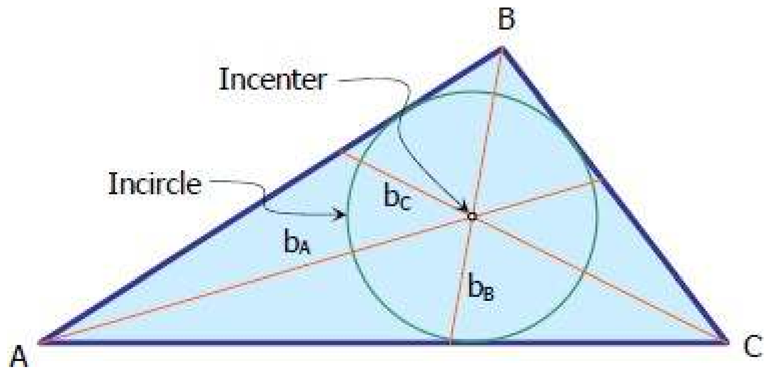
TRIANGLE:

Centers:

Incenter
Incenter is the center of the inscribed circle (incircle) of the triangle, it is the point of intersection of the angle bisectors of the triangle.

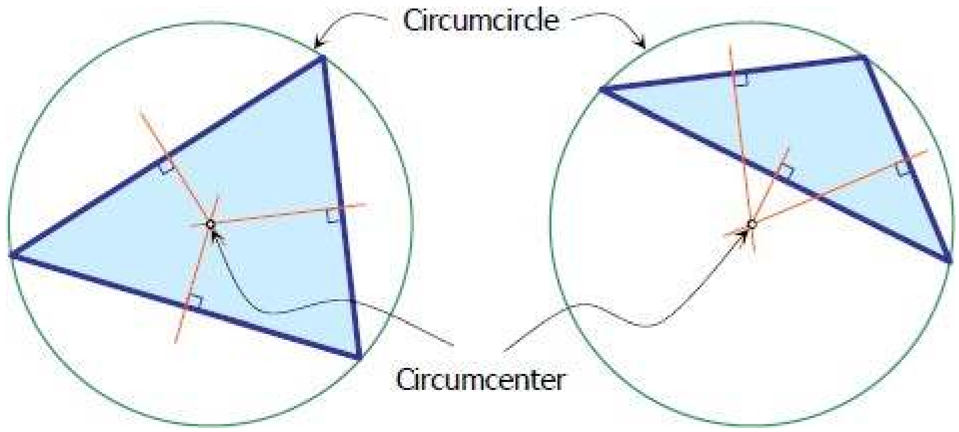


The radius of incircle is given by the formula

r=At/s

where A_t = area of the triangle and s = ½ (a + b + c). See the derivation of formula for radius of incircle.

Circumcenter
Circumcenter is the point of intersection of perpendicular bisectors of the triangle. It is also the center of the circumscribing circle (circumcircle).

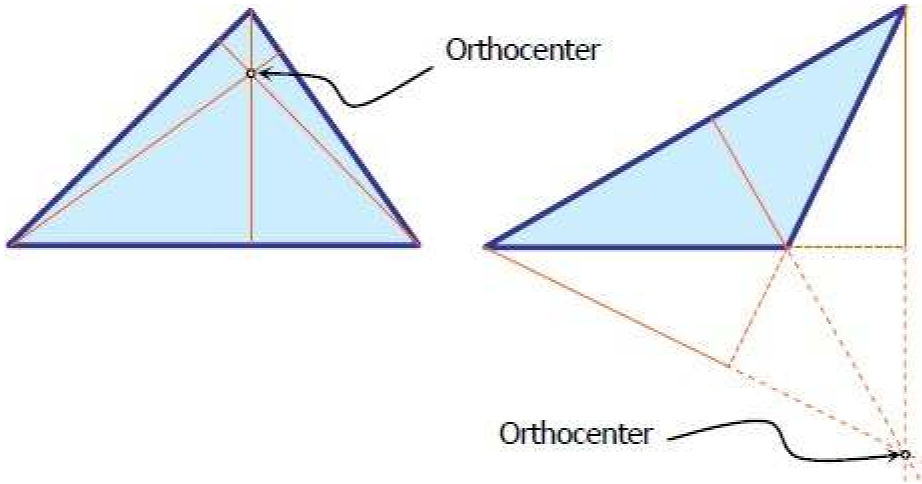


As you can see in the figure above, circumcenter can be inside or outside the triangle. In the case of the right triangle, circumcenter is at the midpoint of the hypotenuse. Given the area of the triangle A_t, the radius of the circumscribing circle is given by the formula

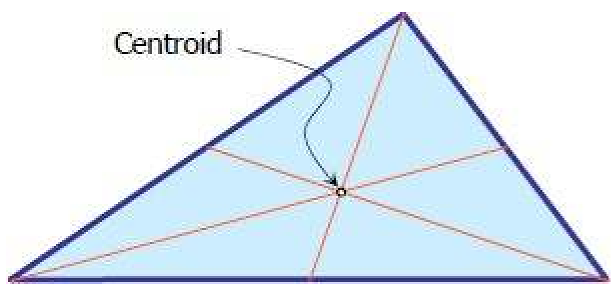
$R=abc/4A_t$

You may want to take a look for the derivation of formula for radius of circumcircle.

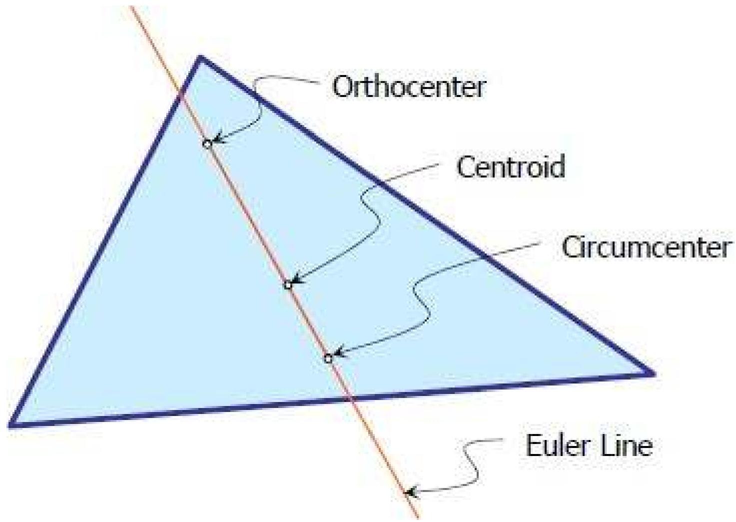
Orthocenter
Orthocenter of the triangle is the point of intersection of the altitudes. Like circumcenter, it can be inside or outside the triangle as shown in the figure below.



Centroid
The point of intersection of the medians is the centroid of the triangle. Centroid is the geometric center of a plane figure.



Euler Line
The line that would pass through the orthocenter, circumcenter, and centroid of the triangle is called the Euler line.



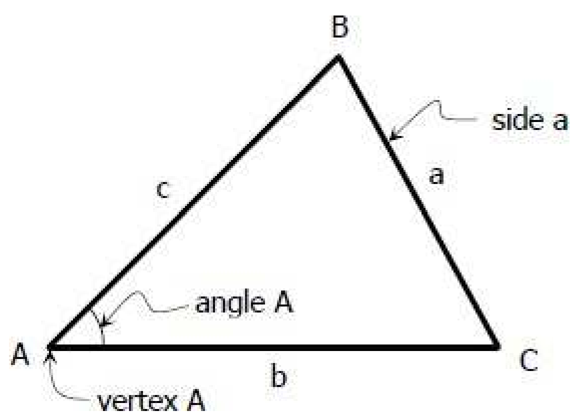
Properties:

Side
Side of a triangle is a line segment that connects two vertices. Triangle has three sides, it is denoted by a, b, and c in the figure below.

Vertex
Vertex is the point of intersection of two sides of triangle. The three vertices of the triangle are denoted by A, B, and C in the figure below. Notice that the opposite of vertex A is side a, opposite to vertex B is side B, and opposite to vertex C is side c.

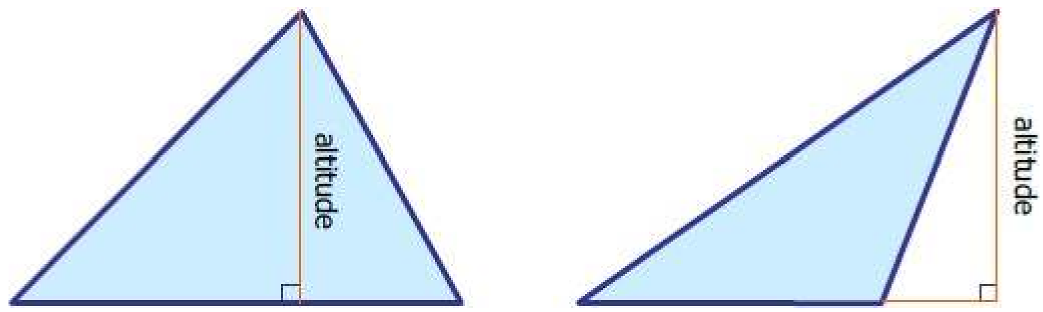
Included Angle or Vertex Angle
Included angle is the angle subtended by two sides at the vertex of the triangle. It is also called vertex angle. For convenience, each included angle has the same notation to that of the vertex, ie. angle A is the included angle at vertex A, and so on. The sum of the included angles of the triangle is always equal to 180°.

A+B+C=180



Altitude, h

Altitude is a line from vertex perpendicular to the opposite side. The altitudes of the triangle will intersect at a common point called orthocenter.

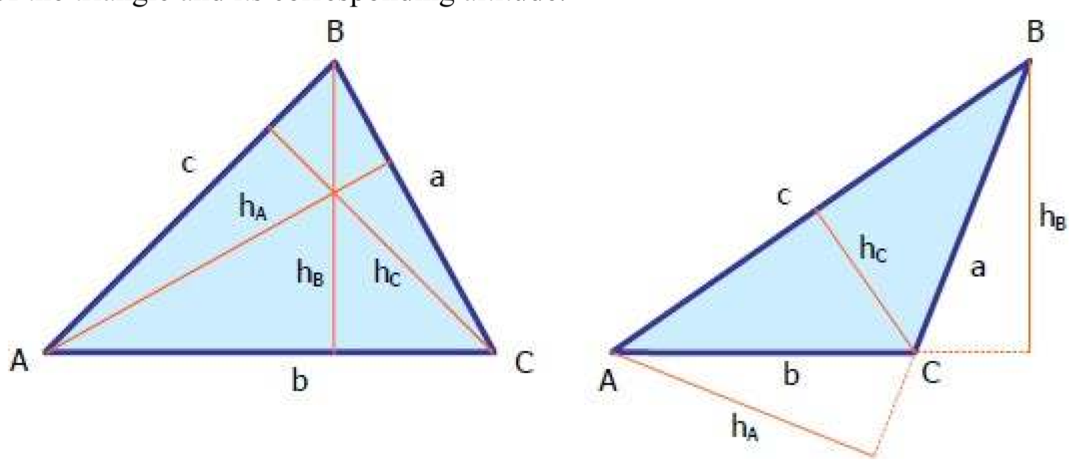


If sides a, b, and c are known, solve one of the angles using Cosine Law then solve the altitude of the triangle by functions of a right triangle. If the area of the triangle A_t is known, the following formulas are useful in solving for the altitudes.

$$h_A = \frac{2A_t}{a}\ ;\ h_B = \frac{2A_t}{b}\ ;\ h_C = \frac{2A_t}{c}$$

Base

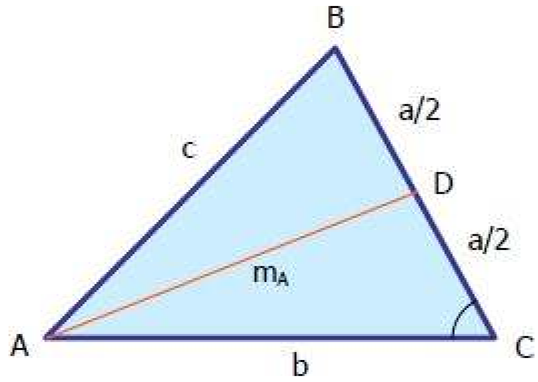
The base of the triangle is relative to which altitude is being considered. Figure below shows the bases of the triangle and its corresponding altitude.



- If h_A is taken as altitude then side a is the base
- If h_B is taken as altitude then side b is the base
- If h_C is taken as altitude then side c is the base

Median, m

Median of the triangle is a line from vertex to the midpoint of the opposite side. A triangle has three medians, and these three will intersect at the centroid. The figure below shows the median through A denoted by m_A.



Given three sides of the triangle, the median can be solved by two steps.

1. Solve for one included angle, say angle C, using Cosine Law. From the figure above, solve for C in triangle ABC.
2. Using triangle ADC, determine the median through A by Cosine Law.

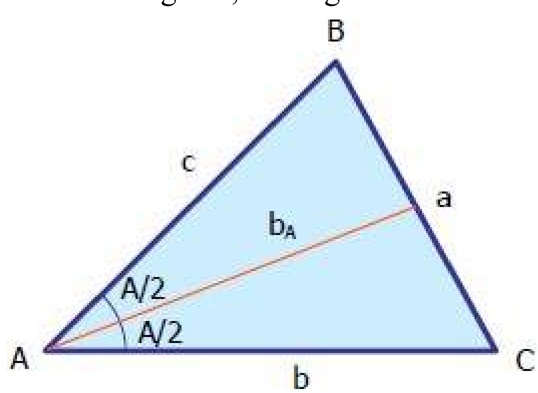
The formulas below, though not recommended, can be used to solve for the length of the medians.

$$4{m_A}^2 = 2b^2 + 2c^2 - a^2$$
$$4{m_B}^2 = 2a^2 + 2c^2 - b^2$$
$$4{m_C}^2 = 2a^2 + 2b^2 - c^2$$

Where m_A, m_B, and m_C are medians through A, B, and C, respectively.

Angle Bisector

Angle bisector of a triangle is a line that divides one included angle into two equal angles. It is drawn from vertex to the opposite side of the triangle. Since there are three included angles of the triangle, there are also three angle bisectors, and these three will intersect at the incenter. The figure shown below is the bisector of angle A, its length from vertex A to side a is denoted as b_A.



The length of angle bisectors is given by the following formulas:

$$b_A = \frac{2\sqrt{bcs\,(s-a)}}{b+c}$$

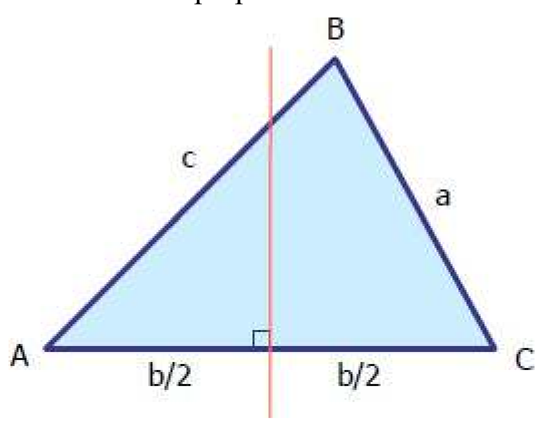
$$b_B = \frac{2\sqrt{acs\,(s-b)}}{a+c}$$

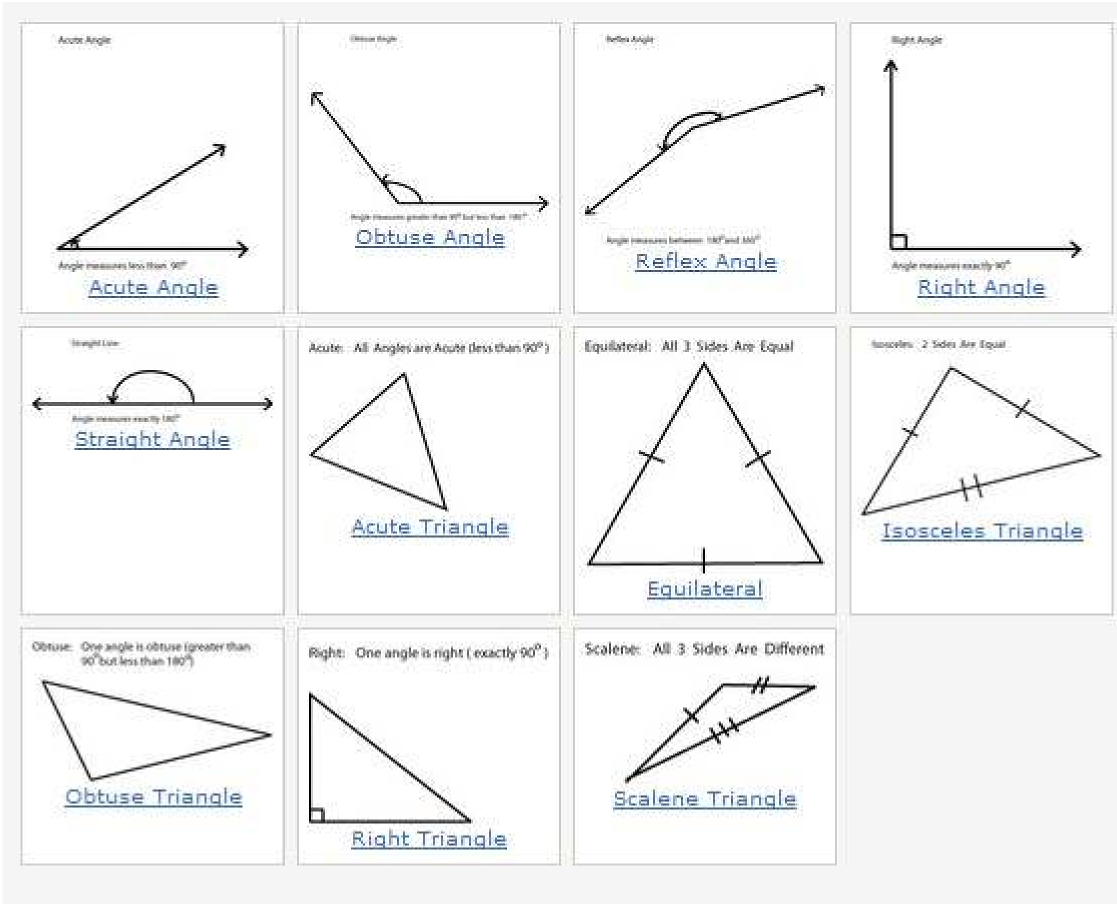
$$b_C = \frac{2\sqrt{abs\,(s-c)}}{a+b}$$

where s=21(a+b+c) called the semi-perimeter and b_A, b_B, and b_C are bisectors of angles A, B, and C, respectively. The given formulas are not worth memorizing for if you are given three sides, you can easily solve the length of angle bisectors by using the Cosine and Sine Laws.

Perpendicular Bisector

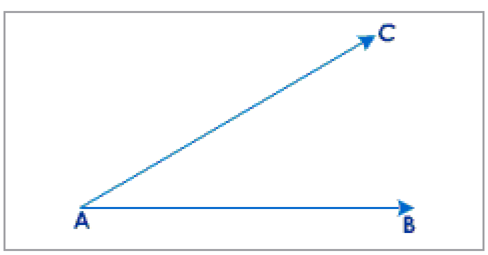
Perpendicular bisector of the triangle is a perpendicular line that crosses through midpoint of the side of the triangle. The three perpendicular bisectors are worth noting for it intersects at the center of the circumscribing circle of the triangle. The point of intersection is called the circumcenter. The figure below shows the perpendicular bisector through side b.





Angle

When two straight lines meet at a common point, they form an angle.



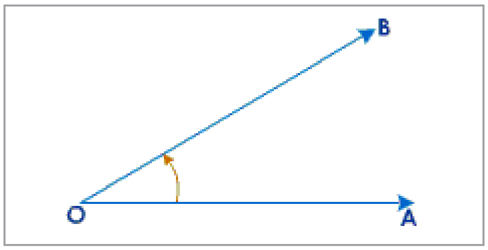
$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called arms of the angle BAC. The point 'A' is called the vertex. The angle formed by the two rays AB and AC is denoted by the symbol $\angle$ BAC. If there is only one angle at A, then the angle BAC may be denoted by $\angle A$.

Note:

In a given plane two lines intersect at only one point. The angles are measured with the help of a protractor. The measure of an angle is called its magnitude.

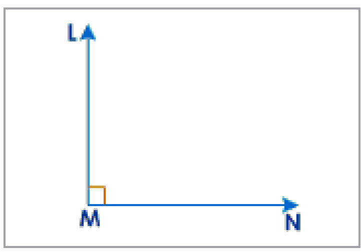
Acute Angle

An angle whose measure is less than 90º is called an acute angle.



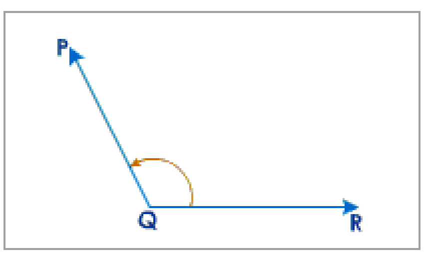
Right angle

It is an angle whose measure is equal to 90º.



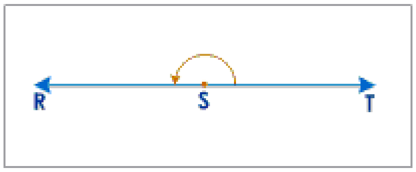
Obtuse angle

An angle whose measure is greater than 90º but less than 180º is called an obtuse angle.



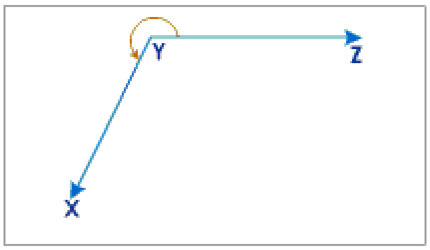
Straight angle

An angle with measure equal to 180º is called a straight angle.



Reflex angle

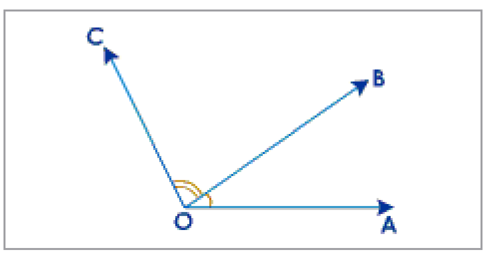
It is an angle with measure greater than 180º but less than 360º.



Angle relations

Adjacent angles

A pair of angles having a common arm and a common vertex.



$\angle AOB$ and $\angle BOC$ are adjacent angles.

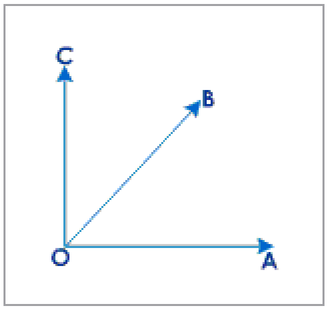
OB is the common arm.

O is the common vertex.

Complementary angles

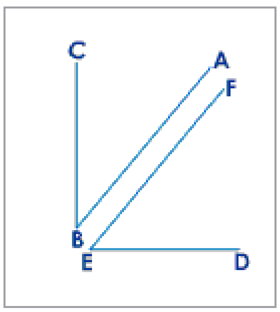
They are a pair of angles, the sum of whose measures is equal to 90º. These may be adjacent angles as well.

$\angle AOB + \angle BOC = 90^o$



(These are adjacent angles)

$\angle ABC + \angle DEF = 90^0$



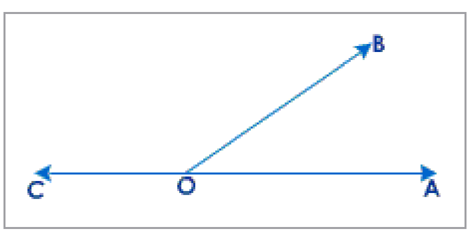
(These are not adjacent angles)

Supplementary angles

They are a pair of angles, the sum of whose measures is equal to 180^o. These may be adjacent angles as well.

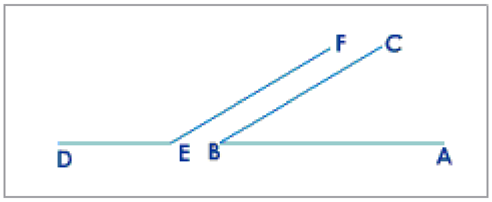
$\angle AOB + \angle BOC = 180^0$

[See figure below]



$\angle ABC + \angle DEF = 180^0$

[See figure below]



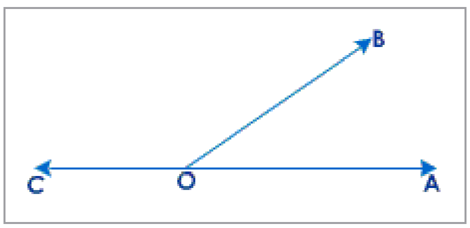
(non-adjacent angles)

Linear pair of adjacent supplementary angles

If a pair of angles are adjacent as well as supplementary then they are said to form a linear pair.

$\angle AOB$ and $\angle BOC$ are adjacent angles,

and $\angle AOB + \angle BOC = 180^0$ [See figure below]



Statement

A mathematical sentence which can be judged to be true or false is called a statement.

Example:

5 + 3 = 8 is a statement.

$x + 2 > 7$ is a sentence but not a statement.

Proof

The course of reasoning, which establishes the truth or falsity of a statement is called proof.

Axioms

The self-evident truths or the basic facts which are accepted without any proof are called axioms.

Example:

- A line contains infinitely many points.
- Things which are equal to the same things are equal to each other.

Theorem

A statement that requires a proof is called a theorem.

Corollary

A statement whose truth can be easily deduced from a theorem is a corollary.

Proposition

A statement of something to be done or considered is called proposition.

You are familiar with statements such as

“If two straight lines intersect each other then the vertically opposite angles are equal”.

“The angles opposite to equal sides of a triangle are equal”.

Proposition is a discussion and is complete in itself. A later proposition depends on the earlier one.

In geometry there are many such statements and they are called propositions.

Theorems and Propositions

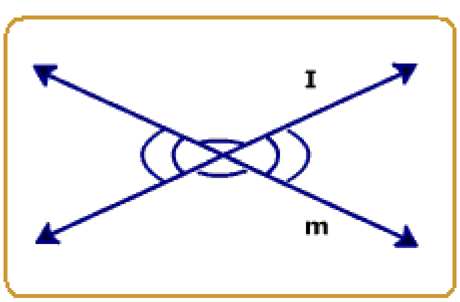
Propositions are of two kinds namely

- Theorem and
- Problems

A theorem is a generalised statement, which can be proved logically. A theorem has two parts, a hypothesis, which states the given facts and a conclusion which states the property to be proved. The two statements given above are examples of theorems.

Theorems are proved using undefined terms, definitions, postulates and occasionally some axioms from algebra.

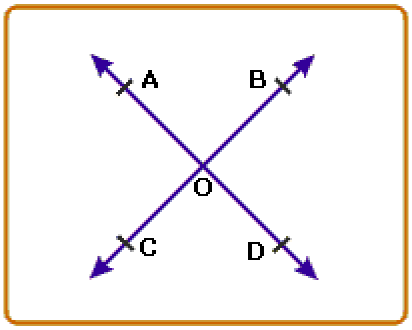
A theorem is a generalised statement because it is always true. For example the statement or the proposition “If two straight lines intersect, then the vertically opposite angles are equal” is true for any two straight lines intersecting at a point. Such a statement is called the general enunciation.



In the theorem stated above, “two lines intersect” is the hypothesis and “vertically opposite angles are equal” is the conclusion. It is the conclusion part that is to be proved logically. To prove a theorem is to demonstrate that the statement follows logically from other accepted statements, undefined terms, definitions or previously proved theorems.

Converse of a theorem

If two statements are such that the hypothesis of one is the conclusion of the other and vice-versa then either of the statement is said to be the converse of the other.



Examples:

Consider the statement of a theorem

"If a transversal intersects two parallel lines, then pairs of corresponding angles are equal”.

This theorem has two parts. If (hypothesis) and then (conclusion).

Let us interchange the hypothesis and conclusion and write the statement.

"If a transversal cuts two other straight lines such that a pair of corresponding angles are equal, then the straight lines are parallel". Such a statement with the hypothesis and conclusion interchanged is called the converse of a given theorem.

Steps to be followed while providing a theorem logically:

- Read the statement of the theorem carefully.
- Identify the data and what is to be proved.
- Draw a diagram for the given data.
- Write the data and what is to be proved by using suitable symbols, applicable to the figure drawn.
- Analyse the logical steps to be followed in proving the theorem.

- Based on the analysis, if there is need for the construction, do it with the help of dotted lines and write it under the step 'Construction'.
- Write the logical proof step by step by stating reasons for each step.

Postulates

A statement whose validity is accepted without proof is called a postulate.

In addition to point, line plane etc, it is also necessary to start with certain other basic statements that are accepted without proof. In geometry these are called postulates.

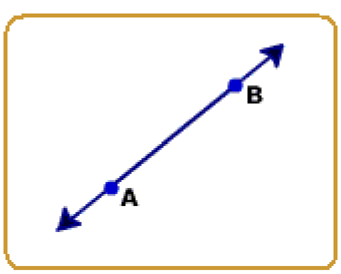
A postulate, though itself is an unproved statement, can be cited as a reason to support a step in a proof. Postulates are just like axioms in arithmetic and algebra, that they are accepted without proof.

Some of the postulates we use often are:

- The line containing any two points in a plane lies wholly in that plane.
- An angle has only one and only one bisector.
- Through any point outside a line, one and only one perpendicular can be drawn to the given line.
- A segment has one and only one mid point.
- Linear pair postulate: If a ray stands on a line, then the sum of the two adjacent angles so formed is 180^{o}.

Activity 1:

Mark two distinct points A and B on the plane of your note book. Can you draw a line passing through A and B?



By experience we find that we can draw only one line through two distinct points A and B.

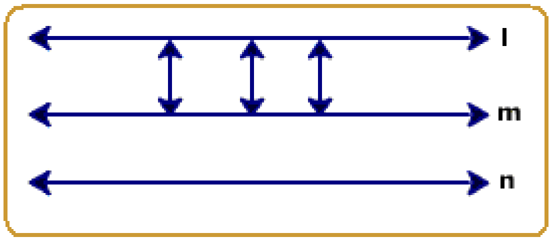
Hence "Given any two distinct points in a plane, there exists one and only line containing them".

This is a self-evident truth. Hence it is a postulate.

Activity 2:

Draw m || l, draw n|| l.

Measure the perpendicular distance between m and n at many points.



We will find this to be same at all points. This means by the property of parallel line m||n.

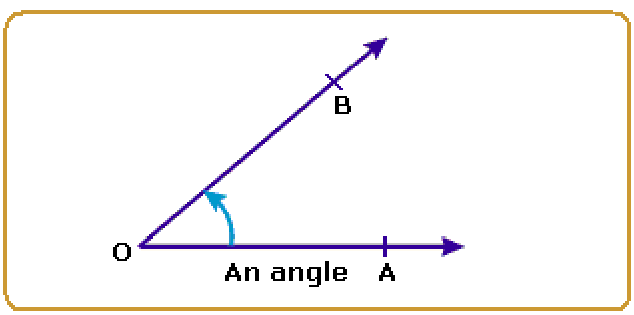
From the above activity we can conclude that

"Two lines which are parallel to the same line, are parallel to themselves".

This is a postulate on parallel lines.

Definition

When two straight lines meet at a point they form an angle.



They are represented as $\angle AOB$ or $A\hat{O}B$.

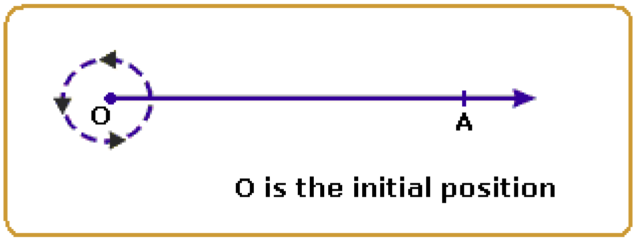
- $\overline{OA}$ and $\overline{OB}$ are called the arms of $\angle AOB$.

- The point at which the arms meet (O) is known as the vertex of the angle.

- The amount of turning from one arm (OA) to other (OB) is called the measure of the angle (ÐAOB) and written as m ÐAOB.

An angle is measured in degrees, minutes and seconds.

If a ray rotates about the starting initial position, in anticlockwise direction, comes back to its original position after 1 complete revolution then it has rotated through 360°.



⇒ 1 complete rotation is divided into 360 equal parts. Each part is 1°.

Each part (1°) is divided into 60 equal parts, each part measures one minute, written as 1'.

1' is divided into 60 equal parts, each part measures 1 second, written as 1".

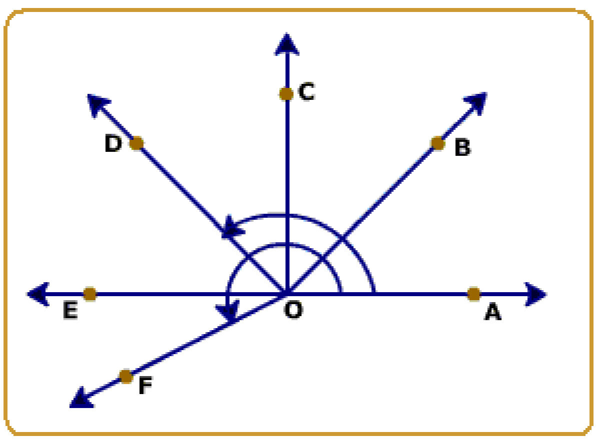
Degrees -----> minutes --------> seconds

1° = 60'

1' = 60"

Recall that the union of two rays forms an angle.

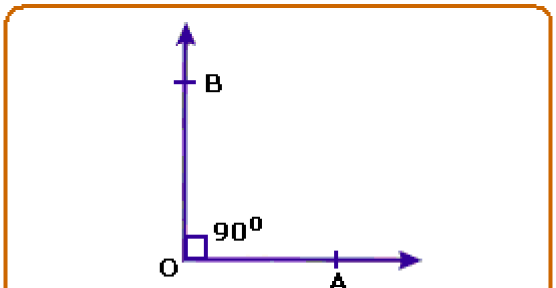
In the figure, observe the different types of angles:



- $A\hat{O}B$ is an acute angle ($0^0 < A\hat{O}B < 90^0$)
- $A\hat{O}C$ is a right angle (an angle equal to 90^0)
- $A\hat{O}D$ is an obtuse angle ($90^0 < A\hat{O}D < 180^0$)
- $A\hat{O}E$ is a straight angle (an angle equal to 180^0)
- $A\hat{O}F$ (measured in anticlock wise direction) is a reflex angle ($180^0 < A\hat{O}F < 360^0$)

Right angle

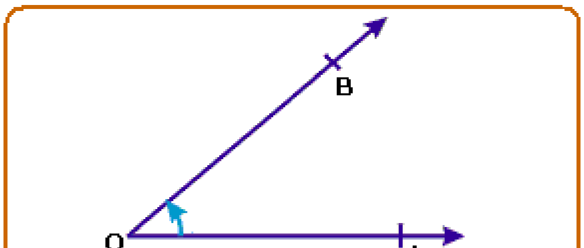
An angle whose measure is 90º is called a right angle.



Right angle

Acute angle

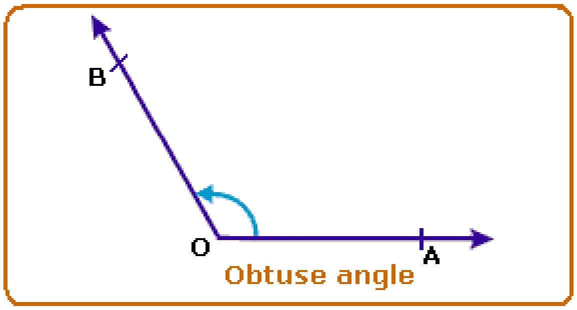
An angle whose measure is less then one right angle (i.e., less than 90º), is called an acute angle.



Acute angle

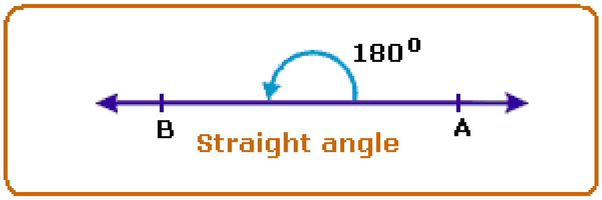
Obtuse angle

An angle whose measure is more than one right angle and less than two right angles (i.e., less than 180º and more than 90º) is called an obtuse angle.



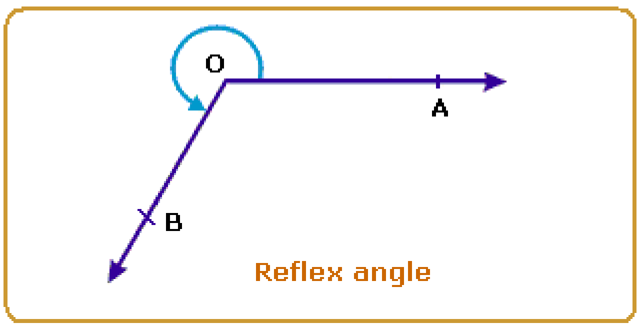
Straight angle

An angle whose measure is 180º is called a straight angle.



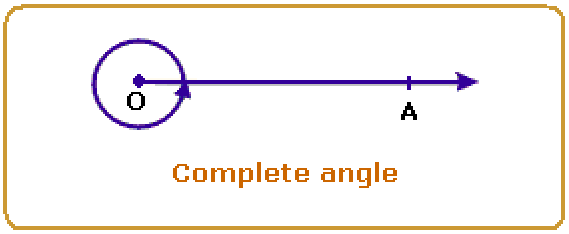
Reflex angle

An angle whose measure is more than 180º and less than 360º is called a reflex angle.
It is written as ref.∠AOB.



Complete angle

An angle whose measure is 360º is called a complete angle.

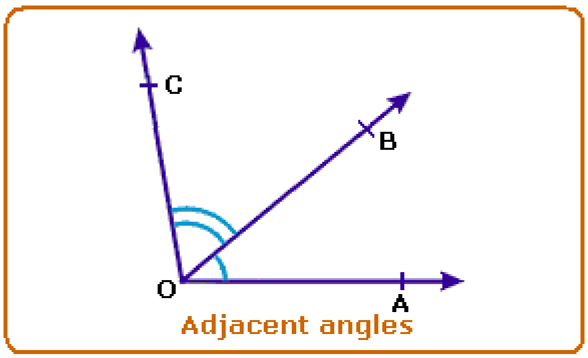


Equal angles

Two angles are said to be equal, if they have the same measure.

Adjacent angles

Two angles having a common vertex and a common arm, such that the other arms of these angles are on opposite sides of the common arm, are called adjacent angles.



Adjacent angles

- O is the common vertex.
- $A\hat{O}B$ and $B\hat{O}C$ are adjacent angles.
- Arm BO separates the two angles.

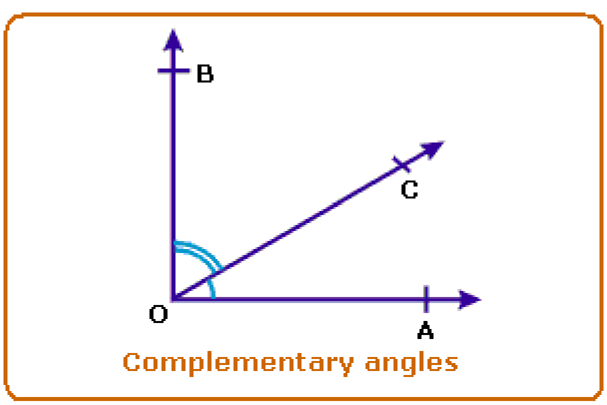
Complementary angles

If the sum of the two angles is one right angle (i.e., 90º), they are called complementary angles.

If the measure of $A\hat{O}C = a^0$, $C\hat{O}B = b^0$, then $a^0 + b^0 = 90^0$.

Therefore $A\hat{O}C$ and $C\hat{O}B$ are complementary angles.

$A\hat{O}C$ is complement of $C\hat{O}B$.



Complementary angles

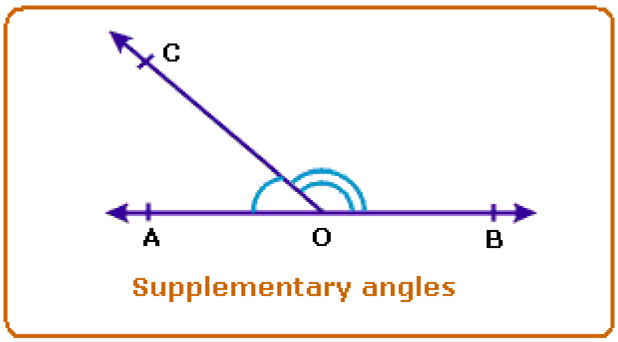
Supplementary angles

Two angles are said to be supplementary, if the sum of their measures is 180º.

Example:

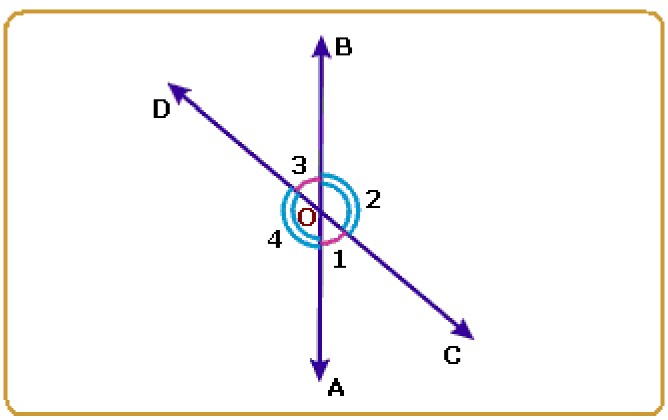
Angles measuring 130º and 50º are supplementary angles.

Two supplementary angles are the supplement of each other.



Vertically opposite angles

When two straight lines intersect each other at a point, the pairs of opposite angles so formed are called vertically opposite angles.



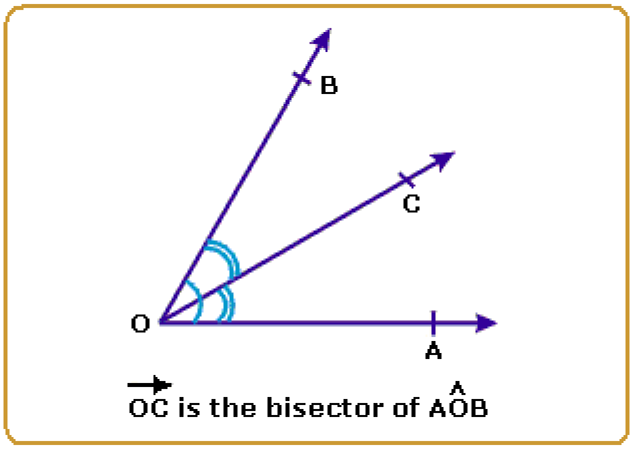
Angles Ð1 and Ð3 and angles Ð2 and Ð4 are vertically opposite angles.

Note:

Vertically opposite angles are always equal.

Bisector of an angle

If a ray or a straight line passing through the vertex of that angle, divides the angle into two angles of equal measurement, then that line is known as the Bisector of that angle.



$$B\hat{O}C = C\hat{O}A$$

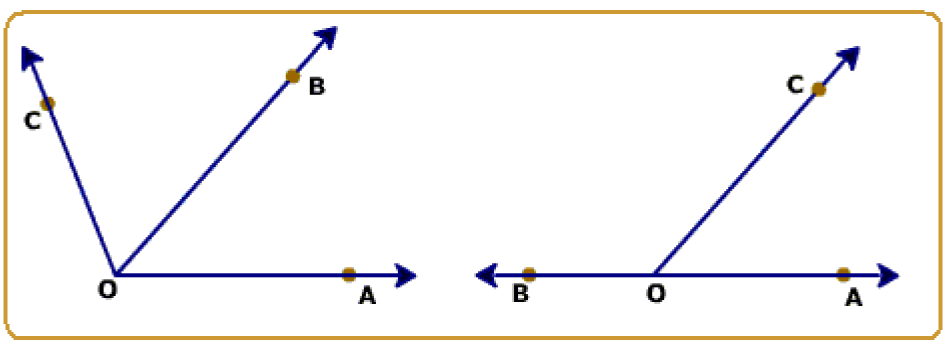
and $B\hat{O}C + C\hat{O}A = A\hat{O}B$

and $A\hat{O}B = 2B\hat{O}C = 2C\hat{O}A$

Linear pair of angles

Two adjacent angles are said to form a linear pair of angles, if their non-common arms are two opposite rays.

Recall adjacent angles. Now observe the pairs of angles in the figure.



In fig, $A\hat{O}B$ and $B\hat{O}C$ are adjacent angles. $\overrightarrow{OB}$ is the common arm. $\overrightarrow{OA}$ and $\overrightarrow{OC}$ are two non-common (or distinct) arms. Observe that the angles have the same vertex O.

Idempotent laws

If A is any set,then

i. $A \cup A = A$

ii. $A \cap A = A$

Identity laws

i. $A \cup \phi = A$

ii. $A \cap U = A$

Commutative laws

If A, B are two sets, then:

i. $A \cup B = B \cup A$

ii. $A \cap B = B \cap A$

Associative laws

If A, B, C are three sets, then:

i. $A \cup (B \cup C) = (A \cup B) \cup C$

ii. $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive laws

If A, B, C are three sets, then:

i. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Laws

- $(A \cup B)' = A' \cap B'$

Proof:

Let $x \in (A \cup B)^{I}$

$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \quad \text{or} \quad x \notin B$

$\Leftrightarrow x \in A^{I} \text{ and } x \in B^{I}$

$\Leftrightarrow x \in A^{I} \cap B^{I}$

$\therefore (A \cup B)^{I} = A^{I} \cup B^{I}$

- $(A \cap B)^{I} = A^{I} \cup B^{I}$

Proof:

Let $x \in (A \cap B)^{I}$

$\Leftrightarrow x \notin A \cap B$

$\Leftrightarrow x \notin A \quad \text{and} \quad x \notin B$

$\Leftrightarrow x \in A^{I} \text{ or } x \in B^{I}$

$\Leftrightarrow x \in A^{I} \cup B^{I}$

$\therefore (A \cap B)^{I} = A^{'} \cup B^{I}$

Laws of Indices

If m and n are positive integers, and $a \neq 0,$ then

(i) $a^m \times a^n = a^{m+n}$ [Product Law]

(ii) $a^m \div a^n = a^{m-n}$ [Quotient Law]

(iii) $(a^m)^n = a^{mn}$ [Power Law]

(iv) $(ab)^m = a^m \cdot b^m$

(v) $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$

(vi) $a^o = 1$

$a^{-1} = \frac{1}{a}$
(vii)

Surd

An irrational root of a positive rational number is called a surd. Consider a number with base 'a' as a positive rational number with power of a fraction, say $\frac{1}{n}$, then

$a^{1/n} = \sqrt[n]{a}$

Since $\sqrt[n]{a}$ is an n^{th} root, it is called a surd of order n, if it is irrational.

e.g., (i) $\sqrt[3]{10}$ is a surd of order 3.

(ii) $\sqrt{53}$ is a surd of order 2.

(iii) $\sqrt[4]{81}$ is NOT a surd because $\sqrt[4]{81} = 3$, and 3 is NOT an irrational number.

Example 1:

Without using tables, simplify:

(i) $36^{-1/2}$ (ii) $(8x^2)^{1/3} \div x^{-1/3}$

Suggested answer:

(i) $36^{-1/2}$

$= (6^2)^{-1/2} = 6^{-1}$

$= \frac{1}{6}$

(ii) $(8x^2)^{1/3} \div x^{-1/3}$

$$= \frac{(8)^{1/3} \cdot x^{2/3}}{x^{-1/3}}$$

$$= 2.x^{\frac{2}{3}+\frac{1}{3}} = 2x$$

Example 2:

$$\sqrt[4]{\frac{256}{625}}$$

Evaluate:

Suggested answer:

$$\left(\frac{256}{625}\right)^{1/4} = \left(\frac{4^4}{5^4}\right)^{1/4}$$

Given expression

$$= \frac{4^{4\times 1/4}}{5^{4\times 1/4}}$$

$$= \frac{4}{5}$$

Example 3:

Simplify:

$$\frac{\sqrt[3]{a^2b}}{\sqrt[3]{a^{-1}b^4}}$$

(i)

(ii) If $2^{x+2} = 128$, find the value of x.

$$4x^{-3}y^2 \div (8xy)^2$$

(iii) Simplify:

$$\sqrt[4]{x^{3a}\ y^6} \times (x^{2/3} \times y^{-1})^a$$

(iv) Simplify:

(v) Show that $(x^{p-q})^{p+q}(x^{q-r})^{q+r}(x^{r-s})^{r+s} = 1$

(vi) 49 × $7^x = (343)^{2x-5}$ find 'x'.

Suggested answer:

(i) Given expression:

$$\frac{(a^2b)^{1/3}}{(a^{-1}b^4)^{1/3}} = \frac{a^{2/3}.b^{1/3}}{a^{-1/3}.b^{4/3}}$$

$$= \frac{a^{2/3+1/3}}{b^{4/3-1/3}} = \frac{a}{b}$$

(ii) Since 128 = 2 × 2 × 2 × 2 × 2 × 2 × 2

$= 2^7$

We have $2^{x+2} = 2^7$ [$\because$ bases are equal]

$\therefore$ x + 2 = 7 [$\therefore$ powers are equal]

$\therefore$ x = 5

(iii)

$$\frac{4x^{-3}y^2}{(8xy)^2} = \frac{4}{x^3} \cdot \frac{y^2}{64x^2y^2}$$

$$= \frac{y^2}{x^3.16x^2y^2}$$

$$= \frac{1}{16x^5} \qquad [\because y^2 \div y^2 = y^{2-2} = y^0 = 1]$$

(iv)

$$= \sqrt[4]{x^{3a}\ y^6} \times (x^{2/3} \times y^{-1})^a$$

$= (x^{3a}y^6)^{1/4} \times (x^{2/3}y^{-1})^a$ $[\because \sqrt[4]{a} = a^{1/4}]$

$= x^{3a/4}y^{3/2} \times x^{2a/3}y^{-a}$ [$\because$ $(x^{3a})^{1/4} = x^{3a/4}$ (power law)]

$$= x^{\frac{3a}{4}+\frac{2a}{3}} \times y^{\frac{3}{2}-a}$$

$$= x^{\frac{17a}{12}} \times y^{\frac{3}{2}-a}$$

(v) LHS = $x^{(p - q)(p + q)}\, x^{(q - r)(q + r)}\, x^{(r - s)(r + s)}$

$$= x^{p^2 - q^2}\ x^{q^2 - r^2}\ x^{r^2 - s^2}$$

$$= x^{p^2 - q^2 + q^2 - r^2 + r^2 - s^2}$$

= x^0 = 1 = R.H.S

(vi) $49 \times 7^x = (343)^{2x - 5}$

$\Rightarrow$ $7^2 \times 7^x = (7^3)^{2x - 5}$

$\Rightarrow$ $7^{2+x} = 7^{6x - 15}$

As the bases are equal, the powers are also equal.

$\therefore$ $2 + x = 6x - 15$

or $5x = 17$

or $x = \frac{17}{5}$
