{2\nu}.$$

This representation is certainly valid for all sufficiently small $|z|$.[1] The precise determination of the radius of convergence (it is $= \pi$) of the power series obtained requires somewhat heavier application of the theory of functions (*cf.* the partial-fractions decomposition of $\pi z \cdot$ $\cdot \cot \pi z$ below; further, 7.3,3, as well as *Elem.*, § 43, and *Th. F. I.*,[2] § 31).

With the help of the formula $\tan z = \cot z - 2 \cot 2z$, we now easily obtain the representation

$$\tan z = \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \frac{2^{2\nu}(2^{2\nu} - 1) B_{2\nu}}{(2\nu)!} z^{2\nu-1},$$

[1] For $z = 0$ too, if one then defines the left side as $\lim_{z \to 0} z \cdot \cot z = 1$.

[2] This refers to volume I of the author's *Theory of Functions*, listed in the Bibliography at the end of this book.

which again is certainly valid for all small $|z|$. (Its exact radius is $= \pi/2$.)

Somewhat deeper methods are required for the derivation of the so-called partial-fractions decomposition of the function $\cot z$. It will, nevertheless, be quoted here, although without proof (for a proof, *cf.* *Th. F. II*, § 6), and several applications will also be made of it in the next chapter: For all $z \neq \pm 1, \pm 2, \ldots$ we have the representation

$$\pi z \cot \pi z = 1 + \sum_{\nu=1}^{\infty} \frac{2z^2}{z^2-\nu^2}. \tag{2}$$

This expansion leads, by means of simple calculations, to further important representations of a similar kind: Since

$$\pi \tan \pi z = \pi \cot \pi z - 2\pi \cot 2\pi z,$$

we obtain, first of all,

$$\pi \tan \pi z = \sum_{\nu=0}^{\infty} \frac{8z}{(2\nu+1)^2 - 4z^2}, \qquad (2z \neq \pm 1, \ \pm 3, \ \ldots). \tag{3}$$

From $1/\sin z = \cot z + \tan \dfrac{z}{2}$ we find, further, that

$$\frac{\pi}{\sin \pi z} = \frac{1}{z} - \frac{2z}{z^2-1^2} + \frac{2z}{z^2-2^2} - + \ldots, \quad (z \neq 0, \ \pm 1, \ \pm 2, \ldots), \tag{4}$$

and finally, replacing z here by $\frac{1}{2} - z$,

$$\frac{\pi}{4 \cos \pi z} = \frac{1}{1^2-(2z)^2} - \frac{3}{3^2-(2z)^2} + \frac{5}{5^2-(2z)^2} - + \ldots, \tag{5}$$
$$(2z \neq \pm 1, \ \pm 3, \ \ldots).$$

6.4. The logarithmic function

The inverse of the function $e^z = w$ is called the natural logarithm. If we interchange the letters: w is called a natural logarithm of z, if $e^w = z$. According to 6.3,**6** and **9**, we can immediately assert more precisely: Every number z different from 0 (and only such a z) possesses precisely one natural logarithm w whose imaginary part satisfies the condition $-\pi < \mathfrak{I}(w) \leqq +\pi$. With this so-called *principal value* w of the natural logarithm of z, all numbers $w + 2k\pi i$, $(k = 0, \pm 1,$

$\pm 2, \ldots$), and only these, are also natural logarithms of the number z. From now on, we shall denote only this principal value of the logarithm of z by $\log z$.

Since $z = e^w$ can be expanded in a power series for any center w_0 (we set $e^{w_0} = z_0$):

$$z - z_0 = \frac{z_0}{1!}(w - w_0) + \frac{z_0}{2!}(w - w_0)^2 + \ldots,$$

it is also possible, inversely, to expand $w = \log z$ in a power series about any center $z_0 \neq 0$:

$$w - w_0 = \frac{1}{z_0}(z - z_0) + c_2(z - z_0)^2 + \ldots,$$

which converges for all sufficiently small $|z - z_0|$.[1] We infer from this, first of all, that, for every $z \neq 0$,

$$\frac{d}{dz}\log z = \frac{1}{z} \quad \text{and consequently} \quad \frac{d^\nu}{dz^\nu}\log z = (-1)^{\nu-1}\frac{(\nu-1)!}{z^\nu}.$$

If we choose $z_0 = 1$, that is if we take $w_0 = 0$, then $c_\nu = \frac{1}{\nu!}\left(\frac{d^\nu}{dz^\nu}\log z\right)_{z=1} =$ $= \frac{(-1)^{\nu-1}}{\nu}$ and we obtain

$$w = (z-1) - \tfrac{1}{2}(z-1)^2 + - \ldots = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu}(z-1)^\nu$$

as the expansion of $\log z$ in a power series with the center $z_0 = 1$, which converges for all sufficiently small $|z-1|$ and represents the principal value of $\log z$. The series obviously converges, however, for all $|z-1| < 1$,[2] and since, according to the origin of the series, its sum w satisfies $e^w = z$, its sum is always some logarithm of z. It is easy to show that it is invariably the principal value: For every z in $|z-1| < 1$, $\operatorname{am} z$ has exactly one value ψ for which $-\pi/2 < \psi < +\pi/2$. With this ψ, then,

$$\mathfrak{I}(w) = \mathfrak{I}(\log z) = \psi + 2k\pi i \quad \text{with a} \quad k = 0, \pm 1, \pm 2, \ldots.$$

---

[1] At the moment it is open to question whether it invariably furnishes the principal value $w = \log z$.

[2] We shall consider the boundary points in just a moment.

Now w, and hence also $\mathfrak{S}(w)$, is a continuous function in $|z-1|<1$. Therefore k must always have the same value in this equation. Hence, since for $z = 1$ we have $\log z = 0$ and therefore $k = 0$, we must take $k = 0$ for all z in $|z-1|<1$, *i.e.*, our series represents the principal value $\log z$ for all these z.

If we replace z by $1+z$, we obtain the expansion

$$\log(1+z) = z - \frac{z^2}{2} + - \dots = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} z^\nu, \qquad (|z|<1), \tag{1}$$

from which we get, by replacing z by $-z$ and changing sign, the expansion

$$\log \frac{1}{1-z} = z + \frac{z^2}{2} + \dots = \sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu}, \qquad (|z|<1), \tag{2}$$

and, by addition, the expansion

$$\tfrac{1}{2} \log \frac{1+z}{1-z} = z + \frac{z^3}{3} + \frac{z^5}{5} + \dots, \qquad (|z|<1). \tag{3}$$

The series obtained are called for brevity the *logarithmic series*.

It was shown already in 5.5,2 that the series (2) also converges for all z of the boundary $|z| = 1$ that are different from $z = +1$. It does not yet follow from this, however, that it also represents the principal value of the logarithm for these z. This will not be proved until we come to 7.3,2.

6.5. The general power and the binomial series

If a is a fixed complex number, then the general power z^a in the function-theoretical sense is by its very nature a multiple-valued function. Only artificially, by means of restrictive supplementary conditions, can it be made into a single-valued function. It is customary to regard as the *principal value* of z^a the value uniquely defined by $\exp(a \cdot \log z)$, where, as in 6.4, $\log z$ denotes the principal value of the natural logarithm.[1] In what follows, z^a always stands for this principal value. It is defined only for $z \neq 0$.

[1] *E.g.*, according to this, the principal value of $i^i = e^{i \log i} = e^{-\pi/2}$ —a real number!

It will be more convenient to replace z by $1+z$. Then $(1+z)^a = \exp\big(a\cdot\log(1+z)\big)$ is uniquely defined for all $z \neq -1$, in particular, for $|z|<1$. As a composite function, it possesses a power-series expansion

$$(1+z)^a = 1+a_1z+a_2z^2+\ldots,$$

which converges for all $|z|<1$, since the "outer" power series is everywhere convergent (see 4.3,**4**). It also represents, as it ought to, the principal value $(1+z)^a$, if, for the inner function, the expansion (1) in 6.4 representing the principal value is taken. According to 4.2,(12), the coefficients are determined by

$$a_\nu = \frac{1}{\nu!}\,\frac{d^\nu}{dz^\nu}\,(1+z)^a \qquad \text{evaluated at } z=0.$$

By 4.3,**1**, however,

$$\frac{d}{dz}\,(1+z)^a = \frac{d}{dz}\,\exp\big(a\log(1+z)\big) = \exp\big(a\log(1+z)\big)\cdot\frac{a}{1+z}$$
$$= a(1+z)^{a-1},$$

and consequently

$$\frac{d^\nu}{dz^\nu}\,(1+z)^a = a(a-1)\cdots(a-\nu+1)\;(1+z)^{a-\nu}.$$

Therefore

$$a_\nu = \frac{a(a-1)\cdots(a-\nu+1)}{1\cdot 2\cdots\nu} = \binom{a}{\nu}.$$ [1]

Thus first of all for all $|z|<1$, the principal value of

$$(1+z)^a = 1+\binom{a}{1}z+\ldots+\binom{a}{\nu}z^\nu+\ldots = \sum_{\nu=0}^{\infty}\binom{a}{\nu}z^\nu,$$

(a fixed, arbitrary, complex; $|z|<1$).

The power series thus obtained is called the ***binomial series***. The convergence behavior of this series on the boundary of its circle of convergence $|z|<1$ was ascertained in 5.5,**6**.

[1] For $\nu=0$, set $\binom{a}{\nu}=\binom{a}{0}=1$ for every a, including $a=0$

6.6. The cyclometric functions

The inverses of the functions $\sin z$ and $\tan z$ are denoted by $\arcsin z$ and $\arctan z$ and are designated as *cyclometric* (or *inverse trigonometric*) *functions*. An analogous statement holds for the inverses of the functions $\cos z$ and $\cot z$, which, however, hardly require separate consideration. Thus we have $w = \arcsin z$ if $\sin w = z$. If z is given arbitrarily, then, by 6.3,**10**, there always exists precisely one number w lying in the strip $-\pi/2 \leqq \Re(w) \leqq +\pi/2$, provided that the part of the boundary of this strip lying below the axis of reals is omitted. The thereby uniquely determined number w is called the *principal value* of $\arcsin z$.[1] Only this principal value will be taken into consideration in what follows.

As the inverse of $z = w - \frac{w^3}{3!} + \frac{w^5}{5!} - + \ldots$, $\arcsin z$ possesses a power-series expansion of the form

$$w = \arcsin z = z + a_2 z^2 + a_3 z^3 + \ldots,$$

which certainly converges for all sufficiently small $|z|$. The procedure for finding the coefficients a_ν is analogous to that used in the preceding cases: As the inverse of $z = \sin w$, the function $w = \arcsin z$ has (according to 4.4, Theorem 3) the derivative $\frac{1}{\cos w} = \frac{1}{\sqrt{1-z^2}}$, where the principal value of the square root, *i.e.*, the value lying close to $+1$ for small $|z|$, is to be taken. Hence, by 6.5,

$$\frac{d \arcsin z}{dz} = \frac{1}{\sqrt{1-z^2}} = 1 + \frac{1}{2} z^2 + \frac{1 \cdot 3}{2 \cdot 4} z^4 + \ldots,$$

and consequently, by 4.2,(16), since $\arcsin 0 = 0$, we have

$$(1) \qquad \arcsin z = z + \frac{1}{2} \cdot \frac{z^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{z^5}{5} + \ldots.$$

Both expansions converge absolutely for $|z| < 1$. The second series is

---

[1] All the remaining values w for which $\sin w = z$, are then given by $w + 2k\pi$ and $\pi - w + 2k\pi$, $(k = 0, \pm 1, \pm 2, \ldots)$.

also still absolutely convergent for $|z| = 1$. For,

$$1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{5} + \ldots$$

is still convergent, as is proved here most quickly as follows: For $0 \leqq x < 1$, the ν^{th} partial sum is

$$x + \frac{1}{2}\cdot\frac{x^3}{3} + \ldots + \frac{1\cdot 3\cdot \ldots \cdot(2\nu-1)}{2\cdot 4\cdot \ldots \cdot (2\nu)}\cdot\frac{x^{2\nu+1}}{2\nu+1} < \text{arc sin } x < \frac{\pi}{2},$$

because all the coefficients are positive. Therefore the ν^{th} partial sum of the preceding series is also $< \pi/2$ for all $\nu = 0, 1, \ldots$; the partial sums are bounded, and hence the series is convergent.

Since, according to this, we have, moreover, for $|z| \leqq 1$, $|\text{arc sin } z| \leqq \text{arc sin } 1 = \pi/2$, and hence, *a fortiori*, $\Re$ (arc sin z) $\leqq \pi/2$, it follows, further, that our series actually represents the principal value for $|z| \leqq 1$. The arc-tan series is found quite analogously. We have $w = \text{arc tan } z$ if $\tan w = z$. For a given $z \neq \pm i$, there exists (see *Elem.*, § 43) precisely one w with $-\pi/2 < \Re(w) \leqq +\pi/2$ for which $\tan w = z$. This number w is the *principal value* of arc tan z, which alone will be considered from now on.[1] As the inverse of $z = \tan w$, the function arc tan z possesses (again by 4.4, Theorem 3) the derivative $\cos^2 w = \dfrac{1}{1+\tan^2 w} = \dfrac{1}{1+z^2}$. Therefore

$$\frac{d}{dz}\text{ arc tan } z = \frac{1}{1+z^2} = 1 - z^2 + z^4 - + \ldots,$$

and consequently, since arc tan $0 = 0$,

$$\text{arc tan } z = z - \frac{z^3}{3} + \frac{z^5}{5} - + \ldots. \tag{2}$$

Both expansions converge absolutely for $|z| < 1$. That (2) is also still convergent on the boundary $|z| = 1$ except at the two points $z = \pm i$ was shown already in 5.5,2.

---

[1] All the remaining values w for which $\tan w = z$, are given in terms of the principal value by $w + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Finally, that (2) also represents the principal value of arc tan z for $|z| < 1$ can be proved as follows: Since $\tan w = z$, we have

$$-i\frac{e^{iw}-e^{-iw}}{e^{iw}+e^{-iw}} = z \qquad \text{or} \qquad e^{2iw} = \frac{1+iz}{1-iz}.$$

Hence,

$$w = \frac{1}{2i}\log(1+iz) + \frac{1}{2i}\log\frac{1}{1-iz}, \tag{3}$$

and therefore the series (2) is also obtained by expanding the logarithms in (3) according to 6.4,(1) and (2). Since for these series the imaginary part of their sum lies between $-\pi/2$ and $+\pi/2$, the same holds for the real part of w in (3), and hence also for the real part of the sum of the series in (2). This series thus represents the principal value of arc tan z.

Chapter 7

NUMERICAL AND CLOSED EVALUATION OF SERIES

7.1. Statement of the problem

If we are given an arbitrary series Σa_ν or an arbitrary sequence $\{s_\nu\}$, then we are always concerned chiefly, first of all, with the question, whether the series (sequence) is convergent or not, and, if it does converge, with the further question, what value it possesses. We shall regard the first question as having been settled by the preceding chapters. The second question, however, requires some explanation: If we establish, say, that $\sum_{\nu=1}^{\infty} \frac{1}{\nu(\nu+1)} = 1$, then this assertion of the value of the series is final, and leaves open no further question regarding the value. If, however, we say that $\sum_{\nu=0}^{\infty} \frac{1}{\nu!} = e$, then—depending on the way in which the number e was originally introduced—this is either (as in our development, in 3.1.2,**2**) merely an abbreviation for the value of the series, which is not yet known any more closely, or else, if e is defined as the limit of $\left(1+\frac{1}{\nu}\right)^\nu$ (see 6.3,**8** for $z = 1$), the assertion that a certain limiting value coincides with a certain other limiting value.

In the first case, we have an evaluation of the series in the strict sense. This case occurs only if the value, s, of the series is a definitely assignable rational number. In the other case, which is by far the more common one, the problem is to express the value of the series or of the sequence in terms of numbers which are already known or familiar to us through other connections—in particular, in terms of numbers which, like, say, the values of the elementary and of many nonelementary functions, can be found in easily accessible books of tables—or to calculate the value of the series numerically. Thus does it come about that the significance of an equation of the form $\Sigma a_\nu = s$, say the binomial series

$$\sqrt{11} = \tfrac{10}{3}\left(1 - \tfrac{1}{100}\right)^{\frac{1}{2}} = \tfrac{10}{3}\left(1 - \tfrac{1}{2} \cdot \tfrac{1}{100} - \tfrac{1}{8} \cdot \tfrac{1}{100^2} - \ldots\right),$$

is greater when read from left to right or when read from right to left, depending on the circumstances of the case. If we read it from right to left, and if we regard $\sqrt{11}$ as "known" (this value is easy to find in many tables), then it gives the value of the series in closed form, and at the same time the numerical character of this value is extensively revealed. If we read the equation from left to right, it furnishes a way (which is actually quite favorable for calculation) to calculate the value $\sqrt{11}$ numerically. It is customary, in this connection, to regard invariably as a *numerical calculation*, the representation of the number in question in the form of a decimal fraction. Note, however, that this number is thereby represented merely by means of another limiting process. For, decimal fractions are nothing but (convergent) infinite series or sequences (*cf.* the remark in 3.2.1,**10**). This representation is also by no means always the better one, because in most cases the succession of digits obeys no recognizable law (as, say, in $\sqrt{2} = 1.4142\ldots$). The advantage of decimal fractions lies solely in the fact that they can be compared easily with respect to magnitude, and give one directly, on the basis of long practice, a feeling for the (approximate) position on the number axis, where the number to be calculated lies, and further, that one knows that, in breaking the decimal fraction off after n digits, the error is nonnegative and smaller than a unit in the last decimal place. For, this tells one how far to carry out a calculation in order to attain, with certainty, a certain accuracy.

One usually undertakes such a numerical calculation of the value of a series by calculating a partial sum s_n by means of direct addition of the initial terms up to a_n, and estimating the "error", *i.e.*, the remainder r_n that has to be added to s_n to yield the value of the series itself. This estimation of the remainder (for examples, see below) is carried out first, and then the index n, up to which the terms are summed, is determined so that the error corresponds to the desired accuracy. Those series are regarded as favorable for this purpose, for which the remainder is already very small for small or moderately large n. The calculation of the partial sums s_n, finally, must be carried out by simple addition—a task which is no longer terrifying in this age of giant calculating machines, but which, at one time, could only be accomplished by assiduous labor.

Following these general remarks, we shall now list a number of numerical and closed calculations of the values of infinite series. In the case of the former, we shall confine ourselves to sketching the method of calculation. The calculation itself must be left to the reader, who is earnestly advised to carry it out. We shall invariably regard the series in question as being denoted by Σa_ν, its sum by s, the partial sums by s_ν, and the remainders by r_ν, so that $s_\nu + r_\nu = s$.

7.2. Numerical evaluations and estimations of remainders

1. *The calculation of the number e* is based on the very rapidly convergent series $\sum_{\nu=0}^{\infty} \frac{1}{\nu!}$. Here we have

$$r_\nu = \frac{1}{(\nu+1)!} + \frac{1}{(\nu+2)!} + \cdots$$
$$< \frac{1}{(\nu+1)!}\left(1 + \frac{1}{\nu+1} + \frac{1}{(\nu+1)^2} + \cdots\right) = \frac{1}{\nu!\nu},$$

so that the error is already very small for moderately large values of ν. We find for $\nu = 12$, say, that

$$2.718281826 < e < 2.718281832.$$

With the aid of the modern calculating machine, e has been calculated in this way to more than 2500 decimal places.[1]

2. *The calculation of the number* π is best based on the arc-tan series in 6.6,(2). That the representation

$$(1) \qquad \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$$

[1] *Cf. G. W. Reitwiesner*, Math. Tables and other Aids to Computation 4 (1950), p. 11–15; *N. C. Metropolis, G. Reitwiesner*, and *J. von Neumann, ibid.*, pp. 109–111.

obtained for $z = 1$ is correct will be shown in 7.3,(2). This series, which from a theoretical standpoint is especially beautiful because it represents π in a particularly simple form, converges too slowly to be useful for the numerical calculation of π. Various artifices lead more quickly to the goal. The following (*J. Machin*, 1706) is especially favorable: The number

$$\alpha = \text{arc tan}\,\frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - + \cdots$$

can easily be calculated from the series. For if, as in the present case, the remainder is an alternating series, then, by 3.4, Theorem 2, we have the rule that the remainder has the same sign as, but is smaller in absolute value than, *the first neglected term.*[1] From $\tan\alpha = \frac{1}{5}$ we obtain, further, $\tan 2\alpha = \frac{5}{12}$, $\tan 4\alpha = \frac{120}{119}$. According to this, 4α is only a little larger than $\pi/4$. We set $4\alpha - \pi/4 = \beta$, and find that

$$\tan\beta = \frac{1}{239}, \quad \beta = \text{arc tan}\,\frac{1}{239} = \frac{1}{239} - \frac{1}{3\cdot 239^3} + - \cdots.$$

From the series for α and β, we calculate $\pi = 4(4\alpha - \beta)$. If we use five terms to calculate α and two terms to calculate β from their respective series, we obtain

$$\pi = 3.1415926\ldots,$$

which is already correct to seven decimal places. With the aid of the modern calculating machine, π has been calculated in this way to more than 2000 decimal places (see footnote, p. 165).

3. *The calculation of natural logarithms* is based on the series (3) in 6.4. For $z = 1/3$, it immediately yields

$$\log 2 = 2\left[\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \cdots\right],$$

a representation which is quite useful for numerical purposes, and furnishes the value $0.6931471\ldots$ correct to seven decimal places if the terms of the series up to $1/(15\cdot 3^{15})$ are used. The remainder then does not affect the seventh decimal place any more.

[1] Thus, the remainder corresponding to the partial sum written down is negative and $< \frac{1}{7\cdot 5^7}$ in absolute value.

Suppose that we have already calculated $\log k$ for an integer $k \geqq 2$. Then, for $z = \frac{1}{2k+1}$, the series just employed yields

$$\log(k+1) = \log k + 2\left[\frac{1}{2k+1} + \frac{1}{3(2k+1)^3} + \cdots\right],$$

which converges very rapidly for $k = 2$, and even more so for the values of k that follow. At most for the logarithms of 2, 3, 5, and 7 (only the logarithms of the prime numbers have to be calculated) would even more rapidly convergent series be desirable. They can be obtained by means of special devices; *cf. J. C. Adams*, Proc. Royal Soc. London, 27 (1878), pp. 88–94.

With $\log 2$ and $\log 5$, we also have $\log 10$, and therewith the modulus

$$M = \frac{1}{\log 10} = 0.43429448\ldots$$

of the system of *Briggs*ian or common logarithms, by which number one has to multiply the natural logarithms in order to obtain the logarithms to the base 10.

4. *The calculation of roots* by the direct method is hardly of any practical importance any more if one is already in possession of logarithms. It is based on the binomial series (*cf.* 6.5). The smaller $|z|$ is, the better this series converges. Therefore, in order to calculate, say, the root $w = \sqrt[p]{k} = k^{1/p}$ $(k, p \geqq 2;$ integers), this value is brought into the form $a(1+x)^{1/p}$, with a simple rational a and a small $|x|$. To this end, choose any rational number a and set $w = a\left(\frac{k}{a^p}\right)^{1/p}$. If a is chosen as a (rough) approximation to w, then k/a^p lies close to 1, *i.e.*, $=1+x$ with a small $|x|$. Then, if, for brevity, we set $1/p = \alpha$, we have

$$w = a\left[1 + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots\right],$$

and the series converges rapidly. Thus, the representations

$$\sqrt{2} = \tfrac{141}{100}\left(1 - \tfrac{119}{20000}\right)^{-\frac{1}{2}}, \quad \sqrt[3]{2} = \tfrac{5}{4}\left(1 + \tfrac{3}{125}\right)^{\frac{1}{3}},$$

$$\sqrt{3} = \tfrac{173}{100}\left(1 - \tfrac{71}{30000}\right)^{-\frac{1}{2}}, \quad \sqrt[3]{3} = \tfrac{10}{7}\left(1 + \tfrac{29}{1000}\right)^{\frac{1}{3}},$$

say, are very rapidly convergent and numerically convenient series for the values of the respective roots.

5. *The calculation of the trigonometric functions* $\cos x$ and $\sin x$ is based naturally on the series (2) and (3) in 6.1, which converge very rapidly for moderately large $|z|$. We must bear in mind, however, that x denotes the radian measure of the angle. Thus, if we wish to calculate $\cos 1°$, say, we have to calculate $\cos x$ for $x = \frac{\pi}{180} = 0.017\ldots.$ We therefore need precise values for π and its powers. For small $|x|$, it is more convenient to calculate first $\sin x$, and then $\cos x$ as $(1-\sin^2 x)^{\frac{1}{2}}$.

The functions $\tan x$ and $\cot x$ are then obtained from $\cos x$ and $\sin x$ by division, or else directly from the power series in 6.3,**11**, which likewise converge rapidly for small $|x|$. The logarithms of the functions are more important, in many respects, than the functions themselves. From the expansion of $z \cot z$ in 6.3,**11**, we obtain, first of all,—we confine ourselves to real $z = x$—a representation of $\cot x - \dfrac{1}{x}$ which shows that this function is also still continuous at $x = 0$ if we define it to have the value 0 there. By integration we than obtain, further,

$$\log \sin x = \log x + \int_0^x \left(\cot t - \frac{1}{t}\right) dt,$$

which leads to the series representation

$$\log \sin x = \log x + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{2^{2\nu} B_{2\nu}}{2\nu \cdot (2\nu)!} x^{2\nu}.$$

A corresponding representation of $\log \cos x$ can easily be obtained by integrating the tan-series.

7.3. Closed evaluations

The foregoing considerations, which have referred exclusively to numerical practice, will now be followed by some theoretically important matters:

1. *Direct formation of partial sums.*

a) If $a \neq 0, -1, -2, \ldots,$ then $\displaystyle\sum_{\nu=0}^{\infty} \frac{1}{(a+\nu)\,(a+\nu+1)} = \frac{1}{a}$. For we have $a_\nu = \dfrac{1}{a+\nu} - \dfrac{1}{a+\nu+1}$.

b) If $a \neq 0, -1, -2, \ldots$, then $\sum_{\nu=0}^{\infty} \frac{1}{(a+\nu)(a+\nu+1)(a+\nu+2)} = \frac{1}{2a(a+1)}$; proof as in a).

c) $\sum_{\nu=2}^{\infty} \frac{1}{\nu^2-1} = \frac{3}{4}$. For we have $a_\nu = \frac{1}{2}\left(\frac{1}{\nu-1} - \frac{1}{\nu+1}\right)$.

In these examples, a_ν was brought into the form $z_\nu - z_{\nu+1}$ or $z_\nu - z_{\nu+q}$ ($q \geq 1$, integral), where $\{z_\nu\}$ was a null sequence. The reader will easily be able to generalize this principle. Numerous further examples are to be found in the works of *Fabry*, *Bromwich* and *Knopp*, mentioned in III of the Bibliography.

2. *Application of Abel's limit theorem.* In 6.4 we found that, for $|z| < 1$, the principal value of $\log(1+z)$ has the representation

$$\log(1+z) = z - \frac{z^2}{2} + - \ldots + (-1)^{\nu-1}\frac{z^\nu}{\nu} + \ldots, \tag{1}$$

and it can be shown as in 5.5,2 that the series also converges for all z on the boundary of the unit circle except at $z = -1$. Does it also follow from these facts that, for these z on the boundary, the value of this series is equal to the principal value $\log(1+z)$? We do not see that this is the case if we go through the proof of (1), because it is only valid for $|z| < 1$. For, in applying Theorem 1 of 4.3, use is made of the absolute convergence of the "inner" power series, which in this case is the series (1). The representation (1) is, nevertheless, still valid for the z in question on the boundary of the circle of convergence. This, however, requires proof. It is rendered possible here and in similar cases by *Abel*'s limit theorem (4.2, Theorem 6). For if $z_0 \neq -1$ is a specific point on $|z| = 1$, then this theorem asserts that the right side of (1) tends to the value of the series $\Sigma(-1)^{\nu-1}\frac{z_0^\nu}{\nu}$ as z approaches z_0 radially. At the same time, however, the left side tends to $\log(1+z_0)$, because of the continuity of $\log(1+z)$ at z_0. Therefore, (1) is also valid for $z = z_0$. In particular,

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu} \equiv 1 - \frac{1}{2} + \frac{1}{3} - + \ldots = \log 2.$$

The corresponding considerations applied to the arc-tan series 6.6,(2) show that the representation

$$\text{(2)} \qquad \operatorname{arc\,tan} z = \sum_{\nu=0}^{\infty} (-1)^{\nu} \frac{z^{2\nu+1}}{2\nu+1},$$

valid in the interior of the unit circle, also holds on the boundary $|z| = 1$ of the unit circle for all $z \neq \pm i$. Thus, in particular, we obtain the especially beautiful representation of $\pi/4$ resulting from (2) for $z = 1$, which was already mentioned in 7.2,(1). It follows, likewise, that the binomial series $\Sigma\binom{\alpha}{\nu}z^{\nu}$ represents the (principal) value $(1+z)^{\alpha}$ wherever the series converges. The totality of points z at which the series converges was determined in 5.5,**6**.

Another example of this kind is the following, where we leave the details to the reader: We have

$$s = 1 - \frac{1}{4} + \frac{1}{7} - + \ldots + \frac{(-1)^{\nu}}{3\nu+1} + \ldots = \lim_{x \to 1-0} f(x),$$

if we set $\quad \Sigma(-1)^{\nu}\dfrac{x^{3\nu+1}}{3\nu+1} = f(x).\quad$ Since $f'(x) = \dfrac{1}{1+x^3}$,

$$s = \int_0^1 \frac{dt}{1+t^3} = \frac{1}{3}\log 2 + \frac{\pi}{3\sqrt{3}}.$$

A similar argument would appear to lead to the value of the series $\Sigma\dfrac{1}{\nu^2}$, which we have not yet determined. For, if we set $\Sigma\dfrac{x^{\nu}}{\nu^2} = f(x)$, then $f'(x) = \Sigma\dfrac{x^{\nu-1}}{\nu}$ and $s = \lim\limits_{x\to 1-0} f(x)$. Now for $f(x)$ we have the representation $f(x) = \displaystyle\int_0^x \frac{1}{t}\log\frac{1}{1-t}\,dt$, but this integral, which is improper for $x = 1$ (at $t = 0$ the integrand is still continuous if it is set $= 1$ there), is not immediately evaluable. In **3** below we shall find the value of the series in an altogether different way.

3. *Series transformations.* An especially effective means for evaluating series in closed form is afforded by series transformations. In 5.6,**2**, we saw that a convergent series Σa_ν leads to another convergent series $\Sigma \alpha_n$ with the same sum, if we set

$$(1) \qquad \alpha_n = \frac{1}{2^{n+1}}\left[\binom{n}{0} a_0 + \binom{n}{1} a_1 + \dots + \binom{n}{n} a_n\right].[1]$$

If this *Eulerian transformation* is applied, say, to the series $\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu+1} = \log 2$ and $\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{2\nu+1} = \pi/4$, we obtain the (considerably better convergent) representations

$$\log 2 = \frac{1}{1 \cdot 2^1} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \dots$$

and

$$\frac{\pi}{2} = 1 + \frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots.$$

The transformation based on the following simple idea is called *Kummer's transformation:* If $\Sigma a_\nu = s$ is to be evaluated, choose a series $\Sigma c_\nu = c$ which is already known to converge and whose terms c_ν are asymptotically proportional to the a_ν, so that $a_\nu/c_\nu \to \gamma \neq 0$. Then evidently

$$s = \Sigma a_\nu = \gamma c + \sum_{\nu=0}^{\infty}\left(1 - \gamma \frac{c_\nu}{a_\nu}\right) a_\nu,$$

and the new series converges more rapidly, because $(1 - \gamma c_\nu/a_\nu) \to 0$. Thus, *e.g.*, if we associate with the series $\Sigma \frac{1}{\nu^2}$ the series $\Sigma 1/\nu(\nu+1) = 1$,

$$s = 1 + \sum_{\nu=1}^{\infty} \frac{1}{\nu^2(\nu+1)}.$$

If we associate with the new series the series $\Sigma 1/\nu(\nu+1)(\nu+2) = 1/4$, we find further that

$$s = 1 + \frac{1}{4} + 2\sum_{\nu=1}^{\infty} \frac{1}{\nu^2(\nu+1)(\nu+2)},$$

[1] The value of the expression in brackets can easily be calculated: Underneath each term of the sequence $a_0, a_1, a_2, \dots$, write down first the sum of this term and the succeeding one, in other words, $a_0 + a_1, a_1 + a_2, a_2 + a_3, \dots$, and keep on repeating this step. Then the initial term in the n^{th} row (taking the original sequence as the 0^{th} row) is precisely the sum $\binom{n}{0} a_0 + \binom{n}{1} a_1 + \dots + \binom{n}{n} a_n$.

and we can thus obtain better and better convergent series for the value s of the original series, which will be found in a moment in a different way.

The most effective series transformation is *Markoff's transformation* or, what comes essentially to the same thing, the transformation acquired through Theorem 10 in 3.6. Under the stronger assumptions that all the series that appear are absolutely convergent, it amounts to *Cauchy's double-series theorem* (see 3.6, Theorem 9, Corollary 3). A particularly beautiful application of it is afforded by the determination of the values of the series $\Sigma 1/n^{2\nu}$, which up to now we have not yet found.

For this purpose, we start with the representation

$$\pi z \cot \pi z = 1 + \sum_{\nu=1}^{\infty} (-1)^{\nu} \frac{(2\pi)^{2\nu} B_{2\nu}}{(2\nu)!} z^{2\nu} \tag{2}$$

derived in 6.3,**11** and valid for all sufficiently small $|z|$, and compare it with the expansion

$$\pi z \cot \pi z = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2} \tag{3}$$

mentioned in 6.3,(2) and valid for all $z \neq \pm 1, \pm 2, \ldots$ and hence likewise for all sufficiently small $|z|$. We now expand, in the sense of 3.6, Theorem 9, Corollary 3 just cited, each term of the series in (3), with the exception of the term 1, in an infinite series:

$$\frac{2z^2}{z^2 - n^2} = -2\frac{z^2}{n^2}\frac{1}{1 - \dfrac{z^2}{n^2}} = -2\frac{z^2}{n^2} - 2\frac{z^4}{n^4} - \ldots - 2\frac{z^{2\nu}}{n^{2\nu}} - \ldots . \tag{4}$$

We imagine these series for $n = 1, 2, \ldots$ to be written down in rows, one under another. Summing first by columns, and then forming the series of column sums, we get

$$-2\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) z^2 - 2\left(\sum_{n=1}^{\infty} \frac{1}{n^4}\right) z^4 - \ldots - 2\left(\sum_{n=1}^{\infty} \frac{1}{n^{2\nu}}\right) z^{2\nu} - \ldots .$$

According to the theorem referred to, this expansion must coincide with the one in (2) if we also drop the 1 there, and if the hypotheses of this theorem are satisfied. The latter, however, is certainly the case. For if, in the expansion on the right in (4), we replace all terms by their absolute values, the value of the series becomes $\frac{2|z|^2}{|z|^2-n^2}$, and the series with these expressions as terms is convergent for small $|z|$, just as (3) was. Consequently, for $\nu = 1, 2, \ldots,$

$$-2\sum_{n=1}^{\infty}\frac{1}{n^{2\nu}} = (-1)^{\nu}\frac{(2\pi)^{2\nu}B_{2\nu}}{(2\nu)!}.$$

We have thus evaluated the interesting series on the left in closed form, for we have, for $\nu = 1, 2, \ldots,$

$$(5) \qquad \sum_{n=1}^{\infty}\frac{1}{n^{2\nu}} = (-1)^{\nu-1}\frac{(2\pi)^{2\nu}}{2\cdot(2\nu)!}B_{2\nu},$$

and, in particular,

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}, \ldots.$$

From these beautiful results we may derive various others:

a) If n ranges from 1 on, we have, for every $\alpha > 1$,

$$\sum\frac{1}{n^{\alpha}} = \sum\frac{1}{(2n-1)^{\alpha}} + \sum\frac{1}{(2n)^{\alpha}}$$

and hence

$$\sum\frac{1}{(2n-1)^{\alpha}} = \left(1-\frac{1}{2^{\alpha}}\right)\sum\frac{1}{n^{\alpha}}.$$

Thus, in particular, for $\nu = 1, 2, \ldots,$

$$(6) \qquad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^{2\nu}} = 1+\frac{1}{3^{2\nu}}+\frac{1}{5^{2\nu}}+\cdots = (-1)^{\nu-1}\frac{(2^{2\nu}-1)\pi^{2\nu}}{2\cdot(2\nu)!}B_{2\nu},$$

and, by subtraction, we find that, for $\nu = 1, 2, \ldots,$

$$(7) \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{2\nu}} = 1-\frac{1}{2^{2\nu}}+\frac{1}{3^{2\nu}}-+\cdots = (-1)^{\nu-1}\frac{(2^{2\nu-1}-1)\pi^{2\nu}}{(2\nu)!}B_{2\nu}.$$

b) From (5) we see that $(-1)^{\nu-1}B_{2\nu} > 0$, so that, in particular, the

$B_{2\nu}$ alternate in sign. Since the value of the series on the left in (5) obviously lies between 1 and 2, we now obtain, as a supplement to the results following (23) in 4.3, an assertion concerning the magnitude of the *Bernoulli* numbers: $|B_{2\nu}| = \frac{2\cdot(2\nu)!}{(2\pi)^{2\nu}}\,\theta_\nu$, with $1<\theta_\nu<2$. Thus, the coefficients of the power series (2) are <4 in absolute value, and therefore the power series has at least the radius 1; it cannot have a larger radius, because the function represented by the series is discontinuous at ± 1.

c) In (5), (6), and (7), n has an even, integral exponent. We remark expressly that for the corresponding series with odd, integral exponents >1, one can make no satisfactory assertions concerning their values.

We arrive at similar beautiful results if we compare, in a corresponding manner, the two representations

$$\text{(8)} \qquad \frac{1}{\cos \pi z} = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{E_{2\nu}}{(2\nu)!}(\pi z)^{2\nu}, \qquad \text{(see 4.3, (26))}$$

and

$$\text{(9)} \qquad \frac{1}{\cos \pi z} = \frac{4}{\pi}\sum_{n=0}^{\infty} (-1)^n \frac{2n+1}{(2n+1)^2-(2z)^2}, \qquad (\textit{cf.}\ 6.3,(5)).$$

The n^{th} term of the last series yields the power series

$$(-1)^n \sum_{\nu=0}^{\infty} \frac{(2z)^{2\nu}}{(2n+1)^{2\nu+1}}.$$

If we now sum over n for fixed ν, and compare the result with (8), we obtain, for $\nu = 1, 2, \ldots,$

$$\text{(10)} \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2\nu+1}} \equiv 1 - \frac{1}{3^{2\nu+1}} + \frac{1}{5^{2\nu+1}} - + \cdots = (-1)^\nu \frac{E_{2\nu}}{2^{2\nu+2}\cdot(2\nu)!}\,\pi^{2\nu+1}.$$

Whereas only those series (5), (6), and (7) with even, integral exponents >0 have been mastered, the values of the series (10) are only known if the exponent $2\nu+1$ is an odd integer.

BIBLIOGRAPHY

I. Encyklopädie der mathematischen Wissenschaften

A. Pringsheim, *Irrationalzahlen und Konvergenz unendlicher Prozesse*, Band I, Teil I, Heft 1, Abt. 3, Leipzig, 1898.

A. Pringsheim and G. Faber, *Algebraische Analysis*, Band II, Teil 3, Heft 1, Leipzig, 1909.

II. Encyclopédie des sciences mathématiques

A. Pringsheim and J. Molk, Tome I, Volume 1, 3. *Nombres irrationnels et notion de limite*. 4. *Algorithmes illimités*. Paris and Leipzig, 1904, 1907.

A. Pringsheim, G. Faber, and J. Molk, Tome II, Volume 2, 7. *Analyse algébrique*. Paris and Leipzig, 1911.

III. Textbooks

T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series*, London, 1908; 2nd edition, 1926.

H. Dörrie, *Unendliche Reinen*, Munich, 1951.

E. Fabry, *Théorie des séries à termes constants*, Paris, 1910.

T. Fort, *Infinite Series*, Oxford, 1930.

J. M. Hyslop, *Infinite Series*, Edinburgh and London, 1942; 3rd edition, Edinburgh, London, and New York, 1947.

K. Knopp, *Theorie und Anwendung der unendlichen Reihen*, Berlin, 1922; 4th edition, Berlin and Heidelberg, 1947.

———, *Theory and Application of Infinite Series*, London and Glasgow, 1928; 2nd edition, 1951.

H. v. Mangoldt and K. Knopp, *Einführung in die höhere Mathematik*, vol. II, 9th edition, Stuttgart, 1948.

IV. Sammlung Göschen — Dover Series

K. Knopp, *Elemente der Funktionentheorie*, Berlin and Leipzig, 1937; 4th edition, Berlin, 1955.

———, *Elements of the Theory of Functions*, New York, 1952.

———, *Funktionentheorie*, vols. I and II, Berlin and Leipzig, 1913; 8th edition, Berlin, 1955.

———, *Theory of Functions*, vols. I and II, New York, 1945, 1947.

INDEX