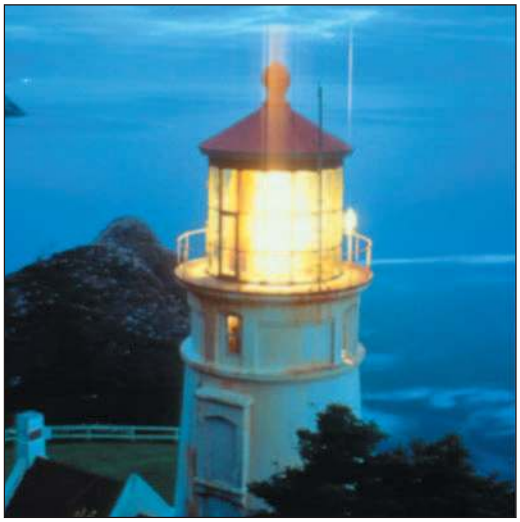
Limits and Continuity

CHAPTER 1



When you enter a darkened room, your eyes adjust to the reduced level of light by increasing the size of your pupils. Enlarging the pupils allows more light to enter the eyes, which makes objects around you easier to see. By contrast, when you enter a brightly lit room, your pupils contract, reducing the amount of light entering the eyes. This is necessary since too much light will overload your visual system.

This visual adjustment mechanism is present in many animals. Researchers study this mechanism by performing experiments and trying to find a mathematical description of the results. In this case, you might want to represent the size of the pupils as a function of the amount of light present. Two basic characteristics of such a *mathematical model* would be

Small pupils

1. As the amount of light (x) increases, the pupil size (y) decreases down to some minimum value p; and
2. As the amount of light (x) decreases, the pupil size (y) increases up to some maximum value P.

Large pupils

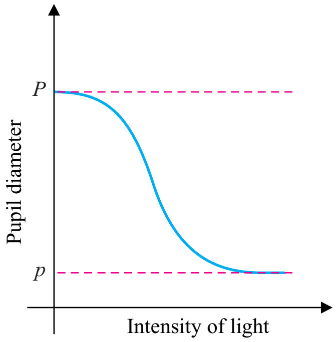


FIGURE 1.1
Size of pupils

Finding a function with these two properties can be a challenge. (Try it!) One possible graph of such a function is shown in Figure 1.1. (See example 5.9 for more.) In this chapter, we develop the concept of *limit,* which can be used to describe functions with specific properties such as those listed above. The limit is the fundamental notion of calculus. This underlying concept is the thread that binds together virtually all of the calculus you are about to study. An investment in carefully studying limits now will have very significant payoffs throughout the remainder of your calculus experience and beyond.

1.1 A BRIEF PREVIEW OF CALCULUS: TANGENT LINES AND THE LENGTH OF A CURVE

In this section, we approach the boundary between precalculus mathematics and the calculus by investigating several important problems requiring the use of calculus. Recall that the slope of a straight line is the change in y divided by the change in x. This fraction is the same regardless of which two points you use to compute the slope. For example, the points (0, 1), (1, 4) and (3, 10) all lie on the line $y = 3x + 1$. The slope of 3 can be obtained from any two of the points. For instance,

$$m = \frac{4-1}{1-0} = 3 \qquad \text{or} \qquad m = \frac{10-1}{3-0} = 3.$$

In the calculus, we generalize this problem to find the slope of a *curve* at a point. For instance, suppose we wanted to find the slope of the curve $y = x^2 + 1$ at the point (1, 2). You might think of picking a second point on the parabola, say (2, 5). The slope of the line through these two points (called a **secant line;** see Figure 1.2a) is easy enough to compute. We have

$$m_{\text{sec}} = \frac{5-2}{2-1} = 3.$$

However, using the points (0, 1) and (1, 2), we get a different slope (see Figure 1.2b):

$$m_{\text{sec}} = \frac{2-1}{1-0} = 1.$$

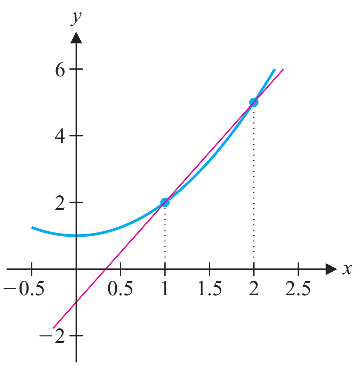


FIGURE 1.2a
Secant line, slope = 3

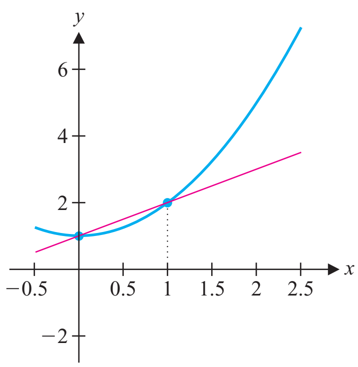


FIGURE 1.2b
Secant line, slope = 1

For curves other than straight lines, the slopes of secant lines joining different points are generally *not* the same, as seen in Figures 1.2a and 1.2b.

If you get different slopes using different pairs of points, then what exactly does it mean for a curve to have a slope at a point? The answer can be visualized by graphically zooming in on the specified point. Take the graph of $y = x^2 + 1$ and zoom in tight on the point (1, 2). You should get a graph something like the one in Figure 1.3. The graph looks very much like a straight line. In fact, the more you zoom in, the straighter the curve appears to be and the less it matters which two points are used to compute a slope. So, here's the strategy: pick several points on the parabola, each closer to the point (1, 2) than the previous one. Compute the slopes of the lines through (1, 2) and each of the points. The closer the second point gets to (1, 2), the closer the computed slope is to the answer you seek.

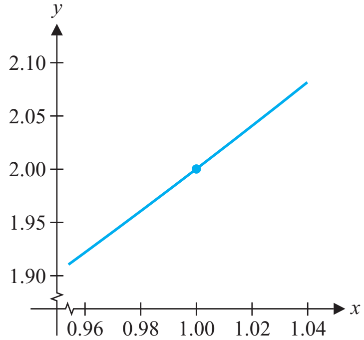


FIGURE 1.3
$y = x^2 + 1$

For example, the point (1.5, 3.25) is on the parabola fairly close to (1, 2). The slope of the line joining these points is

$$m_{\text{sec}} = \frac{3.25 - 2}{1.5 - 1} = 2.5.$$

The point (1.1, 2.21) is even closer to (1, 2). The slope of the secant line joining these two points is

$$m_{\text{sec}} = \frac{2.21 - 2}{1.1 - 1} = 2.1.$$

Continuing in this way, observe that the point (1.01, 2.0201) is closer yet to the point (1, 2). The slope of the secant line through these points is

$$m_{\text{sec}} = \frac{2.0201 - 2}{1.01 - 1} = 2.01.$$

The slopes of the secant lines that we computed (2.5, 2.1 and 2.01) are getting closer and closer to the slope of the parabola at the point (1, 2). Based on these calculations, it seems reasonable to say that the slope of the curve is approximately 2.

Example 1.1 takes our introductory example just a bit further.

EXAMPLE 1.1 Estimating the Slope of a Curve

Estimate the slope of $y = x^2 + 1$ at $x = 1$.

Solution We focus on the point whose coordinates are $x = 1$ and $y = 1^2 + 1 = 2$. To estimate the slope, choose a sequence of points near (1, 2) and compute the slopes of the secant lines joining those points with (1, 2). (We showed sample secant lines in Figures 1.2a and 1.2b.) Choosing points with $x > 1$ (x-values of 2, 1.1 and 1.01) and points with $x < 1$ (x-values of 0, 0.9 and 0.99), we compute the corresponding y-values using $y = x^2 + 1$ and get the slopes shown in the following table.

Second Point	m_{sec}
(2, 5)	$\frac{5-2}{2-1} = 3$
(1.1, 2.21)	$\frac{2.21-2}{1.1-1} = 2.1$
(1.01, 2.0201)	$\frac{2.0201-2}{1.01-1} = 2.01$

Second Point	m_{sec}
(0, 1)	$\frac{1-2}{0-1} = 1$
(0.9, 1.81)	$\frac{1.81-2}{0.9-1} = 1.9$
(0.99, 1.9801)	$\frac{1.9801-2}{0.99-1} = 1.99$

Observe that in both columns, as the second point gets closer to (1, 2), the slope of the secant line gets closer to 2. A reasonable estimate of the slope of the curve at the point (1, 2) is then 2. ■

In Chapter 2, we develop a powerful technique for computing such slopes exactly (and easily). Note what distinguishes the calculus problem from the corresponding algebra problem. The calculus problem involves a process we call a *limit*. While we presently can only estimate the slope of a curve using a sequence of approximations, the limit allows us to compute the slope exactly.

EXAMPLE 1.2 Estimating the Slope of a Curve

Estimate the slope of $y = \sin x$ at $x = 0$.

Solution This turns out to be a very important problem, one that we will return to later. For now, choose a sequence of points on the graph of $y = \sin x$ near (0, 0) and compute the slopes of the secant lines joining those points with (0, 0). The following table shows one set of choices.

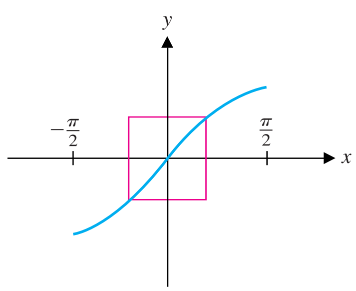


FIGURE 1.4
$y = \sin x$

Second Point	m_{sec}
(1, sin 1)	0.84147
(0.1, sin 0.1)	0.99833
(0.01, sin 0.01)	0.99998

Second Point	m_{sec}
(−1, sin (−1))	0.84147
(−0.1, sin (−0.1))	0.99833
(−0.01, sin (−0.01))	0.99998

Note that as the second point gets closer and closer to (0, 0), the slope of the secant line (m_{sec}) appears to get closer and closer to 1. A good estimate of the slope of the curve at the point (0, 0) would then appear to be 1. Although we presently have no way of computing the slope exactly, this is consistent with the graph of $y = \sin x$ in Figure 1.4. Note that near (0, 0), the graph resembles that of $y = x$, a straight line of slope 1. ■

A second problem requiring the power of calculus is that of computing distance along a curved path. While this problem is of less significance than our first example (both historically and in the development of the calculus), it provides a good indication of the need for mathematics beyond simple algebra. You should pay special attention to the similarities between the development of this problem and our earlier work with slope.

Recall that the (straight-line) distance between two points (x_1, y_1) and (x_2, y_2) is

$$d\{(x_1, y_1), (x_2, y_2)\} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For instance, the distance between the points (0, 1) and (3, 4) is

$$d\{(0, 1), (3, 4)\} = \sqrt{(3 - 0)^2 + (4 - 1)^2} = 3\sqrt{2} \approx 4.24264.$$

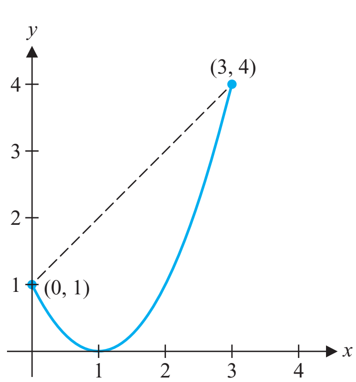


FIGURE 1.5a
$y = (x - 1)^2$

However, this is not the only way we might want to compute the distance between these two points. For example, suppose that you needed to drive a car from (0, 1) to (3, 4) along a road that follows the curve $y = (x - 1)^2$. (See Figure 1.5a.) In this case, you don't care about the straight-line distance connecting the two points, but only about how far you must drive along the curve (the *length* of the curve or *arc length*).

Notice that the distance along the curve must be greater than $3\sqrt{2}$ (the straight-line distance). Taking a cue from the slope problem, we can formulate a strategy for obtaining a sequence of increasingly accurate approximations. Instead of using just one line segment to get the approximation of $3\sqrt{2}$, we could use two line segments, as in Figure 1.5b. Notice that the sum of the lengths of the two line segments appears to be a much better approximation to the actual length of the curve than the straight-line distance used previously.

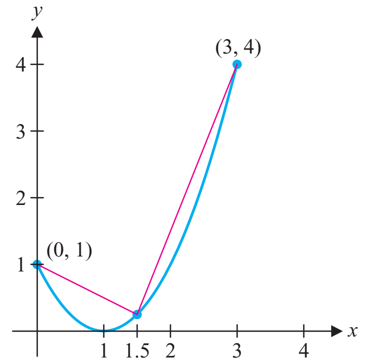


FIGURE 1.5b
Two line segments

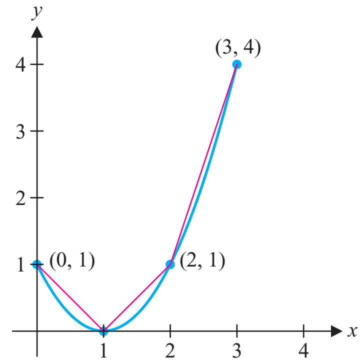


FIGURE 1.5c
Three line segments

This distance is

$$\begin{aligned} d_2 &= d\{(0,\ 1), (1.5, 0.25)\} + d\{(1.5, 0.25), (3, 4)\} \\ &= \sqrt{(1.5-0)^2 + (0.25-1)^2} + \sqrt{(3-1.5)^2 + (4-0.25)^2} \approx 5.71592. \end{aligned}$$

You're probably way ahead of us by now. If approximating the length of the curve with two line segments gives an improved approximation, why not use three or four or more? Using the three line segments indicated in Figure 1.5c, we get the further improved approximation

$$\begin{aligned} d_3 &= d\{(0, 1), (1, 0)\} + d\{(1, 0), (2, 1)\} + d\{(2, 1), (3, 4)\} \\ &= \sqrt{(1-0)^2 + (0-1)^2} + \sqrt{(2-1)^2 + (1-0)^2} + \sqrt{(3-2)^2 + (4-1)^2} \\ &= 2\sqrt{2} + \sqrt{10} \approx 5.99070. \end{aligned}$$

No. of Segments	*Distance*
1	4.24264
2	5.71592
3	5.99070
4	6.03562
5	6.06906
6	6.08713
7	6.09711

Note that the more line segments we use, the better the approximation appears to be. This process will become much less tedious with the development of the definite integral in Chapter 4. For now we list a number of these successively better approximations (produced using points on the curve with evenly spaced x-coordinates) in the table found in the margin. The table suggests that the length of the curve is approximately 6.1 (quite far from the straight-line distance of 4.2). If we continued this process using more and more line segments, the sum of their lengths would approach the actual length of the curve (about 6.126). As in the problem of computing the slope of a curve, the exact arc length is obtained as a limit.

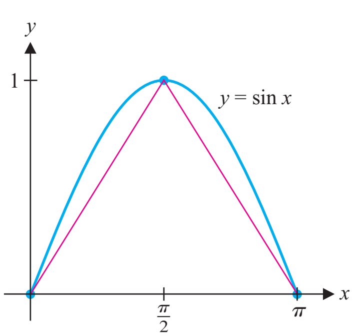


FIGURE 1.6a
Approximating the curve with two line segments

EXAMPLE 1.3 Estimating the Arc Length of a Curve

Estimate the arc length of the curve $y = \sin x$ for $0 \le x \le \pi$. (See Figure 1.6a.)

Solution The endpoints of the curve on this interval are (0, 0) and $(\pi, 0)$. The distance between these points is $d_1 = \pi$. The point on the graph of $y = \sin x$ corresponding to the midpoint of the interval $[0, \pi]$ is $(\pi/2, 1)$. The distance from (0, 0) to $(\pi/2, 1)$ plus the distance from $(\pi/2, 1)$ to $(\pi, 0)$ (illustrated in Figure 1.6a) is

$$d_2 = \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} + \sqrt{\left(\frac{\pi}{2}\right)^2 + 1} \approx 3.7242.$$

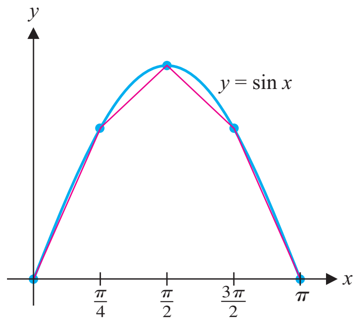


FIGURE 1.6b
Approximating the curve with four line segments

Number of Line Segments	*Sum of Lengths*
8	3.8125
16	3.8183
32	3.8197
64	3.8201

Using the five points $(0, 0)$, $(\pi/4, 1/\sqrt{2})$, $(\pi/2, 1)$, $(3\pi/4, 1/\sqrt{2})$ and $(\pi, 0)$ (i.e., four line segments, as indicated in Figure 1.6b), the sum of the lengths of these line segments is

$$d_4 = 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \frac{1}{2}} + 2\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(1 - \frac{1}{\sqrt{2}}\right)^2} \approx 3.7901.$$

Using nine points (i.e., eight line segments), you need a good calculator and some patience to compute the distance of 3.8125. A table showing further approximations is given in the margin. At this stage, it would be reasonable to estimate the length of the sine curve on the interval $[0, \pi]$ as slightly more than 3.8. ■

BEYOND FORMULAS

In the process of estimating both the slope of a curve and the length of a curve, we make some reasonably obvious (straight-line) approximations and then systematically improve on those approximations. In each case, the shorter the line segments are, the closer the approximations are to the desired value. The essence of this is the concept of *limit,* which separates precalculus mathematics from the calculus. At first glance, this limit idea might seem of little practical importance, since in our examples we never compute the exact solution. In the chapters to come, we will find remarkably simple shortcuts to exact answers. Can you think of ways to find the exact slope in example 1.1?

EXERCISES 1.1

WRITING EXERCISES

1. Explain why each approximation of arc length in example 1.3 is less than the actual arc length.

2. To estimate the slope of $f(x) = x^2 + 1$ at $x = 1$, you would compute the slopes of various secant lines. Note that $y = x^2 + 1$ curves up. Explain why the secant line connecting (1, 2) and (1.1, 2.21) will have slope greater than the slope of the curve at (1, 2). Discuss how the slope of the secant line between (1, 2) and (0.9, 1.81) compares to the slope of the curve at (1, 2).

In exercises 1–12, estimate the slope (as in example 1.1) of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 + 1, a = 1.5$
2. $f(x) = x^2 + 1, a = 2$
3. $f(x) = \cos x, a = 0$
4. $f(x) = \cos x, a = \pi/2$
5. $f(x) = x^3 + 2, a = 1$
6. $f(x) = x^3 + 2, a = 2$
7. $f(x) = \sqrt{x + 1}, a = 0$
8. $f(x) = \sqrt{x + 1}, a = 3$
9. $f(x) = \tan x, a = 0$
10. $f(x) = \tan x, a = 1$

11. Estimate the length of the curve $y = \sqrt{1 - x^2}$ for $0 \le x \le 1$ with (a) $n = 4$ and (b) $n = 8$ line segments. Explain why the exact length is $\pi/2$. How accurate are your estimates?

12. Estimate the length of the curve $y = \sqrt{9 - x^2}$ for $0 \le x \le 3$ with (a) $n = 4$ and (b) $n = 8$ line segments. Explain why the exact length is $3\pi/2$. How would an estimate of π obtained from part (b) of this exercise compare to an estimate of π obtained from part (b) of exercise 11?

In exercises 13–20, estimate the length of the curve $y = f(x)$ on the given interval using (a) $n = 4$ and (b) $n = 8$ line segments. (c) If you can program a calculator or computer, use larger n's and conjecture the actual length of the curve.

13. $f(x) = x^2 + 1, 0 \le x \le 2$
14. $f(x) = x^3 + 2, 0 \le x \le 1$
15. $f(x) = \cos x, 0 \le x \le \pi/2$
16. $f(x) = \sin x, 0 \le x \le \pi/2$
17. $f(x) = \sqrt{x + 1}, 0 \le x \le 3$
18. $f(x) = 1/x, 1 \le x \le 2$

19. $f(x) = x^2 + 1, -2 \le x \le 2$

20. $f(x) = x^3 + 2, -1 \le x \le 1$

21. An important problem in calculus is finding the area of a region. Sketch the parabola $y = 1 - x^2$ and shade in the region above the x-axis between $x = -1$ and $x = 1$. Then sketch in the following rectangles: (1) height $f(-\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = -1$ to $x = -\frac{1}{2}$, (2) height $f(-\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = -\frac{1}{2}$ to $x = 0$, (3) height $f(\frac{1}{4})$ and width $\frac{1}{2}$ extending from $x = 0$ to $x = \frac{1}{2}$ and (4) height $f(\frac{3}{4})$ and width $\frac{1}{2}$ extending from $x = \frac{1}{2}$ to $x = 1$. Compute the sum of the areas of the rectangles. Based on your sketch, does this give you a good approximation of the area under the parabola?

22. To improve the approximation of exercise 21, divide the interval $[-1, 1]$ into 8 pieces and construct a rectangle of the appropriate height on each subinterval. Compared to the approximation in exercise 21, explain why you would expect this to be a better approximation of the actual area under the parabola.

23. Use a computer or calculator to compute an approximation of the area in exercise 21 using (a) 16 rectangles, (b) 32 rectangles and (c) 64 rectangles. Use these calculations to conjecture the exact value of the area under the parabola.

24. Use the technique of exercises 21–23 to estimate the area below $y = \sin x$ and above the x-axis between $x = 0$ and $x = \pi$.

25. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 1$.

26. Use the technique of exercises 21–23 to estimate the area below $y = x^3$ and above the x-axis between $x = 0$ and $x = 2$.

EXPLORATORY EXERCISE

1. Several central concepts of calculus have been introduced in this section. An important aspect of our future development of calculus is to derive simple techniques for computing quantities such as slope and arc length. In this exercise, you will learn how to directly compute the slope of a curve at a point. Suppose you want the slope of $y = x^2$ at $x = 1$. You could start by computing slopes of secant lines connecting the point $(1, 1)$ with nearby points on the graph. Suppose the nearby point has x-coordinate $1 + h$, where h is a small (positive or negative) number. Explain why the corresponding y-coordinate is $(1 + h)^2$. Show that the slope of the secant line is $\dfrac{(1+h)^2 - 1}{1 + h - 1} = 2 + h$. As h gets closer and closer to 0, this slope better approximates the slope of the tangent line. Letting h approach 0, show that the slope of the tangent line equals 2. In a similar way, show that the slope of $y = x^2$ at $x = 2$ is 4 and find the slope of $y = x^2$ at $x = 3$. Based on your answers, conjecture a formula for the slope of $y = x^2$ at $x = a$, for any unspecified value of a.

1.2 THE CONCEPT OF LIMIT

In this section, we develop the notion of limit using some common language and illustrate the idea with some simple examples. The notion turns out to be a rather subtle one, easy to think of intuitively, but a bit harder to pin down in precise terms. We present the precise definition of limit in section 1.6. There, we carefully define limits in considerable detail. The more informal notion of limit that we introduce and work with here and in sections 1.3, 1.4 and 1.5 is adequate for most purposes.

As a start, consider the functions

$$f(x) = \frac{x^2 - 4}{x - 2} \quad \text{and} \quad g(x) = \frac{x^2 - 5}{x - 2}.$$

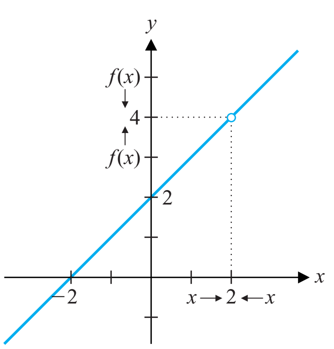


FIGURE 1.7a

$$y = \frac{x^2 - 4}{x - 2}$$

Notice that both functions are undefined at $x = 2$. So, what does this mean, beyond saying that you cannot substitute 2 for x? We often find important clues about the behavior of a function from a graph. (See Figures 1.7a and 1.7b.)

Notice that the graphs of these two functions look quite different in the vicinity of $x = 2$. Although we can't say anything about the value of these functions at $x = 2$ (since this is outside the domain of both functions), we can examine their behavior in the vicinity of

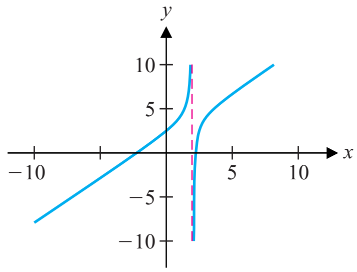


FIGURE 1.7b
$y = \dfrac{x^2 - 5}{x - 2}$

this point. We consider these functions one at a time. First, for $f(x) = \dfrac{x^2 - 4}{x - 2}$, we compute some values of the function for x close to 2, as in the following tables.

x	$f(x) = \dfrac{x^2 - 4}{x - 2}$
1.9	3.9
1.99	3.99
1.999	3.999
1.9999	3.9999

x	$f(x) = \dfrac{x^2 - 4}{x - 2}$
2.1	4.1
2.01	4.01
2.001	4.001
2.0001	4.0001

Notice that as you move down the first column of the table, the x-values get closer to 2, but are all less than 2. We use the notation $x \to 2^-$ to indicate that x *approaches 2 from the left side*. Notice that the table and the graph both suggest that as x gets closer and closer to 2 (with $x < 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the left** is 4, written

$$\lim_{x \to 2^-} f(x) = 4.$$

Likewise, we need to consider what happens to the function values for x close to 2 but larger than 2. Here, we use the notation $x \to 2^+$ to indicate that x *approaches 2 from the right side*. We compute some of these values in the second table.

Again, the table and graph both suggest that as x gets closer and closer to 2 (with $x > 2$), $f(x)$ is getting closer and closer to 4. In view of this, we say that the **limit of $f(x)$ as x approaches 2 from the right** is 4, written

$$\lim_{x \to 2^+} f(x) = 4.$$

We call $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ **one-sided limits.** Since the two one-sided limits of $f(x)$ are the same, we summarize our results by saying that the **limit of $f(x)$ as x approaches 2** is 4, written

$$\lim_{x \to 2} f(x) = 4.$$

The notion of limit as we have described it here is intended to communicate the behavior of a function *near* some point of interest, but not actually *at* that point. We finally observe that we can also determine this limit algebraically, as follows. Notice that since the expression in the numerator of $f(x) = \dfrac{x^2 - 4}{x - 2}$ factors, we can write

$$\begin{aligned}\lim_{x \to 2} f(x) &= \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} && \text{Cancel the factors of } (x - 2). \\ &= \lim_{x \to 2} (x + 2) = 4, && \text{As } x \text{ approaches 2, } (x + 2) \text{ approaches 4.}\end{aligned}$$

where we can cancel the factors of $(x - 2)$ since in the limit as $x \to 2$, x is *close* to 2, but $x \neq 2$, so that $x - 2 \neq 0$.

x	$g(x) = \dfrac{x^2-5}{x-2}$
1.9	13.9
1.99	103.99
1.999	1003.999
1.9999	10,003.9999

x	$g(x) = \dfrac{x^2-5}{x-2}$
2.1	−5.9
2.01	−95.99
2.001	−995.999
2.0001	−9995.9999

Similarly, we consider one-sided limits for $g(x) = \dfrac{x^2-5}{x-2}$, as $x \to 2$. Based on the graph in Figure 1.7b and the table of approximate function values shown in the margin, observe that as x gets closer and closer to 2 (with $x < 2$), $g(x)$ increases without bound. Since there is no number that $g(x)$ is approaching, we say that the *limit of g(x) as x approaches 2 from the left does not exist,* written

$$\lim_{x\to 2^-} g(x) \text{ does not exist.}$$

Similarly, the graph and the table of function values for $x > 2$ (shown in the margin) suggest that $g(x)$ decreases without bound as x approaches 2 from the right. Since there is no number that $g(x)$ is approaching, we say that

$$\lim_{x\to 2^+} g(x) \text{ does not exist.}$$

Finally, since there is no common value for the one-sided limits of $g(x)$ (in fact, neither limit exists), we say that the *limit of g(x) as x approaches 2 does not exist,* written

$$\lim_{x\to 2} g(x) \text{ does not exist.}$$

Before moving on, we should summarize what we have said about limits.

> A limit exists if and only if both corresponding one-sided limits exist and are equal. That is,
>
> $$\lim_{x\to a} f(x) = L, \text{ for some number } L, \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

In other words, we say that $\lim_{x\to a} f(x) = L$ if we can make $f(x)$ as close as we might like to L, by making x sufficiently close to a (on either side of a), but not equal to a.

Note that we can think about limits from a purely graphical viewpoint, as in example 2.1.

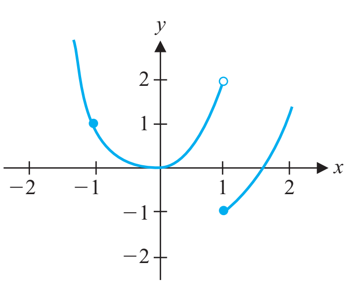


FIGURE 1.8
$y = f(x)$

EXAMPLE 2.1 Determining Limits Graphically

Use the graph in Figure 1.8 to determine $\lim_{x\to 1^-} f(x)$, $\lim_{x\to 1^+} f(x)$, $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$.

Solution For $\lim_{x\to 1^-} f(x)$, we consider the y-values as x gets closer to 1, with $x < 1$. That is, we follow the graph toward $x = 1$ *from the left* ($x < 1$). Observe that the graph dead-ends into the open circle at the point (1, 2). Therefore, we say that $\lim_{x\to 1^-} f(x) = 2$. For $\lim_{x\to 1^+} f(x)$, we follow the graph toward $x = 1$ *from the right* ($x > 1$). In this case, the graph dead-ends into the solid circle located at the point (1, −1). For this reason, we say that $\lim_{x\to 1^+} f(x) = -1$. Because $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, we say that $\lim_{x\to 1} f(x)$ does not exist. Finally, we have that $\lim_{x\to -1} f(x) = 1$, since the graph approaches a y-value of 1 as x approaches −1 both from the left and from the right. ■

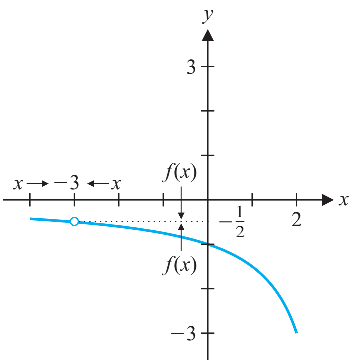


FIGURE 1.9

$$\lim_{x\to -3}\frac{3x+9}{x^2-9}=-\frac{1}{2}$$

x	$\frac{3x+9}{x^2-9}$
−3.1	−0.491803
−3.01	−0.499168
−3.001	−0.499917
−3.0001	−0.499992

x	$\frac{3x+9}{x^2-9}$
−2.9	−0.508475
−2.99	−0.500835
−2.999	−0.500083
−2.9999	−0.500008

EXAMPLE 2.2 A Limit Where Two Factors Cancel

Evaluate $\lim_{x\to -3}\frac{3x+9}{x^2-9}$.

Solution We examine a graph (see Figure 1.9) and compute some function values for x near −3. Based on this numerical and graphical evidence, it's reasonable to conjecture that

$$\lim_{x\to -3^+}\frac{3x+9}{x^2-9}=\lim_{x\to -3^-}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$

Further, note that

$$\lim_{x\to -3^-}\frac{3x+9}{x^2-9}=\lim_{x\to -3^-}\frac{3(x+3)}{(x+3)(x-3)}$$ Cancel factors of $(x+3)$.

$$=\lim_{x\to -3^-}\frac{3}{x-3}=-\frac{1}{2},$$

since $(x-3)\to -6$ as $x\to -3$. Again, the cancellation of the factors of $(x+3)$ is valid since in the limit as $x\to -3$, x is *close* to −3, but $x\neq -3$, so that $x+3\neq 0$. Likewise,

$$\lim_{x\to -3^+}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$

Finally, since the function approaches the *same* value as $x\to -3$ both from the right and from the left (i.e., the one-sided limits are equal), we write

$$\lim_{x\to -3}\frac{3x+9}{x^2-9}=-\frac{1}{2}.$$ ■

In example 2.2, the limit exists because both one-sided limits exist and are equal. In example 2.3, neither one-sided limit exists.

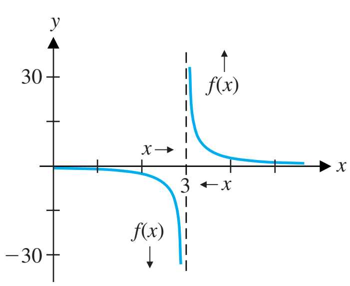


FIGURE 1.10

$$y=\frac{3x+9}{x^2-9}$$

x	$\frac{3x+9}{x^2-9}$
3.1	30
3.01	300
3.001	3000
3.0001	30,000

EXAMPLE 2.3 A Limit That Does Not Exist

Determine whether $\lim_{x\to 3}\frac{3x+9}{x^2-9}$ exists.

Solution We first draw a graph (see Figure 1.10) and compute some function values for x close to 3.

Based on this numerical and graphical evidence, it appears that, as $x\to 3^+$, $\frac{3x+9}{x^2-9}$ is increasing without bound. Thus,

$$\lim_{x\to 3^+}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Similarly, from the graph and the table of values for $x<3$, we can say that

$$\lim_{x\to 3^-}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Since neither one-sided limit exists, we say

$$\lim_{x\to 3}\frac{3x+9}{x^2-9} \text{ does not exist.}$$

Here, we considered both one-sided limits for the sake of completeness. Of course, you should keep in mind that if *either* one-sided limit fails to exist, then the limit does not exist. ■

x	$\frac{3x+9}{x^2-9}$
2.9	−30
2.99	−300
2.999	−3000
2.9999	−30,000

Many limits cannot be resolved using algebraic methods. In these cases, we can approximate the limit using graphical and numerical evidence, as we see in example 2.4.

EXAMPLE 2.4 Approximating the Value of a Limit

Evaluate $\lim_{x\to 0} \frac{\sin x}{x}$.

Solution Unlike some of the limits considered previously, there is no algebra that will simplify this expression. However, we can still draw a graph (see Figure 1.11) and compute some function values.

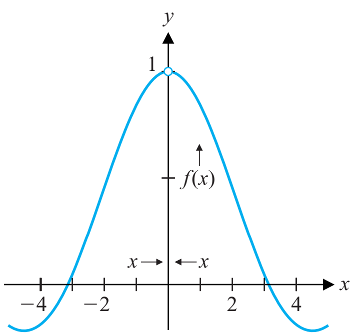


FIGURE 1.11

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

x	$\frac{\sin x}{x}$
0.1	0.998334
0.01	0.999983
0.001	0.99999983
0.0001	0.9999999983
0.00001	0.999999999983

x	$\frac{\sin x}{x}$
−0.1	0.998334
−0.01	0.999983
−0.001	0.99999983
−0.0001	0.9999999983
−0.00001	0.999999999983

The graph and the tables of values lead us to the conjectures:

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{\sin x}{x} = 1,$$

from which we conjecture that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

In Chapter 2, we examine these limits with greater care (and prove that these conjectures are correct). ■

REMARK 2.1

Computer or calculator computation of limits is unreliable. We use graphs and tables of values only as (strong) evidence pointing to what a plausible answer might be. To be certain, we need to obtain careful verification of our conjectures. We see how to do this in sections 1.3–1.7.

EXAMPLE 2.5 A Case Where One-Sided Limits Disagree

Evaluate $\lim_{x\to 0} \frac{x}{|x|}$.

Solution The computer-generated graph shown in Figure 1.12a is incomplete. Since $\frac{x}{|x|}$ is undefined at $x = 0$, there is no point at $x = 0$. The graph in Figure 1.12b correctly shows open circles at the intersections of the two halves of the graph with the y-axis. We also have

$$\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} \frac{x}{x} \quad \text{Since } |x| = x, \text{ when } x > 0.$$
$$= \lim_{x\to 0^+} 1$$
$$= 1$$

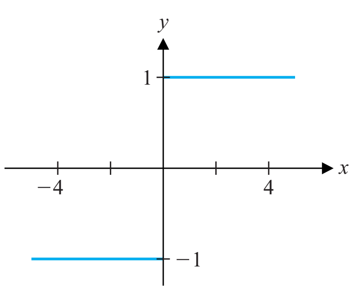


FIGURE 1.12a

$y = \frac{x}{|x|}$

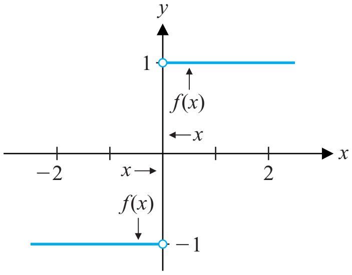


FIGURE 1.12b

$\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

and

$$\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} \frac{x}{-x} \qquad \text{Since } |x| = -x, \text{ when } x < 0.$$
$$= \lim_{x\to 0^-} -1$$
$$= -1.$$

It now follows that $\lim_{x\to 0} \frac{x}{|x|}$ does not exist,

since the one-sided limits are not the same. You should also keep in mind that this observation is entirely consistent with what we see in the graph. ■

EXAMPLE 2.6 A Limit Describing the Movement of a Baseball Pitch

The knuckleball is one of the most exotic pitches in baseball. Batters describe the ball as unpredictably moving left, right, up and down. For a typical knuckleball speed of 60 mph, the left/right position of the ball (in feet) as it crosses the plate is given by

$$f(\omega) = \frac{1.7}{\omega} - \frac{5}{8\omega^2}\sin(2.72\omega)$$

(derived from experimental data in Watts and Bahill's book *Keeping Your Eye on the Ball*), where ω is the rotational speed of the ball in radians per second and where $f(\omega) = 0$ corresponds to the middle of home plate. Folk wisdom among baseball pitchers has it that the less spin on the ball, the better the pitch. To investigate this theory, we consider the limit of $f(\omega)$ as $\omega \to 0^+$. As always, we look at a graph (see Figure 1.13) and generate a table of function values. The graphical and numerical evidence suggests that $\lim_{\omega\to 0^+} f(\omega) = 0$.

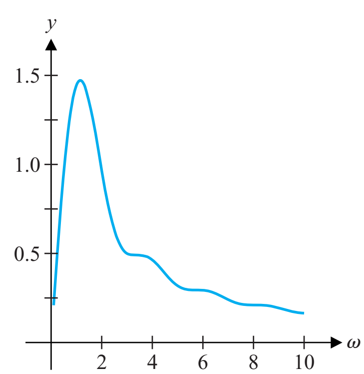


FIGURE 1.13

$y = \frac{1.7}{\omega} - \frac{5}{8\omega^2}\sin(2.72\omega)$

ω	$f(\omega)$
10	0.1645
1	1.4442
0.1	0.2088
0.01	0.021
0.001	0.0021
0.0001	0.0002

The limit indicates that a knuckleball with absolutely no spin doesn't move at all (and therefore would be easy to hit). According to Watts and Bahill, a very slow rotation rate of about 1 to 3 radians per second produces the best pitch (i.e., the most movement). Take another look at Figure 1.13 to convince yourself that this makes sense. ■

EXERCISES 1.2

WRITING EXERCISES

1. Suppose your professor says, "You can think of the limit of $f(x)$ as x approaches a as *what* $f(a)$ *should be*." Critique this statement. What does it mean? Does it provide important insight? Is there anything misleading about it? Replace the phrase in italics with your own best description of what the limit is.

2. Your friend's professor says, "The limit is a *prediction* of what $f(a)$ will be." Compare and contrast this statement to the one in exercise 1. Does the inclusion of the word *prediction* make the limit idea seem more useful and important?

3. We have observed that $\lim_{x\to a} f(x)$ does not depend on the actual value of $f(a)$, or even on whether $f(a)$ exists. In principle, functions such as $f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 13 & \text{if } x = 2 \end{cases}$ are as "normal" as functions such as $g(x) = x^2$. With this in mind, explain why it is important that the limit concept is independent of how (or whether) $f(a)$ is defined.

4. The most common limit encountered in everyday life is the *speed limit*. Describe how this type of limit is very different from the limits discussed in this section.

---

1. For the function graphed below, identify each limit or state that it does not exist.

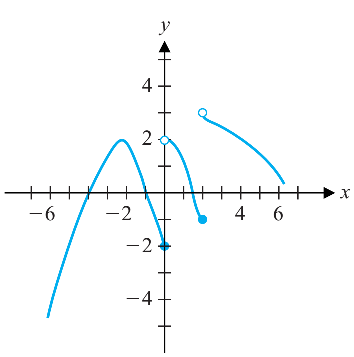
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 1^-} f(x)$

(e) $\lim_{x\to -1} f(x)$ (f) $\lim_{x\to 2^-} f(x)$

(g) $\lim_{x\to 2^+} f(x)$ (h) $\lim_{x\to 2} f(x)$

(i) $\lim_{x\to -2} f(x)$ (j) $\lim_{x\to 3} f(x)$



2. For the function graphed below, identify each limit or state that it does not exist.

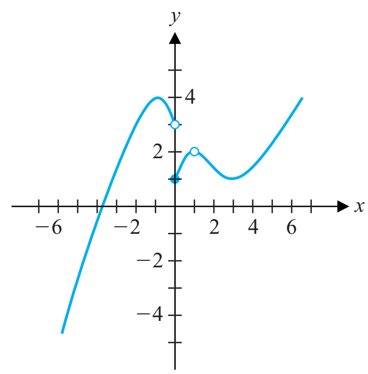
(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$ (d) $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x\to -2} f(x)$ (f) $\lim_{x\to 1^-} f(x)$

(g) $\lim_{x\to 1^+} f(x)$ (h) $\lim_{x\to 1} f(x)$

(i) $\lim_{x\to -1} f(x)$ (j) $\lim_{x\to 3} f(x)$



3. Sketch the graph of $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \geq 2 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to 2} f(x)$ (d) $\lim_{x\to 1} f(x)$

4. Sketch the graph of $f(x) = \begin{cases} x^3 - 1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ \sqrt{x+1} - 2 & \text{if } x > 0 \end{cases}$ and identify each limit.

(a) $\lim_{x\to 0^-} f(x)$ (b) $\lim_{x\to 0^+} f(x)$ (c) $\lim_{x\to 0} f(x)$

(d) $\lim_{x\to -1} f(x)$ (e) $\lim_{x\to 3} f(x)$

5. Sketch the graph of $f(x) = \begin{cases} x^2 + 1 & \text{if } x < -1 \\ 3x + 1 & \text{if } x \geq -1 \end{cases}$ and identify each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 1} f(x)$

6. Sketch the graph of $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 \le x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$ and identify each limit.

(a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$ (c) $\lim_{x\to -1} f(x)$
(d) $\lim_{x\to 1} f(x)$ (e) $\lim_{x\to 0} f(x)$

7. Evaluate $f(1.5)$, $f(1.1)$, $f(1.01)$ and $f(1.001)$, and conjecture a value for $\lim_{x\to 1^+} f(x)$ for $f(x) = \dfrac{x-1}{\sqrt{x}-1}$. Evaluate $f(0.5)$, $f(0.9)$, $f(0.99)$ and $f(0.999)$, and conjecture a value for $\lim_{x\to 1^-} f(x)$ for $f(x) = \dfrac{x-1}{\sqrt{x}-1}$. Does $\lim_{x\to 1} f(x)$ exist?

8. Evaluate $f(-1.5)$, $f(-1.1)$, $f(-1.01)$ and $f(-1.001)$, and conjecture a value for $\lim_{x\to -1^-} f(x)$ for $f(x) = \dfrac{x+1}{x^2-1}$. Evaluate $f(-0.5)$, $f(-0.9)$, $f(-0.99)$ and $f(-0.999)$, and conjecture a value for $\lim_{x\to -1^+} f(x)$ for $f(x) = \dfrac{x+1}{x^2-1}$. Does $\lim_{x\to -1} f(x)$ exist?

In exercises 9–14, use numerical and graphical evidence to conjecture values for each limit.

9. $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

10. $\lim_{x\to -1} \dfrac{x^2+x}{x^2-x-2}$

11. $\lim_{x\to 0} \dfrac{x^2+x}{\sin x}$

12. $\lim_{x\to \pi} \dfrac{\sin x}{x-\pi}$

13. $\lim_{x\to 0} \dfrac{\tan x}{\sin x}$

14. $\lim_{x\to 0} x \csc 2x$

In exercises 15–22, use numerical and graphical evidence to conjecture whether $\lim_{x\to a} f(x)$ exists. If not, describe what is happening at $x = a$ graphically.

15. $\lim_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$

16. $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

17. $\lim_{x\to 1} \dfrac{\sqrt{5-x}-2}{\sqrt{10-x}-3}$

18. $\lim_{x\to 0} \dfrac{x^2+4x}{\sqrt{x^3+x^2}}$

19. $\lim_{x\to 0} \sin\left(\dfrac{1}{x}\right)$

20. $\lim_{x\to 0} x \sin\left(\dfrac{1}{x}\right)$

21. $\lim_{x\to 2} \dfrac{x-2}{|x-2|}$

22. $\lim_{x\to -1} \dfrac{|x+1|}{x^2-1}$

23. Compute $\lim_{x\to 1} \dfrac{x^2+1}{x-1}$, $\lim_{x\to 2} \dfrac{x+1}{x^2-4}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are polynomials with $g(a) = 0$ and $f(a) \ne 0$. What can you conjecture about $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

24. Compute $\lim_{x\to -1} \dfrac{x+1}{x^2+1}$, $\lim_{x\to \pi} \dfrac{\sin x}{x}$ and similar limits to investigate the following. Suppose that $f(x)$ and $g(x)$ are functions with $f(a) = 0$ and $g(a) \ne 0$. What can you conjecture about $\lim_{x\to a} \dfrac{f(x)}{g(x)}$?

In exercises 25–28, sketch a graph of a function with the given properties.

25. $f(-1) = 2$, $f(0) = -1$, $f(1) = 3$ and $\lim_{x\to 1} f(x)$ does not exist.

26. $f(x) = 1$ for $-2 \le x \le 1$, $\lim_{x\to 1^+} f(x) = 3$ and $\lim_{x\to -2} f(x) = 1$.

27. $f(0) = 1$, $\lim_{x\to 0^-} f(x) = 2$ and $\lim_{x\to 0^+} f(x) = 3$.

28. $\lim_{x\to 0} f(x) = -2$, $f(0) = 1$, $f(2) = 3$ and $\lim_{x\to 2} f(x)$ does not exist.

29. As we see in Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is given by $m = \lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$. Estimate the slope m. Graph $y = \sqrt{x}$ and the line with slope m through the point (1, 1).

30. As we see in Chapter 2, the velocity of an object that has traveled $\sqrt{x}$ miles in x hours at the $x = 1$ hour mark is given by $v = \lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$. Estimate this limit.

31. Consider the following arguments concerning $\lim_{x\to 0^+} \sin \dfrac{\pi}{x}$. First, as $x > 0$ approaches 0, $\dfrac{\pi}{x}$ increases without bound; since $\sin t$ oscillates for increasing t, the limit does not exist. Second: taking $x = 1, 0.1, 0.01$ and so on, we compute $\sin \pi = \sin 10\pi = \sin 100\pi = \cdots = 0$; therefore the limit equals 0. Which argument sounds better to you? Explain. Explore the limit and determine which answer is correct.

32. Consider the following arguments concerning $\lim_{x\to 0^+} \dfrac{x^{-0.1}+2}{x^{-0.1}-1}$. First, as x approaches 0, $x^{-0.1}$ approaches 0 and the function values approach -2. Second, as x approaches 0, $x^{-0.1}$ increases and becomes much larger than 2 or -1. The function values approach $\dfrac{x^{-0.1}}{x^{-0.1}} = 1$. Explore the limit and determine which argument is correct.

33. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists but $f(0)$ does not exist. Give an example of a function g such that $g(0)$ exists but $\lim_{x\to 0} g(x)$ does not exist.

34. Give an example of a function f such that $\lim_{x\to 0} f(x)$ exists and $f(0)$ exists, but $\lim_{x\to 0} f(x) \ne f(0)$.

35. In the text, we described $\lim_{x\to a} f(x) = L$ as meaning "as x gets closer and closer to a, $f(x)$ is getting closer and closer to L." As x gets closer and closer to 0, it is true that x^2 gets closer and closer to -0.01, but it is certainly not true that $\lim_{x\to 0} x^2 = -0.01$. Try to modify the description of limit to

make it clear that $\lim_{x\to 0} x^2 \neq -0.01$. We explore a very precise definition of limit in section 1.6.

36. In Figure 1.13, the final position of the knuckleball at time $t = 0.68$ is shown as a function of the rotation rate ω. The batter must decide at time $t = 0.4$ whether to swing at the pitch. At $t = 0.4$, the left/right position of the ball is given by $h(\omega) = \frac{1}{\omega} - \frac{5}{8\omega^2}\sin(1.6\omega)$. Graph $h(\omega)$ and compare to Figure 1.13. Conjecture the limit of $h(\omega)$ as $\omega \to 0$. For $\omega = 0$, is there any difference in ball position between what the batter sees at $t = 0.4$ and what he tries to hit at $t = 0.68$?

37. A parking lot charges \$2 for each hour or portion of an hour, with a maximum charge of \$12 for all day. If $f(t)$ equals the total parking bill for t hours, sketch a graph of $y = f(t)$ for $0 \le t \le 24$. Determine the limits $\lim_{t\to 3.5} f(t)$ and $\lim_{t\to 4} f(t)$, if they exist.

38. For the parking lot in exercise 37, determine all values of a with $0 \le a \le 24$ such that $\lim_{t\to a} f(t)$ does not exist. Briefly discuss the effect this has on your parking strategy (e.g., are there times where you would be in a hurry to move your car or times where it doesn't matter whether you move your car?).

EXPLORATORY EXERCISES

1. In a situation similar to that of example 2.6, the left/right position of a knuckleball pitch in baseball can be modeled by $P = \frac{5}{8\omega^2}(1 - \cos 4\omega t)$, where t is time measured in seconds $(0 \le t \le 0.68)$ and ω is the rotation rate of the ball measured in radians per second. In example 2.6, we chose a specific t-value and evaluated the limit as $\omega \to 0$. While this gives us some information about which rotation rates produce hard-to-hit pitches, a clearer picture emerges if we look at P over its entire domain. Set $\omega = 10$ and graph the resulting function $\frac{1}{160}(1 - \cos 40t)$ for $0 \le t \le 0.68$. Imagine looking at a pitcher from above and try to visualize a baseball starting at the pitcher's hand at $t = 0$ and finally reaching the batter, at $t = 0.68$. Repeat this with $\omega = 5$, $\omega = 1$, $\omega = 0.1$ and whatever values of ω you think would be interesting. Which values of ω produce hard-to-hit pitches?

2. In this exercise, the results you get will depend on the accuracy of your computer or calculator. Work this exercise and compare your results with your classmates' results. We will investigate $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$. Start with the calculations presented in the table (your results may vary):

x	$f(x)$
0.1	$-0.499583\ldots$
0.01	$-0.49999583\ldots$
0.001	$-0.4999999583\ldots$

Describe as precisely as possible the pattern shown here. What would you predict for $f(0.0001)$? $f(0.00001)$? Does your computer or calculator give you this answer? If you continue trying powers of 0.1 (0.000001, 0.0000001 etc.) you should eventually be given a displayed result of -0.5. Do you think this is exactly correct or has the answer just been rounded off? Why is rounding off inescapable? It turns out that -0.5 is the exact value for the limit, so the round-off here is somewhat helpful. However, if you keep evaluating the function at smaller and smaller values of x, you will eventually see a reported function value of 0. This round-off error is not so benign; we discuss this error in section 1.7. For now, evaluate $\cos x$ at the current value of x and try to explain where the 0 came from.

1.3 COMPUTATION OF LIMITS

Now that you have an idea of what a limit is, we need to develop some means of calculating limits of simple functions. In this section, we present some basic rules for dealing with common limit problems. We begin with two simple limits.

For any constant c and any real number a,

$$\lim_{x\to a} c = c. \tag{3.1}$$

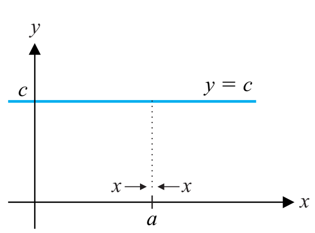


FIGURE 1.14
$\lim_{x\to a} c = c$

In other words, the limit of a constant is that constant. This certainly comes as no surprise, since the function $f(x) = c$ does not depend on x and so, stays the same as $x \to a$. (See Figure 1.14.) Another simple limit is the following.

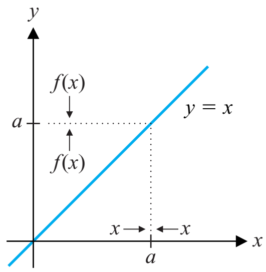


FIGURE 1.15

$\lim_{x \to a} x = a$

For any real number a,

$$\lim_{x \to a} x = a. \tag{3.2}$$

Again, this is not a surprise, since as $x \to a$, x will approach a. (See Figure 1.15.) Be sure that you are comfortable enough with the limit notation to recognize how obvious the limits in (3.1) and (3.2) are. As simple as they are, we use them repeatedly in finding more complex limits. We also need the basic rules contained in Theorem 3.1.

THEOREM 3.1

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and let c be any constant. The following then apply:

(i) $\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$,

(ii) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right]$ and

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \left(\text{if } \lim_{x \to a} g(x) \neq 0\right)$.

The proof of Theorem 3.1 is found in Appendix A and requires the formal definition of limit discussed in section 1.6. You should think of these rules as sensible results that you would certainly expect to be true, given your intuitive understanding of what a limit is. Read them in plain English. For instance, part (ii) says that the limit of a sum (or a difference) equals the sum (or difference) of the limits, *provided the limits exist*. Think of this as follows. If as x approaches a, $f(x)$ approaches L and $g(x)$ approaches M, then $f(x) + g(x)$ should approach $L + M$.

Observe that by applying part (iii) of Theorem 3.1 with $g(x) = f(x)$, we get that, whenever $\lim_{x \to a} f(x)$ exists,

$$\begin{aligned}\lim_{x \to a} [f(x)]^2 &= \lim_{x \to a} [f(x) \cdot f(x)] \\ &= \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} f(x)\right] = \left[\lim_{x \to a} f(x)\right]^2.\end{aligned}$$

Likewise, for any positive integer n, we can apply part (iii) of Theorem 3.1 repeatedly, to yield

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n \tag{3.3}$$

(see exercises 60 and 61).

Notice that taking $f(x) = x$ in (3.3) gives us that for any integer $n > 0$ and any real number a,

$$\lim_{x \to a} x^n = a^n. \tag{3.4}$$

That is, to compute the limit of any positive power of x, you simply substitute in the value of x being approached.

EXAMPLE 3.1 Finding the Limit of a Polynomial

Apply the rules of limits to evaluate $\lim_{x\to 2}(3x^2 - 5x + 4)$.

Solution We have

$$\begin{aligned}\lim_{x\to 2}(3x^2 - 5x + 4) &= \lim_{x\to 2}(3x^2) - \lim_{x\to 2}(5x) + \lim_{x\to 2} 4 && \text{By Theorem 3.1 (ii).}\\ &= 3\lim_{x\to 2} x^2 - 5\lim_{x\to 2} x + 4 && \text{By Theorem 3.1 (i).}\\ &= 3\cdot(2)^2 - 5\cdot 2 + 4 = 6. && \text{By (3.4).}\end{aligned}$$

■

EXAMPLE 3.2 Finding the Limit of a Rational Function

Apply the rules of limits to evaluate $\lim_{x\to 3}\dfrac{x^3 - 5x + 4}{x^2 - 2}$.

Solution We get

$$\begin{aligned}\lim_{x\to 3}\frac{x^3 - 5x + 4}{x^2 - 2} &= \frac{\lim_{x\to 3}(x^3 - 5x + 4)}{\lim_{x\to 3}(x^2 - 2)} && \text{By Theorem 3.1 (iv).}\\ &= \frac{\lim_{x\to 3} x^3 - 5\lim_{x\to 3} x + \lim_{x\to 3} 4}{\lim_{x\to 3} x^2 - \lim_{x\to 3} 2} && \text{By Theorem 3.1 (i) and (ii).}\\ &= \frac{3^3 - 5\cdot 3 + 4}{3^2 - 2} = \frac{16}{7}. && \text{By (3.4).}\end{aligned}$$

■

You may have noticed that in examples 3.1 and 3.2, we simply ended up substituting the value for x, after taking many intermediate steps. In example 3.3, it's not quite so simple.

EXAMPLE 3.3 Finding a Limit by Factoring

Evaluate $\lim_{x\to 1}\dfrac{x^2 - 1}{1 - x}$.

Solution Notice right away that

$$\lim_{x\to 1}\frac{x^2 - 1}{1 - x} \neq \frac{\lim_{x\to 1}(x^2 - 1)}{\lim_{x\to 1}(1 - x)},$$

since the limit in the denominator is zero. (Recall that the limit of a quotient is the quotient of the limits *only* when both limits exist *and* the limit in the denominator is *not* zero.) We can resolve this problem by observing that

$$\begin{aligned}\lim_{x\to 1}\frac{x^2 - 1}{1 - x} &= \lim_{x\to 1}\frac{(x - 1)(x + 1)}{-(x - 1)} && \text{Factoring the numerator and factoring } -1 \text{ from the denominator.}\\ &= \lim_{x\to 1}\frac{(x + 1)}{-1} = -2, && \text{Simplifying and substituting } x = 1.\end{aligned}$$

where the cancellation of the factors of $(x - 1)$ is valid because in the limit as $x \to 1$, x is close to 1, but $x \neq 1$, so that $x - 1 \neq 0$. ■

In Theorem 3.2, we show that the limit of a polynomial at a point is simply the value of the polynomial at that point; that is, to find the limit of a polynomial, we simply substitute in the value that x is approaching.

THEOREM 3.2

For any polynomial $p(x)$ and any real number a,

$$\lim_{x\to a} p(x) = p(a).$$

PROOF

Suppose that $p(x)$ is a polynomial of degree $n \geq 0$,

$$p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0.$$

Then, from Theorem 3.1 and (3.4),

$$\begin{aligned}\lim_{x\to a} p(x) &= \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)\\ &= c_n \lim_{x\to a} x^n + c_{n-1}\lim_{x\to a} x^{n-1} + \cdots + c_1 \lim_{x\to a} x + \lim_{x\to a} c_0\\ &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = p(a). \quad \blacksquare\end{aligned}$$

Evaluating the limit of a polynomial is now easy. Many other limits are evaluated just as easily.

THEOREM 3.3

Suppose that $\lim_{x\to a} f(x) = L$ and n is any positive integer. Then,

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

where for n even, we assume that $L > 0$.

The proof of Theorem 3.3 is given in Appendix A. Notice that this result says that we may (under the conditions outlined in the hypotheses) bring limits "inside" nth roots. We can then use our existing rules for computing the limit inside.

EXAMPLE 3.4 Evaluating the Limit of an nth Root of a Polynomial

Evaluate $\lim_{x\to 2} \sqrt[5]{3x^2 - 2x}$.

Solution By Theorems 3.2 and 3.3, we have

$$\lim_{x\to 2} \sqrt[5]{3x^2 - 2x} = \sqrt[5]{\lim_{x\to 2}(3x^2 - 2x)} = \sqrt[5]{8}. \quad \blacksquare$$

REMARK 3.1

In general, in any case where the limits of both the numerator and the denominator are 0, you should try to algebraically simplify the expression, to get a cancellation, as we do in examples 3.3 and 3.5.

EXAMPLE 3.5 Finding a Limit by Rationalizing

Evaluate $\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$.

Solution First, notice that both the numerator ($\sqrt{x+2}-\sqrt{2}$) and the denominator (x) approach 0 as x approaches 0. Unlike example 3.3, we can't factor the numerator. However, we can rationalize the numerator, as follows:

$$\frac{\sqrt{x+2}-\sqrt{2}}{x} = \frac{(\sqrt{x+2}-\sqrt{2})(\sqrt{x+2}+\sqrt{2})}{x(\sqrt{x+2}+\sqrt{2})} = \frac{x+2-2}{x(\sqrt{x+2}+\sqrt{2})}$$
$$= \frac{x}{x(\sqrt{x+2}+\sqrt{2})} = \frac{1}{\sqrt{x+2}+\sqrt{2}},$$

where the last equality holds if $x \neq 0$ (which is the case in the limit as $x \to 0$). So, we have

$$\lim_{x\to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

So that we are not restricted to discussing only the algebraic functions (i.e., those that can be constructed by using addition, subtraction, multiplication, division, exponentiation and by taking nth roots), we state the following result now, without proof.

THEOREM 3.4

For any real number a, we have

(i) $\lim_{x\to a} \sin x = \sin a$,

(ii) $\lim_{x\to a} \cos x = \cos a$ and

(iii) if p is a polynomial and $\lim_{x\to p(a)} f(x) = L$, then $\lim_{x\to a} f(p(x)) = L$.

Notice that Theorem 3.4 says that limits of the sine and cosine functions are found simply by substitution. A more thorough discussion of functions with this property (called *continuity*) is found in section 1.4.

EXAMPLE 3.6 Evaluating a Limit of a Trigonometric Function

Evaluate $\lim_{x\to 0} \sin\left(\frac{x^3+\pi}{2}\right)$.

Solution By Theorem 3.4, we have

$$\lim_{x\to 0} \sin\left(\frac{x^3+\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

So much for limits that we can compute using elementary rules. Many limits can be found only by using more careful analysis, often by an indirect approach. For instance, consider the problem in example 3.7.

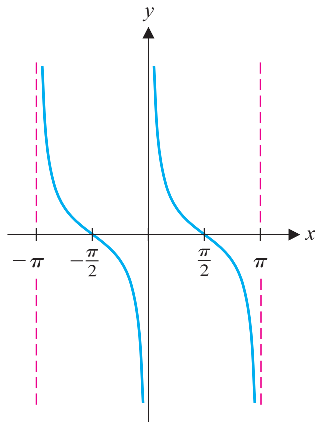


FIGURE 1.16
$y = \cot x$

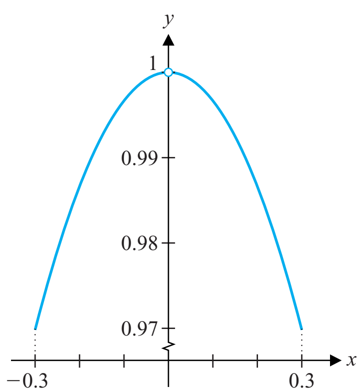


FIGURE 1.17
$y = x \cot x$

x	$x \cot x$
± 0.1	0.9967
± 0.01	0.999967
± 0.001	0.99999967
± 0.0001	0.9999999967
± 0.00001	0.999999999967

EXAMPLE 3.7 A Limit of a Product That Is Not the Product of the Limits

Evaluate $\lim_{x\to 0}(x \cot x)$.

Solution Your first reaction might be to say that this is a limit of a product and so, must be the product of the limits:

$$\lim_{x\to 0}(x \cot x) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \cot x\right) \quad \text{This is incorrect!}$$
$$= 0 \cdot ? = 0, \tag{3.5}$$

where we've written a "?" since you probably don't know what to do with $\lim_{x\to 0} \cot x$. Since the first limit is 0, do we really need to worry about the second limit? The problem here is that we are attempting to apply the result of Theorem 3.1 in a case where the hypotheses are not satisfied. Specifically, Theorem 3.1 says that the limit of a product is the product of the respective limits *when all of the limits exist*. The graph in Figure 1.16 suggests that $\lim_{x\to 0} \cot x$ does not exist. You should compute some function values, as well, to convince yourself that this is in fact the case. So, equation (3.5) does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph (see Figure 1.17) and compute some function values. Based on these, we conjecture that

$$\lim_{x\to 0}(x \cot x) = 1,$$

which is definitely not 0, as you might have initially suspected. You can also think about this limit as follows:

$$\lim_{x\to 0}(x \cot x) = \lim_{x\to 0}\left(x\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\left(\frac{x}{\sin x}\cos x\right)$$
$$= \left(\lim_{x\to 0}\frac{x}{\sin x}\right)\left(\lim_{x\to 0}\cos x\right)$$
$$= \frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1,$$

since $\lim_{x\to 0}\cos x = 1$ and where we have used the conjecture we made in example 2.4 that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. (We verify this last conjecture in section 2.6, using the Squeeze Theorem, which follows.) ■

At this point, we introduce a tool that will help us determine a number of important limits.

THEOREM 3.5 (Squeeze Theorem)

Suppose that

$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$ and that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x\to a} g(x) = L, \text{ also.}$$

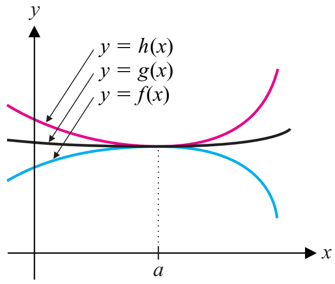


FIGURE 1.18
The Squeeze Theorem

The proof of Theorem 3.5 is given in Appendix A, since it depends on the precise definition of limit found in section 1.6. However, if you refer to Figure 1.18, you should clearly see that if $g(x)$ lies between $f(x)$ and $h(x)$, except possibly at a itself and both $f(x)$ and $h(x)$ have the same limit as $x \to a$, then $g(x)$ gets *squeezed* between $f(x)$ and $h(x)$ and therefore should also have a limit of L. The challenge in using the Squeeze Theorem is in finding appropriate functions f and h that bound a given function g from below and above, respectively, and that have the same limit as $x \to a$.

REMARK 3.2

The Squeeze Theorem also applies to one-sided limits.

EXAMPLE 3.8 Using the Squeeze Theorem to Verify the Value of a Limit

Determine the value of $\lim_{x \to 0} \left[x^2 \cos\left(\frac{1}{x}\right) \right]$.

Solution Your first reaction might be to observe that this is a limit of a product and so, might be the product of the limits:

$$\lim_{x \to 0} \left[x^2 \cos\left(\frac{1}{x}\right) \right] \stackrel{?}{=} \left(\lim_{x \to 0} x^2 \right) \left[\lim_{x \to 0} \cos\left(\frac{1}{x}\right) \right]. \quad \text{This is incorrect!} \tag{3.6}$$

However, the graph of $y = \cos\left(\frac{1}{x}\right)$ found in Figure 1.19 suggests that $\cos\left(\frac{1}{x}\right)$ oscillates back and forth between -1 and 1. Further, the closer x gets to 0, the more rapid the oscillations become. You should compute some function values, as well, to convince yourself that $\lim_{x \to 0} \cos\left(\frac{1}{x}\right)$ does not exist. Equation (3.6) then does not hold and we're back to square one. Since none of our rules seem to apply here, the most reasonable step is to draw a graph and compute some function values in an effort to see what is going on. The graph of $y = x^2 \cos\left(\frac{1}{x}\right)$ appears in Figure 1.20 and a table of function values is shown in the margin.

x	$x^2 \cos(1/x)$
± 0.1	-0.008
± 0.01	8.6×10^{-5}
± 0.001	5.6×10^{-7}
± 0.0001	-9.5×10^{-9}
± 0.00001	-9.99×10^{-11}

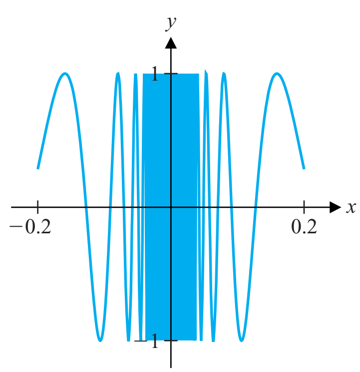


FIGURE 1.19
$y = \cos\left(\frac{1}{x}\right)$

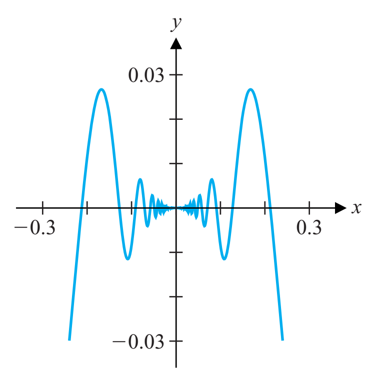


FIGURE 1.20
$y = x^2 \cos\left(\frac{1}{x}\right)$

The graph and the table of function values suggest the conjecture

$$\lim_{x \to 0} \left[x^2 \cos\left(\frac{1}{x}\right) \right] = 0,$$

which we prove using the Squeeze Theorem. First, we need to find functions f and h such that

$$f(x) \le x^2 \cos\left(\frac{1}{x}\right) \le h(x),$$

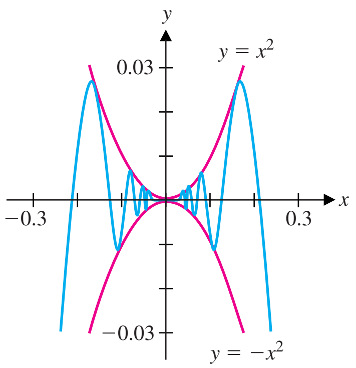


FIGURE 1.21

$y = x^2 \cos\left(\frac{1}{x}\right)$, $y = x^2$ and $y = -x^2$

for all $x \neq 0$ and where $\lim_{x \to 0} f(x) = \lim_{x \to 0} h(x) = 0$. Recall that

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1, \qquad (3.7)$$

for all $x \neq 0$. If we multiply (3.7) through by x^2 (notice that since $x^2 \geq 0$, this multiplication preserves the inequalities), we get

$$-x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2,$$

for all $x \neq 0$. We illustrate this inequality in Figure 1.21. Further,

$$\lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0} x^2.$$

So, from the Squeeze Theorem, it now follows that

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0,$$

also, as we had conjectured. ■

BEYOND FORMULAS

To resolve the limit in example 3.8, we could not apply the rules for limits contained in Theorem 3.1. So, we resorted to an indirect method of finding the limit. This tour de force of graphics plus calculation followed by analysis is sometimes referred to as the **Rule of Three.** (The Rule of Three presents a general strategy for attacking new problems. The basic idea is to look at problems graphically, numerically and analytically.) In the case of example 3.8, the first two elements of this "rule" (the graphics in Figure 1.20 and the accompanying table of function values) suggest a plausible conjecture, while the third element provides us with a careful mathematical verification of the conjecture. In what ways does this sound like the scientific method?

TODAY IN MATHEMATICS

Michael Freedman (1951–)
An American mathematician who first solved one of the most famous problems in mathematics, the four-dimensional Poincaré conjecture. A winner of the Fields Medal, the mathematical equivalent of the Nobel Prize, Freedman says, "Much of the power of mathematics comes from combining insights from seemingly different branches of the discipline. Mathematics is not so much a collection of different subjects as a way of thinking. As such, it may be applied to any branch of knowledge." Freedman finds mathematics to be an open field for research, saying that, "It isn't necessary to be an old hand in an area to make a contribution."

Functions are often defined by different expressions on different intervals. Such **piecewise-defined** functions are important and we illustrate such a function in example 3.9.

EXAMPLE 3.9 A Limit for a Piecewise-Defined Function

Evaluate $\lim_{x \to 0} f(x)$, where f is defined by

$$f(x) = \begin{cases} x^2 + 2\cos x + 1, & \text{for } x < 0 \\ \sec x - 4, & \text{for } x \geq 0 \end{cases}.$$

Solution Since f is defined by different expressions for $x < 0$ and for $x \geq 0$, we must consider one-sided limits. We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x^2 + 2\cos x + 1) = 2\cos 0 + 1 = 3,$$

by Theorem 3.4. Also, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\sec x - 4) = \sec 0 - 4 = 1 - 4 = -3.$$

Since the one-sided limits are different, we have that $\lim_{x \to 0} f(x)$ does not exist. ■

We end this section with an example of the use of limits in computing velocity. In section 2.1, we see that for an object moving in a straight line, whose position at time t is given by the function $f(t)$, the instantaneous velocity of that object at time $t = 1$ (i.e., the velocity at the *instant* $t = 1$, as opposed to the average velocity over some period of time) is given by the limit

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}.$$

EXAMPLE 3.10 Evaluating a Limit Describing Velocity

Suppose that the position function for an object at time t (seconds) is given by

$$f(t) = t^2 + 2 \text{ (feet)}.$$

Find the instantaneous velocity of the object at time $t = 1$.

Solution Given what we have just learned about limits, this is now an easy problem to solve. We have

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[(1+h)^2 + 2] - 3}{h}.$$

While we can't simply substitute $h = 0$ (why not?), we can write

$$\lim_{h \to 0} \frac{[(1+h)^2 + 2] - 3}{h} = \lim_{h \to 0} \frac{(1 + 2h + h^2) - 1}{h}$$ Expanding the squared term.

$$= \lim_{h \to 0} \frac{2h + h^2}{h} = \lim_{h \to 0} \frac{h(2+h)}{h}$$

$$= \lim_{h \to 0} \frac{2+h}{1} = 2.$$ Canceling factors of h.

So, the instantaneous velocity of this object at time $t = 1$ is 2 feet per second. ■

EXERCISES 1.3

WRITING EXERCISES

1. Given your knowledge of the graphs of polynomials, explain why equations (3.1) and (3.2) and Theorem 3.2 are obvious. Name five non-polynomial functions for which limits can be evaluated by substitution.
2. Suppose that you can draw the graph of $y = f(x)$ without lifting your pencil from your paper. Explain why $\lim_{x \to a} f(x) = f(a)$, for every value of a.
3. In one or two sentences, explain the Squeeze Theorem. Use a real-world analogy (e.g., having the functions represent the locations of three people as they walk) to indicate why it is true.
4. Given the graph in Figure 1.20 and the calculations in the accompanying table, it may be unclear why we insist on using the Squeeze Theorem before concluding that $\lim_{x \to 0} [x^2 \cos(1/x)]$ is indeed 0. Review section 1.2 to explain why we are being so fussy.

In exercises 1–34, evaluate the indicated limit, if it exists. Assume that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

1. $\lim_{x \to 0} (x^2 - 3x + 1)$
2. $\lim_{x \to 2} \sqrt[3]{2x + 1}$
3. $\lim_{x \to 0} \tan(x^2)$
4. $\lim_{x \to 2} \frac{x - 5}{x^2 + 4}$
5. $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$
6. $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2}$
7. $\lim_{x \to 2} \frac{x^2 - x - 2}{x^2 - 4}$
8. $\lim_{x \to 1} \frac{x^3 - 1}{x^2 + 2x - 3}$
9. $\lim_{x \to 0} \frac{\sin x}{\tan x}$
10. $\lim_{x \to 0} \frac{\tan x}{x}$
11. $\lim_{x \to 0} \frac{x \cos(-2x + 1)}{x^2 + x}$
12. $\lim_{x \to 0^+} x^2 \csc^2 x$

13. $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

14. $\lim\limits_{x\to 0} \dfrac{2x}{3-\sqrt{x+9}}$

15. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x}-1}$

16. $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$

17. $\lim\limits_{x\to 1} \left(\dfrac{1}{x-1} - \dfrac{2}{x^2-1}\right)$

18. $\lim\limits_{x\to 0} \left(\dfrac{2}{x} - \dfrac{2}{|x|}\right)$

19. $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{1-\cos x}$

20. $\lim\limits_{x\to 0} \sin\left(\dfrac{1}{x^2}\right)$

21. $\lim\limits_{x\to 0} \dfrac{\sin|x|}{x}$

22. $\lim\limits_{x\to 0} \dfrac{\sin^2(x^2)}{x^4}$

23. $\lim\limits_{x\to 2} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

24. $\lim\limits_{x\to -1} f(x)$, where $f(x) = \begin{cases} x^2+1 & \text{if } x < -1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

25. $\lim\limits_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2+2 & \text{if } x < -1 \\ 3x+1 & \text{if } x \ge -1 \end{cases}$

26. $\lim\limits_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x & \text{if } x < 2 \\ x^2 & \text{if } x \ge 2 \end{cases}$

27. $\lim\limits_{x\to -1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

28. $\lim\limits_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x+1 & \text{if } x < -1 \\ 3 & \text{if } -1 < x < 1 \\ 2x+1 & \text{if } x > 1 \end{cases}$

29. $\lim\limits_{h\to 0} \dfrac{(2+h)^2-4}{h}$

30. $\lim\limits_{h\to 0} \dfrac{(1+h)^3-1}{h}$

31. $\lim\limits_{h\to 0} \dfrac{h^2}{\sqrt{h^2+h+3}-\sqrt{h+3}}$

32. $\lim\limits_{x\to 0} \dfrac{\sqrt{x^2+x+4}-2}{x^2+x}$

33. $\lim\limits_{t\to -2} \dfrac{\frac{1}{2}+\frac{1}{t}}{2+t}$

34. $\lim\limits_{x\to 0} \dfrac{\tan 2x}{5x}$

35. Use numerical and graphical evidence to conjecture the value of $\lim\limits_{x\to 0} x^2 \sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions f and h, show graphically that $f(x) \le x^2 \sin(1/x) \le h(x)$ and justify $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} h(x)$.

36. Why can't you use the Squeeze Theorem as in exercise 35 to prove that $\lim\limits_{x\to 0} x^2 \sec(1/x) = 0$? Explore this limit graphically.

37. Use the Squeeze Theorem to prove that $\lim\limits_{x\to 0^+} [\sqrt{x}\cos^2(1/x)] = 0$. Identify the functions f and h, show graphically that $f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$ for all $x > 0$ and justify $\lim\limits_{x\to 0^+} f(x) = 0$ and $\lim\limits_{x\to 0^+} h(x) = 0$.

38. Suppose that $f(x)$ is bounded: that is, there exists a constant M such that $|f(x)| \le M$ for all x. Use the Squeeze Theorem to prove that $\lim\limits_{x\to 0} x^2 f(x) = 0$.

In exercises 39–42, either find the limit or explain why it does not exist.

39. $\lim\limits_{x\to 4^+} \sqrt{16-x^2}$

40. $\lim\limits_{x\to 4^-} \sqrt{16-x^2}$

41. $\lim\limits_{x\to -2^-} \sqrt{x^2+3x+2}$

42. $\lim\limits_{x\to -2^+} \sqrt{x^2+3x+2}$

43. Given that $\lim\limits_{x\to 0^+} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$, quickly evaluate $\lim\limits_{x\to 0^+} \dfrac{\sqrt{1-\cos x}}{x}$.

44. Given that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$, quickly evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2}$.

45. Suppose $f(x) = \begin{cases} g(x) & \text{if } x < a \\ h(x) & \text{if } x > a \end{cases}$ for polynomials $g(x)$ and $h(x)$. Explain why $\lim\limits_{x\to a^-} f(x) = g(a)$ and determine $\lim\limits_{x\to a^+} f(x)$.

46. Explain how to determine $\lim\limits_{x\to a} f(x)$ if g and h are polynomials and $f(x) = \begin{cases} g(x) & \text{if } x < a \\ c & \text{if } x = a \\ h(x) & \text{if } x > a \end{cases}$.

47. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim\limits_{x\to 2}(x^2-3x+1)$ (b) $\lim\limits_{x\to 0} \dfrac{x-2}{x^2+1}$

48. Evaluate each limit and justify each step by citing the appropriate theorem or equation.

(a) $\lim\limits_{x\to -1} [(x+1)\sin x]$ (b) $\lim\limits_{x\to 1} \dfrac{x\cos x}{\tan x}$

In exercises 49–52, use the given position function $f(t)$ to find the velocity at time $t = a$.

49. $f(t) = t^2+2, a = 2$

50. $f(t) = t^2+2, a = 0$

51. $f(t) = t^3, a = 0$

52. $f(t) = t^3, a = 1$

53. In Chapter 2, the slope of the tangent line to the curve $y = \sqrt{x}$ at $x = 1$ is defined by $m = \lim\limits_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$. Compute the slope m. Graph $y = \sqrt{x}$ and the line with slope m through the point $(1, 1)$.

54. In Chapter 2, an alternative form for the limit in exercise 53 is given by $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$. Compute this limit.

55. Use numerical evidence to conjecture the value of $\lim\limits_{x\to 0^+} \cot x$ if it exists. Check your answer with your Computer Algebra System (CAS). If you disagree, which one of you is correct?

In exercises 56–59, use $\lim\limits_{x\to a} f(x) = 2$, $\lim\limits_{x\to a} g(x) = -3$ and $\lim\limits_{x\to a} h(x) = 0$ to determine the limit, if possible.

56. $\lim\limits_{x\to a}[2f(x)-3g(x)]$

57. $\lim\limits_{x\to a}[3f(x)g(x)]$

58. $\lim\limits_{x\to a}\left[\dfrac{f(x)+g(x)}{h(x)}\right]$

59. $\lim\limits_{x\to a}\left[\dfrac{3f(x)+2g(x)}{h(x)}\right]$

60. Assume that $\lim\limits_{x\to a} f(x) = L$. Use Theorem 3.1 to prove that $\lim\limits_{x\to a}[f(x)]^3 = L^3$. Also, show that $\lim\limits_{x\to a}[f(x)]^4 = L^4$.

61. How did you work exercise 60? You probably used Theorem 3.1 to work from $\lim\limits_{x\to a}[f(x)]^2 = L^2$ to $\lim\limits_{x\to a}[f(x)]^3 = L^3$

and then used $\lim_{x\to a}[f(x)]^3 = L^3$ to get $\lim_{x\to a}[f(x)]^4 = L^4$. Going one step at a time, we should be able to reach $\lim_{x\to a}[f(x)]^n = L^n$, for any positive integer n. This is the idea of **mathematical induction.** Formally, we need to show the result is true for a specific value of $n = n_0$ [we show $n_0 = 2$ in the text], then assume the result is true for a general $n = k \geq n_0$. If we show that we can get from the result being true for $n = k$ to the result being true for $n = k + 1$, we have proved that the result is true for any positive integer n. In one sentence, explain why this is true. Use this technique to prove that $\lim_{x\to a}[f(x)]^n = L^n$, for any positive integer n.

62. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{1}{x} = \lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0} x \lim_{x\to 0}\frac{1}{x^2} = 0 \cdot ? = 0.$$

63. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \frac{0}{0} = 1.$$

64. Give an example of functions f and g such that $\lim_{x\to 0}[f(x)+g(x)]$ exists, but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

65. Give an example of functions f and g such that $\lim_{x\to 0}[f(x)\cdot g(x)]$ exists, but at least one of $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

66. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, is it always true that $\lim_{x\to a}[f(x) + g(x)]$ does not exist? Explain.

67. Is the following true or false? If $\lim_{x\to 0} f(x)$ does not exist, then $\lim_{x\to 0}\frac{1}{f(x)}$ does not exist. Explain.

68. Suppose a state's income tax code states the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0.14x & \text{if } 0 \leq x < 10{,}000 \\ 1500 + 0.21x & \text{if } 10{,}000 \leq x \end{cases}.$$

Compute $\lim_{x\to 0^+} T(x)$; why is this good? Compute $\lim_{x\to 10{,}000} T(x)$; why is this bad?

69. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} a + 0.12x & \text{if } x \leq 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $\lim_{x\to 0^+} T(x) = 0$ and $\lim_{x\to 20{,}000} T(x)$ exists. Why is it important for these limits to exist?

70. The **greatest integer function** is denoted by $f(x) = [x]$ and equals the greatest integer that is less than or equal to x. Thus, $[2.3] = 2$, $[-1.2] = -2$ and $[3] = 3$. In spite of this last fact, show that $\lim_{x\to 3}[x]$ does not exist.

71. Investigate the existence of (a) $\lim_{x\to 1}[x]$, (b) $\lim_{x\to 1.5}[x]$, (c) $\lim_{x\to 1.5}[2x]$ and (d) $\lim_{x\to 1}(x - [x])$.

EXPLORATORY EXERCISES

1. The value $x = 0$ is called a **zero of multiplicity** $\boldsymbol{n}$ $(n \geq 1)$ for the function f if $\lim_{x\to 0}\frac{f(x)}{x^n}$ exists and is nonzero but $\lim_{x\to 0}\frac{f(x)}{x^{n-1}} = 0$. Show that $x = 0$ is a zero of multiplicity 2 for x^2, $x = 0$ is a zero of multiplicity 3 for x^3 and $x = 0$ is a zero of multiplicity 4 for x^4. For polynomials, what does multiplicity describe? The reason the definition is not as straightforward as we might like is so that it can apply to nonpolynomial functions, as well. Find the multiplicity of $x = 0$ for $f(x) = \sin x$; $f(x) = x\sin x$; $f(x) = \sin x^2$. If you know that $x = 0$ is a zero of multiplicity m for $f(x)$ and multiplicity n for $g(x)$, what can you say about the multiplicity of $x = 0$ for $f(x) + g(x)$? $f(x)\cdot g(x)$? $f(g(x))$?

2. We have conjectured that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Using graphical and numerical evidence, conjecture the value of $\lim_{x\to 0}\frac{\sin 2x}{x}$, $\lim_{x\to 0}\frac{\sin 3x}{x}$, $\lim_{x\to 0}\frac{\sin \pi x}{x}$ and $\lim_{x\to 0}\frac{\sin x/2}{x}$. In general, conjecture the value of $\lim_{x\to 0}\frac{\sin cx}{x}$ for any constant c. Given that $\lim_{x\to 0}\frac{\sin cx}{cx} = 1$, for any constant $c \neq 0$, prove that your conjecture is correct.

1.4 CONTINUITY AND ITS CONSEQUENCES

When you describe something as *continuous,* just what do you have in mind? For example, if told that a machine has been in *continuous* operation for the past 60 hours, most of us would interpret this to mean that the machine has been in operation *all* of that time, without any interruption at all, even for a moment. Mathematicians mean much the same thing when we say that a function is continuous. A function is said to be *continuous* on an interval if its graph on that interval can be drawn without interruption, that is, without lifting your pencil from the paper.

It is helpful for us to first try to see what it is about the functions whose graphs are shown in Figures 1.22a–1.22d that makes them *discontinuous* (i.e., not continuous) at the point $x = a$.

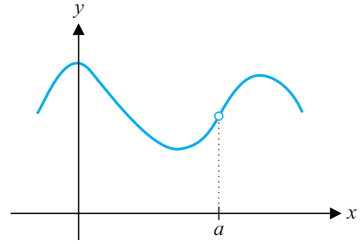


FIGURE 1.22a
$f(a)$ is not defined (the graph has a hole at $x = a$).

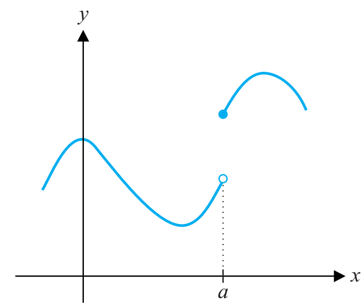


FIGURE 1.22b
$f(a)$ is defined, but $\lim_{x\to a} f(x)$ does not exist (the graph has a jump at $x = a$).

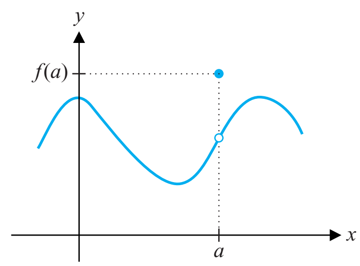


FIGURE 1.22c
$\lim_{x\to a} f(x)$ exists and $f(a)$ is defined, but $\lim_{x\to a} f(x) \neq f(a)$ (the graph has a hole at $x = a$).

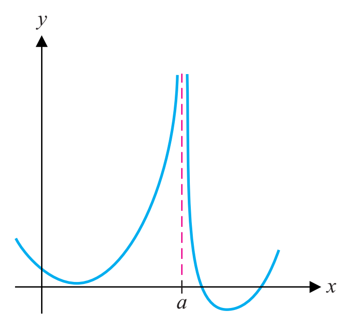


FIGURE 1.22d
$\lim_{x\to a} f(x)$ does not exist (the function "blows up" at $x = a$).

This suggests the following definition of continuity at a point.

REMARK 4.1

The definition of continuity all boils down to the one condition in (iii), since conditions (i) and (ii) must hold whenever (iii) is met. Further, this says that a function is continuous at a point exactly when you can compute its limit at that point by simply substituting in.

DEFINITION 4.1

A function f is **continuous** at $x = a$ when
(i) $f(a)$ is defined, (ii) $\lim_{x\to a} f(x)$ exists and (iii) $\lim_{x\to a} f(x) = f(a)$.
Otherwise, f is said to be **discontinuous** at $x = a$.

For most purposes, it is best for you to think of the intuitive notion of continuity that we've outlined above. Definition 4.1 should then simply follow from your intuitive understanding of the concept.

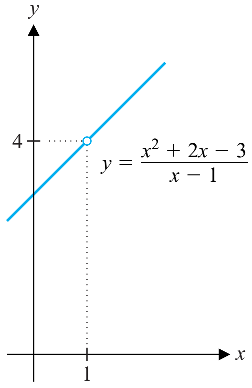


FIGURE 1.23
$y = \frac{x^2 + 2x - 3}{x - 1}$

REMARK 4.2

You should be careful not to confuse the continuity of a function at a point with its simply being defined there. A function can be defined at a point without being continuous there. (Look back at Figures 1.22b and 1.22c.)

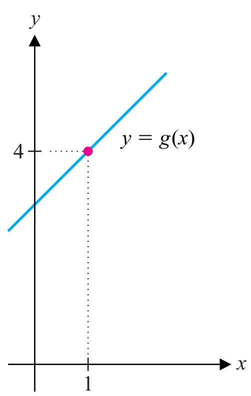


FIGURE 1.24
$y = g(x)$

EXAMPLE 4.1 Finding Where a Rational Function Is Continuous

Determine where $f(x) = \frac{x^2 + 2x - 3}{x - 1}$ is continuous.

Solution Note that

$$f(x) = \frac{x^2 + 2x - 3}{x - 1} = \frac{(x - 1)(x + 3)}{x - 1} \qquad \text{Factoring the numerator.}$$
$$= x + 3, \text{ for } x \neq 1. \qquad \text{Canceling common factors.}$$

This says that the graph of f is a straight line, but with a hole in it at $x = 1$, as indicated in Figure 1.23. So, f is discontinuous at $x = 1$, but continuous elsewhere. ■

EXAMPLE 4.2 Removing a Discontinuity

Make the function from example 4.1 continuous everywhere by redefining it at a single point.

Solution In example 4.1, we saw that the function is discontinuous at $x = 1$, since it is undefined there. So, suppose we just go ahead and define it, as follows. Let

$$g(x) = \begin{cases} \frac{x^2 + 2x - 3}{x - 1}, & \text{if } x \neq 1 \\ a, & \text{if } x = 1, \end{cases}$$

for some real number a.

Notice that $g(x)$ is defined for all x and equals $f(x)$ for all $x \neq 1$. Here, we have

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$$
$$= \lim_{x \to 1} (x + 3) = 4.$$

Observe that if we choose $a = 4$, we now have that

$$\lim_{x \to 1} g(x) = 4 = g(1)$$

and so, g is continuous at $x = 1$.

Note that the graph of g is the same as the graph of f seen in Figure 1.23, except that we now include the point (1, 4). (See Figure 1.24.) Also, note that there's a very simple way to write $g(x)$. (Think about this.) ■

When we can remove a discontinuity by redefining the function at that point, we call the discontinuity **removable.** Not all discontinuities are removable, however. Carefully examine Figures 1.22a–1.22d and convince yourself that the discontinuities in Figures 1.22a and 1.22c are removable, while those in Figures 1.22b and 1.22d are nonremovable. Briefly, a function f has a removable discontinuity at $x = a$ if $\lim_{x \to a} f(x)$ exists and either $f(a)$ is undefined or $\lim_{x \to a} f(x) \neq f(a)$.

EXAMPLE 4.3 Nonremovable Discontinuities

Find all discontinuities of $f(x) = \frac{1}{x^2}$ and $g(x) = \cos\left(\frac{1}{x}\right)$.

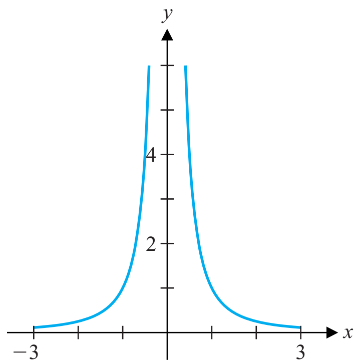


FIGURE 1.25a
$y = \frac{1}{x^2}$

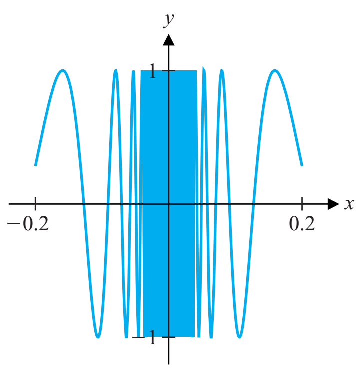


FIGURE 1.25b
$y = \cos(1/x)$

Solution You should observe from Figure 1.25a (also, construct a table of function values) that

$$\lim_{x \to 0} \frac{1}{x^2} \text{ does not exist.}$$

Hence, f is discontinuous at $x = 0$.

Similarly, observe that $\lim_{x \to 0} \cos(1/x)$ does not exist, due to the endless oscillation of $\cos(1/x)$ as x approaches 0 (see Figure 1.25b).

In both cases, notice that since the limits do not exist, there is no way to redefine either function at $x = 0$ to make it continuous there. ■

From your experience with the graphs of some common functions, the following result should come as no surprise.

THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$ and $\cos x$ are continuous everywhere, $\sqrt[n]{x}$ is continuous for all x, when n is odd and for $x > 0$, when n is even.

PROOF

We have already established (in Theorem 3.2) that for any polynomial $p(x)$ and any real number a,

$$\lim_{x \to a} p(x) = p(a),$$

from which it follows that p is continuous at $x = a$. The rest of the theorem follows from Theorem 3.3 and 3.4 in a similar way. ■

From these very basic continuous functions, we can build a large collection of continuous functions, using Theorem 4.2.

THEOREM 4.2

Suppose that f and g are continuous at $x = a$. Then all of the following are true:

(i) $(f \pm g)$ is continuous at $x = a$,
(ii) $(f \cdot g)$ is continuous at $x = a$ and
(iii) (f/g) is continuous at $x = a$ if $g(a) \neq 0$.

Simply put, Theorem 4.2 says that a sum, difference or product of continuous functions is continuous, while the quotient of two continuous functions is continuous at any point at which the denominator is nonzero.

PROOF

(i) If f and g are continuous at $x = a$, then

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \quad \text{From Theorem 3.1.}$$
$$= f(a) \pm g(a) \quad \text{Since } f \text{ and } g \text{ are continuous at } a.$$
$$= (f \pm g)(a),$$

by the usual rules of limits. Thus, $(f \pm g)$ is also continuous at $x = a$.

Parts (ii) and (iii) are proved in a similar way and are left as exercises. ■

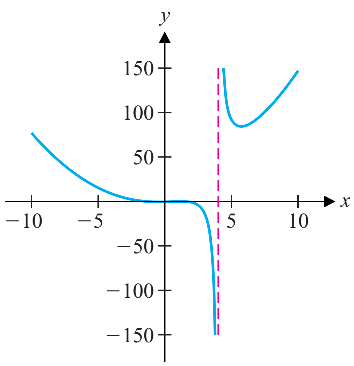


FIGURE 1.26

$$y = \frac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$$

EXAMPLE 4.4 Continuity for a Rational Function

Determine where f is continuous, for $f(x) = \dfrac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$.

Solution Here, f is a quotient of two polynomial (hence continuous) functions. The graph of the function indicated in Figure 1.26 suggests a vertical asymptote at around $x = 4$, but doesn't indicate any other discontinuity. From Theorem 4.2, f will be continuous at all x where the denominator is not zero, that is, where

$$x^2 - 3x - 4 = (x + 1)(x - 4) \neq 0.$$

Thus, f is continuous for $x \neq -1, 4$. (Think about why you didn't see anything peculiar about the graph at $x = -1$.) ■

With the addition of the result in Theorem 4.3, we will have all the basic tools needed to establish the continuity of most elementary functions.

THEOREM 4.3

Suppose that $\lim_{x\to a} g(x) = L$ and f is continuous at L. Then,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

A proof of Theorem 4.3 is given in Appendix A.

Notice that this says that if f is continuous, then we can bring the limit "inside." This should make sense, since as $x \to a$, $g(x) \to L$ and so, $f(g(x)) \to f(L)$, since f is continuous at L.

COROLLARY 4.1

Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a.

PROOF

From Theorem 4.3, we have

$$\begin{aligned}\lim_{x\to a}(f\circ g)(x) &= \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)\\ &= f(g(a)) = (f\circ g)(a). \quad \text{Since } g \text{ is continuous at } a. \ ■\end{aligned}$$

EXAMPLE 4.5 Continuity for a Composite Function

Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

Solution Note that

$$h(x) = f(g(x)),$$

where $g(x) = x^2 - 5x + 2$ and $f(x) = \cos x$. Since both f and g are continuous for all x, h is continuous for all x, by Corollary 4.1. ■

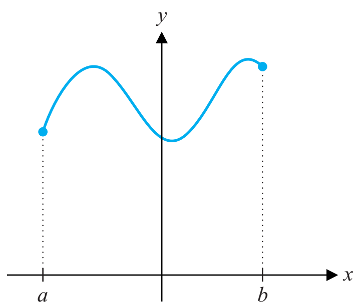


FIGURE 1.27
f continuous on $[a, b]$

DEFINITION 4.2

If f is continuous at every point on an open interval (a, b), we say that f is **continuous on (a, b).** Following Figure 1.27, we say that f is **continuous on the closed interval $[a, b]$,** if f is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is **continuous.** (That is, when we don't specify an interval, we mean continuous everywhere.)

For many functions, it's a simple matter to determine the intervals on which the function is continuous. We illustrate this in example 4.6.

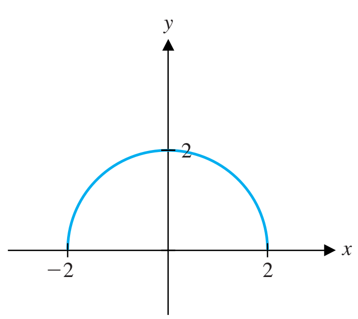


FIGURE 1.28
$y = \sqrt{4 - x^2}$

EXAMPLE 4.6 Continuity on a Closed Interval

Determine the interval(s) where f is continuous, for $f(x) = \sqrt{4 - x^2}$.

Solution First, observe that f is defined only for $-2 \le x \le 2$. Next, note that f is the composition of two continuous functions and hence, is continuous for all x for which $4 - x^2 > 0$. We show a graph of the function in Figure 1.28. Since

$$4 - x^2 > 0$$

for $-2 < x < 2$, we have that f is continuous for all x in the interval $(-2, 2)$, by Theorem 4.1 and Corollary 4.1. Finally, we test the endpoints to see that $\lim_{x\to 2^-} \sqrt{4 - x^2} = 0 = f(2)$ and $\lim_{x\to -2^+} \sqrt{4 - x^2} = 0 = f(-2)$, so that f is continuous on the closed interval $[-2, 2]$. ■

The Internal Revenue Service presides over some of the most despised functions in existence. Look up the current Tax Rate Schedules. In 2002, the first few lines (for single taxpayers) looked like:

For taxable amount over	*but not over*	*your tax liability is*	*minus*
\$0	\$6000	10%	\$0
\$6000	\$27,950	15%	\$300
\$27,950	\$67,700	27%	\$3654

Where do the numbers \$300 and \$3654 come from? If we write the tax liability $T(x)$ as a function of the taxable amount x (assuming that x can be any real value and not just a whole dollar amount), we have

$$T(x) = \begin{cases} 0.10x & \text{if } 0 < x \le 6000 \\ 0.15x - 300 & \text{if } 6000 < x \le 27{,}950 \\ 0.27x - 3654 & \text{if } 27{,}950 < x \le 67{,}700. \end{cases}$$

Be sure you understand our translation so far. Note that it is important that this be a continuous function: think of the fairness issues that would arise if it were not!

EXAMPLE 4.7 Continuity of Federal Tax Tables

Verify that the federal tax rate function $T(x)$ is continuous at the "joint" $x = 27{,}950$. Then, find a to complete the table. (You will find b and c as exercises.)

For taxable amount over	*but not over*	*your tax liability is*	*minus*
\$67,700	\$141,250	30%	a
\$141,250	\$307,050	35%	b
\$307,050	—	38.6%	c

Solution For $T(x)$ to be continuous at $x = 27{,}950$, we must have

$$\lim_{x \to 27{,}950^-} T(x) = \lim_{x \to 27{,}950^+} T(x).$$

Since both functions $0.15x - 300$ and $0.27x - 3654$ are continuous, we can compute the one-sided limits by substituting $x = 27{,}950$. Thus,

$$\lim_{x \to 27{,}950^-} T(x) = 0.15(27{,}950) - 300 = 3892.50$$

and

$$\lim_{x \to 27{,}950^+} T(x) = 0.27(27{,}950) - 3654 = 3892.50.$$

Since the one-sided limits agree and equal the value of the function at that point, $T(x)$ is continuous at $x = 27{,}950$. We leave it as an exercise to establish that $T(x)$ is also continuous at $x = 6000$. (It's worth noting that the function could be written with equal signs on all of the inequalities; this would be incorrect if the function were discontinuous.) To complete the table, we choose a to get the one-sided limits at $x = 67{,}700$ to match. We have

$$\lim_{x \to 67{,}700^-} T(x) = 0.27(67{,}700) - 3654 = 14{,}625,$$

while

$$\lim_{x \to 67{,}700^+} T(x) = 0.30(67{,}700) - a = 20{,}310 - a.$$

So, we set the one-sided limits equal, to obtain

$$14{,}625 = 20{,}310 - a$$

or

$$a = 20{,}310 - 14{,}625 = 5685.$$

Theorem 4.4 should seem an obvious consequence of our intuitive definition of continuity.

HISTORICAL NOTES

Karl Weierstrass (1815–1897) A German mathematician who proved the Intermediate Value Theorem and several other fundamental results of the calculus, Weierstrass was known as an excellent teacher whose students circulated his lecture notes throughout Europe, because of their clarity and originality. Also known as a superb fencer, Weierstrass was one of the founders of modern mathematical analysis.

THEOREM 4.4 (Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.

Theorem 4.4 says that if f is continuous on $[a, b]$, then f must take on *every* value between $f(a)$ and $f(b)$ at least once. That is, a continuous function cannot skip over any numbers between its values at the two endpoints. To do so, the graph would need to leap across the horizontal line $y = W$, something that continuous functions cannot do. (See Figure 1.29a.) Of course, a function may take on a given value W more than once. (See Figure 1.29b.) We must point out that, although these graphs make this result seem reasonable, like any other result, Theorem 4.4 requires proof. The proof is more complicated than you might imagine and we must refer you to an advanced calculus text.

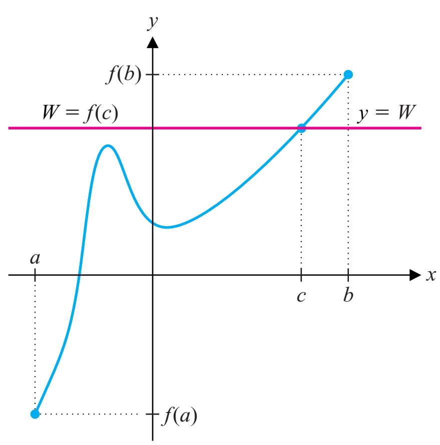


FIGURE 1.29a
An illustration of the Intermediate Value Theorem

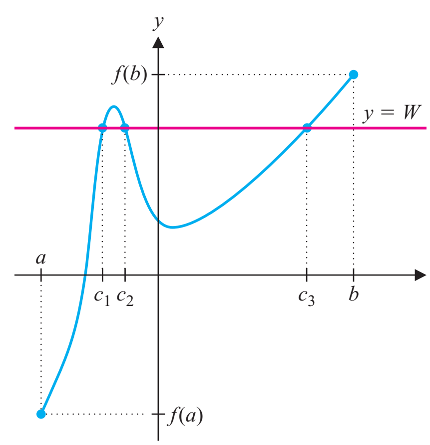


FIGURE 1.29b
More than one value of c

In Corollary 4.2, we see an immediate and useful application of the Intermediate Value Theorem.

COROLLARY 4.2

Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs [i.e., $f(a) \cdot f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$. (Recall that c is then a *zero* of f.)

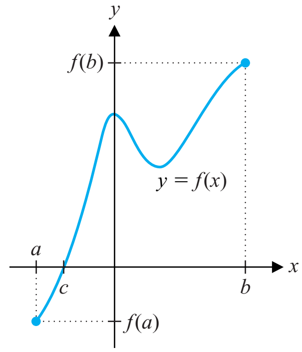


FIGURE 1.30
Intermediate Value Theorem where c is a zero of f

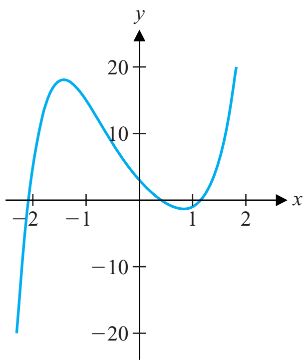


FIGURE 1.31
$y = x^5 + 4x^2 - 9x + 3$

Notice that Corollary 4.2 is simply the special case of the Intermediate Value Theorem where $W = 0$. (See Figure 1.30.) The Intermediate Value Theorem and Corollary 4.2 are examples of *existence theorems;* they tell you that there *exists* a number c satisfying some condition, but they do *not* tell you what c is.

The Method of Bisections

In example 4.8, we see how Corollary 4.2 can help us locate the zeros of a function.

EXAMPLE 4.8 Finding Zeros by the Method of Bisections

Find the zeros of $f(x) = x^5 + 4x^2 - 9x + 3$.

Solution If f were a quadratic polynomial, you could certainly find its zeros. However, you don't have any formulas for finding zeros of polynomials of degree 5. The only alternative is to approximate the zeros. A good starting place would be to draw a graph of $y = f(x)$ like the one in Figure 1.31. There are three zeros visible on the graph. Since f is a polynomial, it is continuous everywhere and so, Corollary 4.2 says that there must be a zero on any interval on which the function changes sign. From the graph, you can see that there must be zeros between -3 and -2, between 0 and 1 and between 1 and 2. We could also conclude this by computing say, $f(0) = 3$ and $f(1) = -1$. Although we've now found intervals that contain zeros, the question remains as to how we can *find* the zeros themselves.

While a rootfinding program can provide an accurate approximation, the issue here is not so much to get an answer as it is to understand how to find one. We suggest a simple yet effective method, called the **method of bisections.**

For the zero between 0 and 1, a reasonable guess might be the midpoint, 0.5. Since $f(0.5) \approx -0.469 < 0$ and $f(0) = 3 > 0$, there must be a zero between 0 and 0.5. Next, the midpoint of [0, 0.5] is 0.25 and $f(0.25) \approx 1.001 > 0$, so that the zero is on the interval (0.25, 0.5). We continue in this way to narrow the interval on which there's a zero until the interval is sufficiently small so that any point in the interval can serve as an adequate approximation to the actual zero. We do this in the following table.

a	*b*	*f(a)*	*f(b)*	*Midpoint*	*f(midpoint)*
0	1	3	−1	0.5	−0.469
0	0.5	3	−0.469	0.25	1.001
0.25	0.5	1.001	−0.469	0.375	0.195
0.375	0.5	0.195	−0.469	0.4375	−0.156
0.375	0.4375	0.195	−0.156	0.40625	0.015
0.40625	0.4375	0.015	−0.156	0.421875	−0.072
0.40625	0.421875	0.015	−0.072	0.4140625	−0.029
0.40625	0.4140625	0.015	−0.029	0.41015625	−0.007
0.40625	0.41015625	0.015	−0.007	0.408203125	0.004

If you continue this process through 20 more steps, you ultimately arrive at the approximate zero $x = 0.40892288$, which is accurate to at least eight decimal places. ■

This method of bisections is a tedious process, if you're working it with pencil and paper. It is interesting because it's reliable and it's a simple, yet general method for finding approximate zeros. Computer and calculator rootfinding utilities are very useful, but our purpose here is to provide you with an understanding of how basic rootfinding works. We discuss a more powerful method for finding roots in Chapter 3.

EXERCISES 1.4

WRITING EXERCISES

1. Think about the following "real-life" functions, each of which is a function of the independent variable time: the height of a falling object, the velocity of an object, the amount of money in a bank account, the cholesterol level of a person, the heart rate of a person, the amount of a certain chemical present in a test tube and a machine's most recent measurement of the cholesterol level of a person. Which of these are continuous functions? For each function you identify as discontinuous, what is the real-life meaning of the discontinuities?

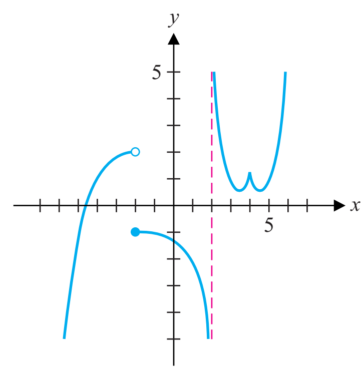
2. Whether a process is continuous or not is not always clear-cut. When you watch television or a movie, the action seems to be continuous. This is an optical illusion, since both movies and television consist of individual "snapshots" that are played back at many frames per second. Where does the illusion of continuous motion come from? Given that the average person blinks several times per minute, is our perception of the world actually continuous? (In what cognitive psychologists call **temporal binding,** the human brain first decides whether a stimulus is important enough to merit conscious consideration. If so, the brain "predates" the stimulus so that the person correctly identifies when the stimulus actually occurred.)

3. When you sketch the graph of the parabola $y = x^2$ with pencil or pen, is your sketch (at the molecular level) actually the graph of a continuous function? Is your calculator or computer's graph actually the graph of a continuous function? On many calculators, you have the option of a connected or disconnected graph. At the pixel level, does a connected graph show the graph of a function? Does a disconnected graph show the graph of a continuous function? Do we ever have problems correctly interpreting a graph due to these limitations? In exploratory exercise 2 in section 1.7, we examine one case where our perception of a computer graph depends on which choice is made.

4. For each of the graphs in Figures 1.22a–1.22d, describe (with an example) what the formula for $f(x)$ might look like to produce the given discontinuity.

In exercises 1–6, use the given graph to identify all discontinuities of the functions.

1.



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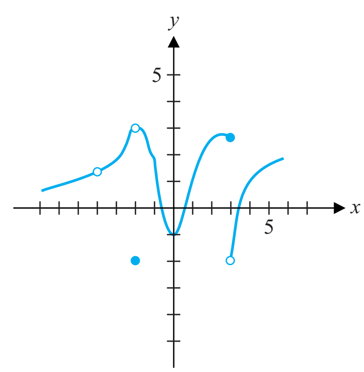

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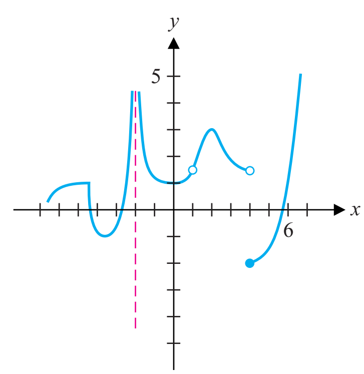

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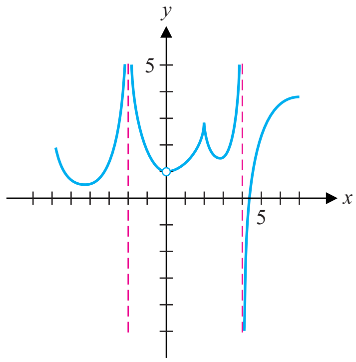

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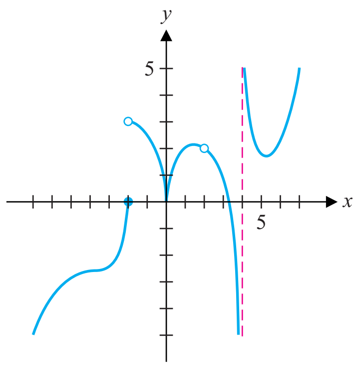

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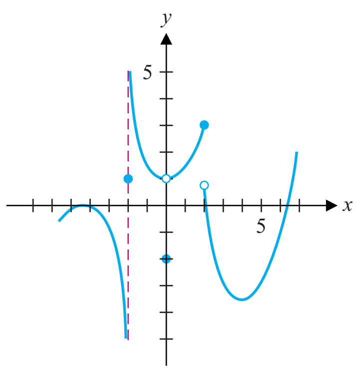


In exercises 7–12, explain why each function is discontinuous at the given point by indicating which of the three conditions in Definition 4.1 are not met.

7. $f(x) = \dfrac{x}{x-1}$ at $x = 1$

8. $f(x) = \dfrac{x^2-1}{x-1}$ at $x = 1$

9. $f(x) = \sin\dfrac{1}{x}$ at $x = 0$

10. $f(x) = \dfrac{2x}{\sqrt{x^3+x^2}}$ at $x = 0$

11. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$ at $x = 2$

12. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$ at $x = 2$

In exercises 13–24, find all discontinuities of $f(x)$. For each discontinuity that is removable, define a new function that removes the discontinuity.

13. $f(x) = \dfrac{x-1}{x^2-1}$

14. $f(x) = \dfrac{4x}{x^2+x-2}$

15. $f(x) = \dfrac{4x}{x^2+4}$

16. $f(x) = \dfrac{3x}{x^2-2x-4}$

17. $f(x) = x^2 \tan x$

18. $f(x) = x \cot x$

19. $f(x) = \dfrac{3x^2}{\sqrt{x^3-x^2}}$

20. $f(x) = \dfrac{3}{\sqrt{1+4/x^2}}$

21. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 & \text{if } x \ge 1 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

23. $f(x) = \begin{cases} 3x-1 & \text{if } x \le -1 \\ x^2+5x & \text{if } -1 < x < 1 \\ 3x^3 & \text{if } x \ge 1 \end{cases}$

24. $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ \sin x & \text{if } 0 < x \le \pi \\ x-\pi & \text{if } x > \pi \end{cases}$

In exercises 25–30, determine the intervals on which $f(x)$ is continuous.

25. $f(x) = \sqrt{x+3}$

26. $f(x) = \sqrt{x^2-4}$

27. $f(x) = \dfrac{6}{\sqrt{x+1}}$

28. $f(x) = (x-1)^{3/2}$

29. $f(x) = \sin(x^2+2)$

30. $f(x) = \cos\left(\dfrac{1}{x}\right)$

In exercises 31–33, determine values of a and b that make the given function continuous.

31. $f(x) = \begin{cases} \dfrac{2\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos x & \text{if } x > 0 \end{cases}$

32. $f(x) = \begin{cases} a\cos x + 1 & \text{if } x < 0 \\ \sin\left(\dfrac{\pi}{2}x\right) & \text{if } 0 \le x \le 2 \\ x^2 - x + b & \text{if } x > 2 \end{cases}$

33. $f(x) = \begin{cases} a\sqrt{9-x} & \text{if } x < 0 \\ \sin bx + 1 & \text{if } 0 \le x \le 3 \\ \sqrt{x-2} & \text{if } x > 3 \end{cases}$

34. Prove Corollary 4.1.

35. Suppose that a state's income tax code states that the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 0.14x & \text{if } 0 < x < 10{,}000 \\ c + 0.21x & \text{if } 10{,}000 \le x. \end{cases}$$

Determine the constant c that makes this function continuous for all x. Give a rationale why such a function should be continuous.

36. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} 0 & \text{if } x \le 0 \\ a + 0.12x & \text{if } 0 < x \le 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $T(x)$ is continuous for all x.

37. In example 4.7, find b and c to complete the table.

38. In example 4.7, show that $T(x)$ is continuous for $x = 6000$.

In exercises 39–44, use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

39. $f(x) = x^2 - 7$, $[2, 3]$

40. $f(x) = x^3 - 4x - 2$, $[2, 3]$

41. $f(x) = x^3 - 4x - 2$, $[-1, 0]$

42. $f(x) = x^3 - 4x - 2$, $[-2, -1]$

43. $f(x) = \cos x - x$, $[0, 1]$

44. $f(x) = \cos x + x$, $[-1, 0]$

A function is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$. In exercises 45–48, determine whether $f(x)$ is continuous from the right at $x = 2$.

45. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 1 & \text{if } x \ge 2 \end{cases}$

46. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

47. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$

49. Define what it means for a function to be **continuous from the left** at $x = a$ and determine which of the functions in exercises 45–48 are continuous from the left at $x = 2$.

50. Suppose that $f(x) = \dfrac{g(x)}{h(x)}$ and $h(a) = 0$. Determine whether each of the following statements is always true, always false or maybe true/maybe false. Explain. (a) $\lim_{x \to a} f(x)$ does not exist. (b) $f(x)$ is discontinuous at $x = a$.

51. The sex of newborn Mississippi alligators is determined by the temperature of the eggs in the nest. The eggs fail to develop unless the temperature is between 26°C and 36°C. All eggs between 26°C and 30°C develop into females, and eggs between 34°C and 36°C develop into males. The percentage of females decreases from 100% at 30°C to 0% at 34°C. If $f(T)$ is the percentage of females developing from an egg at T°C, then

$$f(T) = \begin{cases} 100 & \text{if } 26 \le T \le 30 \\ g(T) & \text{if } 30 < T < 34 \\ 0 & \text{if } 34 \le T \le 36, \end{cases}$$

for some function $g(T)$. Explain why it is reasonable that $f(T)$ be continuous. Determine a function $g(T)$ such that $0 \le g(T) \le 100$ for $30 \le T \le 34$ and the resulting function $f(T)$ is continuous. [Hint: It may help to draw a graph first and make $g(T)$ linear.]

52. If $f(x) = \begin{cases} x^2, & \text{if } x \ne 0 \\ 4, & \text{if } x = 0 \end{cases}$ and $g(x) = 2x$, show that

$$\lim_{x \to 0} f(g(x)) \ne f\left(\lim_{x \to 0} g(x)\right).$$

53. If you push on a large box resting on the ground, at first nothing will happen because of the static friction force that opposes motion. If you push hard enough, the box will start sliding, although there is again a friction force that opposes the motion. Suppose you are given the following description of the friction force. Up to 100 pounds, friction matches the force you apply to the box. Over 100 pounds, the box will move and the friction force will equal 80 pounds. Sketch a graph of friction as a function of your applied force based on this description. Where is this graph discontinuous? What is significant physically about this point? Do you think the friction force actually ought to be continuous? Modify the graph to make it continuous while still retaining most of the characteristics described.

54. For $f(x) = 2x - \dfrac{400}{x}$, we have $f(-1) > 0$ and $f(2) < 0$. Does the Intermediate Value Theorem guarantee a zero of $f(x)$ between $x = -1$ and $x = 2$? What happens if you try the method of bisections?

55. On Monday morning, a saleswoman leaves on a business trip at 7:13 A.M. and arrives at her destination at 2:03 P.M. The following morning, she leaves for home at 7:17 A.M. and arrives at 1:59 P.M. The woman notices that at a particular stoplight along the way, a nearby bank clock changes from 10:32 A.M. to 10:33 A.M. on both days. Therefore, she must have been at the same location at the same time on both days. Her boss doesn't believe that such an unlikely coincidence could occur. Use the Intermediate Value Theorem to argue that it *must* be true that at some point on the trip, the saleswoman was at exactly the same place at the same time on both Monday and Tuesday.

56. Suppose you ease your car up to a stop sign at the top of a hill. Your car rolls back a couple of feet and then you drive through the intersection. A police officer pulls you over for not coming to a complete stop. Use the Intermediate Value Theorem to argue that there was an instant in time when your car was stopped. (In fact, there were at least two.) What is the difference between this stopping and the stopping that the police officer wanted to see?

57. Suppose a worker's salary starts at \$40,000 with \$2000 raises every 3 months. Graph the salary function $s(t)$; why is it discontinuous? How does the function $f(t) = 40{,}000 + \frac{2000}{3}t$ (t in months) compare? Why might it be easier to do calculations with $f(t)$ than $s(t)$?

58. Prove the final two parts of Theorem 4.2.

59. Suppose that $f(x)$ is a continuous function with consecutive zeros at $x = a$ and $x = b$; that is, $f(a) = f(b) = 0$ and $f(x) \neq 0$ for $a < x < b$. Further, suppose that $f(c) > 0$ for some number c between a and b. Use the Intermediate Value Theorem to argue that $f(x) > 0$ for all $a < x < b$.

60. Use the method of bisections to estimate the other two zeros in example 4.8.

61. Suppose that $f(x)$ is continuous at $x = 0$. Prove that $\lim_{x \to 0} x f(x) = 0$.

62. The **converse** of exercise 61 is not true. That is, the fact $\lim_{x \to 0} x f(x) = 0$ does not guarantee that $f(x)$ is continuous at $x = 0$. Find a counterexample; that is, find a function f such that $\lim_{x \to 0} x f(x) = 0$ and $f(x)$ is not continuous at $x = 0$.

63. If $f(x)$ is continuous at $x = a$, prove that $g(x) = |f(x)|$ is continuous at $x = a$.

64. Determine whether the converse of exercise 63 is true. That is, if $|f(x)|$ is continuous at $x = a$, is it necessarily true that $f(x)$ must be continuous at $x = a$?

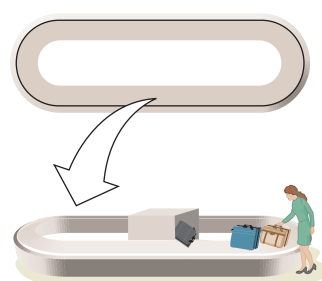
65. Let $f(x)$ be a continuous function for $x \geq a$ and define $h(x) = \max_{a \leq t \leq x} f(t)$. Prove that $h(x)$ is continuous for $x \geq a$. Would this still be true without the assumption that $f(x)$ is continuous?

66. Graph $f(x) = \dfrac{\sin|x^3 - 3x^2 + 2x|}{x^3 - 3x^2 + 2x}$ and determine all discontinuities.

EXPLORATORY EXERCISES

1. In the text, we discussed the use of the method of bisections to find an approximate solution of equations such as $f(x) = x^3 + 5x - 1 = 0$. We can start by noticing that $f(0) = -1$ and $f(1) = 5$. Since $f(x)$ is continuous, the Intermediate Value Theorem tells us that there is a solution between $x = 0$ and $x = 1$. For the method of bisections, we guess the midpoint, $x = 0.5$. Is there any reason to suspect that the solution is actually closer to $x = 0$ than to $x = 1$? Using the function values $f(0) = -1$ and $f(1) = 5$, devise your own method of guessing the location of the solution. Generalize your method to using $f(a)$ and $f(b)$, where one function value is positive and one is negative. Compare your method to the method of bisections on the problem $x^3 + 5x - 1 = 0$; for both methods, stop when you are within 0.001 of the solution, $x \approx 0.198437$. Which method performed better? Before you get overconfident in your method, compare the two methods again on $x^3 + 5x^2 - 1 = 0$. Does your method get close on the first try? See if you can determine graphically why your method works better on the first problem.

2. You have probably seen the turntables on which luggage rotates at the airport. Suppose that such a turntable has two long straight parts with a semicircle on each end. (See the figure.) We will model the left/right movement of the luggage. Suppose the straight part is 40 ft long, extending from $x = -20$ to $x = 20$. Assume that our luggage starts at time $t = 0$ at location $x = -20$, and that it takes 60 s for the luggage to reach $x = 20$. Suppose the radius of the circular portion is 5 ft and it takes the luggage 30 s to complete the half-circle. We model the straight-line motion with a linear function $x(t) = at + b$. Find constants a and b so that $x(0) = -20$ and $x(60) = 20$. For the circular motion, we use a cosine (Why is this a good choice?) $x(t) = 20 + d \cdot \cos(et + f)$ for constants d, e and f. The requirements are $x(60) = 20$ (since the motion is continuous), $x(75) = 25$ and $x(90) = 20$. Find values of d, e and f to make this work. Find equations for the position of the luggage along the backstretch and the other semicircle. What would the motion be from then on?



Luggage carousel

3. Determine all x's for which each function is continuous.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases},$$

$$g(x) = \begin{cases} x^2 + 3 & \text{if } x \text{ is irrational} \\ 4x & \text{if } x \text{ is rational} \end{cases} \quad \text{and}$$

$$h(x) = \begin{cases} \cos 4x & \text{if } x \text{ is irrational} \\ \sin 4x & \text{if } x \text{ is rational} \end{cases}.$$

1.5 LIMITS INVOLVING INFINITY; ASYMPTOTES

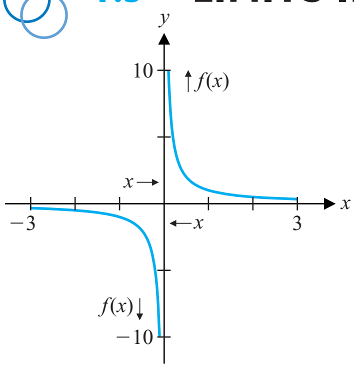


FIGURE 1.32

$\lim_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

x	$\frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10,000
0.00001	100,000

x	$\frac{1}{x}$
−0.1	−10
−0.01	−100
−0.001	−1000
−0.0001	−10,000
−0.00001	−100,000

In this section, we revisit some old limit problems to give more informative answers and examine some related questions.

EXAMPLE 5.1 A Simple Limit Revisited

Examine $\lim_{x\to 0} \frac{1}{x}$.

Solution Of course, we can draw a graph (see Figure 1.32) and compute a table of function values easily, by hand. (See the tables in the margin.)

While we say that the limits $\lim_{x\to 0^+} \frac{1}{x}$ and $\lim_{x\to 0^-} \frac{1}{x}$ do not exist, the behavior of the function is clearly quite different for $x > 0$ than for $x < 0$. Specifically, as $x \to 0^+$, $\frac{1}{x}$ increases without bound, while as $x \to 0^-$, $\frac{1}{x}$ decreases without bound. To communicate more about the behavior of the function near $x = 0$, we write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \tag{5.1}$$

and

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty. \tag{5.2}$$

Graphically, this says that the graph of $y = \frac{1}{x}$ approaches the vertical line $x = 0$, as $x \to 0$, as seen in Figure 1.32. When this occurs, we say that the line $x = 0$ is a **vertical asymptote.** It is important to note that while the limits (5.1) and (5.2) *do not exist,* we say that they "equal" ∞ and $-\infty$, respectively, only to be specific as to *why* they do not exist. Finally, in view of the one-sided limits (5.1) and (5.2), we say that

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

■

REMARK 5.1

It may at first seem contradictory to say that $\lim_{x\to 0^+} \frac{1}{x}$ does not exist and then to write $\lim_{x\to 0^+} \frac{1}{x} = \infty$. Note that since ∞ is *not* a real number, there is no contradiction here. (When we say that a limit "does not exist," we are saying that there is no real number L that the function values are approaching.) We say that $\lim_{x\to 0^+} \frac{1}{x} = \infty$ to indicate that as $x \to 0^+$, the function values are increasing without bound.

EXAMPLE 5.2 A Function Whose One-Sided Limits Are Both Infinite

Evaluate $\lim_{x\to 0} \frac{1}{x^2}$.

Solution The graph in Figure 1.33 (on the following page) seems to indicate a vertical asymptote at $x = 0$. A table of values is easily constructed by hand. (See the accompanying tables.)

x	$\frac{1}{x^2}$
0.1	100
0.01	10,000
0.001	1×10^6
0.0001	1×10^8
0.00001	1×10^{10}

x	$\frac{1}{x^2}$
−0.1	100
−0.01	10,000
−0.001	1×10^6
−0.0001	1×10^8
−0.00001	1×10^{10}

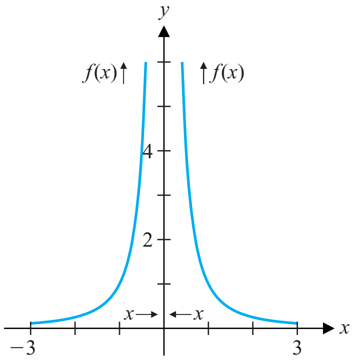


FIGURE 1.33
$\lim_{x\to 0} \frac{1}{x^2} = \infty$

From this, we can see that

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty$$

and
$$\lim_{x\to 0^-} \frac{1}{x^2} = \infty.$$

Since both one-sided limits agree (i.e., both tend to ∞), we say that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

This one concise statement says that the limit does not exist, but also that $f(x)$ has a vertical asymptote at $x = 0$, where $f(x) \to \infty$ as $x \to 0$ from either side. ■

REMARK 5.2

Mathematicians try to convey as much information as possible with as few symbols as possible. For instance, we prefer to say $\lim_{x\to 0} \frac{1}{x^2} = \infty$ rather than $\lim_{x\to 0} \frac{1}{x^2}$ does not exist, since the first statement not only says that the limit does not exist, but also says that $\frac{1}{x^2}$ increases without bound as x approaches 0, with $x > 0$ or $x < 0$.

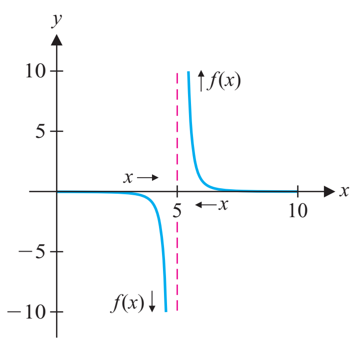


FIGURE 1.34
$\lim_{x\to 5^+} \frac{1}{(x-5)^3} = \infty$ and
$\lim_{x\to 5^-} \frac{1}{(x-5)^3} = -\infty$

EXAMPLE 5.3 A Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x\to 5} \frac{1}{(x-5)^3}$.

Solution In Figure 1.34, we show a graph of the function. From the graph, you should get a pretty clear idea that there's a vertical asymptote at $x = 5$ and just how the function is blowing up there (to ∞ from the right side and to $-\infty$ from the left). You can verify this behavior algebraically, by noticing that as $x \to 5$, the denominator approaches 0, while the numerator approaches 1. This says that the fraction grows large in absolute value, without bound as $x \to 5$. Specifically,

$$\text{as } x \to 5^+, \quad (x-5)^3 \to 0 \quad \text{and} \quad (x-5)^3 > 0.$$

We indicate the sign of each factor by printing a small "+" or "−" sign above or below each one. This enables you to see the signs of the various terms at a glance. In this case, we have

$$\lim_{x\to 5^+} \frac{\overset{+}{1}}{\underset{+}{(x-5)^3}} = \infty. \quad \text{Since } (x-5)^3 > 0, \text{ for } x > 5.$$

Likewise, as $x \to 5^-$, $(x-5)^3 \to 0$ and $(x-5)^3 < 0$.

In this case, we have

$$\lim_{x\to 5^-} \frac{\overset{+}{1}}{\underset{-}{(x-5)^3}} = -\infty. \quad \text{Since } (x-5)^3 < 0, \text{ for } x < 5.$$

Finally, we say that $\lim_{x\to 5} \frac{1}{(x-5)^3}$ does not exist,

since the one-sided limits are different. ■

Learning from the lessons of examples 5.1, 5.2 and 5.3, you should recognize that if the denominator tends to 0 and the numerator does not, then the limit in question does not exist. In this event, we can determine whether the limit tends to ∞ or $-\infty$ by carefully examining the signs of the various factors.

EXAMPLE 5.4 Another Case Where Infinite One-Sided Limits Disagree

Evaluate $\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)}$.

Solution First, notice from the graph of the function shown in Figure 1.35 that there appears to be a vertical asymptote at $x = -2$.

Further, the function appears to tend to ∞ as $x \to -2^+$ and to $-\infty$ as $x \to -2^-$. You can verify this behavior, by observing that

$$\lim_{x\to -2^+} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{+}{(x+2)}} = \infty$$

Since $(x+1) < 0$, $(x-3) < 0$ and $(x+2) > 0$, for $-2 < x < -1$.

and

$$\lim_{x\to -2^-} \frac{\overset{-}{x+1}}{\underset{-}{(x-3)}\underset{-}{(x+2)}} = -\infty.$$

Since $(x+1) < 0$, $(x-3) < 0$ and $(x+2) < 0$, for $x < -2$.

So, we can see that $x = -2$ is indeed a vertical asymptote and that

$$\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)} \text{ does not exist.}$$

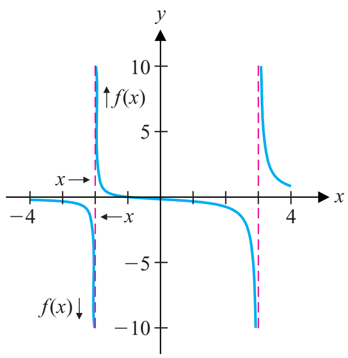


FIGURE 1.35
$\lim_{x\to -2} \frac{x+1}{(x-3)(x+2)}$ does not exist.

EXAMPLE 5.5 A Limit Involving a Trigonometric Function

Evaluate $\lim_{x\to \frac{\pi}{2}} \tan x$.

Solution Notice from the graph of the function shown in Figure 1.36 that there appears to be a vertical asymptote at $x = \frac{\pi}{2}$.

You can verify this behavior by observing that

$$\lim_{x\to \frac{\pi}{2}^-} \tan x = \lim_{x\to \frac{\pi}{2}^-} \frac{\overset{+}{\sin x}}{\underset{+}{\cos x}} = \infty$$

Since $\sin x > 0$ and $\cos x > 0$ for $0 < x < \frac{\pi}{2}$.

and

$$\lim_{x\to \frac{\pi}{2}^+} \tan x = \lim_{x\to \frac{\pi}{2}^+} \frac{\overset{+}{\sin x}}{\underset{-}{\cos x}} = -\infty.$$

Since $\sin x > 0$ and $\cos x < 0$ for $\frac{\pi}{2} < x < \pi$.

So, we can see that $x = \frac{\pi}{2}$ is indeed a vertical asymptote and that

$$\lim_{x\to \frac{\pi}{2}} \tan x \text{ does not exist.}$$

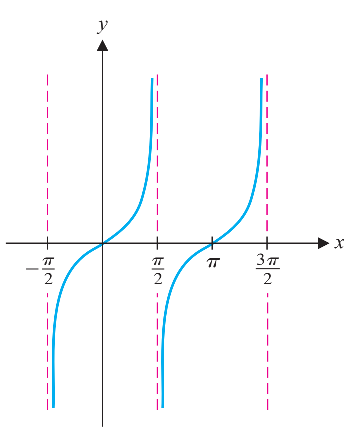


FIGURE 1.36
$y = \tan x$

Limits at Infinity

We are also interested in examining the limiting behavior of functions as x increases without bound (written $x \to \infty$) or as x decreases without bound (written $x \to -\infty$).

Returning to $f(x) = \frac{1}{x}$, we can see that as $x \to \infty$, $\frac{1}{x} \to 0$. In view of this, we write

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

Similarly,

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

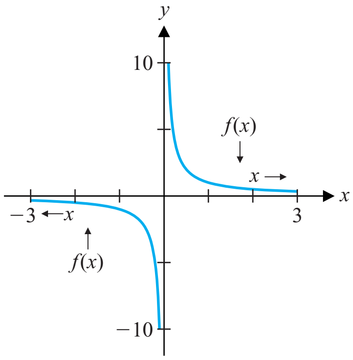


FIGURE 1.37

$\lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to-\infty} \frac{1}{x} = 0$

Notice that in Figure 1.37, the graph appears to approach the horizontal line $y = 0$, as $x \to \infty$ and as $x \to -\infty$. In this case, we call $y = 0$ a **horizontal asymptote.**

EXAMPLE 5.6 Finding Horizontal Asymptotes

Look for any horizontal asymptotes of $f(x) = 2 - \frac{1}{x}$.

Solution We show a graph of $y = f(x)$ in Figure 1.38. Since as $x \to \pm\infty$, $\frac{1}{x} \to 0$, we get that

$$\lim_{x\to\infty} \left(2 - \frac{1}{x}\right) = 2$$

and

$$\lim_{x\to-\infty} \left(2 - \frac{1}{x}\right) = 2.$$

Thus, the line $y = 2$ is a horizontal asymptote. ■

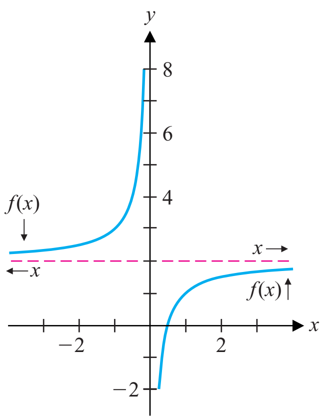


FIGURE 1.38

$\lim_{x\to\infty} \left(2 - \frac{1}{x}\right) = 2$ and

$\lim_{x\to-\infty} \left(2 - \frac{1}{x}\right) = 2$

As you can see in Theorem 5.1, the behavior of $\frac{1}{x^t}$, for any positive rational power t, as $x \to \pm\infty$, is largely the same as we observed for $f(x) = \frac{1}{x}$.

THEOREM 5.1

For any rational number $t > 0$,

$$\lim_{x\to\pm\infty} \frac{1}{x^t} = 0,$$

where for the case where $x \to -\infty$, we assume that $t = \frac{p}{q}$ where q is odd.

A proof of Theorem 5.1 is given in Appendix A. Be sure that the following argument makes sense to you: for $t > 0$, as $x \to \infty$, we also have $x^t \to \infty$, so that $\frac{1}{x^t} \to 0$.

REMARK 5.3

All of the usual rules for limits stated in Theorem 3.1 also hold for limits as $x \to \pm\infty$.

In Theorem 5.2, we see that the behavior of a polynomial at infinity is easy to determine.

THEOREM 5.2

For a polynomial of degree $n > 0$, $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have

$$\lim_{x\to\infty} p_n(x) = \begin{cases} \infty, & \text{if } a_n > 0 \\ -\infty, & \text{if } a_n < 0 \end{cases}.$$

PROOF

We have

$$\lim_{x\to\infty} p_n(x) = \lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0)$$
$$= \lim_{x\to\infty} \left[x^n\left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right)\right]$$
$$= \infty,$$

if $a_n > 0$, since $\lim_{x\to\infty} \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_0}{x^n}\right) = a_n$

and $\lim_{x\to\infty} x^n = \infty$. The result is proved similarly for $a_n < 0$. ■

Observe that you can make similar statements regarding the value of $\lim_{x\to-\infty} p_n(x)$, but be careful: the answer will change depending on whether n is even or odd. (We leave this as an exercise.)

In example 5.7, we again see the need for caution when applying our basic rules for limits (Theorem 3.1), which also apply to limits as $x \to \infty$ or as $x \to -\infty$.

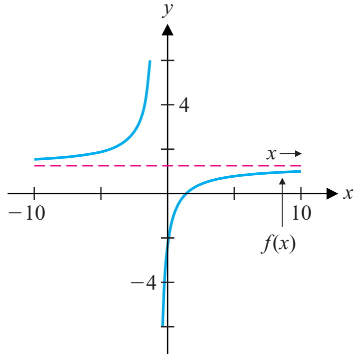


FIGURE 1.39

$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \frac{5}{4}$

x	$\frac{5x-7}{4x+3}$
10	1
100	1.223325
1000	1.247315
10,000	1.249731
100,000	1.249973

EXAMPLE 5.7 A Limit of a Quotient That Is Not the Quotient of the Limits

Evaluate $\lim_{x\to\infty} \frac{5x-7}{4x+3}$.

Solution You might be tempted to write

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \frac{\lim_{x\to\infty}(5x-7)}{\lim_{x\to\infty}(4x+3)}$$

This is an incorrect use of Theorem 3.1 (iv), since the limits in the numerator and the denominator do not exist.

$$= \frac{\infty}{\infty} = 1. \quad (5.3)$$

This is incorrect!

The graph in Figure 1.39 and some function values (see the accompanying table) suggest that the conjectured value of 1 is incorrect. Recall that the limit of a quotient is the quotient of the limits only when *both* limits exist (and the limit in the denominator is nonzero). Since both the limit in the denominator and that in the numerator tend to ∞, the limits *do not exist*.

Further, when a limit looks like $\frac{\infty}{\infty}$, the actual value of the limit can be anything at all. For this reason, we call $\frac{\infty}{\infty}$ an **indeterminate form,** meaning that the value of the expression cannot be determined solely by noticing that both numerator and denominator tend to ∞.

Rule of Thumb: When faced with the indeterminate form $\frac{\infty}{\infty}$ in calculating the limit of a rational function, divide numerator and denominator by the highest power of x appearing in the *denominator.*

Here, we have

$$\lim_{x\to\infty} \frac{5x-7}{4x+3} = \lim_{x\to\infty} \left[\frac{5x-7}{4x+3} \cdot \frac{(1/x)}{(1/x)}\right]$$

Multiply numerator and denominator by $\frac{1}{x}$.

$$= \lim_{x\to\infty} \frac{5-7/x}{4+3/x}$$

Multiply through by $\frac{1}{x}$.

$$= \frac{\lim_{x\to\infty}(5-7/x)}{\lim_{x\to\infty}(4+3/x)}$$

By Theorem 3.1 (iv).

$$= \frac{5}{4} = 1.25,$$

which is consistent with what we observed both graphically and numerically earlier. ■

In example 5.8, we apply our rule of thumb to a common limit problem.

EXAMPLE 5.8 Finding Slant Asymptotes

Evaluate $\lim_{x\to\infty} \frac{4x^3+5}{-6x^2-7x}$ and find any slant asymptotes.

Solution As usual, we first examine a graph. (See Figure 1.40a.) Note that here, the graph appears to tend to $-\infty$ as $x \to \infty$. Further, observe that outside of the interval $[-2, 2]$, the graph looks very much like a straight line. If we look at the graph in a somewhat larger window, this linearity is even more apparent. (See Figure 1.40b.)

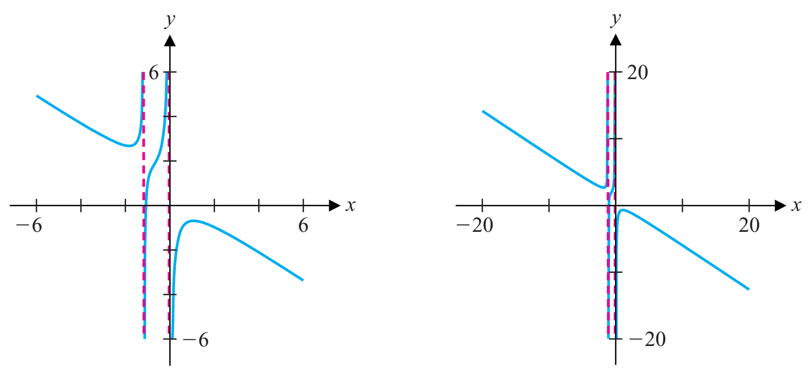


FIGURE 1.40a
$y = \frac{4x^3+5}{-6x^2-7x}$

FIGURE 1.40b
$y = \frac{4x^3+5}{-6x^2-7x}$

Using our rule of thumb, we have

$$\lim_{x\to\infty} \frac{4x^3+5}{-6x^2-7x} = \lim_{x\to\infty}\left[\frac{4x^3+5}{-6x^2-7x}\cdot\frac{(1/x^2)}{(1/x^2)}\right] \quad \text{Multiply numerator and denominator by } \frac{1}{x^2}.$$

$$= \lim_{x\to\infty} \frac{4x+5/x^2}{-6-7/x} \quad \text{Multiply through by } \frac{1}{x^2}.$$

$$= -\infty,$$

since as $x \to \infty$, the numerator tends to ∞ and the denominator tends to -6.

To further explain the behavior seen in Figure 1.40b, we perform a long division. We have

$$\frac{4x^3+5}{-6x^2-7x} = -\frac{2}{3}x + \frac{7}{9} + \frac{5+49/9x}{-6x^2-7x}.$$

Since the third term in this expansion tends to 0 as $x \to \infty$, the function values approach those of the linear function

$$-\frac{2}{3}x + \frac{7}{9},$$

as $x \to \infty$. For this reason, we say that the function has a **slant (or oblique) asymptote.** That is, instead of approaching a vertical or horizontal line, as happens with vertical or horizontal asymptotes, the graph is approaching the slanted straight line $y = -\frac{2}{3}x + \frac{7}{9}$. (This is the behavior we're seeing in Figure 1.40b.) ■

In example 5.9, we consider a model of the size of an animal's pupils. Recall that in bright light, pupils shrink to reduce the amount of light entering the eye, while in dim light, pupils dilate to allow in more light. (See the chapter introduction.)

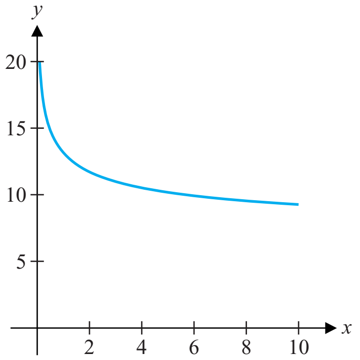


FIGURE 1.41a
$y = f(x)$

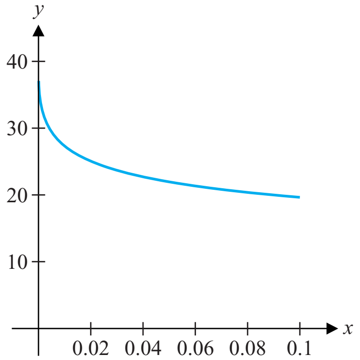


FIGURE 1.41b
$y = f(x)$

EXAMPLE 5.9 Finding the Size of an Animal's Pupils

Suppose that the diameter of an animal's pupils is given by $f(x)$ mm, where x is the intensity of light on the pupils. If $f(x) = \dfrac{160x^{-0.4}+90}{4x^{-0.4}+15}$, find the diameter of the pupils with (a) minimum light and (b) maximum light.

Solution For part (a), notice that $f(0)$ is undefined, since $0^{-0.4}$ indicates a division by 0. We therefore consider the limit of $f(x)$ as x approaches 0, but we compute a one-sided limit, since x cannot be negative. A computer-generated graph of $y = f(x)$ with $0 \le x \le 10$ is shown in Figure 1.41a. It appears that the y-values approach 20 as x approaches 0. To compute the limit, we multiply numerator and denominator by $x^{0.4}$ (to eliminate the negative exponents). We then have

$$\lim_{x\to 0^+} \frac{160x^{-0.4}+90}{4x^{-0.4}+15} = \lim_{x\to 0^+} \frac{160x^{-0.4}+90}{4x^{-0.4}+15} \cdot \frac{x^{0.4}}{x^{0.4}}$$
$$= \lim_{x\to 0^+} \frac{160+90x^{0.4}}{4+15x^{0.4}} = \frac{160}{4} = 40 \text{ mm}.$$

This limit does not seem to match our graph, but notice that Figure 1.41a shows a gap near $x = 0$. In Figure 1.41b, we have zoomed in so that $0 \le x \le 0.1$. Here, a limit of 40 looks more reasonable.

For part (b), we consider the limit as x tends to ∞. From Figure 1.41a, it appears that the graph has a horizontal asymptote at a value close to $y = 10$. We compute the limit

$$\lim_{x\to\infty} \frac{160x^{-0.4}+90}{4x^{-0.4}+15} = \frac{90}{15} = 6 \text{ mm}.$$

So, the pupils have a limiting size of 6 mm, as the intensity of light tends to ∞. ■

EXERCISES 1.5

WRITING EXERCISES

1. It may seem odd that we use ∞ in describing limits but do not count ∞ as a real number. Discuss the existence of ∞: is it a number or a concept?

2. In example 5.7, we dealt with the "indeterminate form" $\frac{\infty}{\infty}$. Thinking of a limit of ∞ as meaning "getting very large" and a limit of 0 as meaning "getting very close to 0," explain why the following are indeterminate forms: $\frac{\infty}{\infty}$, $\frac{0}{0}$, $\infty - \infty$, and $\infty \cdot 0$. Determine what the following non-indeterminate forms represent: $\infty + \infty$, $-\infty - \infty$, $\infty + 0$ and $0/\infty$.

3. On your computer or calculator, graph $y = 1/(x-2)$ and look for the horizontal asymptote $y = 0$ and the vertical asymptote $x = 2$. Most computers will draw a vertical line at $x = 2$ and will show the graph completely flattening out at $y = 0$ for large x's. Is this accurate? misleading? Most computers will compute the locations of points for adjacent x's and try to connect the points with a line segment. Why might this result in a vertical line at the location of a vertical asymptote?

4. Many students learn that asymptotes are lines that the graph gets closer and closer to without ever reaching. This is true for many asymptotes, but not all. Explain why vertical asymptotes are never reached or crossed. Explain why horizontal or slant asymptotes may, in fact, be crossed any number of times; draw one example.

In exercises 1–4, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

1. (a) $\lim_{x\to 1^-} \frac{1-2x}{x^2-1}$ (b) $\lim_{x\to 1^+} \frac{1-2x}{x^2-1}$
   (c) $\lim_{x\to 1} \frac{1-2x}{x^2-1}$
2. (a) $\lim_{x\to -1^-} \frac{1-2x}{x^2-1}$ (b) $\lim_{x\to -1^+} \frac{1-2x}{x^2-1}$
   (c) $\lim_{x\to -1} \frac{1-2x}{x^2-1}$
3. (a) $\lim_{x\to 2^-} \frac{x-4}{x^2-4x+4}$ (b) $\lim_{x\to 2^+} \frac{x-4}{x^2-4x+4}$
   (c) $\lim_{x\to 2} \frac{x-4}{x^2-4x+4}$
4. (a) $\lim_{x\to -1^-} \frac{1-x}{(x+1)^2}$ (b) $\lim_{x\to -1^+} \frac{1-x}{(x+1)^2}$
   (c) $\lim_{x\to -1} \frac{1-x}{(x+1)^2}$

In exercises 5–20, determine each limit (answer as appropriate, with a number, ∞, $-\infty$ or does not exist).

5. $\lim_{x\to 2^-} \frac{-x}{\sqrt{4-x^2}}$
6. $\lim_{x\to -1^-} (x^2-2x-3)^{-2/3}$
7. $\lim_{x\to -\infty} \frac{-x}{\sqrt{4+x^2}}$
8. $\lim_{x\to \infty} \frac{-x}{\sqrt{4+x^2}}$
9. $\lim_{x\to \infty} \frac{x^3-2}{3x^2+4x-1}$
10. $\lim_{x\to \infty} \frac{2x^2-1}{4x^3-5x-1}$
11. $\lim_{x\to \infty} \frac{2x^2-x+1}{4x^2-3x-1}$
12. $\lim_{x\to \infty} \frac{2x-1}{x^2+4x+1}$
13. $\lim_{x\to \infty} \sin 2x$
14. $\lim_{x\to 0^+} \cot 2x$
15. $\lim_{x\to 0^+} \frac{3-2/x}{2+1/x}$
16. $\lim_{x\to \infty} \frac{3-2/x}{2+1/x}$
17. $\lim_{x\to \infty} \frac{3x+\sin x}{4x-\cos 2x}$
18. $\lim_{x\to \infty} \frac{2x^2\sin x}{x^2+4}$
19. $\lim_{x\to \pi/2^+} \frac{\tan x - x}{\tan^2 x+3}$
20. $\lim_{x\to \infty} \frac{\sin x - x}{\sin^2 x+3x}$

In exercises 21–30, determine all horizontal and vertical asymptotes. For each vertical asymptote, determine whether $f(x)\to\infty$ or $f(x)\to-\infty$ on either side of the asymptote.

21. $f(x)=\frac{x}{\sqrt{4+x^2}}$
22. $f(x)=\frac{x}{\sqrt{4-x^2}}$
23. $f(x)=\frac{x}{4-x^2}$
24. $f(x)=\frac{x^2}{4-x^2}$
25. $f(x)=\frac{3x^2+1}{x^2-2x-3}$
26. $f(x)=\frac{1-x}{x^2+x-2}$
27. $f(x)=\cot(1-\cos x)$
28. $f(x)=\frac{\tan x}{1-\sin 2x}$
29. $f(x)=\frac{4\sin x}{x}$
30. $f(x)=\sin\left(\frac{x^2+4}{x^2-4}\right)$

In exercises 31–34, determine all vertical and slant asymptotes.

31. $y=\frac{x^3}{4-x^2}$
32. $y=\frac{x^2+1}{x-2}$
33. $y=\frac{x^3}{x^2+x-4}$
34. $y=\frac{x^4}{x^3+2}$

In exercises 35–38, use graphical and numerical evidence to conjecture a value for the indicated limit.

35. $\lim_{x\to\infty} \frac{x\cos(1/x)}{x-2}$
36. $\lim_{x\to\infty} \frac{x\sin(1/x)}{x+3}$
37. $\lim_{x\to -1} \frac{x-\cos(\pi x)}{x+1}$
38. $\lim_{x\to 0^+} \frac{x}{\cos x-1}$

In exercises 39–42, use graphical and numerical evidence to conjecture the value of the limit. Then, verify your conjecture by finding the limit exactly.

39. $\lim_{x\to\infty}(\sqrt{4x^2-2x+1}-2x)$ (Hint: Multiply and divide by the conjugate expression: $\sqrt{4x^2-2x+1}+2x$ and simplify.)
40. $\lim_{x\to\infty}(\sqrt{x^2+3}-x)$ (See the hint for exercise 39.)
41. $\lim_{x\to\infty}(\sqrt{5x^2+4x+7}-\sqrt{5x^2+x+3})$ (See the hint for exercise 39.)
42. $\lim_{x\to-\infty}\sqrt{x^2+3x+1}+x$
43. Explain why it is reasonable that $\lim_{x\to\infty} f(x)=\lim_{x\to 0^+} f(1/x)$ and $\lim_{x\to-\infty} f(x)=\lim_{x\to 0^-} f(1/x)$.
44. One of the reasons for saying that infinite limits do not exist is that we would otherwise invalidate Theorem 3.1 in section 1.3. Find examples of functions with infinite limits such that parts (ii) and (iv) of Theorem 3.1 do not hold.
45. Suppose that the size of the pupil of a certain animal is given by $f(x)$ (mm), where x is the intensity of the light on the pupil. If $f(x)=\frac{80x^{-0.3}+60}{2x^{-0.3}+5}$, find the size of the pupil with no light and the size of the pupil with an infinite amount of light.
46. Repeat exercise 45 with $f(x)=\frac{80x^{-0.3}+60}{8x^{-0.3}+15}$.
47. Modify the functions in exercises 45 and 46 to find a function f such that $\lim_{x\to 0^+} f(x)=8$ and $\lim_{x\to\infty} f(x)=2$.
48. After an injection, the concentration of a drug in a muscle varies according to a function of time $f(t)$. Suppose that t is measured in hours and $f(t)=\frac{t}{\sqrt{t^2+1}}$. Find the limit of $f(t)$, both as $t\to 0$ and $t\to\infty$, and interpret both limits in terms of the concentration of the drug.
49. Suppose an object with initial velocity $v_0=0$ ft/s and (constant) mass m slugs is accelerated by a constant force F pounds for t seconds. According to Newton's laws of motion, the object's speed will be $v_N=Ft/m$. According to Einstein's theory of relativity, the object's speed will be

$v_E = Fct/\sqrt{m^2c^2 + F^2t^2}$, where c is the speed of light. Compute $\lim_{t\to\infty} v_N$ and $\lim_{t\to\infty} v_E$.

50. According to Einstein's theory of relativity, the mass of an object traveling at speed v is given by $m = m_0/\sqrt{1 - v^2/c^2}$, where c is the speed of light (about 9.8×10^8 ft/s). Compute $\lim_{v\to 0} m$ and explain why m_0 is called the "rest mass." Compute $\lim_{v\to c^-} m$ and discuss the implications. (What would happen if you were traveling in a spaceship approaching the speed of light?) How much does the mass of a 192-pound man ($m_0 = 6$) increase at the speed of 9000 ft/s (about 4 times the speed of sound)?

51. Ignoring air resistance, the maximum height reached by a rocket launched with initial velocity v_0 is $h = \dfrac{v_0^2 R}{19.6R - v_0^2}$ m/s, where R is the radius of the earth. In this exercise, we interpret this as a function of v_0. Explain why the domain of this function must be restricted to $v_0 \geq 0$. There is an additional restriction. Find the (positive) value v_e such that h is undefined. Sketch a possible graph of h with $0 \leq v_0 < v_e$ and discuss the significance of the vertical asymptote at v_e. (Explain what would happen to the rocket if it is launched with initial velocity v_e.) Explain why v_e is called the **escape velocity.**

52. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$ with the degree (largest exponent) of $p(x)$ less than the degree of $q(x)$. Determine the horizontal asymptote of $y = f(x)$.

53. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$ with the degree of $p(x)$ greater than the degree of $q(x)$. Determine whether $y = f(x)$ has a horizontal asymptote.

54. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. If $y = f(x)$ has a horizontal asymptote $y = 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

55. Suppose that $f(x)$ is a rational function $f(x) = \dfrac{p(x)}{q(x)}$. If $y = f(x)$ has a slant asymptote $y = x + 2$, how does the degree of $p(x)$ compare to the degree of $q(x)$?

56. Find a quadratic function $q(x)$ such that $f(x) = \dfrac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = 2$ and two vertical asymptotes $x = \pm 3$.

57. Find a quadratic function $q(x)$ such that $f(x) = \dfrac{x^2 - 4}{q(x)}$ has one horizontal asymptote $y = -\frac{1}{2}$ and exactly one vertical asymptote $x = 3$.

58. Find a function $g(x)$ such that $f(x) = \dfrac{x - 4}{g(x)}$ has two horizontal asymptotes $y = \pm 1$ and no vertical asymptotes.

In exercises 59–64, label the statement as true or false (not always true) for real numbers *a* and *b*.

59. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} [f(x) + g(x)] = a + b$.

60. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$, then $\lim_{x\to\infty} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{a}{b}$.

61. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) - g(x)] = 0$.

62. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} [f(x) + g(x)] = \infty$.

63. If $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\dfrac{f(x)}{g(x)}\right] = 0$.

64. If $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty} \left[\dfrac{f(x)}{g(x)}\right] = 1$.

In exercises 65 and 66, determine all vertical and horizontal asymptotes.

65. $f(x) = \begin{cases} \dfrac{4x}{x-4} & \text{if } x < 0 \\ \dfrac{x^2}{x-2} & \text{if } 0 \leq x < 4 \\ \dfrac{\cos x}{x+1} & \text{if } x \geq 4 \end{cases}$

66. $f(x) = \begin{cases} \dfrac{x+3}{x^2-4x} & \text{if } x < 0 \\ \cos x + 1 & \text{if } 0 \leq x < 2 \\ \dfrac{x^2-1}{x^2-7x+10} & \text{if } x \geq 2 \end{cases}$

67. Explain why $\lim_{t\to\infty} \dfrac{1}{t^2+1}\sin(at) = 0$ for any positive constant a. Although this is theoretically true, it is not necessarily useful in practice. The function $\dfrac{1}{t^2+1}\sin(at)$ is a simple model for a spring-mass system, such as the suspension system on a car. Suppose t is measured in seconds and the car passengers cannot feel any vibrations less than 0.1 (inches). If suspension system A has the vibration function $\dfrac{t}{t^2+1}\sin t$ and suspension system B has the vibration function $\dfrac{t}{t^4+1}\sin t$, determine graphically how long it will take before the vibrations damp out, that is, $|f(t)| < 0.1$. Is the result $\lim_{t\to\infty} \dfrac{t}{t^2+1}\sin t = 0$ much consolation to the owner of car A?

68. (a) State and prove a result analogous to Theorem 5.2 for $\lim_{x\to-\infty} p_n(x)$, for n odd.

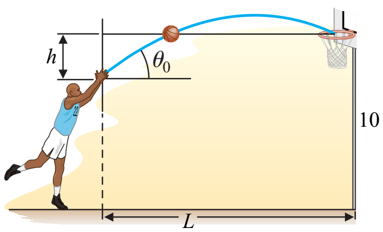
(b) State and prove a result analogous to Theorem 5.2 for $\lim_{x\to-\infty} p_n(x)$, for n even.

69. It is very difficult to find simple statements in calculus that are always true; this is one reason that a careful development of the theory is so important. You may have heard the simple rule: to find the vertical asymptotes of $f(x) = \dfrac{g(x)}{h(x)}$, simply set the denominator equal to 0 [i.e., solve $h(x) = 0$]. Give an example where $h(a) = 0$ but there is *not* a vertical asymptote at $x = a$.

70. In exercise 69, you needed to find an example indicating that the following statement is not (necessarily) true: if $h(a) = 0$, then $f(x) = \dfrac{g(x)}{h(x)}$ has a vertical asymptote at $x = a$. This is not true, but perhaps its converse is true: if $f(x) = \dfrac{g(x)}{h(x)}$ has a vertical asymptote at $x = a$, then $h(a) = 0$. Is this statement true? What if g and h are polynomials?

EXPLORATORY EXERCISES

1. Suppose you are shooting a basketball from a (horizontal) distance of L feet, releasing the ball from a location h feet below the basket. To get a perfect swish, it is necessary that the initial velocity v_0 and initial release angle θ_0 satisfy the equation



$v_0 = \sqrt{gL}/\sqrt{2\cos^2\theta_0(\tan\theta_0 - h/L)}$. For a free throw, take $L = 15$, $h = 2$ and $g = 32$ and graph v_0 as a function of θ_0. What is the significance of the two vertical asymptotes? Explain in physical terms what type of shot corresponds to each vertical asymptote. Estimate the minimum value of v_0 (call it v_{min}). Explain why it is easier to shoot a ball with a small initial velocity. There is another advantage to this initial velocity. Assume that the basket is 2 ft in diameter and the ball is 1 ft in diameter. For a free throw, $L = 15$ ft is perfect. What is the maximum horizontal distance the ball could travel and still go in the basket (without bouncing off the backboard)? What is the minimum horizontal distance? Call these numbers L_{max} and L_{min}. Find the angle θ_1 corresponding to v_{min} and L_{min} and the angle θ_2 corresponding to v_{min} and L_{max}. The difference $|\theta_2 - \theta_1|$ is the angular margin of error. Peter Brancazio has shown that the angular margin of error for v_{min} is larger than for any other initial velocity.

2. A different type of limit at infinity that will be very important to us is the limit of a sequence. Investigating the area under a parabola in Chapter 4, we will compute the following approximations: $\dfrac{2(3)}{6(1)} = 1$, $\dfrac{3(5)}{6(4)} = 0.625$, $\dfrac{4(7)}{6(9)} \approx 0.519$, $\dfrac{5(9)}{6(16)} \approx 0.469$ and so on. Do you see a pattern? If we name our approximations a_1, a_2, a_3 and a_4, verify that $a_n = \dfrac{(n+1)(2n+1)}{6n^2}$. The area under the parabola is the limit of these approximations as n gets larger and larger. Find the area. In Chapter 8, we will need to find limits of the following sequences. Estimate the limit of

(a) $a_n = \dfrac{2(n+1)^2 - 3(n+1) + 4}{n^2 + 3n + 4}$,

(b) $a_n = (1 + 1/n)^n$ and

(c) $a_n = \dfrac{n^3 + 2}{n!}$.

1.6 FORMAL DEFINITION OF THE LIMIT

We have now spent many pages discussing various aspects of the computation of limits. This may seem a bit odd, when you realize that we have never actually *defined* what a limit is. Oh, sure, we have given you an *idea* of what a limit is, but that's about all. Once again, we have said that

$$\lim_{x \to a} f(x) = L,$$

if $f(x)$ gets closer and closer to L as x gets closer and closer to a.

So far, we have been quite happy with this somewhat vague, although intuitive, description. In this section, however, we will make this more precise and you will begin to see how **mathematical analysis** (that branch of mathematics of which the calculus is the most elementary study) works.

Studying more advanced mathematics without an understanding of the precise definition of limit is somewhat akin to studying brain surgery without bothering with all that background work in chemistry and biology. In medicine, it has only been through a careful examination of the microscopic world that a deeper understanding of our own macroscopic world has developed, and good surgeons need to understand what they are doing *and why* they are doing it. Likewise, in mathematical analysis, it is through an understanding of the microscopic behavior of functions (such as the precise definition of limit) that a deeper understanding of the mathematics will come about.

We begin with the careful examination of an elementary example. You should certainly believe that

$$\lim_{x \to 2}(3x + 4) = 10.$$

Suppose that you were asked to explain the meaning of this particular limit to a fellow student. You would probably repeat the intuitive explanation we have used so far: that as x gets closer and closer to 2, $(3x + 4)$ gets arbitrarily close to 10. But, exactly what do we mean by *close?* One answer is that if $\lim_{x \to 2}(3x + 4) = 10$, we should be able to make $(3x + 4)$ as close as we like to 10, just by making x sufficiently close to 2. But can we actually do this? For instance, can we force $(3x + 4)$ to be within distance 1 of 10? To see what values of x will guarantee this, we write an inequality that says that $(3x + 4)$ is within 1 unit of 10:

$$|(3x + 4) - 10| < 1.$$

Eliminating the absolute values, we see that this is equivalent to

$$-1 < (3x + 4) - 10 < 1$$

or $$-1 < 3x - 6 < 1.$$

Since we need to determine how close x must be to 2, we want to isolate $x - 2$, instead of x. So, dividing by 3, we get

$$-\frac{1}{3} < x - 2 < \frac{1}{3}$$

or $$|x - 2| < \frac{1}{3}. \tag{6.1}$$

Reversing the steps that lead to inequality (6.1), we see that if x is within distance $\frac{1}{3}$ of 2, then $(3x + 4)$ will be within the specified distance (1) of 10. (See Figure 1.42 for a graphical interpretation of this.) So, does this convince you that you can make $(3x + 4)$ as close as you want to 10? Probably not, but if you used a smaller distance, perhaps you'd be more convinced.

HISTORICAL NOTES

Augustin Louis Cauchy (1789–1857) A French mathematician who developed the ε-δ definitions of limit and continuity, Cauchy was one of the most prolific mathematicians in history, making important contributions to number theory, linear algebra, differential equations, astronomy, optics and complex variables. A difficult man to get along with, a colleague wrote, "Cauchy is mad and there is nothing that can be done about him, although right now, he is the only one who knows how mathematics should be done."

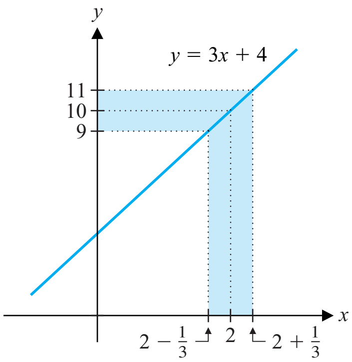


FIGURE 1.42

$2 - \frac{1}{3} < x < 2 + \frac{1}{3}$ guarantees that $|(3x + 4) - 10| < 1$.

EXAMPLE 6.1 Exploring a Simple Limit

Find the values of x for which $(3x + 4)$ is within distance $\frac{1}{100}$ of 10.

Solution We want

$$|(3x + 4) - 10| < \frac{1}{100}.$$

Eliminating the absolute values, we get

$$-\frac{1}{100} < (3x + 4) - 10 < \frac{1}{100}$$

or $$-\frac{1}{100} < 3x - 6 < \frac{1}{100}.$$

Dividing by 3 yields $$-\frac{1}{300} < x - 2 < \frac{1}{300},$$

which is equivalent to $$|x - 2| < \frac{1}{300}.$$ ■

So, based on example 6.1, are you now convinced that we can make $(3x + 4)$ as close as desired to 10? All we've been able to show is that we can make $(3x + 4)$ pretty close to 10. So, how close do we need to be able to make it? The answer is *arbitrarily close*, as close as anyone would ever demand. We can show that this is possible by repeating the arguments in example 6.1, this time for an unspecified distance, call it ε (*epsilon*, where $\varepsilon > 0$).

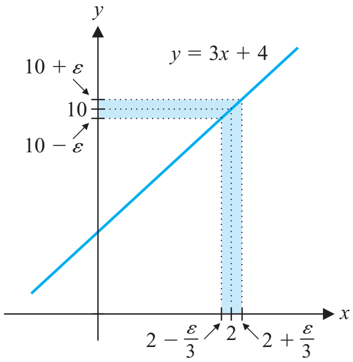


FIGURE 1.43
The range of x-values that keep $|(3x + 4) - 10| < \varepsilon$

EXAMPLE 6.2 Verifying a Limit

Show that we can make $(3x + 4)$ within any specified distance ε of 10 (no matter how small ε is), just by making x sufficiently close to 2.

Solution The objective is to determine the range of x-values that will guarantee that $(3x + 4)$ stays within ε of 10. (See Figure 1.43 for a sketch of this range.) We have

$$|(3x + 4) - 10| < \varepsilon.$$

This is equivalent to
$$-\varepsilon < (3x + 4) - 10 < \varepsilon$$
or
$$-\varepsilon < 3x - 6 < \varepsilon.$$
Dividing by 3, we get
$$-\frac{\varepsilon}{3} < x - 2 < \frac{\varepsilon}{3}$$
or
$$|x - 2| < \frac{\varepsilon}{3}.$$

Notice that each of the preceding steps is reversible, so that $|x - 2| < \frac{\varepsilon}{3}$ also implies that $|(3x + 4) - 10| < \varepsilon$. This says that as long as x is within distance $\frac{\varepsilon}{3}$ of 2, $(3x + 4)$ will be within the required distance ε of 10. That is,

$$|(3x + 4) - 10| < \varepsilon \text{ whenever } |x - 2| < \frac{\varepsilon}{3}.$$

Take a moment or two to recognize what we've done in example 6.2. By using an *unspecified* distance, ε, we have verified that we can indeed make $(3x + 4)$ as close to 10 as might be demanded (i.e., arbitrarily close; just name whatever $\varepsilon > 0$ you would like), simply by making x sufficiently close to 2. Further, we have explicitly spelled out what "sufficiently close to 2" means in the context of the present problem. Thus, no matter how close we are asked to make $(3x + 4)$ to 10, we can accomplish this simply by taking x to be in the specified interval.

Next, we examine this more precise notion of limit in the case of a function that is not defined at the point in question.

EXAMPLE 6.3 Proving That a Limit Is Correct

Prove that $\lim_{x \to 1} \frac{2x^2 + 2x - 4}{x - 1} = 6$.

Solution It is easy to use the usual rules of limits to establish this result. It is yet another matter to verify that this is correct using our new and more precise notion of limit. In this case, we want to know how close x must be to 1 to ensure that

$$f(x) = \frac{2x^2 + 2x - 4}{x - 1}$$

is within an unspecified distance $\varepsilon > 0$ of 6.

First, notice that f is undefined *at* $x = 1$. So, we seek a distance δ (*delta*, $\delta > 0$), such that if x is within distance δ of 1, but $x \neq 1$ (i.e., $0 < |x - 1| < \delta$), then this guarantees that $|f(x) - 6| < \varepsilon$.

Notice that we have specified that $0 < |x - 1|$ to ensure that $x \neq 1$. Further, $|f(x) - 6| < \varepsilon$ is equivalent to

$$-\varepsilon < \frac{2x^2 + 2x - 4}{x - 1} - 6 < \varepsilon.$$

Finding a common denominator and subtracting in the middle term, we get

$$-\varepsilon < \frac{2x^2 + 2x - 4 - 6(x - 1)}{x - 1} < \varepsilon \quad \text{or} \quad -\varepsilon < \frac{2x^2 - 4x + 2}{x - 1} < \varepsilon.$$

Since the numerator factors, this is equivalent to

$$-\varepsilon < \frac{2(x - 1)^2}{x - 1} < \varepsilon.$$

Since $x \neq 1$, we can cancel two of the factors of $(x - 1)$ to yield

$$-\varepsilon < 2(x - 1) < \varepsilon$$

or

$$-\frac{\varepsilon}{2} < x - 1 < \frac{\varepsilon}{2},$$ Dividing by 2.

which is equivalent to $|x - 1| < \varepsilon/2$. So, taking $\delta = \varepsilon/2$ and working backward, we see that requiring x to satisfy

$$0 < |x - 1| < \delta = \frac{\varepsilon}{2}$$

will guarantee that

$$\left| \frac{2x^2 + 2x - 4}{x - 1} - 6 \right| < \varepsilon.$$

We illustrate this graphically in Figure 1.44. ■

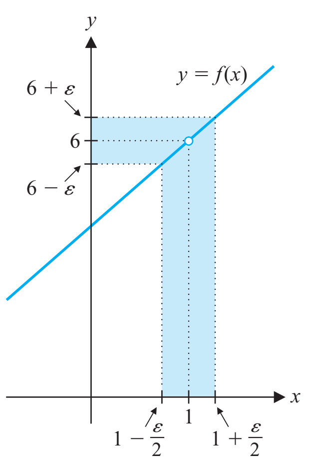


FIGURE 1.44
$0 < |x - 1| < \frac{\varepsilon}{2}$ guarantees that $6 - \varepsilon < \frac{2x^2 + 2x - 4}{x - 1} < 6 + \varepsilon$.

What we have seen so far motivates us to make the following general definition, illustrated in Figure 1.45.

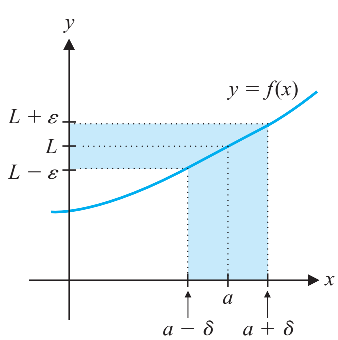


FIGURE 1.45
$a - \delta < x < a + \delta$ guarantees that $L - \varepsilon < f(x) < L + \varepsilon$.

DEFINITION 6.1 (Precise Definition of Limit)

For a function f defined in some open interval containing a (but not necessarily *at a* itself), we say

$$\lim_{x \to a} f(x) = L,$$

if given any number $\varepsilon > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $|f(x) - L| < \varepsilon$.

Notice that example 6.2 amounts to an illustration of Definition 6.1 for $\lim_{x \to 2}(3x + 4)$. There, we found that $\delta = \varepsilon/3$ satisfies the definition.

TODAY IN MATHEMATICS

Paul Halmos (1916–) A Hungarian-born mathematician who earned a reputation as one of the best mathematical writers ever. For Halmos, calculus did not come easily, with understanding coming in a flash of inspiration only after a long period of hard work. "I remember standing at the blackboard in Room 213 of the mathematics building with Warren Ambrose and suddenly I understood epsilons. I understood what limits were, and all of that stuff that people had been drilling into me became clear. . . . I could prove the theorems. That afternoon I became a mathematician."

REMARK 6.1

We want to emphasize that this formal definition of limit is not a new idea. Rather, it is a more precise mathematical statement of the same intuitive notion of limit that we have been using since the beginning of the chapter. Also, we must in all honesty point out that it is rather difficult to explicitly find δ as a function of ε, for all but a few simple examples. Despite this, learning how to work through the definition, even for a small number of problems, will shed considerable light on a deep concept.

Example 6.4, although only slightly more complex than the last several problems, provides an unexpected challenge.

EXAMPLE 6.4 Using the Precise Definition of Limit

Use Definiton 6.1 to prove that $\lim_{x\to 2}(x^2+1)=5$.

Solution If this limit is correct, then given any $\varepsilon > 0$, there must be a $\delta > 0$ for which $0 < |x-2| < \delta$ guarantees that

$$|(x^2+1)-5| < \varepsilon.$$

Notice that

$$\begin{aligned}|(x^2+1)-5| &= |x^2-4| && \text{Factoring the difference}\\ &= |x+2||x-2|. && \text{of two squares.} \qquad (6.2)\end{aligned}$$

Our strategy is to isolate $|x-2|$ and so, we'll need to do something with the term $|x+2|$. Since we're interested only in what happens near $x=2$, anyway, we will only consider x's within a distance of 1 from 2, that is, x's that lie in the interval [1, 3] (so that $|x-2| < 1$). Notice that this will be true if we require $\delta \le 1$ and $|x-2| < \delta$. In this case, we have

$$|x+2| \le 5, \quad \text{Since } x \in [1, 3].$$

and so, from (6.2),

$$\begin{aligned}|(x^2+1)-5| &= |x+2||x-2|\\ &\le 5|x-2|. \qquad (6.3)\end{aligned}$$

Finally, if we require that

$$5|x-2| < \varepsilon, \qquad (6.4)$$

then we will also have from (6.3) that

$$|(x^2+1)-5| \le 5|x-2| < \varepsilon.$$

Of course, (6.4) is equivalent to

$$|x-2| < \frac{\varepsilon}{5}.$$

So, in view of this, we now have two restrictions: that $|x-2| < 1$ *and* that $|x-2| < \frac{\varepsilon}{5}$. To ensure that both restrictions are met, we choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ (i.e., the **minimum** of 1 and $\frac{\varepsilon}{5}$). Working backward, we get that for this choice of δ,

$$0 < |x-2| < \delta$$

will guarantee that

$$|(x^2+1)-5| < \varepsilon,$$

as desired. We illustrate this in Figure 1.46. ■

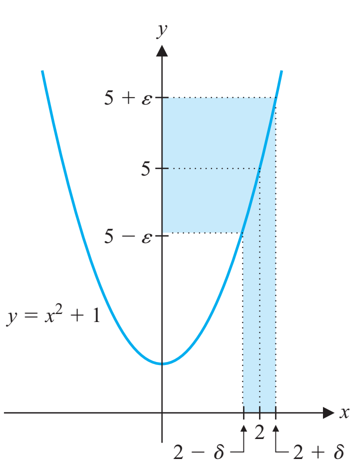


FIGURE 1.46
$0 < |x-2| < \delta$ guarantees that $|(x^2+1)-5| < \varepsilon$.

Exploring the Definition of Limit Graphically

As you can see from example 6.4, this business of finding δ's for a given ε is not easily accomplished. There, we found that even for the comparatively simple case of a quadratic polynomial, the job can be quite a challenge. Unfortunately, there is no procedure that will work for all problems. However, we can explore the definition graphically in the case of more complex functions. First, we reexamine example 6.4 graphically.

EXAMPLE 6.5 Exploring the Precise Definition of Limit Graphically

Explore the precise definition of limit graphically, for $\lim_{x\to 2}(x^2+1)=5$.

Solution In example 6.4, we discovered that for $\delta=\min\left\{1, \frac{\varepsilon}{5}\right\}$,

$$0<|x-2|<\delta \text{ implies that } |(x^2+1)-5|<\varepsilon.$$

This says that (for $\varepsilon \le 5$) if we draw a graph of $y=x^2+1$ and restrict the x-values to lie in the interval $\left(2-\frac{\varepsilon}{5}, 2+\frac{\varepsilon}{5}\right)$, then the y-values will lie in the interval $(5-\varepsilon, 5+\varepsilon)$. Take $\varepsilon=\frac{1}{2}$, for instance. If we draw the graph in the window defined by $2-\frac{1}{10}\le x\le 2+\frac{1}{10}$ and $4.5\le y\le 5.5$, then the graph will not run off the top or bottom of the screen. (See Figure 1.47.) Of course, we can draw virtually the same picture for any given value of ε, since we have an explicit formula for finding δ given ε. For most limit problems, we are not so fortunate. ■

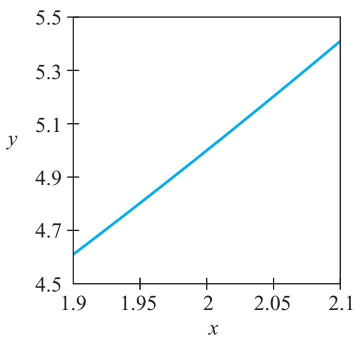


FIGURE 1.47
$y=x^2+1$

EXAMPLE 6.6 Exploring the Definition of Limit for a Trigonometric Function

Graphically find a $\delta>0$ corresponding to (a) $\varepsilon=\frac{1}{2}$ and (b) $\varepsilon=0.1$ for

$\lim_{x\to 2}\sin\frac{\pi x}{2}=0.$

Solution This limit seems plausible enough. After all, $\sin\frac{2\pi}{2}=0$ and $f(x)=\sin x$ is a continuous function. However, the point is to verify this carefully. Given any $\varepsilon>0$, we want to find a $\delta>0$, for which

$$0<|x-2|<\delta \text{ guarantees that } \left|\sin\frac{\pi x}{2}-0\right|<\varepsilon.$$

Note that since we have no algebra for simplifying $\sin\frac{\pi x}{2}$, we cannot accomplish this symbolically. Instead, we'll try to graphically find δ's corresponding to the specific ε's given. First, for $\varepsilon=\frac{1}{2}$, we would like to find a $\delta>0$ for which if $0<|x-2|<\delta$, then

$$-\frac{1}{2}<\sin\frac{\pi x}{2}-0<\frac{1}{2}.$$

Drawing the graph of $y=\sin\frac{\pi x}{2}$ with $1\le x\le 3$ and $-\frac{1}{2}\le y\le\frac{1}{2}$, we get Figure 1.48a.

If you trace along a calculator or computer graph, you will notice that the graph stays on the screen (i.e., the y-values stay in the interval $[-0.5, 0.5]$) for

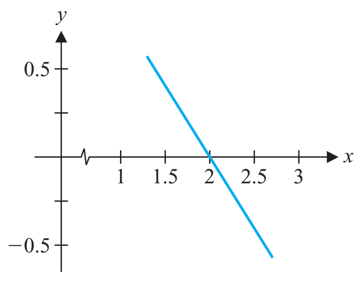


FIGURE 1.48a
$y=\sin\frac{\pi x}{2}$

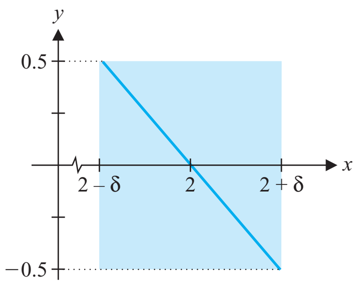


FIGURE 1.48b

$y = \sin \dfrac{\pi x}{2}$

$x \in [1.666667, 2.333333]$. Thus, we have determined experimentally that for $\varepsilon = \dfrac{1}{2}$,

$$\delta = 2.333333 - 2 = 2 - 1.666667 = 0.333333$$

will work. (Of course, any value of δ smaller than 0.333333 will also work.) To illustrate this, we redraw the last graph, but restrict x to lie in the interval [1.67, 2.33]. (See Figure 1.48b.) In this case, the graph stays in the window over the entire range of displayed x-values. Taking $\varepsilon = 0.1$, we look for an interval of x-values that will guarantee that $\sin \dfrac{\pi x}{2}$ stays between -0.1 and 0.1. We redraw the graph from Figure 1.48a, with the y-range restricted to the interval $[-0.1, 0.1]$. (See Figure 1.49a.) Again, tracing along the graph tells us that the y-values will stay in the desired range for $x \in [1.936508, 2.063492]$. Thus, we have experimentally determined that

$$\delta = 2.063492 - 2 = 2 - 1.936508 = 0.063492$$

will work here. We redraw the graph using the new range of x-values (see Figure 1.49b), since the graph remains in the window for all values of x in the indicated interval.

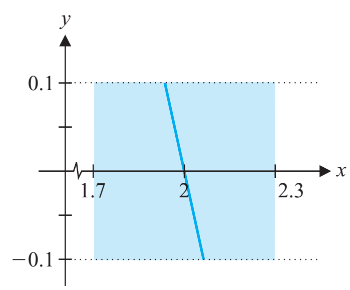


FIGURE 1.49a

$y = \sin \dfrac{\pi x}{2}$

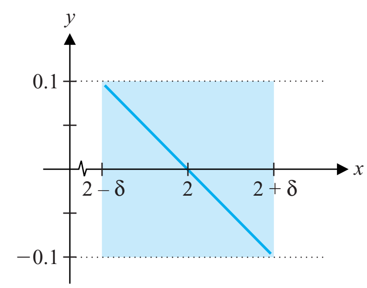


FIGURE 1.49b

$y = \sin \dfrac{\pi x}{2}$

It is important to recognize that we are not *proving* that the above limit is correct. To prove this requires us to symbolically find a δ for *every* $\varepsilon > 0$. The idea here is to use these graphical illustrations to become more familiar with the definition and with what δ and ε represent. ■

EXAMPLE 6.7 Exploring the Definition of Limit Where the Limit Does Not Exist

Determine whether or not $\displaystyle\lim_{x \to 0} \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} = 1$.

Solution We first construct a table of function values. From the table alone, we might be tempted to conjecture that the limit is 1. However, we would be making a *huge* error, as we have not considered negative values of x or drawn a graph. This kind of carelessness is dangerous. Figure 1.50a (on the following page) shows the default graph drawn by our computer algebra system. In this graph, the function values do not quite look like they are approaching 1 as $x \to 0$ (at least as $x \to 0^-$). We now investigate the limit graphically for $\varepsilon = \frac{1}{2}$. We need to find a $\delta > 0$ for which $0 < |x| < \delta$ guarantees that

$$1 - \frac{1}{2} < \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} < 1 + \frac{1}{2}$$

x	$\dfrac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$
0.1	1.03711608
0.01	1.0037461
0.001	1.00037496
0.0001	1.0000375

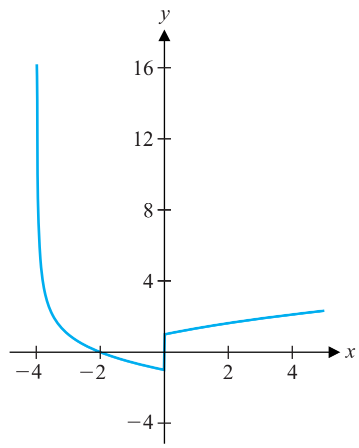


FIGURE 1.50a
$y = \dfrac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$

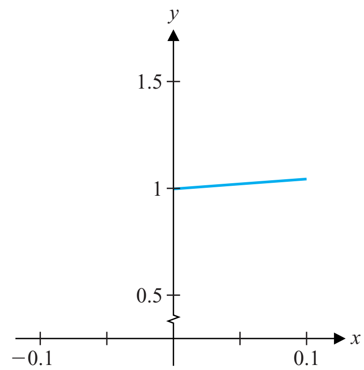


FIGURE 1.50b
$y = \dfrac{x^2 + 2x}{\sqrt{x^3 + 4x^2}}$

or
$$\frac{1}{2} < \frac{x^2 + 2x}{\sqrt{x^3 + 4x^2}} < \frac{3}{2}.$$

We try $\delta = 0.1$ to see if this is sufficiently small. So, we set the x-range to the interval $[-0.1, 0.1]$ and the y-range to the interval $[0.5, 1.5]$ and redraw the graph in this window. (See Figure 1.50b.) Notice that no points are plotted in the window for any $x < 0$. According to the definition, the y-values must lie in the interval $(0.5, 1.5)$ for *all* x in the interval $(-\delta, \delta)$, except possibly for $x = 0$. Further, you can see that $\delta = 0.1$ clearly does not work since $x = -0.05$ lies in the interval $(-\delta, \delta)$, but $f(-0.05) \approx -0.981$ is not in the interval $(0.5, 1.5)$. You should convince yourself that no matter how small you make δ, there is an x in the interval $(-\delta, \delta)$ such that $f(x) \notin (0.5, 1.5)$. (In fact, notice that for all x's in the interval $(-1, 0)$, $f(x) < 0$.) That is, there is no choice of δ that makes the defining inequality true for $\varepsilon = \frac{1}{2}$. Thus, the conjectured limit of 1 is incorrect.

You should note here that, while we've only shown that the limit is not 1, it's somewhat more complicated to show that the limit does not exist. ■

○ Limits Involving Infinity

Recall that we had observed that $\lim\limits_{x \to 0} \dfrac{1}{x^2}$ does not exist, but to be more descriptive, we had written
$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

By this statement, we mean that the function increases without bound as $x \to 0$. Just as with our initial intuitive notion of $\lim\limits_{x \to a} f(x) = L$, this description is imprecise and needs to be more carefully defined. When we say that $\dfrac{1}{x^2}$ increases without bound as $x \to 0$, we mean that we can make $\dfrac{1}{x^2}$ as large as we like, simply by making x sufficiently close to 0. So, given any large positive number, M, we must be able to make $\dfrac{1}{x^2} > M$, for x sufficiently close to 0. We measure closeness here the same way as we did before and arrive at the following definition.

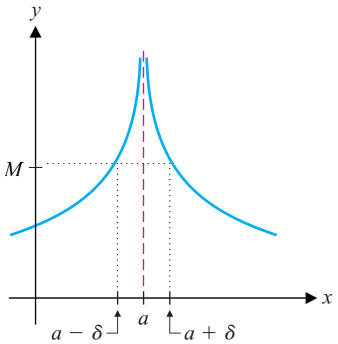


FIGURE 1.51
$\lim_{x \to a} f(x) = \infty$

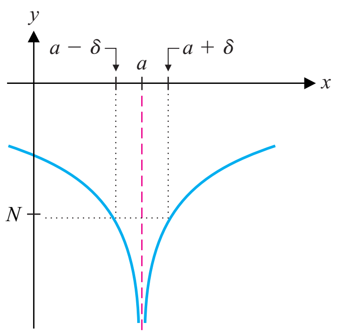


FIGURE 1.52
$\lim_{x \to a} f(x) = -\infty$

DEFINITION 6.2

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = \infty,$$

if given any number $M > 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) > M$. (See Figure 1.51 for a graphical interpretation of this.)

Similarly, we had said that if f decreases without bound as $x \to a$, then $\lim_{x \to a} f(x) = -\infty$. Think of how you would make this more precise and then consider the following definition.

DEFINITION 6.3

For a function f defined in some open interval containing a (but not necessarily at a itself), we say

$$\lim_{x \to a} f(x) = -\infty,$$

if given any number $N < 0$, there is another number $\delta > 0$, such that $0 < |x - a| < \delta$ guarantees that $f(x) < N$. (See Figure 1.52 for a graphical interpretation of this.)

It's easy to keep these definitions straight if you think of their meaning. Don't simply memorize them.

EXAMPLE 6.8 Using the Definition of Limit Where the Limit Is Infinite

Prove that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solution Given any (large) number $M > 0$, we need to find a distance $\delta > 0$ such that if x is within δ of 0 (but not equal to 0) then

$$\frac{1}{x^2} > M. \tag{6.5}$$

Since both M and x^2 are positive, (6.5) is equivalent to

$$x^2 < \frac{1}{M}.$$

Taking the square root of both sides and recalling that $\sqrt{x^2} = |x|$, we get

$$|x| < \sqrt{\frac{1}{M}}.$$

So, for any $M > 0$, if we take $\delta = \sqrt{\frac{1}{M}}$ and work backward, we have that $0 < |x - 0| < \delta$ guarantees that

$$\frac{1}{x^2} > M,$$

as desired. Note that this says, for instance, that for $M = 100$, $\frac{1}{x^2} > 100$, whenever $0 < |x| < \sqrt{\frac{1}{100}} = \frac{1}{10}$. (Verify that this works, as an exercise.) ■

There are two remaining limits that we have yet to place on a careful footing. Before reading on, try to figure out for yourself what appropriate definitions would look like.

If we write $\lim_{x \to \infty} f(x) = L$, we mean that as x increases without bound, $f(x)$ gets closer and closer to L. That is, we can make $f(x)$ as close to L as we like, by choosing x sufficiently large. More precisely, we have the following definition.

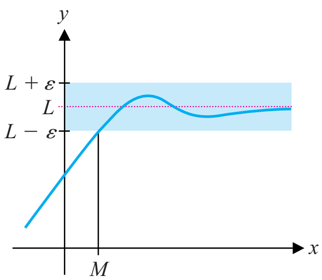


FIGURE 1.53
$\lim_{x \to \infty} f(x) = L$

DEFINITION 6.4

For a function f defined on an interval (a, ∞), for some $a > 0$, we say

$$\lim_{x \to \infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $M > 0$ such that $x > M$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 1.53 for a graphical interpretation of this.)

Similarly, we have said that $\lim_{x \to -\infty} f(x) = L$ means that as x decreases without bound, $f(x)$ gets closer and closer to L. So, we should be able to make $f(x)$ as close to L as desired, just by making x sufficiently large in absolute value and negative. We have the following definition.

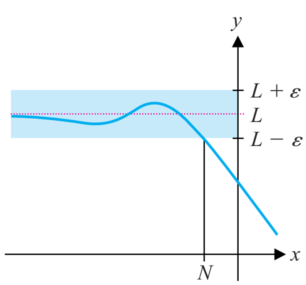


FIGURE 1.54
$\lim_{x \to -\infty} f(x) = L$

DEFINITION 6.5

For a function f defined on an interval $(-\infty, a)$, for some $a < 0$, we say

$$\lim_{x \to -\infty} f(x) = L,$$

if given any $\varepsilon > 0$, there is a number $N < 0$ such that $x < N$ guarantees that

$$|f(x) - L| < \varepsilon.$$

(See Figure 1.54 for a graphical interpretation of this.)

We use Definitions 6.4 and 6.5 essentially the same as we do Definitions 6.1–6.3, as we see in example 6.9.

EXAMPLE 6.9 Using the Definition of Limit Where x Tends to $-\infty$

Prove that $\lim_{x \to -\infty} \frac{1}{x} = 0$.

Solution Here, we must show that given any $\varepsilon > 0$, we can make $\frac{1}{x}$ within ε of 0, simply by making x sufficiently large in absolute value and negative. So, we need to determine those x's for which

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

or

$$\left| \frac{1}{x} \right| < \varepsilon. \tag{6.6}$$

REMARK 6.2

You should take care to note the commonality among the definitions of the five limits we have given. All five deal with a precise description of what it means to be "close." It is of considerable benefit to work through these definitions until you can provide your own words for each. Don't just memorize the formal definitions as stated here. Rather, work toward understanding what they mean and come to appreciate the exacting language mathematicians use.

Since $x < 0$, $|x| = -x$ and so (6.6) becomes

$$\frac{1}{-x} < \varepsilon.$$

Dividing both sides by ε and multiplying by x (remember that $x < 0$ and $\varepsilon > 0$, so that this will change the direction of the inequality), we get

$$-\frac{1}{\varepsilon} > x.$$

So, if we take $N = -\frac{1}{\varepsilon}$ and work backward, we have satisfied the definition and thereby proved that the limit is correct. ■

We don't use the limit definitions to prove each and every limit that comes along. Actually, we use them to prove only a few basic limits and to prove the limit theorems that we've been using for some time without proof. Further use of these theorems then provides solid justification of new limits. As an illustration, we now prove the rule for a limit of a sum.

THEOREM 6.1

Suppose that for a real number a, $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Then,

$$\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2.$$

PROOF

Since $\lim_{x\to a} f(x) = L_1$, we know that given any number $\varepsilon_1 > 0$, there is a number $\delta_1 > 0$ for which

$$0 < |x - a| < \delta_1 \text{ guarantees that } |f(x) - L_1| < \varepsilon_1. \tag{6.7}$$

Likewise, since $\lim_{x\to a} g(x) = L_2$, we know that given any number $\varepsilon_2 > 0$, there is a number $\delta_2 > 0$ for which

$$0 < |x - a| < \delta_2 \text{ guarantees that } |g(x) - L_2| < \varepsilon_2. \tag{6.8}$$

Now, in order to get

$$\lim_{x\to a}[f(x) + g(x)] = (L_1 + L_2),$$

we must show that, given any number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$0 < |x - a| < \delta \text{ guarantees that } |[f(x) + g(x)] - (L_1 + L_2)| < \varepsilon.$$

Notice that

$$\begin{aligned} |[f(x) + g(x)] - (L_1 + L_2)| &= |[f(x) - L_1] + [g(x) - L_2]| \\ &\le |f(x) - L_1| + |g(x) - L_2|, \end{aligned} \tag{6.9}$$

by the triangle inequality. Of course, both terms on the right-hand side of (6.9) can be made arbitrarily small, from (6.7) and (6.8). In particular, if we take $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, then as long as

$$0 < |x - a| < \delta_1 \quad \textit{and} \quad 0 < |x - a| < \delta_2,$$

we get from (6.7), (6.8) and (6.9) that

$$|[f(x) + g(x)] - (L_1 + L_2)| \le |f(x) - L_1| + |g(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. Of course, this will happen if we take

$$0 < |x - a| < \delta = \min\{\delta_1, \delta_2\}. \blacksquare$$

The other rules for limits are proven similarly, using the definition of limit. We show these in Appendix A.

EXERCISES 1.6

WRITING EXERCISES

1. In his 1687 masterpiece *Mathematical Principles of Natural Philosophy,* which introduces many of the fundamentals of calculus, Sir Isaac Newton described the important limit $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ (which we study at length in Chapter 2) as "the limit to which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor ever reach until the quantities vanish." If you ever get weary of all the notation that we use in calculus, think of what it would look like in words! Critique Newton's definition of limit, addressing the following questions in the process. What restrictions do the phrases "never go beyond" and "never reach" put on the limit process? Give an example of a simple limit, not necessarily of the form $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$, that violates these restrictions. Give your own (English language) description of the limit, avoiding restrictions such as Newton's. Why do mathematicians consider the ε–δ definition simple and elegant?
2. You have computed numerous limits before seeing the definition of limit. Explain how this definition changes and/or improves your understanding of the limit process.
3. Each word in the ε–δ definition is carefully chosen and precisely placed. Describe what is wrong with each of the following slightly incorrect "definitions" (use examples!):
   (a) There exists $\varepsilon > 0$ such that there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
   (b) For all $\varepsilon > 0$ and for all $\delta > 0$, if $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.
   (c) For all $\delta > 0$ there exists $\varepsilon > 0$ such that $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.
4. In order for the limit to exist, given every $\varepsilon > 0$, we must be able to find a $\delta > 0$ such that the if/then inequalities are true. To prove that the limit does not exist, we must find a particular $\varepsilon > 0$ such that the if/then inequalities are not true for any choice of $\delta > 0$. To understand the logic behind the swapping of the "for every" and "there exists" roles, draw an analogy with the following situation. Suppose the statement, "Everybody loves somebody" is true. If you wanted to verify the statement, why would you have to talk to every person on earth? But, suppose that the statement is not true. What would you have to do to disprove it?

In exercises 1–8, numerically and graphically determine a δ corresponding to (a) $\varepsilon = 0.1$ and (b) $\varepsilon = 0.05$. Graph the function in the $\varepsilon - \delta$ window [x-range is $(a - \delta, a + \delta)$ and y-range is $(L - \varepsilon, L + \varepsilon)$] to verify that your choice works.

1. $\lim_{x\to 0}(x^2 + 1) = 1$
2. $\lim_{x\to 1}(x^2 + 1) = 2$
3. $\lim_{x\to 0} \cos x = 1$
4. $\lim_{x\to \pi/2} \cos x = 0$
5. $\lim_{x\to 1} \sqrt{x+3} = 2$
6. $\lim_{x\to -2} \sqrt{x+3} = 1$
7. $\lim_{x\to 1} \frac{x+2}{x^2} = 3$
8. $\lim_{x\to 2} \frac{x+2}{x^2} = 1$

In exercises 9–20, symbolically find δ in terms of ε.

9. $\lim_{x\to 0} 3x = 0$
10. $\lim_{x\to 1} 3x = 3$
11. $\lim_{x\to 2}(3x + 2) = 8$
12. $\lim_{x\to 1}(3x + 2) = 5$
13. $\lim_{x\to 1}(3 - 4x) = -1$
14. $\lim_{x\to -1}(3 - 4x) = 7$
15. $\lim_{x\to 1} \frac{x^2 + x - 2}{x - 1} = 3$
16. $\lim_{x\to -1} \frac{x^2 - 1}{x + 1} = -2$
17. $\lim_{x\to 1}(x^2 - 1) = 0$
18. $\lim_{x\to 1}(x^2 - x + 1) = 1$
19. $\lim_{x\to 2}(x^2 - 1) = 3$
20. $\lim_{x\to 0}(x^3 + 1) = 1$
21. Determine a formula for δ in terms of ε for $\lim_{x\to a}(mx + b)$. (Hint: Use exercises 9–14.) Does the formula depend on the value of a? Try to explain this answer graphically.
22. Based on exercises 17 and 19, does the value of δ depend on the value of a for $\lim_{x\to a}(x^2 + b)$? Try to explain this graphically.

23. Modify the ε-δ definition to define the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

24. Symbolically find the largest δ corresponding to $\varepsilon = 0.1$ in the definition of $\lim_{x\to 1^-} 1/x = 1$. Symbolically find the largest δ corresponding to $\varepsilon = 0.1$ in the definition of $\lim_{x\to 1^+} 1/x = 1$. Which δ could be used in the definition of $\lim_{x\to 1} 1/x = 1$? Briefly explain. Then prove that $\lim_{x\to 1} 1/x = 1$.

In exercises 25–30, find a δ corresponding to $M = 100$ or $N = -100$ (as appropriate) for each limit.

25. $\lim_{x\to 1^+} \dfrac{2}{x-1} = \infty$

26. $\lim_{x\to 1^-} \dfrac{2}{x-1} = -\infty$

27. $\lim_{x\to 0^+} \cot x = \infty$

28. $\lim_{x\to \pi^-} \cot x = -\infty$

29. $\lim_{x\to 2^-} \dfrac{2}{\sqrt{4-x^2}} = \infty$

30. $\lim_{x\to 1^-} \dfrac{x}{x^2-1} = -\infty$

In exercises 31–36, find an M or N corresponding to $\varepsilon = 0.1$ for each limit at infinity.

31. $\lim_{x\to \infty} \dfrac{x^2-2}{x^2+x+1} = 1$

32. $\lim_{x\to \infty} \dfrac{x-2}{x^2+x+1} = 0$

33. $\lim_{x\to -\infty} \dfrac{x^2+3}{4x^2-4} = 0.25$

34. $\lim_{x\to -\infty} \dfrac{3x^2-2}{x^2+1} = 3$

35. $\lim_{x\to \infty} \dfrac{x}{\sqrt{x^2+10}} = 1$

36. $\lim_{x\to \infty} \dfrac{x^2+x}{x^2+2x+1} = 1$

In exercises 37–46, prove that the limit is correct using the appropriate definition (assume that k is an integer).

37. $\lim_{x\to \infty} \dfrac{2}{x^3} = 0$

38. $\lim_{x\to -\infty} \dfrac{3}{x^3} = 0$

39. $\lim_{x\to \infty} \dfrac{1}{x^k} = 0,$ for $k > 0$

40. $\lim_{x\to -\infty} \dfrac{1}{x^{2k}} = 0,$ for $k > 0$

41. $\lim_{x\to \infty} \left(\dfrac{1}{x^2+2} - 3\right) = -3$

42. $\lim_{x\to \infty} \dfrac{1}{(x-7)^2} = 0$

43. $\lim_{x\to -3} \dfrac{-2}{(x+3)^4} = -\infty$

44. $\lim_{x\to 7} \dfrac{3}{(x-7)^2} = \infty$

45. $\lim_{x\to 5} \dfrac{4}{(x-5)^2} = \infty$

46. $\lim_{x\to -4} \dfrac{-6}{(x+4)^6} = -\infty$

In exercises 47–50, identify a specific $\varepsilon > 0$ for which no $\delta > 0$ exists to satisfy the definition of limit.

47. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2+3 & \text{if } x > 1 \end{cases}, \quad \lim_{x\to 1} f(x) \neq 2$

48. $f(x) = \begin{cases} x^2-1 & \text{if } x < 0 \\ -x-2 & \text{if } x > 0 \end{cases}, \quad \lim_{x\to 0} f(x) \neq -2$

49. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ 5-x^2 & \text{if } x > 1 \end{cases}, \quad \lim_{x\to 1} f(x) \neq 2$

50. $f(x) = \begin{cases} x-1 & \text{if } x < 2 \\ x^2 & \text{if } x > 2 \end{cases}, \quad \lim_{x\to 2} f(x) \neq 1$

51. A metal washer of (outer) radius r inches weighs $2r^2$ ounces. A company manufactures 2-inch washers for different customers who have different error tolerances. If the customer demands a washer of weight $8 \pm \varepsilon$ ounces, what is the error tolerance for the radius? That is, find δ such that a radius of r within the interval $(2-\delta, 2+\delta)$ guarantees a weight within $(8-\varepsilon, 8+\varepsilon)$.

52. A fiberglass company ships its glass as spherical marbles. If the volume of each marble must be within ε of $\pi/6$, how close does the radius need to be to $1/2$?

53. Prove Theorem 3.1 (i).

54. Prove Theorem 3.1 (ii).

55. Prove the Squeeze Theorem, as stated in Theorem 3.5.

56. Given that $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, prove that $\lim_{x\to a} f(x) = L$.

57. Prove: if $\lim_{x\to a} f(x) = L$, then $\lim_{x\to a} [f(x) - L] = 0$.

58. Prove: if $\lim_{x\to a} [f(x) - L] = 0$, then $\lim_{x\to a} f(x) = L$.

59. In this exercise, we explore the definition of $\lim_{x\to 2} x^2 = 4$ with $\varepsilon = 0.1$. Show that $x^2 - 4 < 0.1$ if $2 < x < \sqrt{4.1}$. This indicates that $\delta_1 = 0.02484$ works for $x > 2$. Show that $x^2 - 4 > -0.1$ if $\sqrt{3.9} < x < 2$. This indicates that $\delta_2 = 0.02515$ works for $x < 2$. For the limit definition, is $\delta = \delta_1$ or $\delta = \delta_2$ the correct choice? Briefly explain.

60. Generalize exercise 59 to find a δ of the form $\sqrt{4+\varepsilon}$ or $\sqrt{4-\varepsilon}$ corresponding to any $\varepsilon > 0$.

EXPLORATORY EXERCISES

1. We hope that working through this section has provided you with extra insight into the limit process. However, we have not yet solved any problems we could not already solve in previous sections. We do so now, while investigating an unusual function. Recall that rational numbers can be written as fractions p/q, where p and q are integers. We will assume that p/q has been simplified by dividing out common factors (e.g., $1/2$ and not $2/4$). Define $f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = \frac{p}{q} \text{ is rational} \end{cases}$. We will try to show that $\lim_{x\to 2/3} f(x)$ exists. Without graphics, we need a good definition to answer this question. We know that $f(2/3) = 1/3$, but recall that the limit is independent of the actual function value. We need to think about x's close to $2/3$. If such an x is irrational, $f(x) = 0$. A simple hypothesis would then be $\lim_{x \to 2/3} f(x) = 0$. We'll try this out for $\varepsilon = 1/6$. We would like to guarantee that $|f(x)| < 1/6$ whenever $0 < |x - 2/3| < \delta$. Well, how many x's have a function value greater than $1/6$? The only possible function values are $1/5$, $1/4$, $1/3$, $1/2$ and 1. The x's with function value $1/5$ are $1/5$, $2/5$, $3/5$, $4/5$ and so on. The closest of these x's to $2/3$ is $3/5$. Find the closest x (not counting $x = 2/3$) to $2/3$ with function value $1/4$. Repeat for $f(x) = 1/3$, $f(x) = 1/2$ and $f(x) = 1$. Out of all these closest x's, how close is the absolute closest? Choose δ to be this number, and argue that if $0 < |x - 2/3| < \delta$, we are guaranteed that $|f(x)| < 1/6$. Argue that a similar process can find a δ for any ε.

2. State a definition for "$f(x)$ is continuous at $x = a$" using Definition 6.1. Use it to prove that the function in exploratory exercise 1 is continuous at every irrational number and discontinuous at every rational number.

1.7 LIMITS AND LOSS-OF-SIGNIFICANCE ERRORS

"Pay no attention to that man behind the curtain . . . ." (from *The Wizard of Oz*)

Things are not always what they appear to be. We spend much time learning to distinguish reality from mere appearances. Along the way, we develop a healthy level of skepticism. You may have already come to realize that mathematicians are a skeptical lot. This is of necessity, for you simply can't accept things at face value.

People tend to accept a computer's answer as a fact not subject to debate. However, when we use a computer (or calculator), we must always keep in mind that these devices perform most computations only approximately. Most of the time, this will cause us no difficulty whatsoever. Modern computational devices generally carry out calculations to a very high degree of accuracy. Occasionally, however, the results of round-off errors in a string of calculations are disastrous. In this section, we briefly investigate these errors and learn how to recognize and avoid some of them.

We first consider a relatively tame-looking example.

EXAMPLE 7.1 A Limit with Unusual Graphical and Numerical Behavior

Evaluate $\lim_{x \to \infty} \frac{(x^3 + 4)^2 - x^6}{x^3}$.

Solution At first glance, the numerator looks like $\infty - \infty$, which is indeterminate, while the denominator tends to ∞. Algebraically, the only reasonable step to take is to multiply out the first term in the numerator. Before we do that, let's draw a graph and compute some function values. (Different computers and different software will produce somewhat different results, but for large values of x, you should see results similar to those shown here.) In Figure 1.55a, the function appears nearly constant, until it begins oscillating around $x = 40{,}000$. Notice that the accompanying table of function values is inconsistent with Figure 1.55a.

The last two values in the table may have surprised you. Up until that point, the function values seemed to be settling down to 8.0 very nicely. So, what happened here and what is the correct value of the limit? Obviously, something unusual has occurred

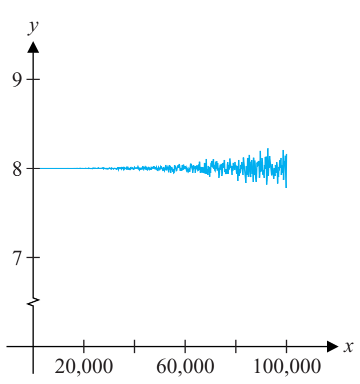


FIGURE 1.55a

$y = \frac{(x^3 + 4)^2 - x^6}{x^3}$

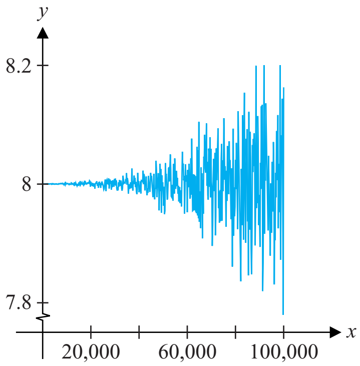


FIGURE 1.55b

$$y = \frac{(x^3+4)^2 - x^6}{x^3}$$

between $x = 1 \times 10^4$ and $x = 1 \times 10^5$. We should look carefully at function values in that interval. A more detailed table is shown below to the right.

Incorrect calculated values

x	$\frac{(x^3+4)^2 - x^6}{x^3}$
10	8.016
100	8.000016
1×10^3	8.0
1×10^4	8.0
1×10^5	0.0
1×10^6	0.0

x	$\frac{(x^3+4)^2 - x^6}{x^3}$
2×10^4	8.0
3×10^4	8.14815
4×10^4	7.8125
5×10^4	0

In Figure 1.55b, we have blown up the graph to enhance the oscillation observed between $x = 1 \times 10^4$ and $x = 1 \times 10^5$. The picture that is emerging is even more confusing. The deeper we look into this limit, the more erratically the function appears to behave. We use the word *appears* because all of the oscillatory behavior we are seeing is an illusion, created by the finite precision of the computer used to perform the calculations or draw the graph. ■

○ Computer Representation of Real Numbers

The reason for the unusual behavior seen in example 7.1 boils down to the way in which computers represent real numbers. Without getting into all of the intricacies of computer arithmetic, it suffices to think of computers and calculators as storing real numbers internally in scientific notation. For example, the number 1,234,567 would be stored as 1.234567×10^6. The number preceding the power of 10 is called the **mantissa** and the power is called the **exponent.** Thus, the mantissa here is 1.234567 and the exponent is 6.

All computing devices have finite memory and consequently have limitations on the size mantissa and exponent that they can store. (This is called **finite precision.**) Many calculators carry a 14-digit mantissa and a 3-digit exponent. On a 14-digit computer, this would suggest that the computer would retain only the first 14 digits in the decimal expansion of any given number.

EXAMPLE 7.2 Computer Representation of a Rational Number

Determine how $\frac{1}{3}$ is stored internally on a 10-digit computer and how $\frac{2}{3}$ is stored internally on a 14-digit computer.

Solution On a 10-digit computer, $\frac{1}{3}$ is stored internally as $\underbrace{3.333333333}_{10\text{ digits}} \times 10^{-1}$. On a 14-digit computer, $\frac{2}{3}$ is stored internally as $\underbrace{6.6666666666667}_{14\text{ digits}} \times 10^{-1}$. ■

For most purposes, such finite precision presents no problem. However, we do occasionally come across a disastrous error caused by finite precision. In example 7.3, we subtract two relatively close numbers and examine the resulting catastrophic error.

EXAMPLE 7.3 A Computer Subtraction of Two "Close" Numbers

Compare the exact value of

$$1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{18} - 1.\underbrace{0000000000000}_{13\text{ zeros}}1 \times 10^{18}$$

with the result obtained from a calculator or computer with a 14-digit mantissa.

Solution Notice that

$$\begin{aligned} 1.\underbrace{0000000000000}_{13\text{ zeros}}4\times 10^{18} - 1.\underbrace{0000000000000}_{13\text{ zeros}}1\times 10^{18} &= 0.\underbrace{0000000000000}_{13\text{ zeros}}3\times 10^{18} \\ &= 30{,}000. \end{aligned} \tag{7.1}$$

However, if this calculation is carried out on a computer or calculator with a 14-digit (or smaller) mantissa, both numbers on the left-hand side of (7.1) are stored by the computer as 1×10^{18} and hence, the difference is calculated as 0. Try this calculation for yourself now. ■

EXAMPLE 7.4 Another Subtraction of Two "Close" Numbers

Compare the exact value of

$$1.\underbrace{0000000000000}_{13\text{ zeros}}6 \times 10^{20} - 1.\underbrace{0000000000000}_{13\text{ zeros}}4 \times 10^{20}$$

with the result obtained from a calculator or computer with a 14-digit mantissa.

Solution Notice that

$$\begin{aligned} 1.\underbrace{0000000000000}_{13\text{ zeros}}6\times 10^{20} - 1.\underbrace{0000000000000}_{13\text{ zeros}}4\times 10^{20} &= 0.\underbrace{0000000000000}_{13\text{ zeros}}2\times 10^{20} \\ &= 2{,}000{,}000. \end{aligned}$$

However, if this calculation is carried out on a calculator with a 14-digit mantissa, the first number is represented as $1.0000000000001 \times 10^{20}$, while the second number is represented by 1.0×10^{20}, due to the finite precision and rounding. The difference between the two values is then computed as $0.0000000000001 \times 10^{20}$ or 10,000,000, which is, again, a very serious error. ■

In examples 7.3 and 7.4, we witnessed a gross error caused by the subtraction of two numbers whose significant digits are very close to one another. This type of error is called a **loss-of-significant-digits error** or simply a **loss-of-significance error.** These are subtle, often disastrous computational errors. Returning now to example 7.1, we will see that it was this type of error that caused the unusual behavior noted.

EXAMPLE 7.5 A Loss-of-Significance Error

In example 7.1, we considered the function $f(x) = \dfrac{(x^3+4)^2 - x^6}{x^3}$.
Follow the calculation of $f(5 \times 10^4)$ one step at a time, as a 14-digit computer would do it.

Solution We have

$$f(5 \times 10^4) = \frac{[(5 \times 10^4)^3 + 4]^2 - (5 \times 10^4)^6}{(5 \times 10^4)^3}$$
$$= \frac{(1.25 \times 10^{14} + 4)^2 - 1.5625 \times 10^{28}}{1.25 \times 10^{14}}$$
$$= \frac{(125{,}000{,}000{,}000{,}000 + 4)^2 - 1.5625 \times 10^{28}}{1.25 \times 10^{14}}$$
$$= \frac{(1.25 \times 10^{14})^2 - 1.5625 \times 10^{28}}{1.25 \times 10^{14}} = 0,$$

since 125,000,000,000,004 is rounded off to 125,000,000,000,000.

Note that the real culprit here was not the rounding of 125,000,000,000,004, but the fact that this was followed by a subtraction of a nearly equal value. Further, note that this is not a problem unique to the numerical computation of limits, but one that occurs in numerical computation, in general. ■

REMARK 7.1

If at all possible, avoid subtractions of nearly equal values. Sometimes, this can be accomplished by some algebraic manipulation of the function.

In the case of the function from example 7.5, we can avoid the subtraction and hence, the loss-of-significance error by rewriting the function as follows:

$$f(x) = \frac{(x^3 + 4)^2 - x^6}{x^3}$$
$$= \frac{(x^6 + 8x^3 + 16) - x^6}{x^3}$$
$$= \frac{8x^3 + 16}{x^3},$$

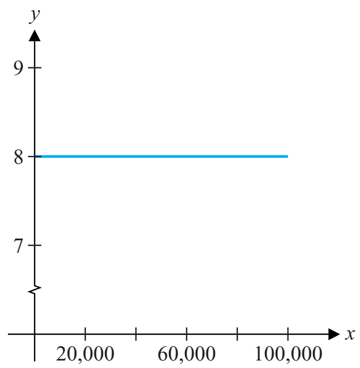


FIGURE 1.56

$y = \frac{8x^3 + 16}{x^3}$

x	$\frac{8x^3 + 16}{x^3}$
10	8.016
100	8.000016
1×10^3	8.000000016
1×10^4	8.00000000002
1×10^5	8.0
1×10^6	8.0
1×10^7	8.0

where we have eliminated the subtraction. Using this new (and equivalent) expression for the function, we can compute a table of function values reliably. Notice, too, that if we redraw the graph in Figure 1.55a using the new expression (see Figure 1.56), we no longer see the oscillation present in Figures 1.55a and 1.55b.

From the rewritten expression, we easily obtain

$$\lim_{x \to \infty} \frac{(x^3 + 4)^2 - x^6}{x^3} = 8,$$

which is consistent with Figure 1.56 and the corrected table of function values.

In example 7.6, we examine a loss-of-significance error that occurs for x close to 0.

EXAMPLE 7.6 Loss-of-Significance Involving a Trigonometric Function

Evaluate $\lim_{x \to 0} \frac{1 - \cos x^2}{x^4}$.

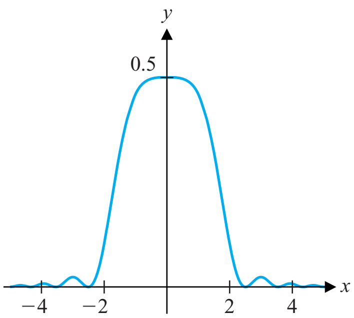


FIGURE 1.57
$y = \frac{1 - \cos x^2}{x^4}$

x	$\frac{\sin^2(x^2)}{x^4(1 + \cos x^2)}$
±0.1	0.499996
±0.01	0.4999999996
±0.001	0.5
±0.0001	0.5
±0.00001	0.5

Solution As usual, we look at a graph (see Figure 1.57) and some function values.

x	$\frac{1 - \cos x^2}{x^4}$
0.1	0.499996
0.01	0.5
0.001	0.5
0.0001	0.0
0.00001	0.0

x	$\frac{1 - \cos x^2}{x^4}$
−0.1	0.499996
−0.01	0.5
−0.001	0.5
−0.0001	0.0
−0.00001	0.0

As in example 7.1, note that the function values seem to be approaching 0.5, but then suddenly take a jump down to 0.0. Once again, we are seeing a loss-of-significance error. In this particular case, this occurs because we are subtracting nearly equal values ($\cos x^2$ and 1). We can again avoid the error by eliminating the subtraction. Note that

$$\frac{1 - \cos x^2}{x^4} = \frac{1 - \cos x^2}{x^4} \cdot \frac{1 + \cos x^2}{1 + \cos x^2}$$

Multiply numerator and denominator by $(1 + \cos x^2)$.

$$= \frac{1 - \cos^2(x^2)}{x^4(1 + \cos x^2)}$$

$1 - \cos^2(x^2) = \sin^2(x^2)$.

$$= \frac{\sin^2(x^2)}{x^4(1 + \cos x^2)}.$$

Since this last (equivalent) expression has no subtraction indicated, we should be able to use it to reliably generate values without the worry of loss-of-significance error. Using this to compute function values, we get the accompanying table.

Using the graph and the new table, we conjecture that

$$\lim_{x \to 0} \frac{1 - \cos x^2}{x^4} = \frac{1}{2}.$$ ■

We offer one final example where a loss-of-significance error occurs, even though no subtraction is explicitly indicated.

EXAMPLE 7.7 A Loss-of-Significance Error Involving a Sum

Evaluate $\lim_{x \to -\infty} x[(x^2 + 4)^{1/2} + x]$.

Solution Initially, you might think that since there is no subtraction (explicitly) indicated, there will be no loss-of-significance error. We first draw a graph (see Figure 1.58) and compute a table of values.

x	$x[(x^2 + 4)^{1/2} + x]$
−100	−1.9998
-1×10^3	−1.999998
-1×10^4	−2.0
-1×10^5	−2.0
-1×10^6	−2.0
-1×10^7	0.0
-1×10^8	0.0

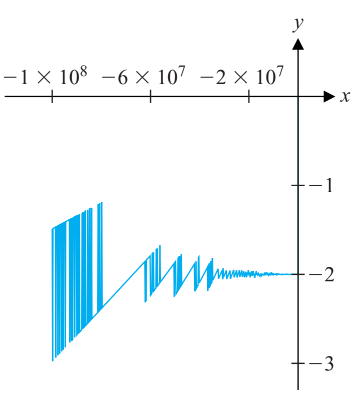


FIGURE 1.58
$y = x[(x^2 + 4)^{1/2} + x]$

You should quickly notice the sudden jump in values in the table and the wild oscillation visible in the graph. Although a subtraction is not explicitly indicated, there is indeed a subtraction here, since we have $x < 0$ and $(x^2+4)^{1/2} > 0$. We can again remedy this with some algebraic manipulation, as follows.

$$x\left[(x^2+4)^{1/2}+x\right] = x\left[(x^2+4)^{1/2}+x\right]\frac{\left[(x^2+4)^{1/2}-x\right]}{\left[(x^2+4)^{1/2}-x\right]}$$

Multiply numerator and denominator by the conjugate.

$$= x\frac{\left[(x^2+4)-x^2\right]}{\left[(x^2+4)^{1/2}-x\right]}$$

Simplify the numerator.

$$= \frac{4x}{\left[(x^2+4)^{1/2}-x\right]}.$$

We use this last expression to generate a graph in the same window as that used for Figure 1.58 and to generate the accompanying table of values. In Figure 1.59, we can see none of the wild oscillation observed in Figure 1.58 and the graph appears to be a horizontal line.

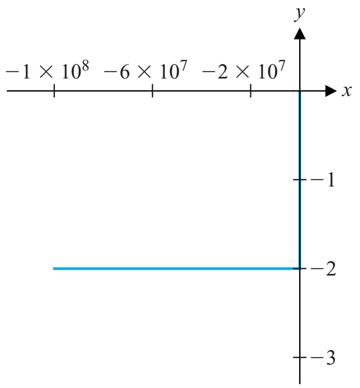


FIGURE 1.59

$$y = \frac{4x}{[(x^2+4)^{1/2}-x]}$$

x	$\dfrac{4x}{\left[(x^2+4)^{1/2}-x\right]}$
-100	-1.9998
-1×10^3	-1.999998
-1×10^4	-1.99999998
-1×10^5	-1.9999999998
-1×10^6	-2.0
-1×10^7	-2.0
-1×10^8	-2.0

Further, the values displayed in the table no longer show the sudden jump indicative of a loss-of-significance error. We can now confidently conjecture that

$$\lim_{x\to-\infty} x[(x^2+4)^{1/2}+x] = -2.$$

BEYOND FORMULAS

In examples 7.5–7.7, we demonstrated calculations that suffered from catastrophic loss-of-significance errors. In each case, we showed how we could rewrite the expression to avoid this error. We have by no means exhibited a general procedure for recognizing and repairing such errors. Rather, we hope that by seeing a few of these subtle errors, and by seeing how to fix even a limited number of them, you will become a more skeptical and intelligent user of technology.

EXERCISES 1.7

WRITING EXERCISES

1. It is probably clear that caution is important in using technology. Equally important is redundancy. This property is sometimes thought to be a negative (wasteful, unnecessary), but its positive role is one of the lessons of this section. By redundancy, we mean investigating a problem using graphical, numerical and symbolic tools. Why is it important to use multiple methods? Answer this from a practical perspective (think of the problems in this section) and a theoretical perspective (if you have learned multiple techniques, do you understand the mathematics better?).

2. The drawback of caution and redundancy is that they take extra time. In computing limits, when should you stop and take extra time to make sure an answer is correct and when is it safe to go on to the next problem? Should you always look at a graph? compute function values? do symbolic work? an ε–δ proof? Prioritize the techniques in this chapter.

3. The limit $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ will be very important in Chapter 2. For a specific function and specific a, we could compute a table of values of the fraction for smaller and smaller values of h. Why should we be wary of loss-of-significance errors?

4. Notice that we rationalized the numerator in example 7.7. The old rule of rationalizing the denominator is another example of rewriting an expression to try to minimize computational errors. Before computers, square roots were very difficult to compute. To see one reason why you might want the square root in the numerator, suppose that you can get only one decimal place of accuracy, so that $\sqrt{3} \approx 1.7$. Compare $\frac{6}{1.7}$ to $\frac{6}{\sqrt{3}}$ and then compare $2(1.7)$ to $\frac{6}{\sqrt{3}}$. Which of the approximations could you do in your head?

In exercises 1–12, (a) use graphics and numerics to conjecture a value of the limit. (b) Find a computer or calculator graph showing a loss-of-significance error. (c) Rewrite the function to avoid the loss-of-significance error.

1. $\lim_{x \to \infty} x\left(\sqrt{4x^2+1} - 2x\right)$

2. $\lim_{x \to -\infty} x\left(\sqrt{4x^2+1} + 2x\right)$

3. $\lim_{x \to \infty} \sqrt{x}\left(\sqrt{x+4} - \sqrt{x+2}\right)$

4. $\lim_{x \to \infty} x^2\left(\sqrt{x^4+8} - x^2\right)$

5. $\lim_{x \to \infty} x\left(\sqrt{x^2+4} - \sqrt{x^2+2}\right)$

6. $\lim_{x \to \infty} x\left(\sqrt{x^3+8} - x^{3/2}\right)$

7. $\lim_{x \to 0} \frac{1 - \cos 2x}{12x^2}$

8. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

9. $\lim_{x \to 0} \frac{1 - \cos x^3}{x^6}$

10. $\lim_{x \to 0} \frac{1 - \cos x^4}{x^8}$

11. $\lim_{x \to \infty} x^{4/3}\left(\sqrt[3]{x^2+1} - \sqrt[3]{x^2-1}\right)$

12. $\lim_{x \to \infty} x^{2/3}\left(\sqrt[3]{x+4} - \sqrt[3]{x-3}\right)$

In exercises 13 and 14, compare the limits to show that small errors can have disastrous effects.

13. $\lim_{x \to 1} \frac{x^2 + x - 2}{x - 1}$ and $\lim_{x \to 1} \frac{x^2 + x - 2.01}{x - 1}$

14. $\lim_{x \to 2} \frac{x - 2}{x^2 - 4}$ and $\lim_{x \to 2} \frac{x - 2}{x^2 - 4.01}$

15. Compare $f(x) = \sin \pi x$ and $g(x) = \sin 3.14x$ for $x = 1$ (radian), $x = 10$, $x = 100$ and $x = 1000$.

16. For exercise 1, follow the calculation of the function for $x = 10^5$ as it would proceed for a machine computing with a 10-digit mantissa. Identify where the round-off error occurs.

In exercises 17 and 18, compare the exact answer to one obtained by a computer with a six-digit mantissa.

17. $(1.000003 - 1.000001) \times 10^7$

18. $(1.000006 - 1.000001) \times 10^7$

19. If you have access to a CAS, test it on the limits of examples 7.1, 7.6 and 7.7. Based on these results, do you think that your CAS does precise calculations or numerical estimates?

EXPLORATORY EXERCISES

1. In this exercise, we look at one aspect of the mathematical study of chaos. First, iterate the function $f(x) = x^2 - 2$ starting at $x_0 = 0.5$. That is, compute $x_1 = f(0.5)$, then $x_2 = f(x_1)$, then $x_3 = f(x_2)$ and so on. Although the sequence of numbers stays bounded, the numbers never repeat (except by the accident of round-off errors). An impressive property of chaotic functions is the **butterfly effect** (more properly referred to as *sensitive dependence on initial conditions*). The butterfly effect applies to the chaotic nature of weather and states that the amount of air stirred by a butterfly flapping its wings in Brazil can create or disperse a tornado in Texas a few days later. Therefore, long-range weather prediction is impossible. To illustrate the butterfly effect, iterate $f(x)$ starting at $x_0 = 0.5$ and $x_0 = 0.51$. How many iterations does it take before the iterations are more than 0.1 apart? Try this again with $x_0 = 0.5$ and $x_0 = 0.501$. Repeat this exercise for the function $g(x) = x^2 - 1$. Even though the functions are almost identical, $g(x)$ is not chaotic and behaves very differently. This represents an important idea in modern medical research called dynamical diseases: a small change in

one of the constants in a function (e.g., the rate of an electrical signal within the human heart) can produce a dramatic change in the behavior of the system (e.g., the pumping of blood from the ventricles).

2. Just as we are subject to round-off error in using calculations from a computer, so are we subject to errors in a computer-generated graph. After all, the computer has to compute function values before it can decide where to plot points. On your computer or calculator, graph $y = \sin x^2$ (a disconnected graph—or point plot—is preferable). You should see the oscillations you expect from the sine function, but with the oscillations getting faster as x gets larger. Shift your graphing window to the right several times. At some point, the plot will become very messy and almost unreadable. Depending on your technology, you may see patterns in the plot. Are these patterns real or an illusion? To explain what is going on, recall that a computer graph is a finite set of pixels, with each pixel representing one x and one y. Suppose the computer is plotting points at $x = 0$, $x = 0.1$, $x = 0.2$ and so on. The y-values would then be $\sin 0^2$, $\sin 0.1^2$, $\sin 0.2^2$ and so on. Investigate what will happen between $x = 15$ and $x = 16$. Compute all the points $(15, \sin 15^2)$, $(15.1, \sin 15.1^2)$ and so on. If you were to graph these points, what pattern would emerge? To explain this pattern, argue that there is approximately half a period of the sine curve missing between each point plotted. Also, investigate what happens between $x = 31$ and $x = 32$.

Review Exercises

WRITING EXERCISES

The following list includes terms that are defined and theorems that are stated in this chapter. For each term or theorem, (1) give a precise definition or statement, (2) state in general terms what it means and (3) describe the types of problems with which it is associated.

Secant line	Limit	Infinite limit
One-sided limit	Continuous	Loss-of-significance error
Removable discontinuity	Horizontal asymptote	Slant asymptote
Vertical asymptote	Squeeze Theorem	Intermediate Value Theorem
Method of bisections	Length of line segment	
Slope of curve		

TRUE OR FALSE

State whether each statement is true or false and briefly explain why. If the statement is false, try to "fix it" by modifying the given statement to make a new statement that is true.

1. In calculus, problems are often solved by first approximating the solution and then improving the approximation.
2. If $f(1.1) = 2.1$, $f(1.01) = 2.01$ and so on, then $\lim_{x\to 1} f(x) = 2$.
3. $\lim_{x\to a}[f(x)\cdot g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$
4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
5. If $f(2) = 1$ and $f(4) = 2$, then there exists an x between 2 and 4 such that $f(x) = 0$.
6. For any polynomial $p(x)$, $\lim_{x\to\infty} p(x) = \infty$.
7. If $f(x) = \dfrac{p(x)}{q(x)}$ for polynomials p and q with $q(a) = 0$, then f has a vertical asymptote at $x = a$.
8. Small round-off errors typically have only small effects on a calculation.
9. $\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a} \sqrt{f(x)} = \sqrt{L}$.

In exercises 1 and 2, numerically estimate the slope of $y = f(x)$ at $x = a$.

1. $f(x) = x^2 - 2x$, $a = 2$
2. $f(x) = \sin 2x$, $a = 0$

In exercises 3 and 4, numerically estimate the length of the curve using (a) $n = 4$ and (b) $n = 8$ line segments and evenly spaced x-coordinates.

3. $f(x) = \sin x$, $0 \le x \le \frac{\pi}{4}$
4. $f(x) = x^2 - x$, $0 \le x \le 2$

In exercises 5–10, use numerical and graphical evidence to conjecture the value of the limit.

5. $\lim_{x\to 0} \dfrac{\tan(x^3)}{x^2}$
6. $\lim_{x\to 1} \dfrac{x^2 - 1}{\cos \pi x + 1}$
7. $\lim_{x\to -2} \dfrac{x+2}{|x+2|}$
8. $\lim_{x\to 0} \tan \dfrac{1}{x}$
9. $\lim_{x\to -\infty} \dfrac{\sqrt{x^2+4}}{3x+1}$
10. $\lim_{x\to\infty} \dfrac{4x^2 + x - 1}{\sqrt{x^4+6}}$

Review Exercises

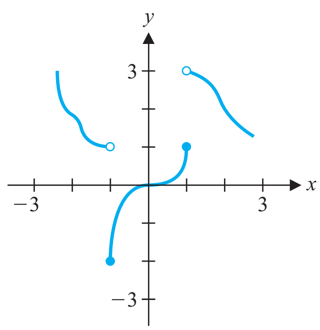
In exercises 11 and 12, identify the limits from the graph of f.

11. (a) $\lim_{x\to -1^-} f(x)$ (b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$ (d) $\lim_{x\to 0} f(x)$

12. (a) $\lim_{x\to 1^-} f(x)$ (b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$ (d) $\lim_{x\to 2} f(x)$



13. Identify the discontinuities in the function graphed above.

14. Sketch a graph of a function f with $f(-1) = 0$, $f(0) = 0$, $\lim_{x\to 1^-} f(x) = 1$ and $\lim_{x\to 1^+} f(x) = -1$.

In exercises 15–34, evaluate the limit. Answer with a number, ∞, $-\infty$ or does not exist.

15. $\lim_{x\to 2} \frac{x^2 - x - 2}{x^2 - 4}$

16. $\lim_{x\to 1} \frac{x^2 - 1}{x^2 + x - 2}$

17. $\lim_{x\to 0} \frac{x^2 + x}{\sqrt{x^4 + 2x^2}}$

18. $\lim_{x\to 0} \frac{x^3 + 2x^2}{\sqrt{x^8 + 4x^4}}$

19. $\lim_{x\to 0} (2 + x)\sin(1/x)$

20. $\lim_{x\to 0} \frac{\sin x^2}{x^2}$

21. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 3x - 1 & \text{if } x < 2 \\ x^2 + 1 & \text{if } x \ge 2 \end{cases}$

22. $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ x^2 + 1 & \text{if } x \ge 1 \end{cases}$

23. $\lim_{x\to 0} \frac{\sqrt[3]{1 + 2x} - 1}{x}$

24. $\lim_{x\to 1} \frac{x - 1}{\sqrt{10 - x} - 3}$

25. $\lim_{x\to 0} \cot(x^2)$

26. $\lim_{x\to 1} \tan\left(\frac{x}{x^2 - 2x + 1}\right)$

27. $\lim_{x\to \infty} \frac{x^2 - 4}{3x^2 + x + 1}$

28. $\lim_{x\to \infty} \frac{2x}{\sqrt{x^2 + 4}}$

29. $\lim_{x\to \pi/2} - \tan^2 x$

30. $\lim_{x\to 3} \frac{x^2 - 2x - 3}{x^2 + 6x + 9}$

31. $\lim_{x\to -\infty} \frac{2x}{x^2 + 3x - 5}$

32. $\lim_{x\to -2} \frac{2x}{x^2 + 3x + 2}$

33. $\lim_{x\to 0} \frac{\sin x}{|\sin x|}$

34. $\lim_{x\to 0} \frac{2x - |x|}{|3x| - 2x}$

35. Use the Squeeze Theorem to prove that $\lim_{x\to 0} \frac{2x^3}{x^2 + 1} = 0$.

36. Use the Intermediate Value Theorem to verify that $f(x) = x^3 - x - 1$ has a zero in the interval [1, 2]. Use the method of bisections to find an interval of length 1/32 that contains a zero.

In exercises 37–40, find all discontinuities and determine which are removable.

37. $f(x) = \frac{x - 1}{x^2 + 2x - 3}$

38. $f(x) = \frac{x + 1}{x^2 - 4}$

39. $f(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x^2 & \text{if } 0 \le x \le 2 \\ 4x - 3 & \text{if } x > 2 \end{cases}$

40. $f(x) = x \cot x$

In exercises 41–44, find all intervals of continuity.

41. $f(x) = \frac{x + 2}{x^2 - x - 6}$

42. $f(x) = \frac{2x}{\sqrt{3x - 4}}$

43. $f(x) = \sin(1 + \cos x)$

44. $f(x) = \sqrt{x^2 - 4}$

In exercises 45–52, determine all vertical, horizontal and slant asymptotes.

45. $f(x) = \frac{x + 1}{x^2 - 3x + 2}$

46. $f(x) = \frac{x + 2}{x^2 - 2x - 8}$

47. $f(x) = \frac{x^2}{x^2 - 1}$

48. $f(x) = \frac{x^3}{x^2 - x - 2}$

49. $f(x) = \frac{x^3}{x^2 + x + 1}$

50. $f(x) = \frac{2x^2}{x^2 + 4}$

51. $f(x) = \frac{3}{\cos x - 1}$

52. $f(x) = \frac{\cos x - 1}{x + 3}$

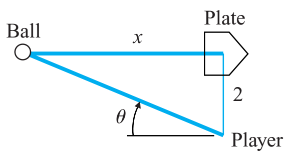
In exercises 53 and 54, (a) use graphical and numerical evidence to conjecture a value for the indicated limit. (b) Find a computer or calculator graph showing a loss-of-significance error. (c) Rewrite the function to avoid the loss-of-significance error.

53. $\lim_{x\to 0} \frac{1 - \cos x}{2x^2}$

54. $\lim_{x\to \infty} x\left(\sqrt{x^2 + 1} - x\right)$

Review Exercises

55. You've heard the sports cliche "keep your eye on the ball." In the diagram, x is the distance from the ball to home plate and θ is an angle indicating the direction of the player's gaze. We denote the speed of the ball by x' and the rate of change of the player's gaze by θ'. For a 90-mph baseball pitch, $x' = -132$ ft/s. It can be shown that $\theta' = \frac{264}{4+x^2}$ radians/second. From this formula, explain why θ' increases as x decreases. Explain the same result from physical principles. Finally, compute $\lim_{x\to 0} \theta'$, the maximum rate of change of the player's gaze. This is not an infinite limit. However, given that human beings cannot maintain focus at a rate of more than about 3 radians/second, how big is the maximum θ'? Is it possible for a baseball player to keep his or her eyes on the ball?



EXPLORATORY EXERCISES

1. For $f(x) = \frac{2x^2 - 2x - 4}{x^2 - 5x + 6}$, do the following. (a) Find all values of x at which f is not continuous. (b) Determine which value in (a) is a removable discontinuity. For this value, find the limit of f as x approaches this value. Sketch a portion of the graph of f near this x-value showing the behavior of the function. (c) For the value in part (a) that is not removable, find the two one-sided infinite limits and sketch the graph of f near this x-value. (d) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ and sketch the portion of the graph of f corresponding to these values. (e) Connect the pieces of your graph as simply as possible. If available, compare your graph to a computer-generated graph.

2. Let $f(t)$ represent the price of an autograph of a famous person at time t (years after 2000). Interpret each of the following (independently) in financial terms: (a) horizontal asymptote $y = 1000$, (b) vertical asymptote at $t = 10$, (c) $\lim_{t\to 4^-} f(t) = 500$ and $\lim_{t\to 4^+} f(t) = 800$ and (d) $\lim_{t\to 8} f(t) = 950$.

3. As discussed in this chapter, the limit of a function is the single number, if one exists, that the function approaches as x approaches its limiting value. The limit concept can be generalized to limiting behavior more complicated than a single number. The study of **chaos** (more properly called **nonlinear dynamics**) makes use of this extended concept. We will explore chaos theory at various times during our calculus journey. For now, we look at some examples of different limiting behaviors. We first *iterate* the function $f_2(x) = x(2-x)$. That means we start at some initial x-value, say $x_0 = 0.5$ and compute $x_1 = f_2(x_0)$, then compute $x_2 = f_2(x_1)$, then $x_3 = f_2(x_2)$ and so on. Use a calculator or computer to verify that $x_1 = 0.5(2-0.5) = 0.75$, $x_2 = 0.9375$, $x_3 = 0.99609375$, $x_4 = 0.9999847412$ and so on. You should conclude that the limit of this sequence of calculations is 1. Now, try iterating $f_{3.2}(x) = x(3.2-x)$ starting at $x_0 = 0.5$. Verify that $x_1 = 1.35$, $x_2 = 2.4975$, $x_3 = 1.75449375$, $x_4 = 2.536131681$ and so on. If you continue this process, you will see a different type of limiting behavior: alternation between the values of (approximately) 1.641742431 and 2.558257569. In what way might you compare this limiting behavior to a periodic function? To find other periodic limits, try the functions $f_{3.48}(x) = x(3.48-x)$, $f_{3.555}(x)$, $f_{3.565}(x)$ and $f_{3.569}(x)$. What is the pattern of the size of the periods? Note that the parameter (subscript) of this family of functions is being changed less and less to produce the higher periods. The limit of these subscripts is also of interest. We explore this in exercise 5.

4. In exercise 3, we looked at some examples from the family of functions $f_c(x) = x(c-x)$ for various values of the parameter c. In particular, as we gradually increased c from $c = 2$ to $c = 3.57$, we saw the limiting behavior (called the **attractor**) change from convergence to a single number (called a **one-cycle**) to alternation between two numbers (a **two-cycle**) to alternation among four numbers (a **four-cycle**) to, eventually, chaos (bounded but aperiodic). The transitions from one type of limiting behavior to another occur at special values of c called **bifurcation points.** By trial and error, find the first bifurcation point; that is, find the number b such that $f_c(x)$ has an attracting one-cycle if $c < b$ and an attracting two-cycle if $c > b$.

5. In this exercise, we look at another aspect of the mathematical study of chaos. In the language of exercises 3 and 4, we start by iterating the function $f(x) = x(4-x)$ starting at $x_0 = 0.5$. That is, compute $x_1 = f(0.5)$, then $x_2 = f(x_1)$, then $x_3 = f(x_2)$ and so on. Although the sequence of numbers stays bounded, the numbers never repeat (except by the accident of round-off errors). This is called **mathematical chaos:** bounded but not periodic. The weather is one example of a natural process that is thought to be chaotic. In what sense is the weather (take, for example, the local temperature) bounded but not periodic? Explain why it is inherently impossible to have accurate long-range weather forecasts.