Algebra and Geometry in several Complex variables

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17.10.2002-16.01.2003

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Chapter 1

Holomorphic functions of several variables

1.1   Analysis in complex plane

The notation $\mathbb{C}$ is used for the domain of one complex variable $z = x + \imath y$, called also complex line, which is isomorphic to the real Euclidean plane with the coordinates $x = \operatorname{Re} z,\; y = \operatorname{Im} z$ and the norm $\|z\|^2 = \|(x, y)\|^2 = x^2 + y^2$. The symbol $\imath = \sqrt{-1}$ means a choice of one of square roots of -1. The tangent space $T(\mathbb{C}) \cong \mathbb{C}$ is generated by the fields

$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \imath \frac{\partial}{\partial y}\right),\; \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y}\right)$$

A function $f : \Omega \to \mathbb{C}$ defined in an open set $\Omega \subset \mathbb{C}$, is called *holomorphic* if

$$\frac{\partial f}{\partial \bar{z}} = 0$$

This equation for $f = u + \imath v$ is equivalent to the system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

For an arbitrary 1-differentiable function f we have

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z},\; dz \doteq dx + \imath dy,\; d\bar{z} \doteq dx - \imath dy$$

Therefore f is holomorphic, if and only if $df = \partial f/\partial z\; dz$. Note that $d\bar{z} \wedge dz = 2\imath dx \wedge dy$.

Theorem 1 *[Cauchy-Green]. For a bounded domain $\Omega \subset C$ with piecewise C^1-boundary and an arbitrary function $f \in C^1 \left(\overline{\Omega} \right)$ and any point $z \in \Omega$ the equation holds*

$$2\pi \imath f(z) = \int_{\partial \Omega} \frac{f(w)\, dw}{w - z} + \int_{\Omega} \frac{\partial f}{\partial \bar{w}} \frac{dw \wedge d\bar{w}}{w - z} \tag{1.1}$$

PROOF. Denote by B the disc of small radius ε centered at z and apply the integral "Stokes" formula (Gauss-Ostrogradski-Green-Stokes-Poincaré...) to the form $\theta(w) \doteq (w - z)^{-1} f(w)\, dw$ and to the chain $\Omega \backslash B$:

$$\int_{\partial \Omega} \theta - \int_{\partial B} \theta = \int_{\partial(\Omega \backslash B)} \theta = \int_{\Omega \backslash B} d\theta$$

We have $d\theta = (w - z)^{-1} \partial f / \partial \bar{w} dw \wedge d\bar{w}$. On the other hand, we have in B, $f(w) = f(z) + O(|w - z|)$, hence

$$\int_{\partial B} \theta = f(z) \int_{\partial B} \frac{dw}{w - z} + \int_{\partial B} O(1)$$

The first integral is equal to $2\pi \imath$, the second tends to zero as $\varepsilon \to 0$. $\boxdot$

If the function f is holomorphic, the second terms vanishes; then (1.1) is called *Cauchy* formula.

1.2   Holomorphic functions of several variables

One denotes $\mathbb{C}^n = \overbrace{\mathbb{C} \times ... \times \mathbb{C}}^{n}$ for an arbitrary integer n; it is a complex vector space of dimension n. If we fix coordinates $z_j = x_j + \imath y_j$, $j = 1, ..., n$ in the factors and get a coordinate system in $\mathbb{C}^n$ complex space. It can be considered as a real vector space of dimension $2n$ with the coordinates x, y where $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$. The space $\mathbb{C}^n$ is endowed with the structure of Euclidean space, where $\langle z, w \rangle_{\mathbb{R}} = \operatorname{Re} \sum z_j \bar{w}_j = \sum x_j u_j + y_j v_j$, $w_j = u_j + \imath v_j$.
Definition. Let $\mathbb{U} \subset \mathbb{C}^n$ be an open set; a function $f : \mathbb{U} \to \mathbb{C}$ is called *holomorphic,* if it satisfies the system of differential equations (Cauchy-Riemann system):

$$\frac{\partial f(z)}{\partial \bar{z}_1} = ... = \frac{\partial f(z)}{\partial \bar{z}_n} = 0 \tag{1.2}$$

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We need not to fix a distinguished coordinate system, if we replace by another coordinate system of the form $w_j = \sum a_j^k z_k + c_j, \ j = 1, ..., n$, where $a_{jk}, c_j \in \mathbb{C}$ are arbitrary such that the matrix $\{a_j^k\}$ is invertible, we turn $\mathbb{C}^n$ to an affine complex space. We have

$$\frac{\partial}{\partial \bar{w}_j} = \sum \bar{b}_k^j \frac{\partial}{\partial \bar{z}_k}$$

where $\{b_k^J\}$ is the inverse matrix. Therefore any function satisfying (1.2) is holomorphic with respect to the affine coordinate system $w_1, ..., w_n$.

Example 1. Any polynomial is holomorphic function in $\mathbb{C}^N$. The sum of a power series

$$\sum_{j=(j_1, ..., j_n)} c_j (z - \zeta)^j$$

is a holomorphic function in any open *polydisc* $P \doteq \{|z_j - \zeta_j| < \varepsilon_j, \ j = 1, ..., n\}$ where this series converges.

Definition. A function $f : \mathbb{U} \to \mathbb{C}$ is called *analytic* at a point $z \in \mathbb{U}$, if it is equal to the sum of a power series that converges in a polydisc P that contains z.

Theorem 2 *Any holomorphic function is analytic (and vice versa).*

PROOF. Assume for simplicity that $n = 2$. Take an arbitrary point $a = (a_1, a_2) \in \mathbb{U}$ a bidisc $P_a \subset \mathbb{U}$ and write by means of the Cauchy formula for an arbitrary $z = (z_1, z_2) \in P$

$$(2\pi\imath)^2 f(z_1, z_2) = \int_{|w_2 - a_2| = r_2} \frac{dw_2}{w_2 - z_2} \int_{|w_1 - a_1| = r_1} \frac{f(w_1, w_2) \, dw_1}{w_1 - z_1}$$
$$= \int_T \frac{f(w_1, w_2) \, dw_1 \wedge dw_2}{(w_1 - z_1)(w_2 - z_2)}$$

where $T = \partial^2 P = \{w : |w_1 - a_1| = r_1, |w_2 - a_2| = r_2\}$ is the 2-chain in $\mathbb{U}$, which is called the *skeleton* of the bidisc P. We have $|z_j - a_j| < r_j = |w_j - a_j|, j = 1, 2$, hence the series

$$\sum_{j_1 \geq 0, j_2 \geq 0} \frac{(z_1 - a_1)^{j_1} (z_2 - a_2)^{j_2}}{(w_1 - a_1)^{j_1 + 1} (w_2 - a_2)^{j_2 + 1}}$$

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converges to $(w_1 - z_1)^{-1} (w_2 - z_2)^{-1}$ as $z \in P_a$. Apply it for the previous formula:

$$(2\pi\iota)^2 f(z_1, z_2) = \sum_{j_1, j_2} (z_1 - a_1)^{j_1} (z_2 - a_2)^{j_2} \int_T \frac{f(w_1, w_2) \, dw_1 \wedge dw_2}{(w_1 - a_1)^{j_1+1} (w_2 - a_2)^{j_2+1}}. \quad \Box$$

Proposition 3 *If f is a holomorphic function in a connected domain $\mathbb{U}$ and $f = 0$ in a neighborhood of a point $z_0 \in \mathbb{U}$. Then $f \equiv 0$ in $\mathbb{U}$.*

If $f = g + \iota h$ is a holomorphic function, then $g = g(x, y)$ and $h = h(x, y)$ are harmonic functions, i.e.

$$\Delta g = \Delta h = 0, \ \Delta \doteq \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} = \sum \frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_j}$$

The same is true for any antiholomorphic function, i.e. for a function f^* such that $\bar{f}^*$ is holomorphic.

Corollary 4 *If f is a holomorphic function in a connected domain $\mathbb{U}$, then the function $|f|$ can not reach its maximal value if $f \neq \text{const}$.*

1.3   Hartogs' phenomenon

Theorem 5 *Let $n > 1, \mathbb{U} \subset \mathbb{C}^n$ be an open set and $\mathbb{K}$ be a compact subset of $\mathbb{U}$ such that $\mathbb{U} \backslash \mathbb{K}$ is connected. Then an arbitrary holomorphic function f in $\mathbb{U} \backslash \Theta$ has a unique holomorphic continuation to $\mathbb{U}$.*

SKETCH OF PROOF. Suppose for simplicity that $n = 2$. Take a complex line $L_b = \{z_2 = b\}$ that does not meet Θ. The function $f_b(z_1) \doteq f(z_1, b)$ is holomorphic in the domain $L_b \cap \mathbb{U}$ of the complex line L_b. We can move the point b in an open set $\omega \subset \mathbb{C}$ which does not meet $\mathbb{K}$. Take the maximal set ω such that this property holds; then there exists a point $\beta \in \partial\omega$ such that L_β meet $\mathbb{K}$. The intersection $\mathbb{K}_* \doteq L_\beta \cap \mathbb{K}$ is a compact set; take a compact $V \subset \mathbb{U} \cap L_\beta$ with smooth boundary that contains $\mathbb{K}_*$. Consider the integral for an arbitrary integer k :

$$\int_{\partial V} w_1^k f(w_1, b) \, dw_1$$

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It vanishes for an arbitrary $b \in \omega$; it is well defined and holomorphic in a connected neighborhood ω_* of the point b_*. It must vanish for $b \in \omega_*$ since the intersection $\omega \cap \omega_*$ is non empty. This implies that the function $f(z_1, b)$ has holomorphic continuation $\tilde{f}$ to V for any $b \in \omega_*$. By Cauchy formula

$$2\pi \imath \tilde{f}(z_1, z_2) = \int \frac{f(w_1, z_2)\, dw_1}{w_1 - z_1}$$

we see that $\tilde{f}$ is holomorphic in $V \times \omega_*$ too. The function $\tilde{f}$ coincides with f in the intersection $\mathbb{U} \backslash \mathbb{K} \cap V \times \omega_*$, if this intersection is connected. It follows that f is holomorphic in $\Omega \backslash \mathbb{K}'$, where $\mathbb{K}' \doteq \mathbb{K} \backslash V \times \omega_*$. We can continue this process of cutting out of the compact $\mathbb{K}$. The condition of theorem implies that we can go on until the set $\mathbb{K}$ disappears. $\boxdot$

Below we give a rigorous proof based on an integral formula.

1.4   Differential forms in $\mathbb{C}^n$

Let $\mathbb{U}$ be a domain in $\mathbb{R}^m$ and $\Omega^q(\mathbb{U})$, $q = 0, 1, ..., 2n$ be the space of smooth differential forms in of degree q in $\mathbb{U}$. The exterior differential defines the linear mappings $d = d_q : \Omega^q(\mathbb{U}) \to \Omega^{q+1}(\mathbb{U})$, $q = 0, ..., 2n$; $d_m = 0$. such that $d_{q+1} d_q = 0$. A form α is called *closed*, if $d\alpha = 0$ and *exact*, if $\alpha = d\beta$; the form β is called *primitive* of α. Any exact form is closed. The q-th de Rham cohomology of $\mathbb{U}$ is the space

$$H^q(\mathbb{U}) \doteq Z^q(\mathbb{U}) / B^q(\mathbb{U})$$

where $Z^q(\mathbb{U})$ the space of closed forms in $\mathbb{U}$ and $B^q(\mathbb{U})$ is the space of exact forms.

Owing to the complex structure of $\mathbb{C}^n \cong \mathbb{R}^{2n}$, an arbitrary form $\alpha \in \Omega^q(\mathbb{U})$ can be written as follows

$$\alpha = \alpha^{q,0} + \alpha^{q-1,1} + ... + \alpha^{0,q}$$

where $\alpha^{r,s} \in \Omega^q(\mathbb{U})$ is a form of type (r, s), $r + s = q$:

$$\alpha^{r,s} = \sum_{i_1 < ... < i_r; j_1 < ... < j_s} a_{i_1,...,i_r; j_1,...,j_s}(x, y)\, dz_{i_1} \wedge ... \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_s}$$

where $a_{i_1,...,j_s}(x, y)$ are smooth functions in $\mathbb{U}$. Therefore there is isomorphism

$$\Omega^q(\mathbb{U}) = \Omega^{q,0}(\mathbb{U}) \oplus ... \oplus \Omega^{0,q}(\mathbb{U})$$

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where $\Omega^{r,s}(\mathbb{U})$ means the space of differential forms of type (r,s). In particular, we have $\Omega^{2n}(\mathbb{U}) = \Omega^{n,n}(\mathbb{U})$. We can write

$$df = \partial f + \bar{\partial} f, \ \partial f = \sum \frac{\partial f}{\partial z_j}, \ \bar{\partial} f = \sum \frac{\partial f}{\partial \bar{z}}$$

and in general $d = \partial + \bar{\partial}$, where

$$\partial : \Omega^{r,s}(\mathbb{U}) \to \Omega^{r+1,s}(\mathbb{U}), \ \partial^* : \Omega^{r,s}(\mathbb{U}) \to \Omega^{r,s+1}(\mathbb{U})$$

Contraction. Given a differential form α of degree q and a vector $t = \sum t_j \partial_j + s_j \bar{\partial}_j$, the contraction is the form β of degree $q-1$:

$$\beta = t \dashv \alpha = \sum (-1)^\rho t_j a_{i_1,\ldots,i_r;j_1,\ldots,j_s} \overbrace{dz_{i_1} \wedge \ldots \widehat{dz_j}\ldots}^{\rho} \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_s}$$

$$+ \sum (-1)^\sigma s_j a_{i_1,\ldots,i_r;j_1,\ldots,j_s} \overbrace{dz_{i_1} \wedge \ldots \wedge dz_{i_r} \wedge d\bar{z}_{j_1} \wedge \ldots \widehat{d\bar{z}_j}\ldots}^{\sigma} \wedge d\bar{z}_{j_s}$$

Lie-derivative of a form with respect to a vector field $t = t(x,y)$:

$$L_t \alpha = d(t \dashv \alpha) + t \dashv d\alpha \tag{1.3}$$

1.5 Bochner-Martinelli formula

Define the form in $\mathbb{C}^n$ of type (n,n):

$$\rho \doteq \frac{d\bar{z} \wedge dz}{|z|^{2n}}, \ dz \doteq dz_1 \wedge \ldots \wedge dz_n, \ d\bar{z} \doteq d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$$

where $|z|^2 = \sum z_j \bar{z}_j$. It is homogeneous of degree 0. Take the antiholomorphic Euler field $\bar{e} \doteq \sum \bar{z}_j \partial/\partial \bar{z}_j$ and define the contraction

$$\omega \doteq \frac{(n-1)!}{(2\pi\imath)^n} \bar{e} \dashv \rho$$

It is called *Bochner-Martinelli* form. It is of type $(n, n-1)$. Fix a point $\zeta \in \mathbb{C}^n$ and replace the argument z to $z - \zeta$ in the form ω. We get the form

$$\omega_\zeta = \frac{(n-1)!}{(2\pi\imath)^n} \sum (-1)^{j-1} \frac{\bar{z}_j - \bar{\zeta}_j}{|z - \zeta|^{2n}} d\bar{z}_1 \wedge \ldots \widehat{d\bar{z}_j}\ldots \wedge d\bar{z}_n \wedge dz$$

Properties:
1. For $n = 1, \omega_\zeta = (2\pi\imath\,(z - \zeta))^{-1}\,dz$ is the Cauchy kernel.
2. The form ω_ζ is closed in $\mathbb{C}^n\backslash\{\zeta\}$.,i.e. $d\omega_\zeta = 0$.

$$\alpha = \sum a_{i_1...i_q}(x)\,dx_{i_1} \wedge ... \wedge dx_{i_q},\ d\alpha = \sum da_{i_1...i_q}(x) \wedge dx_{i_1} \wedge ... \wedge dx_{i_q}$$

It can be checked by a straightforward calculation.
3. For the sphere $S_r = \{|z - \zeta| = r\}$ in $\mathbb{C}^n$ we have

$$\int_{S_r} \omega_\zeta = 1$$

Indeed, take $\zeta = 0$; since ω is homogeneous of degree 0, we can take $r = 1$. By "Stokes" formula

$$\begin{aligned}
\int_{S_1} \omega &= \frac{(n-1)!}{(2\pi\imath)^n} \int_{S_1} \bar{e} \dashv (d\bar{z} \wedge dz)\\
&= \frac{(n-1)!}{(2\pi\imath)^n} \int_{B} d\,(\bar{e} \dashv (d\bar{z} \wedge dz))\\
&= \frac{n!}{(2\pi\imath)^n} \int_{B} d\bar{z} \wedge dz = 1,
\end{aligned}$$

where B is the unit ball, since

$$d\,(\bar{e} \dashv (d\bar{z} \wedge dz)) = nd\bar{z} \wedge dz = n\,(2\imath)^n\,dx \wedge dy$$

and $\int_B dx \wedge dy = \pi^n/n!$.
4. For an arbitrary bounded domain $\mathbb{U}$ with smooth boundary $\partial\mathbb{U}$ and a point $\zeta \in \mathbb{U}$

$$\int_{\partial\mathbb{U}} \omega_\zeta = 1$$

i.e. $d\omega_\zeta = \bar{\partial}\omega_\zeta = \delta_\zeta$, where δ_ζ is the delta-distribution in $\mathbb{C}^n$.
5. Let $\mathbb{U}$ be a bounded domain with smooth boundary $\partial\mathbb{U}$ in $\mathbb{C}^n$ and f be a continuous function on $\partial\mathbb{U}$. Then the integral

$$G(\zeta) = \int_{\partial\mathbb{U}} f\omega_\zeta$$

is a harmonic function in $\mathbb{U}$. For a proof, we note that $\Delta\omega_\zeta = \widetilde{\partial}^*\partial\omega = 0$. (If $n > 1$, $G(z)$ need not to be a holomorphic function.)

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6. For any $j = 1, ..., n$ the form $\partial \omega_\zeta / \partial \bar{\zeta}_j$ is exact in $\mathbb{C}^n \backslash \{\zeta\}$. This derivative is equal to the derivative $\bar{\partial}_j \omega_\zeta$ since ω_ζ only depends on $\zeta - z$. The last one is, in fact, the Lie-derivative, which is defined by (1.3):

$$\frac{\partial \omega_\zeta}{\partial \bar{\zeta}_j} = L_{\bar{\partial}_j} \omega_\zeta = d\left(\bar{\partial}_j \dashv \omega_\zeta\right) + \bar{\partial}_j \dashv d\omega_\zeta = d\left(\bar{\partial}_j \dashv \omega_\zeta\right)$$

since of **4.** This equation can, of course, be checked by direct calculation.

7. If $n > 1$, then for an arbitrary $j = 1, ..., n$ the form ω_ζ is exact in $\mathbb{C}^n \backslash \{z_j \neq \zeta_j\}$. Indeed, $\omega_\zeta = d\left(t_j \dashv \omega_\zeta\right)$, where

$$t_j = -\frac{|z - \zeta|^2}{(n-1)(z_j - \zeta_j)} \frac{\partial}{\partial \bar{z}_j}$$

Theorem 6 *[Bochner-Martinelli] Let $\mathbb{U}$ be a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary and f be a continuous function in $\overline{\mathbb{U}}$ that is holomorphic in $\mathbb{U}$. Then for arbitrary $\zeta \in \mathbb{U}$ we have*

$$f(\zeta) = \int_{\partial \mathbb{U}} f\omega_\zeta$$

PROOF. We have $d(f\omega_\zeta) = df \wedge \omega_\zeta + f d\omega_\zeta = 0 + 0$ in $\mathbb{U} \backslash \mathbb{B}_\zeta$, where $\mathbb{B}_\zeta$ stands for a ball centered in ζ of radius r. Therefore

$$G(\zeta) = \int_{\partial \mathbb{U}} f\omega_\zeta = \int_{\partial \mathbb{B}_\zeta} f\omega_\zeta \to f(\zeta) \int_{\partial \mathbb{B}_\zeta} \omega_\zeta = f(\zeta)$$

as $r \to 0$, since f is a continuous function. $\square$

1.6 Analytic continuation from boundary

Definition. A tangent vector t in $\mathbb{C}^n$ is called *antiholomorphic,* if $t = \sum t_j \partial / \partial \bar{z}_j$ for some $t_j \in \mathbb{C}$. Let S be a real smooth submanifold in $\mathbb{C}^n$. We say that a function $g \in C^1(S)$ satisfies the tangent Cauchy-Riemann equation, if $t(f) = \sum t_j \bar{\partial}_j f = 0$ for an arbitrary antiholomorphic tangent vector t to S.

This is equivalent to the equation $df \wedge dz = 0$ in $T(S)$, if S is a manifold of dimension $2n - 1$.

Theorem 7 *[Bochner-Severi] Let $n > 1$ and $\mathbb{U}$ be a bounded domain in $\mathbb{C}^n$ with smooth connected boundary $\partial \mathbb{U}$. If a function $f \in C^1(\partial \mathbb{U})$ satisfies the tangent Cauchy-Riemann equation, then there exists a continuous function F in $\overline{\mathbb{U}} = \mathbb{U} \cup \partial \mathbb{U}$ that coincides with f on the boundary and is holomorphic in $\mathbb{U}$.*

PROOF. (i) Show that the Bochner-Martinelli integral

$$F(\zeta) \doteq \int_{\partial \mathbb{U}} f \omega_\zeta$$

gives the continuation. First check, that F is holomorphic in $\mathbb{C}^n \backslash \partial \mathbb{U}$. By the property **6,** since $\zeta \in \mathbb{U}$

$$\frac{\partial F}{\partial \bar{\zeta}_j} = \int_{\partial \mathbb{U}} f \frac{\partial \omega_\zeta}{\partial \bar{\zeta}_j} = \int f d \left(\bar{\partial}_j \dashv \omega_\zeta \right) = \int_{\partial \mathbb{U}} d \left(f \bar{\partial}_j \dashv \omega_\zeta \right) = 0$$

We have $d \left(f \bar{\partial}_j \dashv \omega_\zeta \right) = f d \left(\bar{\partial}_j \dashv \omega_\zeta \right) + df \wedge \left(\bar{\partial}_j \dashv \omega_\zeta \right)$, because of $df \wedge dz = 0$ in $\partial \mathbb{U}$.

(ii) Prove that $F = 0$ in $\mathbb{C}^n \backslash \mathbb{U}$. Take a point $\zeta \in \mathbb{C}^n \backslash \mathbb{K}$, where $\mathbb{K}$ is the convex hull of $\overline{\mathbb{U}}$. Then for some j we have $z_j \neq \zeta_j$ for $z \in \overline{\mathbb{U}}$. By **7** we find

$$F(\zeta) = \int_{\partial \mathbb{U}} f d \left(t_j \dashv \omega_\zeta \right) = \int d \left(t_j \dashv f \omega_\zeta \right) = 0$$

Then $F = 0$ in the connected open set $\mathbb{C}^n \backslash \overline{\mathbb{U}}$ by the uniqueness theorem.

(iii) Now we show that for an arbitrary point $z_0 \in \partial \mathbb{U}$ the function $F(\zeta)$ tends to $f(z_0)$ as $\zeta \to z_0$. Let χ be the indicator function of $\mathbb{U}$. We have

$$\chi(\zeta) = \int_{\partial \mathbb{U}} \omega_\zeta$$

in virtue of (ii) and Bochner-Martinelli theorem. Therefore

$$F(\zeta) - f(z_0) \chi(\zeta) = \int_{\partial \mathbb{U}} [f(z) - f(z_0)] \omega_\zeta(z) \tag{1.4}$$

We have $[f(z) - f(z_0)] \omega_\zeta(z) = O\left(|z - z_0|^{-2n+2} dS \right)$ uniformly as $\zeta \to z_0$ and the positive density $|z - z_0|^{-2n+2} dS$ is integrable in the hypersurface $\partial \mathbb{U}$. Therefore (1.4) is a continuous function of ζ according to Lebesgue' convergence theorem. The left side vanishes in $\mathbb{C}^n \backslash \mathbb{U}$ by (ii), hence $F(z_0) = \chi(z_0) f(z_0) = f(z_0)$. $\square$

Corollary 8 *Let $\mathbb{U}$ be an open set in $\mathbb{C}^n$ and $\mathbb{K} \subset \mathbb{U}$ such that $\mathbb{U}\backslash\mathbb{K}$ is connected. Then any holomorphic function $f : \mathbb{U}\backslash\mathbb{K} \to \mathbb{C}$ has unique analytic continuation to $\mathbb{U}$.*

For proof we choose an open set $\mathbb{V}$ with smooth boundary $\partial\mathbb{V}$ such that $\mathbb{K} \subset \mathbb{V} \subset \mathbb{U}$. We have $df \wedge dz = 0$ on the boundary since the function f is holomorphic in $\mathbb{V}$ and continuous in the closure $\bar{\mathbb{V}}$ together with first derivatives. By Bochner-Severi theorem f has analytic continuation $g : \mathbb{V} \to \mathbb{C}$. Define the function F in $\mathbb{U}$ that is equal to f in $\mathbb{U}\backslash\mathbb{V}$ and g in $\bar{\mathbb{V}}$. It is holomorphic in $\mathbb{U}\backslash\partial\mathbb{V}$ and continuous in any point of $\partial\mathbb{V}$. It is holomorphic everywhere in $\mathbb{U}$ in virtue of Problem 1. The function F coincides with f in $\mathbb{U}\backslash\mathbb{V}$ and consequently in $\mathbb{U}\backslash\mathbb{K}$ because of the uniqueness property of holomorphic functions.

1.7 Problems

1. Let f be a continuous function in an open set $\mathbb{U} \subset \mathbb{C}^n$ that is holomorphic in $\mathbb{U}\backslash S$, where S is a C^1-hypersurface. Show that f is holomorphic in $\mathbb{U}$.

2. Let $\mathbb{U}$ be an open connected set in $\mathbb{C}^n$ and a subset Z is given by the equation $\{g = 0\}$, where $g \not\equiv 0$ is a holomorphic function in $\mathbb{U}$. Prove that holomorphic function f in $\mathbb{U}\backslash Z$ that is locally bounded in a neighborhood of each point $z \in Z$, has holomorphic continuation to $\mathbb{U}$.

3. Show that any holomorphic function in $\mathbb{C}^2\backslash\mathbb{R}^2$ has analytic continuation to $\mathbb{C}^2$.

1.8 Leray' formula

Definition. We call *Leray* map any C^1-mapping $\lambda : \mathbb{U} \times \partial\mathbb{U} \to \mathbb{C}^n$ such that $\langle \lambda(z, \zeta), \zeta - z \rangle \neq 0$. Take another copy of $\mathbb{C}^n$ and the differential form $\pi \doteq e \dashv dw = \sum (-1)^j w_j dw_1 \wedge ...\widehat{dw_j}... \wedge dw_n$ in this space. Note the following property:

8. If $a(z) \neq 0$ be a smooth function, then $(az) = \omega_0(z)$. For a proof we show that $\pi = w_1^n d(w_2/w_1) \wedge ... \wedge d(w_n/w_1)$ or that

$$\pi = \frac{1}{(n-1)!} \det \begin{pmatrix} w_1 & ... & w_n \\ dw_1 & ... & dw_n \\ ... & ... & ... \\ dw_1 & ... & dw_n \end{pmatrix}$$

Theorem 9 *[Leray-Fantappiè] Let $\mathbb{U}$ be a bounded domain in $\mathbb{C}^n$ with piecewise smooth boundary and λ be a Leray map. For any holomorphic function $f : \mathbb{U} \to \mathbb{C}$ that is continuous up to the boundary, the following representation holds*

$$f(z) = \int_{\partial \mathbb{U}} f(\zeta) \frac{\pi(\lambda(z,\zeta))}{\langle \lambda(z,\zeta), \zeta - z \rangle^n} \wedge d\zeta \qquad (1.5)$$

Remark. If $\lambda = \overline{\zeta} - \overline{z}$, this formula turns to the Bochner-Martinelli formula. On the other hand, if λ is a holomorphic with respect to z the kernel in (1.5) is a holomorphic in z too.

Problem 4. Show that for any bounded *convex* $\mathbb{U}$ one can choose a Leray map that does not depend on z.

PROOF. Fix $z \in \mathbb{U}$ and consider the quadric $Q \doteq \{\zeta, w : \langle w, \zeta - z \rangle = 1\} \subset \mathbb{C}^n \times \mathbb{C}^n$. This is a complex algebraic manifold of dimension $2n-1$ (see Chapter 2). The $2n-1$-cycle $W_1 \doteq \left\{ w = \frac{\lambda(\zeta)}{\langle \lambda(\zeta), \zeta - z \rangle}, \zeta \in \partial \mathbb{U} \right\}$ is contained in Q as well as the cycle $W_0 \doteq \left\{ w = \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2}, \zeta \in \partial \mathbb{U} \right\}$. There is a $2n$-chain

$$W \doteq \left\{ \zeta, w, t \in [0,1] : w = t \frac{\lambda(\zeta)}{\langle \lambda(\zeta), \zeta - z \rangle} + (1-t) \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^2} \right\}$$

whose image is contained in Q such that $\partial W = W_0 - W_1$. The form $e \dashv dw \wedge d\zeta$ is of type $(2n-1, 0)$ and has holomorphic coefficients. Therefore it is closed in Q. The form $f e \dashv dw \wedge d\zeta$ is closed too, since $d(f e \dashv dw \wedge d\zeta) = df \wedge d\zeta \wedge (e \dashv dw) = 0$ and $df \wedge d\zeta = 0$ in the boundary $\partial \mathbb{U}$. By Stokes Theorem we conclude that

$$\int_{W_1} f \pi \wedge d\zeta = \int_{W_0} f \pi \wedge d\zeta$$

By **8** we have $\pi(w) = \langle \lambda, \zeta - z \rangle^{-n} \pi(\lambda)$ in the left side, hence the left side is equal to (1.4). The same arguments show that the right side coincides with the Bochner-Martinelli formula.

References

[1] B.V.Shabat, Introduction to Complex Analysis, V.2
[2] Gunning, Introduction to holomorphic functions of several variables, V.I,II

Chapter 2

Analytic manifolds

2.1 Submanifolds in $\mathbb{C}^n$

The set of all tangent vectors to the space $\mathbb{C}^n = \mathbb{R}^{2n}$ with complex coefficients at a point w is called the *tangent space* and denote $T_w(\mathbb{C}^n)$. It is generated by the vectors $\partial/\partial x_j, \partial_{y_j}, \ j = 1, ..., n$ or by the vectors

$$\partial_j \doteq \frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - \imath\frac{\partial}{\partial y_j}\right), \ \overline{\partial}_j \doteq \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + \imath\frac{\partial}{\partial y_j}\right)$$

A tangent vector t is called *holomorphic,* or *antiholomorphic,* if it is a linear combination of vectors ∂_j and vectors $\overline{\partial}_j$, respectively. A C^1-function $f : \mathbb{U} \to \mathbb{C}$ is holomorphic, if it is annihilated by all antiholomorphic vectors. The union $T(\mathbb{C}^n) = \cup\{T_w(\mathbb{C}^n), w \in \mathbb{C}^n\}$ is called the *tangent bundle* of the space.

Define the linear operator $\mathbf{J}$ in $T(\mathbb{C}^n)$ by

$$\mathbf{J}\left(\partial_{x_j}\right) = \partial_{y_j}, \ \mathbf{J}\left(\partial_{y_j}\right) = -\partial_{x_j}$$

It is called the operator of complex structure. Obviously $\mathbf{J}(\partial_j) = \imath\partial_j$, $\mathbf{J}\left(\overline{\partial}_j\right) = -\imath\overline{\partial}_j$, $\mathbf{J}^2 = -1$, hence the holomorphic and antiholomorphic vectors are eigenvectors of $\mathbf{J}$ with eigenvalues $\imath$ and $-\imath$, respectively.

Reminder. Let $\mathbb{U}$ be an open subset of $\mathbb{C}^n$, and $S \subset \mathbb{U}$ be the set given by equations

$$h_1(x,y) = ... = h_p(x,y) = 0 \tag{2.1}$$

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where $h_1, ..., h_p$ are real smooth functions in S such that the Jacobian $p \times 2n$-matrix

$$J \doteq \left\{ \frac{\partial h_j}{\partial (x_k, y_k)} \right\}_{k=1,...,n}^{j=1,..,p}$$

has rank p everywhere in S. It is a smooth manifold of dimension $2n - p$. Let $w = u + \imath w \in S$; the space of real tangent vectors $t = \sum t_j \partial/\partial x_j + s_j \partial/\partial y_j$ that annihilate the functions $h_1, .., h_p$ is called the tangent space to S at w and denoted $T_w(S)$. The union $T(S) = \cup \{T_w(S), w \in S\}$ is called the tangent bundle to S. An arbitrary real submanifold $S \subset \mathbb{U}$ is a subset that is locally of form (2.1).

Definition. A complex analytic manifold in $\mathbb{U} \subset \mathbb{C}^N$ is a closed subset $Z \subset \mathbb{U}$ such that for an arbitrary point $w \in Z$ there exist a neighbourhood $\mathbb{W}$ of w and holomorphic functions $f_1, ..., f_q$ in $\mathbb{W}$ such that

$$Z \cap \mathbb{W} = \{z : f_1(z) = ... = f_q(z) = 0\} \tag{2.2}$$

and

$$\mathrm{rank} \left\{ \frac{\partial f_k(z)}{\partial z_j} \right\}_{j=1,...,n}^{k=1,...,q} = q \text{ for } z \in Z \tag{2.3}$$

The space of tangent holomorphic vectors $t = \sum t_j \partial_j$ such that $t(f_k) = 0$ for $k = 1, ..., q$, $z \in Z$ is called the *holomorphic tangent* space; it is denoted $T_{\mathbb{C}}(Z)$.

If $\mathbb{V}$ is an open subset of $\mathbb{U}$, then $Z \cap \mathbb{V}$ is analytic submanifold of $\mathbb{V}$.

Example. The hypersurface $Q \doteq \{p(z) = 0\}$ is called quadric in $\mathbb{C}^n$ (affine quadric), if p is a polynomial of order 2.

Proposition. *Any complex analytic submanifold Z is a real submanifold of dimension $2n - 2q$. The operator of complex structure $\mathbf{J}$ acts in the tangent bundle $T(Z)$.*

PROOF. Write $f_k = g_k + \imath h_k$, the system (2.2) is equivalent to

$$g_1 = h_1 = ... = g_q = h_q = 0$$

The real Jacobian matrix

$$\left\{ \frac{\partial (g_k, h_k)}{\partial (x_j, y_j)} \right\}_{j=1,...,n}^{k=1,...,q}$$

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has rank $2q$ in each point $w \in Z$. Indeed, assume the opposite; then we have

$$\sum_{k=1}^{q} a_k \frac{\partial g_k}{\partial x_j} + b_k \frac{\partial h_k}{\partial x_j} = 0 \qquad (2.4)$$

$$\sum_k a_k \frac{\partial g_k}{\partial y_j} + b_k \frac{\partial h_k}{\partial y_j} = 0 \text{ for } j = 1, .., n$$

for some real coefficients such that $\sum |a_k| + |b_k| \neq 0$. Take the sum of the first equation (2.4) and of the second with the factor $-\imath$: the result can be written in the form

$$\operatorname{Re} \sum (a_k - \imath b_k) \left(\frac{\partial g_k}{\partial x_j} + \imath \frac{\partial h_k}{\partial x_j} - \imath \frac{\partial g_k}{\partial y_j} + \frac{\partial h_k}{\partial y_j} \right) = 0$$

Taking in account the Cauchy-Riemann equations

$$\frac{\partial h_k}{\partial x_j} = -\frac{\partial g_k}{\partial y_j}, \; \frac{\partial h_k}{\partial y_j} = \frac{\partial g_k}{\partial x_j}$$

we find that also $\operatorname{Im} \sum (a_k - \imath b_k) (...) = 0$. Both relations together give

$$\sum_k (a_k - \imath b_k) \partial_j f_k = 0$$

which contradicts (2.3). Therefore Z is a smooth manifold of dimension $2n - 2q$ (or codimension $2q$). The analytic tangent bundle $T_{\mathbb{C}}(Z)$ is generated by the holomorphic fields of the form $t = t_1 \partial_1 + ... + t_n \partial_n$, where $\sum t_j \partial f_k / \partial z_j = 0$ for $k = 1, ..., p$. The holomorphic functions f_k satisfy the equation $\bar{t}(f_k) = 0$. Therefore the space $\bar{T}_{\mathbb{C}}(Z)$ is contained in $T(Z)$, hence the direct sum $T_{\mathbb{C}}(Z) \oplus \bar{T}_{\mathbb{C}}(Z)$ is contained in the space $T(Z)$. Both spaces are of complex dimension $2n - 2q$, consequently, they coincide. We have $\mathbf{J}(t) = \imath t$ and $\mathbf{J}$ transform the bundle $T(Z)$ to itself.

Definition. The number $m \doteq n - q$ is called *complex dimension* of Z, $\dim_{\mathbb{C}} Z = m$. The real dimension of Z, $\dim_{\mathbb{R}} Z$ is equal to $2m$.

Now we formulate the inverse statement:

Theorem 1 *Let $S \subset \mathbb{U} \subset \mathbb{C}^n$ be a real submanifold such that the operator $\mathbf{J}$ acts in $T(Z)$. Then S is a complex analytic submanifold of $\mathbb{U}$.*

Problem. To prove this theorem. Hint: suppose that $n = 2, m = 1$; write S in the form $z_2 = F(x_1, y_1)$ and check that the function F is holomorphic.

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2.2 Simplectic structure in $\mathbb{C}^n$

Definitions. The differential form

$$\sigma \doteq \frac{\imath}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum dx_j \wedge dy_j$$

is closed $d\sigma = 0$ and the n-th power

$$\sigma^{\wedge n} = \left(\sum dx_j \wedge dy_j \right)^{\wedge n} = (-1)^{n(n-1)/2} \, n! \, dx \wedge dy, \; dx = dx_1 \wedge ... \wedge dx_n, \; dy = ... \tag{2.5}$$

is a non vanishing volume form in $\mathbb{C}^n$. Therefore the form σ defines a simplectic structure in $\mathbb{C}^n$. A submanifold $S \subset \mathbb{C}^n$ is called *isotropic*, if $\sigma | S = 0$; it is called *Lagrangian*, if it is isotropic and $\dim S = n$.

Examples. The subspaces $\mathbb{R}^n, \imath \mathbb{R}^n$ are Lagrangian, but no complex linear subspace is Lagrangian.

Exercise. Check that for any holomorphic (or antiholomorphic) tangent vector t the equations hold $\sigma\left(t, \mathbf{J}\left(t\right)\right) = 0$, $\sigma\left(t, \mathbf{J}\left(\bar{t}\right)\right) = |t|^2$.

Proposition 2 *If Z is a complex analytic submanifold in $\mathbb{C}^n$ of dimension m, $K \subset Z$ is a compact set, the $2m$-volume of K can be calculated by means of the simplectic form*

$$Vol_{2m}\left(K\right) = \frac{(-1)^{m(m-1)/2}}{m!} \int_{K} \sigma^{\wedge m} \tag{2.6}$$

PROOF. It is sufficient to check this equation for an arbitrary complex linear subspace L, since any tangent space $T_w\left(Z\right)$ is complex linear too. Use a coordinate change $w = Uz$, where U is an unitary $n \times n$-matrix, i.e. a matrix whose entries are complex numbers, such that $\overline{U}'U = U\overline{U}' = I$, where I is the unit matrix. This generates an orthogonal transformation of $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and does not change the Euclidean metric. Indeed, the interior product in the Euclidean space $\mathbb{R}^{2n}$ can be written in the form

$$\langle z, \zeta \rangle = \text{Re} \sum z_j \overline{\zeta}_j$$

We have

$$\langle Uz, U\zeta \rangle = \text{Re} \sum (Uz)_j \overline{(U\zeta)_j} = \text{Re} \sum \left(\overline{U}'Uz\right)_j \overline{\zeta}_j = \text{Re} \sum z_j \overline{\zeta}_j = \langle z, \zeta \rangle$$

By a choice of U we can make $L = \{w_{n-m+1} = ... = w_n = 0\}$, hence $w_j = u_j + iv_j, j = 1, ..., m$ are complex coordinates in L. Then we have

$$Vol_{2m}(K) = \left| \int_K du_1 \wedge ... du_m \wedge dv_1 \wedge ... \wedge dv_m \right|$$

On the other hand, by (2.5) $\sigma^{\wedge m} = (-1)^{m(m-1)/2} m! \, du_1 \wedge ... du_m \wedge dv_1 \wedge ... \wedge dv_m$. This proves (2.6).

Corollary 3 *The volume of K is equal to the sum of volumes of projections of K to all the coordinate subspaces of $\mathbb{C}^n$ of dimension m.*

PROOF. In terms of the coordinate system z we have

$$\sigma^{\wedge m} = (-1)^{m(m-1)/2} m! \sum_{j_1 < ... < j_m} dx_{j_1} \wedge ... dx_{j_m} \wedge dy_{j_1} \wedge ... \wedge dy_{j_m},$$

which yields

$$Vol_{2m}(K) = \sum_{j_1 < ... < j_m} \int_K dx_{j_1} \wedge ... dx_{j_m} \wedge dy_{j_1} \wedge ... \wedge dy_{j_m}$$

and the integral in the right side equals to the integral of the same form over the projection of K to the coordinate subspace spanned by $z_{j_1}, ..., z_{j_m}$.

Theorem 4 [Wirtinger] *Let S be a real submanifold of $\mathbb{U} \subset \mathbb{C}^n$ of dimension $2m$. Then the inequality*

$$Vol_{2m}(K) \geq \frac{1}{m!} \left| \int_K \sigma^{\wedge m} \right|$$

holds for an arbitrary compact set $K \subset S$. The inequality turns to the equation for any K, if and only, if S is a complex analytic submanifold.

See a proof in the book B.Shabat, Chapter II, Sec.6.

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2.3 Analytic manifolds

Definition. A structure of complex analytic manifold in a metric space M is give by a complex analytic atlas $\{M_\alpha, \phi_\alpha, \alpha \in A\}$. An element of the atlas is a chart (M_α, ϕ_α), where $\phi_\alpha : M_\alpha \to \mathbb{C}^n$ is a homeomorphism of an open subset $M_\alpha \subset M$ onto an open subset $\mathbb{U}_\alpha \subset \mathbb{C}^n$. The maps ϕ_α are consistent in the following sense: for any $\alpha, \beta \in A$ such that $M_\alpha \cap M_\beta \neq \emptyset$ the *connecting* mapping

$$\phi_{\beta\alpha} : \quad \begin{array}{ccc} \phi_\alpha(M_\alpha) & \to & \phi_\beta(M_\beta) \\ \cap & & \cap \\ \mathbb{U}_\alpha & & \mathbb{U}_\beta \end{array}$$

ϕ (i.e. $\phi_{\beta\alpha}\phi_\alpha = \phi_\beta$) is holomorphic with respect to complex coordinates in $\mathbb{C}^n$. The number $n = n_\alpha$ may depend on α; if the space M is connected, n_α is constant. It is called dimension of M. Any other atlas $\{M'_\beta, \phi'_\beta, \beta \in B\}$ in M defines the same analytic structure, if the union $\{M_\alpha, \phi_\alpha\} \cup \{M'_\beta, \phi'_\beta\}$ is an analytic atlas in M.

Definition. Let M, N be analytic manifolds. A continuous mapping $F : M \to N$ is called analytic, if for any chart (M_α, ϕ_α) in M and any chart (N_β, ψ_β) in N the mapping

$$F_{\beta\alpha} : \phi_\alpha\left(M_\alpha \cap F^{-1}(N_\beta)\right) \to \psi_\beta(N_\beta)$$

such that $F_{\beta\alpha}\phi_\alpha = \psi_\beta$, is holomorphic. Composition of analytic mappings is again an analytic mapping. An analytic mapping F is called invertible or *isomorphism* if F^{-1} is defined and is also analytic. A mapping $A : M \to M$ is called *endomorphism*; A is called *automorphism* if it is a isomorphism.

2.4 Examples

1. Riemann sphere, Riemann surfaces.
2. Tori. Take $\lambda \in \mathbb{C}$ such that $\operatorname{Im}\lambda > 0$. The numbers $1, \lambda$ generates the subgroup $L_\lambda = \{n + m\lambda, \ n, m \in \mathbb{Z}\}$ of the additive group $\mathbb{C}$; it is called *lattice*. A coset of this subgroup is a set of the form $z + L_\lambda$. The quotient $\mathbb{T}_\lambda \doteq \mathbb{C}/L_\lambda$ is the set of all cosets. This a metric space homeomorphic to the topological torus. It has the structure of complex analytic manifold of dimension 1 such that the natural mapping $\mathbb{C} \to \mathbb{C}/L_\lambda = \mathbb{T}_\lambda$ is analytic. This structure depends on λ. If $\mu = \frac{a\lambda+b}{c\lambda+d}$ for some integers a, b, c, d such that

$ad - bc = \pm 1$, we have an analytic isomorphism $T_\mu \cong T_\lambda$, since $L_\mu = L_\lambda$. Otherwise there is no analytic isomorphism of the tori.

3. Complex projective space $\mathbb{CP}^n$. This is the (metric) space M of all 1-dimensional subspaces of $\mathbb{C}^{n+1}$. Let $(z_0, ..., z_n)$ be system of linear coordinates in $\mathbb{C}^{n+1}$. The analytic structure in M is defined by means of the atlas $\{M_j, \phi_j, \ j = 0, ..., n\}$, where M_j is the set of lines L that do not belong to the coordinate subspace $\{z_j = 0\}$. The chart $\phi_j : M_j \to \mathbb{C}^n$ is defined as follows: $\phi_j(L) = (w_0, ..., w_{j-1}, \widehat{w_j}, w_{j+1}, ..., w_n)$, where $(w_0, ..., w_{j-1}, 1, w_{j+1}, ..., w_n) \in L$. The point $\phi_j(L)$ is uniquely defined, since the line L can not contain two different points with j-th coordinate equal 1. Check that each connecting mapping $\phi_{jk} \ k \neq j$ is holomorphic. According to the construction, we have $\phi_k(L) = (u_0, .., \widehat{u_k}, ..., u_n)$, where $(u_0, .., u_{k-1}, 1, u_{k+1}..., u_n) \in L$. The points in the same line are proportional: $(w_0, ..., w_{j-1}, 1, w_{j+1}, ..., w_n) = \lambda(u_0, .., u_{k-1}, 1, u_{k+1}..., u_n)$, which implies

$$\lambda = u_j^{-1}, w_i = u_i u_j^{-1}, i \neq j, k; w_k = u_j^{-1}$$

i.e.

$$\phi_{jk}(u_0, .., \widehat{u_k}, ..., u_n) = u_0 u_j^{-1}, ..., u_j^{-1}, ..., u_n u_j^{-1}$$

The mapping ϕ_{jk} is obviously holomorphic.

Chapter 3

Division of holomorphic functions

3.1 Weierstrass' theorems

Weierstrass' theorem is an analogue for holomorphic functions of the Euclides' division algorithm. Let f be a holomorphic function defined in a nbd $\mathbb{U}$ of a point $a \in \mathbb{C}^n$, L be a complex line through this point. We say that L is *suitable* for f, if f does not vanish identically near the point a in $\mathbb{U} \cap L$. We say that a coordinate system $z_1, z_2, ..., z_n$ in $\mathbb{C}^n$ is *suitable* for f in a, if the line $L \doteq \{z' = z'(a)\}$ is suitable for f, where $z' = (z_2, ..., z_n)$. In a suitable coordinate system there exists a natural number k such that

$$f(a) = \partial_1 f(a) = ... = \partial_1^{k-1} f(a) = 0, \partial_1^k f(a) \neq 0$$

Proposition 1 *For an arbitrary holomorphic f that is not identically zero in a nbd of a point a there exists a suitable system of coordinates. Moreover, the set of suitable coordinate systems is open and dense in the set of all linear coordinate systems.*

PROOF. Find $k \geq 0$ such that $f(a) = d^1 f(a) = ... = d^{k-1} f(a) = 0$, but $d^k f(a) \neq 0$. Take for L any line such that $d^k f(a)|L \neq 0$. e

Theorem 2 *[Weierstrass] If a coordinate system $z_1, ..., z_n$ is suitable for f in a then*
(i) [factorization] there exists a function

$$p(z) = (z_1 - a_1)^k + p_1(z')(z_1 - a_1)^{k-1} + ... + p_k(z')$$

where $p_1(z'), ..., p_k(z')$ are holomorphic functions defined in a nbd of the point $b = z'(a) \in \mathbb{C}^{n-1}$ that vanish in this point and

$$f = \phi p$$

where ϕ is holomorphic function in a nbd of a. The functions p and ϕ are uniquely defined.
(ii) [division] For an arbitrary function g holomorphic in a nbd of a we have

$$g = qf + r$$

where again q, r are holomorphic functions in a nbd of a and

$$r(z) = r_0(z') z_1^{k-1} + r_1(z') z_1^{k-2} + ... + r_{k-1}(z')$$

where the functions r_j are holomorphic in a nbd of b. These functions are uniquely defined.

Remark. We have $\phi(a) = (k!)^{-1} \partial_1^k f(a) \neq 0$; therefore the function ϕ is invertible in a nbd of a.

Definition. A function of the form $q(z) = q_0(z') z_1^l + q_1(z') z_1^{l-1} + ... + q_l(z')$ is called *pseudopolynomial* (pp) in z_1. The pseudopolynomial is called *unitary* if $q_0 = 1$; it is called *distinguished* at a point $a \in \mathbb{C}^n$, if it is unitary and $q_1(z'(a)) = ... = q_l(z'(a)) = 0$.

According to (i), p is a distinguished polynomial at a.

Reminder. Let $V \subset \mathbb{C}$ be a bounded set with smooth boundary ∂V and ϕ be a Lifschitz function on ∂V, i.e. $|\phi(z) - \phi(z')| \leq C |z - z'|^\alpha$ for arbitrary $z, z' \in \partial V$ and some positive a, C. The Cauchy integral

$$F(z) \doteq \frac{1}{2\pi \imath} \int_{\partial V} \frac{\phi(w)\, dw}{w - z}$$

with the standard orientation of the boundary ∂V is well defined and holomorphic in the complement.

Theorem 3 *[Sokhozky-Plemelj] The function F is continuous up to ∂V from inside of V and from outside and $\phi = F_+ - F_-$, where F_+, F_- are the boundary values of F.*

PROOF. (i) Assume for simplicity that $a = 0$ and $k > 0$. Take a small disc $\mathbb{D} \subset L$ centered at the origin such that there are no roots of f in the closure of $\mathbb{D}$ except for the origin, which is root of multiplicity k. There exists a ball $\mathbb{B} \subset \mathbb{C}^{n-1}$ centered in $0 = z'(0)$ such that $f(z_1, z') \neq 0$ for $z_1 \in \partial\mathbb{D}, z' \in \mathbb{B}$. Consider the function $h(z) \doteq z_1^{-k} f(z)$; the logarithm $\ln h$ is well defined in a nbd of $\partial\mathbb{D} \times \mathbb{B}$. Write the Cauchy integral

$$H(\lambda, z') \doteq \frac{1}{2\pi\imath} \int_{\partial\mathbb{D}} \frac{\ln h(z)\, dz_1}{z_1 - \lambda}$$

It is holomorphic for $\lambda \in \mathbb{C}\backslash\partial\mathbb{D}$ and $z' \in \mathbb{B}$ and has boundary values in $\partial\mathbb{D}$ from both sides; denote the boundary values H_+, H_- from inside and outside of the disc $\mathbb{D}$, respectively. We have

$$\ln h(z) = H_+(z) - H_-(z)$$

for $z_1 \in \partial\mathbb{D}$. Set $h_+ = \exp(H_+)$; this function is holomorphic and does not vanish in $\mathbb{D} \times \mathbb{B}$ and continuous in the boundary $\partial\mathbb{D} \times \mathbb{B}$. The function $h_- = \exp(-H_-)$ is holomorphic, does not vanish in $\mathbb{C}\backslash\mathbb{D} \times \mathbb{B}$ and tends to 1 as $z_1 \to \infty$. We have $h_+ h_- = h = z_1^{-k} f(z)$, consequently the function $p \doteq z_1^k h_- = f/h_+$ has holomorphic continuation to $\mathbb{D} \times \mathbb{B}$. Therefore $p = p(z_1, z')$ is a polynomial in z_1. For $z' = 0$ we have the function $h(z_1, 0)$ is holomorphic in $\mathbb{D}$, hence $H_- = 0$ and $h_- = 1$ and $p(z_1, 0) = z_1^k$. The equation $f = p h_+$ completes the proof of (i).

(ii) We can replace f by p in virtue of (i). Write for $z' \in \mathbb{B}$

$$2\pi\imath g(z_1, z') = p(z_1, z') \int_{\partial D} \frac{g(\lambda, z')\, d\lambda}{p(\lambda, z')(\lambda - z_1)} + \int_{\partial D} \frac{g(\lambda, z')}{p(\lambda, z')} \frac{p(\lambda, z') - p(z_1, z')}{\lambda - z_1} d\lambda$$

The functions

$$q(z_1, z') = \frac{1}{2\pi\imath} \int \frac{g(\lambda, z')\, d\lambda}{p(\lambda, z')(\lambda - z_1)},$$

$$r(z_1, z') = \int \frac{g(\lambda, z')}{p(\lambda, z')} \frac{p(\lambda, z') - p(z_1, z')}{\lambda - z_1} d\lambda$$

are holomorphic in a nbd of the origin and r is obviously a polynomial in z_1 of order $< k$.

Exercise. By Rouché Theorem for each $z' \in \mathbb{B}$ the function $f(\cdot, z')$ has precisely k roots in $\mathbb{D}$ (counted with multiplicities), say $\zeta_1(z'), ..., \zeta_k(z')$. Show that $p(z_1, z') = (z_1 - \zeta_1(z')) ... (z_1 - \zeta_k(z'))$.

3

Definition. For a point $a \in \mathbb{C}^n$ we denote by $\mathcal{O}_a^n = \mathcal{O}_a(\mathbb{C}^n)$ the algebra of convergent power series

$$a(z) = \sum a_i (z-a)^i = \sum a_{i_1,...,i_n} (z_1 - a_1)^{i_1} ... (z_n - a_n)^{i_n}$$

Any holomorphic function f defined in a nbd of a defines an element of $\mathcal{O}_a^n$. Two functions f, g defines the same element if $f = g$ in a nbd of a. Thus $\mathcal{O}_a^n$ is the $\mathbb{C}$-algebra of *germs* of holomorphic functions at a. The algebra $\mathcal{O}_a^n$ has only one maximal ideal $\mathfrak{m}_a$; this is the ideal of series with zero free term: $a_0 = 0$.

3.2 Stabilization and finiteness properties

Definition. A commutative algebra A is called *Nötherian,* if any ascending chain of ideals

$$I_1 \subset I_2 \subset ... \subset I_r \subset ... \subset A \tag{3.1}$$

stabilizes, i.e. it is constant for sufficiently large k.

This is equivalent to the following property. An A-module M is called *finitely generated,* if there exists a finite set $m_1, ..., m_k \in M$ such that an arbitrary element $m \in M$ can be written as the sum $m = a_1 m_1 + ... + a_k m_k$ with some coefficients $a_1, ..., a_k \in A$ (the coefficients need not to be unique).

Proposition 4 *A commutative algebra A with unit is Nötherian, if and only if*
(*) for any finitely generated module M any submodule $N \subset M$ is also finitely generated.

PROOF. First we check that the property (*) implies stabilization of (3.1). Note that the algebra A is generated by the unit element. Consider the union $I \doteq \cup I_k$. This is an ideal in A, hence is generated by a finite set of elements $b_1, ..., b_m$ since of (*). These elements are contained in I_k for sufficiently large k, hence $I_l = I_k$ for any $l > k$.

Show that the stabilization property implies that N is finitely generated. We use an induction with respect to the minimal number k of generators of the module M. If $k = 0$, $M = 0$ and the statement is trivial. Suppose that $k > 0$ and $m_1, ..., m_k$ is a set of generators. Consider the mapping $p : A \to M$ given by $a \mapsto am_1$. Then $p(A)$ is a submodule in M and $p(A) \cap N$ is a

submodule in N. The quotient $N' \doteq N/N \cap p(A)$ is a submodule in $M' \doteq M/p(A)$. The module M' is generated by $k-1$ elements which are images of $m_2, ..., m_k$. By induction the submodule N' is finitely generated. Let $n_1', ..., n_l'$ be generators of N' and $n_1, ..., n_l \in N$ some elements whose images in N' are $n_1', ..., n_l'$. Consider the subset $I \subset A$ of elements a such that $p(a) \in N$. This is an ideal, hence it is generated by some elements $i_1, ..., i_r$. The submodule $N \cap p(A)$ is generated by $p(i_1), ..., p(i_r)$. Now we claim that N is generated by $n_1, ..., n_l; p(i_1), ..., p(i_r)$. Indeed, for an arbitrary $n \in N$ the image in N' can be written as $a_1 n_1' + ... + a_l n_l'$, whence $b \doteq n - a_1 n_1 - ... - a_l n_l$ is contained in $N \cap p(A)$. It equals $b_1 p(i_1) + ... + b_r p(i_r)$ for some $b_1, ..., b_r \in A$. This completes the induction.

Theorem 5 *The algebra $\mathcal{O}_a^n$ is Nötherian.*

PROOF. We can assume that $a = 0$ and use induction in n. We have $\mathcal{O}^0 \cong \mathbb{C}$ and our statement is trivial for $n = 0$. Now we suppose that $n > 0$. Let I be an ideal in $\mathcal{O}^n$. If $I \neq 0$ there exists a non trivial germ $f \in I$. Choose a suitable coordinate system and write $f = \phi p$ for a pseudopolynomial p of order k and invertible element $h \in \mathcal{O}^n$. Take an arbitrary $g \in I$ and apply Weierstrass division theorem: $g = qp + r$. The remainder r is a uniquely defined pseudopolynomial of order $< k$; the vector of coefficients of r belongs to the set $M = [\mathcal{O}^{n-1}]^k$. The latter is a module over the algebra $\mathcal{O}^{n-1}$ and the union $\{r\}$ is a submodule $N \subset M$. The algebra $\mathcal{O}^{n-1}$ is Nötherian by the inductive assumption, hence N is generated by a finite set of elements. Let $r_1, ..., r_l$ be the pseudopolynomials that correspond to these elements. That the set $\{f, r_1, ..., r_l\}$ generates I. $\square$

Example 1. The maximal ideal $\mathfrak{m}_a$ in $\mathcal{O}_a^n$ is generated by the linear functions $z_1 - a_1, ..., z_n - a_n$. The ideal $\mathfrak{m}_a^k$ is generated by polynomials $(z - a)^i$, $|i| = k$.

Theorem 6 *[Hilbert] If A is a Nötherian algebra, then the polynomial algebra $A[x]$ is Nötherian.*

Problem. To prove this theorem.

Chapter 4

Analytic sets

4.1 Analytic sets and germs

Definition. Let $\mathbb{U} \subset \mathbb{C}^n$ be an open set. A closed subset $Z \subset \mathbb{U}$ is called complex *analytic set*, if for any point $w \in \mathbb{U}$ there exists a nbd $\mathbb{W}$ of w and holomorphic functions $f_1, ..., f_q$ in $\mathbb{W}$ such that $Z \cap \mathbb{W} = \{f_1(z) = ... = f_q(z) = 0\}$. (No assumption on the Jacobian matrix!) If $q = 1$, this set is called *hypersurface*. If M is an analytic manifold, a subset $Z \subset M$ is called analytic, if $\phi(Z)$ is analytic subset of $\mathbb{U}$ for any any chart $\phi : M' \to \mathbb{U} \subset \mathbb{C}^n$ in the analytic structure of M.

Any analytic submanifold is, of course, an analytic set; it is called *regular* analytic set. Otherwise Z is called *singular* analytic set.

Example 1. If $p_1, ..., p_k$ are polynomials $\mathbb{C}^n$, the set $Z = \{p_1(z) = ... = p_k(z) = 0\}$ is an *affine* algebraic variety.

Example 2. If $p_1, ..., p_k$ are homogeneous polynomials, Z is a cone. The set of (complex) lines $\mathbb{P}(Z)$ of Z is analytic subset of the projective space $\mathbb{PC}_{n-1}$; it is called a *projective* algebraic variety.

Proposition 1 *Any finite union of analytic sets is an analytic set. Any intersection of analytic sets is again an analytic set.*

Example 3. The union of coordinate lines in $\mathbb{C}^3$ is an analytic set. It is given by the equations $z_1 z_2 = z_2 z_3 = z_3 z_1 = 0$.

Definition. Two analytic sets $Z, W \subset M$ define the same *germ* at a point $a \in M$ if there exists a nbd $\mathbb{U}$ of a such that $Z \cap \mathbb{U} = W \cap \mathbb{U}$. This is an equivalence relation. Any equivalence class is called analytic *germ*. The

class containing Z is called the *germ* of Z at a. The set of gems of analytic functions $f \in \mathcal{O}_a$ that vanishes in Z is an ideal, denoted $I(Z)$.

Definition. A complex analytic set (or germ) Z is called *reducible*, if it can be represented as an union $Z_1 \cup Z_2$ of analytic sets (germs) such that $Z_i \not\subset Z_j$. Otherwise Z is called *irreducible*.

Proposition 2 *A germ Z at a point a to be irreducible if and only if the ideal $I = I(Z)$ is prime, i.e. the inclusion $fg \in I$ implies that $f \in I$ or $g \in I$.*

Example 4. The set $Z \subset \mathbb{C}^2$, given by the equation $x^3 + y^3 - 3xy = 0$,(Decartes' leaf) is irreducible, but the germ of Z at the origin is reducible. The intersection $Z \cap \{y < b\}$ is reducible, if and only if $b \le 3^{1/3}$.

Proposition 3 *An arbitrary analytic germ Z at point $a \in \mathbb{C}^n$ can be represented in the form*

$$Z = Z_1 \cup ... \cup Z_r \tag{4.1}$$

where $Z_1, ..., Z_r$ are irreducible analytic germs. These germs are uniquely defined, if the representation is minimal, i.e. no term Z_i can be excluded.

PROOF. Suppose that in an arbitrary minimal representation like (4.1) one of the germs $Z_1, ..., Z_r$ is reducible, for instance, $Z_1 = W \cup W'$, where W and W' are smaller than Z_r. Then we get another representation

$$Z = W \cup W' \cup Z_2 \cup ... \cup Z_r$$

It may be not minimal, but then one of the germs W or W' is not contained in $Z_2 \cup ... \cup Z_r$. If W is not contained, we have the minimal representation $Z = W \cup Z_2 \cup ... \cup Z_r$. If W or W' is reducible, we apply the same arguments to it and so on. Then we apply these arguments to the germ Z_2 and so on. The result is a strictly decreasing tree-graph of germs $\{Z_\alpha\}$ by inclusion. The corresponding ideals $I(Z_\alpha)$ form a strictly increasing tree. Any ascending chain of ideals $I(Z_1) \subset I(W) \subset ...$ in this tree stabilizes, since the algebra $\mathcal{O}_a$ is Nötherian. Therefore any chain is finite. Therefore the three is finite too. The last elements in this tree are irreducible. This proves the existence of (1).

We say that a *prime* ideal J is associated to I if there exists an element $f \in \mathcal{O}_a$ such that $J = I : \mathcal{O}_a$, i.e. J is just the set of elements g of the algebra such that $gf \in I$. If the representation (1) is minimal, than an ideal J is associated to I if $J = I(Z_k)$ for some k and vice versa. This proves the uniqueness.

Theorem 4 *Let Z be an irreducible analytic set in $\mathbb{U}$. There exists an analytic subset $\mathrm{Sing}\,(Z) \subset Z$ such that*
(i) Z is an analytic submanifold in a nbd of a point w, if and only if $w \in Z \setminus \mathrm{Sing}\,(Z)$;
(ii) $\mathrm{Sing}\,(Z)$ is nowhere dense in Z;
(iii) $Z \setminus \mathrm{Sing}\,(Z)$ is locally and globally connected and dense in Z.

Corollary 5 *If Z is a irreducible analytic set in $\mathbb{U}$ and the germ of Z at a point $a \in \mathbb{U}$ coincides with the germ of an analytic set W, then $Z \subset W$.*

PROOF. We have $Z \setminus \mathrm{Sing}\,(Z) \subset W$ by analytic continuation; $\mathrm{Sing}\,(Z) \subset W$ since W is closed.

Definition. It follows that $Z \setminus \mathrm{Sing}\,(Z)$ is a submanifold, whose complex dimension m is the same in any point z. This number is called the *dimension* of the analytic set Z. If Z is an arbitrary analytic set and $a \in Z$, then the dimension of Z in a in by definition the maximal of dimensions of the irreducible components $Z_1, ..., Z_q$ that contain a.

Theorem 6 *If the germ of an analytic set Z is irreducible at z, then the set $Z \cap \mathbb{U}$ is irreducible for any sufficiently small ball $\mathbb{U}$ with the centre in z.*

We postpone the proofs.

4.2 Resultant

Let A be a commutative ring. Take two polynomials of one variable t with coefficients in A :

$$p = a_0 + a_1 t + ... + a_m t^m; \; q = b_0 + b_1 t + ... + b_n t^n$$

The determinant of the $(n+m)$-matrix

$$S\,(p,q) \doteq \begin{pmatrix} a_0 & a_1 & ... & a_{m-1} & a_m & 0 & ... & 0 \\ 0 & a_0 & a_1 & ... & a_{m-1} & a_m & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & 0 & a_0 & a_1 & ... & ... & a_m \\ b_0 & ... & ... & b_{n-1} & b_n & 0 & ... & 0 \\ 0 & b_0 & ... & ... & b_{n-1} & b_n & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & 0 \\ 0 & ... & 0 & b_0 & ... & ... & b_{n-1} & b_n \end{pmatrix}$$

is called *Sylvester resultant* of the polynomials: $R(p,q) = \det S(p,q)$. It is an irreducible polynomial of coefficients $a_0, ..., a_m, b_0, ..., b_n$ with integer coefficients.

Proposition 7 *If A is a field, then $R(p,q) = 0$ if and only if there exist polynomials $u, v \in A[t]$ such that $uv \neq 0$ and $up + vq = 0$ $\deg u < n$, $\deg v < m$.*

Proposition 8 *If the field A is algebraically closed, then $R(p,q) = 0$ if and only if there exists, at least, one common root of p and q in A or at infinity.*

Proposition 9 *Suppose that $\alpha_i, i = 1, ..., m$ and $\beta_j, j = 1, ..., n$ are roots in A of p and q, respectively. Then*

$$R(p,q) = a_m^n b_n^m \prod_{i,j} (\alpha_i - \beta_j) = (-1)^{nm} b_n^m \prod_j p(\beta_j) = a_m^n \prod_i q(\alpha_i)$$

Properties

1. Homogeneity: $R(a_0, \lambda a_1, ..., \lambda^m a_m, b_0, \lambda b_1, ..., \lambda^n b_n) = \lambda^{mn} R(a_0, ..., a_m, b_0, ..., b_n)$.

2. Multiplicativity: $R(p_1 p_2, q) = R(p_1, q) R(p_2, q)$.

3. Projective covariance: Take an arbitrary non-degenerate matrix $u = \begin{matrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{matrix}$ and define the projective transformation of the polynomials

$$u^*(p) = (u_{21}t + u_{22})^m p\left(\frac{u_{11}t + u_{12}}{u_{21}t + u_{22}}\right).$$

We have

$$R(u^*(p), u^*(q)) = (\det u)^{mn} R(p,q)$$

4. Bezout-Cayley formula: If $m = n$, consider the polynomial

$$r(t,s) = \frac{p(t)q(s) - p(s)q(t)}{t - s} = \sum_{i,j=0}^{n-1} r_{ij} t^i s^j$$

Then $R(p,q) = \det \|r_{ij}\|$.

5. $R = pu + qv$ where $u, v \in A[t]$.

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4.3 Discriminant

Sylvester discriminant of a polynomial p is the resultant of p and p' :

$$\text{Disc}(p) = R(p, p') = \det S(p, p')$$

$$\text{Disc}(p) = \det \begin{pmatrix} a_0 & a_1 & \dots & a_{m-1} & a_m & 0 & \dots & 0 \\ 0 & a_0 & a_1 & \dots & a_{m-1} & a_m & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & a_0 & a_1 & \dots & \dots & a_m \\ a_1 & \dots & \dots & (m-1)a_{m-1} & ma_n & 0 & \dots & 0 \\ 0 & a_1 & \dots & \dots & (m-1)a_{m-1} & ma_m & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & a_1 & \dots & \dots & (m-1)a_{m-1} & ma_m \end{pmatrix}$$

Proposition 10 *If $\alpha_i, i = 1, ..., m$ are roots of p, then*

$$\text{Disc}(p) = (-1)^{m(m-1)/2} a_m^{2m-2} \prod_{i<j} (\alpha_i - \alpha_j)^2 \qquad (4.2)$$

Explicit formulae: For $n = 2$, $p = a_2 t^2 + a_1 t + a_0$ we have $\text{Disc}(p) = 4a^2 c - ab^2 = a_2 (4a_2 a_0 - a_1^2)$

For $p = t^3 + at + b$

$$S = \begin{pmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & 0 & a & b \\ 0 & 0 & 3 & 0 & a \\ 0 & 3 & 0 & a & 0 \\ 3 & 0 & a & 0 & 0 \end{pmatrix}$$

$\text{Disc}(p) = -4a^3 - 27b^2$.

Example 2. $p = t^4 + at^2 + bt + c$

$$S = \begin{pmatrix} 1 & 0 & a & b & c & 0 & 0 \\ 0 & 1 & 0 & a & b & c & 0 \\ 0 & 0 & 1 & 0 & a & b & c \\ 0 & 0 & 0 & 4 & 0 & 2a & b \\ 0 & 0 & 4 & 0 & 2a & b & 0 \\ 0 & 4 & 0 & 2a & b & 0 & 0 \\ 4 & 0 & 2a & b & 0 & 0 & 0 \end{pmatrix}$$

$\text{Disc}(p) = -4a^3 b^2 - 27b^4 + 16a^4 c + 144ab^2 c - 128a^2 c^2 + 256c^3$.

Proposition 11 *If A is an algebraically closed field, $D(p) = 0$ if and only if p has a multiple root in A or at infinity.*

Write the Sylvester matrix in a different way:

$$
\left(
\begin{pmatrix}
& & & & 0 \ ... \ 0 \ 0 \ a_0 \ a_1 \ ... \ a_m & \\
& & & & 0 \ ... \ 0 \ a_0 \ a_1 \ ... \ a_m \ 0 & \\
& & & & ... & \\
\begin{pmatrix}
& a_0 \ a_1 \ ... \ a_m \ 0 \ 0 \ 0 \ ... \ 0 & \\
a_1 \ 2a_2 \ ... \ (m-1)\,a_{m-1} \ ma_m \ 0 \ 0 \ ... \ 0 \\
0 \ a_1 \ 2a_2 \ ... \ (m-1)\,a_{m-1} \ ma_m \ 0 \ ... \ 0 \\
... \\
... \\
0 \ 0 \ ... \ 0 \ a_1 \ 2a_2 \ ... \ (m-1)\,a_{m-1} \ ma_m
\end{pmatrix}
\end{pmatrix}
\right)
$$

We have the sequence of submatrices $S_1, S_2, ...,$ where S_k is of size $2m+1 - 2k \times 2m-1$.

Proposition 12 *A polynomial p of order m has common divisor of order $> k$ with the derivative p' if and only if $\operatorname{rank} S_k(p) < 2m+1-2k$.*

Let $S_{kj}, j = 1, ...$ are all higher minors of S_k. The condition means that $S_{kj}(p) = 0$ for all j.

4.4   Hypersurfaces

Now we prove Theorem 1 for germs of hypersurfaces.

Take a holomorphic function f in a nbd of a point $a \in \mathbb{C}^n$ and consider the hypersurface $Z = \{f = 0\}$. Choose a suitable coordinate system (z_1, z') for this point and function f. We call a nbd $\mathbb{V}$ of $a = (a_1, a')$ *suitable* if $\mathbb{V} = \mathbb{D} \times \mathbb{B}$, where $\mathbb{D} = \{|z_1 - a_1| \leq r\}$ and $\mathbb{B}$ is a closed connected nbd of the point a' and f is defined in $\mathbb{V}$. The number r and $\mathbb{B}$ are so small that
(i) $f(z_1, z') \neq 0$ for $z' \in \mathbb{B}$ and $z_1 \in \partial\mathbb{D}$ and
(ii) $f(z_1, a')$ has only one root $z_1 = a_1$ in $\mathbb{D}$. Let m be the multiplicity of this root. We study the analytic set

$$Z = \{f(z) = 0, z \in \mathbb{V}\}$$

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By Weierstrass Lemma we have $f = \phi P$ where ϕ is holomorphic and does not vanish in $\mathbb{V}$ and P is a distinguished pseudopolynomial (pp) of order m at a for $z' \in \mathbb{B}$ which means that

$$P(z_1, z') = (z_1 - a_1)^m + A_{m-1}(z')(z_1 - a_1)^{m-1} + ... + A_0(z')$$

and $A_{m-1}(a') = ... = A_0(a') = 0$. The function P is a polynomial with coefficients in the algebra $\mathcal{O}(\mathbb{B})$ of holomorphic functions in $\mathbb{B}$. Substitute the pp P in the Sylvester minors; we obtain the sequence of functions $S_1(P), S_{2j}(P), S_{3j}(P), ... \in \mathcal{O}(\mathbb{B})$.

Theorem 13 *Let k be the minimal number such that $S_{kj}(P) \;/\!\!\!\equiv 0$ for some j and $\mathbb{B}$ be a suitable nbd of a' for S_{kj}. Then*
(i) we have in $\mathbb{V}$
$$P = P_1^{m_1} ... P_s^{m_s} \tag{4.3}$$
where $P_1, ..., P_s$ are irreducible distinguished pp-s in $\mathbb{V}$ and $\max m_j = k$.
(ii) For each $j = 1, ..., q$ the analytic set $Z_j = \{P_j = 0\}$ is irreducible,
(iii) the set $\mathrm{Sing}(Z_j)$ is analytic set of dimension $< n - 1$ and $Z_j \backslash \mathrm{Sing}(Z_j)$ is a connected analytic manifold of dimension $n - 1$ which is dense in Z_j.

PROOF. Assume, first, that $k = 1$. We call $\Delta = \{S_1(P) = 0, z' \in \mathbb{B}\}$ the discriminant set of P. The complement $\mathbb{B}\backslash\Delta$ is connected. Note that the set $Z \backslash P^{-1}(\Delta)$ is an analytic manifold of dimension $n - 1$. Fix a point $b \in \mathbb{B}\backslash\Delta$ and consider the fundamental group $\pi_1(\mathbb{B}\backslash\Delta)$ of loops through b. The polynomial $P(z_1, b)$ has m different roots $\alpha_1, ..., \alpha_m$ in $\mathbb{D}$ since of (4.2). Take a loop $\gamma \subset \mathbb{B}\backslash\Delta$ and construct analytic continuation $\alpha_1(z'), ..., \alpha_m(z')$ along this loop. To the end of γ we return to the point b and obtain the order set of roots $\alpha_1^*, ..., \alpha_m^*$. They are different; hence this set is a substitution of the order set $\alpha_1, ..., \alpha_m$. Thus, the loop γ defines a substitution σ of the ordered set $[1, ..., m]$. This is a group morphism

$$\mu : \pi_1(\mathbb{B}\backslash\Delta) \to S_m \tag{4.4}$$

called the *monodromy*; S_m is the group of substitutions.

Suppose that the monodromy is irreducible, i.e. is no subset of $[1, ..., m]$ invariant with respect of the action of $\pi_1(\mathbb{B}\backslash\gneqq)$. This means that if we start with the root α_1 and take analytic continuation along all loops, we can obtain any of the roots $\alpha_2, ..., \alpha_m$ for $z' = b$ and arbitrary root β of the polynomial $P(z_1, z')$ for any $z' \in \mathbb{B}$. The continuation of α_1 defines a curve in the set

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$Z \backslash P^{-1} (\Delta)$ where $p : \mathbb{D} \times \mathbb{B} \to \mathbb{B}$ is the projection. This set is connected, since we can reach any point (β, z') by a curve from (α_1, b). It is dense in Z since $\mathbb{B} \backslash \Delta$ is dense in $\mathbb{V}$. The set $\mathrm{Sing}\,(Z)$ of singular points of Z is given by the system

$$P = \frac{\partial P}{\partial z_1} = ... = \frac{\partial P}{\partial z_n} = 0$$

It is contained in $Z \cap P^{-1}(\Delta)$. Therefore the set $Z \backslash \mathrm{Sing}\,(Z)$ is connected too and dense in Z. This proves Lemma for the case $q = m_1 = 1$.

Consider the opposite case: the monodromy is reducible, i.e. there exists an irreducible subset, say $\alpha_1, ..., \alpha_l$ of roots of $P(z_1, b)$ that stays invariant under the action of the monodromy. Take an arbitrary point $z' \in \mathbb{B} \backslash \Delta$ a curve $\gamma \subset \mathbb{B} \backslash \Delta$ that joins b and z' and denote by $\beta_1(z'), ..., \beta_l(z')$ the analytic continuations of $\alpha_1, ..., \alpha_l$ along this curve. Set

$$Q(z_1, z') = \prod_1^l (z_1 - \beta_j(z')) \tag{4.5}$$

If we choose another curve γ' instead of γ we come to the same set of roots with, may different order. This does not effect the function Q since it is symmetric in $\beta_1, ..., \beta_l$. Therefore the function is well defined an holomorphic in $\mathbb{C} \times \mathbb{B} \backslash \Delta$. It is bounded since $|\beta_j| \leq r$ for any j. Therefore Q is a distinguished pp of order l at a. The same true for the pp

$$Q'(z_1, z') = \prod_{l+1}^m (z_1 - \beta_j(z'))$$

and we have $P = QQ'$. Hence P is reducible. We apply the same arguments to Q, Q' and so on. We come eventually to the representation (4.3) with $m_1 = ... = m_s = 1$ and irreducible $P_1, ..., P_s$.

Now we consider the case $k > 1$. Let $\mathbb{B}$ be a suitable nbd of a' for the function $D = S_{kj}(P)$ Now we call $\Delta = \{D = 0\}$ the discriminant set of P. Choose a point $b \in \mathbb{B} \backslash \Delta$; there exists a root, say α_1 of multiplicity exactly k of the polynomial $P(z_1, b)$. Choose a curve $\gamma \subset \mathbb{B} \backslash \Delta$ that joins b and a point z' and construct analytic continuation of α_1 along γ. Denote by $\beta_1, ..., \beta_l$ all the roots of $P(z_1, z')$ that can be obtained in this way. Then we define the pp Q as in (4.5) and Q' by taking the product of binomials $z_1 - \beta_j$ over all other roots in the point z'. We have then $P = Q^k Q'$ and the factor Q is an irreducible distinguished pp. Then we apply the same arguments to Q' and so on.

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Corollary 14 *We have*

$$\mathrm{Sing}\,(Z) = \cup_j \,\mathrm{Sing}\,(Z_j) \cup \cup_{i \neq j}\,(Z_i \cap Z_j)$$

4.5 Geometry of irreducible germs

4.5.1 Suitable neighbourhoods

Let Z be an irreducible germ of analytic set at a point $a \in \mathbb{C}^n$. We say that a linear subspace L is *suitable* for Z if $L \cap Z = \{a\}$ and no other subspace $L' \supset L$ satisfies this condition. Choose a coordinate system $z = z_1, ..., z_n$ in $\mathbb{C}^n$ in such a way that $L = \{z_{d+1} = ... = z_n = 0\}$ for some d. We shall see later that the number $n - d$ is equal to $\dim Z$. The subspace $B \doteq \{z_1 = ... = z_d = 0\}$ is complementary, i.e. the mappings $\pi_L\,(z) = (z_1, ..., z_d, 0, ..., 0)$ and $\pi_B\,(z) = (0, ..., 0, z_{d+1}, ..., z_n)$ define the analytic isomorphism $\mathbb{C}^n \cong L \oplus B$. We denote by Z a representative of the germ. We call a nbd $\mathbb{V}$ of a *suitable* for Z if $\mathbb{V} = \mathbb{D} \times \mathbb{B}$ where $\mathbb{D} \subset L$ is a nbd of a such that $\mathbb{D} \cap Z = \{a\}$ and $\mathbb{B} \subset B$ is a compact connected nbd of a such that the mapping

$$\pi_B : Z \cap \mathbb{V} \to \mathbb{B} \tag{4.6}$$

is *proper (*i.e. the pull back of an arbitrary compact set $K \subset \mathbb{B}$ is a compact set) and

$$I_d \doteq I\,(Z) \cap \mathcal{O}\,(\mathbb{B}) = \{0\} \tag{4.7}$$

Proposition 15 *For an arbitrary nbd U of a and arbitrary suitable subspace L through a there exists a suitable nbd $\mathbb{V} \subset U$. Moreover, the projection (6) is open, finite (i.e. the set $\pi_B^{-1}\,(b)$ is finite for any $b \in \mathbb{B}$) and surjective.*

PROOF. Let L be a subspace through a that satisfies (4.7) of maximal dimension. Choose a coordinate system $z = z_1, ..., z_n$ in $\mathbb{C}^n$ such that $L = \{z' = 0\}$ where $z' = (z_{d+1}, ..., z_n)$. The complementary space is given by $B = \{z_1 = ... = z_d = 0\}$. Choose small discs $\mathbb{D}_1, ..., \mathbb{D}_d$ and a ball $\mathbb{B} \subset B$ such that the set $\partial\,(\mathbb{D}) \times \mathbb{B}$ has no common point with Z where $\mathbb{D} = \mathbb{D}_1 \times ... \times \mathbb{D}_d$. Fix an integer $j, 1 \leq j \leq d$ and consider the subspace $L_j = \{z_j = z_{d+1} = ... = z_n = 0\}$ and the complementary space B_j spanned by B and the z_j-axis. Let $\pi_j : \mathbb{C}^n \to B_j$ be the coordinate projection. We claim that the set $\pi_j\,(Z \cap \mathbb{V})$ is a hypersurface in $\mathbb{V}_j \doteq \mathbb{D}_j \times \mathbb{B}$ if $\mathbb{D}_j$ and $\mathbb{B}$ are

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sufficiently small. Indeed, there is a function $f \in I_j \backslash \{0\}$, where $I_j \doteq I(Z)$ $\cap \mathcal{O}(\mathbb{V}_j)$ otherwise L_{d-1} is not a maximal subspace that fulfils (4.7). By Weierstrass Lemma 1 we can replace f by a pp p with respect to a suitable coordinate system. Apply the previous Theorem; one of the factors belongs to $I(Z)$ because of this ideal is prime. We can assume that p is irreducible. This pp generates the ideal I_j. Indeed, if $g \in I_j$, applying Weierstrass Lemma (ii) we reduce g to a pp q. The resultant $R(p, q)$ belongs to I_d and does not vanish identically. This contradicts the maximality of dimension of L_d. The polynomial q is multiple of p because of p is irreducible. This that the ideal I_j is principal and moreover the coordinate system $z_j; z'$ is suitable for a generator of this ideal. Therefore there we can take a distinguished pp p_j as a generator. It is irreducible pp, consequently the discriminant $D_j = D_j(z')$ does not vanish identically. Consider the analytic set

$$P \doteq \{p_1(z_1; z') = p_2(z_2; z') = ... = p_d(z_d; z') = 0\}$$

It contains Z because of (iii). Now we see that the projection π_d is finite for a suitable choice of $\mathbb{D}_1, ..., \mathbb{D}_d$ and $\mathbb{B}$. We only need to check that π_d is a surjection. Suppose the opposite: take a point $w' \in \mathbb{B} \backslash \pi_d(Z)$. No point of the form $\zeta(w') = (z_1, ..., z_d; w') \in P$ belongs to Z because of the assumption. There are no more then $m_1 ... m_d$ such points. Therefore there exists a function $g \in I(Z)$ that does not vanish in any such point $\zeta(w')$. Apply Weierstrass Lemma (ii) to g and p_1 and obtain the remainder $g_1(z_1; z)$; it also does not vanish in the points $\zeta(w')$. Consider the resultant $g_2 = R(p_1, g_1)$ with respect to the variable z_1. It belongs to $I(Z)$, does not depend on z_1 and vanish in no point $\zeta(w)$. We apply Weierstrass Lemma to g_2 and p_2 and so on. To the end we obtain a function $g_d(z') \in I(Z)$ that does not vanish in w'. This contradicts to the previous construction and proves surjectivity of π_d. The set P is proper and finite over $\mathbb{B}_d$ since of the choice of discs $\mathbb{D}_1, ..., \mathbb{D}_d$. Therefore $Z \subset P$ is also proper since of the continuity property of the set root of a polynomial.

4.5.2   Geometry of irreducible analytic sets

Let L_d, B_d be the suitable linear subspace as above. The set $\Delta' = \{D' = 0\}$ is a hypersurface in $\mathbb{B}$ where $D' \doteq D_1 ... D_d$. Take a point $w \in \mathbb{B}_d \backslash \Delta'$ and denote by $\zeta_1(w), ..., \zeta_m(w)$ all the points of the set Z such that $\pi_d(\zeta) = w$. Choose a linear function $l = l(z_1, ..., z_d)$ in L_d in such a way that the values

$$l(\zeta_1(w)), ..., l(\zeta_m(w)) \text{ are different} \qquad (4.8)$$

By changing the coordinate system we can take $l = z_d$. Let m be the order of the pp $p = p_d$ and $D = D(z')$ be the discriminant of p. It does vanish identically because of the order m is minimal; we denote $\Delta = \{D = 0\}$.

Theorem 16 *Let $\mathbb{V}$ be a suitable nbd for Z. For any $j = 1, ..., d - 1$ there exists a pp $q_j(z_d; z')$ or order $< m$ in $\mathbb{C} \times \mathbb{B}$ such that the set*

$$p(z_d, z') = 0; \ Dz_j - q_j = 0, j = 1, ..., d - 1, z' \in \mathbb{B}$$

is contained in $Z \cap \mathbb{V}$ and contains $Z \cap \mathbb{V}_d \backslash \pi_d^{-1}(\Delta)$.

PROOF. Fix $j < d$ and consider the function

$$h_j(z_d, z') = z_j\left(\pi_{d-1}^{-1}(z_d, z')\right)$$

defined on the root set $Z(p) = \{p(z_d, z') = 0, D'(z') \neq 0\}$. It is well-defined because of (4.8) and is analytic. There are m roots $\zeta_1(z'), ..., \zeta_m(z')$ in $Z(p)$. Take the interpolating pp for Dh_j

$$q_j(z_d, z') = D(z') \sum h_j(\zeta_k(z')) \frac{\prod_{l \neq k}(z_d - \zeta_l(z'))}{\prod_{l \neq k}(\zeta_k(z') - \zeta_l(z'))}$$

The sum is a pp and is equal $h_j(\zeta_k)$ as $z_d = \zeta_k$, consequently the pp q coincides with Dz_j in the set $Z(p)$. It is a pp with coefficients that are analytic in $\mathbb{B}_d \backslash \Delta$. The coefficients are bounded because of the formula $D = \text{const} \, \Pi(\zeta_k - \zeta_l)^2$. Therefore the coefficients of q are analytic in $\mathbb{B}_d$.

4.5.3 Continuity of roots of a polynomial

Definition. Let (M, r) be a metric space, $C(M)$ the set of compact subsets of M. The Hausdorf distance in $C(M)$ is defined as follows

$$\text{dist}(X, Y) = \max\left\{\max_{x \in X} \min_{y \in Y} r(x, y), \max_{y \in Y} \min_{x \in X} r(x, y)\right\}$$

Proposition 17 *Let $Z(p)$ be the set of roots of an unitary polynomial $p(t) = t^n + a_1 t^{n-1} + ... + a_{n-1} t + a_n$ in $\mathbb{C}$. This set as a function of the point $(a_1, ..., a_n) \in \mathbb{C}^n$ is a Lipschitz function of order $1/n$. Moreover, we have*

$$\max_{Z(p)} |\zeta| \leq n \max_{j=1,...n} |a_j|^{1/j} \tag{4.9}$$

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PROOF. It is sufficient to prove (4.9). By scaling it is reduced to the case $\max |a_j|^{1/j} = 1$. Therefore we can assume that $|a_j| \leq 1$ for all j. The polynomial does not vanish if $|t| > n$, because of $|t^n|/n > |a_j t^{n-j}|$ for all $j > 0$. Therefore $|\zeta| \leq n$ for all its roots.

Chapter 5

Analytic and polynomial algebras

5.1   Local and global algebras

Our basic field will be always $\mathbb{C}$. Here are some classes of commutative $\mathbb{C}$-algebras:

1. The algebra of polynomials $\mathbb{C}[z_1, ..., z_n]$ of transcendent elements $z_1, ..., z_n$. On the other hand a polynomial $p = p(z)$ can be considered as a holomorphic function in $\mathbb{C}^n$. Vice versa, a holomorphic function f with polynomial growth rate $f(z) = O(|z|^q)$ at infinity is a polynomial of order $\leq q$. This algebra is Nötherian by Hilbert's theorem. For any point $a \in \mathbb{C}^n$ the set $\mathfrak{m}_a$ of polynomials that vanishes in a is a maximal ideal in $\mathbb{C}[z_1, ..., z_n]$. We shall prove that an arbitrary maximal ideal is of this form.

2. For an arbitrary open set $\mathbb{U} \subset \mathbb{C}^n$ the algebra $\mathcal{O}(\mathbb{U})$ of holomorphic functions in $\mathbb{U}$.

3. We use the notation $\mathcal{O}_a^n$ for the algebra of germs of holomorphic functions at a point $a \in \mathbb{C}^n$ and $\mathcal{O}^n = \mathcal{O}_0^n$. If we fix a coordinate system in $\mathbb{C}^n$, it is isomorphic to the algebra of convergent power series

$$f(z) = \sum f_{i_1, ..., i_n} (z_1 - a_1)^{i_1} ... (z_n - a_n)^{i_n} \equiv \sum f_i (z - a)^i \qquad (5.1)$$

with complex coefficients f_i. The condition of convergence is the polidisc $\{|z_j - a_j| < r_j, j = 1, ..., n\}$ is fulfilling of the estimate

$$|f_{i_1, ..., i_n}| \leq C r_1^{-i_1} ... r_n^{-i_n}$$

for some constant C. The estimate $|f_i| \leq Cr^{-|i|}$, $|i| = i_1 + \ldots i_n$ is equivalent for $r_1 = \ldots = r_n = r$. We have obviously for any point a, $\mathcal{O}_a \cong \mathcal{O}_0 = \mathbb{C}\{z_1, \ldots, z_n\}$.

Definition. An *analytic algebra* is an algebra of the form $A = \mathcal{O}^n/I$, where $\mathcal{O}^n = \mathcal{O}_0(\mathbb{C}^n) = \mathbb{C}\{z_1, \ldots, z_n\}$ is the algebra of convergent power series, and I is an ideal in this algebra. It is *local,* i.e. there is only one maximal ideal $\mathfrak{m} = \mathfrak{m}(A)$. It is equal to the image of the maximal ideal $\mathfrak{m}(\mathcal{O}^n)$ under the surjection $p : \mathcal{O}^n \to A$. The canonical mapping $A \to A/\mathfrak{m}(A) \cong \mathbb{C}$ is called the *residue* morphism.

The number n is not an invariant of the algebra A; for example the field $\mathbb{C}$ admits the surjection $\mathcal{O}^n \to \mathbb{C}$ for arbitrary n. The minimal number n is called *embedding* dimension $\mathrm{emdim}\, A$ of the analytic algebra A. For any analytic algebra A and ideal I in A the *residue* algebra A/I is also an analytic algebra.

Any analytic algebra is Nötherian, since $\mathcal{O}^n$ is Nötherian (Ch.3).

4. The algebra $\mathcal{F}_a$ of formal power series (5.1); no assumptions on growth of coefficients. It is a local algebra.

Problem 1. To prove that the algebra $\mathcal{F}_a$ is Nötherian. Hint: to check Weierstrass Lemma for formal series.

Definition. Let A be a commutative algebra. The *(Krull)* dimension $\dim A$ is the largest length d of strictly ascending chains of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \ldots \subset \mathfrak{p}_d \subset A$$

Theorem 1 *The dimension of an analytic algebra $A = \mathcal{O}^n/I$ is equal to $\dim_{\mathbb{C}} Z$ where $Z = Z(I)$ is the set of common roots of elements $a \in I$.*

Example 1. $A = \mathcal{O}^3/(z_1 z_2, z_2 z_3, z_3 z_1)$, $\mathrm{emdim}\, A = 3$ and $\dim A = 1$; $\mathfrak{p}_0 = (z_1, z_2) \subset \mathfrak{p}_1 = \mathfrak{m} \subset A$.

5.2 Primary decomposition

Definition. Let A be a commutative ring, I is an ideal in A. The *radical* $\mathrm{rad}\, I$ of and ideal I is the set of elements $a \in A$ such that $a^k \in I$ for some natural k. The radical is always an ideal: if $a^k \in I, b^l \in I$, then $(a+b)^{k+l-1} \in I$.

An ideal $I \subset A$ is called *primary* if the inclusion $ab \in I$ implies either $a \in I$ or $b \in \operatorname{rad} I$. The radical of a primary ideal I is always a prime ideal. The inverse is not true. The prime ideal $\operatorname{rad} I$ is called *associated* to the primary ideal I.

Example 2. Consider the ideal $I \subset \mathcal{O}^4$ generated by the entries of the matrix

$$\left(\begin{array}{cc} z_1 & z_2 \\ z_3 & z_4 \end{array} \right)^2 = \left(\begin{array}{cc} z_1^2 + z_2 z_3 & z_2 (z_1 + z_4) \\ z_3 (z_1 + z_4) & z_2 z_3 + z_4^2 \end{array} \right)$$

The set $Z(I)$ of roots is given by $\{z_1 + z_4 = z_1 z_4 - z_2 z_3 = 0\}$. The ideal $J = (z_1 + z_4, z_1 z_4 - z_2 z_3)$ is prime and $I \neq J$ is primary associated to J.

Proposition 2 *If I_1, I_2 are primary ideals associated to a prime ideal $\mathfrak{p}$, then $I = I_1 \cap I_2$ is again a primary ideal associated to $\mathfrak{p}$.*

PROOF. If $ab \in I$, but $a \notin I$, then $a \notin I_1$ (or $a \notin I_2$) and $b \in \operatorname{rad} I_1 = \mathfrak{p}$.$\square$

Theorem 3 *In any commutative Nötherian algebra every irreducible ideal is primary.*

PROOF. Suppose that an ideal I is not primary. Then there exists elements $a, b \in A$ such that

$$ab \in I, \ a \notin I, \ b \notin \operatorname{rad} I \tag{5.2}$$

If $c \in A$ we denote by $(I : c)$ the ideal of elements e such that $ec \in I$. Consider the ascending chain of ideals

$$(I : b) \subset (I : b^2) \subset ... \subset (I : b^k) \subset ...$$

It stabilizes for some k, which means that $(I : b^k) = (I : b^{k+1})$. We assert that

$$I = (I, a) \cap (I, b^k) \tag{5.3}$$

Both ideals (I, a) and (I, b^k) are strictly larger than I and our assertion will imply that I is reducible. To check (5.3) we show that any element c of the right side belongs to I. We have $c = i + u b^k$ for some $i \in I$ and $u \in A$. On the other hand, $c \in (I, a)$; therefore $cb \in I$ because of (5.2). This implies

$$ib + ub^{k+1} = cb \in I, \ ub^{k+1} \in I$$

consequently $u \in (I : b^{k+1}) = (I : b^k)!$ and $ub^k \in I, c = i + ub^k \in I$, q.e.d. $\square$

Corollary 4 *An arbitrary ideal in a Nötherian algebra is equal to intersection of primary ideals.*

PROOF. By the method of Proposition of Sec.4.1 we prove that I can be written as intersection of irreducible ideals

$$I = I_1 \cap ... \cap I_s \tag{5.4}$$

Each ideal I_r is primary according to the previous Theorem.□

This statement can be made sharper. A representation (5.4) is called *irredundant* if there is no ideal I_r in (5.4) that contains the intersection of other ideals. The following result is due to Emanuil Lasker and Emmy Nöther:

Theorem 5 *[Main decomposition theorem] Every ideal I in a Nötherian algebra admits an irredundant representation (5.4) where all the ideals I_r are primary and the associated prime ideals are distinct.*

PROOF. Take a representation (5.4), remove some ideals if it is redundant, then take intersection of primary ideals with equal radicals.

Primary decomposition is not unique!

Example 3. Take the ideal $I = (zw, w^2) \subset \mathcal{O}^2$ (the line with the "thick" point at the origin). It has the primary decomposition for an arbitrary $t \in \mathbb{C}$:

$$I = \left(z + tw, w^2\right) \cap (w)$$

Note that the associated prime ideals $\mathfrak{m}$ and (w) are uniquely defined.

Theorem 6 *[Uniqueness 1] The set of prime ideals $\mathfrak{p}_1, ..., \mathfrak{p}_s$ associated to the primary components of a irredundant decomposition (5.4) is uniquely defined.*

PROOF. $Ass\,(I) \doteq \{\mathfrak{p}_j\}$. We show that a prime ideal $\mathfrak{p}$ is associated to I if and only if there exists an element $a \in A$ such that $(I : a) = \mathfrak{p}$. Check, first, that any of ideals $\mathfrak{p}_j$ possesses this property. Let $j = 1$; take an element $b \in I_2 \cap ... \cap I_s \backslash I_1$ and consider the largest natural k such that $b\mathfrak{p}^k \not\subset I$. Take an element $a \in b\mathfrak{p}^k \backslash I$; we have $a\mathfrak{p} \subset I$, i.e. $(I : a) \supset \mathfrak{p}$; on the other hand $(I : a) \subset \mathfrak{p}$ since I_1 is primary associated to $\mathfrak{p}$. Therefore $(I : a) = \mathfrak{p}$.

Inversely, suppose that $(I : a)$ is equal to a prime ideal $\mathfrak{p}$. Show that $\mathfrak{p}$ coincides with one of the ideals $\mathfrak{p}_1, ..., \mathfrak{p}_s$. If it is not the case, we can find

elements $b_j \in \mathfrak{p}_j \backslash \mathfrak{p}$, and natural numbers $k_j, j = 1, ..., s$ such that $\mathfrak{p}_j^{k_j} \subset I_j$. Then $b_1^{k_1} ... b_s^{k_s} a \in I$, but $b_1^{k_1} ... b_s^{k_s} \notin \mathfrak{p}$. This is a contradiction. $\square$

A component I_j of (5.4) is called *embedded*, if the corresponding ideal $\mathfrak{p}_j$ contains another ideal $\mathfrak{p}_k \neq \mathfrak{p}_j$.

Theorem 7 *[Uniqueness 2] All non embedded components of (5.4) are uniquely defined.*

These results are generalized for arbitrary A-module E of finite type instead of a and an arbitrary submodule F of E.

5.3 Complete intersection ideals

Definition. An ideal $I \subset \mathcal{O}^n$ is called *complete intersection* ideal (c.i.i.) if it possesses a system of generators $f_1, ..., f_s$ such that $s = \operatorname{codim} I \doteq n - \dim I$.

  Examples 4. Any principal ideal is c.i.i.

  5. The maximal ideal $\mathfrak{m}$ in $\mathcal{O}^n$ is c.i.i.

  6. The ideal $(z_1 z_2, z_2 z_3, z_3 z_1) \subset \mathcal{O}^3$ is not a c.i.i.

  Problem 2. If $(f_1, ..., f_s)$ is a complete intersection ideal, then $(f_1, ..., f_r)$ is so for any $r < s$.

  Problem 3. If $I = (f_1, ..., f_s)$ is an arbitrary ideal in $\mathcal{O}^n$, then $s \geq n - \dim I$.

Theorem 8 *[Lasker-Macaulay-Cohen] If I is c.i.i. of dimension d, all prime ideals associated to I are of the same dimension.*

  Example 7. The ideal $(f_1, f_2) \subset \mathcal{O}^3$ is c.i.i. if f_1 is a linear and f_2 is non-singular quadratic function. It is primary if the plane $f_1 = 0$ is tangent to the cone $f_2 = 0$. Otherwise, $(f_1, f_2) = I(Z_1) \cap I(Z_2)$, where $I(Z_{1,2})$ are prime ideals; $Z_{1,2}$ are lines.

5.4 Zero-dimensional ideals

Definition. An analytic algebra A is called *Artinian* algebra if $\dim_{\mathbb{C}} A < \infty$. If I is an ideal in an analytic algebra A; the residue algebra A/I is Artinian if and only if I is has finite index in A.

  The maximal ideal $\mathfrak{m}(A)$ is of index 1; any power $\mathfrak{m}^k(A)$ is also of finite index. Therefore any ideal I of Krull dimension 0 is of finite index, because it contains a power of the maximal ideal. The inverse is also true:

Proposition 9 *Any ideal I of finite index in an analytic algebra A is primary and is associated to the maximal ideal $\mathfrak{m}(A)$.*

In other terms any ideal of finite index is dimension zero and vice versa. A local is ring is a commutative ring A with unit 1 that possesses only one maximal ideal $\mathfrak{m}$. Any element of the form $a = 1 + m$, $m \in \mathfrak{m}$ is invertible since it does not belong to any maximal ideal.

Lemma 10 *[Nakayama's Lemma] Let B be a commutative local ring with the maximal ideal $\mathfrak{m}$ and F be a finitely generated B-module such that $\mathfrak{m}F = F$. Then $F = 0$.*

PROOF OF LEMMA. Let $f_1, ..., f_s$ be a set of generators of F; by the condition for any $i = 1, ..., s$ there exist elements $m_{ij} \in \mathfrak{m}$ such that

$$f_i = \sum m_{ij} f_j, i = 1, ..., s$$

This system of equation can be written as follows $(E - M) f = 0$ for the column $f = (f_1, ..., f_s)$ unit matrix E and the matrix $M = \{m_{ij}\}$. We have $\det (E - M) = 1 + m$ where $m \in \mathfrak{m}$, which implies that the matrix $E - M$ is invertible over B. This implies $f = 0$. $\square$

PROOF OF PROPOSITION. From $ab \in I$, $a \notin I$ follows that $b \in \mathfrak{m}(A)$. Consider the descending chain of ideals $\mathfrak{m}^j(B)$, $j = 0, 1, ...$ in the analytic algebra $B = A/I$. Suppose it stabilizes for $j = k$, i.e. $\mathfrak{m}^k(B) = \mathfrak{m}^{k+1}(B)$. By Nakayama's lemma we conclude that $\mathfrak{m}^k(B) = 0$ which implies the inclusion $\mathfrak{m}^k(A) \subset I$. This completes the proof.

Problem 4. Show that always $\mathfrak{m}^k(A) \subset I$ holds for $k = \dim_\mathbb{C} A/I$.

5.5   Classical theory of polynomial ideals

This theory shows the relation between ideals in the algebra $\mathbb{C}[z_1, ..., z_n]$ and algebraic varieties $Z \subset \mathbb{C}^n$.

Theorem 11 *If the set Z of common roots of elements of an ideal I is empty, then I contains the unit element.*

PROOF. Take a system of generators $p_1, ..., p_s$ of I. They have no common root. We wish to show that there exist polynomials $g_1, ..., g_s$ such that $g_1 p_1 +$

$... + g_s p_s = 1.$ We use induction in n. For $n = 0$ the statement is trivial. For $n > 0$ we choose a coordinate system in such a way that all the highest power of z_n in p_j has constant coefficient for all j. Take the resultants $R(p_i, p_j)$ with respect to z_n. They belong to the algebra $\mathbb{C}[z_1, ..., z_{n-1}]$. It is easy to check that they have no common root in $\mathbb{C}^{n-1}$. By inductive assumption the ideal generated by the resultants contains 1. On the other hand each resultant belongs to I (see property 5, Ch.4). Therefore $1 \in I$. $\square$

Theorem 12 *[Hilbert's Nullstellensatz] For an arbitrary polynomial ideal I there exists a number k such that $p^k \in I$ for an arbitrary polynomial p that vanishes in $Z(I)$.*

PROOF. We use arguments of A.Rabinowitsch. Let $p_1, ..., p_s$ be a set of generators of I. Consider the polynomial algebra $B = \mathbb{C}[z_1, ..., z_n, w]$, where w is a transcendent element. The polynomials $p_1, ..., p_s, 1 - wp$ belong to B and does not vanish simultaneously. By the previous Theorem there exist $g_1, ..., g_s, g \in B$ such that

$$g_1 p_1 + ... + g_s p_s + g(1 - wp) = 1$$

Substitute $w = p^{-1}$ and remove the resulting dominators multiplying by p^k for k equal the maximal of orders of $g_1, ..., g_n$. $\square$

More precise description of zero-dimensional ideals is given by

Theorem 13 *[M.Nöther, Macaulay, E.Lasker] Let I be an ideal in the algebra $A = \mathbb{C}[z_1, ..., z_n]$ such that $\dim Z(I) = 0$. Than a polynomial f belongs to I if and only if for each point $a \in Z(I)$ the local condition*

$$f \in I + \mathfrak{m}_a^k(A) \qquad (5.5)$$

is satisfied, where k is the minimal number such that $\mathfrak{m}_a^k(A) \subset I + \mathfrak{m}_a^{k+1}(A)$.

The condition (5.5) is called Max Nöther condition. It contains a finite number of scalar equations since the algebra $A_z = \mathcal{O}_a / \mathfrak{m}_a^k$ is an Artin algebra and the set $Z(I)$ is finite.

For the general case a description can be done by means of infinite number of conditions:

Theorem 14 *[Artin-Riess] Let I be an arbitrary ideal in the algebra $\mathcal{O}_a$. A germ $f \in \mathcal{O}_a$ belongs to I if $f \in I + \mathfrak{m}_a^k(A)$ for any k. For an ideal I in the polynomial algebra $\mathbb{C}[z_1, ..., z_n]$ the inclusion $f \in I$ holds if the above conditions is fulfilled for each point $a \in \mathbb{C}^n$.*

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This condition is consistent only if $z \in Z(I)$ and is obviously necessary. It can be formulated as follows: f belongs to the ideal I_z in the formal algebra $\mathcal{F}_z$ generated by elements of I. The last ideal is called *localization* of I at z.

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Chapter 6

Nöther operators, residue and bases

6.1 Differential operators in modules

Definition. Let $\mathcal{A}$ be a commutative algebra, $\mathcal{E}, \mathcal{F}$ be $\mathcal{A}$-modules, $l \geq 0$ is an integer; a *differential operator* $q : \mathcal{E} \to \mathcal{F}$ of order $\leq l$ is a linear operator such that for an arbitrary $a \in \mathcal{A}$

$$(\operatorname{ad} a)\, q \doteq aq - qa$$

is a differential of order $\leq l - 1$. A differential operator of order 0 is, by definition, an $\mathcal{A}$-morphism.

If q is $\mathcal{A}$-differential operator of order l and $z_1, ..., z_n \in \mathcal{A}$, then for an arbitrary $i \in \mathbb{N}^n$ the mapping $q^{(i)} : \mathcal{E} \to \mathcal{F}$

$$(\operatorname{ad} z)^i\, q \doteq (\operatorname{ad} z_1)^{i_1} ... (\operatorname{ad} z_n)^{(i_n)}\, q$$

is a d.o. of order $\leq l - |i|$. (Note that the operations $\operatorname{ad} a$ and $\operatorname{ad} b$ commute for any $a, b \in \mathcal{A}$.) The kernel of a d.o. need not to be a submodule of $\mathcal{E}$.

Example 1. A partial differential operator

$$q\left(z, D\right) = \sum_{|i| \leq m} q_i\left(z\right) D^i, \ D^i \doteq \left(\frac{\partial}{\partial z_1}\right)^{i_1} ... \left(\frac{\partial}{\partial z_n}\right)^{i_n} \tag{6.1}$$

in a domain $\mathbb{U} \subset \mathbb{C}^n$ with holomorphic coefficients $q_i\left(z\right)$ defines for any point $z \in \mathbb{U}$ a $\mathcal{O}_z$-differential operator $q_z : \mathcal{O}_z \to \mathcal{O}_z$ of order m. The function

$$q\left(z, \xi\right) \doteq \sum q_i\left(z\right) \xi^i, \ \xi^i \doteq \xi_1^{i_1} ... \xi_n^{i_n}$$

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is called *symbol* of the operator q. Denote by $q^{(i)}(z, D)$ d.o. with the symbol $q^{(i)}(z, \xi) = D_\xi^i q(z, \xi)$.

Problem 1. Show the inverse: any $\mathcal{O}_w$-differential operator in $\mathcal{O}_w$ can be written in the form (6.1) and $q^{(i)} = (\mathrm{ad}\, z)^i q$.

Proposition 1 *The generalized Leibniz formula holds for arbitray $a, b \in \mathcal{O}$*

$$q(z, D)(ab) = \sum \frac{D^i a}{i!} q^{(i)}(z, D) b$$

A proof is left to reader.

Definition. A $\mathcal{A}$-differential operator $q : \mathcal{E} \to \mathcal{F}$ is called operator of Nöther type, if for any element $a \in \mathcal{A}$ there exists a $\mathcal{A}$-morphism $b : \mathcal{F} \to \mathcal{F}$ such that $qa = bq$.

If q is of Nöther type then $\mathcal{S} = \mathrm{Ker}\, q$ is a submodule of $\mathcal{E}$: if $e \in \mathrm{Ker}\, q$, $a \in \mathcal{A}$, then $q(ae) = bq(e) = 0$. The operator q is called *Nöther* operator for $\mathcal{S}$.

Problem 2. Let $q_1, ..., q_l \in \mathbb{C}[\xi_1, ..., \xi_n]$ and

$$q : f \mapsto q_i(\partial_1, ..., \partial_n) f(0), \quad i = 1, ..., l; \ \partial_j = \partial/\partial z_j$$

is a d.o. $q : \mathcal{O}_0^n \to \mathbb{C}^l$. The operator q is of Nöther type if and only if the polynomial $\partial q_i / \partial \xi_j$ belongs to the linear span of $q_1, ..., q_l$ for any i and $j = 1, ..., n$.

Problem 3. Let $\mathcal{A}$ be an Artin $\mathbb{C}$-algebra. Show that any linear bijection

$$q : \mathcal{A} \to \mathbb{C}^l$$

is a $\mathcal{A}$-operator of Nöther type.

This fact is generalized as follows:

Theorem 2 *Let $\mathcal{I}$ be a primary ideal in $\mathcal{O}^n$ associated to a prime ideal $\mathbf{p}$. Then there exists an integer l and a Nöther differential operator $q : \mathcal{O}^n \to [\mathcal{O}^n/\mathbf{p}]^l$ for $\mathcal{I}$.*

Corollary 3 *Let $\mathcal{I}$ be an arbitrary ideal in $\mathcal{O}^n$, $\mathcal{I} = \mathcal{I}_1 \cap ... \cap \mathcal{I}_s$ be a primary decomposition and $\mathbf{p}_1, ..., \mathbf{p}_s$ be the associated prime ideals. Then we have $\cap \mathrm{Ker}\, q_r = \mathcal{I}$ where*

$$q_r : \mathcal{O}^n \to [\mathcal{O}^n/\mathbf{p}_r]^{l(r)}$$

is a Nöther operator for $\mathcal{I}_r$, $r = 1, ..., s$.

2

For a proof we take a primary decomposition of I and apply the above Theorem to each component. This theory is generalized for arbitrary modules of finite type.

Theorem 4 *Let $\mathcal{M}$ be an arbitrary $\mathcal{O}^n$-module of finite type, $\mathcal{N}$ is a submodule*

$$\mathcal{N} = \mathcal{N}_1 \cap ... \cap \mathcal{N}_s$$

be a primary decomposition and $\mathsf{p}_1, ..., \mathsf{p}_s$ are associated prime ideals. For each j there exists an integer l_j and an $\mathcal{O}^n$-differential operator $q_j : \mathcal{M} \to [\mathcal{O}^n/\mathsf{p}]^{l_j}$ of Nöther type such that $\cap \operatorname{Ker} q_j = \mathcal{N}$.

See [2] for a proof.

Problem 4. Let $F : [\mathcal{O}^n]^s \to [\mathcal{O}^n]^t$ be a morphism of $\mathcal{O}^n$-modules. Show that the submodule $\operatorname{Ker} F$ is primary associated to $\mathsf{p} = 0$ and F is a Nöther operator for this submodule.

6.2   Polynomial ideals revisited

A similar statement are true also for the some nonlocal algebras, in particular, for the polynomial algebra.

Theorem 5 *Let $\mathcal{I}$ be a primary ideal in $\mathcal{A} = \mathbb{C}[z_1, ..., z_n]$ and $\mathcal{P}$ the associated prime ideal. Then there exists an integer l and a $\mathcal{A}$-differential operator $q : \mathcal{A} \to [\mathcal{A}/\mathcal{P}]^l$ of Nöther type such that $\operatorname{Ker} q = \mathcal{I}$.*

This is a generalization of the M.Nöther-Macaulay-Lasker theorem as above.

Example 2. If $\mathsf{I} = (f^k)$ where f is a nonzero element of A, then I is primary associated to $\mathsf{p} = (f)$. Let the z_1-axis is suitable for f; then the differential operator

$$q : \mathsf{A} \to [\mathsf{A}/\mathsf{p}]^k, \ qa = \left[a, \partial_1 a, ..., \partial_1^{k-1} a\right] \ (\operatorname{mod} \mathsf{p})$$

is of Nöther type and $\operatorname{Ker} q = \mathsf{I}$.

Example 3. The ideal $\mathsf{I} = (z_2^2, z_3^2, z_2 + z_1 z_3) \subset \mathsf{A} = \mathbb{C}[z_1, z_2, z_3]$ is a primary ideal associated to $\mathsf{p} = (z_2, z_3)$. The operator

$$q : \mathsf{A} \to [\mathsf{A}/\mathsf{p}]^2, qa = [a, (\partial_3 - z_1 \partial_2) a] \ (\operatorname{mod} \mathsf{p})$$

3

is a Nöther operator for I. This operator can be defined in the analytic algebra $\mathcal{O}^3$. It is again an operator of Nöther type and his kernel coincides with the ideal $\mathcal{I} = I\mathcal{O}^3$.

Problem 5. Show that the operator

$$q : \mathbb{C}[z_1, ..., z_n] \to \mathbb{C}^{n+2}, \ qa = [a(0), a_1'(0), ..., a_n'(0), \Delta a(0)]$$

is of Nöther type. Find a finite system of generators of the ideal $\operatorname{Ker} q$.

Problem 6. Let I be the ideal in $\mathsf{A} = \mathbb{C}[z_1, ..., z_n]$ or in $\mathsf{A} = \mathcal{O}^n$ generated by the basic symmetric polynomials $s_1, ..., s_n$ of $z_1, ..., z_n$. Show

(i) $Z(I) = \{0\}$;

(ii) $\dim \mathsf{A}/I = n!$;

(iii) Show that the d.o.

$$q : \mathsf{A} \to \mathbb{C}^N, \ f \mapsto \left\{ J^{(\alpha)}(D_z) f(0), |\alpha| \leq n! - n \right\}$$

is a Nöther operator for I, where

$$J(\xi) = \det \frac{\partial(s_1(\xi), ..., s_n(\xi))}{\partial(\xi_1, ..., \xi_n)}, \ \xi = \xi_1, ..., \xi_n;$$

$$J^{(\alpha)}(\xi) = D_\xi^\alpha J(\xi)$$

6.3 Residue

Fix a coordinate system $z_1, ..., z_n$ in $\mathbb{C}^n$; let $\mathbb{U} \subset \mathbb{C}^n$ be an open set and $w \in \mathbb{U}$. Let $f_1, ..., f_n$ be holomorphic functions in $\mathbb{U}$ that vanishes at the only point w that define a zero dimensional ideal $\mathcal{I} \subset \mathcal{O}_w^n$, i.e. $Z = \{f_1 = ... = f_n = 0\} = \{w\}$. Take positive numbers $\varepsilon_1, ..., \varepsilon_n$ and consider the variety

$$\Gamma(\varepsilon) = \{|f_1(z)| = \varepsilon_1, ..., |f_n(z)| = \varepsilon_n, z \in \mathbb{U}\}$$

It is compact if the numbers $\varepsilon_1, ..., \varepsilon_n$ are sufficiently small. The set singular set $\Gamma(\varepsilon) \cap Z(J)$ is of real codimension ≥ 1 in $\Gamma(\varepsilon)$ where $Z(J)$ is the hypersurface of roots of the Jacobian

$$J = \det \frac{\partial(f_1, ..., f_n)}{\partial(z_1, ..., z_n)}$$

The set $\Gamma(\varepsilon) \setminus Z(J)$ is a smooth real manifold of dimension n. It is orientated by the form

$$\delta \doteq d(\arg f_1) \wedge ... \wedge d(\arg f_n)$$

Let a be an arbitrary holomorphic function in $\mathbb{U}$. The integral

$$\operatorname{Res} \omega = (2\pi i)^{-n} \int_{\Gamma(\varepsilon)} \frac{adz}{f_1 \ldots f_n}, \ dz = dz_1 \wedge \ldots \wedge dz_n$$

is called the (multiple) *residue* of the form $\omega = (f_1 \ldots f_n)^{-1} adz$.
Properties:
1. The integral does not depend on $\varepsilon_1, \ldots, \varepsilon_n$. The form ω is holomorphic and therefore closed in $\mathbb{U} \backslash Z (f_{1 \ldots} f_n)$.
2. The residue vanishes for $g \in \mathcal{I}_w = (f_1, \ldots, f_n) \subset \mathcal{O}_w$. Indeed, for $g = af_j$, $a \in \mathsf{O}_w$ we take $\varepsilon_j \to 0$ and check that the integral tends to zero. It follows that the residue is defined on the Artin algebra $\mathcal{A} \doteq \mathcal{O}_w / \mathcal{I}_w$

Proposition 6 *We have*

$$\operatorname{Res} \frac{Jadz}{f_1 \ldots f_n} = \mu\, a\,(w), \ \mu \doteq \dim \mathcal{A}$$

Theorem 7 *For an arbitrary non zero element $g \in \mathcal{A}$ there exists an element $h \in \mathcal{A}$ such that $B(g, h) = 1$.*

Corollary 8 *The bilinear form*

$$B(g, h) = \operatorname{Res} \frac{ghdz}{f_1 \ldots f_n}$$

defined on the Artin algebra $\mathcal{A}$ is non degenerated.

Corollary 9 *The linear functional $q : \mathcal{A} \to \mathbb{C}$ defined by*

$$q(a) = \operatorname{Res} (f_1 \ldots f_n)^{-1} adz$$

is a differential operator of order $< m$. The operator

$$Q : \mathcal{A} \to \mathbb{C}^N, \ Q(a) = \left\{ q^{(i)}(a), |i| < m \right\}$$

is a Nöther operator for $\mathcal{I}_w$.

Problem 7. Suppose that $g_i = a_i^j f_j, i = 1, \ldots, n$ where $a_i^j \in \mathcal{O}_w$ and the matrix $a \doteq \left\{ a_i^j \right\}$ is invertible. Show that for any $b \in \mathcal{O}_w$

$$\operatorname{Res} \frac{(\det a)\, bdz}{g_1 \ldots g_n} = \operatorname{Res} \frac{bdz}{f_1 \ldots f_n}$$

5

6.4 Linear bases in ideals

Let I be an ideal in the $\mathbb{C}$-algebra (or $\mathbb{R}$-algebra) F of formal power series of n independent $z_1, ..., z_n$. We wish to choose a dense linear free system in I. Take, first the monomials

$$z^i = z_1^{i_1}...z_n^{i_n}, \ i = (i_1, ..., i_n) \in \mathbb{N}^n$$

where $\mathbb{N}$ is the set of naturals (i.e. of nonnegative integers). The monomials are linearly free and the span is equal to the subalgebra of polynomials. It is dense in F in the sense that an arbitrary series a is equal to a polynomial up to an element of $\mathfrak{m}^k$ for arbitrary k. Introduce the *grad-lexicographic* order in $\mathbb{N}^n$: we say that i follows (after) j and write $i \succ j$ if either $|i| > |j|$ or

$$|i| = |j| \ \text{and} \ i_1 = j_1, i_2 = j_2, ..., i_{m-1} = j_{m-1}, i_m > j_m \ \text{for some} \ m, 1 \le m \le n$$

This makes a complete ordering in the set $\mathbb{N}^n$, i.e. for any two different elements we have either $i \succ j$ or $i \prec j$. Note the property: if $i, j, k \in \mathbb{N}^n$ are arbitrary, then $i \succ j$ is equivalent to $i + k \succ j + k$.

Now for an arbitrary $i \in \mathbb{N}^n$ the subspace $\mathsf{F}(i) \subset \mathsf{F}$ of series that contains only monomials z^j for $j \succeq i$, is an ideal in F. Consider the quotient

$$Q(i) \doteq \left[\mathsf{I} \cap \mathsf{F}(i) + \mathsf{F}\left(i^+\right)\right] / \mathsf{F}\left(i^+\right)$$

where i^+ is the next to i vector. The dimension of $Q(i)$ is equal to 0 or 1. Now we run from $i = 0$ in ascending order of vector $i \in \mathbb{N}^n$ and distinguish some of vectors. We skip if $\dim Q(i) = 0$, or of $i \in j + \mathbb{N}^n$ for some distinguished vector j. If it is not the case, we distinguish an element $f_i \in \mathsf{I} \cap \mathsf{F}(i)$ whose class in $Q(i)$ is not zero. As the result, we get a sequence of elements

$$f_{i(1)}, f_{i(2)}, ..., f_{i(m)}, ... \in \mathsf{I}$$

with the following properties:

Proposition 10 *The set* $V \doteq \{i(m)\} \subset \mathbb{N}^n$ *is finite.*

PROOF. The algebra F is Nötherian. $\square$
Denote for convenience $f_m = f_{i(m)}, m = 1, ..., s$.

Proposition 11 *The elements $f_1, ..., f_s$ generate the ideal I, moreover for any $i \in \mathbb{N}^n$, an arbitrary element $a \in I \cap F(i)$ can be written in the form*

$$a = a_1 f_1 + ... + a_s f_s$$

where for each $j = 1, ..., s$ we have $a_j f_j \in I \cap F(i_j)$ where $i_j \succeq i$.

Denote $B = V + \mathbb{N}^n$ and by F_B the space of formal power series whose coefficients c_b vanish for $b \in B$. Let $f_1, ..., f_s$ be generators of the ideal I. Consider the morphism of F-modules

$$\phi : F^s \to F : (g_1, ..., g_s) \mapsto g_1 f_1 + .. + g_s f_s$$

Theorem 12 *The spaces I and F_B are complementary, moreover there exist linear operators*

$$D : F \to F_B, \quad G : F \to F^s$$

such that
(i)

$$E = D + \phi G, \qquad (6.2)$$

where E means the identity operator in F,
(ii) D vanishes in I and G vanishes in F_B.
(iii) If $f_1, ..., f_s \in \mathcal{O}$, then $D : \mathcal{O} \to \mathcal{O}_B, \quad G : \mathcal{O} \to \mathcal{O}^s$, which imply the equation (6.2) in $\mathcal{O}$.

Examples 3. Let $a = 0$, $I = (f)$ where $f \in \mathfrak{m}^k$ and

$$f(z) = z_1^k + \phi_1(z') z_1^{k-1} + ... + \phi_k(z'); \quad z' = (z_2, ..., z_n)$$

By the above algorithm we get $V = \{(k, 0, ..., 0)\}$.
Example 4. Let $f_1 = z_1^{m_1}, ..., f_k = z_k^{m_k}$ and $I = (f_1, ..., f_k)$. Then V consists of vectors $i(1) = (m_1, 0, ..., 0), ..., i(k) = (0, ..., m_k, 0, ..., 0)$.
  Problem 8. Let I be a complete intersection ideal generated by homogeneous polynomials, such that $\dim A/I = 0$. Then the set $\Gamma = \mathbb{N}^n \setminus B$ is symmetric in the following sense: there exists an integer q such that $\#\Gamma_p = \#\Gamma_{q-p}$ for $p = 0, ..., q$ where Γ_p is the set of elements $i \in \Gamma, |i| = p$ and $\Gamma_k = 0$ for $k > q$.
  Problem 9. Is it true that for an arbitrary complete intersection ideal I in $\mathcal{O}^n$ of dimension d the set B is generated by $n - d$ vertices?

6.5 Bases and division in a module

Let $\mathcal{A}$ be a commutative Nötherian algebra with a unit element, $\mathcal{M}$ be a $\mathcal{A}$-module of finite type. Then there exists a surjection (epimorphism) of A-modules $\mu : \mathcal{A}^t \to \mathcal{M}$. The kernel $\operatorname{Ker}\mu$ is a submodule of $\mathcal{A}^t$; it is again a module of finite type and there exists a morphism $\phi : \mathcal{A}^s \to \mathcal{A}^t$ such that $\operatorname{Im}\phi = \operatorname{Ker}\mu$. This means that $\operatorname{Coker}\phi = \mathcal{A}^t / \operatorname{Im}\phi \cong \mathcal{M}$. If we write elements of $\mathcal{A}^t$ as columns, the morphism ϕ can be written by a $t \times s$-matrix whose entries belong to $\mathcal{A}$.

Now let $\mathcal{A} = \mathsf{F}$; construct a grad-lexicographic order in F^t. For this we choose the standard basis $e_1, ..., e_t$ of this F-module where $e_1, ..., e_t$ are rows (or columns) of the unit $t \times t$-matrix E. The monomials $z^i e_j, i \in \mathbb{N}^n$, $j = 1, ..., t$ form a dense free system in F^t. We write $z^i e_k \succ z^j e_l$ if $i \succ j$ or $i = j$ and $k > l$. For each pair (k, i) we denote by $\mathsf{F}(i, k)$ the subspace of F^t spanned by monomials that follow or equal to $z^i e_k$. Denote $\mathsf{G} = \operatorname{Im}\phi$ and take the vector space

$$Q(i, k) = \mathsf{G} \cap \mathsf{F}(i, k) / \mathsf{F}\left((i, k)^+\right)$$

where $(i, k)^+$ means the pair that follow immediately to (i, k). This is a space of dimension 1 or 0. We repeat the construction of the previous section and get a finite distinguised system $f_{b(1)}, f_{b(2)}, ...$ where $b(m) \in [\mathbb{N}^n]^t$ which possesses the properties to the above Propositions. This basis is called "standard" or "Gröbner" basis. Also the result of the previous section is generalized as follows. Denote by F_B the subspace of F^t of columns

$$a = \sum c_{(i,k)} z^i e_k$$

where $c_{(i,k)} \neq 0$ only if $(i, k) \in B$.

Theorem 13 *The spaces G and F_B are complementary, moreover there exist linear operators*

$$D : \mathsf{F}^t \to \mathsf{F}_B, \ G : \mathsf{F}^t \to \mathsf{F}^s$$

such that
(i)

$$E = D + \phi G, \tag{6.3}$$

where E means the identity operator in F^t,
(ii) D vanishes in I and G vanishes in F_B.

To get more details on (6.3) we consider the family of functionals $\delta_{i,k} : \mathsf{F}^t \to \mathbb{C}$; $\delta_{i,k}\left(z^i e_k\right) = 1$ and $\delta_{i,k}\left(z^j e_l\right) = 0$ otherwise.

Theorem 14 *[continuation] (iii) If $\phi \in \mathcal{O}^{txs}$, there exist number numbers κ, C such that*

$$\delta_{j,l} D\left(z^i e_k\right) = 0 \ for \ |j| < |i|$$
$$\delta_{j,l} G\left(z^i e_k\right) = 0 \ for \ |j| < |i| + \kappa$$

and

$$\left|\delta_{j,l} D\left(z^i e_k\right)\right| + \left|\delta_{j,l} G\left(z^i e_k\right)\right| \leq C^{|j|-|i|+\kappa}$$

It follows that $D : \mathcal{O} \to \mathcal{O}_B$, $G : \mathcal{O} \to \mathcal{O}^s$, which imply the equation (6.3) in $\mathcal{O}^s$.

This is a generalization of the Weierstrass division theorem. See [1] for a proof.

References

[1] V.P.Palamodov, Linear differential operators with constant coefficients, Ch.2,Sec.4, 1967 (Russian), 1970 (English)
[2] V.P.Palamodov, Differential operators on coherent analytic sheaves, Math.Sb. **77(119)** (1968), N3, 390-422

Chapter 7

Sheaves

7.1 Categories and functors

Definition. A category $\mathbf{C}$ is a class of objects and "morphisms" of objects with the following properties:

(1) for any objects X, Y all morphisms $f : X \to Y$ form a set denoted $\hom_{\mathbf{C}}(X, Y)$;

(2) for any object X an element $\mathrm{id}_X \in \hom_{\mathbf{C}}(X, X)$ is distinguished; it is called the identity morphism;

(3) for arbitrary objects X, Y, Z the set mapping is defined

$$\hom_{\mathbf{C}}(X, Y) \times \hom_{\mathbf{C}}(Y, Z) \to \hom_{\mathbf{C}}(X, Z)$$

For morphisms $g \in \hom_{\mathbf{C}}(X, Y), h \in \hom_{\mathbf{C}}(Y, Z)$ the image of the pair (g, h) is called the *composition*; it is denoted hg. The composition operation is associative.

(4) for any $h \in \hom_{\mathbf{C}}(X, Y)$ we have $\mathrm{id}_Y \, f = f \, \mathrm{id}_X = f$.

A morphism f is called *isomorphism* if there exists a morphism $g \in \hom_{\mathbf{C}}(Y, X)$ such that $gf = \mathrm{id}_X, \ fg = \mathrm{id}_Y$.

For an arbitrary category $\mathbf{C}$ the *dual* category $\mathbf{C}^*$ is defined in the following way: the objects are the same, but $\hom_{\mathbf{C}^*}(X, Y) \doteq \hom_{\mathbf{C}}(Y, X)$ (inversion of all arrows).

Examples 1. A class of sets and set mappings is a category.

2. A class of topological spaces and continuous mappings.

3. For a topological space $\mathbb{X}$ the set $Top(\mathbb{X})$ of all open subsets and inclusions $\mathbb{V} \subset \mathbb{U}$ is a category.

4. A class of vector spaces and linear mappings over a field K.

5. A class of modules and morphisms over an algebra A.

6. The category of analytic algebras and so on.

Definition. Let $\mathbf{C}_1, \mathbf{C}_2$ be categories; a covariant *functor* $\mathbf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ is a class mapping that transforms objects to objects and morphisms to morphisms preserving compositions and identity morphisms:

$$\mathbf{F}\,(\mathrm{id}_X) = \mathrm{id}_{\mathbf{F}(X)}; \mathbf{F}\,(gf) = \mathbf{F}\,(g)\,\mathbf{F}\,(f)$$

A contravariant *functor* $\mathbf{F} : \mathbf{C}_1 \to \mathbf{C}_2$ is, by definition, a covariant functor $\mathbf{C}_1 \to \mathbf{C}_2^*$.

Functor morphism $\phi : \mathbf{F} \Rightarrow \mathbf{G}$ is a mapping $\phi : \mathbf{C}_1 \to \mathrm{hom}\,(\mathbf{C}_2)$ such that for any object $X \in \mathbf{C}_1$ we have $\phi\,(X) \in \mathrm{hom}\,(\mathbf{F}\,(X), \mathbf{G}\,(X))$ and for any morphism $f : X \to Y$ of category $\mathbf{C}_1$ the relation holds

$$\phi\,(Y)\,\mathbf{F}\,(X) = \mathbf{F}\,(Y)\,\phi\,(X)$$

Definition. Given two objects X, Y of a category $\mathbf{C}$, the *direct product* $X \times Y$ in $\mathbf{C}$ is defined as an object $X \times Y$ together with some morphisms p_X to X and p_Y to Y that satisfies the following condition: for any object U and morphisms $f : U \to X, g : U \to Y$ there exists and is unique a morphism $U \to X \times Y$ such that the diagram commutes

$$
\begin{array}{ccc}
U & \to & Y \\
\downarrow & \searrow & \uparrow p_X \\
X & \overset{p_X}{\Leftarrow} & X \times Y
\end{array}
$$

The k-th direct power $X \times ... \times X$ is denoted X^k or $[X]^k$.

A more general notion is given below:

Definition. The *fibre product* of morphisms $f : X \to Z$ and $g : Y \to Z$ in a category $\mathbf{C}$ (shortly, the fibre product of X and Y over Z) is an object, denoted $f \times_Z g$ or $X \times_Z Y$, together with morphisms f' to Y and g' to X that commute with f and g :

$$
\begin{array}{ccc}
X \times_Z Y & \overset{f'}{\to} & Y \\
\downarrow g' & & \downarrow g \\
X & \overset{f}{\to} & Z
\end{array}
$$

that possesses the following property. For any object U and morphisms $U \to X$ and $U \to Y$ that commute with f and g there exists and is unique

2

a morphism $h : U \to X \times_Z Y$ such that the diagram commutes:

$$
\begin{array}{ccccccc}
U & - & - & & \to & Y \\
 & \searrow^{h} & & \nearrow^{f'} & & | \\
| & & X \times_Z Y & & & | \\
 & \swarrow^{g'} & & & & \\
\downarrow & & & & & \downarrow g \\
X & - & - & \xrightarrow{f} & & Z
\end{array}
$$

The fibre product over an object Z is the direct product in the relative category $\mathbf{C}_Z$ where the objects are morphisms $X \to Z$ and morphisms are corresponding triangle commutative diagrams. The fibre product need not to exists in an arbitrary category.

By inverting arrows we come to the notion of *fibre coproduct* (amalgamated sum).

Examples 6. The fibre product exists in the category of topological spaces.

7. The fibre product does not exists in the category $\mathbf{C}$ of smooth (or of complex analytic) manifolds. For example, take the mappings $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy$ and $g : \mathbb{R} \to \mathbb{R}, g(z) \equiv 0$. The fibre product $f \times_{\mathbb{R}} g$ must be the singular curve $\{xy = 0\} \times \mathbb{R}$ which does not present in $\mathbf{C}$.

8. We shall show later that the fibre product exists in the category of complex analytic spaces. This is one of the reasons to introduce this category.

7.2 Abelian categories

Definition. Let $\mathbf{C}$ be a category with the following property (A_1) each set $\hom(X, Y)$ has a structure of abelian (i.e. commutative) group such that the composition operation is bi-additive. We assume also there exists a *zero* object 0, i.e. such an object that the group $\hom(0, 0)$ is trivial. For a morphism $f : E \to F$ an object $\operatorname{Ker} f$ of $\mathbf{C}$ together with a morphism $k : \operatorname{Ker} f \to E$ is called the *kernel* of f if $fk = 0$ and for any morphism $g : K \to E$ in $\mathbf{C}$ such that $fg = 0$ there exists and is unique a morphism $g' : G \to \operatorname{Ker} f$ such that $g = kg'$:

$$
\begin{array}{ccccc}
 & & K & & \\
 & \nearrow^{g'} & g \downarrow & & \\
\operatorname{Ker} f & \xrightarrow{k} & E & \xrightarrow{f} & F
\end{array}
$$

3

The standard notation is $\ker f = k$. By inversion all arrows, i.e. by replacing $\mathbf{C}$ by the dual category $\mathbf{C}^*$ we get the notion of *cokernel*; the standard notation

$$\operatorname{cok} f : F \to \operatorname{Cok} f$$

Any morphism $h : F \to L$ such that $hf = 0$ can be lifted to the kernel, i.e. there exists and unique a morphism $h' : \operatorname{Cok} f \to L$ such that $h' \operatorname{cok} f = h$:

$$
\begin{array}{ccccc}
 & & L & & \\
 & & & {\scriptstyle h'} & \\
 & & \uparrow h & \nwarrow & \\
E & \xrightarrow{f} & F & \xrightarrow{\operatorname{cok} f} & \operatorname{Cok} f
\end{array}
$$

We assume further that the condition (A_2) is fulfilled: in the category $\mathbf{C}$ each morphism has kernel and cokernel. A morphism with zero kernel or cokernel is called injective, respectively, surjective. A bijection is a injective and surjective morphism. Another standard constructions are $\operatorname{Im} \phi = \operatorname{Ker} \operatorname{cok} \phi$ and $\operatorname{Coim} \phi = \operatorname{Cok} \ker \phi$. There exists a canonical morphism $\hat{\phi} : \operatorname{Coim} \phi \to \operatorname{Im} \phi$. It is a bijection. A category is called *Abelian* if also the condition (A_3) is fulfilled: $\hat{\phi}$ is always an isomorphism. Sometimes it is also assumed that the direct products and coproducts (direct sum) are always defined.

A sequence of morphisms

$$E \xrightarrow{f} F \xrightarrow{g} G \tag{7.1}$$

is *semiexact* if $gf = 0$. If the kernels and cokernels are defined in $\mathbf{Ab}$, then there exist morphisms

$$f' : E \to \operatorname{Ker} g, \ g' : \operatorname{Cok} f \to G$$

and the morphism

$$i : \operatorname{Cok} f' \to \operatorname{Ker} g' \tag{7.2}$$

All of them are uniquely defined. If i is an isomorphism; the object (7.2) is called the *homology* of (7.1). This is always true for arbitrary Abelian category.

A morphism f of $\mathbf{Ab}$ is called monomorphism (injective morphism) if $\operatorname{Ker} f$ exists and is equal to the trivial group (zero group). If $\operatorname{Cok} f$ exists and is equal to zero, f is called epimorphism (surjective).

7.3 Sheaves

Definition. Let $\mathbb{X}$ be a topological space and $\mathbf{C}$ be a category whose objects are sets; a contravariant functor $\mathbf{F} : Top\,(\mathbb{X}) \to \mathbf{C}$ is called *presheaf* on $\mathbb{X}$ with values in $\mathbf{C}$. For $\mathbb{V} \subset \mathbb{U} \in Top\,(\mathbb{X})$ the morphism $\mathbf{F}\,(\mathbb{U}) \to \mathbf{F}\,(\mathbb{V})$ is called *restriction* mapping. A *sheaf* with values in $\mathbf{C}$ is an arbitrary presheaf that satisfies the condition (sheaf axiom):

(*) if $\mathbb{V} = \cup_A \mathbb{V}_\alpha, \mathbb{V}_\alpha \in Top\,(\mathbb{X})$ and elements $f_\alpha \in \mathbf{F}\,(\mathbb{V}_\alpha)\,, \alpha \in A$ are such that for arbitrary $\alpha, \beta \in A$,

$$f_\alpha | \mathbb{V}_{\alpha\beta} = f_\beta | \mathbb{V}_{\alpha\beta} \text{ where } \mathbb{V}_{\alpha\beta} = \mathbb{V}_\alpha \cap \mathbb{V}_\beta,$$

then there exists and unique an element $f \in \mathbf{F}\,(\mathbb{V})$ such that $f|\mathbb{V}_\alpha = f_\alpha$ for each $\alpha \in A$.

An element of $\mathbf{F}\,(\mathbb{V})$ is called *section* of the sheaf over $\mathbb{V}$. The sheaf axiom means that sections can be glued together.

The *bundle* of a sheaf $\mathbf{F}$ is the mapping $\phi : F \to X$ where

$$\phi^{-1}\,(x) = F_x \doteq \lim_{x \in \mathbb{U}} \mathbf{F}\,(\mathbb{U})$$

The space F_x is called the *stalk* of $\mathbf{F}$ at x. For an element $f \in \mathbf{F}\,(\mathbb{U})$ its image $f_x \in F_x$ is called the *germ* of f in x. The topology in F is defined as follows: for each $f \in \mathbf{F}\,(\mathbb{U})$ the set $\{f_x, x \in \mathbb{U}\} \subset F$ is open. An arbitrary open set is a union of sets of this form.

Properties: 1. The mapping ϕ is locally a homeomorphism of topological spaces. This is a characteristic property of sheaves. Indeed, let $\phi : F \to \mathbb{X}$ be a continuous mapping that is locally a homeomorphism. For an open $\mathbb{U} \subset \mathbb{X}$ we consider the set $\mathbf{F}\,(\mathbb{U})$ of all continuous sections, i.e. of continuous mappings $s : \mathbb{U} \to F$ such that $\phi s = \mathrm{id}_{\mathbb{U}}$. The functor $\mathbb{U} \Rightarrow \mathbf{F}\,(\mathbb{U})$ is a presheaf which satisfies the sheaf axiom.

2. Any functor morphism $\mathbf{F} \Rightarrow \mathbf{G}$ generates a continuous mapping of bundles $\varphi : F \to G$ such that the diagram commutes

$$
\begin{array}{ccc}
F & \xrightarrow{\;\varphi\;} & G \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
$$

It is called *sheaf* morphism (mapping).

3. Given a sheaf $\pi : F \to \mathbb{Y}$ and a continuous mapping $f : \mathbb{X} \to \mathbb{Y}$ of topological spaces, the fibre product $\pi \times_{\mathbb{Y}} f$ is a topological space denoted

<center>5</center>

$f^*(F)$ together with a continuous mapping $f^*(\pi) : f^*(F) \to \mathbb{X}$. This mapping is locally a homeomorphism, consequently $f^*(F)$ is a sheaf. It is called the inverse image (or pull back) of the sheaf π.

Examples 1. Let $\mathbb{X}$ be a topological space; for any open $\mathbb{U} \subset \mathbb{X}$ we consider the space $C(\mathbb{U})$ of continuous functions $f : \mathbb{U} \to \mathbb{C}$. For $\mathbb{V} \subset \mathbb{U}$ the restriction mapping $C(\mathbb{U}) \to C(\mathbb{V}) : f \mapsto f|\mathbb{V}$ is defined. This is a contravariant functor from $Top(\mathbb{X})$ to the category of $\mathbb{C}$-vector spaces. This functor is a sheaf, denoted $\mathcal{C}(\mathbb{X})$.

2. Let M be a complex analytic manifold. The sheaf of (germs) holomorphic functions is defined on M; we use the notation $\mathcal{O}(M)$.

3. Let A be a commutative ring and $\mathrm{Spec}(A)$ be the set of all prime ideals of A. This set is endowed with the *Zariski* topology: for an element $a \in A$ the set $\mathbb{U}(a)$ of all prime ideals p that does not contain a is Zariski open. A set in $\mathrm{Spec}(A)$ is open if it is a union of sets $\mathbb{U}(a)$, $a \in A$. The topological space $\mathrm{Spec}(A)$ is supplied with the presheaf $\mathcal{A}$: by definition $\mathcal{A}(U(a)) = A_a$ where A_a is the *localization* of A with respect to a, i.e. the algebra of quotients $f/a^k, f \in A$, k is a natural number. If $b = ac$ for some $c \in A$, we have $\mathbb{U}(b) \subset \mathbb{U}(a)$; and a morphism $\mathcal{A}(\mathbb{U}(a)) \to \mathcal{A}(\mathbb{U}(b))$ is canonically defined. The presheaf $\mathcal{A}$ satisfies the sheaf axiom, i.e. is a sheaf. The space $\mathrm{Spec}(A)$ with the sheaf $\mathcal{A}$ is called the *affine scheme* of the ring A.

Chapter 8

Coherent analytic sheaves and analytic spaces

8.1 Analytic sheaves

Definition. Let $\mathbb{U}$ be an open set in $\mathbb{C}^n$, $\mathcal{O} = \mathcal{O}\left(\mathbb{U}\right)$ be the sheaf of holomorphic functions in $\mathbb{U}$. An *analytic* sheaf $\mathcal{M}$ in $\mathbb{U}$ (or $\mathcal{O}$-sheaf) is a sheaf of $\mathbb{C}$-vector spaces where a continuous action of $\mathcal{O}$ is defined, i.e. for any point $z \in \mathbb{U}$ the stalk $\mathcal{M}_z$ has a structure of module over the algebra $\mathcal{O}_z$ and the mapping $\mathcal{M} \times \mathcal{O} \to \mathcal{M}$ is continuous. In other words, for arbitrary open $\mathbb{V} \subset \mathbb{U}$ the space $\mathcal{M}\left(\mathbb{V}\right)$ has structure of module over $\mathcal{O}\left(\mathbb{V}\right)$ which is compatible with the restriction mappings, i.e. the diagram is commutative for arbitrary open $\mathbb{W} \subset \mathbb{V}$:

$$
\begin{array}{ccccc}
\mathcal{M}\left(\mathbb{V}\right) & \times & \mathcal{O}\left(\mathbb{V}\right) & \to & \mathcal{M}\left(\mathbb{V}\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}\left(\mathbb{W}\right) & \times & \mathcal{O}\left(\mathbb{W}\right) & \to & \mathcal{M}\left(\mathbb{W}\right)
\end{array}
$$

A morphism of analytic sheaves is any morphism of sheaves of vector spaces $\phi : \mathcal{M} \to \mathcal{N}$ that is compatible with the action of the sheaf $\mathcal{O}$. For a morphism ϕ the kernel $\mathrm{Ker}\,\phi$ and cokernel $\mathrm{Cok}\,\phi$ are defined as $\mathcal{O}$-sheaves. Direct sums and products of $\mathcal{O}$-sheaves are well defined.

A sequence of $\mathcal{O}$-morphisms

$$
\mathcal{L} \xrightarrow{\psi} \mathcal{M} \xrightarrow{\phi} \mathcal{N}
$$

is called *exact* if $\mathrm{Im}\,\psi = \mathrm{Ker}\,\phi$.

Definition. Let $\mathcal{M}$ be an analytic sheaf in $\mathbb{U}$; it is called $\mathcal{O}$-sheaf of finite type, if there exists a surjection $\pi : \mathcal{O}^s \to \mathcal{M}$ for some natural s. An exact sequence of $\mathcal{O}$-sheaves

$$\mathcal{O}^t \xrightarrow{\phi} \mathcal{O}^s \xrightarrow{\pi} \mathcal{M} \to 0$$

is called finite representation of the $\mathcal{O}$-sheaf $\mathcal{M}$ if s, t are some natural numbers. An analytic sheaf $\mathcal{M}$ is called *coherent* if it has locally a finite representation, i.e. if any point z has a nbd $\mathbb{V}$ such that $\mathcal{M}|\mathbb{V}$ is has a finite representation.

The *support* $\operatorname{supp} \mathcal{M}$ of the analytic sheaf is the set of points z such that $\mathcal{M}_z \neq 0$. The support of an arbitrary coherent sheaf in $\mathbb{U}$ is a closed analytic subset $Z \subset \mathbb{U}$.

Examples 1. For an arbitrary point $w \in \mathbb{C}^n$ the sheaf $\mathbb{C}_{(w)}$ is coherent whose stalk at the point w is equal to $\mathbb{C}$ and to zero otherwise.

2. For an arbitrary morphism of $\mathcal{O}$-sheaves

$$\phi : \mathcal{O}^t \to \mathcal{O}^s \tag{8.1}$$

the sheaf $\operatorname{Cok} \phi$ is coherent by definition. We shall show that $\operatorname{Ker} \phi$ is coherent too. This sheaf is called the *syzygy* of ϕ.

3. For a point $w \in \mathbb{C}^n$ consider the sheaf $\mathcal{O}_{(w)}$ whose stalk at $z = w$ is equal to $\mathcal{O}_w$ and to 0 otherwise. This is an analytic, but not coherent, sheaf.

Problem. To prove this assertion.

Theorem 1 *The sheaf* $\operatorname{Ker} \phi$ *is locally of finite type.*

Lemma 2 *Let p a pseudopolynomial of order m in a suitable domain $\mathbb{U} = \mathbb{C} \times \mathbb{V} \subset \mathbb{C}^n$ that is distinguished in a point $w = (w_1, w')$. Then Weierstrass' Lemma (ii) holds also locally in $\mathbb{U}$, i.e. for the sheaves $\mathcal{O} = \mathcal{O}^n|\mathbb{U}$ and $\mathcal{O}' = \mathcal{O}^{n-1}|\mathbb{V}$.*

PROOF OF LEMMA. We show that for any $z \in \mathbb{U}$ and a germ $g \in \mathcal{O}_z$ the equation holds $g = ph + r$, where $h \in \mathcal{O}_z$ and r is a pp. of order $< m$ in z_1. This is obviously true if $p(z) \neq 0$ with $r = 0$. In the case $p(z) = 0$, let k be the first number such that $(\partial/\partial z_1)^k p(z) \neq 0$; we have always $k \leq m$. We apply Weierstrass Lemma (i) to p at z which yields $p = bq$, where q is a distinguished pp. of order k in z and $b \in \mathcal{O}_z$, $b(z) \neq 0$. Then we apply Lemma (ii) and get $g = qh' + r'$ where $h' \in \mathcal{O}_z$ and r' is a pp. of order $< k$. This gives $g = ph + r'$ where $h = b^{-1}h'$ and our Lemma follows. $\square$

PROOF OF THEOREM. First we prove that the syzygy sheaf $\operatorname{Ker}\phi$ is locally of finite type for any morphism of the form (8.1). We use induction in n and in s. The case $n = 0$ is trivial. Assume that $n > 0$. Consider the case $s = 1$. Write $\phi(a) = f_1 a_1 + ... + f_t a_t$ and take an arbitrary point $w \in \mathbb{U}$. Suppose that the germ of function f_1 at w is not zero. Choose a suitable coordinate system (z_1, z') in a nbd of w and write $f_1 = gp$ by Weierstrass' Lemma, where $g(w) \neq 0$ and p is a distinguished pseudopolynomial of order m in z_1. By the same Lemma we can write $g^{-1} f_j = g_j p + h_j$, where $h_j, j = 2, ..., t$ is pp. of order $< m$. The equation $\phi(a) = 0$ is equivalent to

$$p\tilde{a}_1 + h_2 a_2 + ... + h_t a_t = 0$$

in a nbd of w where $\tilde{a}_1 = a_1 + g_2 a_2 + ... + g_t a_t$. Apply Weierstrass Lemma again and get $a_j = c_j p + b_j$ where b_j is a pp. of order $< m$. Rewrite the syzygy relation in the form

$$pb_1 = h \doteq -h_2 b_2 - .. - h_t b_t, \ b_1 = \tilde{a}_1 + c_2 h_2 + ... + c_t h_t \qquad (8.2)$$

The right side is a pp. of order $\leq 2m - 2$. The function b_1 is a pp. of order $\leq m - 2$. Indeed, we can blow up the circle ∂D in the formula

$$b_1(z_1, z') = \frac{1}{2\pi i} \int_{\partial D} \frac{h(\lambda, z') \, d\lambda}{p(\lambda, z')(\lambda - z_1)}$$

Therefore it is easy to see that $b_1 = O\left(|z_1|^{m-2}\right)$ for large $|z_1|$. Let B_j be the m-row of coefficients of the pp. $b_j, j = 1, 2, ..., t$. Joining these rows, we get a row B of length mt of elements of the sheaf $\mathcal{O}' \doteq \mathcal{O}^{n-1}|\mathbb{V}$. Equation (8.2) means that B belongs to the kernel of a $\mathcal{O}'$-morphism

$$P : [\mathcal{O}']^{mt} \to [\mathcal{O}']^{2m-1}$$

in a nbd of $w' = z'(w)$. By the inductive assumption the sheaf $\operatorname{Ker} P$ is finitely generated in a nbd $\mathbb{W}'$ of this point, which means that there exist sections $B^{(1)}, .., B^{(r)}$ of this sheaf in $\mathbb{W}$ that generate the syzygy. The corresponding rows of pp.-s

$$\left(b_1^{(1)}, ..., b_t^{(1)}\right), ..., \left(b_1^{(r)}, ..., b_t^{(r)}\right)$$

generate the solutions of the equation (8.2) over the sheaf $\mathcal{O}'$ and moreover the sheaf $\mathcal{O}$. Returning back, we see that

$$(\tilde{a}_1, a_2, ..., a_t) = c_2 (-g_2, p, 0..., 0) + ... + c_t (-g_t, 0, ..., 0, p) + (b_1, ..., b_t)$$

3

which means that the left side belongs to $\mathcal{O}$-span of $t-1+r$ sections of the sheaf $[\mathcal{O}]^t$. The same true for the kernel of (8.1) where $s = 1$.

Now we suppose that the kernel any mapping (8.1) with the rank $s-1$ instead of s is of finite type. We show this for ϕ. Consider the truncated mapping

$$\phi' : \mathcal{O}^t \to \mathcal{O}^{s-1}$$

where the first row of the matrix ϕ is omitted. By induction $\operatorname{Ker}\phi'$ is locally of finite type, i.e. there exists a nbd $\mathbb{W}$ of w and a $\mathcal{O}$-morphism $\psi : \mathcal{O}^r \to \mathcal{O}^t$ in $\mathbb{W}$ such that $\operatorname{Im}\psi = \operatorname{Ker}\phi'|\mathbb{W}$. Take the first row ϕ_1 of ϕ and consider the mapping

$$\phi_1\psi : \mathcal{O}^r \to \mathcal{O}$$

The sheaf $\operatorname{Ker}\phi_1\psi$ is generated by some sections $\beta_1, ..., \beta_v$ of $\mathcal{O}^r$ in a nbd $\mathbb{W}'$ of w. The the sections $\psi(\beta_1), ..., \psi(\beta_v)$ generate $\operatorname{Ker}\phi$. $\square$

Theorem 3 *For an arbitrary morphism of coherent analytic sheaves in $\mathbb{U} \subset \mathbb{C}^n$*

$$\phi : \mathcal{M} \to \mathcal{N}$$

the sheaves $\operatorname{Cok}\phi$ and $\operatorname{Ker}\phi$ are coherent too.

Proof. The first assertion follows directly from the definition. Take a point $z \in \mathbb{U}$, some representations of the sheaves in a nbd $\mathbb{V}$ of z and construct a commutative diagram

$$
\begin{array}{ccccccc}
\operatorname{Ker}(f \times r) & \to & \mathcal{O}^w & \overset{\sigma}{\to} & \operatorname{Ker}\phi & \to 0 & \\
\downarrow & & r\downarrow & & \downarrow & & \\
\mathcal{O}^t & \overset{f}{\to} & \mathcal{O}^s & \overset{\pi}{\to} & \mathcal{M} & \to & 0 \\
\beta\downarrow & & \alpha\downarrow & & \phi\downarrow & & \\
\mathcal{O}^v & \overset{g}{\to} & \mathcal{O}^u & \overset{\rho}{\to} & \mathcal{N} & \to & 0 \\
& & & \searrow & \downarrow & & \\
& & & & \operatorname{Cok}\phi & &
\end{array}
$$

To construct the mapping α, we define, first, the values of α on the free generators $e_1, ..., e_s$ of the sheaf $\mathcal{O}^s$, keeping the relation $\rho\alpha(e_j) = \phi\pi(e_j)$; then we extend α on the $\mathcal{O}$-span by the obvious rule

$$\alpha\left(\sum a_j e_j\right) = \sum a_j \alpha(e_j), \, a_j \in \mathcal{O}$$

4

We construct β in a similar way. It is easy to see that sequence

$$\mathcal{O}^v \times \mathcal{O}^s \overset{g \times \alpha}{\to} \mathcal{O}^u \to \operatorname{Cok}\phi \to 0 \qquad (8.3)$$

is exact. It follows that $\operatorname{Cok}\phi$ is coherent.

By the previous Theorem the sheaf $\operatorname{Ker} g \oplus \alpha$ in (8.3) is finitely generated in a nbd $\mathbb{W}$ of z, i.e. there exists a $\mathcal{O}$-morphism $R = (r, r') : \mathcal{O}^w \to \mathcal{O}^v \times \mathcal{O}^s$ such that $\operatorname{Im} R = \operatorname{Ker} g \times \alpha$. We have $\phi\pi r' = \rho\alpha r' = -\rho g r = 0$, consequently there exists a $\mathcal{O}$-morphism σ which makes the diagram commute. It is easy to check that σ is surjective. Now we only need to check that $\operatorname{Ker}\sigma$ is of finite type. For this we consider the mapping

$$f \times r : \mathcal{O}^w \times \mathcal{O}^t \to \mathcal{O}^s$$

The kernel is locally of finite type by the previous Theorem. On the other hand the sequence

$$\operatorname{Ker}(f \times r) \to \mathcal{O}^w \overset{\sigma}{\to} \operatorname{Ker}\phi \to 0$$

is exact. This completes the proof. $\square$

Problem 1. Let $f : \mathcal{M} \to \mathcal{N}$ be a morphism of coherent sheaves such that $\operatorname{Ker} f = 0, \operatorname{Cok} f = 0$. Show that f is a isomorphism.

Corollary 4 *The category of coherent analytic sheaves on an arbitrary complex analytic manifold is Abelian.*

Problem 2. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a morphism of coherent sheave in an open set $\mathbb{U}$ such that the morphism of $\mathcal{O}_z$-modules $\phi_z : \mathcal{M}_z \to \mathcal{N}_z$ is injective (bijective) for a point $z \in \mathbb{U}$. Show that there exists a nbd $\mathbb{V}$ of z such that $\phi|\mathbb{V}$ is injective (bijective).

8.2 Coherent sheaves of ideals

Theorem 5 [H.CARTAN] *For an arbitrary closed analytic set $Z \subset \mathbb{U} \subset \mathbf{C}^n$ the sheaf $\mathcal{I}(Z)$ if germs of holomorphic functions vanishing in Z is coherent.*

Theorem 6 *Let I be an ideal in the algebra $A = \mathcal{O}_w$, $p_1 = I(Z_1), ..., p_r = I(Z_r)$ be associated prime ideals and $q_1, ..., q_r$ be corresponding Nöther operators. Then the sheaf $\mathcal{I}$ of germs of holomorphic functions a satisfying the differential equations*

$$q_1 a | Z_1 = 0, ..., q_r a | Z_r = 0$$

is coherent analytic sheaf in a nbd of the point w. The stalk of $\mathcal{I}$ in w coincides with I.

Cartan's Theorem is the particular case, if we take irreducible components of Z for Z_j and the canonical morphisms $\mathsf{A} \to \mathsf{A}/\mathsf{p}_j$ for the Nöther operators. q, $j = 1, ..., r$.

8.3   Category of analytic algebras

Consider the category $\mathbf{A}$ of analytic algebras; let A, B be analytic algebras a morphism in the category $\mathbf{A}$ is a morphism of $\mathbb{C}$-algebras $\beta : B \to A$ i.e. a morphism of rings such that the diagram commutes

$$
\begin{array}{ccc}
B & \overset{\beta}{-\!-\!\longrightarrow} & A \\
& \underset{\pi}{\searrow} \quad \underset{\pi'}{\swarrow} & \\
& \mathbb{C} &
\end{array}
$$

where π, π' are the residue morphisms. It follows that $\beta\left(m\left(B\right)\right) \subset m(A)$.

Let $A \in \mathbf{A}$; we have $A = O_a^m / I$ where I is an ideal in the algebra O_a^n of germs of analytic functions in $(\mathbb{C}^m, a)$. Let $B = O_b^n / J$ be another analytic algebra. Consider a holomorphic mapping

$$
w = f(z) : w_1 = f_1(z_1, ... z_m), ..., w_n = f_n(z_1, ..., z_m) : (\mathbb{C}^m, a) \to \mathbb{C}^n
$$

defined in a nbd of a such that $f(a) = b$. It generates a morphism of analytic algebras in the usual way:

$$
f^* : O_b^n \to O_a^m; \ g(w) \mapsto g(f(z))
$$

If $f^*(J) \subset I$ this mapping defines also a morphism of $\mathbb{C}$-algebras $\beta : B \to A$.

Problem 3. Show that an arbitrary morphism in the category $\mathbf{A}$ can be obtained by means of the above construction.

8.4   Complex analytic spaces

Definition. A *model* complex analytic space is a closed analytic set $Z \subset \mathbb{U} \subset \mathbb{C}^n$ together with a coherent sheaf $\mathcal{A}$ of analytic algebras in $\mathbb{U}$ such that $\operatorname{supp} \mathcal{A} = Z$.

An arbitrary (complex) analytic space is a topological space X endowed with a sheaf $\mathcal{O}_X$ of $\mathbb{C}$-algebras such that the pair $(X, \mathcal{O}_X)$ is locally isomorphic to a model complex analytic space, i.e. for an arbitrary $x \in X$ there exists a nbd Y and an isomorphism of sheaves of $\mathbb{C}$-algebras

$$(Y, \mathcal{O}_X|Y) \cong (Z, \mathcal{A})$$

The space X is called the *support* of the analytic space, the sheaf $\mathcal{O}_X$ is called the *structure sheaf* of the space.

Examples 1. Any complex analytic manifold M with the sheaf $\mathcal{O}_M$ of germs of holomorphic functions is a complex analytic space.

2. The point $0 \in \mathbb{C}$ endowed by the algebra $\mathbb{C}$ is called simple point. The same point supplied with the algebra $\mathbb{D} = \mathbb{C}[z]/(z^2)$ is again a complex analytic space; it is called *double* point.

3. For an arbitrary analytic algebra $\mathbb{A}$ of embedding dimension n there exists a nbd $\mathbb{U}$ of a point, say $0 \in \mathbb{C}^n$, analytic subset $Z \subset \mathbb{U}$ and coherent sheaf $\mathcal{A}|\mathbb{U}$ of analytic algebras such that $\mathcal{A}_0 = \mathbb{A}$.

Definition. Let $(X, \mathcal{O}_X)$ be a complex analytic space, $\mathcal{M}$ be a $\mathcal{O}_X$-sheaf in X. It is called coherent sheaf if the sheaf $\mathcal{M}|Z$ is a coherent $\mathcal{O}_Z$-sheaf for any local model complex analytic space $(\mathbb{U}, Z, \mathcal{O}_Z)$. Let $\mathcal{I}$ be a coherent subsheaf of $\mathcal{O}_X$, i.e. a coherent sheaf of ideals. The quotient sheaf $\mathcal{A} = \mathcal{O}_X/\mathcal{I}$ is a coherent sheaf of analytic algebras. Let $S \subset X$ be the support of the sheaf $\mathcal{A}$. The pair $(S, \mathcal{A})$ is again a complex analytic space. It is called closed *subspace* of $(X, \mathcal{O}_X)$.

If $Y \subset X$ is an open subset, then the pair $(Y, \mathcal{O}_X|Y)$ is an analytic space. It is called open subspace of $(X, \mathcal{O}_X)$.

Definition. A morphism of complex analytic spaces $F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (f, ϕ), where $f : X \to Y$ is a continuous mapping of topological spaces and

$$\phi : f^*(\mathcal{O}_Y) \to \mathcal{O}_X$$

is morphism of sheaves of $\mathbb{C}$-algebras. The last condition means that for any point $x \in X$ the morphism $\phi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ belongs to the category **A**.

In particular, if $(S, \mathcal{O}_S)$ is a subspace of $(X, \mathcal{O}_X)$, the *embedding* mapping $i : (S, \mathcal{O}_S) \to (X, \mathcal{O}_X)$ is well defined: $i^*(\mathcal{O}_{X,x}) \to \mathcal{O}_{S,x}$ is the canonical morphism to the quoteint algebra.

Example 4. Let $(X, \mathcal{O})$ be an arbitrary analytic space; a morphism $(f, \phi) : (0, \mathbb{D}) \to (X, \mathcal{O})$ is defined by the point $x = f(0)$ and by a morphism $\phi : \mathcal{O}_{X,x} \to \mathbb{D}$. Take the functionional $t : \mathbb{D} \to \mathbb{C}$ defined by $t(1) = 0, t(z) = 1$

(the standard tangent vector). The composition $\tau = t\phi : \mathcal{O}_{X,x} \to \mathbb{C}$ is a functional satisfying the Leibniz formula

$$\tau(ab) = \tau(a)\, b(x) + a(x)\, \tau(b)$$

i.e. τ is a tangent vector in x. Vice versa, arbitrary tangent vector in $(X, \mathcal{O})$ is defined by a morphism of the dual point to this space.

Theorem 7 *For an arbitrary analytic algebra $\mathbb{A}$ there exists a model analytic space $(X, \mathcal{O})$ with a distinguished point $z \in X$ and a isomorphism $\mathcal{O}_z \cong \mathbb{A}$ of $\mathbb{C}$-algebras.*
If there are two analytic spaces $(X, z, \mathcal{O})$ $(X', z', \mathcal{O}')$, then there exist neighbourhoods Y of z and Y' of z' and an isomorphism $(f, \phi) : (Y, z, \mathcal{O}|Y) \to (Y', z', \mathcal{O}'|Y')$ such that $f(z) = z'$ and the diagram commutes

$$
\begin{array}{ccccc}
\mathcal{O}'_{z'} & \overset{\phi_z}{-\,-\,-} & \to & \mathcal{O}_z \\
& \searrow & & \swarrow & \\
& & \mathbb{A} & &
\end{array}
$$

The relation described in this Proposition is an equivalence relation in the class of (complex) analytic spaces with distinguished points $(X, z, \mathcal{O})$. A class of equivalent analytic spaces is called *germ* of analytic spaces. Any element of a class is called representative of this germ. The class of germs is a category denoted **G**. Morphisms of germs are defined in a natural way. For a germ represented by $(X, z, \mathcal{O})$ the analytic algebra $\mathbb{A} = \mathcal{O}_z$ is well defined. This makes a contravariant functor $\mathbf{Al} : \mathbf{G} \Rightarrow \mathbf{A}$. We denote by $\mathbb{G}(\mathbb{A})$ the class of equivalent analytic spaces constructed from an analytic algebra $\mathbb{A}$. Thus we have a contravariant functor $\mathbf{Ge} : \mathbf{A} \Rightarrow \mathbf{G}; \mathbb{A} \mapsto \mathbb{G}(\mathbb{A})$. The functors **Al** and **Ge** are inverse one to another. This makes the category **A** and **G** dual one to another. Therefore the algebraic and geometric languages are equivalent in this theory.

Definition. A morphism $F = (f, \phi)$ of complex analytic spaces is called *finite* if f is proper and the sheaf $\mathcal{O}_X$ is locally of finite type over the sheaf $f^*(\mathcal{O}_Y)$.

Example 5. Let $f : \mathbb{U} \to \mathbb{C}^n$ be a holomorphic mapping defined in a nbd $\mathbb{U}$ of the origin. If $f(0) = 0$ and $f^{-1}(0) = \{0\}$ (i.e. the ideal $(f_1, ..., f_n)$ is c.i.i.) then the mapping $(f, \phi) : (\mathbb{U}, \mathcal{O}) \to (\mathbb{C}^n, \mathcal{O})$ is finite.

8.5 Fibre products

We shall prove

Theorem 8 *The fibre product exists in the category of complex analytic spaces.*

First note the fibre product exists in the category **Top** of topological spaces; the direct product in **Top** is denoted $X \times Y$. For arbitrary continuous mappings $f : X \to Z$ and $g : Y \to Z$ we define subset

$$X \times_Z Y = \{(x, y) \,; f(x) = g(y)\} \subset X \times Y$$

and endow it with the topology induced from the direct product.

We show that the fibre product is also defined in the category **G** of germs of analytic spaces. This category is dual to the category **A** of analytic algebras, where the corresponding construction is the fibre coproduct.

Theorem 9 *The fibre coproduct is well defined in the category* **A**.

PROOF. Given morphisms β and γ in **A**, we need to define the commutative diagram

$$
\begin{array}{ccc}
B \otimes_A \Gamma & \stackrel{\beta'}{\leftarrow} & \Gamma \\
\uparrow \gamma' & & \uparrow \gamma \\
B & \stackrel{\beta}{\leftarrow} & A
\end{array}
$$

which possesses the property of fibre coproduct, see Chapter 7. According to the previous section the morphisms β, γ can be constructed from some holomorphic mappings $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, $g : (\mathbb{C}^p, 0) \to (\mathbb{C}^m, 0)$ such that

$$f^*(I) \subset J, g^*(I) \subset K, \ A = O^m/I, B = O^n/J, \Gamma = O^p/K$$

Let $z_1, ..., z_m$ be the coordinate functions in $\mathbb{C}^m$; the functions $f^*(z_j) = z_j(f(w))$ and $g^*(z_j) = z_j(g(v)), j = 1, ..., m$ are holomorphic in $(\mathbb{C}^n, 0)$ and $(\mathbb{C}^p, 0)$, respectively. Take the algebra O^{n+p} which corresponds to $\mathbb{C}^n \times \mathbb{C}^p$ and consider the ideals $(I), (J)$ generated by elements of I and of J. Define

$$B \otimes_A \Gamma \doteq O^{n+p}/(I) + (J) + (f^*(z_1) - g^*(z_1), ..., f^*(z_m) - g^*(z_m))$$

This is an analytic algebra. The morphisms β' and γ' are defined in the obvious way. We skip checking the universal property of fibre coproduct. $\square$

Remark. The algebra $B \otimes_A \Gamma$ is called the tensor product of analytic algebras. If one of the algebras B, Γ is Artinian, this tensor product coincides with the algebraic one.

PROOF OF THEOREM. The fibre product of germs of analytic spaces is the object dual to the coproduct of the corresponding analytic algebras. This, in fact, gives a proof of Theorem for model analytic spaces. Indeed, let

$$(f, \phi) : (X, \mathcal{O}_X) \to (Z, \mathcal{O}_Z), \, (g, \psi) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$$

be morphisms of model spaces where $Z \subset \mathbb{W}$ and $\mathbb{W}$ is an open set in $\mathbb{C}^m$. Let $z_1, ..., z_m$ be the coordinates in $\mathbb{C}^m$. They are sections of the sheaf $\mathcal{O}_Z = \mathcal{O}^m / \mathcal{K}$ over Z. By shrinking X, Y and Z, we can suppose that the sections $\phi(z_1), ..., \phi(z_m)$ are defined in an open nbd $\mathbb{U} \subset \mathbb{C}^n$ of X. We assume also that the sections $\psi(z_1), ..., \psi(z_m)$ are defined in an open nbd $\mathbb{V} \subset \mathbb{C}^n$ of Y. We have $\mathcal{O}_X = \mathcal{O}^n / \mathcal{I}$, $\mathcal{O}_Y = \mathcal{O}^p / \mathcal{J}$ and take the sheaf

$$\mathcal{O}_P = \mathcal{O}^{n+p} / (\mathcal{I}) + (\mathcal{J}) + \mathcal{G}$$

where $(\mathcal{I})$ the sheaf of ideals generated by the sheaf $\mathcal{I}$ in $\mathcal{O}^{n+p}$; $(\mathcal{J})$ has the similar meaning; the sheaf $\mathcal{G}$ is generated by the functions

$$\psi(z_1) - \phi(z_1), ..., \psi(z_m) - \phi(z_m)$$

The sheaf $\mathcal{O}_P$ is a coherent sheaf of analytic algebras supported by the analytic subset $P = X \times_Z Y \subset \mathbb{U} \times \mathbb{V}$, whence $(P, \mathcal{O}_P)$ is a model complex space. The morphisms of this space to $(X, \mathcal{O}_X)$ and to $(Y, \mathcal{O}_Y)$ are defined in an obvious way. The local constructions glue together to a global construction that gives the fibre product of arbitrary complex analytic spaces. $\square$

Remark. The support of the fibre product of analytic spaces $(X, \mathcal{O}_X), (Y, \mathcal{O}_Y)$ is the fibre product $X \times_Z Y$ of the supports.

The fibre product $f \times_Z g$ is sometimes called the pull back of f by the action of g.

Definition. Let $f : X \to Y$ be a morphism of analytic spaces, $(y, \mathbb{C}) \in Y$ be a point. The *fibre* of f over x in the fibre product $f \times_X i(y)$, where $i(y) : y \to Y$ is the embedding.

Example 6. Let $f : \mathbb{U} \to \mathbb{C}$ be a holomorphic function in a connected open set $\mathbb{U} \subset \mathbb{C}$. Consider f as a morphism of analytic spaces. For an arbitrary point $w \in \mathbb{C}$ the fibre $f \times \{w\}$ is the subspace of $\mathbb{U}$ supported by the set $f^{-1}(w)$. The structure sheaf equals $\mathcal{O}' = \mathcal{O}_{\mathbb{U}} / \mathcal{G}$ where $\mathcal{G}$ is the sheaf of

ideals generated by the function $w - f(z)$. If $f \not\equiv w$ in $\mathbb{U}$, we have for any $a \in f^{-1}(w)$

$$\mathcal{O}'_a \cong \mathbb{C}[z] / ((z-a)^m)$$

where m is the *multiplicity* of the root a of the function $w - f(z)$, i.e.

$$f(a) - w = f'(a) = \ldots = f^{(m-1)}(a) = 0, \ f^{(m)}(a) \neq 0.$$

Note that $\dim_{\mathbb{C}} \mathcal{O}'_a = m$.

Chapter 9

Elements of homological algebra

9.1 Complexes and homology

Definition. Let $\mathbf{C}$ be a category with the properties (A_1) and (A_2). A complex in $\mathbf{C}$ is a semi-exact sequence of morphisms of $\mathbf{C}$:

$$\ldots \overset{d_{k-1}}{\to} E^k \overset{d_k}{\to} E^{k+1} \overset{d_{k+1}}{\to} E^{k+2} \to \ldots$$

We can represent a complex as a graded object $E^* \doteq \oplus_k E^k$ with the endomorphism $d = \oplus d_k$ of degree 1 which satisfies the equation $d^2 = dd = 0$. It is called *differential* in the complex. The *(co)homology* of the complex is the graded object

$$H(E^*) \doteq \oplus H^k(E)$$

where $H^k(E)$ is the (co)homology at the k-th place; it is equal to the object

$$i : \operatorname{Cok} d'_{k-1} \overset{\cong}{\to} \operatorname{Ker} d'_k$$

The morphisms

$$d'_{k-1} : E^{k-1} \to \operatorname{Ker} d_k, \ d'_k : \operatorname{Cok} d_{k-1} \to E^{k+1}$$

follow from the equation $d_k d_{k-1} = 0$. The complex E^* is *exact* if $H^*(E) = 0$.

A *morphism* of complexes $e : (E^*, d_E) \to (F^*, d_F)$ is a sequence of morphisms $e^k : E^k \to F^k$ that commute with the differentials, which means

1

that $ed_E = d_F e$, i.e. $d_{F,k+1} e_k = e_{k+1} d_{E,k}$ for each $k \in \mathbb{N}$. Any morphism of complexes induces the morphism of the homology:

$$H\left(e^k\right) : H^k\left(E\right) \to H^k\left(F\right), \ k \in \mathbb{N}$$

Example 1. In the category **Ab** of abelian groups $H\left(E\right) = \operatorname{Ker} d / \operatorname{Im} d$.

Let A be a commutative ring; denote by $\mathbf{C}\left(\mathsf{A}\right)$ the category of modules over A.

Theorem 1 *Let*

$$0 \to E \xrightarrow{e} F \xrightarrow{f} G \to 0$$

be an exact sequence of complexes in $\mathbf{C}\left(\mathsf{A}\right)$*, i.e. it is exact in each grading. Then there exist morphisms*

$$\delta^k : H^k\left(G\right) \to H^{k+1}\left(E\right), \ k \in \mathbb{N}$$

such that the "long" sequence is exact

$$... \to H^{k-1}\left(G\right) \xrightarrow{\delta} H^k\left(E\right) \to H^k\left(F\right) \to H^k\left(G\right) \xrightarrow{\delta} H^{k+1}\left(E\right) \to ... \quad (9.1)$$

Lemma 2 *[Snake Lemma] For an arbitrary commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \to & E_0 & \to & F_0 & \to & G_0 & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & E_1 & \to & F_1 & \to & G_1 & \to & 0
\end{array}
$$

with exact lines there exists a morphism $\delta : \operatorname{Ker} \gamma \to \operatorname{Cok} \alpha$ *which makes the sequence exact*

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \xrightarrow{\kappa} \operatorname{Ker}\gamma \xrightarrow{\delta} \operatorname{Cok}\alpha \xrightarrow{\zeta} \operatorname{Cok}\beta \to \operatorname{Cok}\gamma \quad (9.2)$$

PROOF. Consider the diagram

$$
\begin{array}{ccccccccc}
 & & \operatorname{Ker}\alpha & \to & \operatorname{Ker}\beta & \xrightarrow{\kappa} & \operatorname{Ker}\gamma & \to & \\
 & & \downarrow & & \downarrow & & \downarrow & & \searrow \\
 & & E_0 & \xrightarrow{i} & F_0 & \xrightarrow{p} & G_0 \to & 0 & \nearrow \\
 & & \leftarrow & & \leftarrow & & \leftarrow & \leftarrow & \\
 & \nearrow & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
\downarrow & 0 \to & E_1 & \xrightarrow{i} & F_1 & \xrightarrow{p} & G_1 & & \\
 & \searrow & \downarrow \xi & & \downarrow \eta & & \downarrow & & \\
 & & \operatorname{Cok}\alpha & \xrightarrow{\zeta} & \operatorname{Cok}\beta & \to & \operatorname{Cok}\gamma & &
\end{array}
$$

2

Take an element $g_0 \in \operatorname{Ker} \gamma$ and find $f_0 \in F_0$ and $e_1 \in E_1$ such that

$$g_0 = p\left(f_0\right), \ \beta\left(f_0\right) = i\left(e_1\right)$$

and set $\delta\left(g_0\right) = \xi\left(e_1\right)$. Check that this element is well defined; choose another element f_0' such that $p\left(f_0'\right) = g_0$. We have $p\left(f_0' - f_0\right) = 0$, therefore there exists an element e_0 such that $f_0' - f_0 = i\left(e_0\right)$. It follows that $\beta\left(f_0'\right) - \beta\left(f_0\right) = \beta i\left(e_0\right) = i\alpha\left(e_0\right)$ and $\beta\left(f_0'\right) = i\left(e_1 + \alpha\left(e_0\right)\right)$. This choice leads to the element $\xi\left(e_1 + \alpha\left(e_0\right)\right)$ which coincides with $\xi\left(e_1\right)$ since of $\xi\alpha = 0$. This shows that the element $\delta\left(g\right)$ it does not depend on choice of e_2. It follows that δ is homomorphism of modules.

We have $\zeta\delta\left(g\right) = \zeta\left(\xi\left(e_1\right)\right) = \eta i\left(e_1\right) = \eta\beta i\left(e_1\right) = 0$. On the other hand, if $g = \kappa\left(f\right)$ for an element $f \in \operatorname{Ker}\beta$, then $g = p\left(f\right)$ for some $f \in \operatorname{Ker}\beta$ which implies that $\delta\left(g\right) = 0$. This implies that (1) is a complex.

Check that this complex is exact. Take an element in $\operatorname{Ker}\zeta$. Write this element in the form $\xi\left(e_1\right)$. We have $\eta i\left(e_1\right) = 0$, which means that there exists an element f_1 such that $i\left(e_1\right) = \beta\left(f_0\right)$. We have $\gamma p\left(f_0\right) = p\beta\left(f_0\right) = pi\left(e_1\right) = 0$ which means that $p\left(f_0\right) \in \operatorname{Ker}\gamma$. Since $\delta\left(p\left(f_0\right)\right) = \xi\left(e_1\right)$, this shows exactness in the term $\operatorname{Cok}\alpha$. We can check exactness in other terms by similar arguments, which are called "diagram chasing". $\square$

Draw another snake diagram with exact rows and lines

$$
\begin{array}{ccccccccc}
 & & & & & & 0 & & 0 & & 0 \\
 & & & & & & \downarrow & & \downarrow & & \downarrow \\
E & \to & F & \xrightarrow{p} & G \to 0 & & H\left(E\right) & \to & H\left(F\right) & \to & H\left(G\right) \\
\downarrow \alpha' & & \downarrow \beta' & & \downarrow \gamma' & \nearrow & \downarrow & & \downarrow & & \downarrow \\
\operatorname{Ker}\alpha & \xrightarrow{\lambda} & \operatorname{Ker}\beta & \xrightarrow{\kappa} & \operatorname{Ker}\gamma & -\uparrow \xrightarrow{\delta} & \operatorname{Cok}\alpha & \xrightarrow{\xi} & \operatorname{Cok}\beta & \to & \operatorname{Cok}\gamma \\
\downarrow \pi & & \downarrow \rho & & \downarrow & \nearrow & \downarrow \alpha'' & & \downarrow \beta'' & & \downarrow \gamma'' \\
H\left(E\right) & \xrightarrow{\eta} & H\left(F\right) & \to & H\left(G\right) & 0 \to & E & \xrightarrow{i} & F & \to & G \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
$$

We show that the "snake" mapping makes the long line exact. First note that the morphism $\delta : \operatorname{Ker}\gamma \to \operatorname{Cok}\alpha$ vanishes on the image $\operatorname{Ker}\beta$. Indeed, $\operatorname{Im}\gamma' \subset \operatorname{Im}\kappa$ because of p is surjective. Therefore δ can be lifted to a mapping $H\left(G\right) \to \operatorname{Cok}\alpha$. On the other hand, the image of this mapping is contained in $\operatorname{Ker}\alpha''$ because of the equation $i\alpha''\delta = \beta''\zeta\delta = 0$ where i is injective. The kernel of α'' is isomorphic to $H\left(E\right)$, whence δ makes the "snake" mapping

3

from $H(G)$ to $H(E)$. Show that this morphism makes the sequence (9.1) exact. Exactness in the terms $H(E)$ and $H(G)$ follows from exactness of the middle line. Check exactness in the term $H(F)$. Let h be an element of the module Ker$\{H(F) \to H(G)\}$. Take an element $f \in \text{Ker}\,\beta$ such that $\rho(f) = h$ and an element $g \in G$ such that $\gamma'(g) = \kappa(f)$. By surjectivity of p we can find $f' \in F$ such that $p(f') = g$. We have $\kappa(f - \beta'(f')) = \kappa(f) - \gamma'(p(f')) = 0$. We have $f - \beta'(f') = \lambda(e)$ for some $e \in \text{Ker}\,\alpha$ because of the middle line is exact. Then $h = \eta(\pi(e))$, which completes the proof. $\square$

9.2   Exact functors

Definition. Let $\mathbf{C}$ be a category satisfying (A_1, A_2) and $\mathbf{Ab}$ be the category of abelian groups. Let $\mathbf{F} : \mathbf{C} \to \mathbf{Ab}$ be a covariant functor. For any exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$$

in $\mathbf{C}$ the sequence

$$0 \to \mathbf{F}(X) \xrightarrow{\mathbf{F}(f)} \mathbf{F}(Y) \xrightarrow{\mathbf{F}(g)} \mathbf{F}(Z) \tag{9.3}$$

is semi-exact. This follows from the definition: $\mathbf{F}(g)\mathbf{F}(f) = \mathbf{F}(gf) = \mathbf{F}(0) = 0$. The functor is called *left exact* if (9.3) is always exact. It is called *right exact* if for any exact sequence

$$X \to Y \to Z \to 0 \tag{9.4}$$

the sequence

$$\mathbf{F}(X) \to \mathbf{F}(Y) \to \mathbf{F}(Z) \to 0$$

is exact. For a contravariant functor this definition is given by replacing the category $\mathbf{C}$ by the dual category.

Example 2. The functors $\mathbf{h}_V(\cdot) = \hom(V, \cdot)$, $\mathbf{h}^V(\cdot) = \hom(\cdot, V)$ are left exact for arbitrary object $V \in \mathbf{C}$.

Let $\mathbf{C}$ be an arbitrary category. For an arbitrary object X the mapping $\mathbf{h}_X(\cdot)$ is defined as covariant functor from $\mathbf{C}$ to the category $\mathbf{Sets}$ of sets. The mapping $\mathbf{h}^X(\cdot)$ is a contravariant functor in the same categories.

Definition. Let $\mathbf{F} : \mathbf{C} \Rightarrow \mathbf{Sets}$ be a covariant functor. It is called *representable* if there exist an object R of the category $\mathbf{C}$ and an element $f \in \mathbf{F}(R)$ such that the morphism of functors $\phi : \mathbf{h}_R \Longrightarrow \mathbf{F}$ given by

$$\phi(\alpha) = \mathbf{F}(\alpha) f, \ \alpha \in \mathbf{h}_R(X) = \hom(R, X)$$

is an isomorphism.

A contravariant functor is called representable if it is representable as a functor form the dual category $\mathbf{C}^*$ to $\mathbf{Sets}$.

9.3 Tensor product

Let $\mathbf{C}(\mathsf{A})$ be again a category of modules over a ring A; for arbitrary $X, Y \in \mathbf{C}$ the set $\hom_{\mathbf{C}}(X, Y)$ has the structure of A-module.

Definition. Fix some objects E, F of $\mathbf{C}$ and consider the functor $\mathbf{B}_{E,F}$ in $\mathbf{C}$ defined as follows. For $X \in \mathbf{C}$, $\mathbf{B}_{E,F}(X)$ is the modules of all morphisms $f : E \times F \to X$ that are A-linear in both arguments. Another way to define this module is

$$\mathbf{B}_{E,F}(X) = \hom_{\mathbf{C}}(E, \hom_{\mathbf{C}}(F, X))$$

Indeed, if we fix the first argument $e \in E$, then $f(e, \cdot)$ is an A-morphism in the second argument. Suppose that the functor $\mathbf{B}$ is representable by an object T and a morphism $b : E \times F \to T$. The object T is called the *tensor product* of E and F and is denoted

$$T = E \otimes_{\mathbf{C}} F$$

The standard notation for the morphism b is $b(e, f) = e \otimes f$.

The pair (T, b) is unique up to isomorphism. Tensor product depends, of course, on the category $\mathbf{C}$.

Example 3. The tensor product in the category A of analytic algebras was defined in Chapter 7. Moreover for an analytic algebra A the category $\mathsf{A}(\mathsf{A})$ of analytic algebras over A possesses the tensor product: for two objects $\mathsf{B} \to \mathsf{A}$ and $\mathsf{C} \to \mathsf{A}$ the tensor product is isomorphic to the analytic algebra $\mathsf{B} \otimes_{\mathsf{A}} \mathsf{C}$.

Let again A be a commutative ring with unit element 1, $\mathbf{C}(\mathsf{A})$ be the category of all A-modules.

Proposition 3 *In the category* $\mathbf{C}(\mathsf{A})$ *the tensor product is always defined.*

PROOF. For a set G we consider the free A-module A^G generated by elements of G. For arbitrary A-modules E, F, define the free A-module $\mathsf{A}^{E \times F}$ generated by the set $E \times F$. Consider the submodule R generated by the elements

$$
\begin{aligned}
&(e + e', f) - (e, f) - (e', f), e, e' \in E, f, f' \in F &(9.5)\\
&(e, f + f') - (e, f) - (e, f'), \\
&(ae, f) - a\,(e, f),\ (e, af) - a\,(e, f), a \in \mathsf{A}
\end{aligned}
$$

The quotient module $T = \mathsf{A}^{E \times F}/R$ is the tensor product $E \otimes_{\mathbf{C}} F$. The mapping $b : E \times F \to E \otimes_{\mathbf{C}} F$ is obvious: $(e, f) \mapsto (e, f) \pmod{R}$ (here (e, f) is considered as a generator of $\mathsf{A}^{E \times F}$). Any A-bilinear morphism $f : E \times F \to M$ is extended to $\mathsf{A}^{E \times F}$ as A-morphism. It vanishes on elements (9.5) and has a unique lifting to the quotient $\mathsf{A}^{E \times F}/R$. $\square$

The tensor product in this category is denoted $E \otimes_{\mathsf{A}} F$.

Proposition 4 *The tensor product defines a covariant functor in $\mathbf{C}\,(\mathsf{A})$ in each argument.*

PROOF. Let $f : F \to G$ be a morphism in the category. It generates a morphism $\mathsf{A}^{E \times F} \to \mathsf{A}^{E \times G}$. Obviously $R\,(E, F)$ is mapped to $R\,(E, G)$ whence we get a morphism

$$
\mathsf{A}^{E \times F}/R\,(E, F) \to \mathsf{A}^{E \times G}/R\,(E, G)
$$

Problem. To prove this fact only using the abstract definition of tensor product.

Proposition 5 *We have for arbitrary A-modules E, F, G*

$$
\begin{aligned}
E \otimes_{\mathsf{A}} \mathsf{A} &\cong E, &(9.6)\\
E \otimes_{\mathsf{A}} F &\cong F \otimes_{\mathsf{A}} E &(9.7)\\
(E \times F) \otimes_{\mathsf{A}} G &\cong (E \otimes_{\mathsf{A}} G) \times (F \otimes_{\mathsf{A}} G) &(9.8)
\end{aligned}
$$

PROOF. Any coset in $\mathsf{A}^{E \times \mathsf{A}} \pmod{R}$ has a unique element $(e, 1)$ where 1 is the unit element of A. This proves (9.6). The relations (9.7) and (9.8) can be proved in the same way. $\square$

Proposition 6 *The tensor product functor in the category $\mathbf{C}\,(\mathsf{A})$ of A-modules is right exact in each of its arguments.*

6

PROOF. The functor hom is left exact and we have the isomorphism

$$\hom\left(X \otimes_{\mathsf{A}} U, V\right) \cong \hom\left(X, \hom\left(U, V\right)\right)$$

for arbitrary objects X, U, V of $\mathbf{C}\left(\mathsf{A}\right)$. Therefore for any exact sequence (9.4) the sequence

$$0 \to \hom\left(Z \otimes_{\mathsf{A}} U, V\right) \xrightarrow{\eta^*} \hom\left(Y \otimes_{\mathsf{A}} U, V\right) \xrightarrow{\xi^*} \hom\left(X \otimes_{\mathsf{A}} U, V\right) \qquad (9.9)$$

is also exact. From this we can conclude that the sequence

$$X \otimes_{\mathsf{A}} U \xrightarrow{\xi} Y \otimes_{\mathsf{A}} U \xrightarrow{\eta} Z \otimes_{\mathsf{A}} U \to 0 \qquad (9.10)$$

is exact. It is semi-exact because of tensor product is a functor. Check exactness; take $V = \operatorname{Cok}\eta$ and denote $\pi : Z \otimes_{\mathsf{A}} U \to V$ the canonical projection. The morphism $\eta^*\left(\pi\right)$ belongs to the middle term of (9.9) and vanishes, whence $\pi = 0$ because of η^* is injection. This implies that η is surjective. Check exactness of (9.10) in the second term. Take $V = \operatorname{Cok}\xi = Y \otimes_{\mathsf{A}} U/\operatorname{Im}\xi$ and consider the canonical projection $\rho : Y \otimes_{\mathsf{A}} U \to V$. We have $\xi^*\left(\rho\right) = 0$, whence $\rho = \eta^*\left(\sigma\right)$ for a morphism σ that belongs to the first term of (9.9). This means that $\rho = \sigma\eta$ consequently ρ vanishes on $\operatorname{Ker}\eta$ and therefore $\operatorname{Im}\xi \supset \operatorname{Ker}\eta$, q.e.d.

This proves right exactness with respect to the first argument and implies exactness in the second argument because of the tensor product functor is symmetric. $\square$

9.4   Projective resolvents

Definition. Let $\mathbf{C}$ be a category with the properties (A$_1$,A$_2$). An object P is call
ed *projective* if for any surjective morphism p and morphism f in the diagram

$$
\begin{array}{ccccc}
E & \xrightarrow{p} & F & \to & 0 \\
& \searrow e & \uparrow f & & \\
& & P & &
\end{array}
$$

there exists a morphism e which makes it commutative, i.e. $f = pe$. It is called *lifting* of f.

The direct sum of projective objects is projective. The zero object is projective.

Example 4. In the category $\mathbf{C}(\mathsf{A})$ the ring A is a projective object. A module of the form A^n where n is finite or not, is called *free* A-module. Each free module of finite type is a projective object in the category $\mathbf{C}(\mathsf{A})$.

Definition. A *projective resolvent* of an object X is an exact sequence

$$... \to P_k \xrightarrow{d_k} P_{k-1} \to ... \to P_1 \to P_0 \to X \to 0$$

where $P_0, P_1, ..., P_k, ...$ are projective objects. We say that a category $\mathbf{C}$ has many projective objects, if for any object X there exists a projective object and a surjection $P \to X$. If the category has many projective objects, then any object has a projective resolvent.

Example 5. If A is a Nötherian algebra, then any A-module M of finite type has a projective resolvent.

Proposition 7 *(I) For any morphism $f : X \to Y$ in $\mathbf{C}$ and arbitrary projective resolvent $(P, d) \to X$, $(Q, e) \to Y$ there exists a morphism of complexes $i : P \to Q$ which makes a commutative diagram with e.*
(II) Any two morphisms i, j as in part (I) are homotopically equivalent, i.e. there exists a morphism of graded objects $h : P \to Q$ of degree 1 such that

$$j - i = eh + hd$$

PROOF. We construct the morphism i to make the following diagram commutative:

$$
\begin{array}{ccccccccc}
...\to & Q_2 & \to & Q_1 & \xrightarrow{e_1} & Q_0 & \xrightarrow{e_0} & Y & \to 0 \\
& \uparrow i_2 & & \uparrow i_1 & & \uparrow i_0 & & \uparrow f & \\
...\to & P_2 & \to & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & X & \to 0
\end{array}
\qquad (9.11)
$$

Since e_0 is surjective, we can lift the morphism d_0 to a morphism i_0 such that $e_0 i_0 = f d_0$. We have $e_0 i_0 d_1 = d_0 d_1 = 0$. Therefore the image of $i_0 d_1$ is contained in $\operatorname{Ker} e_0 = \operatorname{Im} e_1$. We lift $i_0 d_1$ to a morphism i_1 such that $e_1 i_1 = i_0 d_1$ and so on. The morphism j can be chosen in the same way.

Now consider the composition

$$
\begin{array}{ccccccccc}
...\to & Q_2 & \to & Q_1 & \xrightarrow{e_1} & Q_0 & \xrightarrow{e_0} & Y & \to 0 \\
& \uparrow j_2 - i_2 & \searrow h_1 \; \uparrow j_1 - i_1 & & \searrow h_0 \; \uparrow j_0 - i_0 & & \uparrow f & \\
...\to & P_2 & \to & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{d_0} & X & \to 0
\end{array}
$$

Consider the morphism $k_0 = j_0 - i_0$; we have $e_0 k_0 = f d_0 - f d_0 = 0$. Therefore there exists a morphism h_0 such that $e_1 h_0 = k_0$. Take $k_1 = j_1 - i_1 - h_0 d_1$; we have

$$e_1 k_1 = e_1 (j_1 - i_1) - e_1 h_0 d_1 = (j_0 - i_0) d_1 - k_0 d_1 = k_0 d_1 - k_0 d_1 = 0$$

Therefore there exists a morphism h_1 such that $e_2 h_1 = k_1$ and so on. $\square$

9

Chapter 10

Derived functors, Ext and Tor

10.1 Derived functors

Definition. [A.Grothendieck] Let $\mathbf{C}$ be an Abelina category with many projective objects and $\mathbf{F} : \mathbf{C} \Rightarrow \mathbf{Ab}$ be a covariant right exact functor (or contravariant left exact functor). The *derived* functors of $\mathbf{F}$ are:

$$X \Rightarrow \mathbf{F}_k(X) = H_k(\mathbf{F}(P_*)), k = 0, 1, ... \qquad (10.1)$$

where $P \to X$ is an arbitrary projective resolvent of X and P_* is considered as the complex

$$P_* : ... \to P_k \xrightarrow{d} P_{k-1} \to ... \to P_1 \xrightarrow{d_0} P_0 \to 0$$

Proposition 1 *There is natural isomorphism* $\mathbf{F}_0 = \mathbf{F}$.

PROOF. By definition $\mathbf{F}_0(X) = \mathrm{Cok}\,\mathbf{F}(d_0)$. On the other hand, the sequence $P_1 \to P_0 \to X \to 0$ is exact and the functor is right exact. Therefore the sequence

$$\mathbf{F}(P_1) \xrightarrow{\mathbf{F}(d_0)} \mathbf{F}(P_0) \to \mathbf{F}(X) \to 0$$

is exact too which means that $\mathbf{F}(X) \cong \mathrm{Cok}\,\mathbf{F}(d_0)$. $\square$

Proposition 2 *(I) The groups* $H_k(\mathbf{F}(P)), k = 1, 2, ...$ *do not depend on the choice of projective resolvent.*
(II) The mappings (10.1) are covariant functors.

PROOF. Let $f : X \to Y$ be morphism in $\mathbf{C}$. Choose projective resolvents P and Q and a morphism of complexes $i : P \to Q$ that makes the commutative diagram (11) of Ch.9. Applying the functor we obtain the morphism of complexes $\mathbf{F}(i) : \mathbf{F}(P) \to \mathbf{F}(Q)$ which induces a morphism of homology $H_*(\mathbf{F}(P)) \to H_*(\mathbf{F}(Q))$. According to Ch.9, any other morphism of complexes $j : P \to Q$ that commutes with f is homotopic equivalent, i.e. $i - j = d_Q h + h d_P$. This implies

$$\mathbf{F}(i) - \mathbf{F}(j) = \mathbf{F}(d_Q)\mathbf{F}(h) + \mathbf{F}(h)\mathbf{F}(d_P)$$

The morphism $\mathbf{F}(d_P)$ vanishes in $H_*(\mathbf{F}(P))$ and the contribution of $\operatorname{Im}\mathbf{F}(d_Q)$ in $\mathbf{H}_*(\mathbf{F}(Q))$ is equal to zero. Therefore $\mathbf{F}(i)$ and $\mathbf{F}(j)$ induces the same morphism in homology $H_*(\mathbf{F}(X)) \to H_*(\mathbf{F}(Y))$. In particular, if $f = \operatorname{id}_X$,and $i = \operatorname{id}_P$ this shows that $\mathbf{F}(j)$ is equal to the identity morphism of the groups (10.1). This proves (I), which means that the graded groups $\mathbf{F}_*(X)$ are well define. For an arbitrary f the graded morphism $\mathbf{F}_*(f) \doteq \mathbf{F}(i)$ of graded objects $\mathbf{F}_*(X) \to \mathbf{F}_*(Y)$ does not depend on i. For an arbitrary morphism $g : Y \to Z$ we can choose a projective resolvent R and a related morphism of complexes $j : Q \to R$ and so on. We get the equation

$$\mathbf{F}_*(gf) = \mathbf{F}(ji) = \mathbf{F}(j)\mathbf{F}(i) = \mathbf{F}_*(g)\mathbf{F}_*(f)$$

which proves (II). $\square$

Definition.[Continuation] A *injective* object in an Abelian category $\mathbf{C}$ is defined as a projective object in the dual category. This means that I is injective if for any injective morphism $i : X \to Y$ and arbitrary morphism $k : X \to I$ there exists a morphism $l : Y \to I$ such that $li = k$. A category $\mathbf{C}$ has many injective objects, if for an arbitrary object X there is an injective morphism $i : X \to I$ in an injective object I. An injective resolvent of X is defined in the similar way:

$$0 \to X \to I_0 \to I_1 \to \ldots$$

Let $\mathbf{C}$ be a category with many injective objects and $\mathbf{F} : \mathbf{C} \Rightarrow \mathbf{Ab}$ be a covariant left exact functor (or contravariant right exact functor). The *derived* functors are defined as follows:

$$\mathbf{F}^k(X) = H^k(\mathbf{F}(I)), k = 0, 1, \ldots$$

where X is an arbitrary object and I is an arbitrary injective resolvent of X. We have $\mathbf{F}^0 = \mathbf{F}$ because this functor is left exact.

2

10.2 Examples

1. *Ext.* Let A be a ring and $\mathbf{C}(A)$ be the category of modules over A. The functor $\hom(\cdot, \cdot)$ is defined in $\mathbf{C}(A)$ which is left exact contravariant in the first argument and covariant in the second one. The category $\mathbf{C}(A)$ has many projective and injective objects. Therefore the derivatives of the functor hom are well defined. They are denoted

$$N \times M \mapsto \operatorname{Ext}_A^k(N, M)_{1,2}, k = 0, 1, 2, ...$$

(Ext=extension). These partial derivatives functors coincide: $\operatorname{Ext}(\cdot, \cdot)_1 \cong \operatorname{Ext}(\cdot, \cdot)_2$.

2. *Cohomology of a sheaf.* Let $\mathbf{Sh}_X$ be the category of sheaves of abelian groups on a topological space X. The functor $\Gamma : \mathcal{F} \mapsto F(X, \mathcal{F})$ with values in the category $\mathbf{Ab}$ is covariant and left exact. The $\mathbf{Sh}_X$ has many injective objects. The derivatives $H^k(X, \mathcal{F})$, $k = 0, 1, 2$ are well defined and called cohomology groups of the sheaf $\mathcal{F}$.

3. *Direct image of a sheaf.* Let $f : X \to Y$ be a continuous mapping of topological spaces, $\mathcal{F} \in \mathbf{Sh}_X$. Take an open set $V \subset Y$ and consider the cohomology group $H^*(f^{-1}(V), \mathcal{F})$. For any open $W \subset V$ the restriction homomorphism

$$H^*(f^{-1}(V), \mathcal{F}) \to H^*(f^{-1}(W), \mathcal{F})$$

is defined, which means that $R^*(\cdot, \mathcal{F}) \doteq H^*(f^{-1}(\cdot), \mathcal{F})$ is a contravariant functor $\mathbf{Top}(Y) \Longrightarrow \mathbf{Ab}$, i.e. a presheaf on Y. The sheaf generated by this presheaf is called *dirived direct image* (pull down) of the sheaf $\mathcal{F}$; it is denoted $\mathsf{R}_f^*(\mathcal{F})$. We have $\mathsf{R}_f^*(\mathcal{F}) = H^*(X, \mathcal{F})$, if Y is a point.

4. *Tangent cohomology.* Let A be a commutative algebra, M be a A-module. A derivation $t : A \to M$ is a linear mapping that satisfies the Leibniz formula $t(ab) = t(a)b + at(b)$ (which is a differential operator of order 1). Let $T(A, M) \doteq \operatorname{Der}(A, M)$ be the A-module of derivations valued in M. The functor $\mathbf{F} : M \mapsto \operatorname{Der}(A, M)$ is covariant and left exact in the category $\mathbf{C}(A)$. The derived functor $T^*(A, M)$ is called tangent cohomology of the algebra with values in M.

5. *Tor.* The functor of tensor product is covariant and right exact. Applying this construction to the functors $\mathbf{F}(X, \cdot) = X \otimes_A \cdot$ and $\mathbf{F}(\cdot, Y) = \cdot \otimes_A Y$, the derivatives are defined

$$\operatorname{Tor}_k^A(X, \cdot)_1 = \mathbf{F}_k(X, \cdot), \ \operatorname{Tor}_k^A(\cdot, Y)_2 = \mathbf{F}_k(\cdot, Y), k = 0, 1, 2, ...$$

3

(Tor=torsion). They are, again, isomorphic: $\mathrm{Tor}\,()_1 \cong \mathrm{Tor}\,()_2$.

Problem. To check this isomorphism for $A=\mathbb{Z}$.

10.3 Exact sequence

Proposition 3 *Let* **C** *be an Abelian category with many projective objects and* $\mathbf{F} : \mathbf{C} \to \mathbf{Ab}$ *be a covariant right exact functor. Then for any "short" exact sequence*

$$0 \to X \to Y \to Z \to 0 \tag{10.2}$$

there is "long" exact sequence for derivatives of the functor

$$... \to \mathbf{F}^{k+1}(Z) \to \mathbf{F}^k(X) \to \mathbf{F}^k(Y) \to \mathbf{F}^k(Z) \to ...$$
$$... \to \mathbf{F}^1(Y) \to \mathbf{F}^1(Z) \to \mathbf{F}(X) \to \mathbf{F}(Y) \to \mathbf{F}(Z) \to 0$$

PROOF. Construct a bicomplex

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
...... & \to & P_2 & \to & P_1 & \to & P_0 & \to & X & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
..... & \to & Q_2 & \to & Q_1 & \to & Q_0 & \to & Y & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
..... & \to & R_2 & \to & R_1 & \to & R_0 & \to & Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

where the columns are exact and the lines are projective resolvents of X, Y, Z. For this we take, first, arbitrary projective resolvents P of X and R of Z and define the graded object $Q = P \oplus R$. The morphisms $P \to Q \to R$ are canonical; the differential in Q is to be defined to make this diagram commutative. Then we replace the right column by zero objects and apply the functor $\mathbf{F}$ to this bicomplex. We get another bicomplex where the columns are again exact and apply Theorem 1 of Ch.9. $\square$

10.4 Properties of Tor

Proposition 4 *There are canonical isomorphisms of functors*

$$\mathrm{Tor}_k^{\mathsf{A}}(X,Y)_1 \cong \mathrm{Tor}_k^{\mathsf{A}}(X,Y)_2, \, k = 0, 1, 2, ...$$

PROOF. Take a free resolution P of X and a free resolution Q of Y and manufacture the tensor product $P \otimes_{\mathsf{A}} Q$. The homology in the last row are isomorphic to the homology of the right column. $\square$

For an arbitrary exact sequence of modules (10.2) and a module E we have the long exact sequence

$$... \to \operatorname{Tor}_{k+1}(Z, E) \to \operatorname{Tor}_k(X, E) \to \operatorname{Tor}_k(Y, E) \to \operatorname{Tor}_k(Z, E) \to ...$$
$$\to \operatorname{Tor}_2(Z, E) \to \operatorname{Tor}_1(X, E) \to \operatorname{Tor}_1(Y, E) \to$$
$$\to \operatorname{Tor}_1(Z, E) \to X \otimes E \to Y \otimes E \to Z \otimes E \to 0$$

where we omit the low index A. The roles of arguments can be exchanged.

Example. Let $\mathsf{A}= \mathbb{Z}$ be the ring of integers, $E = \mathbb{Z}/p\mathbb{Z}$, $F = \mathbb{Z}/q\mathbb{Z}$ where p, q are some non zero integers. The sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to E \to 0$$

is exact. Therefore we have the exact sequence

$$0 = \operatorname{Tor}(\mathbb{Z}, F) \to \operatorname{Tor}_1(E, F) \to F \xrightarrow{p \otimes F} F \to E \otimes F \to 0$$

If the numbers p, q are mutually prime, the mapping $p \otimes F$ is bijective and we have
$$\operatorname{Tor}_1(E, F) = E \otimes F = 0$$
Otherwise,

$$E \otimes F = \mathbb{Z}/p\mathbb{Z} + q\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}, \ \operatorname{Tor}_1(E, F) = r\mathbb{Z}/q\mathbb{Z} = \mathbb{Z}/d\mathbb{Z}$$

where d is the greatest common divisor of p and q; $r = q/d$.

Problem. Let J, K be ideals in A. Show that

$$\operatorname{Tor}_1^{\mathsf{A}}(\mathsf{A}/J, \mathsf{A}/K) = J \cap K / JK$$

10.5 Flat modules

Definition. A A-module M is called *flat* if $\operatorname{Tor}_k^{\mathsf{A}}(M, E) = 0$ for $k = 1, 2, ...$ and arbitrary A-module E.

Proposition 5 *If M is a flat A-module and $E \xrightarrow{e} F \xrightarrow{f} G$ is an arbitrary exact sequence of A-modules, then the sequence*

$$E \otimes M \to F \otimes M \to G \otimes M \qquad\qquad (10.3)$$

is also exact.

PROOF. Include the sequence in the commutative diagram

where all the lines are exact and apply the functor $\otimes M$ to each one. The tilted lines keep to be exact since M is flat. Therefore the horizontal line is also exact. It coincides with (10.3). $\square$

Proposition 6 *Let E, M be A-modules and*

$$... \to F_2 \to F_1 \to F_0 \to E \to 0$$

be an exact sequence of modules where all F_k are flat. Then there is an isomorphism

$$\mathrm{Tor}_* (E, M) \cong H_* (F \otimes M) \ \text{ where } F \doteq ... \to F_2 \to F_1 \to F_0 \to 0$$

Theorem 7 *Let A be a local Nötherian $\mathbb{C}$-algebra and M be a A-module of finite type. Then the following conditions are equivalent*
(i) M is free,
(ii) M is projective,
(iii) M is flat,
(iv) $\mathrm{Tor}_1 (M, A/m(A)) = 0$.

PROOF. Obviously, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$. Show that $(iv) \Rightarrow (i)$. The module $\mathsf{A}/m\,(\mathsf{A})$ is isomorphic to the basic field $\mathbb{C}$ and the module $M \otimes \mathbb{C}$ is a vector space of finite dimension. Choose elements $m_1, ..., m_d \in M$ whose images in $M \otimes \mathbb{C}$ make a basis and define the morphism $p : \mathsf{A}^d \to M$ such that $p\,(e_k) = m_k, k = 1, ..., d$. From the exact sequence

$$0 \to \operatorname{Ker} p \to \mathsf{A}^d \xrightarrow{p} M \to \operatorname{Cok} p \to 0 \qquad (10.4)$$

we find the exact sequence

$$\mathsf{A}^d \otimes \mathbb{C} \xrightarrow{p \otimes \mathbb{C}} M \otimes \mathbb{C} \to \operatorname{Cok} p \otimes \mathbb{C} \to 0$$

The mapping $p \otimes \mathbb{C}$ is an isomorphism according to the construction. Therefore $\operatorname{Cok} p \otimes \mathbb{C} = 0$. By Nakayama Lemma this implies that $\operatorname{Cok} p = 0$. From (10.4) we now conclude that the sequence

$$0 = \operatorname{Tor}_1\,(M, \mathbb{C}) \to \operatorname{Ker} p \otimes \mathbb{C} \to \mathsf{A}^d \otimes \mathbb{C} \xrightarrow{p \otimes \mathbb{C}} M \otimes \mathbb{C} \to 0$$

is exact. It follows that $\operatorname{Ker} p \otimes \mathbb{C} = 0$. The module $\operatorname{Ker} p$ is of finite type because the algebra is Nötherian. Again by Nakayama Lemma we find that $\operatorname{Ker} p = 0$. We see from (10.4) that p isomorphism which proves (i). $\square$

Remark. The field $\mathbb{C}$ can be replaced by another field, for example, by $\mathbb{R}$.

10.5.1 Restriction and extension of scalars

Let $\phi : \mathsf{A} \to \mathsf{B}$ be a homomorphism of commutative rings and M be a B-module. It has also an A-module structure as follows: the action of an element $a \in \mathsf{A}$ in M is defined as follows: $m \mapsto \phi\,(a)\,m$. This operation is called *restriction of scalars.*

If K be an A-module then the tensor product $K_\mathsf{B} = \mathsf{B} \otimes_\mathsf{A} K$ has structure of B-module: $b_1\,(b_2 \otimes k) = b_1 b_2 \otimes k$. The operation $K \Rightarrow K_\mathsf{B}$ is called *extension of scalars.*

Proposition 8 *If K is a flat A-module, then K_B is a flat B-module.*

PROOF follows from isomorphisms for an arbitrary B-module E :

$$E \otimes_\mathsf{B} K_\mathsf{B} \cong (E \otimes_\mathsf{B} \mathsf{B}) \otimes_\mathsf{A} K \cong E \otimes_\mathsf{A} K$$

10.6 Syzygy theorem

An analytic algebra A is called *regular* of dimension n, if $A \cong O^n$.

Theorem 9 *For any regular algebra A of dimension n and arbitrary A-modules M, N we have $\mathrm{Tor}_k^A (M, N) = 0$ for $k > n$.*

PROOF. We assume for simplicity that M is of finite type. Take a free resolvent
$$F_j \xrightarrow{d_j} F_{j-1} \to ... \to F_1 \xrightarrow{d_1} F_0 \xrightarrow{p} M \to 0$$
We shall show that $K_{n-1} = \mathrm{Ker}\, d_{n-1}$ is a free module. Then we can write the complete free resolvent of M as follows

$$0 \to K_{n-1} \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$$

Applying the functor $\otimes N$ to this resolvent, we get a complex which vanishes for degrees $\geq n$. The homology of this complex vanishes in the same degrees which implies the Theorem.

Now set $K_j \doteq \mathrm{Ker}\, d_j, j = 1, 2, ..., K_0 \doteq \mathrm{Ker}\, p, K_{-1} \doteq M$. From the exact sequence
$$0 \to K_j \to F_j \xrightarrow{d_j} K_{j-1} \to 0, \ j = -1, 0, 1, ...$$
we find another sequence

$$0 = \mathrm{Tor}_{k+1} (F_j, \mathbb{C}) \to \mathrm{Tor}_{k+1} (K_{j-1}, \mathbb{C}) \to \mathrm{Tor}_k (K_j, \mathbb{C}) \to \mathrm{Tor}_k (F_j, \mathbb{C}) = 0$$

where $k = 1, 2, ...$ This implies $\mathrm{Tor}_{k+1} (K_{j-1}, \mathbb{C}) \cong \mathrm{Tor}_k (K_j, \mathbb{C})$, whence this module only depends on $k + j$, in particular,

$$\mathrm{Tor}_1 (K_{n-1}, \mathbb{C}) \cong \mathrm{Tor}_{n+1} (M, \mathbb{C})$$

The right side vanishes by the following Lemma which implies $\mathrm{Tor}_1 (K_{n-1}, \mathbb{C}) = 0$. By Theorem 7 this implies our statement. $\square$

Lemma 10 *There exists a free A-resolvent of $\mathbb{C}$ of length n.*

PROOF OF LEMMA. Let $z_1, ..., z_n$ be generators of the maximal ideal $\mathfrak{m}(A)$. Consider the free A-algebra K generated by n elements $e_1, ..., e_n$ with commutation relations
$$e_j e_i = -e_i e_j, \ e_i^2 = 0$$

(the Grassman algebra with n odd generators). It is free A-module of rank $=$ 2^n generated by products

$$e_{j_1} e_{j_2} ... e_{j_k}, \ j_1 < j_2 < ... < j_k$$

It is graded if we set $\deg a = 0$ for $a \in A$ and $\deg e_{j_1} e_{j_2} ... e_{j_k} = k$. Thus we have $K = \sum K_j$ where $K_j \cong A^{\binom{n}{j}}$. Define the endomorphism d in K by $d(e_j) = z_j, j = 1, ..., n$ and extend it to K by means of Leibniz formula with alternating sign:
$$d(ab) = d(a) b + (-1)^{\deg a} a d(b)$$

i.e. d is a derivative of degree -1 in the graded commutative algebra K. The morphism is a differential because of the formula

$$d^2(e_i e_j) = d(d(e_i) e_j - e_i d(e_j)) = z_i z_j - z_j z_i = 0$$

The complex (K,d) is called *Koszul* complex:

$$0 \to A \to A^n \to A^{\binom{n}{2}} \to ... A^{\binom{n}{j}} \to A^{\binom{n}{j-1}} \to ... \to A^n \to A \to 0$$

We have $H_0(K) \cong \mathbb{C}$.

Show that $H_j(K) = 0$ for $j > 0$. This is obvious for $n = 1$. Apply induction in n. Let (L, δ) be the Koszul complex over a regular algebra B of variables $z' = (z_2, ..., z_n)$. There is the natural injection $L \to K$. Take an element f of positive degree in the kernel of d. The vector $\phi(z') = f(0, z') |e_1 = 0$ belongs to kernel of δ; by inductive assumption there exists an element $\psi \in L$ such that $\phi = \delta \psi$. The vector $f' \doteq f - d\psi$ vanishes as $z_1 = 0, e_1 = 0$; write $f' = z_1 g + e_1 \xi$, where $g \in K, \xi \in L$. This implies that $\delta \xi = 0$ in the case $\deg \xi > 0$ and $\xi(0) = 0$ if $\deg \xi = 0$. Again by induction we have $\xi = \delta \eta$. Write $f'' = f' + d(e_1 \eta)$ and have $f'' = z_1 g', dg' = 0$. Define $h \doteq e_1 g'$ and have $dh = z_1 g' - e_1 dg' = f''$. Finally, $f = d(\psi + e_1 \eta + h)$. $\square$

Let A,B be analytic algebras. Consider the direct coproduct $E = A \otimes_{\mathbb{C}} B$ in the category of analytic algebras (see Chapter 8). Note that $E \otimes_A \mathbb{C} \cong B$.

Proposition 11 *For arbitrary analytic algebras A and B the A-module E is flat.*

Remark. The B-module E is of course, flat too because the coproduct is symmetric.

9

PROOF. We have the homomorphism of algebras $A \to E$ by $a \mapsto a \otimes 1$. An arbitrary morphism $f : A^s \to A^r$ generates the morphism $F : E^s \to E^r$ which operates as multiplication by the matrix f. We only need to check that for any exact sequence

$$A^t \xrightarrow{q} A^s \xrightarrow{p} A^r \qquad (10.5)$$

the sequence

$$E^t \xrightarrow{Q} E^s \xrightarrow{P} E^r$$

is exact too. Assume for simplicity that the algebra B is regular $B \cong O^n$. Then E is the algebra of convergent power series of some variables $w_1, ..., w_n$ with coefficients in A. Take an arbitrary $e \in \operatorname{Ker} P$ and write

$$e = \sum_{j=(j_1,...,j_n)} e_j w^j, \ e_j \in \operatorname{Ker} p$$

By exactness of (10.5) we have $e_j = qc_j$ for each j and some $c_j \in A^t$. We have $e = Qc$ where $c = \sum c_j w^j$ provided this series converges. We can not guarantee convergence because the solution c_j may be not unique. On the other hand, any finite sum belongs to E^t; therefore we know that $e \in \operatorname{Im} Q + \mathfrak{m}^k (A) E^s$ for any k. This implies $e \in \operatorname{Im} Q$ in virtue of Artin-Rees Theorem, Ch.5. $\square$

Chapter 11

Deformation of complex spaces

11.1 Flat morphisms

Definition. Let $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of complex analytic spaces. It is called *flat* at a point $x \in X$ if the morphism $\phi_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. The morphism (f, ϕ) is flat if it is so at each point. We discuss now a criterium of flatness:

Theorem 1 *Let $\phi : A \to B$ be a morphism of analytic algebras and* **M** *be a B-module of finite type. Then the following conditions are equivalent:*
(i) M is a flat A-module,
(ii) $\mathrm{Tor}_1^A(M, \mathbb{C}) = 0, \mathbb{C} = A/m(A)$;
(iii) If

$$E^{r_2} \overset{D_2}{\to} E^{r_1} \overset{D_1}{\to} E^{r_0} \to \mathbf{M} \to 0 \tag{11.1}$$

is an exact sequence of A-modules where $E = A \otimes_{\mathbf{C}} R$ and R is an analytic algebra, then the sequence

$$R^{r_2} \overset{d_2}{\to} R^{r_1} \overset{d_1}{\to} R^{r_0}, \; d_j \doteq D_j \otimes \mathbb{C} \tag{11.2}$$

is exact.

PROOF. The implication $(i) \Rightarrow (ii)$ is obvious. The implication $(i) \Rightarrow (iii)$ follows from Proposition 6, Ch.10. Assume (iii) and extend (11.1) to the an infinite exact sequence of modules $E^{r_j}, j = 3, 4, ...$The A-modules E^r are flat by Proposition 10, Ch.10. By Proposition 6,Ch.10, the functor $\mathrm{Tor}_*^{\mathsf{A}}(\cdot, \mathbb{C})$ can be calculated by applying the functor $\mathbb{C}$ to this sequence.

Since $E \otimes \mathbb{C} \cong R,$ we obtain a sequence of $\mathbb{C}$-modules R^{r_j} which fragment is (11.2). Exactness of (11.2) gives (ii). The proof of $(iii) \Rightarrow (i)$ is more technical, see [2],Ch.IV,Sec.2. $\square$

11.2 Examples

Example 1. Direct product: $X = Y \times Z,$ $f : X \to Y$ is the canonical projection, and $\phi : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,(y,z)} = \mathcal{O}_{Y,y} \otimes_{\mathbb{C}} \mathcal{O}_{Z,z}$ is given by $\phi(b) = b \otimes 1$. This makes $\mathcal{O}_{X,(y,z)}$ a flat $\mathcal{O}_{Y,y}$-module by Proposition 9, Ch.10.

Example 2. A holomorphic function $f : \mathbb{U} \to \mathbb{C}$ defined in a connected open set $\mathbb{U} \subset \mathbb{C}^n$ is a flat morphism, provided f is not a constant. Indeed, consider the model complex analytic manifold $\mathbb{V} = \mathbb{U} \times \mathbb{C}$ with coordinates z, w, endowed by the sheaf of holomorphic functions $\mathcal{O}_{\mathbb{V}}$ and sheaf $\mathcal{I}$ of ideals in $\mathcal{O}_{\mathbb{V}}$ generated by the function $a(z,w) = f(z) - w$. We have the diagram

$$
\begin{array}{ccc}
(\mathbb{V}, \mathcal{O}_{\mathbb{V}}/\mathcal{I}) & \cong & (\mathbb{U}, \mathcal{O}_{\mathbb{U}}) \\
j \downarrow & & f \downarrow \\
(\mathbb{V}, \mathcal{O}_{\mathbb{V}}) & \overset{q}{\to} & (\mathbb{C}, \mathcal{O}_{\mathbb{C}})
\end{array}
$$

where $j = (\mathrm{id}_{\mathbb{V}}, \pi)$ and $\pi : \mathcal{O}_{\mathbb{V}} \to \mathcal{O}_{\mathbb{V}}/\mathcal{I}$ is canonical, q is the canonical projection to the direct factor. Therefore we have the exact sequence of $\mathcal{O}_{\mathbb{C}}$-modules

$$
0 \to \mathcal{O}_{\mathbb{V}} \overset{(f-w)}{\to} \mathcal{O}_{\mathbb{V}} \overset{\pi}{\to} \mathcal{O}_{\mathbb{V}}/\mathcal{I} \cong \mathcal{O}_{\mathbb{U}} \tag{11.3}
$$

where the morphism $(f - w)$ is multiplication by $f - w$. We have $\mathcal{O}_{\mathbb{V},(z,w)} = \mathcal{O}_{\mathbb{U},z} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C},w}$. By Theorem 1 we can check flatness of the morphism f in a point a by observing the sequence

$$
0 \to \mathcal{O}_a^n \overset{f-b}{\to} \mathcal{O}_a^n
$$

where $b = f(a)$. This sequence is exact, because of $f - b \ /\equiv 0$ and this function is not zero divisor in the algebra $\mathcal{O}_{\mathbb{U},a}$.

Example 3. Let $f : \mathbb{U} \to \mathbb{C}^m$ be a holomorphic mapping defined in $\mathbb{U} \subset \mathbb{C}^n$ such that the analytic set $f^{-1}(0)$ is of dimension $n - m$. Then the morphism $f : (\mathbb{U}, \mathcal{O}^n) \to (\mathbb{C}, \mathcal{O}^m)$ is flat. Set $\mathbb{V} = \mathbb{U} \times \mathbb{C}^m$ and follow the arguments of Example 2. Instead of (11.3) we need to write a resolvent of the form

$$
K(\mathcal{O}_{\mathbb{V}}, f - w) \to \mathcal{O}_{\mathbb{V}}/\mathcal{I} \to 0 \tag{11.4}
$$

where $\mathcal{I}$ is the sheaf of ideal generated by the functions $f_j - w_j$, $j = 1, ..., m$ where $f = (f_1, ..., f_m)$ and $K(\mathcal{O}_{\mathbb{V}}, f - w)$ is the Koszul complex generated over $\mathcal{O}_{\mathbb{V}}$ by m elements $e_1, ..., e_m$ and the differential is given by $d(e_j) = f_j - w_j$ (see Ch.10).

Theorem 2 *The Koszul complex $K(\mathcal{O}_0^n, g)$ is exact if and only if the elements $g_1, ..., g_m$ form the complete intersection ideal in $\mathcal{O}_0^n$, i.e. $\dim Z(g_1, ..., g_m) = n - m$.*

This is a corollary of the fact that the algebra $\mathcal{O}^n$ is a Cohen-Macaulay module, see [1],Ch.IVB,Th.3 for a proof. Fix a point $a \in \mathbb{U}$ and set $w = b = f(a)$ in (11.4). We get the Koszul complex $K(\mathcal{O}_a^n, f - b)$; it is exact by Theorem 2. The mapping f is flat by Theorem 1.

11.3 Pull back

Proposition 3 *Let $(f, \phi) : X \to Y$ be a flat morphism and $g : Z \to Y$ an arbitrary morphism of analytic spaces. Then the morphism $g^*(f)$ in the fibre product diagram is also flat*

$$
\begin{array}{ccc}
X & \leftarrow & X \times_Y Z \\
f \downarrow & & \downarrow g^*(f) \\
Y & \overset{g}{\leftarrow} & Z
\end{array}
$$

PROOF. Set $W = X \times_Y Z$; we have

$$\mathcal{O}_{W,(x,z)} = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Z,z}$$

where $\otimes$ means fibre coproduct in the category $\mathbf{C}$ of analytic algebras. We apply Proposition 8, Ch.10 for the homomorphism $\phi_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. It follows that the algebra $\mathcal{O}_{W,(x,z)}$ is flat over $\mathcal{O}_{Z,z}$.

11.4 Deformation of analytic spaces

We consider two classes of analytic objects: germs and compact spaces.
Definition. Let (X_0, a) be a germ of analytic space at the distinguished point a. A deformation of the germ with a base (S, s_0) is an arbitrary flat

morphism of germs $F : (X, b) \to (S, s_0)$ together with an isomorphism $i : (X_0, a) \overset{\cong}{\to} F^{-1}(s_0)$ where $F^{-1}(s_0)$ is the fibre of F.

The mapping f in Example 3 generates a deformation $(\mathbb{U}, a) \to (\mathbb{C}^m, b)$ of the germ $Z(f_1 - b_1, ..., f_m - b_m)$.

Definition. Let X_0 be a compact analytic space and (S, s_0) be a germ of analytic spaces. A deformation of X_0 with the base (S, s_0) is an arbitrary flat proper morphism $F : X \to (S, s_0)$ together with an isomorphism $i : X_0 \overset{\cong}{\to} F^{-1}(s_0)$.

Definition. The notion of isomorphism and pull back look similar in both cases. Let $F : X \to S$, $F' : X' \to S$ be deformations of a space (germ) X_0. They are isomorphic if there exist nbd Y of $F^{-1}(s_0)$ and a nbd Y' of $F'^{-1}(s_0)$ and a commutative diagram

$$
\begin{array}{ccccc}
Y & - & \overset{\overset{j}{\cong}}{} & \to & Y' \\
F| & i \searrow & & \nearrow i' & |F' \\
\downarrow & & X_0 & & \downarrow \\
S & === & == & === & S
\end{array}
$$

In virtue of Proposition 3 the pull back $g^*(f)$ of f by a morphism of germs $g : (T, t_0) \to (S, s_0), g(t_0) = s_0$ is a deformation of the same space (germ) with the base (T, t_0).

Definition. Fix an analytic space (or germ) X_0; for an arbitrary germ of analytic space S we consider the set $\mathbf{Def}(X_0, S)$ of isomorphism classes of deformations of X_0 with the base S. The symbol $\mathbf{Def}(X_0, \cdot)$ is a contravariant functor from the category of germs to the category $\mathbf{Sets}$ (or a covariant functor from the category of analytic algebras). For a morphism of germs $g : S \to T$ the set morphism $g^* : \mathbf{Def}(X_0, S) \to \mathbf{Def}(X_0, T)$ is given by the pull back operation.

Definition. Let X_0 be a germ or a compact analytic space. A deformation $F : X \to S$ of X_0 is called *versal* if an arbitrary deformation $G : Y \to T$ of X_0 is isomorphic to the pull back $h^*(F)$ by a morphism of germs $h : T \to S$. A versal deformation is called *minimal*, if $\dim S$ takes its minimum. Any two minimal versal deformations are isomorphic, but the isomorphism is need not to be canonical. The deformation F is called *universal* if the morphism h is unique. The latter means that the functor $\mathbf{Def}(X_0, \cdot)$ is representable by (S, F).

This functor is representable in very seldom cases. In particular, any compact complex analytic manifolds of dimension 1, i.e. Riemann surfaces

X_0 there exists a universal deformation. Its base S is a germ of $\mathbb{C}^d$ where d is equal to genus g of X_0 for $g = 0, 1$ and $d = 3g - 3$ for $g > 1$. Moreover some singular Riemann surfaces possesses universal deformation.

Example 4. Consider the family of elliptic curves $X_{a,b}$ in $\mathbb{CP}_2$ which are given in affine coordinates by

$$w^2 = z^3 + az + b$$

The parameter $p = a^3 b^{-2} \in \mathbb{C}$ takes equal values for isomorphic curves and distinguishes non isomorphic ones. It defines the embedding of the set of isomorphism classes to $\mathbb{CP}_1$ if we set $p = \infty$ in the case $b = 0, a \neq 0$. The point $p = -27/4$ is the image of the singular curve $X_{-3,2}$ that can be written by the equation $w^2 = (z - 1)^2 (z + 2)$. From geometrical point of view this is a complex projective line with a point of transversal self-intersection. These curves are fibres of a proper flat morphism $X \to \mathbb{CP}_1$. This morphism is a universal deformation of each fibre.

Example 5. Let X_0 be the point $0 \in \mathbb{C}$ with the Artin algebra $\mathbb{C}[z] / (z^{k+1})$. The versal deformation of X_0 is given by the morphism $F : Z \to S$ where $S = (\mathbb{C}^k, 0)$ and Z is the subset of $\mathbb{C} \times S$ given by the equation

$$p(z, s) = z^{k+1} + s_{k-1} z^{k-1} + s_{k-2} z^{k-2} + \ldots + s_1 z + s_0$$

where $s_0, s_1, \ldots, s_{k-1}$ are coordinate functions in S.

Example 6. Let X_0 be the set given by $q(z) = 0$ in $\mathbb{C}^n$ where q is non singular quadratic form. The family $X_s = \{q(z) - s = 0\}$ is a versal deformation of X_0.
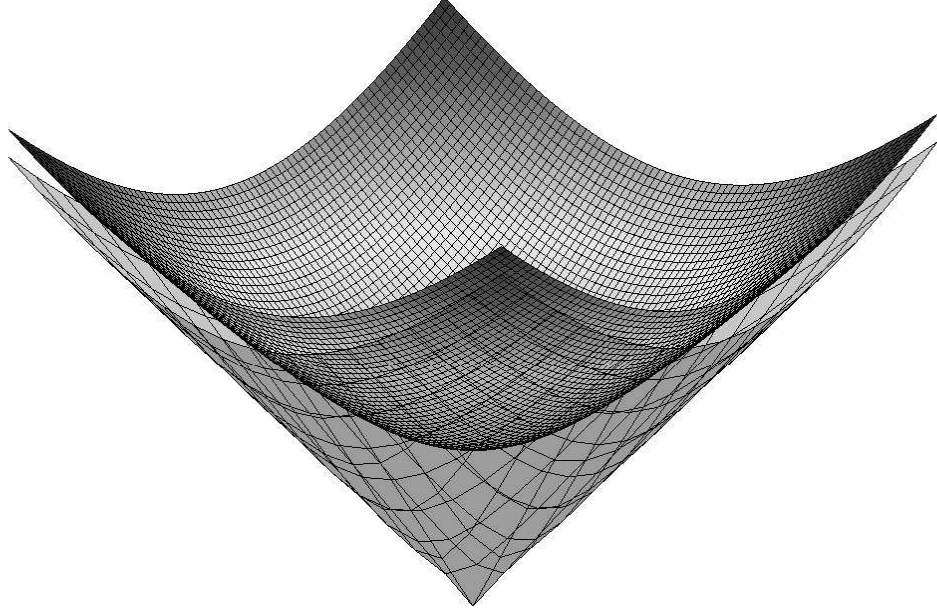
Deformation of a quadratic cone

Example 7. Let X_0 be the germ given by the ideal $I = (f_1, f_2, f_3)$ in the algebra $\mathcal{O}_0^3$ where $f_1 = z_2 z_3, f_2 = z_1 z_3, f_3 = z_1 z_2$. The support of X_0 is the union of three coordinate axis in $\mathbb{C}^3$. Set $S = \mathbb{C}^3, s_0 = 0$ and consider the subspace $X \subset \mathbb{C}^3 \times S$ given by the equations $F_1 = F_2 = F_3 = 0$ where

$$F_1 = z_2 z_3 + s_3 z_2 + s_3 s_2, F_2 = z_3 z_1 + s_1 z_3 + s_1 s_3, F_3 = z_1 z_2 + s_2 z_1 + s_1 s_2$$

The structure sheaf of X is $\mathcal{O}_X = E/I$, where $E = \mathcal{O}(\mathbb{C}^3 \times S)$ and $I = (F_1, F_2, F_3)$. Take the projection $\mathbb{C}^3 \times S \to S$ and consider its restriction $p : X \to S, \phi : \mathcal{O}_S \to E \to E/I$. We claim that (p, ϕ) is flat. Indeed, we have the exact sequence
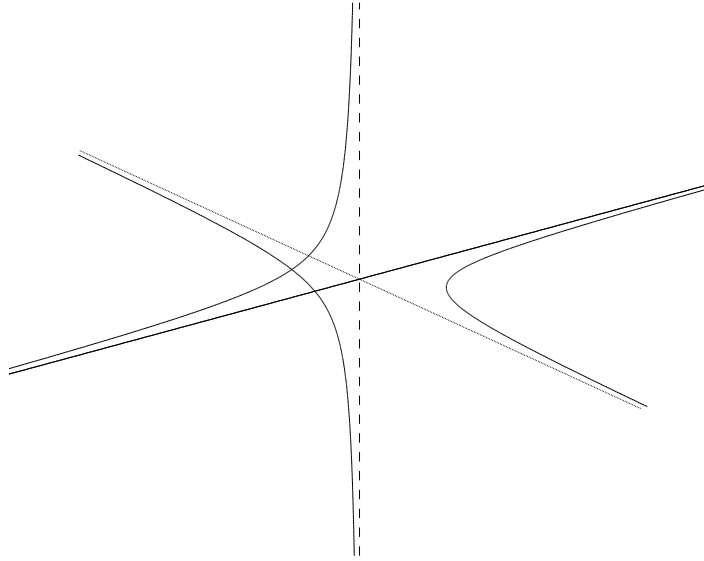
$$0 \to E \to E^3 \xrightarrow{G} E^3 \xrightarrow{F} E \to \mathcal{O}_X \to 0 \tag{11.5}$$

where $FA = F_1 A_1 + F_2 A_2 + F_3 A_3$,

$$GB = B_1 (z_1 + s_1, -z_2, -s_3) + B_2 (-s_1, z_2 + s_2, -z_3) + B_3 (-z_1, -s_2, z_3 + s_3)$$

Since $E_0 = \mathcal{O}_0^3 \otimes_{\mathbf{C}} A, A = \mathcal{O}_{S,0}$ we can apply Theorem 1. Multiplying by $\mathbb{C}$ over $\mathcal{O}_S$ we get the sequence

$$A^3 \xrightarrow{g} A^3 \xrightarrow{f} A$$

where $fa = f_1a_1 + f_2a_2 + f_3a_3 = z_2z_3a_1 + z_3z_1a_2 + z_1z_2a_3$

$$gb = b_1(z_1, -z_2, 0) + b_2(0, z_2, -z_3) + b_3(-z_1, 0, z_3) \qquad (11.6)$$

This sequence is obviously exact which implies flatness of (p, ψ).

Look at the fibres X_s of this morphism. If only one of coordinates s_1, s_2, s_3 is not zero, X_s is union of three straight lines in $\mathbb{C}^3$ with two intersection points. If only one of the coordinates vanishes, X_s is union of a plane hyperbola and a orthogonal line which meets this hyperbola. Finally, if $s_1s_2s_3 \neq 0$ the fibre is a smooth irreducible curve.

Theorem 4 *For an arbitrary germ of complex analytic space (X_0, a) that is a manifold except for the distinguished point a, there exists a minimal versal deformation.*

This means that there exists, essentially, only one minimal versal deformation $X \rightarrow (S, 0)$. The number emdim S can be calculated in advance in algebraic terms: there exists a canonical isomorphism $T_0(S) \cong T^1(X_0)$. Here $T_0(S)$ is the tangent space to S, i.e. the space of functionals $t : \mathcal{O}_S \rightarrow \mathbb{C}$ that satisfy the Leibniz condition. The space $T^1(X_0)$ is the first derived functor of Der (A, A), $\mathsf{A} = \mathcal{O}_{X_0}$ (see Example 4 Ch.10). For example, any germ of hypersurface $X_0 = \{f(z) = 0, z \in \mathbb{C}^n\}$ we have

$$T^1(X_0) = \mathcal{O}_0^n / \left(f, f'_{z_1}, ..., f'_{z_n}\right)$$

7

This space has finite dimension if X_0 has the only singular point $z = 0$.

Theorem 5 *For an arbitrary compact complex analytic space there exists a minimal versal deformation.*

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Chapter 12

Finite morphisms

12.1 Direct image

Definition. A morphism $f : X \to Y$ of topological spaces is called *finite* if it is proper and each fibre $f^{-1}(y), y \in Y$ is a finite set.

Theorem 1 *Let $(f, \phi) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a finite morphism of complex analytic spaces. The direct image $Rf(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-sheaf.*

PROOF. This property is local with respect to Y and we can assume that Y is a model analytic space in an open set $\mathbb{V} \subset \mathbb{C}^n$ and $\mathcal{O}_Y = \mathcal{O}_{\mathbb{V}}/\mathcal{J}$. Choose a point $y_0 \in Y$, say $y_0 = 0$, the set $f^{-1}(0)$ consists of n points. Since f is proper we can choose a nbd Y' of y_0 such that the space $X' \doteq f^{-1}(Y')$ is union of n disjoint components $X_1, ..., X_n$. We have $Rf(\mathcal{O}_X) = \oplus Rf(\mathcal{O}_{X_j})$. Therefore the statement of Theorem is reduced to the case of one component, i.e. we can assume that X is a model space and the set $f^{-1}(0)$ contains only one point, say, $x_0 = 0$.

Thus we have $X \subset \mathbb{U}, \mathcal{O}_X = \mathcal{O}_{\mathbb{U}}/\mathcal{I}$ where $\mathbb{U}$ is an open nbd of the point x_0 in $\mathbb{C}^m$, $\mathcal{I}$ is a coherent subsheaf of $\mathcal{O}_{\mathbb{U}}$ generated by some functions $a_1, ..., a_l$. Let $w_1, ..., w_n$ be coordinate functions in $\mathbb{C}^n$ and $f_j = \phi(w_j), j = 1, ..., n$ are sections of $\mathcal{O}_{\mathbb{U}}$ in a nbd of the origin. Shrinking nbd $\mathbb{U}$ we can assume that these sections are defined in $\mathbb{U}$. Let Z be the subspace of $\mathbb{U} \times \mathbb{V}$ with the structure sheaf $\mathcal{O}_{\mathbb{U} \times \mathbb{V}}/\mathcal{K}$, where $\mathcal{K}$ is the sheaf of ideal generated by functions $a_i(z)$ and $w_j - f_j(z)$. We have $X \cong Z$ and the commutative diagram of morphisms of analytic spaces

$$
\begin{array}{ccccc}
X & \cong & Z & \subset & \mathbb{U} \times \mathbb{V} \\
f \downarrow & & g \downarrow & & p \downarrow \\
0 \in Y & = & Y & \subset & \mathbb{V}
\end{array}
$$

where p is the canonical projection. Therefore we can replace f by g. The support of Z is the closed analytic set in $\mathbb{U} \times \mathbb{V}$ and the intersection of Z with the subspace $L = \{w = 0\}$ contains only the origin $z = 0$. This implies that the coordinate system (z, w) is convenient for Z and we can apply Theorem 16, Ch.4. According this theorem, there exists for any $j = 1, ..., m$ a distinguished pp. $p_j(z_j, w)$ in z_j that vanishes in Z. By the decomposition theorem (Ch.5) the pp. p_j^k belongs to the ideal $\mathcal{K}_0$ for some natural k and arbitrary j. We can assume that $k = 1$. Let d_j be the order of $p_j, j = 1, ..., m$ and $J \subset \mathbb{Z}^m$ is the set of indices $j = (j_1, ..., j_m)$ such that $0 \leq j_k < d_k, k = 1, ..., m$. The set J contains $d = d_1...d_m$ elements. Consider the morphism of $\mathcal{O}_Y$-sheaves

$$
\sigma : \mathcal{O}_Y^d \to Rg_* (\mathcal{O}_Z), \{a_j, j \in J\} \mapsto \sum_J z^j a_j (w) \tag{12.1}
$$

Lemma 2 *The morphism σ is surjective in a neighborhood of the origin.*

PROOF OF LEMMA. We need to show that for a nbd $\mathbb{V}' \subset \mathbb{V}$ of the origin and for arbitrary $w_0 \in \mathbb{W} \subset \mathbb{V}'$ and a section $S \in \Gamma(p^{-1}(\mathbb{W}), \mathcal{O}_Z)$ there exists sections $a_j \in \Gamma(\mathbb{W}', \mathcal{O}_Y), j \in J$ such that

$$
S = \sum z^j a_j \pmod{\mathcal{K}}
$$

where $\mathbb{W}'$ is a nbd of w_0. Consider the system of equations

$$
p_1(z_1, w) = ... = p_m(z_m, w) = 0 \tag{12.2}
$$

For each $w \in \mathbb{V}$ the number of solutions $z = \zeta_1(w), ..., \zeta_d(w)$ is equal to d counting multiplicity. The set Z is contained in the set P given by (12.2). The value S_ζ of the section S in a point $\zeta \in g^{-1}(w)$ belongs to $\mathcal{O}_{Z,\zeta} = \mathcal{O}_{\mathbb{U} \times \mathbb{V}, \zeta}/\mathcal{K}_\zeta$. Choose an element $s_\zeta \in \mathcal{O}_{\mathbb{U} \times \mathbb{V}, \zeta}$ in the coset S_ζ. If a point $(\zeta, w_0) \in P$ does not belong to $g^{-1}(w_0)$ we set $s_\zeta = 0$. The germs s_ζ define a holomorphic function $s(z, w)$ in a nbd of the set $\{\zeta_1(w_0), ..., \zeta_d(w_0)\}$. We interpolate this function by means of functions $r(z, w)$ that are pp. with respect to z of order $< d_j$ in z_j. We call such functions *truncated* pp. In the case $m = 1$ we argue

as in Ch.3 and set

$$r\left(z,w\right)=\frac{1}{2\pi\imath}\int_{\partial F}\frac{s\left(\lambda,w\right)}{p\left(\lambda,w\right)}\frac{p\left(\lambda,w\right)-p\left(z,w\right)}{\lambda-z}d\lambda \qquad (12.3)$$

$$q\left(z,w\right)=\frac{1}{2\pi\imath}\int_{\partial F}\frac{s\left(\lambda,w\right)d\lambda}{p\left(\lambda,w\right)\left(\lambda-z\right)}$$

where $F\subset\mathbb{U}$ is an open nbd of the set $\{\zeta_1\left(w_0\right),...,\zeta_d\left(w_0\right)\}$ with smooth boundary such that the function s is holomorphic in the closure of F. The function r is a pp. in z with coefficients that are holomorphic in a nbd of w_0, q is holomorphic in $F\times\mathbb{W}'$ where $\mathbb{W}'$ is a small nbd of w_0. We have $s=pq+r$; function p is a section of the sheaf $\mathcal{K}$, consequently the image of r in $\Gamma\left(p^{-1}\left(\mathbb{W}'\right),\mathcal{O}_Z\right)$ coincides with σ.

In the general case we need to apply the interpolation (12.3) m times. $\square$

Now we only need to show that the kernel of σ is finitely generated. Since the pseudopolynomials (12.2) are section of the sheaf $\mathcal{K}$, we can apply the division Theorem 2(I),Ch.3 to a system of generators $\mathcal{K}$ and replace them by the remainders $q_1,...,q_t$ modulo the system (12.2) which are truncated pp.. Thus the pp. $p_1,...,p_d;q_1,...,q_t$ are generators of $\mathcal{K}$. We can now write the inclusion $\{a_j\}\in\operatorname{Ker}\sigma$ in the form

$$\sum z^j a_j\left(w\right)=\sum b_k\left(z,w\right)q_k\left(z,w\right)+\sum c_l\left(z,w\right)p_l\left(z,w\right) \qquad (12.4)$$

where all b_j are also truncated pp.-s. It follows that c_j are truncated pp. too (follow arguments of Lemma 2,Ch.8. Therefore both side of (12.4) are truncated pp. of degree $<2d_k$ in z_k. Equalizing coefficients at the monomials $z^j, 0\le j_k<2d_k$, we get a finite system of linear equations for the a_j and coefficients of pp. b_k,c_l; all these coefficients are elements of the algebra $\mathcal{O}_{Y,w_0}$. The coefficients of pp. q_k and p_l are entries of the system. By Theorem 1, Ch.8 the sheaf of solutions of this system is finitely generated over $\mathcal{O}_Y$. $\square$

12.2   Multiplicity of flat morphisms

Theorem 3 *If $(f,\phi):(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ is an arbitrary finite morphism of complex analytic spaces, then*
(I) the function

$$\mu\left(y\right)\doteq\dim_{\mathbb{C}}\Gamma\left(X_y,\mathcal{O}_{X_y}\right)$$

is upper semi-continuous where $X_y=f^{-1}\left(y\right)$.
(II) If the morphism is flat, this function is locally constant.

PROOF. We have for any $y \in Y$ and $x \in X_y$

$$\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y.y}} \mathbb{C}$$

whence

$$\Gamma\left(X_y, \mathcal{O}_{X_y}\right) = Rf\left(X, \mathcal{O}_X\right)_y \otimes_{\mathcal{O}_{Y.y}} \mathbb{C} \tag{12.5}$$

We apply the following Lemma to the sheaf $\mathcal{F} \doteq Rf\left(\mathcal{O}_X\right)$ which gives (I).

Lemma 4 *For an arbitrary coherent $\mathcal{O}$-sheaf $\mathcal{F}$ in $\mathbb{C}^n$ the function*

$$d\left(z\right) \doteq \dim_{\mathbb{C}} \mathcal{F}_z \otimes_{\mathcal{O}_z} \mathbb{C}$$

is upper semi-continuous.

PROOF OF LEMMA. Take an exact sequence of $\mathcal{O}$-sheaves $\mathcal{O}^t \to \mathcal{O}^s \to \mathcal{F} \to 0$ in an open set $\mathbb{U}$. The first morphism is given by a holomorphic $s \times t$-matrix A. For any point $z \in \mathbb{U}$ we have the exact sequence

$$
\begin{array}{ccccccc}
\mathcal{O}^t \otimes_{\mathcal{O}_z} \mathbb{C} & \to & \mathcal{O}^s \otimes_{\mathcal{O}_z} \mathbb{C} & \to & \mathcal{F} \otimes_v \mathbb{C} & \to & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
\mathbb{C}^t & \overset{A(z)}{\to} & \mathbb{C}^s & \to & \mathcal{F} \otimes \mathbb{C} & \to & 0
\end{array}
$$

Therefore $d\left(z\right) = s - \operatorname{rank} A\left(z\right)$ and Lemma follows. $\square$

Now we prove (II). By Theorem 7, Ch.10 we have $\mathcal{F}_0 \cong \mathcal{O}_{Y,0}^r$ for some natural r. This implies that $\operatorname{rank} A = r$ and $d\left(z\right) = s - r$ in a nbd $\mathbb{V}'$ of the origin. $\square$

Definition. Let $f : \mathbb{U} \to \mathbb{V} \subset \mathbb{C}^n$ be a proper holomorphic mapping where $\mathbb{V} \subset \mathbb{C}^n$. For any $w \in \mathbb{V}$ the fibre $f^{-1}\left(w\right)$ is a finite set because it is a compact analytic subset of $\mathbb{U}$. We call multiplicity $\mu_f\left(z\right)$ of f in a point z the dimension of the structure algebra of the fibre $f^{-1}\left(w\right), w = f\left(z\right)$ in this points, i.e.

$$\mu_f\left(z\right) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{U},z} / \left(f_1 - w_1, ..., f_n - w_n\right)$$

Corollary 5 *For any proper mapping $f : \mathbb{U} \to \mathbb{V}$ the sum of multiplicities $\mu_f\left(z\right)$ taken over a fibre $f^{-1}\left(w\right)$ is locally constant.*

Corollary 6 *Any finite flat holomorphic mapping is open.*

The general result is as follows

Theorem 7 *[Remmert-Grauert] For an arbitrary proper morphism of analytic space $f : X \to Y$ and an arbitrary coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ each direct image sheaf $R^q f (\mathcal{F})$ is a $\mathcal{O}_Y$-coherent sheaf.*

Corollary 8 *[Cartan-Serre] For any compact complex space X and an arbitrary coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ the space $H^q (X, \mathcal{F})$ has finite dimension for $k \geq 0$.*

12.3 Example

Let X_0 be a point with the structure algebra $A = \mathcal{O}_0^2 / \mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}^2$. The ideal $\mathfrak{m}^2$ is generated by the monomials z^2, zw, w^2 where z, w are coordinates in $\mathbb{C}^2$. Consider the space $Z = \mathbb{C}^4$ with coordinates s, t, u, v and the subspace $Y \subset \mathbb{C}^2 \times Z$ given by the equations $F_1 = F_2 = F_3 = 0$ where

$$F_1 = z^2 + sz + tw, \ F_2 = zw, \ F_3 = w^2 + uz + vw$$

The projection $\mathbb{C}^2 \times Z \to Z$ induces the morphism $p : Y \to Z$. It is not flat because f.e. the syzygy relation $w(z^2) - z(zw) = 0$ can not be extended to a syzygy for F_1, F_2, F_3. Now we restrict the morphism to the subspace $S \subset Z$ given by the ideal $J \subset \mathcal{O}_Z$, $J = (tu, tv, us)$. The support of S is the union of three planes

$$P_1 = \{t = u = 0\}, P_2 = \{u = v = 0\}, P_3 = \{s = t = 0\}$$

Let $X \doteq p^{-1}(S) = Y \cap (\mathbb{C}^2 \times S)$; $\mathcal{O}_X = \mathcal{O}(\mathbb{C}^2 \times Z) / (F_1, F_2, F_3, tu, tv, us)$. We claim that the projection $\pi : X \to S$ is a flat morphism....

Let us count multiplicities. For the point $s = t = u = v = 0$ we have $\mu_f = 3$. For a point in P_1 the system of equations looks as follows $z^2 + sz = zw = w^2 + vw = 0$. There are three solutions: $(-s, 0), (0, 0), (0, -v)$. For a point in P_2 the system is $z^2 + sz + tw = zw = w^2 = 0$. The solutions are the points $(-s, 0)$ and $(0, 0)$ where $\mu = 2$. The plane P_3 is symmetric to P_2

12.4 Residue revisited

Now we proof properties of the residue functional Res mentioned in Sec.6.3. Let $f : \mathbb{U} \to \mathbb{C}^n$ be a holomorphic mapping defined in an open set $\mathbb{U} \subset \mathbb{C}^n$

such that $f^{-1}(0) = \{a\}$ for the origin $0 \in \mathbb{C}^n$. Denote $\mathcal{A} = \mathcal{O}_a/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by $f_1, ..., f_n$.

Proposition 9 *We have*

$$\operatorname{Res} \frac{Jgdz}{f_1...f_n} = \mu\, g(a)\,, \ \mu \doteq \dim \mathsf{A},\ g \in \mathsf{A} \tag{12.6}$$

Theorem 10 *For an arbitrary non zero element $g \in \mathsf{A}$ there exists an element $h \in \mathsf{A}$ such that*

$$\operatorname{Res} \frac{ghdz}{f_1...f_n} = 1$$

PROOF OF PROPOSITION. If $\mathbb{W}$ is a sufficiently small nbd of the origin, then the mapping $f : \mathbb{U}' \to \mathbb{W}$ is finite and proper where $\mathbb{U}' = \mathbb{U} \cap f^{-1}(\mathbb{W})$. The number Let $\mu(\zeta)$ be the multiplicity of $f - f(\zeta)$ in $z \in \mathbb{U}$. For any point $w \in \mathbb{W}$ the sum of multiplicities $\mu_f(z)$ over is equal $\mu(a) = \mu$ according to Corollary 5. By Sard's Theorem we conclude that there exists a zero measure subset $Z \subset \mathbb{W}$ such that any point $w \in \mathbb{W} \backslash Z$ is non critical value of f, i.e. $m(z) = 1$ for each point $z \in f^{-1}(w)$. This implies that the fibre consists of μ non critical points. Now we replace f_j by $f_j - w_j$.in the construction of the residue integral. We obtain the family of integrals

$$\operatorname{Res} \frac{gdz}{(f_1 - w_1)...(f_n - w_n)} = (2\pi\imath)^{-n} \int_{\Gamma(w,\varepsilon)} \frac{gdz}{(f_1 - w_1)...(f_n - w_n)}$$

where

$$\Gamma(w,\varepsilon) = \{|f_1(z) - w_1| = \varepsilon_1, ..., |f_n(z) - w_n| = \varepsilon_n, z \in \mathbb{U}\}$$

which is defined and holomorphic for $\varepsilon > 0$ and for $w \in \mathbb{W}'$ for sufficiently small nbd $\mathbb{W}'$ of the origin. Fix $w \in \mathbb{W}' \backslash Z$ and make $\varepsilon \to 0$. Since the integral does not depend on ε we get

$$\operatorname{Res} \frac{gJdz}{(f_1 - w_1)...(f_n - w_n)} = \sum_{f(z)=w} g(z)$$

The left side is holomorphic as function of w; the number of terms is equal μ and each term tends to $g(a)$ as $w \to 0$. This implies (12.6). $\square$

6

PROOF OF THEOREM. Take a system e of polynomials $e_1, ..., e_\mu$ which generates a linear basis in A and consider the $\mu \times \mu$-matrix

$$B(f, e) = \left\{ \text{Res} \frac{e_i e_j dz}{f_1 ... f_n} \right\}_\mu^{i,j=1}$$

The inequality $\det B \neq 0$ will imply Theorem. We shall prove this inequality by means of double induction with respect to numbers μ and $\nu = \text{emdim}\mathsf{A} = \dim \mathfrak{m}(\mathsf{A}) / \mathfrak{m}^2(\mathsf{A})$. The case $\mu = 1$ coincides with the case $\nu = 0$; the statement is trivial because $\text{Res}\, gdz = g(a)$. Otherwise the statement can be reduced to case $n = \nu$. Indeed, if $n > \nu$, we can change coordinates in $\mathbb{V}$ and in $\mathbb{U}$ in such a way that the forms $df_{\nu+1}(a), ..., df_n(a)$ are independent. This implies that $\mathcal{O}_a^n / (f_{\nu+1}, ..., f_n) \cong \mathcal{O}^\nu$, whence A is a quotient of $\mathcal{O}^\nu$. The case $\nu = 1$is easy to treat by means of the explicit formula

$$\text{Res} \frac{gdz}{f} = \left(\frac{d}{dz} \right)^{\mu-1} \left(\frac{(z-a)^\mu g}{f} \right) |_{z=a}$$

Consider the case $\nu > 1$ and construct a two dimensional deformation F_s of the function $F_0 \doteq f$ such that $\det B(F_s) \neq 0$ except for occasionally $s = 0$. We shall see that the function $b(s) \doteq \det B(F_s)$ is holomorphic. The inequality $b(s) \neq 0$ for $s \neq 0$ shows that $b(0) \neq 0$ because of any theorem on removing singularity (see Ch.1). This will imply our statement.

Now we construct the deformation F_s. Since $\nu \geq 2$, we can change coordinates in $\mathbb{U}$ and $\mathbb{V}$ in such a way that $f_1, f_2 \in \mathfrak{m}^2(\mathcal{O}_a)$, and the functions $z_1 - a_1, z_2 - a_2$ are linearly independent in $\mathfrak{m}(\mathcal{O}_a) / \mathfrak{m}^2(\mathcal{O}_a) + \mathcal{I}$. We set

$$F_1(s, z) = f_1(z) + s_1(z_1 - a_1), F_2(s, z) = f_2(z) + s_2(z_2 - a_2), F_3 = f_3, ..., F_n = f_n$$

Take a small ball B centered in a. The function $\sum |f_j|^2$ does not vanish in ∂B hence the function $\sum |F_j|$ does not vanish too for sufficiently small $|s_1| + |s_2|$. Therefore the set $F^{-1}(s, 0)$ is finite and the morphism $\pi : X \to \mathbb{S} \subset \mathbb{C}^2$ given by the equations $F_1 = ... = F_n = 0$ is proper for a nbd $\mathbb{S}$ of the origin. It is flat y Theorem 1.Ch.11 because these functions form a complete intersection ideal. Now we consider the point $s \in \mathbb{S} \backslash \{0\}$ and the fibre $X_s = F^{-1}(s)$. We have the option: $\sharp X_s = 1$ or $\sharp X_s > 1$. In the first case a is the only point of the fibre. We have either $s_1 \neq 0$ or $s_2 \neq 0$. In both cases $\text{emdim}\, \mathcal{O}_a / \mathcal{I}_{a,s} < n$ where $\mathcal{I}_{a,s} \doteq (F_1(s, \cdot), ...F_n(s, \cdot))$. By inductive assumption we conclude that $b(s) \neq 0$. In the second option the set X_s

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consists of several points $a_1 = a, a_2, ..., a_q, q > 1$. By Corollary 5 we have $\mu = \sum \mu_{F_s}(a_k)$ where $\mu_{F_s}(a_k) > 0$ and therefore $\mu_{F_s}(a_k) < \mu_f$. By induction we conclude that each local function $b_j(s)$ does not vanish.

Next we show that the matrix $B(F_s)$ can be transformed to a block-diagonal form. Take a polynomial basis $\{e_{k,j}, j = 1, ..., \mu(a_k)\}$ in the local algebra $\mathsf{A}_k \doteq \mathcal{O}_{a_k}/\mathcal{I}_{a_k,s}$ for each $k = 1, ..., q$. Take a polynomial $h_k = h_k(z)$ that is equal to 1 in a_k and belongs to $\mathcal{I}_{a_j,s}$ for $j \neq k$. The system of polynomials $e' \doteq \{h_k e_{k,j}\}$ contains bases for all points a_k and the matrix $B(F_s)$ has block-diagonal form in this basis with the blocks $B(F_s, a_k), k = 1, .., q$. This implies

$$\det B(F_s, e') = \prod \det B(F_s, a_j) \neq 0$$

On the other hand, the system e generates the free sheaf $R\pi(\mathcal{O}_X)$ in a nbd of origin in $\mathbb{S}$ (Theorem 1). By (12.5) the image of e in the space $\Gamma(X_s, \mathcal{O}(X_s))$ is a linear basis. Therefore $e' = Le$ where L is a linear transformation and $\det B(F_s, e') = (\det L)^2 \det B(F_s, e)$ which implies that $\det B(F_s, e) \neq 0$. $\square$

12.5 Example

Consider the mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ given by basic symmetric polynomials $f_1 = s_1(z_1, ..., z_n), ..., f_n = s_n(z_1, ..., z_n)$. It is proper, $\mu_0 = \dim\mathsf{A} = n!$. The number of points in $f^{-1}(w)$ equals $n!$ if w does not belong to the discriminant set of the polynomial $t^n + w_1 t^{n-1} + ... + w_{n-1}t + w_n$. The system of monomials

$$e_j = z^j, j_k \leq k - j, k = 1, ..., n$$

generates a basis in $\mathsf{A} = \mathcal{O}_0/(f_1, ..., f_n)$. We have

$$J(z) = \prod_{i<j} (z_i - z_j) = \sum_{\pi \in S(n)} \varepsilon(\pi) \pi(e_*)$$

where $e_* = z_1^{n-1} z_2^{n-2}...z_{n-1}$ and $S(n)$ is the permutation group of n elements, $\varepsilon(\pi) = \pm$ is the sign of the permutation. Therefore we have

$$\mathrm{Res}\, \frac{e_* dz}{f_1...f_n} = 1$$

For any element e_j there exists a unique element e_k such that $e_j e_k = e_*$, whence $\mathrm{Res}\, e_j e_k / f_1...f_n = 1$.