Chapter 7

Surface Integrals and Vector Analysis

7.1 Parametrized Surfaces

1. (a) To find a normal vector we calculate

$$\mathbf{T}_s(s,t) = (2s, 1, 2s) \quad \text{so} \quad \mathbf{T}_s(2,-1) = (4,1,4)$$
$$\mathbf{T}_t(s,t) = (-2t, 1, 3) \quad \text{so} \quad \mathbf{T}_t(2,-1) = (2,1,3).$$

Then a normal vector is

$$\mathbf{N}(2,-1) = \mathbf{T}_s(2,-1) \times \mathbf{T}_t(2,-1) = (-1,-4,2).$$

(b) We find an equation for the tangent plane using

$$0 = \mathbf{N}(2,-1)\cdot(\mathbf{x} - (3,1,1)) = (-1,-4,2)\cdot(\mathbf{x}-(3,1,1)) = -x+3-4y+4+2z-2.$$

This is equivalent to $x + 4y - 2z = 5$.

2. First we figure that since $2\sin t = 1$, either $t = \pi/6$ or $5\pi/6$. Since $2\cos t < 0$ we know that $t = 5\pi/6$. Then we can see that $\sin s = \sqrt{2}/2$ so $s = \pi/4$. Next, find a normal vector to the surface at the given point by calculating

$$\mathbf{T}_s(s,t) = (-(5+2\cos t)\sin s, (5+2\cos t)\cos s, 0) \quad \text{and}$$
$$\mathbf{T}_t(s,t) = (-2\sin t\cos s, -2\sin t\sin s, 2\cos t) \quad \text{so}$$
$$\mathbf{N}(s,t) = \mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = 2(5+2\cos t)(\cos s\cos t, \sin s\cos t, \sin t). \quad \text{Therefore,}$$
$$\mathbf{N}(\pi/4, 5\pi/6) = \frac{\sqrt{3}-5}{\sqrt{2}}(\sqrt{3},\sqrt{3},-\sqrt{2}).$$

We calculate an equation for the tangent plane by writing $\mathbf{N}\cdot(\mathbf{x} - (x_0,y_0,z_0)) = 0$ or, equivalently in this case,

$$0 = (\sqrt{3},\sqrt{3},-\sqrt{2})\cdot\left(\mathbf{x} - \left(\frac{5-\sqrt{3}}{\sqrt{2}}, \frac{5-\sqrt{3}}{\sqrt{2}}, 1\right)\right) \quad \text{or} \quad \sqrt{3}x + \sqrt{3}y - \sqrt{2}z = 5\sqrt{6} - 4\sqrt{2}.$$

3. Since $x = e^s$ at $x = 1$, we know that $s = 0$. Also since $z = 2e^{-s} + t$, when $z = 0$ and $s = 0$, we have $t = -2$. As above we calculate,

$$\mathbf{T}_s(s,t) = (e^s, 2t^2e^{2s}, -2e^{-s}) \quad \text{and} \quad \mathbf{T}_t(s,t) = (0, 2te^{2s}, 1).$$

Thus, $\mathbf{N}(0,-2) = \mathbf{T}_s(0,-2)\times\mathbf{T}_t(0,-2) = (1,8,-2)\times(0,-4,1) = (0,-1,-4)$. Then an equation of the tangent plane is $0 = \mathbf{N}(0,-2)\cdot(\mathbf{x}-(1,4,0)) = (0,-1,-4)\cdot(\mathbf{x}-(1,4,0))$. We can simplify this to $y + 4z = 4$.

4. (a) $\mathbf{T}_s(s,t) = (2s\cos t, 2s\sin t, 1)$ so $\mathbf{T}_s(-1,0) = (-2,0,1)$. Also, $\mathbf{T}_t(s,t) = (-s^2\sin t, s^2\cos t, 0)$ so $\mathbf{T}_t(-1,0) = (0,4,0)$. Therefore, $\mathbf{N}(-1,0) = (-2,0,1)\times(0,4,0) = (-4,0,-8)$.

(b) An equation of the tangent plane is $(-4,0,-8)\cdot(\mathbf{x}-(1,0,-1)) = 0$. This simplifies to $x + 2z = -1$.

(c) Note that the x-component of **X** is $s^2\cos t$ and the y-component is $s^2\sin t$ and the z-component is a function of s. We can eliminate the t by looking at x^2+y^2. So without much work we have found that an equation for the image of **X** is $x^2+y^2-z^4 = 0$.

5. (a) Using *Mathematica* and the command:

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ParametricPlot3D[{s, s^2 + t, t^2}, {s, −2, 2}, {t, −2, 2}, AxesLabel → {x,y,z}],
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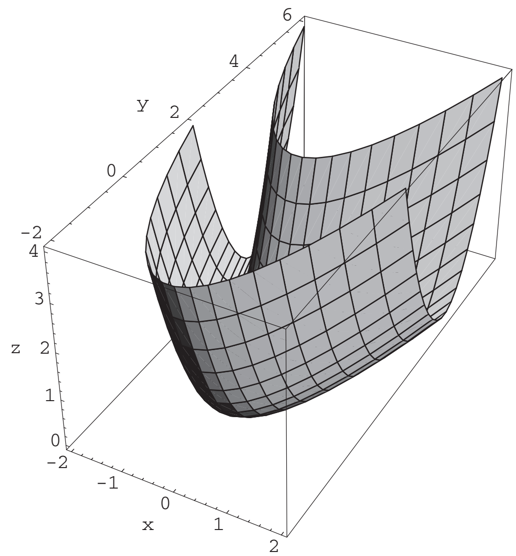
we obtain the image



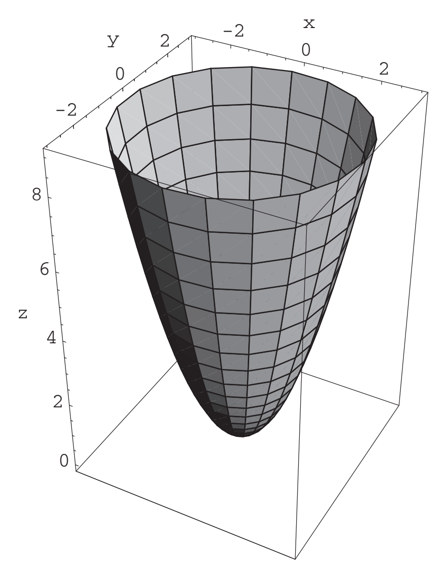
(b) To determine whether the surface is smooth we need to calculate **N**. First, $\mathbf{T}_s(s,t) = (1, 2s, 0)$, and $\mathbf{T}_t(s,t) = (0, 1, 2t)$ so $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (4st, -2t, 1)$. We conclude that $\mathbf{N} \neq \mathbf{0}$ for any (s,t) so **N** is smooth.

(c) If $(s, s^2 + t, t^2) = (1, 0, 1)$, then $s = 1$ and $t = -1$. So $\mathbf{N}(1, -1) = (-4, 2, 1)$ and an equation of the tangent plane at this point is $(-4, 2, 1) \cdot (\mathbf{x} - (1, 0, 1)) = 0$ or more simply, $4x - 2y - z = 3$.

6. In Exercise 1, $x = s^2 - t^2, y = s + t$, and so if we note that $x = (s+t)(s-t) = y(s-t)$, then $x/y = s - t$. This allows us to solve for s and t separately: $2s = y + x/y$ and $2t = y - x/y$. This means that $z = s^2 + 3t$ can be written as $z = (y + x/y)^2/4 + 3(y - x/y)$.

7. (a) For the surface, we have $\mathbf{X}(s,t) = (s\cos t, s\sin t, s^2)$, where $s \geq 0$ and $0 \leq t \leq 2\pi$. This means that $\mathbf{T}_s(s,t) = (\cos t, \sin t, 2s)$ and $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$. Then a normal vector is given by $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (-2s^2\cos t, -2s^2\sin t, s)$. This means that the surface is smooth except when $s = 0$. In other words, S is smooth (as a parametrized surface) except at the origin. Note that the point $(1, \sqrt{3}, 4) = \mathbf{X}(2, \frac{\pi}{3})$. Thus $\mathbf{N}(2, \frac{\pi}{3}) = (-8\cos\frac{\pi}{3}, -8\sin\frac{\pi}{3}, 2) = (-4, -4\sqrt{3}, 2)$ and thus an equation of the tangent plane is given by $(-4, -4\sqrt{3}, 2)\cdot(x - 1, y - \sqrt{3}, z - 4) = 0$ or, equivalently, by $2x + 2\sqrt{3}y - z = 4$.

(b) See the figure below and note that $z = x^2 + y^2$ so we see that S is a paraboloid.



(c) Again, $z = x^2 + y^2$.

(d) Part (a) above takes care of every point except the origin. At the origin $\mathbf{N} = \mathbf{0}$, but we easily see that the tangent plane there is the horizontal plane $z = 0$. Thus smoothness in the sense defined in Section 7.1 depends on the parametrization as well as the geometry of the underlying surface.

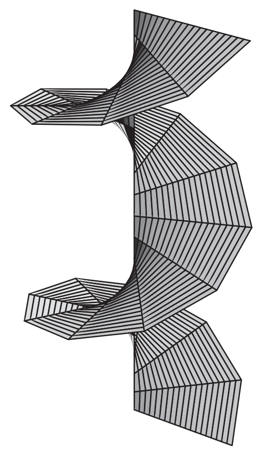
8. Really there's not much to show. You know that if the image of the parametrized surface is to be an ellipsoid, you need $a(2\sin s\cos t)^2 + b(3\sin s\sin t)^2 + c(\cos s)^2 = 1$. So $a = 1/4$, $b = 1/9$, and $c = 1$. Therefore the image satisfies $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$.

9. For $t = t_0, \mathbf{X}(s, t_0) = ((a + b\cos t_0)\cos s, (a + b\cos t_0)\sin s, b\sin t_0)$. So z is constant and $x^2 + y^2 = (a + b\cos t_0)^2$. This is a circle of radius $a + b\cos t_0$ centered at $(0, 0, b\sin t_0)$.

10. (a) When $\theta = \pi/3$, the r-coordinate curve is given by $(r/2, r\sqrt{3}/2, \pi/3)$ where $r \geq 0$. This is the ray $y = \sqrt{3}x$ where $x \geq 0$ and $z = \pi/3$. In general, the r-coordinate curve when $\theta = \theta_0$ is a ray in the $z = \theta_0$ plane. The solution is simpler than the following four cases make it seem. If $\cos\theta_0 \neq 0$ then $y = (\tan\theta_0)x$ where $x \geq 0$ if $\cos\theta_0 > 0$ and $x \leq 0$ if $\cos\theta_0 < 0$. If $\cos\theta_0 = 0$, then the ray is $x = 0$ with $y \geq 0$ if $\sin\theta_0 > 0$ and $y \leq 0$ if $\sin\theta_0 < 0$.

(b) When $r = 1$ the θ-coordinate curve is the helix $(\cos\theta, \sin\theta, \theta)$. In general, when $r = r_0$ the θ-coordinate curve is the helix $(r_0\cos\theta, r_0\sin\theta, \theta)$.

(c) You can see that the helicoids are made up of the helices that are the θ-coordinate curves.



11. (a) First we consider the sphere as the graph of the function $f(x, y) = \sqrt{4 - (x-2)^2 - (y+1)^2}$. The partial derivatives are

$$f_x = \frac{-(x-2)}{\sqrt{4 - (x-2)^2 - (y+1)^2}} \quad f_y = \frac{-(y+1)}{\sqrt{4 - (x-2)^2 - (y+1)^2}}.$$

So $f_x(1, 0, \sqrt{2}) = 1/\sqrt{2}$ and $f_y(1, 0, \sqrt{2}) = -1/\sqrt{2}$. By Theorem 3.3 of Chapter 2, $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. In this case, this is $z = \sqrt{2} + (1/\sqrt{2})(x - 1) - (1/\sqrt{2})y$, or equivalently, $-x + y + \sqrt{2}z = 1$.

(b) Now we look at the sphere as a level surface of $F(x, y, z) = (x-2)^2 + (y+1)^2 + z^2$. The gradient $\nabla F(x, y, z) = 2(x - 2, y + 1, z)$ and, therefore, $\nabla F(1, 0, \sqrt{2}) = (-2, 2, 2\sqrt{2})$. By formula (5) of Section 2.6, the tangent plane is given by

$$0 = \nabla F(1, 0, \sqrt{2}) \cdot (\mathbf{x} - (1, 0, \sqrt{2})) = (-2, 2, 2\sqrt{2}) \cdot (\mathbf{x} - (1, 0, \sqrt{2})).$$

This too is equivalent to $-x + y + \sqrt{2}z = 1$.

(c) Now we'll use the results of this section. Considering the z-component, we see $2\cos s = \sqrt{2}$ so $\cos s = \sqrt{2}/2$. Considering the y- and x-components, $2\sin s\sin t = 1$ and $\sin s\cos t = -1$. Thus we have that $s = \pi/4$ and $t = 3\pi/4$. Also $\mathbf{T}_s(s, t) = (2\cos s\cos t, 2\cos s\sin t, -2\sin s)$ and $\mathbf{T}_t(s, t) = (-2\sin s\sin t, 2\sin s\cos t, 0)$. A normal vector to the sphere at the specified point is

$$\mathbf{N}(\pi/4, 3\pi/4) = \mathbf{T}_s(\pi/4, 3\pi/4) \times \mathbf{T}_t(\pi/4, 3\pi 4) = (-1, 1, -\sqrt{2}) \times (-1, -1, 0) = (-\sqrt{2}, \sqrt{2}, 2).$$

The tangent plane is given by $(-\sqrt{2}, \sqrt{2}, 2) \cdot (\mathbf{x} - (1, 0, \sqrt{2})) = 0$ which is also equivalent to $-x + y + \sqrt{2}z = 1$.

12. The sphere of radius 3 is parametrized as $\mathbf{X}(s, t) = (3\cos s\sin t, 3\sin s\sin t, 3\sin t)$, where $0 \leq s < 2\pi$ and $0 \leq t \leq \pi$. To obtain the lower hemisphere, we need the z-coordinate to be nonpositive. Thus we may use the same expression for $\mathbf{X}(s, t)$, only with $0 \leq s < 2\pi$ and $\pi/2 \leq t \leq \pi$.

13. We may let $x = 2\cos s$, $z = 2\sin s$, and $y = t$, where $0 \leq s < 2\pi$, to parametrize the entire, infinitely long cylinder. To obtain the desired finite cylinder, we just let $D = \{(s, t) \mid 0 \leq s \leq 2\pi, \ -1 \leq t \leq 3\}$ and define $\mathbf{X}\colon D \to \mathbf{R}^3$, $\mathbf{X}(s, t) = (2\cos s, t, 2\sin s)$.

14. Note that the region we are describing is the part of the plane having equation $5x+10y+2z=10$, or $z=5-\frac{5}{2}x-5y$ lying in the first octant. The projection of the triangle in the xy-plane is the triangular region

$$\{(x,y) \mid x\geq 0,\ y\geq 0,\ 5x+10y\leq 10\} = \left\{(x,y) \mid 0\leq y\leq 1-\frac{x}{2},\ 0\leq x\leq 2\right\}.$$

(This was found by setting $z=0$ in the equation for the plane.) Hence the desired surface may be parametrized as $\mathbf{X}\colon D\to \mathbf{R}^3$, $\mathbf{X}(s,t)=\left(s,t,5-\frac{5}{2}s-5t\right)$, where $D=\left\{(s,t) \mid 0\leq t\leq 1-\frac{s}{2},\ 0\leq s\leq 2\right\}$.

15. If we rewrite the equation for the hyperboloid as $z^2=x^2+y^2+1$, then we see that we must have $z=\pm\sqrt{x^2+y^2+1}$. Therefore, the hyperboloid may be parametrized with two maps as $\mathbf{X}_1\colon \mathbf{R}^2\to\mathbf{R}^3$, $\mathbf{X}_1(s,t)=\left(s,t,\sqrt{s^2+t^2+1}\right)$ and $\mathbf{X}_2\colon \mathbf{R}^2\to\mathbf{R}^3$, $\mathbf{X}_2(s,t)=\left(s,t,-\sqrt{s^2+t^2+1}\right)$.

16. (a) $\mathbf{X}(1,-1)=(1,-1,-1)$ and we have $\mathbf{T}_s=(3s^2,0,t), \mathbf{T}_t=(0,3t^2,s)$. Hence the normal at (1, −1, −1), which is when $s=1, t=-1$, is $\mathbf{N}(1,-1)=\mathbf{T}_s(1,-1)\times\mathbf{T}_t(1,-1)=(3,0,-1)\times(0,3,1)=(3,-3,9)$. So an equation for the tangent plane is

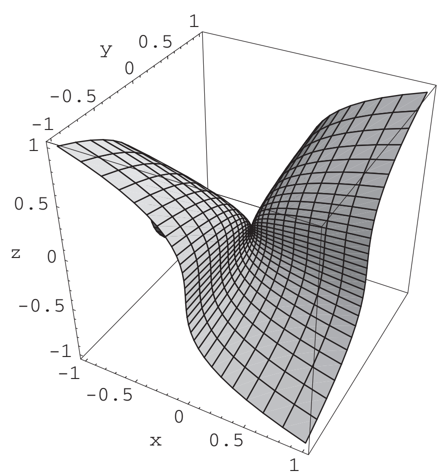
$$3(x-1)-3(y+1)+9(z+1)=0 \quad \text{or} \quad x-y+3z=-1.$$

(b) In general we have that the standard normal is given by

$$\mathbf{N}(s,t)=\mathbf{T}_s\times\mathbf{T}_t=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3s^2 & 0 & t \\ 0 & 3t^2 & s \end{vmatrix}=(-3t^3,-3s^3,9s^2t^2).$$

Note that $\mathbf{N}=\mathbf{0}$ when $s=t=0$, i.e., at (0, 0, 0). So the surface fails to be smooth there.

(c) A computer graph is shown.



(d) With $x=s^3$, $y=t^3$, $z=st$, $\sqrt[3]{xy}=\sqrt[3]{s^3t^3}=st=z$. Sometimes a computer will graph $z=\sqrt[3]{xy}$ for points only where x and y are nonnegative (or sometimes where $xy\geq 0$).

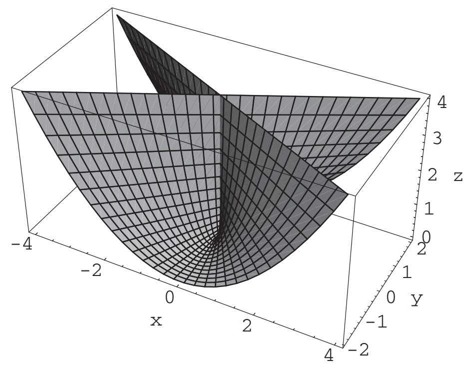
17. (a) $y^2z=t^2\cdot s^2=(st)^2=x^2$

(b) The standard normal is

$$\mathbf{N}(s,t)=\mathbf{T}_s\times\mathbf{T}_t=(t,0,2s)\times(s,1,0)$$
$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 0 & 2s \\ s & 1 & 0 \end{vmatrix}=(-2s,2s^2,t).$$

So $\mathbf{N}=\mathbf{0}$ when $s=t=0$, i.e., at (0, 0, 0). At this point $\mathbf{X}$ fails to be smooth.

(c) A computer graph is shown.



(d) $\mathbf{X}(s_1, t_1) = \mathbf{X}(s_2, t_2)$ when $s_1 t_1 = s_2 t_2$, $t_1 = t_2$, $s_1^2 = s_2^2$. Thus if $t_1 = t_2 = 0$ and $s_1 = \pm s_2$ we get the same image—i.e., $\mathbf{X}(s, 0) = \mathbf{X}(-s, 0) = (0, 0, s^2)$. Thus the positive z-axis (which lies on the image of **X**) is *not* uniquely determined.

(e) Note that $(2, 1, 4) = \mathbf{X}(2, 1)$. From work in part (b), $\mathbf{N}(2, 1) = (-4, 8, 1)$ so an equation for the tangent plane is $-4(x-2) + 8(y-1) + 1(z-4) = 0$ or $-4x + 8y + z = 4$.

(f) $(0, 0, 1) = \mathbf{X}(-1, 0) = \mathbf{X}(1, 0)$.

$$\mathbf{N}(-1, 0) = (2, 2, 0) \quad \mathbf{N}(1, 0) = (-2, 2, 0)$$

So the corresponding tangent planes have equations $x + y = 0$ and $x - y = 0$ respectively.
(If you look at the graph in part (c), you can see two parts of the surface intersecting, so this makes sense.)

18. Here we generalize the results of parts (a) and (c) of Exercise 11. If we view S as the graph of a function $f(x, y)$ then we can apply formula (4) of Section 2.3:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We can rewrite this equation as $0 = (f_x(a, b), f_y(a, b), -1) \cdot (\mathbf{x} - (a, b, f(a, b)))$, where $a = x(s_0, t_0), b = y(s_0, t_0)$, and $f(a, b) = z(s_0, t_0)$. In other words, we are also considering S to be a surface that is parametrized by $\mathbf{X}(s, t) = (x(s, t), y(s, t), z(s, t))$ and so, using the Chain Rule,

$$\mathbf{T}_s(s, t) = (x_s(s, t), y_s(s, t), f_x(x, y)x_s(s, t) + f_y(x, y)y_s(s, t)) \quad \text{and}$$
$$\mathbf{T}_t(s, t) = (x_t(s, t), y_t(s, t), f_x(x, y)x_t(s, t) + f_y(x, y)y_t(s, t)).$$

We calculate the normal vector **N** by taking the cross product $\mathbf{T}_s \times \mathbf{T}_t$ and simplifying to obtain

$$\mathbf{N}(s, t) = [x_t(s, t)y_s(s, t) - x_s(s, t)y_t(s, t)](f_x(x, y), f_y(x, y), -1).$$

So an equation of the tangent plane at (s_0, t_0) is $\mathbf{N} \cdot (\mathbf{x} - \mathbf{X}(s_0, t_0)) = 0$ which in this case is

$$[x_t(s_0, t_0)y_s(s_0, t_0) - x_s(s_0, t_0)y_t(s_0, t_0)](f_x(a, b), f_y(a, b), -1) \cdot (\mathbf{x} - (a, b, f(a, b))) = 0 \quad \text{or}$$
$$(f_x(a, b), f_y(a, b), -1) \cdot (\mathbf{x} - (a, b, f(a, b))) = 0.$$

So we see that in this case the results of the two methods agree.

19. (a) To find an equation for the tangent plane to a surface described by the equation $y = g(x, z)$ at the point $(a, g(a, c), c)$ we basically permute the case detailed in the text and in Exercise 18 to obtain either

$$(g_x(a, c), -1, g_z(a, c)) \cdot (\mathbf{x} - (a, g(a, c), c)) = 0 \quad \text{or}$$
$$g_x(a, c)(x - a) - (y - g(a, c)) + g_z(a, c)(z - c) = 0.$$

(b) Similarly, an equation for the tangent plane to a surface described by the equation $x = h(y, z)$ at the point $(h(b, c), b, c)$ is either

$$(-1, h_y(b, c), h_z(b, c)) \cdot (\mathbf{x} - (h(b, c), b, c)) = 0 \quad \text{or}$$
$$-(x - h(b, c)) + h_y(b, c)(y - b) + h_z(b, c)(z - c) = 0.$$

20. We have $\mathbf{X}\colon D \to \mathbf{R}^3$ and by Definition 3.8 of Chapter 2, the linear approximation is given by

$$\mathbf{x} = \mathbf{X}(s_0, t_0) + D\mathbf{X}(s_0, t_0)\begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix}.$$

Here $D\mathbf{X}(s_0, t_0)$ is the matrix

$$D\mathbf{X}(s_0, t_0) = \begin{bmatrix} x_s(s_0, t_0) & x_t(s_0, t_0) \\ y_s(s_0, t_0) & y_t(s_0, t_0) \\ z_s(s_0, t_0) & z_t(s_0, t_0) \end{bmatrix} = [(\mathbf{T}_s(s_0, t_0))^T \quad (\mathbf{T}_t(s_0, t_0))^T].$$

Thus the tangent plane to the surface is given by

$$\begin{aligned}(x, y, z) &= \mathbf{X}(s_0, t_0) + [(\mathbf{T}_s(s_0, t_0))^T \quad (\mathbf{T}_t(s_0, t_0))^T]\begin{bmatrix} s - s_0 \\ t - t_0 \end{bmatrix} \\ &= \mathbf{X}(s_0, t_0) + \mathbf{T}_s(s_0, t_0)(s - s_0) + \mathbf{T}_t(s_0, t_0)(t - t_0).\end{aligned}$$

21. By Exercise 20,

$$\begin{aligned}(x, y, z) &= (1, 0, 1) + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} s - 1 \\ t + 1 \end{bmatrix} \\ (x, y, z) &= (s, 2s + t - 1, -2t - 1).\end{aligned}$$

We check this against our result for Exercise 5(c):

$$4x - 2y - z = 4s - 2(2s + t - 1) - (-2t - 1) = 3.$$

22. In Exercise 3 we parametrized a cylinder of radius a and height h by $\mathbf{X}(s, t) = (a\cos s, a\sin s, t)$ for $0 \le t \le h$ and $0 \le s < 2\pi$. Then $\mathbf{T}_s(s, t) = (-a\sin s, a\cos s, 0)$, $\mathbf{T}_t(s, t) = (0, 0, 1)$, and $\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t) = (a\cos s, a\sin s, 0)$. Then, by formula (6), the surface area of S is

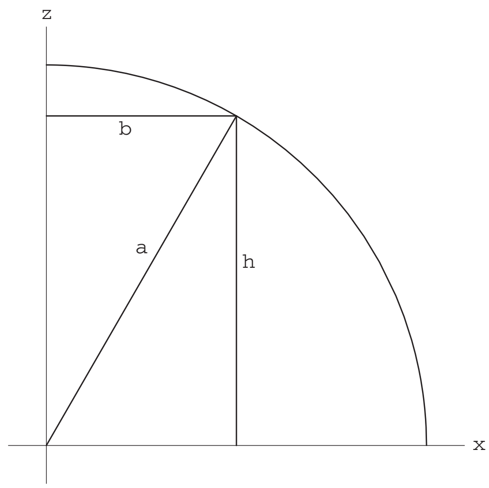
$$\int_0^{2\pi}\int_0^h \|\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t)\|\, dt\, ds = \int_0^{2\pi}\int_0^h a\, dt\, ds = 2\pi ah.$$

23. As in Exercise 22 we need to calculate $\|\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t)\|$. We have that $\mathbf{X}(s, t) = (s + t, s - t, s)$ for $-1 \le s \le 1$ and $-\sqrt{1 - s^2} \le t \le \sqrt{1 - s^2}$. Therefore, $\mathbf{T}_s(s, t) = (1, 1, 1), \mathbf{T}_t(s, t) = (1, -1, 0), \mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t) = (1, 1, -2)$ and $\|\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t)\| = \sqrt{6}$. So we are integrating $\sqrt{6}$ over the unit disk in the *st*-plane. Therefore, the surface area of $\mathbf{X}(D) = \iint_D \sqrt{6}\, dt\, ds = \sqrt{6}\pi$.

24. For the parametrization of the helicoid, $\mathbf{X}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$ so $\mathbf{T}_r(r, \theta) = (\cos\theta, \sin\theta, 0)$, $\mathbf{T}_\theta(r, \theta) = (-r\sin\theta, r\cos\theta, 1)$, $\mathbf{T}_r(r, \theta) \times \mathbf{T}_\theta(r, \theta) = (\sin\theta, -\cos\theta, r)$ and $\|\mathbf{T}_r(r, \theta) \times \mathbf{T}_\theta(r, \theta)\| = \sqrt{1 + r^2}$. Then the surface area of n "turns" of the helicoid is

$$\begin{aligned}\int_0^{2\pi n}\int_0^1 \sqrt{1 + r^2}\, dr\, d\theta &= \int_0^{2\pi n} \frac{1}{2}[\sqrt{2} + \sinh^{-1}(1)]\, d\theta = \int_0^{2\pi n} \frac{1}{2}[\sqrt{2} + \ln(\sqrt{2} + 1)]\, d\theta \\ &= [\sqrt{2} + \ln(\sqrt{2} + 1)]\pi n.\end{aligned}$$

25. A quick look at the figure below shows a cutaway of a quarter of the *xz*-plane intersection of the cylindrical hole of radius b bored in a sphere of radius a. The height of the hole is $2\sqrt{a^2 - b^2}$. The top half of the ring is the region swept out by the portion of the diagram containing the letter 'h'.



If $\mathbf{X}(s,t) = (a\sin s\cos t, a\sin s\sin t, a\cos s)$, then $\mathbf{T}_s(s,t) = (a\cos s\cos t, a\cos s\sin t, -a\sin s)$, $\mathbf{T}_t(s,t) = (-a\sin s\sin t, a\sin s\cos t, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = a^2\sin s(\sin s\cos t, \sin s\sin t, \cos s)$ and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = a^2\sin s$. Notice that the angle s made with the z-axis has lower limit $\cos^{-1}(h/a) = \cos^{-1}(\sqrt{a^2-b^2}/a)$ and upper limit $\pi/2$. So the surface area is

$$2\int_0^{2\pi}\int_{\cos^{-1}(\sqrt{a^2-b^2}/a)}^{\pi/2} a^2\sin s\,ds\,dt = 2\int_0^{2\pi} a^2\left(\frac{\sqrt{a^2-b^2}}{a}\right)dt = 4\pi a\sqrt{a^2-b^2}.$$

26. The parametrization of the paraboloid is $\mathbf{X}(s,t) = (s\cos t, s\sin t, 9-s^2)$ where $0\le t\le 2\pi$ and $0\le s\le 3$. So $\mathbf{T}_s(s,t) = (\cos t, \sin t, -2s)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (2s^2\cos t, 2s^2\sin t, s)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = s\sqrt{4s^2+1}$. The surface area is then

$$\int_0^{2\pi}\int_0^3 s\sqrt{4s^2+1}\,ds\,dt = \frac{1}{12}\int_0^{2\pi}\left[(1+4s^2)^{3/2}\right]\Big|_0^3\,dt = \frac{\pi}{6}(37^{3/2}-1).$$

27. We'll parametrize the surface by $\mathbf{X}(s,t) = (s\cos t, s\sin t, 2s^2)$ for $0\le t\le 2\pi$ and $1\le s\le 2$. So $\mathbf{T}_s(s,t) = (\cos t, \sin t, 4s)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (4s^2\cos t, 4s^2\sin t, s)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = s\sqrt{16s^2+1}$. So the surface area is

$$\int_0^{2\pi}\int_1^2 s\sqrt{16s^2+1}\,ds\,dt = \frac{1}{48}\int_0^{2\pi}(65^{3/2}-17^{3/2})\,dt = \frac{\pi}{24}(65^{3/2}-17^{3/2}).$$

28. **(a)** First we use the parametrization $\mathbf{X}(s,t) = (s, t, a-s-t)$ and calculate $\mathbf{T}_s(s,t) = (1,0,-1)$, $\mathbf{T}_t(s,t) = (0,1,-1)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (1,1,1)$ and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = \sqrt{3}$. The surface area is then the integral of $\sqrt{3}$ over the disk of radius a, which is $\iint_D \sqrt{3}\,ds\,dt = \sqrt{3}\pi a^2$.

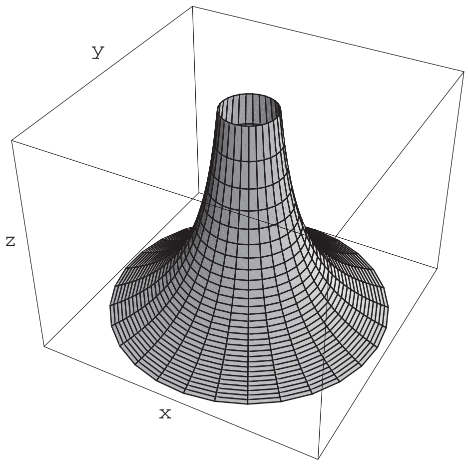
(b) To use formula (9), we view the surface as $z = f(x,y) = a-x-y$, so $f_x(x,y) = -1$ and $f_y(x,y) = -1$. Therefore, formula (9) gives the surface area as

$$\iint_D \sqrt{(-1)^2+(-1)^2+1}\,dx\,dy = \iint_D \sqrt{3}\,dx\,dy = \sqrt{3}\pi a^2.$$

29. We have $z = f(x,y)$ and $f_x^2+f_y^2 = a$ so, by formula (9), the surface area is

$$\iint_D \sqrt{f_x^2+f_y^2+1}\,dx\,dy = \iint_D \sqrt{a+1}\,dx\,dy = \sqrt{a+1}(\text{area of } D).$$

30. **(a)** Here is a sketch of the surface for $z \geq 1$.



(b) We can calculate the volume under the infinite funnel by disks:

$$\int_1^\infty \frac{\pi}{z^2}\, dz = \lim_{b\to\infty} -\frac{\pi}{z}\bigg|_1^b = \pi.$$

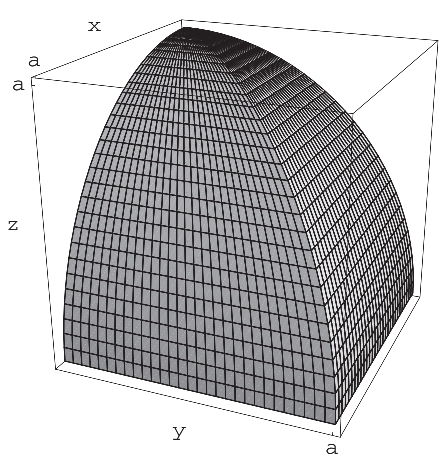
(c) To calculate the surface area, we'll parametrize the funnel as $\mathbf{X}(s,t) = \left(s\cos t, s\sin t, \frac{1}{s}\right)$, where $0 < s \leq 1$ and $0 \leq t < 2\pi$. Then $\mathbf{T}_s(s,t) = \left(\cos t, \sin t, -\frac{1}{s^2}\right)$, $\mathbf{T}_t(s,t) = (-s\sin t, s\cos t, 0)$, $\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = \left(\frac{1}{s}\cos t, \frac{1}{s}\sin t, s\right)$ and $\|\mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t)\| = \sqrt{\frac{1}{s^2} + s^2}$. Therefore, using tables or a computer algebra system, we see that the surface area is given by

$$\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{s^2}+s^2}\, ds\, dt = \lim_{a\to 0+}\int_0^{2\pi}\int_0^1 \sqrt{\frac{1}{s^2}+s^2}\, ds\, dt$$

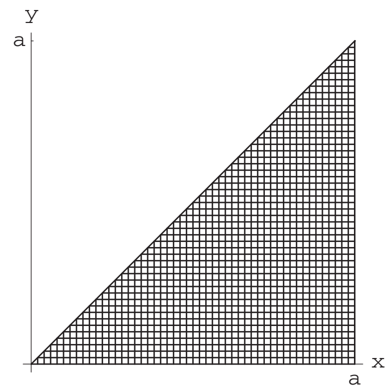
$$\pi \lim_{a\to 0+}[\sqrt{2} - \ln(\sqrt{2}+1) - (\sqrt{a^4+1} - \ln(1+\sqrt{1+a^4}) + \ln(a^2))].$$

Each term in this last expression possesses a finite limit except $\ln(a^2)$. Since $\lim_{a\to 0+}\ln(a^2) = -\infty$, we see that the surface area is infinite.

31. The first octant portion of the intersection is shown below.



Note that 1/16 of the total surface area is that of the graph of $z = \sqrt{a^2 - x^2}$ lying over the triangular region bounded by $y = x, x = a$, and $y = 0$.



For $z = \sqrt{a^2 - x^2}$, $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{a^2 - x^2}}$ and $\dfrac{\partial z}{\partial y} = 0$. Hence

$$\begin{aligned}\text{Surface area} &= 16\int_0^a \int_0^x \sqrt{\frac{x^2}{a^2 - x^2} + 0 + 1}\, dy\, dx = 16\int_0^a x\sqrt{\frac{x^2 + a^2 - x^2}{a^2 - x^2}}\, dx \\ &= 16\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\, dx.\end{aligned}$$

Let $u = a^2 - x^2$ so $du = -2x\, dx$. Then

$$\begin{aligned}\text{Surface area} &= -8a\int_{a^2}^0 \frac{du}{\sqrt{u}} = 8a\int_0^{a^2} u^{-1/2}\, du = 8a \cdot 2u^{1/2}\Big|_0^{a^2} \\ &= 16a^2.\end{aligned}$$

32. We have $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = f(r,\theta) \end{cases}$ $(r,\theta) \in D$. Therefore,

$$\mathbf{T}_r = \left(\cos\theta, \sin\theta, \frac{\partial f}{\partial r}\right), \quad \mathbf{T}_\theta = \left(-r\sin\theta, r\cos\theta, \frac{\partial f}{\partial\theta}\right).$$

So

$$\mathbf{N}(r,\theta) = \mathbf{T}_r \times \mathbf{T}_\theta = \left(\sin\theta\frac{\partial f}{\partial\theta} - r\cos\theta\frac{\partial f}{\partial r}, -r\sin\theta\frac{\partial f}{\partial r} - \cos\theta\frac{\partial f}{\partial\theta}, r\cos^2\theta + r\sin^2\theta\right).$$

Hence

$$\begin{aligned}\|\mathbf{N}\|^2 &= \sin^2\theta\left(\frac{\partial f}{\partial\theta}\right)^2 - 2r\sin\theta\cos\theta\frac{\partial f}{\partial r}\frac{\partial f}{\partial\theta} + r^2\cos^2\theta\left(\frac{\partial f}{\partial r}\right)^2 \\ &\quad + \cos^2\theta\left(\frac{\partial f}{\partial\theta}\right)^2 + 2r\sin\theta\cos\theta\frac{\partial f}{\partial r}\frac{\partial f}{\partial\theta} + r^2\sin^2\theta\left(\frac{\partial f}{\partial r}\right)^2 + r^2 \\ &= \left(\frac{\partial f}{\partial\theta}\right)^2 + r^2\left(\left(\frac{\partial f}{\partial r}\right)^2 + 1\right).\end{aligned}$$

Thus we have

$$\|\mathbf{N}\| = r\sqrt{\frac{1}{r^2}\left(\frac{\partial f}{\partial\theta}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 + 1} \quad \text{and}$$

Surface area $= \iint_D r\sqrt{\frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 + \left(\frac{\partial f}{\partial r}\right)^2 + 1}\, dr\, d\theta$, using formula (6) in §7.1.

33. We have $\begin{cases} x = f(\varphi,\theta)\sin\varphi\cos\theta \\ y = f(\varphi,\theta)\sin\varphi\sin\theta \\ z = f(\varphi,\theta)\cos\varphi \end{cases}$ from spherical/Cartesian conversions. From this,

$$\mathbf{T}_\varphi = (f_\varphi \sin\varphi\cos\theta + f\cos\varphi\cos\theta, f_\varphi\sin\varphi\sin\theta + f\cos\varphi\sin\theta, f_\varphi\cos\varphi - f\sin\varphi)$$
$$\mathbf{T}_\theta = (f_\theta\sin\varphi\cos\theta - f\sin\varphi\sin\theta, f_\theta\sin\varphi\sin\theta + f\sin\varphi\cos\theta, f_\theta\cos\varphi).$$

After some careful computation and using $\cos^2\alpha + \sin^2\alpha = 1$, we find $\mathbf{N}(\varphi,\theta) = \mathbf{T}_\varphi \times \mathbf{T}_\theta = (ff_\theta\sin\theta - ff_\varphi\sin\varphi\cos\varphi\cos\theta + f^2\sin^2\varphi\cos\theta, f^2\sin^2\varphi\sin\theta - ff_\varphi\sin\varphi\cos\varphi\sin\theta - ff_\theta\cos\theta, f^2\sin\varphi\cos\varphi + ff_\varphi\sin^2\varphi)$. After still more computation, one finds $\|\mathbf{N}\|^2 = (ff_\theta)^2 + (ff_\varphi)^2\sin^2\varphi + f^4\sin^2\varphi$, so that using formula (6) in §7.1,

$$\begin{aligned}\text{Surface area} &= \iint_D \sqrt{(ff_\theta)^2 + (ff_\varphi)^2\sin^2\varphi + f^4\sin^2\varphi}\, d\varphi\, d\theta \\ &= \iint_D f(\varphi,\theta)\sqrt{f_\theta^2 + \sin^2\varphi(f_\varphi + f^2)}\, d\varphi\, d\theta.\end{aligned}$$

7.2 Surface Integrals

Many of your students will apply the formulas and techniques introduced in this section by first finding a parametrization for the given surface. In many cases, as was shown in the text, if they examine the geometry of the surface, an easier solution might present itself. In several of the solutions below, each approach is outlined.

1. We will use Definition 2.1 to calculate the integral: $\iint_{\mathbf{X}} f\, dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{T}_s \times \mathbf{T}_t\|\, ds\, dt$. Here $\mathbf{X}(s,t) = (s, s+t, t)$, $\mathbf{T}_s(s,t) = (1,1,0)$, $\mathbf{T}_t(s,t) = (0,1,1)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (1,-1,1)$, and $\|\mathbf{N}(s,t)\| = \sqrt{3}$. Also, $f(\mathbf{X}(s,t)) = s^2 + (s+t)^2 + t^2 = 2(s^2 + st + t^2)$. So

$$\begin{aligned}\iint_{\mathbf{X}} (x^2+y^2+z^2)\, dS &= 2\sqrt{3}\int_0^2\int_0^1 (s^2+st+t^2)\, ds\, dt = 2\sqrt{3}\int_0^2 \left(\frac{1}{3} + \frac{t}{2} + t^2\right) dt \\ &= 2\sqrt{3}\left(\frac{2}{3} + 1 + \frac{8}{3}\right) = \frac{26}{\sqrt{3}}.\end{aligned}$$

2. (a) Since $\mathbf{X}(s,t) = (s+t, s-t, st)$, we can calculate $\mathbf{T}_s(s,t) = (1,1,t)$, $\mathbf{T}_t(s,t) = (1,-1,s)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = (s+t, t-s, -2)$, and $\|\mathbf{N}(s,t)\| = \sqrt{2s^2+2t^2+4}$. Using polar coordinates in the double integral, we obtain

$$\begin{aligned}\iint_{\mathbf{X}} 4\, dS &= \iint_D 4\sqrt{2s^2+2t^2+4}\, ds\, dt = \int_0^{\pi/2}\int_0^1 4r\sqrt{2r^2+4}\, dr\, d\theta \\ &= \frac{2}{3}\int_0^{\pi/2} [6\sqrt{6} - 8]\, d\theta = \frac{\pi}{3}[6\sqrt{6} - 8].\end{aligned}$$

(b) By Definition 2.2, $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)\, ds\, dt$. Here $\mathbf{F}(\mathbf{X}(s,t)) = (s+t, s-t, st)$ and so, from part (a), we know that $\mathbf{N}(s,t) = (s+t, t-s, -2)$. This means that $\mathbf{F}\cdot\mathbf{N} = (s+t)^2 - (s-t)^2 - 2st = 2st$. Therefore,

$$\begin{aligned}\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} &= \iint_D 2st\, ds\, dt = \int_0^1\int_0^{\sqrt{1-t^2}} 2st\, ds\, dt \\ &= \int_0^1 (s^2t)\Big|_0^{\sqrt{1-t^2}} dt = \int_0^1 (t - t^3)\, dt = \left(\frac{t^2}{2} - \frac{t^4}{4}\right)\Big|_0^1 = \frac{1}{4}.\end{aligned}$$

3. We need to calculate $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S}$. The surface is given by a level set of $f(x,y,z) = 2x - 2y + z$. Since $\nabla f = (2,-2,1)$, the upward-pointing unit normal is $\frac{1}{3}(2,-2,1)$. So, since $2x-2y+z=2$,

$$\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \frac{1}{3}\iint_S (x,y,z)\cdot(2,-2,1)\,dS = \frac{1}{3}\iint_S (2x-2y+z)\,dS$$
$$= \frac{1}{3}\iint_S (2)dS = \frac{2}{3}\iint_S dS = \frac{2}{3}\|\mathbf{N}\|(\text{area of } D) = 2(\text{area of } D).$$

Here D is the "shadow" of S in the *xy*-plane. D is a right triangle in the *xy*-plane with legs each of length 1. Hence

$$\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = 2(\text{area of } D) = (2)((1/2)(1)(1)) = 1.$$

4. (a) You can easily verify that both **X** and **Y** parametrize the surface $z = 3x^2+3y^2$ for $0 \le x^2+y^2 \le 4$. The major difference is that **X** covers the surface once while **Y** covers the surface twice.

(b) For **X**, the standard normal **N** is

$$(\cos t, \sin t, 6s)\times(-s\sin t, s\cos t, 0) = (-6s^2\cos t, -6s^2\sin t, s)$$

so

$$\iint_{\mathbf{X}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^2 (s\sin t, -s\cos t, 9s^4)\cdot(-6s^2\cos t, -6s^2\sin t, s)\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^2 9s^5\,ds\,dt = \int_0^{2\pi} \left.\frac{9s^6}{6}\right|_0^2 dt = \int_0^{2\pi} 96\,dt = 192\pi.$$

For **Y**, the standard normal **N** is

$$(2\cos t, 2\sin t, 24s)\times(-2s\sin t, 2s\cos t, 0) = (-48s^2\cos t, -48s^2\sin t, 4s)$$

so

$$\iint_{\mathbf{Y}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k})\cdot d\mathbf{S} = \int_0^{4\pi}\int_0^1 (2s\sin t, -2s\cos t, 144s^4)\cdot(-48s^2\cos t, -48s^2\sin t, 4s)\,ds\,dt$$
$$= \int_0^{4\pi}\int_0^1 576s^5\,ds\,dt = \int_0^{4\pi} \left.\frac{576s^6}{6}\right|_0^1 dt = \int_0^{4\pi} 96\,dt = 384\pi.$$

As noted in part (a), the integral over **Y** should be twice the integral over **X** since they both parametrize the same space but **Y** covers the space twice.

5. We will parametrize the six faces of the cube as follows (in each case $-2 \le s, t \le 2$):

i	$\mathbf{X}(s,t)$ for S_i	face
1	$(s,t,2)$	top
2	$(s,t,-2)$	bottom
3	$(s,2,t)$	right
4	$(s,-2,t)$	left
5	$(2,s,t)$	front
6	$(-2,s,t)$	back

Note that in each case $\|\mathbf{N}(s,t)\| = 1$, so $\iint_{S_i} [x(s,t)]^2\|\mathbf{N}(s,t)\|\,ds\,dt = \iint_{S_i} [x(s,t)]^2 ds\,dt$ for $1 \le i \le 6$. Also,

$\iint_{S_i} [x(s,t)]^2 \, ds\, dt = \int_{-2}^{2}\int_{-2}^{2} s^2 \, ds\, dt$ for $i = 1, 2, 3, 4$ and $\iint_{S_i} [x(s,t)]^2 \, ds\, dt = \int_{-2}^{2}\int_{-2}^{2} 4 \, ds\, dt$ for $i = 5, 6$. Then

$$\begin{aligned}\iint_S x^2 \, dS &= \sum_{i=1}^{6} \iint_{S_i} [x(s,t)]^2 \|\mathbf{N}(s,t)\| \, ds\, dt \\ &= 4\int_{-2}^{2}\int_{-2}^{2} s^2 \, ds\, dt + 2\int_{-2}^{2}\int_{-2}^{2} 4 \, ds\, dt \\ &= 4\int_{-2}^{2} \left.\frac{s^3}{3}\right|_{-2}^{2} dt + 8\int_{-2}^{2} s\big|_{-2}^{2} \, dt \\ &= 4\int_{-2}^{2} \frac{16}{3} \, dt + 8\int_{-2}^{2} 4 \, dt = \frac{256}{3} + 128 = \frac{640}{3}.\end{aligned}$$

6. We parametrize the lateral surface of the cylinder by $\mathbf{X}(s,t) = (a\cos s, a\sin s, t)$ where $0 \le s \le 2\pi$ and $0 \le t \le h$. So we have $\mathbf{T}_s(s,t) = (-a\sin s, a\cos s, 0)$, $\mathbf{T}_t(s,t) = (0,0,1)$, $\mathbf{N}(s,t) = \mathbf{T}_s(s,t) \times \mathbf{T}_t(s,t) = (a\cos s, a\sin s, 0)$, and $\|\mathbf{N}(s,t)\| = a$. So

$$\iint_S (x^2+y^2) \, dS = \int_0^{2\pi}\int_0^h (a^2\cos^2 s + a^2 \sin^2 s) a \, dt\, ds = \int_0^{2\pi}\int_0^h a^3 \, dt\, ds = 2\pi h a^3.$$

A quicker approach is to note that on the cylinder $x^2 + y^2 = a^2$, so

$$\iint_S (x^2+y^2) \, dS = \iint_S a^2 \, dS = a^2 \cdot \text{area of } S = a^2(2\pi a h) = 2\pi h a^3.$$

7. (a) Because $x^2 + y^2 + z^2 = a^2$ on the surface,

$$\iint_S (x^2+y^2+z^2) \, dS = a^2(\text{surface area of } S) = 4\pi a^4.$$

(b) Here we note that by part (a)

$$\iint_S x^2 \, dS + \iint_S y^2 \, dS + \iint_S z^2 \, dS = 4\pi a^4$$

and by the symmetries of the sphere

$$\iint_S x^2 \, dS = \iint_S y^2 \, dS = \iint_S z^2 \, dS. \quad \text{So} \quad \iint_S y^2 \, dS = 4\pi a^4/3.$$

8. (a) The sphere is symmetric about the plane $x = 0$. Hence $\iint_S x \, dS = 0$ as for each small piece of the sphere with coordinate $x > 0$ (and $x \le a$), there is a corresponding piece with coordinate $x < 0$. Hence contributions in an appropriate Riemann sum will cancel.

(b) For $x^2 + y^2 + z^2 = a^2$ the outward unit normal is given by $\mathbf{n} = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}$. Thus

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \frac{1}{a}(x+y+z) \, dS \\ &= \frac{1}{a}\left(\iint_S x \, dS + \iint_S y \, dS + \iint_S z \, dS\right) = 0\end{aligned}$$

since each surface integral is zero via reasoning as in part (a).

9. (a) We parametrize the cylinder as $\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = s \end{cases} \quad 0 \le t < 2\pi, -2 \le s \le 2.$

Then

$$\begin{aligned}\|\mathbf{T}_s \times \mathbf{T}_t\| &= \|(0,0,1) \times (-2\sin t, 2\cos t, 0)\| = \|(-2\cos t, -2\sin t, 0)\| \\ &= 2.\end{aligned}$$

Hence

$$\iint_S (z - x^2 - y^2)\, dS = \int_0^{2\pi} \int_{-2}^{2} (s-4)\cdot 2\, ds\, dt = \int_0^{2\pi} (s^2 - 8s)\Big|_{s=-2}^{2} dt$$
$$= \int_0^{2\pi} -32\, dt = -64\pi.$$

(b) $\iint_S (z - x^2 - y^2)\, dS = \iint_S z\, dS - \iint_S (x^2+y^2)\, dS$. S is symmetric about the $z = 0$ plane and $x^2 + y^2 = 4$ on S. Hence $\iint_S z\, dS = 0$ and $-\iint_S (x^2+y^2)\, dS = -\iint_S 4\, dS = -4\cdot$ (surface area of S) $= -4(4\pi\cdot 4) = -64\pi$.

The following calculations are useful for Exercises 10–18. Let's parametrize the surface of the cylinder in three pieces:

- S_1 = the lateral surface, $\mathbf{X}(s,t) = (3\cos s, 3\sin s, t)$ for $0 \le s \le 2\pi$ and $0 \le t \le 4$.
- S_2 = the bottom surface, $\mathbf{X}(s,t) = (t\cos s, t\sin s, 0)$ for $0 \le s \le 2\pi$ and $0 \le t \le 3$.
- S_3 = the top surface, $\mathbf{X}(s,t) = (t\cos s, t\sin s, 4)$ for $0 \le s \le 2\pi$ and $0 \le t \le 3$.

For S_1, $\mathbf{T}_s(s,t) = (-3\sin s, 3\cos s, 0)$, $\mathbf{T}_t(s,t) = (0,0,1)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (3\cos s, 3\sin s, 0)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = 3$. For both S_2 and S_3, $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0)$, $\mathbf{T}_t(s,t) = (\cos s, \sin s, 0)$, $\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t) = (0,0,-t)$, and $\|\mathbf{T}_s(s,t)\times\mathbf{T}_t(s,t)\| = t$. Because we are orienting with outward normals, $\mathbf{N}(s,t) = (0,0,-t)$ on S_2 and $\mathbf{N}(s,t) = (0,0,t)$ on S_3.

In Exercises 10–13 we use Definition 2.1: $\iint_{\mathbf{X}} f\, dS = \iint_D f(\mathbf{X}(s,t))\|\mathbf{N}(s,t)\|\, ds\, dt$. And we'll break down the integral as $\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$.

10. $\iint_S z\, dS = \int_0^{2\pi}\int_0^4 3t\, dt\, ds + \int_0^{2\pi}\int_0^3 0\, dt\, ds + \int_0^{2\pi}\int_0^3 4t\, dt\, ds = 48\pi + 36\pi = 84\pi$.

11. $\iint_S y\, dS = \int_0^4\int_0^{2\pi} 9\cos s\, ds\, dt + 2\int_0^3\int_0^{2\pi} t^2\cos s\, ds\, dt = 0 + 0 = 0$. Alternatively, you could notice that we are integrating an odd function of y over a region that is symmetric with respect to y.

12. $\iint_S xyz\, dS = \int_0^4\int_0^{2\pi} 27t\cos s\sin s\, ds\, dt + \int_0^3\int_0^{2\pi} 0\, ds\, dt + \int_0^3\int_0^{2\pi} 4t^3\cos s\sin s\, ds\, dt = 0$. Use the substitution $u = \sin s$. Again, alternatively, you could use a symmetry argument. We are again integrating an odd function of y over a region that is symmetric with respect to y.

13.

$$\iint_S x^2\, dS = \int_0^4\int_0^{2\pi} 27\cos^2 s\, ds\, dt + 2\int_0^3\int_0^{2\pi} t^3\cos^2 s\, ds\, dt$$
$$= 27\int_0^4 \left[\frac{s}{2} + \frac{1}{9}\sin 2s\right]\Big|_0^{2\pi} dt + 2\int_0^3 t^3\left[\frac{s}{2} + \frac{1}{4}\sin 2s\right]\Big|_0^{2\pi} dt = 27\int_0^4 \pi\, dt + 2\int_0^3 \pi t^3\, dt$$
$$= 108\pi + \frac{81\pi}{2} = \frac{297\pi}{2}.$$

For Exercises 14–18, we use Definition 2.2: $\iint_{\mathbf{X}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)\, ds\, dt$. *For another way of solving these exercises, recall from Section 2.6, that if S is a surface in $\mathbf{R}^3$ defined by an equation of the form $f(x,y,z) = c$, then if $\mathbf{x}_0 \in X$, the gradient vector $\nabla f(\mathbf{x}_0)$ is a vector normal to the plane tangent to S at $\mathbf{x}_0$. Therefore the unit normal to S_1 (a surface given by $x^2 + y^2 = 9$) is $\mathbf{n} = (x\mathbf{i} + y\mathbf{j})/3$, while the unit normal to S_2 is $-\mathbf{k}$ and the unit normal to S_3 is $\mathbf{k}$.*

14.

$$\iint_S (x\mathbf{i} + y\mathbf{j})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi} (3\cos s, 3\sin s, 0)\cdot(3\cos s, 3\sin s, 0)\, ds\, dt$$
$$+ \int_0^3\int_0^{2\pi} (t\cos s, t\sin s, 0)\cdot(0,0,-t)\, ds\, dt + \int_0^3\int_0^{2\pi} (t\cos s, t\sin s, 0)\cdot(0,0,t)\, ds\, dt$$
$$= \int_0^4\int_0^{2\pi} 9\, ds\, dt = 72\pi.$$

A different approach would be to observe that as the unit normals for S_2 and S_3 are $\pm\mathbf{k}$ then $\mathbf{F}\cdot\mathbf{n} = 0$ on S_2 and S_3. On S_1 the unit normal is $(x\mathbf{i} + y\mathbf{j})/3$ So $\mathbf{F}\cdot\mathbf{n} = (x^2+y^2)/3 = 9/3 = 3$. Therefore we obtain, $\iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = 3(\text{area of } S_1) = 3(2\pi(3)(4)) = 72\pi$.

15.

$$\iint_S (z\mathbf{k})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(0,0,t)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt + \int_0^3\int_0^{2\pi}(0,0,0)\cdot(0,0,-t)\,ds\,dt$$
$$+ \int_0^3\int_0^{2\pi}(0,0,4)\cdot(0,0,t)\,ds\,dt = \int_0^3\int_0^{2\pi} 4t\,ds\,dt = \int_0^3 8\pi t\,dt = 36\pi.$$

A different approach would have been to notice that, since the unit normal vector to the lateral surface S_1 has no $\mathbf{k}$ component, $\iint_{S_1} z\mathbf{k}\cdot d\mathbf{S} = 0$. Also, $z = 0$ on S_2 so $\iint_{S_2} z\mathbf{k}\cdot d\mathbf{S} = 0$. Finally, $z = 4$ on S_3 and therefore

$$\iint_S z\mathbf{k}\cdot d\mathbf{S} = \iint_{S_3} z\mathbf{k}\cdot d\mathbf{S} = \iint_{S_3} 4\mathbf{k}\cdot\mathbf{k}\,dS = \iint_{S_3} 4\,dS = 4\cdot(\text{area of } S_3) = 4(\pi 3^2) = 36\pi.$$

16.

$$\iint_S (y^3\mathbf{i})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(27\sin^3 s, 0, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+ \int_0^3\int_0^{2\pi}(t^3\sin^3 s, 0, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi}(t^3\sin^3 s,0,0)\cdot(0,0,t)\,ds\,dt$$
$$= 81\int_0^4\int_0^{2\pi}\sin^3 s\cos s\,ds\,dt = \frac{81}{4}\int_0^4 \sin^4 s\Big|_0^{2\pi}\,dt = 0.$$

Again, a careful student should have noticed that there is no $\mathbf{k}$ component and so the integrals over S_2 and S_3 are each 0.

17.

$$\iint_S (-y\mathbf{i} + x\mathbf{j})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(27\sin^3 s, 0, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+ \int_0^3\int_0^{2\pi}(t^3\sin^3 s, 0, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi}(t^3\sin^3 s,0,0)\cdot(0,0,t)\,ds\,dt$$
$$= 81\int_0^4\int_0^{2\pi}\sin^3 s\cos s\,ds\,dt = \frac{81}{4}\int_0^4 \sin^4 s\Big|_0^{2\pi}\,dt = 0.$$

Again, a careful student should have noticed that there is no $\mathbf{k}$ component and so the integrals over S_2 and S_3 are each 0. Therefore, a different approach would be to calculate

$$\iint_S (-y\mathbf{i} + x\mathbf{j})\cdot d\mathbf{S} = \iint_{S_1} (-y\mathbf{i} + x\mathbf{j})\cdot(x\mathbf{i} + y\mathbf{j})/3\,dS = \iint_{S_1} 0\,dS = 0.$$

18.

$$\iint_S (x^2\mathbf{i})\cdot d\mathbf{S} = \int_0^4\int_0^{2\pi}(9\cos^2 s, 0, 0)\cdot(3\cos s, 3\sin s, 0)\,ds\,dt$$
$$+ \int_0^3\int_0^{2\pi}(t^2\cos^2 s, 0, 0)\cdot(0,0,-t)\,ds\,dt + \int_0^3\int_0^{2\pi}(t^2\cos^2 s,0,0)\cdot(0,0,t)\,ds\,dt$$
$$= 27\int_0^4\int_0^{2\pi}\cos^3 s\,ds\,dt = 27\int_0^4\int_0^{2\pi}(1-\sin^2 s)\cos s\,ds\,dt = 27\int_0^4 [\sin s - (\sin^3 s)/3]\Big|_0^{2\pi}\,dt = 0.$$

Again, a careful student should have noticed that there is no $\mathbf{k}$ component and so the integrals over S_2 and S_3 are each 0.

We calculate the flux from $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{N}(s,t)\,ds\,dt$. *For Exercises 19–22 we have that*

$$\begin{aligned}
\mathbf{X}(\varphi,\theta) &= (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi) \quad \text{for } 0\le\theta<2\pi \text{ and } 0\le\varphi\le\pi,\\
\mathbf{T}_\varphi(\varphi,\theta) &= (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi),\\
\mathbf{T}_\theta(\varphi,\theta) &= (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0), \quad \text{and}\\
\mathbf{N}(\varphi,\theta) &= \mathbf{T}_\varphi(\varphi,\theta)\times\mathbf{T}_\theta(\varphi,\theta) = (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\\
&= a^2\sin\varphi(\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).
\end{aligned}$$

19.

$$\begin{aligned}
\iint_S (y\mathbf{j})\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^{\pi/2}(0, a\sin\varphi\sin\theta, 0)\cdot(a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\,d\varphi\,d\theta\\
&= \int_0^{2\pi}\int_0^{\pi/2}(a^3\sin^3\varphi\sin^2\theta)\,d\varphi\,d\theta = a^3\int_0^{2\pi}\int_0^{\pi/2}(1-\cos^2\varphi)\sin\varphi\sin^2\theta\,d\varphi\,d\theta\\
&= a^3\int_0^{2\pi}\left[-\cos\varphi+\frac{\cos^3\varphi}{3}\right]\Bigg|_0^{\pi/2}\sin^2\theta\,d\varphi\,d\theta = \frac{2a^3}{3}\int_0^{2\pi}\sin^2\theta\,d\theta\\
&= \frac{2a^3}{3}\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta = \frac{2a^3}{3}\left[\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]\Bigg|_0^{2\pi} = \frac{2\pi a^3}{3}.
\end{aligned}$$

20.

$$\begin{aligned}
&\iint_S (y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS\\
&= \int_0^{2\pi}\int_0^{\pi/2}(a\sin\varphi\sin\theta, -a\sin\varphi\cos\theta, 0)\cdot(a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\,d\varphi\,d\theta\\
&= a^3\int_0^{2\pi}\int_0^{\pi/2}(\sin^3\varphi\sin\theta\cos\theta - \sin^3\varphi\cos\theta\sin\theta)\,d\varphi\,d\theta = 0.
\end{aligned}$$

Actually it is simpler not to resort to the parametrization. Since $\mathbf{n} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/a$ for the sphere we see that $(y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n} = 0$ and so $\iint_S (y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS = 0$.

21.

$$\begin{aligned}
\iint_S(-y\mathbf{i}+x\mathbf{j}-\mathbf{k})\cdot\mathbf{n}\,dS &= -\iint_S \mathbf{k}\cdot\mathbf{n}\,dS - \iint_S(y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS\\
&= -\iint_S \mathbf{k}\cdot\mathbf{n}\,dS \quad \left(\text{since, by Exercise 20, } \iint_S(y\mathbf{i}-x\mathbf{j})\cdot\mathbf{n}\,dS = 0\right)\\
&= -\int_0^{2\pi}\int_0^{\pi/2}(0,0,1)\cdot(a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)\,d\varphi\,d\theta\\
&= -a^2\int_0^{2\pi}\int_0^{\pi/2}(\cos\varphi\sin\varphi)\,d\varphi\,d\theta = -a^2\int_0^{2\pi}\frac{\sin^2\varphi}{2}\Bigg|_0^{\pi/2}d\theta\\
&= -\frac{a^2}{2}\int_0^{2\pi}d\theta = -\pi a^2.
\end{aligned}$$

22.

$$\begin{aligned}
&\iint_S (x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}) \cdot \mathbf{n}\, dS \\
&= \int_0^{\pi/2}\int_0^{2\pi} [(a^2\sin^2\varphi\cos^2\theta, a^2\sin^2\varphi\cos\theta\sin\theta, a^2\cos\varphi\sin\varphi\cos\theta) \\
&\qquad \cdot (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi\sin\varphi)]\, d\theta\, d\varphi \\
&= a^4\int_0^{\pi/2}\int_0^{2\pi} (\sin^4\varphi\cos^3\theta + \sin^4\varphi\cos\theta\sin^2\theta + \cos^2\varphi\sin^2\varphi\cos\theta)\, d\theta\, d\varphi \\
&= a^4\int_0^{\pi/2}\int_0^{2\pi} (\sin^4\varphi\cos\theta + \sin^2\varphi\cos^2\varphi\cos\theta)\, d\theta\, d\varphi \\
&= a^4\int_0^{\pi/2}\int_0^{2\pi} (\sin^2\varphi\cos\theta)\, d\theta\, d\varphi = a^4\int_0^{\pi/2} [\sin^2\varphi\sin\theta]\Big|_0^{2\pi} d\theta = 0.
\end{aligned}$$

A different approach would be to see that

$$\begin{aligned}
\iint_S (x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}) \cdot \mathbf{n}\, dS &= \iint_S x(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}\, dS \\
&= \iint_S x\frac{a^2}{a}\, dS = a\iint_S x\, dS.
\end{aligned}$$

The integrand is an odd function of x which is being integrated over a region which is symmetric with respect to x; therefore $\iint_S (x^2\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}) \cdot \mathbf{n}\, dS = 0$.

23. We have $\mathbf{T}_s = (\cos t, \sin t, 0)$ and $\mathbf{T}_t = (-s\sin t, s\cos t, 1)$, so that the standard normal is

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix} = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + s\,\mathbf{k}.$$

Therefore, the flux of $\mathbf{F}$ is given by

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi}\int_0^2 \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt \\
&= \int_0^{2\pi}\int_0^2 \left(s\sin t, s\cos t, t^3\right) \cdot (\sin t, -\cos t, s)\ ds\, dt \\
&= \int_0^{2\pi}\int_0^2 \left(s(\sin^2 t - \cos^2 t) + st^3\right)\ ds\, dt \\
&= \int_0^{2\pi}\int_0^2 \left(st^3 - s\cos 2t\right)\ ds\, dt = \int_0^{2\pi} \left(\tfrac{1}{2}s^2t^3 - \tfrac{1}{2}s^2\cos 2t\right)\Big|_{s=0}^2\ dt \\
&= \int_0^{2\pi} \left(2t^3 - 2\cos 2t\right)\ dt = \left(\tfrac{1}{2}t^4 - \sin 2t\right)\Big|_0^{2\pi} = 8\pi^4.
\end{aligned}$$

24. We may parametrize the cone by $\mathbf{X}(s,t) = (s\cos t, s\sin t, s)$, where $-2 \le s \le 1$, $0 \le t \le 2\pi$. Then the standard normal

$$\mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{vmatrix} = -s\cos t\,\mathbf{i} - s\sin t\,\mathbf{j} + s\,\mathbf{k}$$

points the wrong way. (It points upward when $z = s > 0$ and downward when $z = s < 0$.) Thus we take $\mathbf{N}$ to be

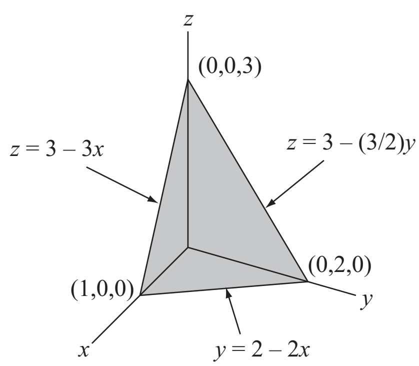
$s \cos t\, \mathbf{i} + s \sin t\, \mathbf{j} - s\, \mathbf{k}$. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_{-2}^{1} \left(2s \cos t, 2s \sin t, s^2\right) \cdot (s \cos t, s \sin t, -s)\; ds\, dt$$
$$= \int_0^{2\pi} \int_{-2}^{1} \left(2s^2 \sin^2 t + 2s^2 \cos^2 t - s^3\right)\; ds\, dt$$
$$= \int_0^{2\pi} \int_{-2}^{1} \left(2s^2 - s^3\right)\; ds\, dt = \frac{39\pi}{2}.$$

25. The surface $z = g(x, y) = ye^x$ has upward normal $\mathbf{N} = -g_x(x, y)\, \mathbf{i} - g_y(x, y)\, \mathbf{j} + \mathbf{k} = -ye^x\, \mathbf{i} - e^x\, \mathbf{j} + \mathbf{k}$. Therefore, the flux of $\mathbf{F} = y^3 z\, \mathbf{i} - xy\, \mathbf{j} + (x + y + z)\, \mathbf{k}$ is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 \left(y^4 e^x, -xy, x + y + ye^x\right) \cdot \left(-ye^x, -e^x, 1\right)\; dy\, dx$$
$$= \int_0^1 \int_0^1 \left(-y^5 e^{2x} + yxe^x + x + y + ye^x\right)\; dy\, dx$$
$$= \int_0^1 \int_0^1 \left(-\tfrac{1}{6} e^{2x} + \tfrac{1}{2} xe^x + x + \tfrac{1}{2} + \tfrac{1}{2} e^x\right)\; dx$$
$$= \left(-\tfrac{1}{12} e^{2x} + \tfrac{1}{2}(xe^x - e^x) + \tfrac{1}{2} x^2 + \tfrac{1}{2} x + \tfrac{1}{2} e^x\right)\Big|_0^1$$
$$= \tfrac{13}{12} - \tfrac{1}{12} e^2 + \tfrac{1}{2} e.$$

26. The tetrahedron has four triangular faces; we must consider surface integrals over each of them and then add the results.



The top slanted face is the first octant part of the plane through the points $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$. This plane has equation $6x + 3y + 2z = 6$, or $z = 3 - 3x - \frac{3}{2}y$ and upward normal $\mathbf{N} = (3, 3/2, 1)$. The "shadow" of this region in the xy-plane is the triangular region $\{(x, y, 0) \mid 0 \le y \le 2 - 2x,\ 0 \le x \le 1\}$; the shadow in the yz-plane is $\{(0, y, z) \mid 0 \le z \le 3 - \frac{3}{2}y,\ 0 \le y \le 2\}$; the shadow in the xz-plane is $\{(x, 0, z) \mid 0 \le z \le 3 - 3x,\ 0 \le x \le 1\}$. These three shadow regions determine the other three faces of the tetrahedron.

Now we calculate. For the top face S_1, we have $z = 3 - 3x - \frac{3}{2}y$, so that

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^{2-2x} \left(x^2, 12 - 12x - 6y, y - x\right) \cdot \left(3, \tfrac{3}{2}, 1\right)\; dy\, dx$$
$$= \int_0^1 \int_0^{2-2x} \left(3x^2 + 18 - 19x - 8y\right)\; dy\, dx$$
$$= \int_0^1 \left((3x^2 - 19x + 18)(2 - 2x) - 4(2 - 2x)^2\right)\; dx$$
$$= 2 \int_0^1 \left(-3x^3 + 22x^2 - 37x + 18 - 8(1 - x)^2\right)\; dx = \frac{41}{6}.$$

The bottom face S_2 is the portion of the plane $z = 0$ over the triangular region $\{(x, y, 0) \mid 0 \le y \le 2 - 2x,\ 0 \le x \le 1\}$. To have an overall outward normal, we must take the normal here to be $\mathbf{N} = -\mathbf{k}$. Therefore, with $z = 0$, we have

$$\begin{aligned}\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{2-2x} \left(x^2, 0, y - x\right) \cdot (0, 0, -1)\ dy\, dx \\ &= \int_0^1 \int_0^{2-2x} (x - y)\, dy\, dx \\ &= \int_0^1 \left(x(2 - 2x) - \tfrac{1}{2}(2 - 2x)^2\right)\ dx = \int_0^1 \left(2x - 2x^2 - 2(1 - x)^2\right)\ dx \\ &= \left(x^2 - \tfrac{2}{3}x^3 + \tfrac{2}{3}(1 - x)^3\right)\Big|_0^1 = -\frac{1}{3}.\end{aligned}$$

The left face S_3 is the portion of the plane $y = 0$ over the triangular region $\{(x, 0, z) \mid 0 \le z \le 3 - 3x,\ 0 \le x \le 1\}$. To have an overall outward normal, we must here take the normal to be $\mathbf{N} = -\mathbf{j}$. Therefore, with $y = 0$, we have

$$\begin{aligned}\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_0^{3-3x} \left(x^2, 4z, -x\right) \cdot (0, -1, 0)\ dz\, dx \\ &= \int_0^1 \int_0^{3-3x} -4z\, dz\, dx \\ &= \int_0^1 -2(3 - 3x)^2\, dx = -6.\end{aligned}$$

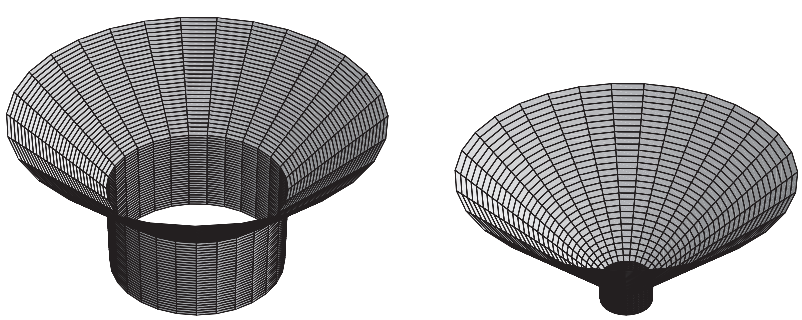
Finally, the right face S_4 is the portion of the plane $x = 0$ over the triangular region $\{(0, y, z) \mid 0 \le z \le 3 - \frac{3}{2}y,\ 0 \le y \le 2\}$. For an overall outward normal, we must take the normal to be $\mathbf{N} = -\mathbf{i}$. Therefore, with $x = 0$, we have

$$\begin{aligned}\iint_{S_4} \mathbf{F} \cdot d\mathbf{S} &= \int_0^2 \int_0^{3-(3/2)y} (0, 4z, -x) \cdot (-1, 0, 0)\ dz\, dy \\ &= \int_0^2 \int_0^{3-(3/2)y} 0\, dz\, dy = 0.\end{aligned}$$

Thus our final result is

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_3} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_4} \mathbf{F} \cdot d\mathbf{S} \\ &= \frac{41}{6} - \frac{1}{3} - 6 + 0 = \frac{1}{2}.\end{aligned}$$

27. (a) Below left is just the portion of S for $0 \le z \le 2$ so that you can more clearly see the funnel shape. Below right is a sketch of S.



(b) For the cylindrical portion of S, $\mathbf{X}(s, t) = (\cos s, \sin s, t)$ for $0 \le s < 2\pi$ and $0 \le t \le 1$. In that case $\mathbf{T}_s(s, t) = (-\sin s, \cos s, 0)$, $\mathbf{T}_t(s, t) = (0, 0, 1)$, $\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t) = (\cos s, \sin s, 0)$ and so the outward pointing unit normal for this portion is $\mathbf{n} = (\cos s, \sin s, 0) = x\mathbf{i} + y\mathbf{j}$.
For the conical portion of S, $\mathbf{X}(s, t) = (t \cos s, t \sin s, t)$ for $0 \le s < 2\pi$ and $1 \le t \le 9$. In that case $\mathbf{T}_s(s, t) = (-t \sin s, t \cos s, 0)$, $\mathbf{T}_t(s, t) = (\cos s, \sin s, 1)$, $\mathbf{T}_s(s, t) \times \mathbf{T}_t(s, t) = (t \cos s, t \sin s, -t)$ and so the outward pointing unit normal for this portion is $\mathbf{n} = (1/\sqrt{2})(\cos s, \sin s, -1) = (1/\sqrt{2})((x/z)\mathbf{i} + (y/z)\mathbf{j} - \mathbf{k})$.

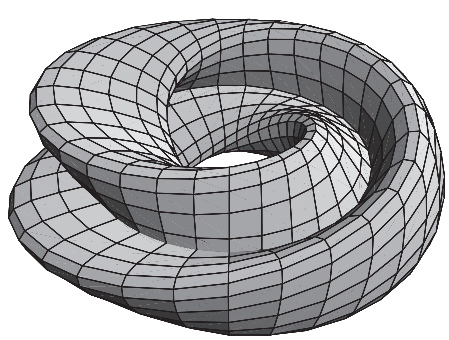
(c)

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S (-y\mathbf{i}+x\mathbf{j}+z\mathbf{k})\cdot d\mathbf{S} = \int_0^{2\pi}\int_1^9 (-t\sin s, t\cos s, t)\cdot(t\cos s, t\sin s, -t)\,dt\,ds$$
$$+\int_0^{2\pi}\int_0^1 (-\sin s, \cos s, t)\cdot(\cos s, \sin s, 0)\,dt\,ds = \int_0^{2\pi}\int_1^9 -t^2\,dt\,ds + \int_0^{2\pi}\int_0^1 0\,dt\,ds$$
$$= \int_0^{2\pi} -\frac{t^3}{3}\bigg|_1^9 ds = \int_0^{2\pi}\left[-\frac{729}{3}+\frac{1}{3}\right] ds = -\frac{1456\pi}{3}.$$

28. We know that the heat flux density $\mathbf{H} = -k\nabla T = -k(2x, 2y, 6z-12)$. On the ground $k = 3$ and $\mathbf{X}(s,t) = (t\cos s, t\sin s, 0)$ for $0 \le t \le 2$ and $0 \le s \le 2\pi$. Also, $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0)$, $\mathbf{T}_t(s,t) = (\cos s, \sin s, 0)$ and so $\mathbf{N}(s,t) = (0,0,-t)$. Along the glass we have $k = 1$ and $\mathbf{X}(s,t) = (t\cos s, t\sin s, 8-2t^2)$ for $0 \le t \le 2$ and $0 \le s \le 2\pi$. Also, $\mathbf{T}_s(s,t) = (-t\sin s, t\cos s, 0)$, $\mathbf{T}_t(s,t) = (\cos s, \sin s, -4t)$ and therefore $\mathbf{N}(s,t) = (-4t^2\cos s, -4t^2\sin s, -t)$. The outward normal must be $-\mathbf{N}(s,t) = (4t^2\cos s, 4t^2\sin s, t)$.

$$\iint_S \mathbf{H}\cdot d\mathbf{S} = \iint_{S_1} \mathbf{H}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{H}\cdot d\mathbf{S}$$
$$= \int_0^{2\pi}\int_0^2 -3(2t\cos s, 2t\sin s, -12)\cdot(0,0,-t)\,dt\,ds$$
$$- \int_0^{2\pi}\int_0^2 (2t\cos s, 2t\sin s, 36-12t^2)\cdot(4t^2\cos s, 4t^2\sin s, t)\,dt\,ds$$
$$= \int_0^{2\pi}\int_0^2 -36t\,dt\,ds + \int_0^{2\pi}\int_0^2 (-8t^3 - 36t + 12t^3)\,dt\,ds$$
$$= \int_0^{2\pi}\int_0^2 (4t^3 - 72t)\,dt\,ds = \int_0^{2\pi} [t^4 - 36t^2]\bigg|_0^2 ds$$
$$= \int_0^{2\pi} [16-144]\,ds = -256\pi.$$

29. (a) A sketch of the surface for $a = 2$ using *Mathematica* is:



(b) At $t = 0$ we have that $\sin t = \sin 2t = 0$ and so the s-coordinate curve is given by $(x,y,z) = (a\cos s, a\sin s, 0)$. This is a circle of radius a in the *xy*-plane.

(c) A computer algebra system would help the following calculation. It is not difficult; it is just very easy to drop a term here or there.

$$\mathbf{T}_s(s,t) = \left(\cos s\left[-\frac{1}{2}\sin\frac{s}{2}\sin t - \frac{1}{2}\cos\frac{s}{2}\sin 2t\right] - \sin s\left[a + \cos\frac{s}{2}\sin t - \sin\frac{s}{2}\sin 2t\right],\right.$$
$$\sin s\left[-\frac{1}{2}\sin\frac{s}{2}\sin t - \frac{1}{2}\cos\frac{s}{2}\sin 2t\right] + \cos s\left[a + \cos\frac{s}{2}\sin t - \sin\frac{s}{2}\sin 2t\right],$$
$$\left.\frac{1}{2}\cos\frac{s}{2}\sin t - \frac{1}{2}\sin\frac{s}{2}\sin 2t\right), \quad \text{so}$$
$$\mathbf{T}_s(s,0) = (-a\sin s, a\cos s, 0).$$
$$\mathbf{T}_t(s,t) = \left(\cos s\left[\cos\frac{s}{2}\cos t - 2\cos 2t\sin\frac{s}{2}\right], \sin s\left[\cos\frac{s}{2}\cos t - 2\cos 2t\sin\frac{s}{2}\right],\right.$$
$$\left.2\cos\frac{s}{2}\cos 2t + \cos t\sin\frac{s}{2}\right) \quad \text{so}$$
$$\mathbf{T}_t(s,0) = \left(\cos s\left[\cos\frac{s}{2} - 2\sin\frac{s}{2}\right], \sin s\left[\cos\frac{s}{2} - 2\sin\frac{s}{2}\right], 2\cos\frac{s}{2} + \sin\frac{s}{2}\right).$$

Calculate the cross product $\mathbf{T}_s(s,0) \times \mathbf{T}_t(s,0)$ to obtain

$$\mathbf{N}(s,0) = \left(a\cos s\left[2\cos\frac{s}{2} + \sin\frac{s}{2}\right], a\sin s\left[2\cos\frac{s}{2} + \sin\frac{s}{2}\right], 2a\sin\frac{s}{2} - a\cos\frac{s}{2}\right).$$

We note that

$$\mathbf{X}(0,0) = (a,0,0) = \mathbf{X}(2\pi,0)$$

but

$$\mathbf{N}(0,0) = (2a,0,-a) \quad \text{while} \quad \mathbf{N}(2\pi,0) = (-2a,0,a).$$

When you travel around the s-coordinate curve at $t = 0$ once, you find that the normal vector is now pointing in the opposite direction. The conclusion is that the Klein bottle cannot be orientable.

7.3 Stokes's and Gauss's Theorems

Exercises 1–4 are similar to Example 1 from the text. Recall from Section 2.6 that if S is a surface in $\mathbf{R}^3$ defined by an equation of the form $f(x,y,z) = c$, then if $\mathbf{x}_0 \in X$, the gradient vector $\nabla f(\mathbf{x}_0)$ is a vector normal to the plane tangent to S at $\mathbf{x}_0$.

1. Calculate

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & x^2+y^2 \end{vmatrix} = (2y - y)\mathbf{i} + (-2x + x)\mathbf{j} = y\mathbf{i} - x\mathbf{j}.$$

By symmetry we can see that the integral will be zero; however, let's follow the instructions. View the surface as a level set at height 1 of $f(x,y,z) = x^2 + y^2 + 5z$ Then $\mathbf{N} = \nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 5\mathbf{k}$. So,

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (y\mathbf{i} - x\mathbf{j}) \cdot (2x\mathbf{i} + 2y\mathbf{j} + 5\mathbf{k})\,dx\,dy$$
$$= \iint_D (2xy - 2xy)\,dx\,dy = 0.$$

On the other hand, ∂S consists of $C = \{(x,y,z) | x^2+y^2 = 1 \text{ and } z = 0\}$ which we parametrize by $\mathbf{x}(t) = (\cos t, \sin t, 0)$. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,dt = \int_0^{2\pi} (0,0,1) \cdot (-\sin t, \cos t, 0)\,dt = 0.$$

These two answers agree.

2. S is a helicoid. We begin by calculating

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

We calculated a normal vector in Exercise 24 of Section 7.1: $\mathbf{N} = (\sin t, -\cos t, s)$. So,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (\sin t\mathbf{i} - \cos t\mathbf{j} + s\mathbf{k})\, dt\, ds \\ &= \int_0^1 \int_0^{\pi/2} (\sin t - \cos t + s)\, dt\, ds = \int_0^1 \frac{\pi}{2} s\, ds \\ &= \frac{\pi}{4} s^2 \Big|_0^1 = \frac{\pi}{4}.\end{aligned}$$

On the other hand, ∂S consists of four pieces which we parametrize by $\mathbf{x}_1(s) = (s, 0, 0)$ for $0 \le s \le 1, \mathbf{x}_2(t) = (\cos t, \sin t, t)$ for $0 \le t \le \pi/2, \mathbf{x}_3(s) = (0, 1 - s, \pi/2)$ for $0 \le s \le 1$, and $\mathbf{x}_4(t) = (0, 0, \pi/2 - t)$ for $0 \le t \le \pi/2$. Then,

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 (0, s, 0) \cdot (1, 0, 0)\, ds + \int_0^{\pi/2} (t, \cos t, \sin t) \cdot (-\sin t, \cos t, 1)\, dt \\ &\quad + \int_0^1 (\pi/2, 0, 1 - s) \cdot (0, -1, 0)\, ds + \int_0^{\pi/2} (\pi/2 - t, 0, 0) \cdot (0, 0, -1)\, dt \\ &= \int_0^1 0\, ds + \int_0^{\pi/2} (-t \sin t + \cos^2 t + \sin t)\, dt + \int_0^1 0\, ds + \int_0^{\pi/2} 0\, dt \\ &= \frac{\pi}{4}.\end{aligned}$$

These two answers agree.

3. We see that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \mathbf{0} \quad \text{so} \quad \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = 0.$$

On the other hand, ∂S consists of $C = \{(x, y, z) | y^2 + z^2 = 16 \text{ and } x = 0\}$ which we parametrize by $\mathbf{x}(t) = (0, 4\cos t, 4\sin t)$ for $0 \le t \le 2\pi$. Then,

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_0^{2\pi} (0, 4\cos t, 4\sin t) \cdot (0, -4\sin t, 4\cos t)\, dt = 0.$$

These two answers agree.

4. For S,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y - z & x + y^2 - z & 4y - 3x \end{vmatrix} = (4 - (-1))\mathbf{i} + (3 - 1)\mathbf{j} + (1 - 2)\mathbf{k} = 5\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

If we parametrize S by $\mathbf{X}(s, t) = (2\cos s \sin t, 2\sin s \sin t, 2\cos t)$, a downward normal vector is given by $\mathbf{N} = (4\cos s \sin^2 t, 4\sin s \sin^2 t, 4\sin t \cos t)$. So,

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D (5\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (4\cos s \sin^2 t\mathbf{i} + 4\sin s \sin^2 t\mathbf{j} + 4\sin t \cos t\mathbf{k})\, ds\, dt \\ &= \int_{\pi/2}^{\pi} \int_0^{2\pi} (20\cos s \sin^2 t\mathbf{i} + 8\sin s \sin^2 t\mathbf{j} - 4\sin t \cos t\mathbf{k})\, ds\, dt \\ &= \int_{\pi/2}^{\pi} (4\pi \sin(2t))\, dt = 4\pi.\end{aligned}$$

On the other hand, ∂S consists of $C = \{(x, y, z) | y^2 + z^2 = 4 \text{ and } z = 0\}$ which we parametrize by $\mathbf{x}(t) = (2\cos t, -2\sin t, 0)$.

Then,

$$\begin{aligned}\oint_{\partial S} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)\,dt\\ &= \int_0^{2\pi} (-4\sin t, 2\cos t + 4\sin^2 t, -8\sin t - 6\cos t)\cdot(-2\sin t, -2\cos t, 0)\,dt\\ &= \int_0^{2\pi} (8\sin^2 t - 4\cos^2 t - 8\sin^2 t\cos t)\,dt\\ &= \int_0^{2\pi} (8 - 6(1+\cos 2t) - 8\sin^2 t\cos t)\,dt\\ &= \int_0^{2\pi} (2 - 6\cos 2t - 8\sin^2 t\cos t)\,dt = 4\pi.\end{aligned}$$

These two answers agree.

5. Stokes's Theorem implies that we don't need to be concerned that S is defined as the union of S_1 and S_2 if we choose the calculation along the boundary. Then ∂S is parametrized by $\mathbf{x}(t) = (3\cos t, 3\sin t, 0)$ where $0 \le t \le 2\pi$, and so

$$\begin{aligned}\oint_{\partial S} \mathbf{F}\cdot d\mathbf{s} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t))\cdot \mathbf{x}'(t)\,dt\\ &= \int_0^{2\pi} (27\cos^3 t\mathbf{i} + 3^7\sin^7 t\mathbf{j})\cdot(-3\sin t, 3\cos t, 0)\,dt\\ &= -3^4\int_0^{2\pi} \cos^3 t\sin t\,dt + 3^8\int_0^{2\pi}\sin^7 t\cos t\,dt = 0.\end{aligned}$$

6. Note that $\nabla\cdot\mathbf{F} = 3$ so

$$\begin{aligned}\iiint_D \nabla\cdot\mathbf{F}\,dV = 3\iiint_D dV &= 3\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (9 - x^2 - y^2)\,dy\,dx\\ &= 3\int_{-3}^{3} \frac{4}{3}(9-x^2)^{3/2}\,dx = \frac{243\pi}{2}.\end{aligned}$$

On the other hand, the boundary of D is in two pieces: S_1 = the disk at height $z = 0$ and S_2 = the portion of the paraboloid about the xy-plane. Parametrize S_1 by $\mathbf{X}_1(s,t) = (t\cos s, t\sin s, 0)$ for $0 \le s \le 2\pi$ and $0 \le t \le 3$. Then $\mathbf{N}_1(s,t) = (0,0,-t)$. Also parametrize S_2 by $\mathbf{X}_2(s,t) = (t\cos s, t\sin s, 9-t^2)$. Then $\mathbf{N}_2(s,t) = (2t^2\cos s, 2t^2\sin s, t)$. So

$$\begin{aligned}\oiint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}\\ &= \int_0^{2\pi}\int_0^3 (t\cos s, t\sin s, 0)\cdot(0,0,-t)\,dt\,ds\\ &\quad + \int_0^{2\pi}\int_0^3 (t\cos s, t\sin s, 9-t^2)\cdot(2t^2\cos s, 2t^2\sin s, t)\,dt\,ds\\ &= \int_0^{2\pi}\int_0^3 (9t + t^3)\,dt\,ds = \int_0^{2\pi}\left[\frac{9t^2}{2} + \frac{t^4}{4}\right]\Big|_0^3 ds\\ &= \int_0^{2\pi}\frac{243}{4}\,ds = \frac{243\pi}{2}.\end{aligned}$$

These two answers agree.

7. Here $\nabla\cdot\mathbf{F} = 0$ so $\iiint_D \nabla\cdot\mathbf{F}\,dV = 0$. As for the integral over the surface, because the normal vectors of each of the three opposite pairs of sides are equal and opposite, everything will cancel. So

$$\begin{aligned}
&\oiint_S \mathbf{F}\cdot d\mathbf{S}\\
&= \iint_{\text{top}} (y-x, y-1, x-y)\cdot(0,0,1)\,dS + \iint_{\text{bottom}} (y-x, y-(-1), x-y)\cdot(0,0,-1)\,dS\\
&\quad + \iint_{\text{front}} (y-1, y-z, 1-y)\cdot(1,0,0)\,dS + \iint_{\text{back}} (y-(-1), y-z, -1-y)\cdot(-1,0,0)\,dS\\
&\quad + \iint_{\text{right}} (1-x, 1-z, x-1)\cdot(0,1,0)\,dS + \iint_{\text{left}} (-1-x, -1-z, x-(-1))\cdot(0,-1,0)\,dS\\
&= \iint_{\text{top}} (x-y)\,dS + \iint_{\text{bottom}} (y-x)\,dS + \iint_{\text{front}} (-1)\,dS + \iint_{\text{back}} (-1)\,dS\\
&\quad + \iint_{\text{right}} (1)\,dS + \iint_{\text{left}} (1)\,dS = 0.
\end{aligned}$$

These two answers agree.

8. Note that $\nabla\cdot\mathbf{F} = 2x+2$ so

$$\begin{aligned}
\iiint_D \nabla\cdot\mathbf{F}\,dV &= 2\iiint_D (x+1)\,dV\\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2+1}^{5} (x+1)\,dz\,dy\,dx\\
&= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(x+1)(4-x^2-y^2)]\,dy\,dx\\
&= \frac{8}{3}\int_{-2}^{2} [(x+1)(4-x^2)^{3/2}]\,dx = \frac{8}{3}(6\pi) = 16\pi.
\end{aligned}$$

On the other hand, the boundary of D can be split into two pieces: the flat top piece S_1 and the surface of the paraboloid S_2. A parametrization of S_1 is $\mathbf{X}_1(s,t) = (t\cos s, t\sin s, 5)$ for $0\le s\le 2\pi$ and $0\le t\le 2$. Then a normal vector is $\mathbf{N}_1(s,t) = (0,0,t)$. A parametrization of S_2 is $\mathbf{X}_2(s,t) = (t\cos s, t\sin s, t^2+1)$ for $0\le s\le 2\pi$ and $0\le t\le 2$. Then a normal vector is $\mathbf{N}_2(s,t) = (2t^2\cos s, 2t^2\sin s, -t)$. So,

$$\begin{aligned}
\oiint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}\\
&= \int_0^{2\pi}\int_0^2 (t^2\cos^2 s, t\sin s, 5)\cdot(0,0,t)\,dt\,ds\\
&\quad + \int_0^{2\pi}\int_0^2 (t^2\cos^2 s, t\sin s, t^2+1)\cdot(2t^2\cos s, 2t^2\sin s, -t)\,dt\,ds\\
&= \int_0^{2\pi}\int_0^2 (2t^4\cos^3 s + 2t^3\sin^2 s - t^3 + 4t) = \int_0^{2\pi}\left[8 + \frac{64\cos^3 s}{5} - 4\cos 2s\right] ds = 16\pi.
\end{aligned}$$

These two answers agree.

9. Since $\dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}\right) = \dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ we see that

$$\nabla\cdot\mathbf{F} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}, \quad \text{and}$$

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \frac{2}{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{2}{\rho}(\rho^2\sin\varphi)\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_a^b (2\rho\sin\varphi)\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}[\rho^2\sin\varphi]\Big|_a^b\,d\varphi\,d\theta$$
$$= (b^2-a^2)\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta = (b^2-a^2)\int_0^{2\pi}2\,d\theta = 4\pi(b^2-a^2).$$

On the other hand the boundary consists of two pieces: S_1 is the sphere of radius a and S_2 is the sphere of radius b. Parametrize S_1 by $\mathbf{X}_1(s,t) = (a\sin s\cos t, a\sin s\sin t, a\cos s)$ for $0\le s\le\pi$ and $0\le t\le 2\pi$. Then a normal vector is $\mathbf{N}_1(s,t) = -a^2\sin s(\sin s\cos t,\sin s\sin t,\cos s)$. A similar calculation for S_2 yields $\mathbf{N}_2(s,t) = b^2\sin s(\sin s\cos t,\sin s\sin t,\cos s)$. Note that $\mathbf{N}_1$ is oriented pointing inward and $\mathbf{N}_2$ is oriented pointing outward. Then,

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}$$
$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{a}((a\sin s\cos t, a\sin s\sin t, a\cos s)\cdot[-a^2\sin s(\sin s\cos t,\sin s\sin t,\cos s)]\,ds\,dt$$
$$+ \int_0^{2\pi}\int_0^{\pi}\frac{1}{b}(b\sin s\cos t, b\sin s\sin t, b\cos s)\cdot[b^2\sin s(\sin s\cos t,\sin s\sin t,\cos s)]\,ds\,dt$$
$$= \int_0^{2\pi}\int_0^{\pi}[-a^2\sin s]\,ds\,dt + \int_0^{2\pi}\int_0^{\pi}[b^2\sin s]\,ds\,dt$$
$$= \int_0^{2\pi}[-2a^2]\,dt + \int_0^{2\pi}[2b^2]\,dt = 4\pi(b^2-a^2).$$

These two answers agree.

10. For Stokes's theorem we assume that S is a bounded, piecewise smooth, oriented surface in $\mathbf{R}^3$. To specialize to Green's theorem we must further assume that S is in the *xy*-plane. In each case we assume that the boundary $C = \partial S$ consists of finitely many simple, closed curves which are oriented so that S is on the left as you traverse C. In each case, $\mathbf{F}$ is a vector field of class C^1 whose domain includes S. In general, this would mean that $\mathbf{F}(x,y,z) = m(x,y,z)\mathbf{i} + n(x,y,z)\mathbf{j} + p(x,y,z)\mathbf{k}$ but because S is planar we assume that $\mathbf{F}$ is independent of z and that its $\mathbf{k}$-component is identically zero. In other words, we take $\mathbf{F}(x,y,z) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. Then

$$\iint_S\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}.$$

But by Stokes's theorem,

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \oint_{\partial S}\mathbf{F}\cdot d\mathbf{s}.$$

Then, by the formula for the differential form of the line integral given in Section 6.1,

$$\oint_{\partial S}\mathbf{F}\cdot d\mathbf{s} = \int_C M\,dx + N\,dy.$$

And so we get Green's theorem from Stokes's theorem.

11. Begin by calculating

$$\nabla\times\mathbf{F} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 2xyz+5z & e^x\cos(yz) & x^2y\end{vmatrix} = (x^2+e^xy\sin(yz))\mathbf{i} + 5\mathbf{j} + (e^x\cos(yz)-2xz)\mathbf{k}.$$

As in Example 2, we see that this looks difficult, but that Stokes's theorem implies that

$$\iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\nabla\times\mathbf{F}\cdot d\mathbf{S}$$

where S and S_1 have the same boundary. So let S_1 be the disk in the $y = 1$ plane bounded by the circle $x^2 + z^2 = 9$. The rightward pointing unit normal to S_1 is (0, 1, 0) and so

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S} \\ &= \iint_{S_1} ((x^2 + e^x y \sin(yz))\mathbf{i} + 5\mathbf{j} + (e^x \cos(yz) - 2xz)\mathbf{k}) \cdot (0, 1, 0)\, dS \\ &= \iint_{S_1} 5\, dS = 5(\text{area of } S_1) = 5(\pi 3^2) = 45\pi.\end{aligned}$$

12. The boundary of S is the ellipse $4x^2 + y^2 = 4$ in the $z = 0$ plane. By Stokes's theorem

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{S} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S}$$

where S' is any piecewise smooth, orientable surface with $\partial S' = \partial S$ (subject to appropriate orientation). One computes that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & e^{y^2} & ze^{xy} \end{vmatrix} = xze^{xy}\mathbf{i} - yze^{xy}\mathbf{j}.$$

This has no **k**-component. So let us take for S' the portion of the $z = 0$ plane inside the ellipse. Hence $\mathbf{n} = \mathbf{k}$ so that

$$\begin{aligned}\iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_{S'} (xze^{xy}\mathbf{i} - yze^{xy}\mathbf{j}) \cdot \mathbf{k}\, dS \\ &= \iint_{S'} 0\, dS = 0.\end{aligned}$$

13. (a) By the double angle formula we have $z = \sin 2t = 2 \sin t \cos t = 2xy$.

(b) $\oint_C (y^3 + \cos x)\, dx + (\sin y + z^2)\, dy + x\, dz = \oint_C \mathbf{F} \cdot d\mathbf{s}$ where $\mathbf{F} = (y^3 + \cos x)\mathbf{i} + (\sin y + z^2)\mathbf{j} + x\mathbf{k}$. By Stokes's theorem we may calculate the line integral by evaluating $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where S is the portion of $z = 2xy$ bounded by C. (Note that S lies over the unit disk in the xy-plane.) Now $\nabla \times \mathbf{F} = -4xy\mathbf{i} - \mathbf{j} - 3y^2\mathbf{k} = -2z\mathbf{i} - \mathbf{j} - 3y^2\mathbf{k}$ on S. Note that the orientation of C is compatible with an upward orientation of S. So we may take for normal

$$\mathbf{n} = \frac{-2y\mathbf{i} - 2x\mathbf{j} + \mathbf{k}}{\sqrt{4x^2 + 4y^2 + 1}} \qquad \text{(unit normal of } S\text{)}.$$

Hence $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_D (8xy^2 + 2x - 3y^2)\, dx\, dy$ (D = unit disk in xy-plane).
Now use polar coordinates, so that the integral becomes

$$\begin{aligned}&\int_0^{2\pi} \int_0^1 (8r^3 \sin^2\theta \cos\theta + 2r\cos\theta - 3r^2 \sin^2\theta) r\, dr\, d\theta \\ &= \int_0^{2\pi} \left(\frac{8}{5} \sin^2\theta \cos\theta + \frac{2}{3}\cos\theta - \frac{3}{4}\left(\frac{1}{2}(1 - \cos 2\theta)\right)\right) d\theta \\ &= \left(\frac{8}{15}\sin^3\theta + \frac{2}{3}\sin\theta - \frac{3}{8}\theta + \frac{3}{16}\sin 2\theta \right)\Bigg|_0^{2\pi} = -\frac{3\pi}{4}.\end{aligned}$$

14. First note that $\nabla \times \mathbf{F} = \left(xze^x \cos yz, 3x^2yz^2 - (1 + x)e^x \sin yz, 2xy - x^2z^3\right)$. Stokes's theorem implies

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

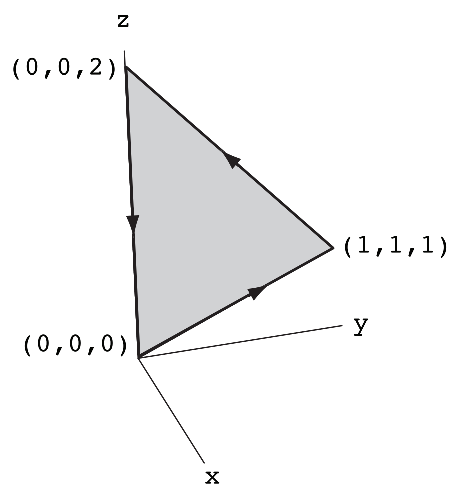
where S' is the top face ($z = a$) of the cube, oriented by downward normal $-\mathbf{k}$. This gives

$$\begin{aligned}\iint_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_{-a}^{a} \int_{-a}^{a} (2xy - a^3x^2)(-1)\, dx\, dy \\ &= \int_{-a}^{a} \left(\frac{a^3}{3}x^3 - yx^2 \right)\Bigg|_{x=-a}^{a} dy = \int_{-a}^{a} \frac{2a^6}{3}\, dy = \frac{4a^7}{3}.\end{aligned}$$

15. Note that the path lies in the plane $x = y$. Thus, by Stokes's theorem

$$\text{Work} = \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

where S is the triangular part of the plane $x = y$ enclosed by C. The configuration looks as follows:



Thus S is given by

$$\begin{cases} x = s \\ y = s \\ z = t \end{cases},$$

where $s \le t \le 2 - s$ and $0 \le s \le 1$. The appropriate normal vector to S is

$$\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j}.$$

Direct calculation reveals that $\nabla \times \mathbf{F} = (xy + x^2 z)\mathbf{i} + (xy - 2xyz)\mathbf{j} - (xz + yz)\mathbf{k}$, so that

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_0^1 \int_s^{2-s} (\nabla \times \mathbf{F})(s, s, t) \cdot \mathbf{N}\, dt\, ds \\ &= \int_0^1 \int_s^{2-s} (s^2 + s^2 t, s^2 - 2s^2 t, -2st) \cdot (1, -1, 0)\, dt\, ds \\ &= \int_0^1 \int_s^{2-s} 3s^2 t\, dt\, ds = \int_0^1 \frac{3}{2} s^2 t^2 \Big|_{t=s}^{2-s} ds \\ &= \int_0^1 \frac{3}{2} s^2 \left((2-s)^2 - s^2 \right) ds = \frac{3}{2} \int_0^1 \left(4s^2 - 4s^3 \right) ds \\ &= \frac{3}{2} \left(\frac{4}{3} - 1 \right) = \frac{1}{2}. \end{aligned}$$

16. Let $\mathbf{F} = (3\cos x + z)\mathbf{i} + (5x - e^y)\mathbf{j} - 3y\,\mathbf{k}$. Then, by Stokes's theorem

$$\oint_C (3\cos x + z)\, dx + (5x - e^y)\, dy - 3y\, dz = \oint_C \mathbf{F} \cdot d\mathbf{s} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S},$$

where S is the portion of the plane $2x - 3y + 5z = 17$ enclosed by C, oriented consistently with the orientation of C. A unit normal to S is given by $\mathbf{n} = (2, -3, 5)/\sqrt{38}$ and

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3\cos x + z & 5x - e^y & -3y \end{vmatrix} = (-3, 1, 5).$$

Therefore, we have

$$\begin{aligned}\oint_C (3\cos x{+}z)\,dx + (5x - e^y)\,dy - 3y\,dz \\ &= \pm\iint_S (-3,1,5)\cdot\frac{(2,-3,5)}{\sqrt{38}}\,dS = \pm\iint_S \frac{-6-3+25}{\sqrt{38}}\,dS \\ &= \pm\frac{16}{\sqrt{38}}(\text{area of } S) = \pm\frac{16}{\sqrt{38}}(\text{area inside } C),\end{aligned}$$

where the $\pm$ sign depends on the orientation of C.

17. The key to this problem is to recall that the volume of a solid region W may be calculated using a surface integral:

$$\text{Volume of } W = \frac{1}{3}\oiint_{\partial W}(x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k})\cdot d\mathbf{S}.$$

Now we calculate. The top of the solid is bounded by the paraboloid given by $z = 9 - x^2 - y^2$; if we write $\mathbf{X}_1(x,y) = (x, y, 9 - x^2 - y^2)$, then the standard (upward) normal is given by $\mathbf{N}_1 = (2x, 2y, 1)$. The bottom of the solid is bounded by the paraboloid given by $z = 3x^2 + 3y^2 - 16$; if we write $\mathbf{X}_2(x,y) = (x, y, 3x^2 + 3y^2 - 16)$, then the standard normal is given by $(-6x, -6y, 1)$. However, to put top and bottom surfaces S_1 and S_2 together to give ∂W a consistent, outward-pointing normal, we need to take $\mathbf{N}_2 = (6x, 6y, -1)$ for the correct orientation. Now the paraboloids intersect when $3x^2 + 3y^2 - 16 = 9 - x^2 - y^2$, or when $x^2 + y^2 = 25/4$; hence we have that $\partial W = S_1 \cup S_2$, where S_1 and S_2 are the respective portions of the top and bottom paraboloids with x- and y-coordinates in the disk $D = \{(x,y) \mid x^2 + y^2 \le 25/4\}$. Thus, with $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$, we have

$$\oiint_{\partial W}\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1}\mathbf{F}\cdot d\mathbf{S} + \iint_{S_2}\mathbf{F}\cdot d\mathbf{S}.$$

For the top boundary, we have

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}_1(x,y))\cdot\mathbf{N}_1(x,y)\,dx\,dy \\ &= \iint_D (x, y, 9 - x^2 - y^2)\cdot(2x, 2y, 1)\,dx\,dy \\ &= \iint_D \left(x^2 + y^2 + 9\right)\,dx\,dy.\end{aligned}$$

This last integral is most easily calculated using polar coordinates. Therefore,

$$\begin{aligned}\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \left(x^2 + y^2 + 9\right)\,dx\,dy \\ &= \int_0^{5/2}\int_0^{2\pi}\left(r^2 + 9\right)r\,d\theta\,dr = 2\pi\left(\frac{1}{4}r^4 + \frac{9}{2}r^2\right)\bigg|_0^{5/2} = \frac{2425\pi}{32}.\end{aligned}$$

We make similar calculations for the bottom boundary:

$$\begin{aligned}\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{X}_2(x,y))\cdot\mathbf{N}_2(x,y)\,dx\,dy \\ &= \iint_D (x, y, 3x^2 + 3y^2 - 16)\cdot(6x, 6y, -1)\,dx\,dy \\ &= \iint_D \left(3x^2 + 3y^2 + 16\right)\,dx\,dy = \int_0^{5/2}\int_0^{2\pi}\left(3r^2 + 16\right)r\,d\theta\,dr \\ &= 2\pi\left(\frac{3}{4}r^4 + 8r^2\right)\bigg|_0^{5/2} = \frac{5075\pi}{32}.\end{aligned}$$

Hence

$$\text{Volume of } W = \frac{1}{3}\left(\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{S_2} \mathbf{F}\cdot d\mathbf{S}\right)$$
$$= \frac{1}{3}\left(\frac{2425\pi}{32} + \frac{5075\pi}{32}\right) = \frac{625\pi}{8}.$$

18. S is the portion of the "bell" surface for which $z = e^{1-x^2-y^2}$ and $z \geq 1$. Take S_2 to be the disk in the plane $z = 1$ bounded by the circle $x^2 + y^2 = 1$. Then $S \bigcup S_2$ is the boundary of a solid V. S is oriented with an upward pointing normal and S_2 is oriented with a downward pointing normal.

$$\nabla\cdot\mathbf{F} = 0 \quad \text{so} \quad \iiint_V \nabla\cdot\mathbf{F}\, dV = 0.$$

Also,

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_2} (x, y, 2-2z)\cdot(0,0,-1)\, dS = \iint_{S_2} (2z-2)\, dS.$$

But along $S_2, z = 1$, so $\iint_{S_2} (2z-2)\, dS = \iint_{S_2} (2-2)\, dS = 0$. So

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \nabla\cdot\mathbf{F}\, dV - \iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = 0 - 0 = 0.$$

19. Let $\mathbf{X}\colon D \to \mathbf{R}^3$, $\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$ parametrize S and $(u(t), v(t))$, $a \leq t \leq b$ parametrize ∂D so that $\mathbf{X}(u(t), v(t))$ parametrizes ∂S. (Note the assumption that ∂D can be parametrized by a single path—this is not a problem.) Write $\mathbf{F}$ as $M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. We need to show that

$$(*) \quad \oint_{\partial S} (M\mathbf{i} + N\mathbf{j} + P\mathbf{k})\cdot d\mathbf{s} = \iint_S \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\cdot d\mathbf{S}.$$

Consider the line integral in $(*)$. We may write it in differential form as $\oint_{\partial S} M\, dx + N\, dy + P dz$. Consider, for the moment, just the piece $\oint_{\partial S} M\, dx$. By the chain rule, $\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$. Hence,

$$\oint_{\partial S} M\, dx = \int_a^b M(\mathbf{X}(u(t), v(t)))\left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}\right) dt = \int_{\partial D} M\circ\mathbf{X}\frac{\partial x}{\partial u}\, du + M\circ\mathbf{X}\frac{\partial x}{\partial v}\, dv.$$

The last line integral in just an integral in the *uv*-plane and so we may apply Green's theorem to find

$$\oint_{\partial S} M\, dx = \iint_D \left[\frac{\partial}{\partial u}\left(M\circ\mathbf{X}\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(M\circ\mathbf{X}\frac{\partial x}{\partial u}\right)\right] du\, dv.$$

We need to apply the chain rule again, along with the product rule:

$$\frac{\partial}{\partial u}\left(M\circ\mathbf{X}\frac{\partial x}{\partial v}\right) = \left(\frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial v} + M\circ\mathbf{X}\frac{\partial^2 x}{\partial u\,\partial v}$$
$$\frac{\partial}{\partial v}\left(M\circ\mathbf{X}\frac{\partial x}{\partial u}\right) = \left(\frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v}\right)\frac{\partial x}{\partial u} + M\circ\mathbf{X}\frac{\partial^2 x}{\partial v\,\partial u}.$$

Since the exercise allows us to assume that $\mathbf{X}$ is of class C^2, the mixed partials are equal: $\frac{\partial^2 x}{\partial u \partial v} = \frac{\partial^2 x}{\partial v \partial u}$. Therefore, our double integral becomes, after cancellation,

$$(**) \quad \iint_D \left[\frac{\partial M}{\partial y}\left(\frac{\partial x}{\partial v}\frac{\partial y}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\right) + \frac{\partial M}{\partial z}\left(\frac{\partial x}{\partial v}\frac{\partial z}{\partial u} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)\right] du\, dv.$$

Now consider the surface integral in $(*)$. Using the parametrization $\mathbf{X}$, and calculating the normal, we have that it is equal to

$$\iint_D (P_y - N_z, M_z - P_x, N_x - M_y)\cdot(y_u z_v - y_v z_u, z_u x_v - z_v x_u, x_u y_v - x_v y_u)\, du\, dv.$$

Next, calculate the dot product and isolate just those terms that contain M. Then the piece of the surface integral in $(*)$ that involves just M is

$$\iint_D \left[\frac{\partial M}{\partial z}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) - \frac{\partial M}{\partial y}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\right] du\,dv.$$

This is the same as the double integral in $(**)$.

In an entirely analogous way, we may show that $\oint_{\partial S} N\,dy$ and $\oint_{\partial S} P dz$ are equal to the remaining pieces of the surface integral in $(*)$, completing the proof.

20. We will calculate $\iiint_V \nabla\cdot\mathbf{F}\,dV$ for the closed cube and then subtract $\oiint_{S_2}\mathbf{F}\cdot d\mathbf{S}$ where S_2 is the bottom. Orient all of the faces of the cube with an outward pointing normal. In particular this means that the normal to S_2 is downward pointing.

$$\begin{aligned}\iiint_V \nabla\cdot\mathbf{F}\,dV &= \int_0^1\int_0^1\int_0^1 (2xze^{x^2} + 3 - 7yz^6)\,dx\,dy\,dz \\ &= \int_0^1\int_0^1 [ze^{x^2} + 3x - 7xyz^6]\Big|_{x=0}^{x=1} dy\,dz = \int_0^1\int_0^1 [ez - z + 3 - 7yz^6]\,dy\,dz \\ &= \int_0^1 \left[ezy - zy + 3y - \frac{7}{2}y^2z^6\right]\Big|_{y=0}^{y=1} dz = \int_0^1 \left[ez - z + 3 - \frac{7}{2}z^6\right] dz \\ &= \left[\frac{z^2}{2}e - \frac{z^2}{2} + 3z - \frac{1}{2}z^7\right]\Big|_0^1 = \frac{e}{2} + 2.\end{aligned}$$

Also, $\oiint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \int_0^1\int_0^1 (0,3y,2)\cdot(0,0,-1)\,dy\,dx = \int_0^1\int_0^1 (-2)\,dy\,dx = -2$. Therefore,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \left(\frac{e}{2}+2\right) - (-2) = \frac{e}{2} + 4.$$

21. **(a)** If $\mathbf{F} = f\mathbf{a}$, then

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{\partial}{\partial x}(fa_1) + \frac{\partial}{\partial y}(fa_2) + \frac{\partial}{\partial z}(fa_3) \\ &= a_1\frac{\partial f}{\partial x} + a_2\frac{\partial f}{\partial y} + a_3\frac{\partial f}{\partial z} \\ &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\cdot(a_1,a_2,a_3) = \nabla f\cdot\mathbf{a}.\end{aligned}$$

(b) With $\mathbf{F} = f\,\mathbf{i}$, we may apply Gauss's theorem:

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

The left side is

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S (\mathbf{F}\cdot\mathbf{n})\,dS = \oiint_S (f\,\mathbf{i})\cdot\mathbf{n})\,dS = \oiint_S (fn_1)\,dS.$$

Using part (a) with $\mathbf{a} = \mathbf{i}$, the right side is

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \nabla f\cdot\mathbf{i}\,dV = \iiint_D \frac{\partial f}{\partial x}\,dV.$$

Similarly, with $\mathbf{a} = \mathbf{j}$, we have

$$\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_S (f\,\mathbf{j})\cdot\mathbf{n})\,dS = \oiint_S (fn_2)\,dS$$

and

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \nabla f\cdot\mathbf{j}\,dV = \iiint_D \frac{\partial f}{\partial y}\,dV.$$

Finally, with $\mathbf{a} = \mathbf{k}$ we obtain

$$\oiint_S (fn_3)\,dS = \iiint_D \frac{\partial f}{\partial z}\,dV.$$

(c) Using part (b),

$$\oiint_S f\mathbf{n}\,dS = \left(\oiint_S fn_1\,dS, \oiint_S fn_2\,dS, \oiint_S fn_3\,dS\right)$$
$$= \left(\iiint_D \frac{\partial f}{\partial x}\,dV, \iiint_D \frac{\partial f}{\partial y}\,dV, \iiint_D \frac{\partial f}{\partial z}\,dV\right) = \iiint_D \nabla f\,dV.$$

22. Using the previous exercise, we have

$$\begin{aligned}\mathbf{B} &= -\oiint_{\partial D} p\mathbf{n}\,dS \\ &= -\iiint_D \nabla p\,dV = -\iiint_D \delta g\mathbf{k}\,dV = -\left(\iiint_D 1\,dV\right)(\delta g\mathbf{k}) \\ &= -(\text{volume of } D)(\delta g\mathbf{k}) = -(\text{mass of liquid displaced})(g\mathbf{k}) \\ &= -(\text{weight of liquid displaced})\mathbf{k}.\end{aligned}$$

Note that the negative sign is correct—the buoyant force should point *upwards* and it does, since the z-axis is oriented down.

23. The proof is outlined in the proof of Theorem 3.5 of Chapter 6. One direction has already been proved in Theorem 4.3 of Chapter 3. There it was established that if $\mathbf{F} = \nabla f$, then $\nabla \times \mathbf{F} = \mathbf{0}$. Now suppose that $\nabla \times \mathbf{F} = \mathbf{0}$. We show that then $\oint_C \mathbf{F}\cdot d\mathbf{s} = 0$ where C is any piecewise C^1, simple closed curve in $R \subseteq \mathbf{R}^3$. The idea is to "fill in C", that is, to find a surface $S \subseteq R$ whose boundary is C. Since R is simply-connected, this is possible. If we orient S consistently with C, then we may apply Stokes's theorem to conclude

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

This shows, among other things, that $\mathbf{F}$ has path-independent integrals over curves in R. Therefore, by Theorem 3.3 of Chapter 6, $\mathbf{F} = \nabla f$ for some function f on R.

24. (a) Note that the boundary of D is made up of two components:

- $S_5 =$ the sphere centered at the origin of radius 5 oriented with outward pointing normal and
- $S_7 =$ the sphere centered at the origin of radius 7 oriented with outward pointing normal.

Then by Gauss's theorem

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \oiint_{S_7} \mathbf{F}\cdot d\mathbf{S} - \oiint_{S_5} \mathbf{F}\cdot d\mathbf{S} = (7a+b) - (5a+b) = 2a.$$

(b) By Theorem 4.4 of Section 3.4, $\nabla\cdot(\nabla\times\mathbf{G}) = 0$. So if D is the solid sphere centered at the origin with radius r then, since $\mathbf{F} = \nabla\times\mathbf{G}$,

$$\begin{aligned}ar + b &= \oiint_{S_r} \mathbf{F}\cdot d\mathbf{S} \quad \text{(next apply Gauss's theorem)} \\ &= \iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D \nabla\cdot(\nabla\times\mathbf{G})\,dV = 0.\end{aligned}$$

Therefore $ar + b = 0$ for all values of r. We conclude that $a = b = 0$.

25. (a) If $f(x,y,z) = \ln(x^2+y^2+z^2)$ then $\nabla f(x,y,z) = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{x^2+y^2+z^2} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{a^2}$ on S. Also, the unit normal to the sphere that points away from the origin is $\mathbf{n}(x,y,z) = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{\sqrt{4(x^2+y^2+z^2)}} = \dfrac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{2a} =$

$\dfrac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}$. So,

$$\begin{aligned}\iint_S \frac{\partial f}{\partial n}\,dS &= \iint_S \nabla f\cdot\mathbf{n}\,dS\\ &= \iint_S \frac{2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}}{x^2+y^2+z^2}\cdot\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}\,dS\\ &= \iint_S \frac{2}{a}\,dS = \frac{2}{a}(\text{surface area of } S)\\ &= \frac{2}{a}\left(\frac{4\pi a^2}{8}\right) = \pi a.\end{aligned}$$

(b) First calculate that $\nabla\cdot(\nabla f) = \dfrac{2}{x^2+y^2+z^2}$. We'll use spherical coordinates to integrate.

$$\begin{aligned}\iiint_D \nabla\cdot(\nabla f)\,dV &= \iiint_D \frac{2}{x^2+y^2+z^2}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^a\left(\frac{2}{\rho^2}\right)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= 2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \sin\varphi\,d\rho\,d\varphi\,d\theta = 2a\int_0^{\pi/2}\int_0^{\pi/2}\sin\varphi\,d\varphi\,d\theta\\ &= 2a\int_0^{\pi/2} d\theta = \pi a.\end{aligned}$$

(c) By Gauss's theorem, $\iiint_D \nabla\cdot(\nabla f)\,dV = \oiint_{\partial D}(\nabla f)\cdot d\mathbf{S}$. The boundary of D consists of four pieces: S, the surface from part (a); S_x, the intersection of D and the plane $x=0$; S_y, the intersection of D and the plane $y=0$; and S_z, the intersection of D and the plane $z=0$. On S_x we know that $\nabla f(0,y,z) = \dfrac{2y\mathbf{j}+2z\mathbf{k}}{y^2+z^2}$ and $\mathbf{n}=(-1,0,0)$ so

$$\iint_{S_x}\nabla f\cdot d\mathbf{S} = \iint_{S_x}\nabla f\cdot\mathbf{n}\,dS = \iint_{S_x} 0\,dS = 0.$$

A similar analysis gives us $\iint_{S_y}\nabla f\cdot d\mathbf{S}=0$ and $\iint_{S_z}\nabla f\cdot d\mathbf{S}=0$. Therefore,

$$\iiint_D \nabla\cdot(\nabla f)\,dV = \oiint_{\partial D}(\nabla f)\cdot d\mathbf{S} = \iint_S \nabla f\cdot d\mathbf{S} = \iint_S \frac{\partial f}{\partial n}\,dS.$$

26. By Gauss's theorem, $\iiint_D \nabla\cdot(\nabla f)dV = \oiint_{\partial D}(\nabla f)\cdot d\mathbf{S}$. Here the boundary of D consists of finitely many piecewise smooth, closed orientable surfaces S_i. By assumption, $\oiint_{S_i}(\nabla f)\cdot d\mathbf{S}=0$ and so $\iiint_D \nabla\cdot(\nabla f)dV = 0$. This is true for any solid D, so $\nabla\cdot(\nabla f)=0$. As we saw earlier in the text $\nabla\cdot(\nabla f) = \dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2}$. So f is harmonic.

27. We will shrink the region D specified in the problem down to a point P. The volume decreases monotonically as we shrink the solid. Let D_V be the shrunken version of D which is the solid of volume V and let $S_V = \partial D_V$ for $0\le V\le$ the volume of D. Then, by Gauss's theorem,

$$\oiint_{S_V}\mathbf{F}\cdot d\mathbf{S} = \iiint_{D_V}\nabla\cdot\mathbf{F}\,dV.$$

By the mean value for triple integrals, there exists a $Q_V\in D_V$ so that

$$\iiint_{D_V}\nabla\cdot\mathbf{F}\,dV = \iiint_{D_V}\nabla\cdot\mathbf{F}(Q_V)\,dV = \nabla\cdot\mathbf{F}(Q_V)(\text{volume of } D).$$

So

$$\lim_{V\to 0}\frac{1}{V}\oiint_{S_V}\mathbf{F}\cdot d\mathbf{S} = \lim_{V\to 0}\nabla\cdot\mathbf{F}(Q_V) = \nabla\cdot\mathbf{F}(P) = \text{div}\,\mathbf{F}(P).$$

28. The six faces of the cube S are given by planes with equations

$$x = x_0 \pm \frac{a}{2} \text{ (front and back)}, \qquad y = y_0 \pm \frac{a}{2} \text{ (right and left)}, \qquad z = z_0 \pm \frac{a}{2} \text{ (top and bottom)}.$$

The respective outward unit normal vectors to these faces are $\pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$.

From Exercise 27, we have that the divergence of $\mathbf{F}$ at P may be computed as

$$\operatorname{div}\mathbf{F}(P) = \lim_{V\to 0}\frac{1}{V}\oint\!\!\!\oint_S \mathbf{F}\cdot d\mathbf{S} = \lim_{a\to 0^+}\frac{1}{a^3}\oint\!\!\!\oint_S \mathbf{F}\cdot d\mathbf{S}.$$

To calculate $\oint\!\!\oint_S \mathbf{F}\cdot d\mathbf{S}$, we add the contributions of the six surface integrals over each of the six square faces. Consider first just the integrals over the faces given by $x = x_0 + \frac{a}{2}$ and $x = x_0 - \frac{a}{2}$. These integrals contribute

$$\begin{aligned}\iint_{\text{front}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\text{back}} \mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{front}} (\mathbf{F}\cdot\mathbf{i})\, dS + \iint_{\text{back}} (\mathbf{F}\cdot(-\mathbf{i}))\, dS \\ &= \iint_{\text{front}} F_1\, dS + \iint_{\text{back}} -F_1\, dS.\end{aligned}$$

The faces are parametrized as

$$\text{front: } \mathbf{X}_1(y,z) = \left(x_0 + \tfrac{a}{2}, y, z\right) \qquad \text{back: } \mathbf{X}_2(y,z) = \left(x_0 - \tfrac{a}{2}, y, z\right),$$

where (y,z) varies over the square $D = \left[y_0 - \frac{a}{2}, y_0 + \frac{a}{2}\right] \times \left[z_0 - \frac{a}{2}, z_0 + \frac{a}{2}\right]$. Hence

$$\begin{aligned}\iint_{\text{front}} F_1\, dS + \iint_{\text{back}} -F_1\, dS &= \iint_{\mathbf{X}_1} F_1\, dS + \iint_{\mathbf{X}_2} -F_1\, dS \\ &= \iint_D F_1\left(x_0 + \tfrac{a}{2}, y, z\right)\, dy\, dz + \iint_D -F_1\left(x_0 - \tfrac{a}{2}, y, z\right)\, dy\, dz \\ &= \iint_D \left[F_1\left(x_0 + \tfrac{a}{2}, y, z\right) - F_1\left(x_0 - \tfrac{a}{2}, y, z\right)\right]\, dy\, dz.\end{aligned}$$

By the mean value theorem for double integrals, there is a point $(y_1, z_1) \in D$ such that

$$\begin{aligned}&\iint_D \left[F_1\left(x_0 + \tfrac{a}{2}, y, z\right) - F_1\left(x_0 - \tfrac{a}{2}, y, z\right)\right]\, dy\, dz \\ &\qquad = \left[F_1\left(x_0 + \tfrac{a}{2}, y_1, z_1\right) - F_1\left(x_0 - \tfrac{a}{2}, y_1, z_1\right)\right] (\text{area of } D) \\ &\qquad = a^2\left[F_1\left(x_0 + \tfrac{a}{2}, y_1, z_1\right) - F_1\left(x_0 - \tfrac{a}{2}, y_1, z_1\right)\right].\end{aligned}$$

In a similar manner, the two surface integrals over the faces given by $y = y_0 + \frac{a}{2}$ and $y = y_0 - \frac{a}{2}$ contribute

$$a^2\left[F_2\left(x_2, y_0 + \tfrac{a}{2}, z_2\right) - F_1\left(x_2, y_0 - \tfrac{a}{2}, z_2\right)\right]$$

to $\oint\!\!\oint_S \mathbf{F}\cdot d\mathbf{S}$, where (x_2, z_2) is a suitable point in the square $\left[x_0 - \frac{a}{2}, x_0 + \frac{a}{2}\right] \times \left[z_0 - \frac{a}{2}, z_0 + \frac{a}{2}\right]$. And, finally, the two surface integrals over the faces given by $z = z_0 + \frac{a}{2}$ and $z = z_0 - \frac{a}{2}$ contribute

$$a^2\left[F_3\left(x_3, y_3, z_0 + \tfrac{a}{2}\right) - F_1\left(x_3, y_3, z_0 - \tfrac{a}{2}\right)\right],$$

where (x_3, y_3) is a suitable point in the square $\left[x_0 - \frac{a}{2}, x_0 + \frac{a}{2}\right] \times \left[y_0 - \frac{a}{2}, y_0 + \frac{a}{2}\right]$.

Putting all of this together, we have

$$\begin{aligned}
\operatorname{div}\mathbf{F}(P) &= \lim_{a\to 0^+}\frac{1}{a^3}\oiint_S \mathbf{F}\cdot d\mathbf{S}\\
&= \lim_{a\to 0^+}\frac{1}{a^3}\left\{a^2\left[F_1\left(x_0+\tfrac{a}{2},y_1,z_1\right)-F_1\left(x_0-\tfrac{a}{2},y_1,z_1\right)\right]\right.\\
&\qquad + a^2\left[F_2\left(x_2,y_0+\tfrac{a}{2},z_2\right)-F_1\left(x_2,y_0-\tfrac{a}{2},z_2\right)\right]\\
&\qquad \left. + a^2\left[F_3\left(x_3,y_3,z_0+\tfrac{a}{2}\right)-F_1\left(x_3,y_3,z_0-\tfrac{a}{2}\right)\right]\right\}\\
&= \lim_{a\to 0^+}\frac{F_1\left(x_0+\frac{a}{2},y_1,z_1\right)-F_1\left(x_0-\frac{a}{2},y_1,z_1\right)}{a}\\
&\quad + \lim_{a\to 0^+}\frac{F_2\left(x_2,y_0+\frac{a}{2},z_2\right)-F_1\left(x_2,y_0-\frac{a}{2},z_2\right)}{a}\\
&\quad + \lim_{a\to 0^+}\frac{F_3\left(x_3,y_3,z_0+\frac{a}{2}\right)-F_1\left(x_3,y_3,z_0-\frac{a}{2}\right)}{a}.
\end{aligned}$$

Note that as $a \to 0^+$, each of the square faces shrinks down to the point $P(x_0, y_0, z_0)$. In particular, we have $(y_1, z_1) \to (y_0, z_0)$, $(x_2, z_2) \to (x_0, z_0)$, and $(x_3, y_3) \to (x_0, y_0)$. Thus, using the remark about partial derivatives, we see that the sum of the limits above is

$$\frac{\partial F_1}{\partial x}(x_0,y_0,z_0)+\frac{\partial F_2}{\partial y}(x_0,y_0,z_0)+\frac{\partial F_3}{\partial z}(x_0,y_0,z_0),$$

as desired.

29. (a) $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_z\mathbf{e}_z$. The area of the top face is

$$(\Delta\theta/2\pi)[\pi(r+\Delta r/2)^2-\pi(r-\Delta r/2)^2] = (\Delta\theta/2)(2r\Delta r) = r\Delta\theta\Delta r.$$

Therefore,

$$\begin{aligned}
\iint_{\text{top}}\mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{top}}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{top}}\mathbf{F}\cdot\mathbf{e}_z\,dS = \iint_{\text{top}}F_z\,dS\\
&\approx F_z(r,\theta,z+\Delta z/2)(\text{area of top}) = F_z(r,\theta,z+\Delta z/2)r\Delta\theta\Delta r.
\end{aligned}$$

The calculation for the bottom face is similar. The differences are that the normal vector points down and F_z is evaluated at a different point. The result is that

$$\iint_{\text{bottom}}\mathbf{F}\cdot d\mathbf{S} \approx -F_z(r,\theta,z-\Delta z/2)r\Delta\theta\Delta r.$$

The area of the outer face is

$$(\Delta z)(\Delta\theta/2\pi)[2\pi(r+\Delta r/2)] = \Delta\theta\Delta z(r+\Delta r/2).$$

Therefore,

$$\begin{aligned}
\iint_{\text{outer}}\mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{outer}}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{outer}}\mathbf{F}\cdot\mathbf{e}_r\,dS = \iint_{\text{outer}}F_r\,dS\\
&\approx F_r(r+\Delta r/2,\theta,z)(\text{area of outer}) = F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2)\Delta\theta\Delta z.
\end{aligned}$$

The calculation for the inner face is similar. The differences are that the normal vector points inward, F_r is evaluated at a different point, and the area of the face is slightly different. The result is that

$$\iint_{\text{inner}}\mathbf{F}\cdot d\mathbf{S} \approx -F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)\Delta\theta\Delta z.$$

The area of either the left or right face is just $\Delta r\Delta z$. Therefore, the integral along the left face (looking from the origin out at the solid) is

$$\begin{aligned}
\iint_{\text{left}}\mathbf{F}\cdot d\mathbf{S} &= \iint_{\text{left}}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{\text{left}}\mathbf{F}\cdot\mathbf{e}_\theta\,dS = \iint_{\text{left}}F_\theta\,dS\\
&\approx F_\theta(r,\theta+\Delta\theta/2,z)(\text{area of left}) = F_\theta(r,\theta+\Delta\theta/2,z)\Delta r\Delta z.
\end{aligned}$$

The calculation for the right face is similar. The differences are that the normal vector points the opposite direction and F_θ is evaluated at a different point. The result is that

$$\iint_{\text{right}} \mathbf{F}\cdot d\mathbf{S} \approx -F_\theta(r,\theta-\Delta\theta/2,z)\Delta r\Delta z.$$

We sum these to obtain

$$\begin{aligned}\oiint_S \mathbf{F}\cdot d\mathbf{S} \approx{}& F_z(r,\theta,z+\Delta z/2)r\Delta\theta\Delta r - F_z(r,\theta,z-\Delta z/2)r\Delta\theta\Delta r\\ &+ F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2)\Delta\theta\Delta z - F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)\Delta\theta\Delta z\\ &+ F_\theta(r,\theta+\Delta\theta/2,z)\Delta r\Delta z - F_\theta(r,\theta-\Delta\theta/2,z)\Delta r\Delta z.\end{aligned}$$

(b) To calculate the divergence using the results of Exercise 27 we will divide the answer to part (a) by $V \approx r\Delta\theta\Delta r\Delta z$ and take the limit as $V \to 0$. Two notes before the calculation: 1) We can replace $\approx$ with $=$ because in the limit our approximation assumptions are true and 2) in evaluating each of the limits we use the remark given in the text at the end of Exercise 28 (although you may want to break the argument of the middle limit down further to see what is going on).

$$\begin{aligned}\operatorname{div}\mathbf{F}(P) &= \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S}\\ &= \lim_{V\to 0}\left[\frac{F_z(r,\theta,z+\Delta z/2)r\Delta\theta\Delta r - F_z(r,\theta,z-\Delta z/2)r\Delta\theta\Delta r}{r\Delta\theta\Delta r\Delta z}\right]\\ &\quad+\lim_{V\to 0}\left[\frac{F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2)\Delta\theta\Delta z - F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)\Delta\theta\Delta z}{r\Delta\theta\Delta r\Delta z}\right]\\ &\quad+\lim_{V\to 0}\left[\frac{F_\theta(r,\theta+\Delta\theta/2,z)\Delta r\Delta z - F_\theta(r,\theta-\Delta\theta/2,z)\Delta r\Delta z}{r\Delta\theta\Delta r\Delta z}\right]\\ &= \lim_{\Delta z\to 0}\left[\frac{F_z(r,\theta,z+\Delta z/2) - F_z(r,\theta,z-\Delta z/2)}{\Delta z}\right]\\ &\quad+\lim_{\Delta r\to 0}\left[\frac{F_r(r+\Delta r/2,\theta,z)(r+\Delta r/2) - F_r(r-\Delta r/2,\theta,z)(r-\Delta r/2)}{r\Delta r}\right]\\ &\quad+\lim_{\Delta\theta\to 0}\left[\frac{F_\theta(r,\theta+\Delta\theta/2,z) - F_\theta(r,\theta-\Delta\theta/2,z)}{r\Delta\theta}\right]\\ &= \left[\frac{\partial F_z}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}\right]\Bigg|_P.\end{aligned}$$

30. Follow the steps from Exercise 29. This time $\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\theta\mathbf{e}_\theta + F_\varphi\mathbf{e}_\varphi$. Again, for each face, $\iint_S \mathbf{F}\cdot d\mathbf{S}$ is approximately the product of the component of **F** in the normal direction evaluated at the center point of the face and the area of that face. So summing up we have that

$$\begin{aligned}\oiint_S \mathbf{F}\cdot d\mathbf{S} \approx{}& F_\varphi(\rho,\theta,\varphi+\Delta\varphi/2)\rho\sin(\varphi+\Delta\varphi/2)\Delta\theta\Delta\rho - F_\varphi(\rho,\theta,\varphi-\Delta\varphi/2)\rho\sin(\varphi-\Delta\varphi/2)\Delta\theta\Delta\rho\\ &+ F_\rho(\rho+\Delta\rho/2,\theta,\varphi)(\rho+\Delta\rho/2)^2\sin\varphi\Delta\theta\Delta\varphi - F_\rho(\rho-\Delta\rho/2,\theta,\varphi)(\rho-\Delta\rho/2)^2\sin\varphi\Delta\theta\Delta\varphi\\ &+ F_\theta(\rho,\theta+\Delta\theta/2,\varphi)\rho\Delta\rho\Delta\varphi - F_\theta(\rho,\theta-\Delta\theta/2,\varphi)\rho\Delta\rho\Delta\varphi.\end{aligned}$$

Divide through by $V \approx \rho^2\sin\varphi\Delta\rho\Delta\theta\Delta\varphi$ and simplify to obtain

$$\begin{aligned}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S} \approx{}& \left[\frac{F_\varphi(\rho,\theta,\varphi+\Delta\varphi/2)\sin(\varphi+\Delta\varphi/2) - F_\varphi(\rho,\theta,\varphi-\Delta\varphi/2)\sin(\varphi-\Delta\varphi/2)}{\rho\sin\varphi\Delta\varphi}\right]\\ &+\left[\frac{F_\rho(\rho+\Delta\rho/2,\theta,\varphi)(\rho+\Delta\rho/2)^2 - F_\rho(\rho-\Delta\rho/2,\theta,\varphi)(\rho-\Delta\rho/2)^2}{\rho^2\Delta\rho}\right]\\ &+\left[\frac{F_\theta(\rho,\theta+\Delta\theta/2,\varphi) - F_\theta(\rho,\theta-\Delta\theta/2,\varphi)}{\rho\sin\varphi\Delta\theta}\right].\end{aligned}$$

Take the limit as $V \to 0$ to conclude

$$\begin{aligned}\text{div}\,\mathbf{F}(P) &= \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S}\\ &= \lim_{\Delta\varphi\to 0}\left[\frac{F_\varphi(\rho,\theta,\varphi+\Delta\varphi/2)\sin(\varphi+\Delta\varphi/2) - F_\varphi(\rho,\theta,\varphi-\Delta\varphi/2)\sin(\varphi-\Delta\varphi/2)}{\rho\sin\varphi\Delta\varphi}\right]\\ &\quad + \lim_{\Delta\rho\to 0}\left[\frac{F_\rho(\rho+\Delta\rho/2,\theta,\varphi)(\rho+\Delta\rho/2)^2 - F_\rho(\rho-\Delta\rho/2,\theta,\varphi)(\rho-\Delta\rho/2)^2}{\rho^2\Delta\rho}\right]\\ &\quad + \lim_{\Delta\theta\to 0}\left[\frac{F_\theta(\rho,\theta+\Delta\theta/2,\varphi) - F_\theta(\rho,\theta-\Delta\theta/2,\varphi)}{\rho\sin\varphi\Delta\theta}\right]\\ &= \left[\frac{1}{\rho\sin\varphi}\frac{\partial}{\partial\varphi}(\sin\varphi F_\varphi) + \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 F_\rho) + \frac{1}{\rho\sin\varphi}\frac{\partial F_\theta}{\partial\theta}\right]\bigg|_P.\end{aligned}$$

31. Let $\mathbf{F}$, P, $\mathbf{n}$, S and C be as described in the text. As in Exercise 27, we will assume that C shrinks down to the point P so that the area of the surface bounded decreases monotonically. We will then refer to S_A and C_A as the surface and bounding curve that corresponds to area A. Then by Stokes's theorem,

$$\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} = \iint_{S_A}\nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_{S_A}(\nabla\times\mathbf{F}\cdot\mathbf{n})\,dS.$$

By the mean value theorem for surface integrals, there is some point $Q_A \in S_A$ such that

$$\iint_{S_A}(\nabla\times\mathbf{F}\cdot\mathbf{n})\,dS = (\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})(\text{area of } S_A) = (\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})\,A.$$

Therefore,

$$\begin{aligned}\lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} &= \lim_{A\to 0}\frac{1}{A}[(\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})A] = \lim_{A\to 0}(\nabla\times\mathbf{F}(Q_A)\cdot\mathbf{n})\\ &= \mathbf{n}\cdot(\nabla\times\mathbf{F}(P)) = \mathbf{n}\cdot\text{curl}\,\mathbf{F}(P).\end{aligned}$$

32. **(a)** By Exercise 31, $\mathbf{e}_z\cdot\text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\dfrac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}$. Here $A \approx r\Delta r\Delta\theta$.

$$\begin{aligned}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} \approx{}& -F_r\left(r,\theta+\frac{\Delta\theta}{2},z\right)\Delta r - F_\theta\left(r-\frac{\Delta r}{2},\theta,z\right)\left(r-\frac{\Delta r}{2}\right)\Delta\theta\\ &+ F_r\left(r,\theta-\frac{\Delta\theta}{2},z\right)\Delta r + F_\theta\left(r+\frac{\Delta r}{2},\theta,z\right)\left(r+\frac{\Delta r}{2}\right)\Delta\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{e}_z\cdot\text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\ &= \lim_{\Delta\theta\to 0}\left[-\frac{F_r\left(r,\theta+\frac{\Delta\theta}{2},z\right) - F_r\left(r,\theta-\frac{\Delta\theta}{2},z\right)}{r\Delta\theta}\right]\\ &\quad + \lim_{\Delta r\to 0}\left[\frac{F_\theta\left(r+\frac{\Delta r}{2},\theta,z\right)\left(r+\frac{\Delta r}{2}\right) - F_\theta\left(r-\frac{\Delta r}{2},\theta,z\right)\left(r-\frac{\Delta r}{2}\right)}{r\Delta r}\right]\\ &= -\frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial r}(rF_\theta).\end{aligned}$$

(b) Again by Exercise 31, $\mathbf{e}_r\cdot\text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\dfrac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}$. Here $A \approx r\,\Delta z\,\Delta\theta$.

$$\begin{aligned}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s} \approx{}& F_z\left(r,\theta+\frac{\Delta\theta}{2},z\right)\Delta z - F_\theta\left(r,\theta,z+\frac{\Delta z}{2}\right)r\,\Delta\theta\\ &- F_z\left(r,\theta-\frac{\Delta\theta}{2},z\right)\Delta z + F_\theta\left(r,\theta,z-\frac{\Delta z}{2}\right)r\,\Delta\theta.\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{e}_r \cdot \operatorname{curl} \mathbf{F}(P) &= \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} \\
&= \lim_{\Delta\theta\to 0}\left[\frac{F_z\left(r,\theta+\frac{\Delta\theta}{2},z\right)-F_z\left(r,\theta-\frac{\Delta\theta}{2},z\right)}{r\Delta\theta}\right] \\
&\quad + \lim_{\Delta z\to 0}\left[-\frac{F_\theta\left(r,\theta,z+\frac{\Delta z}{2}\right)-F_\theta\left(r,\theta,z-\frac{\Delta z}{2}\right)}{\Delta z}\right] \\
&= \frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}.
\end{aligned}$$

(c) Again by Exercise 31, $\mathbf{e}_\theta \cdot \operatorname{curl} \mathbf{F}(P) = \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s}$. Here $A = \Delta r \Delta z$.

$$\begin{aligned}
\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} \approx\ & F_z\left(r-\frac{\Delta r}{2},\theta,z\right)\Delta z + F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)\Delta r \\
& - F_z\left(r+\frac{\Delta r}{2},\theta,z\right)\Delta z - F_r\left(r,\theta,z-\frac{\Delta z}{2}\right)\Delta r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{e}_\theta \cdot \operatorname{curl} \mathbf{F}(P) &= \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} \\
&= \lim_{\Delta r\to 0}\left[-\frac{F_z\left(r+\frac{\Delta r}{2},\theta,z\right)-F_z\left(r-\frac{\Delta r}{2},\theta,z\right)}{\Delta r}\right] \\
&\quad + \lim_{\Delta z\to 0}\left[\frac{F_r\left(r,\theta,z+\frac{\Delta z}{2}\right)-F_r\left(r,\theta,z-\frac{\Delta z}{2}\right)}{\Delta z}\right] \\
&= -\frac{\partial F_z}{\partial r} + \frac{\partial F_r}{\partial z}.
\end{aligned}$$

The final conclusion is just a matter of putting the three pieces together and checking that the sum agrees with the determinant given.

33. This is similar to Exercise 32. By Exercise 31, $\mathbf{e}_\rho \cdot \operatorname{curl} \mathbf{F}(P) = \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s}$. Here $A \approx \rho^2 \sin\varphi \Delta\varphi \Delta\theta$.

$$\begin{aligned}
\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} =\ & -F_\varphi\left(\rho,\theta+\frac{\Delta\theta}{2},\varphi\right)\rho\Delta\varphi - F_\theta\left(\rho,\theta,\varphi-\frac{\Delta\varphi}{2}\right)\rho\sin\left(\varphi-\frac{\Delta\varphi}{2}\right)\Delta\theta \\
& + F_\varphi\left(\rho,\theta-\frac{\Delta\theta}{2},\varphi\right)\rho\Delta\varphi + F_\theta\left(\rho,\theta,\varphi+\frac{\Delta\varphi}{2}\right)\rho\sin\left(\varphi+\frac{\Delta\varphi}{2}\right)\Delta\theta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{e}_\rho \cdot \operatorname{curl} \mathbf{F}(P) &= \lim_{A\to 0} \frac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} \\
&= \lim_{\Delta\theta\to 0}\left[-\frac{F_\varphi\left(\rho,\theta+\frac{\Delta\theta}{2},\varphi\right)-F_\varphi\left(\rho,\theta-\frac{\Delta\theta}{2},\varphi\right)}{\rho\sin\varphi\Delta\theta}\right] \\
&\quad + \lim_{\Delta\varphi\to 0}\left[\frac{F_\theta\left(\rho,\theta,\varphi+\frac{\Delta\varphi}{2}\right)\sin\left(\varphi+\frac{\Delta\varphi}{2}\right)-F_\theta\left(\rho,\theta,\varphi-\frac{\Delta\varphi}{2}\right)\sin\left(\varphi-\frac{\Delta\varphi}{2}\right)}{\rho\sin\varphi\Delta\varphi}\right] \\
&= \frac{1}{\rho\sin\varphi}\left[-\frac{\partial F_\varphi}{\partial\theta} + \frac{\partial}{\partial\varphi}(\sin\varphi F_\theta)\right].
\end{aligned}$$

Again, by Exercise 31, $\mathbf{e}_\theta \cdot \text{curl}\,\mathbf{F}(P) = \lim_{A\to 0} \dfrac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s}$. Here $A \approx \rho\Delta\varphi\Delta\rho$.

$$\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} = F_\varphi\left(\rho+\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho+\frac{\Delta\rho}{2}\right)\Delta\varphi - F_\rho\left(\rho,\theta,\varphi+\frac{\Delta\varphi}{2}\right)\Delta\rho$$
$$- F_\varphi\left(\rho-\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho-\frac{\Delta\rho}{2}\right)\Delta\varphi + F_\rho\left(\rho,\theta,\varphi-\frac{\Delta\varphi}{2}\right)\Delta\rho.$$

Therefore,

$$\begin{aligned}\mathbf{e}_\theta \cdot \text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\ &= \lim_{\Delta\rho\to 0}\left[\frac{F_\varphi\left(\rho+\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho+\frac{\Delta\rho}{2}\right) - F_\varphi\left(\rho-\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho-\frac{\Delta\rho}{2}\right)}{\rho\Delta\rho}\right]\\ &\quad+\lim_{\Delta\varphi\to 0}\left[-\frac{F_\rho\left(\rho,\theta,\varphi+\frac{\Delta\varphi}{2}\right) - F_\rho\left(\rho,\theta,\varphi-\frac{\Delta\varphi}{2}\right)}{\rho\Delta\varphi}\right]\\ &= \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho F_\varphi) - \frac{\partial F_\rho}{\partial\varphi}\right].\end{aligned}$$

Again, by Exercise 31, $\mathbf{e}_\varphi \cdot \text{curl}\,\mathbf{F}(P) = \lim_{A\to 0} \dfrac{1}{A}\oint_{C_A} \mathbf{F}\cdot d\mathbf{s}$. Here $A \approx \rho\sin\varphi\Delta\rho\Delta\theta$.

$$\oint_{C_A} \mathbf{F}\cdot d\mathbf{s} = F_\theta\left(\rho-\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho-\frac{\Delta\rho}{2}\right)\sin\varphi\Delta\theta + F_\rho\left(\rho,\theta+\frac{\Delta\theta}{2},\varphi\right)\Delta\rho$$
$$- F_\theta\left(\rho+\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho+\frac{\Delta\rho}{2}\right)\sin\varphi\Delta\theta - F_\rho\left(\rho,\theta-\frac{\Delta\theta}{2},\varphi\right)\Delta\rho.$$

Therefore,

$$\begin{aligned}\mathbf{e}_\varphi \cdot \text{curl}\,\mathbf{F}(P) &= \lim_{A\to 0}\frac{1}{A}\oint_{C_A}\mathbf{F}\cdot d\mathbf{s}\\ &= \lim_{\Delta\rho\to 0}\left[-\frac{-F_\theta\left(\rho+\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho+\frac{\Delta\rho}{2}\right) + F_\theta\left(\rho-\frac{\Delta\rho}{2},\theta,\varphi\right)\left(\rho-\frac{\Delta\rho}{2}\right)}{\rho\Delta\rho}\right]\\ &\quad+\lim_{\Delta\theta\to 0}\left[\frac{F_\rho\left(\rho,\theta+\frac{\Delta\theta}{2},\varphi\right) - F_\rho\left(\rho,\theta-\frac{\Delta\theta}{2},\varphi\right)}{\rho\sin\varphi\Delta\theta}\right]\\ &= \frac{1}{\rho}\left[-\frac{\partial}{\partial\rho}(\rho F_\theta) + \frac{1}{\sin\varphi}\frac{\partial F_\rho}{\partial\theta}\right].\end{aligned}$$

Again, the final conclusion is just a matter of assembling the pieces above and checking that the sum agrees with the determinant.

34. We use the results of Exercises 27 and 31:

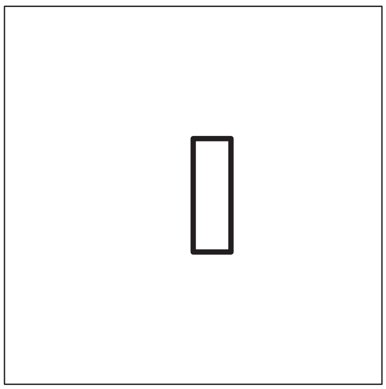
$$\text{div}\,\mathbf{F}(P) = \lim_{V\to 0}\frac{1}{V}\oiint_S \mathbf{F}\cdot d\mathbf{S}$$
$$\mathbf{n}\cdot\text{curl}\,\mathbf{F}(P) = \lim_{A\to 0}\frac{1}{A}\oint_C \mathbf{F}\cdot d\mathbf{s}.$$

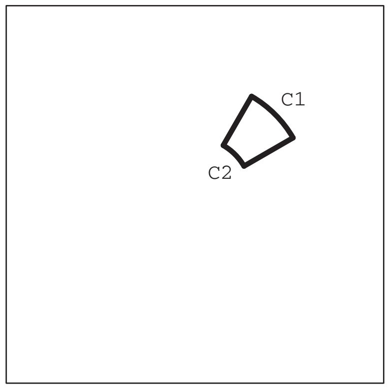
The vector fields to be considered are planar, so the divergence results should actually be interpreted as

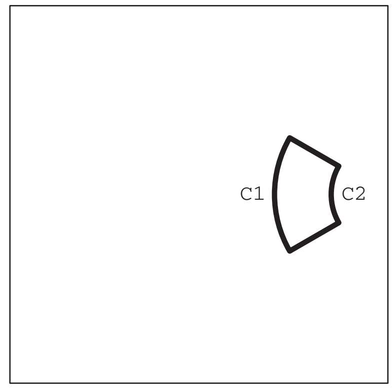
$$\text{div}\,\mathbf{F}(P) = \lim_{A\to 0}\frac{1}{A}\oint_C (\mathbf{F}\cdot\mathbf{n})\,ds.$$

(See the discussion regarding two-dimensional flux in Section 6.2.) Here **n** is the outward unit normal to C that lies in the plane. We need to find the four fields for which the divergence is identically zero. Intuitively, you can see in figures (b) and (e) by looking at symmetric neighborhoods of the center point that at the center point the divergence is not zero. We will be more precise than that. For the curl result, we need only take **n** to be the unit vector pointing up out of the plane of the vector field. Using these results, we may categorize the vector fields by drawing appropriate paths.

(a) Draw a rectangular path C with sides parallel to the x- and y-axes (see below left). Along such a path, $\oint_C \mathbf{F} \cdot d\mathbf{s} \neq 0$, since the path is tangent to the vector field along vertical segments and **F** has different magnitudes along these segments. The integrals along the horizontal segments will be equal and opposite. This will be true in the limit, so curl $\mathbf{F} \neq \mathbf{0}$. On the other hand, $\oint_C (\mathbf{F} \cdot \mathbf{n}) ds = 0$ because $\mathbf{F} \cdot \mathbf{n}$ vanishes on vertical parts of C and has opposite sign on the two horizontal segments. Therefore, div $\mathbf{F} = 0$.





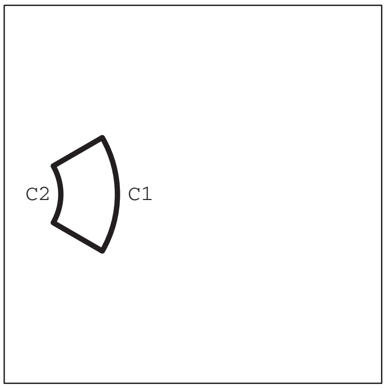


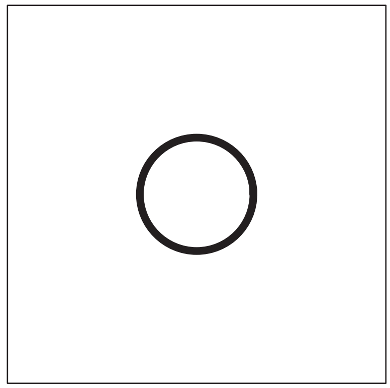
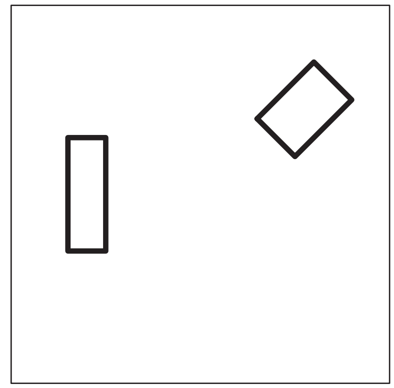
(b) Draw a path contained in the upper right quarter of the diagram that is a "polar rectangle" (see above center). In other words, we draw the path so that two of the sides are tangent to the vector field (one in the same direction, one in the opposite direction) and the remaining two sides are sides each of whose distance to the center of the figure is constant. Note that once the path is oriented, the segments labelled C_1 and C_2 will receive "opposite" orientations. Here $\left| \int_{C_1} (\mathbf{F} \cdot \mathbf{n})\, ds \right| > \left| \int_{C_2} (\mathbf{F} \cdot \mathbf{n})\, ds \right|$ and $\left| \int (\mathbf{F} \cdot \mathbf{n})\, ds \right| = 0$ along the radial segments. Therefore, div $\mathbf{F} \neq 0$. On the other hand, $\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0$, since **F** is perpendicular to C_1 and C_2. However, $\mathbf{F} \cdot \mathbf{T}$ has opposite signs on the radial pieces so $\oint_C \mathbf{F} \cdot d\mathbf{s} = \oint_C (\mathbf{F} \cdot \mathbf{T}) ds = 0$. Hence curl $\mathbf{F} = 0$.

(c) Again our path will be a polar rectangle (see above right). This time orient the path clockwise and picture the center of the coordinate system to be at the center of the right border of the figure. Denote the left-most, "vertical" side C_1 and the right-most, "vertical" side C_2. Orient the path either way. C_1 and C_2 will receive "opposite" orientations. The idea here is that $\int_{C_1} \mathbf{F} \cdot d\mathbf{s}$ is cancelled by $\int_{C_2} \mathbf{F} \cdot d\mathbf{s}$ because the integral of the smaller magnitude of **F** along the longer segment C_1 is balanced by the integral of the larger magnitude of **F** along the shorter segment C_2. Integrals along the other segments are 0 because **F** is perpendicular to those segments. Hence, curl $\mathbf{F} = \mathbf{0}$. The path is also arranged so $\oint_C (\mathbf{F} \cdot \mathbf{n})\, ds = 0$. It is zero along C_1 and C_2 and cancels on the other segments. Hence, div $\mathbf{F} = 0$.

(d) Again choose a polar rectangle for our path (see below left). This time picture the center of the coordinate system to be at the center of the left border of the figure. What makes this different from the vector field in (c) is that here $\|\mathbf{F}\|$ is constant. For this reason, $\oint_C \mathbf{F} \cdot d\mathbf{s} \neq 0$ and, therefore, curl $\mathbf{F} \neq \mathbf{0}$. On the other hand, div $\mathbf{F} = 0$ for the same reasons as in part (c).

(e) Let our path be an oriented circle centered at the center of the figure (see below center). It is clear that $\oint_C (\mathbf{F} \cdot \mathbf{n}) ds \neq 0$ and therefore div $\mathbf{F} \neq 0$. Likewise, $\oint_C \mathbf{F} \cdot d\mathbf{s} \neq 0$, so curl $\mathbf{F} \neq \mathbf{0}$.



(f) Well, by elimination we must have div $\mathbf{F} = 0$ and curl $\mathbf{F} = \mathbf{0}$. For the divergence argument, choose a rectangular path in the upper right quarter of the diagram with two sides parallel to and symmetric about the diagonal from the lower left corner to the upper right corner of the diagonal. For the curl argument, use a rectangular path with sides parallel to the coordinate axes (see above right).

7.4 Further Vector Analysis; Maxwell's Equations

1. Notice the similarities between this exercise and Exercise 28 in the Miscellaneous Exercises for Chapter 6. By Gauss's theorem (Theorem 3.3),

$$\oiint_{\partial D} f\nabla g \cdot d\mathbf{S} = \iiint_D \nabla \cdot (f\nabla g)\, dV.$$

By the product rule,

$$\iiint_D \nabla \cdot (f\nabla g)\, dV = \iiint_D (\nabla f \cdot \nabla g + f\nabla^2 g)\, dV = \iiint_D (\nabla f \cdot \nabla g)\, dV + \iiint_D (f\nabla^2 g)\, dV.$$

2. Let $f \equiv 1$ in Green's first formula. Then $\nabla f = \mathbf{0}$ so the first term in Green's first formula is 0, so

$$\iiint_D \nabla^2 g\, dV = \oiint_S \nabla g \cdot d\mathbf{S}.$$

We assumed that g is harmonic so $\nabla^2 g = 0$. Also we know that $S = \partial D$. Therefore, by the definition of the normal derivative,

$$0 = \oiint_{\partial D} \nabla g \cdot d\mathbf{S} = \oiint_{\partial D} (\nabla g \cdot \mathbf{n})\, dS = \oiint_{\partial D} \frac{\partial g}{\partial n}\, dS.$$

3. **(a)** Using Green's first formula with $f = g$, we obtain

$$\iiint_D \nabla f \cdot \nabla f\, dV + \iiint_D f\nabla^2 f\, dV = \oiint_S f\nabla f \cdot d\mathbf{S}.$$

We are assuming that f is harmonic, so the second integral on the left side is 0. Therefore,

$$\iiint_D \nabla f \cdot \nabla f\, dV = \oiint_{\partial D} f\nabla f \cdot d\mathbf{S} = \oiint_{\partial D} f(\nabla f \cdot \mathbf{n})\, dS = \oiint_{\partial D} f\frac{\partial f}{\partial n}\, dS.$$

(b) If $f = 0$ on the boundary of D, then part (a) implies that

$$0 = \oiint_{\partial D} f\frac{\partial f}{\partial n}\, dS = \iiint_D \nabla f \cdot \nabla f\, dV.$$

But $\nabla f \cdot \nabla f = \|\nabla f\|^2 \geq 0$. So the right-hand integral was of a non-negative, continuous integrand. For this to be zero, the integrand must have been identically zero. In other words, $\nabla f \cdot \nabla f$ is zero on D. We conclude that ∇f is zero on D and so f is constant on D. Since $f(x, y, z) = 0$ on ∂D and f is constant on D, we must have that $f \equiv 0$ on D.

4. Use the hint and consider $f = f_1 - f_2$. Then, since $f_1 = f_2$ on ∂D, we have that $f = 0$ on ∂D. Note that if f_1 and f_2 are harmonic on D, then f is harmonic on D. Therefore, by Exercise 3(b), $f \equiv 0$ on all of D so $f_1 = f_2$ on D.

5. (a) Using the hint we see that the rate of fluid flowing into W is $\iiint_W \frac{\partial\rho}{\partial t}dV$ and the rate of fluid flowing out of W is $\oiint_S \rho\mathbf{F}\cdot d\mathbf{S}$. Hence we have $\oiint_S \rho\mathbf{F}\cdot d\mathbf{S} = -\iiint_W \frac{\partial\rho}{\partial t}dV$. Also, by Gauss's theorem, we have $\iiint_W \nabla\cdot(\rho\mathbf{F})dV = \oiint_S \rho\mathbf{F}\cdot d\mathbf{S}$; therefore $\iiint_W \nabla\cdot(\rho\mathbf{F})dV = -\iiint_W \frac{\partial\rho}{\partial t}dV$. Finally, as in the arguments in the text, we point out that the equation

$$\iiint_R \nabla\cdot(\rho\mathbf{F})\,dV = -\iiint_R \frac{\partial\rho}{\partial t}\,dV$$

holds for *any* solid region $R \subseteq W$ by the same argument. Thus, by shrinking R to a point, we can conclude $\nabla\cdot(\rho\mathbf{F}) = -\frac{\partial\rho}{\partial t}$.

(b) From (14) in the text, the current density field $\mathbf{J}$ is $\rho\mathbf{v}$. Therefore, $\iiint_W \frac{\partial\rho}{\partial t}dV$ represents the current flowing into W and $\oiint_S \mathbf{J}\cdot d\mathbf{S}$ represents the current flowing out of W (across S). Hence, the same argument as that given in part (a) shows that $\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$.

6. We are given that the total heat leaving D per unit time is $-\iiint_D \sigma\rho\frac{\partial T}{\partial t}dV$. This is equal to the flux $\oiint_S \mathbf{H}\cdot d\mathbf{S}$ which, by the definition of $\mathbf{H}$, is the same as $\oiint_S -k\nabla T\cdot d\mathbf{S}$. By Gauss's theorem, we have $\oiint_S -k\nabla T\cdot d\mathbf{S} = \iiint_D -k\nabla\cdot\nabla T\,dV$. Therefore, $-\iiint_D \sigma\rho\frac{\partial T}{\partial t}dV = \iiint_D -k\nabla\cdot\nabla T\,dV$. Since D is arbitrary, shrink it to a point to conclude that $-\sigma\rho\frac{\partial T}{\partial t} = -k\nabla\cdot\nabla T$ or $\sigma\rho\frac{\partial T}{\partial t} = k\nabla\cdot\nabla T$.

7. We know from the argument in Exercise 6 that $-\iiint_D \sigma\rho\frac{\partial T}{\partial t}dV = \iiint_D -\nabla\cdot(k\nabla T)dV$. Use the product rule to conclude that this equals $\iiint_D -(\nabla k\cdot\nabla T + k\nabla^2 T)dV$. As before, shrink to a point to conclude $\sigma\rho\frac{\partial T}{\partial t} = k\nabla^2 T + \nabla k\cdot\nabla T$.

8. This is immediate from the heat equation since $\partial T/\partial t = 0$ and σ, ρ, k are constants.

9. (a)

$$\oiint_{\partial D} \mathbf{H}\cdot d\mathbf{S} = \oiint_{\partial D} -k\nabla T\cdot d\mathbf{S} = \iiint_D \nabla\cdot(-k\nabla T)\,dV \quad \text{by Gauss's theorem}$$
$$= -k\iiint_D \nabla^2 T\,dV = 0 \quad \text{by Exercise 8.}$$

(b) By part (a), there can be no net inflow or outflow of heat. Thus, heat must be flowing into D from the inner (hotter) sphere and out of D through the outer sphere at the same rate.

10. (a) Since $w = T_1 - T_2$, $\nabla^2 w = \nabla^2(T_1 - T_2)$. But T_1 and T_2 each satisfy the heat equation given in the exercise, so

$$\nabla^2 w = \nabla^2(T_1 - T_2) = \frac{\partial T_1}{\partial t} - \frac{\partial T_2}{\partial t} = \frac{\partial}{\partial t}(T_1 - T_2) = \frac{\partial w}{\partial t}.$$

So w satisfies the heat equation. Now for $(x, y, z) \in D$ we have

$$w(x, y, z, 0) = T_1(x, y, z, 0) - T_2(x, y, z, 0) = \alpha(x, y, z) - \alpha(x, y, z) = 0.$$

So the first condition holds. Also for all $(x, y, z) \in \partial D$ and $t \geq 0$ we see

$$w(x, y, z, t) = T_1(x, y, z, t) - T_2(x, y, z, t) = \phi(x, y, z, t) - \phi(x, y, z, t) = 0.$$

So w satisfies the second condition.

(b) We take the derivative

$$E'(t) = \frac{d}{dt}\left[\frac{1}{2}\iiint_D w^2\,dV\right] = \frac{1}{2}\iiint_D \frac{\partial}{\partial t}(w^2)\,dV = \iiint_D w\frac{\partial w}{\partial t}\,dV.$$

From part (a) we know that w satisfies the heat equation so

$$\iiint_D w\frac{\partial w}{\partial t}\,dV = \iiint_D w\nabla^2 w\,dV.$$

Using Green's first formula with $f = g = w$, we have

$$\iiint_D w\nabla^2 w\,dV = \oiint_{\partial D} w\nabla w\cdot d\mathbf{S} - \iiint_D \nabla w\cdot\nabla w\,dV = -\iiint_D \nabla w\cdot\nabla w\,dV$$

since we showed in part (a) that $w \equiv 0$ on ∂D. Thus, $E'(t) = -\iiint_D \|\nabla w\|^2 dV \le 0$.

(c) In part (a) we showed that $w(x,y,z,0) = 0$ on D. Therefore,

$$E(0) = \frac{1}{2}\iiint_D [w(x,y,z,0)]^2\,dV = 0.$$

Now $E(t)$ is the integral of a non-negative integrand so $E(t) \ge 0$. On the other hand, from part (b) we know that E is nonincreasing. Therefore, E is a nonincreasing, nonnegative function such that $E(0) = 0$. Hence $E(t) = 0$ for all $t \ge 0$.

(d) By part (c), $\iiint_D w^2 dV = 0$ for all $t \ge 0$. Since $w^2 \ge 0$, we must have $[w(x,y,z,t)]^2 = 0$ for all $(x,y,z) \in D$ and $t \ge 0$. Therefore $w(x,y,z,t) = 0$ for all $(x,y,z) \in D$ and $t \ge 0$. Hence $T_1(x,y,z,t) = T_2(x,y,z,t)$ for all $(x,y,z) \in D$ and $t \ge 0$.

11. From Ampère's law we have $\mathbf{J} = \dfrac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$. Therefore,

$$\begin{aligned}\nabla\times\mathbf{J} &= \frac{1}{\mu_0}\nabla\cdot(\nabla\times\mathbf{B}) - \epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t} = -\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}\\ &= -\epsilon_0\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = -\epsilon_0\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) \quad \text{by Gauss's law,}\\ &= -\frac{\partial\rho}{\partial t}.\end{aligned}$$

12. We find where $\nabla\cdot\mathbf{E} = 0$.

$$\nabla\cdot\mathbf{E} = \frac{\partial}{\partial x}(x^3 - x) + \frac{\partial}{\partial y}\left(\frac{1}{4}y^3\right) + \frac{\partial}{\partial z}\left(\frac{1}{9}z^3 - 2z\right).$$

So $\nabla\cdot\mathbf{E} = 3x^2 + \frac{3}{4}y^2 + \frac{1}{3}z^2 - 3$. This is zero for points on the ellipsoid $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$.

13. First we check that $\nabla\cdot\mathbf{F} = 0$ wherever $\mathbf{F}$ is defined (i.e., away from the origin):

$$\begin{aligned}\nabla\cdot\mathbf{F} &= k\left(\frac{\partial}{\partial x}\frac{x}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial}{\partial z}\frac{z}{(x^2+y^2+z^2)^{3/2}}\right)\\ &= k\left(\frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}} + \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}}\right.\\ &\qquad\left. + \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^{3/2}}\right).\end{aligned}$$

Multiply numerator and denominator by $(x^2+y^2+z^2)^{1/2}$:

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{k}{(x^2+y^2+z^2)^{7/2}}\Big(3(x^2+y^2+z^2)^2 - 3x^2(x^2+y^2+z^2) - 3y^2(x^2+y^2+z^2)\\ &\qquad - 3z^2(x^2+y^2+z^2)\Big)\\ &= \frac{k(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}(3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2) \equiv 0.\end{aligned}$$

Thus, by Gauss's theorem, $\oiint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\,dV = 0$ if $S = \partial D$ and S does not enclose the origin. If S does enclose the origin, let D be the solid region between S and a small sphere S_b of radius b that encloses the origin and is inside S (as in Figure 7.54). Then

$$0 = \iiint_D \nabla\cdot\mathbf{F}\,dV = \oiint_S \mathbf{F}\cdot d\mathbf{S} - \oiint_{S_b}\mathbf{F}\cdot d\mathbf{S}$$

where S and S_b are both oriented by *outward* normals.

Hence $\oiint_S \mathbf{F}\cdot d\mathbf{S} = \oiint_{S_b} \mathbf{F}\cdot d\mathbf{S} = \oiint_{S_b} \dfrac{k\mathbf{x}}{\|\mathbf{x}\|^3}\cdot \dfrac{1}{b}\mathbf{x}\, dS$ (outward normal $\mathbf{n}$ to S_b is $\frac{1}{b}\mathbf{x}$)

$$= \oiint_{S_b} \frac{k\|\mathbf{x}\|^2}{b\|\mathbf{x}\|^3}\, dS = \oiint_{S_b} \frac{kb^2}{b^4}\, dS \quad (\|\mathbf{x}\| = b \text{ on } S_b)$$

$$= \frac{k}{b^2}\cdot(\text{surface area of } S_b) = \frac{k}{b^2}(4\pi b^2) = 4\pi k.$$

14. **(a)** We may write $\mathbf{E}(\mathbf{x}) = E_\rho(\mathbf{x})\mathbf{e}_\rho + E_\varphi(\mathbf{x})\mathbf{e}_\varphi + E_\theta(\mathbf{x})\mathbf{e}_\theta$. The field of a point charge at the origin must be symmetric about the origin. Thus $E_\varphi = E_\theta = 0$, so $\mathbf{E}(\mathbf{x}) = E_\rho(\mathbf{x})\mathbf{e}_\rho = E(\mathbf{x})\mathbf{e}_\rho$. Once again, by symmetry, $E(\mathbf{x})$ must be constant on any sphere centered at the origin, so E can only depend on ρ. Hence $\mathbf{E}(\mathbf{x}) = E(\rho)\mathbf{e}_\rho$.

(b) We have

$$\begin{aligned}\oiint_S E(\rho)\mathbf{e}_\rho\cdot d\mathbf{S} &= \oiint_S \mathbf{E}\cdot d\mathbf{S} \quad \text{from part (a)},\\ &= \iiint_D \nabla\cdot\mathbf{E}\, dV \quad \text{using Gauss's theorem},\\ &= \iiint_D \frac{\rho}{\epsilon_0}\, dV \quad \text{using Gauss's law},\\ &= \frac{q}{\epsilon_0} \quad \text{by definition of } \rho \text{ and } q.\end{aligned}$$

(c) We have $\oiint_S E(\rho)\mathbf{e}_\rho\cdot d\mathbf{S} = \oiint_S E(\rho)\mathbf{e}_\rho\cdot\mathbf{n}\, dS = \oiint_S E(\rho)\mathbf{e}_\rho\cdot\mathbf{e}_\rho\, dS = \oiint_S E(\rho)\, dS$. Since, by part (b), $\oiint_S E(\rho)\mathbf{e}_\rho\cdot d\mathbf{S} = \dfrac{q}{\epsilon_0}$, we have $\oiint_S E(\rho)\, dS = \dfrac{q}{\epsilon_0}$.

(d) By part (c), $q/\epsilon_0 = \oiint_S E(\rho)\, dS$. But, obviously, ρ is constant on the sphere of radius a and so on that sphere $q/\epsilon_0 = \oiint_S E(\rho)\, dS = E(a)\cdot 4\pi a^2$. Thus we see that $E(\rho) = q/(4\pi\epsilon_0\rho^2)$. Hence,

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0\rho^2}\mathbf{e}_\rho = \frac{q}{4\pi\epsilon_0\|\mathbf{x}\|^2}\frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \text{ as desired.}$$

15. **(a)** This is just a straightforward calculation. Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. Then

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix} = (P_y - N_z)\mathbf{i} + (M_z - P_x)\mathbf{j} + (N_x - M_y)\mathbf{k}$$

and

$$\begin{aligned}\nabla\times(\nabla\times\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P_y - N_z & M_z - P_x & N_x - M_y \end{vmatrix}\\ &= (N_{xy} - M_{yy} - M_{zz} + P_{xz})\mathbf{i} + (P_{yz} - N_{zz} - N_{xx} + M_{yx})\mathbf{j}\\ &\quad + (M_{zx} - P_{xx} - P_{yy} + N_{zy})\mathbf{k}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
\nabla(\nabla\cdot\mathbf{F}) &= \nabla(M_x+N_y+P_z)\\
&= (M_{xx}+N_{yx}+P_{zx})\mathbf{i}+(M_{xy}+N_{yy}+P_{zy})\mathbf{j}+(M_{xz}+N_{yz}+P_{zz})\mathbf{k}\\
\text{and } (\nabla\cdot\nabla)\mathbf{F} &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{F}\\
&= (M_{xx}+M_{yy}+M_{zz})\mathbf{i}+(N_{xx}+N_{yy}+N_{zz})\mathbf{j}+(P_{xx}+P_{yy}+P_{zz})\mathbf{k}.\\
\text{Hence, } \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F} &= (N_{yx}+P_{zx}-M_{yy}-M_{zz})\mathbf{i}+(M_{xy}+P_{zy}-N_{xx}-N_{zz})\mathbf{j}\\
&\quad +(M_{xz}+N_{yz}-P_{xx}-P_{yy})\mathbf{k}.
\end{aligned}$$

By assumption **F** is of class C^2 and so the mixed partials are equal; thus we have the result:

$$\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}.$$

(b) First we show that **E** satisfies the wave equation.

$$\begin{aligned}
\nabla^2\mathbf{E} &= \nabla(\nabla\cdot\mathbf{E})-\nabla\times(\nabla\times\mathbf{E}) \quad \text{from part (a),}\\
&= \nabla\left(\frac{\rho}{\epsilon_0}\right)-\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{using Gauss's and Faraday's laws,}\\
&= \frac{1}{\epsilon_0}\nabla\rho+\frac{\partial}{\partial t}(\nabla\times\mathbf{B})\\
&= \frac{1}{\epsilon_0}\nabla\rho+\frac{\partial}{\partial t}\left(\mu_0\mathbf{J}+\epsilon_0\,\mu_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{using Ampère's law}\\
&= \mathbf{0}+\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2} \quad \text{since there are no charges or currents (so } \rho\equiv 0 \text{ and } \mathbf{J}\equiv\mathbf{0}).
\end{aligned}$$

Thus $\nabla^2\mathbf{E} = k\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ where $k=\epsilon_0\,\mu_0$.

Next we show that **B** satisfies the wave equation.

$$\begin{aligned}
\nabla^2\mathbf{B} &= \nabla(\nabla\cdot\mathbf{B})-\nabla\times(\nabla\times\mathbf{B}) \quad \text{from part (a),}\\
&= \mathbf{0}-\nabla\times\left(\mu_0\,\mathbf{J}+\epsilon_0\,\mu_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{using Maxwell's equations,}\\
&= -\epsilon_0\,\mu_0\nabla\times\frac{\partial\mathbf{E}}{\partial t} \text{ since } \mathbf{J}\equiv\mathbf{0} \quad \text{(no currents),}\\
&= -\epsilon_0\,\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E})\\
&= -\epsilon_0\,\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{from Faraday's law,}\\
&= \epsilon_0\,\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2}.
\end{aligned}$$

So $\nabla^2\mathbf{B} = k\dfrac{\partial^2 B}{\partial t^2}$ where $k=\epsilon_0\,\mu_0$.

(c) By part (a),

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{E})-(\nabla\cdot\nabla)\mathbf{E}&=\nabla\times(\nabla\times\mathbf{E})\\&=\nabla\times\left(-\frac{\partial\mathbf{B}}{\partial t}\right)\quad\text{by Faraday's law,}\\&=-\frac{\partial}{\partial t}(\nabla\times\mathbf{B})\\&=-\frac{\partial}{\partial t}\left[\mu_0\mathbf{J}+\epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\right]\quad\text{by Ampère's law,}\\&=-\mu_0\frac{\partial}{\partial t}\left[\mathbf{J}+\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right].\end{aligned}$$

(d) Again from part (a),

$$\begin{aligned}\nabla^2\mathbf{E}&=\nabla(\nabla\cdot\mathbf{E})-\nabla\times(\nabla\times\mathbf{E})\\&=\mathbf{0}-\nabla\times(\nabla\times\mathbf{E})\quad\text{by Gauss's law and the fact that }\rho=0,\\&=\mu_0\frac{\partial\mathbf{J}}{\partial t}+\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}\quad\text{from the argument in part (c).}\end{aligned}$$

16. Start with the non-static version of Ampère's law.

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{B})&=\nabla\cdot\left(\mu_0\,\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)=\nabla\cdot(\mu_0\mathbf{J})+\nabla\cdot\left(\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\\&=-\mu_0\frac{\partial\rho}{\partial t}+\mu_0\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}\quad\text{from the continuity equation}\\&=-\mu_0\frac{\partial\rho}{\partial t}+\mu_0\frac{\partial\rho}{\partial t}\quad\text{from Gauss's law}\\&=0.\end{aligned}$$

17. **(a)** From Ampère's law in the static case, $\nabla\times\mathbf{B}-\mu_0\mathbf{J}$ must be $\mathbf{0}$ when $\mathbf{J}$ does not depend on time. Otherwise, the difference must depend on time. If $\mathbf{F}_1$ is a time-varying vector field then $\partial\mathbf{F}_1/\partial t\neq\mathbf{0}$. If, on the other hand, $\mathbf{F}_1$ does not depend on time, then $\partial\mathbf{F}_1/\partial t=\mathbf{0}$. Hence, if we take $\nabla\times\mathbf{B}-\mu_0\mathbf{J}=\partial\mathbf{F}_1/\partial t$, then we will have an equation that is valid in both the static and the non-static cases.

(b) This is similar to our calculation in Exercise 16.

$$\begin{aligned}\nabla\cdot(\nabla\times\mathbf{B})&=\nabla\cdot(\mu_0\mathbf{J})+\nabla\cdot\frac{\partial\mathbf{F}_1}{\partial t}=-\mu_0\frac{\partial\rho}{\partial t}+\nabla\cdot\frac{\partial\mathbf{F}_1}{\partial t}\quad\text{from the continuity equation}\\&=-\mu_0\epsilon_0\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}+\nabla\cdot\frac{\partial\mathbf{F}_1}{\partial t}\quad\text{from Gauss's law.}\end{aligned}$$

So to have $\nabla\cdot(\nabla\times\mathbf{B})=0$ we conclude that $\mu_0\epsilon_0\nabla\cdot\dfrac{\partial\mathbf{E}}{\partial t}=\nabla\cdot\dfrac{\partial\mathbf{F}_1}{\partial t}$.

(c) If $\nabla\cdot\dfrac{\partial\mathbf{F}_1}{\partial t}=\mu_0\,\epsilon_0\nabla\dfrac{\partial\mathbf{E}}{\partial t}$, then by part (b),

$$\frac{\partial\mathbf{F}_1}{\partial t}=\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mathbf{F}_2\quad\text{where}\quad\nabla\cdot\mathbf{F}_2=0.$$

Therefore the most general formulation is

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}+\mathbf{F}_2.$$

18. We first show that $\mathbf{E}$ satisfies the telegrapher's equation. From Exercise 15(d) we know that

$$\nabla^2\mathbf{E}=\mu_0\frac{\partial\mathbf{J}}{\partial t}+\epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2},$$

but here $\mathbf{J} = \sigma\mathbf{E}$, so

$$\nabla^2\mathbf{E} = \mu_0\sigma\frac{\partial \mathbf{E}}{\partial t} + \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}.$$

Next we show that $\mathbf{B}$ satisfies the telegrapher's equation. Now,

$$\begin{aligned}
\nabla^2\mathbf{B} &= \nabla(\nabla\cdot\mathbf{B}) - \nabla\times(\nabla\times\mathbf{B}) \quad \text{from Exercise 15(a)} \\
&= \mathbf{0} - \nabla\times\left(\mu_0\,\mathbf{J} + \epsilon_0\,\mu_0\frac{\partial\mathbf{E}}{\partial t}\right) \quad \text{by Maxwell's equations,} \\
&= -\nabla\times\left(\mu_0\sigma\mathbf{E} + \epsilon_0\,\mu_0\frac{\partial\mathbf{E}}{\partial t}\right) \\
&= -\mu_0\sigma(\nabla\times\mathbf{E}) - \epsilon_0\mu_0\frac{\partial}{\partial t}(\nabla\times\mathbf{E}) \\
&= -\mu_0\sigma\left(-\frac{\partial\mathbf{B}}{\partial t}\right) - \epsilon_0\mu_0\frac{\partial}{\partial t}\left(-\frac{\partial\mathbf{B}}{\partial t}\right) \quad \text{by Faraday's law,} \\
&= \mu_0\sigma\frac{\partial\mathbf{B}}{\partial t} + \epsilon_0\mu_0\frac{\partial^2\mathbf{B}}{\partial t^2}.
\end{aligned}$$

19. Since $\mathbf{P} = \mathbf{E}\times\mathbf{B}$,

$$\begin{aligned}
\iint_S \mathbf{P}\cdot d\mathbf{S} &= \iint_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S} \\
&= \iiint_D \nabla\cdot(\mathbf{E}\times\mathbf{B})\,dV \quad \text{by Gauss's theorem,} \\
&= \iiint_D (\mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B}))\,dV \\
&= \iiint_D \left[-\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\left(\mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}\right)\right] dV \quad \text{from Faraday and Ampère's laws.}
\end{aligned}$$

Since $\mathbf{B}$ and $\mathbf{E}$ are both assumed to be constant in time, $\dfrac{\partial\mathbf{B}}{\partial t} = \dfrac{\partial\mathbf{E}}{\partial t} = \mathbf{0}$. Therefore, we get the desired result:

$$\iint_S \mathbf{P}\cdot d\mathbf{S} = \iiint_D -\mu_0\mathbf{E}\cdot\mathbf{J}\,dV.$$

20. (a) If $\mathbf{r} = (r_1, r_2, r_3)$ and $\mathbf{x} = (x, y, z)$,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\iiint_D \rho(\mathbf{x})\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV \\
&= \frac{1}{4\pi\epsilon_0}\left(\iiint_D \rho(x,y,z)\frac{r_1 - x}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV, \iiint_D \rho(x,y,z)\frac{r_2 - y}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV, \iiint_D \rho(x,y,z)\frac{r_3 - z}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV\right).
\end{aligned}$$

(b) Look at the first component of $\mathbf{E}$. (The arguments for the other two components are similar.) We have

$$\left|\frac{\rho(x,y,z)}{4\pi\epsilon_0}\frac{r_1 - x}{\|\mathbf{r}-\mathbf{x}\|^3}\right| \le \frac{|\rho(x,y,z)|}{4\pi\epsilon_0}\frac{\|\mathbf{r}-\mathbf{x}\|}{\|\mathbf{r}-\mathbf{x}\|^3} \le \frac{K}{\|\mathbf{r}-\mathbf{x}\|^2}$$

where K may be taken to be the maximum value of $|\rho|$ on D divided by $4\pi\epsilon_0$. Thus,

$$\left|\frac{1}{4\pi\epsilon_0}\iiint_D \rho(x,y,z)\frac{r_1 - x}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV\right| \le \iiint_D \frac{K}{\|\mathbf{r}-\mathbf{x}\|^2}\,dV.$$

(c) Use spherical coordinates with $\mathbf{r}$ as the origin so that the spherical coordinate ρ is $\|\mathbf{r}-\mathbf{x}\|$. Then

$$\iiint_D \frac{K}{\|\mathbf{r}-\mathbf{x}\|^2}\,dV = \iiint_D \frac{K}{\rho^2}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \iiint_D K\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Note that $K\sin\varphi$ is a bounded, continuous integrand. Since D is a bounded region, this last integral must converge. Hence, by the remarks in the exercise, the original triple integral must converge.

21. We are given $\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_D \mathbf{J}\times\frac{\mathbf{r}-\mathbf{x}}{\|\mathbf{r}-\mathbf{x}\|^3}\,dV$. Now,

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ J_1 & J_2 & J_3 \\ r_1 - x & r_2 - y & r_3 - z \end{vmatrix} = [(r_3-z)J_2 - (r_2-y)J_3]\,\mathbf{i} + [(r_1-x)J_3 - (r_3-z)J_1]\,\mathbf{j}$$

$$+ [(r_2-y)J_1 - (r_1-x)J_2]\,\mathbf{k}.$$

Hence the first component of the triple integral for **B** is

$$\frac{\mu_0}{4\pi}\iiint_D \left(J_2\frac{r_3-z}{\|\mathbf{r}-\mathbf{x}\|^3} - J_3\frac{r_2-y}{\|\mathbf{r}-\mathbf{x}\|^3}\right)dV.$$

(The other components are of the same form.) Note that each term in the integrand is of the form described in Exercise 20. Thus, using the arguments in Exercise 20, each component integral of **B** must converge.

True/False Exercises for Chapter 7

1. True.
2. False. (Note that the parametrization only gives $y \geq 3$.)
3. True. (Let $u = s^3$ and $v = \tan t$.)
4. False. (The standard normal vanishes when s or t is zero.)
5. False. (The limits of integration are not correct.)
6. True. (Use symmetry.)
7. False. (The value of the integral is 24.)
8. True. (Use symmetry.)
9. True.
10. True. ($\mathbf{F}\cdot\mathbf{n} = 0$.)
11. False. (The integral has value 32π.)
12. True.
13. False. (The value is 0.)
14. True.
15. False. (The surface must be connected.)
16. False. (Consider the Möbius strip.)
17. True. (The result follows from Stokes's theorem.)
18. False. (The value is the same only up to sign.)
19. True. (Use Gauss's theorem.)
20. True. (Apply Gauss's theorem.)
21. False. (Gauss's theorem implies that the integral is *at most* twice the surface area.)
22. False.
23. True.
24. True.
25. False. (Should be the flux of the *curl* of **F**.)
26. True. (This is what Gauss's theorem says.)
27. True. (Apply Green's first formula.)
28. False. (The negative sign is incorrect.)
29. False. (f is determined up to addition of a harmonic function.)
30. False. (Only if S doesn't enclose the origin.)

Miscellaneous Exercises for Chapter 7

1. Here are the matches:

(a) C (b) E (c) A

(d) D (e) F (f) B

Brief reasons:

(c) The projection of **X** into the *xy*-plane, for fixed s, is a circle centered at the origin of radius $2 + \cos\ s$.

(b) Note that $x^2 + y^2 = z^2$, so we have a conical surface.

(a) Since $y = s$, the intersection of the surface with the plane $y = 0$ is the parametrized curve $x = -t^3, y = 0, z = -t^2$ or $z = -x^{2/3}, y = 0$, which is a cuspidal curve.

(d) Let $t = \pi/2$. Then $x = 0, y = s, z = \sin s$. So the intersection of the surface by the $x = 0$ plane is a sinusoidal curve.

(f) For constant values of s we have a helix, so the surface should be a helicoid.

(e) By elimination, this must correspond to F.

2. (a) Consider all the lines through (0, 0, 1). Either such a line is tangent to the sphere, or else it passes through another point of the sphere S. The lines tangent to S at (0, 0, 1) fill out the tangent plane $z = 1$. All the other lines therefore have "slope vectors" with nonzero **k**-components. Hence they intersect the $z = 0$ plane somewhere. Thus any line joining (0, 0, 1) and (s, t, 0) intersects S at a point other than (0, 0, 1).

(b) The line joining (0, 0, 1) and (s, t, 0) is given parametrically by

$$\mathbf{y}(u) = (1-u)(0,0,1) + u(s,t,0) = (us, ut, 1-u).$$

To see where the line intersects the sphere, we insert the parametric equations $\begin{cases} x = us \\ y = ut \\ z = 1-u \end{cases}$ into the equation for S and solve for u. Thus:

$$\begin{aligned}(us)^2 + (ut)^2 + (1-u)^2 = 1 &\Leftrightarrow u^2(s^2+t^2+1) - 2u + 1 = 1 \\ &\Leftrightarrow u((s^2+t^2+1)u - 2) = 0.\end{aligned}$$

So either $u = 0$ (which corresponds to (0, 0, 1)) or $u = \dfrac{2}{s^2+t^2+1}$. For this second value of u, we may define $\mathbf{X}(s,t)$ as

$$\begin{aligned}\mathbf{X}(s,t) = \mathbf{y}\left(\frac{2}{s^2+t^2+1}\right) &= \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, 1 - \frac{2}{s^2+t^2+1}\right) \\ &= \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1}\right).\end{aligned}$$

(c) Check that the coordinates of **X**(s, t) satisfy the equation for S, i.e., that

$$\begin{aligned}&\left(\frac{2s}{s^2+t^2+1}\right)^2 + \left(\frac{2t}{s^2+t^2+1}\right)^2 + \left(\frac{s^2+t^2-1}{s^2+t^2+1}\right)^2 \\ &= \frac{4s^2+4t^2+s^4+t^4+2s^2t^2-2s^2-2t^2+1}{(s^2+t^2+1)^2} \\ &= \frac{s^4+t^4+2s^2t^2+2s^2+2t^2+1}{(s^2+t^2+1)^2} = \frac{(s^2+t^2+1)^2}{(s^2+t^2+1)^2} \equiv 1.\end{aligned}$$

Note that there are no values for s and t so that $\mathbf{X}(s,t) = (0,0,1)$. (To see thus, look at the first two coordinates—we must have $s = t = 0$. But then $\mathbf{X}(0,0) = (0,0,-1)$). Hence the parametrization misses the north pole.

3. (a) If we use cylindrical coordinates $x = r\cos\theta, y = r\sin\theta, z = z$, then the equation $x^2+y^2-z^2 = 1$ becomes $r^2-z^2 = 1$ or, since $r \geq 0, r = \sqrt{z^2+1}$. Hence the desired parametrization is $\mathbf{X}(z,\theta) = (\sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, z)$ where $z \in \mathbf{R}$ and $0 \leq \theta \leq 2\pi$.

(b) Modify the cylindrical coordinate substitution by letting $x = ar\cos t, y = br\sin t, z = cs$. Substitution into the equation for the hyperboloid yields $r^2 - s^2 = 1$ so $r = \sqrt{s^2+1}$. Hence a parametrization is $\mathbf{X}(s,t) = (a\sqrt{s^2+1}\cos t, b\sqrt{s^2+1}\sin t, cs)$, where $s \in \mathbf{R}$ and $0 \leq t \leq 2\pi$.

(c) Substitute the parametric equations for $\mathbf{l}_1$ into the left side of the equation for the hyperboloid:

$$\begin{aligned}\frac{a^2(x_0-y_0t)^2}{a^2} + \frac{b^2(x_0t+y_0)^2}{b^2} - \frac{c^2t^2}{c^2} &= x_0^2 - 2x_0y_0t + y_0^2t^2 + x_0^2t^2 + 2x_0y_0t + y_0^2 - t^2 \\ &= x_0^2 + y_0^2 + (x_0^2+y_0^2)t^2 - t^2 \\ &= 1 + t^2 - t^2 = 1,\end{aligned}$$

since $x_0^2 + y_0^2 = 1$. Thus $\mathbf{l}_1$ lies on the hyperboloid. The calculation for $\mathbf{l}_2$ is similar.

(d) The plane tangent to the hyperboloid at the point $(ax_0, by_0, 0)$ is given by

$$\nabla F(ax_0, by_0, 0)\cdot(x - ax_0, y - by_0, 0) = 0 \quad \text{where} \quad F = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2}.$$

That is, the tangent plane is

$$(*) \quad \frac{x_0}{a}(x - ax_0) + \frac{y_0}{b}(y - by_0) = 0.$$

If we substitute the parametric equations for $\mathbf{l}_1$ into the left side of $(*)$, we find

$$\frac{x_0}{a}(a(x_0 - y_0 t) - ax_0) + \frac{y_0}{b}(b(x_0 t + y_0) - by_0) = -x_0 y_0 t + y_0 x_0 t = 0$$

for all t. Therefore, the line $\mathbf{l}_1$ lies in the plane. A similar calculation can be made for $\mathbf{l}_2$.

4. Reconsider the parametrization from Exercise 3(a), $\mathbf{X}(z,\theta) = (\sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, z)$ where $z \in \mathbf{R}$ and $0 \le \theta \le 2\pi$. Then we have,

$$\mathbf{T}_z = \left(\frac{z}{\sqrt{z^2+1}}\cos\theta, \frac{z}{\sqrt{z^2+1}}\sin\theta, 1\right),$$
$$\mathbf{T}_\theta = (-\sqrt{z^2+1}\sin\theta, \sqrt{z^2+1}\cos\theta, 0).$$
$$\text{Thus} \quad \mathbf{T}_z \times \mathbf{T}_\theta = (-\sqrt{z^2+1}\cos\theta, -\sqrt{z^2+1}\sin\theta, z),$$
$$\text{so that} \quad \|\mathbf{T}_z \times \mathbf{T}_\theta\| = \sqrt{(z^2+1)+z^2} = \sqrt{2z^2+1}.$$

Therefore,

$$\text{Surface area} = \int_0^{2\pi}\int_{-a}^{a} \sqrt{2z^2+1}\,dz\,d\theta = \pi(\sqrt{2}\ln(\sqrt{2a^2+1} + \sqrt{2}a) + 2a\sqrt{2a^2+1}).$$

(Let $\tan u = \sqrt{2}z$ in the z-integral.)

5. (a) This is similar to Exercise 3(b). First, consider a variant of spherical coordinates: $x = a\rho\cos\theta\sin\varphi, y = b\rho\sin\theta\sin\varphi$, and $z = c\rho\cos\varphi$. If we set $\rho = 1$, we get the desired parametrization: $x = a\sin\varphi\cos\theta, y = b\sin\theta\sin\varphi$, and $z = c\cos\varphi$ where $0 \le \varphi \le \pi$ and $0 \le \theta \le 2\pi$.

(b) Here we have

$$\mathbf{T}_\varphi = (a\cos\varphi\cos\theta, b\cos\varphi\sin\theta, -c\sin\varphi) \quad \text{and}$$
$$\mathbf{T}_\theta = (-a\sin\varphi\sin\theta, b\sin\varphi\cos\theta, 0).$$

Therefore,

$$\mathbf{N} = \mathbf{T}_\varphi \times \mathbf{T}_\theta = (bc\sin^2\varphi\cos\theta, ac\sin^2\varphi\sin\theta, ab\cos\varphi\sin\varphi) \quad \text{and}$$
$$\|\mathbf{N}\| = b^2c^2\sin^4\varphi\cos^2\theta + a^2c^2\sin^4\varphi\sin^2\theta + a^2b^2\cos^2\varphi\sin^2\varphi.$$

Therefore,

$$\text{Surface area} = \int_0^{2\pi}\int_0^{\pi} \sqrt{b^2c^2\sin^4\varphi\cos^2\theta + a^2c^2\sin^4\varphi\sin^2\theta + a^2b^2\cos^2\varphi\sin^2\varphi}\,d\varphi\,d\theta.$$

In the special case where $a = b = c$, we find that

$$\begin{aligned}
\text{Surface area} &= \int_0^{2\pi}\int_0^{\pi} a^2\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \cos^2\varphi\sin^2\varphi}\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}\int_0^{\pi}\sqrt{\sin^4\varphi + \cos^2\varphi\sin^2\varphi}\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}\int_0^{\pi}\sqrt{\sin^2\varphi}\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta \\
&= a^2\int_0^{2\pi}(-\cos\varphi)\Big|_0^{\pi}\,d\theta \\
&= a^2\int_0^{2\pi}2\,d\theta = 4\pi a^2.
\end{aligned}$$

6. **(a)** The t-coordinate curve is $(s_0, f(s_0)\cos t, f(s_0)\sin t)$, which is a circle of radius $|f(s_0)|$ in the $x = s_0$ plane. That is, the radius of this cross-sectional circle depends on $f(s_0)$.

(b) $\mathbf{T}_s = (1, f'(s)\cos t, f'(s)\sin t)$ and $\mathbf{T}_t = (0, -f(s)\sin t, f(s)\cos t)$, so $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (f(s)f'(s), -f(s)\cos t, -f(s)\sin t)$. Thus $\|\mathbf{N}\| = \sqrt{[f(s)]^2[f'(s)]^2 + [f(s)]^2} = |f(s)|\sqrt{[f'(s)]^2 + 1}$. So

$$\begin{aligned}
\text{Surface area} &= \int_0^{2\pi}\int_a^b |f(x)|\sqrt{[f'(x)]^2+1}\,dx\,dt \\
&= \int_a^b\int_0^{2\pi} |f(x)|\sqrt{[f'(x)]^2+1}\,dt\,dx \\
&= 2\pi\int_a^b |f(x)|\sqrt{[f'(x)]^2+1}\,dx.
\end{aligned}$$

7. **(a)** This should remind students of when they were using washer and shell methods for surfaces of revolution. Of course, here we are finding a surface area and not volume. If you look at the specific value $s = s_0$ then, since we are revolving around the y-axis, we are sweeping out a circle of radius s_0 in the plane $y = f(s_0)$. Therefore, a parametrization is $\mathbf{X}(s,t) = (s\cos t, f(s), s\sin t), a \le s \le b, 0 \le t \le 2\pi$. Compare this with Exercise 6(a).

(b) Using the parametrization in (a), $\mathbf{T}_s = (\cos t, f'(s), \sin t)$ and $\mathbf{T}_t = (-s\sin t, 0, s\cos t)$. Therefore, $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (sf'(s)\cos t, -s, sf'(s)\sin t)$, so $\|\mathbf{N}\| = \sqrt{s^2[f'(s)]^2 + s^2}$. Hence

$$\begin{aligned}
\text{Surface area} &= \int_a^b\int_0^{2\pi}\sqrt{s^2[f'(s)]^2+s^2}\,dt\,ds \\
&= 2\pi\int_a^b s\sqrt{[f'(s)]^2+1}\,ds \\
&= 2\pi\int_a^b x\sqrt{[f'(x)]^2+1}\,dx
\end{aligned}$$

by changing the variable of integration. Compare this result with that of Exercise 6(b).

8. **(a)** It would be helpful for you to first draw a picture. The surface is the curve $z = f(x)$ in the *xz*-plane extended so that the derivative in the y direction is identically zero (i.e., we're dragging the curve in the y direction). Then S_1, the portion of S lying over D, may be parametrized as $\mathbf{X}(x,y) = (x, y, f(x)), (x,y) \in D$. Then $\mathbf{T}_x = (1, 0, f'(x))$ and $\mathbf{T}_y = (0, 1, 0)$, so that $\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_y = (-f'(x), 0, 1)$, and so $\|\mathbf{N}\| = \sqrt{1 + [f'(x)]^2}$. Hence,

$$\text{Surface area} = \iint_D \sqrt{1+[f'(x)]^2}\,dx\,dy = \iint_D \sqrt{1+[f'(x)]^2}\,dA.$$

Since $s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\,dt$, then, by the fundamental theorem of calculus, we have that $s'(x) = \sqrt{1+[f'(x)]^2}$. Thus the surface area is $\iint_D s'(x)\,dA$.

(b) From Green's theorem, $\oint_C s(x)\,dy = \iint_D s'(x)\,dA$, which, by part (a), is the surface area.

(c) Here we are working a specific example of what we worked out in part (a). The rectangle D is $[1,3]\times[-2,2]$. Using part (a), we compute the surface area as $\iint_D s'(x)\,dA$. Now $s'(x) = \sqrt{1+[f'(x)]^2}$ where $z = f(x) = \dfrac{x^3}{3} + \dfrac{1}{4x}$, and so $f'(x) = x^2 - \dfrac{1}{4x^2}$. Therefore,

$$1+[f'(x)]^2 = 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right) = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2.$$

Hence,

$$\begin{aligned}
\text{Surface area} &= \iint_D \sqrt{1+[f'(x)]^2}\,dA = \iint_D \sqrt{\left(x^2+\frac{1}{4x^2}\right)^2}\,dA \\
&= \int_{-2}^{2}\int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx\,dy = \int_{-2}^{2} \left(\frac{1}{3}x^3 - \frac{1}{4x}\right)\Bigg|_{x=1}^{3} dy \\
&= \int_{-2}^{2} \left(9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}\right) dy = \int_{-2}^{2} \frac{53}{6}\,dy = \frac{106}{3}.
\end{aligned}$$

9. (a) The surface integral $\iint_S f\,dS$, roughly speaking, represents the "sum" of all the values of f on S. The area of S is a measure of the size of S. So the quotient can be thought of as the "total" amount of f divided by the size of the region being sampled.

(b) Parametrize the sphere as $\mathbf{X}(s,t) = (7\cos s\sin t, 7\sin s\sin t, 7\cos t)$, $0 \le s \le 2\pi$, $0 \le t \le \pi$. Then, following Example 11 in Section 7.1, $\|\mathbf{T}_s\times\mathbf{T}_t\| = 49\sin t$. Note that, on the surface S, the temperature $T(x,y,z) = x^2+y^2-3z = 49 - z^2 - 3z$. As a result, we can calculate

$$\begin{aligned}
\iint_S T(x,y,z)\,dS &= \int_0^{2\pi}\int_0^{\pi} (49 - 49\cos^2 t - 21\cos t)49\sin t\,dt\,ds \\
&= 49\int_0^{2\pi} \left(-49\cos t + \frac{49}{3}\cos^3 t + \frac{21}{2}\cos^2 t\right)\Bigg|_0^{\pi} ds \\
&= 49\int_0^{2\pi} \left(49(2) - \frac{49}{3}(2) + \frac{21}{2}(1-1)\right) ds = \frac{(49)^2(4)(2\pi)}{3}.
\end{aligned}$$

Now, since the surface area of a sphere of radius 7 is $4\pi(49)$, we have

$$[T]_{\text{avg}} = \frac{\iint_S T(x,y,z)\,dS}{\text{surface area}} = \frac{(49)^2(4)(2\pi)}{3}\,\frac{1}{4\pi(49)} = \frac{98}{3}.$$

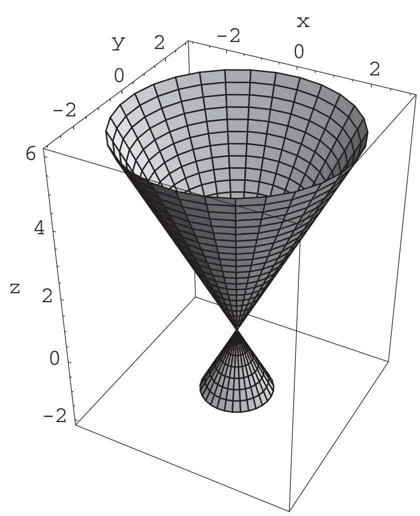
10. The surface area of the cylinder is $2\pi\cdot 2\cdot 3 = 12\pi$. If we parametrize the surface as

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \quad 0\le t < 2\pi, \quad 0 \le s \le 3 \\ z = s \end{cases}$$

then $\|\mathbf{T}_s \times \mathbf{T}_t\| = 2$. Hence

$$\begin{aligned}[f]_{\text{avg}} &= \frac{1}{12\pi}\iint_S f\,dS = \frac{1}{12\pi}\int_0^3\int_0^{2\pi}(4e^s\cos^2 t - 4s\sin^2 t)\cdot 2\,dt\,ds \\ &= \frac{1}{3\pi}\int_0^3\int_0^{2\pi}[e^s(1+\cos 2t) - s(1-\cos 2t)]\,dt\,ds \\ &= \frac{1}{3\pi}\int_0^3\left(e^s\left(t+\frac{1}{2}\sin 2t\right) - s\left(t-\frac{1}{2}\sin 2t\right)\right)\Bigg|_{t=0}^{2\pi} ds \\ &= \frac{1}{3\pi}\int_0^3(2\pi e^s - 2\pi s)\,ds = \frac{2}{3}\int_0^3(e^s - s)\,ds \\ &= \frac{2}{3}\left(e^s - \frac{1}{2}s^2\right)\Bigg|_0^3 = \frac{2}{3}\left(e^3 - \frac{9}{2} - 1\right) = \frac{2e^3 - 11}{3}.\end{aligned}$$

11. The cone looks as follows.



The upper nappe has a height of 6 and radius of 3; the lower nappe has a height of 2 and a radius of 1. Hence the total surface area is

$$\pi\cdot 1\cdot\sqrt{5} + \pi\cdot 3\cdot 3\sqrt{5} = 10\sqrt{5}\pi.$$

Next, parametrize the surface as $\begin{cases} x = s\cos t \\ y = s\sin t \\ z = 2s \end{cases}$ with $-1 \le s \le 3, 0 \le t < 2\pi$. Then

$$\|\mathbf{N}\| = \|\mathbf{T}_s \times \mathbf{T}_t\| = \|(-2s\cos t, -2s\sin t, s)\| = \sqrt{5}\,|s|.$$

Therefore,

$$\begin{aligned}
[f]_{\text{avg}} &= \frac{1}{10\sqrt{5}\pi}\iint_S f\,dS = \frac{1}{10\sqrt{5}\pi}\int_0^{2\pi}\int_{-1}^{3}(s^2-3)\sqrt{5}|s|\,ds\,dt \\
&= \frac{1}{10\pi}\int_{-1}^{3}\int_0^{2\pi}(s^2-3)|s|\,dt\,ds \\
&= \frac{1}{10\pi}\int_{-1}^{3}2\pi(s^2-3)|s|\,ds = \frac{1}{5}\left[\int_{-1}^{0}(s^2-3)(-s)\,ds + \int_0^3(s^2-3)s\,ds\right] \\
&= \frac{1}{5}\left[\left(-\frac{1}{4}s^4+\frac{3}{2}s^2\right)\Big|_{-1}^{0} + \left(\frac{1}{4}s^4-\frac{3}{2}s^2\right)\Big|_0^3\right] \\
&= \frac{1}{5}\left(\frac{1}{4}-\frac{3}{2}+\frac{81}{4}-\frac{27}{2}\right) = \frac{11}{10}.
\end{aligned}$$

12. The total mass is $\iint_{\mathbf{X}} \delta\,dS$. For the helicoid, $\mathbf{T}_s = (\cos t, \sin t, 0)$ and $\mathbf{T}_t = (-s\sin t, s\cos t, 1)$. Then $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (\sin t, -\cos t, s)$ and $\|\mathbf{N}\| = \sqrt{1+s^2}$. Hence,

$$\begin{aligned}
\text{Total mass} &= \iint_{\mathbf{X}} \sqrt{x^2+y^2}\,dS \\
&= \int_0^{4\pi}\int_0^1 \sqrt{(s\cos t)^2+(s\sin t)^2}\sqrt{1+s^2}\,ds\,dt \\
&= \int_0^{4\pi}\int_0^1 s\sqrt{1+s^2}\,ds\,dt \\
&= 4\pi\left(\frac{1}{2}\cdot\frac{2}{3}(1+s^2)^{3/2}\right)\Big|_{s=0}^{1} = \frac{4\pi}{3}(2\sqrt{2}-1).
\end{aligned}$$

13. By the symmetry of the surface, we must have $\bar{x} = \bar{y} = \bar{z}$. We compute $\bar{z}$ explicitly. Since δ is constant, it will cancel from the center of mass integrals:

$$\bar{z} = \frac{\iint_S z\delta\,dS}{\iint_S \delta\,dS} = \frac{\delta\iint_S z\,dS}{\delta\iint_S dS} = \frac{\iint_S z\,dS}{\text{surface area of } S}.$$

The surface area of the first octant portion of a sphere of radius a is $\frac{1}{8}(4\pi a^2) = \frac{1}{2}\pi a^2$. Therefore, $\bar{z} = \frac{2}{\pi a^2}\iint_S z\,dS$. We may parametrize the first octant portion of the sphere as $\mathbf{X}(\varphi,\theta) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, a\cos\varphi)$, $0 \le \varphi \le \pi/2$, $0 \le \theta \le \pi/2$. Hence,

$$\begin{aligned}
\mathbf{T}_\varphi &= (a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi), \\
\mathbf{T}_\theta &= (-a\sin\varphi\sin\theta, a\sin\varphi\cos\theta, 0).
\end{aligned}$$

Therefore,

$$\mathbf{N} = (a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\theta) \quad \text{and} \quad \|\mathbf{N}\| = a^2\sin\varphi.$$

Thus,

$$\begin{aligned}
\bar{z} &= \frac{2}{\pi a^2}\int_0^{\pi/2}\int_0^{\pi/2}(a\cos\varphi)a^2\sin\varphi\,d\varphi\,d\theta \\
&= \frac{2a^3}{\pi a^2}\int_0^{\pi/2}\int_0^{\pi/2}\cos\varphi\sin\varphi\,d\varphi\,d\theta \\
&= \frac{2a}{\pi}\int_0^{\pi/2}\left(\frac{1}{2}\sin^2\varphi\right)\Big|_{\varphi=0}^{\pi/2}d\theta \\
&= \frac{2a}{\pi}\int_0^{\pi/2}\frac{1}{2}\,d\theta = \frac{a}{\pi}\cdot\frac{\pi}{2} = \frac{a}{2}.
\end{aligned}$$

14. A quick sketch should convince you that, by symmetry, $\bar{x} = 0$ and $\bar{y} = \dfrac{a}{2}$. The equation for the surface may be written as $z = \sqrt{a^2 - x^2}$, so that $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{a^2 - x^2}}$ and $\dfrac{\partial z}{\partial y} = 0$. Then

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy = \sqrt{1 + \frac{x^2}{a^2 - x^2}}\, dx\, dy = \sqrt{\frac{a^2}{a^2 - x^2}}\, dx\, dy.$$

Hence,

$$\bar{z} = \frac{\iint_S z\delta\, dS}{\iint_S \delta\, dS} = \frac{\iint_S z\, dS}{\iint_S dS} = \frac{\iint_S z\, dS}{\pi a^2},$$

since the surface area of half a cylinder is πa^2. Now we calculate

$$\begin{aligned} \bar{z} &= \frac{1}{\pi a^2} \iint_S z\, dS = \frac{1}{\pi a^2} \int_0^a \int_{-a}^a \sqrt{a^2 - x^2}\sqrt{\frac{a^2}{a^2 - x^2}}\, dx\, dy \\ &= \frac{1}{\pi a^2} \int_0^a \int_{-a}^a a\, dx\, dy = \frac{a}{\pi a^2}(2a^2) = \frac{2a}{\pi}. \end{aligned}$$

15. By symmetry $\bar{x} = \bar{y} = 0$, so we only need to calculate $\bar{z} = \frac{\iint_S z\delta\, dS}{\iint_S \delta\, dS}$. Now

$$\delta(x, y, z) = x^2 + y^2 + (z + a)^2.$$

If we parametrize the sphere: $\begin{cases} x = a\cos s \sin t & 0 \le s < 2\pi \\ y = a \sin s \sin t & 0 \le t \le \pi \\ z = a\cos t \end{cases}$, then $\|\mathbf{N}\| = a^2 \sin t$ (see Example 1 of §7.2). We therefore have

$$\begin{aligned} \iint_S \delta\, dS &= \iint_S \left(x^2 + y^2 + (z + a)^2\right)\, dS \\ &= \int_0^\pi \int_0^{2\pi} \left(a^2 \sin^2 t + (a\cos t + a)^2\right) a^2 \sin t\, ds\, dt \\ &= 2\pi a^2 \int_0^\pi \left(a^2 \sin^2 t + a^2 \cos^2 t + 2a^2 \cos t + a^2\right) \sin t\, dt \\ &= 4\pi a^4 \int_0^\pi (1 + \cos t) \sin t\, dt \\ &= 4\pi a^4 \left(-\cos t + \tfrac{1}{2}\sin^2 t\right)\Big|_0^\pi = 8\pi a^4 \end{aligned}$$

$$\begin{aligned} \iint_S z\delta\, dS &= \iint_S z\left(x^2 + y^2 + (z + a)^2\right)\, dS \\ &= \int_0^\pi \int_0^{2\pi} a\cos t\left(a^2 \sin^2 t + (a\cos t + a)^2\right) a^2 \sin t\, ds\, dt \\ &= 4\pi a^5 \int_0^\pi (1 + \cos t)\cos t \sin t\, dt \\ &= 4\pi a^5 \left(\tfrac{1}{2}\sin^2 t - \tfrac{1}{3}\cos^3 t\right)\Big|_0^\pi = \frac{8\pi a^5}{3}. \end{aligned}$$

Hence,

$$\bar{z} = \frac{8\pi a^5/3}{8\pi a^4} = \frac{a}{3}.$$

16. Parametrize the cylinder as $\begin{cases} x = a\cos s \\ y = t \\ z = a \sin s \end{cases} \quad 0 \le s < 2\pi, 0 \le t \le 2.$

Then $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t = (-a\sin s, 0, a\cos s) \times (0,1,0) = (-a\cos s, 0, -a\sin s)$ so $\|\mathbf{N}\| = a$. Hence

$$\begin{aligned} M &= \iint_S \delta\, dS = \int_0^2 \int_0^{2\pi} (a^2\cos^2 s + t)\cdot a\, ds\, dt \\ &= \int_0^2 \int_0^{2\pi} \left(at + \frac{a^3}{2}(1+\cos^2 s)\right) ds\, dt = \int_0^2 (2\pi a t + \pi a^3)\, dt \\ &= (\pi a t^2 + \pi a^3 t)\Big|_0^2 = 4\pi a + 2\pi a^3 = 2\pi a(a^2+2). \end{aligned}$$

Symmetry implies $\bar{z} = 0$, so we calculate

$$\begin{aligned} \bar{x} &= \frac{1}{2\pi a(a^2+2)} \int_0^2 \int_0^{2\pi} a\cos s\,(a(t + a^2\cos^2 s))\, ds\, dt \\ &= \frac{a}{2\pi(a^2+2)} \int_0^2 \int_0^{2\pi} (t\cos s + a^2\cos^3 s)\, ds\, dt \\ &= \frac{a}{2\pi(a^2+2)} \int_0^2 \int_0^{2\pi} (t\cos s + a^2(1-\sin^2 s)\cos s)\, ds\, dt \\ &= \frac{a}{2\pi(a^2+2)} \int_0^2 \left(t\sin s + a^2\sin s - \frac{a^2}{3}\sin^3 s\right)\Big|_0^{2\pi} dt = 0. \end{aligned}$$

(Actually, you can really see this from symmetry.)

$$\begin{aligned} \bar{y} &= \frac{1}{2\pi a(a^2+2)} \iint_S y(x^2+y)\, dS = \frac{1}{2\pi a(a^2+2)} \int_0^2 \int_0^{2\pi} t\,(t + a^2\cos^2 s)\cdot a\, ds\, dt \\ &= \frac{1}{2\pi(a^2+2)} \int_0^2 \int_0^{2\pi} \left[t^2 s + a^2 t\left(\frac{1}{2}s + \frac{1}{4}\sin 2s\right)\Big|_{s=0}^{2\pi}\right] dt \\ &= \frac{1}{2\pi(a^2+2)} \int_0^2 (2\pi t^2 + \pi a^2 t)\, dt = \frac{1}{2(a^2+2)} \int_0^2 (2t^2 + a^2 t)\, dt \\ &= \frac{1}{2(a^2+2)} \left(\frac{2}{3}t^3 + \frac{a^2}{2}t^2\right)\Big|_0^2 = \frac{1}{2(a^2+2)}\left(\frac{16}{3} + 2a^2\right) \\ &= \frac{1}{a^2+2}\left(\frac{8}{3} + a^2\right) = \frac{3a^2+8}{3a^2+6}. \end{aligned}$$

So $(\bar{x}, \bar{y}, \bar{z}) = \left(0, \dfrac{3a^2+8}{3a^2+6}, 0\right)$.

17. (a) Parametrize the frustum $z^2 = 4x^2 + 4y^2$, $2 \le z \le 4$, as $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 2r)$, $0 \le \theta \le 2\pi$, $1 \le r \le 2$. Then

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \\ \frac{\partial(x,z)}{\partial(r,\theta)} &= \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2 & 0 \end{vmatrix} = 2r\sin\theta \\ \frac{\partial(y,z)}{\partial(r,\theta)} &= \begin{vmatrix} \sin\theta & r\cos\theta \\ 2 & 0 \end{vmatrix} = -2r\cos\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} I_z &= \iint_S (x^2+y^2)\, dS = \int_0^{2\pi} \int_1^2 r^2\sqrt{r^2 + 4r^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \int_1^2 \sqrt{5}r^3\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{5}}{4}(15)\, d\theta = \frac{15\sqrt{5}\pi}{2}. \end{aligned}$$

(b) The radius of gyration is given by $r_z = \sqrt{\dfrac{I_z}{M}}$. Assuming, as in part (a), that the density is 1, the total mass is just the surface area of the frustum. This can be computed from the surface area of the cone without much trouble. We view the frustum as a large cone (of height 4) with the tip (a similar cone of height 2) removed and note that the surface area of a cone is π(radius)(slant height). Then

$$\text{Surface area of frustum} = \pi(2)(2\sqrt{5}) - \pi(1)(\sqrt{5}) = 3\sqrt{5}\pi.$$

Hence

$$r_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{15\sqrt{5}\pi}{2}\frac{1}{3\sqrt{5}\pi}} = \sqrt{\frac{5}{2}}.$$

(Note: you can also compute the surface area as $\int_0^{2\pi}\int_1^2 \sqrt{5}r\,dr\,d\theta$.)

(c) We recompute the integral for I_z with $\delta = kr$. Thus

$$\begin{aligned} I_z &= \iint_S (x^2+y^2)\delta\,dS \\ &= \int_0^{2\pi}\int_1^2 r^2kr\sqrt{5r^2}\,dr\,d\theta \\ &= \int_0^{2\pi}\int_1^2 \sqrt{5}kr^4\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{5}k}{5}(2^5-1)\,d\theta = \frac{62\sqrt{5}\pi k}{5}. \end{aligned}$$

The total mass of the frustum is

$$\begin{aligned} M &= \iint_S \delta\,dS = \int_0^{2\pi}\int_1^2 kr\sqrt{5}r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{5}}{3}(2^3-1)\,d\theta = \frac{14\sqrt{5}\pi k}{3}. \end{aligned}$$

Hence

$$r_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{62\sqrt{5}\pi k}{5}\frac{3}{14\sqrt{5}\pi k}} = \sqrt{\frac{93}{35}}.$$

18. (a)

$$\begin{aligned} I_z = \iint_S (x^2+y^2)\,\delta\,dS = \delta\iint_S a^2\,dS &= \delta a^2\cdot\text{surface area} \\ &= \delta a^2\cdot 2\pi a\cdot 2b = 4\pi\delta a^3 b \end{aligned}$$

(b) $M = \iint_S \delta\,dS = \delta\cdot 4\pi ab$, so $r_z = \sqrt{\dfrac{I_z}{M}} = \sqrt{\dfrac{4\pi\delta a^3 b}{4\pi\delta ab}} = a.$

19. (a) $I_x = \iint_S (y^2+z^2)\delta\,dS$. If we parametrize S by $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = s \end{cases}$ $-b \le s \le b, 0 \le t < 2\pi$, then $\|\mathbf{N}\| = \|\mathbf{T}_s\times\mathbf{T}_t\| = a$ and so

$$\begin{aligned} I_x &= \int_{-b}^{b}\int_0^{2\pi}(a^2\sin^2 t + s^2)\delta a\,dt\,ds = \delta a\int_{-b}^{b}\left(\frac{a^2}{2}\left(t - \frac{1}{2}\sin 2t\right) + s^2t\Big|_{t=0}^{2\pi}\right)ds \\ &= \delta a\int_{-b}^{b}(\pi a^2 + 2\pi s^2)\,ds = \pi\delta a\left(2a^2b + \frac{4}{3}b^3\right) = \frac{2\pi ab\delta}{3}(3a^2+2b^2) \\ I_y &= \iint_S (x^2+z^2)\delta\,dS = \int_{-b}^{b}\int_0^{2\pi}(a^2\cos^2 t + s^2)\delta a\,dt\,ds \\ &= \delta a\int_{-b}^{b}\left(\frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + s^2t\right)\Big|_{t=0}^{2\pi}ds = \pi\delta a\left(2a^2b + \frac{4}{3}b^3\right) \quad \text{as before.} \end{aligned}$$

(b) From Exercise 18, $M = 4\pi ab\delta$, so

$$r_x = r_y = \sqrt{\frac{\pi\delta a\left(2a^2b + \frac{4}{3}b^3\right)}{4\pi ab\delta}} = \sqrt{\frac{2a^2 + \frac{4}{3}b^2}{4}} = \sqrt{\frac{a^2 + \frac{2}{3}b^2}{2}}$$
$$= \sqrt{\frac{3a^2 + 2b^2}{6}}.$$

20. (a) Let M be the maximum value of f on D and m the minimum value. (The numbers M and m must exist since D is compact.) Then

$$m = \frac{\iint_D mg\,dA}{\iint_D g\,dA} \le \frac{\iint_D fg\,dA}{\iint_D g\,dA} \le \frac{\iint_D Mg\,dA}{\iint_D g\,dA} = M.$$

Hence by the intermediate value theorem, there must be some point P in D such that

$$f(P) = \frac{\iint_D fg\,dA}{\iint_D g\,dA},$$

which gives the result, provided $\iint_D g\,dA \ne 0$.
If $\iint_D g\,dA = 0$ then we have

$$0 = m\iint_D g\,dA = \iint_D mg\,dA \le \iint_D fg\,dA \le \iint_D Mg\,dA = M\iint_D g\,dA = 0,$$

so $\iint_D fg\,dA = 0$ and any P in D gives the desired result.

(b) Assume that S may be parametrized by a single function $\mathbf{X}$. Then

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}(\mathbf{X}(s,t))\cdot\mathbf{n}(s,t)\|\mathbf{N}(s,t)\|\,ds\,dt$$
$$= \mathbf{F}(\mathbf{X}(s_0,t_0))\cdot\mathbf{n}(s_0,t_0)\iint_D \|\mathbf{N}(s,t)\|\,ds\,dt \quad \text{by part (a),}$$
$$= \mathbf{F}(P)\cdot\mathbf{n}(P)(\text{area of } S)$$

where $P = \mathbf{X}(s_0,t_0)$.

21. (a) Let $\mathbf{a} = (a_1, a_2, a_3)$ and assume $\mathbf{x}(t) = (x(t), y(t), z(t))$ parametrizes C. Then

$$\oint_C \mathbf{a}\cdot d\mathbf{s} = \int_a^b \mathbf{a}\cdot\mathbf{x}'(t)\,dt$$
$$= \int_a^b (a_1x'(t) + a_2y'(t) + a_3z'(t))\,dt$$
$$= (a_1x(t) + a_2y(t) + a_3z(t))\Big|_a^b$$
$$= \mathbf{a}\cdot\mathbf{x}(t)\Big|_a^b = \mathbf{a}\cdot\mathbf{x}(b) - \mathbf{a}\cdot\mathbf{x}(a)$$
$$= 0$$

since $\mathbf{x}(a) = \mathbf{x}(b)$ because C is a closed curve.

(b) Let S be any smooth, orientable surface with boundary curve C. If we orient S appropriately and use Stokes's theorem, we have

$$\oint_C \mathbf{a}\cdot d\mathbf{s} = \iint_S \nabla\times\mathbf{a}\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0.$$

22. Note that C lies in the surface $z = x^2 - y^2$. The line integral is

$$\oint_C \mathbf{F}\cdot d\mathbf{s}, \text{ where } \mathbf{F} = (x^2 + z^2)\,\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Therefore, Stokes's theorem implies that

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S},$$

where S is the portion of $z = x^2 - y^2$ bounded by C. Note that S lies over the unit disk in the xy-plane. We may take for unit normal

$$\mathbf{n} = \frac{-2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{4x^2+4y^2+1}} \quad \text{and}$$

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+z^2 & y & z \end{vmatrix} = 2z\mathbf{j} = 2(x^2-y^2)\,\mathbf{j} \text{ on } S.$$

Thus $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_D 4y(x^2-y^2)\,dA$ where D is the unit disk. This is

$$\begin{aligned}&\int_0^{2\pi}\int_0^1 4r^4(\cos^2\theta\sin\theta - \sin^3\theta)\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{4}{5}(\cos^2\theta\sin\theta - \sin^3\theta)\,d\theta = \frac{4}{5}\int_0^{2\pi}(\cos^2\theta\sin\theta - (1-\cos^2\theta)\sin\theta)\,d\theta \\ &= \frac{4}{5}\left(-\frac{2}{3}\cos^3\theta + \cos\theta\right)\Big|_0^{2\pi} = 0.\end{aligned}$$

23. By Stokes's theorem

$$\begin{aligned}\oint_{\partial S}(f\nabla g)\cdot d\mathbf{s} &= \iint_S \nabla\times(f\nabla g)\cdot d\mathbf{S} \\ &= \iint_S (\nabla f\times\nabla g + f\nabla\times(\nabla g))\cdot d\mathbf{S} = \iint_S (\nabla f\times\nabla g)\cdot d\mathbf{S},\end{aligned}$$

since $\nabla\times(\nabla g) = \mathbf{0}$ (see §3.4).

24. Using the result of Exercise 23 (twice):

$$\oint_{\partial S}(f\nabla g + g\nabla f)\cdot d\mathbf{s} = \iint_S (\nabla f\times\nabla g + \nabla g\times\nabla f)\cdot d\mathbf{S} = \iint_S \mathbf{0}\cdot d\mathbf{S} = 0$$

because $\nabla f\times\nabla g = -\nabla g\times\nabla f$.

25.

$$\begin{aligned}\oint_{\partial S}(f\nabla f)\cdot d\mathbf{s} &= \oint_{\partial S}\frac{1}{2}(f\nabla f + f\nabla f)\cdot d\mathbf{s} \\ &= 0 \quad \text{by Exercise 24.}\end{aligned}$$

26. **(a)** First apply Stokes's theorem:

$$\begin{aligned}&\frac{1}{2}\oint_C (bz-cy)\,dx + (cx-az)\,dy + (ay-bx)\,dz \\ &= \frac{1}{2}\iint_D \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ bz-cy & cx-az & ay-bx \end{vmatrix}\cdot d\mathbf{S} \quad (D \text{ is the region enclosed by } C) \\ &= \frac{1}{2}\iint_D (2a,2b,2c)\cdot d\mathbf{S} = \iint_D (a,b,c)\cdot d\mathbf{S} \\ &= \iint_D \mathbf{n}\cdot\mathbf{n}\,dS = \iint_D dS \quad \text{since } \mathbf{n} \text{ is a unit vector,} \\ &= \text{area enclosed by } C.\end{aligned}$$

(b) If C is contained in the *xy*-plane, then $\mathbf{n} = \mathbf{k}$, so $a = b = 0$ and $c = 1$ in the notation above. Hence the result reduces to

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \text{area enclosed by } C.$$

27. By Faraday's law

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = -\frac{d}{dt}\iint_S \mathbf{B} \cdot d\mathbf{S}.$$

On the other hand, using Stokes's theorem,

$$\iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{E} \cdot d\mathbf{s} = \int_{\partial S} (\mathbf{E} \cdot \mathbf{T})\,ds = 0,$$

since $\mathbf{E}$ is everywhere perpendicular to ∂S. Thus $\dfrac{d}{dt}\displaystyle\iint_S \mathbf{B} \cdot d\mathbf{S} = 0$, so the magnetic flux does not vary with time.

28. For Gauss's theorem to apply to the situation, S must be closed. Hence ∂S is empty. But then there really is no line integral $\displaystyle\int_{\partial S} \mathbf{G} \cdot d\mathbf{s}$. If we try to apply Stokes's theorem in general (i.e., to surfaces with nonempty boundary) then we cannot also apply Gauss's theorem.

29. Note that the boundary ∂W of W consists of three parts: S, $\tilde{S}_a$ and the lateral surfaces L of ∂W. With ∂W oriented by outward normal, and if we take S and $\tilde{S}_a$ to be oriented in the same way,

$$\oiint_{\partial W} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm\left(\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} - \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}\right) + \iint_L \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$$

(The $\pm$ sign depends on how S, $\tilde{S}_a$ are oriented with respect to the orientation of ∂W.) Now L consists of a collection of segments of the rays defining $\Omega(S, O)$. Thus L is *tangent* to $\mathbf{x}$. Hence $\mathbf{x} \cdot \mathbf{n} = 0$ where $\mathbf{n}$ is the appropriate unit normal to L. Thus $\displaystyle\iint_L \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = 0$. Thus

$$\oiint_{\partial W} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm\left(\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} - \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}\right).$$

Gauss's theorem implies

$$\oiint_{\partial W} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iiint_W \left(\nabla \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|^3}\right) dV = \iiint_W 0\,dV.$$

Hence $\displaystyle\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$. On $\tilde{S}_a$, $\mathbf{n} = \dfrac{\mathbf{x}}{\|\mathbf{x}\|}$, so

$$\begin{aligned}\Omega(S, O) &= \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iint_{\tilde{S}_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|}\,dS \\ &= \iint_{\tilde{S}_a} \frac{\|\mathbf{x}\|^2}{\|\mathbf{x}\|^4}\,dS = \iint_{\tilde{S}_a} \frac{1}{\|\mathbf{x}\|^2}\,dS.\end{aligned}$$

But on $\tilde{S}_a$, $\|\mathbf{x}\| = a$, so

$$\Omega(S, O) = \iint_{\tilde{S}_a} \frac{1}{a^2}\,dS = \frac{1}{a^2}\,(\text{surface area of } \tilde{S}_a).$$

30. From the definition of $\Omega(S, O)$, we calculate $\Omega(S, O) = \displaystyle\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$. Now $\mathbf{x} = (x(s,t), y(s,t), z(s,t))$, so that $\|\mathbf{x}\| = \sqrt{x^2 + y^2 + z^2}$. Moreover, the standard normal $\mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t$ is

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial y}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial y}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix} \mathbf{i} - \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial z}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial z}{\partial t} \end{vmatrix} \mathbf{j} + \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix} \mathbf{k}.$$

Hence

$$\iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \iint_D \frac{1}{(x^2+y^2+z^2)^{3/2}} \mathbf{x} \cdot (\mathbf{T}_s \times \mathbf{T}_t)\, ds\, dt$$

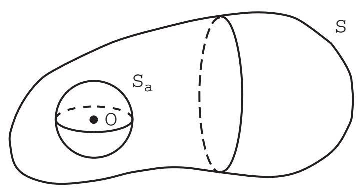
$$= \iint_D \frac{1}{(x^2+y^2+z^2)^{3/2}} \begin{vmatrix} x & y & z \\ \partial x/\partial s & \partial y/\partial s & \partial z/\partial s \\ \partial x/\partial t & \partial y/\partial t & \partial z/\partial t \end{vmatrix} ds\, dt \quad \text{as desired.}$$

31. First, if S does not enclose the origin then, by Gauss's theorem

$$\Omega(S,O) = \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm \iiint_W \nabla \cdot \left(\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \right) dV = \iiint_W 0\, dV = 0.$$

Here the $\pm$ sign depends on the orientation of S and W is the region enclosed by S.

Next, if S does enclose the origin, let S_a be the sphere of radius a centered at O and contained inside S. Let D be the solid region in $\mathbf{R}^3$ between S_a and S.



Note that $\nabla \cdot \left(\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \right)$ throughout D since D doesn't contain O. If S_a is oriented by *inward* normal (which points away from D), then, by Gauss's theorem, we have:

$$0 = \iiint_D \nabla \cdot \left(\frac{\mathbf{x}}{\|\mathbf{x}\|^3} \right) dV = \oiint_{\partial D} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} = \pm \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S} + \iint_{S_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}.$$

Hence $\Omega(S,O) = \pm \iint_{S_a} \frac{\mathbf{x}}{\|\mathbf{x}\|^3} \cdot d\mathbf{S}$. On S_a, $\mathbf{n} = -\frac{\mathbf{x}}{\|\mathbf{x}\|} = -\frac{1}{a}$ so

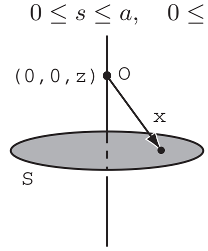
$$\Omega(S,O) = \pm \iint_{S_a} \frac{\mathbf{x}}{a^3} \cdot \left(-\frac{1}{a}\mathbf{x} \right) dS = \pm \iint_{S_a} -\frac{1}{a^4} (\mathbf{x} \cdot \mathbf{x})\, dS$$

$$= \pm \iint_{S_a} -\frac{a^2}{a^4}\, dS = \pm \frac{1}{a^2} \text{ (surface area of } S_a)$$

$$= \pm \frac{1}{a^2}(4\pi a^2) = \pm 4\pi.$$

32. We may parametrize S as

$$\begin{cases} x = s\cos t \\ y = s\sin t \\ z = 0 \end{cases} \quad 0 \le s \le a, \quad 0 \le t < 2\pi\ .$$



Then one way to orient S is with unit normal $\mathbf{n} = \mathbf{k}$. Also, we have the vector $\mathbf{x}$ from O to a point of S given by

$$\mathbf{x} = (s\cos t, s\sin t, -z) \Rightarrow \|\mathbf{x}\| = \sqrt{s^2 + z^2}.$$

Hence

$$\begin{aligned}\Omega(S,O) &= \iint_S \frac{\mathbf{x}}{\|\mathbf{x}\|^3}\cdot \mathbf{n}\,dS = \int_0^{2\pi}\int_0^a \frac{-z}{(s^2+z^2)^{3/2}}\,s\,ds\,dt\\ &= -z\int_0^a\int_0^{2\pi}\frac{s}{(s^2+z^2)^{3/2}}\,dt\,ds\\ &= -\pi z\int_0^a \frac{2s}{(s^2+z^2)^{3/2}}\,ds = -\pi z(s^2+z^2)^{-1/2}(-2)\Big|_0^a\\ &= 2\pi z\left((a^2+z^2)^{-1/2}-\frac{1}{|z|}\right) = 2\pi z\left(\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{|z|}\right)\\ &= 2\pi z\left(\frac{1}{\sqrt{a^2+z^2}}-\frac{1}{\sqrt{z^2}}\right) = 2\pi z\left(\frac{\sqrt{z^2}-\sqrt{a^2+z^2}}{\sqrt{z^2}\sqrt{a^2+z^2}}\right).\end{aligned}$$

Now

$$\frac{z}{\sqrt{z^2}} = \begin{cases} +1 & \text{if } z>0\\ -1 & \text{if } z<0\end{cases}$$

and

$$\sqrt{z^2} = \begin{cases} z & \text{if } z\ge 0\\ -z & \text{if } z<0\end{cases}.$$

Thus

$$\Omega(S,O) = \begin{cases} 2\pi\left(\dfrac{z-\sqrt{a^2+z^2}}{\sqrt{a^2+z^2}}\right) & \text{if } z>0\\[2ex] 2\pi\left(\dfrac{z+\sqrt{a^2+z^2}}{\sqrt{a^2+z^2}}\right) & \text{if } z<0.\end{cases}$$

($z\neq 0$ because O should not be a point of S.)

Note that if $z>0, z-\sqrt{a^2+z^2}<0$ and $|z-\sqrt{a^2+z^2}|<\sqrt{a^2+z^2}$. Hence $0>\Omega(S,O)>-2\pi$. If $z<0$, then $z+\sqrt{a^2+z^2}>0$ and $z+\sqrt{a^2+z^2}<\sqrt{a^2+z^2}$. Hence $0<\Omega(S,O)<2\pi$. Either way $-2\pi<\Omega(S,O)<2\pi$. Now as $z\to 0^+, \Omega(S,O)\to -2\pi$ and as $z\to 0^-, \Omega(S,O)\to 2\pi$. Hence as O passes through S, there is a jump of 4π.

33. We have

$$\begin{aligned}\nabla\times\mathbf{G} &= \nabla\times\int_0^1 t\mathbf{F}(t\mathbf{r})\times\mathbf{r}\,dt \quad \text{where } \mathbf{r}=(x,y,z),\\ &= \int_0^1 \nabla\times(t\mathbf{F}(t\mathbf{r})\times\mathbf{r})\,dt\\ &= \int_0^1 t\nabla\times(\mathbf{F}(t\mathbf{r})\times\mathbf{r})\,dt \quad \text{since } t \text{ behaves as a constant with respect to } \nabla,\\ &= \int_0^1 t\{\mathbf{F}(t\mathbf{r})\nabla\cdot\mathbf{r}-\mathbf{r}\nabla\cdot\mathbf{F}(t\mathbf{r})+(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r})-(\mathbf{F}(t\mathbf{r})\cdot\nabla)\mathbf{r}\}\,dt \quad \text{by the first identity,}\\ &= \int_0^1 t\{3\mathbf{F}(t\mathbf{r})-\mathbf{r}\nabla\cdot\mathbf{F}(t\mathbf{r})+(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r})-\mathbf{F}(t\mathbf{r})\}\,dt\\ &= \int_0^1 t\{2\mathbf{F}(t\mathbf{r})-\mathbf{r}\nabla\cdot\mathbf{F}(t\mathbf{r})+(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r})\}\,dt.\end{aligned}$$

To compute $\nabla\cdot\mathbf{F}(t\mathbf{r})$, note that $\dfrac{\partial}{\partial x}\mathbf{F}(t\mathbf{r}) = t\dfrac{\partial\mathbf{F}}{\partial\mathbf{X}}$ by the hint. This implies that $\nabla\cdot\mathbf{F}(t\mathbf{r}) = t\nabla_{\mathbf{X,Y,Z}}\mathbf{F}(\mathbf{X,Y,Z})$ where $\nabla_{\mathbf{X,Y,Z}}$ signifies that all partials are to be taken with respect to $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ where $\mathbf{X}=tx, \mathbf{Y}=ty$, and $\mathbf{Z}=tz$. Thus $\nabla\cdot\mathbf{F}(t\mathbf{r})=0$ since $\mathbf{F}$ is assumed to be divergenceless. By the second identity given in the hint,

$$(\mathbf{r}\cdot\nabla)\mathbf{F}(t\mathbf{r}) = \frac{d}{dt}[t\mathbf{F}(t\mathbf{r})]-\mathbf{F}(t\mathbf{r}).$$

Hence,

$$\begin{aligned}
\nabla \times \mathbf{G} &= \int_0^1 t\left\{2\mathbf{F}(t\mathbf{r}) + \frac{d}{dt}[t\mathbf{F}(t\mathbf{r})] - \mathbf{F}(t\mathbf{r})\right\} dt \\
&= \int_0^1 t\left\{\mathbf{F}(t\mathbf{r}) + \frac{d}{dt}[t\mathbf{F}(t\mathbf{r})]\right\} dt \\
&= \int_0^1 \frac{d}{dt}[t^2\mathbf{F}(t\mathbf{r})]\, dt \quad \text{by the last identity in the hint,} \\
&= t^2\mathbf{F}(t\mathbf{r})\Big|_{t=0}^{1} = \mathbf{F}(\mathbf{r}).
\end{aligned}$$

34. Note $\nabla \cdot \mathbf{F} = 2 - 1 - 1 = 0$ so, by the result of Exercise 33, a vector potential for $\mathbf{F}$ must exist. We can compute it by

$$\begin{aligned}
\mathbf{G} &= \int_0^1 t(2xt, -yt, -zt) \times (x, y, z)\, dt = \int_0^1 t(0, -3xzt, 3xyt)\, dt \\
&= \int_0^1 (0, -3xzt^2, 3xyt^2)\, dt = (0, -xzt^3, xyt^3)\Big|_{t=0}^{1} \\
&= (0, -xz, xy).
\end{aligned}$$

35. $\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3 \neq 0$, so, by the result of Exercise 33, $\mathbf{F}$ has no vector potential.

36. $\nabla \cdot \mathbf{F} = 0 + 0 + 0 = 0$, so, by the result of Exercise 33, a vector potential for $\mathbf{F}$ must exist. We compute it as follows.

$$\begin{aligned}
\mathbf{G} &= \int_0^1 t(3yt, 2xzt^2, -7x^2yt^3) \times (x, y, z)\, dt \\
&= \int_0^1 (2xz^2t^3 + 7x^2y^2t^4, -7x^3yt^4 - 3yzt^2, 3y^2t^2 - 2x^2zt^3)\, dt \\
&= \left(\frac{1}{2}xz^2 + \frac{7}{5}x^2y^2, -\frac{7}{5}x^3y - yz, y^2 - \frac{1}{2}x^2z\right).
\end{aligned}$$

37. Since $\nabla \times (\nabla\phi) = \mathbf{0}$ for any C^2 function, we have

$$\nabla \times (\mathbf{G} + \nabla\phi) = \nabla \times \mathbf{G} + \nabla \times (\nabla\phi) = \nabla \times \mathbf{G} + \mathbf{0} = \mathbf{F}.$$

Thus $\mathbf{G} + \nabla\phi$ is a vector potential for $\mathbf{F}$.

38. (a) Write $\mathbf{F} = -\dfrac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$. Then

$$\begin{aligned}
\frac{\partial F_1}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&= \frac{(x^2+y^2+z^2)^2 - 3x^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial F_2}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(x^2+y^2+z^2)^2 - 3y^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}} \\
\frac{\partial F_3}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right] = \frac{(x^2+y^2+z^2)^2 - 3z^2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}}.
\end{aligned}$$

Thus

$$\nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \frac{3(x^2+y^2+z^2)^2 - (3x^2+3y^2+3z^2)(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{7/2}} = 0.$$

(b) Let S be a sphere of radius a enclosing the origin. Consider S to be the union of hemispheres S_1 and S_2, each oriented so that the normal vector points away from the center of the sphere. If $\mathbf{F} = \nabla \times \mathbf{G}$, then

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \nabla \times \mathbf{G} \cdot d\mathbf{S} = \iint_{S_1} \nabla \times \mathbf{G} \cdot d\mathbf{S} + \iint_{S_2} \nabla \times \mathbf{G} \cdot d\mathbf{S} \\ &= \oint_{\partial S_1} \mathbf{G} \cdot d\mathbf{s} + \oint_{\partial S_2} \mathbf{G} \cdot d\mathbf{s} \quad \text{by Stokes's theorem} \\ &= 0,\end{aligned}$$

since ∂S_1 and ∂S_2 inherit opposite orientations from S_1 and S_2 and are equal as unoriented curves. On the other hand $\mathbf{n} = \mathbf{r}/\|\mathbf{r}\|$, so

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S -\frac{GMm}{\|\mathbf{r}\|^3}\mathbf{r} \cdot \frac{\mathbf{r}}{\|\mathbf{r}\|}\, dS = -GMm \iint_S \frac{\|\mathbf{r}\|^2}{\|\mathbf{r}\|^4}\, dS \\ &= -GMm \iint_S \frac{1}{\|\mathbf{r}\|^2}\, dS = -GMm \iint_S \frac{1}{a^2}\, dS \quad \text{since } \|\mathbf{r}\| = a \text{ on } S, \\ &= -GMm\frac{4\pi a^2}{a^2} = -4\pi GMm \neq 0.\end{aligned}$$

Hence, it cannot be that $\mathbf{F} = \nabla \times \mathbf{G}$.

(c) $\mathbf{F}$ is not of class C^1 on $\mathbf{R}^3$; $\mathbf{F}$ is undefined at the origin. The C^1 hypothesis is assumed in Exercise 33, so there's no contradiction.

39. We calculate the curl:

$$\begin{aligned}\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) &= \nabla \times \mathbf{E} + \nabla \times \frac{\partial \mathbf{A}}{\partial t} \\ &= \nabla \times \mathbf{E} + \frac{\partial}{\partial t}\nabla \times \mathbf{A} \\ &= \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}\end{aligned}$$

by Faraday's law. Since **E, B** and thus **A** are all defined on a simply-connected region, we must have that $\mathbf{E} + \partial\mathbf{A}/\partial t$ is conservative.

40. Substituting $\nabla \times \mathbf{A}$ for $\mathbf{B}$ in Ampère's law, we have

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

From the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, we have

$$\mu_0\mathbf{J} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}.$$

Since $\mathbf{E} + \partial\mathbf{A}/\partial t$ is conservative, $\mathbf{E} = \nabla f - \dfrac{\partial \mathbf{A}}{\partial t}$, so that

$$\begin{aligned}\mu_0\mathbf{J} &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\nabla f - \frac{\partial \mathbf{A}}{\partial t}\right) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - \mu_0\epsilon_0\left(\nabla\left(\frac{\partial f}{\partial t}\right) - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \quad \text{since } f \text{ is of class } C^2.\end{aligned}$$

Thus

$$\mu_0\mathbf{J} = \nabla\left(\nabla \cdot \mathbf{A} - \mu_0\epsilon_0\frac{\partial f}{\partial t}\right) - \nabla^2\mathbf{A} + \mu_0\epsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2},$$

which is equivalent to the desired formula.

41. Again we have $\mathbf{E} = \nabla f - \dfrac{\partial \mathbf{A}}{\partial t}$ so that Gauss's law becomes $\rho/\epsilon_0 = \nabla \cdot \mathbf{E} = \nabla \cdot \left(\nabla f - \dfrac{\partial \mathbf{A}}{\partial t}\right) = \nabla^2 f - \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$ or $\nabla^2 f = \rho/\epsilon_0 + \dfrac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$.

42. **(a)** If $\tilde{\mathbf{A}} = \mathbf{A} + \nabla\phi$, then in order to have

$$\begin{aligned}\nabla \tilde{f} &= \mathbf{E} + \frac{\partial \tilde{\mathbf{A}}}{\partial t} \\ &= \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} + \nabla \frac{\partial \phi}{\partial t} \\ &= \nabla f + \nabla \frac{\partial \phi}{\partial t},\end{aligned}$$

we must have $\nabla \tilde{f} = \nabla \left(f + \dfrac{\partial \phi}{\partial t}\right)$. Thus, up to addition of a constant, $\tilde{f} = f + \dfrac{\partial \phi}{\partial t}$.

(b) The condition that $\nabla \cdot \tilde{\mathbf{A}} = \mu_0\epsilon_0 \dfrac{\partial \tilde{f}}{\partial t}$ is equivalent to

$$\begin{aligned}&\nabla \cdot (\mathbf{A} + \nabla\phi) = \mu_0\epsilon_0 \frac{\partial}{\partial t}\left(f + \frac{\partial \phi}{\partial t}\right) \qquad \text{or} \\ &\nabla \cdot \mathbf{A} + \nabla^2\phi = \mu_0\epsilon_0 \left(\frac{\partial f}{\partial t} + \frac{\partial^2 \phi}{\partial t^2}\right) \quad \Leftrightarrow \\ &\nabla^2\phi - \mu_0\epsilon_0 \frac{\partial^2 \phi}{\partial t^2} = -\nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial f}{\partial t}.\end{aligned}$$

43. If the final equation in part (b) above can be solved for ϕ, then we may arrange things so that $\nabla \cdot \mathbf{A} = \mu_0\epsilon_0 \dfrac{\partial f}{\partial t}$. Then the equation in Exercise 40 is

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \nabla \overbrace{\left(\nabla \cdot \mathbf{A} - \mu_0\epsilon_0 \frac{\partial f}{\partial t}\right)}^{0} = -\mu_0 \mathbf{J}$$

and the equation in Exercise 41 is

$$\nabla^2 f = \frac{\rho}{\epsilon_0} + \mu_0\epsilon_0 \frac{\partial^2 f}{\partial t} \quad \text{or} \quad \nabla^2 f - \mu_0\epsilon_0 \frac{\partial^2 f}{\partial t^2} = \frac{\rho}{\epsilon_0}.$$

44. We check all the equations, given the assumptions.

$$\begin{aligned}\nabla \cdot \mathbf{E} = \nabla \cdot \underbrace{\left(-\frac{\partial \mathbf{A}}{\partial t} + \nabla f\right)}_{\mathbf{E}} &= -\frac{\partial}{\partial t} \nabla \cdot \mathbf{A} + \nabla^2 f = -\frac{\partial}{\partial t}\left(\mu_0\epsilon_0 \frac{\partial f}{\partial t}\right) + \nabla^2 f \\ &= -\mu_0\epsilon_0 \frac{\partial^2 f}{\partial t^2} + \nabla^2 f = \frac{\rho}{\epsilon_0}\end{aligned}$$

from the second equation in Exercise 43.

$$\begin{aligned}\nabla \times \mathbf{E} &= -\nabla \times \left(\frac{\partial \mathbf{A}}{\partial t} - \nabla f\right) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= \nabla \cdot (\nabla \times \mathbf{A}) = 0 \\ \nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad \text{(identity)} \\ &= \nabla(\nabla \cdot \mathbf{A}) - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} + \mu_0 \mathbf{J}\end{aligned}$$

by the equation in part (b) of Exercise 42

$$= \nabla\left(\mu_0\epsilon_0\frac{\partial f}{\partial t}\right) - \mu_0\epsilon_0\frac{\partial}{\partial t}(-\mathbf{E} + \nabla f) + \mu_0\mathbf{J},$$

using the condition $\nabla\cdot\mathbf{A} = \mu_0\epsilon_0\dfrac{\partial f}{\partial t}$, and that $\dfrac{\partial\mathbf{A}}{\partial t} = \nabla f - \mathbf{E}$

$$= \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mu_0\mathbf{J}$$

since we may assume f to be of class C^2.