LIMITS AND CONTINUITY

1. The concept of limit

Example 1.1. Let $f(x) = \dfrac{x^2-4}{x-2}$. Examine the behavior of $f(x)$ as x approaches 2.

Solution. Let us compute some values of $f(x)$ for x close to 2, as in the tables below.

x	$f(x) = \dfrac{x^2-4}{x-2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x) = \dfrac{x^2-4}{x-2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

We see from the first table that $f(x)$ is getting closer and closer to 4 as x approaches 2 from the left side. We express this by saying that "the limit of $f(x)$ as x approaches 2 from left is 4", and write

$$\lim_{x\to 2^-} f(x) = 4.$$

Similarly, by looking at the second table, we say that "the limit of $f(x)$ as x approaches 2 from right is 4", and write

$$\lim_{x\to 2^+} f(x) = 4.$$

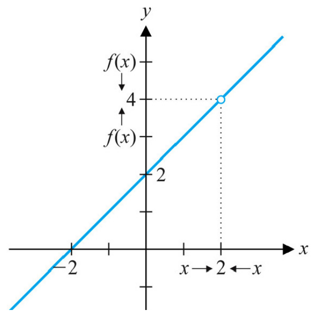
We call $\lim\limits_{x\to 2^-} f(x)$ and $\lim\limits_{x\to 2^+} f(x)$ *one-sided limits.* Since the two one-sided limits of $f(x)$ are the same, we can say that "the limit of $f(x)$ as x approaches 2 is 4", and write

$$\lim_{x\to 2} f(x) = 4.$$

Note that since $x^2-4 = (x-2)(x+2)$, we can write

$$\begin{aligned}\lim_{x\to 2} f(x) &= \lim_{x\to 2} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2}(x+2) = 4,\end{aligned}$$

where we can cancel the factors of $(x-2)$ since in the limit as $x \to 2$, x is close to 2, but $x \neq 2$, so that $x-2 \neq 0$. Below, find the graph of $f(x)$, from which it is also clear that $\lim_{x\to 2} f(x) = 4$.

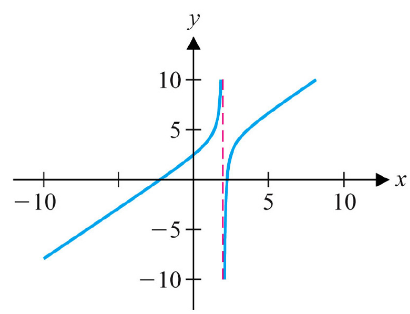


Example 1.2. Let $g(x) = \dfrac{x^2-5}{x-2}$. Examine the behavior of $g(x)$ as x approaches 2.

Solution. Based on the graph and tables of approximate function values shown below,

x	$g(x) = \dfrac{x^2-5}{x-2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10,003.9999

x	$g(x) = \dfrac{x^2-5}{x-2}$
2.1	−5.9
2.01	−95.99
2.001	−995.999
2.0001	−9995.9999



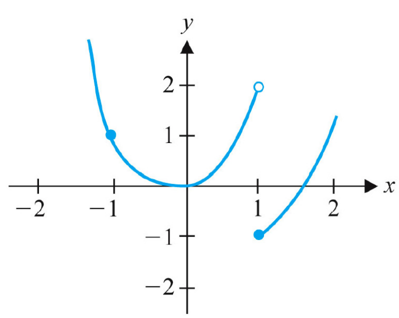
observe that as x gets closer and closer to 2 from the left, $g(x)$ increases without bound and as x gets closer and closer to 2 from the left, $g(x)$ decreases without bound. We express this situation by saying that the limit of $g(x)$ as x approaches 2 from the left is ∞, and $g(x)$ as x approaches 2 from the right is $-\infty$ and write

$$\lim_{x\to 2^-} g(x) = \infty, \qquad \lim_{x\to 2^+} g(x) = -\infty.$$

Since there is no common value for the one-sided limits of $g(x)$, we say that the limit of $g(x)$ as x approaches 2 does not exists and write

$$\lim_{x\to 2} g(x) \text{ does not exits.}$$

Example 1.3. Use the graph below to determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$.



Solution. It is clear from the graph that

$$\lim_{x\to 1^-} f(x) = 2 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = -1.$$

Since $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ does not exist. It is also clear from the graph that

$$\lim_{x\to -1^-} f(x) = 1 \quad \text{and} \quad \lim_{x\to -1^+} f(x) = 1.$$

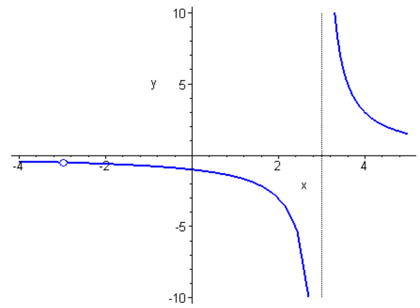
Since $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$, $\lim_{x\to -1} f(x) = 1$.

Example 1.4. (1) Graph $\dfrac{3x+9}{x^2-9}$.

(2) Evaluate $\lim_{x\to -3} \dfrac{3x+9}{x^2-9}$.

(3) Evaluate $\lim_{x\to 3} \dfrac{3x+9}{x^2-9}$.

Solution. (1) Note that $f(x) = \dfrac{3x+9}{x^2-9} = \dfrac{3}{x-3}$ for $x \neq -3$. Then, by shifting and scaling the graph of $y = \dfrac{1}{x}$, we obtain

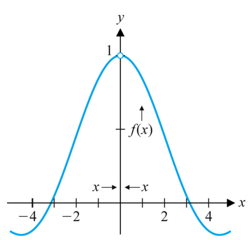


(2) Since $f(x) = \dfrac{3x+9}{x^2-9} = \dfrac{3}{x-3}$ for $x \neq -3$, $\lim_{x\to -3} \dfrac{3x+9}{x^2-9} = \lim_{x\to -3} \dfrac{3}{x-3} = -\dfrac{1}{2}$.

(3) It is seen from the graph that $\lim_{x\to 3^\pm} \dfrac{3x+9}{x^2-9} = \pm\infty$. Hence, $\lim_{x\to 3} \dfrac{3x+9}{x^2-9}$ does not exist.

Example 1.5. Evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$.

Solution. From the following tables and the graph



x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.99999983
0.0001	0.9999999983
0.00001	0.999999999983

x	$\frac{\sin x}{x}$
−0.1	0.998334
−0.01	0.999983
−0.001	0.99999983
−0.0001	0.9999999983
−0.00001	0.999999999983

one can conjecture that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

From now on, we will use the following fact without giving its proof.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

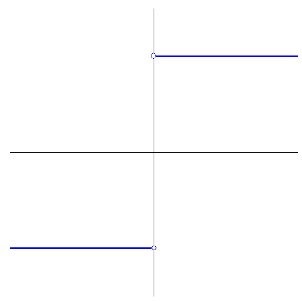
Example 1.6. Evaluate $\lim_{x \to 0} \frac{x}{|x|}$.

Solution. Note that

$$\frac{x}{|x|} = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x > 0 \end{cases}$$

So, $\lim_{x \to 0^+} \frac{x}{|x|} = 1$ while $\lim_{x \to 0^-} \frac{x}{|x|} = -1$. Since the left limit is not equal to the right limit,

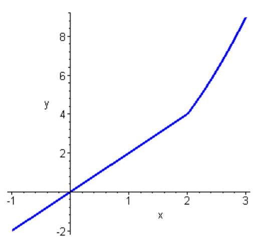
$$\lim_{x \to 0} \frac{x}{|x|} \quad \text{does not exist.}$$



Example 1.7. Sketch the graph of $f(x) = \begin{cases} 2x & \text{if} \quad x < 2 \\ x^2 & \text{if} \quad x \geq 2 \end{cases}$ and identify each limit.

(a) $\lim_{x \to 2^-} f(x)$
(b) $\lim_{x \to 2^+} f(x)$
(c) $\lim_{x \to 2} f(x)$
(d) $\lim_{x \to 1} f(x)$

Solution.
The graph is shown below.



And,

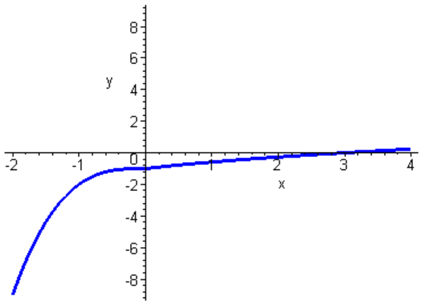
(a) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 2x = 4$
(b) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x^2 = 4$
(c) $\lim_{x \to 2} f(x) = 4$
(d) $\lim_{x \to 1} f(x) = \lim_{x \to 1} 2x = 2$

Example 1.8. Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ \sqrt{x+1} - 2 & \text{if} \quad x > 0 \end{cases}$ and identify each limit.

(a) $\lim_{x \to 0^-} f(x)$
(b) $\lim_{x \to 0^+} f(x)$
(c) $\lim_{x \to 0} f(x)$
(d) $\lim_{x \to -1} f(x)$

(e) $\lim_{x \to 3} f(x)$

Solution.
The graph is shown below.



And,

(a) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^3 - 1 = -1$
(b) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x+1} - 2 = -1$
(c) $\lim_{x \to 0} f(x) = -1$
(d) $\lim_{x \to -1} f(x) = \lim_{x \to -1} x^3 - 1 = -2$
(e) $\lim_{x \to 3} f(x) = \lim_{x \to 3} \sqrt{x+1} - 2 = 0$

2. Computation of Limits

It is easy to see that for any constant c and any real number a,

$$\lim_{x \to a} c = c,$$

and

$$\lim_{x \to a} x = a.$$

The following theorem lists some basic rules for dealing with common limit problems

Theorem 2.1 *Suppose that* $\lim_{x \to a} f(x)$ *and* $\lim_{x \to a} g(x)$ *both exist and let* c *be any constant. Then,*

(i) $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$,
(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,
(iii) $\lim_{x \to a} [f(x)g(x)] = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right]$, *and*
(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ *provided* $\lim_{x \to a} g(x) \neq 0$.

By using (iii) of Theorem 2.1, whenever $\lim_{x \to a} f(x)$ exits,

$$\lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x)f(x)]$$
$$= \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} f(x)\right] = \left[\lim_{x \to a} f(x)\right]^2.$$

Repeating this argument, we get that

$$\lim_{x \to a} [f(x)]^2 = \left[\lim_{x \to a} f(x)\right]^n,$$

for any positive integer n. In particular, for any positive integer n and any real number a,

$$\lim_{x \to a} x^n = a^n.$$

Example 2.1. Evaluate

(1) $\lim_{x \to 2} (3x^2 - 5x + 4)$.

(2) $\lim_{x\to 3} \dfrac{x^3 - 5x + 4}{x^2 - 2}$.

(3) $\lim_{x\to 1} \dfrac{x^2 - 1}{1 - x}$.

Theorem 2.2 *For any polynomial $p(x)$ and any real number a,*

$$\lim_{x\to a} p(x) = p(a).$$

Theorem 2.3 *Suppose that $\lim_{x\to a} f(x) = L$ and n is any positive integer. Then,*

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

where for n even, we assume that $L > 0$.

Example 2.2. Evaluate

(1) $\lim_{x\to 2} \sqrt[5]{3x^2 - 2x}$.

(2) $\lim_{x\to 0} \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$.

Theorem 2.4 *For any real number a, we have*

(i) $\lim_{x\to a} \sin x = \sin a$,

(ii) $\lim_{x\to a} \cos x = \cos a$,

(iii) $\lim_{x\to a} e^x = e^a$,

(iv) $\lim_{x\to a} \ln x = \ln a$, *for* $a > 0$,

(v) $\lim_{x\to a} \sin^{-1} x = \sin^{-1} a$, *for* $-1 < a < 1$,

(vi) $\lim_{x\to a} \cos^{-1} x = \cos^{-1} a$, *for* $-1 < a < 1$,

(vii) $\lim_{x\to a} \tan^{-1} x = \tan^{-1} a$, *for* $-\infty < a < \infty$,

(viii) *if p is a polynomial and* $\lim_{x\to p(a)} f(x) = L$, *then* $\lim_{x\to a} f(p(x)) = L$.

Example 2.3. Evaluate $\lim_{x\to 0} \sin^{-1}\left(\dfrac{x+1}{2}\right)$.

Example 2.4. Evaluate $\lim_{x\to 0}(x \cot x)$.

Theorem 2.5 (Sandwich Theorem) *Suppose that*

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x\to a} g(x) = L, \text{ too.}$$

Example 2.5. Evaluate $\lim_{x\to 0}\left[x^2\cos\left(\frac{1}{x}\right)\right]$.

Example 2.6. Evaluate $\lim_{x\to 0} f(x)$, where f is defined by

$$f(x) = \begin{cases} x^2 + 2\cos x + 1 & \text{if } x < 0 \\ e^x - 4 & \text{if } x \geq 0 \end{cases}.$$

Example 2.7. Evaluate.

(1) $\lim_{x\to 0} \frac{1-e^{2x}}{1-e^x}$.
(2) $\lim_{x\to 1} \frac{x^3-1}{x^2+2x-3}$.
(3) $\lim_{x\to 0} \frac{\sin x}{\tan x}$.
(4) $\lim_{x\to 0} \frac{\tan 2x}{5x}$.
(5) $\lim_{x\to 0} \frac{xe^{-2x+1}}{x^2+x}$.
(6) $\lim_{x\to 0^+} x^2\csc^2 x$.
(7) $\lim_{x\to 1} \left(\frac{1}{x-1} - \frac{2}{x^2-1}\right)$.
(8) $\lim_{x\to 0} \frac{(1+x)^3-1}{x}$.
(9) $\lim_{x\to 0} \frac{\sin|x|}{x}$.
(10) $\lim_{x\to 1} [\![x]\!]$.
(11) $\lim_{x\to 1.5} [\![x]\!]$.
(12) $\lim_{x\to 1} (x - [\![x]\!])$.

3. Continuity and Its Consequences

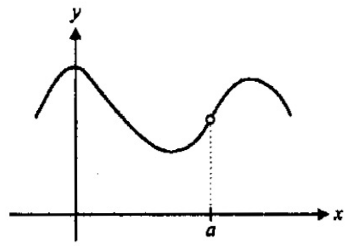
A function f is *continuous* at $x = a$ when

(i) $f(a)$ is defined,
(ii) $\lim_{x\to a} f(x)$ exists, and
(iii) $\lim_{x\to a} f(x) = f(a)$.

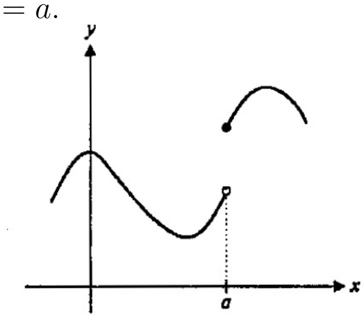
Otherwise f is said to be *discontinuous* at $x = a$.

Example 3.1. Let us see some examples of functions that are discontinuous at $x = a$.

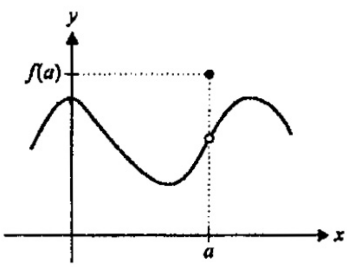
(1) The function is not defined at $x = a$. The graph has a hole at $x = a$.



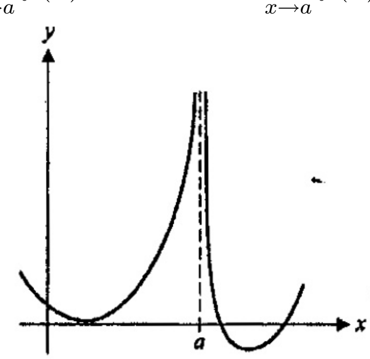
(2) The function is defined at $x = a$, but $\lim_{x \to a} f(x)$ does not exist. The graph has a jump at $x = a$.



(3) $\lim_{x \to a} f(x)$ exists and $f(a)$ is defined but $\lim_{x \to a} f(x) \neq f(a)$. The graph has a hole at $x = a$.



(4) $\lim_{x \to a} f(x) = \infty$ and so $\lim_{x \to a} f(x) = f(a)$ never holds. The function blows up at $x = a$.



Example 3.2. Determine where $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ is continuous.

The point $x = a$ is called a *removable* discontinuity of a function f if one can remove the discontinuity by redefining the function at that point. Otherwise, it is called a *nonremovable* or an *essential* discontinuity of f. Clearly, a function has a removable discontinuity at $x = a$ if and only if $\lim_{x \to a} f(x)$ exists and is finite.

Example 3.3. Classify all the discontinuities of

(1) $f(x) = \dfrac{x^2+2x-3}{x-1}$.
(2) $f(x) = \dfrac{1}{x^2}$.
(3) $f(x) = \cos\dfrac{1}{x}$.

Theorem 3.1 *All polynomials are continuous everywhere. Additionally,* $\sin x$, $\cos x$, $\tan^{-1} x$ *and* e^x *are continuous everywhere,* $\sqrt[n]{x}$ *is continuous for all* x*, when* n *is odd and for* $x > 0$*, when* n *is even. We also have* $\ln x$ *is continuous for* $x > 0$ *and* $\sin^{-1} x$ *and* $\cos^{-1} x$ *are continuous for* $-1 < x < 1$.

Theorem 3.2 *Suppose that* f *and* g *are continuous at* $x = a$*. Then all of the following are true:*

(1) $(f \pm g)$ *is continuous at* $x = a$,
(2) $(f \cdot g)$ *is continuous at* $x = a$*, and*
(3) (f/g) *is continuous at* $x = a$ *if* $g(a) \neq 0$.

Example 3.4. Find and classify all the discontinuities of $\dfrac{x^4-3x^2+2}{x^2-3x-4}$.

Theorem 3.3 *Suppose that* $\lim_{x\to a} g(x) = L$ *and* f *is continuous at* L*. Then,*

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

Corollary 3.4 *Suppose that* g *is continuous at* a *and* f *is continuous at* $g(a)$*. Then the composition* $f \circ g$ *is continuous at* a.

Example 3.5. Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

If f is continuous at every point on an open interval (a, b), we say that f is *continuous on* (a, b). We say that f is *continuous on the closed interval* $[a, b]$, if f is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is *continuous*.

Example 3.6. Determine the interval(s) where f is continuous, for

(1) $f(x) = \sqrt{4-x^2}$,
(2) $f(x) = \ln(x-3)$.

Example 3.7. For what value of a is

$$f(x) = \begin{cases} x^2-1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

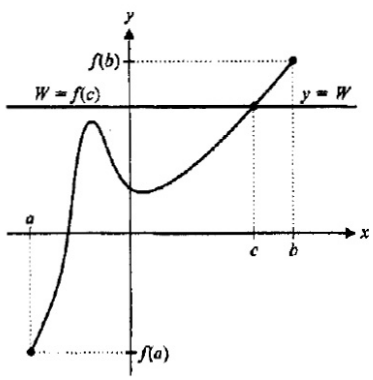
Example 3.8. Let

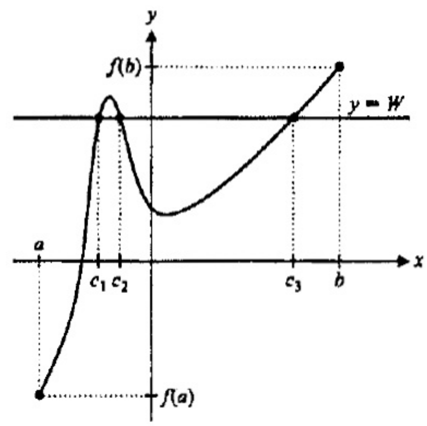
$$f(x) = \begin{cases} 2\,\mathrm{sgn}(x-1), & x > 1, \\ a, & x = 1, \\ x + b, & x < 1. \end{cases}$$

If f is continuous at $x = 1$, find a and b.

Theorem 3.5 (Intermediate Value Theorem) *Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.*

Example 3.9. Two illustrations of the intermediate value theorem:





Corollary 3.6 *Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$.*

4. Limits involving infinity; asymptotes

If the values of f grow without bound, as x approaches a , we say that $\lim_{x\to a} f(x) = \infty$. Similarly, if the values of f become arbitrarily large and negative as x approaches a, we say that $\lim\limits_{x\to a} f(x) = -\infty$.

A line $x = a$ is a *vertical asymptote* of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \pm\infty.$$

Example 4.1. Evaluate

(1) $\lim\limits_{x\to 0} \dfrac{1}{x}$.

(2) $\lim\limits_{x\to -3} \dfrac{1}{(x+3)^2}$.

(3) $\lim\limits_{x\to 2} \dfrac{(x-2)^2}{x^2-4}$.

(4) $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$.

(5) $\lim\limits_{x\to 2^+} \dfrac{x-3}{x^2-4}$.

(6) $\lim_{x\to 2^-} \dfrac{x-3}{x^2-4}$.
(7) $\lim_{x\to 2} \dfrac{x-3}{x^2-4}$.
(8) $\lim_{x\to 2} \dfrac{2-x}{(x-2)^3}$.
(9) $\lim_{x\to 5} \dfrac{1}{(x-5)^3}$.
(10) $\lim_{x\to -2} \dfrac{x+1}{(x-3)(x+2)}$.
(11) $\lim_{x\to \frac{\pi}{2}} \tan x$.

Intuitively, $\lim_{x\to\infty} f(x) = L$ (or, $\lim_{x\to-\infty} f(x) = L$ if x moves increasingly far from the origin in the positive direction (or, in the negative direction), $f(x)$ gets arbitrarily close to L.

Example 4.2. Clearly, $\lim_{x\to\infty} \dfrac{1}{x} = 0$ and $\lim_{x\to-\infty} \dfrac{1}{x} = 0$.

A line $y = b$ is a *horizontal asymptote* of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$

Example 4.3. Evaluate $\lim_{x\to\infty} 5 + \dfrac{1}{x}$.

Theorem 4.1 *For any rational number $t > 0$,*

$$\lim_{x\to\pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = p/q$ where q is odd.

Theorem 4.2 *For any polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have*

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

Example 4.4. Evaluate

(1) $\lim_{x\to\infty} \dfrac{2x^3+7}{x^3-x^2+x+7}$.
(2) $\lim_{x\to\infty} \dfrac{3x+7}{x^2-2}$.
(3) $\lim_{x\to\infty} \sin\dfrac{1}{x}$.
(4) $\lim_{x\to\infty} \sqrt{x^2+2x+3} - x$.
(5) $\lim_{x\to\infty} \dfrac{\sin x}{x}$.
(6) $\lim_{x\to\infty} \sin x$.

Example 4.5. Find the horizontal asymptote(s) of $f(x) = \dfrac{2-x+\sin x}{x+\cos x}$

Let $f(x) = \dfrac{P(x)}{Q(x)}$. If (the degree of P) = (the degree of Q)+1, then the graph of f has a oblique (slant) asymptote. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to 0 as $x \to \pm\infty$.

Example 4.6. Find the asymptotes of the graph of f, if

(1) $f(x) = \dfrac{x^2 - 3}{2x - 4}$.
(2) $f(x) = \dfrac{2x}{x + 1}$.

Example 4.7. Evaluate

(1) $\lim\limits_{x \to 0^-} e^{\frac{1}{x}}$.
(2) $\lim\limits_{x \to 0^+} e^{\frac{1}{x}}$.
(3) $\lim\limits_{x \to \infty} \tan^{-1} x$.
(4) $\lim\limits_{x \to -\infty} \tan^{-1} x$.
(5) $\lim\limits_{x \to 0^+} \ln x$.
(6) $\lim\limits_{x \to \infty} \ln x$.
(7) $\lim\limits_{x \to 0} \sin\left(e^{-\frac{1}{x^2}}\right)$.
(8) $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{x^2 + 1}}$.
(9) $\lim\limits_{x \to -\infty} \dfrac{x}{\sqrt{x^2 + 1}}$.