MAT 1361: Notes on Induction

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Let $\mathbb{N} = \{0, 1, 2, \cdots, \}$ be the natural numbers. In mathematics, we often wish to prove statements of the following forms:

(i) *All natural numbers have some property* P, i.e. $\forall n \in \mathbb{N}.P(n)$, or else

(ii) *All natural numbers* $\geq k$ *have some property* P, i.e. $\forall n \geq k \in \mathbb{N}.P(n)$

To prove such statements, we introduce the method of *mathematical induction*. It is based on the idea that natural numbers are built by starting from 0, and then continually adding 1 to the previous number: $0, 1, 2, 3, \ldots, n-1, n, n+1, \ldots$ We now show how this leads to a fundamental new proof technique.

> We now describe Simple and Strong Induction, and some examples of proofs. More examples were given in class and labs. You are expected to give **all details** and **complete discussion**, as in the examples below, in homework and tests,

1 Simple Induction

This proof technique looks like the following inference rule:

$$\begin{array}{cl} P(0) & \text{(Base Case)} \\ \forall n \in \mathbb{N} \ (\ P(n) \rightarrow P(n+1)\ ) & \text{(Inductive Step)} \\ \hline \forall n \in \mathbb{N}\ P(n) & \text{(Conclusion)} \end{array}$$

This rule says: *if the Base Case and the Inductive Step are true, then we conclude* $\forall n \in \mathbb{N}\ P(n)$. The hypothesis $P(n)$ of the inductive step is called the **Inductive Hypothesis (I.H.)**.

Simple Induction says the following:

- Start by checking whether $P(0)$ is true, i.e. if "0 has property P"
- Next, for each natural number $n \in \mathbb{N}$, prove the implication $P(n) \rightarrow P(n+1)$.
- Then conclude, by mathematical induction, that $\forall n \in \mathbb{N}\ P(n)$.

What does this accomplish? As shown in class, if we start with the truth of $P(0)$ (base case), and using the Inductive Step, we iteratively observe that $P(1)$, $P(2)$, $P(3)$, $P(4)$, etc. are all true. Hence each of the $P(n)$'s is true, for any $n \in \mathbb{N}$. So we conclude $\forall n \in \mathbb{N}\ P(n)$ is true, which is what we wanted to prove.

Example 1.1 Let $5 \mid m$ mean "5 evenly divides m", i.e. m is a multiple of 5, so $\exists q \in \mathbb{Z}\ m = 5{\cdot}q$. Prove by induction:

$$\forall n \in \mathbb{N}\ (5 \mid (n^5 - n))$$

Proof: Let $P(n) = 5 \mid (n^5 - n)$. We wish to prove $\forall n \in \mathbb{N}\ P(n)$.

(Base Case): Notice $0^5 - 0 = 0$. So $P(0)$ says: $5 \mid 0$, i.e. 0 is a multiple of 5. This is true because $0 = 5{\cdot}0$.

(Inductive Step): We must prove $\forall n \in \mathbb{N}\ (\ P(n) \rightarrow P(n+1)\ )$.

$$\text{Suppose } P(n): \qquad \boxed{5 \mid (n^5 - n)} \tag{1}$$

Equation (1) is called the Induction Hypothesis (I.H.)

$$\text{We wish to prove } P(n+1): \qquad \boxed{5 \mid (\ (n+1)^5 - (n+1)\ )} \tag{2}$$

Method: start with the right hand side of (2), and manipulate it algebraically until you can make use of the I.H. (1) to get the result. Here's the algebra:

$$\begin{aligned} (n+1)^5 - (n+1) &= n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 \\ &= (n^5 - n) + (5n^4 + 10n^3 + 10n^2 + 5n) \\ &= \underbrace{(n^5 - n)}_{(*)} + \underbrace{5(n^4 + 2n^3 + 2n^2 + n)}_{(**)} \qquad (3) \end{aligned}$$

In line (3), we must show that 5 evenly divides each summand $(*)$ and $(**)$. First observe $5 \mid (*)$, that is $5 \mid (n^5 - n)$, by the induction hypothesis (1). On the other hand, obviously $5 \mid 5(n^4 + 2n^3 + 2n^2 + n)$, so $5 \mid (**)$. Thus 5 divides both summands in line (3), so (2) holds; that is, $P(n+1)$ holds. Hence we have shown $P(n) \to P(n+1)$. This finishes the inductive step.

From the Base Case and the Inductive Step, we conclude $\forall n \in \mathbb{N}\ P(n)$, by mathematical induction.

We can increase the range of Simple induction, by starting the induction from any point k, in order to prove $\forall n \geq k\ P(n)$. Here is this version of induction (here $k, n \in \mathbb{N}$):

$$\begin{array}{cl} P(k) & \text{(Base Case)} \\ \forall n \geq k\ (\ P(n) \to P(n+1)\ ) & \text{(Inductive Step)} \\ \hline \forall n \geq k\ P(n) & \text{(Conclusion)} \end{array}$$

Examples will be given in class and labs.

2 Strong Induction

Another form of induction, called *Strong Mathematical Induction*, is an important technique. This is used in cases when you wish to prove $P(n+1)$ in the inductive step, but need to use a stronger Inductive Hypothesis than just $P(n)$.

2.1 Strong Induction: Basic Form

$$\begin{array}{cl} P(0) & \text{(Base Case)} \\ \forall n\ (\ (\ P(0) \wedge P(1) \wedge \cdots \wedge\ P(n)\ ) \to P(n+1)\ ) & \text{(Strong Inductive Step)} \\ \hline \forall n\ P(n) & \text{(Conclusion)} \end{array}$$

Here, the base case is $P(0)$ as usual. But the inductive hypothesis is stronger: in order to prove $P(n+1)$, you may assume more than just $P(n)$. In fact, you may assume as inductive hypothesis *all the previous values* $P(0), P(1), \cdots, P(n)$. This is called the *strong induction hypothesis* (S.I.H.). This gives much more flexibility.

Of course, there is an obvious extension of the above Strong Induction scheme, in which we start at any point k:

$$\begin{array}{cl} P(k) & \text{(Base Case)} \\ \forall n \geq k\ (\ (\ P(k) \wedge P(k+1) \wedge \cdots \wedge\ P(n)\ ) \to P(n+1)\ ) & \text{(Strong Inductive Step)} \\ \hline \forall n \geq k\ P(n) & \text{(Conclusion)} \end{array}$$

Example 2.1 Prove the following by strong induction:

Every integer $n \geq 2$ can be written as the product of primes i.e. $n = p_1 \cdot p_2 \cdot \ldots \cdot p_m$, for some m, where the p_i are all primes.

Proof: Let $S(n)$ be the statement that "n can be written as a product of prime numbers". We prove $\forall n \geq 2\ S(n)$ by strong induction.

- Base Case: $S(2)$, since 2 is a prime (therefore 2 is a product of a sequence of primes of length 1 !).
- Strong Inductive Step: Suppose (Strong Induction Hypothesis) $S(2) \wedge S(3) \wedge \cdots \wedge S(n)$. We must prove $S(n+1)$. There are two cases:

  Case 1: $n+1$ is a prime. In that case, $S(n+1)$ already holds.

  Case 2: $n+1$ is not a prime, so it is a composite number. Therefore $n+1 = a \cdot b$, where $2 \leq a < n+1$ and $2 \leq b < n+1$. So by S.I.H. , we know $S(a)$ and $S(b)$ are true. Therefore both a and b can be written as a product of primes, say $a = p_1 \cdot \cdots \cdot p_m$ and $b = q_1 \cdot \cdots \cdot q_r$, where the p_i and q_j are primes. Hence $n+1 = a \cdot b = p_1 \cdot \cdots \cdot p_m \cdot q_1 \cdot \cdots \cdot q_r$ is also a product of primes, so $S(n+1)$.
- From the base case and the strong induction step, we conclude $\forall n \geq 2\ S(n)$ by strong induction.

2.2 Strong Induction: More Advanced Forms

Sometimes we want to use the method of strong induction, but the Strong Induction Hypothesis is only true for large values of n. In that case we have to check the base cases by hand , until the restrictions on the induction variable n force $P(n)$ to make sense. For example, we cannot assume $P(n-25)$ is true, if $n-25$ is negative! So the problem and the inductive assumptions must be carefully set-up.

Example 2.2

Prove : *Every natural number $n \geq 12$ can be written as a sum of 4's and 5's*

Proof: Let $P(n)$ be the statement "n can be written as a sum of 4's and 5's. We want to prove $\forall n \geq 12\ P(n)$.

Basic Discussion: observe we want to prove, for an arbitrary $n \geq 12$, that $P(n+1)$, assuming (by S.I.H.) that we know the property $P(k)$ holds for all smaller numbers $12 \leq k \leq n$.

Naive idea: Let $n \geq 12$. We write $n+1$ in terms of smaller numbers, so we can make use of property P. So observe:

$$n + 1 = (n - 3) + 4$$

Using this fact, if we are assuming (by S.I.H.) the property $P(n-3)$ then $n-3$ can be written as a sum of 4's and 5's, hence adding one more 4, $n+1$ can also be written as a sum of 4's and 5's. Hence $P(n+1)$ is true.

Patching up the base cases: The above idea is great, except *we must assume $P(n-3)$ makes sense!* What if $n = 12$. Then $P(n-3)$ is $P(9)$ which is outside the range of induction (Now in fact 9 can be written as as a sum of 4's and 5's . But if $n = 14$, again $P(n-3)$ is $P(11)$, which is also outside the range of induction and $P(11)$ is False.). What we *really* have to do is start looking at $P(n)$ when $n \geq 15$, since then $n - 3 \geq 12$. We then check the small values (the base cases) by hand: check $P(12), P(13), P(14), P(15)$. So when we prove the Strong Inductive Step

$$(\ P(12) \wedge \cdots \wedge P(15) \wedge P(16) \wedge \cdots \wedge P(n)\ ) \rightarrow P(n+1)$$

we need only look at values of $n \geq 15$, since if $12 \leq n-3 \leq 15$, we know we have already checked $P(n-3)$ as part of the Base Cases.

Here is the final proof, using the property P above:

Base Cases: We check by hand P(12), P(13), P(14), P(15). This is easy, since $12 = 3{\cdot}4 = 4+4+4, 13 = 2{\cdot}4+5, 14 = 4+2{\cdot}5, 15 = 3{\cdot}5$.

Strong Inductive Step: Assume for arbitrary $n \geq 12$, that $P(k)$ is true for all k where $12 \leq k \leq n$. We must prove $P(n+1)$. We consider $n \geq 15$ (the smaller cases, where the $n = k \leq 15$ are checked in the Base Cases). Observe $n+1 = (n-3)+4$ and $12 \leq n-3 \leq n$. So by the S. I. H. $P(n-3)$ holds (and by the above remarks, we know $P(n-3)$ is well-defined). Therefore $n-3$ can be written as a sum of 4's and 5's, so $n+1$ can too. Hence $P(n+1)$.

Example 2.3 Consider the integer sequence $a_0, a_1, a_2, \cdots$, where $a_0 = 1, a_1 = 2, a_2 = 3$ and where

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} \qquad \forall n \geq 3$$

Prove: $a_n \leq 3^n$, for all $n \geq 0$.
Proof: Let $P(n) = a_n \leq 3^n$. We must prove $\forall n \in \mathbb{N}\ P(n)$.

Base Cases: In the inductive step, we will need the formula for a_{n+1}, and this will involve a_{n-2}. Hence, we must assume that $n-2 \geq 0$. Hence, in the inductive step, we will only consider $n \geq 2$, so the base cases are $n = 0, 1, 2$.

- $P(0)$ is true: it says $a_0 \leq 3^0$ and $P(1)$ is true: it says $a_1 = 2 \leq 3$
- $P(2)$ is true: it says $a_2 = 3 \leq 3^2 = 9$.

Strong Induction Step: We must prove

$$\forall n \in \mathbb{N}\ (\ (P(0) \wedge \cdots \wedge P(n)) \rightarrow P(n+1)\ )$$

We assume for arbitary n the S.I.H., i.e. $P(0) \wedge \cdots \wedge P(n))$ is true, so $P(k)$ is true for $0 \leq k \leq n$. We wish to prove $P(n+1)$ is true. Notice that this says: $a_{n+1} \leq 3^{n+1}$. Hence we need to calculate a_{n+1}. Go to the original formula above and plug-in $n+1$ for n. We get

$$a_{n+1} = a_n + a_{n-1} + a_{n-2}$$

We've already checked $P(k)$ is true for the base cases $P(0), P(1), P(2)$. The S.I.H. says that if $2 \leq k \leq n$ we can also assume $P(k)$ is well-defined and true. Then

$$\begin{aligned} a_{n+1} &= a_n + a_{n-1} + a_{n-2} \\ &\leq 3^n + 3^{n-1} + 3^{n-2} \qquad \text{by S.I.H.} \\ &\leq 3^n + 3^n + 3^n = 3{\cdot}3^n = 3^{n+1} \end{aligned}$$

Hence $a_{n+1} \leq 3^{n+1}$, so $P(n+1)$ holds. This is true for arbitrary $n \in \mathbb{N}$. So by strong induction, $\forall n \in \mathbb{N}\ P(n)$.

Remark 2.4 What the calculations above really say is that for $n \geq 2$,

$$(\ P(n-2) \wedge P(n-1) \wedge P(n)\ ) \rightarrow P(n+1)$$

And of course the S.I.H. implies the hypothesis of this implication is true.