Chapter 6

Line Integrals

6.1 Scalar and Vector Line Integrals

1. (a) $\mathbf{x}'(t) = (-3, 4)$ and $\|\mathbf{x}'(t)\| = 5$ so by Definition 1.1,

$$\int_{\mathbf{x}} f\, ds = \int_0^2 (x+2y)(5)\, dt = 5\int_0^2 [(2-3t)+(8t-2)]\, dt = 5\int_0^2 5t\, dt = \frac{25}{2}t^2\Big|_0^2 = 50.$$

(b) $\mathbf{x}'(t) = (-\sin t, \cos t)$ and $\|\mathbf{x}'(t)\| = 1$ so by Definition 1.1,

$$\int_{\mathbf{x}} f\, ds = \int_0^{\pi} (x+2y)(1)\, dt = \int_0^{\pi} [\cos t + 2\sin t]\, dt = [\sin t - 2\cos t]|_0^{\pi} = 4.$$

For Exercises 2–7 we will use Definition 1.1. For each calculate $\mathbf{x}'$, $\|\mathbf{x}'\|$, *and* $f(\mathbf{x})$.

2.

$$\int_{\mathbf{x}} f\, ds = \int_0^2 [(t)(2t)(3t)\sqrt{1^2+2^2+3^2}]\, dt = 6\sqrt{14}\int_0^2 t^3 dt = \frac{6\sqrt{14}}{4}t^4\Big|_0^2 = 24\sqrt{14}.$$

3.

$$\int_{\mathbf{x}} f\, ds = \int_1^3 \left[\frac{t+t^{3/2}}{t+t^{3/2}}\sqrt{1+1+\frac{9}{4}t}\right] dt = \int_1^3 \sqrt{2+\frac{9}{4}t}\, dt = \frac{8}{27}\left(2+\frac{9}{4}t\right)^{3/2}\Bigg|_1^3$$
$$= (35\sqrt{35} - 17\sqrt{17})/27.$$

4.

$$\int_{\mathbf{x}} f\, ds = \sqrt{16+9}\int_0^{2\pi} (3\cos 4t + \cos 4t \sin 4t + 27t^3)\, dt = 5\int_0^{2\pi}\left(3\cos 4t + \frac{1}{2}\sin 8t + 27t^3\right) dt$$
$$= 5\left(\frac{3}{4}\sin 4t - \frac{1}{16}\cos 8t + \frac{27}{4}t^4\right)\Bigg|_0^{2\pi} = 540\pi^4.$$

5.

$$\int_{\mathbf{x}} f\, ds = \int_0^5 \frac{e^{2t}}{e^{4t}}\sqrt{17}e^{2t}\, dt = \int_0^5 \sqrt{17}\, dt = 5\sqrt{17}.$$

6.

$$\int_{\mathbf{x}} f\, ds = \int_0^1 2t \cdot 2\, dt + \int_1^2 (3t-1)\cdot 3\, dt + \int_2^3 (2t+1)\cdot 2\, dt$$
$$= 2t^2\Big|_0^1 + \left(\frac{9}{2}t^2 - 3t\right)\Bigg|_1^2 + (2t^2+2t)\Big|_2^3 = 2 + \frac{21}{2} + 12 = \frac{49}{2}.$$

7.

$$\int_{\mathbf{x}} f\, ds = \int_0^1 [(2t-t)\sqrt{1+4t^2}]\, dt + \int_1^3 (2-1+2t^2-4t+2)\, dt$$
$$= (5^{3/2}-1)/12 + 22/3 = (5^{3/2}+87)/12.$$

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For Exercises 8–16 we will use Definition 1.2.

8.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t+1, t, 3t-1) \cdot (2,1,3)\, dt = \int_0^1 (14t-1)\, dt = (7t^2 - t)\Big|_0^1 = 6.$$

9.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} (2-\cos t, \sin t) \cdot (\cos t, \sin t)\, dt = \int_0^{\pi/2} (2\cos t - \cos^2 t + \sin^2 t)\, dt$$
$$= (2\sin t - (\sin 2t)/2)\Big|_0^{\pi/2} = 2.$$

10.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t+1, t+2) \cdot (2,1)\, dt = \int_0^1 (5t+4)\, dt = \left(\frac{5}{2}t^2 + 4t\right)\Big|_0^1 = \frac{13}{2}.$$

11.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^1 (t^3 - t^2, t^{11}) \cdot (2t, 3t^2)\, dt$$
$$= \int_{-1}^1 (2t^4 - 2t^3 + 3t^{13})\, dt = \left(\frac{2}{5}t^5 - \frac{1}{2}t^4 + \frac{3}{14}t^{14}\right)\Big|_{-1}^1 = \frac{4}{5}.$$

12.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} (3\cos t, 6\cos t \sin t, 30t\cos t \sin t) \cdot (-3\sin t, 2\cos t, 5)\, dt$$
$$= \int_0^{2\pi} (-9\cos t \sin t + 12\cos^2 t \sin t + 150t\cos t \sin t)\, dt$$
$$= \int_0^{2\pi} -9\cos t \sin t\, dt + \int_0^{2\pi} 12\cos^2 t \sin t\, dt + \int_0^{2\pi} 75t\sin 2t\, dt.$$

In the first two integrals, let $w = \cos t$; in the last integrate by parts. Thus

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \left(\frac{9}{2}\cos^2 t - 4\cos^3 t - \frac{75}{2}t\cos 2t + \frac{75}{4}\sin 2t\right)\Big|_0^{2\pi}$$
$$= 0 - 0 - 75\pi + 0 = -75\pi.$$

13.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (-3(t^2+t), 2t+1, 3e^{2t}) \cdot (2, 2t+1, e^t)\, dt$$
$$= \int_0^1 (-2t^2 - 2t + 1 + 3e^{3t})\, dt = \left(\tfrac{2}{3}t^3 - t^2 + t + e^{3t}\right)\Big|_0^1$$
$$= \frac{2}{3} - 1 + 1 + e^3 - 1 = \frac{3e^3 - 5}{3}.$$

14.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{-1}^1 (t, 3t^2, -2t^3) \cdot (1, 6t, 6t^2)\, dt = \int_{-1}^1 (t + 18t^3 - 12t^5)\, dt = 0.$$

15.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{4\pi} (t, \sin^2 t, 2t) \cdot (-\sin t, \cos t, 1/3)\, dt = \int_0^{4\pi} (-t\sin t + \sin^2 t\cos t + 2t/3)\, dt$$
$$= \left(t\cos t - \sin t + \frac{\sin^3 t}{3} + \frac{t^2}{3}\right)\Big|_0^{4\pi} = \frac{12\pi + 16\pi^2}{3}.$$

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16.

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (t^2 \cos t^3, t \sin t^3, t^3 \sin t^6) \cdot (1, 2t, 3t^2)\, dt = \int_0^1 (t^2 \cos t^3 + 2t^2 \sin t^3 + 3t^5 \sin t^6)\, dt$$
$$= \left(\frac{\sin t^3}{3} - \frac{2\cos t^3}{3} - \frac{\cos t^6}{2} \right) \Bigg|_0^1 = \frac{7 - 7\cos 1 + 2 \sin 1}{6}.$$

Assign at least one of Exercises 17 and 19 so that the students are exposed to the notation before they encounter Green's theorem in the next section.

17. $\int_{\mathbf{x}} x\, dy - y\, dx = \int_0^\pi [3(\cos 3t)^2 + 3(\sin 3t)^2]\, dt = 3 \int_0^\pi dt = 3\pi.$

18. $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^2 (2t, 1, 0) \cdot (1, 6t, 0)\, dt = \int_0^2 8t\, dt = 16.$

19. The good news is: there is a ton of cancellation.

$$\int_{\mathbf{x}} \frac{x\, dx + y\, dx}{(x^2+y^2)^{3/2}} = \int_0^{2\pi} \frac{e^{2t}\cos 3t(2e^{2t}\cos 3t - 3e^{2t}\sin 3t) + e^{2t}\sin 3t(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)}{(e^{4t}\cos^2 3t + e^{4t}\sin^2 3t)^{3/2}}\, dt$$
$$= \int_0^{2\pi} 2e^{-2t}\, dt = -e^{-2t}\Big|_0^{2\pi} = 1 - e^{-4\pi}.$$

20. Note that $\mathbf{x} = (t, 2\sqrt{t})$ and $\mathbf{x}' = (1, t^{-1/2})$, so

$$\int_C 3y\, ds = \int_1^9 6t^{1/2}\sqrt{1 + \frac{1}{t}}\, dt = 40\sqrt{10} - 8\sqrt{2}.$$

21. (a)

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t^2, t^2 - t) \cdot (1, 2t)\, dt = \int_0^1 2t^3\, dt = \frac{1}{2}.$$
$$\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{1/2} (2 - 8t + 8t^2, 4t^2 - 2t) \cdot (-2, 8t - 4)\, dt$$
$$= \int_0^{1/2} (32t^3 - 48t^2 + 24t - 4)\, dt = -\frac{1}{2}.$$

(b) The path $\mathbf{y}$ is an orientation-reversing reparametrization of $\mathbf{x}$.

22. We write the path as $\mathbf{x}(t) = (t+1, -4t+1, 2t+1)$, $0 \le t \le 1$. This means that $\mathbf{x}'(t) = (1, -4, 2)$, therefore

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 ((1+t)^2(1-4t) - 4(1+2t) + 2(2(1+t) - (1-4t)))\, dt$$
$$= \int_0^1 (-4t^3 - 7t^2 + 2t - 1)\Big|_0^1 = -\frac{10}{3}.$$

23. First we organize the information we need for each of the four paths (each is for $0 \le t \le 1$).

i	$\mathbf{x}_i$	$\mathbf{x}_i'(t)$	$\mathbf{F}(\mathbf{x}_i(t)) \cdot \mathbf{x}_i'(t)$
1	$(1-2t, 1, 3)$	$(-2, 0, 0)$	$-2(486 - 3(1-2t))$
2	$(-1, 1-2t, 3)$	$(0, -2, 0)$	$-2(-1)$
3	$(-1+2t, -1, 3)$	$(2, 0, 0)$	$2(486 - 3(1-2t))$
4	$(1, -1+2t, 3)$	$(0, 2, 0)$	$2(-1)$

So

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{s} = \sum_{i=1}^4 \int_{\mathbf{x}_i} \mathbf{F}(\mathbf{x}_i(t)) \cdot \mathbf{x}_i'(t)\, dt$$
$$= \int_0^1 (-2(486 - 3(1-2t)))\, dt + \int_0^1 (2)\, dt + \int_0^1 (2(486 - 3(1-2t)))\, dt + \int_0^1 (-2)\, dt$$
$$= 0.$$

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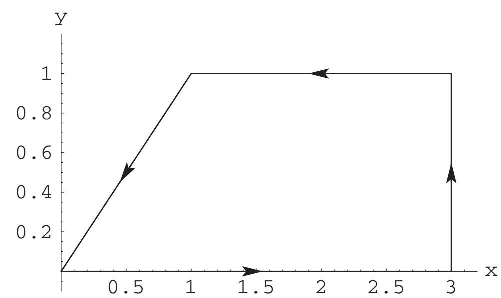
24. The path is $\mathbf{x}(t) = (2t+1, 4t+1)$, $0 \le t \le 1$. The integral is

$$\int_C (x^2 - y)\,dx + (x - y^2)\,dy = \int_0^1 [4t^2(2) + (-16t^2 - 6t)(4)]\,dt$$
$$= \int_0^1 (-56t^2 - 24t)\,dt = -\frac{92}{3}.$$

25. The curve C looks like



Then $\displaystyle\int_C x^2y\,dx - (x+y)\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4}$

- C_1 is the segment from (0, 0) to (3, 0), given as $\mathbf{x}(t) = (t, 0)$, $0 \le t \le 3 \Rightarrow \mathbf{x}'(t) = (1, 0)$.
  Then $\displaystyle\int_{C_1} = \int_0^3 0\,dt - t\cdot 0 = 0.$
- C_2 is the segment from (3, 0) to (3, 1), given by $\mathbf{x}(t) = (3, t)$, $0 \le t \le 1 \Rightarrow \mathbf{x}'(t) = (0, 1)$.
  Then $\displaystyle\int_{C_2} = \int_0^1 0 - (3+t)\,dt = \left(-3t - \frac{1}{2}t^2\right)\Big|_0^1 = -7/2.$
- C_3 is the segment from (3, 1) to (1, 1), given by $\mathbf{x}(t) = (3-t, 1)$, $0 \le t \le 2 \Rightarrow \mathbf{x}'(t) = (-1, 0)$.
  Then $\displaystyle\int_{C_3} = \int_0^2 (3-t)^2(-1)\,dt - (4-t)\cdot 0 = \frac{1}{3}(3-t)^3\Big|_0^2 = \frac{1}{3}(1-27) = -\frac{26}{3}.$
- C_4 is the segment from (1, 1) to (0, 0), given by $\mathbf{x}(t) = (1-t, 1-t)$, $0 \le t \le 1 \Rightarrow \mathbf{x}'(t) = (-1, -1)$.

$$\int_{C_4} = \int_0^1 [(1-t)^3(-1) + (2-2t)]\,dt = \left(\frac{1}{4}(1-t)^4 + 2t - t^2\right)\Big|_0^1$$
$$= 2 - 1 - \frac{1}{4} = \frac{3}{4}.$$

So $\displaystyle\int_C = -\frac{7}{2} - \frac{26}{3} + \frac{3}{4} = -\frac{137}{12}.$

26. Parametrize C as $\mathbf{x}(t) = (t^2, t^3)$, $-1 \le t \le 1$, so that $\mathbf{x}'(t) = (2t, 3t^2)$. Then

$$\int_C x^2y\,dx - xy\,dy = \int_{-1}^1 (t^7(2t) - t^5(3t^2))\,dt = \int_{-1}^1 (2t^8 - 3t^7)\,dt = \left(\frac{2}{9}t^9 - \frac{3}{8}t^8\right)\Big|_{-1}^1 = \frac{4}{9}.$$

27. Parametrize C as $\mathbf{x}(t) = (3-t, (3-t)^2)$, $0 \le t \le 3$, so that the parabola is oriented correctly. Then

$$\int_C y\,dx - x\,dy = \int_0^3 \left[(3-t)^2(-1) - (3-t)(-2(3-t))\right]\,dt$$
$$= \int_0^3 (3-t)^2\,dt = -\tfrac{1}{3}(3-t)^3\Big|_0^3 = 9.$$

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28. We parametrize C in two parts: $\mathbf{x}(t) = (t, -t)$ for $-2 \le t \le 0$ and $\mathbf{x}(t) = (t, t)$ for $0 \le t \le 1$. Therefore,

$$\int_C (x-y)^2\,dx + (x+y)^2\,dy = \int_{-2}^{0} ((2t)^2 + 0)\,dt + \int_0^1 (0 + (2t)^2)\,dt$$
$$= \int_{-2}^{1} 4t^2\,dt = \frac{4}{3}t^3\Big|_{-2}^{1} = 12.$$

29. In order to obtain the correct direction, we parametrize C as $\mathbf{x}(t) = (2\sin t, 2\cos t)$, $0 \le t \le \pi$. Then

$$\int_C xy^2\,dx - xy\,dy = \int_0^{\pi} \left[(8\sin t\cos^2 t)(2\cos t) - (4\sin t\cos t)(-2\sin t)\right]\,dt$$
$$= \int_0^{\pi} \left[16\cos^3 t\sin t + 8\sin^2 t\cos t\right]\,dt$$
$$= -4\cos^4 t + \frac{8}{3}\sin^3 t\Big|_0^{\pi} = -4 - (-4) = 0.$$

30. We parametrize the circle as $\mathbf{x}(t) = (4\cos t, 4\sin t)$, $0 \le t \le 2\pi$. Then the circulation is given by

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} (16\cos^2 t - 4\sin t, 16\cos t\sin t + 4\cos t)\cdot(-4\sin t, 4\cos t)\,dt$$
$$= \int_0^{2\pi} (-64\cos^2 t\sin t + 16\sin^2 t + 64\cos^2 t\sin t + 16\cos^2 t)\,dt$$
$$= \int_0^{2\pi} 16\,dt = 32\pi.$$

31. The path is $\mathbf{x}(t) = (4t+1, 2t+1, -t+2)$, $0 \le t \le 1$. The integral is

$$\int_C yz\,dx - xz\,dy + xy\,dz = \int_0^1 [4(-2t^2 + 3t + 2) - 2(-4t^2 + 7t + 2) - (8t^2 + 6t + 1)]\,dt$$
$$= \int_0^1 [-8t^2 - 8t + 3]\,dt = -\frac{11}{3}.$$

32. We must parametrize C. Along the cylinder we may take $x = 2\cos t$, $y = 2\sin t$. Then $z = x^2$ so we have $z = 4\cos^2 t$. The curve is traced once as t varies from 0 to 2π, so we have

$$\int_C z\,dx + x\,dy + y\,dz = \int_0^{2\pi} [(4\cos^2 t)(-2\sin t) + (2\cos t)(2\cos t) + (2\sin t)8\cos t(-\sin t)]\,dt$$
$$= \int_0^{2\pi} (8\cos^2 t(-\sin t) + 2(1+\cos 2t) - 16\sin^2 t\cos t)\,dt$$
$$= \left(\frac{8}{3}\cos^3 t + 2t + \sin 2t - \frac{16}{3}\sin^3 t\right)\Big|_0^{2\pi} = 4\pi.$$

33. Using formula (3) in §6.1, we have

$$\int_{\mathbf{x}} \mathbf{T}\cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{T}\cdot\mathbf{T})\,ds = \int_{\mathbf{x}} 1\,ds = \text{length of } \mathbf{x}.$$

34. Of course it's left to Becky Thatcher to figure out that the path is $\mathbf{x}(t) = (5\cos t, 5\sin t)$, $0 \le t \le \pi/2$, so the area of one side of the fence is

$$\int_C (10 - x - y)\,ds = \int_0^{\pi/2} 5(10 - 5\cos t - 5\sin t)\,dt = 25[\pi - 2] \approx 28.54 \text{ ft}^2.$$

35. (a) The force that Sisyphus is applying is $50\mathbf{x}'(t)/\|\mathbf{x}'(t)\|$. The path is given as $\mathbf{x}(t) = (5\cos 3t, 5\sin 3t, 10t)$ and so $\mathbf{x}'(t) = (-15\sin 3t, 15\cos 3t, 10)$ and $\|\mathbf{x}'(t)\| = \sqrt{325}$. The total work done is

$$\int_0^{10} \frac{50\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}\cdot\mathbf{x}'(t)\,dt = \int_0^{10} 50\|\mathbf{x}'(t)\|\,dt = \int_0^{10} 50\sqrt{325}\,dt = 2500\sqrt{13} \text{ ft-lb}.$$

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(b) This time the 75 pounds is applied straight down. The total work done is

$$\int_0^{10} (0,0,75)\cdot(-15\sin 3t, 15\cos 3t, 10)\,dt = \int_0^{10} 750\,dt = 7500 \text{ ft-lb}.$$

36. The force is applied in the direction of $(24, 32-14t)$. Force is applied in the opposite direction to the tension. The total work done is

$$\int_0^1 25\frac{(24, 32-14t)}{\sqrt{24^2+(32-14t)^2}}\cdot(0,-14)\,dt = -(7)(25)\int_0^1 \frac{32-14t}{\sqrt{400-224t+49t^2}}\,dt = -250 \text{ ft-lb}.$$

37. The path is $\mathbf{x}(t) = (t, f(t))$, $a \le t \le b$, and $\mathbf{x}'(t) = (1, f'(t))$. Since $\mathbf{F} = y\mathbf{i}$,

$$\int_C \mathbf{F}\cdot d\mathbf{s} = \int_a^b (f(t),0)\cdot(1,f'(t))\,dt = \int_a^b f(t)\,dt.$$

38. We take the sphere to be of radius c, so that $x^2+y^2+z^2=c^2$. Begin with the hint and take the derivative with respect to t of $[x(t)]^2+[y(t)]^2+[z(t)]^2=c^2$. Divide the result by 2 to obtain: $x(t)x'(t)+y(t)y'(t)+z(t)z'(t)=0$. Now we are ready to calculate the integral.

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} &= \int_a^b (x(t),y(t),z(t))\cdot(x'(t),y'(t),z'(t))\,dt \\ &= \int_a^b (x(t)x'(t)+y(t)y'(t)+z(t)z'(t))\,dt \\ &= \int_a^b 0\,dt = 0.\end{aligned}$$

39. If $\mathbf{x}$ is a parametrization of C, then formula (3) of §6.1 gives $\int_C \nabla f\cdot d\mathbf{s} = \int_{\mathbf{x}} (\nabla f\cdot\mathbf{T})\,ds$, where $\mathbf{T} = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$. But $\mathbf{T}$ is tangent to C and ∇f is perpendicular to level sets of f (including C), so $\nabla f\cdot\mathbf{T} = 0$, and thus the integral must be zero.

40. **(a)** We have $\dfrac{ds}{dt} = v(s)$, so $dt = \dfrac{ds}{v(s)}$. Hence the total time for the trip is $\int dt = \int \dfrac{ds}{v(s)} = 2\int_0^{20}\dfrac{ds}{2s+20}$ (where I've used symmetry) $= \ln(2s+20)\big|_0^{20} = \ln 60 - \ln 20 = \ln 3 \approx 1.0986$ hours or 65.92 min.

(b) On a semicircular path you can travel at a maximum constant speed of 60 mph. You must do so for 20π miles, so the trip will take $\dfrac{20\pi}{60} = \pi/3 \approx 1.047$ hrs or 62.83 min.

(c) Traveling through the center of Cleveland (as in part (a)) will take

$$\begin{aligned}2\int_0^{20}\frac{ds}{s^2/16+25} &= 2\int_0^{20}\frac{16\,ds}{s^2+20^2} = 32\int_0^{\pi/4}\frac{20\sec^2\theta\,d\theta}{20^2\sec^2\theta} \\ &= \frac{32}{20}\int_0^{\pi/4} d\theta = \frac{8}{5}\cdot\frac{\pi}{4} = \frac{2\pi}{5} \approx 1.2566 \text{ hrs} \quad \text{or} \quad 75.40 \text{ min}.\end{aligned}$$

Going around Cleveland will take $\dfrac{20\pi}{50} = \dfrac{2\pi}{5}$—same time!

41. **(a)** Newton's second law gives $m\mathbf{a} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Take the dot product with $\mathbf{v}$: $m\mathbf{a}\cdot\mathbf{v} = q(\mathbf{E}\cdot\mathbf{v}+(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}) = q\mathbf{E}\cdot\mathbf{v}$ since $\mathbf{v}\times\mathbf{B}\perp\mathbf{v}$.

(b)

$$\begin{aligned}\text{Work} &= \int_{\mathbf{x}} \mathbf{E}\cdot d\mathbf{s} = \int_a^b \mathbf{E}(\mathbf{x}(t))\cdot\mathbf{x}'(t)\,dt = \int_a^b \mathbf{E}(\mathbf{x}(t))\cdot\mathbf{v}(t)\,dt \\ &= \int_a^b m\mathbf{a}(t)\cdot\mathbf{v}(t)\,dt \text{ by part (a).}\end{aligned}$$

If the path has constant speed, then $\|\mathbf{v}(t)\|$ is constant. Hence $\mathbf{v}\cdot\mathbf{v}$ is constant so that $\dfrac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 0 \Leftrightarrow 2\mathbf{a}\cdot\mathbf{v} = 0 \Leftrightarrow \mathbf{a}\cdot\mathbf{v} = 0$. Therefore the integrand of the work integral is zero.

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42. (a) In this case all $\Delta x_k = \Delta x = \dfrac{1}{4}$, while $\Delta y_1 = \dfrac{1}{16}$, $\Delta y_2 = \dfrac{3}{16}$, $\Delta y_3 = \dfrac{5}{16}$, $\Delta y_4 = \dfrac{7}{16}$. Then

$$T_4 = \left[0^3 + 2\left(\frac{1}{16}\right)^3 + 2\left(\frac{1}{4}\right)^3 + 2\left(\frac{9}{16}\right)^3 + 1^3\right]\frac{1/4}{2}$$
$$+ \left(-0^2 - \left(\frac{1}{4}\right)^2\right)\frac{1/16}{2} + \left(-\left(\frac{1}{4}\right)^2 - \left(\frac{1}{2}\right)^2\right)\frac{3/16}{2} + \left(-\left(\frac{1}{2}\right)^2 - \left(\frac{3}{4}\right)^2\right)\frac{5/16}{2}$$
$$+ \left(-\left(\frac{3}{4}\right)^2 - 1^2\right)\frac{7/16}{2} = -\frac{2675}{8192} \approx -0.326538.$$

(b) With $y = x^2$ we have $dy = 2x\,dx$ so that

$$\int_C y^3\,dx - x^2\,dy = \int_0^1 (x^2)^3\,dx - x^2(2x\,dx) = \int_0^1 (x^6 - 2x^3)\,dx$$
$$= \left(\frac{1}{7}x^7 - \frac{1}{2}x^4\right)\Big|_0^1 = -5/14 = -0.357143.$$

43. (a) We have $\mathbf{x}_0 = (0,0,0)$, $\mathbf{x}_1 = \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right)$, $\mathbf{x}_2 = \left(\frac{1}{2}, 1, \frac{3}{2}\right)$, $\mathbf{x}_3 = \left(\frac{3}{4}, \frac{3}{2}, \frac{9}{4}\right)$, $\mathbf{x}_4 = (1,2,3)$. Then all $\Delta x_k = \dfrac{1}{4}$, $\Delta y_k = \dfrac{1}{2}$, $\Delta z_k = \dfrac{3}{4}$. Then

$$T_4 = \left(0 + 2\cdot\frac{3}{8} + 2\cdot\frac{3}{2} + 2\cdot\frac{27}{8} + 6\right)\frac{1/4}{2} + (0 + 2\cdot 1 + 2\cdot 2 + 2\cdot 3 + 4)\frac{1/2}{2}$$
$$+ \left(0 + 2\cdot\frac{1}{32} + 2\cdot\frac{1}{4} + 2\cdot\frac{27}{32} + 2\right)\frac{3/4}{2} = \frac{245}{32} = 7.65625.$$

(b) Parametrize C as $\begin{cases} x = t \\ y = 2t, \quad 0 \le t \le 1. \\ z = 3t \end{cases}$ Then

$$\int_C yz\,dx + (x+z)\,dy + x^2y\,dz = \int_0^1 (6t^2 + 4t\cdot 2 + 2t^3\cdot 3)\,dt$$
$$= \left(2t^3 + 4t^2 + \frac{3}{2}t^4\right)\Big|_0^1 = \frac{15}{2} = 7.5.$$

44. (a) We have: $\Delta x_1 = 1$, $\Delta x_2 = 0$, $\Delta x_3 = 1$, $\Delta x_4 = \Delta x_5 = \Delta x_6 = \Delta x_7 = 0$, $\Delta x_8 = -1$;
$\Delta y_1 = 3$, $\Delta y_2 = 1$, $\Delta y_3 = \Delta y_4 = 0$, $\Delta y_5 = -1$, $\Delta y_6 = 0$, $\Delta y_7 = -1$, $\Delta y_8 = 0$;
$\Delta z_1 = 0$, $\Delta z_2 = 1$, $\Delta z_3 = \Delta z_4 = 1$, $\Delta z_5 = 0$, $\Delta z_6 = \Delta z_7 = -1$, $\Delta z_8 = 0$. Then

$$T_8 = (0+0)\frac{\Delta x_1}{2} + (0+1)\frac{\Delta x_2}{2} + (1+2)\frac{\Delta x_3}{2} + (2+2)\frac{\Delta x_4}{2} + (2+2)\frac{\Delta x_5}{2}$$
$$+ (2+3)\frac{\Delta x_6}{2} + (3+4)\frac{\Delta x_7}{2} + (4+4)\frac{\Delta x_8}{2} + (0+1)\frac{\Delta y_1}{2}$$
$$+ (1+1)\frac{\Delta y_2}{2} + (1+1)\frac{\Delta y_3}{2} + (1+2)\frac{\Delta y_4}{2} + (2+3)\frac{\Delta y_5}{2}$$
$$+ (3+3)\frac{\Delta y_6}{2} + (3+3)\frac{\Delta y_7}{2} + (3+3)\frac{\Delta y_8}{2}$$
$$+ (1+2)\frac{\Delta z_1}{2} + (2+2)\frac{\Delta z_2}{2} + (2+2)\frac{\Delta z_3}{2} + (2+2)\frac{\Delta z_4}{2} + (2+3)\frac{\Delta z_5}{2}$$

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$$+ (3+3)\frac{\Delta z_6}{2} + (3+3)\frac{\Delta z_7}{2} + (3+4)\frac{\Delta z_8}{2}$$
$$= \frac{3}{2} + (-4) + \frac{3}{2} + 1 + \left(-\frac{5}{2}\right) + (-3) + 2 + 2 + 2 + (-3) + (-3) = -\frac{11}{2}.$$

(b) Now

$$\begin{array}{lll} \Delta x_1 = x_2 - x_0 = 1 & \Delta y_1 = 4 & \Delta z_1 = 1 \\ \Delta x_2 = x_4 - x_2 = 1 & \Delta y_2 = 0 & \Delta z_2 = 2 \\ \Delta x_3 = x_6 - x_4 = 0 & \Delta y_3 = -1 & \Delta z_3 = -1 \\ \Delta x_4 = x_8 - x_6 = -1 & \Delta y_4 = -1 & \Delta z_4 = -1 \end{array}$$

Then

$$\begin{aligned} T_4 &= (0+1)\frac{\Delta x_1}{2} + (1+2)\frac{\Delta x_2}{2} + (2+3)\frac{\Delta x_3}{2} + (3+4)\frac{\Delta x_4}{2} \\ &\quad + (0+1)\frac{\Delta y_1}{2} + (1+2)\frac{\Delta y_2}{2} + (2+3)\frac{\Delta y_3}{2} + (3+3)\frac{\Delta y_4}{2} \\ &\quad + (1+2)\frac{\Delta z_1}{2} + (2+2)\frac{\Delta z_2}{2} + (2+3)\frac{\Delta z_3}{2} + (3+4)\frac{\Delta z_4}{2} \\ &= \frac{1}{2} + \frac{3}{2} + \left(-\frac{7}{2}\right) + 2 + \left(-\frac{5}{2}\right) + (-3) + \frac{3}{2} + 4 + \left(-\frac{5}{2}\right) + \left(-\frac{7}{2}\right) \\ &= -\frac{11}{2}. \end{aligned}$$

6.2 Green's Theorem

1. $M(x,y) = -x^2y$ and $N(x,y) = xy^2$.

- For the line integral the path is $\mathbf{x}(t) = (2\cos t, 2\sin t)$, $0 \le t \le 2\pi$.

$$\begin{aligned} \oint_{\partial D} M\,dx + N\,dy &= \int_0^{2\pi} (-8\cos^2 t \sin t, 8\cos t \sin^2 t) \cdot (-2\sin t, 2\cos t)\,dt \\ &= 32\int_0^{2\pi} \sin^2 t \cos^2 t\,dt \\ &= (4t - \sin 4t)\Big|_0^{2\pi} = 8\pi. \end{aligned}$$

- For the area calculation, we use polar coordinates:

$$\begin{aligned} \iint_D (N_x - M_y)\,dA = \iint_D (y^2 + x^2)\,dA &= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta \\ &= \int_0^{2\pi} 4\,d\theta = 8\pi. \end{aligned}$$

2. $M(x,y) = x^2 - y$ and $N(x,y) = x + y^2$.

- For the line integral, the path is split into four pieces, in each case $0 \le t \le 1$: $\mathbf{x}_1(t) = (2t, 0)$, $\mathbf{x}_2(t) = (2, t)$, $\mathbf{x}_3(t) = (2-2t, 1)$, and $\mathbf{x}_4(t) = (0, 1-t)$.

$$\begin{aligned} \oint_{\partial D} M\,dx + N\,dy &= \int_0^1 [2(4t^2) + (2+t^2) - 2(4t^2 - 8t + 3) - (t^2 - 2t + 1)]\,dt \\ &= \int_0^1 [18t - 5]\,dt = 4. \end{aligned}$$

- The area calculation is straightforward:

$$\iint_D (N_x - M_y)\,dA = \iint_D 2\,dA = \int_0^1\int_0^2 2\,dx\,dy = 4.$$

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3. $M(x,y) = y$ and $N(x,y) = x^2$.

- For the line integral, the path is again split into four pieces, in each case $0 \le t \le 1$: $\mathbf{x}_1(t) = (1-2t, 1)$, $\mathbf{x}_2(t) = (-1, 1-2t)$, $\mathbf{x}_3(t) = (-1+2t, -1)$, and $\mathbf{x}_4(t) = (1, -1+2t)$.

$$\oint_{\partial D} M\,dx + N\,dy = \int_0^1 [-2(1) + -2(1) + 2(-1) + 2(1)]\,dt$$
$$= \int_0^1 -4\,dt = -4.$$

- The area calculation is again straightforward:

$$\iint_D (N_x - M_y)\,dA = \iint_D (2x-1)\,dA = \int_{-1}^1 \int_{-1}^1 (2x-1)\,dx\,dy = \int_{-1}^1 -2\,dy = -4.$$

4. $M(x,y) = 2y$ and $N(x,y) = x$.

- For the line integral, the path is split into two pieces: $\mathbf{x}_1(t) = (a\cos t, a\sin t)$, $0 \le t \le \pi$, and $\mathbf{x}_2(t) = (-a+2at, 0)$, $0 \le t \le 1$.

$$\oint_{\partial D} M\,dx + N\,dy = \int_0^\pi (2a\sin t, a\cos t)\cdot(-a\sin t, a\cos t)\,dt + \int_0^1 a(0)\,dt$$
$$= a^2\int_0^\pi (-2\sin^2 t + \cos^2 t)\,dt = a^2\int_0^\pi (-2 + 3\cos^2 t)\,dt = -\frac{\pi a^2}{2}.$$

- We'll use polar coordinates for the area calculation:

$$\iint_D (N_x - M_y)\,dA = \iint_D (1-2)\,dA = \int_0^\pi \int_0^a -r\,dr\,d\theta = \int_0^\pi -\frac{a^2}{2}\,d\theta = -\frac{\pi a^2}{2}.$$

5. $M(x,y) = 3y$ and $N(x,y) = -4x$.

- For the line integral, the path is $\mathbf{x}(t) = (2\cos t, \sqrt{2}\sin t)$, $0 \le t \le 2\pi$.

$$\oint_{\partial D} M\,dx + N\,dy = \int_0^{2\pi} \left[(3\sqrt{2}\sin t)(-2\sin t) - (8\cos t)(\sqrt{2}\cos t)\right]\,dt$$
$$= \int_0^{2\pi} \left(-6\sqrt{2}\sin^2 t - 8\sqrt{2}\cos^2 t\right)\,dt$$
$$= -2\sqrt{2}\int_0^{2\pi} \left(3\sin^2 t + 4\cos^2 t\right)\,dt = -2\sqrt{2}\int_0^{2\pi} \left(3 + \cos^2 t\right)\,dt$$
$$= -2\sqrt{2}\int_0^{2\pi} \left(3 + \tfrac{1}{2}(1+\cos 2t)\right)\,dt$$
$$= -2\sqrt{2}\left(\frac{7}{2}t + \frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} = -14\sqrt{2}\pi.$$

- For the double integral calculation, we have:

$$\iint_D (N_x - M_y)\,dA = \iint_D (-4-3)\,dA = -7\iint_D dA$$
$$= -7\cdot(\text{area of } D) = -7\cdot 2\cdot\sqrt{2} = -14\sqrt{2}\pi.$$

(See Example 3 in §6.2.) Alternatively, we can let $x = 2u$, $y = \sqrt{2}v$ so that the ellipse $x^2 + 2y^2 = 4$ transforms to $u^2 + v^2 = 1$. The Jacobian of this transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2}\end{bmatrix} = 2\sqrt{2}.$$

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Then, using the change of variables theorem from §5.5,

$$-7\iint_D dA = -7\iint_{D^*} 2\sqrt{2}\,du\,dv = -14\sqrt{2}\cdot(\text{area of } D^*) = -14\sqrt{2}\pi,$$

since D^* is just the unit disk.

6. $M(x,y) = x^2y + x$ and $N(x,y) = y^3 - xy^2$.

- In order to make the line integral calculation along the boundary of D, we need *two* parametrized paths:

$$\mathbf{x}_1(t) = (3\cos t, 3\sin t),\ 0 \le t \le 2\pi \qquad \text{and} \qquad \mathbf{x}_2(t) = (2\cos t, 2\sin t),\ 0 \le t \le 2\pi.$$

Note, however, that the path $\mathbf{x}_2$ goes counterclockwise, which is the wrong orientation for Green's theorem. We must take this into account and compute

$$\oint_{\partial D}(x^2y+x)\,dx + (y^3 - xy^2)\,dy$$
$$= \int_{\mathbf{x}_1}(x^2y+x)\,dx + (y^3-xy^2)\,dy - \int_{\mathbf{x}_2}(x^2y+x)\,dx + (y^3-xy^2)\,dy.$$

Thus we calculate

$$\int_{\mathbf{x}_1}(x^2y+x)\,dx + (y^3-xy^2)\,dy$$
$$= \int_0^{2\pi}\left[(27\cos^2 t\sin t + 3\cos t)(-3\sin t) + (27\sin^3 t - 27\cos t\sin^2 t)(3\cos t)\right]\,dt$$
$$= \int_0^{2\pi}\left(-162\cos^2 t\sin^2 t - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi}\left(-\frac{81}{2}(1+\cos 2t)(1-\cos 2t) - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi}\left(-\frac{81}{2}(1-\cos^2 2t) - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi}\left(-\frac{81}{2} + \frac{81}{4}(1+\cos 4t) - 9\cos t\sin t + 81\sin^3 t\cos t\right)\,dt$$
$$= \left(-\frac{81}{4}t + \frac{81}{16}\sin 4t - \frac{9}{2}\sin^2 t + \frac{81}{4}\sin^4 t\right)\Big|_0^{2\pi} = -\frac{81\pi}{2}.$$

Similarly, we have

$$\int_{\mathbf{x}_2}(x^2y+x)\,dx + (y^3-xy^2)\,dy$$
$$= \int_0^{2\pi}\left[(8\cos^2 t\sin t + 2\cos t)(-2\sin t) + (8\sin^3 t - 8\cos t\sin^2 t)(2\cos t)\right]\,dt$$
$$= \int_0^{2\pi}\left(-32\cos^2 t\sin^2 t - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi}\left(-8(1+\cos 2t)(1-\cos 2t) - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi}\left(-8(1-\cos^2 2t) - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \int_0^{2\pi}\left(-8 + 4(1+\cos 4t) - 4\cos t\sin t + 16\sin^3 t\cos t\right)\,dt$$
$$= \left(-4t + \sin 4t - 2\sin^2 t + 4\sin^4 t\right)\Big|_0^{2\pi} = -8\pi.$$

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Therefore,

$$\oint_{\partial D} (x^2y + x)\,dx + (y^3 - xy^2)\,dy = -\frac{81\pi}{2} + 8\pi = -\frac{65\pi}{2}.$$

- For the double integral calculation, making use of polar coordinates, we have:

$$\iint_D (N_x - M_y)\,dA = \iint_D (-y^2 - x^2)\,dA = \int_0^{2\pi}\int_2^3 -r^2 \cdot r\,dr\,d\theta$$
$$= \int_0^{2\pi} -\tfrac{1}{4}(3^4 - 2^4)\,d\theta = -\frac{\pi}{2}(81 - 16) = -\frac{65\pi}{2}.$$

7. (a) By Green's theorem, we have

$$\oint_C y^2\,dx + x^2\,dy = \iint_D \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(y^2)\right] dA = \int_0^1\int_0^1 (2x - 2y)\,dx\,dy$$
$$= \int_0^1 (x^2 - 2xy)\Big|_{x=0}^1 dy = \int_0^1 (1 - 2y)\,dy = (y - y^2)\Big|_0^1 = 0.$$

(b) Our path is made up of four straight-line pieces with $0 \le t \le 1$ on each: $\mathbf{x}_1(t) = (t, 0)$ (so $dx = dt$, $dy = 0$), $\mathbf{x}_2(t) = (1, t)$ (so $dx = 0$, $dy = dt$), $\mathbf{x}_3(t) = (1 - t, 1)$ (so $dx = -dt$, $dy = 0$), and $\mathbf{x}_4(t) = (0, 1 - t)$ (so $dx = 0$, $dy = -dt$). Therefore,

$$\oint_C y^2\,dx + x^2\,dy = \int_{\mathbf{x}_1} + \int_{\mathbf{x}_2} + \int_{\mathbf{x}_3} + \int_{\mathbf{x}_4}$$
$$= \int_0^1 (0 + t^2(0))\,dt + \int_0^1 (t^2(0) + 1)\,dt$$
$$+ \int_0^1 (1(-1) + (1-t)^2(0))\,dt + \int_0^1 ((1-t)^2(0) + 0)\,dt$$
$$= 0 + 1 - 1 + 0 = 0.$$

8. $M(x, y) = 3xy$ and $N(x, y) + 2x^2$.

- For the line integral, the path is split into four pieces: $\mathbf{x}_1(t) = (0, -2t)$, $\mathbf{x}_2(t) = (2t, -2)$, and $\mathbf{x}_3(t) = (2, -2 + 2t)$, with $0 \le t \le 1$, and $\mathbf{x}_4(t) = (\cos t + 1, \sin t)$ with $0 \le t \le \pi$. So

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_0^1 [-2(0) + 2(-12t) + 2(8)]\,dt + \int_0^\pi [-3\sin^2 t(\cos t + 1) + 2\cos t(\cos t + 1)^2]\,dt$$
$$= \int_0^1 [-24t + 16]\,dt + \int_0^\pi [2\cos t + 4\cos^2 t + 2\cos^3 t - 3\sin^2 t - 3\cos t\sin^2 t]\,dt$$
$$= 4 + \pi/2.$$

- If D is the region bounded by C, then

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \iint_D (4x - 3x)\,dA = \iint_D x\,dA$$
$$= \int_{-2}^0\int_0^2 x\,dx\,dy + \int_0^\pi\int_0^1 r(r\cos\theta + 1)\,dr\,d\theta$$
$$= \int_{-2}^0 2\,dy + \int_0^\pi \left[\frac{1}{3}\cos\theta + \frac{1}{2}\right] d\theta = 4 + \pi/2.$$

9. Note that the curve is oriented clockwise so the square lies on the right side of the curve.

$$\oint_C (x^2 - y^2)\,dx + (x^2 + y^2)\,dy = -\int_0^1\int_0^1 (2x + 2y)\,dy\,dx = -\int_0^1 (2x + 1)\,dx = -2.$$

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10. As we saw in Section 6.1, Work $= \oint_C \mathbf{F}\cdot d\mathbf{s}$. If D is the ellipse $x^2+4y^2=4$ and its boundary is C, then by Green's theorem

$$\oint_C (4y-3x, x-4y)\cdot d\mathbf{s} = \iint_D (1-4)\,dA = \int_{-2}^{2}\int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}} -3\,dy\,dx$$
$$= \int_{-2}^{2}\left[-6\sqrt{1-x^2/4}\right]dx = -6\pi.$$

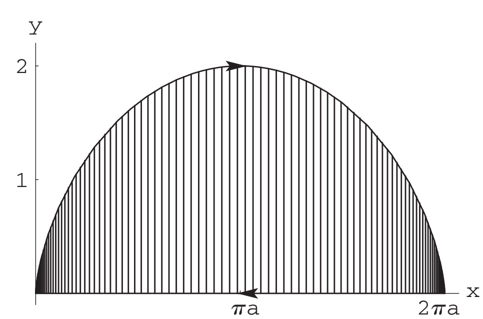
For Exercises 11, 12, 14, 15, 16 we will calculate the area of a region using formula (1):

$$\textit{area of } D = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy.$$

11. By formula (1), the area is $\frac{1}{2}\oint_{\partial R} -y\,dx + x\,dy$. We can work this out, as in the case of Exercises 2 and 3, by enumerating the paths along the four sides and calculating the integral. We can, however, eliminate a lot of work by first noting that $dy = 0$ along both horizontal parts of the path and that $x = 0$ along the left vertical portion of the path. Also, $dx = 0$ along both vertical parts of the path and $y = 0$ along the bottom portion. So

$$\frac{1}{2}\oint_{\partial R} -y\,dx + x\,dy = \frac{1}{2}\left[\int_0^a b\,dx + \int_0^b a\,dy\right] = ab.$$

12. One arch of the cycloid is produced from $t = 0$ to $t = 2\pi$.



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Because of the orientation shown,

$$\text{Area} = \frac{1}{2}\oint_C y\,dx - x\,dy = \frac{1}{2}\int_{C_1} + \frac{1}{2}\int_{C_2}.$$

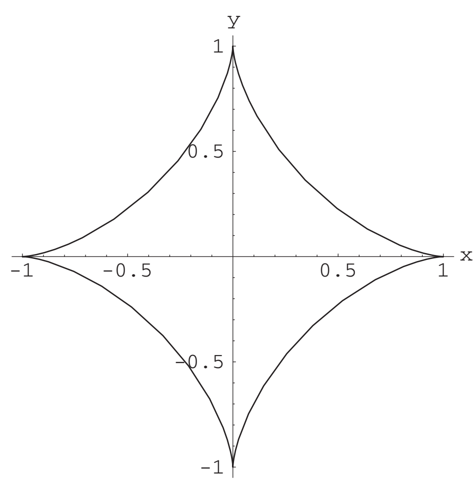
$$\begin{aligned}\frac{1}{2}\int_{C_1} &= \frac{1}{2}\int_0^{2\pi} (a(1-\cos t)\cdot a(1-\cos t) - a(t-\sin t)\cdot a\sin t)\,dt \\ &= \frac{a^2}{2}\int_0^{2\pi} ((1-\cos t)^2 - t\sin t + \sin^2 t)\,dt = \frac{a^2}{2}\int_0^{2\pi} (1 - 2\cos t + \cos^2 t - t\sin t + \sin^2 t)\,dt \\ &= \frac{a^2}{2}\int_0^{2\pi} (2 - 2\cos t - t\sin t)\,dt = \frac{a^2}{2}(2t - 2\sin t + t\cos t - \sin t)\Big|_0^{2\pi} = \frac{a^2}{2}(4\pi + 2\pi) = 3\pi a^2.\end{aligned}$$

13. By Green's theorem:

$$\begin{aligned}\oint_C (x^4y^5 - 2y)\,dx + (3x + x^5y^4)\,dy &= -\iint_D ((3 + 5x^4y^4) - (5x^4y^4 - 2))\,dA \\ &= -\iint_D 5\,dA = -5\cdot\text{area of } D = -5(2+3+4) = -45.\end{aligned}$$

(Note the minus sign because of the orientation of the curve.)

14. A sketch of a hypocycloid with $a = 1$ is:



Let D be the interior of the hypocycloid and let C be the bounding curve traced by the path $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$. Then by Green's theorem,

$$\begin{aligned}\iint_D dy\,dx = \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy &= \int_0^{2\pi} [a\sin^3 t(3a\cos^3 t\sin t) + a\cos^3 t(3a\sin^2 t\cos t)]\,dt \\ &= \frac{3a^2}{2}\int_0^{2\pi} (\cos^2 t\sin^4 t + \cos^4 t\sin^2 t)\,dt = \frac{3\pi a^2}{8}.\end{aligned}$$

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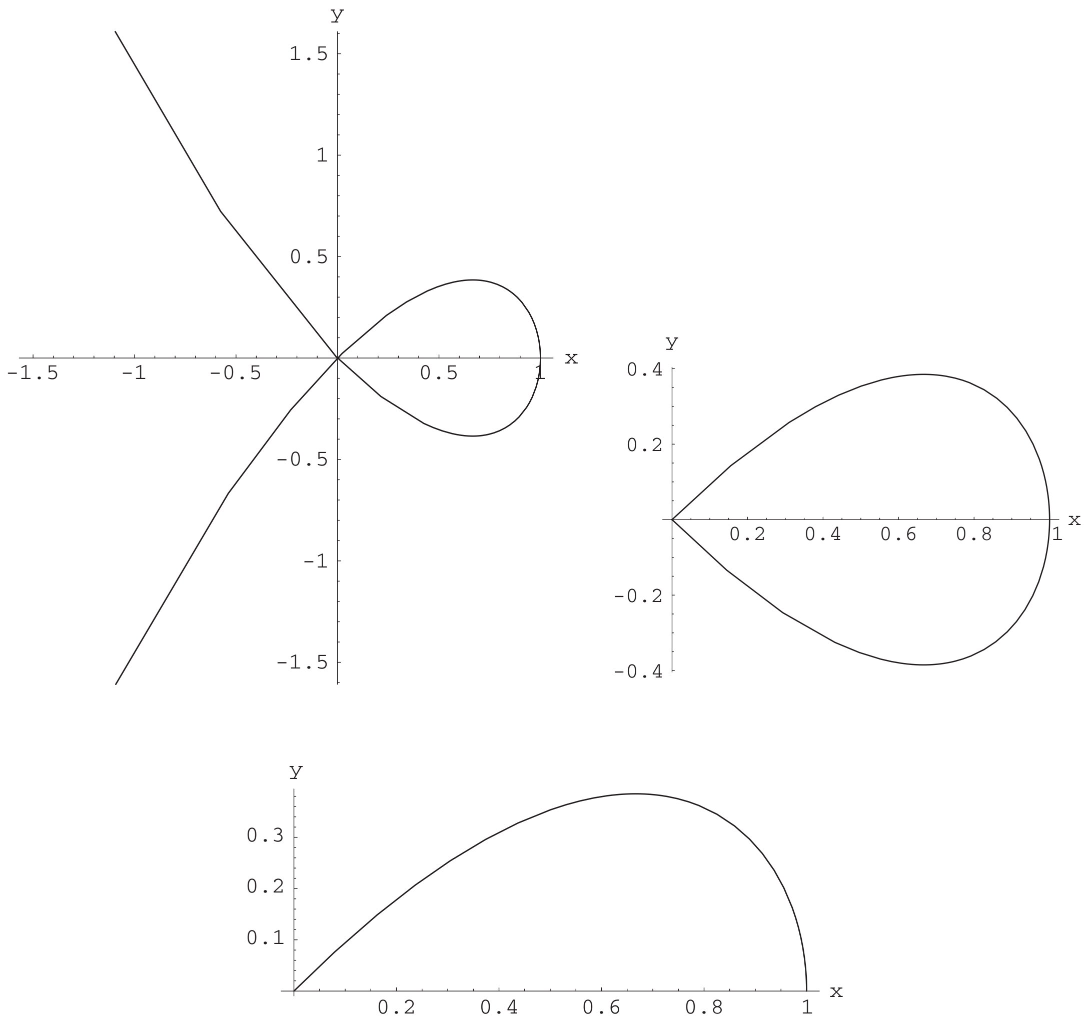
15. **(a)** Shown below are three views of the curve $\mathbf{x}(t) = (1 - t^2, t^3 - t)$.



The figure on the top left is for $-3 \leq t \leq 3$, the top right figure is for $-1 \leq t \leq 1$, and the figure on the bottom is for $-1 \leq t \leq 0$. The first gives a feel for the curve, the second isolates the closed portion of the curve and the third gives us the orientation: that as t increases from -1 to 1 the path moves clockwise.

(b) We again must make an adjustment because the path moves clockwise. The area is

$$\frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy = -\int_{-1}^{1} [(t^3 - t)(2t) + (1 - t^2)(3t^2 - 1)]\,dt$$

$$= \int_{-1}^{1} (t^4 - 2t^2 + 1)\,dt = \frac{8}{15}.$$

16. In this exercise, we are finding the area of the region D that is outside the ellipse and inside the circle.

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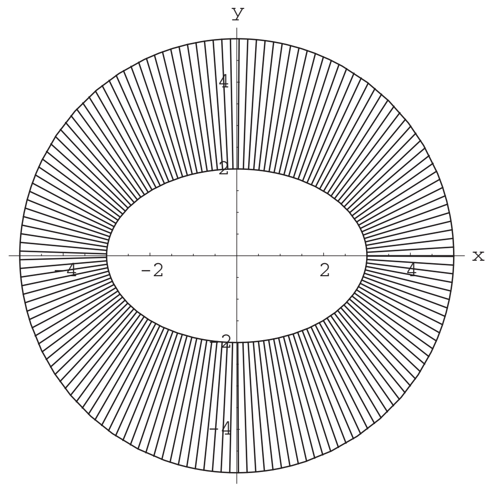

We need to orient the boundary curve C so that the path travels counterclockwise around the circle and clockwise around the ellipse. We split this path into two pieces, each with $0 \le t \le 2\pi$, $\mathbf{x}_1(t) = (5\cos t, 5\sin t)$ and $\mathbf{x}_2(t) = (3\cos t, -2\sin t)$. By Green's theorem,

$$\begin{aligned}\iint_D dA &= \frac{1}{2}\oint_{\partial D} -y\,dx + x\,dy \\ &= \frac{1}{2}\int_0^{2\pi} [(-5\sin t)(-5\sin t) + (5\cos t)(5\cos t)]\,dt \\ &\quad + \frac{1}{2}\int_0^{2\pi} [(2\sin t)(-3\sin t) + (3\cos t)(-2\cos t)]\,dt \\ &= \frac{1}{2}\int_0^{2\pi} 25\,dt + \frac{1}{2}\int_0^{2\pi} (-6)\,dt = 19\pi.\end{aligned}$$

17. It is easier if we work from the line integral to the double integral. By Green's theorem,

$$\oint_{\partial D} x\,dy = \iint_D \left(\frac{\partial x}{\partial x}\right) dA = \iint_D dA = \text{Area of } D.$$

Similarly, also by Green's theorem,

$$-\oint_{\partial D} y\,dx = -\iint_D \left(-\frac{\partial y}{\partial y}\right) dA = \iint_D dA = \text{Area of } D.$$

Note: Assign Exercises 17 and 18 together to point out that the students cannot mix the two line integrals given in Exercise 17. The quadrilateral in Exercise 18 has one vertical side and one horizontal side so there is a temptation to use the integral with a dx in it along the vertical side and the integral with a dy in it along the horizontal side so that they disappear. You must choose one or the other for the entire problem.

18. We'll use the results of Exercise 17. If we use the formula $\oint_{\partial D} x\,dy$, then for the side connecting (1, 1) to $(-1, 1)$, since there is no change in y, this integral is 0. Therefore,

$$\text{Area of } D = \oint_C x\,dy = \int_0^1 ((2-t)(2) + 1(-1) + 0 + (-1+3t)(-1))\,dt = \int_0^1 (4-5t)\,dt = \frac{3}{2}.$$

19. The area inside the polygon may be computed from

$$\frac{1}{2}\oint_C -y\,dx + x\,dy.$$

The key is to parametrize the boundary C of the polygon. This may be done in n line segment pieces. For $k = 1, \ldots, n-1$, the line segment from (a_k, b_k) to (a_{k+1}, b_{k+1}) is

$$\mathbf{x}_k(t) = ((a_{k+1} - a_k)t + a_k, (b_{k+1} - b_k)t + b_k), \quad 0 \le t \le 1,$$

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while the last segment from (a_n, b_n) to (a_1, b_1) is

$$\mathbf{x}_n(t) = ((a_1 - a_n)t + a_n, (b_1 - b_n)t + b_n), \quad 0 \le t \le 1.$$

Thus, for $k = 1, \ldots, n-1$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{x}_k} - y\,dx + x\,dy \\
&\quad = \frac{1}{2}\int_0^1 [((b_k - b_{k+1})t - b_k)(a_{k+1} - a_k) + ((a_{k+1} - a_k)t + a_k)(b_{k+1} - b_k)]\;dt \\
&\quad = \frac{1}{2}\int_0^1 [(b_k - b_{k+1})(a_{k+1} - a_k)t - b_k(a_{k+1} - a_k) \\
&\qquad\qquad + (a_{k+1} - a_k)(b_{k+1} - b_k)t + a_k(b_{k+1} - b_k)]\;dt \\
&\quad = \frac{1}{2}\int_0^1 (-a_{k+1}b_k + a_kb_k + a_kb_{k+1} - a_kb_k)\;dt \\
&\quad = \frac{1}{2}\int_0^1 (-a_{k+1}b_k + a_kb_{k+1})\;dt = \frac{1}{2}(-a_{k+1}b_k + a_kb_{k+1}) = \frac{1}{2}\begin{vmatrix} a_k & b_k \\ a_{k+1} & b_{k+1}\end{vmatrix}.
\end{aligned}$$

For the last segment, the calculation is very similar, so we abbreviate the steps:

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{x}_k} - y\,dx + x\,dy \\
&\quad = \frac{1}{2}\int_0^1 [((b_n - b_1)t - b_n)(a_1 - a_n) + ((a_1 - a_n)t + a_n)(b_1 - b_n)]\;dt \\
&\quad = \frac{1}{2}\int_0^1 (-a_1b_n + a_nb_1)\;dt = \frac{1}{2}(-a_1b_n + a_nb_1) = \frac{1}{2}\begin{vmatrix} a_n & b_n \\ a_1 & b_1\end{vmatrix}.
\end{aligned}$$

Adding the results of these calculations, we obtain

$$\frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\left(\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2\end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3\end{vmatrix} + \cdots + \begin{vmatrix} a_{n-1} & b_{n-1} \\ a_n & b_n\end{vmatrix} + \begin{vmatrix} a_n & b_n \\ a_1 & b_1\end{vmatrix}\right),$$

as desired.

20. (a) Using the hint, we see that $\|\mathbf{x}(t)\|^2 = a^2$ when

$$((a+1)\cos t - \cos{(a+1)t})^2 + ((a+1)\sin t - \sin{(a+1)t})^2 = a^2.$$

Expanding and simplifying the left side gives

$$\begin{aligned}
(a+1)^2 + 1 - 2(a+1)\cos t\cos{(a+1)t} - 2(a+1)\sin t\sin{(a+1)t} &= a^2 \\
\iff 2a + 2 - 2(a+1)\left(\cos t\cos{(a+1)t} + \sin t\sin{(a+1)t}\right) &= 0 \\
\iff \cos t\cos{(a+1)t} + \sin t\sin{(a+1)t} &= 1.
\end{aligned}$$

Using the subtraction formula for cosine, this last equation becomes

$$\cos{(a+1)t - t} = 1 \iff \cos at = 1 \iff t = \frac{2\pi n}{a}.$$

The graphs for of the epicycloids for $a = 5$ and $a = 6$ are shown.

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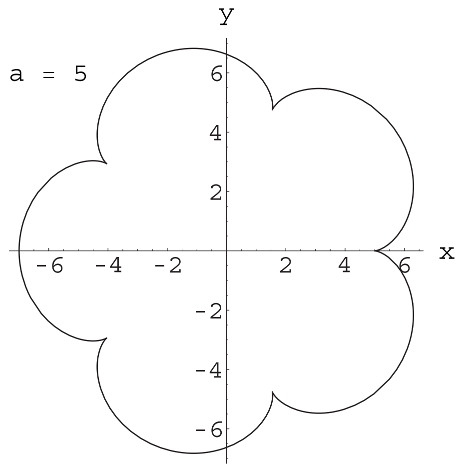

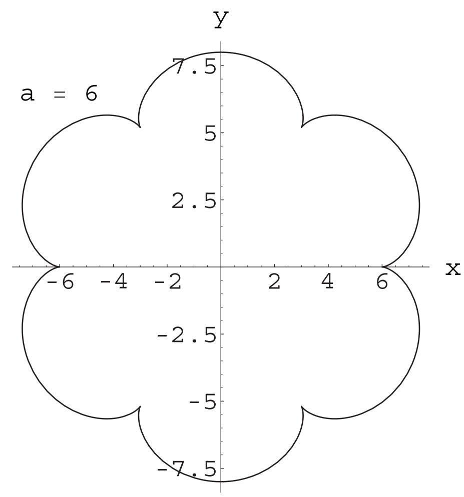

(b) When a is an integer larger than 1, the epicycloid traces its complete image once for t in $[0, 2\pi)$. To compute the enclosed area, we use the line integral $\frac{1}{2}\oint_C -y\,dx + x\,dy$. Thus the area is

$$\begin{aligned}
&\frac{1}{2}\int_0^{2\pi} [((a+1)\sin t - \sin(a+1)t)\,(-(a+1)\sin t + (a+1)\sin(a+1)t) \\
&\qquad + ((a+1)\cos t - \cos(a+1)t)\,((a+1)\cos t - (a+1)\cos(a+1)t)]\;dt \\
&= \frac{1}{2}\int_0^{2\pi} \left[(a+1)^2 - \left((a+1) + (a+1)^2\right)(\cos(a+1)t\cos t + \sin(a+1)t\sin t)\right. \\
&\qquad \left. + (a+1)\right]\;dt \\
&= \frac{(a+1)}{2}\int_0^{2\pi} [(a+1) - (a+2)(\cos(a+1)t\cos t + \sin(a+1)t\sin t) + 1]\;dt \\
&= \frac{(a+1)(a+2)}{2}\int_0^{2\pi} [1 - (\cos(a+1)t\cos t + \sin(a+1)t\sin t)]\;dt
\end{aligned}$$

after expansion and some simplification. Next we use the subtraction formula for cosine:

$$\begin{aligned}
\text{Area} &= \frac{(a+1)(a+2)}{2}\int_0^{2\pi} (1 - \cos at)\;dt \\
&= \frac{(a+1)(a+2)}{2}\left(t - \frac{1}{a}\sin at\right)\Big|_0^{2\pi} = \pi(a+1)(a+2).
\end{aligned}$$

(Note that we used the fact that a is an integer when evaluating the final integral.)

(c) The area of the fixed circle is πa^2. Thus

$$\lim_{a\to\infty} \frac{\pi(a+1)(a+2)}{\pi a^2} = \lim_{a\to\infty}\left(1 + \frac{1}{a}\right)\left(1 + \frac{2}{a}\right) = 1.$$

Hence, in the limit, the epicycloid's area approaches that of the fixed circle.

21. By Green's theorem,

$$\oint_C 5y\,dx - 3x\,dy = \iint_D (-3-5)\,dA = \iint_D -8\,dA = -8(\text{area of } D),$$

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where D is the region in the plane enclosed by the cardioid. We may evaluate the double integral using polar coordinates.

$$\begin{aligned}\iint_D -8\,dA &= -8\int_0^{2\pi}\int_0^{1-\sin\theta} r\,dr\,d\theta\\ &= -8\int_0^{2\pi} \tfrac{1}{2}r^2\Big|_{r=0}^{r=1-\sin\theta} d\theta = -4\int_0^{2\pi}(1-\sin\theta)^2\,d\theta\\ &= -4\int_0^{2\pi}(1-2\sin\theta+\sin^2\theta)\,d\theta = -4\int_0^{2\pi}\left(1-2\sin\theta+\tfrac{1}{2}(1-\cos 2\theta)\right)\,d\theta\\ &= -4\left(\tfrac{1}{2}\theta+2\cos\theta-\tfrac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} = -12\pi.\end{aligned}$$

22. (a) Note that we have

$$\frac{\partial}{\partial x}\frac{y}{x^2+y^2} = -\frac{2xy}{x^2+y^2} = \frac{\partial}{\partial y}\frac{x}{x^2+y^2}.$$

Therefore, Green's theorem implies that, for the region D enclosed by C

$$\oint_C \frac{x\,dx+y\,dy}{x^2+y^2} = \pm\iint_D\left(-\frac{2xy}{x^2+y^2}-\left(-\frac{2xy}{x^2+y^2}\right)\right)dA = \pm\iint_D 0\,dA = 0.$$

(The $\pm$ sign is due to the fact that we do not know the orientation of C, not that it ultimately matters.)

(b) Green's theorem does *not* apply since M and N are not defined at the origin, which is in the region D enclosed by C.

(b) If C_1 and C_2 both enclose the origin and don't cross or touch, then one of the curves must lie entirely inside the other. Let's assume that C_2 lies inside C_1. Together, C_1 and C_2 make up the boundary of a region D that does *not* contain the origin. Thus we may apply Green's theorem to D and its boundary:

$$\begin{aligned}0 &= \iint_D\left(\frac{\partial}{\partial x}\frac{y}{x^2+y^2}-\frac{\partial}{\partial y}\frac{x}{x^2+y^2}\right)dA = \oint_{\partial D}\frac{x\,dx+y\,dy}{x^2+y^2}\\ &= \oint_{C_1}\frac{x\,dx+y\,dy}{x^2+y^2} - \oint_{C_2}\frac{x\,dx+y\,dy}{x^2+y^2}.\end{aligned}$$

Hence the desired result follows. Note that the orientation of ∂D requires a counterclockwise orientation on the outer curve, but a *clockwise* orientation on the inner curve.

(c) Find a circle C' of some small radius a so that C' lies entirely inside C and is oriented the same way that C is. Then, from part (c), we know that

$$\oint_C \frac{x\,dx+y\,dy}{x^2+y^2} = \oint_{C'}\frac{x\,dx+y\,dy}{x^2+y^2}.$$

We may evaluate this last integral using the parametrization $\mathbf{x}(t) = (a\cos t, a\sin t)$, $0 \le t \le 2\pi$. Thus

$$\oint_{C'}\frac{x\,dx+y\,dy}{x^2+y^2} = \pm\int_0^{2\pi}\frac{(a\cos t)(-a\sin t)+(a\sin t)(a\cos t)}{a^2} = \pm\int_0^{2\pi} 0\,dt = 0.$$

(Once again the $\pm$ sign is due to the fact that we do not know the actual orientation of C'.)

23. (a) By the divergence theorem:

$$\oint_C (2y\mathbf{i}-3x\mathbf{j})\cdot\mathbf{n}\,ds = \iint_D[(2y)_x+(-3x)_y]\,dA = \iint_D 0\,dA = 0.$$

(b) For direct computation, $\mathbf{n} = (\cos\theta, \sin\theta)$ and $x = \cos\theta$ and $y = \sin\theta$. Therefore,

$$\mathbf{F}\cdot\mathbf{n} = (2y,-3x)\cdot(\cos\theta,\sin\theta) = 2\cos\theta\sin\theta - 3\cos\theta\sin\theta = -\cos\theta\sin\theta = -\frac{1}{2}\sin 2\theta.$$

Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = -\frac{1}{2}\int_0^{2\pi}\sin 2\theta\,d\theta = 0.$$

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24. Similar to what was done in the proof of the divergence theorem, we will calculate the line integral $\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds$ along a C^1 segment of ∂D. Recall that $\mathbf{T}(t) = \mathbf{x}'(t)/\|\mathbf{x}'(t)\|$.

$$\int_a^b (\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T}(t))\, \|\mathbf{x}'(t)\|\, dt = \int_a^b (\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t))\, dt$$

$$= \int_a^b ((M(x(t), y(t))\, x'(t) + N(x(t), y(t))\, y'(t))\, dt = \int_{\mathbf{x}} M\, dx + N\, dy.$$

We extend this result to the entire curve and apply Green's theorem.

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial D} M\, dx + N\, dy = \iint_D (N_x - M_y)\, dA.$$

25. By Green's Theorem, if D is the region bounded by C,

$$\oint_C 3x^2 y\, dx + x^3\, dy = \iint_D (3x^2 - 3x^2)\, dA = 0.$$

(Note that in this case the orientation of C is not important as $-(3x^2 - 3x^2) = 3x^2 - 3x^2$.)

26. If C is oriented as required and D is the region bounded by C, then by Green's Theorem,

$$\oint_C -y^3\, dx + (x^3 + 2x + y)\, dy = \iint_D (3x^2 + 2 + 3y^2)\, dA > 0.$$

27. Let $\delta = 1$ if C is oriented counterclockwise and $\delta = -1$ if C is oriented clockwise. Let D be the region bounded by C. Then by Green's Theorem,

$$\oint_C (x^2y^3 - 3y)\, dx + x^3y^2\, dy = \delta \iint_D (3x^2y^2 - 3x^2y^2 + 3)\, dA = 3\delta \text{ (the area of the rectangle).}$$

28.

$$\text{Flux} = \oint_C (\mathbf{r} \cdot \mathbf{n})\, ds = \int_a^b (x\mathbf{i} + y\mathbf{j}) \cdot \frac{y'\mathbf{i} - x'\mathbf{j}}{\sqrt{x'^2 + y'^2}} \sqrt{x'^2 + y'^2}\, dt$$

$$= \int_a^b \left(x\frac{dy}{dt} - y\frac{dx}{dt} \right) dt = \int_C -y\, dx + x\, dy$$

$$= \iint_D (1 - (-1))\, dA \quad \text{by Green's theorem}$$

$$= \iint_D 2\, dA = 2 \cdot (\text{area inside } C).$$

29. We have $u\, \nabla v = \left(u\frac{\partial v}{\partial x}, u\frac{\partial v}{\partial y} \right)$ so that

$$\oint_C (u\, \nabla v) \cdot d\mathbf{s} = \oint_C u\frac{\partial v}{\partial x}\, dx + u\frac{\partial v}{\partial y}\, dy$$

$$= \iint_D \left(\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial y} \right) - \frac{\partial}{\partial y}\left(u\frac{\partial v}{\partial x} \right) \right) dA \quad \text{by Green's theorem}$$

$$= \iint_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial x\, \partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - u\frac{\partial^2 v}{\partial y\, \partial x} \right) dA$$

$$= \iint_D \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \right) dA \quad \text{since } v \text{ is of class } C^2$$

$$= \iint_D \frac{\partial(u, v)}{\partial(x, y)}\, dA.$$

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30. Let D be the region bounded by C. By Green's theorem,

$$\oint_C \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dx = -\iint_D \left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2}\right) dA = -\iint_D 0\,dA = 0.$$

31. First, $\oint_{\partial D} \frac{\partial f}{\partial n}\,ds = \oint_{\partial D} \nabla f \cdot \mathbf{n}\,ds = \oint_{\partial D} \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \mathbf{n}\,ds$. You can continue the calculation or note that this is the same computation done in the proof of the divergence theorem with $\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$. Therefore, applying Green's theorem,

$$\oint_{\partial D} \frac{\partial f}{\partial n}\,ds = \oint_{\partial D} \frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial x}\,dy = \iint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dA = \iint_D \nabla^2 f\,dA.$$

6.3 Conservative Vector Fields

1. (a) Let C be the path parametrized by $\mathbf{x}(t) = (t, t, t)$ with $0 \le t \le 1$. Then

$$\int_C z^2\,dx + 2y\,dy + xz\,dz = \int_0^1 (t^2 + 2t + t^2)\,dt = 2\int_0^1 (t^2 + t)\,dt = \frac{5}{3}.$$

(b) Let $\mathbf{x}(t) = (t, t^2, t^3)$ with $0 \le t \le 1$. Then

$$\int_C z^2\,dx + 2y\,dy + xz\,dz = \int_0^1 (t^6 + 2t^2(2t) + t^4(3t^2))\,dt = 4\int_0^1 (t^6 + t^3)\,dt = \frac{11}{7}.$$

(c) Parts (a) and (b) show that line integrals are not path-independent. By Theorem 3.3, therefore, **F** is not conservative.

2. (a) Let C be the path parameterized by $\mathbf{x}(t) = (t^2, t^3, t^5)$ with $0 \le t \le 1$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t^5, t^4 + t^{10}, 2t^8) \cdot (2t, 3t^2, 5t^4)\,dt = \int_0^1 (7t^6 + 13t^{12})\,dt = 2.$$

(b) Let C be comprised of the two paths: $\mathbf{x}_1(t) = (t, 0, 0)$ and $\mathbf{x}_2(t) = (1, t, t)$ each with $0 \le t \le 1$. The integral along $\mathbf{x}_1$ is easily seen to be zero (y, z, and dx are all identically zero along $\mathbf{x}_1$). We have that

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^1 (2t, 1 + t^2, 2t) \cdot (0, 1, 1)\,dt = 2.$$

(c) Obviously the fact that our answers to parts (a) and (b) are the same is not enough to convince us that **F** is conservative. We can, however, easily see that $\mathbf{F} = \nabla\left(x^2 y + yz^2\right)$ so **F** is conservative.

In Exercises 3–9, we will check to see whether $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is conservative by checking to see whether $\partial N/\partial x = \partial M/\partial y$ (formula (1)).

3. $\frac{\partial N}{\partial x} = ye^{xy} \neq e^{x+y} = \frac{\partial M}{\partial y}$, so **F** is not conservative.

4. $\frac{\partial N}{\partial x} = 2x\cos y = \frac{\partial M}{\partial y}$, so **F** is conservative. We want to find f where $\mathbf{F} = \nabla f(x, y)$. We find that the indefinite integral of $2x\sin y$ with respect to x is $x^2 \sin y$. To see whether any adjustments need to be made, we check to make certain that $\frac{\partial}{\partial y}(x^2 \sin y) = x^2 \cos y$. It does, so we conclude that $f(x, y) = \nabla(x^2 \sin y)$.

5. $\frac{\partial N}{\partial x} = 3x^2 \sin y + \frac{1 - x^2y^2}{(1 + x^2y^2)^2} \neq -3x^2 \sin y + \frac{1 - x^2y^2}{(1 + x^2y^2)^2} = \frac{\partial M}{\partial y}$, so **F** is not conservative.

6. $\frac{\partial N}{\partial x} = \frac{2xy}{(1 + x^2)^2} = \frac{\partial M}{\partial y}$, so **F** is conservative. $\mathbf{F} = \nabla\left(\frac{x^2y^2}{2(1 + x^2)}\right)$.

7. Note that $\frac{\partial}{\partial y}(e^{-y} - y\sin(xy)) = -e^{-y} - \sin xy - xy\cos xy = \frac{\partial}{\partial x}(-xe^{-y} - x\sin xy)$. Since the domain of **F** is all of $\mathbf{R}^2$, the vector field is conservative. Thus $\mathbf{F} = \nabla f$, so $\frac{\partial f}{\partial x} = e^{-y} - y\sin xy \Rightarrow f(x, y) = xe^{-y} + \cos xy + g(y)$ for some g. Hence $\frac{\partial f}{\partial y} = -xe^{-y} - x\sin xy + g'(y) = -xe^{-y} - x\sin xy$ so $g'(y) = 0$. Thus $f(x, y) = xe^{-y} + \cos xy + C$ is a potential for any C.

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8. $\dfrac{\partial N}{\partial x} = 12xy - y \neq 12xy + 6y = \dfrac{\partial M}{\partial y}$, so **F** is *not* conservative.

9. $\dfrac{\partial N}{\partial x} = 12xy = \dfrac{\partial M}{\partial y}$, so **F** is conservative. $\mathbf{F} = \nabla\left(3x^2y^2 - x^3 + \frac{1}{3}y^3\right)$.

In Exercises 10–18, we will check to see whether $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ *is conservative by checking whether* $\nabla \times \mathbf{F} = \mathbf{0}$. *This amounts to checking whether* $\partial N/\partial x = \partial M/\partial y, \partial P/\partial x = \partial M/\partial z$, *and* $\partial P/\partial y = \partial N/\partial z$. *We also need to check that the domain of* **F** *is simply-connected. This last condition is only an issue in Exercise 16.*

10. $\nabla \times \mathbf{F} = (6x^2 - 2yz - 6x^2z^2 - 2y)\,\mathbf{i} + (3xyz^2 - 2z - 12xy)\,\mathbf{j} + (3xz^3 - x)\,\mathbf{i} \neq \mathbf{0}$. Hence **F** is *not* conservative.

11. We see that $\partial N/\partial x = 4xz^3 - 2x = \partial M/\partial y$, $\partial P/\partial x = 12xyz^2 = \partial M/\partial z$, and $\partial P/\partial y = 6x^2z^2 + 2y = \partial N/\partial z$. Thus **F** is conservative. $\mathbf{F} = \nabla\left(2x^2yz^3 - x^2y + y^2z\right)$.

12. $\nabla \times \mathbf{F} = (2xe^{xyz} + 2x^2yze^{xyz})\mathbf{i} - (2ze^{xyz} + 2xyz^2e^{xyz})\mathbf{k} \neq \mathbf{0}$. Hence **F** is *not* conservative.

13. We see that $\partial N/\partial x = 1 = \partial M/\partial y$, $\partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = \cos yz - yz\sin yz = \partial N/\partial z$. So **F** is conservative. $\mathbf{F} = \nabla(x^2 + xy + \sin yz)$.

14. Here, $\partial N/\partial x = 0 \neq 1 = \partial M/\partial y$, so **F** is not conservative.

15. We see that $\partial N/\partial x = e^x\cos y = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 0 = \partial N/\partial z$. So **F** is conservative. $\mathbf{F} = \nabla(e^x\sin y + z^3 + 2z)$.

16. We see that $\partial N/\partial x = 0 = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 2z/y = \partial N/\partial z$. So **F** is conservative in each of the simply-connected regions on which it is defined: $\{(x,y,z)|y > 0\}$ and $\{(x,y,z)|y < 0\}$. On each, $\mathbf{F} = \nabla(x^3 + z^2\ln|y|)$.

17. We see that $\partial N/\partial x = ze^{-yz} + e^{xyz}(xyz^2 + z) = -\partial M/\partial y$, so **F** is not conservative.

18. We see that, for **G**, $\partial N/\partial x = 2x = \partial M/\partial y, \partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = 2y = \partial N/\partial z$. So $\mathbf{G} = (2xy, x^2 + 2yz, y^2)$ is conservative and $\mathbf{G} = \nabla(x^2y + y^2z)$. We know, therefore, that **F** is not conservative because the wording of the problem assured us that exactly one of **F** and **G** was conservative. It may be more satisfying to verify that for **F**, $\partial M/\partial y = 2xyz^3$ while $\partial N/\partial x = 4xy$. These are different so **F** is not conservative.

19. **(a)** We have, for $i = 1, \ldots, n$, that $f_{x_i}(\mathbf{x}) = 0$ for all $\mathbf{x}$ in the domain of f. Taking these results one at a time, we have

$$f_{x_1}(\mathbf{x}) = 0 \Longrightarrow f \text{ is a function of } x_2, \ldots, x_n \text{ only.}$$
$$f_{x_2}(\mathbf{x}) = 0 \Longrightarrow \text{in addition } f \text{ is a function of } x_3, \ldots, x_n \text{ only.}$$
$$\vdots$$

Continuing in this way, we see that f must be independent of all variables, and so must be a constant function.

(b) We have $\nabla g = \nabla h = \mathbf{F}$. Consider $f = g - h$. Then

$$\nabla f = \nabla g - \nabla h = \mathbf{F} - \mathbf{F} = \mathbf{0}.$$

Therefore, by part (a), $f = g - h$ is constant.

20. For **F** to be conservative, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x\sin y - y\cos x) = \sin y + y\sin x.$$

Thus

$$M(x,y) = -\cos y + \tfrac{1}{2}y^2\sin x + u(x),$$

where u is any C^1 function of x.

21. For **F** to be conservative, we must have $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial y}(ye^{2x} + 3x^2e^y) = e^{2x} + 3x^2e^y$. Thus $N(x,y) = \dfrac{1}{2}e^{2x} + x^3e^y + u(y)$ where u is any C^1 function of y.

22. Note that the constant function $g(x) = 0$ is a trivial solution. Otherwise, we must have

$$\frac{\partial}{\partial y}\left[(xe^x + y^2)g(x)\right] = \frac{\partial}{\partial x}\left[xyg(x)\right].$$

Thus means that

$$2yg(x) = yg(x) + xyg'(x) \iff \frac{g'(x)}{g(x)} = \frac{1}{x}.$$

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Integrating this last equation, we have

$$\ln|g(x)| = \ln|x| + C \qquad \text{or} \qquad \ln\left|\frac{g(x)}{x}\right| = C.$$

Exponentiating, we have

$$\left|\frac{g(x)}{x}\right| = k,$$

where $k = e^C$. Thus $g(x) = \pm kx$. If we allow k to be completely arbitrary (i.e., positive, negative, or zero), then we may simply say $g(x) = kx$ for *any* constant k gives a solution.

23. For $\mathbf{F}$ to be conservative, we must have $\nabla \times \mathbf{F} = \mathbf{0}$. Thus we demand

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3y - 3x^2z & N(x,y,z) & 2yz - x^3 \end{vmatrix}$$
$$= (2z - N_z)\,\mathbf{i} + (-3x^2 + 3x^2)\,\mathbf{j} + (N_x - x^3)\,\mathbf{k}.$$

From this, we see that N must satisfy $\partial N/\partial x = x^3$ and $\partial N/\partial z = 2z$. The first equation implies that $N(x,y,z) = \frac{1}{4}x^4 + g(y,z)$, and so $2z = \partial N/\partial z = \partial g/\partial z$, which in turn implies that $g(y,z) = z^2 + h(y)$. From here it is easy to check that the curl condition above is satisfied when $N(x,y,z) = \frac{1}{4}x^4 + z^2 + h(y)$, where h is any function of class C^1 defined on a simply-connected domain.

24. For $\mathbf{F}$ to be conservative, we must have $\nabla \times \mathbf{F} = \mathbf{0}$. Thus we impose

$$\mathbf{0} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3x^2 + 3y^2z\sin xz & ay\cos xz + bz & 3xy^2\sin xz + 5y \end{vmatrix}$$
$$= (6xy\sin xz + 5 + axy\sin xz - b)\mathbf{i}$$
$$+ (3y^2\sin xz + 3xy^2z\cos xz - 3y^2\sin xz - 3xy^2z\cos xz)\mathbf{j}$$
$$+ (-ayz\sin xz - 6yz\sin xz)\mathbf{k}.$$

From this, it is easy to see that only the choices $a = -6$, $b = 5$ will work. Moreover, the resulting vector field is clearly defined on all of $\mathbf{R}^3$ (a simply-connected region), so the vanishing of the curl is enough to guarantee that $\mathbf{F}$ is conservative.

25. (a) As above we check that $\partial N/\partial x = 0 = \partial M/\partial y$, $\partial P/\partial x = 0 = \partial M/\partial z$, and $\partial P/\partial y = \cos y \cos z = \partial N/\partial z$. So $\mathbf{F}$ is conservative. $\mathbf{F} = \nabla(x^3/3 + \sin y \sin z)$.

(b) By Theorem 3.3,

$$\int_x \mathbf{F}\cdot d\mathbf{s} = f(\mathbf{x}(1)) - f(\mathbf{x}(0)) = f(2,e,e^2) - f(1,1,1) = 7/3 + \sin e\sin(e^2) - \sin^2 1.$$

26. Since $(7y-5x)_x = -5 = (3x-5y)_y$, $\mathbf{F} = (3x-5y)\mathbf{i} + (7y-5x)\mathbf{j}$ is conservative and the integral is path independent. We'll integrate along the path $\mathbf{x}(t) = (4t+1, -t+3)$ for $0 \le t \le 1$.

$$\int_C (3x-5y)\,dx + (7y-5x)\,dy = \int_0^1 [4(3(4t+1) - 5(-t+3)) - ((7(-t+3) - 5(4t+1))]\,dt$$
$$= \int_0^1 (95t - 64)\,dt = -\frac{33}{2}.$$

Using Theorem 3.3,

$$\int_C (3x-5y)\,dx + (7y-5x)\,dy = f(5,2) - f(1,3), \quad \text{where } \mathbf{F} = \nabla f.$$

In this case, $f(x,y) = 3x^2/2 - 5xy + 7y^2/2$. Therefore,

$$\int_C (3x-5y)\,dx + (7y-5x)\,dy = \left(\frac{3}{2}(25) - 5(5)(2) + \frac{7}{2}(4)\right) - \left(\frac{3}{2}(1) - 5(1)(3) + \frac{7}{2}(9)\right) = -\frac{33}{2}.$$

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27. Here

$$\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = -\frac{xy}{(x^2+y^2)^{3/2}} = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right).$$

So **F** is conservative so long as we restrict the domain. Our domain must be simply-connected and must contain the upper half of the circle of radius 2 centered at the origin. Our domain also must not contain the origin as **F** is not defined at the origin. We can choose, for example, the upper half disk of radius 3 centered at the origin minus the upper half disk of radius one centered at the origin. This "semi-annular" region meets all of our conditions. Therefore, the given integral is path independent. We'll integrate along the path $\mathbf{x}(t) = (2\cos t, 2\sin t)$, $0 \le t \le \pi$. The integral

$$\int_C \frac{x\,dy + y\,dx}{\sqrt{x^2+y^2}} = \int_0^\pi \frac{4\cos^2 t - 4\sin^2 t}{4\cos^2 t + 4\sin^2 t}\,dt = \int_0^\pi \cos 2t\,dt = \left.\frac{\sin 2t}{2}\right|_0^\pi = 0.$$

Using Theorem 3.3, and the fact that $\mathbf{F} = \nabla f$ where $f(x,y) = \sqrt{x^2+y^2}$,

$$\int_C \frac{x\,dy + y\,dx}{\sqrt{x^2+y^2}} = f(-2,0) - f(2,0) = \sqrt{(-2)^2+0} - \sqrt{2^2+0} = 0.$$

28. This time we check that three pairs of partial derivatives are equal:

$$\frac{\partial}{\partial x}(2x+z) = 2 = \frac{\partial}{\partial y}(2y-3z)$$
$$\frac{\partial}{\partial x}(y-3x) = -3 = \frac{\partial}{\partial z}(2y-3z)$$
$$\frac{\partial}{\partial y}(y-3z) = 1 = \frac{\partial}{\partial z}(2x+z).$$

We conclude that **F** is conservative, because the domain of **F** is all of $\mathbf{R}^3$. The given integral, therefore, is path independent. We'll integrate along the paths $\mathbf{x}_1(t) = (0,t,t)$, $0 \le t \le 1$, and $\mathbf{x}_2(t) = (t, t+1, 2t+1)$, $0 \le t \le 1$. The integral

$$\begin{aligned}\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz \\ &= \int_0^1 (0(-t) + 1(t) + 1(t))\,dt + \int_0^1 ((-4t-1) + (4t+1) + 2(-2t+1))\,dt \\ &= \int_0^1 2t\,dt + \int_0^1 (-4t+2)\,dt = 1 + 0 = 0.\end{aligned}$$

Using Theorem 3.3, and the fact that $\mathbf{F} = \nabla f$ where $f(x,y,z) = 2xy - 3xz + yz$, we obtain

$$\int_C (2y-3z)\,dx + (2x+z)\,dy + (y-3x)\,dz = f(1,2,3) - f(0,0,0) = 1.$$

In Exercises 29–32, to determine the work, we need to calculate line integrals of the form $\int_C \mathbf{F}\cdot d\mathbf{s}$, where C is an appropriate curve from A to B. To do this, we use the result of Theorem 3.3, since all of the vector fields in these exercises are conservative.

29. A potential function for **F** is easily calculated to be $f(x,y) = x^3y - xy^2$. Thus, for any curve C from $(0,0)$ to $(2,1)$ the work is

$$\int_C \mathbf{F}\cdot d\mathbf{s} = f(2,1) - f(0,0) = 6 - 0 = 6.$$

30. A potential function for **F** is $f(x,y) = 2x^{3/2}y$. Thus the work is

$$f(9,1) - f(1,2) = 54 - 4 = 50.$$

31. A potential function for **F** is $f(x,y,z) = x^2yz - xy^2z^3$. Therefore, the work is

$$f(6,4,2) - f(1,1,1) = -480 - 0 = -480.$$

32. A potential function for **F** is $f(x,y,z) = x^2y\cos z$. Hence the work is

$$f(2,3,0) - f(1,1,\pi/2) = 12 - 0 = 12.$$

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33. (a) We'll check to see where $N_x = M_y$.

$$\frac{\partial}{\partial x}\left(\frac{x^2+1}{y^3}\right) = \frac{2x}{y^3} = \frac{\partial}{\partial y}\left(\frac{x+xy^2}{y^2}\right),$$

therefore, **F** is conservative on each of the two simply-connected sets on which it is defined. More precisely, **F** is conservative on $\{(x,y)|y>0\}$ and on $\{(x,y)|y<0\}$.

(b) The scalar potential is $f(x,y) = \dfrac{x^2+x^2y^2+1}{2y^2}$.

(c) As the particle moves from (0, 1) to (1, 1) along the parabola $y = 1 + x - x^2$ we note that $y > 0$ and so the path lies entirely in one of the simply-connected regions. We can, therefore, apply Theorem 3.3 and calculate the work done as $f(1,1) - f(0,1) = 3/2 - 1/2 = 1$.

34. (a) We need to check that three pairs of partial derivatives are equal:

$$\frac{\partial}{\partial x}(x+g(y)+z) = 1 = \frac{\partial}{\partial y}(f(x)+y+z)$$
$$\frac{\partial}{\partial x}(x+y+h(z)) = 1 = \frac{\partial}{\partial z}(f(x)+y+z)$$
$$\frac{\partial}{\partial y}(x+y+h(z)) = 1 = \frac{\partial}{\partial z}(x+g(y)+z).$$

(b) $\mathbf{F} = \nabla\phi(x,y,z)$ where, for constants a, b, and c,

$$\phi(x,y,z) = xy + xz + yz + \int_a^x f(t)\,dt + \int_b^y g(t)\,dt + \int_c^z h(t)\,dt.$$

(c) Using Theorem 3.3,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \phi(x_1,y_1,z_1) - \phi(x_0,y_0,z_0)\\ &= x_1y_1 - x_0y_0 + x_1z_1 - x_0z_0 + y_1z_1 - y_0z_0 + \int_{x_0}^{x_1} f(t)\,dt + \int_{y_0}^{y_1} g(t)\,dt + \int_{z_0}^{z_1} h(t)\,dt.\end{aligned}$$

35. (a) **F** is conservative since $\mathbf{F} = \nabla f$ where $f(x,y,z) = \sin(x^2+xz) + \cos(y+yz)$.

(b) Since we have a potential function,

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} \nabla f\cdot d\mathbf{s} &= f(\mathbf{x}(1)) - f(\mathbf{x}(0))\\ &= f(1,1,\pi-1) - f(0,0,0) = -2.\end{aligned}$$

36. (a) $\mathbf{G} = \mathbf{F} + x\mathbf{j}$, where **F** is given in Exercise 35. Now $\nabla\times\mathbf{G} = \nabla\times\mathbf{F} + \nabla\times(x\mathbf{j}) = \mathbf{k} \neq \mathbf{0}$, so **G** is not conservative.

(b) Here we have that

$$\int_{\mathbf{x}} \mathbf{G}\cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{F}+x\mathbf{j})\cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{x}} x\mathbf{j}\cdot d\mathbf{s}.$$

From Exercise 35, we have that $\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = -2$, so

$$\begin{aligned}\int_{\mathbf{x}} \mathbf{G}\cdot d\mathbf{s} &= -2 + \int_0^1 (0,t^3,0)\cdot\left(3t^2, 2t, \pi - \frac{\pi}{2}\cos\frac{\pi t}{2}\right)dt\\ &= -2 + \int_0^1 2t^4\,dt = -2 + 2/5 = -\frac{8}{5}.\end{aligned}$$

37. You could check that **F** is conservative by confirming that $\nabla\times\mathbf{F} = \mathbf{0}$ on any simply-connected region that misses the origin. It is, however, easy enough to find the scalar potential for **F** is $f(x,y,z) = GMm(x^2+y^2+z^2)^{-1/2}$. So the work done by **F** as a particle of mass m moves from $\mathbf{x}_0$ to $\mathbf{x}_1$ is

$$f(x_1,y_1,z_1) - f(x_0,y_0,z_0) = \frac{GMm}{\sqrt{x_1^2+y_1^2+z_1^2}} - \frac{GMm}{\sqrt{x_0^2+y_0^2+z_0^2}} = GMm\left(\frac{1}{\|\mathbf{x}_1\|} - \frac{1}{\|\mathbf{x}_0\|}\right).$$

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True/False Exercises for Chapter 6

1. True.
2. False. (The value is 2.)
3. False. (The integral is negative.)
4. True.
5. False. (The integral is 0.)
6. True.
7. False. (There is equality only up to sign.)
8. False.
9. True.
10. True.
11. True.
12. False. (∇f is everywhere normal to C.)
13. True.
14. False. (The work is at most 3 times the length of C.)
15. False. (The line integral could be $\pm \int_C \|\mathbf{F}\|\, ds$, depending on whether **F** points in the same or the opposite direction as C.)
16. True. (Just use Green's theorem.)
17. False. (Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and consider Green's theorem.)
18. True. (Use the divergence theorem in the plane.)
19. False. (Under appropriate conditions, the integral is $f(B) - f(A)$.)
20. True.
21. True.
22. False. (There's a negative sign missing.)
23. False. (For the vector field to be conservative, the line integral must be zero for *all* closed curves, not just a particular one.)
24. True.
25. False. (The vector field $(e^x \cos y \sin z, e^x \sin y \sin z, e^x \cos y \cos z)$ is not conservative.)
26. False. (**F** must be of class C^1 on a simply-connected region.)
27. False. (The domain is not simply-connected.)
28. True.
29. False. (f is only defined up to a constant.)
30. True.

Miscellaneous Exercises for Chapter 6

1. Partition the curve into n pieces each of length $\Delta s_k = (\text{length of } C)/n$. The right side of the given formula is just our calculation of arclength for a rectifiable curve:

$$\frac{\int_C f\, ds}{\text{length of } C} = \frac{\int_C f\, ds}{\int_C ds}.$$

Now, if on the kth sub-interval we choose any $\mathbf{c}_k$, then on the interval $f(\mathbf{x}) \approx f(\mathbf{c}_k)$. Therefore,

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n} f(\mathbf{c}_k)\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{f(\mathbf{c}_k)}{n} = \frac{\lim_{n\to\infty}\sum_{k=1}^{n} f(\mathbf{c}_k)\Delta s_k}{\text{length of } C} = \frac{\int_C f\, ds}{\text{length of } C}.$$

For each value of n we are calculating an average of n values of f at points on the curve. As n grows large, if this limit exists, it is reasonable to define it as the **average value** of f along C.

2. Here $f(\mathbf{x}(t)) = 2 + 2t^2$, and $\|\mathbf{x}'(t)\| = \sqrt{2}$. Therefore,

$$[f]_{\text{avg}} = \frac{\int_C f\, ds}{\int_C ds} = \frac{\int_0^{3\pi}((2t^2+2)\sqrt{2})\, dt}{\int_0^{3\pi}\sqrt{2}\, dt} = \frac{2}{3\pi}\int_0^{3\pi}(t^2+1)\, dt = \frac{2}{3\pi}(9\pi^3 + 3\pi) = 6\pi^2 + 2.$$

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3. We may parametrize the semicircle as $\mathbf{x}(t) = (a\cos t, a\sin t)$, where $0 \le t \le \pi$. Therefore, $\|\mathbf{x}'(t)\| = a$. The length of the semicircle is πa and so the average y-coordinate may be found by calculating

$$\frac{1}{\pi a}\int_0^\pi a\sin t\cdot a\,dt = \frac{a}{\pi}\int_0^\pi \sin t\,dt = \frac{2a}{\pi}.$$

4. Calculate $[z]_{\text{avg}}$ as $\dfrac{\int_C z\,ds}{\text{length of } C}$. The total length of C is just the sum of the lengths of four straight segments:

$$2 + 1 + \sqrt{4+0+1} + \sqrt{1+1+1} = 3 + \sqrt{5} + \sqrt{3}.$$

Now $\int_C z\,ds = \int_{C_1} z\,ds + \cdots + \int_{C_4} z\,ds$, but z is clearly zero on two of the four segments.

The segment C_3 joining (2, 1, 0) and (0, 1, 1) may be parametrized as

$$\begin{aligned}\mathbf{x}(t) &= (1-t)(2,1,0) + t(0,1,1), \quad 0 \le t \le 1\\ &= (2-2t, 1, t).\end{aligned}$$

Thus $\mathbf{x}'(t) = (-2, 0, 1)$ and $\|\mathbf{x}'(t)\| = \sqrt{5}$. Therefore,

$$\int_{C_3} z\,ds = \int_0^1 t\cdot\sqrt{5}\,dt = \frac{\sqrt{5}}{2}.$$

The segment C_4 joining (0, 1, 1) and (1, 0, 2) may be parametrized as

$$\begin{aligned}\mathbf{x}(t) &= (1-t)(0,1,1) + t(1,0,2), \quad 0 \le t \le 1\\ &= (t, 1-t, t+1).\end{aligned}$$

Thus $\mathbf{x}'(t) = (1, -1, 1)$ and $\|\mathbf{x}'(t)\| = \sqrt{3}$. Hence

$$\int_{C_4} z\,ds = \int_0^1 (t+1)\sqrt{3}\,dt = \sqrt{3}\left(\frac{t^2}{2} + t\right)\Big|_0^1 = \frac{3\sqrt{3}}{2}.$$

Putting all this together, we find

$$[z]_{\text{avg}} = \frac{\sqrt{5}/2 + 3\sqrt{3}/2}{3+\sqrt{5}+\sqrt{3}} = \frac{\sqrt{5}+3\sqrt{3}}{2(3+\sqrt{5}+\sqrt{3})} \approx 0.5333.$$

5. The curve may be parametrized as $\mathbf{x}(t) = (\sqrt{5}\cos t, \sin t, 2\sin t), 0 \le t \le 2\pi$. Then $\|\mathbf{x}'(t)\| = \sqrt{(-\sqrt{5}\sin t)^2 + (\cos t)^2 + (2\cos t)^2} = \sqrt{5}$ so the length of C is $\int_0^{2\pi}\sqrt{5}\,dt = 2\pi\sqrt{5}$. Now

$$\begin{aligned}\int_C f\,ds &= \int_0^{2\pi}(4\sin^2 t + \sqrt{5}\cos t\cdot e^{\sin t})\sqrt{5}\,dt\\ &= \sqrt{5}\int_0^{2\pi}(2(1-\cos 2t) + \sqrt{5}\,e^{\sin t}\cdot\cos t)\,dt\\ &= \sqrt{5}(2t - \sin 2t + \sqrt{5}\,e^{\sin t})\Big|_0^{2\pi} = \sqrt{5}(4\pi + \sqrt{5} - 0 + 0 - \sqrt{5})\\ &= 4\pi\sqrt{5}.\end{aligned}$$

Hence $[f]_{\text{avg}} = \dfrac{4\sqrt{5}\pi}{2\sqrt{5}\pi} = 2.$

6. (a) For the total mass we integrate the density along the curve:

$$\int_C (3-y)\,ds = \int_0^\pi 2(3 - 2\sin t)\,dt = 6\pi - 8.$$

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(b) The density depends only on y and the wire is symmetric with respect to x so $\bar{x} = 0$ (if you write out the formula you'll see that the numerator is an integral of an odd function of x over a curve that is symmetric with respect to x). Also, since $z \equiv 0$, we quickly conclude that $\bar{z} = 0$. What remains is to calculate

$$\bar{y} = \frac{\int_C y\delta(x,y,z)\,ds}{\int_C \delta(x,y,z)\,ds} = \frac{\int_0^\pi (2\sin t(3 - 2\sin t)2)\,dt}{6\pi - 8} = \frac{24 - 4\pi}{6\pi - 8} = \frac{12 - 2\pi}{3\pi - 4}.$$

7. Locate the wire in the first quadrant of the xy-plane. Then the center is at $\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ and $\delta(x,y,z) = \left(x - \frac{a}{\sqrt{2}}\right)^2 + \left(y - \frac{a}{\sqrt{2}}\right)^2$. From symmetry considerations, we must have $\overline{x} = \overline{y}$. Now parametrize the quarter circle as

$$\begin{cases} x = a\cos t \\ y = a\sin t, \quad 0 \le t \le \pi/2. \end{cases}$$

Then $\|\mathbf{x}'(t)\| = a$. We have

$$\begin{aligned}
M &= \int_C \delta\,ds = \int_0^{\pi/2} \left(\left(a\cos t - \frac{a}{\sqrt{2}}\right)^2 + \left(a\sin t - \frac{a}{\sqrt{2}}\right)^2\right) a\,dt \\
&= a^3 \int_0^{\pi/2} \left(\cos^2 t - \sqrt{2}\cos t + \frac{1}{2} + \sin^2 t - \sqrt{2}\sin t + \frac{1}{2}\right) dt \\
&= a^3 \int_0^{\pi/2} (2 - \sqrt{2}\cos t - \sqrt{2}\sin t)\,dt = (\pi - 2\sqrt{2})a^3 \\
M_{yz} &= \int_C x\delta\,ds = \int_0^{\pi/2} a\cos t\left(\left(a\cos t - \frac{a}{\sqrt{2}}\right)^2 + \left(a\sin t - \frac{a}{\sqrt{2}}\right)^2\right) \cdot a\,dt \\
&= a^4 \int_0^{\pi/2} \cos t(2 - \sqrt{2}\cos t - \sqrt{2}\sin t)\,dt = a^4 \int_0^{\pi/2} \left(2\cos t - \frac{\sqrt{2}}{2}(1 + \cos 2t) - \sqrt{2}\sin t\cos t\right) dt \\
&= a^4 \left(2\sin t - \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{4}\sin 2t - \frac{\sqrt{2}}{2}\sin^2 t\right)\Big|_0^{\pi/2} \\
&= a^4 \left(2 - \frac{\sqrt{2}\pi}{4} - 0 - \frac{\sqrt{2}}{2} - 0\right) \\
&= \left(\frac{8 - \sqrt{2}\pi - 2\sqrt{2}}{4}\right) a^4.
\end{aligned}$$

Hence

$$\bar{x} = \bar{y} = \frac{(8 - \sqrt{2}\pi - 2\sqrt{2})a^4}{4} \Big/ (\pi - 2\sqrt{2})a^3 = \left(\frac{8 - \sqrt{2}\pi - 2\sqrt{2}}{4(\pi - 2\sqrt{2})}\right) a.$$

8. (a) By symmetry $\bar{x} = \bar{y} = 0$ and $\bar{z} = 8\pi$.
(b) $\|\mathbf{x}'\| = \sqrt{9 + 16} = 5$; $\delta(x,y,z) = x^2 + y^2 + z^2$. Hence the mass of the wire is

$$\begin{aligned}
M &= \int_{\mathbf{x}} \delta\,ds = \int_0^{4\pi} (9 + 16t^2) \cdot 5\,dt = \frac{20\pi}{3}(27 + 256\pi^2) \\
\bar{x} &= \frac{1}{M}\int_{\mathbf{x}} x\delta\,ds = \frac{1}{M}\int_0^{4\pi} 3\cos t(9 + 16t^2) \cdot 5\,dt = \frac{1}{M} \cdot 1920\pi \\
\bar{y} &= \frac{1}{M}\int_{\mathbf{x}} y\delta\,ds = \frac{1}{M}\int_0^{4\pi} 3\sin t(9 + 16t^2) \cdot 5\,dt = \frac{1}{M}(-3840\pi^2) \\
\bar{z} &= \frac{1}{M}\int_{\mathbf{x}} z\delta\,ds = \frac{1}{M}\int_0^{4\pi} 4t(9 + 16t^2) \cdot 5\,dt = \frac{1}{M}(160\pi^2)(9 + 128\pi^2).
\end{aligned}$$

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Thus

$$\bar{x} = \frac{3 \cdot 1920\pi}{20\pi(27 + 256\pi^2)} = \frac{288}{27 + 256\pi^2} \approx 0.112781$$
$$\bar{y} = \frac{3(-3840\pi^2)}{20\pi(27 + 256\pi^2)} = -\frac{576\pi}{27 + 256\pi^2} \approx -0.708625$$
$$\bar{z} = \frac{3 \cdot 160\pi^2(9 + 128\pi^2)}{20\pi(27 + 256\pi^2)} = \frac{24\pi(9 + 128\pi^2)}{27 + 256\pi^2} \approx 37.5662.$$

9. **(a)** Parametrize the wire as $\mathbf{x}(t) = (2\cos t, 2\sin t), 0 \le t \le \pi$. Then $\|\mathbf{x}'\| = 2$ and

$$\begin{aligned} I_y &= \int_C x^2 \delta\, ds = \int_0^\pi 4\cos^2 t(3 - 2\sin t) \cdot 2\, dt \\ &= \int_0^\pi (24\cos^2 t - 16\cos^2 t \sin t)\, dt = \int_0^\pi (12(1 + \cos 2t) + 16\cos^2 t(-\sin t))\, dt \\ &= 12\pi + \frac{16}{3}(-1) - \frac{16}{3}(1) = 12\pi - \frac{32}{3} = \frac{36\pi - 32}{3}. \end{aligned}$$

(b) The square of the distance between a point on the wire and the z-axis is $x^2 + y^2$. Thus $I_z = \int_C (x^2 + y^2)\delta(x, y, z)\, ds$. Using the given information,

$$I_z = \int_0^\pi 4 \cdot (3 - 2\sin t) 2\, dt = 8(3\pi - 4) = 24\pi - 32.$$

The total mass was found in Exercise 6 to be $6\pi - 8$. Hence the radius of gyration is

$$r_z = \sqrt{\frac{24\pi - 32}{6\pi - 8}} = 2.$$

10. Parametrize the curve as $\mathbf{x}(t) = \left(t, \frac{t}{2} + 2\right), -2 \le t \le 2$. Then $\|\mathbf{x}'\| = \frac{\sqrt{5}}{2}$ and

$$\begin{aligned} I_x &= \int_C y^2 \delta\, ds = \int_{-2}^2 \left(\frac{t}{2} + 2\right)^2 \left(\frac{t}{2} + 2\right) \cdot \frac{\sqrt{5}}{2}\, dt = \sqrt{5}\int_{-2}^2 \left(\frac{t}{2} + 2\right)^3 \cdot \frac{1}{2}\, dt \\ &= \sqrt{5}\int_1^3 u^3\, du = \frac{\sqrt{5}}{4}u^4\Big|_1^3 = 20\sqrt{5} \\ M &= \int_C \delta\, ds = \int_{-2}^2 \left(\frac{t}{2} + 2\right)\frac{\sqrt{5}}{2}\, dt = \frac{\sqrt{5}}{2}\left(\frac{1}{4}t^2 + 2t\right)\Big|_{-2}^2 = 4\sqrt{5} \end{aligned}$$

Hence

$$r_x = \sqrt{I_x/M} = \sqrt{\frac{20\sqrt{5}}{4\sqrt{5}}} = \sqrt{5}.$$

11. We use x as parameter so $\mathbf{x}(t) = (t, t^2),\ 0 \le t \le 2$, and $\|\mathbf{x}'\| = \sqrt{1 + 4t^2}$. Then

$$I_x = \int_C y^2 \delta\, ds = \int_0^2 t^4 \cdot t\sqrt{1 + 4t^2}\, dt = \int_0^2 t^5\sqrt{1 + 4t^2}\, dt.$$

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Now let $2t = \tan\theta$ so $dt = \dfrac{1}{2}\sec^2\theta\,d\theta$. Then

$$\begin{aligned}
I_x &= \int_0^{\tan^{-1}4} \frac{1}{32}\tan^5\theta\ \sec\theta\left(\frac{1}{2}\sec^2\theta\,d\theta\right)\\
&= \frac{1}{64}\int_0^{\tan^{-1}4}\tan^4\theta\ \sec^2\theta(\sec\theta\ \tan\theta\,d\theta)\\
&= \frac{1}{64}\int_0^{\tan^{-1}4}(\sec^2\theta-1)^2\sec^2\theta(\sec\theta\ \tan\theta\,d\theta)\\
&= \frac{1}{64}\int_0^{\tan^{-1}4}(\sec^6\theta-2\sec^4\theta+\sec^2\theta)\,d(\sec\theta)\\
&= \frac{1}{64}\left(\frac{1}{7}\sec^7\theta-\frac{2}{5}\sec^5\theta+\frac{1}{3}\sec^3\theta\right)\Big|_0^{\tan^{-1}4}\\
&= \frac{1}{64}\left(\frac{1}{7}\,17^{7/2}-\frac{2}{5}\,17^{5/2}+\frac{1}{3}\,17^{3/2}-\frac{1}{7}+\frac{2}{5}-\frac{1}{3}\right)\\
&= \frac{7769\sqrt{17}-1}{840}\approx 38.1326.
\end{aligned}$$

We also have

$$\begin{aligned}
M = \int_C \delta\,ds = \int_0^2 t\sqrt{1+4t^2}\,dt &= \frac{1}{8}\cdot\frac{2}{3}(1+4t^2)^{3/2}\Big|_0^2\\
&= \frac{1}{12}(17^{3/2}-1)\approx 5.75773.
\end{aligned}$$

Hence

$$\begin{aligned}
r_x = \sqrt{\frac{I_x}{M}} &= \sqrt{\frac{7769\sqrt{17}-1}{840}\cdot\frac{12}{17\sqrt{17}-1}} = \sqrt{\frac{7769\sqrt{17}-1}{1190\sqrt{17}-70}}\\
&\approx 2.57349.
\end{aligned}$$

12. **(a)** $I_x = \int_C (y^2+z^2)\delta(x,y,z)\,ds$, $I_y = \int_C (x^2+z^2)\delta(x,y,z)\,ds$, $I_z = \int_C (x^2+y^2)\delta(x,y,z)\,ds$

(b) For the given parametrization, $\|\mathbf{x}'\| = \sqrt{9+16} = 5$.

$$\begin{aligned}
I_x &= 5\delta\int_0^{4\pi}(9\sin^2 t+16t^2)\,dt = 5\delta\int_0^{4\pi}\left(\frac{9}{2}(1-\cos 2t)+16t^2\right)\,dt\\
&= 5\delta\left(18\pi+\frac{1024\pi^3}{3}\right) = \frac{10\pi(27+512\pi^2)\delta}{3}\\
I_y &= 5\delta\int_0^{4\pi}(9\cos^2 t+16t^2)\,dt = 5\delta\int_0^{4\pi}\left(\frac{9}{2}(1+\cos 2t)+16t^2\right)\,dt\\
&= \frac{10\pi(27+512\pi^2)\delta}{3}\\
I_z &= 5\delta\int_0^{4\pi}9\,dt = 180\pi\delta
\end{aligned}$$

Now $M = \displaystyle\int_C \delta\,ds = \int_0^{4\pi}5\delta\,dt = 20\delta$. Thus

$$r_x = r_y = \sqrt{\frac{\pi(27+512\pi^2)}{6}},\quad r_z = \sqrt{\frac{180\pi\delta}{20\delta}} = 3\sqrt{\pi}.$$

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13. We may parametrize the segment as $\mathbf{x}(t) = (1-t)(-1,1,2) + t(2,2,3)$, $0 \le t \le 1$, or $\mathbf{x}(t) = (3t-1, t+1, t+2)$. Then $\|\mathbf{x}'\| = \sqrt{9+1+1} = \sqrt{11}$.

$$\begin{aligned} I_z &= \int_C (x^2+y^2)\delta\, ds = \int_0^1 [(3t-1)^2 + (t+1)^2][1+(t+2)^2] \cdot \sqrt{11}\, dt \\ &= \sqrt{11}\int_0^1 (10t^4 + 36t^3 + 36t^2 - 12t + 10)\, dt = \sqrt{11}(2t^5 + 9t^4 + 12t^3 - 6t^2 + 10t)\Big|_0^1 \\ &= \sqrt{11}(2+9+12-6+10) = 27\sqrt{11} \\ M &= \int_C \delta\, ds = \int_0^1 (1+(t+2)^2)\sqrt{11}\, dt = \sqrt{11}\left(t + \frac{1}{3}(t+2)^3\right)\Big|_0^1 \\ &= \sqrt{11}\left(1+9-\frac{8}{3}\right) = \frac{22\sqrt{11}}{3}. \end{aligned}$$

Hence $r_z = \sqrt{I_z/M} = \sqrt{\dfrac{81}{22}} = \dfrac{9\sqrt{22}}{22}$.

Exercises 14, 15, and 23 explore polar versions of results we've seen in this Chapter.

14. (a) The path is

$$\mathbf{x}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta).$$

Using the product rule, we find that

$$\mathbf{x}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta).$$

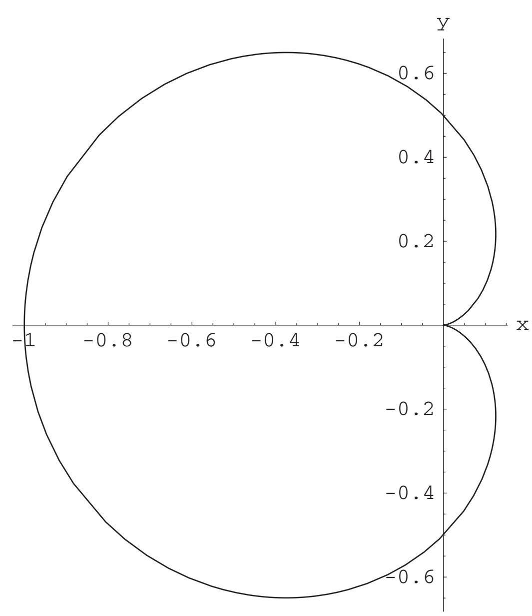
The length of $\mathbf{x}'(\theta)$ is a straightforward calculation:

$$\|\mathbf{x}'(t)\| = \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2} = \sqrt{(f'(\theta))^2 + (f(\theta))^2}.$$

We conclude that the arclength of the curve between $(f(a), a)$ and $(f(b), b)$ is

$$\int_C ds = \int_{\mathbf{x}(\theta)} \|\mathbf{x}'(\theta)\|\, d\theta = \int_a^b \sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$$

(b) The sketch of $r = \sin^2(\theta/2)$ is



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The length is

$$\int_0^{2\pi} \sqrt{\sin^4(\theta/2) + \sin^2(\theta/2)\cos^2(\theta/2)}\, d\theta = \int_0^{2\pi} \sin(\theta/2)\, d\theta = -2(-1-1) = 4.$$

15. (a) $\oint_C g(x,y)\, ds = \int_{\mathbf{x}(\theta)} g(\mathbf{x}(\theta))\|\mathbf{x}'(\theta)\|\, d\theta = \int_a^b g(f(\theta)\cos\theta, f(\theta)\sin\theta)\sqrt{(f(\theta))^2 + (f'(\theta))^2}\, d\theta.$

(b) We'll use the formula from part (a).

$$\begin{aligned}\int_C g\, ds &= \int_0^{2\pi} ([(e^{3\theta}\cos\theta)^2 + (e^{3\theta}\sin\theta)^2 - 2(e^{3\theta}\cos\theta)]\sqrt{e^{6\theta} + 9e^{6\theta}})\, d\theta \\ &= \int_0^{2\pi} ([e^{6\theta} - 2e^{3\theta}\cos\theta]\sqrt{10}e^{3\theta})\, d\theta \\ &= \frac{\sqrt{10}}{333}(37e^{18\pi} - 108e^{12\pi} + 71).\end{aligned}$$

In this text κ is always non-negative. In cases where the curvature is signed, differential geometers are often interested in the **total squared curvature**: $\int_C \kappa^2\, ds$.

16. In Section 3.2 it was shown that

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}.$$

Here $\mathbf{v} = \mathbf{x}'$ and $\mathbf{a} = \mathbf{x}''$. So

$$K = \int_C \kappa\, ds = \int_a^b \left(\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3}\|\mathbf{x}'\|\right) dt = \int_a^b \left(\frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^2}\right) dt.$$

17. We use the results of Exercise 16:

$$\begin{aligned}K &= \int_0^{10\pi} \frac{\|(-3\sin t, 3\cos t, 4) \times (-3\cos t, -3\sin t, 0)\|}{\|(-3\sin t, 3\cos t, 4)\|^2}\, dt \\ &= \int_0^{10\pi} \frac{\|(-12\sin t, -12\cos t, 9)\|}{25}\, dt = \int_0^{10\pi} \frac{3}{5}\, dt = 6\pi.\end{aligned}$$

18. We use the results of Exercise 16 with $\mathbf{x}(t) = (t, At^2, 0)$:

$$\begin{aligned}K &= \int_a^b \frac{\|(1, 2At, 0) \times (0, 2A, 0)\|}{\|(1, 2At, 0)\|^2}\, dt = \int_a^b \frac{\|(0,0,2A)\|}{1 + 4A^2t^2}\, dt \\ &= \int_a^b \frac{2A}{1 + 4A^2t^2}\, dt = \tan^{-1}(2At)\Big|_a^b = \tan^{-1}(2Ab) - \tan^{-1}(2Aa).\end{aligned}$$

19. We parameterize the ellipse by the path $\mathbf{x}(t) = (a\cos t, b\sin t, 0)$ for $0 \le t \le 2\pi$. Then, using Exercise 16,

$$\begin{aligned}K &= \int_0^{2\pi} \frac{\|(-a\sin t, b\cos t, 0) \times (-a\cos t, -b\sin t, 0)\|}{\|(-a\sin t, b\cos t, 0)\|^2}\, dt = \int_0^{2\pi} \frac{\|(0,0,ab)\|}{a^2\sin^2 t + b^2\cos^2 t}\, dt \\ &= \int_0^{2\pi} \frac{ab}{a^2\sin^2 t + b^2\cos^2 t}\, dt = 2\pi.\end{aligned}$$

This verifies Fenchel's Theorem for the given ellipse. This final integral was calculated using *Mathematica*. With work it can also be done by hand.

20. (a) By Fenchel's theorem (see Exercise 19), we know that for C (a simple, closed C^1 curve in $\mathbf{R}^3$), $K \ge 2\pi$, so

$$K = \int_C \kappa\, ds \ge 2\pi.$$

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But $0 \le \kappa \le 1/a$, therefore

$$K = \int_C \kappa\, ds \le \int_C \frac{1}{a}\, ds = \frac{1}{a}\int_C ds = \frac{L}{a}.$$

Putting these two inequalities together we see that

$$\frac{L}{a} \ge K \ge 2\pi \quad \text{so} \quad L \ge 2\pi a.$$

(b) To conclude that $L = 2\pi a$ we would need both of the preliminary inequalities to be equalities. As we saw in Exercise 19, we have $K = 2\pi$ when C is also a plane convex curve. Also, as we saw above, $K = L/a$ when $\kappa = 1/a$. Together, these two conditions imply that C is a circle of radius a.

21. The work done is

$$\int_{\mathbf{x}} \mathbf{F}\cdot ds = \int_0^1 (\sin(t^3), \cos(-t^2), t^4)\cdot(3t^2, -2t, 1)\, dt = \int_0^1 (3t^2\sin(t^3) - 2t\cos(-t^2) + t^4)\, dt$$

$$= (-\cos(t^3) + \sin(-t^2) + t^5/5)\Big|_0^1 = 6/5 - \cos 1 - \sin 1.$$

22. The first thing to note is that we are traversing the path in the wrong direction to apply Green's theorem. If C_1 is the triangular path described in the problem from the origin, to (0, 1), to (1, 0), back to the origin, then let C_2 be the path traversed in the opposite direction and let D be the region bounded by C_1 and C_2. Then

$$\oint_{C_1} x^2y\, dx + (x+y)y\, dy = -\oint_{C_2} x^2y\, dx + (x+y)y\, dy = -\iint_D (y - x^2)\, dA$$

$$= -\int_0^1\int_0^{1-x} (y - x^2)\, dy\, dx = -\int_0^1 (y^2/2 - x^2y)\Big|_0^{1-x}$$

$$= -\int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} + x^3\right) dx = -\frac{1}{2}\left(x - x^2 - \frac{x^3}{3} + \frac{x^4}{2}\right)\Big|_0^1 = -\frac{1}{12}.$$

23. In Section 6.2 we saw that Green's theorem implied the formula

$$\text{Area} = \frac{1}{2}\oint_{\partial D} -y\, dx + x\, dy.$$

In general the boundary ∂D of the region D consists of the curve $r = f(\theta)$, which may be parametrized by $\mathbf{x}(\theta) = (x(\theta), y(\theta)) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$, and possibly straight line segments along $\theta = a$ and $\theta = b$. The line $\theta = a$ may be parametrized by $\mathbf{y}(r) = (x(r), y(r)) = (r\cos a, r\sin a)$ and the line $\theta = b$ may be parametrized similarly. Note that, along the straight segment C_1 given by $\theta = a$, we have

$$\frac{1}{2}\int_{C_1} -y\, dx + x\, dy = \frac{1}{2}\int_0^{f(a)} (-r\sin a\cos a + r\cos a\sin a)\, dr = 0.$$

An identical result holds for the straight segment C_2 given by $\theta = b$. Therefore, the area of D may be evaluated by computing the line integral over the path $\mathbf{x}$ described above:

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\mathbf{x}} -y\, dx + x\, dy \\ &= \frac{1}{2}\int_a^b ((-f(\theta)\sin\theta)(f'(\theta)\cos\theta - f(\theta)\sin\theta) \\ &\qquad + (f(\theta)\cos\theta)(f'(\theta)\sin\theta + f(\theta)\cos\theta))\, d\theta \\ &= \frac{1}{2}\int_a^b (f(\theta))^2\, d\theta.\end{aligned}$$

24. By Green's theorem, if D is the region with $C = \partial D$,

$$\oint_C f(x)\, dx + g(y)\, dy = \iint_D \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x))\right) dx\, dy = \iint_D 0\, dx\, dy = 0.$$

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25. We begin by applying Green's theorem (here D has constant density δ):

$$\frac{1}{2\cdot\text{area of }D}\oint_{\partial D} x^2\,dy = \frac{1}{2\cdot\text{area of }D}\iint_D 2x\,dx\,dy = \frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy}$$
$$= \frac{\iint_D x\,\delta\,dx\,dy}{\iint_D \delta\,dx\,dy} = \bar{x}.$$

Similarly,

$$\frac{1}{\text{area of }D}\oint_{\partial D} xy\,dy = \frac{1}{\text{area of }D}\iint_D y\,dx\,dy = \frac{\iint_D y\,dx\,dy}{\iint_D dx\,dy}$$
$$= \frac{\iint_D y\,\delta\,dx\,dy}{\iint_D \delta\,dx\,dy} = \bar{y}.$$

For the second pair of formulas, we proceed in an entirely similar manner with Green's theorem.

$$-\frac{1}{\text{area of }D}\oint_{\partial D} xy\,dx = -\frac{1}{\text{area of }D}\iint_D -x\,dx\,dy$$
$$= \frac{1}{\text{area of }D}\iint_D x\,dx\,dy = \frac{\iint_D x\,dx\,dy}{\iint_D dx\,dy}$$
$$= \frac{\iint_D x\,\delta\,dx\,dy}{\iint_D \delta\,dx\,dy} = \bar{x}.$$

Also,

$$-\frac{1}{2\cdot\text{area of }D}\oint_{\partial D} y^2\,dx = -\frac{1}{2\cdot\text{area of }D}\iint_D -2y\,dx\,dy$$
$$= \frac{1}{\text{area of }D}\iint_D y\,dx\,dy = \frac{\iint_D y\,\delta\,dx\,dy}{\iint_D \delta\,dx\,dy} = \bar{y}.$$

26. Along the bottom of the triangle, dy is zero and along the left side x is zero, so the first pair of line integrals in Exercise 25 must be zero except along the side connecting (1, 0) to (0, 2). Parametrize this side by $\mathbf{x}(t) = (1-t, 2t)$ with $0 \le t \le 1$. Note also that the area of the triangle is 1. Using the results of Exercise 25, we have

$$\bar{x} = \frac{1}{2}\oint_{\partial D} x^2\,dy = \frac{1}{2}\int_0^1 (1-t)^2 2\,dt = \int_0^1 (t^2 - 2t + 1)\,dt$$
$$= \frac{1}{3} - 1 + 1 = \frac{1}{3}$$

and

$$\bar{y} = \oint_{\partial D} xy\,dy = \int_0^1 (1-t)(2t)2\,dt = 2\int_0^1 (-2t^2 + 2t)\,dt$$
$$= 2\left(-\frac{2}{3} + 1\right) = \frac{2}{3}.$$

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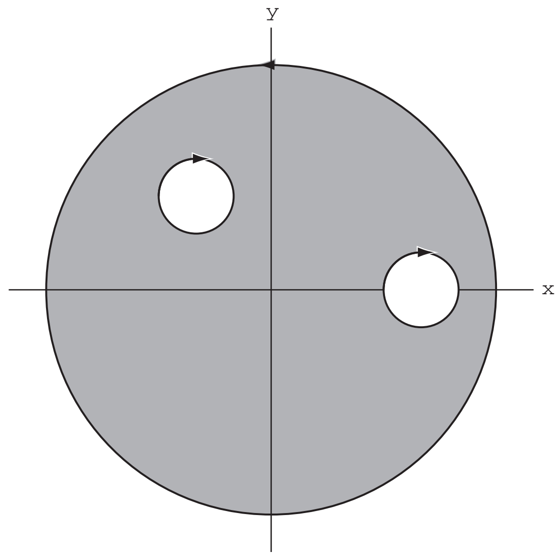
27. The region in question looks like:



The area of this region is

$$36\pi - \pi - \pi = 34\pi.$$

Using the result of Exercise 25, we calculate

$$\bar{x} = \frac{1}{2 \cdot \text{area of } D} \oint_{\partial D} x^2\, dy \quad \text{and} \quad \bar{y} = -\frac{1}{2 \cdot \text{area of } D} \oint_{\partial D} y^2\, dx$$

(other computations are possible).

We may parametrize the outer boundary of the region by

$$\mathbf{x}(t) = (6\cos t, 6\sin t), \quad 0 \le t \le 2\pi$$

and the inner two circles by

$$\mathbf{y}(t) = (4 + \sin t, \cos t), 0 \le t \le 2\pi \text{ and}$$
$$\mathbf{z}(t) = (\sin t - 2, \cos t + 2), 0 \le t \le 2\pi.$$

Hence

$$\begin{aligned}
\bar{x} &= \frac{1}{68\pi} \oint_{\partial D} x^2\, dy \\
&= \frac{1}{68\pi}\left[\int_0^{2\pi} 36\cos^2 t \cdot 6\cos t\, dt + \int_0^{2\pi} (\sin t + 4)^2(-\sin t)\, dt + \int_0^{2\pi} (\sin t - 2)^2(-\sin t)\, dt\right] \\
&= \frac{1}{68\pi}\left[\int_0^{2\pi} 216(1 - \sin^2 t)\cos t\, dt - \int_0^{2\pi} (\sin^3 t + 8\sin^2 t + 16\sin t)\, dt\right. \\
&\qquad \left. - \int_0^{2\pi} (\sin^3 t - 4\sin^2 t + 4\sin t)\, dt\right] \\
&= \frac{1}{68\pi}\left[\int_0^{2\pi} 216(1 - \sin^2 t)\cos t\, dt - \int_0^{2\pi} (2\sin^3 t + 4\sin^2 t + 20\sin t)\, dt\right] \\
&= \frac{1}{68\pi}\left[(216\sin t - 72\sin^3 t)\Big|_0^{2\pi} - \int_0^{2\pi} (2(1 - \cos^2 t)\sin t + 2(1 - \cos 2t) + 20\sin t)\, dt\right]
\end{aligned}$$

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$$= \frac{1}{68\pi}\left[0 + \left(2\cos t - \frac{2}{3}\cos^3 t - 2t + \sin 2t + 20\cos t\right)\Big|_0^{2\pi}\right]$$

$$= \frac{1}{68\pi}(-4\pi) = -\frac{1}{34}$$

and

$$\bar{y} = -\frac{1}{68\pi}\oint_{\partial D} y^2\,dx = -\frac{1}{68\pi}\left[\int_0^{2\pi} 36\sin^2 t\cdot(-6\sin t)\,dt + \int_0^{2\pi}\cos^2 t\cdot\cos t\,dt + \int_0^{2\pi}(\cos t + 2)^2\cos t\,dt\right]$$

$$= -\frac{1}{68\pi}\left[\int_0^{2\pi} 216(1-\cos^2 t)(-\sin t)\,dt + \int_0^{2\pi}\cos^3 t\,dt + \int_0^{2\pi}(\cos^3 t + 4\cos^2 t + 4\cos t)\,dt\right]$$

$$= -\frac{1}{68\pi}\left[(216\cos t - 72\cos^3 t)\Big|_0^{2\pi} + \int_0^{2\pi}(2(1-\sin^2 t)\cos t + 2(1+\cos 2t) + 4\cos t)\,dt\right]$$

$$= -\frac{1}{68\pi}\left[0 + \left(2\sin t - \frac{2}{3}\sin^3 t + 2t + \sin 2t + 4\sin t\right)\Big|_0^{2\pi}\right]$$

$$= -\frac{1}{68\pi}[4\pi] = -\frac{1}{34}.$$

28. We can write $f\nabla g$ as $(f\partial g/\partial x, f\partial g/\partial y)$. Now apply the divergence theorem and collect the appropriate terms.

$$\oint_C f\nabla g\cdot\mathbf{n}\,ds = \iint_D\left(\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)\right)dA$$

$$= \iint_D\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right)dA$$

$$= \iint_D\left(\left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] + \left[f\frac{\partial^2 g}{\partial x^2} + f\frac{\partial^2 g}{\partial y^2}\right]\right)dA$$

$$= \iint_D(\nabla f\cdot\nabla g + f\nabla^2 g)\,dA.$$

29. Apply the results of Exercise 28 to both parts of the line integral.

$$\oint_C(f\nabla g - g\nabla f)\cdot\mathbf{n}\,ds = \oint_C f\nabla g\cdot\mathbf{n}\,ds - \oint_C g\nabla f\cdot\mathbf{n}\,ds$$

$$= \iint_D(f\nabla^2 g + \nabla f\cdot\nabla g)\,dA - \iint_D(g\nabla^2 f + \nabla g\cdot\nabla f)\,dA$$

$$= \iint_D(f\nabla^2 g - g\nabla^2 f)\,dA$$

30. With $f(x,y)\equiv 1$ in Green's first identity, we have $\nabla f\equiv\mathbf{0}$, so

$$\iint_D(f\nabla^2 g + \nabla f\cdot\nabla g)\,dA = \iint_D\nabla^2 g\,dA = \oint_{\partial D}\nabla g\cdot\mathbf{n}\,ds = \oint_{\partial D}\frac{\partial g}{\partial n}\,ds.$$

But if g is harmonic, $\nabla^2 g = 0$, so $\oint_{\partial D}\frac{\partial g}{\partial n}\,ds = 0$.

31. Now use Green's first identity with $f = g$ and f harmonic to obtain $\iint_D(\nabla f\cdot\nabla f)\,dA = \oint_C(f\nabla f\cdot\mathbf{n})\,ds$. Since $C = \partial D$ and $\nabla f\cdot\mathbf{n} = \frac{\partial f}{\partial n}$, the desired result follows.

32. If f is zero on the boundary of D, then Exercise 31 implies that $0 = \oint_{\partial D} f\frac{\partial f}{\partial n}\,ds = \iint_D\nabla f\cdot\nabla f\,dA$. But $\nabla f\cdot\nabla f = \|\nabla f\|^2 \geq 0$. Thus the right integral is of a nonnegative, continuous integrand. For it to be zero, the integrand must be identically zero. That is, $\nabla f\cdot\nabla f$ vanishes an D. We conclude that ∇f is zero on D and so f must be constant. Since $f(x,y) = 0$ on ∂D and f is constant on D, we must have $f\equiv 0$ on D.

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33. Let $f = f_1 - f_2$. Then since $f_1 = f_2$ on ∂D, $f = 0$ on ∂D. Also f is harmonic if f_1 and f_2 are. Hence, by Exercise 32, $f \equiv 0$ on D so $f_1 = f_2$ on D.

34. (a) Exercise 37 from Section 6.3 is a particularly nice example of a nontrivial radially symmetric vector field because there is a compelling physical reason for the field **F** to be radially symmetric. There **F** is the gravitational force field of a mass M on a particle of mass m.

$$\mathbf{F} = -\frac{GMm}{(x^2+y^2+z^2)^{3/2}}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = -\frac{GMm}{(x^2+y^2+z^2)}\frac{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\|(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\|} = -\frac{GMm}{\rho^2}\mathbf{e}_\rho.$$

(b) Apply the formula for the curl in spherical coordinates found in Theorem 4.6 in Chapter 3.

$$\nabla\times\mathbf{F} = \frac{1}{\rho^2\sin\varphi}\begin{vmatrix} \mathbf{e}_\rho & \rho\,\mathbf{e}_\varphi & \rho\sin\varphi\,\mathbf{e}_\theta \\ \partial/\partial\rho & \partial/\partial\varphi & \partial/\partial\theta \\ f(\rho) & 0 & 0 \end{vmatrix} = \left(0, 0, \frac{1}{\rho}\mathbf{e}_\theta \begin{vmatrix} \partial/\partial\rho & \partial/\partial\varphi \\ f(\rho) & 0 \end{vmatrix}\right) = (0,0,0).$$

When students get to complex analysis and learn to integrate around poles, texts often refer to their experience with Green's theorem in multivariable calculus. At least one of Exercises 35 and 36 should be assigned so that this reference might ring a bell.

35. (a) The boundary is in two pieces which we separately parametrize as $\mathbf{x}_1(\theta) = (\cos\theta, \sin\theta)$ and $\mathbf{x}_2(\theta) = (a\cos\theta, -a\sin\theta)$, each for $0 \le \theta \le 2\pi$. The line integral is then

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_0^{2\pi}(\sin^2\theta+\cos^2\theta)\,d\theta + \int_0^{2\pi}\left(-\frac{a^2\sin^2\theta}{a^2} - \frac{a^2\cos^2\theta}{a^2}\right)d\theta$$
$$= \int_0^{2\pi}(1-1)\,d\theta = 0.$$

The double integral is

$$\iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]dx\,dy = \iint_D \left[\frac{-x^2+y^2}{(x^2+y^2)^2} + \frac{-y^2+x^2}{(x^2+y^2)^2}\right) dx\,dy$$
$$= \iint_D 0\,dx\,dy = 0.$$

Thus the conclusion of Green's theorem holds for **F** in the given annular region.

(b) This time the line integral is only taken over the outer boundary and so

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi.$$

The same cancellation takes place in the double integral as in part (a), so

$$\iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right]dx\,dy = 0.$$

The problem is that **F** is not defined at the origin.

(c) Let D be the region so that ∂D consists of the given curve C oriented counterclockwise and also the curve C_a, the circle of radius a centered at the origin oriented clockwise. Then **F** is defined everywhere in the region D. Green's theorem holds so

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy + \oint_{C_a} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \oint_{C\cup C_a} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
$$= \iint_D \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)dx\,dy = 0, \quad \text{but}$$

$$\oint_{C_a} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = -2\pi. \quad \text{Therefore,} \quad \oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = 2\pi.$$

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36. (a)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} & 0 \end{vmatrix} = (0,0,0).$$

(b) Here the path is $\mathbf{x}(\theta) = (\cos\theta, \sin\theta)$ for $0 \le \theta \le 2\pi$. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} ((-\sin\theta)(-\sin\theta) + (\cos\theta)(\cos\theta))\,d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

(c) We saw in part (b) that the line integral around a closed path is not zero, so **F** cannot be conservative on its domain.

(d) The conditions are not met for the theorem as the domain of **F** is not a simply-connected region.

37. (a) By the divergence theorem, the flux

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_0^1\int_0^5 \left(\frac{\partial}{\partial x}[e^y] + \frac{\partial}{\partial y}[x^4]\right) dy\,dx = 0.$$

(b) Again, by the divergence theorem, the flux

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D \left(\frac{\partial}{\partial x}[f(y)] + \frac{\partial}{\partial y}[f(x)]\right) dy\,dx = 0.$$

38. Over the path $\mathbf{x}(t)$,

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} m\mathbf{a}\cdot d\mathbf{s} = \int_a^b m\mathbf{x}''(t)\cdot\mathbf{x}'(t)\,dt = m\int_a^b \frac{1}{2}\frac{d}{dt}[\mathbf{x}'(t)\cdot\mathbf{x}'(t)]\,dt$$
$$= \frac{1}{2}m\|\mathbf{x}'(t)\|^2\Big|_a^b = \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2.$$

39. We'll first replace **F** with $-\nabla V$ and apply Theorem 3.3.

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} -\nabla V\cdot d\mathbf{s} = -V(B) + V(A),$$

where $A = \mathbf{x}(a)$ and $B = \mathbf{x}(b)$. However, in Exercise 38 we showed that

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2.$$

Therefore,

$$\begin{aligned} -V(B) + V(A) &= \frac{1}{2}m[v(b)]^2 - \frac{1}{2}m[v(a)]^2, \text{ or} \\ V(A) + \frac{1}{2}m[v(a)]^2 &= V(B) + \frac{1}{2}m[v(b)]^2. \end{aligned}$$

We see, therefore, that the sum of the potential and kinetic energies of the particle remains constant.

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