## Chapter 8

# Vector Analysis in Higher Dimensions

### 8.1 An Introduction to Differential Forms

**1.** $(dx_1 - 3\,dx_2)(7,3) = dx_1(7,3) - 3\,dx_2(7,3) = 7 - 3(3) = -2.$

**2.**

$$\begin{aligned}&(2\,dx + 6\,dy - 5\,dz)(1,-1,-2)\\ &\quad = 2\,dx(1,-1,2) + 6\,dy(1,-1,-2) - 5\,dz(1,-1,-2)\\ &\quad = 2(1) + 6(-1) - 5(-2) = 6.\end{aligned}$$

**3.** $(3\,dx_1 \wedge dx_2)((4,-1),(2,0)) = 3 \det \begin{bmatrix} dx_1(4,-1) & dx_1(2,0) \\ dx_2(4,-1) & dx_2(2,0) \end{bmatrix} = 3 \det \begin{bmatrix} 4 & 2 \\ -1 & 0 \end{bmatrix} = 3(2) = 6.$

**4.**

$$\begin{aligned}&(4\,dx \wedge dy - 7\,dy \wedge dz)((0,1,-1),(1,3,2))\\ &\quad = 4\,dx \wedge dy((0,1,-1),(1,3,2)) - 7\,dy \wedge dz((0,1,-1),(1,3,2))\\ &\quad = 4 \det \begin{bmatrix} dx(0,1,-1) & dx(1,3,2) \\ dy(0,1,-1) & dy(1,3,2) \end{bmatrix} - 7 \det \begin{bmatrix} dy(0,1,-1) & dy(1,3,2) \\ dz(0,1,-1) & dz(1,3,2) \end{bmatrix}\\ &\quad = 4 \det \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix} - 7 \det \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} = 4(-1) - 7(5) = -39.\end{aligned}$$

**5.** We have

$$\begin{aligned}7\,dx \wedge dy \wedge dz(\mathbf{a},\mathbf{b},\mathbf{c}) &= 7 \det \begin{bmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) & dx(\mathbf{c}) \\ dy(\mathbf{a}) & dy(\mathbf{b}) & dy(\mathbf{c}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) & dz(\mathbf{c}) \end{bmatrix}\\ &= 7 \det \begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} = 7(-1 + 12 + 15) = 182.\end{aligned}$$

**6.** We have

$$\begin{aligned}&(dx_1 \wedge dx_2 + 2\,dx_2 \wedge dx_3 + 3\,dx_3 \wedge dx_4)(\mathbf{a},\mathbf{b})\\ &\quad = \det \begin{bmatrix} dx_1(\mathbf{a}) & dx_1(\mathbf{b}) \\ dx_2(\mathbf{a}) & dx_2(\mathbf{b}) \end{bmatrix} + 2 \det \begin{bmatrix} dx_2(\mathbf{a}) & dx_2(\mathbf{b}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) \end{bmatrix} + 3 \det \begin{bmatrix} dx_3(\mathbf{a}) & dx_3(\mathbf{b}) \\ dx_4(\mathbf{a}) & dx_4(\mathbf{b}) \end{bmatrix}\\ &\quad = \det \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} + 3 \det \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5 + 2(-5) + 3(-5) = -30.\end{aligned}$$

**7.**

$$\begin{aligned}
&(2\,dx_1 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_3 \wedge dx_5)(\mathbf{a}, \mathbf{b}, \mathbf{c}) \\
&= 2 \det \begin{bmatrix} dx_1(\mathbf{a}) & dx_1(\mathbf{b}) & dx_1(\mathbf{c}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) & dx_3(\mathbf{c}) \\ dx_4(\mathbf{a}) & dx_4(\mathbf{b}) & dx_4(\mathbf{c}) \end{bmatrix} + \det \begin{bmatrix} dx_2(\mathbf{a}) & dx_2(\mathbf{b}) & dx_2(\mathbf{c}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) & dx_3(\mathbf{c}) \\ dx_5(\mathbf{a}) & dx_5(\mathbf{b}) & dx_5(\mathbf{c}) \end{bmatrix} \\
&= 2 \det \begin{bmatrix} 1 & 0 & 5 \\ -1 & 9 & 0 \\ 4 & 1 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & 0 \\ -1 & 9 & 0 \\ 2 & -1 & -2 \end{bmatrix} \\
&= 2(-185) + 0 = -370.
\end{aligned}$$

**8.** $\omega_{(3,-1,4)}(\mathbf{a}) = (-9\,dx + 4\,dy + 192\,dz)(a_1, a_2, a_3) = -9a_1 + 4a_2 + 192a_3.$

**9.**

$$\begin{aligned}
\omega_{(2,-1,-3,1)}(\mathbf{a}, \mathbf{b}) &= (-6\,dx_1 \wedge dx_3 + dx_2 \wedge dx_4)(\mathbf{a}, \mathbf{b}) \\
&= -6 \det \begin{bmatrix} dx_1(\mathbf{a}) & dx_1(\mathbf{b}) \\ dx_3(\mathbf{a}) & dx_3(\mathbf{b}) \end{bmatrix} + \det \begin{bmatrix} dx_2(\mathbf{a}) & dx_2(\mathbf{b}) \\ dx_4(\mathbf{a}) & dx_4(\mathbf{b}) \end{bmatrix} \\
&= -6(a_1b_3 - a_3b_1) + a_2b_4 - a_4b_2.
\end{aligned}$$

**10.**

$$\begin{aligned}
\omega_{(0,-1,\pi/2)}(\mathbf{a}, \mathbf{b}) &= (1\,dx \wedge dy - 1\,dy \wedge dz + 4\,dx \wedge dz)(\mathbf{a}, \mathbf{b}) \\
&= \begin{vmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) \\ dy(\mathbf{a}) & dy(\mathbf{b}) \end{vmatrix} - \begin{vmatrix} dy(\mathbf{a}) & dy(\mathbf{b}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) \end{vmatrix} + 4 \begin{vmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) \end{vmatrix} \\
&= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} - \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} + 4 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \\
&= a_1b_2 - a_2b_1 - (a_2b_3 - a_3b_2) + 4(a_1b_3 - a_3b_1)
\end{aligned}$$

**11.**

$$\begin{aligned}
\omega_{(x,y,z)}((2,0,-1),\ (1,7,5)) &= \cos x \begin{vmatrix} 2 & 1 \\ 0 & 7 \end{vmatrix} - \sin z \begin{vmatrix} 0 & 7 \\ -1 & 5 \end{vmatrix} + (y^2+3) \begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix} \\
&= 14\ \cos x - 7 \sin z + 11(y^2 + 3)
\end{aligned}$$

**12.** We have

$$\begin{aligned}
\omega_{(0,0,0)}(\mathbf{a}, \mathbf{b}, \mathbf{c}) &= (3\,dx \wedge dy \wedge dz)(\mathbf{a}, \mathbf{b}, \mathbf{c}) \\
&= 3 \det \begin{bmatrix} dx(\mathbf{a}) & dx(\mathbf{b}) & dx(\mathbf{c}) \\ dy(\mathbf{a}) & dy(\mathbf{b}) & dy(\mathbf{c}) \\ dz(\mathbf{a}) & dz(\mathbf{b}) & dz(\mathbf{c}) \end{bmatrix} = 3 \det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \\
&= 3(a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3).
\end{aligned}$$

**13.** We have

$$\begin{aligned}
\omega_{(x,y,z)}\left((1,0,0),(0,2,0),(0,0,3)\right) &= \left(e^x \cos y + (y^2+2)e^{2z}\right) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&= 6\left(e^x \cos y + (y^2+2)e^{2z}\right).
\end{aligned}$$

**14.** From Definition 1.3 of exterior product,

$$\begin{aligned}&(3\,dx + 2\,dy - x\,dz) \wedge (x^2\,dx - \cos y\,dy + 7\,dz)\\ &= 3x^2\,dx \wedge dx + 2x^2\,dy \wedge dx - x^3\,dz \wedge dx - 3\cos y\,dx \wedge dy - 2\cos y\,dy \wedge dy + x\cos y\,dz \wedge dy\\ &\quad + 21\,dx \wedge dz + 14\,dy \wedge dz - 7x\,dz \wedge dz\\ &= 2x^2\,dy \wedge dx - x^3\,dz \wedge dx - 3\cos y\,dx \wedge dy + x\cos y\,dz \wedge dy\\ &\quad + 21\,dx \wedge dz + 14\,dy \wedge dz \quad \text{using (4)},\\ &= -(2x^2 + 3\cos y)\,dx \wedge dy + (x^3 + 21)\,dx \wedge dz\\ &\quad + (14 - x\cos y)\,dy \wedge dz \quad \text{using (3)}.\end{aligned}$$

**15.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}&(y\,dx - x\,dy) \wedge (z\,dx \wedge dy + y\,dx \wedge dz + x\,dy \wedge dz)\\ &= yz\,dx \wedge dx \wedge dy - xz\,dy \wedge dx \wedge dy + y^2\,dx \wedge dx \wedge dz - xy\,dy \wedge dx \wedge dz\\ &\quad + xy\,dx \wedge dy \wedge dz - x^2\,dy \wedge dy \wedge dz\\ &= 2xy\,dx \wedge dy \wedge dz \quad \text{using (3) and (4)}.\end{aligned}$$

**16.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}&(2\,dx_1 \wedge dx_2 - x_3\,dx_2 \wedge dx_4) \wedge (2x_4\,dx_1 \wedge dx_3 + (x_3 - x_2)\,dx_3 \wedge dx_4)\\ &= 4x_4\,dx_1 \wedge dx_2 \wedge dx_1 \wedge dx_3 - 2x_3x_4\,dx_2 \wedge dx_4 \wedge dx_1 \wedge dx_3\\ &\quad + 2(x_3 - x_2)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - x_3(x_3 - x_2)\,dx_2 \wedge dx_4 \wedge dx_3 \wedge dx_4\\ &= -2x_3x_4\,dx_2 \wedge dx_4 \wedge dx_1 \wedge dx_3 + 2(x_3 - x_2)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \quad \text{using (4)},\\ &= 2(x_3x_4 + x_3 - x_2)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \quad \text{using (3)}.\end{aligned}$$

**17.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}&(x_1\,dx_1 + 2x_2\,dx_2 + 3x_3\,dx_3) \wedge ((x_1 + x_2)\,dx_1 \wedge dx_2 \wedge dx_3 + (x_3 - x_4)\,dx_1 \wedge dx_2 \wedge dx_4)\\ &= x_1(x_1 + x_2)\,dx_1 \wedge dx_1 \wedge dx_2 \wedge dx_3 + 2x_2(x_1 + x_2)\,dx_2 \wedge dx_1 \wedge dx_2 \wedge dx_3\\ &\quad + 3x_2(x_1 + x_2)\,dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_3 + x_1(x_3 - x_4)\,dx_1 \wedge dx_1 \wedge dx_2 \wedge dx_4\\ &= 2x_2(x_3 - x_4)\,dx_2 \wedge dx_1 \wedge dx_2 \wedge dx_4 + 3x_3(x_3 - x_4)\,dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4.\end{aligned}$$

Using equation (4), this last expression is equal to

$$0 + 0 + 0 + 0 + 0 + 3x_3(x_3 - x_4)\,dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4 = 3x_3(x_3 - x_4)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4,$$

using equation (3).

**18.** We can work everything out, or note that $\omega$ and $\eta$ in this problem are $\eta$ and $\omega$ (respectively) in Exercise 17. Thus anticommutativity (property 2 of Proposition 1.4) may thus be applied to give

$$\omega \wedge \eta = (-1)^{3\cdot 1} 3x_3(x_3 - x_4)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = -3x_3(x_3 - x_4)\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

**19.** Again from Definition 1.3 of exterior product,

$$\begin{aligned}&(x_1\,dx_2 \wedge dx_3 - x_2x_3\,dx_1 \wedge dx_5) \wedge (e^{x_4x_5}\,dx_1 \wedge dx_4 \wedge dx_5 - x_1\cos x_5\,dx_2 \wedge dx_3 \wedge dx_4)\\ &= x_1e^{x_4x_5}\,dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_4 \wedge dx_5 - x_2x_3e^{x_4x_5}\,dx_1 \wedge dx_5 \wedge dx_1 \wedge dx_4 \wedge dx_5\\ &\quad - x_1^2\cos x_5\,dx_2 \wedge dx_3 \wedge dx_2 \wedge dx_3 \wedge dx_4 + x_1x_2x_3\cos x_5\,dx_1 \wedge dx_5 \wedge dx_2 \wedge dx_3 \wedge dx_4\\ &= x_1e^{x_4x_5}\,dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_4 \wedge dx_5 + x_1x_2x_3\cos x_5\,dx_1 \wedge dx_5 \wedge dx_2 \wedge dx_3 \wedge dx_4 \quad \text{using (4)},\\ &= (x_1e^{x_4x_5} - x_1x_2x_3\cos x_5)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5 \quad \text{using (3)}.\end{aligned}$$

**20.** Using Definition 1.1,

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$$

$$= \det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \ldots & dx_{i_1}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_j}(\mathbf{a}_1) & dx_{i_j}(\mathbf{a}_2) & \ldots & dx_{i_j}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_l}(\mathbf{a}_1) & dx_{i_l}(\mathbf{a}_2) & \ldots & dx_{i_l}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \ldots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix}$$

$$= -\det \begin{bmatrix} dx_{i_1}(\mathbf{a}_1) & dx_{i_1}(\mathbf{a}_2) & \ldots & dx_{i_1}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_l}(\mathbf{a}_1) & dx_{i_l}(\mathbf{a}_2) & \ldots & dx_{i_l}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_j}(\mathbf{a}_1) & dx_{i_j}(\mathbf{a}_2) & \ldots & dx_{i_j}(\mathbf{a}_k) \\ \vdots & \vdots & & \vdots \\ dx_{i_k}(\mathbf{a}_1) & dx_{i_k}(\mathbf{a}_2) & \ldots & dx_{i_k}(\mathbf{a}_k) \end{bmatrix}$$

(since switching rows $l$ and $j$ changes the sign of the determinant)

$$= -dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k).$$

**21.** This is easier to show in person, but the point is that if you switch the two identical forms then, on the one hand, nothing has changed and, on the other hand, formula (3) says that you now have the negative of what you started with. So

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k} = -dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k}$$

and therefore

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_k} = 0.$$

**22.** A $k$-form $\omega$ on $\mathbf{R}^n$ may be written as $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. For each summand, each of the $k\, dx_{i_j}$'s is one of $dx_1, dx_2, \ldots dx_n$. If $k > n$, then, by the pigeon hole principle, there must be at least one repeated term $dx_l$ in $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ (i.e., it will look like $dx_{i_1} \wedge \cdots \wedge dx_l \wedge \cdots \wedge dx_l \wedge \cdots \wedge dx_{i_k}$). And so, by formula (4), we have that $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$. Hence every term of $\omega$ is zero.

**23.** Let $\omega_1 = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\omega_2 = \sum G_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\eta = \sum H_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$. Then

$$\begin{aligned}
(\omega_1 + \omega_2) \wedge \eta &= \left[ \sum_{i_1, \ldots, i_k} (F_{i_1 \ldots i_k} + G_{i_1 \ldots i_k})\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right] \wedge \sum_{j_1, \ldots, j_l} H_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_l}} (F_{i_1 \ldots i_k} + G_{i_1 \ldots i_k}) H_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_l}} F_{i_1 \ldots i_k} H_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&\quad + \sum_{\substack{i_1, \ldots, i_k \\ j_1, \ldots, j_l}} G_{i_1 \ldots i_k} H_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\
&= \omega_1 \wedge \eta + \omega_2 \wedge \eta.
\end{aligned}$$

**24.** Let $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\eta = \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$. Then

$$\omega \wedge \eta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1\ldots i_k} G_{j_1\ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Now move $dx_{j_1}$ to the front by switching, in reverse order, with each of the $dx_{i_p}$'s. There are $k$ switches so, by formula (3), there are $k$ sign changes and this last equation becomes

$$\omega \wedge \eta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1\ldots i_k} G_{j_1\ldots j_l} (-1)^k\, dx_{j_1} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_l}.$$

Similarly, we use $k$ more interchanges to move $dx_{j_2}$ into the second position. We repeat this for each of the $l$ $dx_{j_q}$'s. and our equation becomes

$$\omega \wedge \eta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1\ldots i_k} G_{j_1\ldots j_l} \underbrace{(-1)^k(-1)^k\cdots(-1)^k}_{l \text{ times}}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = (-1)^{kl}\eta \wedge \omega.$$

**25.** Let $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\eta = \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$, and $\tau = \sum H_{u_1 \ldots u_m}\, dx_{u_1} \wedge \cdots \wedge dx_{u_m}$. Then

$$\begin{aligned}(\omega \wedge \eta) \wedge \tau &= \left[\sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1\ldots i_k} G_{j_1\ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}\right] \\ &\quad \wedge \sum_{u_1,\ldots,u_m} H_{u_1\ldots u_m}\, dx_{u_1} \wedge \cdots \wedge dx_{u_m} \\ &= \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l \\ u_1,\ldots,u_m}} F_{i_1\ldots i_k} G_{j_1\ldots j_l} H_{u_1\ldots u_m}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \wedge dx_{u_1} \wedge \cdots \wedge dx_{u_m}.\end{aligned}$$

Similarly, calculate $\omega \wedge (\eta \wedge \tau)$ and you will obtain the same result.

**26.** Here $\omega = \sum F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, and $\eta = \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ and $f$ is a function (or 0-form). First we note that

$$\begin{aligned}(f\omega) \wedge \eta &= \left(\sum_{i_1,\ldots,i_k} fF_{i_1\ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \wedge \left(\sum_{j_1,\ldots,j_l} G_{j_1\ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}\right) \\ &= \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} fF_{i_1\ldots i_k} G_{j_1\ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= f \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} F_{i_1\ldots i_k} G_{j_1\ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= f(\omega \wedge \eta).\end{aligned}$$

We will use this result to establish the second equality,

$$\begin{aligned}f(\omega \wedge \eta) &= (-1)^{kl} f(\eta \wedge \omega) \quad \text{by property 2 of Proposition 1.4,} \\ &= (-1)^{kl}(f\eta) \wedge \omega \quad \text{by the result established above,} \\ &= (-1)^{kl}(-1)^{kl}\omega \wedge (f\eta) \quad \text{by property 2 of Proposition 1.4,} \\ &= \omega \wedge (f\eta).\end{aligned}$$

Therefore, $(f\omega) \wedge \eta = f(\omega \wedge \eta) = \omega \wedge (f\eta)$.

## 8.2 Manifolds and Integrals of $k$-Forms

**1.** Here the map is $\mathbf{X}(\theta_1, \theta_2, \theta_3) = (3\cos\theta_1,\ 3\sin\theta_1,\ 3\cos\theta_1 + 2\cos\theta_2,\ 3\sin\theta_1 + 2\sin\theta_2,\ 3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3,\ 3\sin\theta_1 + 2\sin\theta_2 + \sin\theta_3)$.
Follow the lead of Example 2 from the text. Each component function is at least $C^1$ so the mapping is at least $C^1$. To see one-one, consider the equation $\mathbf{X}(\theta_1, \theta_2, \theta_3) = \mathbf{X}(\hat\theta_1, \hat\theta_2, \hat\theta_3)$. The first two component equations would then have $\cos\theta_1 = \cos\hat\theta_1$ and $\sin\theta_1 = \sin\hat\theta_1$. Since $0 \le \theta_1, \hat\theta_1 < 2\pi$ we see that $\theta_1 = \hat\theta_1$. Using this information in the next two component functions, we make the same conclusion for $\theta_2$ and $\hat\theta_2$. Finally, use all of this information in the last set of equations to see that $\theta_3 = \hat\theta_3$. So $\mathbf{X}$ is one-one and $C^1$. What is left to show is that the tangent vectors $\mathbf{T}_{\theta_1}$, $\mathbf{T}_{\theta_2}$, and $\mathbf{T}_{\theta_3}$ are linearly independent.

$$\mathbf{T}_{\theta_1} = (-3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1)$$
$$\mathbf{T}_{\theta_2} = (0, 0, -2\sin\theta_2, 2\cos\theta_2, -2\sin\theta_2, 3\cos\theta_2)$$
$$\mathbf{T}_{\theta_3} = (0, 0, 0, 0, \sin\theta_3, \cos\theta_3)$$

Because of the leading pair of zeros in $\mathbf{T}_{\theta_2}$ and $\mathbf{T}_{\theta_3}$ we can see that if $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{\theta_2} + c_3\mathbf{T}_{\theta_3} = \mathbf{0}$, then $c_1 = 0$. Looking at the second pair of zeros in $\mathbf{T}_{\theta_3}$ we can then see that $c_2 = 0$. This would then force $c_3 = 0$. So $\mathbf{T}_{\theta_1}$, $\mathbf{T}_{\theta_2}$, and $\mathbf{T}_{\theta_3}$ are linearly independent. We have shown that the parametrized 3-manifold is a smooth parametrized 3-manifold.

**2.** As in Example 3, let's begin by describing the location of the point $(x_1, y_1)$. It is anywhere in the annular region described by $(l_1\cos\theta_1, l_1\sin\theta_1)$ where $1 \le l_1 \le 3$ and $0 \le \theta_1 < 2\pi$. You can now describe $(x_2, y_2)$ as being this same annular region centered at $(x_1, y_1)$. Together this means that the locus of $(x_2, y_2)$ is the interior of a disk of radius 6. Using variables $l_2$ and $\theta_2$ such that $1 \le l_2 \le 3$ and $0 \le \theta_2 < 2\pi$, the mapping is

$$\mathbf{X}(l_1, \theta_1, l_2, \theta_2) = (l_1\cos\theta_1, l_1\sin\theta_1, l_1\cos\theta_1 + l_2\cos\theta_2, l_1\sin\theta_1 + l_2\sin\theta_2).$$

As before, the component functions are at least $C^1$ so the mapping is at least $C^1$. As for one-one, consider $\mathbf{X}(l_1, \theta_1, l_2, \theta_2) = \mathbf{X}(\hat l_1, \hat\theta_1, \hat l_2, \hat\theta_2)$. From the first component functions we see that $(x_1, y_1)$ lies on a circle of radius $l_1$ and $(\hat x_1, \hat y_1)$ lies on a circle of radius $\hat l_1$ so $l_1 = \hat l_1$. Then, as in Exercise 1, $\cos\theta_1 = \cos\hat\theta_1$ and $\sin\theta_1 = \sin\hat\theta_1$. As $0 \le \theta_1, \hat\theta_1 < 2\pi$, we see that $\theta_1 = \hat\theta_1$. Now the rest of the argument follows in exactly the same way since $(x_2, y_2)$ is related to $(x_1, y_1)$ in the same way that $(x_1, y_1)$ is related to the origin. We now need to show that the four tangent vectors are linearly independent.

$$\mathbf{T}_{l_1} = (\cos\theta_1, \sin\theta_1, \cos\theta_1, \sin\theta_1)$$
$$\mathbf{T}_{\theta_1} = (-l_1\sin\theta_1, l_1\cos\theta_1, -l_1\sin\theta_1, l_1\cos\theta_1)$$
$$\mathbf{T}_{l_2} = (0, 0, \cos\theta_2, \sin\theta_2)$$
$$\mathbf{T}_{\theta_2} = (0, 0, -l_2\sin\theta_2, l_2\cos\theta_2)$$

Look at the equation $c_1\mathbf{T}_{l_1} + c_2\mathbf{T}_{\theta_1} + c_3\mathbf{T}_{l_2} + c_4\mathbf{T}_{\theta_2} = \mathbf{0}$. Because of the leading pair of zeros in $\mathbf{T}_{l_2}$ and $\mathbf{T}_{\theta_2}$ we can see that $c_1\cos\theta_1 = c_2 l_1\sin\theta_1$ and $c_1\sin\theta_1 = -c_2 l_1\cos\theta_1$. Solve for $c_1$ in the first equation and substitute into the second equation to get $c_2 l_1\sin^2\theta_1 = -c_2 l_1\cos^2\theta_1$. Because $l_1$ cannot be zero, this implies that $c_2 = 0$. This then implies that $c_1 = 0$. Given that, we can make the same argument to show $c_3 = c_4 = 0$. Therefore the four tangent vectors are linearly independent and we have described the states of the robot arm as a smooth parametrized 4-manifold in $\mathbf{R}^4$.

**3.** This is a combination of Example 3 and Exercise 2. Let's begin by describing the location of the point $(x_1, y_1)$. It is anywhere on a circle of radius 3 centered at the origin. So $(x_1, y_1) = (3\cos\theta_1, 3\sin\theta_1)$ where $0 \le \theta_1 < 2\pi$. We can then describe $(x_2, y_2)$ as being this same annular region centered at $(x_1, y_1)$. Together this means that the locus of $(x_2, y_2)$ is $(3\cos\theta_1 + l_2\cos\theta_2, 3\sin\theta_1 + l_2\sin\theta_2)$ where $1 \le l_2 \le 2$ and $0 \le \theta_2 < 2\pi$. Similarly we describe $(x_3, y_3)$ in terms of $(x_2, y_2)$ using variables $l_3$ and $\theta_3$ such that $1 \le l_3 \le 2$ and $0 \le \theta_2 < 2\pi$. The mapping is

$$\begin{aligned}\mathbf{X}(\theta_1, l_2, \theta_2, l_3, \theta_3) = (&3\cos\theta_1, 3\sin\theta_1, 3\cos\theta_1 + l_2\cos\theta_2, 3\sin\theta_1 + l_2\sin\theta_2,\\ &3\cos\theta_1 + l_2\cos\theta_2 + l_3\cos\theta_3, 3\sin\theta_1 + l_2\sin\theta_2 + l_3\sin\theta_3).\end{aligned}$$

As before, the component functions are at least $C^1$ so the mapping is at least $C^1$. As for one-one, consider $\mathbf{X}(\theta_1, l_2, \theta_2, l_3, \theta_3) = \mathbf{X}(\hat\theta_1, \hat l_2, \hat\theta_2, \hat l_3, \hat\theta_3)$. From the first two component functions we see that $\cos\theta_1 = \cos\hat\theta_1$ and $\sin\theta_1 = \sin\hat\theta_1$ and $0 \le \theta_1, \hat\theta_1 < 2\pi$ so $\theta_1 = \hat\theta_1$. Now, $(x_2, y_2)$ lies on a circle of radius $l_2$ and $(\hat x_2, \hat y_2)$ lies on a circle of radius $\hat l_2$ with each circle centered at the same point $(x_1, y_1) = (\hat x_1, \hat y_1)$. So $l_2 = \hat l_2$. Then, as above, $\cos\theta_2 = \cos\hat\theta_2$ and $\sin\theta_2 = \sin\hat\theta_2$. As $0 \le \theta_2, \hat\theta_2 < 2\pi$, we see that $\theta_2 = \hat\theta_2$. Now the rest of the argument follows in exactly the same way since $(x_3, y_3)$ is related to $(x_2, y_2)$ in the same way that $(x_2, y_2)$ is related to $(x_1, y_1)$.

We now need to show that the five tangent vectors are linearly independent.

$$\begin{aligned}
\mathbf{T}_{\theta_1} &= (-3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1, -3\sin\theta_1, 3\cos\theta_1)\\
\mathbf{T}_{l_2} &= (0, 0, \cos\theta_2, \sin\theta_2, \cos\theta_2, \sin\theta_2)\\
\mathbf{T}_{\theta_2} &= (0, 0, -l_2\sin\theta_2, l_2\cos\theta_2, -l_2\sin\theta_2, l_2\cos\theta_2)\\
\mathbf{T}_{l_3} &= (0, 0, 0, 0, \cos\theta_3, \sin\theta_3)\\
\mathbf{T}_{\theta_3} &= (0, 0, 0, 0, -l_3\sin\theta_3, l_3\cos\theta_3)
\end{aligned}$$

Look at the equation $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{l_2} + c_3\mathbf{T}_{\theta_2} + c_4\mathbf{T}_{l_3} + c_5\mathbf{T}_{\theta_3} = \mathbf{0}$. Because of the leading pair of zeros in all but the vector $\mathbf{T}_{\theta_1}$ we conclude that $c_1 = 0$. The remainder of the argument is exactly as in Exercise 2. Because the first four components of $\mathbf{T}_{l_3}$ and $\mathbf{T}_{\theta_3}$ are zero, we can see that $c_2\cos\theta_2 = c_3 l_2\sin\theta_2$ and $c_2\sin\theta_2 = -c_3 l_2\cos\theta_2$. Solve for $c_2$ in the first equation and substitute into the second equation to get $c_3 l_2\sin^2\theta_2 = -c_3 l_2\cos^2\theta_2$. Because $l_2$ cannot be zero, $c_3 = 0$. This then implies that $c_2 = 0$. Given that, we can make the same argument to show $c_4 = c_5 = 0$. Therefore the five tangent vectors are linearly independent and we have described the states of the robot arm as a smooth parametrized 5-manifold in $\mathbf{R}^6$.

**4.** We can use spherical coordinates to describe the parametrized space. The point $(x_1, y_1, z_1)$ can be written as $(2\sin\varphi_1\cos\theta_1, 2\sin\varphi_1\sin\theta_1, 2\cos\varphi_1)$ where $0 \le \varphi_1 \le \pi$ and $0 \le \theta_1 < 2\pi$. We can then write $(x_2, y_2, z_2)$ as $(x_1 + \sin\varphi_2\cos\theta_2, y_1 + \sin\varphi_2\sin\theta_2, z_1 + \cos\varphi_2)$ where $0 \le \varphi_2 \le \pi$ and $0 \le \theta_2 < 2\pi$. In other words, our mapping is

$$\begin{aligned}
\mathbf{X}(\theta_1, \varphi_1, \theta_2, \varphi_2) = (&2\sin\varphi_1\cos\theta_1, 2\sin\varphi_1\sin\theta_1, 2\cos\varphi_1,\\
&2\sin\varphi_1\cos\theta_1 + \sin\varphi_2\cos\theta_2, 2\sin\varphi_1\sin\theta_1 + \sin\varphi_2\sin\theta_2, 2\cos\varphi_1 + \cos\varphi_2).
\end{aligned}$$

As in the previous exercises, the fact that the component functions are at least $C^1$ tells us that the mapping is at least $C^1$. Checking one-one is a little more interesting than in the above exercises. Consider the implications of the equation $\mathbf{X}(\theta_1, \varphi_1, \theta_2, \varphi_2) = \mathbf{X}(\hat\theta_1, \hat\varphi_1, \hat\theta_2, \hat\varphi_2)$. By the third component functions we see that $\cos\varphi_1 = \cos\hat\varphi_1$. Because $0 \le \varphi_1, \hat\varphi_1 \le \pi$ we see that $\varphi_1 = \hat\varphi_1$. Substituting this into the sixth component function implies that $\varphi_2 = \hat\varphi_2$. Now comparing the equations from the first two component functions we see that if $\varphi_1 = 0$ or $\pi$ then $\theta_1$ need not be the same as $\hat\theta_1$. This is allowed—recall that the mapping might not be one-one on the boundary of the domain. Other than on the boundary, $\cos\theta_1 = \cos\hat\theta_1$ and $\sin\theta_1 = \sin\hat\theta_1$ and so, as before $\theta_1 = \hat\theta_1$. Again, substitute this into the equations that arise from the fourth and fifth component functions to conclude that, except when $\varphi_2$ is 0 or $\pi$, we must have $\theta_2 = \hat\theta_2$.

We now need to show that the four tangent vectors are linearly independent.

$$\begin{aligned}
\mathbf{T}_{\theta_1} &= (-2\sin\varphi_1\sin\theta_1, 2\sin\varphi_1\cos\theta_1, 0, -2\sin\varphi_1\sin\theta_1, 2\sin\varphi_1\cos\theta_1, 0)\\
\mathbf{T}_{\varphi_1} &= (2\cos\varphi_1\cos\theta_1, 2\cos\varphi_1\sin\theta_1, -2\sin\varphi_1, 2\cos\varphi_1\cos\theta_1, 2\cos\varphi_1\sin\theta_1, -2\sin\varphi_1)\\
\mathbf{T}_{\theta_2} &= (0, 0, 0, -\sin\varphi_2\sin\theta_2, \sin\varphi_2\cos\theta_2, 0)\\
\mathbf{T}_{\varphi_2} &= (0, 0, 0, \cos\varphi_2\cos\theta_2, \cos\varphi_2\sin\theta_2, -\sin\varphi_2)
\end{aligned}$$

Look at the equation $c_1\mathbf{T}_{\theta_1} + c_2\mathbf{T}_{\varphi_1} + c_3\mathbf{T}_{\theta_2} + c_4\mathbf{T}_{\varphi_2} = \mathbf{0}$. There is a zero in the third component of all of the tangent vectors except for $\mathbf{T}_{\varphi_1}$. This tells us that $c_2 = 0$. If that is the case, then there is a zero in the sixth component of all of the remaining tangent vectors except for $\mathbf{T}_{\varphi_2}$ so $c_4 = 0$. But then the leading trio of zeros in $\mathbf{T}_{\theta_2}$ implies that $c_1 = 0$ which in turn would mean that $c_3 = 0$. Therefore the four tangent vectors are linearly independent and we have described the states of the robot arm as a smooth parametrized 4-manifold in $\mathbf{R}^6$.

**5.** This is just an exercise in linear algebra. If $\mathbf{x} \in \mathbf{R}^n$ is orthogonal to $\mathbf{v}_i$ for $i = 1, \dots, k$, then $\mathbf{x}\cdot\mathbf{v}_i = 0$ for $i = 1, \dots, k$. An arbitrary vector $\mathbf{v}$ in Span$\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is of the form $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$ for scalars $c_1, \dots, c_k \in \mathbf{R}$. The calculation is straightforward:

$$\mathbf{x}\cdot\mathbf{v} = \mathbf{x}\cdot(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) = c_1(\mathbf{x}\cdot\mathbf{v}_1) + \cdots + c_k(\mathbf{x}\cdot\mathbf{v}_k) = c_1(0) + \cdots + c_k(0) = 0.$$

In other words, $\mathbf{x}$ is orthogonal to $\mathbf{v}$.

**6.** By Definition 2.1, $\displaystyle\int_{\mathbf{x}}\omega = \int_0^\pi \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\,dt$. We have, $\mathbf{x}'(t) = (-a\sin t, b\cos t, c)$ and also $\omega = b\,dx - a\,dy + xy\,dz$ so that

$$\int_{\mathbf{x}} \omega = \int_0^{\pi} [b(-a \sin t) - a(b \cos t) + (ab \cos t \sin t)c]\, dt$$
$$= ab \int_0^{\pi} [-\sin t - \cos t + c \sin t \cos t]\, dt$$
$$= ab \left( \cos t - \sin t + \frac{c}{2} \sin^2 t \right) \Big|_0^{\pi} = -2ab.$$

**7.** Parametrize the unit circle $C$ by $\mathbf{x}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$. Then

$$\int_C \omega = \int_0^{2\pi} \omega_{\mathbf{x}(t)}(-\sin t, \cos t)\, dt = \int_0^{2\pi} (\sin t\, dx - \cos t\, dy)(-\sin t, \cos t)\, dt$$
$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\, dt = \int_0^{2\pi} -1\, dt = -2\pi.$$

**8.** Parametrize the segment as $\mathbf{x}(t) = (t, t, \ldots, t), 0 \le t \le 3$. Then $\mathbf{x}'(t) = (1, 1, \ldots, 1)$ and so

$$\omega_{\mathbf{x}(t)}(\mathbf{x}'(t)) = (t\, dx_1 + t^2\, dx_2 + \cdots + t^n\, dx_n)(1, 1, \ldots, 1) = t + t^2 + \cdots + t^n.$$

Hence,

$$\int_C \omega = \int_0^3 (t + t^2 + \cdots + t^n)\, dt = \left( \frac{1}{2}t^2 + \frac{1}{3}t^3 + \cdots + \frac{1}{n+1} t^{n+1} \right) \Big|_0^3 = \sum_{k=2}^{n+1} \frac{3^k}{k} = \sum_{k=1}^{n} \frac{3^{k+1}}{k+1}.$$

**9.** By Definition 2.3, $\int_S \omega = \iint_D \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t)\, ds\, dt$. For $\mathbf{X}(s,t) = (s \cos t, s \sin t, t)$, we have $\mathbf{T}_s = (\cos t, \sin t, 0)$, and $\mathbf{T}_t = (-s \sin t, s \cos t, 1)$. Then

$$\omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) = (t\, dx \wedge dy + 3\, dz \wedge dx - s \cos t\, dy \wedge dz)(\mathbf{T}_s, \mathbf{T}_t)$$
$$= t \begin{vmatrix} \cos t & -s \sin t \\ \sin t & s \cos t \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 \\ \cos t & -s \sin t \end{vmatrix} - s \cos t \begin{vmatrix} \sin t & s \cos t \\ 0 & 1 \end{vmatrix}$$
$$= st - 3 \cos t - \frac{s}{2} \sin 2t.$$

Thus

$$\int_S \omega = \int_0^{4\pi} \int_0^1 \left( st - 3 \cos t - \frac{s}{2} \sin 2t \right) ds\, dt = \int_0^{4\pi} \left( \frac{1}{2} t - 3 \cos t - \frac{1}{4} \sin 2t \right) dt$$
$$= \left( \frac{1}{4} t^2 - 3 \sin t + \frac{1}{8} \cos 2t \right) \Big|_0^{4\pi} = 4\pi^2.$$

**10. (a)** First calculate the two tangent vectors for this parametrization of the helicoid. We have $\mathbf{T}_{u_1} = (\cos 3u_2, \sin 3u_2, 0)$ and $\mathbf{T}_{u_2} = (-3u_1 \sin 3u_2, 3u_1 \cos 3u_2, 5)$. Then

$$\Omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} -5 \sin 3u_2 & \cos 3u_2 & -3u_1 \sin 3u_2 \\ 5 \cos 3u_2 & \sin 3u_2 & 3u_1 \cos 3u_2 \\ -3u_1 & 0 & 5 \end{bmatrix} = -9u_1^2 - 25 < 0$$

for all $(u_1, u_2)$. Therefore this particular parametrization is incompatible with $\Omega$.

**(b)** There is more than one solution. One possible way to do this is to switch the ordering of the variables so that the resulting determinant is positive. Try the parametrization $\mathbf{Y}(u_1, u_2) = \mathbf{X}(u_2, u_1) = (u_2 \cos 3u_1, u_2 \sin 3u_1, 5u_1)$ for $0 \le u_1 \le 2\pi$ and $0 \le u_2 \le 5$. Then the tangent vectors are $\mathbf{T}_{u_1} = (-3u_2 \sin 3u_1, 3u_2 \cos 3u_1, 5)$ and $\mathbf{T}_{u_2} = (\cos 3u_1, \sin 3u_1, 0)$. Then

$$\Omega_{\mathbf{Y}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} -5 \sin 3u_1 & -3u_2 \sin 3u_1 & \cos 3u_1 \\ 5 \cos 3u_1 & 3u_2 \cos 3u_1 & \sin 3u_1 \\ -3u_2 & 5 & 0 \end{bmatrix} = 9u_1^2 + 25 > 0$$

for all $(u_1, u_2)$. Therefore this particular parametrization is now compatible with $\Omega$.

**(c)** Since the goal is to change the sign of the resulting determinant, we can change $\Omega$ to $\Phi$ where

$$\Phi_{\mathbf{X}(u_1,u_2)}(\mathbf{a},\mathbf{b}) = -\det\begin{bmatrix} -5\sin 3u_2 & a_1 & b_1 \\ 5\cos 3u_2 & a_2 & b_2 \\ -3u_1 & a_3 & b_3 \end{bmatrix}.$$

**(d)** The discussion following Theorem 2.11 tells us what to do if the parametrization is compatible. Since the parametrization $\mathbf{X}$ is incompatible with $\Omega$ we make the following simple adjustment: $\int_S \omega = -\int_{\mathbf{X}} \omega$. We pause to calculate

$$\begin{aligned}\omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2}) &= 5u_2\begin{vmatrix} \cos 3u_2 & -3u_1\sin 3u_2 \\ \sin 3u_2 & 3u_1\cos 3u_2 \end{vmatrix} \\ &\quad - (u_1^2\cos^2 3u_2 + u_1^2\sin^2 3u_2)\begin{vmatrix} \sin 3u_2 & 3u_1\cos 3u_2 \\ 0 & 5 \end{vmatrix} \\ &= 15u_1u_2 - 5u_1^2\sin 3u_2.\end{aligned}$$

Hence,

$$\begin{aligned}\int_S \omega &= -\int_0^5\int_0^{2\pi}(15u_1u_2 - 5u_1^2\sin 3u_2)\,du_2\,du_1 = 5\int_0^5\int_0^{2\pi}(u_1^2\sin 3u_2 - 3u_1u_2)\,du_2\,du_1 \\ &= 5\int_0^5\left[u_1^2\left(\frac{-\cos 3u_2}{3}\right) - \frac{3u_1u_2^2}{2}\right]\Bigg|_{u_2=0}^{u_2=2\pi} du_1 = 5\int_0^5(-6\pi^2u_1)\,du_1 \\ &= -30\pi^2\int_0^5 u_1\,du_1 = -15\pi^2u_1^2\Big|_0^5 = -375\pi^2.\end{aligned}$$

**11. (a)** For the parametrization given, we calculate the tangent vectors as $\mathbf{T}_{u_1} = (\cos u_2, \sin u_2, 0)$, $\mathbf{T}_{u_2} = (-u_1\sin u_2, u_1\cos u_2, 0)$, and $\mathbf{T}_{u_3} = (0,0,1)$. Then

$$\Omega_{\mathbf{X}(u)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2},\mathbf{T}_{u_3}) = \det\begin{bmatrix} \cos u_2 & -u_1\sin u_2 & 0 \\ \sin u_2 & u_1\cos u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = u_1.$$

As $0 \le u_1 \le \sqrt{5}$, this is positive when $u_1 \neq 0$. Note that when $u_1 = 0$ the parametrization is not one-one and also that $\mathbf{T}_{u_2} = \mathbf{0}$ so $\mathbf{T}_{u_1}$, $\mathbf{T}_{u_2}$, and $\mathbf{T}_{u_3}$ are not linearly independent. In other words, the parametrization is not smooth when $u_1 = 0$. It is, however, smooth when $u_1 \neq 0$. You can easily see that the mapping is one-one and at least $C^1$. To see that the tangent vectors are linearly independent, consider the equation $c_1\mathbf{T}_{u_1} + c_2\mathbf{T}_{u_2} + c_3\mathbf{T}_{u_3} = \mathbf{0}$. We see from the third components that $c_3 = 0$. Look at the remaining equations and we see that

$$\begin{cases} (\cos u_2)c_1 - (u_1\sin u_2)c_2 = 0 \\ (\sin u_2)c_1 + (u_1\cos u_2)c_2 = 0. \end{cases}$$

Multiply the first equation by $-\sin u_2$ and the second by $\cos u_2$ and add to obtain $u_1c_2 = 0$. Because we are assuming that $u_1 \neq 0$, this implies that $c_2 = 0$ and therefore $c_1 = 0$. This shows that the tangent vectors are linearly independent and hence the parametrization is smooth when $u_1 \neq 0$. The conclusion is then that the parametrization given is compatible with the orientation when it is smooth.

**(b)** We can read the boundary pieces right off of the original parametrization: they are paraboloids that intersect at $z = -1$ in a circle in the plane $z = -1$ of radius $\sqrt{5}$ centered at $(0,0,-1)$. The boundary is

$$\partial M = \{(x,y,z)|z = x^2 + y^2 - 6, z \le -1\} \cup \{(x,y,z)|z = 4 - x^2 - y^2, z \ge -1\}.$$

We can easily adapt the parametrization to each of these pieces. For the bottom, use

$$\mathbf{Y}_1 : [0,\sqrt{5}] \times [0,2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_1(s_1,s_2) = (s_1\cos s_2, s_1\sin s_2, s_1^2 - 6).$$

For the top, use

$$\mathbf{Y}_2 : [0,\sqrt{5}] \times [0,2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_2(s_1,s_2) = (s_1\cos s_2, s_1\sin s_2, 4 - s_1^2).$$

**(c)** On the bottom part of $\partial M$ the outward-pointing unit vector

$$\mathbf{V}_1 = \frac{(2x, 2y, -1)}{\sqrt{4x^2+4y^2+1}}. \text{ In terms of } \mathbf{Y}_1, \text{ this is } \mathbf{V}_1 = \frac{(2s_1\cos s_2, 2s_1 \sin s_2, -1)}{\sqrt{4s_1^2+1}}.$$

On the top part of $\partial M$ the outward-pointing unit vector

$$\mathbf{V}_2 = \frac{(2x, 2y, 1)}{\sqrt{4x^2+4y^2+1}}. \text{ In terms of } \mathbf{Y}_2, \text{ this is } \mathbf{V}_2 = \frac{(2s_1\cos s_2, 2s_1 \sin s_2, 1)}{\sqrt{4s_1^2+1}}.$$

**12.** The paraboloid can be parametrized as $\mathbf{X}(s,t) = (s, t, s^2+t^2)$ where $0 \le s^2+t^2 \le 4$. Therefore, $\mathbf{T}_s = (1, 0, 2s)$ and $\mathbf{T}_t = (0, 1, 2t)$. Note that this parametrization is compatible with the orientation $\Omega$ derived from the normal $\mathbf{N} = (-2x, -2y, 1)$ as

$$\Omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) = \det[\mathbf{N} \quad \mathbf{T}_s \quad \mathbf{T}_t] = \det \begin{bmatrix} -2s & 1 & 0 \\ -2t & 0 & 1 \\ 1 & 2s & 2t \end{bmatrix} = 2s^2+2t^2+1 > 0.$$

Therefore, we may compute $\int_S \omega$ as $\int_{\mathbf{X}} \omega$. So we begin by calculating

$$\begin{aligned} \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) &= (e^{s^2+t^2}\, dx \wedge dy + t\, dz \wedge dx + s\, dy \wedge dz)(\mathbf{T}_s, \mathbf{T}_t) \\ &= e^{s^2+t^2} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + t \begin{vmatrix} 2s & 2t \\ 1 & 0 \end{vmatrix} + s \begin{vmatrix} 0 & 1 \\ 2s & 2t \end{vmatrix} \\ &= e^{s^2+t^2} - 2t^2 - 2s^2. \end{aligned}$$

Use this in the calculation:

$$\begin{aligned} \int_S \omega &= \iint_{0 \le s^2+t^2 \le 4} [e^{s^2+t^2} - 2(s^2+t^2)]\, ds\, dt \\ &= \int_0^{2\pi} \int_0^2 (e^{r^2} - 2r^2) r\, dr\, d\theta \quad \text{using polar coordinates,} \\ &= \int_0^{2\pi} \left( \frac{1}{2} e^{r^2} - \frac{1}{2} r^4 \right) \Big|_{r=0}^{2} d\theta \\ &= \int_0^{2\pi} \left( \frac{1}{2} e^4 - \frac{1}{2} - 8 \right) d\theta = \pi(e^4 - 17). \end{aligned}$$

**13.** The cylinder can be parametrized as $\mathbf{X}(s,t) = (2\cos t, s, 2\sin t)$ where $-1 \le s \le 3$ and $0 \le t \le 2\pi$. Therefore, $\mathbf{T}_s = (0, 1, 0)$ and $\mathbf{T}_t = (-2\sin t, 0, 2\cos t)$. This parametrization turns out to be compatible with the orientation $\Omega$ derived from the normal $\mathbf{N} = (x, 0, z)$ as

$$\Omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) = \det\left[\mathbf{N} \quad \mathbf{T}_s \quad \mathbf{T}_t\right] = \begin{vmatrix} 2\cos t & 0 & -2\sin t \\ 0 & 1 & 0 \\ 2\sin t & 0 & 2\cos t \end{vmatrix} = 4\cos^2 t + 4\sin^2 t = 4 > 0.$$

Therefore, we may compute $\int_S \omega$ as $\int_{\mathbf{X}} \omega$. Hence we calculate

$$\begin{aligned} \omega_{\mathbf{X}(s,t)}(\mathbf{T}_s, \mathbf{T}_t) &= \left( 2\sin t\, dx \wedge dy + e^{s^2}\, dz \wedge dx + 2\cos t\, dy \wedge dz \right) (\mathbf{T}_s, \mathbf{T}_t) \\ &= 2\sin t \begin{vmatrix} 0 & -2\sin t \\ 1 & 0 \end{vmatrix} + e^{s^2} \begin{vmatrix} 0 & 2\cos t \\ 0 & -2\sin t \end{vmatrix} + 2\cos t \begin{vmatrix} 1 & 0 \\ 0 & 2\cos t \end{vmatrix} \\ &= 4\sin^2 t + 0 + 4\cos^2 t = 4. \end{aligned}$$

Thus

$$\int_S \omega = \iint_{[-1,3]\times[0,2\pi]} 4\, ds\, dt = 32\pi.$$

**14.** We have, for the given parametrization, that

$$\mathbf{T}_s = \left(\frac{\cos t}{2\sqrt{s}},\ -\frac{\sin t}{2\sqrt{4-s}},\ \frac{\sin t}{2\sqrt{s}},\ -\frac{\cos t}{2\sqrt{4-s}}\right)$$

and

$$\mathbf{T}_t = \left(-\sqrt{s}\sin t, \sqrt{4-s}\cos t, \sqrt{s}\cos t, -\sqrt{4-s}\sin t\right).$$

Thus,

$$\begin{aligned}\omega_{\mathbf{X}(s,t)}(\mathbf{T}_s,\mathbf{T}_t) &= \big(((4-s)\sin^2 t + (4-s)\cos^2 t)\,dx_1\wedge dx_3 \\ &\quad - (2s\cos^2 t + 2s\sin^2 t)\,dx_2\wedge dx_4\big)(\mathbf{T}_s,\mathbf{T}_t) \\ &= (4-s)\begin{vmatrix}\frac{\cos t}{2\sqrt{s}} & -\sqrt{s}\sin t \\ \frac{\sin t}{2\sqrt{s}} & \sqrt{s}\cos t\end{vmatrix} - 2s\begin{vmatrix}-\frac{\sin t}{2\sqrt{4-s}} & \sqrt{4-s}\cos t \\ -\frac{\cos t}{2\sqrt{4-s}} & -\sqrt{4-s}\sin t\end{vmatrix} \\ &= (4-s)\left(\tfrac{1}{2}\cos^2 t + \tfrac{1}{2}\sin^2 t\right) - 2s\left(\tfrac{1}{2}\sin^2 t + \tfrac{1}{2}\cos^2 t\right) = 2 - \tfrac{3}{2}s.\end{aligned}$$

Hence,

$$\int_{\mathbf{X}}\omega = \int_0^{2\pi}\int_1^3 \left(2-\tfrac{3}{2}s\right)\,ds\,dt = \int_0^{2\pi}\left(2s - \tfrac{3}{4}s^2\right)\Big|_{s=1}^{s=3} dt = \int_0^{2\pi}(-2)\,dt = -4\pi.$$

**15.** We have, for the given parametrization, that $\mathbf{T}_{u_1} = (1,0,0,4(2u_1-u_3)), \mathbf{T}_{u_2} = (0,1,0,0)$, and $\mathbf{T}_{u_3} = (0,0,1,2(u_3-2u_1))$. Thus,

$$\begin{aligned}\omega_{\mathbf{X}(u_1,u_2,u_3)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2},\mathbf{T}_{u_3}) &= (u_2\,dx_2\wedge dx_3\wedge dx_4 + 2u_1u_3\,dx_1\wedge dx_2\wedge dx_3)(\mathbf{T}_{u_1},\mathbf{T}_{u_2},\mathbf{T}_{u_3}) \\ &= u_2\begin{vmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 4(2u_1-u_3) & 0 & 2(u_3-2u_1)\end{vmatrix} + 2u_1u_3\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} \\ &= u_2(8u_1-4u_3) + 2u_1u_3 = 8u_1u_2 - 4u_2u_3 + 2u_1u_3.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{\mathbf{X}}\omega &= \int_0^1\int_0^1\int_0^1 (8u_1u_2 - 4u_2u_3 + 2u_1u_3)\,du_1\,du_2\,du_3 \\ &= \int_0^1\int_0^1 (4u_2 - 4u_2u_3 + u_3)\,du_2\,du_3 \\ &= \int_0^1 (2 - 2u_3 + u_3)\,du_3 = 2 - \frac{1}{2} = \frac{3}{2}.\end{aligned}$$

## 8.3 The Generalized Stokes's Theorem

**1.** Using Definition 3.1,

$$d(e^{xyz}) = \frac{\partial}{\partial x}(e^{xyz})\,dx + \frac{\partial}{\partial y}(e^{xyz})\,dy + \frac{\partial}{\partial z}(e^{xyz})\,dz + e^{xyz}(yz\,dx + xz\,dy + xy\,dz).$$

**2.** Using Definition 3.1,

$$d(x^3y - 2xz^2 + xy^2z) = (3x^2y - 2z^2 + y^2z)\,dx + (x^3 + 2xyz)\,dy + (xy^2 - 4xz)\,dz.$$

**3.** Again, using Definition 3.1,

$$\begin{aligned}d((x^2+y^2)\,dx + xy\,dy) &= d(x^2+y^2)\wedge dx + d(xy)\wedge dy \\ &= (2x\,dx + 2y\,dy)\wedge dx + (y\,dx + x\,dy)\wedge dy \\ &= 2y\,dy\wedge dx + y\,dx\wedge dy \quad \text{using (4) from Section 8.1,} \\ &= -y\,dx\wedge dy \quad \text{using (3) from Section 8.1.}\end{aligned}$$

**4.** Again, using Definition 3.1,

$$\begin{aligned} &d(x_1\,dx_2 - x_2\,dx_1 + x_3x_4\,dx_4 - x_4x_5\,dx_5) \\ &\quad = dx_1 \wedge dx_2 - dx_2 \wedge dx_1 + (x_4\,dx_3 + x_3\,dx_4) \wedge dx_4 - (x_5\,dx_4 + x_4\,dx_5) \wedge dx_5 \\ &\quad = dx_1 \wedge dx_2 - dx_2 \wedge dx_1 + x_4\,dx_3 \wedge dx_4 - x_5\,dx_4 \wedge dx_5 \quad \text{using (4) from Section 8.1,} \\ &\quad = 2\,dx_1 \wedge dx_2 + x_4\,dx_3 \wedge dx_4 - x_5\,dx_4 \wedge dx_5 \quad \text{using (3) from Section 8.1.} \end{aligned}$$

**5.** Again, using Definition 3.1,

$$\begin{aligned} d(xz\,dx \wedge dy - y^2 z\,dx \wedge dz) &= (z\,dx + x\,dz) \wedge dx \wedge dy - (2yz\,dy + y^2\,dz) \wedge dx \wedge dz \\ &= x\,dz \wedge dx \wedge dy - 2yz\,dy \wedge dx \wedge dz \quad \text{using (4) from Section 8.1,} \\ &= (x + 2yz)\,dx \wedge dy \wedge dz \quad \text{using (3) from Section 8.1.} \end{aligned}$$

**6.** Again, using Definition 3.1,

$$\begin{aligned} &d(x_1x_2x_3\,dx_2 \wedge dx_3 \wedge dx_4 + x_2x_3x_4\,dx_1 \wedge dx_2 \wedge dx_3) \\ &\quad = (x_2x_3\,dx_1 + x_1x_3\,dx_2 + x_1x_2\,dx_3) \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &\qquad + (x_3x_4\,dx_2 + x_2x_4\,dx_3 + x_2x_3\,dx_4) \wedge dx_1 \wedge dx_2 \wedge dx_3 \\ &\quad = x_2x_3\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + x_2x_3\,dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 \quad \text{using (4) from Section 8.1,} \\ &\quad = 0 \quad \text{using (3) from Section 8.1.} \end{aligned}$$

**7.** For this solution $\widehat{dx_i}$ means that the term $dx_i$ is omitted.

$$\begin{aligned} d\omega &= \sum_{i=1}^{n} d(x_i)^2 \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} 2x_i\,dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\ &= \sum_{i=1}^{n} (-1)^{i-1} 2x_i\,dx_1 \wedge \cdots \wedge dx_n \quad \text{using equation (3) of Section 8.1 repeatedly} \\ &= 2(x_1 - x_2 + x_3 - \cdots + (-1)^{n-1}x_n)\,dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

**8.** Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$; then

$$\begin{aligned} df_{\mathbf{x}_0}(\mathbf{u}) &= (f_{x_1}(\mathbf{x}_0)\,dx_1 + f_{x_2}(\mathbf{x}_0)\,dx_2 + \cdots + f_{x_n}(\mathbf{x}_0)\,dx_n)(\mathbf{u}) \\ &= f_{x_1}(\mathbf{x}_0)u_1 + f_{x_2}(\mathbf{x}_0)u_2 + \cdots + f_{x_n}(\mathbf{x}_0)u_n \\ &= (f_{x_1}(\mathbf{x}_0), f_{x_2}(\mathbf{x}_0), \ldots, f_{x_n}(\mathbf{x}_0)) \cdot \mathbf{u} \\ &= \nabla f(\mathbf{x}_0) \cdot \mathbf{u} \\ &= D_{\mathbf{u}} f(\mathbf{x}_0) \quad \text{by Theorem 6.2 of Chapter 2.} \end{aligned}$$

**9.** For $\omega = F(x,z)\,dy + G(x,y)\,dz$, we have $d\omega = (F_x\,dx + F_z\,dz) \wedge dy + (G_x\,dx + G_y\,dy) \wedge dz$. Expanding, this gives $d\omega = F_x\,dx \wedge dy + G_x\,dx \wedge dz + (G_y - F_z)\,dy \wedge dz$. But we are told that $d\omega = z\,dx \wedge dy + y\,dx \wedge dz$ so

$$\frac{\partial F}{\partial x} = z, \ \frac{\partial G}{\partial x} = y, \ \text{and} \ \frac{\partial G}{\partial y} - \frac{\partial F}{\partial z} = 0.$$

The first equation implies that $F(x,z) = xz + f(z)$ for some differentiable function $f$ of $z$ alone. Similarly, the second equation implies that $G(x,y) = xy + g(y)$ for some differentiable function $g$ of $y$ alone. Using these results together with the third equation we see that $x + g'(y) = x + f'(z)$ or $g'(y) = f'(z)$. This can only be true if their common value is a constant $C$. So if $g'(y) = f'(z) = C$, then $f(z) = Cz + D_1$ and $g(y) = Cy + D_2$ for arbitrary constants $C$, $D_1$, and $D_2$. We conclude that $F(x,z) = xz + Cz + D_1$ and $G(x,y) = xy + Cy + D_2$.

**10.** If $\omega = 2x\,dy \wedge dz - z\,dx \wedge dy$, then $d\omega = 2\,dx \wedge dy \wedge dz - dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$. From Exercise 11 of Section 8.2, $M$ is parametrized as $\mathbf{X}\colon D \to \mathbf{R}^3; \mathbf{X}(u_1, u_2, u_3) = (u_1 \cos u_2, u_1 \sin u_2, u_3)$ where $D = \{(u_1, u_2, u_3) | u_1^2 - 6 \le u_3 \le 4 - u_1^2, 0 \le u_1 \le \sqrt{5}, 0 \le u_2 < 2\pi\}$. If we orient $M$ by the 3-form $\Omega = dx \wedge dy \wedge dz$, then

$$\Omega_{\mathbf{X(u)}}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}) = \det \begin{bmatrix} \cos u_2 & -u_1 \sin u_2 & 0 \\ \sin u_2 & u_1 \cos u_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = u_1 \ge 0.$$

As before, this is strictly positive when the parametrization is smooth so the parametrization is compatible with the orientation. Therefore, using this orientation,

$$\int_M d\omega = \int_{\mathbf{X}} d\omega = \int_0^{2\pi} \int_0^{\sqrt{5}} \int_{u_1^2-6}^{4-u_1^2} u_1\, du_3\, du_1\, du_2 = 2\pi \int_0^{\sqrt{5}} u_1(10 - 2u_1^2)\, du_1$$

$$= 4\pi \left( \frac{5}{2} u_1^2 - \frac{1}{4} u_1^4 \right) \Bigg|_0^{\sqrt{5}} = 4\pi \left( \frac{25}{2} - \frac{25}{4} \right) = 25\pi.$$

On the other hand, $\partial M$ is parametrized on the bottom surface as

$$\mathbf{Y}_1 : [0, \sqrt{5}] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_1(s_1, s_2) = (s_1 \cos s_2, s_1 \sin s_2, s_1^2 - 6)$$

with tangent vector normal to $\partial M$

$$\mathbf{V}_1 = \frac{(2s_1 \cos s_2, 2s_1 \sin s_2, -1)}{\sqrt{4s_1^2 + 1}}.$$

The boundary $\partial M$ is parametrized on the top surface as

$$\mathbf{Y}_2 : [0, \sqrt{5}] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{Y}_2(s_1, s_2) = (s_1 \cos s_2, s_1 \sin s_2, 4 - s_1^2)$$

with tangent vector normal to $\partial M$

$$\mathbf{V}_2 = \frac{(2s_1 \cos s_2, 2s_1 \sin s_2, 1)}{\sqrt{4s_1^2 + 1}}.$$

Then we have that the induced orientation on $\partial M$ is given by $\Omega^{\partial M}(\mathbf{a}_1, \mathbf{a}_2) = \Omega(\mathbf{V}, \mathbf{a}_1, \mathbf{a}_2)$. Therefore we see that on the bottom part of $\partial M$

$$\begin{aligned} \Omega^{\partial M}_{\mathbf{Y}_1(\mathbf{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) &= \Omega_{\mathbf{X}(s_1, s_2, s_1^2 - 6)}(\mathbf{V}_1, \mathbf{T}_{s_1}, \mathbf{T}_{s_2}) \\ &= \det \begin{bmatrix} \dfrac{2s_1 \cos s_2}{\sqrt{4s_1^2+1}} & \cos s_2 & -s_1 \sin s_2 \\ \dfrac{2s_1 \sin s_2}{\sqrt{4s_1^2+1}} & \sin s_2 & s_1 \cos s_2 \\ \dfrac{-1}{\sqrt{4s_1^2+1}} & 2s_1 & 0 \end{bmatrix} \\ &= -\frac{4s_1^3 + s_1}{\sqrt{4s_1^2+1}} \le 0. \end{aligned}$$

The parametrization $\mathbf{Y}_1$ is incompatible with the induced orientation on $\partial M$. Along the top part of $\partial M$

$$\begin{aligned} \Omega^{\partial M}_{\mathbf{Y}_2(\mathrm{s})}(\mathbf{T}_{s_1}, \mathbf{T}_{s_2}) &= \Omega_{\mathbf{X}(s_1, s_2, 4 - s_1^2)}(\mathbf{V}_2, \mathbf{T}_{s_1}, \mathbf{T}_{s_2}) \\ &= \det \begin{bmatrix} \dfrac{2s_1 \cos s_2}{\sqrt{4s_1^2+1}} & \cos s_2 & -s_1 \sin s_2 \\ \dfrac{2s_1 \sin s_2}{\sqrt{4s_1^2+1}} & \sin s_2 & s_1 \cos s_2 \\ \dfrac{1}{\sqrt{4s_1^2+1}} & -2s_1 & 0 \end{bmatrix} \\ &= \frac{4s_1^3 + s_1}{\sqrt{4s_1^2+1}} \ge 0. \end{aligned}$$

This parametrization is compatible with the induced orientation on $\partial M$.

Therefore we set up our integral (changing signs in the first integrand because of the incompatibility of the parametrization) to obtain the following.

$$\begin{aligned}
\int_{\partial M}\omega &= -\int_{\mathbf{Y}_1}\omega + \int_{\mathbf{Y}_2}\omega \\
&= -\int_0^{2\pi}\int_0^{\sqrt{5}}\left\{2s_1\cos s_2\begin{vmatrix}\sin s_2 & s_1\cos s_2\\ 2s_1 & 0\end{vmatrix} - (s_1^2-6)\begin{vmatrix}\cos s_2 & -s_1\sin s_2\\ \sin s_2 & s_1\cos s_2\end{vmatrix}\right\}ds_1\,ds_2 \\
&\quad + \int_0^{2\pi}\int_0^{\sqrt{5}}\left\{2s_1\cos s_2\begin{vmatrix}\sin s_2 & s_1\cos s_2\\ -2s_1 & 0\end{vmatrix} - (4-s_1^2)\begin{vmatrix}\cos s_2 & -s_1\sin s_2\\ \sin s_2 & s_1\cos s_2\end{vmatrix}\right\}ds_1\,ds_2 \\
&= \int_0^{2\pi}\int_0^{\sqrt{5}}[8s_1^3\cos^2 s_2 + (2s_1^2-10)s_1]\,ds_1\,ds_2 \\
&= \int_0^{2\pi}\left[s_1^4\cos 2s_2 + \frac{3}{2}s_1^4 - 5s_1^2\right]\Bigg|_{s_1=0}^{s_1=\sqrt{5}} ds_2 \\
&= \int_0^{2\pi}[25\cos 2s_2 + 25/2]\,ds_2 = [(25/2)\sin 2s_2 + 25s_2/2]\Big|_0^{2\pi} = 25\pi.
\end{aligned}$$

**11.** One integral is easy. Since $\omega = xy\,dz\wedge dw$ and $\partial M = \{(x,y,z,w)|x=0, 8-2y^2-2z^2-2w^2=0\}$, we see that $x=0$ along $\partial M$ so $\int_{\partial M}\omega = \int_{\partial M}0 = 0$.

Now $d\omega = d(xy)\wedge dz\wedge dw = x\,dy\wedge dz\wedge dw + y\,dx\wedge dz\wedge dw$. We can orient $M$ any way we wish, so we won't worry about this—we'll choose the orientation to be compatible with the parametrization.

$$\mathbf{X}: D\to\mathbf{R}^4,\quad \mathbf{X}(u_1,u_2,u_3) = (8-2u_1^2-2u_2^2-2u_3^2, u_1, u_2, u_3)$$

where $D = \{(u_1,u_2,u_3)|u_1^2+u_2^2+u_3^2\le 4\}$ (i.e., the solid ball of radius 2). Then

$$\begin{aligned}
\int_M d\omega &= \int_{\mathbf{X}} d\omega = \iiint_B d\omega_{\mathbf{X}(u)}(\mathbf{T}_{u_1},\mathbf{T}_{u_2},\mathbf{T}_{u_3})\,du_1\,du_2\,du_3 \\
&= \iiint_B\left\{(8-2u_1^2-2u_2^2-2u_3^2)\begin{vmatrix}1&0&0\\0&1&0\\0&0&1\end{vmatrix} + u_1\begin{vmatrix}-4u_1 & -4u_2 & -4u_3\\ 0&1&0\\0&0&1\end{vmatrix}\right\}du_1\,du_2\,du_3 \\
&= \iiint_B(8-2(u_1^2+u_2^2+u_3^2)-4u_1^2)\,du_1\,du_2\,du_3.
\end{aligned}$$

At this point it is helpful to switch to spherical coordinates. The previous quantity is then

$$
\begin{aligned}
&= \int_0^{2\pi}\int_0^{\pi}\int_0^{2}(8-2\rho^2-4\rho^2\sin^2\varphi\cos^2\theta)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= 8\cdot(\text{volume of } B) - 2\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^4(\sin\varphi+2\sin^3\varphi\cos^2\theta)\,d\rho\,d\varphi\,d\theta \\
&= 8\cdot\left(\frac{4}{3}\pi 2^3\right) - 2\int_0^{2\pi}\int_0^{\pi}\frac{32}{5}(\sin\varphi+2\sin\varphi(1-\cos^2\varphi)\cos^2\theta)\,d\varphi\,d\theta \\
&= \frac{256\pi}{3} - \frac{64}{5}\int_0^{2\pi}\{(-\cos\varphi)\Big|_0^{\pi} + 2\cos^2\theta(-\cos\varphi+(\cos^3\varphi)/3)\Big|_{\varphi=0}^{\pi}\}\,d\theta \\
&= \frac{256\pi}{3} - \frac{64}{5}\int_0^{2\pi}\left\{2+2\cos^2\theta\cdot\left(2-\frac{2}{3}\right)\right\}d\theta = \frac{256\pi}{3} - \frac{128}{5}\int_0^{2\pi}\left(1+\frac{4}{3}\cos^2\theta\right)d\theta \\
&= \frac{256\pi}{3} - \frac{128}{5}\int_0^{2\pi}\left(\frac{5}{3}+\frac{2}{3}\cos 2\theta\right)d\theta \quad \text{(using the half angle formula)} \\
&= \frac{256\pi}{3} - \frac{128}{5}\left(\frac{5}{3}(2\pi)+\frac{1}{3}\sin 2\theta\Big|_0^{2\pi}\right) = \frac{256\pi}{3} - \frac{256\pi}{3} = 0.
\end{aligned}
$$

**12. (a)** Using the generalized version of Stokes's theorem (Theorem 3.2), we have

$$
\begin{aligned}
\frac{1}{3}\int_{\partial M} x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy &= \frac{1}{3}\int_M d(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) \\
&= \frac{1}{3}\int_M dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy \\
&= \frac{1}{3}\int_M 3\,dx\wedge dy\wedge dz \quad \text{using formula (3) of Section 8.1}, \\
&= \int_M dx\wedge dy\wedge dz = \iiint_M dx\,dy\,dz = \text{volume of } M.
\end{aligned}
$$

(See Definition 2.6 and Example 6 of Section 8.2.)

**(b)** This generalizes the result demonstrated in part (a). Notice that the $k$th summand is $(-1)^{k-1}x_k$ multiplied by the $(n-1)$-form which is the wedge product of the $dx_i$'s in order with $dx_k$ missing. In other words, the $k$th summand is

$$(-1)^{k-1}x_k\,dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n$$

where $\widehat{dx_k}$ means that $dx_k$ is omitted. (Make the obvious adjustments to the expression if it is the first or last term that is omitted.) Then

$$d(\text{of the } k\text{th summand}) = (-1)^{k-1}\,dx_k\wedge dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n.$$

Let $\omega$ denote the $(n-1)$-form in the integrand. Then, using the generalized Stokes's theorem,

$$
\begin{aligned}
\frac{1}{n}\int_{\partial M}\omega &= \frac{1}{n}\int_M d\omega \\
&= \frac{1}{n}\int_M\left(\sum_{k=1}^{n}(-1)^{k-1}\,dx_k\wedge dx_1\wedge\cdots\wedge\widehat{dx_k}\wedge\cdots\wedge dx_n\right).
\end{aligned}
$$

Use formula (3) of Section 8.1 to "move" each $dx_k$ back into the slot from which it has been omitted and collect terms to obtain

$$\frac{1}{n}\int_{\partial M}\omega = \frac{1}{n}\int_M n\,dx_1\wedge\cdots\wedge dx_n = \int\cdots\int_M dx_1\cdots dx_n.$$

It is entirely reasonable to take this last $n$-dimensional integral to represent the $n$-dimensional volume of $M$.

## True/False Exercises for Chapter 8

1. True.
2. False. (There is a negative sign missing.)
3. True.
4. False.
5. True.
6. False. (A negative sign is missing.)
7. True.
8. False. (There should be no negative sign.)
9. True.
10. True.
11. False. ($\mathbf{X}(1,1,-1) = \mathbf{X}(1,1,1)$, so $\mathbf{X}$ is not one-one on $D$.)
12. True. (Both manifolds are the same helicoid.)
13. False. (The agreement is only up to sign.)
14. True.
15. False. (This is only true if $n$ is even.)
16. False. (A negative sign is missing.)
17. True.
18. False. ($d\omega = 0$.)
19. True. ($d\omega$ would be an $(n+1)$-form, and there are no nonzero ones on $\mathbf{R}^n$.)
20. True. (This is the generalized Stokes's theorem, since $\partial M = \emptyset$.)

## Miscellaneous Exercises for Chapter 8

**1.** **(a)** First, by definition of the exterior product and derivative

$$\begin{aligned} d(f \wedge g) &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(fg)\, dx_i = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \right) dx_i \quad \text{by the product rule,} \\ &= g \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i + f \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}\, dx_i \\ &= g \wedge df + f \wedge dg \\ &= df \wedge g + (-1)^0 f \wedge dg. \end{aligned}$$

**(b)** If $k = 0$, then write $\omega = f$ so that

$$\begin{aligned} d(\omega \wedge \eta) = d(f \wedge \eta) &= d\left( \sum f G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \right) = \sum d(f G_{j_1 \ldots j_l}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= \sum (df \wedge G_{j_1 \ldots j_l} + f \wedge dG_{j_1 \ldots j_l}) \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \quad \text{from (a),} \\ &= df \wedge \sum G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} + f \wedge \sum dG_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= df \wedge \eta + (-1)^0 f \wedge d\eta. \end{aligned}$$

**(c)** If $l = 0$, then write $\eta = g$ so that

$$\begin{aligned} d(\omega \wedge \eta) &= d(\omega \wedge g) = d\left(\sum gF_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) \\ &= \sum d(gF_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= \sum (dg \wedge F_{i_1 \ldots i_k} + g \wedge dF_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\ &= dg \wedge \omega + g \wedge d\omega \\ &= (-1)^k \omega \wedge dg + d\omega \wedge g \end{aligned}$$

by part 2 of Proposition 1.4 (recall *dg* is a 1-form).

**(d)** In general,

$$\begin{aligned} d(\omega \wedge \eta) &= d\left(\sum_{1 \le i_1 < \cdots < i_k \le n} F_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge \sum_{1 \le j_1 < \cdots < j_l \le n} G_{j_1 \ldots j_l}\, dx_{j_1} \wedge \cdots \wedge dx_{j_l}\right) \\ &= d\left(\sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_l \le n}} F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}\right) \\ &= \sum d(F_{i_1 \ldots i_k} G_{j_1 \ldots j_l}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \quad \text{so by part (a),} \\ &= \sum (dF_{i_1 \ldots i_k} \wedge G_{j_1 \ldots j_l} + F_{i_1 \ldots i_k} \wedge dG_{j_1 \ldots j_l}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= \sum dF_{i_1 \ldots i_k} \wedge G_{j_1 \ldots j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &\quad + \sum F_{i_1 \ldots i_k} \wedge dG_{j_1 \ldots j_l} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &= \sum dF_{i_1 \ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge G_{j_1 \ldots j_l} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \\ &\quad + \sum F_{i_1 \ldots i_k} \wedge (-1)^k dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dG_{j_1 \ldots j_l} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l} \end{aligned}$$

since $G_{j_1 \ldots j_l}$ is a 0-form and $dG_{j_1 \ldots j_l}$ is a 1-form,

$$= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

**2. (a)** Define $\mathbf{X} : [0,1] \times [0,1] \times [0,1] \times [0,1] \to \mathbf{R}^5, \mathbf{X}(u_1, u_2, u_3, u_4, u_5) = (u_1, u_2, u_3, u_4, u_1u_2u_3u_4)$. Then $\mathbf{T}_{u_1} = (1, 0, 0, 0, u_2u_3u_4), \mathbf{T}_{u_2} = (0, 1, 0, 0, u_1u_3u_4), \mathbf{T}_{u_3} = (0, 0, 1, 0, u_1u_2u_4)$, and $\mathbf{T}_{u_4} = (0, 0, 0, 1, u_1u_2u_3)$. From this we see that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}, \mathbf{T}_{u_3}, \mathbf{T}_{u_4}) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1.$$

**(b)** We can now calculate

$$\begin{aligned} &\int_M dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ u_2u_3u_4 & u_1u_3u_4 & u_1u_2u_4 & u_1u_2u_3 \end{bmatrix} du_1\, du_2\, du_3\, du_4 \\ &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 u_1u_2u_3\, du_1\, du_2\, du_3\, du_4 = \frac{1}{8}. \end{aligned}$$

**3. (a)** The curve $C$ may be parametrized as $\mathbf{x}(t) = (t, f(t)), a \le t \le b$. Then $\mathbf{x}'(t) = (1, f'(t))$ and this is compatible with the orientation of $C$. By Definition 2.1, we have

$$\int_C \omega = \int_{\mathbf{x}} \omega = \int_a^b \omega_{\mathbf{x}(t)}(\mathbf{x}'(t))\, dt.$$

For $\omega = y\, dx$ this is

$$\int_a^b f(t) \cdot 1\, dt = \int_a^b f(t)\, dt = \text{area under the graph.}$$

**(b)** Parametrize $S$ by

$$\mathbf{X} : [a, b] \times [c, d] \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2) = (u_1, u_2, f(u_1, u_2)).$$

The upward unit normal $\mathbf{N}$ is given by

$$\mathbf{N} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}.$$

The parametrization is compatible with the orientation since

$$\mathbf{T}_{u_1} \times \mathbf{T}_{u_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_{u_1} \\ 0 & 1 & f_{u_2} \end{vmatrix} = (-f_{u_1}, -f_{u_2}, 1)$$

is parallel to $\mathbf{N}$ (when $\mathbf{N}$ is expressed in terms of the parametrization). Thus,

$$\int_S \omega = \int_{\mathbf{X}} \omega = \int_c^d \int_a^b \omega_{\mathbf{X}(u_1,u_2)}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2})\, du_1\, du_2.$$

For $\omega = z\, dx \wedge dy$, this is

$$\int_c^d \int_a^b f(u_1, u_2) \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} du_1\, du_2 = \int_c^d \int_a^b f(u_1, u_2)\, du_1\, du_2 = \text{area under the graph.}$$

**(c)** Parametrize $M$ using

$$\mathbf{X} : D \to \mathbf{R}^n, \quad \mathbf{X}(u_1, \ldots, u_{n-1}) = (u_1, \ldots, u_{n-1}, f(u_1, \ldots, u_{n-1})).$$

Then, depending on how $M$ is oriented,

$$\begin{aligned} \int_M \omega = \pm \int_{\mathbf{X}} \omega &= \pm \int \cdots \int_D \omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \ldots, \mathbf{T}_{u_{n-1}})\, du_1 \cdots du_{n-1} \\ &= \pm \int \cdots \int_D f(u_1, \ldots, u_{n-1}) \det \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} du_1 \cdots du_{n-1} \\ &= \pm \int \cdots \int_D f(u_1, \ldots, u_{n-1})\, du_1 \cdots du_{n-1} = \pm(n\text{-dimensional volume under the graph}). \end{aligned}$$

If you orient $M$ with the unit normal

$$\mathbf{N} = (-1)^n \frac{(f_{x_1}, \ldots, f_{x_{n-1}}, -1)}{\sqrt{(f_{x_1})^2 + \cdots + (f_{x_{n-1}})^2 + 1}}$$

we can guarantee a $+$ sign above.

**4. (a)** Define a parametrization

$$\mathbf{X} : [0, 3] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2) = (\cos u_2, u_1, \sin u_2).$$

Then we may define $\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}, \mathbf{b}) = \det[\mathbf{N}\ \ \mathbf{a}\ \ \mathbf{b}]$. Note that $\mathbf{X}$ is compatible with this orientation as $\mathbf{T}_{u_1} = (0, 1, 0)$ and $\mathbf{T}_{u_2} = (-\sin u_2, 0, \cos u_2)$ so that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} \cos u_2 & 0 & -\sin u_2 \\ 0 & 1 & 0 \\ \sin u_2 & 0 & \cos u_2 \end{bmatrix} = 1 > 0.$$

(Note that the first column is the normal $\mathbf{N}$ in terms of the parametrization.)

**(b)** The boundary $\partial M$ consists of two disjoint pieces. The left piece is $\{(x,0,z)|x^2+z^2=1\}$, parametrized by $\mathbf{Y}_1 : [0,2\pi) \to \mathbf{R}^3, \mathbf{Y}_1(t) = (\cos t, 0, \sin t)$. The right piece is $\{(x,3,z)|x^2+z^2=1\}$, parametrized by $\mathbf{Y}_2 : [0,2\pi) \to \mathbf{R}^3, \mathbf{Y}_2(t) = (\cos t, 3, \sin t)$.

**(c)** We must first determine $\mathbf{V}$, a unit vector tangent to $M$, normal to $\partial M$, and pointing away from $M$. If you think about the boundary pieces we looked at in part (b), a vector corresponding to the left side is $\mathbf{V}_1 = (0,-1,0)$ and corresponding to the right side is $\mathbf{V}_2 = (0,1,0)$. Then, along the left circle of $\partial M$,

$$\Omega^{\partial M}_{\mathbf{Y}_1(t)}(\mathbf{a}) = \Omega_{\mathbf{X}(0,t)}(\mathbf{V}_1, \mathbf{a})$$

and along the right circle of $\partial M$,

$$\Omega^{\partial M}_{\mathbf{Y}_2(t)}(\mathbf{a}) = \Omega_{\mathbf{X}(3,t)}(\mathbf{V}_2, \mathbf{a}).$$

Note that

$$\Omega^{\partial M}_{\mathbf{Y}_1(t)}(\mathbf{T}_t) = \det \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & -1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix} = -1.$$

So the parametrization $\mathbf{Y}_1$ is incompatible with $\Omega^{\partial M}$. However,

$$\Omega^{\partial M}_{\mathbf{Y}_2(t)}(\mathbf{T}_t) = \det \begin{bmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{bmatrix} = 1.$$

So the parametrization $\mathbf{Y}_2$ is compatible with $\Omega^{\partial M}$.

**(d)** If $\omega = z\,dx + (x+y+z)\,dy - x\,dz$, we have

$$d\omega = dz \wedge dx + (dx + dy + dz) \wedge dy - dx \wedge dz = dx \wedge dy - dy \wedge dz - 2\,dx \wedge dz.$$

Then using the orientation $\Omega$ and the parametrization $\mathbf{X}$ from part (a), we have

$$\int_M d\omega = \int_{\mathbf{X}} d\omega = \int_0^{2\pi}\int_0^3 \left( \begin{vmatrix} 0 & -\sin u_2 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 0 & \cos u_2 \end{vmatrix} - 2 \begin{vmatrix} 0 & -\sin u_2 \\ 0 & \cos u_2 \end{vmatrix} \right) du_1\,du_2$$
$$= \int_0^{2\pi}\int_0^3 (\sin u_2 - \cos u_2)\,du_1\,du_2 = 3(-\cos u_2 - \sin u_2)|_0^{2\pi} = 0.$$

On the other hand, using the parametrizations $\mathbf{Y}_1$ and $\mathbf{Y}_2$ for $\partial M$ in parts (b) and (c), we have (after reversing the sign for the left piece because of the incompatibility with $\Omega^{\partial M}$)

$$\int_{\partial M} \omega = -\int_{\mathbf{Y}_1} \omega + \int_{\mathbf{Y}_2} \omega$$
$$= -\int_0^{2\pi} [\sin t(-\sin t) + (\cos t + \sin t)\cdot 0 - \cos t(\cos t)]\,dt$$
$$+ \int_0^{2\pi} [\sin t(-\sin t) + (\cos t + 3 + \sin t)\cdot 0 - \cos t(\cos t)]\,dt = 0.$$

**5.** If $S^4$ is the unit 4-sphere in $\mathbf{R}^5$, then let $B$ denote the 5-dimensional unit ball

$$B = \{x_1, x_2, x_3, x_4, x_5)|x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 1\}.$$

Note that $\partial B = S^4$. Then using the generalized Stokes's theorem, we have

$$\int_{S^4} \omega = \int_B d\omega.$$

For $\omega = x_3\,dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + x_4\,dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$ we have $d\omega = dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4 \wedge dx_5 + dx_4 \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 = dx_1 \wedge \cdots \wedge dx_5 - dx_1 \wedge \cdots \wedge dx_5 = 0$. Hence $\int_{S^4} \omega = \int_B 0 = 0$.

**6. (a)** Let $\omega = f$. Then $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ and

$$d(df) = \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i = \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_j \partial x_i} \wedge dx_j\right) \wedge dx_i$$
$$= \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i + \sum_{i>j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i,$$

since the terms where $i = j$ contain $dx_i \wedge dx_i = 0$. By exchanging the roles of $i$ and $j$ in the second sum, we find

$$d(df) = \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i + \sum_{i<j} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j$$
$$= \sum_{i<j} \left(-\frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial^2 f}{\partial x_j \partial x_i}\right) dx_i \wedge dx_j = 0$$

since the mixed partials are equal because $f$ is of class $C^2$.

**(b)** Now

$$d(d\omega) = d\left(d\left(\sum_{1\le i_1<\cdots<i_k\le n} F_{i_1\ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)\right)$$
$$= d\left(\sum_{1\le i_1<\cdots<i_k\le n} dF_{i_1\ldots i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right)$$
$$= \sum_{1\le i_1<\cdots<i_k\le n} [d(dF_{i_1\ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$
$$+ (-1)^1\, dF_{i_1\ldots i_k} \wedge d(dx_{i_1} \wedge \cdots \wedge dx_{i_k})] \quad \text{from Exercise 1,}$$
$$= -\sum_{1\le i_1<\cdots<i_k\le n} dF_{i_1\ldots i_k} \wedge d(dx_{i_1} \wedge \cdots \wedge dx_{i_k})$$

since $d(dF_{i_1\ldots i_k}) = 0$ from part (a). But

$$d(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = d(1\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = d(1) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0.$$

Hence $d(d\omega) = 0$, as desired.

**7. (a)** If $\omega$ is a 0-form, write $\omega = f$. Then, using the first row of the chart, the 1-form $d\omega$ corresponds to the vector field $\nabla f$. Hence, from the second row of the chart, $d(d\omega)$ is the 2-form that corresponds to $\nabla \times \nabla f$. Thus $d(d\omega) = 0$ "translates" to the statement $\nabla \times (\nabla f) = \mathbf{0}$.

**(b)** If $\omega$ is a 1-form, it corresponds to the vector field $\mathbf{F}$ and, using the second row of the chart, $d\omega$ is the 2-form that corresponds to $\nabla \times \mathbf{F}$, another vector field. Then, using the third row of the chart, $d(d\omega)$ is the 3-form that corresponds to $\nabla \cdot (\nabla \times \mathbf{F})$. Hence, $d(d\omega) = 0$ "translates" to the statement that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

**8. (a)** The outward unit normal $\mathbf{N} = (x, y, z)$ gives orientation form $\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{a}_1, \mathbf{a}_2) = \det[\mathbf{N}\ \ \mathbf{a}_1\ \mathbf{a}_2]$ where $\mathbf{X}$ is a parametrization of $S$. For a specific parametrization we can use

$$\mathbf{X} : [0, \pi] \times [0, 2\pi) \to \mathbf{R}^3; \quad \mathbf{X}(u_1, u_2) = (\sin u_1 \cos u_2, \sin u_1 \sin u_2, \cos u_1).$$

Then $\mathbf{T}_{u_1} = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, -\sin u_1)$ and $\mathbf{T}_{u_2} = (-\sin u_1 \sin u_2, \sin u_1 \cos u_2, 0)$, so that

$$\Omega_{\mathbf{X}(\mathbf{u})}(\mathbf{T}_{u_1}, \mathbf{T}_{u_2}) = \det \begin{bmatrix} \sin u_1 \cos u_2 & \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ \sin u_1 \sin u_2 & \cos u_1 \sin u_2 & \sin u_1 \cos u_2 \\ \cos u_1 & -\sin u_1 & 0 \end{bmatrix} = \sin u_1 \ge 0.$$

In fact, this quantity is strictly greater than 0 when the parametrization is smooth and so the parametrization is compatible with the orientation.

Next we note that on $S$ we have $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$ as the denominators in $\omega$ are all 1 on $S$. Therefore,

$$\begin{aligned}
\int_S \omega &= \int_{\mathbf{X}} \omega \\
&= \int_0^{2\pi}\int_0^{\pi} \left\{ \sin u_1 \cos u_2 \begin{vmatrix} \cos u_1 \sin u_2 & \sin u_1 \cos u_2 \\ -\sin u_1 & 0 \end{vmatrix} \right. \\
&\quad + \sin u_1 \sin u_2 \begin{vmatrix} -\sin u_1 & 0 \\ \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \end{vmatrix} \\
&\quad \left. + \cos u_1 \begin{vmatrix} \cos u_1 \cos u_2 & -\sin u_1 \sin u_2 \\ \cos u_1 \sin u_2 & \sin u_1 \cos u_2 \end{vmatrix} \right\} du_1\,du_2 \\
&= \int_0^{2\pi}\int_0^{\pi} (\sin^3 u_1 + \cos^2 u_1 \sin u_1)\,du_1\,du_2 = \int_0^{2\pi}\int_0^{\pi} \sin u_1\,du_1\,du_2 \\
&= 2\pi(-\cos u_1)\big|_0^{\pi} = 4\pi.
\end{aligned}$$

**(b)** For $\omega$ as given we calculate

$$d\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(y^2+z^2-2x^2)\,dx - 3xy\,dy - 3xz\,dz}{(x^2+y^2+z^2)^{5/2}}$$

$$d\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{-3xy\,dx - (x^2+z^2-2y^2)\,dy - 3yz\,dz}{(x^2+y^2+z^2)^{5/2}}$$

$$d\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{-3xz\,dx - 3yz\,dy - (x^2+y^2-2z^2)dz}{(x^2+y^2+z^2)^{5/2}}$$

Hence,

$$\begin{aligned}
d\omega = \frac{1}{(x^2+y^2+z^2)^{5/2}} &\left[(y^2+z^2-2x^2)\,dx \wedge dy \wedge dz \right. \\
&+ (x^2-2y^2+z^2)\,dy \wedge dz \wedge dx \\
&\left. + (x^2+y^2-2z^2)\,dz \wedge dx \wedge dy\right]
\end{aligned}$$

This is identically equal to 0 wherever it is defined.

**(c)** Since $M$ does not include the origin, we have $\int_M d\omega = \int_M 0 = 0$ from part (b).

$\partial M$ consists of two pieces. The outer piece $S_1$ is the unit sphere $x^2 + y^2 + z^2 = 1$, oriented by the outward unit normal $\mathbf{n}_1 = (x, y, z)$. The inner piece is the sphere $x^2 + y^2 + z^2 = a^2$ of radius $a$, oriented by inward unit normal $\mathbf{n}_2 = (-x, -y, -z)/a$. Then, using Proposition 2.4, we have

$$\int_{\partial M} \omega = \iint_{\partial M} \mathbf{F} \cdot d\mathbf{S} \text{ where } \mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}.$$

In the following calculation we will use the fact that $x^2 + y^2 + z^2$ is 1 on $S_1$ and is $a^2$ on $S_2$.

$$\begin{aligned}
\int_{\partial M} \omega &= \iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1\,dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2\,dS \\
&= \iint_{S_1} 1\,dS + \iint_{S_2} \frac{1}{a^2}\,dS \\
&= (1)(\text{surface area of } S_1) - \frac{1}{a^2}(\text{surface area of } S_2) \\
&= 4\pi - \frac{1}{a^2}(4\pi a^2) = 0.
\end{aligned}$$

This verifies Theorem 3.2.

**(d)** No—since $\omega$ is not defined at the origin, Theorem 3.2 does not apply.

**(e)** Let $M$ be the 3-manifold bounded on the outside by $S$, oriented with the outward normal, and on the inside by $S_\epsilon$, oriented by the inward normal. Then $\mathbf{0} \notin M$, so we have

$$0 = \int_M d\omega = \int_{\partial M} \omega = \int_S \omega + \int_{S_\epsilon} \omega = \int_S \omega - 4\pi.$$

The last equality follows from part (c). The conclusion is that $\int_S \omega = 4\pi$.

**9.** Because $\partial M = \emptyset$, the note following Theorem 3.2 advises us to take $\int_{\partial M} \omega \wedge \eta$ to be 0 in the equation $\int_{\partial M} \omega \wedge \eta = \int_M d(\omega \wedge \eta)$. Now substitute the results of Exercise 1 to get

$$0 = \int_M d(\omega \wedge \eta) = \int_M d\omega \wedge \eta + (-1)^k \omega \wedge d\eta = \int_M d\omega \wedge \eta + (-1)^k \int_M \omega \wedge d\eta.$$

Pull this last piece to the other side to obtain the result

$$(-1)^{k+1} \int_M \omega \wedge d\eta = \int_M d\omega \wedge \eta.$$

**10.** By the generalized Stokes's theorem,

$$\begin{aligned}\int_{\partial M} f\omega &= \int_M d(f\omega)\\ &= \int_M (df \wedge \omega + f \wedge d\omega) \quad \text{by the result of Exercise 1,}\\ &= \int_M (df \wedge \omega + f\, d\omega).\end{aligned}$$

Hence

$$\int_M f\, d\omega = \int_{\partial M} f\omega - \int_M df \wedge \omega.$$