Nonlinear Dynamics

Some exercises and solutions

S. Strogatz – Nonlinear dynamics and chaos

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Please note: The following exercises should but mustn't be correct. If you are convinced to have found an error, feel free to contact me. The Matlab codes below need some extra scripts which can be found at http://seriousjr.kyomu.43-1.org/notizen/.

version: 19 June 2013

correction(s): solution to exercise 3.1.4

Contents

Exercises for Chapter 2

2.1 A Geometric Way of Thinking

In the next three exercises, interpret $\dot{\boldsymbol{x}} = \mathbf{sin}\,(\boldsymbol{x})$ as a flow on the line.

2.1.1 Find all the fixed points of the flow.

At a fixed point, the flow has to be zero.
$\dot{x} \overset{!}{=} 0 \;\;\Leftrightarrow\;\; \sin(x) = 0 \;\;\Rightarrow\;\; x^* = n\pi \quad \forall n \in \mathbb{N}.$

There are infinitely many fixed points.

2.1.2 At which points x does the flow have greatest velocity to the right?

The velocity and its direction are determined by the value of $\dot{x}$. So, at the maximum positive value of the function ist the greatest velocity to the right.
$\sin(x) = 1 \;\;\Leftrightarrow\;\; x^* = \frac{\pi}{2} + n \cdot 2\pi \quad \forall n \in \mathtt{N}.$

The flow has the greatest velocity to the right at all values x^*.

2.2 Fixed Points and Stability

Analyze the following equations graphically. In each case, sketch the vector field on the real line, find all the fixed points, classify their stability, and sketch the graph of $x(t)$.

2.2.1 $\dot{x} = 4x^2 - 16$

The analytical solution is:

$$\dot{x} = 4x^2 - 16 \quad \Leftrightarrow \quad \int \frac{1}{x^2-4}\,\mathrm{d}x = \int 4\,\mathrm{d}t \quad \Leftrightarrow \quad \frac{1}{4}\ln\left(\frac{x-2}{x+2}\right) = 4t + C_1$$

$$\Leftrightarrow \quad x = 2\frac{1+C_2 e^{16t}}{1-C_2 e^{16t}} \qquad C_2(t=0) = \frac{x-2}{x+2}$$

There are two fixed points: $x_1^* = -2$ (which is stable) and $x_2^* = 2$ (unstable).

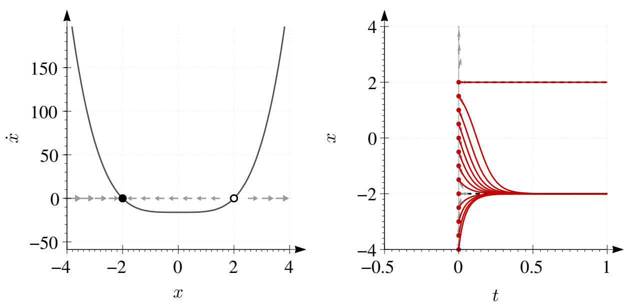


Fig. 2.1: Left: Phase space of $\dot{x} = 4x^2 - 16$, right: time-dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -4 : 0.5 : 2$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.1 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-4:0.1:4],0);

% Differentialgleichung
dx=x.^4-16;

ylim_extra=[1/6 -1/6];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
skizze_zeitverlauf(x,dx,5,1);

hold on                             % Analytische Lösung
t=0:0.0025:1;
for startval=-4:0.5:2
   C=(startval-2)/(startval+2);
   plot(0,startval,'o','MarkerFaceColor',[0.75 0 0], ...
        'MarkerEdgeColor',[0.75 0 0])
   plot(t,2*(1+exp(16*t)*C)./(1-exp(16*t)*C), ...
        'LineWidth',2,'Color',[0.75 0 0])
end
```

2.2.2 $\dot{x} = 1 - x^{14}$

No analytical solution found. The fixed points are $x_1^* = -1$ (stable) and $x_2^* = 1$ (unstable).

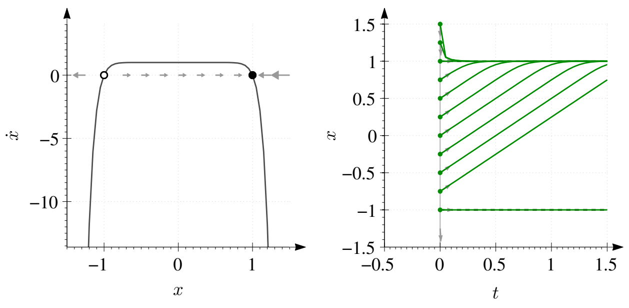


Fig. 2.2: Left: Phase space of $\dot{x} = 1 - x^{14}$, right: time-dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -1 : 0.5 : 1.5$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.2 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-1.5:0.05:1.5],0);

% Differentialgleichung
dx=1-x.^14;

ylim_extra=[-0.95 1/96];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
skizze_zeitverlauf(x,dx,5);

hold on                             % Numerische Lösung
t=0:0.05:2;
for startval=-1:0.25:1.5
   [t_s,x_s]=ode23(inline('1-x.^14','t','x'),t,startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,x_s,'LineWidth',2,'Color',[0 0.55 0])
end
```

2.2.3 $\dot{x} = x - x^3$

The analytical solution is:

$$\dot{x} = x - x^3 \quad \Leftrightarrow \quad \int \mathrm{d}t = \int \frac{1}{x(1-x^2)}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{1-x}\,\mathrm{d}x - \frac{1}{2}\int \frac{1}{1+x}\,\mathrm{d}x$$

$$\Leftrightarrow \quad x = \pm\frac{Ce^t}{\sqrt{1+C^2e^{2t}}} \qquad C(t=0) = \frac{x}{\sqrt{1-x^2}}$$

There are three fixed points: $x^*_{1,3} = \pm 1$ (stable) and $x^*_2 = 0$ (unstable).

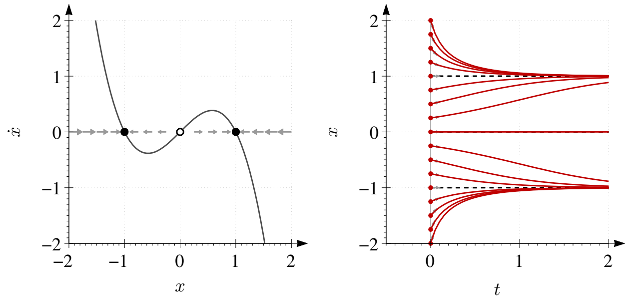


Fig. 2.3: Left: Phase space of $\dot{x} = x - x^3$, right: time–dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -2 : 0.25 : 2$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.3 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-2:0.05:2],0);

% Differentialgleichung
dx=x-x.^3;

ylim_extra=[-1/3 -1/3];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
[substatusflag,handle]=stabilitaetsanalyse(x,y,dx, ...
                    zeros(size(y)),[],[],5,ylim_extra);
for i_count=2:4
   set(handle(i_count),'XTick',[-2:2])
end

skizze_zeitverlauf(x,dx,5);

hold on                         % Analytische Lösung
t=0:0.05:2;
for startval=-2:0.25:2
   C=(startval)/sqrt(1-startval^2);
   plot(0,startval,'o','MarkerFaceColor',[0.75 0 0], ...
        'MarkerEdgeColor',[0.75 0 0])
   plot(t,exp(t)*C./sqrt(1+exp(2*t)*C^2), ...
        'LineWidth',2,'Color',[0.75 0 0])
end
```

2.2.4 $\dot{x} = e^{-x}\sin(x)$

No analytical solution found. The stable fixed points are $x^*_{\mathrm{s}} = (2k-1)\pi \quad \forall k \in \mathtt{N}$ and the unstable fixed points are $x^*_{\mathrm{u}} = 2k\pi \quad \forall k \in \mathtt{N}$.

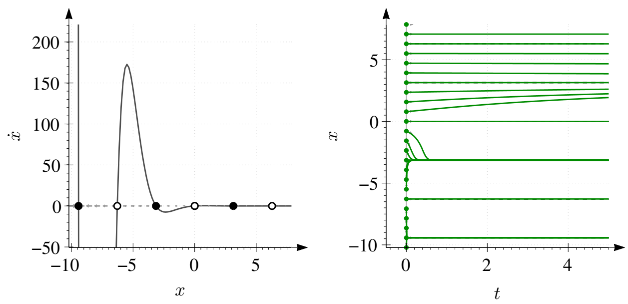


Fig. 2.4: Left: Phase space of $\dot{x} = e^{-x}\sin(x)$, right: time–dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -\frac{13}{4}\pi : \frac{\pi}{4} : \frac{9}{4}\pi$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.4 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-3.25*pi:0.05*pi:2.5*pi],0);

% Differentialgleichung
dx=exp(-x).*sin(x);

ylim_extra=[-87/512 -419/512];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],4, ...
                    ylim_extra);
skizze_zeitverlauf(x,dx,5,5);

hold on                         % Numerische Lösung
t=0:0.01:5;
for startval=-3.25*pi:0.25*pi:2.5*pi
   [t_s,x_s]=ode23s(inline('exp(-x).*sin(x)','t','x'),...
                    t,startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,x_s,'LineWidth',2,'Color',[0 0.55 0])
end
```

Systems of the form $\dot{x} = a + b\cos(x)$.

The analytical solution of a a system $\dot{x} = a + b\cos(x)$ can be obtained with some tricks. First, we substitute $s = \tan\left(\frac{x}{2}\right)$ and get

$$\cos(x) = \frac{1-s^2}{1+s^2} \qquad \text{and} \qquad \mathrm{d}x = \frac{2}{1+s^2}\mathrm{d}s.$$

Inserting and integrating yields

$$\begin{aligned}\int \mathrm{d}t = t + C = \int \frac{1}{a+b\cos(x)}\,\mathrm{d}x \quad &= \quad \int \frac{1}{a+b\frac{1-s^2}{1+s^2}} \cdot \frac{2}{1+s^2}\,\mathrm{d}s \\ &= \frac{2}{\sqrt{a^2-b^2}}\arctan\left(\frac{\sqrt{a-b}}{\sqrt{a+b}}\tan\left(\frac{x}{2}\right)\right).\end{aligned}$$

Having this form, it is straightforward to show the analytical solutions of the following two integrals. However, due to the definition of $\arctan(\varphi)$, this analytical solutions are restricted to the interval $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$.

2.2.5 $\dot{x} = 1 + \frac{1}{2}\cos(x)$

Using the formula from above, the analytical solution is

$$x = 2\arctan\left(\sqrt{3}\tan\left(\frac{\sqrt{3}}{4}(t+C)\right)\right), \qquad C(t=0) = \frac{4}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{2}\right)\right).$$

Analyzing the phase portrait (or the formula) reveals: There are no fixed points.

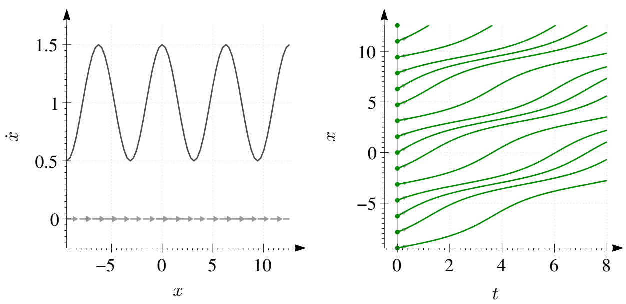


Fig. 2.5: Left: Phase space of $\dot{x} = 1 + \frac{1}{2}\cos(x)$, right: time–dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -3\pi : \frac{\pi}{2} : 4\pi$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.5 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-3*pi:0.1*pi:4*pi],0);

% Differentialgleichung
dx=1+0.5*cos(x);

ylim_extra=[3/4 1/6];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],4, ...
                    ylim_extra);
skizze_zeitverlauf(x,dx,5,8);

hold on                          % Analytische Lösung
t=0:0.02:8;
for startval=-3*pi:0.5*pi:4*pi
   % An der Stelle t=0 gilt
   C=4/sqrt(3)*(atan(tan(startval/2)/sqrt(3))+ ...
     pi*floor((startval+pi)/2/pi)); % grafische Korrektur
   plot(0,startval,'o','MarkerFaceColor',[0.75 0 0], ...
        'MarkerEdgeColor',[0.75 0 0])
   plot(t,2*(atan(sqrt(3)*tan(sqrt(3)/4*(t+C)))+ ...
        ... % Grafischer Korrekturterm
        pi*floor(((t+C)*sqrt(3)/2+pi)/2/pi)), ...
        'LineWidth',2,'Color',[0.75 0 0])
end

skizze_zeitverlauf(x,dx,5,8);

hold on                          % Numerische Lösung
t=0:0.02:8;
for startval=-3*pi:0.5*pi:4*pi
   [t_s,x_s]=ode23(inline('1+0.5*cos(x)','t','x'),t, ...
                   startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,x_s,'LineWidth',2,'Color',[0 0.55 0])
end
```

2.2.6 $\dot{x} = 1 - 2\cos(x)$

Using the formula from above, the analytical solution is

$$x = 2\arctan\left(\frac{i}{\sqrt{3}}\tan\left(\frac{\sqrt{3}i}{2}(t+C)\right)\right), \qquad C(t=0) = \frac{2}{\sqrt{3}i}\arctan\left(\frac{\sqrt{3}}{i}\tan\left(\frac{x}{2}\right)\right).$$

The stable fixed points are $x_\mathrm{s}^* = 2k\pi - \arccos\left(\frac{1}{2}\right) \quad \forall k \in \mathtt{N}$ and the unstbale fixed points are $x_\mathrm{u}^* = 2k\pi + \arccos\left(\frac{1}{2}\right) \quad \forall k \in \mathtt{N}$.

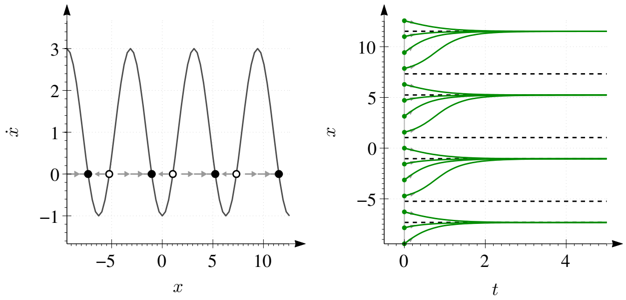


Fig. 2.6: Left: Phase space of $\dot{x} = 1 - 2\cos(x)$, right: time–dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -3\pi : \frac{\pi}{2} : 4\pi$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.6 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-3*pi:0.1*pi:4*pi],0);

% Differentialgleichung
dx=1-2*cos(x);

ylim_extra=[1/6 1/6];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],4, ...
                    ylim_extra);
skizze_zeitverlauf(x,dx,5,5);

hold on                          % Analytische Lösung
t=0:0.05:5;
for startval=-3*pi:0.5*pi:4*pi
   % An der Stelle t=0 gilt
   C=2/sqrt(3)/i*atan(tan(startval/2)/i*sqrt(3));
   plot(0,startval,'o','MarkerFaceColor',[0.75 0 0], ...
        'MarkerEdgeColor',[0.75 0 0])
   plot(t,2*atan(i/sqrt(3)*tan(sqrt(3)/2*i*(t+C))), ...
        'LineWidth',2,'Color',[0.75 0 0])
end

skizze_zeitverlauf(x,dx,5,5);

hold on                          % Numerische Lösung
t=0:0.05:5;
for startval=-3*pi:0.5*pi:4*pi
   [t_s,x_s]=ode23(inline('1-2*cos(x)','t','x'),t, ...
                   startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,x_s,'LineWidth',2,'Color',[0 0.55 0])
end
```

2.2.7 $\dot{x} = e^x - \cos(x)$

No analytical solution found. There is an unstable fixed point at zero and no fixed point for $x > 0$. In the left half plane, the space between stable and unstable fixed points is approaching a constant value (π) as $x \to -\infty$.

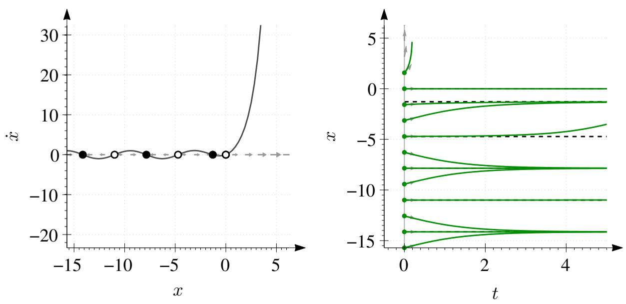


Fig. 2.7: Left: Phase space of $\dot{x} = e^x - \cos(x)$, right: time–dependent behaviour $x(t)$ with numerical solutions (start values $x(0) = -5\pi : \frac{\pi}{2} : \frac{\pi}{2}$).

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.7 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-5*pi:0.1*pi:2*pi],0);

% Differentialgleichung
dx=exp(x)-cos(x);

ylim_extra=[1/24 -15/16];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],4, ...
                    ylim_extra);
skizze_zeitverlauf(x,dx,5,5);

hold on                          % Numerische Lösung
t=0:0.05:5;
for startval=-5*pi:0.5*pi:0.5*pi
   if (startval > 1)
      t=0:0.01:0.199;
   end
   [t_s,x_s]=ode23(inline('exp(x)-cos(x)','t','x'),t, ...
                   startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,x_s,'LineWidth',2,'Color',[0 0.55 0])
end
```

2.2.10 Fixed points

For each of (a)–(e), find an equation $\dot{x} = f(x)$ with the stated properties or if there are no examples, explain why not. (In all cases, assume that $f(x)$ is a smooth function.)

a) Every real number is a fixed point.

At a fixed point, the flow has to be zero. If the flow should be zero for all values of x $\Leftrightarrow \quad \dot{x} = 0$.

b) Every integer is a fixed point.

The flow must be zero et every integer, which requires a (smooth) periodic function. One choice of an adjusted, periodic function is $\dot{x} = \sin(\pi x)$.

c) There are precisely three fixed points, and all of them are stable.

A stable or unstable fixed point implies changing the sign of the function values locally. Between any two fixed point of the same type (stable, unstable) must be a fixed point of the other type, because of the mean value theorem at a smooth function. Thus, this property cannot be fulfilled.

d) There are no fixed points.

Any function whose flow is never zero. All constant functions $\dot{x} = c \quad \forall c \in \mathtt{R}\backslash\{0\}$ have this property.

e) There are precisely 100 fixed points.

Without assembling functions or restricting periodic functions to intervals, one could use a polynomial with 100 zeros, e. g. $\prod_{k=1}^{100}(x-k)$.

2.2.13 Terminal velocity

The velocity $v(t)$ of a skydiver falling to the ground is governed by $m\dot{v} = mg - kv^2$, where m is the mass of the dkydiver, g is the acceleration due to gravity, and $k > 0$ is a constant related to the amount of air resistance.

a) Obtain the analytical solution for $v(t)$, assuming that $v(0) = 0$.

Separate the variables and integrate using $\int \frac{1}{x^2-a^2} = \frac{1}{2a} \ln\left(\frac{x-a}{x+a}\right) + C$.

$$m\dot{v} = mg - kv^2 \quad \Leftrightarrow \quad -\frac{m}{k}\int \frac{1}{v^2 - \frac{m}{k}g}\,\mathrm{d}x = \int \mathrm{d}t$$

$$\Leftrightarrow \quad -\frac{1}{2}\sqrt{\frac{m}{gk}}\ln\left(\frac{v-\sqrt{mgk}}{v+\sqrt{mgk}}\right) = t + C \quad \Leftrightarrow \quad v = \sqrt{\frac{mg}{k}}\left(\frac{1+C_2e^{-2\sqrt{\frac{gk}{m}}t}}{1-C_2e^{-2\sqrt{\frac{gk}{m}}t}}\right)$$

$$v(0) = 0 \quad \Rightarrow \quad C_2 = -1 \quad \Rightarrow \quad v(t) = \sqrt{\frac{mg}{k}}\left(\frac{1-e^{-2\sqrt{\frac{gk}{m}}t}}{1+e^{-2\sqrt{\frac{gk}{m}}t}}\right)$$

Due to $\tanh(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$, the result can also be written as $v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{gk}{m}}t\right)$.

b) Find the limit of $v(t)$ as $t \to \infty$. This limiting velocity is called the *terminal velocity*.

As $t \to \infty$, $v(t) \to \sqrt{\frac{mg}{k}}$. So the terminal velocity is $v_\infty = \sqrt{\frac{mg}{k}}$.

c) Give a graphical analysis of this problem, and thereby re-derive a formula for the terminal velocity.

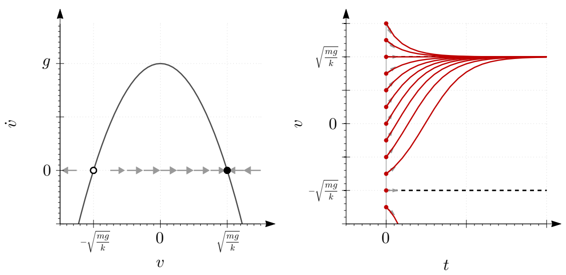


Fig. 2.8: Left: Phase space of $\dot{v} = g - \frac{k}{m}v^2$, right: time–dependent behaviour $v(t)$ with numerically obtained trajectories.

As can be seen, physically meaningful solutions ($v > 0$) approach the stable fixed point $v^* = \sqrt{\frac{mg}{k}}$ as $t \to \infty$. Therefore, v^* is the terminal velocity.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.2.13 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-1.5:0.0625:1.5],0);

% Funktion und Parameter
dx=-2*x.^2+2;                  % Differentialgleichung

ylim_extra=[-1/3 1/6];

% Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],4, ...
                   ylim_extra);

% Achsenbeschriftung anpassen
ebenen=get(gcf,'Children');
renameaxis(ebenen(2),'$$v$$','$$\dot{v}$$',[], ...
          {'$$-\sqrt{\frac{mg}{k}}$$';'0'; ...
          '$$\sqrt{\frac{mg}{k}}$$'}, ...
          {'';'0';'';'$$g$$'},26,0);

skizze_zeitverlauf(x,dx,4,2);

hold on                          % Analytische Lösung
for startval=-1.25:0.25:1.5
   if (startval < -1)
      t=0:0.05:0.4;
   else
      t=0:0.1:2;
   end
   C=(startval-1)/(startval+1);
   plot(0,startval,'o','MarkerFaceColor',[0.75 0 0], ...
        'MarkerEdgeColor',[0.75 0 0])
   plot(t,(1+exp(-4*t)*C)./(1-exp(-4*t)*C), ...
        'LineWidth',2,'Color',[0.75 0 0])
end

% Achsenbeschriftung anpassen
ebenen=get(gcf,'Children');
renameaxis(ebenen(2),'$$t$$','$$v$$',[],{'0';'';''}, ...
          {'';'$$-\sqrt{\frac{mg}{k}}$$';'';'0';''; ...
          '$$\sqrt{\frac{mg}{k}}$$';''},26,0);
```

2.4 Linear Stability Analysis

Use linear stability analysis to classify the fixed points of the following systems. If linear stability analysis fails because $f'(x^*) = 0$, use graphical argument to decide the stability.

Linear stability analysis means calculating the derivative and evaluate it at the values of the fixed points. Positive values indicate a positive slope and therefore an instable fixed point. Negative values result in a stable fixed point with the same argumentation. If the derivative is zero at the fixed point, graphical analysis is needed.

2.4.1 $\dot{x} = x(1-x)$

The fixed points are $x_1^* = 0$ and $x_2^* = 1$. The derivative is $\ddot{x} = -2x + 1$. Inserting the x–values of the first fixed point yields $\ddot{x}(x_1^*) = 1$. Therefore, x_1^* is unstable. The second fixed point is stable due to $\ddot{x}(x_2^*) = -1$.

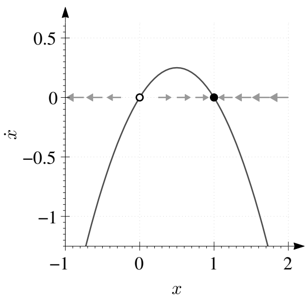


Fig. 2.9: Phase space of $\dot{x} = x(1-x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.1 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-1:0.05:2],0);

% Differentialgleichung
dx=-x.^2+x;

ylim_extra=[-1/3 1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
```

2.4.2 $\dot{x} = x(1-x)(2-x)$

The fixed points are $x_1^* = 0$, $x_2^* = 1$ and $x_3^* = 2$. The derivative is $\ddot{x} = -3x^2 - 6x + 2$. x_1^* is unstable ($\ddot{x}(x_1^*) = 2$), x_2^* is stable ($\ddot{x}(x_2^*) = -1$) and x_3^* is unstable ($\ddot{x}(x_3^*) = 2$) again.

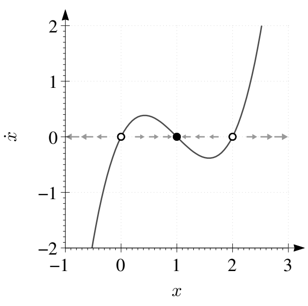
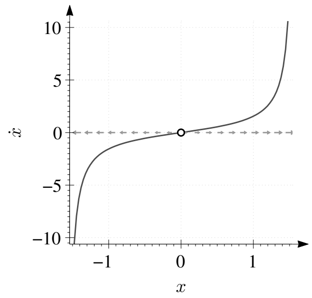


Fig. 2.10: Phase space of $\dot{x} = x(1-x)(2-x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.2 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-1:0.05:3],0);

% Differentialgleichung
dx=x.^3-3*x.^2+2*x;

ylim_extra=[-1/3 -1/3];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
```

2.4.3 $\dot{x} = \tan(x)$

In any interval $[(k-1)\frac{\pi}{2}, k\frac{\pi}{2})$ $\forall k \in \mathtt{N}$ is a fixed point $x^* = k\pi$. As the derivative $\ddot{x} = 1 + \tan(x)^2$ shows, all fixed points are unstable ($\ddot{x}(x^*) = 1 \quad \forall k \in \mathtt{N}$).

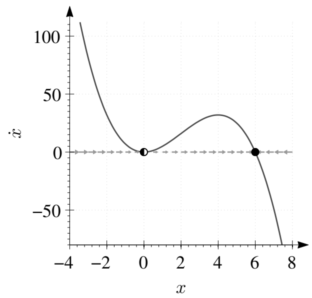



```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.3 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-0.49*pi:0.01*pi:0.49*pi],0);

% Differentialgleichung
dx=tan(x);

ylim_extra=[-1/3 -1/3];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
```

Fig. 2.11: Phase space of $\dot{x} = \tan(x)$.

2.4.4 $\dot{x} = x^2(6 - x)$

The fixed points are $x^*_{1,2} = 0$ and $x^*_3 = 6$. The derivative is $\ddot{x} = -3x^2 + 12x$. Thus, the third fixed point is stable ($\ddot{x}(x^*_3) = -36$) and the stability of $x^*_{1,2}$ cannot be determined by linear stability analysis. Graphical analysis reveals that $x^*_{1,2}$ is semistable.

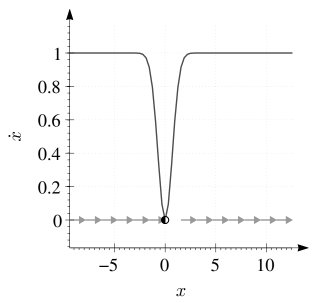



```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.4 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-4:0.1:8],0);

% Differentialgleichung
dx=-x.^3+6*x.^2;

ylim_extra=[-1/6 -1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
```

Fig. 2.12: Phase space of $\dot{x} = x^2(6 - x)$.

2.4.5 $\dot{x} = 1 - e^{-x^2}$

The only fixed point is $x^* = 0$ and the derivative is $\ddot{x} = 2xe^{-x^2}$. Again, the stability cannot be determinded using linear stability analysis. Graphical analysis can be used to classify the fixed point as semistable.

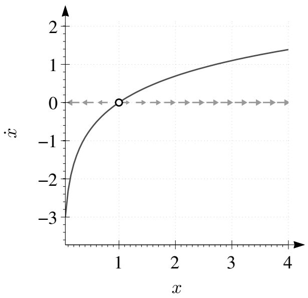



```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.5 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-3*pi:0.1*pi:4*pi],0);

% Differentialgleichung
dx=1-exp(-x.^2);

ylim_extra=[1/6 1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
```

Fig. 2.13: Phase space of $\dot{x} = 1 - e^{-x^2}$.

2.4.6 $\dot{x} = \ln(x)$

The only fixed point is $x^* = 1$. The derivative is $\ddot{x} = \frac{1}{x}$. So, the fixed point is unstable ($\ddot{x}(x^*) = 1$).




```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.6 -- %%');

% Gitter erzeugen
[x y]=meshgrid(0.05:0.05:4,0);

% Differentialgleichung
dx=log(x);

ylim_extra=[1/6 1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
```

Fig. 2.14: Phase space of $\dot{x} = \ln(x)$.

2.4.7 $\dot{x} = ax - x^3$ where a can be positive, negative, or zero. Discuss all three cases

The fixed points vary as the parameter a is varied. The derivative is $\ddot{x} = -3x^2 + a$.

a) $a < 0$: There is only one fixed point $x^* = 0$. This fixed point is stable $\ddot{x}(x^*) = a$.

b) $a = 0$: Again, there is only one fixed point $x^* = 0$ with a multiplicity of three. To determine its stability, linear stability analysis cannot be used.

c) $a > 0$: In this case, three fixed points exist ($x_1^* = -\sqrt{a}$, $x_2^* = 0$ and $x_3^* = \sqrt{a}$). x_2^* is unstable ($\ddot{x}(x_2^*) = a$) while the other ones are stable ($\ddot{x}(x_{1,3}^*) = -2a$).

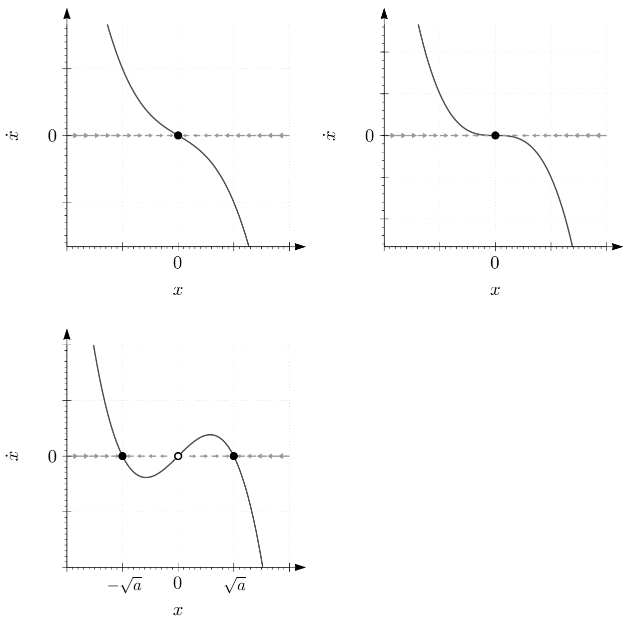


Fig. 2.15: Phase space of $\dot{x} = ax - x^3$. Upper left: $a < 0$, upper right: $a = 0$, lower left: $a > 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.4.7 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-2:0.05:2],0);

for a=-1:1
   % Differentialgleichung
   dx=a*x-x.^3;

   ylim_extra=[-1/3 -1/3];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],4, ...
                       ylim_extra);

   % Achsenbeschriftung anpassen
   ebenen=get(gcf,'Children');
   for i_change_xtick=1:3
      set(ebenen(i_change_xtick),'XTick',[-2 -1 0 1 2])
   end
   switch a
      case -1
         renameaxis(ebenen(2),'keep','keep',[], ...
                    {'';'';'0';'';''},{'';'0';''},26,0);
      case 0
         renameaxis(ebenen(2),'keep','keep',[], ...
               {'';'';'0';'';''},{'';'';'0';'';''},26,0);
      case 1
         renameaxis(ebenen(2),'keep','keep',[], ...
          {'';'$$-\sqrt{a}$$';'0';'$$\sqrt{a}$$';''}, ...
          {'';'';'0';'';''},26,0);
      otherwise
         disp('Warnung: Keine Achsenanpassung ', ...
              'implementiert');
   end
end
```

2.7 Potentials

For each of the following vector fields, plot the potential function $V(x)$ and identify all the equilibrium points and their stability

The potential can be calculated with $\dot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$.

2.7.1 $\dot{x} = x(1 - x)$

The potential of this function is $V(x) = \frac{x^3}{3} - \frac{x^2}{2} + C$. It can be seen, that the function has a local maximum at $V(x_\mathrm{u}^*) = 0$ (indicating an unstable fixed point) and a local minimum at $V(x_\mathrm{s}^*) = 1$ (stable fixed point).

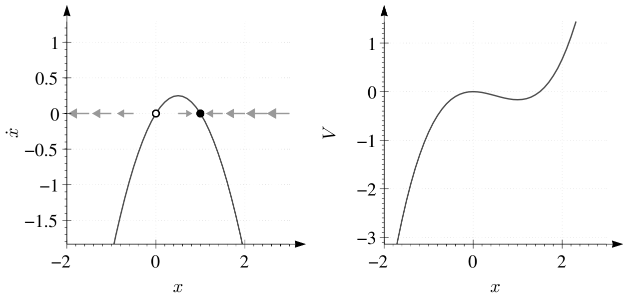


Fig. 2.16: Left: Phase space of $\dot{x} = x(1 - x)$, right: potential function $V(x) = \frac{x^3}{3} - \frac{x^2}{2} + C$ with $C = 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.7.1 -- %%');

% Feld anlegen
[x,y]=meshgrid(-2:0.1:3,0);

% Differentialgleichung
dx=x-x.^2;

% dazugehöriges Potential
V=x.^3/3-x.^2/2;

ylim_extra=[-2/3 1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                     ylim_extra);

ylim_extra=[-1/6 -1/3];

% Potential darstellen
customplot(x',V',[],ylim_extra,[],'$$x$$','$$V$$');
```

2.7.2 $\dot{x} = 3$

The potential of this function is $V(x) = -3x + C$. This function has no extremum (within finite values) and therefore no fixed points.

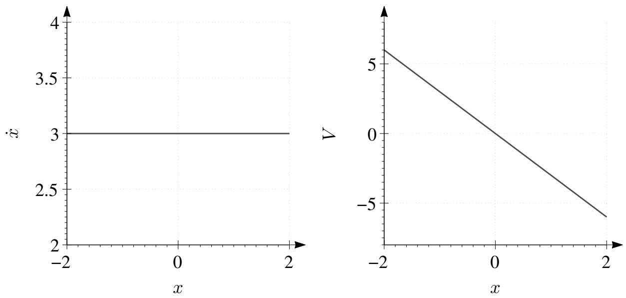


Fig. 2.17: Left: Phase space of $\dot{x} = 3$, right: potential function $V(x) = -3x + C$ with $C = 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.7.2 -- %%');

% Feld anlegen
[x,y]=meshgrid(-2:0.1:2,0);

% Differentialgleichung
dx=3*ones(size(x));

% dazugehöriges Potential
V=-3*x;

ylim_extra=[1/3 1/3];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                     ylim_extra);

ylim_extra=[1/6 1/6];

% Potential darstellen
customplot(x',V',[],ylim_extra,[],'$$x$$','$$V$$');
```

2.7.3 $\dot{x} = \sin(x)$

The potential of this function is $V(x) = \cos(x) + C$. The minima of $V(x)$ (stable fixed points) are $V(x_s^*) = (2k-1)\pi \quad \forall k \in \mathtt{N}$ and the maxima (unstable fixed points) are $V(x_u^*) = 2k\pi \quad \forall k \in \mathtt{N}$.

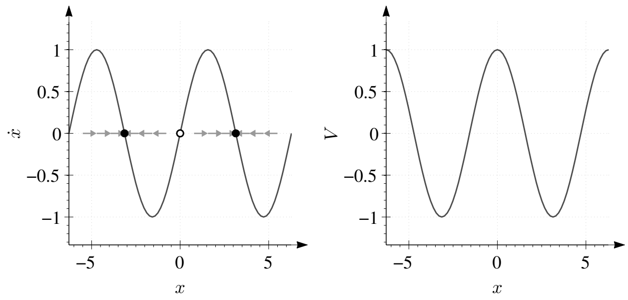


Fig. 2.18: Left: Phase space of $\dot{x} = \sin(x)$, right: potential function $V(x) = \cos(x) + C$ with $C = 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.7.3 -- %%');

% Feld anlegen
[x,y]=meshgrid(-2*pi:0.05*pi:2*pi,0);

% Differentialgleichung
dx=sin(x);

% dazugehöriges Potential
V=cos(x);

ylim_extra=[1/6 1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);

ylim_extra=[1/6 1/6];

% Potential darstellen
customplot(x',V',[],ylim_extra,[],'$$x$$','$$V$$');
```

2.7.4 $\dot{x} = 2 + \sin(x)$

The potential of this function is $V(x) = -2x + \cos(x) + C$. There are no minima/maxima in $V(x)$ and thus no fixed points.

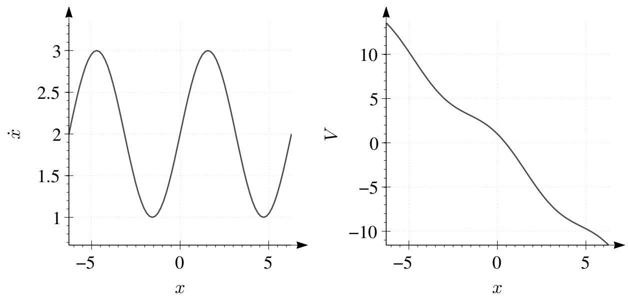


Fig. 2.19: Left: Phase space of $\dot{x} = 2 + \sin(x)$, right: potential function $V(x) = -2x + \cos(x) + C$ with $C = 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.7.4 -- %%');

% Feld anlegen
[x,y]=meshgrid(-2*pi:0.05*pi:2*pi,0);

% Differentialgleichung
dx=2+sin(x);

% dazugehöriges Potential
V=-2*x+cos(x);

ylim_extra=[1/6 1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);

ylim_extra=[0 0];

% Potential darstellen
customplot(x',V',[],ylim_extra,[],'$$x$$','$$V$$');
```

2.7.5 $\dot{x} = -\sinh(x)$

The potential of this function is $V(x) = \cosh(x) + C$. There is one global minimum at $V(x^*) = 0$ (stable fixed point).

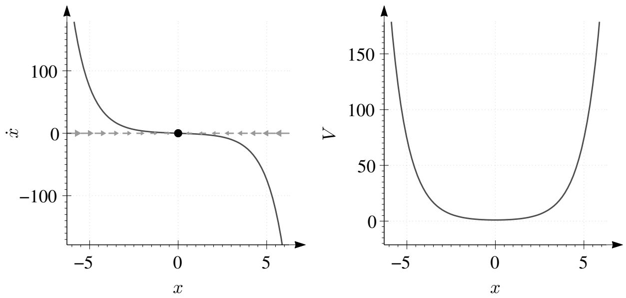


Fig. 2.20: Left: Phase space of $\dot{x} = -\sinh(x)$, right: potential function $V(x) = \cos(x) + C$ with $C = 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.7.5 -- %%');

% Feld anlegen
[x,y]=meshgrid(-2*pi:0.05*pi:2*pi,0);

% Differentialgleichung
dx=-sinh(x);

% dazugehöriges Potential
V=cosh(x);

ylim_extra=[-1/6 -1/6];

% Stabilitätsanalyse und Fixpunkte
stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);

ylim_extra=[1/12 -1/3];

% Potential darstellen
customplot(x',V',[],ylim_extra,[],'$$x$$','$$V$$');
```

2.7.6 $\dot{x} = r + x - x^3$

The potential of this function is $V(x) = \frac{x^4}{4} - \frac{x^2}{2} - rx + C$. For values of $|r| < \sqrt{\frac{4}{27}}$, there are three fixed points. The W–potential indicates the outer fixed points to be stable and the inner to be unstable. At $|r| = \sqrt{\frac{4}{27}}$ two fixed points annihilate each other and only a stable one remains.

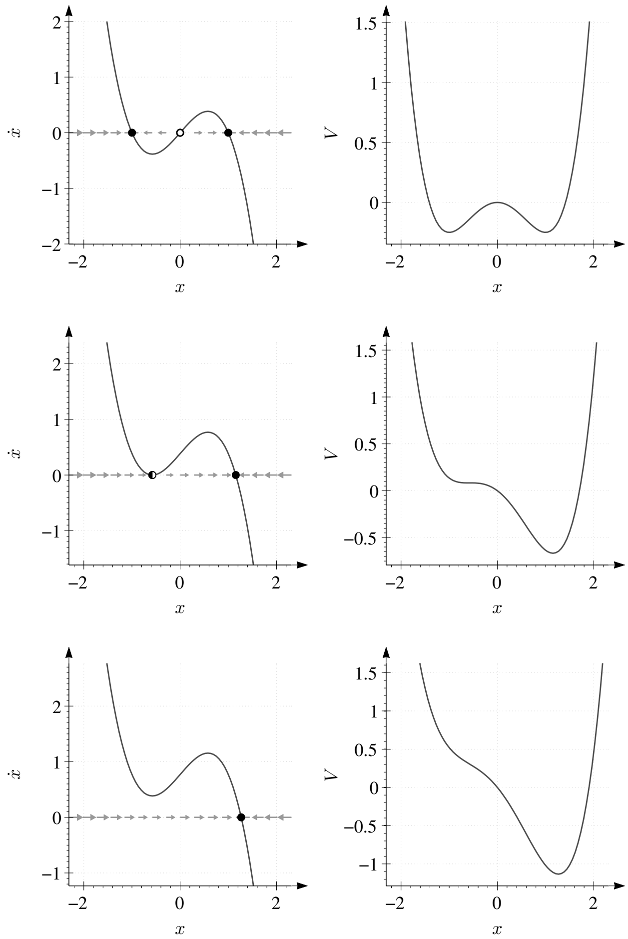


Fig. 2.21: Left column: Phase space of $\dot{x} = r + x - x^3$, right column: potential function $V(x) = \frac{x^4}{4} - \frac{x^2}{2} - rx + C$ with $C = 0$. From top to bottom row: $r = 0$, $r = \sqrt{\frac{4}{27}}$, $r = 2\sqrt{\frac{4}{27}}$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 2.7.6 -- %%');

% Feld anlegen
[x,y]=meshgrid(-4/sqrt(3):0.1/sqrt(3):4/sqrt(3),0);

% Variation des Parameters
for r=0:sqrt(4/27):2*sqrt(4/27)

   % Differentialgleichung
   dx=r+x-x.^3;

   % dazugehöriges Potential
   V=0.25*x.^4-0.5*x.^2-r*x;

   ylim_extra=[-2/5 -2/5];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                     ylim_extra);

   ylim_extra=[1/48 -5/8];

   % Potential darstellen
   customplot(x',V',[],ylim_extra,[],'$$x$$','$$V$$');
end
```

Exercises for Chapter 3

3.1 Saddle–Node Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle–node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

3.1.1 $\dot{x} = 1 + rx + x^2$

A stable and an unstable fixed point exist as $|r| \geq 2$. To see this, set $\dot{x} = 1 + rx + x^2 = 0$ to analyse the curve of the fixed points. Rearranging the terms yields $x_{1,2} = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - 1}$. The argument of the square root has to be nonnegative which is fulfilled for $|r| \geq 2$. Finally, both functions describe the curve of the bifurcation diagram. The curves approach $f_1 = -r$ and $f_2 = 0$ as $|r| \to \infty$.

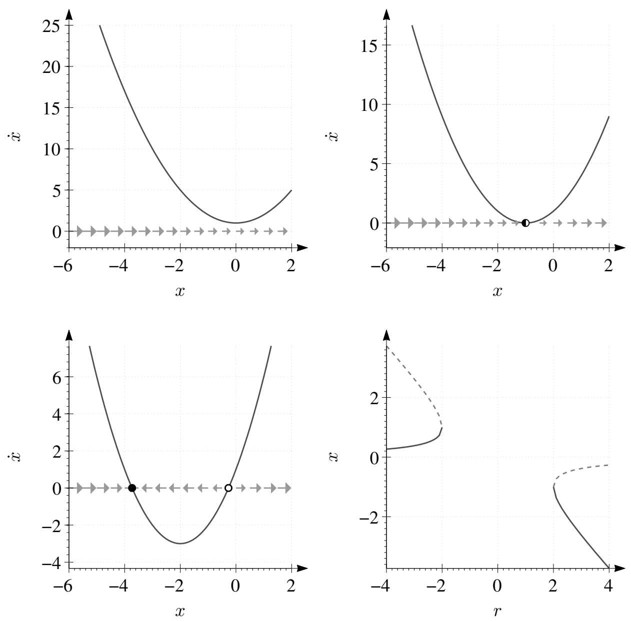


Fig. 3.1: All except bottom right: Phase space of $\dot{x} = 1 + rx + x^2$, top left: $r = 0$, top right: $r = 2$, bottom left: $r = 4$, bottom right: bifurcation diagram of $\dot{x} = 1 + rx + x^2$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.1.1 -- %%');

% Feld anlegen
[x,y]=meshgrid(-6:0.1:2,0);

% Variation des Bifurkationsparameters
for r=0:2:4

   % Differentialgleichung
   dx=1+r*x+x.^2;

   ylim_extra=[1/12 -1/3];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                    ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[-4:0.1:4];
x_1=-r_bf/2-sqrt(r_bf.^2/4-1);
x_2=-r_bf/2+sqrt(r_bf.^2/4-1);

% Bifurkationspunkte finden
ch1=max(find(r_bf<=-2));
ch2=min(find(r_bf>=2));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch2:end]) r_bf([1:ch1 ch2:end])]', ...
   [x_1([1:ch1 ch2:end]) x_2([1:ch1 ch2:end])]', ...
   [min(r_bf) max(r_bf)],[], ...
   [size(x_1(1:ch1),2) size(x_1(ch2:end),2) ...
   size(x_1(1:ch1),2) size(x_1(ch2:end),2);0 0 1 1], ...
   '$$r$$','$$x$$');
```

3.1.2 $\dot{x} = r - \cosh(x)$

A stable and an unstable fixed point exist as $r \geq 1$. We set $\dot{x} = r - \cosh(x) = 0$ to analyse the curve of the fixed points. Rearranging the terms yields $x_{1,2} = \pm \operatorname{arcosh}(r)$. While $\cosh(x)$ can never get smaller than 1, $\operatorname{arcosh}(r)$ must have an argument $r \geq 1$. $x_2 = -\operatorname{arcosh}(r)$ is the unstable fixed point.

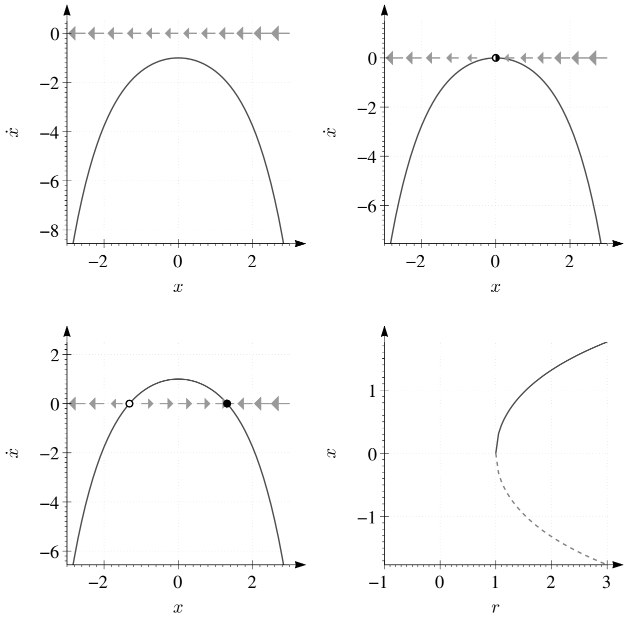


Fig. 3.2: All except bottom right: Phase space of $\dot{x} = r - \cosh(x)$, top left: $r = 0$, top right: $r = 1$, bottom left: $r = 2$, bottom right: bifurcation diagram of $\dot{x} = r - \cosh(x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.1.2 -- %%');

% Feld anlegen
[x,y]=meshgrid(-3:0.1:3,0);

% Variation des Bifurkationsparameters
for r=0:2

   % Differentialgleichung
   dx=r-cosh(x);

   ylim_extra=[-1/6 1/6];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[-1:0.05:3];
x_1=acosh(r_bf);
x_2=-acosh(r_bf);

% Bifurkationspunkte finden
ch1=min(find(r_bf>=1));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf(ch1:end) r_bf(ch1:end)]', ...
   [x_1([ch1:end]) x_2(ch1:end)]', ...
   [min(r_bf) max(r_bf)], ...
   [],[size(x_1(ch1:end),2) size(x_1(ch1:end),2); ...
   0  1],'$$r$$','$$x$$');
```

3.1.3 $\dot{x} = r + x - \ln(1 + x)$

While solving $\dot{x} = 0$ for x is problematic, solving to r results in $r = \ln(1 + x) + x$. $\ln(1 + x)$ has to have values $x \geq -1$. If $x \to -1$ or $x \to \infty$, $r \to -\infty$. So, there are no fixed points for $r > 0$. For $r < 0$, the fixed point approaching $x = -1$ is stable, the other one unstable.

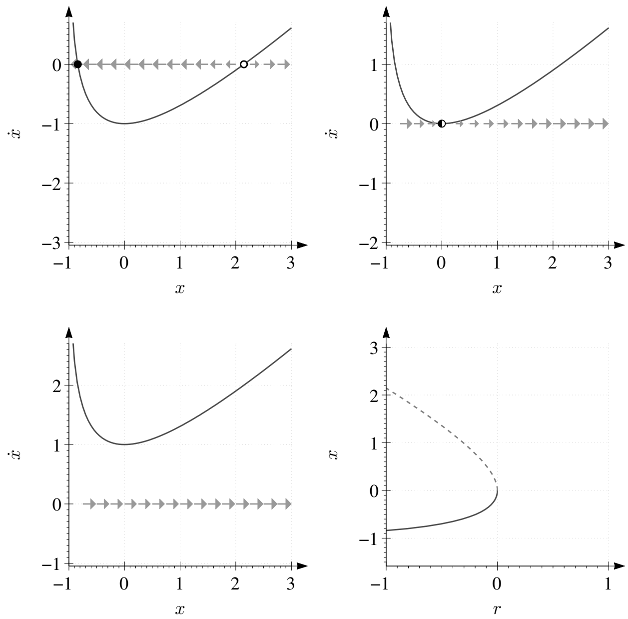


Fig. 3.3: All except bottom right: Phase space of $\dot{x} = r + x - \ln(1 + x)$, top left: $r = -1$, top right: $r = 0$, bottom left: $r = 1$, bottom right: bifurcation diagram of $\dot{x} = r + x - \ln(1 + x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.1.3 -- %%');

% Feld anlegen
[x,y]=meshgrid(-1:0.05:3,0);

% Variation des Bifurkationsparameters
for r=-1:1

   % Differentialgleichung
   dx=r+x-log(1+x);

   ylim_extra=[1 -1/6];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
x_1=[-1:0.05:2.5];
r_bf=log(1+x_1)-x_1;

% Bifurkationspunkte finden
ch1=max(find(x_1<=0));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf(1:ch1) r_bf(ch1:end)]', ...
   [x_1([1:ch1]) x_1([ch1:end])]',[-1 1],...
   [1/6 1/6],[size(x_1(1:ch1),2) size(x_1(ch1:end),2);...
   0 1],'$$r$$','$$x$$');
```

3.1.4 $\dot{x} = r + \frac{1}{2}x - \frac{x}{(1+x)}$

Solving for x yields $x_{1,2} = \frac{1}{2} - r \pm \sqrt{r^2 - 3r + \frac{1}{4}}$. As the root is nonnegative for $|r - \frac{3}{2}| > \sqrt{2}$, the two fixed points cease to exist within this interval. Due to the type of function (asymptotic behaviour for $x \to \pm\infty$), the fixed point farther away from -1 is always the unstable fixed point.

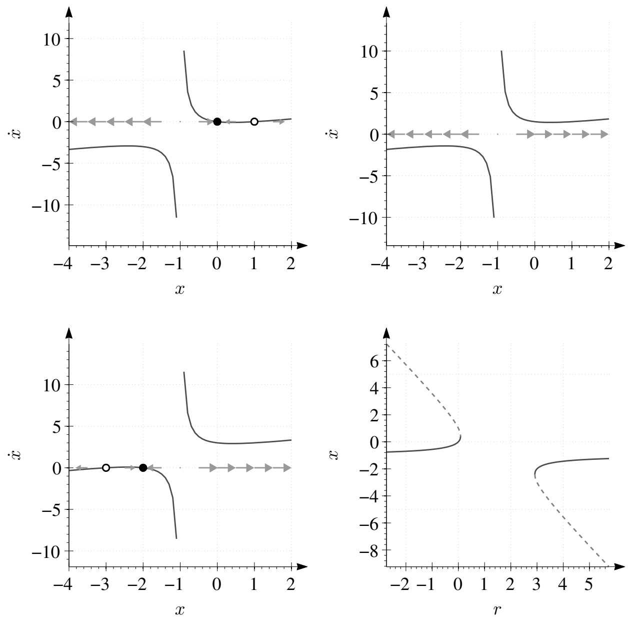


Fig. 3.4: All except bottom right: Phase space of $\dot{x} = r + \frac{1}{2}x - \frac{x}{(1+x)}$, top left: $r = 0$, top right: $r = 1$, bottom left: $r = 2$, bottom right: bifurcation diagram of $\dot{x} = r + \frac{1}{2}x - \frac{x}{(1+x)}$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.1.4 -- %%');

% Feld anlegen
[x,y]=meshgrid(-4:0.1:2,0);

% Variation des Bifurkationsparameters
for r=0:1.5:3

   % Differentialgleichung
   dx=r+0.5*x-x./(1+x);

   ylim_extra=[1/6 1/6];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
   % Übergang von -Inf +Inf ist kein Fixpunkt-> Korrektur
   ebenen=get(gcf,'Children');
   punkt=get(ebenen(3),'Children');
   if (r < 1.5)
      delete(punkt(3))
   else
      delete(punkt(1))
   end
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[1.5-3*sqrt(2):0.01*sqrt(2):1.5+3*sqrt(2)];
x_1=0.5-r_bf+sqrt(r_bf.^2-3*r_bf+0.25);
x_2=0.5-r_bf-sqrt(r_bf.^2-3*r_bf+0.25);

% Bifurkationspunkte finden
ch1=max(find(r_bf<=1.5-sqrt(2)));
ch2=min(find(r_bf>=1.5+sqrt(2)));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch2:end]) r_bf([1:ch1 ch2:end])]', ...
   [x_1([1:ch1 ch2:end]) x_2([1:ch1 ch2:end])]',[],...
   [],[size(x_1(1:ch1),2) size(x_1(ch2:end),2) ...
   size(x_1(1:ch1),2) size(x_1(ch2:end),2); ...
   1 0 0 1],'$$r$$','$$x$$');
```

3.1.5 (Unusual bifurcations)

In discussing the normal form of the saddle–node bifurcation, we mentioned the assumption that $a = \frac{\partial f}{\partial r}\Big|_{(x^*,r_c)} \neq 0$. To see what can happen if $\frac{\partial f}{\partial r}\Big|_{(x^*,r_c)} = 0$, sketch the vector fields for the follwing examples, and then plot the fixed points as a function of r.

a) $\dot{x} = r^2 - x^2$:
There is one stable and one unstable fixed point. Rearranging the terms gives $x_{1,2} = \pm|r|$. So, $x_1 = -r$ is stable for $r < 0$, and for $r > 0$ unstable. Accordingly, $x_2 = r$ is unstable for $r < 0$ and stable otherwise.

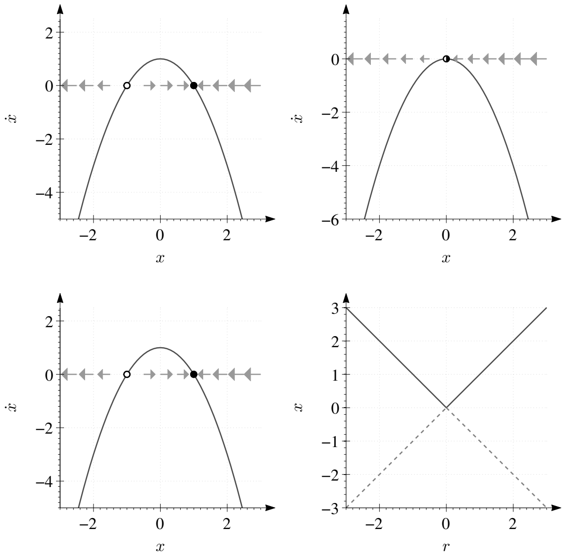


Fig. 3.5: All except bottom right: Phase space of $\dot{x} = r^2 - x^2$, top left: $r = -1$, top right: $r = 0$, bottom left: $r = 1$, bottom right: bifurcation diagram of $\dot{x} = r^2 - x^2$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.1.5a -- %%');

% Feld anlegen
[x,y]=meshgrid(-3:0.1:3,0);

% Variation des Bifurkationsparameters
for r=-1:1

   % Differentialgleichung
   dx=r^2-x.^2;

   ylim_extra=[-1/3 1/6];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[-3:0.1:3];
x_1=sqrt(r_bf.^2);
x_2=-sqrt(r_bf.^2);

% Bifurkationspunkte finden
ch1=max(find(r_bf<=0));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch1:end]) r_bf([1:ch1 ch1:end])]', ...
   [x_1([1:ch1 ch1:end]) x_2([1:ch1 ch1:end])]',[],...
   [],[size(x_1(1:ch1),2) size(x_1(ch1:end),2) ...
   size(x_1(1:ch1),2) size(x_1(ch1:end),2); ...
   0 0 1 1],'$$r$$','$$x$$');
```

b) $\dot{\boldsymbol{x}} = \boldsymbol{r^2} + \boldsymbol{x^2}$:

There is only one halfstable fixed point at $x = 0$ for $r = 0$. Rearranging the terms gives $x_{1,2} = \pm i|r|$, where all terms are purely imaginary except for $r = 0$.

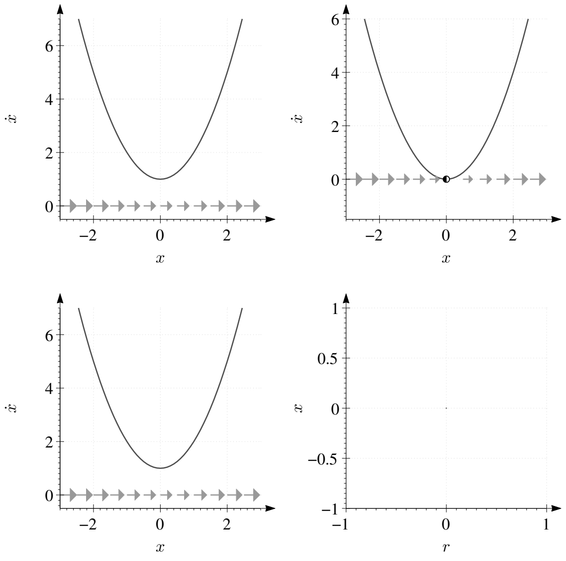


Fig. 3.6: All except bottom right: Phase space of $\dot{x} = r^2 + x^2$, top left: $r = -1$, top right: $r = 0$, bottom left: $r = 1$, bottom right: bifurcation diagram of $\dot{x} = r^2 + x^2$ with only one halfstable fixed point at $x = 0$, $r = 0$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.1.5b -- %%');

% Feld anlegen
[x,y]=meshgrid(-3:0.1:3,0);

% Variation des Bifurkationsparameters
for r=-1:1

   % Differentialgleichung
   dx=r^2+x.^2;

   ylim_extra=[1/6 -1/3];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[-3:0.1:3];
x_1=sqrt(-r_bf.^2);
x_2=-sqrt(-r_bf.^2);

% Bifurkationspunkte finden
ch1=max(find(r_bf==0));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf(ch1) r_bf(ch1)]', [x_1(ch1) x_2(ch1)]',[],...
   [],[size(x_1(ch1),2) size(x_1(ch1),2); ...
   0 1],'$$r$$','$$x$$');
```

3.2 Transcritical Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

3.2.1 $\dot{x} = rx + x^2$

There are two fixed points, described by $x_1 = 0$ and $x_2 = -r$. While $r < 0$, x_1 represents the stable fixed point and x_2 the unstable one. At $r = 0$ they change stability.

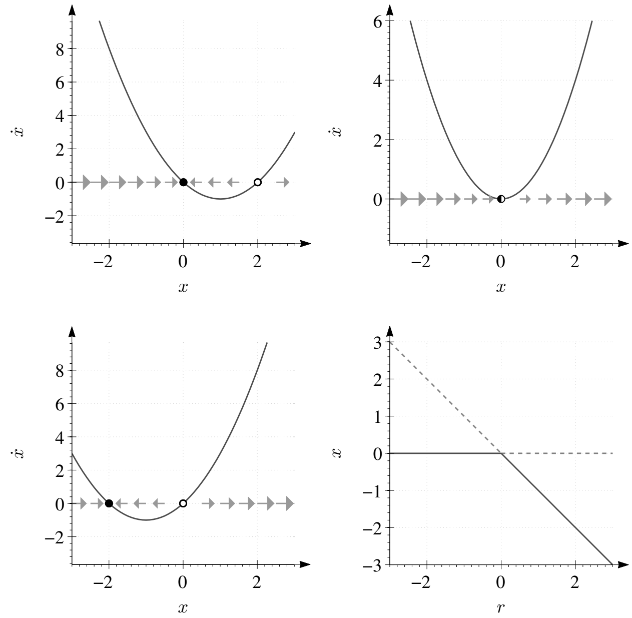


Fig. 3.7: All except bottom right: Phase space of $\dot{x} = rx + x^2$, top left: $r = -2$, top right: $r = 0$, bottom left: $r = 2$, bottom right: bifurcation diagram of $\dot{x} = rx + x^2$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.2.1 -- %%');

% Feld anlegen
[x,y]=meshgrid(-3:0.1:3,0);

% Variation des Bifurkationsparameters
for r=-2:2:2

   % Differentialgleichung
   dx=r*x+x.^2;

   ylim_extra=[1/6 -1/3];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[-3:0.1:3];
x_1=zeros(1,size(r_bf,2));
x_2=-r_bf;

% Bifurkationspunkte finden
ch1=min(find(r_bf>=0));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch1:end]) r_bf([1:ch1 ch1:end])]', ...
   [x_1([1:ch1 ch1:end]) x_2([1:ch1 ch1:end])]',[], ...
   [],[size(x_1(1:ch1),2) size(x_1(ch1:end),2) ...
   size(x_1(1:ch1),2) size(x_1(ch1:end),2); ...
   0 1 1 0],'$$r$$','$$x$$');
```

3.2.2 $\dot{x} = rx - \ln(1 + x)$

Here, one fixed point moves along $x_1 = 0$. It is stable while $r < 1$. At $r = 1$ a second fixed point appears at $x = \infty$ changes its stability from unstable to stable at $r = 1$. Here, $\dot{x}$ cannot be transformed to $x = f(r)$, so $r = \frac{\ln(1+x)}{x}$ is used to describe the behaviour. The stable fixed points approaches $x = -1$ as $r \to \infty$.

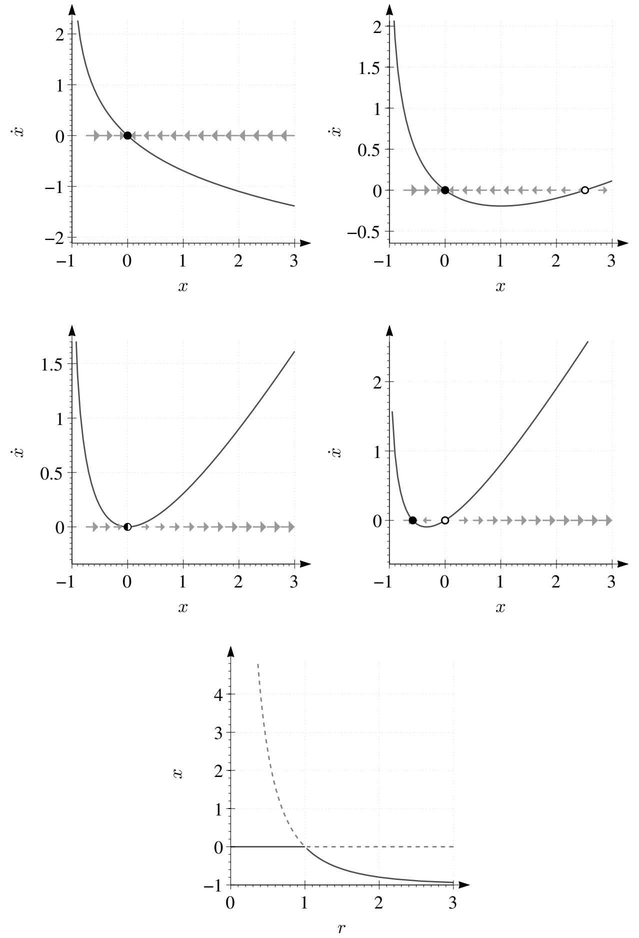
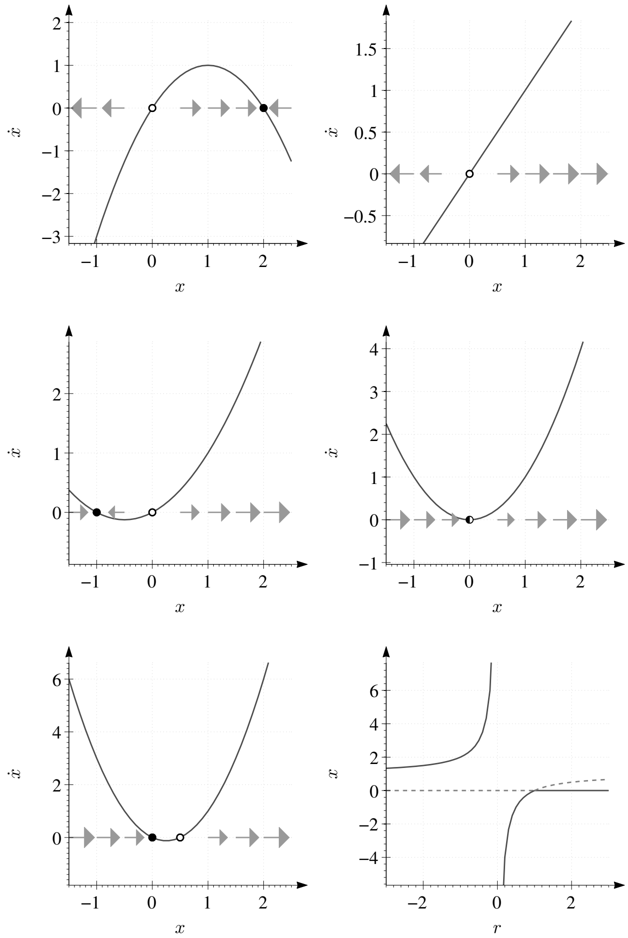


Fig. 3.8: All except bottom center: Phase space of $\dot{x} = rx - \ln(1 + x)$, top left: $r = 0$, top right: $r = 0.5$, middle left: $r = 1$, middle right: $r = 1.5$, bottom center: bifurcation diagram of $\dot{x} = rx - \ln(1 + x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.2.2 -- %%');

% Feld anlegen
[x,y]=meshgrid(-1:0.05:3,0);

% Variation des Bifurkationsparameters
for r=0:0.5:1.5

   % Differentialgleichung
   dx=r*x-log(1+x);

   ylim_extra=[1/6 -1/6];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
x_1=[-1:0.05:6];
x_2=zeros(1,size(x_1,2));
r_bf=log(1+x_1)./x_1;

% Bifurkationspunkte finden
ch1=max(find(x_1<0));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch1:end]) r_bf([1:ch1]) -1 1]', ...
   [x_1([1:ch1 ch1:end]) x_2([1:ch1 ch1 end])]',[0 3],...
   [0 -1/6],[size(x_1(1:ch1),2) size(x_1(ch1:end),2) ...
   size(x_1(1:ch1),2) size(x_1([ch1 end]),2); ...
   0 1 1 0],'$$r$$','$$x$$');
```

3.2.3 $\dot{x} = x - rx(1-x)$

Two fixed points exist and interchange stability at $r = 1$. Therefore, $x_1 = 0$ is stable for $x < 1$ and $x_2 = \frac{r-1}{r}$ for $x > 0$. As can be seen, $x_2 \to \infty$ as $r \to 0$ and x_2 comes from $-\infty$ for $r > 0$, which yields in a different appearance of the fixed points around zero. As $|r| \to \infty$, $x_1 = 0$ and $x_2 \to 1$.




```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.2.3 -- %%');

% Feld anlegen
[x,y]=meshgrid(-1.5:0.1:2.5,0);

% Variation des Bifurkationsparameters
for r=[-1 0 0.5 1 2]

   % Differentialgleichung
   dx=x-r*x+r*x.^2;

   if (r>0)
      ylim_extra=[1/6 -1/3];
   else
      if (r<0)
         ylim_extra=[-1/3 1/6];
      else
         ylim_extra=[-1/6 -1/6];
      end
   end

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                        ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[-3:0.1:3];
x_1=zeros(1,size(r_bf,2));
x_2=(r_bf-1)./r_bf;

% Bifurkationspunkte finden
ch1=min(find(r_bf>=1));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch1:end]) r_bf([1:ch1 ch1:end])]', ...
   [x_1([1:ch1 ch1:end]) x_2([1:ch1 ch1:end])]',[], ...
  [-1/6 -1/6],[size(x_1(1:ch1),2) size(x_1(ch1:end),2)...
   size(x_1(1:ch1),2) size(x_1(ch1:end),2); ...
   1 0 0 1],'$$r$$','$$x$$');
```

Fig. 3.9: All except bottom right: Phase space of $\dot{x} = x - rx(1-x)$, top left: $r = -1$, top right: $r = 0$, middle left: $r = 0.5$, middle right: $r = 1$, bottom left: $r = 1.5$, bottom right: bifurcation diagram of $\dot{x} = x - rx(1-x)$.

3.2.4 $\dot{x} = x(r - e^x)$

Two fixed points exist and interchange stability at $r = 1$. As long as $r < 0$, there is only one (stable) fixed point at $x = 0$. For $r > 0$ another fixed point emerges and merges with the stable fixed point at $r = 1$ to change its stability. So, $x_1 = 0$ is stable for $x < 1$ and $x_2 = \ln(x)$ for $x > 1$.

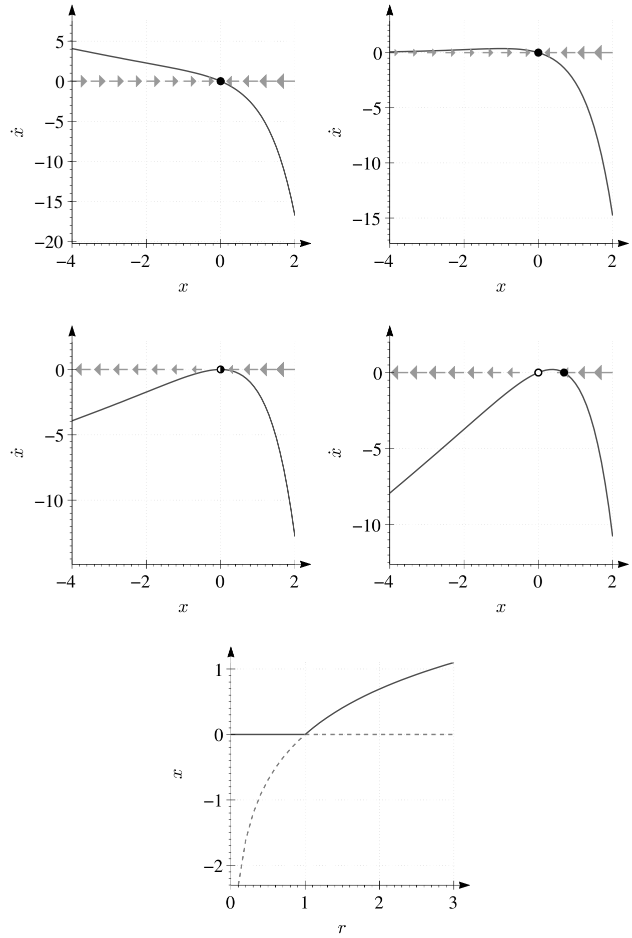


Fig. 3.10: All except bottom center: Phase space of $\dot{x} = x(r - e^x)$, top left: $r = -1$, top right: $r = 0$, middle left: $r = 1$, middle right: $r = 2$, bottom center: bifurcation diagram of $\dot{x} = x(r - e^x)$.

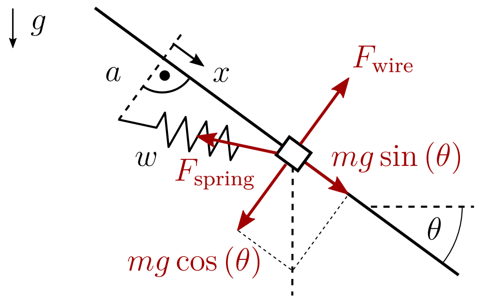
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.2.4 -- %%');

% Feld anlegen
[x,y]=meshgrid(-4:0.1:2,0);

% Variation des Bifurkationsparameters
for r=-1:2

   % Differentialgleichung
   dx=x.*(r-exp(x));

   ylim_extra=[1/6 1/6];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],5, ...
                       ylim_extra);
end

% Parametervariation und dazugehörige Fixpunktgleichungen
r_bf=[0:0.1:3];
x_1=zeros(1,size(r_bf,2));
x_2=log(r_bf);

% Bifurkationspunkte finden
ch1=min(find(r_bf>=1));

% Plotte das Bifurkationsdiagramm
customplot(...
   [r_bf([1:ch1 ch1:end]) r_bf([1:ch1 ch1:end])]', ...
   [x_1([1:ch1 ch1:end]) x_2([1:ch1 ch1:end])]',[], ...
   [],[size(x_1(1:ch1),2) size(x_1(ch1:end),2) ...
   size(x_1(1:ch1),2) size(x_1(ch1:end),2); ...
   0 1 1 0],'$$r$$','$$x$$');
```

3.6 Imperfect Bifurcations and Catastrophes

3.6.5 Mechanical example of imperfect bifurcation and catastrophe

Consider the bead on a tilted wire discussed at the end of section 3.6.

a) **Show that the equilibrium positions of the bead satisfy** $\boldsymbol{mg\sin(\theta) = kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)}$.



At an equilibrium position, the sum of all forces acting on the bead must be zero. Although we don't know the normal force of the wire F_{wire}, we can restrict ourselves to forces in the direction of the wire. While the gravitational force is simply $mg\sin(\theta)$, the spring force requires some more calculation.

The spring force (relaxed length of spring L_0, coefficient k) is linearly dependent on the length of the spring. Thus, $F_{\text{spring}} = k(w - L_0)$. The force projected on the direction of the wire is $F_{\text{spring,proj}} = \frac{x}{w}k(w - L_0)$. Replacing $w = \sqrt{x^2 + a^2}$ yields $F_{\text{spring,proj}} = kx\left(1 - \frac{L_0}{\sqrt{x^2+a^2}}\right)$ which is equal to $mg\sin(\theta)$.

b) **Show that the equilibrium equation can be written in dimensionless form as** $\boldsymbol{1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}}$ **for appropriate choices of** $\boldsymbol{R}$, $\boldsymbol{h}$ **and** $\boldsymbol{u}$.

The variable in the dimensionless form is u ($u \sim x$). Therefore, we need one term without x (which must be made 1), a term with $\frac{1}{x}$ and a term similiar to $\frac{1}{\sqrt{x^2}}$. Dividing by kx and rearranging yields $1 - \frac{mg\sin(\theta)}{kx} = \frac{L_0}{\sqrt{x^2+a^2}}$. Now we have to modify the argument of the square root to get the dependence from u to x and we are done.

In short, choosing $u = \frac{x}{a}$, $R = \frac{L_0}{a}$ and $h = \frac{mg\sin(\theta)}{ak}$ yields $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$.

c) **Give a graphical analysis of the dimensionless equation for the cases $\boldsymbol{R < 1}$ and $\boldsymbol{R > 1}$. How many equilibria can exist in each case?**

The curve is approaching one as $u \to \infty$. There are no oscillations but an overshoot on either side of the vertical axis is possible. If $R < 1$, there is exactly one (unstable) fixed point, which is in the right half plane (close to zero). The location is determinded by the size of R and h (and close to 2 for $R = 1,\ h = 1$). For $R > 1$ the situation is more involved. While the unstable fixed point is moving to infinity, depending on h, two more fixed points can exist (the location is depending on the size of both parameters, again). A numerical investigation for some values is shown below. All parameter values (points) above the R–h curve result in three fixed points, all below in one and values on the curve in two.

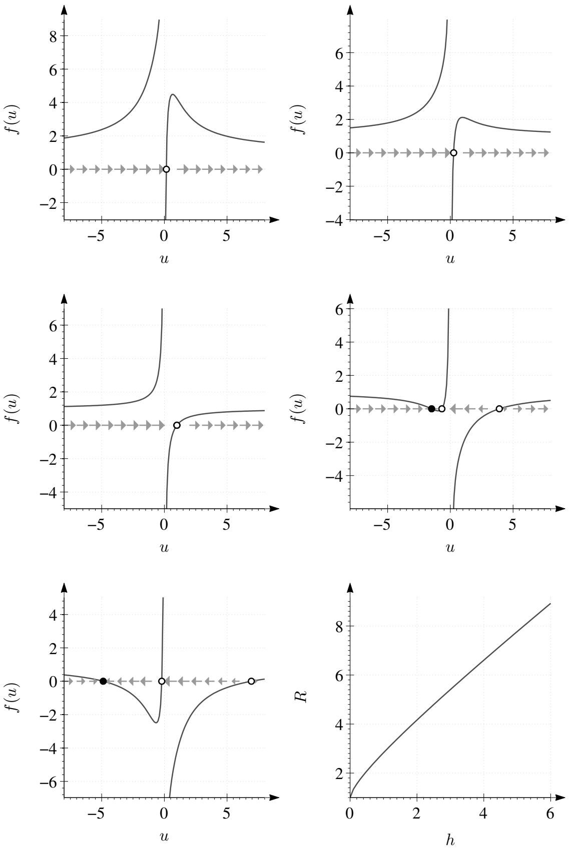


Fig. 3.11: All except bottom right: Plot of $1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$ with $h = 1$, top left: $R = -6$, top right: $R = -3$, middle left: $R = 0$, middle right: $R = 3$, bottom left: $R = 6$, bottom right: Dependence of h and R to have one (below curve), two (on curve) or three (above curve) fixed points.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 3.6.5 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-8:0.1:8],0);

% Parameter festlegen
h=1;

for R=-6:3:6
   % Differentialgleichung
   dx=1-h./x-R./sqrt(1+x.^2);

   ylim_extra=[-1/5-R/30 -1/5+R/30];

   % Stabilitätsanalyse und Fixpunkte
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],10,...
                    ylim_extra);

   % Nachbearbeitung (kein Fixpunkt bei Null)
   bild=get(gcf,'Children');
   elemente=get(bild(3),'Children');
   if (R <0)
      delete(elemente(2));
   else
      delete(elemente(3));
   end
   renameaxis(bild(2),'$$u$$','$$f(u)$$');
end

% Bifurkationspunkt linke Seite für R>1 verfolgen
u=-12:0.01:-0;
fu=inline('1-param./u-((left+right)/2)./sqrt(1+u.^2)',...
          'u','param','left','right');

% Nullstellen der Funktion finden
[sf hR]=funczeros(u,fu,10e-9,0:0.1:6,[0 10]);

% Grafische Ausgabe
customplot(hR(:,1),hR(:,2),[],[0 1/32],[size(hR,1);0],...
           '$$h$$','$$R$$');
```

d) **Let $\boldsymbol{r = R - 1}$. Show that the equilibrium equation reduces to $\boldsymbol{h + ru - \frac{1}{2}u^3 \approx 0}$ for small $\boldsymbol{r,\ h}$ and $\boldsymbol{u}$.**

Using the approximation $\sqrt{1+u^2} \approx 1 + \frac{1}{2}u^2$ for small values of u, we obtain

$$
\begin{aligned}
& 1 - \frac{h}{u} = \frac{r+1}{1+\frac{1}{2}u^2} \\
\Leftrightarrow \quad & (u-h)\left(1 + \frac{1}{2}u^2\right) = ur + u \\
\Leftrightarrow \quad & h + \frac{1}{2}u^2 h + ru - \frac{1}{2}u^3 = 0
\end{aligned}
$$

Ignoring $\frac{1}{2}u^2 h$, we have reduced the equilibrium equation to $h + ru - \frac{1}{2}u^3 \approx 0$.

e) **Find an approximate formula for the saddle–node bifurcation curves in the limit of small $\boldsymbol{r,\ h}$ and $\boldsymbol{u}$.**

The saddle–node bifurcation occur at the local minimum/maximum of our equation $h + ru - \frac{1}{2}u^3 \approx 0$. We get the value where the bifurcation occurs with the help of the derivative

$$
\begin{aligned}
& \frac{\mathrm{d}}{\mathrm{d}u}\left(h + ru - \frac{1}{2}u^3\right) = r - \frac{3}{2}u^2 \stackrel{!}{=} 0 \quad \Leftrightarrow \quad u_{\text{min/max}} = \pm\sqrt{\frac{2}{3}r} \\
& h(u_{\text{min/max}}) = \pm\sqrt{\frac{8}{27}r^3}
\end{aligned}
$$

The approximate formula for the saddle–node bifurcation curves in the parameter space $h,\ r$ is $h_c(r) = \pm\sqrt{\frac{8}{27}r^3}$. Values $|h| < |h_c|$ result in three fixed points, $|h| > |h_c|$ in one.

h) **Interpret your results physically, in terms of the original dimensional variables.**

r can be seen as the length of the spring relative to its relaxed length. A small value means small relative excitation. h is the ratio between the force of the bead along the wire and the spring force perpendicular to the wire. Here, a small value indicates, that the perpendiculat spring force has to be much higher than the force of the bead along the wire. This can also be achieved by having a very small tilt angle.

As the last part suggested, changing the h less than h_c results in one stable equilibrium point. Otherwise the bead will have two stable equilibria (and an unstable one) on the wire.

Exercises for Chapter 4

4.4 Overdamped Pendulum

4.4.4 Torsional spring

Suppose that our overdamped pendulum is connected to a torsional spring. As the pendulum rotates, the spring winds up and generates an opposing torque $-k\theta$. Then the equation of motion becomes $b\dot{\theta} + mgL\sin(\theta) = \Gamma - k\theta$.

a) Does this equation give a well–defined vector field on the circle?

No, because $\dot{\theta}(\theta)$ and $\dot{\theta}(\theta + 2\pi)$ have to be the same (periodicity), which can easily be falsified by the existence of the term $k\theta$. Thus, the angular velocity is not uniquely defined and not on a circle.

b) Nondimensionalize the equation

Dividing by mgL and substituting $\tau = \frac{mgL}{b}t$ with the definition $\theta' = \frac{\mathrm{d}\theta}{\mathrm{d}\tau}$ yields

$$\theta' = \zeta - \xi\theta - \sin(\theta),$$

whereas $\zeta = \frac{\Gamma}{mgL}$ and $\xi = \frac{k}{mgL}$.

c) What does the pendulum do in the long run?

A natural assumption is $mgL > 0$, and for the opposing force $k \geq 0$. If we allow $k = 0$ and therefore $\xi = 0$, there are no fixed points possible for $|\zeta| > 1$ and infinitely for $|\zeta| \leq 1$. On the other hand, if $\xi \geq 1$, we have exactly one (stable) fixed point where the system is driven to. The most interesting intervall is $0 < \xi < 1$, where at least one stable fixed point and up to n stable and $n - 1$ unstable fixed points may exist (n can be arbitrarily large).

To sum it up, if $\xi > 0$, the pendulum will eventually approach a stable fixed point and therefore come to rest.

d) Show that many bifurcations occur as k is varied from 0 to ∞. What kind of bifurcations are they?

If $k = 0$ ($\xi = 0$), the overdamped pendulum would actually describe a well–defined vector field on the circle. There are two fixed points on the circle (infinitely on the line) if $|\zeta| \leq 1$, otherwise none. As described in the previous part, as ξ is greater than one, only one (stable) fixed point exists. As ξ resp. $k \to 0$, more fixed points emerge (always in pairs: a stable and an unstable one). This spontaneous emerging of two fixed points is typical for saddle–node bifurcations and can be seen in the plots below.

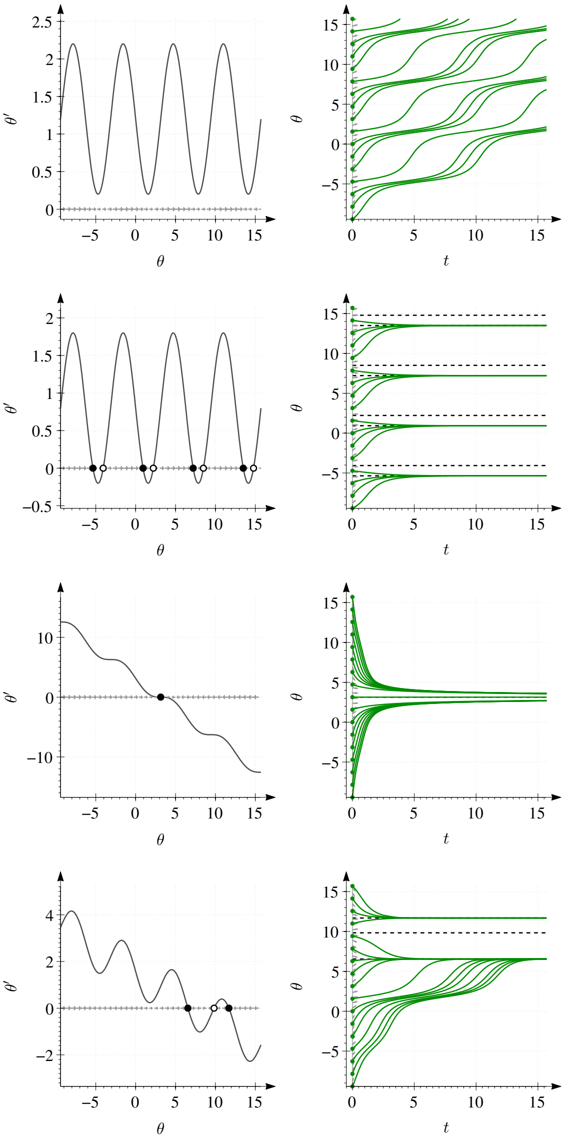


Fig. 4.1: Left side: Phase portrait of $\theta' = \zeta - \xi\theta - \sin(\theta)$, right side: time dependence, first row: $\zeta = 1.2, \quad \xi = 0$, second row: $\zeta = 0.8, \quad \xi = 0$, third row: $\zeta = \pi, \quad \xi = 1$, fourth row: $\zeta = \frac{\pi}{2}, \quad \xi = 0.2$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 4.4.4 -- %%');

% Gitter erzeugen
[x y]=meshgrid([-3*pi:pi/48:5*pi],0);

zeta_vec=[1.2 0.8 pi pi/2];
xi_vec=[0 0 1 0.2];

for index=1:4
   zeta=zeta_vec(index);
   xi=xi_vec(index);

   % Differentialgleichung
   dx=zeta-xi*x-sin(x);
   dgl_sys=@(t,x)[zeta-xi*x-sin(x)];

   ylim_extra=[1/6 1/6];

   % Stabilitätsanalyse, Fixpunkte und zeitlicher Verlauf
   stabilitaetsanalyse(x,y,dx,zeros(size(y)),[],[],10,...
                       ylim_extra);
   bild=get(gcf,'Children');
   renameaxis(bild(2),'$$\theta$$','$$\theta^\prime$$');
   skizze_zeitverlauf(x,dx,10);

   hold on                              % Numerische Lösung
   t=0:0.02:5*pi;
   for startval=-3*pi:pi/2:5*pi
      [t_s,x_s]=ode23(dgl_sys,t,startval);
      plot(0,startval,'o','MarkerFaceColor', ...
           [0 0.55 0],'MarkerEdgeColor',[0 0.55 0])
      plot(t_s,x_s,'LineWidth',2,'Color',[0 0.55 0])
      bild=get(gcf,'Children');
      renameaxis(bild(2),'$$t$$','$$\theta$$');
   end
end
```

4.5 Fireflies

4.5.1 Triangle wave

In the firefly model, the sinusoidal form of the firefly's response function was chosen somewhat arbitrarily. Consider the alternative model $\dot{\Theta} = \Omega,\ \dot{\theta} = \omega + Af(\Theta - \theta)$, where f is given now by a triangle wave, not a sine wave. Specifically, let

$$f(\phi) = \begin{cases} \phi, & -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \\ \pi - \phi, & \frac{\pi}{2} \leq \phi \leq \frac{3}{2}\pi \end{cases}$$

on the interval $-\frac{\pi}{2} \leq \phi \leq \frac{3}{2}\pi$, and extend f periodically outside this interval.

a) Graph $f(\phi)$.

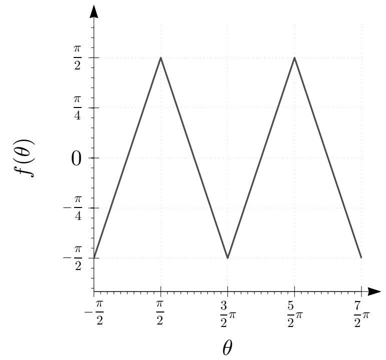


Fig. 4.2: Triangle wave as defined above.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 4.5.1 -- %%');

theta1=-pi/2:0.1*pi/2:pi/2;
theta2=pi/2:0.1*pi/2:3/2*pi;
xval=[theta1,theta2,2*pi+theta1,2*pi+theta2];
yval=[theta1,pi-theta2,theta1,pi-theta2];
customplot(xval',yval',[],[1/6 1/6],[size(xval,2);0], ...
           '$$\theta$$','$$f(\theta)$$');
ebenen=get(gcf,'Children');
for i_correct=1:3
   set(ebenen(i_correct),'XTick',[-pi/2:pi:7/2*pi], ...
                          'YTick',[-pi/2:pi/4:pi/2])
end

renameaxis(ebenen(2),'keep','keep',[], ...
           {'$$-\frac{\pi}{2}$$';'$$\frac{\pi}{2}$$'; ...
           '$$\frac{3}{2}\pi$$';'$$\frac{5}{2}\pi$$'; ...
         '$$\frac{7}{2}\pi$$'},{'$$-\frac{\pi}{2}$$'; ...
           '$$-\frac{\pi}{4}$$';'0'; ...
           '$$\frac{\pi}{4}$$';'$$\frac{\pi}{2}$$'});
```

b) Find the range of entrainment.

In the range of entrainment, the firefly is able to synchronize (match frequency). This implies the difference $\dot{\phi} = \dot{\Theta} - \dot{\theta}$ to be zero and therefore

$$\dot{\phi} = \dot{\Theta} - \dot{\theta} = 0 = \Omega - \omega - Af(\Theta - \theta) \quad \Leftrightarrow \quad \Omega = \omega + Af(\Theta - \theta).$$

$f(\theta)$ ranges from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, so the range of entrainment is $\omega - A\frac{\pi}{2} \leq \Omega \leq \omega + A\frac{\pi}{2}$.

c) Assuming that the firefly is phase–locked to the stimulus, find a formula for the phase difference ϕ^*.

Being phase–locked, $\dot{\phi} = \Omega - \omega - Af(\phi^*) = 0$ which yields $f(\phi^*) = \frac{\Omega - \omega}{A}$. As can be seen from above, $|f(\phi^*)| < \frac{\pi}{2}$.

d) Find a formula for T_{drift}.

Using the integration formula for T_{drift}, inserting $f(\phi)$ and partitioning the integral in smooth intervals yields

$$T_{\text{drift}} = \int\limits_0^{2\pi} \frac{\mathrm{d}t}{\mathrm{d}\phi} \mathrm{d}\phi = \int\limits_0^{2\pi} \frac{1}{\Omega - \omega - Af(\phi)}\, \mathrm{d}\phi$$

$$= \frac{1}{A} \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\frac{\Omega-\omega}{A} - \phi}\, \mathrm{d}\phi + \frac{1}{A} \int\limits_{\frac{\pi}{2}}^{\frac{3}{2}\pi} \frac{1}{\frac{\Omega-\omega}{A} - \pi + \phi}\, \mathrm{d}\phi = \frac{2}{A} \ln \left(\frac{\frac{\Omega-\omega}{A} + \frac{\pi}{2}}{\frac{\Omega-\omega}{A} - \frac{\pi}{2}} \right).$$

Exercises for Chapter 5

5.1 Definitions and Examples

5.1.1 Ellipses and energy conservation for the harmonic oscillator.

Consider the harmonic oscillator $\dot{x} = v, \ \dot{v} = -\omega^2 x$.

a) Show that the orbits are given by ellipses $\omega^2 x^2 + v^2 = C$, where C is any non-negative constant. (Hint: Divide the $\dot{x}$ equation by the $\dot{v}$ equation, separate the v's from the x's, and integrate the resulting separable equation.)

$$
\begin{aligned}
\frac{\dot{x}}{\dot{v}} = \frac{v}{-\omega^2 x} \quad &\Leftrightarrow \quad \int -\omega^2 x \, \mathrm{d}x = \int v \, \mathrm{d}v \quad \Leftrightarrow \quad \frac{1}{2}\omega^2 x^2 + \frac{1}{2}v^2 = C_1 \\
&\Leftrightarrow \quad \omega^2 x^2 + v^2 = C
\end{aligned}
$$

Due to the addition of quadratic terms, the constant C must be nonnegative.

b) Show that this conclusion is equivalent to conservation of energy.

Multiplication with m yields $\frac{1}{2}m\omega^2 x^2 + \frac{1}{2}mv^2 = C_2$. The second term on the left side can be interpreted as the kinetic energy. Substituting $\omega = \sqrt{\frac{k}{m}}$ with the (spring) constant k in the first term gives the spring energy $\frac{1}{2}kx^2$.

5.1.2 Consider the system $\dot{x} = ax, \ \dot{y} = -y$, where $a < -1$.

Show that all trajectories become parallel to the y–direction as $t \to \infty$, and parallel to the x–direction as $t \to -\infty$.

(Hint: Examine the slope $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}$.)

Both equations can be observed independently. Separating the variables and integrating $\frac{\mathrm{d}x}{\mathrm{d}t} = ax$ yields $x(t) = C_1 e^{at} C$. For the second equation we obtain $y(t) = C_2 e^{-t}$. Inserting both results in the slope yields $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{C_2}{aC_1} e^{t(-a-1)}$. Since $a < -1$, the slope will go to zero (parallel to x–axis) as $t \to -\infty$ or to infinity (parallel to y–axis) as $t \to \infty$.

Write the following systems in matrix form.

We make use of the fact, that two first order systems $\dot{x}$ and $\dot{y}$ with linear dependence in either or both variables can be written in the form $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \boldsymbol{A}\begin{pmatrix}x\\ y\end{pmatrix}$.

5.1.3 $\dot{x} = y,\ \dot{y} = -x$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

5.1.4 $\dot{x} = 3x - 2y,\ \dot{y} = 2y - x$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{bmatrix}3 & -2\\ -1 & 2\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

5.1.5 $\dot{x} = 0,\ \dot{y} = x + y$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

5.1.6 $\dot{x} = x,\ \dot{y} = 5x + y$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{bmatrix}1 & 0\\ 5 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

5.2 Classification of Linear Systems

5.2.1 Consider the system $\dot{x} = 4x - y,\ \dot{y} = 2x + y$.

a) Write the system as $\dot{\mathrm{x}} = A\mathrm{x}$. Show that the characteristic polynomial is $\boldsymbol{\lambda^2 - 5\lambda + 6}$, and find the eigenvalues and eigenvectors of $\boldsymbol{A}$.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{bmatrix}4 & -1\\ 2 & 1\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}, \qquad \det(A) = (\lambda-4)(\lambda-1)+2 = 0 \quad\Leftrightarrow\quad \lambda^2 - 5\lambda + 6 = 0$$

$$\lambda_{1,2} = \frac{5 \pm \sqrt{5^2 - 4\cdot 6}}{2} \quad\Rightarrow\quad \lambda_1 = 3, \quad \lambda_2 = 2$$

$$A^*_{(\lambda_1)} = \begin{bmatrix}-1 & 1\\ -2 & 2\end{bmatrix}\mathrm{v}_1 = \begin{pmatrix}0\\ 0\end{pmatrix} \quad\Rightarrow\quad \mathrm{v}_1 = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$$A^*_{(\lambda_2)} = \begin{bmatrix}-2 & 1\\ -2 & 1\end{bmatrix}\mathrm{v}_2 = \begin{pmatrix}0\\ 0\end{pmatrix} \quad\Rightarrow\quad \mathrm{v}_2 = \begin{pmatrix}1\\ 2\end{pmatrix}$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$ and their eigenvectors are $\mathrm{v}_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$ and $\mathrm{v}_2 = \begin{pmatrix} 1 & 2 \end{pmatrix}^T$.

b) Find the general solution of the system.

The general solution can be obtained by inserting the eigenvectors and eigenvalues in the fundamental solution $z(t) = C_1 \mathrm{v}_1 e^{\lambda_1 t} + C_2 \mathrm{v}_2 e^{\lambda_2 t}$ with some constants C_1 and C_2 which yields

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

c) Classify the fixed point at the origin.

Since both eigenvalues are positive, our fixed point at the origin is unstable. Calculating the trace ($\mathrm{tr}\,(A) = 5$) and determinant ($\det\,(A) = 6$) yields an unstable node.

d) Solve the system subject to the initial condition $(\boldsymbol{x_0}, \boldsymbol{y_0}) = (\boldsymbol{3}, \boldsymbol{4})$.

We insert the initial condition in our general solution and determine the constants.

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3\cdot 0} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2\cdot 0} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad \Leftrightarrow \quad C_1 + C_2 = 3, \quad C_1 + 2C_2 = 4$$

$$\Rightarrow \quad C_1 = 2,\; C_2 = 1$$

So our solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{3t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$$

5.3 Love Affairs

5.3.2 Consider the affair described by $\dot{R} = J,\; \dot{J} = -R + J$.

a) Characterize the romantic styles of Romeo $\boldsymbol{R}$ and Juliet $\boldsymbol{J}$.

Romeos affection grows or decays depending on Juliets state and size of affection. The more Juliet loves him, the faster his love for her grows and vice versa. In contrary, the more Romeo loves Juliet, the more Juliet's love is decaying. Additionally, Juliet's growth of affection is depending on her actual state of love. Thus, they pull and push each other in infinitly growing (change of) love and hate.

b) Classify the fixed point at the origin. What does this imply for the affair?

The system can be described as

$$\begin{pmatrix} \dot{R} \\ \dot{J} \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} R \\ J \end{pmatrix},$$

which has the characteristic equation $\lambda^2 - \lambda + 1$. The eigenvalues are $\lambda_{1,1} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$. Two complex eigenvalues with a positive real part imply an unstable spiral as fixed point at the origin.

c) **Sketch $R(t)$ and $J(t)$ as functions of t, assuming $R(0) = 1,\ J(0) = 0$.**

To calculate the functions $R(t)$ and $J(t)$, we first have to obtain the eigenvectors.

$$A^*_{(\lambda_1)} = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 1 \\ -1 & \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{bmatrix} \begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Leftrightarrow \quad v_{1,2} = (\frac{1}{2} + \frac{\sqrt{3}}{2}i)v_{1,1}$$

$$\Leftrightarrow \quad -v_{1,1} + (\frac{1}{4} + \frac{3}{4})v_{1,1} = 0$$

With a compley pair of eigenvalues, one coordinate of the eigenvector can be chosen arbitrarily (second equation yields $0 = 0$). Choosing $v_{1,1} = 1$ results in $v_{1,2} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. For the second eigenvalue, we let $v_{2,1} = 1$ and get $v_{2,2} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$. So our fundamental solution is

$$X(t) = C_1 \begin{pmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix} e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)t} + C_2 \begin{pmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix} e^{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)t}.$$

Using $R(0) = 1$ and $J(0) = 0$, the constants can be calculated as $C_1 = \frac{1}{2} + \frac{\sqrt{3}}{6}i$ and $C_2 = \frac{1}{2} - \frac{\sqrt{3}}{6}i$. Further calculation yields

$$R(t) = \left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)e^{\frac{1}{2}t}$$
$$J(t) = \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)e^{\frac{1}{2}t}.$$

It can easily be seen, that $J(t)$ is zero for $t = n\frac{2}{\sqrt{3}}\pi \quad \forall n \in \mathtt{N}$. $R(t)$ is zero for $t = (n + \frac{1}{3})\frac{2}{\sqrt{3}}\pi \quad \forall n \in \mathtt{N}$. The limiting exponential functions can be obtained by finding the maximum amplitude which yields $f_{\mathrm{lim},R} = \pm\left(\cos\left(\frac{11}{6}\pi\right) - \sin\left(\frac{11}{6}\pi\right)\frac{1}{\sqrt{3}}\right)e^{\frac{1}{2}t}$ and $f_{\mathrm{lim},J} = \pm\frac{2}{\sqrt{3}}e^{\frac{1}{2}t}$. To get $f_{\mathrm{lim},R}$ you may find it useful to calculate the derivative of $R(t)$ and examine it.

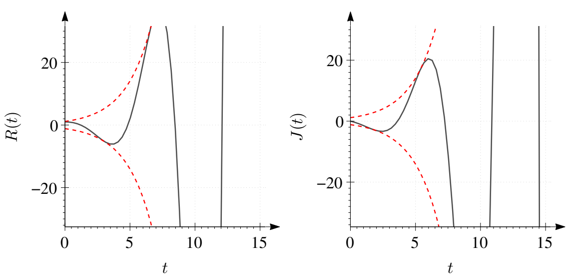


Fig. 5.1: Time behaviour of the solutions and the limiting exponential functions, left: $R(t) = \left(\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)e^{\frac{1}{2}t}$, right: $J(t) = \frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)e^{\frac{1}{2}t}$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 5.3.2c -- %%');

% Laufkoordinate
t=[0:0.1*pi:4.9*pi];

% Zeitabhängige Gleichungen
Rt=exp(0.5*t).*(cos(sqrt(3)/2.*t)- ...
                sin(sqrt(3)/2.*t)/sqrt(3));
Jt=-exp(0.5*t).*sin(sqrt(3)/2.*t)*2/sqrt(3);

% Plotte R(t)
customplot(...
   t', Rt',[], [-46/384 -323/384],[size(t,2); 0], ...
   '$$t$$', '$$R(t)$$');

% Und die begrenzenden Exponentialfunktionen
handles=get(gcf,'Children');
hold(handles(3),'on');
plot(t,(cos(11/6*pi)-sin(11/6*pi)/sqrt(3))* ...
     exp(0.5*t),'r--','LineWidth',1.8);
plot(t,-(cos(11/6*pi)-sin(11/6*pi)/sqrt(3))* ...
     exp(0.5*t),'r--','LineWidth',1.8);

% Plotte J(t)
customplot(...
   t', Jt',[], [-262/384 -112/384],[size(t,2); 0], ...
   '$$t$$', '$$J(t)$$');

% Und die begrenzenden Exponentialfunktionen
handles=get(gcf,'Children');
hold(handles(3),'on');
plot(t,2/sqrt(3)*exp(0.5*t),'r--','LineWidth',1.8);
plot(t,-2/sqrt(3)*exp(0.5*t),'r--','LineWidth',1.8);
```

Exercises for Chapter 6

6.1 Phase Portraits

Computer work: Plot computer–generated phase portraits of the following systems. As always, you may write your own computer programs or use any readymade software, e.g. *MacMath* (Hubbard and West 1992).

6.1.8 van der Pol oscillator

$\dot{x} = y, \quad \dot{y} = -x + y(1 - x^2)$

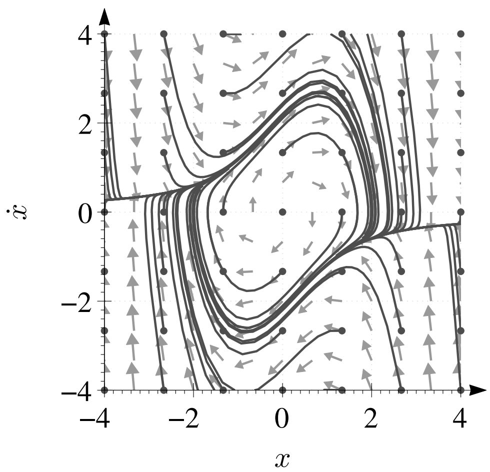


Fig. 6.1: van der Pol oscillator $\dot{x} = y, \quad \dot{y} = -x + y(1 - x^2)$ with numerical results.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 6.1.8 -- %%');

% Feld anlegen
lval=[-4 4];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

% van der Pol Oscillator
dx=y;
dy=-x+y.*(1-x.^2);
dgl_sys=@(t,v)[v(2); -v(1)+v(2)*(1-v(1)^2)];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
           [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=0:0.1:4;

for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
      [t_s,res]=ode23(dgl_sys,ts, ...
                        [startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor', ...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color', ...
           [0.3 0.3 0.3])
   end
end
```

6.1.9 Dipole fixed point

$\dot{x} = 2xy, \quad \dot{y} = y^2 - x^2$

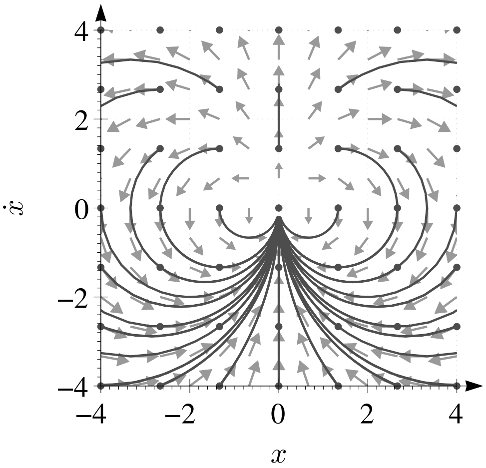


Fig. 6.2: Dipole fixed point $\dot{x} = 2xy, \quad \dot{y} = y^2 - x^2$ with numerical results.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 6.1.9 -- %%');

% Feld anlegen
lval=[-4 4];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

% Dipole fixed point
dx=2*x.*y;
dy=y.^2-x.^2;
dgl_sys=@(t,v)[2*v(1)*v(2); v(2)^2-v(1)^2];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
         [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)

      % Berücksichtige, dass Werte oben mitte divergieren
      if ((startx == 0) && (starty > 0))
         ts=0:0.5*10^(-starty):8*10^(-starty);
      else
         ts=0:0.03:5;
      end
      [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor', ...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color', ...
           [0.3 0.3 0.3])
   end
end
```

6.1.10 Two–eyed monster

$\dot{x} = y + y^2, \quad \dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borelli and Coleman 1987, p. 385)

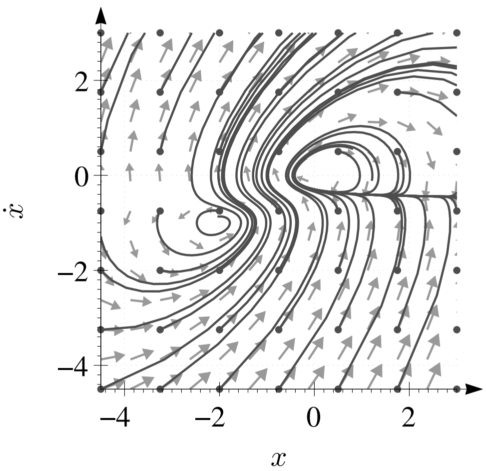


Fig. 6.3: Two–eyed monster $\dot{x} = y + y^2, \quad \dot{y} = -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ with numerical results.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 6.1.10 -- %%');

% Feld anlegen
lval=[-4.5 3];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

% Two--eyed monster
dx=y+y.^2;
dy=-0.5*x+0.2*y-x.*y+1.2*y.^2;
dgl_sys=@(t,v)[v(2)+v(2)^2; ...
              -0.5*v(1)+0.2*v(2)-v(1)*v(2)+1.2*v(2)^2];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
         [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=0:0.1:10;

for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
      [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor', ...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color', ...
           [0.3 0.3 0.3])
   end
end
```

6.1.11 Parrot

$\dot{x} = y + y^2, \quad \dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ (from Borelli and Coleman 1987, p. 384)

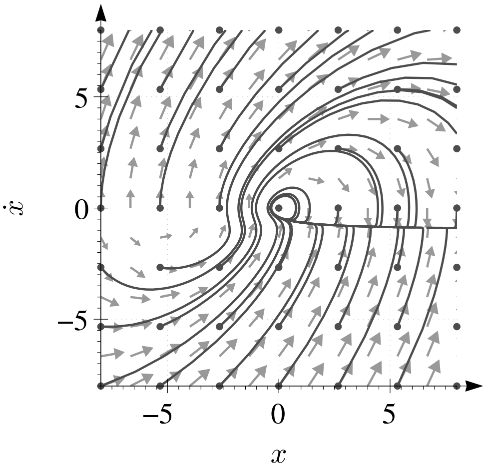


Fig. 6.4: Parrot $\dot{x} = y + y^2, \quad \dot{y} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2$ with numerical results.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 6.1.11 -- %%');

% Feld anlegen
lval=[-8 8];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

% Parrot
dx=y+y.^2;
dy=-x+0.2*y-x.*y+1.2*y.^2;
dgl_sys=@(t,v)[v(2)+v(2)^2; ...
               -v(1)+0.2*v(2)-v(1)*v(2)+1.2*v(2)^2];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
           [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=0:0.03:10;

for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
      [t_s,res]=ode23s(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor', ...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color', ...
           [0.3 0.3 0.3])
   end
end
```

6.7 Pendulum

6.7.2 Pendulum driven by a constant torque

The equation $\ddot{\boldsymbol{\theta}} + \sin(\boldsymbol{\theta}) = \boldsymbol{\gamma}$ describes the dynamics of an undamped pendulum driven by a constant torque, or an undamped Josephson junction driven by a constant bias current.

a) Find all the equilibrium points and classify them as γ varies.

First, we can rewrite the system with two first–order systems

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= -\sin(\theta) + \gamma.\end{aligned}$$

The equilibrium points are $\begin{pmatrix}\theta & v\end{pmatrix}^T = \begin{pmatrix}\arcsin(\gamma) & 0\end{pmatrix}^T$. $\arcsin(\gamma)$ has infinitely many points for one γ (periodicity in vertical direction). To classify them, we calculate the Jacobian insert the values and evaluate its determinant.

$$J = \begin{bmatrix}\frac{\partial\dot{\theta}}{\partial\theta} & \frac{\partial\dot{\theta}}{\partial v} \\ \frac{\partial\dot{v}}{\partial\theta} & \frac{\partial\dot{v}}{\partial v}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\cos(\theta) & 0\end{bmatrix}, \qquad J^* = \begin{bmatrix}0 & 1 \\ -\cos(\arcsin(\gamma)) & 0\end{bmatrix},$$
$$\det(J^*) = \cos(\arcsin(\gamma)).$$

Since $\operatorname{tr}(J^*) = 0$, we have centers for $\det(J^*) > 0$, saddle nodes for $\det(J^*) < 0$ or non–isolated fixed points otherwise.

$\arcsin(\gamma)$ is only defined for $-1 \leq \gamma \leq 1$, but there are infinitely many values for each defined coordinate (due to its vertical periodicity). More precisely, if β is a solution of $\arcsin(\gamma)$, than $2n\pi + \beta$ and $(2n-1)\pi - \beta$ are also. In the cosine function $\cos(\arcsin(\gamma))$, the second set of periodic solutions will produce results with an opposing sign due to the periodicity shift of both functions. As a result, $J = \cos(\arcsin(\gamma))$ yields infinitely many positive (centers) and negative values (saddles) for one β.

In short, $-1 < \gamma < 1$ yields infinitely many centers and saddle nodes, choosing $\gamma = 1$ or $\gamma = -1$, $\det(J^*) = 0$ results in non–isolated fixed points and other γ are not allowed.

b) Sketch the vector field.

see d)

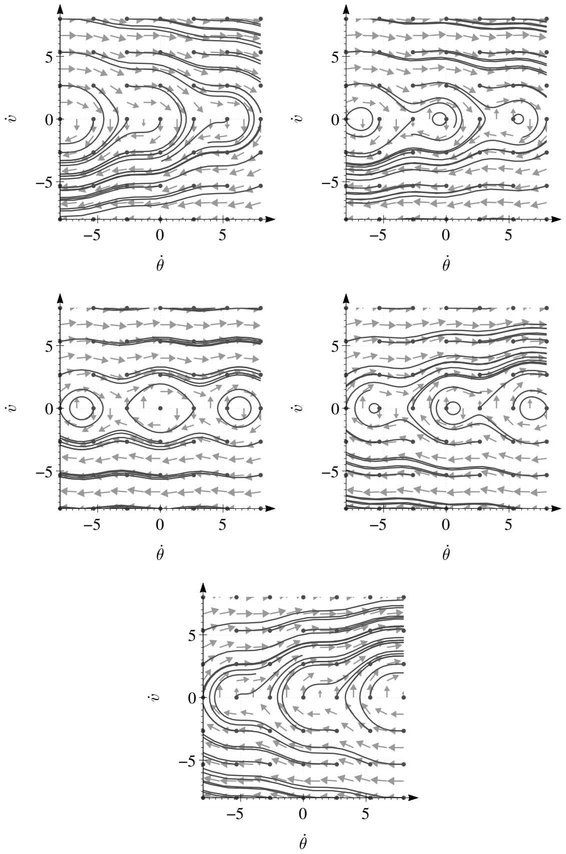
c) Is the system conservative? If so, find a conserved quantity. Is the system reversible?

Multiplying with $\dot{\theta}$ suggests a time derivative. Rearraning to $\frac{\mathrm{d}}{\mathrm{d}t}$ yields

$$\begin{aligned}&\dot{\theta}\ddot{\theta} + \dot{\theta}\sin(\theta) - \gamma\dot{\theta}) = 0 \\ \Leftrightarrow\quad &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{\theta}^2 - \cos(\theta) - \gamma\theta\right) = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2 - mR^2\cos(\theta) - mR^2\gamma\theta\right) = 0\end{aligned}$$

where $v = \dot{\theta}R$. In this equation, $\frac{1}{2}mv^2 + V(\theta)$ represents a constant (energy) and therefore a conservative quantity. Since $\ddot{\theta} + \sin(\theta) = \gamma$ is invariant to $t \to -t$ (second derivative will have the same sign), the system is reversible.

d) Sketch the phase portrait on the plane as γ varies.



```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 6.7.2 -- %%');

% Feld anlegen
lval=[-8 8];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

for gamma=-1:0.5:1
   % DGL
   dx=y;
   dy=-sin(x)+gamma;
   dgl_sys=@(t,v)[v(2); ...
                  -sin(v(1))+gamma];

   % Zeichenebene vorbereiten und Vektorfeld zeichnen
   customplot([lval(1) lval(2)]', ...
              [lval(1) lval(2)]',[],[],[2;-1], ...
              '$$\dot{\theta}$$','$$\dot{v}$$');
   vectorfield(x,y,dx,dy);
   hold on

   % Numerische Lösungen bestimmen und dazu zeichnen
   ts=0:0.2:8;

   for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
      for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
         [t_s,res]=ode23s(dgl_sys,ts,[startx starty]);

         plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
         plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
              [0.3 0.3 0.3])
      end
   end
end
```

Fig. 6.5: Numerical solutions and vector field to $\dot{\theta} = v, \quad \dot{v} = -\sin(\theta) + \gamma$. Top left: $\gamma = -1$, top right: $\gamma = -\frac{1}{2}$, middle left: $\gamma = 0$, middle right: $\gamma = -\frac{1}{2}$, bottom left: $\gamma = 1$.

e) Find the approximate frequency of small oscillations about any centers in the phase portrait.

As we know from our observation of the Jacobian, the origin is on a center $(\det(J^*) > 0)$. Determining its eigenvalues from the characteristic equation yields $\lambda_{1,2} = \pm i\sqrt{\cos(\arcsin(\gamma))}$. To associate the eigenvalues λ with the frequency ω, one can for example calculate the fundamental equation and observe the time behaviour to be $\lambda = i\omega$ and thus $\omega = \sqrt{\cos(\arcsin(\gamma))}$. As our frequency can't get larger than one intervall of the $\arcsin(\gamma)$ function, we can also use the trigonometric identity and write $\omega = \sqrt[4]{1-\gamma^2}$

Exercises for Chapter 7

7.2 Ruling Out Closed Orbits

7.2.10 Show that the system $\dot{x} = y - x^3, \quad \dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

Our Liapunov function must be zero at our equilibrium point $\begin{pmatrix} 0 & 0 \end{pmatrix}^T$ and greater at other points. Calculating the derivative and inserting yields

$$\begin{aligned} \dot{V} &= 2ax\dot{x} + 2by\dot{y} = 2ax(y - x^3) + 2by(-x - y^3) \\ &= -2ax^4 - 2by^4 + 2axy - 2bxy. \end{aligned}$$

For our derivative to be negative for all values, the first two terms are always negative with positive a, b and the second two terms must vanish by choosing $a = b$. So, our Liapunov function is $V = ax^2 + ay^2$ with $a > 0$.

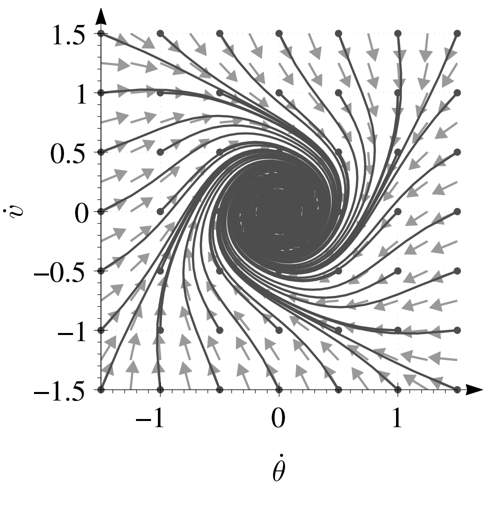


Fig. 7.1: Phase space of $\dot{x} = y - x^3, \quad \dot{y} = -x - y^3$ with numerical solutions.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 7.2.10 -- %%');

% Feld anlegen
lval=[-1.5 1.5];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

% DGL
dx=y-x.^3;
dy=-x-y.^3;
dgl_sys=@(t,v)[v(2)-v(1)^3; ...
               -v(1)-v(2)^3];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
            [lval(1) lval(2)]',[],[],[2;-1], ...
            '$$\dot{\theta}$$','$$\dot{v}$$');
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=[0:0.1:5 6:0.5:10];
for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
      [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
            [0.3 0.3 0.3])
   end
end

% Spirale weiter verfolgen
ts=[0:0.1:5 6:0.5:10 15:1:30 35:5:70 80:10:200];
for startx=lval(1)/9:(lval(2)-lval(1))/27:lval(2)/9
   for starty=lval(1)/9:(lval(2)-lval(1))/27:lval(2)/9
      [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
            [0.3 0.3 0.3])
   end
end
```

7.6 Weakly Nonlinear Oscillators

For each of the following systems $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$ with $0 < \varepsilon \ll 1$, calculate the averaged equations, which are defined by

$$r' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \sin(\theta)\, \mathrm{d}\theta \equiv \langle h \sin(\theta) \rangle$$

$$r\phi' = \frac{1}{2\pi} \int_0^{2\pi} h(\theta) \cos(\theta)\, \mathrm{d}\theta \equiv \langle h \cos(\theta) \rangle .$$

Here, r and ϕ are the slowly–varying amplitude and phase of the approximate (averaged) solution $x_0 = r\cos(\tau + \phi)$ of x. Then analyze the long–term behavior of the averaged system. Find the amplitude of any limit cycles for the original system.

$h(x, \dot{x}) = h(r\cos(\theta), -r\sin(\theta))$

We can make our life easier with the help of some relations:

$$\langle \sin(\varphi) \rangle = \langle \cos(\varphi) \rangle = 0$$

$$\left\langle \sin(\varphi)^{2n} \right\rangle = \left\langle \cos(\varphi)^{2n} \right\rangle = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)}, \qquad n \geq 1$$

$$\left\langle \sin(\varphi)^{2n+1} \right\rangle = \left\langle \cos(\varphi)^{2n+1} \right\rangle = 0, \qquad n \geq 1$$

7.6.6 $h(x, \dot{x}) = x\dot{x}$

Inserting yields $h = r\cos(\theta)(-r\sin(\theta))$. Therefore

$$r' = \left\langle -r^2 \sin(\theta)^2 \cos(\theta) \right\rangle = -r^2 \langle \cos(\theta) \rangle + r^2 \left\langle \cos(\theta)^3 \right\rangle = 0$$

$$r\phi' = \left\langle -r^2 \sin(\theta) \cos(\theta)^2 \right\rangle = -r^2 \langle \sin(\theta) \rangle + r^2 \left\langle \sin(\theta)^3 \right\rangle = 0.$$

Neither our slow–varying amplitude $r(x_0)$, nor our slow–varying phase $\phi(x_0)$ changes (both derivatives are zero). Thus, the the system doesn't change in the long–term and the amplitude of a limit cycles is only depending on the initial condition.

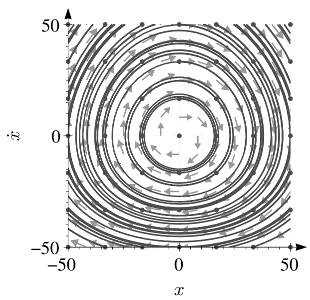


Fig. 7.2: Phase space of $\ddot{x} + x + \varepsilon x\dot{x}$, with $\varepsilon = 0.01$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 7.6.6 -- %%');

% Feld anlegen
lval=[-50 50];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

epsilon=0.01;

% DGL
dx=y;
dy=-epsilon*x.*y-x;
dgl_sys=@(t,v)[v(2); -epsilon*v(1)*v(2)-v(1)];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
           [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=0:0.1:8;
for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
      [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
           [0.3 0.3 0.3])
   end
end
```

7.6.7 $h(x, \dot{x}) = (x^4 - 1)\dot{x}$

Inserting yields $h = -r^5 \cos(\theta)^4 \sin(\theta) + r \sin(\theta)$. Therefore

$$\begin{aligned} r' &= \left\langle -r^5 \cos(\theta)^4 \sin(\theta)^2 + r \sin(\theta)^2 \right\rangle = r^5 \left\langle \cos(\theta)^6 \right\rangle - r^5 \left\langle \cos(\theta)^4 \right\rangle + r \left\langle \sin(\theta) \right\rangle \\ &= \frac{5}{16} r^5 - \frac{3}{8} r^5 + \frac{1}{2} r = \frac{1}{2} r - \frac{1}{16} r^5 \\ r\phi' &= \left\langle -r^5 \cos(\theta)^5 \sin(\theta) + r \sin(\theta) \cos(\theta) \right\rangle = -r^5 \left\langle \cos(\theta)^5 \sin(\theta) \right\rangle + r \left\langle \sin(\theta) \cos(\theta) \right\rangle \\ &= 0 + 0. \end{aligned}$$

There is no long–term phase change. We need to solve $r' = \frac{1}{2} r - \frac{1}{16} r^5$ to get our amplitude.

$$\begin{aligned} & \int \frac{2}{r(1 - \frac{1}{8} r^4)} \, \mathrm{d}r = \int \mathrm{d}t \\ \Leftrightarrow \quad & \ln\left(\frac{r^4}{8 - r^4}\right) = t + C \\ \Leftrightarrow \quad & r = \sqrt[4]{\frac{8}{\frac{e^{-t}}{C_2} + 1}} \end{aligned}$$

There is a stable limit cycle with amplitude $\lim\limits_{t \to \infty} r = \sqrt[4]{8}$.

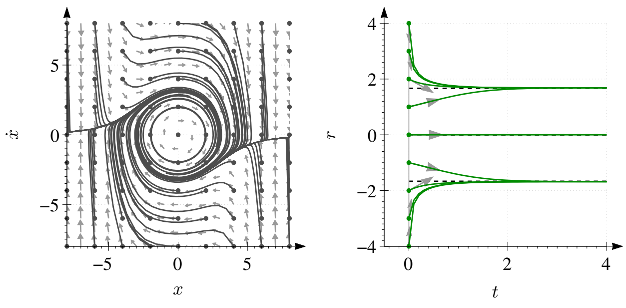


Fig. 7.3: Left: Phase space of $\ddot{x} + x + \varepsilon(x^4 - 1)\dot{x}$, with $\varepsilon = 0.01$, right: time dependency of the averaged amplitude $r(t)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 7.6.7 -- %%');

% Feld anlegen
lval=[-8 8];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/16:lval(2));

epsilon=0.01;

% DGL
dx=y;
dy=-epsilon*(x.^4-1).*y-x;
dgl_sys=@(t,v)[v(2); -epsilon*(v(1)^4-1)*v(2)-v(1)];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
           [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=0:0.1:5;
for startx=lval(1):(lval(2)-lval(1))/8:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/8:lval(2)
      [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
           [0.3 0.3 0.3])
   end
end

% DGL für r
r=-4:0.2:4;
dr=0.5*r-1/16*r.^5;

ylim_extra=[1/6 1/6];

% Zeitlicher Verlauf
skizze_zeitverlauf(r,dr,5);

hold on                              % Numerische Lösung
ts=0:0.1:5;
for startval=-4:1:4
   [t_s,r_s]=ode23(inline('1/2*r-1/16*r^5;','t','r'), ...
                   ts,startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,r_s,'LineWidth',2,'Color',[0 0.55 0])
end

% Achsenbeschriftung anpassen
ebenen=get(gcf,'Children');
renameaxis(ebenen(2),'$$t$$','$$r$$');
```

7.6.8 $h(x, \dot{x}) = (|x| - 1)\dot{x}$

Inserting yields $h = -r^2|\cos(\theta)|\sin(\theta) + r\sin(\theta)$. Before we start, we consider the following:

$$\begin{aligned}\int \sin(\theta)^2 \cos(\theta)\,\mathrm{d}\theta &= \frac{\sin(\theta)^3}{3}\\ \int \sin(\theta)\cos(\theta)^2\,\mathrm{d}\theta &= -\frac{\cos(\theta)^3}{3}.\end{aligned}$$

Inserting the values yields

$$\begin{aligned}
r' &= \left\langle -r^2|\cos(\theta)|\sin(\theta)^2 + r\sin(\theta)^2\right\rangle\\
&= -r^2\left\langle \sin(\theta)^2\cos(\theta)\right\rangle\Big|_{0\le\theta<\frac{\pi}{2},\ \frac{3\pi}{2}\le\theta<2\pi} + r^2\left\langle\sin(\theta)^2\cos(\theta)\right\rangle\Big|_{\frac{\pi}{2}\le\theta<\frac{3\pi}{2}} + r\left\langle\sin(\theta)^2\right\rangle\\
&= -\frac{r^2}{2\pi}\int_0^{\frac{\pi}{2}}\sin(\theta)^2\cos(\theta)\,\mathrm{d}\theta - \frac{r^2}{2\pi}\int_{\frac{3\pi}{2}}^{2\pi}\sin(\theta)^2\cos(\theta)\,\mathrm{d}\theta + \frac{r^2}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin(\theta)^2\cos(\theta)\,\mathrm{d}\theta\\
&\quad + \frac{r}{2\pi}\int_0^{2\pi}\sin(\theta)^2\,\mathrm{d}\theta\\
&= -\frac{r^2}{6\pi} - \frac{r^2}{6\pi} - \frac{r^2}{3\pi} + \frac{r}{2} = \frac{r}{2} - \frac{2r^2}{3\pi}\\
r\phi' &= \left\langle -r^2|\cos(\theta)|\sin(\theta)\cos(\theta) + r\sin(\theta)\cos(\theta)\right\rangle\\
&= -r^2\left\langle\sin(\theta)\cos(\theta)^2\right\rangle\Big|_{0\le\theta<\frac{\pi}{2},\ \frac{3\pi}{2}\le\theta<2\pi} + r^2\left\langle\sin(\theta)\cos(\theta)^2\right\rangle\Big|_{\frac{\pi}{2}\le\theta<\frac{3\pi}{2}} + r\left\langle\sin(\theta)\cos(\theta)\right\rangle\\
&= -\frac{r^2}{2\pi}\int_0^{\frac{\pi}{2}}\sin(\theta)\cos(\theta)^2\,\mathrm{d}\theta - \frac{r^2}{2\pi}\int_{\frac{3\pi}{2}}^{2\pi}\sin(\theta)\cos(\theta)^2\,\mathrm{d}\theta + \frac{r^2}{2\pi}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\sin(\theta)\cos(\theta)^2\,\mathrm{d}\theta + 0\\
&= -\frac{r^2}{3\pi} + \frac{r^2}{3\pi} + 0 + 0 = 0.
\end{aligned}$$

There is no long–term phase change. We need to solve $r' = \frac{1}{2}r - \frac{2}{3\pi}r^2$ to get our amplitude.

$$\begin{aligned}
&\int \frac{2}{r}\,\mathrm{d}r + \int \frac{2}{\frac{3}{4}\pi - r}\,\mathrm{d}r = \int \mathrm{d}t\\
\Leftrightarrow\quad & 2\ln\left(\frac{r^4}{8 - r^4}\right) = t + C\\
\Leftrightarrow\quad & r = \frac{3}{4}\pi\frac{1}{\frac{e^{-\frac{1}{2}t}}{C_2} + 1}
\end{aligned}$$

There is a stable limit cycle with amplitude $\lim\limits_{t\to\infty} r = \frac{3}{4}\pi$.

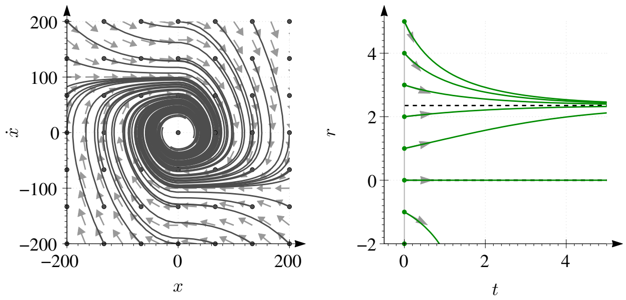


Fig. 7.4: Left: Phase space of $\ddot{x}+x+\varepsilon(|x|-1)\dot{x}$, with $\varepsilon = 0.01$, right: time dependency of the averaged amplitude $r(t)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 7.6.8 -- %%');

% Feld anlegen
lval=[-200 200];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

epsilon=0.01;

% DGL
dx=y;
dy=-epsilon*(abs(x)-1).*y-x;
dgl_sys=@(t,v)[v(2); -epsilon*(abs(v(1))-1)*v(2)-v(1)];

% Zeichenebene vorbereiten und Vektorfeld zeichnen
customplot([lval(1) lval(2)]', ...
           [lval(1) lval(2)]',[],[],[2;-1]);
vectorfield(x,y,dx,dy);
hold on

% Numerische Lösungen bestimmen und dazu zeichnen
ts=0:0.1:10;

for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
   for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
      [t_s,res]=ode23s(dgl_sys,ts,[startx starty]);

      plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0 0 0])
      plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
           [0.3 0.3 0.3])
   end
end

% DGL für r
r=-2:0.2:5;
dr=0.5*r-2/(3*pi)*r.^2;

ylim_extra=[1/6 1/6];

% Zeitlicher Verlauf
skizze_zeitverlauf(r,dr,5);

hold on                          % Numerische Lösung
for startval=-2:1:5
   if (startval < 0)
      ts=0:1/(10*(1-startval)):1;
   else
      ts=0:0.1:8;
   end

   [t_s,r_s]=ode23(inline('1/2*r-2/(3*pi)*r^2;','t',...
                   'r'),ts,startval);
   plot(0,startval,'o','MarkerFaceColor',[0 0.55 0], ...
        'MarkerEdgeColor',[0 0.55 0])
   plot(t_s,r_s,'LineWidth',2,'Color',[0 0.55 0])
end

% Achsenbeschriftung anpassen
ebenen=get(gcf,'Children');
renameaxis(ebenen(2),'$$t$$','$$r$$');
```

Exercises for Chapter 8

8.2 Hopf Bifurcations

8.2.12 Analytical criterion to decide if a Hopf bifurcation is subcritical or supercritical

Any system at a Hopf bifurcation can be put into the following form by suitable changes of variables:

$$\dot{x} = -\omega y + f(x,y), \qquad \dot{y} = \omega x + g(x,y),$$

where f and g contain only higher–order nonlinear terms that vanish at the origin. As shown by Guckenheimer and Holmes (1983, pp. 152–156), one can decide whether the bifurcation is subcritical or supercritical by calculating the sign of the following quantity:

$$16a = f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy} + \frac{1}{\omega}\Big(f_{xy}(f_{xx}+f_{yy}) - g_{xy}(g_{xx}+g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\Big)$$

where the subscripts denote partial derivatives evaluated at $(0,0)$. The criterion is: If $a < 0$, the bifurcation is supercritical; if $a > 0$ the bifurcation is subcritical.

a) Calculate a for the system $\dot{x} = -y + xy^2, \quad \dot{y} = x - x^2$.

Here we have $\omega = 1$, $f(x,y) = xy^2$ and $g(x,y) = -x^2$. Inserting yields

$$\begin{aligned} 16a &= 0 + 2 + 0 + 0 + \frac{1}{1}\Big(2y(0+2x) - 0(2+0) - 0\cdot 2 + 2x\cdot 0\Big) \\ &= 2 + 4xy \end{aligned}$$

Evaluating at $(0,0)$ yields $a = \frac{1}{8}$. Thus, the hopf bifurcation is subcritical.

b) Use part (a) to decide which type of Hopf bifurcation occurs for $\dot{x} = -y + \mu x + xy^2, \quad \dot{y} = x + \mu y - x^2$ at $\mu = 0$.

Since the system at $\mu = 0$ is identical to that of part (a), the Hopf bifurctaion is also subcritical.

c) Verify your results by plotting phase portraits on the computer.

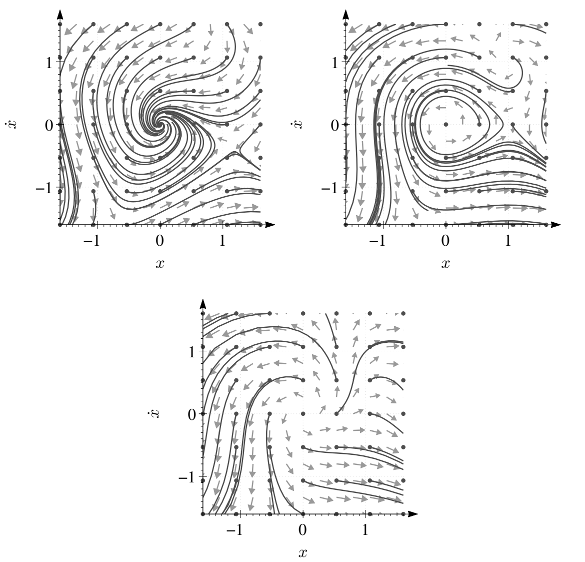


Fig. 8.1: Phase space of $\dot{x} = -y + \mu x + xy^2, \quad \dot{y} = x + \mu y - x^2$, top left: $\mu = -\frac{1}{2}$, top right: $\mu = 0$, bottom: $\mu = \frac{1}{2}$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 8.2.12 -- %%');

% Feld anlegen
lval=[-1.6 1.6];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

for mu=-0.5:0.5:0.5
   % DGL
   dx=-y+mu*x+x.*y.^2;
   dy=x+mu*y-x.^2;
   dgl_sys=@(t,v)[-v(2)+mu*v(1)+v(1)*v(2)^2; ...
                  v(1)+mu*v(2)-v(1)^2];

   % Zeichenebene vorbereiten und Vektorfeld zeichnen
   customplot([lval(1) lval(2)]', ...
              [lval(1) lval(2)]',[],[],[2;-1]);
   vectorfield(x,y,dx,dy);
   hold on

   % Numerische Lösungen bestimmen und dazu zeichnen
   for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
      for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
         if ((abs(startx) > 1) || (abs(starty) > 1))
            ts=0:0.05:5;
         else
            ts=0:0.1:20;
         end

         [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

         plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
         plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
              [0.3 0.3 0.3])
      end
   end
end
```

8.4.3 Homoclinic bifurcation

Using numerical integration, find the value of $\boldsymbol{\mu}$ at which the system $\dot{\boldsymbol{x}} = \boldsymbol{\mu x + y - x^2}$, $\dot{\boldsymbol{y}} = \boldsymbol{-x + \mu y + 2x^2}$ undergoes a homoclinic bifurcation. Sketch the phase portrait just above and below the bifurcation.

Our system has two fixed points,

$$\mathrm{FP}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \mathrm{FP}_2 = \begin{bmatrix} \frac{1+\mu^2}{2+\mu} \\ \frac{1-2\mu+\mu^2-2\mu^3}{4+4\mu+\mu^2} \end{bmatrix}$$

where FP_1 is a spiral and FP_2 a saddle node. Here, before a homoclinic bifurcation takes place, points near FP_1 are attracted by the spiral or repelled to a limit cycle, depending on μ. But if μ approaches the critical value, the points near FP_1 approach the homoclinic orbit of the saddle node. If μ passed the critical value, values near FP_1 can escape the local domain. Numerical investigation reveals $\mu_{\text{crit}} \approx 0.06626$ as shown below.

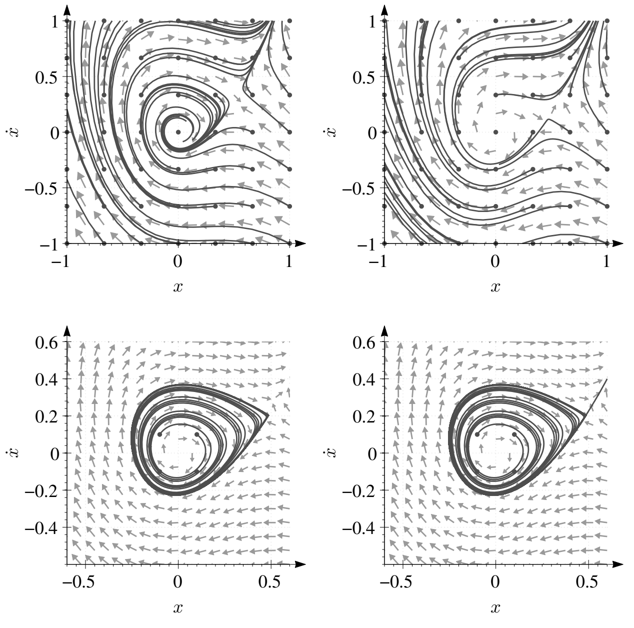


Fig. 8.2: Phase space of $\dot{x} = \mu x + y - x^2$, $\quad \dot{y} = -x + \mu y + 2x^2$, top left: $\mu = -0.1$, top right: $\mu = 0.2$, in the bottom row only the trajectories of four values near to FP_1 are shown. bottom left: $\mu = 0.06626$ before the homoclinic bifurcation (at least within the first 10000 s), bottom right: $\mu = 0.06627$ after the bifurcation.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 8.4.3 -- %%');

% Feld anlegen
lval=[-1 1];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2));

for mu=[-0.1 0.2]
   % DGL
   dx=mu*x+y-x.^2;
   dy=-x+mu*y+2*x.^2;
   dgl_sys=@(t,v)[mu*v(1)+v(2)-v(1)^2; ...
                 -v(1)+mu*v(2)+2*v(1)^2];

   % Zeichenebene vorbereiten und Vektorfeld zeichnen
   customplot([lval(1) lval(2)]', ...
            [lval(1) lval(2)]',[],[],[2;-1]);
   vectorfield(x,y,dx,dy);
   hold on

   % Numerische Lösungen bestimmen und dazu zeichnen
   ts=0:0.1:8;

   for startx=lval(1):(lval(2)-lval(1))/6:lval(2)
      for starty=lval(1):(lval(2)-lval(1))/6:lval(2)
         [t_s,res]=ode23(dgl_sys,ts, ...
                               [startx starty]);

         plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
         plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
              [0.3 0.3 0.3])
      end
   end
end

% Feld anlegen
lval=[-0.6 0.6];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/16:lval(2));

% Und auch den Übergang im Detail zeichnen
for mu=[0.06626 0.06627]
   % DGL
   dx=mu*x+y-x.^2;
   dy=-x+mu*y+2*x.^2;
   dgl_sys=@(t,v)[mu*v(1)+v(2)-v(1)^2; ...
                 -v(1)+mu*v(2)+2*v(1)^2];

   % Zeichenebene vorbereiten und Vektorfeld zeichnen
   customplot([lval(1) lval(2)]', ...
            [lval(1) lval(2)]',[],[],[2;-1]);
   vectorfield(x,y,dx,dy);
   hold on

   % Numerische Lösungen bestimmen und dazu zeichnen
   ts=0:0.1:360; %für mu=0.06626 bis 10000 getestet

   for startx=[-0.1 0.1]
      for starty=[-0.1 0.1]
         [t_s,res]=ode23(dgl_sys,ts, ...
                               [startx starty]);

         plot(res(1,1),res(1,2),'o','MarkerFaceColor',...
           [0.3 0.3 0.3],'MarkerEdgeColor',[0.3 0.3 0.3])
         plot(res(:,1),res(:,2),'LineWidth',2,'Color',...
              [0.3 0.3 0.3])
      end
   end
end
```

8.5.2 Consider the driven pendulum $\theta'' + \alpha\theta' + \sin(\theta) = I$.

By numerical computation of the phase portrait, verify that if α is fixed and sufficiently small, the system's stable limit cycle is destroyed in a homoclinic bifurcation as I decreases. Show that if α is too large, the bifurcation is an infinite–period bifurcation instead.

First, we rewrite the second–order system as two first–order systems $\theta' = v$ and $v' = I - \alpha v - \sin(\theta)$. As I is passing 1, the bifurcations appear. If $\alpha > 1$, we have an infinite–period bifurcation as I passes 1, while for $\alpha < 1$ we have a homoclinic bifurcation, depending on the value of I (which must also be between zero and one).

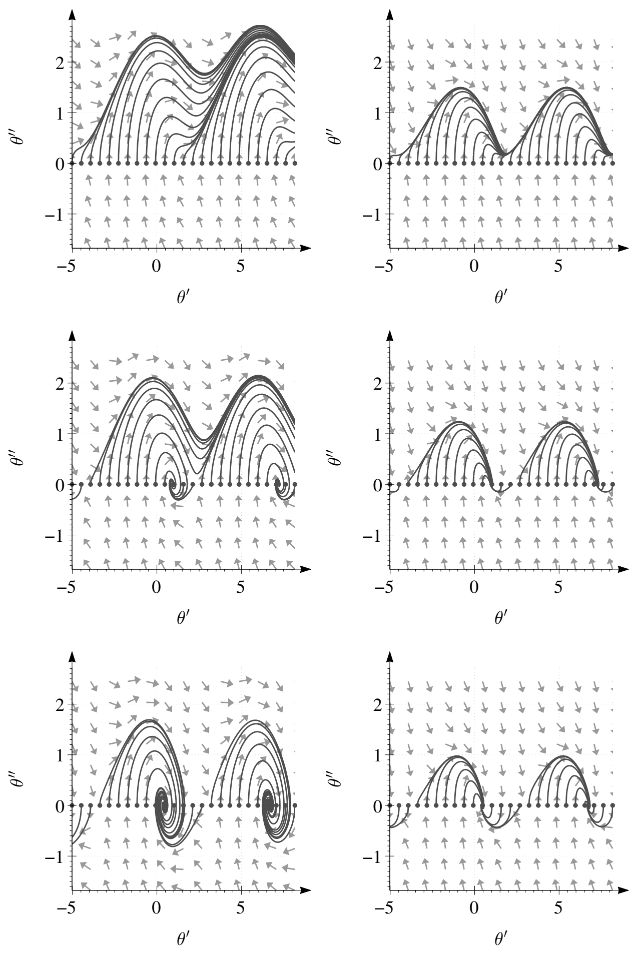


Fig. 8.3: Phase space of $\theta' = v$, $\quad v' = I - \alpha v - \sin(\theta)$. Left column: $\alpha = 0.5$, right column: $\alpha = 1.3$, top row: $I = 1.2$, middle row: $I = 0.8$, bottom row: $I = 0.4$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 8.5.2 -- %%');

% Feld anlegen
lval=[-1.6*pi 2.6*pi];
[x y]=meshgrid(lval(1):(lval(2)-lval(1))/12:lval(2), ...
               lval(1)/3:(lval(2)-lval(1))/32:lval(2)/3);

for alpha=[0.5 1.3]
   for I=[1.2 0.8 0.4]

      % DGL
      dx=y;
      dy=I-alpha*y-sin(x);
      dgl_sys=@(t,v)[v(2); I-alpha*v(2)-sin(v(1))];

      % Zeichenebene vorbereiten und Vektorfeld zeichnen
      customplot([lval(1) lval(2)]', ...
                 [lval(1)/3 lval(2)/3]',[],[],[2;-1]);
      ebenen=get(gcf,'Children');
      vectorfield(x,y,dx,dy,ebenen(1),ebenen(2), ...
               [0.6 0.6 0.6;0.4 75 90;0 0.85 0;2 0.5 4]);
      hold on

      % Numerische Lösungen bestimmen und dazu zeichnen
      ts=0:0.1:18;

      for startx=lval(1):(lval(2)-lval(1))/24:lval(2)
         for starty=[0];
           [t_s,res]=ode23(dgl_sys,ts,[startx starty]);

            plot(res(1,1),res(1,2),'o', ...
                 'MarkerFaceColor',[0.3 0.3 0.3], ...
                 'MarkerEdgeColor',[0.3 0.3 0.3])
            plot(res(:,1),res(:,2),'LineWidth',2, ...
                 'Color',[0.3 0.3 0.3])

            % Achsenbeschriftungen anpassen
            ebenen=get(gcf,'Children');
            set(get(ebenen(2),'XLabel'),'String', ...
                '$$\theta^\prime$$');
            set(get(ebenen(2),'YLabel'),'String', ...
                '$$\theta^{\prime\prime}$$');
         end
      end
   end
end
```

8.6 Coupled Oscillators and Quasiperiodicity

8.6.7 Mechanical example of quasiperiodicity.

The equations

$$m\ddot{r} = \frac{h}{mr^3} - k, \qquad \dot{\theta} = \frac{h}{mr^2}$$

govern the motion of a mass m subject to a central force of constant strength $k > 0$. Here r, θ are polar coordinates and $h > 0$ is a constant (the angular momentum of the particle).

a) Show that the system has a solution of the form $r = r_0$, $\dot{\theta} = \omega_\theta$, corresponding to uniform circular motion of radius r_0 and frequency ω_θ. Find formulas for r_0 and ω_θ.

   Having circular motion, there is no change of the radius ($\ddot{r} = 0$). We can write

$$m\ddot{r} = 0 = \frac{h^2}{mr_0^3} - k \quad \Leftrightarrow \quad r_0 = \sqrt[3]{\frac{h^2}{mk}}.$$

   Afterwards, the second equation can be rearranged as follows

$$\dot{\theta} = \omega_\theta = \frac{h}{mr_0^2} \quad \Leftrightarrow \quad \omega_\theta = \sqrt[3]{\frac{k^2}{hm}}.$$

b) Find the frequency ω_r for small radial oscillations about the circular orbit.

   Taking into account small radial perturbations $r = r_0 + \delta r$ yields

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}(r_0 + \delta r) = \frac{h}{m(r_0 + \delta r)^3} - k$$

c) Show that these small radial oscillations correspond to quasiperiodic motion by calculating the winding number ω_r/ω_θ.

d) Show by a geometric argument that the motion is either periodic or quasiperiodic for any amplitude of radial oscillation. (To say it in a more interesting way, the motion is never chaotic.)

e) Solve the equations on a computer, and plot the particle's path in the plane with polar coordinates r, θ.

8.7 Poincaré Maps

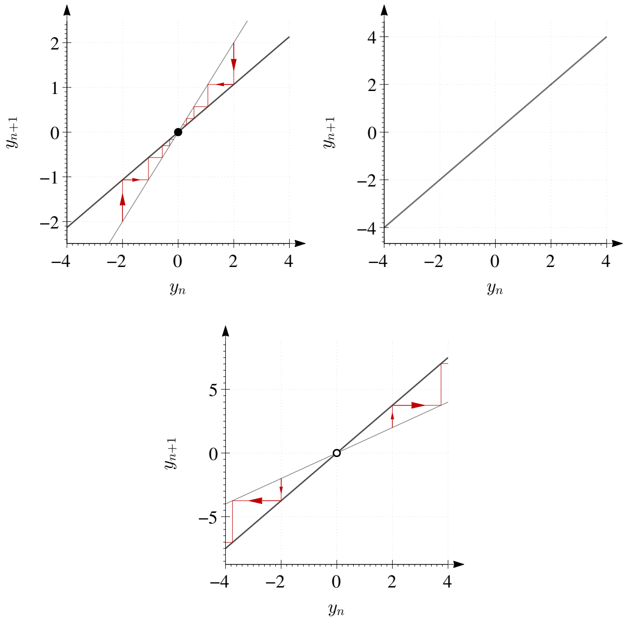
8.7.2 Consider the vector field on the cylinder given by $\dot{\theta} = 1, \quad \dot{y} = ay$.

Define an appropriate Poincaré map and find a formula for it. Show that the system has a periodic orbit. Classify its stability for all real values of a.

Since the time of flight can be seen with $\dot{\theta} = 1$ to be 2π, we can take 0 and 2π as our integration limits for time and evaluate the integral

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} = ay \quad &\Leftrightarrow \quad \frac{1}{a}\int_{y_0}^{y_1} \frac{1}{y}\,\mathrm{d}y = \int_0^{2\pi} \mathrm{d}t \\ &\Leftrightarrow \quad y_1 = y_0 e^{2\pi a}. \end{aligned}$$

So we have a Poincaré map $P(y) = ye^{2\pi a}$. It has a stable fixed point at zero for $a < 0$ (and therefore the original system $\dot{\theta} = 1, \quad \dot{y} = ay$ has a stable limit cycle at $y = 0$). $a > 0$ yields an unstable fixed point and therefore no stable limit cycle. The case $a = 0$ yields neutrally stable limit cycles.




```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 8.7.2 -- %%');

% Variiere a
for a=-0.1:0.1:0.1
   % Poincaré Map der Differentialgleichung
   P=inline(['exp(',num2str(a),'*2*pi)*var'],'var');

   % Zu betrachtendes Intervall
   interval=[-4 4];

   % Startwerte für die Cobweb-Trajektorien
   startval=[-2 2];

   % Anzahl der Iterationen
   steps=4;

   % Zeichne das Cobweb
   cobwebplot(P,interval,startval,steps,1,[1/12 1/12]);

   % Achsenbeschriftungen anpassen
   ebenen=get(gcf,'Children');
   set(get(ebenen(2),'XLabel'),'String','$$y_n$$');
   set(get(ebenen(2),'YLabel'),'String','$$y_{n+1}$$');
end
```

Fig. 8.4: Poincaré map $P(y) = ye^{2\pi a}$ of the system $\dot{\theta} = 1, \quad \dot{y} = ay$, top left: $a = -0.1$, top right: $a = 0$, bottom center: $a = 0.1$.

Exercises for Chapter 9

9.3 Chaos on a Strange Attractor

(Numerical experiments) For each of the values of r given below, use a computer to explore the dynamics of the Lorenz system, assuming $\sigma = 10$ and $b = 8/3$ as usual. In each case plot $x(t),\ y(t)$, and x vs. z. You should investigate the consequences of choosing different initial conditions and lengths of integration. Also in some cases you may want to ignore the transient behavior, and plot only the sustained long–term behavior.

The following initial values were used for the numerical investigation of the Lorenz system:

$$\begin{pmatrix}\alpha\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\\alpha\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\\alpha\end{pmatrix},\quad \alpha = 1,\ 10,\ 100; \qquad \beta\begin{pmatrix}\gamma_x\\\gamma_y\\\gamma_z\end{pmatrix},\quad \beta = 1,\ 20; \quad \gamma_{x,y,z} = \pm 1$$

The system has three fixed points, $\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\begin{pmatrix}\sqrt{b(r-1)}\\\sqrt{b(r-1)}\\r-1\end{pmatrix}$ and $\begin{pmatrix}-\sqrt{b(r-1)}\\-\sqrt{b(r-1)}\\r-1\end{pmatrix}$.

9.3.2 $r = 10$

$\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to the fixed point $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$. All other initial values approach a stable fixed point, either $\begin{pmatrix}\sqrt{24} & \sqrt{24} & 9\end{pmatrix}^T$ or $\begin{pmatrix}-\sqrt{24} & -\sqrt{24} & 9\end{pmatrix}^T$.

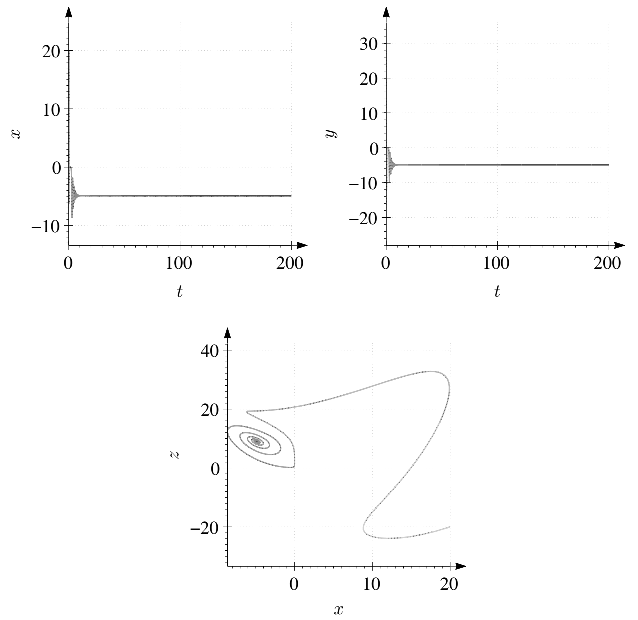


Fig. 9.1: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 10$ for initial values $(20\ -20\ -20)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 9.3.x -- %%');

aufgabe=2;

% Lorenz DGL
sigma=10;
b=8/3;

r_vec=[0 10 22 24.5 100 126.52 400];
r=r_vec(aufgabe);

dgl_sys=@(t,x)[sigma*(x(2)-x(1)); ...
               r*x(1)-x(2)-x(1)*x(3); ...
               x(1)*x(2)-b*x(3)];

%ts=[0:0.005:10 10.01:0.01:40 40.02:0.02:200];
ts=[0:0.005:10 10.01:0.01:120];

startvalmat=[1 0 0;10 0 0;100 0 0; ...
             0 1 0;0 10 0;0 100 0; ...
             0 0 1;0 0 10;0 0 100];
%startvalmat=[1 1 1;  -1 1 1; 1 -1 1; 1 1 -1;
%             1 -1 -1; -1 1 -1; -1 -1 1; -1 -1 -1];
%startvalmat=20*startvalmat;

for count=1:size(startvalmat,1)
   [t_s,res]=ode23(dgl_sys,ts,startvalmat(count,:));

   disp(['start: (',num2str(startvalmat(count,:)), ...
         '): x_e=',num2str(res(end,1)), ...
         '; y_e=',num2str(res(end,2)), ...
         '; z_e=',num2str(res(end,3))]);

   customplot(t_s,res(:,1),[],[1/6 1/6],[size(ts,2); ...
              3.8],'$$t$$','$$x$$');
   customplot(t_s,res(:,2),[],[1/6 1/6],[size(ts,2); ...
              3.8],'$$t$$','$$y$$');
   customplot(res(:,1),res(:,3),[],[1/6 1/6], ...
              [size(ts,2);3.8],'$$x$$','$$z$$');
end
```

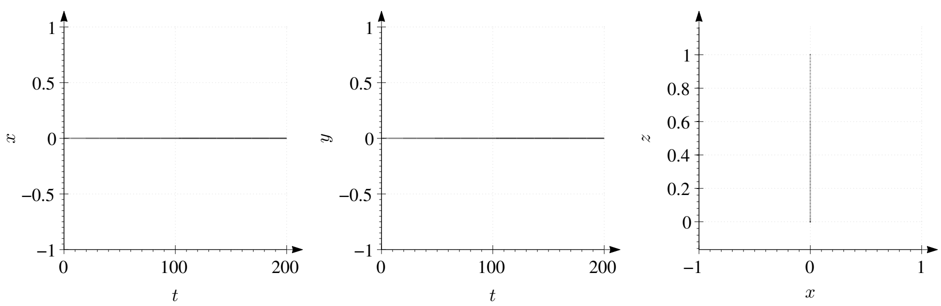


Fig. 9.2: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 10$ for initial values $(0\ 1\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

9.3.3 $r = 22$ (transient chaos)

Some initial values produce a transient chaos (as title suggested) but for all initial values convergence to a fixed point can be observed. Again, $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to the fixed point $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$, while the other values eventually approach $\begin{pmatrix}\sqrt{56} & \sqrt{56} & 21\end{pmatrix}^T$ or $\begin{pmatrix}-\sqrt{56} & -\sqrt{56} & 21\end{pmatrix}^T$.

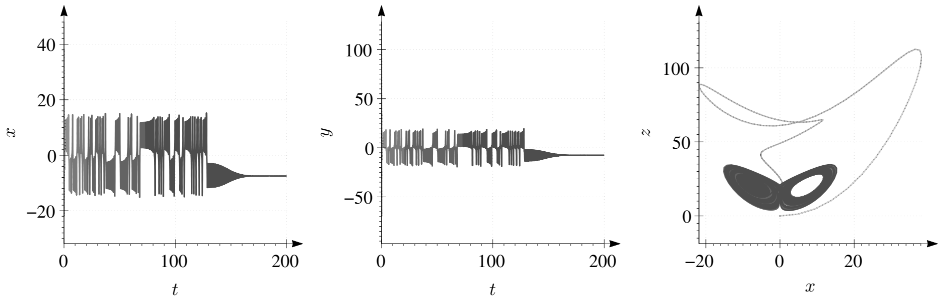


Fig. 9.3: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 22$ for initial values $(0\ 100\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

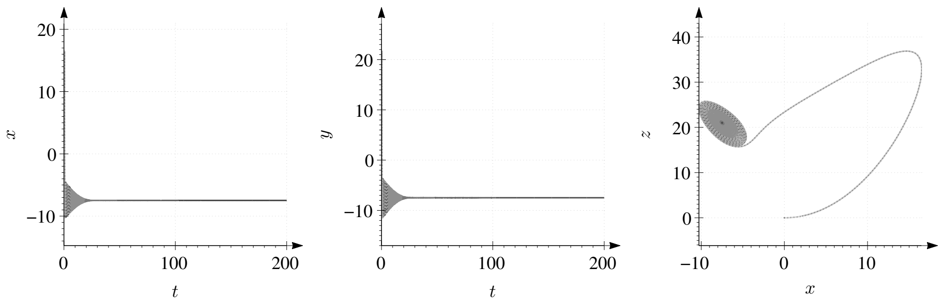


Fig. 9.4: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 22$ for initial values $(0\ 1\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

9.3.4 $r = 24.5$ (chaos and stable point co–exist)

Still, $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$, but other initial values seem to trace out a strange attractor. Eventually, some values spiral down to one fixed point $\left(\begin{pmatrix}\sqrt{\frac{188}{3}} & \sqrt{\frac{188}{3}} & 23.5\end{pmatrix}^T\right.$ or $\left.\begin{pmatrix}-\sqrt{\frac{188}{3}} & -\sqrt{\frac{188}{3}} & 23.5\end{pmatrix}^T\right)$ but others don't (at least in the observed time interval).

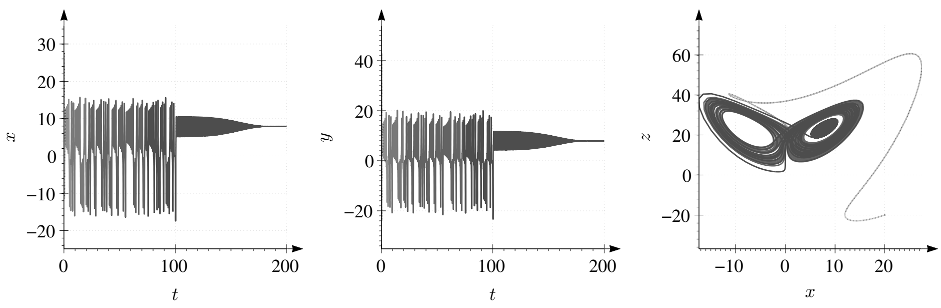


Fig. 9.5: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 24.5$ for initial values $(20\ \ -20\ \ -20)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

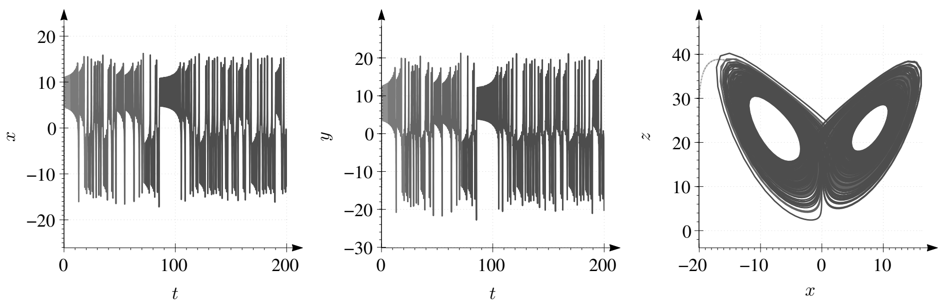


Fig. 9.6: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 24.5$ for initial values $(-20\ \ -20\ 20)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

9.3.5 $r = 100$ (surprise)

As before, $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$, but all other initial values seem to be on a strange attractor. Here, some values don't exhibit the typical ring–like structure but seem to fill out the space around the fixed points $\left(\begin{pmatrix}\sqrt{264} & \sqrt{264} & 99\end{pmatrix}^T, \begin{pmatrix}-\sqrt{264} & -\sqrt{264} & 99\end{pmatrix}^T\right)$.

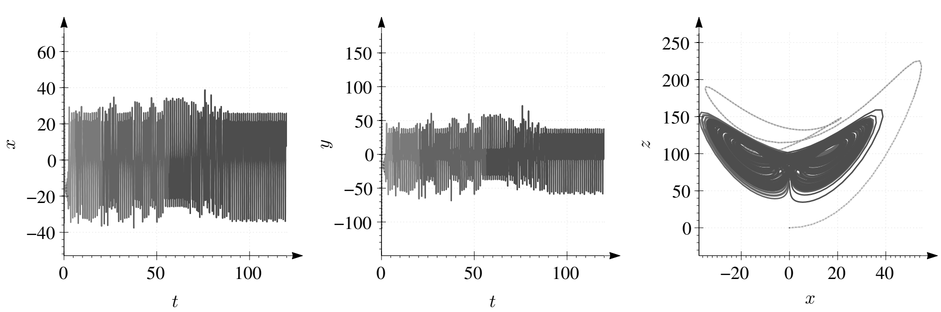


Fig. 9.7: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 100$ for initial values $(0\ 100\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

9.3.6 $r = 126.52$

Again, $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$, but all other initial values seem to be on a strange attractor.

Two fixed points are at $\left(\begin{pmatrix}\sqrt{\frac{8368}{25}} & \sqrt{\frac{8368}{25}} & 125.52\end{pmatrix}^T, \begin{pmatrix}-\sqrt{\frac{8368}{25}} & -\sqrt{\frac{8368}{25}} & 125.52\end{pmatrix}^T\right)$.

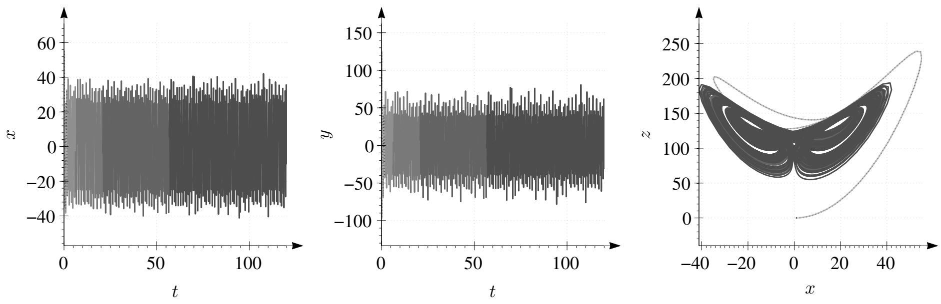


Fig. 9.8: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 126.52$ for initial values $(1\ 0\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

9.3.7 $r = 400$

Again, $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$, but all other initial values are after some transient behavior on a narrow, periodic band. This indicates an attracting limit cycle.

Two fixed points are at $\left(\begin{pmatrix}\sqrt{1064} & \sqrt{1064} & 399\end{pmatrix}^T, \begin{pmatrix}-\sqrt{1064} & -\sqrt{1064} & 399\end{pmatrix}^T\right)$.

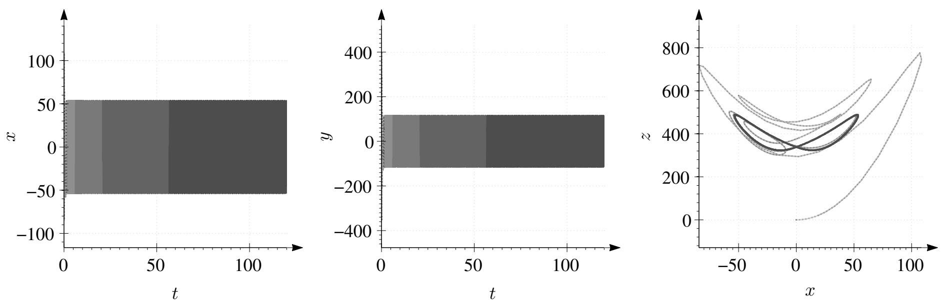


Fig. 9.9: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 400$ for initial values $(0\ 10\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

9.3.8 Practice with the definition of an attractor

Consider the following familiar system in polar coordinates: $\dot{\boldsymbol{r}} = \boldsymbol{r(1-r^2)}, \quad \dot{\boldsymbol{\theta}} = \boldsymbol{1}$. Let $\boldsymbol{D}$ be the disk $\boldsymbol{x^2+y^2 \leq 1}$.

a) Is $\boldsymbol{D}$ an invariant set?

Since $D : r^2 \leq 1$, all initial values in D will stay in D, so it is an invariant set.

b) Does $\boldsymbol{D}$ attract an open set of initial conditions?

The condition can be stated as $\dot{r} = r(1-r^2) \leq 1$. It means, the length of the radii isn't allowed to grow for the value to be attracted. Testing values outside the disk (e.g. $r = 1.1 \;\Rightarrow\; \dot{r} = 0.11 < 1$) shows, that they are also attracted. So D attracts an open set of initial conditions.

c) Is $\boldsymbol{D}$ an attractor? If not, why not? If so, find its basin of attraction.

D is not minimal, since it has a smaller attractor within it. Therefore, it is no attractor.

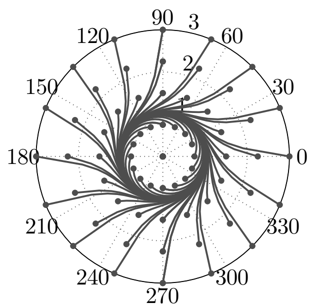


Fig. 9.10: Phase space of D with numerical solutions. The circles are lines of the same radius and the values are shown in degrees.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 9.3.8 -- %%');

ts=[0:0.1:2];

dgl_sys=@(t,r)[r(1)*(1-r(1)^2); 1];

for startr=-3:0.75:3
   for starttheta=-pi/2:pi/8:pi/2
      [t_s,res]=ode23(dgl_sys,ts,[startr starttheta]);

      t1=polar(res(1,2),res(1,1)); hold on;
      t2=polar(res(:,2),res(:,1));

      set(t1,'Marker','o', ...
            'MarkerFaceColor',[0.3 0.3 0.3], ...
            'MarkerEdgeColor',[0.3 0.3 0.3])
      set(t2,'LineWidth',2,'Color',[0.3 0.3 0.3])
      clear t1 t2
   end
end

textfelder=findall(gcf,'Type','text');
for i_resize=1:length(textfelder),
   set(textfelder(i_resize),'FontSize',24)
end
```

d) Repeat part (c) for the circle $\boldsymbol{x^2+y^2 = 1}$.

$D' : r^2 = 1$ is an invariant set because initial values on the circle stay on it $(r = 1 \in D \Rightarrow \dot{r} = 0)$. As for D in part (b), D' attracts an open set of initial conditions but D' is minimal. Since all values without the origin are attracted, $\mathtt{R} \setminus \{0, 0\}$ is the basin of attraction.

9.5 Exploring Parameter Space

(Numerical experiments) For each of the values of r fiven below, use a computer to explore the dynamics of the Lorenz system, assuming $\sigma = 10$ and $b = 8/3$ as usual. In each case, plot $x(t)$, $y(t)$ and x vs. z.

9.5.1 $r = 166.3$ (intermittent chaos)

As in previous parts concerning the Lorenz system, $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$. The other values exhibit transient chaos on a strange attractor, but after time, all seem to be on an attracting limit cycle.

Two fixed points are at $\left(\begin{pmatrix}\sqrt{\frac{2204}{5}} & \sqrt{\frac{2204}{5}} & 165.3\end{pmatrix}^T, \begin{pmatrix}-\sqrt{\frac{2204}{5}} & -\sqrt{\frac{2204}{5}} & 165.3\end{pmatrix}^T\right)$.

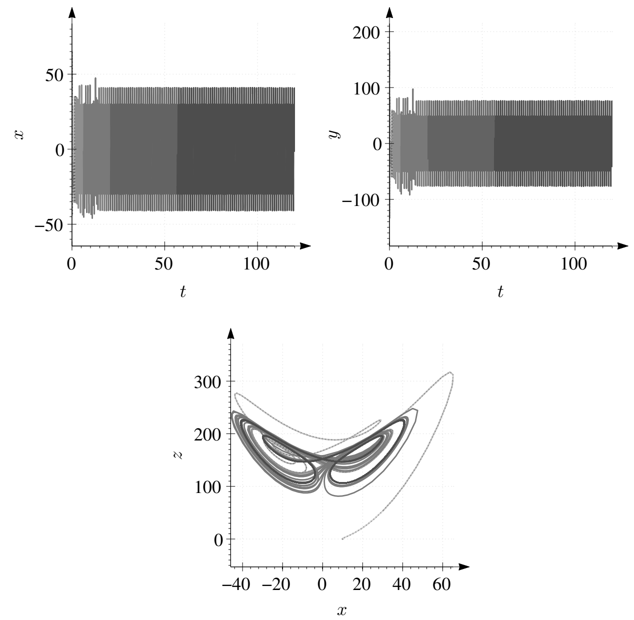


Fig. 9.11: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 166.3$ for initial values $(10\ 0\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
disp('%% -- Aufgabe 9.3.x -- %%');

aufgabe=2;

% Lorenz DGL
sigma=10;
b=8/3;

r_vec=[0 10 22 24.5 100 126.52 400];
r=r_vec(aufgabe);

dgl_sys=@(t,x)[sigma*(x(2)-x(1)); ...
               r*x(1)-x(2)-x(1)*x(3); ...
               x(1)*x(2)-b*x(3)];

%ts=[0:0.005:10 10.01:0.01:40 40.02:0.02:200];
ts=[0:0.005:10 10.01:0.01:120];

startvalmat=[1 0 0;10 0 0;100 0 0; ...
             0 1 0;0 10 0;0 100 0; ...
             0 0 1;0 0 10;0 0 100];
%startvalmat=[1 1 1;  -1 1 1; 1 -1 1; 1 1 -1;
%              1 -1 -1; -1 1 -1; -1 -1 1; -1 -1 -1];
%startvalmat=20*startvalmat;

for count=1:size(startvalmat,1)
   [t_s,res]=ode23(dgl_sys,ts,startvalmat(count,:));

   disp(['start: (',num2str(startvalmat(count,:)), ...
         '): x_e=',num2str(res(end,1)), ...
         '; y_e=',num2str(res(end,2)), ...
         '; z_e=',num2str(res(end,3))]);

   customplot(t_s,res(:,1),[],[1/6 1/6],[size(ts,2); ...
              3.8],'$$t$$','$$x$$');
   customplot(t_s,res(:,2),[],[1/6 1/6],[size(ts,2); ...
              3.8],'$$t$$','$$y$$');
   customplot(res(:,1),res(:,3),[],[1/6 1/6], ...
              [size(ts,2);3.8],'$$x$$','$$z$$');
end
```

9.5.2 $r = 212$ (noisy periodicity)

Again $\begin{pmatrix}0 & 0 & \alpha\end{pmatrix}^T$ converges to $\begin{pmatrix}0 & 0 & 0\end{pmatrix}^T$. This time, the band on the strange attractor is broader but still represents a limit cycle (hence the name noisy periodicity).

Two fixed points are at $\left(\begin{pmatrix}\sqrt{\frac{1688}{3}} & \sqrt{\frac{1688}{3}} & 211\end{pmatrix}^T, \begin{pmatrix}-\sqrt{\frac{1688}{3}} & -\sqrt{\frac{1688}{3}} & 211\end{pmatrix}^T\right)$.

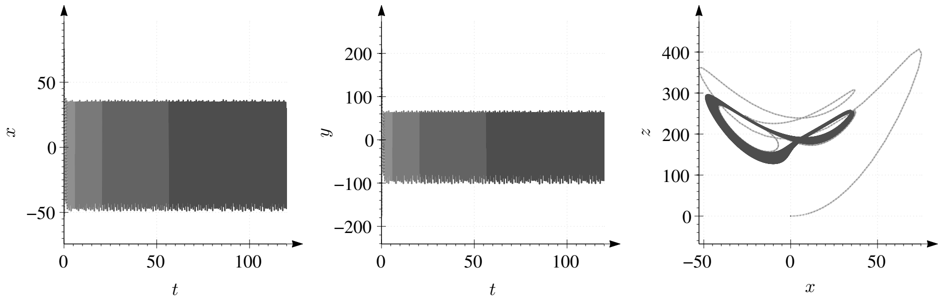


Fig. 9.12: Phase space of Lorenz system ($\sigma = 10$, $b = \frac{8}{3}$) with $r = 212$ for initial values $(0\ 10\ 0)^T$, top left: time dependency $x(t)$, top right: time dependency $y(t)$, bottom: phase space of $z(x)$.