Chapter 5

Multiple Integration

5.1 Introduction: Areas and Volumes

1.

$$\int_0^2\int_1^3 (x^2+y)\,dy\,dx = \int_0^2 (x^2y+y^2/2)\Big|_{y=1}^{y=3} dx = \int_0^2 ((3x^2+9/2)-(x^2+1/2))\,dx$$
$$= \int_0^2 (2x^2+4)\,dx = (2x^3/3+4x)\Big|_0^2 = 40/3.$$

2.

$$\int_0^\pi\int_1^2 (y\sin x)\,dy\,dx = \int_0^\pi \left(\frac{y^2}{2}\sin x\right)\Big|_{y=1}^{y=2} dx = \int_0^\pi \left((2\sin x)-\left(\frac{1}{2}\sin x\right)\right)\,dx$$
$$= \frac{3}{2}\int_0^\pi (\sin x)\,dx = -\frac{3}{2}(\cos x)\Big|_0^\pi = \frac{3}{2}+\frac{3}{2} = 3.$$

3.

$$\int_{-2}^4\int_0^1 (xe^y)\,dy\,dx = \int_{-2}^4 (xe^y)\Big|_{y=0}^{y=1} dx = \int_{-2}^4 (x(e-1))\,dx$$
$$= \frac{x^2}{2}(e-1)\Big|_{-2}^4 = (8-2)(e-1) = 6(e-1).$$

4.

$$\int_0^{\pi/2}\int_0^1 (e^x\cos y)\,dx\,dy = \int_0^{\pi/2} (e^x\cos y)\Big|_{x=0}^{x=1} dy = \int_0^{\pi/2} ((e-1)\cos y)\,dy$$
$$= (e-1)\sin y\Big|_0^{\pi/2} = e-1.$$

5.

$$\int_1^2\int_0^1 (e^{x+y}+x^2+\ln y)\,dx\,dy = \int_1^2\int_0^1 (e^xe^y+x^2+\ln y)\,dx\,dy = \int_1^2 \left(e^xe^y+\frac{x^3}{3}+x\ln y\right)\Big|_{x=0}^{x=1} dy$$
$$= \int_1^2 \left((e-1)e^y+\frac{1}{3}+\ln y\right)\,dy = \left((e-1)e^y+\frac{y}{3}+y\ln y-y\right)\Big|_1^2$$
$$= (e-1)(e^2-e)+\frac{1}{3}+2\ln 2-1 = e^3-2e^2+e-\frac{2}{3}+2\ln 2.$$

6.

$$\int_1^9\int_1^e \left(\frac{\ln\sqrt{x}}{xy}\right)\,dx\,dy = \frac{1}{2}\int_1^9\int_1^e \left(\frac{\ln x}{xy}\right)\,dx\,dy \quad \text{(treat } y \text{ as a constant—use substitution)}$$
$$= \frac{1}{2}\int_1^9 \frac{(\ln x)^2}{2y}\Big|_{x=1}^{x=e} dy = \frac{1}{2}\int_1^9 \left(\frac{1}{2y}\right)\,dy = \frac{1}{4}\ln y\Big|_1^9 = \frac{\ln 9}{4}.$$

7. (a) Here we are fixing x and finding the area of the slices:

$$A(x) = \int_0^2 (x^2 + y^2 + 2)\,dy = \left(x^2 y + \frac{y^3}{3} + 2y\right)\Big|_0^2 = 2x^2 + 20/3.$$

Now we "add up the areas of these slices":

$$V = \int_{-1}^2 A(x)\,dx = \int_{-1}^2 (2x^2 + 20/3)\,dx = \left(\frac{2}{3}x^3 + \frac{20}{3}x\right)\Big|_{-1}^2 = \left(\frac{16}{3} + \frac{40}{3}\right) - \left(-\frac{2}{3} - \frac{20}{3}\right) = 26.$$

(b) Now we fix y and find the area of the slices:

$$\begin{aligned} A(y) &= \int_{-1}^2 (x^2 + y^2 + 2)\,dx = \left(\frac{x^3}{3} + y^2 x + 2x\right)\Big|_{-1}^2 \\ &= \left(\frac{8}{3} + 2y^2 + 4\right) - \left(-\frac{1}{3} - y^2 - 2\right) = 9 + 3y^2. \end{aligned}$$

Adding up the area of these slices:

$$V = \int_0^2 A(y)\,dy = \int_0^2 (9 + 3y^2)\,dy = (9y + y^3)\Big|_0^2 = 26.$$

8. Here we are calculating:

$$\begin{aligned} \int_1^2 \int_0^3 (x + 3y + 1)\,dx\,dy &= \int_1^2 \left(\frac{x^2}{2} + 3yx + x\right)\Big|_0^3 dy = \int_1^2 \left(\frac{9}{2} + 9y + 3\right)\,dy \\ &= \int_1^2 \left(\frac{15}{2} + 9y\right)\,dy = \left(\frac{15}{2}y + \frac{9}{2}y^2\right)\Big|_1^2 \\ &= (15 + 18) - (15/2 + 9/2) = 21. \end{aligned}$$

9. Here we are calculating

$$\begin{aligned} \int_{-1}^2 \int_0^1 (2x^2 + y^4 \sin \pi x)\,dx\,dy &= \int_{-1}^2 \left(\frac{2}{3}x^3 - \frac{y^4}{\pi}\cos \pi x\right)\Big|_0^1 dy = \int_{-1}^2 \left(\frac{2}{3} + \frac{2y^4}{\pi}\right)\,dy \\ &= \left(\frac{2}{3}y + \frac{2y^5}{5\pi}\right)\Big|_{-1}^2 = \left(\frac{4}{3} + \frac{64}{5\pi}\right) - \left(-\frac{2}{3} - \frac{2}{5\pi}\right) = 2 + \frac{66}{5\pi}. \end{aligned}$$

10. This is the volume of the "rectangular box" bounded by the plane $z = 2$, the *xy*-plane, and the planes $x = 1$, $x = 3$, $y = 0$, and $y = 2$. Here we could just calculate the volume of this $2 \times 2 \times 2$ box as 8 without integrating—or

$$V = \int_0^2 \int_1^3 2\,dx\,dy = \int_0^2 2x\Big|_1^3 dy = \int_0^2 4\,dy = 4y\Big|_0^2 = 8.$$

11. This is the volume of the region bounded by the paraboloid $z = 16 - x^2 - z^2$, the *xy*-plane, and the planes $x = 1$, $x = 3$, $y = -2$, and $y = 2$. The volume is

$$\begin{aligned} V &= \int_1^3 \int_{-2}^2 (16 - x^2 - y^2)\,dy\,dx = \int_1^3 \left(16y - x^2 y - \frac{y^3}{3}\right)\Big|_{-2}^2 dx = \int_1^3 \left(64 - 4x^2 - \frac{16}{3}\right)\,dx \\ &= \left(64x - \frac{4}{3}x^3 - \frac{16}{3}x\right)\Big|_1^3 = (192 - 36 - 16) - (64 - 4/3 - 16/3) = 248/3. \end{aligned}$$

12. This is the volume of the region bounded by $z = \sin x \cos y$, the *xy*-plane, and the planes $x = 0$, $x = \pi$, $y = -\pi/2$, and $y = \pi/2$. The volume is

$$\begin{aligned} V &= \int_{-\pi/2}^{\pi/2} \int_0^\pi (\sin x \cos y)\,dx\,dy = \int_{-\pi/2}^{\pi/2} (-\cos x \cos y)\Big|_0^\pi dy \\ &= 2\int_{-\pi/2}^{\pi/2} \cos y\,dy = 2\sin y\Big|_{-\pi/2}^{\pi/2} = 4. \end{aligned}$$

13. This is the volume of the region bounded by $z = 4 - x^2$, the *xy*-plane, and the planes $x = -2,\ x = 2,\ y = 0$, and $y = 5$. The volume is

$$V = \int_0^5 \int_{-2}^2 (4 - x^2)\, dx\, dy = \int_0^5 (4x - x^3/3)\Big|_{-2}^{2}\, dy = \int_0^5 ((8 - 8/3) - (-8 + 8/3))\, dy$$
$$= \int_0^5 \frac{32}{3}\, dy = \frac{32}{3} y\Big|_0^5 = 160/3.$$

14. This is the volume of the region bounded by $z = |x| \sin \pi y$, the *xy*-plane, and the planes $x = -2,\ x = 3,\ y = 0$, and $y = 1$. The volume is

$$V = \int_{-2}^3 \int_0^1 |x| \sin \pi y\, dy\, dx = \int_{-2}^3 -\frac{|x|}{\pi} \cos \pi y\Big|_0^1\, dx = \int_{-2}^3 \frac{2|x|}{\pi}\, dx.$$

At this point we use the definition of absolute value to split this into two quantities:

$$V = \int_{-2}^0 -\frac{2}{\pi} x\, dx + \int_0^3 \frac{2}{\pi} x\, dx = -\frac{x^2}{\pi}\Big|_{-2}^0 + \frac{x^2}{\pi}\Big|_0^3 = \frac{4}{\pi} + \frac{9}{\pi} = \frac{13}{\pi}.$$

15.

$$\int_{-5}^5 \int_{-1}^2 (5 - |y|)\, dx\, dy = \int_{-5}^5 (5 - |y|)x\Big|_{x=-1}^{2}\, dy$$
$$= \int_{-5}^5 (5 - |y|) \cdot 3\, dy = 150 - 3\int_{-5}^5 |y|\, dy$$
$$= 150 - 3\int_{-5}^0 (-y)\, dy - 3\int_0^5 y\, dy$$
$$= 150 + \frac{3}{2} y^2\Big|_{-5}^0 - \frac{3}{2} y^2\Big|_0^5 = 150 - \frac{75}{2} - \frac{75}{2} = 75.$$

The iterated integral gives the volume of the region bounded by the graph of $z = 5 - |y|$, the *xy*-plane, and the planes $x = -1$, $x = 2$, $y = -5$, $y = 5$. (The solid so described is a rectangular prism.)

16. We have $V = \int_a^b \int_c^d f(x, y)\, dy\, dx$. Since $0 \le f(x, y) \le M$, the solid bounded by $y = f(x, y)$, the *xy*-plane, and the planes $x = a$, $x = b$, $y = c$, $y = d$ sits inside the rectangular block of height M and base bounded by $x = a$, $x = b$, $y = c$, $y = d$. Hence $V \le M(b - a)(d - c)$

5.2 Double Integrals

1. Since the integrand $f(x, y) = y^3 + \sin 2y$ is continuous, the double integral $\iint_R (y^3 + \sin 2y)\, dA$ exists by Theorem 2.4. Now consider a Riemann sum corresponding to the double integral that we obtain by partitioning the rectangle $[0, 3] \times [-1, 1]$ symmetrically with respect to the x-axis and by choosing test points $\mathbf{c}_{ij}$ in each subrectangle that are also symmetric with respect to the x-axis. Then

$$S = \sum_{i,j} f(\mathbf{c}_{ij})\, \Delta A_{ij} = \sum_{i,j} (y_{ij}^3 + \sin 2y_{ij})\, \Delta A_{ij}$$

(where y_{ij} denotes the y-coordinate of $\mathbf{c}_{ij}$) must be zero since the terms cancel in pairs because $f(x, -y) = -f(x, y)$. When we shrink the rectangles in the limit, we can arrange to preserve all the symmetry. Hence the limit under such restrictions must be zero and thus the overall limit (which must exist in view of Theorem 2.4) must also be zero.

2. The integrand $f(x, y) = x^5 + 2y$ is continuous, so the double integral exists by Theorem 2.4. Consider a Riemann sum corresponding to the double integral that we obtain by partitioning the rectangle $[-3, 3] \times [-2, 2]$ symmetrically with respect to *both* coordinate axes and by choosing test points $\mathbf{c}_{ij}$ in each subrectangle that are also symmetric with respect to both axes. Then

$$S = \sum_{i,j} f(\mathbf{c}_{ij})\, \Delta A_{ij} = \sum_{i,j} (x_{ij}^5 + 2y_{ij})\, \Delta A_{ij} = \sum_{i,j} x_{ij}^5\, \Delta A_{ij} + \sum_{i,j} 2y_{ij}\, \Delta A_{ij}$$

must be zero since the terms in each sum will cancel in pairs (because $(-x)^5 = -x^5$ and $2(-y) = -2y$). When we shrink the rectangles in the limit, we can arrange to preserve all the symmetry. Hence the limit under such restrictions must be zero and thus the overall limit (which must exist in view of Theorem 2.4) must also be zero.

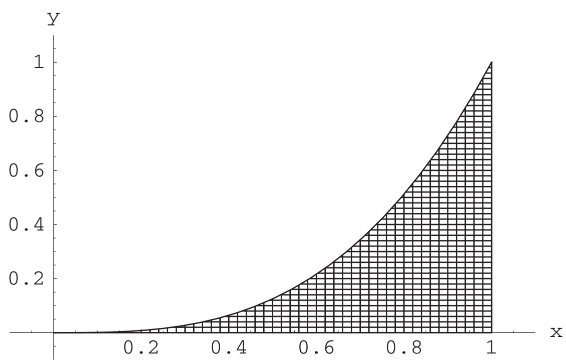
Note: you may want to discuss Exercise 3 (b) before assigning it, to get your students in the habit of looking critically at problems before working on them.

3. (a) We are computing

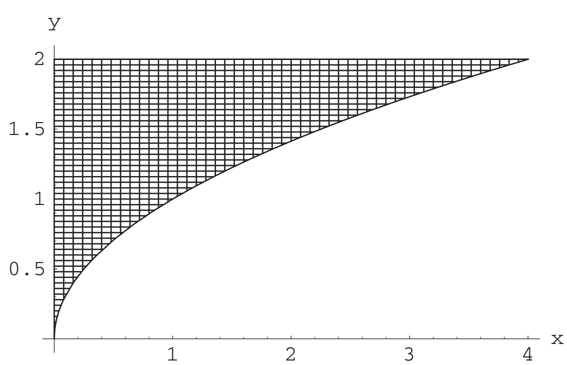
$$\int_{-2}^{2}\int_{0}^{4-x^2} x^3\,dy\,dx = \int_{-2}^{2} x^3 y\Big|_{0}^{4-x^2} dx = \int_{-2}^{2}(4x^3 - x^5)\,dx = (x^4 - x^6/6)\Big|_{-2}^{2} = 0.$$

(b) The integrand is an odd function depending only on x and the region is symmetric about the y-axis. The students encounter this situation when they looked at $\int_{-a}^{a} x^3\,dx$ in first year calculus.

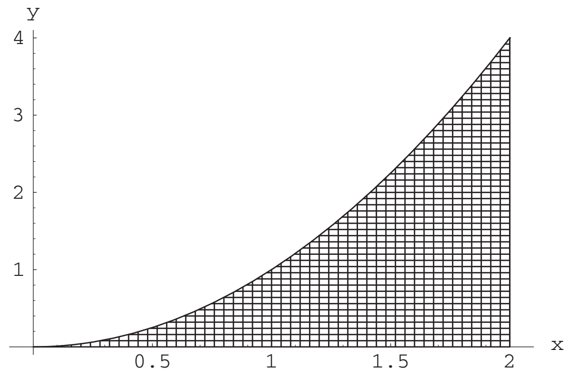
4. $\displaystyle\int_0^1\int_0^{x^3} 3\,dy\,dx = \int_0^1 3y\Big|_0^{x^3} dx = \int_0^1 3x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$. The region over which we are integrating is:



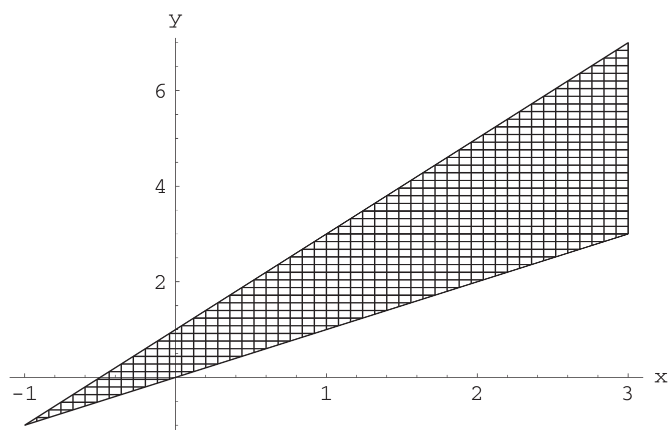
5. $\displaystyle\int_0^2\int_0^{y^2} y\,dx\,dy = \int_0^2 xy\Big|_0^{y^2} dy = \int_0^2 y^3\,dy = \frac{y^4}{4}\Big|_0^2 = 4$. The region over which we are integrating is:



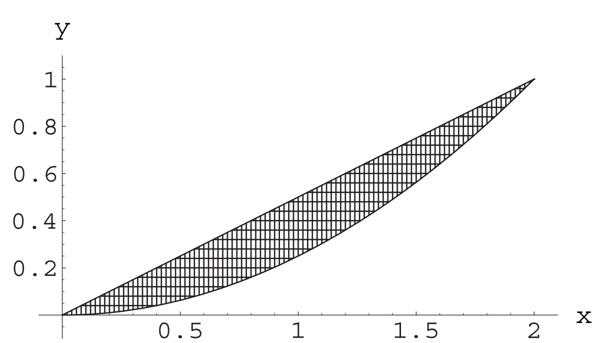
6. $\int_0^2 \int_0^{x^2} y\,dy\,dx = \int_0^2 \frac{y^2}{2}\Big|_0^{x^2} dx = \int_0^2 \frac{x^4}{2}\,dx = \frac{x^5}{10}\Big|_0^2 = \frac{32}{10} = \frac{16}{5}$. The region over which we are integrating is:



7. $\int_{-1}^3 \int_x^{2x+1} xy\,dy\,dx = \int_{-1}^3 \frac{xy^2}{2}\Big|_x^{2x+1} dx = \frac{1}{2}\int_{-1}^3 (3x^3 + 4x^2 + x)\,dx = \frac{1}{2}\left[\frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{x^2}{2}\right]\Big|_{-1}^3 = \frac{152}{3}$. The region over which we are integrating is:



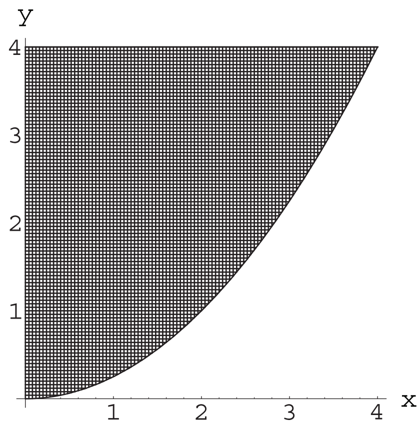
8. $\int_0^2 \int_{x^2/4}^{x/2} (x^2 + y^2)\,dy\,dx = \int_0^2 \left(x^2 y + \frac{y^3}{3}\right)\Big|_{y=x^2/4}^{y=x/2} dx$
$= \int_0^2 \left(\left(\frac{x^3}{2} + \frac{x^2}{24}\right) - \left(\frac{x^4}{4} + \frac{x^6}{192}\right)\right) dx = \left[\frac{13}{96}x^4 - \frac{1}{20}x^5 - \frac{1}{1344}x^7\right]\Big|_0^2 = \frac{33}{70}$. The region over which we are integrating is:



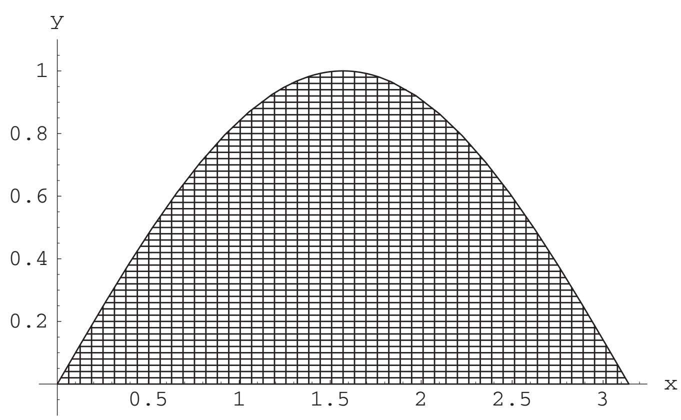
9. $\int_0^4 \int_0^{2\sqrt{y}} x \sin(y^2)\,dx\,dy = \int_0^4 \frac{x^2}{2}\sin(y^2)\Big|_{x=0}^{x=2\sqrt{y}} dy = \int_0^4 2y\sin(y^2)\,dy$. Now let $u = y^2$, so $du = 2y\,dy$. Then this integral becomes

$$\int_0^{16} \sin u \, du = -\cos u \Big|_0^{16} = 1 - \cos 16.$$

The region over which we are integrating is:

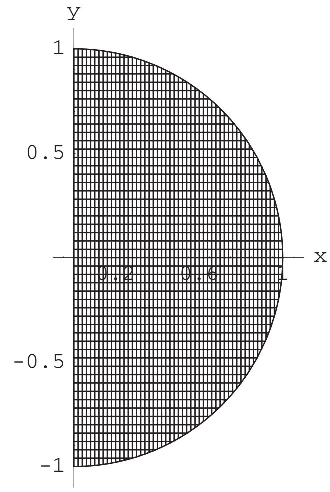


10. $\displaystyle\int_0^{\pi} \int_0^{\sin x} y \cos x \, dy \, dx = \int_0^{\pi} \frac{y^2}{2} \cos x \Big|_0^{\sin x} dx = \frac{1}{2} \int_0^{\pi} (\sin^2 x \cos x) \, dx =$ (using the substitution $u = \sin x$) $\displaystyle\frac{1}{2} \int_{x=0}^{x=\pi} u^2 \, du = \frac{\sin^3 x}{6} \Big|_0^{\pi} = 0$. The region over which we are integrating is:



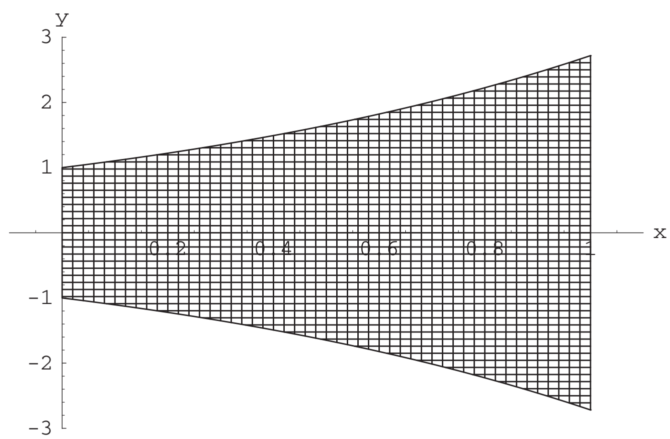
Note: After you assign Exercises 11 and 12, together you can probe to see whether students see that they are the same. This is a nice set-up for Section 5.3 where they will learn about interchanging the order of integration.

11. $\displaystyle\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3 \, dy \, dx = \int_0^1 3y \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_0^1 6\sqrt{1-x^2} \, dx =$ (using the substitution $x = \sin t$) $= 3\pi/2$. You can also see that the region over which we are integrating is a half-circle of radius 1 so we have found the volume of the cylinder over this region of height 3. This figure is:



12. This is the same as Exercise 11 with the limits of integration reversed. The solution is again $3\pi/2$.

13. $\int_0^1 \int_{-e^x}^{e^x} y^3\,dy\,dx = \int_0^1 \frac{y^4}{4}\Big|_{-e^x}^{e^x} dx = \int_0^1 0\,dx = 0$. The region over which we are integrating is:

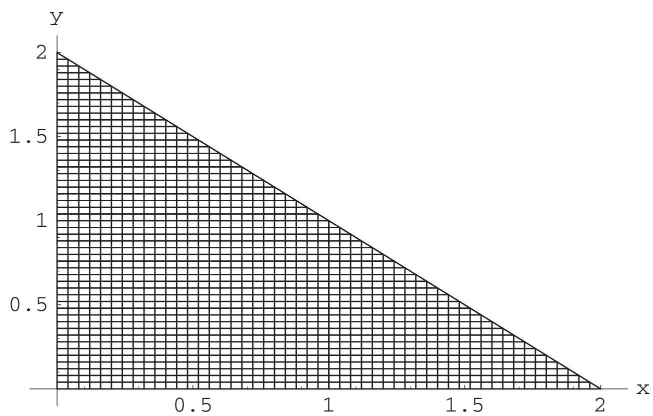


14. For each square in the domain we need to estimate the height of the square and multiply it by the length times the width. For our estimate we will choose the value of the height $f(\mathbf{c}_{ij})$ in the lower right corner of the square in row i column j as our height for the square. The heights are then:

4	5	6	7	8	9	9	10	9	9
4	5	6	7	8	9	10	11	10	9
4	5	6	7	8	9	10	10	10	9
4	5	6	7	8	8	9	9	9	9
4	5	6	7	7	8	8	8	8	8

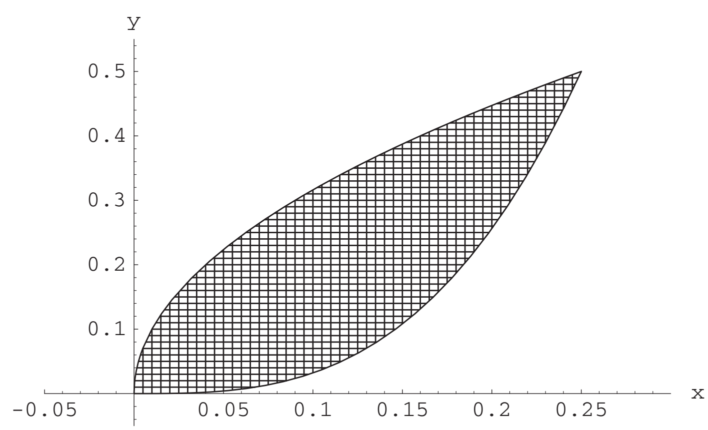
Each box has a base of area 25 so the sum of the products of 25 times the heights is 92500. Of course, this answer depends on what point in each box we chose for our estimate—your mileage may vary.

15. A quick sketch of the region over which we are integrating helps us set up our double integral.



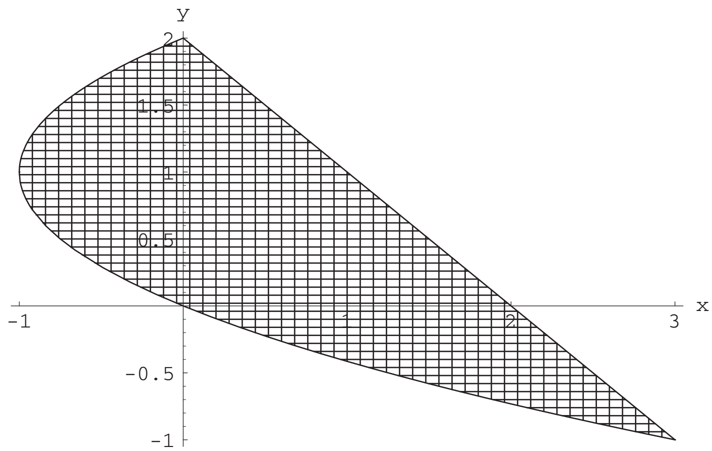
$$\int_0^2 \int_0^{2-x} (1 - xy)\, dy\, dx = \int_0^2 \left(y - \frac{xy^2}{2} \right)\Bigg|_0^{2-x} dx = \int_0^2 (2 - 3x + 2x^2 - x^3/2)\, dx$$
$$= \left(2x - \frac{3}{2}x^2 + \frac{2}{3}x^3 - \frac{x^4}{8} \right)\Bigg|_0^2 = 4 - 6 + 16/3 - 2 = 4/3.$$

16. Again a sketch of the region over which we are integrating helps us set up our double integral. The top bounding curve is $y = \sqrt{x}$ and the bottom curve is $y = 32x^3$.



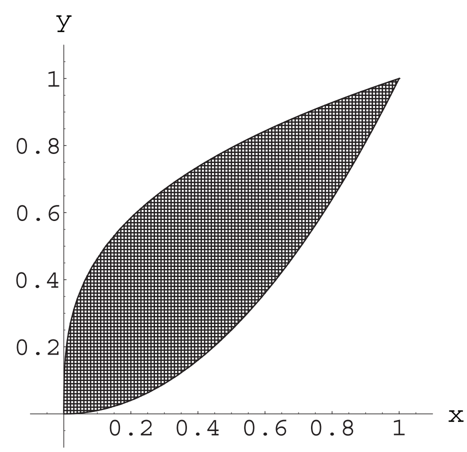
$$\int_0^{1/4} \int_{32x^3}^{\sqrt{x}} 3xy\, dy\, dx = \int_0^{1/4} \left(\frac{3xy^2}{2} \right)\Bigg|_{32x^3}^{\sqrt{x}} dx = \int_0^{1/4} \left(\frac{3}{2}x^2 - 1536x^7 \right) dx$$
$$= \left(\frac{3}{2}x^2 - 192x^8 \right)\Bigg|_0^{1/4} = \frac{1}{128} - \frac{3}{1024} = \frac{5}{1024}.$$

17. We can easily determine the limits of integration from the sketch and/or by solving for where $x+y=2$ intersects the parabola $y^2-2y-x=0$.



$$\int_{-1}^{2}\int_{y^2-2y}^{2-y}(x+y)\,dx\,dy=\int_{-1}^{2}\left(\frac{x^2}{2}+xy\right)\Bigg|_{y^2-2y}^{2-y}dy=\int_{-1}^{2}\left(-\frac{y^4}{2}+y^3-\frac{y^2}{2}+2\right)dy$$
$$=\left(-\frac{y^5}{10}+\frac{y^4}{4}-\frac{y^3}{6}+2y\right)\Bigg|_{-1}^{2}=\frac{99}{20}.$$

18. The region D of integration has top boundary curve $x=y^3$ and bottom boundary curve $y=x^2$ and looks like:

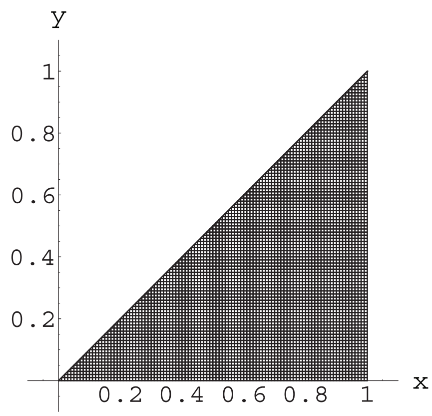


Note that $y=x^2$ may be expressed as $x=\sqrt{y}$ since the region of interest lies in the first quadrant. Hence we have a type 2 elementary region and

$$\iint_D xy\,dA=\int_0^1\int_{y^3}^{\sqrt{y}}xy\,dx\,dy=\int_0^1\frac{x^2}{2}\,y\,\Bigg|_{x=y^3}^{x=\sqrt{y}}dy=\int_0^1\left(\frac{y^2}{2}-\frac{y^7}{2}\right)dy$$
$$=\frac{1}{2}\left(\frac{1}{3}y^3-\frac{1}{8}y^8\right)\Bigg|_0^1=\frac{5}{48}.$$

Note that we may also set up this integral as $\int_0^1\int_{x^2}^{\sqrt[3]{x}}xy\,dy\,dx$.

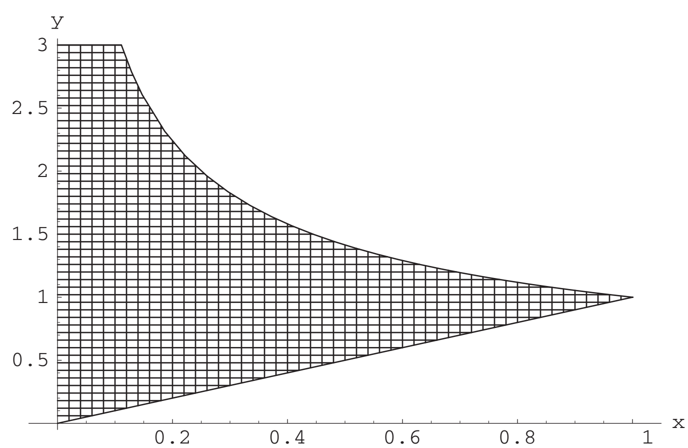
19. The region D is triangular, with top boundary the line $y = x$ and looks like:



Viewing D as a type 1 region we have

$$\iint_D e^{x^2}\, dA = \int_0^1 \int_0^x e^{x^2}\, dy\, dx$$
$$= \int_0^1 x e^{x^2}\, dx = \int_0^1 \frac{1}{2} e^u\, du \qquad \text{where } u = x^2$$
$$= \frac{1}{2} e^u \Big|_0^1 = \frac{1}{2}(e - 1).$$

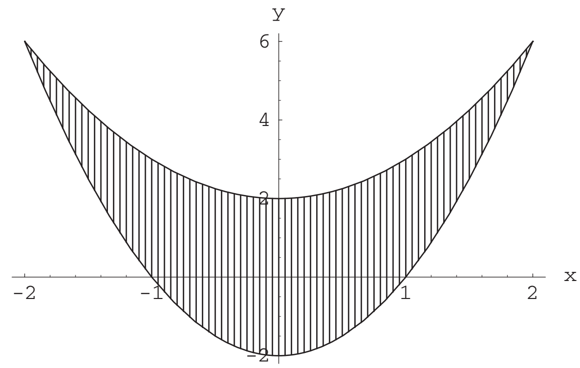
20. We see from the sketch that we need to divide the integral into two pieces. For $0 \le x \le 1/9$ we see that $x \le y \le 3$ and for $1/9 \le x \le 1$ we see that $x \le y \le 1/\sqrt{x}$.



$$\iint_D 3y\, dA = \int_0^{1/9} \int_x^3 3y\, dy\, dx + \int_{1/9}^1 \int_x^{1/\sqrt{x}} 3y\, dy\, dx$$
$$= \int_0^{1/9} \frac{3}{2} y^2 \Big|_x^3\, dx + \int_{1/9}^1 \frac{3}{2} y^2 \Big|_x^{1/\sqrt{x}}\, dx$$
$$= \int_0^{1/9} \left(\frac{27}{2} - \frac{3}{2} x^2 \right) dx + \int_{1/9}^1 \left(\frac{3}{2x} - \frac{3}{2} x^2 \right) dx$$

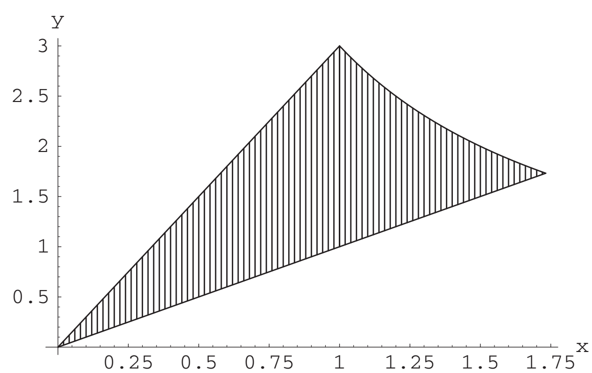
$$= \left(\frac{27}{2}x - \frac{1}{2}x^3\right)\bigg|_0^{1/9} + \left(\frac{3}{2}\ln x - \frac{1}{2}x^3\right)\bigg|_{1/9}^{1}$$
$$= \frac{3}{2} - \frac{1}{2} - \frac{3}{2}\ln(1/9) = 1 + \ln 27.$$

21. From the sketch below we see that this is a fairly straightforward integral.



$$\iint_D (x-2y)\,dA = \int_{-2}^{2}\int_{2x^2-2}^{x^2+2}(x-2y)\,dy\,dx$$
$$= \int_{-2}^{2}(xy - y^2)\bigg|_{2x^2-2}^{x^2+2}dx = \int_{-2}^{2}(3x^4 - x^3 - 12x^2 + 4x)\,dx$$
$$= (3x^5/5 - x^4/4 - 4x^3 + 2x^2)\bigg|_{-2}^{2} = 192/5 - 64 = -128/5$$

22. From the sketch below we see that this integral needs to be done in two pieces.



$$\iint_D (x^2+y^2)\,dA = \int_0^1\int_x^{3x}(x^2+y^2)\,dy\,dx + \int_1^{\sqrt{3}}\int_x^{3/x}(x^2+y^2)\,dy\,dx$$
$$= \int_0^1 (x^2y + y^3/3)\bigg|_x^{3x}dx + \int_1^{\sqrt{3}}(x^2y + y^3/3)\bigg|_x^{3/x}dx$$
$$= \int_0^1 (32/3)x^3\,dx + \int_1^{\sqrt{3}}(9/x^3 + 3x - 4x^3/3)\,dx = 8/3 + 10/3 = 6$$

23. As in the proof of property 1 in the text, we note that the Riemann sum whose limit is

$$\iint_R cf\,dA \text{ is } \sum_{i,j=1}^{n} cf(\mathbf{c}_{ij})\Delta A_{ij} = c\sum_{i,j=1}^{n} f(\mathbf{c}_{ij})\Delta A_{ij} \to c\iint_R f\,dA.$$

24. $\iint_R g\,dA = \iint_R (f+[g-f])\,dA$ which, by property 1, equals $\iint_R f\,dA + \iint_R [g-f]\,dA$. But $g-f \geq 0$ so $\iint_R [g-f]\,dA \geq 0$ and so $\iint_R g\,dA \geq \iint_R f\,dA$.

25. Define $f^+ = \max(f,0)$ and $f^- = \max(-f,0)$. Note that both f^+ and f^- have only non-negative values. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Since $f^\pm \leq |f| = f^+ + f^-$ we can see that $|f|$ is Riemann integrable. Also we can use property 2 to conclude that

$$\begin{aligned}\left|\iint_R f\,dA\right| &= \left|\iint_R (f^+ - f^-)\,dA\right| = \left|\iint_R f^+\,dA - \iint_R f^-\,dA\right| \\ &\leq \iint_R f^+\,dA + \iint_R f^-\,dA = \iint_R |f|\,dA.\end{aligned}$$

26. **(a)** Intuitively, the volume of a figure with constant height should be the area of the base times the height. In this case that is just the area of the base. More formally, by Definition 2.3,

$$\iint_D 1\,dA = \lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} \Delta x_i \Delta y_j.$$

We are assuming that D is an elementary region; let's consider the case of a type 1 region, then we can rewrite the above sum as

$$\lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} (\delta(c_i) - \gamma(c_i))\Delta x_i = \int_a^b (\delta(x) - \gamma(x))\,dx = \text{the area of } D.$$

The proof is not much different for the other elementary regions.

(b) We integrate $\iint_D 1\,dA = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx = 2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$. We've seen this above in Exercises 11 and 12. Let $x = a\sin t$ and integrate to get the desired result.

27. Using Exercise 26, the area is

$$\begin{aligned}\iint_A 1\,dA &= \int_0^1 \int_{x^3}^{x^2} 1\,dy\,dx = \int_0^1 (x^2 - x^3)\,dx \\ &= (x^3/3 - x^4/4)\Big|_0^1 = 1/3 - 1/4 = 1/12.\end{aligned}$$

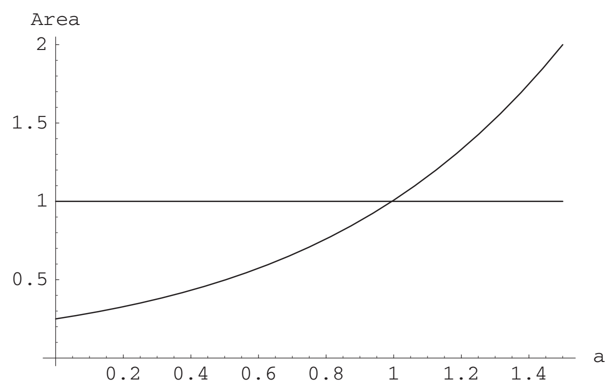
28. Again using Exercise 26, the area is

$$\begin{aligned}\iint_A 1\,dA &= \int_0^{\sqrt{5}-2} \int_{2x}^{1-2x-x^2} 1\,dy\,dx = \int_0^{\sqrt{5}-2} (1 - 4x - x^2)\,dx \\ &= (x - 2x^2 - x^3/3)\Big|_0^{\sqrt{5}-2} = (1/3)(10\sqrt{5} - 22).\end{aligned}$$

29. We integrate $\int_{-a}^{a}\int_{-\sqrt{b^2-b^2x^2/a^2}}^{\sqrt{b^2-b^2x^2/a^2}} dy\,dx = 2\int_{-a}^{a}\sqrt{b^2 - \frac{b^2x^2}{a^2}}\,dx = \frac{2b}{a}\left(\int_{-a}^{a}\sqrt{a^2-x^2}\,dx\right) = \frac{b}{a}(\pi a^2) = \pi ab.$

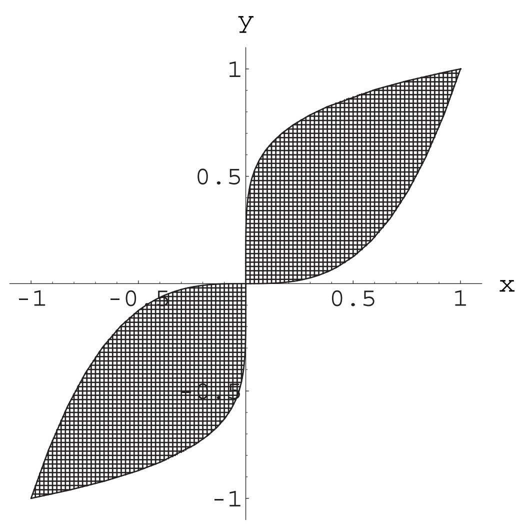
30. **(a)** For $x \geq 0$ the curve $x^3 - x$ lies below the curve $y = ax^2$ between 0 and their positive point of intersection $x = \dfrac{a+\sqrt{a^2+4}}{2}$. So the area is given by $\int_0^{(a+\sqrt{a^2+4})/2}\int_{x^3-x}^{ax^3} dy\,dx.$

(b) The graph of area against a is:



The area is 1 at $a \approx .995$.

31. The region looks like:

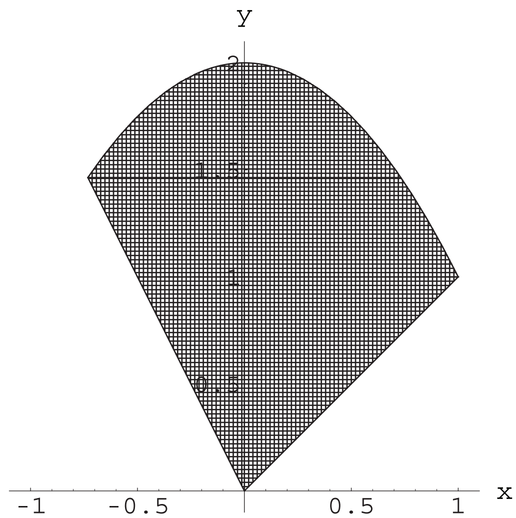


By symmetry, it's enough to calculate the first quadrant area and double it. Thus

$$\text{Total area} = 2\int_0^1 \int_{x^3}^{x^{1/5}} 1\,dy\,dx = 2\int_0^1 \left(x^{1/5} - x^3\right)\,dx$$

$$= 2\left(\frac{5}{6}x^{6/5} - \frac{1}{4}x^4\right)\Big|_0^1 = 2\left(\frac{5}{6} - \frac{1}{4}\right) = \frac{7}{6}.$$

32. The region in question looks like:



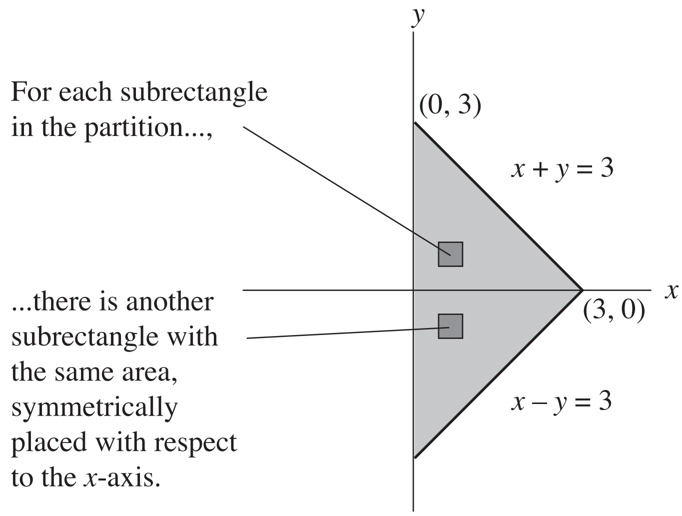
Note that the intersection point of $y = -2x$ and $y = 2 - x^2$ is $(1 - \sqrt{3}, -2 + 2\sqrt{3})$. We use the y-axis to divide the region into two type 1 subregions. Then

$$\begin{aligned}\text{Area} &= \int_{1-\sqrt{3}}^{0} \int_{-2x}^{2-x^2} 1\, dy\, dx + \int_0^1 \int_x^{2-x^2} 1\, dy\, dx \\ &= \int_{1-\sqrt{3}}^{0} (2 + 2x - x^2)\, dx + \int_0^1 (2 - x - x^2)\, dx \\ &= \left(2x + x^2 - \frac{x^3}{3}\right)\Big|_{1-\sqrt{3}}^{0} + \left(2x - \frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{6\sqrt{3} - 8}{3} + \frac{7}{6} = \frac{4\sqrt{3} - 3}{2}.\end{aligned}$$

33. First, note that the integrand is continuous; hence the integral as the limit of Riemann sums must exist. Second, note that the region D is symmetric with respect to the x-axis. Next, note that we can break up the integral as

$$\iint_D (y^3 + e^{x^2} \sin y + 2)\, dA = \iint_D y^3\, dA + \iint_D e^{x^2} \sin y\, dA + \iint_D 2\, dA.$$

Consider first $\iint_D y^3\, dA$ and note that the integrand, y^3, is an odd function. Hence, in a Riemann sum, we can arrange to partition any rectangle that contains D in such a way that for every subrectangle above the x-axis (i.e., where $y > 0$), there is a corresponding "mirror image" subrectangle—with the same area—*below* the x-axis (where $y < 0$). Then the "test points" in each pair of subrectangles may be chosen to have *opposite* y-coordinates. (See the figure below.)



The Riemann sum corresponding to this partition will be

$$\sum_{i,j} y_{ij}^3\,\Delta A_{ij} = 0,$$

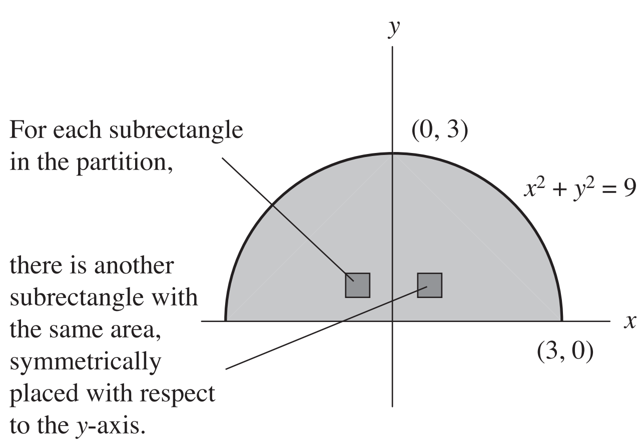
since the terms of the sum will cancel in pairs. Thus, even when we take the limit of this sum as $\Delta A_{ij} \to 0$, we still obtain zero. Therefore, we conclude that $\iint_D y^3\,dA = 0$. Using a similar argument, we find that $\iint_D e^{x^2}\sin y\,dA = 0$ as well. Hence

$$\begin{aligned}\iint_D (y^3 + e^{x^2}\sin y + 2)\,dA &= \iint_D y^3\,dA + \iint_D e^{x^2}\sin y\,dA + \iint_D 2\,dA\\ &= 0 + 0 + 2\iint_D dA = 2(\text{area of } D)\\ &= 2(9) = 18.\end{aligned}$$

34. First, note that the integrand is continuous; hence the integral as the limit of Riemann sums must exist. Second, note that the region D is symmetric with respect to the y-axis. Next, note that we can break up the integral as

$$\iint_D (2x^3 - y^4\sin x + 2)\,dA = \iint_D 2x^3\,dA - \iint_D y^4\sin x\,dA + \iint_D 2\,dA.$$

Consider first $\iint_D 2x^3\,dA$ and note that the integrand, $2x^3$, is an odd function. Hence, in a Riemann sum, we can arrange to partition any rectangle that contains D in such a way that for every subrectangle to the right of the y-axis (i.e., where $x > 0$), there is a corresponding "mirror image" subrectangle—with the same area—to the *left* of the y-axis (where $x < 0$). Then the "test points" in each pair of subrectangles may be chosen to have *opposite* x-coordinates. (See the figure below.)



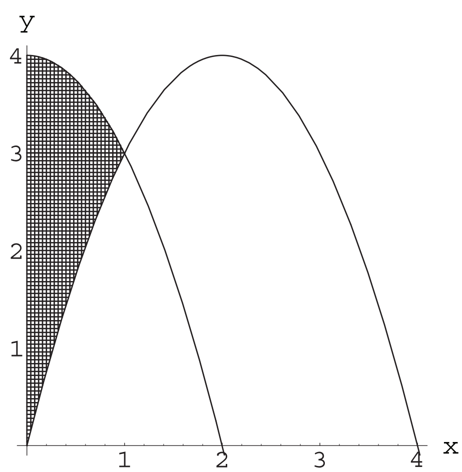
The Riemann sum corresponding to this partition will be

$$\sum_{i,j} 2x_{ij}^3 \, \Delta A_{ij} = 0,$$

since the terms of the sum will cancel in pairs. Thus, even when we take the limit of this sum as $\Delta A_{ij} \to 0$, we still obtain zero. Therefore, we conclude that $\iint_D 2x^3 \, dA = 0$. Using a similar argument, we find that $\iint_D y^4 \sin x \, dA = 0$ as well. Hence

$$\begin{aligned}\iint_D (2x^3 - y^4 \sin x + 2) \, dA &= \iint_D 2x^3 \, dA - \iint_D y^4 \sin x \, dA + \iint_D 2 \, dA \\ &= 0 + 0 + 2 \iint_D dA = 2(\text{area of } D) \\ &= 2\left(\frac{9\pi}{2}\right) = 9\pi.\end{aligned}$$

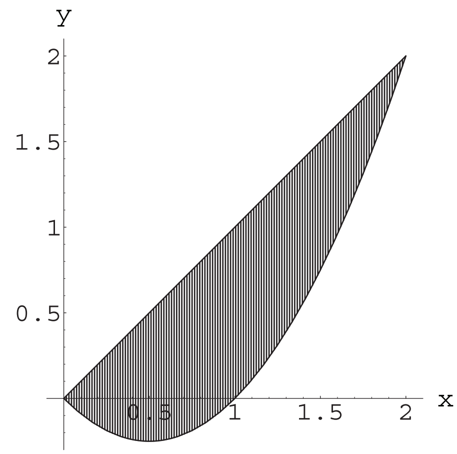
35. The volume is given by $\iint_D (24 - 2x - 6y) \, dA$, where D is the region in the xy-plane bounded by $y = 4 - x^2$, $y = 4x - x^2$, and the y-axis. Now D is a type 1 region that looks like:



Thus the volume is

$$\begin{aligned}\int_0^1 \int_{4x-x^2}^{4-x^2} (24 - 2x - 6y) \, dy \, dx &= \int_0^1 \left[(24 - 2x)y - 3y^2\right] \Big|_{y=4x-x^2}^{y=4-x^2} dx \\ &= \int_0^1 \left[(24 - 2x)(4 - 4x) - 3(4 - x^2)^2 + 3(4x - x^2)^2\right] dx \\ &= \int_0^1 \left[8(x^2 - 13x + 12) - 24x^3 + 72x^2 - 48\right] dx \\ &= \int_0^1 \left[80x^2 - 24x^3 - 104x + 48\right] dx = \frac{50}{3}.\end{aligned}$$

36. The volume is given by $\iint_D (x^2 + 6y^2) \, dA$, where D is the region in the xy-plane bounded by $y = x$ and $y = x^2 - x$. This region D looks like:



Therefore, the volume is

$$\int_0^1 \int_{x^2-x}^{x} (x^2+6y^2)\,dy\,dx = \int_0^2 \left[x^2(2x-x^2)+2\left(x^3-(x^2-x)^3\right)\right]\,dx$$
$$= \int_0^2 \left[-2x^6+6x^5-7x^4+6x^3\right]\,dx$$
$$= \left(-\frac{2}{7}x^7+x^6-\frac{7}{5}x^5+\frac{3}{2}x^4\right)\Bigg|_0^2 = \frac{232}{35}.$$

37. The graphs of $y=x^2-10$ and $y=31-(x-1)^2$ intersect at $x=-4$ and $x=5$ with the graph of $y=x^2-10$ lying below the graph of $y=31-(x-1)^2$ on this interval.

$$\int_{-4}^{5}\int_{x^2-10}^{31-(x-1)^2}(4x+2y+25)\,dy\,dx = \int_{-4}^{5}(4xy+y^2+25y)\Big|_{x^2-10}^{31-(x-1)^2}\,dx$$
$$= \int_{-4}^{5}(-12x^3-78x^2+330x+1800)\,dx$$
$$= (-3x^4-26x^3+165x^2+1800x)\Big|_{-4}^{5} = 11664.$$

38. (a) This is a special case of the region over which we integrated in Exercise 26 (b). The integral is $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(x^2-y^2+5)\,dy\,dx.$

(b) You can use your favorite computer algebra system. Using *Mathematica*, enter the command:

Integrate[Integrate[$x^2-y^2+5, \{y, -\mathrm{Sqrt}[4-x^2], \mathrm{Sqrt}[4-x^2]\}], \{x,-2,2\}$] or

Integrate[$x^2-y^2+5, \{x,-2,2\}, \{y, -\mathrm{Sqrt}[4-x^2], \mathrm{Sqrt}[4-x^2]\}$] and get the answer 20π.

39. By symmetry we see that the volume is four times the volume of the piece over the first quadrant $(x, y \geq 0)$. In this region $|x|=x$ and $|y|=y$ so the volume is

$$4\int_0^2\int_0^{2-x}(2-x-y)\,dy\,dx = 4\int_0^2(2y-xy-y^2/2)\Big|_0^{2-x}\,dx = 4\int_0^2(2-2x+x^2/2)\,dx$$
$$= 4(2x-x^2+x^3/6)\Big|_0^2 = 16/3.$$

The results demonstrated in Exercises 40 and 41 are arrived at easily but worth seeing. In Exercise 40 we have the dream situation where the double integral of a product can be split into the product of integrals. We quickly see that this only works in a very special case. In Exercise 41 we examine a function where $\iint f\,dy\,dx$ exists but $\iint f\,dA$ does not.

40. (a) The function $h(x, y) = f(x)g(y)$ satisfies the conditions of Theorem 2.6 (Fubini's theorem) on $[a, b] \times [c, d]$. So:

$$\iint_R f(x)g(y)\, dA = \int_a^b \int_c^d f(x)g(y)\, dy\, dx.$$

For emphasis, we rewrite this last integral with parentheses and, since $f(x)$ does not depend on y, we have:

$$\int_a^b \left(\int_c^d f(x)g(y)\, dy\right) dx = \int_a^b f(x) \left(\int_c^d g(y)\, dy\right) dx.$$

But $\int_c^d g(y)\, dy$ is constant so we can pull it out of this last integral to get the result:

$$\int_a^b f(x) \left(\int_c^d g(y)\, dy\right) dx = \left(\int_c^d g(y)\, dy\right)\left(\int_a^b f(x)\, dx\right).$$

(b) If D is an elementary region we can perform the first step above, if D is not an elementary region, there's not much we can do. For example, if D is a type 1 region, $D = \{(x, y) | \gamma(x) \le y \le \delta(x), a \le x \le b\}$ then

$$\iint_R f(x)g(y)\, dA = \int_a^b \left(\int_{\gamma(x)}^{\delta(x)} f(x)g(y)\, dy\right) dx = \int_a^b f(x) \left(\int_{\gamma(x)}^{\delta(x)} g(y)\, dy\right) dx.$$

41. (a) If x is rational, then $\int_0^2 f(x, y)\, dy = \int_0^2 1\, dy = 2$. If x is irrational, then $\int_0^2 f(x, y)\, dy = \int_0^1 0\, dy + \int_1^2 2\, dy = 2$.

(b) Using our answer from part (a), $\int_0^1 \int_0^2 f(x, y)\, dy\, dx = \int_0^1 2\, dx = 2$.

(c) If $\mathbf{c}_{ij}$ has a rational x coordinate, then $f(\mathbf{c}_{ij}) = 1$ and so the Riemann sum will converge to the area of the region, which is 2.

(d) In this case $f(\mathbf{c}_{ij}) = 1$ for our points in the region $[0, 1] \times [0, 1]$ and $f(\mathbf{c}_{ij}) = 2$ for our points in the region $[0, 1] \times [1, 2]$. In short, the Riemann sums will converge to $(1)(1) + (2)(1) = 3$.

(e) As we saw in parts (c) and (d), the Riemann sum does not have a well defined limit and so f fails to be integrable on R, even though in part (b) we actually computed the iterated integral.

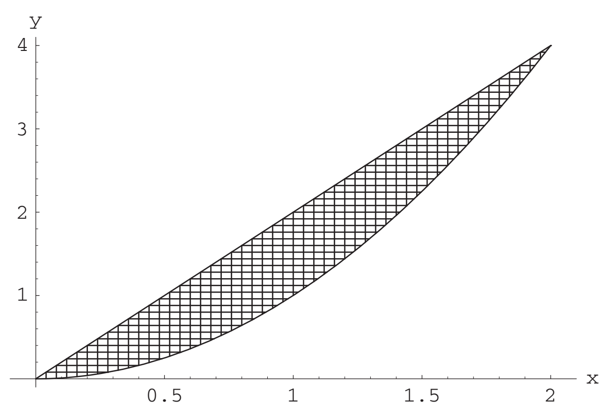
5.3 Changing The Order of Integration

This is a good section in which to encourage students to explore with a computer system.

1. (a)

$$\int_0^2 \int_{x^2}^{2x} (2x + 1)\, dy\, dx = \int_0^2 (2x + 1)(2x - x^2)\, dx = \int_0^2 (-2x^3 + 3x^2 + 2x)\, dx$$
$$= \left(-\frac{x^4}{2} + x^3 + x^2\right)\Big|_0^2 = 4.$$

(b) The region of integration is bounded above by $y = 2x$ and below by $y = x^2$:

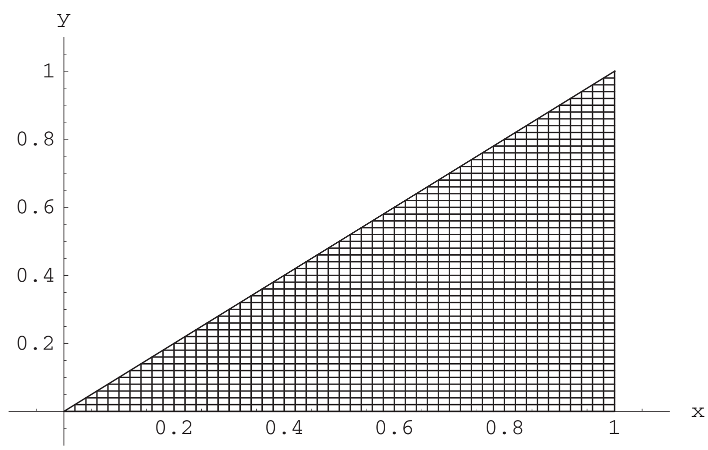


(c)

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (2x+1)\,dx\,dy = \int_0^4 (x^2+x)\Big|_{y/2}^{\sqrt{y}} dy = \int_0^4 \left(-\frac{y^2}{4} + \frac{y}{2} + \sqrt{y}\right) dy$$
$$= \left(-\frac{y^3}{12} + \frac{y^2}{4} + \frac{2y^{3/2}}{3}\right)\Big|_0^4 = 4.$$

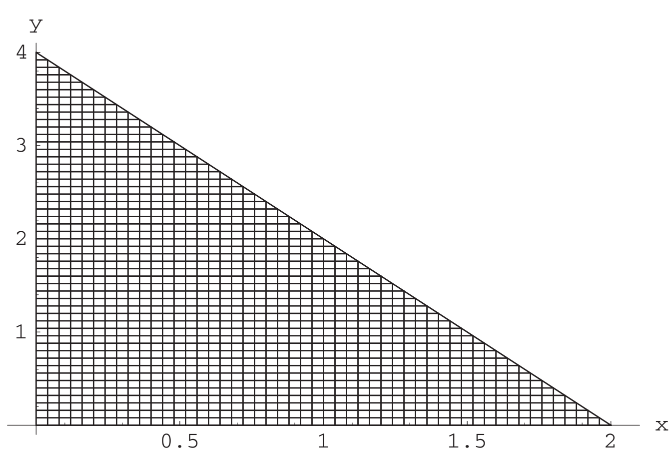
Note: In Exercises 2–9, most students will find the biggest challenge in reversing the order of integration (the topic of this section). You may want to suggest that they reverse the order of integration in all of the exercises, but that they evaluate both iterated integrals only in Exercises 7–9.

2. The region of integration is:



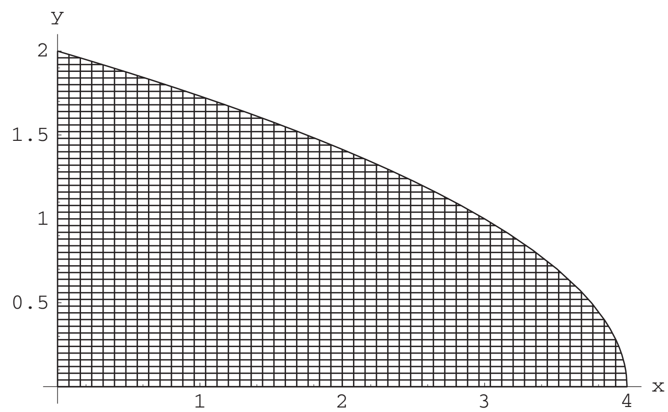
$$\int_0^1 \int_0^x (2-x-y)\,dy\,dx = \int_0^1 (2x - 3x^2/2)\,dx = 1/2 \quad \text{and}$$
$$\int_0^1 \int_y^1 (2-x-y)\,dx\,dy = \int_0^1 \frac{3}{2}(y^2 - 2y + 1)\,dy = 1/2.$$

3. The region of integration is:



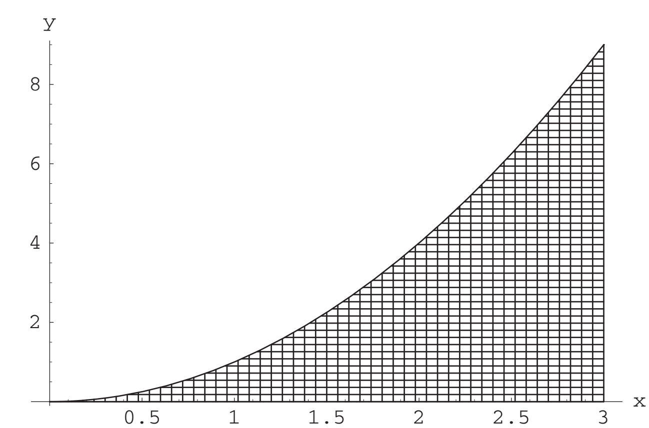
$$\int_0^2 \int_0^{4-2x} y\,dy\,dx = \int_0^2 (2x^2 - 8x + 8)\,dx = 16/3 \quad \text{and}$$
$$\int_0^4 \int_0^{2-y/2} y\,dx\,dy = \int_0^4 (-y^2/2 + 2y)\,dy = 16/3.$$

4. The region of integration is:



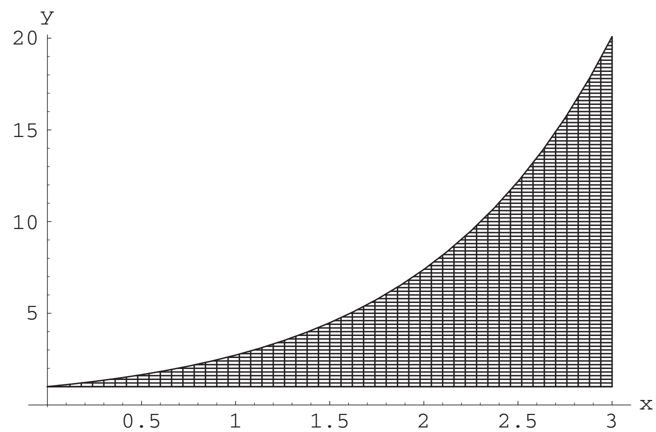
$$\int_0^2 \int_0^{4-y^2} x\,dx\,dy = \int_0^2 \left(\frac{(4-y^2)^2}{2}\right) dy = 128/15 \quad \text{and}$$
$$\int_0^4 \int_0^{\sqrt{4-x}} x\,dy\,dx = \int_0^4 (x\sqrt{4-x})\,dx = 128/15.$$

5. The region of integration is:



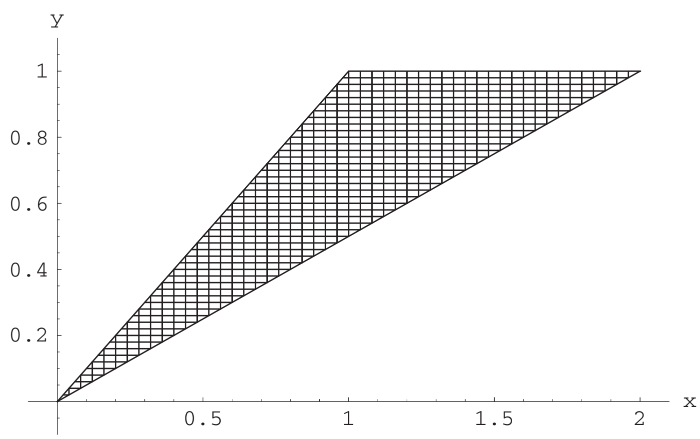
$$\int_0^9 \int_{\sqrt{y}}^3 (x+y)\,dx\,dy = \int_0^9 \frac{1}{2}(-2y^{3/2} + 5y + 9)\,dy = 891/20 \quad \text{and}$$
$$\int_0^3 \int_0^{x^2} (x+y)\,dy\,dx = \int_0^9 (x^4/2 + x^3)\,dx = 891/20.$$

6. The region of integration is:



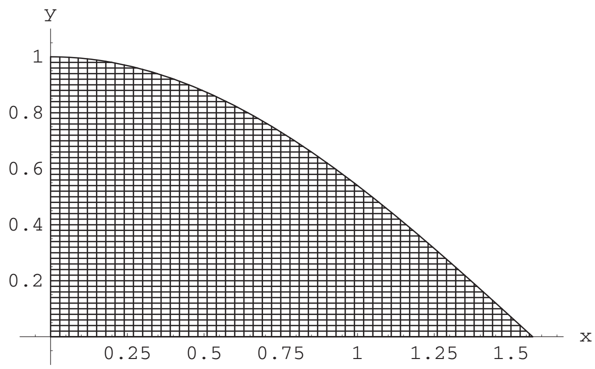
$$\int_0^3 \int_1^{e^x} 2\,dy\,dx = \int_0^3 (2e^x - 2)\,dx = 2e^3 - 8 \quad \text{and}$$
$$\int_1^{e^3} \int_{\ln y}^3 2\,dx\,dy = \int_0^3 (6 - 2\ln y)\,dy = 2e^3 - 8.$$

7. The region of integration is:



$$\int_0^1 \int_y^{2y} e^x\,dx\,dy = \int_0^1 (e^{2y} - e^y)\,dy = \frac{1}{2}(e^2 - 2e + 1) \quad \text{and}$$
$$\int_0^1 \int_{x/2}^x e^x\,dy\,dx + \int_1^2 \int_{x/2}^1 e^x\,dy\,dx = \int_0^1 (xe^x/2)\,dy + \int_1^2 (e^x - xe^x/2)\,dy = \frac{1}{2} + \frac{e^2}{2} - e.$$

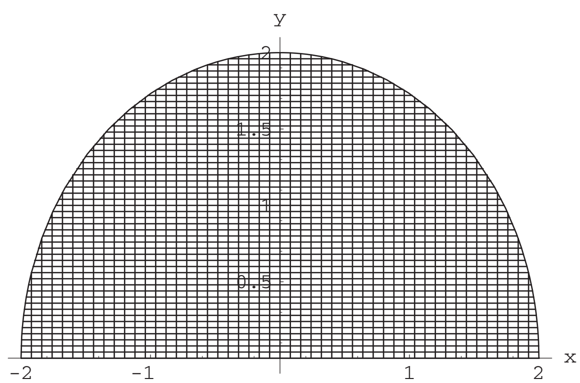
8. The region of integration is:



$$\int_0^{\pi/2}\int_0^{\cos x} \sin x\, dy\, dx = \int_0^{\pi/2} (\cos x \sin x)\, dx = 1/2 \quad \text{and}$$

$$\int_0^1\int_0^{\cos^{-1} y} \sin x\, dx\, dy = \int_0^1 (1-y)\, dy = 1/2.$$

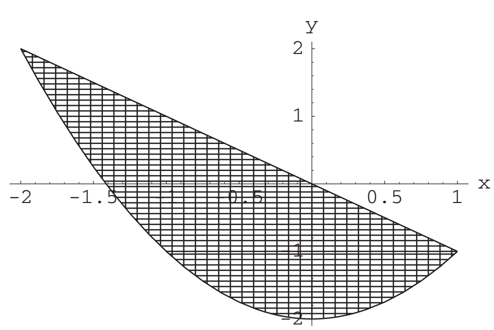
9. The region of integration is:



$$\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\, dx\, dy = \int_0^2 (2y\sqrt{4-y^2})\, dy = 16/3 \quad \text{and}$$

$$\int_{-2}^2\int_0^{\sqrt{4-x^2}} y\, dy\, dx = \int_{-2}^2 (-x^2/2+2)\, dx = 16/3.$$

10. The limits of integration describe a region D bounded on the top by the line $y = -x$ and on the bottom by the parabola $y = x^2 - 2$, as shown in the figure.



To reverse the order of integration we must divide D into two regions by the line $y = -1$. Then the original integral is equivalent to the sum

$$\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\,dx\,dy + \int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\,dx\,dy$$

The first of these integrals is

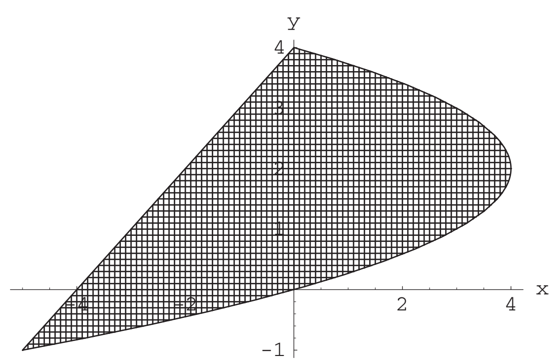
$$\begin{aligned}\int_{-2}^{-1}\int_{-\sqrt{y+2}}^{\sqrt{y+2}}(x-y)\,dx\,dy &= \int_{-2}^{-1} -2y\sqrt{y+2}\,dy\\ &= \int_0^1 -2(u-2)\sqrt{u}\,du = -2\int_0^1 (u^{3/2}-2u^{1/2})\,du\\ &= -2\left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}\right)\Big|_0^1 = -2\left(\frac{2}{5}-\frac{4}{3}\right) = \frac{28}{15}.\end{aligned}$$

The second integral is

$$\begin{aligned}\int_{-1}^{2}\int_{-\sqrt{y+2}}^{-y}(x-y)\,dx\,dy &= \int_{-1}^{2}\left(\frac{1}{2}y^2-\frac{1}{2}(y+2)+y^2-y\sqrt{y+2}\right)dy\\ &= \left(\frac{1}{2}y^3-\frac{1}{4}y^2-y\right)\Big|_{-1}^{2} - \int_{-1}^{2} y\sqrt{y+2}\,dy = \frac{3}{4}-\int_1^4 (u-2)\sqrt{u}\,du\\ &= \frac{3}{4}-\left(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2}\right)\Big|_1^4 = \frac{3}{4}-\frac{64}{5}+\frac{32}{3}+\frac{2}{5}-\frac{4}{3}\\ &= -\frac{139}{60}.\end{aligned}$$

Thus the final answer is $\frac{28}{15}-\frac{139}{60} = -\frac{9}{20}$.

11. The limits of integration describe a region D bounded on the left by $x = y - 4$ and on the right by the parabola $x = 4y - y^2$.



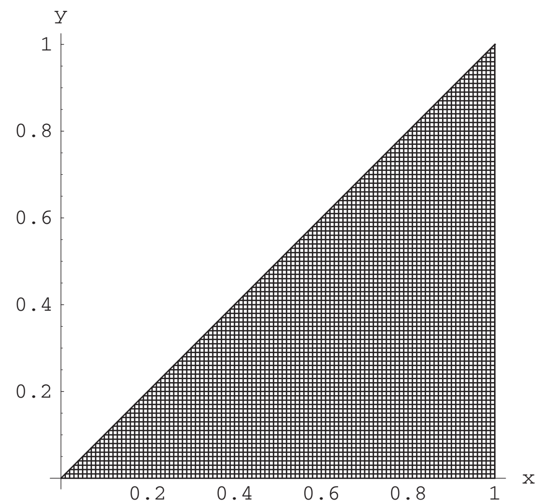
To reverse the order of integration, divide D into two regions by the line $x = 0$ (the y-axis). The original integral is equivalent to

$$\begin{aligned}&\int_{-5}^{0}\int_{2-\sqrt{4-x}}^{x+4}(y+1)\,dy\,dx + \int_0^4\int_{2-\sqrt{4-x}}^{2+\sqrt{4-x}}(y+1)\,dy\,dx\\ &= \int_{-5}^{0}\left(\frac{1}{2}(x+4)^2+(x+4)-\frac{1}{2}(2-\sqrt{4-x})^2-2+\sqrt{4-x}\right)dx\\ &\quad + \int_0^4\left(\frac{1}{2}(2+\sqrt{4-x})^2+(2+\sqrt{4-x})-\frac{1}{2}(2-\sqrt{4-x})^2-(2-\sqrt{4-x})\right)dx\end{aligned}$$

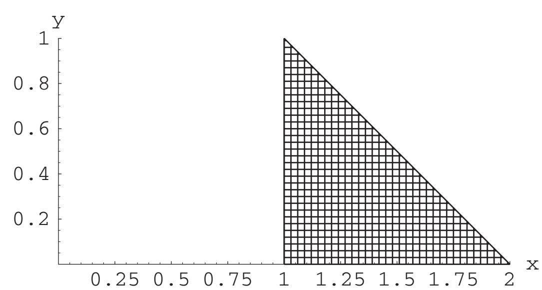
$$= \int_{-5}^{0} \left(6 + 3\sqrt{4-x} + \frac{11}{2}x^2 + \frac{1}{2}x^2\right) dx + \int_{0}^{4} 6\sqrt{4-x}\, dx$$

$$= \frac{241}{12} + 32 = \frac{625}{12}.$$

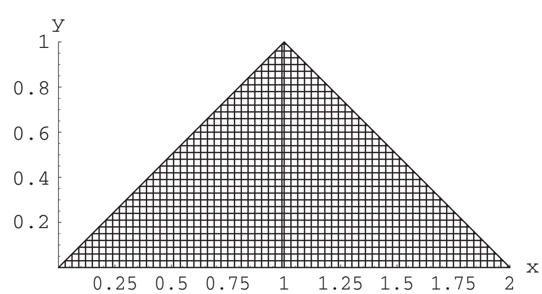
12. The limits of integration of the first integral describe the triangular region D_1 bounded on top by $y = x$:



The limits of integration of the second integral describe the triangular region D_2 bounded by $y = 2 - x$:



Taken together, we obtain the triangular region D below



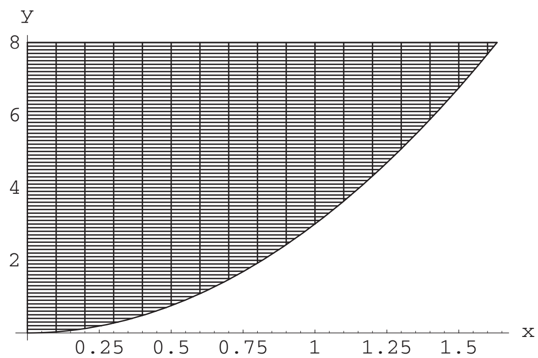
Reversing the order of integration, we find that the sum of the integrals equals

$$\int_0^1 \int_y^{2-y} \sin x \, dx\, dy = \int_0^1 (-\cos(2-y) + \cos y)\, dy$$

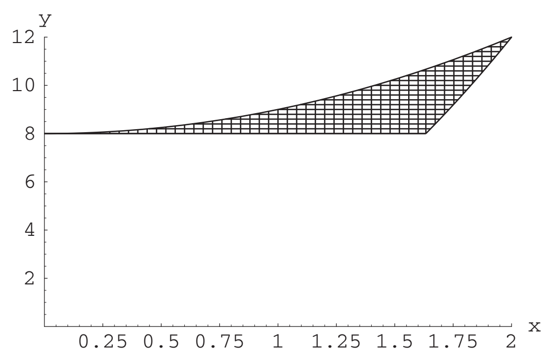
$$= (\sin(2-y) + \sin y)\Big|_0^1 = \sin 1 + \sin 1 - \sin 2$$

$$= 2\sin 1 - \sin 2.$$

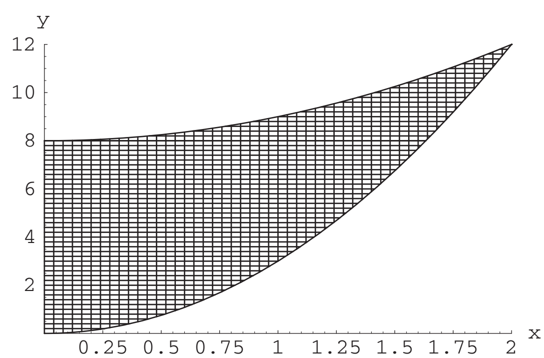
13. The limits of integration of the first integral describe the region D_1 bounded on the left by the x-axis, on the right by $x = \sqrt{y/3}$ (or, equivalently, by $y = 3x^2$) and on top by $x = 8$.



The limits of integration of the second integral describe the region D_2 bounded on the bottom by $y = 8$, on the left by $x = \sqrt{y-8}$ (which is equivalent to $y = x^2 + 8$), and on the right by $x = \sqrt{-y/3}$.



Together, D_1 and D_2 give the full region D of integration.



When we reverse the order of integration, the sum of integrals is equal to

$$\begin{aligned}\int_0^2 \int_{3x^2}^{x^2+8} y\,dy\,dx &= \int_0^2 \frac{1}{2}((x^2+8)^2 - 9x^4)\,dx \\ &= \frac{1}{2}\int_0^2 (-8x^4 + 16x^2 + 64)\,dx \\ &= \frac{1}{2}\left(-\frac{256}{5} + \frac{128}{3} + 128\right) = \frac{896}{15}.\end{aligned}$$

14. We reverse the order of integration:

$$\int_0^1 \int_{3y}^3 \cos x^2\,dx\,dy = \int_0^3 \int_0^{x/3} \cos x^2\,dy\,dx = \int_0^3 (y\cos x^2)\Big|_0^{x/3} dx$$
$$= \frac{1}{3}\int_0^3 x\cos x^2\,dx = \frac{\sin x^2}{6}\Big|_0^3 = \frac{\sin 9}{6}.$$

15. We reverse the order of integration:

$$\int_0^1 \int_y^1 x^2 \sin xy\,dx\,dy = \int_0^1 \int_0^x x^2 \sin xy\,dy\,dx = \int_0^1 (-x\cos xy)\Big|_0^x dx$$
$$= \int_0^1 (x - x\cos x^2)\,dx = \frac{1}{2}(x^2 - \sin x^2)\Big|_0^1 = \frac{1}{2}(1 - \sin 1).$$

16. We reverse the order of integration:

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\,dx\,dy = \int_0^\pi \int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^\pi \frac{y\sin x}{x}\Big|_0^x dx = \int_0^\pi (\sin x)\,dx = -\cos x\Big|_0^\pi = 2.$$

17. We reverse the order of integration:

$$\int_0^3 \int_0^{9-x^2} \frac{xe^{3y}}{9-y}\,dy\,dx = \int_0^9 \int_0^{\sqrt{9-y}} \frac{xe^{3y}}{9-y}\,dx\,dy = \int_0^9 \frac{x^2e^{3y}}{2(9-y)}\Big|_0^{\sqrt{9-y}} dy$$
$$= \int_0^9 (e^{3y}/2)\,dx = (e^{3y}/6)\Big|_0^9 = \frac{e^{27}-1}{6}.$$

18. We reverse the order of integration:

$$\int_0^2 \int_{y/2}^1 e^{-x^2}\,dy\,dx = \int_0^1 \int_0^{2x} e^{-x^2}\,dy\,dx = \int_0^1 e^{-x^2} y\Big|_0^{2x} dx$$
$$= \int_0^1 (2xe^{-x^2})\,dx = (-e^{-x^2})\Big|_0^1 = 1 - \frac{1}{e}.$$

Note: It's kind of interesting to see, in Exercises 19–21, that order of integration matters to us and to computer algebra systems.

19. (a) After churning for a while the program returned a sum of terms that included Bessel functions, Gamma functions and other non-trivial and non-elightening results.

(b) You would use integration by parts twice and then substitute back in to eliminate the integral.

(c) In a blink of an eye you get $\int_0^1 \int_0^{2y} y^2 \cos xy\,dx\,dy = (1/4)(1 - \cos 2)$.

20. (a) Again, the program thought for a while and warned that inverse functions were being used and that values could be lost for multivalued inverses. This time, however, it did come up with the correct answer of $(1/4)(1 - \cos 81)$.

(b) The calculation $\int_0^9 \int_0^{\sqrt{y}} x\sin y^2\,dx\,dy$ resulted in the same answer, but the solution came much more quickly.

21. (a) The software did nothing more than typeset the integral and leave it unevaluated.

(b) This time *Mathematica* quickly calculated the integral $\int_0^{\pi/2} \int_0^{\sin x} e^{\cos x} dy\,dx = e - 1$.

5.4 Triple Integrals

In Exercises 1–3, use Theorem 4.5, Fubini's Theorem, to integrate in the most convenient order. Exercise 4 asks the students to reconsider what happened in Exercise 1. Exercise 3 is a nice opportunity to look back at a result from Section 5.2.

1. If we integrate with respect to x first, the integral simplifies:

$$\iiint_{[-1,1]\times[0,2]\times[1,3]} xyz\,dV = \int_1^3\int_0^2\int_{-1}^1 xyz\,dx\,dy\,dz = \int_1^3\int_0^2 \left.\frac{x^2yz}{2}\right|_{-1}^1 dy\,dz$$
$$= \int_1^3\int_0^2 0\,dy\,dz = 0.$$

2. Here order doesn't matter.

$$\iiint_{[0,1]\times[0,2]\times[0,3]} (x^2+y^2+z^2)\,dV = \int_0^1\int_0^2\int_0^3 (x^2+y^2+z^2)\,dz\,dy\,dx$$
$$= \int_0^1\int_0^2 \left.\left(x^2z+y^2z+\frac{z^3}{3}\right)\right|_0^3 dy\,dx$$
$$= \int_0^1\int_0^2 (3x^2+3y^2+9)\,dy\,dx$$
$$= \int_0^1 \left.(3x^2y+y^3+9y)\right|_0^2 dx$$
$$= \int_0^1 (6x^2+26)\,dx$$
$$= \left.(2x^3+26x)\right|_0^1 = 28.$$

3. You could work this out as in Exercise 2, or suggest to your students that they could extend the result they established in Exercise 40 of Section 5.2:

$$\iiint_{[1,e]\times[1,e]\times[1,e]} \left(\frac{1}{xyz}\right) dV = \left(\int_1^e \frac{1}{x}\,dx\right)\left(\int_1^e \frac{1}{y}\,dy\right)\left(\int_1^e \frac{1}{z}\,dz\right)$$
$$= \left(\int_1^e \frac{1}{x}\,dx\right)^3 = \left(\ln x\Big|_1^e\right)^3 = 1^3 = 1.$$

4. This works for the same reason that Exercise 1 simplified. We are integrating an odd function of z on an interval that is symmetric in the z coordinate and so, since $\int_{-3}^3 z\,dz = 0$, the triple integral will also be 0.

5.

$$\int_{-1}^2\int_1^{z^2}\int_0^{y+z} 3yz^2\,dx\,dy\,dz = \int_{-1}^2\int_1^{z^2} \left.3xyz^2\right|_0^{y+z} dy\,dz = 3\int_{-1}^2\int_1^{z^2} (y^2z^2+yz^3)\,dy\,dz$$
$$= 3\int_{-1}^2 \left.\left(\frac{y^3z^2}{3}+\frac{y^2z^3}{2}\right)\right|_1^{z^2} dz = 3\int_{-1}^2 \left(\frac{z^8}{3}+\frac{z^7}{2}-\frac{z^3}{2}-\frac{z^2}{3}\right) dz$$
$$= 3\left.\left(\frac{z^9}{27}+\frac{z^8}{16}-\frac{z^4}{8}-\frac{z^3}{9}\right)\right|_{-1}^2 = \frac{1539}{16}.$$

6.

$$\int_1^3\int_0^z\int_1^{xz} (x+2y+z)\,dy\,dx\,dz = \int_1^3\int_0^z \left.(xy+y^2+zy)\right|_1^{xz} dx\,dz$$
$$= \int_1^3\int_0^z (x^2z+x^2z^2+xz^2-x-z-1)\,dx\,dz$$
$$= \int_1^3 \left.\left(\frac{x^3z}{3}+\frac{x^3z^2}{3}+\frac{x^2z^2}{2}-\frac{x^2}{2}-xz-x\right)\right|_0^z dz$$
$$= \int_1^3 \left(\frac{z^5}{3}+\frac{5z^4}{6}-\frac{3z^2}{2}-z\right) dz = \frac{574}{9}.$$

7.

$$\int_0^1 \int_{1+y}^{2y} \int_z^{y+z} z\,dx\,dz\,dy = \int_0^1 \int_{1+y}^{2y} xz\Big|_z^{y+z} dz\,dy$$

$$= \int_0^1 \int_{1+y}^{2y} yz\,dz\,dy = \int_0^1 (yz^2/2)\Big|_{1+y}^{2y} dy$$

$$= \int_0^1 \left(\frac{3y^3}{2} - y^2 - \frac{y}{2}\right) dy = -\frac{5}{24}.$$

8. (a) This is a higher-dimensional analogue of Exercise 26 from Section 5.2. Again the idea would be that if we were in four-dimensional space that a figure of constant height would have volume equal to the volume of the base multiplied by the height. In this case that would be just the volume of the base. Somehow this is a lot less physically appealing or intuitive. By Definition 4.3,

$$\iiint_W 1\,dA = \lim_{\text{all } \Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i,j,k=1}^{n} \Delta x_i \Delta y_j \Delta z_k.$$

The intuition follows from examining the formula above on the right. This converges to the volume of W. More formally, we are assuming that W is an elementary region; let's consider the case of a type 1 region, then we can rewrite the sum above as

$$\lim_{\text{all } \Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} \Delta x_i \Delta y_j (\psi(\mathbf{c}_{ij}) - \varphi(\mathbf{c}_{ij})) = \lim_{\text{all } \Delta x_i \to 0} \sum_{i=1}^{n} \Delta x_i \left(\int_{\gamma(c_i)}^{\delta(c_i)} (\psi(y) - \varphi(y))\,dy \right)$$

$$= \int_a^b \int_{\gamma(c_i)}^{\delta(c_i)} (\psi(y) - \varphi(y))\,dy\,dx = \text{volume of } W.$$

The proof is not much different for the other elementary regions.

(b) Work out that the equation of the circle where the two paraboloids intersect is $x^2 + y^2 = 9/2$ so

$$\text{Volume} = \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{-\sqrt{9/2-x^2}}^{\sqrt{9/2-x^2}} \int_{x^2+y^2}^{9-x^2-y^2} 1\,dz\,dy\,dx$$

$$= \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{-\sqrt{9/2-x^2}}^{\sqrt{9/2-x^2}} (9 - 2x^2 - 2y^2)\,dy\,dx$$

$$= \int_{-3/\sqrt{2}}^{3/\sqrt{2}} \left(\left[12 - \frac{8}{3}x^2\right] \sqrt{\frac{9}{2} - x^2} \right) dx$$

$$= \left(\sqrt{\frac{9}{2} - x^2} \left[\frac{15x}{2} - \frac{2x^3}{3}\right] + \frac{81}{4} \arcsin\left[\frac{\sqrt{2}x}{3}\right] \right) \Bigg|_{-3/\sqrt{2}}^{3/\sqrt{2}}$$

$$= \frac{81\pi}{4}.$$

9. Of course there are other ways to calculate the volume of the sphere.

$$\text{Volume} = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\,dz\,dy\,dx = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2 - x^2 - y^2}\,dy\,dx$$

$$= \int_{-a}^{a} \left(y\sqrt{a^2 - x^2 - y^2} - (a^2 - x^2) \arcsin\left[\frac{y}{\sqrt{a^2 - x^2}}\right] \right) \Bigg|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \pi \int_{-a}^{a} (a^2 - x^2)\,dx = \pi(a^2 x - x^3/3)\Big|_{-a}^{a} = \frac{4\pi a^3}{3}.$$

10. The students have seen this as the volume of a solid of revolution. We'll orient the cone so that the vertex is down at the origin and the axis is along the z-axis. Then the horizontal cross sections are circles of radius rz/h. This simplifies the following computation:

$$\text{Volume} = \int_0^h \int_{-rz/h}^{rz/h} \int_{-\sqrt{(rz/h)^2-x^2}}^{\sqrt{(rz/h)^2-x^2}} dy\,dx\,dz = \int_0^h \pi \frac{r^2}{h^2} z^2\,dz = \frac{1}{3}\pi r^2 h.$$

11.

$$\begin{aligned}\int_0^1 \int_{-2}^2 \int_0^{y^2} (2x - y + z)\,dz\,dy\,dx &= \int_0^1 \int_{-2}^2 \left(2xz - yz + \frac{z^2}{2}\right)\Big|_0^{y^2} dy\,dx \\ &= \int_0^1 \int_{-2}^2 (2xy^2 - y^3 + y^4/2)\,dy\,dx \\ &= \int_0^1 \left(\frac{2xy^3}{3} - \frac{y^4}{4} + \frac{y^5}{10}\right)\Big|_{-2}^2 dx \\ &= \int_0^1 \left(\frac{32x}{3} + \frac{64}{10}\right) dx \\ &= \left(\frac{16x^2}{3} + \frac{32}{5}x\right)\Big|_0^1 = \frac{176}{15}.\end{aligned}$$

12.

$$\begin{aligned}\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{2-x-z} y\,dy\,dz\,dx &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\frac{y^2}{2}\right)\Big|_0^{2-x-z} dz\,dx \\ &= \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ((2-x-z)^2/2)\,dz\,dx \\ &= \int_{-1}^1 \left(\frac{1}{3}\sqrt{1-x^2}(2x^2 - 12x + 13)\right) \\ &= \left(\sqrt{1-x^2}\left(\frac{2x^3 - 16x^2 + 25x + 16}{12}\right) + \frac{9}{4}\arcsin x\right)\Big|_{-1}^1 \\ &= \frac{9\pi}{4}.\end{aligned}$$

13. Here $\int_{-3}^3 \int_{x^2}^9 \int_0^{9-y} 8\,xyz\,dz\,dy\,dx = 0$, because we are integrating an odd function in x over an interval that is symmetric in x (see Exercises 1 and 4).

14.

$$\begin{aligned}\int_0^3 \int_x^3 \int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx &= \int_0^3 \int_x^3 \frac{z^2}{2}\Big|_0^{\sqrt{9-y^2}} dy\,dx \\ &= \int_0^3 \int_x^3 ((9-y^2)/2)\,dy\,dx \\ &= \int_0^3 \left(\frac{-y^3 + 27y}{6}\right)\Big|_x^3 dx \\ &= \int_0^3 \left(\frac{x^3 - 27x + 54}{6}\right) dx = \frac{81}{8}.\end{aligned}$$

15. Here we are again integrating a polynomial. The only difficulty is in the set up:

$$\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} (1 - z^2)\,dz\,dy\,dx = \frac{1}{10}.$$

16. Again, the set up and solution are:

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 3x\,dz\,dy\,dx = \frac{64}{5}.$$

17.

$$\int_0^3\int_0^{3-x}\int_{-\sqrt{3-x^2/3}}^{\sqrt{3-x^2/3}}(x+y)\,dz\,dy\,dx = \int_0^3\int_0^{3-x}(2(x+y)\sqrt{3-x^2/3})\,dy\,dx$$
$$= \int_0^3((9-x^2)\sqrt{3-x^2/3})\,dx$$
$$= \left(\sqrt{3-x^2/3}\left(\frac{45x-2x^3}{8}\right)+\frac{81\sqrt{3}}{8}\arcsin(x/3)\right)\Big|_0^3$$
$$= \frac{81\sqrt{3}\pi}{16}.$$

18.

$$\int_{-2}^2\int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}}\int_0^{x+2} z\,dz\,dy\,dx = \int_{-2}^2\int_{-\sqrt{1-x^2/4}}^{\sqrt{1-x^2/4}}((x+2)^2/2)\,dy\,dx$$
$$= \int_{-2}^2((x+2)^2\sqrt{1-x^2/4})\,dx$$
$$= \left(\frac{3x^3+16x^2+18x-64}{12}\sqrt{1-x^2/4}+5\arcsin(x/2)\right)\Big|_{-2}^2$$
$$= 5\pi.$$

19. $\displaystyle\int_0^1\int_{y^2}^y\int_0^y(4x+y)\,dz\,dx\,dy = \int_0^1\int_{y^2}^y(4x+y)y\,dx\,dy = \int_0^1(3y^3-2y^5-y^4)\,dy$
$\displaystyle= \left[\frac{3}{4}y^4-\frac{1}{3}y^6-\frac{1}{5}y^5\right]\Big|_0^1 = \frac{13}{60}.$

20. The surfaces $z = x^2+2y^2$ and $z = 6-x^2-y^2$ intersect where

$$x^2+2y^2 = 6-x^2-y^2 \iff 2x^2+3y^2 = 6.$$

Since we are only interested in the first octant part of the solid, the shadow of the solid in the xy-plane is the region bounded by the ellipse $2x^2+3y^2 = 6$ and the coordinate axes in the first quadrant. Thus we calculate:

$$\int_0^{\sqrt{3}}\int_0^{\sqrt{2-2x^2/3}}\int_{x^2+2y^2}^{6-x^2-y^2} x\,dz\,dy\,dx = \int_0^{\sqrt{3}}\int_0^{\sqrt{2-2x^2/3}} x(6-2x^2-3y^2)\,dy\,dx$$
$$= \int_0^{\sqrt{3}}\left[(6x-2x^3)\sqrt{2-\frac{2}{3}x^2}-x\left(2-\frac{2}{3}x^2\right)^{3/2}\right]\,dx$$
$$= \int_0^{\sqrt{3}} 2x\left(2-\frac{2}{3}x^2\right)^{3/2}\,dx = \int_2^0 -\frac{3}{2}u^{3/2}\,du,$$

where $u = 2-\dfrac{2}{3}x^2$,

$$= \int_0^2\frac{3}{2}u^{3/2}\,du = \frac{3}{5}u^{5/2}\Big|_0^2 = \frac{12\sqrt{2}}{5}.$$

21. The volume is given by

$$\iiint_W 1\,dV = \int_0^2\int_0^{2-x}\int_0^{4-x^2} 1\,dz\,dy\,dx$$
$$= \int_0^2\int_0^{2-x}(4-x^2)\,dy\,dx = \int_0^2(4-x^2)(2-x)\,dx$$
$$= \int_0^2(x^3-2x^2-4x+8)\,dx = \left(\frac{x^4}{4}-\frac{2x^3}{3}-2x^2+8x\right)\Big|_0^2 = \frac{20}{3}.$$

22. The volume is

$$\iiint_W 1\,dV = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}}\int_0^{6-2y} 1\,dz\,dy\,dx = \int_{-3}^{3}\int_0^{\sqrt{9-x^2}}(6-2y)\,dy\,dx$$
$$= \int_{-3}^{3}\left[6\sqrt{9-x^2}-(9-x^2)\right]\,dx = \int_{-3}^{3}6\sqrt{9-x^2}\,dx + \int_{-3}^{3}(x^2-9)\,dx$$
$$= \int_{-3}^{3}6\sqrt{9-x^2}\,dx + \left(\frac{x^3}{3}-9x\right)\Big|_{-3}^{3}$$
$$= \int_{-3}^{3}6\sqrt{9-x^2}\,dx - 36.$$

For the remaining integral, let $x = 3\sin\theta$ so that $dx = 3\cos\theta\,d\theta$. Then

$$\int_{-3}^{3}6\sqrt{9-x^2}\,dx = \int_{-\pi/2}^{\pi/2}6(3\cos\theta)3\cos\theta\,d\theta = 27\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta$$
$$= 27\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_{-\pi/2}^{\pi/2} = 27\pi.$$

(Alternatively, we could have recognized this integral as six times the area of a semicircle of radius 3, or $6(\pi\cdot 3^2/2) = 27\pi$.) Hence the total volume is $27\pi - 36$.

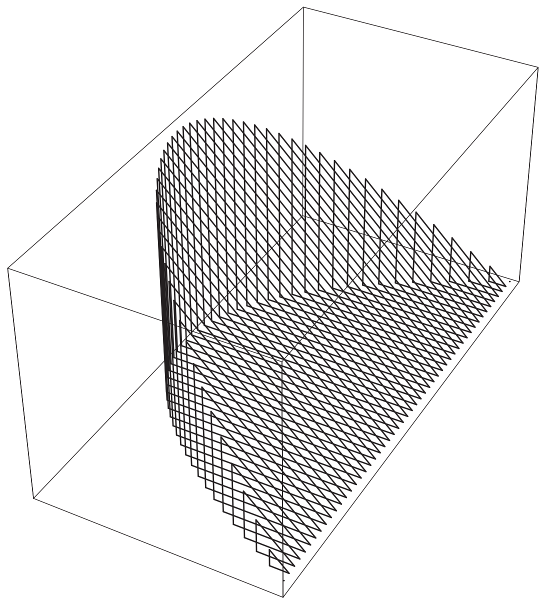
23.

$$\int_{-1}^{1}\int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}}\int_{4x^2+y^2}^{2-y^2} dz\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{(1-y^2)/2}}^{\sqrt{(1-y^2)/2}}(2-4x^2-2y^2)\,dx\,dy$$
$$= \int_{-1}^{1}\left(\frac{4\sqrt{2}}{3}(y^2-1)\sqrt{1-y^2}\right)\,dy$$
$$= \frac{\pi}{\sqrt{2}}.$$

24.

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}(2\sqrt{a^2-x^2})\,dy\,dx$$
$$= \int_{-a}^{a}4(a^2-x^2)\,dx$$
$$= \frac{16a^3}{3}.$$

25. The region looks like a wedge of cheese:



The five other forms are:

$$\begin{aligned}\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{1-x} f(x,y,z)\,dz\,dx\,dy &= \int_{0}^{1}\int_{-\sqrt{x}}^{\sqrt{x}}\int_{0}^{1-x} f(x,y,z)\,dz\,dy\,dx\\ &= \int_{0}^{1}\int_{0}^{1-x}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\,dy\,dz\,dx\\ &= \int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{x}}^{\sqrt{x}} f(x,y,z)\,dy\,dx\,dz\\ &= \int_{-1}^{1}\int_{0}^{1-y^2}\int_{y^2}^{1-z} f(x,y,z)\,dx\,dz\,dy\\ &= \int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{y^2}^{1-z} f(x,y,z)\,dx\,dy\,dz.\end{aligned}$$

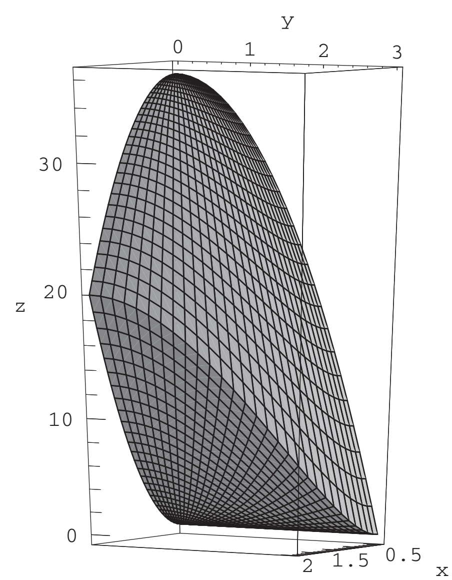
26. The five other forms are:

$$\begin{aligned}\int_{0}^{1}\int_{0}^{1}\int_{0}^{x^2} f(x,y,z)\,dz\,dx\,dy &= \int_{0}^{1}\int_{0}^{1}\int_{0}^{x^2} f(x,y,z)\,dz\,dy\,dx\\ &= \int_{0}^{1}\int_{0}^{1}\int_{\sqrt{z}}^{1} f(x,y,z)\,dx\,dz\,dy\\ &= \int_{0}^{1}\int_{0}^{1}\int_{\sqrt{z}}^{1} f(x,y,z)\,dx\,dy\,dz\\ &= \int_{0}^{1}\int_{0}^{x^2}\int_{0}^{1} f(x,y,z)\,dy\,dz\,dx\\ &= \int_{0}^{1}\int_{\sqrt{z}}^{1}\int_{0}^{1} f(x,y,z)\,dy\,dx\,dz.\end{aligned}$$

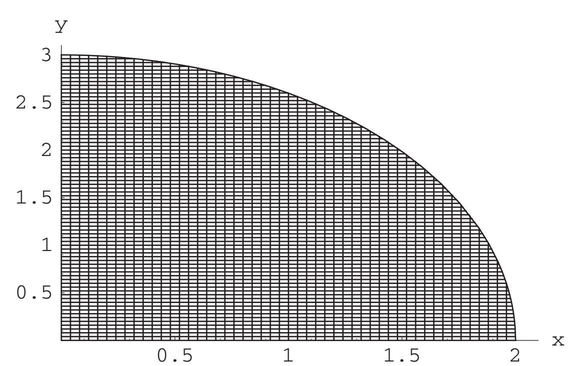
27. The five other forms are:

$$\begin{aligned}\int_0^2 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx &= \int_0^2 \int_y^2 \int_0^y f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^2 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^2 \int_z^2 \int_z^x f(x,y,z)\,dy\,dx\,dz \\ &= \int_0^2 \int_0^y \int_y^2 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^2 \int_z^2 \int_y^2 f(x,y,z)\,dx\,dy\,dz.\end{aligned}$$

28. (a) The solid W is bounded below by the surface $z = 5x^2$, above by the paraboloid $z = 36 - 4x^2 - 4y^2$, on the left by the *xz*-plane (i.e., $y = 0$), and in back by the *yz*-plane (i.e., $x = 0$). The solid is shown below.



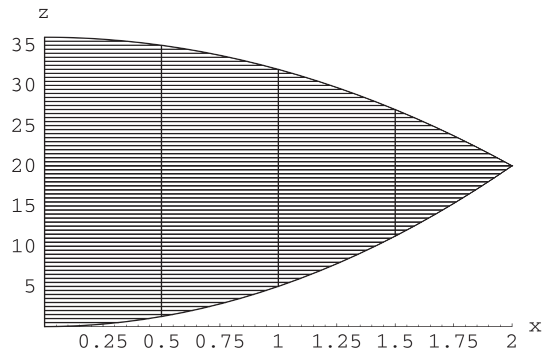
(b) The shadow of the solid in the *xy*-plane is a quarter of the ellipse $9x^2 + 4y^2 = 36$ (obtained by finding the intersection curve of $z = 5x^2$ and $z = 36 - 4x^2 - 4y^2$.) The shadow looks like:



Using the shadow region to reverse the order of integration between x and y, we find that the original integral is equivalent to

$$\int_0^3 \int_0^{\frac{1}{3}\sqrt{36-4y^2}} \int_{5x^2}^{36-4x^2-4y^2} 2\,dz\,dx\,dy.$$

(c) In this case, we need to consider the shadow of W in the *xz*-plane.



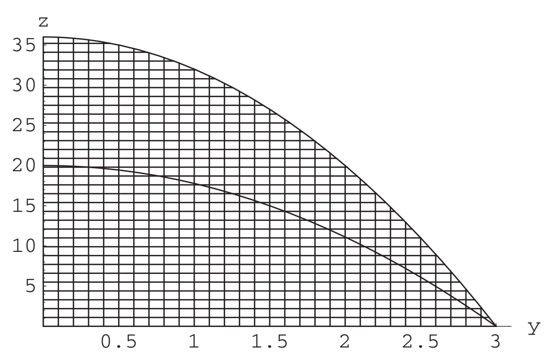
This region is bounded on the left by $x = 0$, on the bottom by $z = 5x^2$, and on top by $z = 36 - 4x^2$ (the section by $y = 0$). Now the full solid W is bounded in the y-direction by $y = 0$ and $y = \frac{1}{2}\sqrt{36 - 4x^2 - z}$ (the latter is just the paraboloid surface). Hence the desired iterated integral is

$$\int_0^2 \int_{5x^2}^{36-4x^2} \int_0^{\frac{1}{2}\sqrt{36-4x^2-z}} 2\, dy\, dz\, dx.$$

(d) Here we use the same shadow in the *xz*-plane as in part (c), only to integrate with respect to x before integrating with respect to z will require dividing the shadow into two regions by the line $z = 20$. (Equivalently, we are dividing the solid W into two solids by the plane $z = 20$.) This is why we need a sum of integrals. They are

$$\int_0^{20} \int_0^{\sqrt{z/5}} \int_0^{\frac{1}{2}\sqrt{36-4x^2-z}} 2\, dy\, dx\, dz + \int_{20}^{36} \int_0^{\frac{1}{2}\sqrt{36-z}} \int_0^{\frac{1}{2}\sqrt{36-4x^2-z}} 2\, dy\, dx\, dz.$$

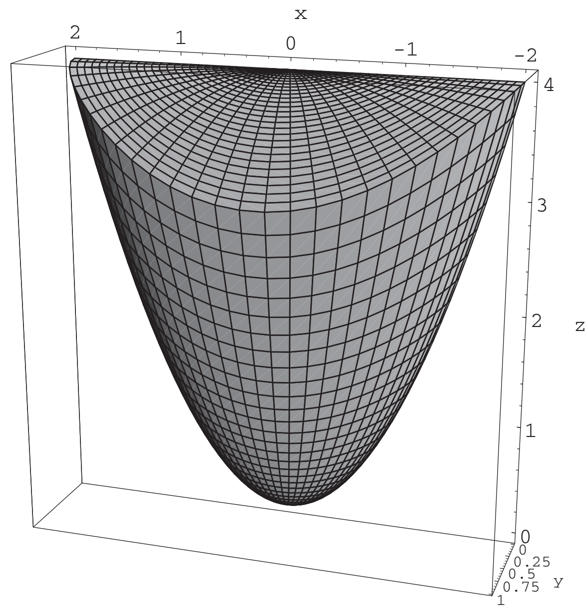
(e) To integrate with respect to x first, we need to divide W in a very different manner. The shadow in the *yz*-plane shows a region bounded on the left by $y = 0$ and above by $z = 36 - 4y^2$ (the section of the paraboloid by $x = 0$). However, the curve of intersection of the surfaces $z = 5x^2$ and $z = 36 - 4x^2 - 4y^2$ with x eliminated yields the equation $z = 20 - \dfrac{20y^2}{9}$. It is along this curve that we must divide the *yz*-shadow and thus the integrals. (Note: This curve is just the shadow of the intersection curve of the two surfaces projected into the *yz*-plane.)



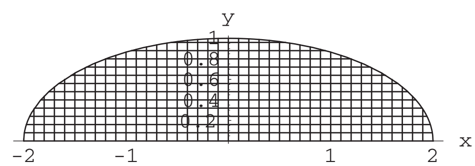
Thus the desired sum of integrals is

$$\int_0^3 \int_0^{20-20y^2/9} \int_0^{\sqrt{z/5}} 2\, dx\, dz\, dy + \int_0^3 \int_{20-20y^2/9}^{36-4y^2} \int_0^{\frac{1}{2}\sqrt{36-4y^2-z}} 2\, dx\, dz\, dy.$$

29. (a) The solid W is bounded below by the paraboloid $z = x^2 + 3y^2$, above by the surface $z = 4 - y^2$ and in back by the plane $y = 0$. The solid is shown below.



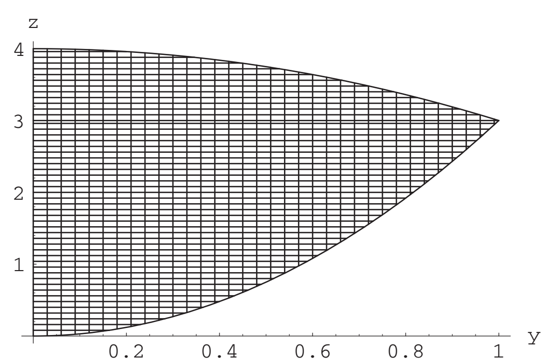
(b) The shadow of W in the *xy*-plane is half of the region inside the ellipse $x^2 + 4y^2 = 4$ (the half with $y \geq 0$). It may be obtained by finding the intersection curve of $z = x^2 + 3y^2$ and $z = 4 - y^2$ and eliminating z. The shadow looks like



Using the shadow to reverse the order of integration between x and y, we find that the original integral is equivalent to

$$\int_0^1 \int_{-2\sqrt{1-y^2}}^{2\sqrt{1-y^2}} \int_{x^2+3y^2}^{4-y^2} (x^3 + y^3)\, dz\, dx\, dy.$$

(c) We need to consider the shadow of W in the *yz*-plane.



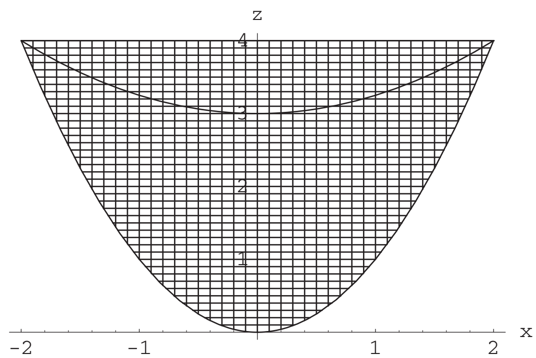
The region is bounded on the left by $y = 0$, on the bottom by $z = 3y^2$ (the section by $x = 0$) and on the top by $z = 4 - y^2$. The full solid W is bounded in the x-direction by the paraboloid $z = x^2 + 3y^2$, which must be expressed in terms of x as $x = \pm\sqrt{z - 3y^2}$. Putting all this information together, we find the desired iterated integral is

$$\int_0^1 \int_{3y^2}^{4-y^2} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dz\, dy.$$

(d) Here we use the same shadow in the *yz*-plane as in part (c), only to integrate with respect to y before integrating with respect to z requires dividing the shadow into two regions by the line $z = 3$. (Equivalently, we are dividing the solid W by the plane $z = 3$.) This is why we need a sum of integrals. They are

$$\int_0^3 \int_0^{\sqrt{z/3}} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dy\, dz + \int_3^4 \int_0^{\sqrt{4-z}} \int_{-\sqrt{z-3y^2}}^{\sqrt{z-3y^2}} (x^3 + y^3)\, dx\, dy\, dz.$$

(e) To integrate with respect to y first, we need to divide W in a different manner. The shadow in the *xz*-plane shows a region bounded by $z = x^2$ (the section of the paraboloid by $y = 0$) and $z = 4$. However, the curve of intersection of the surfaces $z = x^2 + 3y^2$ and $z = 4 - y^2$ with y eliminated yields the equation $z = \frac{x^2}{4} + 3$. It is along this curve that we must divide the *xz*-shadow and thus the integrals. (Note: This curve is just the shadow of the intersection curve of the two surfaces projected into the *xz*-plane.)



Thus the desired sum of integrals is

$$\int_{-2}^{2} \int_{x^2}^{(x^2/4)+3} \int_0^{\sqrt{(z-x^2)/3}} (x^3 + y^3)\, dy\, dz\, dx + \int_{-2}^{2} \int_{(x^2/4)+3}^{4} \int_0^{\sqrt{4-z}} (x^3 + y^3)\, dy\, dz\, dx.$$

5.5 Change of Variables

1. (a)

$$\mathbf{T}(u, v) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

(b) In this case we can see by inspection that the transformation stretches by 3 in the horizontal direction and reflects without a stretch in the vertical direction. Therefore the image $D = \mathbf{T}(D^*)$ where D^* is the unit square is the rectangle $[0, 3] \times [-1, 0]$.

2. (a) This is similar to the map in Example 4 with a scaling factor of $1/\sqrt{2}$. We can also rewrite

$$\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a rotation matrix (the determinant is 1 so there is no stretching) which rotates the unit square counterclockwise by $45°$ leaving the vertex at the origin in place.

(b) We rewrite

$$\mathbf{T}(u, v) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

This is a rotation followed by a reflection. You can apply Proposition 5.1 and see where $\mathbf{T}$ maps each of the vertices to completely determine the image of the unit square. You will see that the vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(1, 0)$ are mapped to $(0, 0)$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(\sqrt{2}, 0)$, and $(1/\sqrt{2}, -1/\sqrt{2})$.

3. Again, since **T** has non-zero determinant, we can apply Proposition 5.1 and see where **T** maps each of the vertices. We conclude that **T** maps D^* to the parallelogram whose vertices are: (0, 0), (11, 2), (4, 3), and (15, 5).

4. We are trying to determine the entries a, b, c, and d in the expression:

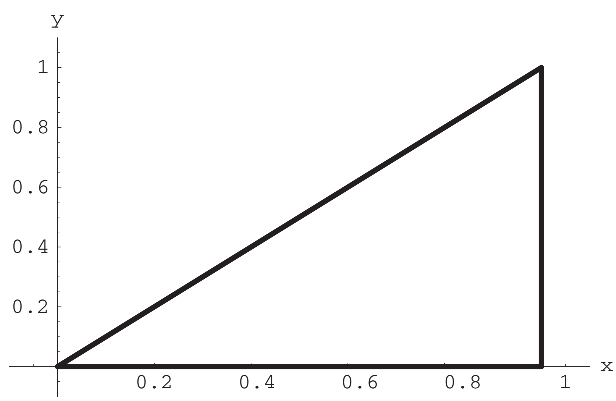
$$\mathbf{T}(u, v) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Since $\mathbf{T}(0, 0) = (0, 0)$ we know that the motion is not a translation. Also $\mathbf{T}(0, 5) = (4, 1)$ so $b = 4/5$ and $d = 1/5$. Now $\mathbf{T}(1, 2) = (1, -1)$ so $a = -3/5$ and $c = -7/5$. We check with the remaining vertex: $\mathbf{T}(-1, 3) = (3, 2)$.

$$\mathbf{T}(u, v) = \begin{bmatrix} -3/5 & 4/5 \\ -7/5 & 1/5 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

5. As noted in the text, we have a result for $\mathbf{R}^3$ that is analogous to Proposition 5.1, so as in Exercises 3 and 2 (b) we can just compute the images of the vertices of W^*. We conclude that W^* maps to the parallelepiped with vertices: (0, 0, 0), (3, 1, 5), $(-1, -1, 3)$, $(0, 2, -1)$, (2, 0, 8), (3, 3, 4), $(-1, 1, 2)$, and (2, 2, 7).

6. You can see that $\mathbf{T}(u, v) = (u, uv)$ is not one-one on D^* by observing that all points of the form (0, v) get mapped to the origin under **T**. In fact, you can imagine the map by picturing the left vertical side of the unit square being shrunk down to a point at the origin. The image is the triangle:



7. This map should be a happy memory for the students:

$$(x, y, z) = \mathbf{T}(\rho, \varphi, \theta) = (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho\cos\varphi)$$

is familiar from their work with spherical coordinates.

(a) This is the unit ball: $D = \{(x, y, z) | x^2 + y^2 + z^2 \le 1\}$.

(b) This is the portion of the unit ball in the first octant: $D = \{(x, y, z) | x^2 + y^2 + z^2 \le 1, x, y, z \ge 0\}$.

(c) You can think of this as the region from part (b) with the portion corresponding to $0 \le \rho < 1/2$ removed. It is the portion in the first octant of the shell 1/2 unit thick around a sphere of radius 1/2: $D = \{(x, y, z) | 1/4 \le x^2 + y^2 + z^2 \le 1, x, y, z \ge 0\}$.

8. (a) $\displaystyle\int_0^1 \int_{y/2}^{(y/2)+2} (2x - y)\, dx\, dy = \int_0^1 (x^2 - xy)\Big|_{y/2}^{(y/2)+2} dy = \int_0^1 4\, dy = 4$. A sketch of D is shown below.

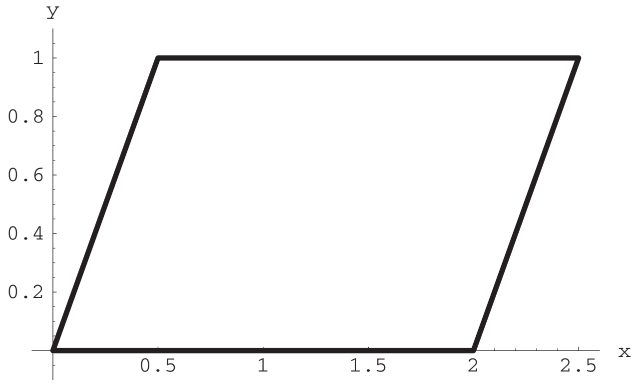
(b) We again can apply Proposition 5.1 and see that the vertices are mapped: $(0, 0) \to (0, 0), (2, 0) \to (4, 0), (1/2, 1) \to (0, 1)$, and $(5/2, 1) \to (4, 1)$ so D^* is $[0, 4] \times [0, 1]$.

(c) First note that

$$\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = 2 \quad \text{so} \quad \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2}.$$

Then, using the change of variables theorem,

$$\int_0^1 \int_{y/2}^{(y/2)+2} (2x - y)\, dx\, dy = \int_0^1 \int_0^4 u(1/2)\, du\, dv = \int_0^1 \frac{u^2}{4}\Big|_0^4 du = \int_0^1 4\, dv = 4.$$



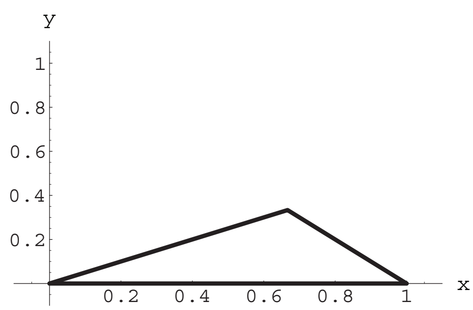
9. First,

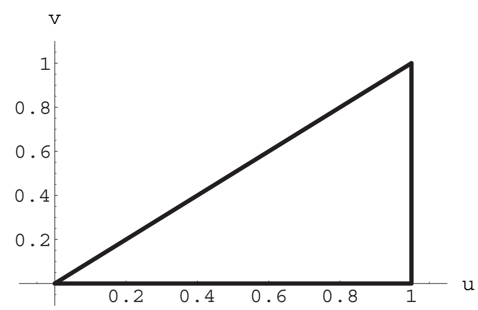
$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = 2 \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}.$$

Also we can rewrite $x^5(2y-x)e^{(2y-x)^2} = u^5 v e^{v^2}$, and the transformed region is $[0,2]\times[0,2]$ so

$$\int_0^2 \int_{x/2}^{(x/2)+1} x^5(2x-y)e^{(2x-y)^2}\,dy\,dx = \int_0^2\int_0^2 u^5 v e^{v^2}(1/2)\,du\,dv = \frac{16}{3}\int_0^2 v e^{v^2}\,dv = \frac{8}{3}(e^4-1).$$

10. The original region D is sketched below left. The transformation $u = x+y$ and $v = x-2y$ maps D to the region D^* sketched below right.





We may find $\partial(x,y)/\partial(u,v)$ in two ways. First, solving for x and y in terms of u and v, we have

$$x = \frac{2u+v}{3}, \qquad y = \frac{u-v}{3}.$$

Thus

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = -\frac{2}{9} - \frac{1}{9} = -\frac{1}{3}.$$

Alternatively, we may calculate

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = -2-1 = -3.$$

Therefore, $\partial(x,y)/\partial(u,v) = (\partial(u,v)/\partial(x,y))^{-1} = -1/3$.

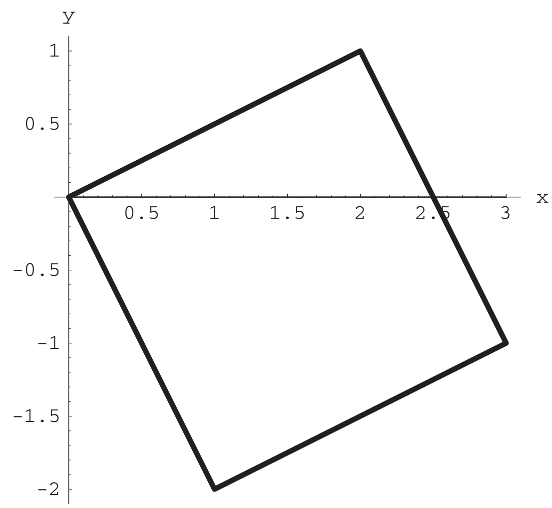
Using the change of variables theorem, our integral becomes

$$\int_0^1\int_0^u \frac{1}{3}\left(\frac{u^{1/2}}{v^{1/2}}\right)dv\,du = \int_0^1 \frac{2}{3}u^{1/2}v^{1/2}\Big|_0^u\,du = \int_0^1 \frac{2}{3}u\,du = \frac{u^2}{3}\Big|_0^1 = \frac{1}{3}.$$

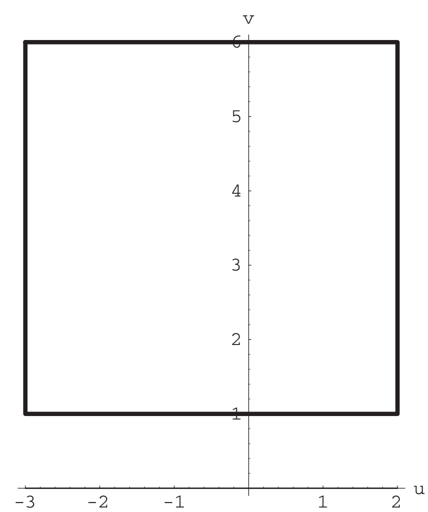
11. Here the problem cries out to you to let $u = 2x+y$ and $v = x-y$. Once you've made that move you can easily figure that $\partial(x,y)/\partial(u,v) = -1/3$ and that the new region is $[1,4]\times[-1,1]$. So the integral is

$$\int_1^4\int_{-1}^1 u^2 e^v(1/3)\,dv\,du = \frac{1}{3}\int_1^4 u^2 e^v\Big|_{-1}^1\,du = (e-e^{-1})\frac{u^3}{9}\Big|_1^4 = 7(e-e^{-1}).$$

12. If we sketch the region we get the square:



The transformation we use is $u = 2x + y - 3$ and $v = 2y - x + 6$ so $\partial(x, y)/\partial(u, v) = 1/5$ and the transformed region is the square:



Our integral is

$$\int_1^6 \int_{-3}^2 \frac{u^2}{5v^2}\, du\, dv = \frac{1}{15} \int_1^6 \left.\frac{u^3}{v^2}\right|_{-3}^2 dv = \frac{7}{3} \int_1^6 v^{-2}\, dv = \frac{7}{3}\left(-\frac{1}{v}\right)\Bigg|_1^6 = \frac{35}{18}.$$

Note: In Exercises 13–17 the Jacobian for the change of variables is r. Assign Exercise 16 so that your students appreciate the role of the extra r.

13. $\displaystyle \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3\, dy\, dx = \int_0^{2\pi} \int_0^1 3r\, dr\, d\theta = \int_0^{2\pi} \frac{3}{2} d\theta = 3\pi.$

14. $\displaystyle \int_0^2 \int_0^{\sqrt{4-x^2}} dy\, dx = \int_0^{\pi/2} \int_0^2 r\, dr\, d\theta = \int_0^{\pi/2} 2\, d\theta = \pi.$

15. $\displaystyle \int_0^{2\pi} \int_0^3 r^4\, dr\, d\theta = \int_0^{2\pi} \left.\frac{r^5}{5}\right|_0^3 d\theta = \int_0^{2\pi} \frac{243}{5}\, d\theta = \frac{486\pi}{5}.$

16. $\displaystyle \int_{-a}^a \int_0^{\sqrt{a^2-y^2}} e^{x^2+y^2}\, dx\, dy = \int_{-\pi/2}^{\pi/2} \int_0^a re^{r^2}\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left.\frac{1}{2}e^{r^2}\right|_0^a d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(e^{a^2} - 1) d\theta = \pi(e^{a^2} - 1)/2.$

17. $\displaystyle\int_0^3\int_0^x \frac{dy\,dx}{\sqrt{x^2+y^2}} = \int_0^{\pi/4}\int_0^{3\sec\theta} dr\,d\theta = \int_0^{\pi/4} 3\sec\theta\,d\theta = 3\ln(\sec\theta+\tan\theta)|_0^{\pi/4} = \ln(1+\sqrt{2})-\ln 1 = \ln(1+\sqrt{2}).$

18. This is a job for polar coordinates. The given disk has boundary circle with equation $x^2+(y-1)^2=1$. In polar coordinates this equation becomes

$$r^2\cos^2\theta + (r\sin\theta-1)^2 = 1 \Leftrightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$$
$$\Leftrightarrow r^2 = 2r\sin\theta.$$

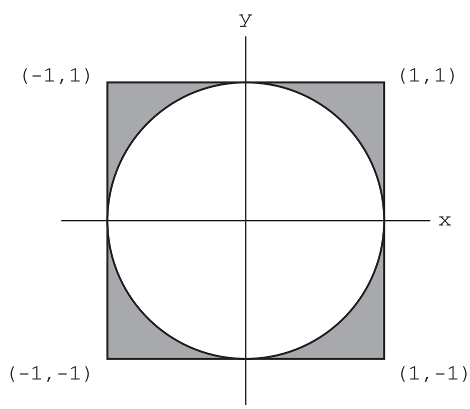
Factoring out r, the boundary circle has equation $r = 2\sin\theta$. In fact, this circle is completely traced by letting θ vary from 0 to π. Thus the region D inside the disk is given by

$$D = \{(r,\theta)|r \le 2\sin\theta, \quad 0\le\theta\le\pi\}.$$

Hence

$$\begin{aligned}
\iint_D \frac{1}{\sqrt{4-x^2-y^2}}\,dA &= \int_0^\pi\int_0^{2\sin\theta} \frac{1}{\sqrt{4-r^2}}\,r\,dr\,d\theta \\
&= \int_0^\pi -\frac{1}{2}(2\sqrt{4-r^2})\Big|_0^{2\sin\theta} d\theta = -\int_0^\pi (\sqrt{4-4\sin^2\theta}-2)\,d\theta \\
&= -2\int_0^\pi \sqrt{\cos^2\theta}\,d\theta + \int_0^\pi 2\,d\theta \\
&= -2\left(\int_0^{\pi/2}\cos\theta\,d\theta + \int_{\pi/2}^\pi (-\cos\theta)\,d\theta\right) + 2\pi \\
&= -2\left[\sin\frac{\pi}{2} - \sin 0 - \sin\pi + \sin\frac{\pi}{2}\right] + 2\pi \\
&= -4 + 2\pi = 2\pi - 4.
\end{aligned}$$

19. The region in question looks like



We find $\displaystyle\iint_D y^2 dA = \iint_{\text{square}} y^2 dA - \iint_{\text{disk}} y^2 dA$

$$\iint_{\text{square}} y^2\,dA = \int_{-1}^1\int_{-1}^1 y^2\,dx\,dy = \int_{-1}^1 2y^2\,dy = \frac{2}{3}y^3\Big|_{-1}^1 = \frac{4}{3}$$

$$\begin{aligned}
\iint_{\text{disk}} y^2\,dA &= \int_0^{2\pi}\int_0^1 r^2\sin^2\theta\cdot r\,dr\,d\theta = \int_0^{2\pi}\frac{1}{4}\sin^2\theta\,d\theta \\
&= \frac{1}{8}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = \frac{1}{8}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{\pi}{4}.
\end{aligned}$$

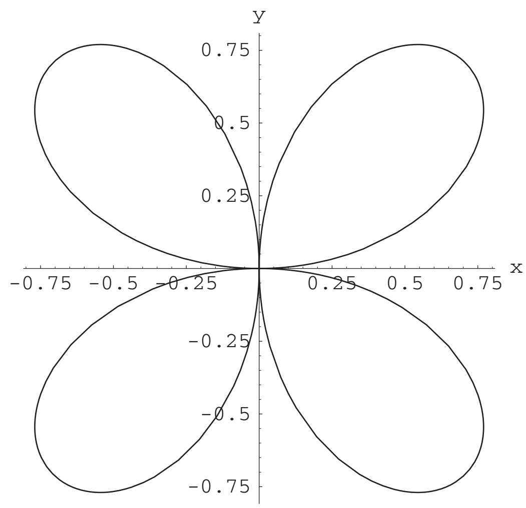
Thus

$$\iint_D y^2\,dA = \frac{4}{3} - \frac{\pi}{4} = \frac{16-3\pi}{12}.$$

20. A sketch of the rose is shown below. One leaf means that $0 \le \theta \le \pi/2$. The area of one leaf is

$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2 2\theta\,d\theta = \frac{1}{16}(4\theta - \sin 4\theta)\Big|_0^{\pi/2} = \frac{\pi}{8}.$$

The total area is then four times this, or $\pi/2$.



21. If n is *odd*, the polar equation $r = a\cos n\theta$ determines an n-leafed rose. Half of one of the n leaves is traced as θ varies from 0 to $\pi/(2n)$. Hence the total area enclosed is

$$\begin{aligned}\iint_D 1\,dA &= 2n\int_0^{\pi/(2n)}\int_0^{a\cos n\theta} r\,dr\,d\theta = 2n\int_0^{\pi/(2n)} \frac{1}{2}(a\cos n\theta)^2\,d\theta \\ &= na^2\int_0^{\pi/(2n)} \cos^2 n\theta\,d\theta = \frac{na^2}{2}\int_0^{\pi/(2n)}(1+\cos 2n\theta)\,d\theta \\ &= \frac{na^2}{2}\left(\theta + \frac{1}{2n}\sin 2n\theta\right)\Big|_0^{\pi/(2n)} = \frac{\pi a^2}{4}.\end{aligned}$$

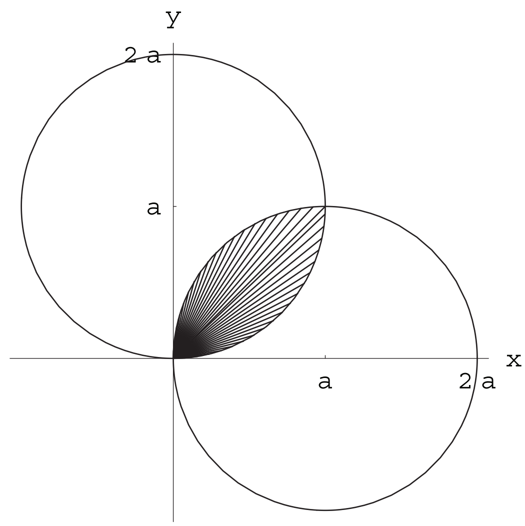
If n is *even*, then the equation $r = a\cos n\theta$ determines a rose with $2n$ leaves. Half of one of these $2n$ leaves is again traced as θ varies from 0 to $\pi/(2n)$. The total area enclosed is

$$\iint_D 1\,dA = 2(2n)\int_0^{\pi/(2n)}\int_0^{a\cos n\theta} r\,dr\,d\theta.$$

Since this is just twice the previous iterated integral, there is no reason to recompute; the result is $\pi a^2/2$.

In each case the answer depends only on a, not the specific value of n other than its parity.

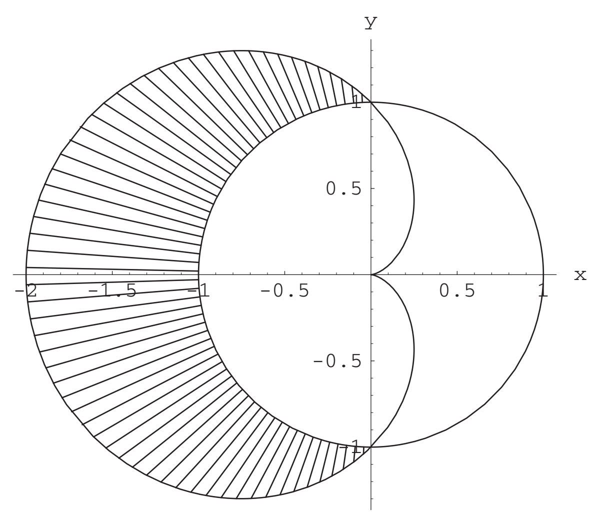
22. The circles $r = 2a\cos\theta$ and $r = 2a\sin\theta$ are both of radius a with respective centers at $(a, 0)$ and $(0, a)$ (in Cartesian coordinates). The region in question looks like:



The circles intersect where $2a\cos\theta = 2a\sin\theta \iff \theta = \pi/4$ (also at the origin where $r = 0$). By symmetry, we have

$$\begin{aligned}\text{Area} &= \iint_D 1\, dA = 2\int_0^{\pi/4}\int_0^{2a\sin\theta} r\, dr\, d\theta \\ &= \int_0^{\pi/4} (2a\sin\theta)^2\, d\theta = \int_0^{\pi/4} 2a^2(1-\cos 2\theta)\, d\theta \\ &= 2a^2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/4} = \frac{\pi a^2}{2} - a^2 = \frac{(\pi-2)a^2}{2}.\end{aligned}$$

23. We sketch the graphs of the cardioid $r = 1 - \cos\theta$ and the circle $r = 1$:

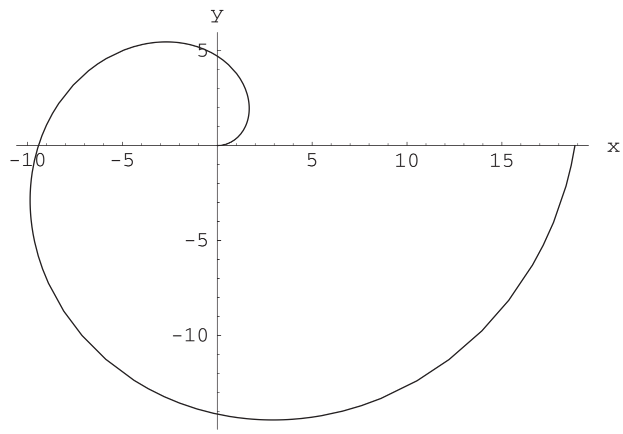


The two curves intersect when $1 - \cos\theta = 1$ which is when $\cos\theta = 0$ so the two points of intersection are $(r, \theta) = (1, \pi/2)$ and $(1, 3\pi/2)$. The region between the two graphs is where $\pi/2 \le \theta \le 3\pi/2$. The area is

$$\begin{aligned}\int_{\pi/2}^{3\pi/2}\int_1^{1-\cos\theta} r\, dr\, d\theta &= \int_{\pi/2}^{3\pi/2}\left(\frac{\cos^2\theta}{2} - \cos\theta\right) d\theta \\ &= \frac{1}{8}(2\theta - 8\sin\theta + \sin 2\theta)\Big|_{\pi/2}^{3\pi/2} = 2 + \frac{\pi}{4}.\end{aligned}$$

24. We want the area "inside" the spiral shown below. The area is

$$\int_0^{2\pi}\int_0^{3\theta} r\,dr\,d\theta = \int_0^{2\pi}\frac{9}{2}\theta^2\,d\theta = \frac{3}{2}\theta^3\Big|_0^{2\pi} = 12\pi^3.$$

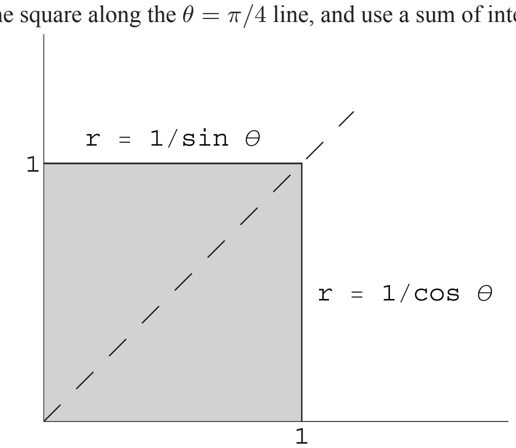


25. The integral is

$$\int_{\pi/3}^{\pi}\int_0^1 r\cos r^2\,dr\,d\theta = \frac{1}{2}\int_{\pi/3}^{\pi}\sin r^2\Big|_0^1\,d\theta = \frac{1}{2}\sin 1\Big|_{\pi/3}^{\pi} = \frac{\pi}{3}(\sin 1).$$

26. $\iint_D \sin(x^2+y^2)\,dA = \int_0^{\pi/2}\int_1^3 \sin(r^2)\cdot r\,dr\,d\theta$
$= \int_0^{\pi/2}\left(-\tfrac{1}{2}\cos(r^2)\right)\Big|_{r=1}^{r=3}\,d\theta = \int_0^{\pi/2}\tfrac{1}{2}(\cos 1-\cos 9)\,d\theta = \frac{\pi}{4}(\cos 1-\cos 9).$

27. Two of the edges of the unit square are given by $x = 1$ (or $r = 1/\cos\theta$ in polar coordinates) and by $y = 1$ (i.e., by $r = 1/\sin\theta$). We need to divide the square along the $\theta = \pi/4$ line, and use a sum of integrals:



Thus

$$\iint_D \frac{x}{\sqrt{x^2+y^2}}\,dA = \int_0^{\pi/4}\int_0^{1/\cos\theta} \frac{r\cos\theta}{r}\cdot r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{1/\sin\theta} r\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/4} \frac{1}{2\cos^2\theta}\cdot\cos\theta\,d\theta + \int_{\pi/4}^{\pi/2}\frac{1}{2\sin^2\theta}\cdot\cos\theta\,d\theta$$
$$= \int_0^{\pi/4}\frac{1}{2}\sec\theta\,d\theta + \int_{\theta=\pi/4}^{\theta=\pi/2}\frac{1}{2(\sin\theta)^2}\,d(\sin\theta)$$
$$= \frac{1}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4} - \frac{1}{2\sin\theta}\Big|_{\pi/4}^{\pi/2}$$
$$= \frac{1}{2}\ln(\sqrt{2}+1) - \frac{1}{2}\ln 1 - \frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{1}{2}(\ln(\sqrt{2}+1)+\sqrt{2}-1).$$

28. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3}\frac{e^z}{\sqrt{x^2+y^2}}\,dz\,dy\,dx = \int_0^{2\pi}\int_0^3\int_r^3\frac{e^z}{r}\cdot r\,dz\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^3(e^3-e^r)\,dr\,d\theta = \int_0^{2\pi}(3e^3-e^3+1)\,d\theta = 2\pi(2e^3+1).$

29. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{4-x^2-y^2} e^{x^2+y^2+z}\,dz\,dx\,dy = \int_0^{2\pi}\int_0^1\int_0^{4-r^2} e^{r^2+z}\,r\,dz\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^1\left(re^{r^2}\cdot e^z\right)\Big|_{z=0}^{z=4-r^2}dr\,d\theta = \int_0^{2\pi}\int_0^1 re^{r^2}\left(e^{4-r^2}-1\right)dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\int_0^1\left(e^4r-re^{r^2}\right)dr\,d\theta = \int_0^{2\pi}\left(\frac{e^4}{2}-\frac{e}{2}+\frac{1}{2}\right)d\theta = \pi(e^4-e+1).$

30. Since B is a ball we will use spherical coordinates:

$$\iiint_B \frac{dV}{\sqrt{x^2+y^2+z^2+3}} = \int_0^{2\pi}\int_0^{\pi}\int_0^2\frac{\rho^2\sin\varphi}{\sqrt{\rho^2+3}}\,d\rho\,d\varphi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\left(\left[\sqrt{7}-\frac{3}{2}\,\text{arcsinh}(2/\sqrt{3})\right]\sin\varphi\right)d\varphi\,d\theta$$
$$= \int_0^{2\pi}(2\sqrt{7}-3\,\text{arcsinh}(2/\sqrt{3}))\,d\theta$$
$$= 4\sqrt{7}\pi-6\pi\,\text{arcsinh}(2/\sqrt{3}) \quad \text{which is the same as the text's solution}$$
$$= (4\sqrt{7}-6\ln(2+\sqrt{7})+3\ln 3)\pi.$$

31. Here we will use cylindrical coordinates:

$$\iiint_W(x^2+y^2+2z^2)\,dV = \int_{-1}^{2}\int_0^{2\pi}\int_0^2 r(r^2+2z^2)\,dr\,d\theta\,dz$$
$$= \int_{-1}^{2}\int_0^{2\pi}(4z^2+4)\,d\theta\,dz$$
$$= \int_{-1}^{2}(8\pi z^2+8\pi)\,dz = 48\pi.$$

32. We use cylindrical coordinates:

$$\iiint_W \frac{z}{\sqrt{x^2+y^2}}\,dV = \int_0^{2\pi}\int_0^3\int_{2r^2-6}^{12} \frac{z}{r}\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \frac{1}{2}\left(144-(2r^2-6)^2\right)\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (54+12r^2-2r^4)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(54r+4r^3-\frac{2}{5}r^5\right)\Big|_0^3\,d\theta = 2\pi\left(\frac{864}{5}\right) = \frac{1728\pi}{5}.$$

33. Again we use cylindrical coordinates:

$$\text{Volume} = \iiint_W 1\,dV = \int_0^{2\pi}\int_0^b\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^b r\sqrt{a^2-r^2}\,dr\,d\theta = \int_0^{2\pi}\int_{a^2}^{a^2-b^2} -\frac{1}{2}\sqrt{u}\,du\,d\theta$$

where $u = a^2 - r^2$,

$$= \int_0^{2\pi}\int_{a^2-b^2}^{a^2} \frac{1}{2}\sqrt{u}\,du\,d\theta = \int_0^{2\pi}\frac{1}{3}\left(a^3-(a^2-b^2)^{3/2}\right)\,d\theta$$

$$= \frac{2\pi}{3}\left[a^3-(a^2-b^2)^{3/2}\right].$$

34. It is natural to use spherical coordinates.

$$\iiint_W \frac{dV}{\sqrt{x^2+y^2+z^2}} = \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho\sin\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}((b^2-a^2)\sin\varphi)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}(b^2-a^2)\,d\theta = 2\pi(b^2-a^2).$$

35. Once again we use spherical coordinates.

$$\iiint_W \sqrt{x^2+y^2+z^2}\,e^{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b (\rho^3 e^{\rho^2}\sin\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}([(1-a^2)e^{a^2}+(b^2-1)e^{b^2}]\sin\varphi)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}((1-a^2)e^{a^2}+(b^2-1)e^{b^2})\,d\theta$$

$$= 2\pi((1-a^2)e^{a^2}+(b^2-1)e^{b^2}).$$

36. We use spherical coordinates:

$$\begin{aligned}\iiint_W (x+y+z)\,dV &= \int_0^{\pi/2}\int_0^{\pi/2}\int_a^b (\rho\sin\varphi\cos\theta+\rho\sin\varphi\sin\theta+\rho\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= \int_0^{\pi/2}\int_0^{\pi/2}\frac{b^4-a^4}{4}\left(\sin^2\varphi(\cos\theta+\sin\theta)+\sin\varphi\cos\varphi\right)\,d\varphi\,d\theta\\ &= \frac{b^4-a^4}{4}\int_0^{\pi/2}\left[(\cos\theta+\sin\theta)\left(\frac{1}{2}\varphi-\frac{1}{4}\sin 2\varphi\right)+\frac{1}{2}\sin^2\varphi\right]\Big|_{\varphi=0}^{\varphi=\pi/2}\,d\theta\\ &= \frac{b^4-a^4}{4}\int_0^{\pi/2}\left(\frac{\pi}{4}(\cos\theta+\sin\theta)+\frac{1}{2}\right)\,d\theta\\ &= \frac{b^4-a^4}{4}\left[\frac{\pi}{4}(\sin\theta-\cos\theta)+\frac{1}{2}\theta\right]\Big|_0^{\pi/2}\\ &= \frac{b^4-a^4}{8}\left[\frac{\pi}{2}(1+1)+\frac{\pi}{2}\right]=\frac{3\pi(b^4-a^4)}{16}.\end{aligned}$$

37. We use spherical coordinates, in which case the cone $z=\sqrt{3x^2+3y^2}$ has equation

$$\rho\cos\varphi=\sqrt{3}\rho\sin\varphi \iff \tan\varphi=\frac{1}{\sqrt{3}} \iff \varphi=\frac{\pi}{6}.$$

The sphere $x^2+y^2+z^2=6z$ has spherical equation

$$\rho^2=6\rho\cos\varphi \iff \rho=6\cos\varphi.$$

Thus

$$\begin{aligned}\iiint_W z^2\,dV &= \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6\cos\varphi}(\rho^2\cos^2\varphi)\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta\\ &= \int_0^{2\pi}\int_0^{\pi/6}\frac{6^5}{5}\cos^7\varphi\sin\varphi\,d\varphi\,d\theta=\int_0^{2\pi}\left(-\frac{7776}{40}\cos^8\varphi\right)\Big|_{\varphi=0}^{\varphi=\pi/6}\,d\theta\\ &= \int_0^{2\pi}\frac{972}{5}\left(1-\frac{81}{256}\right)\,d\theta=\frac{972}{5}\left(\frac{175}{128}\right)\pi=\frac{8505\pi}{32}.\end{aligned}$$

38. We are integrating over a cone with vertex at the origin and base the disk at height 6 with radius 3. We will use cylindrical coordinates.

$$\begin{aligned}\iiint_W (2+\sqrt{x^2+y^2})\,dV &= \int_0^3\int_0^{2\pi}\int_{2r}^6 r(2+r)\,dz\,d\theta\,dr\\ &= \int_0^3\int_0^{2\pi}(-2r^3+2r^2+12r)\,d\theta\,dr\\ &= \int_0^3(2\pi(-2r^3+2r^2+12r))\,dr=63\pi.\end{aligned}$$

You should assign one of Exercises 39 or 40 so that your students see the benefits of using another coordinate system even when it is not explicitly called for. You might want to stress that the symmetries of the problem are what lead you, in this case, to choose cylindrical coordinates. Exercise 41 is fun because students will be tempted to use spherical coordinates—life is much easier if they use cylindrical coordinates.

39. We will use cylindrical coordinates.

$$\begin{aligned}\iiint_W dV &= \int_0^1 \int_0^{2\pi} \int_{-\sqrt{10-2r^2}}^{\sqrt{10-2r^2}} r\,dz\,d\theta\,dr \\ &= \int_0^1 \int_0^{2\pi} (2r\sqrt{10-2r^2})\,d\theta\,dr \\ &= \int_0^1 (4\pi r\sqrt{10-2r^2})\,dr \\ &= \frac{4\pi}{3}(5\sqrt{10}-8\sqrt{2}).\end{aligned}$$

40. We will again use cylindrical coordinates.

$$\begin{aligned}\iiint_W dV &= \int_0^2 \int_0^{2\pi} \int_0^{9-r^2} r\,dz\,d\theta\,dr \\ &= \int_0^2 \int_0^{2\pi} (9r-r^3)\,d\theta\,dr \\ &= \int_0^2 (18\pi r - 2\pi r^3)\,dr \\ &= 28\pi.\end{aligned}$$

41. We will again use cylindrical coordinates.

$$\begin{aligned}\iiint_W (2+x^2+y^2)\,dV &= \int_3^5 \int_0^{2\pi} \int_0^{\sqrt{25-z^2}} (2+r^2)r\,dr\,d\theta\,dz \\ &= \int_3^5 \int_0^{2\pi} \left(\frac{z^4}{4} - \frac{27z^2}{2} + \frac{725}{4}\right) d\theta\,dz \\ &= \int_3^5 \left(2\pi\left[\frac{z^4}{4} - \frac{27z^2}{2} + \frac{725}{4}\right]\right) dz = \frac{656\pi}{5}.\end{aligned}$$

42. You can draw a million pictures, but the easiest way to visualize this is by taking an apple corer and a potato and cutting in the three orthogonal directions. This will provide you with a model that the students can hold and pass around to aid their discussion. They can easily identify symmetries and cut the model along the coordinate planes to set up the integral.
If you do this and look in the first octant, you will see a seam along the line $y = x$. If we split the integral along this line we will have 1/16 of the desired volume. Using cylindrical coordinates this means that $0 \le \theta \le \pi/4$ and the cylinder with axis of symmetry the z-axis gives us that $0 \le r \le a$. The hard one to see is z, but because we are only looking at the wedge on one side of $\theta = \pi/4$ we need only worry about one other cylinder so $0 \le z \le \sqrt{a^2 - r^2\cos^2(\theta)}$.
So the volume is

$$V = 16\int_0^{\pi/4} \int_0^a \int_0^{\sqrt{a^2-r^2\cos^2(\theta)}} r\,dz\,dr\,d\theta = 8a^3(2-\sqrt{2}).$$

5.6 Applications of Integration

Exercises 1–9 concern average value.

1. (a) Let's assume a 30-day month.

$$\begin{aligned}[f]_{\text{avg}} &= \frac{1}{30}\int_0^{30} I(x)\,dx = \frac{1}{30}\int_0^{30} \left(75\cos\frac{\pi x}{15} + 80\right) dx \\ &= \frac{1}{30}\left(\frac{1125}{\pi}\sin\frac{\pi x}{15} + 80x\right)\bigg|_0^{30} = 2400/30 = 80 \text{ cases.}\end{aligned}$$

(b) Here the 2 cents will be a constant that pulls through the integral so the average holding cost is just 2 cents times the average daily inventory, or \$1.60.

2. We will divide the integral by the area:

$$[f]_{\text{avg}} = \frac{1}{(2\pi)(4\pi)}\int_0^{2\pi}\int_0^{4\pi}\sin^2 x\cos^2 y\,dy\,dx = \frac{1}{8\pi^2}\int_0^{2\pi}\left(\frac{\sin^2 x}{4}(\sin 2y + 2y)\right)\Big|_0^{4\pi}\,dx$$
$$= \frac{1}{8\pi^2}\int_0^{2\pi}(2\pi\sin^2 x)\,dx = \frac{1}{8\pi^2}\left(\frac{\pi}{2}(2x - \sin 2x)\right)\Big|_0^{2\pi} = \frac{2\pi^2}{8\pi^2} = \frac{1}{4}.$$

3. Again we will divide the integral by the area:

$$[f]_{\text{avg}} = \frac{1}{1/2}\int_0^1\int_0^{1-x} e^{2x+y}\,dy\,dx = 2\int_0^1 (e^{2x+y})\Big|_0^{1-x}\,dx$$
$$= 2\int_0^1 (e^{x+1} - e^{2x})\,dx = (2e^{1+x} - e^{2x})\Big|_0^1 = e^2 - 2e + 1.$$

4. Here we are finding the average over a ball of volume $4\pi/3$. We'll integrate using cylindrical coordinates because z appears explicitly in the integrand.

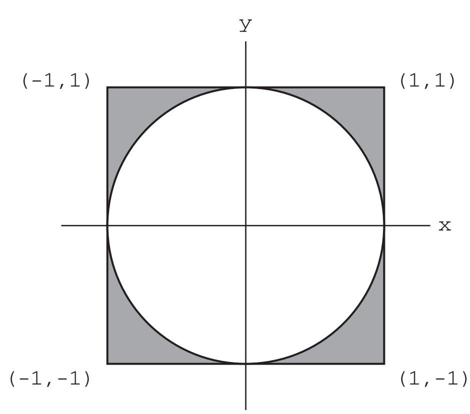
$$[g]_{\text{avg}} = \frac{1}{4\pi/3}\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{1-z^2}} re^z\,dr\,d\theta\,dz = \frac{3}{4\pi}\int_{-1}^1\int_0^{2\pi}\left(\frac{e^z}{2}(1-z^2)\right)\,d\theta\,dz$$
$$= \frac{3}{4\pi}\int_{-1}^1 (\pi e^z(1-z^2))\,dz = \frac{3}{4\pi}\frac{4\pi}{e} = \frac{3}{e}.$$

5. (a) We are told that in the $2\times 2\times 2$ cube centered at the origin, $T(x,y,z) = c(x^2+y^2+z^2)$. The average temperature of the cube is

$$[T]_{\text{avg}} = \frac{c}{8}\int_{-1}^1\int_{-1}^1\int_{-1}^1 (x^2+y^2+z^2)\,dx\,dy\,dz = \frac{c}{8}\int_{-1}^1\int_{-1}^1 (2z^2+2y^2+2/3)\,dy\,dz$$
$$= \frac{c}{8}\int_{-1}^1 (4z^2+8/3)\,dz = \frac{c}{8}(8) = c.$$

(b) $T(x,y,z) = c$ when $x^2+y^2+z^2 = 1$ so the temperature is equal to the average temperature on the surface of the unit sphere.

6. The region looks like



and the area of it is $2^2 - \pi = 4 - \pi$. Hence the average value is

$$\frac{1}{4-\pi}\iint_D (x^2+y^2)\,dA = \frac{1}{4-\pi}\left[\iint_{D_1}(x^2+y^2)\,dA - \iint_{D_2}(x^2+y^2)\,dA\right],$$

where D_1 denotes the square and D_2 the disk.

$$\iint_{D_1} (x^2+y^2)\,dA = \int_{-1}^{1}\int_{-1}^{1} (x^2+y^2)\,dx\,dy = \int_{-1}^{1} \left(\frac{1}{3}x^3 + y^2 x\right)\Big|_{x=-1}^{1} dy$$

$$= \int_{-1}^{1} \left(\frac{2}{3} + 2y^2\right) dy = \left(\frac{2}{3}y + \frac{2}{3}y^3\right)\Big|_{-1}^{1} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$$

$$\iint_{D_2} (x^2+y^2)\,dA = \int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}$$

Therefore the average value is

$$\frac{1}{4-\pi}\left(\frac{8}{3} - \frac{\pi}{2}\right) = \frac{16-3\pi}{24-6\pi} = \frac{3\pi-16}{6\pi-24} \approx 1.2766.$$

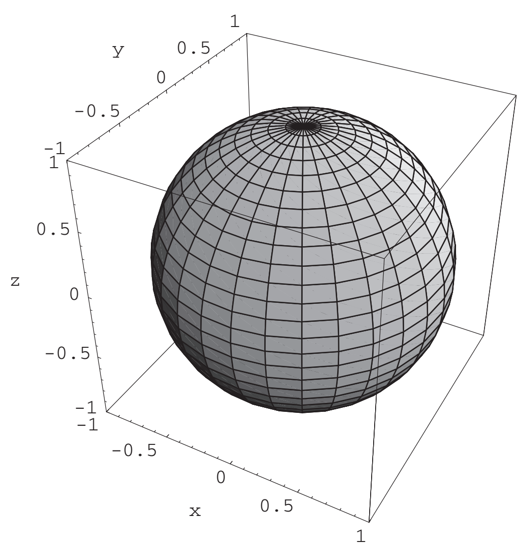
7. The volume of W is $8 - \frac{4}{3}x = \frac{24-4\pi}{3}$. Thus the average value is

$$\frac{3}{24-4\pi}\iiint_W (x^2+y^2+z^2)\,dV = \frac{3}{24-4\pi}\left(\iiint_{W_1} (x^2+y^2+z^2)\,dV - \iiint_{W_2} (x^2+y^2+z^2)\,dV\right)$$

where W_1 denotes the cube and W_2 the ball. Using Cartesian coordinates to integrate over W_1 and spherical coordinates to integrate over W_2, this may be calculated as

$$\frac{3}{24-4\pi}\left(\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x^2+y^2+z^2)\,dz\,dy\,dx - \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^4 \sin\varphi\,d\rho\,d\varphi\,d\theta\right)$$

$$= \frac{3\pi-30}{5\pi-30}.$$



8. We are looking for the average value of the minimum of x and y in the 6×6 box. This is 1/36 times the sum of the average value for x in the region where $x \le y$ and the average value for y in the region where $y \le x$. Because of the symmetry, the average value can be calculated by doubling the result for either region and dividing by 36:

$$[\text{Time}]_{\text{avg}} = \frac{2}{36}\int_0^6\int_0^x y\,dy\,dx = \frac{1}{18}\int_0^6 \frac{x^2}{2}\,dx$$

$$= \frac{1}{18}\left(\frac{x^3}{6}\right)\Big|_0^6 = 2.$$

9. This is an extension of Exercise 8. The domain is $[0,6] \times [0,6] \times [0,6]$. This time there is six-fold symmetry so we will calculate the average value for z in the region where $z \leq y \leq x$ and multiply by 6 and then divide by 6^3 which is the volume of the domain.

$$[\text{Time}]_{\text{avg}} = \frac{6}{216}\int_0^6\int_0^x\int_0^y z\,dz\,dy\,dx = \frac{1}{36}\int_0^6\int_0^x \frac{y^2}{2}\,dy\,dz$$
$$= \frac{1}{36}\int_0^6 \frac{x^3}{6}\,dx = \frac{1}{36}\left(\frac{x^4}{24}\right)\Big|_0^6 = 3/2.$$

So with three train lines the average wait is 90 seconds.

Exercises 10–24 concern centers of mass. We use the formula:

$$\textit{Center of mass} = \frac{\displaystyle\int_a^b x\delta(x)\,dx}{\displaystyle\int_a^b \delta(x)\,dx}$$

and its variants.

10. (a) The curve $y = 8 - 2x^2$ intersects the x-axis at ± 2. So

$$\int_{-2}^2\int_0^{8-2x^2} c\,dy\,dx = c\int_{-2}^2 (8-2x^2)\,dx = c\,(8x - 2x^3/3)\Big|_{-2}^2 = 64c/3$$
$$M_y = \int_{-2}^2\int_0^{8-2x^2} cx\,dy\,dx = c\int_{-2}^2 (8x - 2x^3)\,dx = c\,(4x^2 - x^4/2)\Big|_{-2}^2 = 0 \quad \text{and}$$
$$M_x = \int_{-2}^2\int_0^{8-2x^2} cy\,dy\,dx = (c/2)\int_{-2}^2 (8-2x^2)^2 dx = c\left(\frac{2}{5}x^5 - \frac{16}{3}x^3 + 32x\right)\Big|_{-2}^2 = 1024c/15$$

So $\bar{x} = 0$ and $\bar{y} = \dfrac{1024c/15}{64c/3} = 16/5$.

(b) Again, we see the symmetry with respect to x so $\bar{x} = 0$. The following integrals are straightforward so we leave out the details, but

$$\bar{y} = \frac{\displaystyle\int_{-2}^2\int_0^{8-2x^2} 3cy^2\,dy\,dx}{\displaystyle\int_{-2}^2\int_0^{8-2x^2} 3cy\,dy\,dx} = \frac{32768c/35}{1024c/5} = 32/7.$$

11. We assume that the plate has uniform density and place it so that the center of the straight border is at the origin and the semicircle is symmetric with respect to the y-axis. Once again this means that $\bar{x} = 0$.

$$\bar{y} = \frac{\displaystyle\int_{-a}^a\int_0^{\sqrt{a^2-x^2}} cy\,dy\,dx}{\pi a^2 c/2} = \frac{2a^3c/3}{\pi a^2 c/2} = \frac{4a}{3\pi}.$$

12. First calculate

$$M = \int_0^2\int_{x^2}^{2x} (1 + x + y)\,dy\,dx = \frac{24}{5}$$
$$M_y = \int_0^2\int_{x^2}^{2x} [x(1 + x + y)]\,dy\,dx = \frac{28}{5}$$
$$M_x = \int_0^2\int_{x^2}^{2x} [y(1 + x + y)]\,dy\,dx = \frac{328}{35}$$

Thus,

$$\bar{x} = \frac{28/5}{24/5} = 7/6 \quad \text{and} \quad \bar{y} = \frac{328/35}{24/5} = 41/21.$$

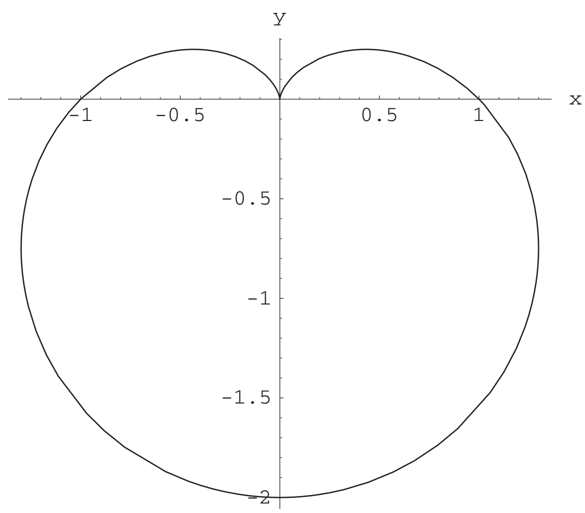
13. Again we first calculate

$$M = \int_0^9 \int_0^{\sqrt{x}} (xy)\,dy\,dx = \frac{243}{2}$$
$$M_y = \int_0^9 \int_0^{\sqrt{x}} (x^2y)\,dy\,dx = \frac{6561}{8}$$
$$M_x = \int_0^9 \int_0^{\sqrt{x}} (xy^2)\,dy\,dx = \frac{1458}{7}$$

so

$$\bar{x} = \frac{6561/8}{243/2} = 27/4 \quad \text{and} \quad \bar{y} = \frac{1458/7}{243/2} = 12/7.$$

14. We'll take δ to be 1. A look at the figure below tells us again that $\bar{x} = 0$. We'll use polar integrals to calculate $\bar{y}$.



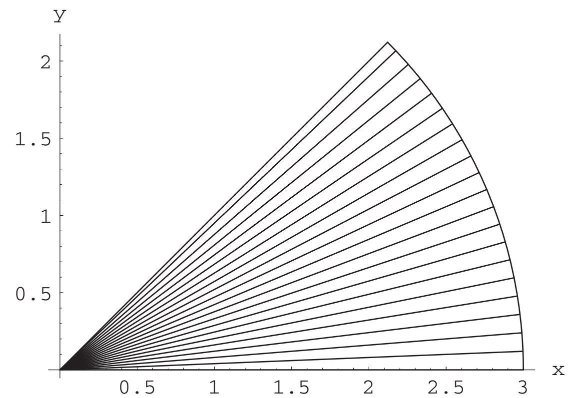
We first calculate $M = \int_0^{2\pi} \int_0^{1-\sin\theta} r\,dr\,d\theta = \frac{3\pi}{2}$ and $M_x = \iint_D y\,dA = \int_0^{2\pi} \int_0^{1-\sin\theta} r^2 \sin\theta\,dr\,d\theta = \frac{-5\pi}{4}$, so $\bar{y} = \frac{-5\pi/4}{3\pi/2} = -5/6$.

15. We first calculate

$$M = \int_0^{\pi/3} \int_0^{4\cos\theta} r\,dr\,d\theta = \sqrt{3} + \frac{4\pi}{3}$$
$$M_y = \iint_D x\,dA = \int_0^{\pi/3} \int_0^{4\cos\theta} r^2 \cos\theta\,dr\,d\theta = \frac{7}{\sqrt{3}} + \frac{8\pi}{3} \quad \text{and}$$
$$M_x = \iint_D y\,dA = \int_0^{\pi/3} \int_0^{4\cos\theta} r^2 \sin\theta\,dr\,d\theta = 5,$$

so $\bar{x} = \frac{7\sqrt{3} + 8\pi}{3\sqrt{3} + 4\pi}$ and $\bar{y} = \frac{15}{3\sqrt{3} + 4\pi}$.

16. The region is a slice of pie:



$$\text{Total mass } M = \iint_D \delta\, dA = \int_0^{\pi/4} \int_0^3 (4-r)r\, dr\, d\theta = \int_0^{\pi/4} \left(2r^2 - \frac{1}{3}r^3\right)\Big|_{r=0}^{3}$$

$$= \int_0^{\pi/4} (18-9)\, d\theta = \frac{9\pi}{4}$$

$$M_y = \iint_D x\delta\, dA = \int_0^{\pi/4} \int_0^3 (4r^2 - r^3)\cos\theta\, dr\, d\theta = \int_0^{\pi/4} \left(36 - \frac{81}{4}\right)\cos\theta\, d\theta$$

$$= \frac{63}{4\sqrt{2}}$$

$$M_x = \iint_D y\delta\, dA = \int_0^{\pi/4} \int_0^3 (4r^2 - r^3)\sin\theta\, dr d\theta = \int_0^{\pi/4} \left(36 - \frac{81}{4}\right)\sin\theta\, d\theta$$

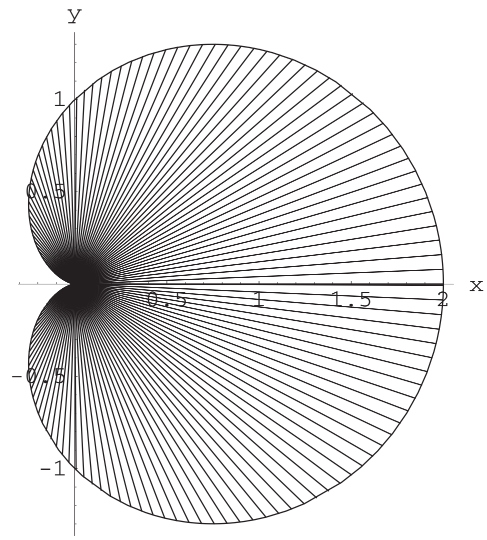
$$= \frac{63}{8}(2-\sqrt{2})$$

Thus

$$\bar{x} = \frac{M_y}{M} = \frac{63}{4\sqrt{2}} \cdot \frac{4}{9\pi} = \frac{7\sqrt{2}}{2\pi}$$

$$\bar{y} = \frac{M_x}{M} = \frac{63(2-\sqrt{2})}{8}\,\frac{4}{9\pi} = \frac{7(2-\sqrt{2})}{2\pi}$$

17. The region in question looks as follows:



$$\text{Total mass } M = \iint_D \delta\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^2 dr\, d\theta$$
$$= \int_0^{2\pi} \frac{1}{3}(1+\cos\theta)^3 d\theta$$
$$= \int_0^{2\pi} \frac{1}{3}(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\, d\theta = \frac{5\pi}{3}$$
$$M_y = \iint_D x\delta\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \cos\theta\, dr\, d\theta = \int_0^{2\pi} \frac{1}{4}(1+\cos\theta)^4 \cos\theta\, d\theta$$
$$= \frac{7\pi}{4} \qquad \text{(after some effort!)}$$
$$M_x = \iint_D y\delta\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r^3 \sin\theta\, dr\, d\theta$$
$$= \int_0^{2\pi} \frac{1}{4}(1+\cos\theta)^4 \sin\theta\, d\theta = -\frac{1}{4}\int_2^2 u^4 du = 0$$

(It's also possible to see this from symmetry.) Thus

$$\bar{x} = \frac{7\pi}{4} \cdot \frac{3}{5\pi} = \frac{21}{20}, \qquad \bar{y} = 0.$$

18. Because the volume of the tetrahedron is 1, we can find the centroid by calculating:

$$\bar{x} = \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} x\, dz\, dy\, dx = \frac{1}{4}$$
$$\bar{y} = \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} y\, dz\, dy\, dx = \frac{1}{2} \quad \text{and,}$$
$$\bar{z} = \int_0^1 \int_0^{2-2x} \int_0^{3-3y/2-3x} z\, dz\, dy\, dx = \frac{3}{4}.$$

19. (a) First calculate:

$$M = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} dz\,dy\,dx = 12$$
$$M_{yz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} x\,dz\,dy\,dx = 6$$
$$M_{xz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} y\,dz\,dy\,dx = 0$$
$$M_{xy} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} z\,dz\,dy\,dx = \frac{108}{5}.$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = (1/2, 0, 9/5)$.

(b) Next we calculate the center of mass with the given density function.

$$M = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} (z + x^2)\,dz\,dy\,dx = \frac{168}{5}$$
$$M_{yz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} x(z + x^2)\,dz\,dy\,dx = \frac{129}{5}$$
$$M_{xz} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} y(z + x^2)\,dz\,dy\,dx = 0$$
$$M_{xy} = \int_{-1}^{2}\int_{-1}^{1}\int_{3y^2}^{3} z(z + x^2)\,dz\,dy\,dx = \frac{2376}{35}.$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = (43/56, 0, 99/49)$.

Note that in Exercises 20–22, the symmetry with respect to the z-axis implies that $\bar{x} = 0$ and $\bar{y} = 0$. We only explicitly set up all of the integrals in the solution of Exercise 20.

20. First calculate:

$$M = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} r\,dz\,d\theta\,dr = 36\sqrt{2}\pi - \frac{63\pi}{2}$$
$$M_{yz} = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} r\ \cos\theta\,dz\,d\theta\,dr = 0$$
$$M_{xz} = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} r\ \sin\theta\,dz\,d\theta\,dr = 0$$
$$M_{xy} = \int_{0}^{3}\int_{0}^{2\pi}\int_{r^2/3}^{\sqrt{18-r^2}} rz\,dz\,d\theta\,dr = \frac{189\pi}{4}.$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \dfrac{21}{2(8\sqrt{2} - 7)}\right)$.

21. As noted above, $\bar{x} = 0$ and $\bar{y} = 0$. First calculate:

$$M = \int_{0}^{5/2}\int_{0}^{2\pi}\int_{3r^2-16}^{9-r^2} r dz\,d\theta\,dr = \frac{625\pi}{8}$$
$$M_{xy} = \int_{0}^{5/2}\int_{0}^{2\pi}\int_{3r^2-16}^{9-r^2} rz\,dz\,d\theta\,dr = -\frac{10625\pi}{96}$$

This means that $(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, -\dfrac{17}{12}\right)$.

22. Note first that, by symmetry, the centroid must lie along the z-axis, so $\bar{x} = \bar{y} = 0$. Since the density is to be assumed constant, we may take it to be equal to 1. Then we have

$$\bar{z} = \frac{\iiint_W z\,dV}{\iiint_W dV}.$$

We use cylindrical coordinates to calculate the integrals. Thus the cone has cylindrical equation $z = 2r$ and the sphere $r^2 + z^2 = 25$. These surfaces intersect where $r^2 + 4r^2 = 25$, or, equivalently, where $r = \sqrt{5}$. Hence

$$\begin{aligned}\iiint_W dV &= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_{2r}^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}}\left(\sqrt{25-r^2}-2r\right) r\,dr\,d\theta \\ &= \int_0^{2\pi}\left(-\frac{1}{2}\cdot\frac{2}{3}(25-r^2)^{3/2} - \frac{2}{3}r^3\right)\Bigg|_{r=0}^{r=\sqrt{5}} d\theta \\ &= \left(-\frac{50\sqrt{5}}{3} + \frac{125}{3}\right)(2\pi) = \frac{(250-100\sqrt{5})\pi}{3}.\end{aligned}$$

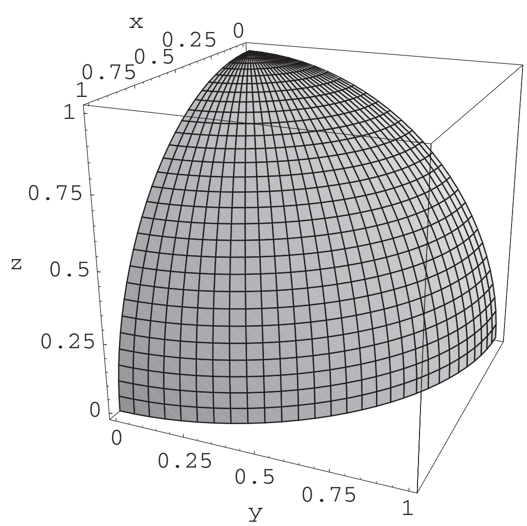
Also,

$$\begin{aligned}\iiint_W z\,dV &= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_{2r}^{\sqrt{25-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}}\frac{1}{2}\left(25-r^2-4r^2\right) r\,dr\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{\sqrt{5}}\left(25r-5r^3\right)dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(\frac{125}{2}-\frac{125}{4}\right)d\theta = \frac{125\pi}{4}.\end{aligned}$$

Therefore,

$$\bar{z} = \frac{125\pi/4}{(250-100\sqrt{5})\pi/3} = \frac{15}{8(5-2\sqrt{5})} \approx 3.55.$$

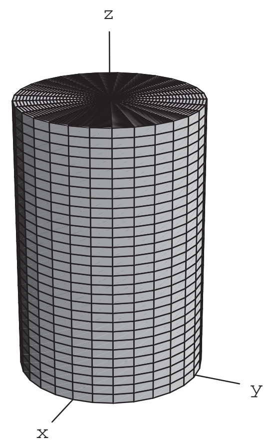
23. By symmetry $\bar{x} = \bar{y} = \bar{z}$. $\bar{z}$ is easiest to find. Volume of W is $\frac{1}{8}\left(\frac{4}{3}\pi a^3\right) = \frac{\pi a^3}{6}$.



Thus

$$\begin{aligned}\bar{z} &= \frac{6}{\pi a^3}\iiint_W z\,dV = \frac{6}{\pi a^3}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho\cos\varphi\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= \frac{6}{\pi a^3}\int_0^{\pi/2}\int_0^{\pi/2}\frac{a^4}{4}\cos\varphi\sin\varphi\,d\varphi\,d\theta = \frac{3a}{2\pi}\int_0^{\pi/2}\frac{1}{2}d\theta \\ &= \frac{3a}{8}.\end{aligned}$$

24. If we put the bottom of the cylinder in the *xy*-plane, then $\delta(x, y, z) = (h - z)^2$.



Therefore the total mass is

$$\begin{aligned} M &= \iiint_W \delta\, dV = \int_0^{2\pi} \int_0^a \int_0^h (h-z)^2 r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^a -\frac{1}{3}(h-z)^3 \Big|_{z=0}^{h} r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^a \frac{h^3}{3} r\, dr\, d\theta = \frac{\pi h^3 a^2}{3}. \end{aligned}$$

$\bar{x} = \bar{y} = 0$ by symmetry, so we compute

$$\begin{aligned} M_{xy} &= \iiint_W z\delta\, dV = \int_0^{2\pi} \int_0^a \int_0^h z(h-z)^2 r\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^a \int_0^h (h^2 z - 2hz^2 + z^3) r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^a r \cdot \frac{1}{12} h^4\, dr\, d\theta = \frac{\pi h^4 a^2}{12}. \end{aligned}$$

Thus

$$\bar{z} = \frac{\pi h^4 a^2}{12} \cdot \frac{3}{\pi h^3 a^2} = \frac{h}{4}.$$

25. (a) By symmetry we can see that the moment of inertia about each of the coordinate axes is the same.

$$I_x = I_y = I_z = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (y^2 + z^2)\, dz\, dy\, dx = \frac{1}{30}.$$

(b) Again, by symmetry we see that the radius of gyration about each of the coordinate axes is the same. We calculate

$$M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\, dy\, dx = \frac{1}{6}.$$

Then $r_x = r_y = r_z = \sqrt{\dfrac{1/30}{1/6}} = 1/\sqrt{5}$.

26. By symmetry we can see that the moment of inertia about each of the coordinate axes is the same

$$I_x = I_y = I_z = \int_0^2 \int_0^2 \int_0^2 (y^2 + z^2)(x + y + z + 1)\, dz\, dy\, dx = 96.$$

Again, by symmetry we see that the radius of gyration about each of the coordinate axes is the same. We calculate

$$M = \int_0^2 \int_0^2 \int_0^2 (x + y + z + 1)\, dz\, dy\, dx = 32.$$

Then $r_x = r_y = r_z = \sqrt{\dfrac{96}{32}} = \sqrt{3}$.

27. (a) The problem cries out to be solved using cylindrical coordinates. For I_z this means that the $x^2 + y^2$ in the integrand is r^2 so

$$\begin{aligned} I_z &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 2zr^3\, dz\, d\theta\, dr = \frac{6561\pi}{4} \quad \text{and} \\ M &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 2zr\, dz\, d\theta\, dr = 486\pi, \quad \text{so} \\ r_z &= \frac{3\sqrt{3}}{2\sqrt{2}}. \end{aligned}$$

(b) This time

$$\begin{aligned} I_z &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 r^4\, dz\, d\theta\, dr = \frac{8748\pi}{35} \quad \text{and} \\ M &= \int_0^3 \int_0^{2\pi} \int_{r^2}^9 r^2\, dz\, d\theta\, dr = \frac{324\pi}{5}, \quad \text{so} \\ r_z &= \frac{3\sqrt{3}}{\sqrt{7}}. \end{aligned}$$

28. Although it may be tempting to move to spherical coordinates, it is nice to have a z-coordinate so we will stay with cylindrical coordinates.

(a)

$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3 c\, dz\, d\theta\, dr = \frac{8\pi c a^5}{15} \quad \text{and} \\ M &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} rc\, dz\, d\theta\, dr = \frac{4\pi c a^3}{3} \quad \text{so} \\ r_z &= a\sqrt{\frac{2}{5}}. \end{aligned}$$

(b)

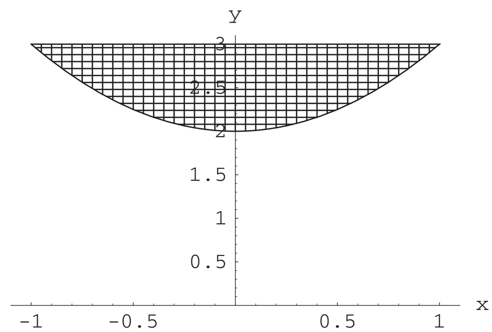
$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3(r^2+z^2)\, dz\, d\theta\, dr = \frac{8\pi a^7}{21} \quad \text{and} \\ M &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r(r^2+z^2)\, dz\, d\theta\, dr = \frac{4\pi a^5}{5} \quad \text{so} \\ r_z &= a\sqrt{\frac{10}{21}}. \end{aligned}$$

(c)

$$\begin{aligned} I_z &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^5\, dz\, d\theta\, dr = \frac{32\pi a^7}{105} \quad \text{and} \\ M &= \int_0^a \int_0^{2\pi} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r^3\, dz\, d\theta\, dr = \frac{8\pi\delta a^5}{15} \quad \text{so} \\ r_z &= \frac{2a}{\sqrt{7}}. \end{aligned}$$

29.

$$
\begin{aligned}
I_x &= \iint_D y^2\delta\, dA = \int_{-1}^{1}\int_{x^2+2}^{3} y^2(x^2+1)\, dy\, dx \\
&= \int_{-1}^{1}\left(9-\frac{1}{3}(x^2+2)^3\right)(x^2+1)\, dx = \int_{-1}^{1}\frac{1}{3}(19+7x^2-18x^4-7x^6-x^8)\, dx \\
&= \frac{1}{3}\left(38+\frac{14}{3}-\frac{36}{5}-2-\frac{2}{9}\right) = \frac{1496}{135}
\end{aligned}
$$



30.

$$
\begin{aligned}
I_z &= \iint_{[0,2]\times[0,1]}(x^2+y^2)\delta\, dA = \int_0^2\int_0^1 (x^2+y^2)(1+y)\, dy\, dx \\
&= \int_0^2\int_0^1 (x^2+y^2+x^2y+y^3)\, dy\, dx = \int_0^2\left(x^2+\frac{1}{3}+\frac{1}{2}x^2+\frac{1}{4}\right) dx \\
&= \frac{8}{3}+\frac{2}{3}+\frac{4}{3}+\frac{1}{2} = \frac{31}{6}
\end{aligned}
$$

31. The only adjustment in the formula for I_x is because we are using the square of the distance from the line $y = 3$ and not the formula given in text which squares the distance from the x-axis. This is a straightforward application of formula (8).

$$
I_{y=3} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(x^2(3-y)^2))\, dy\, dx = \frac{116\pi}{3}.
$$

What follows is preliminary work for Exercises 32–34. You should probably assign all three together.

We will be calculating

$$
V(0,0,r) = -\iiint_W \frac{Gm\delta(x,y,z)\, dV}{\sqrt{x^2+y^2+(z-r)^2}}.
$$

In this special case, W is the shell bound by spheres centered at the origin of radii a and b where $a < b$. The volume of W is therefore $4\pi(b^3-a^3)/3$. The density is assumed to be constant and so the density is mass divided by volume, so

$$
\delta = \frac{M}{[4\pi(b^3-a^3)/3]} = \frac{3M}{4\pi(b^3-a^3)}.
$$

So

$$\begin{aligned}
V(0,0,r) &= -Gm\delta \iiint_W \frac{dV}{\sqrt{x^2+y^2+(z-r)^2}} \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \iiint_W \frac{dV}{\sqrt{x^2+y^2+(z-r)^2}} \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b\int_0^{\pi} \frac{\rho^2\sin\varphi}{\sqrt{\rho^2+r^2-2r\rho\cos\varphi}}\, d\varphi\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b \left[\left(\frac{\rho}{r}\right)\sqrt{\rho^2+r^2-2r\rho\cos\varphi}\right]\Bigg|_{\varphi=0}^{\pi} d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b \left[\left(\frac{\rho}{r}\right)(|\rho+r|-|\rho-r|)\right] d\rho\, d\theta.
\end{aligned}$$

Our final note before proceeding to Exercises 32–34 is that

$$\left(\frac{\rho}{r}\right)(|\rho+r|-|\rho-r|) = \begin{cases} 2\rho & \text{if } \rho \geq r, \text{ and} \\ 2\rho^2/r & \text{if } \rho < r. \end{cases}$$

32. *See preliminary work above.* When $r \geq b$, then in the range $a \leq \rho \leq b$, we have that $\rho \leq r$, so

$$\begin{aligned}
V(0,0,r) &= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b \frac{2\rho^2}{r}\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} \frac{2\rho^3}{3r}\Bigg|_a^b d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} \frac{2(b^3-a^3)}{3r}\, d\theta \\
&= -\frac{GmM}{2\pi r}\int_0^{2\pi} d\theta = -\frac{GmM}{r}.
\end{aligned}$$

33. *See preliminary work above.* When $r \leq a$, then in the range $a \leq \rho \leq b$, we have that $\rho \geq r$, so

$$\begin{aligned}
V(0,0,r) &= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^b 2\rho\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} (b^2-a^2)\, d\theta \\
&= \frac{-3GmM(b^2-a^2)}{2(b^3-a^3)}.
\end{aligned}$$

What is striking about this result is that $V(0,0,r)$ is independent of r. Therefore, since $\mathbf{F} = -\nabla V$, we see that there is no gravitational force.

34. **(b)** Students might consider the connection before they explicitly find it in part (a). If $a < r < b$, then we have a combination of the two cases dealt with in Exercises 32 and 33. For $a \leq \rho \leq r$, we are in a case similar to Exercise 32, and for $r \leq \rho \leq b$ we are in a case similar to Exercise 33.

(a) We must break the integral at $\rho = r$:

$$\begin{aligned}
V(0,0,r) &= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_a^r \frac{2\rho^2}{r}\, d\rho\, d\theta - \frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi}\int_r^b 2\rho\, d\rho\, d\theta \\
&= -\frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} \frac{2(r^3-a^3)}{3r}\, d\theta - \frac{3GmM}{4\pi(b^3-a^3)} \int_0^{2\pi} (b^2-r^2)\, d\theta \\
&= -\frac{3GmM}{2(b^3-a^3)}\left(\frac{2(r^3-a^3)}{3r}\right) - \frac{3GmM}{2(b^3-a^3)}(b^2-r^2) \\
&= -\frac{GmM}{2r(b^3-a^3)}(3b^2r - 2a^3 - r^3).
\end{aligned}$$

5.7 Numerical Approximations of Multiple Integrals

1. (a) We let $\Delta x = \dfrac{3.1-3}{2} = 0.05$, $\Delta y = \dfrac{2.1-1.5}{3} = 0.2$. Thus, using formula (6) with $f(x,y) = x^2 - 6y^2$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.05)(0.2)}{4}\,[&f(3,1.5) + f(3,2.1) + f(3.1,1.5) + f(3.1,2.1) \\ &+ 2\,(f(3,1.7) + f(3,1.9) + f(3.05,1.5) + f(3.05,2.1) + f(3.1,1.7) + f(3.1,1.9)) \\ &+ 4\,(f(3.05,1.7) + f(3.05,1.9))] \\ = -0.621375& \end{aligned}$$

(b) $\displaystyle\int_3^{3.1}\int_{1.5}^{2.1}(x^2-6y^2)\,dy\,dx = \int_3^{3.1}(x^2y-2y^3)\Big|_{y=1.5}^{y=2.1}dx = \int_3^{3.1}\left[0.6x^2 - 2(2.1^3-1.5^3)\right]\,dx$

$= \left[0.2x^3 - 2(2.1^3-1.5^3)x\right]\Big|_3^{3.1} = 0.2(3.1^3-3^3) - 0.2(2.1^3-1.5^3) = -0.619$

2. (a) We let $\Delta x = \dfrac{3.3-3}{2} = 0.15$, $\Delta y = \dfrac{3.3-3}{3} = 0.1$. Thus, using formula (6) with $f(x,y) = xy^2$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.15)(0.1)}{4}\,[&f(3,3) + f(3,3.3) + f(3.3,3) + f(3.3,3.3) \\ &+ 2\,(f(3,3.1) + f(3,3.2) + f(3.15,3) + f(3.15,3.3) + f(3.3,3.1) + f(3.3,3.2)) \\ &+ 4\,(f(3.15,3.1) + f(3.15,3.2))] \\ = 2.81563& \end{aligned}$$

(b) $\displaystyle\int_3^{3.3}\int_3^{3.3}xy^2\,dy\,dx = \int_3^{3.3}\frac{x}{3}y^3\Big|_{y=3}^{y=3.3}dx = \int_3^{3.3}\frac{x}{3}\left(3.3^3-3^3\right)\,dx = 2.979\int_3^{3.3}x\,dx$

$= \left[\dfrac{2.979}{2}x^2\right]\Big|_3^{3.3} = \dfrac{2.979}{2}(3.3^2-3^2) = 2.81516$

3. (a) We let $\Delta x = \dfrac{2.2-2}{2} = 0.1$, $\Delta y = \dfrac{1.6-1}{3} = 0.2$. Thus, using formula (6) with $f(x,y) = x/y$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.1)(0.2)}{4}\,[&f(2,1) + f(2,1.6) + f(2.2,1) + f(2.2,1.6) \\ &+ 2\,(f(2,1.2) + f(2,1.4) + f(2.1,1) + f(2.1,1.6) + f(2.2,1.2) + f(2.2,1.4)) \\ &+ 4\,(f(2.1,1.2) + f(2.1,1.4))] \\ = 0.19825& \end{aligned}$$

(b) $\displaystyle\int_2^{2.2}\int_1^{1.6}\frac{x}{y}\,dy\,dx = \int_2^{2.2}x\ln y\Big|_{y=1}^{y=1.6}dx = \int_2^{2.2}x\cdot\ln(1.6)\,dx$

$= \left[\dfrac{\ln(1.6)}{2}x^2\right]\Big|_2^{2.2} = \dfrac{\ln 1.6}{2}(2.2^2-2^2) = 0.42\ln(1.6) = 0.197402$

4. (a) We let $\Delta x = \dfrac{1.4-1}{2} = 0.2$, $\Delta y = \dfrac{4.3-4}{3} = 0.1$. Thus, using formula (6) with $f(x,y) = \sqrt{x}+\sqrt{y}$, we have

$$\begin{aligned} T_{2,3} = \frac{(0.2)(0.1)}{4}\,[&f(1,4) + f(1,4.3) + f(1.4,4) + f(1.4,4.3) \\ &+ 2\,(f(1,4.1) + f(1,4.2) + f(1.2,4) + f(1.2,4.3) + f(1.4,4.1) + f(1.4,4.2)) \\ &+ 4\,(f(1.2,4.1) + f(1.2,4.2))] \\ = 0.375666& \end{aligned}$$

(b) $\displaystyle\int_1^{1.4}\int_4^{4.3}\left(\sqrt{x}+\sqrt{y}\right)\,dy\,dx = \int_1^{1.4}\left(\sqrt{x}y+\frac{2}{3}y^{3/2}\right)\Bigg|_{y=4}^{y=4.3}dx$

$\displaystyle = \int_1^{1.4}(0.3\sqrt{x}+0.61111318)\,dx = \frac{0.6}{3}\left((1.4)^{3/2}-1\right)+(0.61111318)(0.4)=0.375746$

5. (a) We let $\Delta x = \dfrac{1.1-1}{2}=0.05$, $\Delta y=\dfrac{0.6-0}{3}=0.2$. Thus, using formula (6) with $f(x,y)=e^{x+2y}$, we have

$$\begin{aligned}T_{2,3} &= \frac{(0.05)(0.2)}{4}\left[f(1,0)+f(1,0.6)+f(1.1,0)+f(1.1,0.6)\right.\\&\quad+2\left(f(1,0.2)+f(1,0.4)+f(1.05,0)+f(1.05,0.6)+f(1.1,0.2)+f(1.1,0.4)\right)\\&\quad\left.+4\left(f(1.05,0.2)+f(1.05,0.4)\right)\right]\\&=0.336123\end{aligned}$$

(b) $\displaystyle\int_1^{1.1}\int_0^{0.6}e^{x+2y}\,dy\,dx=\int_1^{1.1}\int_0^{0.6}e^xe^{2y}\,dy\,dx=\int_1^{1.1}e^x\left(\frac{1}{2}e^{2y}\right)\Bigg|_{y=0}^{y=0.6}dx$

$\displaystyle=\int_1^{1.1}\left(\frac{e^{1.2}-1}{2}\right)e^x\,dx=\frac{e^{1.2}-1}{2}(e^{1.1}-1)=0.331642$

6. (a) We let $\Delta x=\dfrac{0.2-1}{2}=0.1$, $\Delta y=\dfrac{\pi/3-\pi/6}{3}=\dfrac{\pi}{18}$. Thus, using formula (6) with $f(x,y)=x\cos y$, we have

$$\begin{aligned}T_{2,3} &= \frac{(0.1)(\pi/18)}{4}\left[f(0,\pi/6)+f(0,\pi/3)+f(0.2,\pi/6)+f(0.2,\pi/3)\right.\\&\quad+2\left(f(0,2\pi/9)+f(0,5\pi/18)+f(0.1,\pi/6)+f(0.1,\pi/3)\right.\\&\qquad\left.+f(0.2,2\pi/9)+f(0.2,5\pi/18)\right)\\&\quad\left.+4\left(f(0.1,2\pi/9)+f(0.1,5\pi/18)\right)\right]\\&=0.00730192\end{aligned}$$

(b) $\displaystyle\int_0^{0.2}\int_{\pi/6}^{\pi/3}x\cos y\,dy\,dx=\int_0^{0.2}x\sin y\Big|_{y=\pi/6}^{y=\pi/3}dx=\int_0^{1/5}\left(\frac{\sqrt{3}-1}{2}\right)x\,dx$

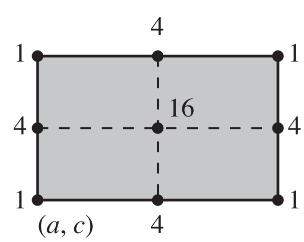
$\displaystyle=\frac{\sqrt{3}-1}{100}=0.00732051$

Note that in all of the solutions to Exercises 7–12 below, the rectangle $R=[a,b]\times[c,d]$ is partitioned as in the figure below and the Simpson's rule approximations $S_{2,2}$ may be written as

$$S_{2,2}=\frac{\Delta x\,\Delta y}{9}\sum_{j=0}^{2}\sum_{i=0}^{2}w_{ij}f(x_i,y_j),$$

where

$$w_{ij}=\begin{cases}1 & \textit{if } (x_i,y_j) \textit{ is one of the four vertices of } R;\\ 4 & \textit{if } (x_i,y_j) \textit{ is a point on an edge of } R\textit{, but not a vertex;}\\ 16 & \textit{if } (x_i,y_j) \textit{ is a point in the interior of } R.\end{cases}$$



7. (a) We let $\Delta x = \dfrac{0.1-(-0.1)}{2} = 0.1$, $\Delta y = \dfrac{0.3-0}{2} = 0.15$. Hence, with $f(x,y) = y^4 - xy^2$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.1)(0.15)}{9}\left[f(-0.1,0)+f(-0.1,0.3)+f(0.1,0)+f(0.1,0.3)\right.\\ &\quad \left.+4\left(f(-0.1,0.15)+f(0,0)+f(0,0.3)+f(0.1,0.15)\right)+16f(0,0.15)\right]\\ &= 0.00010125 \end{aligned}$$

(b) $\displaystyle\int_{-0.1}^{0.1}\int_0^{0.3}\left(y^4-xy^2\right)\,dy\,dx = \int_{-0.1}^{0.1}\left(\tfrac{1}{5}y^5-\tfrac{1}{3}xy^3\right)\Big|_{y=0}^{y=0.3}dx$

$\displaystyle = \int_{-0.1}^{0.1}(0.000486-0.009x)\,dx = \left(0.000486x-0.0045x^2\right)\Big|_{-0.1}^{0.1} = 0.0000972$

8. (a) We let $\Delta x = \dfrac{0.1-0}{2} = 0.05$, $\Delta y = \dfrac{2-1}{2} = 0.5$. Hence, with $f(x,y) = 1/(1+x^2)$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.05)(0.5)}{9}\left[f(0,1)+f(0,2)+f(0.1,1)+f(0.1,2)\right.\\ &\quad \left.+4\left(f(0,1.5)+f(0.05,1)+f(0.05,2)+f(0.1,1.5)\right)+16f(0.05,1.5)\right]\\ &= 0.0996687 \end{aligned}$$

(b) $\displaystyle\int_0^{0.1}\int_1^2\frac{1}{1+x^2}\,dy\,dx = \int_0^{0.1}\frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^{0.1} = \tan^{-1}0.1 = 0.0996687$

(Note that this agrees to seven decimal places with our answer in part (a).)

9. (a) We let $\Delta x = \dfrac{1.1-1}{2} = 0.05$, $\Delta y = \dfrac{0.6-0}{2} = 0.3$. Hence, with $f(x,y) = e^{x+2y}$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.05)(0.3)}{9}\left[f(1,0)+f(1,0.6)+f(1.1,0)+f(1.1,0.6)\right.\\ &\quad \left.+4\left(f(1,0.3)+f(1.05,0)+f(1.05,0.6)+f(1.1,0.3)\right)+16f(1.05,0.3)\right]\\ &= 0.331871 \end{aligned}$$

(b) In part (b) of Exercise 5 we calculated $\displaystyle\int_1^{1.1}\int_0^{0.6}e^{x+2y}\,dy\,dx$ to be 0.331642.

10. (a) We let $\Delta x = \dfrac{\pi/4-0}{2} = \dfrac{\pi}{8}$, $\Delta y = \dfrac{\pi/2-\pi/4}{2} = \dfrac{\pi}{8}$. Hence, with $f(x,y) = \sin 2x\cos 3y$, we have

$$\begin{aligned} S_{2,2} &= \frac{(\pi/8)(\pi/8)}{9}\left[f(0,\pi/4)+f(0,\pi/2)+f(\pi/4,\pi/4)+f(\pi/4,\pi/2)\right.\\ &\quad \left.+4\left(f(0,3\pi/8)+f(\pi/8,\pi/4)+f(\pi/8,\pi/2)+f(\pi/4,3\pi/8)\right)+16f(\pi/8,3\pi/8)\right]\\ &= -0.288808 \end{aligned}$$

(b) $\displaystyle\int_0^{\pi/4}\int_{\pi/4}^{\pi/2}\sin 2x\cos 3y\,dy\,dx = \int_0^{\pi/4}\frac{1}{3}\sin 2x\sin 3y\Big|_{y=\pi/4}^{y=\pi/2}dx$

$\displaystyle = \int_0^{\pi/4}-\left(\frac{2+\sqrt{2}}{6}\right)\sin 2x\,dx = \frac{2+\sqrt{2}}{12}\cos 2x\Big|_0^{\pi/4}$

$\displaystyle = \frac{2+\sqrt{2}}{12}(0-1) = -\frac{2+\sqrt{2}}{12} = -0.284518$

11. (a) We let $\Delta x = \Delta y = \dfrac{\pi/4 - 0}{2} = \dfrac{\pi}{8}$. Hence, with $f(x, y) = \sin(x + y)$, we have

$$\begin{aligned} S_{2,2} &= \frac{(\pi/8)(\pi/8)}{9}\left[f(0,0) + f(0,\pi/4) + f(\pi/4,0) + f(\pi/4,\pi/4)\right. \\ &\quad \left. + 4\left(f(0,\pi/8) + f(\pi/8,0) + f(\pi/8,\pi/4) + f(\pi/4,\pi/8)\right) + 16f(\pi/8,\pi/8)\right] \\ &= 0.414325 \end{aligned}$$

(b) $\displaystyle\int_0^{\pi/4}\int_0^{\pi/4} \sin(x+y)\,dy\,dx = \int_0^{\pi/4} -\cos(x+y)\Big|_{y=0}^{y=\pi/4}\,dx$
$\displaystyle= \int_0^{\pi/4}\left(-\cos\left(x + \frac{\pi}{4}\right) + \cos x\right)\,dx = \left[\sin x - \sin\left(x + \frac{\pi}{4}\right)\right]\Big|_0^{\pi/4}$
$= \sqrt{2} - 1 = 0.414214$

12. (a) We let $\Delta x = \dfrac{1.1 - 1}{2} = 0.05$, $\Delta y = \dfrac{\pi/4 - 0}{2} = \dfrac{\pi}{8}$. Hence, with $f(x, y) = e^x \cos y$, we have

$$\begin{aligned} S_{2,2} &= \frac{(0.05)(\pi/8)}{9}\left[f(1,0) + f(1,\pi/4) + f(1.1,0) + f(1.1,\pi/4)\right. \\ &\quad \left. + 4\left(f(1,\pi/8) + f(1.05,0) + f(1.05,\pi/4) + f(1.1,\pi/8)\right) + 16f(1.05,\pi/8)\right] \\ &= 0.202178 \end{aligned}$$

(b) $\displaystyle\int_0^{1.1}\int_0^{\pi/4} e^x \cos y\,dy\,dx = \int_0^{1.1} e^x \sin y\Big|_{y=0}^{y=\pi/4}\,dx = \int_0^{1.1} \frac{\sqrt{2}}{2}\,e^x\,dx$
$\displaystyle= \frac{\sqrt{2}}{2}\left(e^{1.1} - e\right) = 0.202151$

13. (a) The paraboloid is a portion of the graph of $f(x, y) = 4 - x^2 - 3y^2$. We have $\partial f/\partial x = -2x$, $\partial f/\partial y = -6y$ so that the surface area integral we desire is

$$\int_0^1\int_0^1 \sqrt{4x^2 + 36y^2 + 1}\,dy\,dx.$$

(b) We let $\Delta x = \Delta y = \dfrac{1 - 0}{4} = 0.25$. With $g(x, y) = \sqrt{4x^2 + 36y^2 + 1}$, we have

$$\begin{aligned} T_{4,4} &= \frac{(0.25)(0.25)}{4}\left[g(0,0) + g(0,1) + g(1,0) + g(1,1)\right. \\ &\quad + 2\left(g(0,0.25) + g(0,0.5) + g(0,0.75) + g(0.25,0) + g(0.25,1) + g(0.5,0)\right. \\ &\qquad \left. + g(0.5,1) + g(0.75,0) + g(0.75,1) + g(1,0.25) + g(1,0.5) + g(1,0.75)\right) \\ &\quad + 4\left(g(0.25,0.25) + g(0.25,0.5) + g(0.25,0.75) + g(0.5,0.25) + g(0.5,0.5)\right. \\ &\qquad \left.\left. + g(0.5,0.75) + g(0.75,0.25) + g(0.75,0.5) + g(0.75,0.75)\right)\right] \\ &= 3.52366 \end{aligned}$$

14. (a) We let $\Delta x = \dfrac{1.5 - 1}{2} = 0.25$, $\Delta y = \dfrac{2 - 1.4}{4} = 0.15$. Then

$$\begin{aligned} T_{2,4} &= \frac{(0.25)(0.15)}{4}\left[\ln(2 + 1.4) + \ln(2 + 2) + \ln(3 + 1.4) + \ln(3 + 2)\right. \\ &\quad + 2\left(\ln(2 + 1.55) + \ln(2 + 1.7) + \ln(2 + 1.85) + \ln(2.5 + 1.4) + \ln(2.5 + 2)\right. \\ &\qquad \left. + \ln(3 + 1.55) + \ln(3 + 1.7) + \ln(3 + 1.85)\right) \\ &\quad \left. + 4\left(\ln(2.5 + 1.55) + \ln(2.5 + 1.7) + \ln(2.5 + 1.85)\right)\right] \\ &= 0.429161 \end{aligned}$$

(b) We have

$$\frac{\partial^2}{\partial x^2}\ln(2x+y) = -\frac{4}{(2x+y)^2} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\ln(2x+y) = -\frac{1}{(2x+y)^2}.$$

The maximum magnitude of both of these expressions on the rectangle $[1, 1.5] \times [1.4, 2]$ occurs at $(1, 1.4)$. Hence, from Theorem 7.3, we have that

$$|E_{2,4}| \le \frac{(0.5)(0.6)}{12}\left((0.25)^2 \cdot \frac{4}{(2+1.4)^2} + (0.15)^2 \cdot \frac{1}{(2+1.4)^2}\right) = 0.00589317.$$

Thus the actual value of the integral lies between 0.428572 and 0.42975.

(c) With Δx and Δy as in part (a), we have

$$\begin{aligned} S_{2,4} = \frac{(0.25)(0.15)}{9} & [\ln(2+1.4) + \ln(2+2) + \ln(3+1.4) + \ln(3+2) \\ & + 2(\ln(2+1.7) + \ln(3+1.7)) + 4(\ln(2+1.55) + \ln(2+1.85) \\ & \quad + \ln(2.5+1.4) + \ln(2.5+2) + \ln(3+1.55) + \ln(3+1.85)) \\ & + 8\ln(2.5+1.7) + 16(\ln(2.5+1.55) + \ln(2.5+1.85))] \\ = 0.429552 & \end{aligned}$$

(d) We have

$$\frac{\partial^4}{\partial x^4}\ln(2x+y) = -\frac{96}{(2x+y)^4} \quad \text{and} \quad \frac{\partial^4}{\partial y^4}\ln(2x+y) = -\frac{6}{(2x+y)^4}$$

and, as in part (b), the maximum magnitude of both of these expressions on $[1, 1.5] \times [1.4, 2]$ occurs at $(1, 1.4)$. Hence, from Theorem 7.4, we have that

$$|E_{2,4}| \le \frac{(0.5)(0.6)}{180}\left((0.25)^4 \cdot \frac{96}{(2+1.4)^4} + (0.15)^4 \cdot \frac{6}{(2+1.4)^4}\right) = 6.36068 \times 10^{-6}.$$

Hence the actual value of the integral lies between 0.429546 and 0.429559.

15. To answer the question, we compare the errors of the respective methods as given in Theorems 7.3 and 7.4.

First we consider the error $E_{4,4}$ associated with the trapezoidal rule approximation $T_{4,4}$. In this case we have

$$\Delta x = \frac{1.4-1}{4} = 0.1 \quad \text{and} \quad \Delta y = \frac{0.7-0.5}{4} = 0.05.$$

In addition,

$$\frac{\partial^2}{\partial x^2}\ln(xy) = -\frac{1}{x^2} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\ln(xy) = -\frac{1}{y^2}.$$

Theorem 7.3 says that there exist points (ζ_1, η_1) and (ζ_2, η_2) in the rectangle $[1, 1.4] \times [0.5, 0.7]$ such that

$$\begin{aligned} E_{4,4} &= -\frac{(1.4-1)(0.7-0.5)}{12}\left[(0.1)^2\left(-\frac{1}{\zeta_1^2}\right) + (0.05)^2\left(-\frac{1}{\eta_2^2}\right)\right] \\ &= \frac{(0.4)(0.2)}{12}\left[\frac{(0.1)^2}{\zeta_1^2} + \frac{(0.05)^2}{\eta_2^2}\right]. \end{aligned}$$

Now $1 \le \zeta_1 \le 1.4$ and $0.5 \le \eta_2 \le 0.7$ so that, if we choose $\zeta_1 = 1.4$ and $\eta_2 = 0.7$, we can make the value in the brackets as small as possible; hence $E_{4,4} \ge 0.000068027$.

Next we consider the error $E_{2,2}$ associated with the Simpson's rule approximation $S_{2,2}$. Hence we have

$$\Delta x = \frac{1.4-1}{2} = 0.2 \quad \text{and} \quad \Delta y = \frac{0.7-0.5}{2} = 0.1;$$

also

$$\frac{\partial^4}{\partial x^4}\ln(xy) = -\frac{6}{x^4} \quad \text{and} \quad \frac{\partial^4}{\partial y^4}\ln(xy) = -\frac{6}{y^4}.$$

Theorem 7.4 says that there exist points (ζ_1, η_1) and (ζ_2, η_2) in $[1, 1.4] \times [0.5, 0.7]$ such that

$$E_{2,2} = -\frac{(1.4-1)(0.7-0.5)}{180}\left[(0.2)^4\left(-\frac{6}{\zeta_1^4}\right) + (0.1)^4\left(-\frac{6}{\eta_2^4}\right)\right]$$
$$= \frac{(0.4)(0.2)}{30}\left[\frac{(0.2)^4}{\zeta_1^4} + \frac{(0.1)^4}{\eta_2^4}\right].$$

By choosing $\zeta_1 = 1$ and $\eta_2 = 0.5$ we make the expression as large as possible; hence $E_{2,2} \le 8.5\overline{3} \times 10^{-6}$. Since the *maximum* possible error using Simpson's rule is *less* than the *minimum* possible error using the trapezoidal rule, we see that $S_{2,2}$ will be more accurate than $T_{4,4}$.

16. We calculate the error $E_{n,n}$ associated with the trapezoidal rule approximation $T_{n,n}$. Note first that

$$\frac{\partial^2}{\partial x^2}\left(e^{x^2+2y}\right) = (4x^2+2)e^{x^2+2y} \quad \text{and} \quad \frac{\partial^2}{\partial y^2}\left(e^{x^2+2y}\right) = 4e^{x^2+2y}.$$

The maximum values of these expressions on the rectangle $[0, 0.2] \times [-0.1, 0.1]$ both occur at the point $(0.2, 0.1)$ and are, respectively, $(2.16)e^{0.24}$ and $4e^{0.24}$. Also note that in calculating $T_{n,n}$, we have $\Delta x = \Delta y = 0.2/n$. Thus, from Theorem 7.3, we have that

$$|E_{n,n}| \le \frac{(0.2)(0.2)}{12}\left[\left(\frac{0.2}{n}\right)^2(2.16)e^{0.24} + \left(\frac{0.2}{n}\right)^2\left(4e^{0.24}\right)\right] = \frac{(0.2)^4(6.16)e^{0.24}}{12n^2}.$$

For this last expression to be at most 10^{-4}, we must have

$$\frac{(0.2)^4(6.16)e^{0.24}}{12n^2} \le 10^{-4} \iff n^2 \ge \frac{10^4(0.2)^4(6.16)e^{0.24}}{12} \iff n > 3.23.$$

Hence, since n must be an integer, we should take n to be at least 4.

17. (a) We have

$$\frac{\partial^2}{\partial x^2}\left(e^{x-y}\right) = \frac{\partial^2}{\partial y^2}\left(e^{x-y}\right) = e^{x-y}.$$

The maximum value of e^{x-y} on $[0, 0.3] \times [0, 0.4]$ is $e^{0.3-0} = e^{0.3}$. Furthermore, in computing the approximation $T_{n,n}$ we have $\Delta x = 0.3/n$ and $\Delta y = 0.4/n$. Thus Theorem 7.3 implies that

$$|E_{n,n}| \le \frac{(0.3)(0.4)}{12}\left[\left(\frac{0.3}{n}\right)^2 e^{0.3} + \left(\frac{0.4}{n}\right)^2 e^{0.3}\right] = \frac{(0.3)(0.4)(0.5)^2e^{0.3}}{12n^2}.$$

For this expression to be at most 10^{-5}, we must have

$$\frac{(0.3)(0.4)(0.5)^2e^{0.3}}{12n^2} \le 10^{-5} \iff n^2 \ge \frac{10^5(0.3)(0.4)(0.5)^2e^{0.3}}{12} \iff n > 18.37.$$

Thus we should take n to be at least 19.

(b) In this case, we use Theorem 7.4. First note that we have

$$\frac{\partial^4}{\partial x^4}\left(e^{x-y}\right) = \frac{\partial^4}{\partial y^4}\left(e^{x-y}\right) = e^{x-y},$$

so that, as in part (a), the maximum value of e^{x-y} on $[0, 0.3] \times [0, 0.4]$ is $e^{0.3}$. Moreover, in computing the approximation $S_{2n,2n}$, we have $\Delta x = 0.3/(2n)$ and $\Delta y = 0.4/(2n)$. Therefore, Theorem 7.4 implies that

$$|E_{2n,2n}| \le \frac{(0.3)(0.4)}{180}\left[\left(\frac{0.3}{2n}\right)^4 e^{0.3} + \left(\frac{0.4}{2n}\right)^4 e^{0.3}\right] = \frac{(0.3)(0.4)((0.3)^4+(0.4)^4)e^{0.3}}{180 \cdot 16n^4}.$$

For this expression to be at most 10^{-5}, we must have

$$\frac{(0.3)(0.4)((0.3)^4+(0.4)^4)e^{0.3}}{180 \cdot 16n^4} \le 10^{-5} \iff n^4 \ge \frac{10^5(0.3)(0.4)((0.3)^4+(0.4)^4)e^{0.3}}{180 \cdot 16}$$
$$\iff n > 0.659.$$

Thus, since n must be an integer, we must have n at least 1; that is, $S_{2,2}$ will give an approximation with the desired accuracy.

18. **(a)** Let $\Delta x = \dfrac{2-0}{2} = 1$, $\Delta y = \dfrac{3-0}{2} = 1.5$. With $f(x, y) = 3x + 5y$, we have

$$T_{2,2} = \frac{1(1.5)}{4}\left[f(0,0) + f(0,3) + f(2,0) + f(2,3) \right.$$
$$\left. + 2\left(f(0,1.5) + f(1,0) + f(1,3) + f(2,1.5)\right) + 4f(1,1.5)\right] = 63.$$

(b) $\displaystyle\int_0^2 \int_0^3 (3x+5y)\,dy\,dx = \int_0^2 \left(3xy + \tfrac{5}{2}y^2\right)\Big|_{y=0}^{y=3} dx = \int_0^2 \left(9x + \tfrac{45}{2}\right)\,dx$
$= \left(\tfrac{9}{2}x^2 + \tfrac{45}{2}x\right)\Big|_0^2 = 63.$

This is exactly the same result as in part (a).

(c) Note that, for all (x, y), we have

$$\frac{\partial^2}{\partial x^2}(3x+5y) = \frac{\partial^2}{\partial y^2}(3x+5y) = 0.$$

Hence Theorem 7.3 shows that the error term $E_{2,2}$ must be zero. Hence it's no surprise that the results in parts (a) and (b) are the same.

19. **(a)** Let $\Delta x = \dfrac{0-(-1)}{2} = 0.5$, $\Delta y = \dfrac{1/2-0}{2} = 0.25$. With $f(x, y) = x^3y^3$, we have

$$S_{2,2} = \frac{(0.5)(0.25)}{9}\left[f(-1,0) + f(-1,\tfrac{1}{2}) + f(0,0) + f(0,\tfrac{1}{2}) \right.$$
$$\left. + 4\left(f(-1,\tfrac{1}{4}) + f(-\tfrac{1}{2},0) + f(-\tfrac{1}{2},\tfrac{1}{2}) + f(0,\tfrac{1}{4})\right) + 16f(-\tfrac{1}{2},\tfrac{1}{4})\right]$$
$$= -0.00390625$$

(b) $\displaystyle\int_{-1}^0 \int_0^{1/2} x^3y^3\,dy\,dx = \int_{-1}^0 \frac{x^3}{4}y^4\Big|_{y=0}^{y=1/2} dx = \int_{-1}^0 \frac{x^3}{64}\,dx = \frac{x^4}{256}\Big|_{-1}^0 = -\frac{1}{256}.$

(c) The answers in parts (a) and (b) turn out to be the same. Note that, for all (x, y), we have

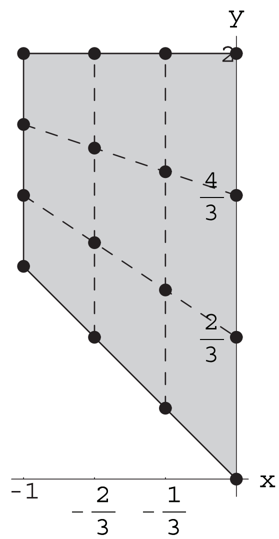
$$\frac{\partial^4}{\partial x^4}\left(x^3y^3\right) = \frac{\partial^4}{\partial y^4}\left(x^3y^3\right) = 0.$$

Hence Theorem 7.4 shows that the error term $E_{2,2}$ must be zero.

20. We let $\Delta x = \dfrac{0-(-1)}{3} = \frac{1}{3}$, so that $x_0 = -1$, $x_1 = -\frac{2}{3}$, $x_2 = -\frac{1}{3}$, $x_3 = 0$. Then $\Delta y(x) = \dfrac{2-(-x)}{3} = \dfrac{x+2}{3}$ so that

$$\begin{aligned}
\Delta y(-1) = \tfrac{1}{3} \quad &\Longrightarrow \quad y_0(x_0) = 1,\ y_1(x_0) = \tfrac{4}{3},\ y_2(x_0) = \tfrac{5}{3},\ y_3(x_0) = 2\\
\Delta y(-\tfrac{2}{3}) = \tfrac{4}{9} \quad &\Longrightarrow \quad y_0(x_1) = \tfrac{2}{3},\ y_1(x_1) = \tfrac{10}{9},\ y_2(x_1) = \tfrac{14}{9},\ y_3(x_1) = 2\\
\Delta y(-\tfrac{1}{3}) = \tfrac{5}{9} \quad &\Longrightarrow \quad y_0(x_2) = \tfrac{1}{3},\ y_1(x_2) = \tfrac{8}{9},\ y_2(x_2) = \tfrac{13}{9},\ y_3(x_2) = 2\\
\Delta y(0) = \tfrac{2}{3} \quad &\Longrightarrow \quad y_0(x_3) = 0\ ,y_1(x_3) = \tfrac{2}{3},\ y_2(x_3) = \tfrac{4}{3},\ y_3(x_3) = 2
\end{aligned}$$

This information is pictured in the figure below.



Therefore, using $f(x, y) = x^3 + 2y^2$, we have

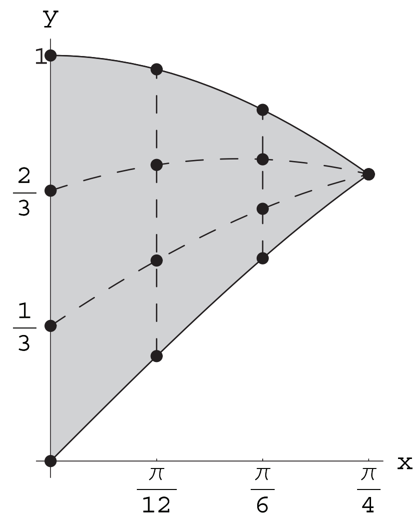
$$
\begin{aligned}
T_{3,3} = &\frac{(1/3)(1/3)}{4}\left[f(-1,1) + 2f(-1,\tfrac{4}{3}) + 2f(-1,\tfrac{5}{3}) + f(-1,2)\right] \\
&+ \frac{(1/3)(4/9)}{4}\left[2f(-\tfrac{2}{3},\tfrac{2}{3}) + 4f(-\tfrac{2}{3},\tfrac{10}{9}) + 4f(-\tfrac{2}{3},\tfrac{14}{9}) + 2f(-\tfrac{2}{3},2)\right] \\
&+ \frac{(1/3)(5/9)}{4}\left[2f(-\tfrac{1}{3},\tfrac{1}{3}) + 4f(-\tfrac{1}{3},\tfrac{8}{9}) + 4f(-\tfrac{1}{3},\tfrac{13}{9}) + 2f(-\tfrac{1}{3},2)\right] \\
&+ \frac{(1/3)(2/3)}{4}\left[f(0,0) + 2f(0,\tfrac{2}{3}) + 2f(0,\tfrac{4}{3}) + f(0,2)\right] \\
= &\ 4.97119
\end{aligned}
$$

(Note that the exact answer is $24/5 = 4.8$.)

21. We let $\Delta x = \dfrac{\pi/4 - 0}{3} = \dfrac{\pi}{12}$, so that $x_0 = 0$, $x_1 = \frac{\pi}{12}$, $x_2 = \frac{\pi}{6}$, $x_3 = \frac{\pi}{4}$. Then $\Delta y(x) = \dfrac{\cos x - \sin x}{3}$, so that

$$
\begin{aligned}
\Delta y(0) = \tfrac{1}{3} \quad &\Longrightarrow \quad y_0(x_0) = 0,\ y_1(x_0) = \tfrac{1}{3},\ y_2(x_0) = \tfrac{2}{3},\ y_3(x_0) = 1 \\
\Delta y\left(\tfrac{\pi}{12}\right) = \tfrac{1}{3\sqrt{2}} \quad &\text{(by use of the half-angle formula)} \\
&\Longrightarrow \quad y_0(x_1) = \sin\left(\tfrac{\pi}{12}\right),\ y_1(x_1) = \sin\left(\tfrac{\pi}{12}\right) + \tfrac{1}{3\sqrt{2}}, \\
&\qquad\quad y_2(x_1) = \sin\left(\tfrac{\pi}{12}\right) + \tfrac{2}{3\sqrt{2}},\ y_3(x_1) = \sin\left(\tfrac{\pi}{12}\right) + \tfrac{1}{\sqrt{2}} \\
\Delta y\left(\tfrac{\pi}{6}\right) = \tfrac{\sqrt{3}-1}{6} \quad &\Longrightarrow \quad y_0(x_2) = \tfrac{1}{2},\ y_1(x_2) = \tfrac{\sqrt{3}+2}{6},\ y_2(x_2) = \tfrac{2\sqrt{3}+1}{6},\ y_3(x_2) = \tfrac{\sqrt{3}}{2} \\
\Delta y\left(\tfrac{\pi}{4}\right) = 0 \quad &\Longrightarrow \quad \text{partition points not needed.}
\end{aligned}
$$

This information is pictured in the figure below.



Thus, using $f(x,y) = 2x\cos y + \sin^2 x$, we have

$$T_{3,3} = \frac{(\pi/12)(1/3)}{4}\left[f(0,0) + 2f(0,\tfrac{1}{3}) + 2f(0,\tfrac{2}{3}) + f(0,1)\right]$$
$$+ \frac{(\pi/12)(1/(3\sqrt{2}))}{4}\Big[2f(\tfrac{\pi}{12}, \sin\tfrac{\pi}{12}) + 4f(\tfrac{\pi}{12}, \sin\tfrac{\pi}{12} + \tfrac{1}{3\sqrt{2}})$$
$$+ 4f(\tfrac{\pi}{12}, \sin\tfrac{\pi}{12} + \tfrac{2}{3\sqrt{2}}) + 2f(\tfrac{\pi}{12}, \cos\tfrac{\pi}{12})\Big]$$
$$+ \frac{(\pi/12)((\sqrt{3}-1)/6)}{4}\Big[2f(\tfrac{\pi}{6}, \tfrac{1}{2}) + 4f(\tfrac{\pi}{6}, \tfrac{\sqrt{3}+2}{6}) + 4f(\tfrac{\pi}{6}, \tfrac{2\sqrt{3}+1}{6}) + 2f(\tfrac{\pi}{6}, \tfrac{\sqrt{3}}{2})\Big]$$
$$= 0.190978$$

(This approximation turns out to be rather low.)

22. Let $\Delta x = \dfrac{0.3-0}{3} = 0.1$, so that $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$. Then $\Delta y(x) = \dfrac{2x - x}{3} = \dfrac{x}{3}$ so that

$$\Delta y(0) = 0 \implies \text{partition points not needed;}$$
$$\Delta y(0.1) = \frac{0.1}{3} \implies y_0(x_1) = 0.1,\ y_1(x_1) = 0.1\overline{3},\ y_2(x_1) = 0.1\overline{6},\ y_3(x_1) = 0.2$$
$$\Delta y(0.2) = \frac{0.2}{3} \implies y_0(x_2) = 0.2,\ y_1(x_2) = 0.2\overline{6},\ y_2(x_2) = 0.\overline{3},\ y_3(x_2) = 0.4$$
$$\Delta y(0.3) = 0.1 \implies y_0(x_3) = 0.3\ , y_1(x_3) = 0.4,\ y_2(x_3) = 0.5,\ y_3(x_1) = 0.6$$

Then, using $f(x,y) = xy - x^2$, we have

$$T_{3,3} = \frac{(0.1)(0.1/3)}{4}\left[2f(0.1,0.1) + 4f(0.1,0.1\overline{3}) + 2f(0.1,0.1\overline{6}) + 2f(0.1,0.2)\right]$$
$$+ \frac{(0.1)(0.2/3)}{4}\left[2f(0.2,0.2) + 4f(0.2,0.2\overline{6}) + 4f(0.2,0.\overline{3}) + 2f(0.2,0.4)\right]$$
$$+ \frac{(0.1)(0.1)}{4}\left[f(0.3,0.3) + 2f(0.3,0.4) + 2f(0.3,0.5) + f(0.3,0.6)\right]$$
$$= 0.001125$$

(Note that the actual value is 0.0010125.)

23. Let $\Delta x = \dfrac{\pi/3 - 0}{3} = \dfrac{\pi}{9}$, so that $x_0 = 0$, $x_1 = \frac{\pi}{9}$, $x_2 = \frac{2\pi}{9}$, $x_3 = \frac{\pi}{3}$. Then $\Delta y(x) = \dfrac{\sin x - 0}{3} = \dfrac{1}{3}\sin x$, so that

$$\begin{aligned}
\Delta y(0) = 0 &\implies \text{partition points not needed;}\\
\Delta y\left(\tfrac{\pi}{9}\right) = \tfrac{1}{3}\sin\tfrac{\pi}{9} &\implies y_0(x_1) = 0,\ y_1(x_1) = \tfrac{1}{3}\sin\tfrac{\pi}{9},\ y_2(x_1) = \tfrac{2}{3}\sin\tfrac{\pi}{9},\ y_3(x_1) = \sin\tfrac{\pi}{9}\\
\Delta y\left(\tfrac{2\pi}{9}\right) = \tfrac{1}{3}\sin\tfrac{2\pi}{9} &\implies y_0(x_2) = 0,\ y_1(x_2) = \tfrac{1}{3}\sin\tfrac{2\pi}{9},\ y_2(x_2) = \tfrac{2}{3}\sin\tfrac{2\pi}{9},\ y_3(x_2) = \sin\tfrac{2\pi}{9}\\
\Delta y\left(\tfrac{\pi}{3}\right) = \tfrac{\sqrt{3}}{6} &\implies y_0(x_3) = 0,\ y_1(x_3) = \tfrac{\sqrt{3}}{6},\ y_2(x_3) = \tfrac{\sqrt{3}}{3},\ y_3(x_3) = \tfrac{\sqrt{3}}{2}
\end{aligned}$$

Thus, using $f(x,y) = x/\sqrt{1-y^2}$, we have

$$\begin{aligned}
T_{3,3} &= \frac{(\frac{\pi}{9})(\frac{1}{3}\sin\frac{\pi}{9})}{4}\left[2f(\tfrac{\pi}{9},0) + 4f(\tfrac{\pi}{9},\tfrac{1}{3}\sin\tfrac{\pi}{9}) + 4f(\tfrac{\pi}{9},\tfrac{2}{3}\sin\tfrac{\pi}{9}) + 2f(\tfrac{\pi}{9},\sin\tfrac{\pi}{9})\right]\\
&\quad + \frac{(\frac{\pi}{9})(\frac{1}{3}\sin\frac{2\pi}{9})}{4}\left[2f(\tfrac{\pi}{9},0) + 4f(\tfrac{2\pi}{9},\tfrac{1}{3}\sin\tfrac{2\pi}{9}) + 4f(\tfrac{2\pi}{9},\tfrac{2}{3}\sin\tfrac{2\pi}{9}) + 2f(\tfrac{2\pi}{9},\sin\tfrac{2\pi}{9})\right]\\
&\quad + \frac{(\pi/9)(\sqrt{3}/6)}{4}\left[f(\tfrac{\pi}{3},0) + 2f(\tfrac{\pi}{3},\tfrac{\sqrt{3}}{6}) + 2f(\tfrac{\pi}{3},\tfrac{\sqrt{3}}{3}) + f(\tfrac{\pi}{3},\tfrac{\sqrt{3}}{2})\right]\\
&= 0.412888
\end{aligned}$$

(This actual value is $\pi^3/81 \approx 0.382794$, so our approximation is not especially good here.)

24. We must first let $\Delta y = \dfrac{\pi - 1}{3}$, so that $y_0 = 1$, $y_1 = \frac{\pi+2}{3}$, $y_2 = \frac{2\pi+1}{3}$, $y_3 = \pi$. Then $\Delta x(y) = \dfrac{y-0}{3} = \dfrac{y}{3}$, so that

$$\begin{aligned}
\Delta x(1) = \tfrac{1}{3} &\implies x_0(y_0) = 0,\ x_1(y_0) = \tfrac{1}{3},\ x_2(y_0) = \tfrac{2}{3},\ x_3(y_0) = 1\\
\Delta x\left(\tfrac{\pi+2}{3}\right) = \tfrac{\pi+2}{9} &\implies x_0(y_1) = 0,\ x_1(y_1) = \tfrac{\pi+2}{9},\ x_2(y_1) = \tfrac{2\pi+4}{9},\ x_3(y_1) = \tfrac{\pi+2}{3}\\
\Delta x\left(\tfrac{2\pi+1}{3}\right) = \tfrac{2\pi+1}{9} &\implies x_0(y_2) = 0,\ x_1(y_2) = \tfrac{2\pi+1}{9},\ x_2(y_2) = \tfrac{4\pi+2}{9},\ x_3(y_2) = \tfrac{2\pi+1}{3}\\
\Delta x(\pi) = \tfrac{\pi}{3} &\implies x_0(y_3) = 0,\ x_1(y_3) = \tfrac{\pi}{3},\ x_2(y_3) = \tfrac{2\pi}{3},\ x_3(y_3) = \pi
\end{aligned}$$

Thus, using $f(x,y) = \sin x$, we have

$$\begin{aligned}
T_{3,3} &= \frac{(\frac{\pi-1}{3})(\frac{1}{3})}{4}\left[f(0,1) + 2f(\tfrac{1}{3},1) + 2f(\tfrac{2}{3},1) + f(1,1)\right]\\
&\quad + \frac{(\frac{\pi-1}{3})(\frac{\pi+2}{9})}{4}\left[2f(0,\tfrac{\pi+2}{3}) + 4f(\tfrac{\pi+2}{9},\tfrac{\pi+2}{3}) + 4f(\tfrac{2\pi+4}{9},\tfrac{\pi+2}{3}) + 2f(\tfrac{\pi+2}{3},\tfrac{\pi+2}{3})\right]\\
&\quad + \frac{(\frac{\pi-1}{3})(\frac{2\pi+1}{9})}{4}\left[2f(0,\tfrac{2\pi+1}{3}) + 4f(\tfrac{2\pi+1}{9},\tfrac{2\pi+1}{3}) + 4f(\tfrac{4\pi+2}{9},\tfrac{2\pi+1}{3}) + 2f(\tfrac{2\pi+1}{3},\tfrac{2\pi+1}{3})\right]\\
&\quad + \frac{(\frac{\pi-1}{3})(\frac{\pi}{3})}{4}\left[f(0,\pi) + 2f(\tfrac{\pi}{3},\pi) + 2f(\tfrac{2\pi}{3},\pi) + f(\pi,\pi)\right]\\
&= 2.78757
\end{aligned}$$

(This actual value is $\sin 1 + \pi - 1 \approx 2.98306$, so this result is quite rough.)

25. We let $\Delta y = \dfrac{1.6-1}{3} = 0.2$, so that $y_0 = 1$, $y_1 = 1.2$, $y_2 = 1.4$, $y_3 = 1.6$. Then $\Delta x(y) = \dfrac{2y - y}{3} = \dfrac{y}{3}$, so that

$$\begin{aligned}
\Delta x(1) = \tfrac{1}{3} &\implies x_0(y_0) = 1,\ x_1(y_0) = \tfrac{4}{3},\ x_2(y_0) = \tfrac{5}{3},\ x_3(y_0) = 2\\
\Delta x(1.2) = 0.4 &\implies x_0(y_1) = 1.2,\ x_1(y_1) = 1.6,\ x_2(y_1) = 2,\ x_3(y_1) = 2.4\\
\Delta x(1.4) = 0.4\overline{6} &\implies x_0(y_2) = 1.4,\ x_1(y_2) = 1.8\overline{6},\ x_2(y_2) = 2.\overline{3},\ x_3(y_2) = 2.8\\
\Delta x(1.6) = 0.5\overline{3} &\implies x_0(y_3) = 1.6,\ x_1(y_3) = 2.1\overline{3},\ x_2(y_3) = 2.\overline{6},\ x_3(y_3) = 3.2
\end{aligned}$$

Thus, using $f(x,y)=\ln(xy)$, we have

$$\begin{aligned}T_{3,3} = {} & \frac{(0.2)(0.\overline{3})}{4}\left[f(1,1)+2f(\tfrac{4}{3},1)+2f(\tfrac{5}{3},1)+f(2,1)\right]\\ & + \frac{(0.2)(0.4)}{4}\left[2f(1.2,1.2)+4f(1.6,1.2)+4f(2,1.2)+2f(2.4,1.2)\right]\\ & + \frac{(0.2)(0.4\overline{6})}{4}\left[2f(1.4,1.4)+4f(1.8\overline{6},1.4)+4f(2.\overline{3},1.4)+2f(2.8,1.4)\right]\\ & + \frac{(0.2)(0.5\overline{3})}{4}\left[f(1.6,1.6)+2f(2.1\overline{3},1.6)+2f(2.\overline{6},1.6)+f(3.2,1.6)\right]\\ = {} & 0.724061\end{aligned}$$

(This actual value is closer to 0.724519.)

26. (a) We have

$$\begin{aligned}\iint_R L\,dA &= \int_a^b\int_c^d (Ax+By+C)\,dy\,dx = \int_a^b \left[(Ax+C)y+\tfrac{1}{2}By^2\right]\Big|_{y=c}^{y=d}dx\\ &= \int_a^b \left[(Ax+C)(d-c)+\tfrac{1}{2}B(d^2-c^2)\right]\,dx\\ &= (d-c)\int_a^b \left[Ax+C+\tfrac{1}{2}B(c+d)\right]\,dx\\ &= (d-c)\left[\tfrac{1}{2}A(b^2-a^2)+\left(C+\tfrac{1}{2}B(c+d)\right)(b-a)\right]\\ &= (b-a)(d-c)\left[\tfrac{1}{2}A(a+b)+\tfrac{1}{2}B(c+d)+C\right]\\ &= \frac{(b-a)(d-c)}{4}\left[2A(a+b)+2B(c+d)+4C\right].\end{aligned}$$

(Note that we drew out factors along the way.) The average of the values of L taken at the vertices of R is

$$\begin{aligned}&\frac{1}{4}\left[L(a,c)+L(a,d)+L(b,c)+L(b,d)\right]\\ &\quad= \frac{1}{4}\left[(Aa+Bc+C)+(Aa+Bd+C)+(Ab+Bc+C)+(Ab+Bd+C)\right]\\ &\quad= \frac{1}{4}\left[2A(a+b)+2B(c+d)+4C\right].\end{aligned}$$

If we multiply this expression by $(b-a)(d-c)$, which is the area of R, we obtain the expression for $\iint_R L\,dA$ calculated above.

(b) To calculate $T_{1,1}$, note that $\Delta x = b-a$, $\Delta y = d-c$, so that $x_0=a$, $x_1=b$, $y_0=c$, $y_1=d$ and formula (6) becomes

$$\begin{aligned}T_{1,1} &= \frac{(b-a)(d-c)}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]\\ &= (\text{area of } R)\cdot(\text{average of values of } f \text{ on vertices of } R).\end{aligned}$$

(c) By part (b), the approximation $T_{1,1}$ to $\iint_{R_{ij}} f\,dA$ is

$$\frac{\Delta x\,\Delta y}{4}\left[f(x_{i-1},y_{j-1})+f(x_{i-1},y_j)+f(x_i,y_{j-1})+f(x_i,y_j)\right].$$

Thus $\iint_R f\,dA = \sum_{j=1}^{n}\sum_{i=1}^{m}\iint_{R_{ij}} f\,dA$ is approximated by

$$\begin{aligned}
&\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{\Delta x\,\Delta y}{4}\left[f(x_{i-1},y_{j-1})+f(x_{i-1},y_j)+f(x_i,y_{j-1})+f(x_i,y_j)\right]\\
&=\frac{\Delta x\,\Delta y}{4}\left[\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_{i-1},y_{j-1})+\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_{i-1},y_j)\right.\\
&\qquad\left.+\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_i,y_{j-1})+\sum_{j=1}^{n}\sum_{i=1}^{m}f(x_i,y_j)\right]\\
&=\frac{\Delta x\,\Delta y}{4}\left[f(x_0,y_0)+\sum_{j=1}^{n-1}f(x_0,y_j)+\sum_{i=1}^{m-1}f(x_i,y_0)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\right.\\
&\qquad+f(x_0,y_n)+\sum_{j=1}^{n-1}f(x_0,y_j)+\sum_{i=1}^{m-1}f(x_i,y_n)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\\
&\qquad+f(x_m,y_0)+\sum_{j=1}^{n-1}f(x_m,y_j)+\sum_{i=1}^{m-1}f(x_i,y_0)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\\
&\qquad\left.+f(x_m,y_n)+\sum_{j=1}^{n-1}f(x_m,y_j)+\sum_{i=1}^{m-1}f(x_i,y_n)+\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)\right]\\
&=\frac{\Delta x\,\Delta y}{4}\left[f(x_0,y_0)+2\sum_{i=1}^{m-1}f(x_i,y_0)+f(x_m,x_0)\right.\\
&\qquad+2\sum_{j=1}^{n-1}f(x_0,y_j)+4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1}f(x_i,y_j)+2\sum_{j=1}^{n-1}f(x_m,y_j)\\
&\qquad\left.+f(x_0,y_n)+2\sum_{i=1}^{m-1}f(x_i,y_n)+f(x_m,y_n)\right]\\
&=T_{m,n}.
\end{aligned}$$

True/False Exercises for Chapter 5

1. False. (Not all rectangles must have sides parallel to the coordinate axes.)
2. True.
3. True.
4. True.
5. False. (Let $f(x,y)=x$, for example.)
6. True.
7. False. (The integral on the right isn't even a number!)
8. True.
9. True.
10. False. (It's a type 1 region.)
11. True.
12. True.
13. False. (The value of the integral is 3.)
14. True. (Use symmetry.)
15. True.

16. False. (The y part of the integrand gives a nonzero value.)
17. True. (The inner integral with respect to z is zero because of symmetry.)
18. False. (The triple integral of y is zero because of symmetry, but not the triple integral of x.)
19. True.
20. False. (The area is 30 square units.)
21. False. (The integrals are opposites of one another.)
22. True.
23. False. (A factor of r should appear in the integrand.)
24. True.
25. False. (A factor of ρ is missing in the integrand.)
26. True.
27. True.
28. False. (The centroid is at $\left(0, 0, \frac{1}{4}h\right)$.)
29. True.
30. True.

Miscellaneous Exercises for Chapter 5

1. First let's split the integrand:

$$\iiint_B (z^3 + 2)\, dV = \iiint_B z^3 dV + \iiint_B 2\, dV = \iiint_B z^3 dV + 2\iiint_B dV.$$

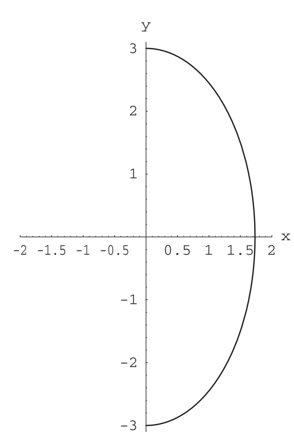
Here B is the ball of radius 3 centered at the origin. The integral of an odd function of z over a region which is symmetric with respect to z is 0. The other integral is twice the volume of a sphere of radius 3 so

$$\iiint_B (z^3 + 2)\, dV = 2\left(\frac{4}{3}\pi 3^3\right) = 72\pi.$$

2. As in Exercise 1 we see that our integrand is the sum of odd functions in x and y and a constant which we are integrating over a region which is symmetric with respect to x and y. Our answer will be -3 times the volume of the hemisphere of radius 2. In symbols,

$$V = \iiint_W (x^3 + y - 3)\, dV = -3\iiint_W dV = -3\left(\frac{1}{2}\right)\left(\frac{4}{3}\pi 2^3\right) = -16\pi.$$

3. (a) We'll use the bounds given for z in both integrals and just reverse the order of integration for x and y. We are integrating over the ellipse:



$$\iiint_W 3\,dV = \int_{-3}^{3}\int_0^{\sqrt{3-y^2/3}}\int_{2x^2+y^2}^{9-x^2} 3\,dz\,dx\,dy \quad \text{and}$$
$$= \int_0^{\sqrt{3}}\int_{-\sqrt{9-3x^2}}^{\sqrt{9-3x^2}}\int_{2x^2+y^2}^{9-x^2} 3\,dz\,dy\,dx.$$

(b) Using *Mathematica*, the result was $(81\sqrt{3}\pi)/4$ in either order.

4. First follow the hint (noting that x' and y' are just dummy variables) and write

$$F(x,y) = \int_a^x g(x',y)\,dx' \quad \text{where} \quad g(x',y) = \int_c^y f(x',y')\,dy'.$$

By the fundamental theorem of calculus,

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\int_a^x g(x',y)\,dx' = g(x,y).$$

Also, again by the fundamental theorem,

$$\frac{\partial^2 F}{\partial y\partial x} = \frac{\partial}{\partial y}[g(x,y)] = \frac{\partial}{\partial y}\int_c^y f(x,y')\,dy' = f(x,y).$$

By Fubini's theorem,

$$\int_a^x\int_b^y f(x',y')\,dy'dx' = \int_c^y\int_a^x f(x',y')\,dx'dy'.$$

As above, write

$$F(x,y) = \int_c^y h(x,y')\,dy' \quad \text{where} \quad h(x,y') = \int_a^x f(x',y')\,dx'.$$

Proceeding as above we see that

$$\frac{\partial F}{\partial y} = h(x,y) \quad \text{and} \quad \frac{\partial^2 F}{\partial x\partial y} = f(x,y).$$

5. I think the given form is the easiest to integrate:

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r\sqrt{9-r^2}\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(9-\frac{16\sqrt{2}}{3}\right)d\theta$$
$$= 2\pi\left(9-\frac{16\sqrt{2}}{3}\right).$$

(a) In Cartesian coordinates, z doesn't really change and for the outer two limits, we are integrating over a unit circle so our answer is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{9-x^2-y^2}} dz\,dy\,dx.$$

(b) The solid is the intersection of the top half of a sphere of radius 3 centered at the origin and a cylinder of radius 1 with axis of symmetry the z-axis. In spherical coordinates this means that we have to split the integral into two pieces: one that corresponds to the spherical cap and one that corresponds to the straight sides. The "cone" of intersection is when $\varphi = \sin^{-1} 1/3$. For the integral that corresponds to the "straight sides", $0 \le r \le 1$. In spherical coordinates that is $0 \le \rho\sin\varphi \le 1$ or $0 \le \rho \le \csc\varphi$. The integrals are, therefore,

$$\int_0^{2\pi}\int_0^{\sin^{-1}1/3}\int_0^3 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + \int_0^{2\pi}\int_{\sin^{-1}1/3}^{\pi/2}\int_0^{\csc\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta.$$

6. (a) This solid is similar to that in Exercise 5. It is the intersection of a cylinder over the circle of radius 2 with center $(0,2)$ (i.e., $x^2 = 4y - y^2$) and the plane $x = 0$ with caps on either end that are portions of the sphere of radius 4 centered at the origin ($z = \pm\sqrt{16-x^2-y^2}$).

(b) In cylindrical coordinates, $-\sqrt{16-r^2} \le z \le \sqrt{16-r^2}$ and we are above the first quadrant so $0 \le \theta \le \pi/2$. Since $x^2+y^2=4y$, $r^2 = 4r\sin\theta$ so in the first quadrant, $r = 4\sin\theta$. The volume is therefore

$$\begin{aligned} V &= \int_0^{\pi/2}\int_0^{4\sin\theta}\int_{-\sqrt{16-r^2}}^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{4\sin\theta} 2r\sqrt{16-r^2}\,dr\,d\theta \\ &= \left(\frac{128}{3}\right)\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta = \frac{64}{9}(3\pi-4). \end{aligned}$$

Exercises 7 and 8 are a good lesson in the advantage of choosing the right coordinate system in which to work. This simple problem in Cartesian coordinates is a pain using either cylindrical or spherical coordinates.

7. Orient the cube so that a vertex is at the origin and the edges that meet at that vertex lie along the x-, y- and z-axes so that the cube is in the first octant. We'll double the volume of half of the cube. In this case $0 \le z \le a$, $0 \le \theta \le \pi/4$ and the only difficulty is with r. The radius varies from 0 to the line $x = a$. In cylindrical coordinates $x = r\cos\theta$ so $r = a\sec\theta$ and our limits for r are $0 \le r \le a\sec\theta$. The volume is

$$V = 2\int_0^a\int_0^{\pi/4}\int_0^{a\sec\theta} r\,dr\,d\theta\,dz = \int_0^a\int_0^{\pi/4} a^2\sec^2\theta\,d\theta\,dz = \int_0^a a^2\,dz = a^3.$$

The above calculation wouldn't change much if you followed the hint in the text and placed the center of the cube at the origin. In this case you would have 1/8 of the figure in the first octant and you would be calculating the volume of a cube with sides $a/2$.

8. We again orient the cube so that a vertex is at the origin and the edges that meet at that vertex lie along the x-, y- and z-axes so that the cube is in the first octant. As in Exercise 7 we will double the volume of half of the cube corresponding to $0 \le \theta \le \pi/4$. We will have to split the integral into two pieces: the piece in which ρ is bounded by the top of the cube ($z = a$ or $\rho = a\sec\varphi$) and the piece in which ρ is bounded by the side of the cube ($x = a$ or $\rho = a\ \csc\ \varphi\sec\theta$). The boundary value of φ depends on θ. Set $a = \rho\cos\varphi$ equal to $a = \rho\sin\varphi\cos\theta$ and solve to obtain $\varphi = \cot^{-1}\cos\theta$. So the volume is

$$\begin{aligned} V &= 2\int_0^{\pi/4}\int_0^{\cot^{-1}\cos\theta}\int_0^{a\ \sec\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta + 2\int_0^{\pi/4}\int_{\cot^{-1}\cos\theta}^{\pi/2}\int_0^{a\ \csc\ \varphi\sec\theta} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= 2\left(\frac{a^3}{6}+\frac{a^3}{3}\right) = a^3. \end{aligned}$$

Exercises 9–17 are examples where a change of variables helps. Exercise 14 depends on Exercise 11 and together they are much less difficult than they may first appear.

9. Here we will let $u = x - 2y$ and $v = x + y$. We calculate

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = 3 \quad\text{so}\quad \frac{\partial(x,y)}{\partial(u,v)} = 1/3.$$

The three boundary lines $x + y = 1, x = 0$, and $y = 0$ correspond to $v = 1, 2v = -u$, and $u = v$. We have all of the pieces to assemble our integral:

$$\begin{aligned} \iint_D \cos\left(\frac{x-2y}{x+y}\right) dA &= \int_0^1\int_{-2v}^{v}\frac{1}{3}\cos\left(\frac{u}{v}\right)du\,dv = \int_0^1 \frac{v}{3}\sin\left(\frac{u}{v}\right)\Big|_{-2v}^{v} du \\ &= \int_0^1 \frac{v}{3}(\sin 1 - \sin(-2))\,dv = \frac{v^2}{6}(\sin 1 + \sin 2)\Big|_0^1 = \frac{1}{6}(\sin 1 + \sin 2). \end{aligned}$$

10. Let $u = y^3$ and $v = x + 2y$. Then $0 \le u \le 216, 0 \le v \le 1$, and

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 0 & 3y^2 \\ 1 & 2 \end{vmatrix} = -3y^2 = -3u^{2/3} \quad\text{so}\quad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{3u^{2/3}}.$$

Then

$$\begin{aligned}\int_0^6\int_{-2y}^{1-2y} y^3(x+2y)^2 e^{(x+2y)^3}\,dx\,dy &= \int_0^{216}\int_0^1 \left(\frac{u}{3u^{2/3}}v^2e^{v^3}\right)dv\,du\\ &= \int_0^{216}\int_0^1\left(\frac{1}{3}u^{1/3}v^2e^{v^3}\right)dv\,du\\ &= \int_0^{216}\left(\frac{1}{9}u^{1/3}e^{v^3}\right)\Big|_{v=0}^{1}du\\ &= \int_0^{216}\left(\frac{1}{9}u^{1/3}(e-1)\right)du\\ &= \left(\frac{1}{9}\left(\frac{3}{4}u^{4/3}\right)(e-1)\right)\Big|_0^{216}\\ &= 108(e-1).\end{aligned}$$

11. (a) As we've seen before, we can write the integral as $\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} dy\,dx$.

(b) When we scale by letting $x = a\bar{x}$ and $y = b\bar{y}$, the ellipse is transformed into the unit circle E^* in the $\bar{x}\bar{y}$-plane. To rewrite the integral we also quickly calculate that $\partial(x,y)/\partial(\bar{x},\bar{y}) = ab$. The transformed integral is $\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}} ab\,d\bar{y}\,d\bar{x}$.

(c) Because we are integrating over a unit circle, we transform to polar coordinates (o.k., really we do it because the text tells us to):

$$\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}} ab\,d\bar{y}\,d\bar{x} = \int_0^{2\pi}\int_0^1 abr\,dr\,d\theta = \int_0^{2\pi}\frac{1}{2}ab\,d\theta = \frac{1}{2}ab(2\pi) = \pi\,ab.$$

12. (a) With $u = 2x - y$, $v = x + y$, we see $u + v = 3x$ so $x = \dfrac{u+v}{3}$ which implies $y = \dfrac{2v-u}{3}$. Substituting these expressions into the equation for the ellipse, we obtain

$$13\left(\frac{u+v}{3}\right)^2 + 14\left(\frac{u+v}{3}\right)\left(\frac{2v-u}{3}\right) + 10\left(\frac{2v-u}{3}\right)^2 = 9.$$

Expanding and simplifying, we find

$$\frac{u^2}{9} + v^2 = 1.$$

(b) Area $= \iint_E 1\,dA = \iint_E 1\,dx\,dy = \iint_{E^*}\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ where E^* denotes the corresponding ellipse in the *uv*-plane given above. Now $\dfrac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix}1/3 & 1/3\\ 2/3 & -1/3\end{bmatrix} = -\dfrac{1}{3}$ so

Area $= \iint_{E^*}\dfrac{1}{3}du\,dv = \dfrac{1}{3}(\text{area of } E^*) = \dfrac{1}{3}(\pi\cdot 3\cdot 1) = \pi$ using the result of part (c) of Exercise 11.

13. With $u = x - y$, $v = x + y$ we find that $x = \dfrac{u+v}{2}$, $y = \dfrac{v-u}{2}$. Substituting these expressions into the equation for E, we find

$$5\left(\frac{u+v}{2}\right)^2 + 6\left(\frac{u+v}{2}\right)\left(\frac{v-u}{2}\right) + 5\left(\frac{v-u}{2}\right)^2 = 4,$$

which simplifies to

$$\frac{u^2}{4} + v^2 = 1.$$

The area of ellipse E^* in the *uv*-plane is 2π. The area of the original ellipse E is

$$\begin{aligned}\iint_E 1\,dA = \iint_E dx\,dy = \iint_{E^*}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv &= \iint_{E^*}\left|\det\begin{bmatrix}1/2 & 1/2\\ -1/2 & 1/2\end{bmatrix}\right| du\,dv\\ &= \iint_{E^*}\left|\frac{1}{2}\right| du\,dv = \frac{1}{2}\text{ area of } E^* = \frac{1}{2}(2\pi) = \pi.\end{aligned}$$

14. We follow the steps in Exercise 11, inserting the same letters for ease in locating the corresponding parts.

(a) First we write the integral in Cartesian coordinates as

$$\int_{-a}^{a}\int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}}\int_{-c\sqrt{1-x^2/a^2-y^2/b^2}}^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz\,dy\,dx.$$

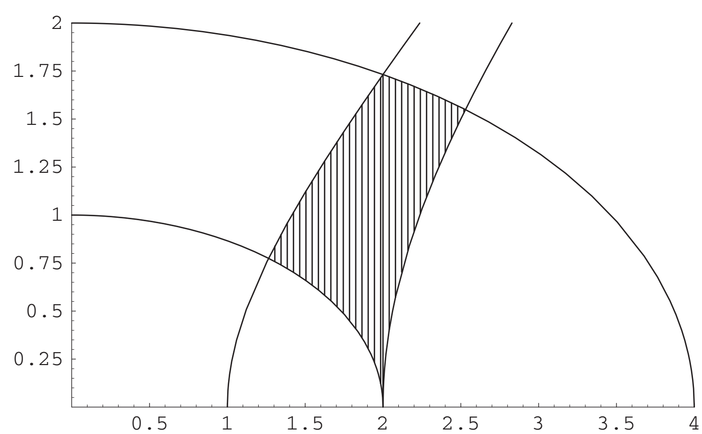
(b) We now scale the variables using $x = a\bar{x}, y = b\bar{y}$, and $z = c\bar{z}$. Note that the ellipsoid E is transformed into the unit sphere E^* and that $\partial(x,y,z)/\partial(\bar{x},\bar{y},\bar{z}) = abc$. The transformed integral is:

$$\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}}\int_{-\sqrt{1-\bar{x}^2-\bar{y}^2}}^{\sqrt{1-\bar{x}^2-\bar{y}^2}} abc\,d\bar{z}\,d\bar{y}\,d\bar{x}.$$

(c) Because we are integrating over a unit sphere, we will transform to spherical coordinates:

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-\bar{x}^2}}^{\sqrt{1-\bar{x}^2}}\int_{-\sqrt{1-\bar{x}^2-\bar{y}^2}}^{\sqrt{1-\bar{x}^2-\bar{y}^2}} abc\,d\bar{z}\,d\bar{y}\,d\bar{x} &= \int_0^{2\pi}\int_0^1\int_0^{\pi} abc\,\rho^2\sin\varphi\,d\varphi\,d\rho\,d\theta \\ &= \int_0^{2\pi}\int_0^1 \left(-\cos\varphi(abc)\rho^2\right)\Big|_0^{\pi} d\rho\,d\theta = \int_0^{2\pi}\int_0^1 (2abc\,\rho^2)\,d\rho\,d\theta \\ &= \int_0^{2\pi}\frac{2}{3}abc\,d\theta = \frac{4}{3}\pi abc.\end{aligned}$$

15. If you didn't first sketch the region you may be tempted to use the numerator and denominator of the integrand as your new variables. The diamond-like shape is bounded on two sides by the hyperbolas $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$ and on the other two sides by the ellipses $x^2/4 + y^2 = 1$ and $x^2/4 + y^2 = 4$.



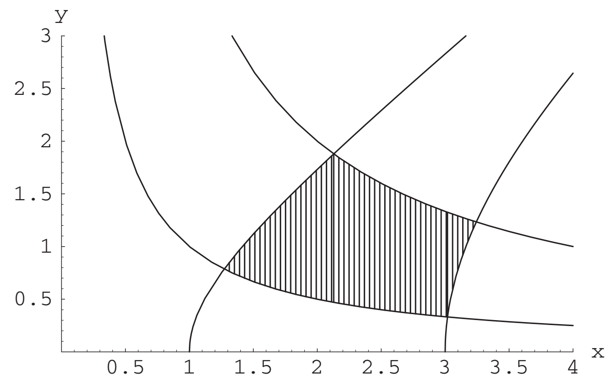
We, therefore, make the change of variables $u = x^2 - y^2$ and $v = x^2/4 + y^2$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ x/2 & 2y \end{vmatrix} = 5xy \quad \text{so} \quad \frac{\partial(x,y)}{\partial(u,v)} = 1/(5xy).$$

The integral greatly simplifies:

$$\begin{aligned}\iint_D \frac{xy}{y^2-x^2}\,dA &= \int_1^4\int_1^4 \left(\frac{xy}{-u}\frac{1}{5xy}\right) du\,dv = -\frac{1}{5}\int_1^4\int_1^4 \frac{1}{u}\,du\,dv \\ &= -\frac{1}{5}\int_1^4 \ln 4\,dv = -\frac{3}{5}\ln 4.\end{aligned}$$

16. The region D is bounded on the left and right by $x^2 - y^2 = 1$ and $x^2 - y^2 = 9$ and on the bottom and top by $xy = 1$ and $xy = 4$. It looks like



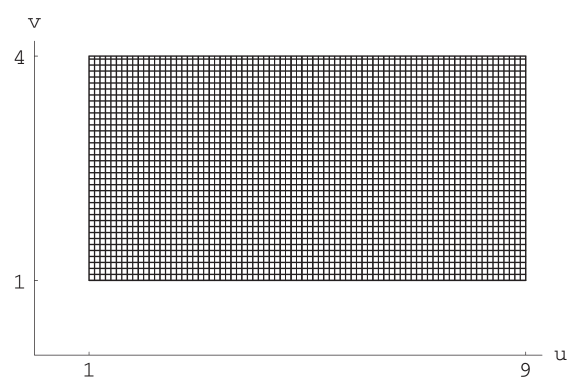
This suggests we try the change of variables $u = x^2 - y^2$, $v = xy$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} 2x & -2y \\ y & x \end{bmatrix} = 2(x^2 + y^2)$$

so that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2(x^2 + y^2)}.$$

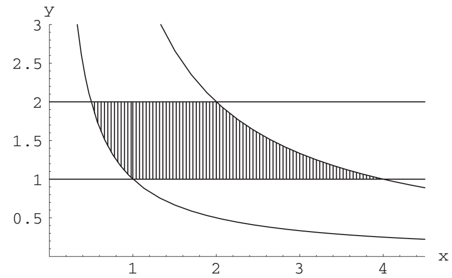
Moreover, the region D^* in the uv-plane corresponding to D is



Thus, using the change of variables theorem, we have

$$\iint_D (x^2 + y^2)e^{x^2 - y^2}\, dA = \iint_{D^*} \frac{1}{2} e^u \, du\, dv = \int_1^4 \int_1^9 \frac{1}{2} e^u \, du\, dv$$
$$= \int_1^4 \frac{1}{2}(e^9 - e)\, dV = \frac{3}{2}(e^9 - e).$$

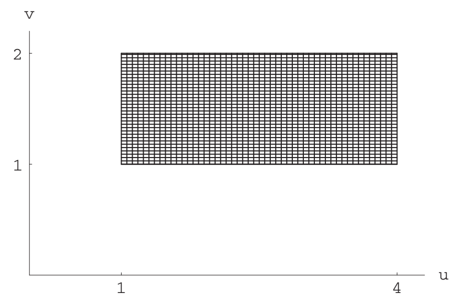
17. The region D is bounded on the bottom and top by $y = 1$ and $y = 2$ and on the left and right by $xy = 1$ and $xy = 4$; the region looks like the following figure.



With this in mind, we try the change of variables $u = xy, v = y$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} = y = v \quad \Longrightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{v}.$$

Moreover, the region D^* in the *uv*-plane corresponding to D is the rectangle $[1,4]\times[1,2]$:



The change of variables theorem tells us that

$$\begin{aligned}\iint_D \frac{1}{x^2y^2+1}\,dA &= \iint_{D^*} \frac{1}{u^2+1}\cdot\frac{1}{v}\,du\,dv = \int_1^4\int_1^2 \frac{1}{u^2+1}\cdot\frac{1}{v}\,dv\,du \\ &= \int_1^4 \frac{\ln 2}{u^2+1}\,du = \ln 2\left(\tan^{-1} u\Big|_1^4\right) \\ &= \ln 2\left(\tan^{-1} 4 - \frac{\pi}{4}\right).\end{aligned}$$

18. (a) Follow the same steps as in defining the double and triple integrals.

- Define a **partition** of $B = [a,b]\times[c,d]\times[p,q]\times[r,s]$ of order n to be four collections of partition points that break up B into a union of n^4 subboxes. See Definition 4.1 and add that $r = w_0 < w_1 < \cdots < w_n = s$ and $\Delta w_l = w_l - w_{l-1}$.
- Define a Riemann sum. For a function f defined on B, partition B as above and let $\mathbf{c}_{ijkl}$ be any point in the subbox

$$B_{ijkl} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]\times[z_{k-1}, z_k]\times[w_{l-1}, w_l].$$

- The Riemann sum of f on B corresponding to the partition is

$$S = \sum_{i,j,k,l=1}^{n} f(\mathbf{c}_{ijkl})\Delta x_i \Delta y_j \Delta z_k \Delta w_l = \sum_{i,j,k,l=1}^{n} f(\mathbf{c}_{ijkl})\Delta V_{ijkl}.$$

- Define the **quadruple integral** of f on B, written

$$\iiiint_B f(x,y,z,w)\,dV = \iiiint_B f(x,y,z,w)\,dx\,dy\,dz\,dw$$

to be

$$\iiiint_B f(x,y,z,w)\,dV = \lim_{\text{all }\Delta x_i,\Delta y_j,\Delta z_k,\Delta w_l\to 0} \sum_{i,j,k,l=1}^{n} f(\mathbf{c}_{ijkl})\Delta x_i \Delta y_j \Delta z_k \Delta w_l.$$

- We extend the definition to compact non-box regions W by defining the function f^{ext} which is f everywhere in W and is 0 everywhere else. Then if B is a box containing W we can define

$$\iiiint_W f dV = \iiiint_B f^{ext} dV.$$

- As in the cases of the double and triple integrals, Fubini's theorem allows us to evaluate the integral as an iterated integral.

(b) We calculate:

$$\begin{aligned}\iiiint_W (x+2y+3z-4w)\,dV &= \int_0^2\int_{-1}^3\int_0^4\int_{-2}^2 (x+2y+3z-4w)\,dw\,dz\,dy\,dx\\ &= 4\int_0^2\int_{-1}^3\int_0^4 (x+2y+3z)\,dz\,dy\,dx\\ &= 4\int_0^2\int_{-1}^3 (4x+8y+24)\,dy\,dx\\ &= 4\int_0^2 (16x+32+96)\,dx = 64\int_0^2 (x+8)\,dx\\ &= 64(2+16) = 1152.\end{aligned}$$

19. (a) We are just generalizing what we've done to set up the area of a circle or the volume of a sphere (more recently see Exercises 11 and 14 from this section). Here our integral is:

$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\int_{-\sqrt{a^2-x^2-y^2-z^2}}^{\sqrt{a^2-x^2-y^2-z^2}} dw\,dz\,dy\,dx.$$

(b) You should get $\pi^2a^2/4$.

(c) For $n=5$ you should get $8\pi^2a^5/15$, and for $n=6$ you should get $\pi^3a^6/6$. If you include the cases for $n=2$ and $n=3$ you may begin to see a pattern for the even exponents. If n is even, the volume of the n-sphere of radius a is $\pi^{n/2}a^n/(n/2)!$. Fitting in the odd terms looks really hard and the pattern shouldn't occur to any of your students. In fact, the general formula depends on the Gamma function which is beyond what we would expect the students to know at this point. For kicks, the volume of the n-sphere of radius a is

$$\frac{\pi^{n/2}a^n}{\Gamma((n/2)+1)}.$$

Note that the volume of an n-sphere of radius a decreases to 0 as n increases.

20. Let $x_1 = a\bar{x}_1,\ x_2 = a\bar{x}_2, \ldots, x_n = a\bar{x}_n$. Then, by substitution,

$$\begin{aligned}B &= \{(x_1,\ldots,x_n) \mid x_1^2+\cdots+x_n^2 \le a^2\}\\ &= \{(\bar{x}_1,\ldots,\bar{x}_n) \mid (a\bar{x}_1)^2+\cdots+(a\bar{x}_n)^2 \le a^2\}\\ &= \{(\bar{x}_1,\ldots,\bar{x}_n) \mid \bar{x}_1^2+\cdots+\bar{x}_n^2 \le 1\},\end{aligned}$$

which is the *unit* ball in $(\bar{x}_1,\ldots,\bar{x}_n)$-coordinates. The Jacobian of this change of variables is

$$\frac{\partial(x_1,\ldots,x_n)}{\partial(\bar{x}_1,\ldots,\bar{x}_n)} = \det\begin{bmatrix} a & 0 & \cdots & 0\\ 0 & a & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a\end{bmatrix} = a^n.$$

Hence

$$V_n(a) = \int\cdots\int_B 1\,dx_1\cdots dx_n = \int\cdots\int_U a^n\,d\bar{x}_1\cdots d\bar{x}_n = C_na^n.$$

21. **(a)** Since a point $(x_1, \dots, x_n)$ in B satisfies the inequality $x_1^2+\cdots+x_n^2 \le a^2$, a point in B of the form $(x_1, x_2, 0, \dots, 0)$ must have $x_1^2 + x_2^2 + 0 \cdots + 0 \le a^2$. Thus (x_1, x_2), considered as a point in $\mathbf{R}^2$, satisfies $x_1^2 + x_2^2 \le a^2$, so that (x_1, x_2) lies in the disk of radius a in $\mathbf{R}^2$.

(b) The point $(x_1, x_2, 0, \dots, 0)$ in B described in part (a) has coordinates that relate to polar coordinates (r, θ) by $x_1^2 + x_2^2 = r^2 \le a^2$. Hence any point $(r, \theta, x_3, \dots, x_n)$ in B lying over the specific point (r, θ) in the disk must satisfy $r^2 + x_3^2 + \cdots + x_n^2 \le a^2 \iff x_3^2 + \cdots + x_n^2 \le a^2 - r^2$. Hence the coordinates $(x_3, \dots, x_n)$ fill out an $(n-2)$-dimensional ball of radius $\sqrt{a^2 - r^2}$.

(c) If D denotes the radius a disk centered at the origin, then, from part (b), we have

$$\begin{aligned} V_n(a) &= \int \cdots \int_B dx_1 \cdots dx_n = \iint_D V_{n-2}\left(\sqrt{a^2 - r^2}\right) dx_1\, dx_2 \\ &= \int_0^{2\pi} \int_0^a V_{n-2}\left(\sqrt{a^2 - r^2}\right) r\, dr\, d\theta. \end{aligned}$$

22. By the previous exercise, we have

$$\begin{aligned} V_n(a) &= \int_0^{2\pi} \int_0^a V_{n-2}\left(\sqrt{a^2 - r^2}\right) r\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^a C_{n-2}\left(a^2 - r^2\right)^{(n-2)/2} r\, dr\, d\theta \qquad \text{from Exercise 20,} \\ &= C_{n-2} \int_0^{2\pi} \int_{a^2}^0 u^{(n-2)/2} \left(-\tfrac{1}{2}\, du\right) d\theta \\ &= \frac{C_{n-2}}{2} \int_0^{2\pi} \frac{2}{n} u^{n/2} \Big|_{u=0}^{a^2} d\theta \\ &= \frac{2\pi}{n} C_{n-2} a^n. \end{aligned}$$

Now $V_{n-2}(a) = C_{n-2}a^{n-2}$, so we have

$$V_n(a) = \left(\frac{2\pi}{n} a^2\right)\left(C_{n-2}a^{n-2}\right) = \left(\frac{2\pi}{n} a^2\right) V_{n-2}(a).$$

23. **(a)** The one-dimensional ball of radius a consists of points in $\mathbf{R}$ described as

$$\{x_1 \in \mathbf{R} \mid x_1^2 \le a^2\} = \{x_1 \in \mathbf{R} \mid -a \le x_1 \le a\} = [-a, a].$$

The one-dimensional volume of this "ball" is the length of the interval; thus $V_1(a) = 2a$. The two-dimensional ball of radius a consists of points $(x_1, x_2) \in \mathbf{R}^2$ such that $x_1^2 + x_2^2 \le a^2$. Such points form a disk of radius a, so the two-dimensional volume of this disk is its area; hence $V_2(a) = \pi a^2$.

(b) By repeatedly using the recursive formula in Exercise 22, we have

$$V_n(a) = \begin{cases} \left(\dfrac{2\pi}{n}\, a^2\right)\left(\dfrac{2\pi}{n-2}\, a^2\right) \cdots \left(\dfrac{2\pi}{4}\, a^2\right) V_2(a) & \text{if } n \text{ is even,} \\[2ex] \left(\dfrac{2\pi}{n}\, a^2\right)\left(\dfrac{2\pi}{n-2}\, a^2\right) \cdots \left(\dfrac{2\pi}{1}\, a^2\right) V_1(a) & \text{if } n \text{ is odd.} \end{cases}$$

In the expressions for $V_n(a)$ above, there are $\frac{n}{2} - 1$ factors appearing before $V_2(a)$ when n is even and $\frac{n-1}{2}$ factors appearing before $V_1(a)$ when n is odd. Hence, using the results of part (a), we have

$$V_n(a) = \begin{cases} \dfrac{2^{(n/2)-1}\pi^{(n/2)-1}\left(a^2\right)^{(n/2)-1}\pi a^2}{n(n-2)\cdots 4} & \text{if } n \text{ is even} \\ \dfrac{2^{(n-1)/2}\pi^{(n-1)/2}\left(a^2\right)^{(n-1)/2} 2a}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{2^{n/2}\pi^{n/2}\left(a^2\right)^{n/2}}{n(n-2)\cdots 4\cdot 2} & \text{if } n \text{ is even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}\left(a^2\right)^{(n-1)/2} a}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{2^{n/2}\pi^{n/2}a^n}{2(n/2)\cdot 2\left((n/2)-1\right)\cdot 2\left((n/2)-2\right)\cdots(2\cdot 1)} & \text{if } n \text{ is even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}a^n}{n!!} & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \dfrac{\pi^{n/2}a^n}{(n/2)!} & \text{if } n \text{ is even} \\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}a^n}{n!!} & \text{if } n \text{ is odd.} \end{cases}$$

24. (a) To obtain the mass we compute the following integral (which is straightforward so the details are omitted):

$$M = \int_0^{2\pi}\int_0^{\pi}\int_3^4 ((.12\rho^2)\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta = 74.976\pi \approx 235.5440508g.$$

(b) Because the shell is sealed, the volume is $V = (4/3)\pi(4^3) = 256\pi/3$ cm^3, so the mass of that volume of water is greater, and so the shell would float.

(c) If the core of the shell fills with water, then the volume that the shell has to displace is $V = (4/3)\pi(4^3 - 3^3) = (4/3)\pi(37)$. The water for that volume would have mass of about 155 grams so the shell would sink.

25. When you average the height of the hemisphere of radius a, first you integrate

$$\int_0^{2\pi}\int_0^a zr\, dr\, d\theta = \int_0^{2\pi}\int_0^a r\sqrt{a^2 - r^2}\, dr\, d\theta = \frac{2}{3}\pi a^3.$$

For the average height, we divide this by the area of the region over which we are integrating:

$$\frac{(2/3)\pi a^3}{\pi a^2} = \frac{2}{3}a.$$

We now solve to see which values of r correspond to this height: $(2/3)a = \sqrt{a^2 - r^2}$ when $(4/9)a^2 = a^2 - r^2$, which is when $r = \sqrt{5}a/3$. Therefore, the pole can be installed at most $\sqrt{5}a/3$ from the center of the floor of the dome.

26. (a) By the fundamental theorem of calculus

$$\frac{d}{dy}\int_c^y G(y')\, dy' = G(y) \quad \text{so} \quad \frac{d}{dy}\int_c^y\int_a^b f_y(x, y')\, dx\, dy' = \int_a^b f_y(x, y)\, dx.$$

(b) On the other hand, by Fubini's theorem,

$$\begin{aligned} \frac{d}{dy}\int_c^y\int_a^b f_y(x, y')\, dx\, dy' &= \frac{d}{dy}\int_a^b\int_c^y f_y(x, y')\, dy' dx \\ &= \frac{d}{dy}\int_a^b (f(x, y) - f(x, c))\, dx = \frac{d}{dy}\int_a^b f(x, y)\, dx. \end{aligned}$$

Combine parts (a) and (b) to obtain the desired results.

27. (a)

$$I(\epsilon,\delta)=\int_\epsilon^{1-\epsilon}\int_\delta^{1-\delta}\frac{1}{\sqrt{xy}}\,dy\,dx=\int_\epsilon^{1-\epsilon}\frac{2}{\sqrt{x}}(\sqrt{1-\delta}-\sqrt{\delta})\,dx$$
$$=4(\sqrt{1-\epsilon}-\sqrt{\epsilon})(\sqrt{1-\delta}-\sqrt{\delta})$$

(b) $\lim_{(\epsilon,\delta)\to(0,0)} I(\epsilon,\delta)=4\cdot 1\cdot 1=4$

28. For $0<\epsilon<\frac{1}{2}, 0<\delta<\frac{1}{2}$ we consider $I(\epsilon,\delta)=\iint_{D_{\epsilon,\delta}}\frac{1}{x+y}dA$ where $D_{\epsilon,\delta}=[\epsilon,1-\epsilon]\times[\delta,1-\delta]$. Then

$$I(\epsilon,\delta)=\int_\epsilon^{1-\epsilon}\int_\delta^{1-\delta}\frac{1}{x+y}\,dy\,dx=\int_\epsilon^{1-\epsilon}\ln(x+y)\Big|_{y=\delta}^{1-\delta}dx$$
$$=\int_\epsilon^{1-\epsilon}(\ln(x+1-\delta)-\ln(x+\delta))\,dx.$$

Using integration by parts, we find that $\int\ln u\,du=u\ln u-u+C$ so that

$$\begin{aligned}I(\epsilon,\delta)&=[(x+1-\delta)\ln(x+1-\delta)-(x+1-\delta)-(x+\delta)\ln(x+\delta)+(x+\delta)|_{x=\epsilon}^{1-\epsilon}\\&=(2-\epsilon-\delta)\ln(2-\epsilon-\delta)-(2-\epsilon-\delta)-(1-\epsilon+\delta)\ln(1-\epsilon+\delta)\\&\quad+(1-\epsilon+\delta)-(\epsilon+1-\delta)\ln(\epsilon+1-\delta)+(\epsilon+1-\delta)\\&\quad+(\epsilon+\delta)\ln(\epsilon+\delta)-(\epsilon+\delta).\end{aligned}$$

To evaluate $\lim_{(\epsilon,\delta)\to(0^+,0^+)} I(\epsilon,\delta)$ we first note that, by l'Hôpital's rule,

$$\lim_{u\to 0^+}u\ln u=\lim_{u\to 0^+}\frac{\ln u}{1/u}=\lim_{u\to 0^+}\frac{1/u}{-1/u^2}=-\lim_{u\to 0^+}u=0.$$

Thus $(\epsilon+\delta)\ln(\epsilon+\delta)\to 0$ as $(\epsilon,\delta)\to(0^+,0^+)$. The other terms in the expression have evident limits so that

$$\lim_{(\epsilon,\delta)\to(0^+,0^+)} I(\epsilon,\delta)=2\ln 2-2-\ln 1+1-\ln 1+1+0-0=2\ln 2.$$

29. For $0<\epsilon<\frac{1}{2}, 0<\delta<\frac{1}{2}$, let $D_{\epsilon,\delta}=[\epsilon,1-\epsilon]\times[\delta,1-\delta]$ and consider

$$I(\epsilon,\delta)=\iint_{D_{\epsilon,\delta}}\frac{x}{y}\,dA=\int_\delta^{1-\delta}\int_\epsilon^{1-\epsilon}\frac{x}{y}\,dx\,dy$$
$$=\int_\delta^{1-\delta}\frac{\left(\frac{1}{2}-\epsilon\right)}{y}\,dy=\left(\frac{1}{2}-\epsilon\right)(\ln(1-\delta)-\ln\delta).$$

Note that $\lim_{(\epsilon,\delta)\to(0,0)} I(\epsilon,\delta)=-\infty$ since $\frac{1}{2}-\epsilon$ and $\ln(1-\delta)$ remain finite, but $\ln\delta\to-\infty$. Thus the improper integral does not converge.

In Exercises 30 and 31 the students will need integration by parts and l'Hôpital's rule.

30.

$$\iint_D\ln\sqrt{x^2+y^2}\,dA=\lim_{\epsilon\to 0}\iint_{D_\epsilon}\ln\sqrt{x^2+y^2}\,dA=\lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1 r\ln r\,dr\,d\theta$$
$$=\lim_{\epsilon\to 0}\int_0^{2\pi}\left[\frac{r^2}{2}\ln r-\frac{r^2}{4}\right]\Big|_\epsilon^1 d\theta=\lim_{\epsilon\to 0}\int_0^{2\pi}\left[-\frac{1}{4}-\frac{\epsilon^2}{2}\ln\epsilon+\frac{\epsilon^2}{4}\right]d\theta$$
$$=\lim_{\epsilon\to 0}2\pi\left[-\frac{1}{4}-\frac{\epsilon^2}{2}\ln\epsilon+\frac{\epsilon^2}{4}\right]=-\pi/2.$$

31. Define $B_\epsilon = \{(x, y, z) | \epsilon \le x^2 + y^2 + z^2 \le 1\}$. Then

$$\begin{aligned}\iiint_B \ln\sqrt{x^2+y^2+z^2}\,dV &= \lim_{\epsilon\to 0}\iiint_{B_\epsilon} \ln\sqrt{x^2+y^2+z^2}\,dV \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1\int_0^\pi ((\ln\rho)\rho^2\sin\varphi)\,d\varphi\,d\rho\,d\theta \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi}\int_\epsilon^1 ((2\rho^2\ln\rho))\,d\rho\,d\theta \\ &= \lim_{\epsilon\to 0}\int_0^{2\pi} ((2/3)\rho^3\ln\rho - 2\rho^3/9)\Big|_\epsilon^1 d\theta \\ &= \lim_{\epsilon\to 0} 2\pi\left(-\frac{2}{9} - \frac{2\epsilon^3}{3}\ln\epsilon + \frac{2\epsilon^3}{9}\right) = -4\pi/9.\end{aligned}$$

32. (a)

$$\begin{aligned}I(a,b) &= \int_1^a\int_1^b \frac{1}{x^2y^3}\,dy\,dx = \int_1^a -\frac{1}{2y^2x^2}\Big|_{y=1}^b dx \\ &= \int_1^a \left(\frac{1}{2} - \frac{1}{2b^2}\right)\frac{1}{x^2}dx = \left(\frac{1}{2} - \frac{1}{2b^2}\right)\left(1 - \frac{1}{a}\right)\end{aligned}$$

(b) As $a, b \to \infty$, $I(a, b) \to \frac{1}{2}$.

33. Let $D_{a,b} = [1, a] \times [1, b]$ and consider, for $p, q \ne 1$,

$$\begin{aligned}I(a,b) &= \iint_{D_{a,b}} \frac{1}{x^py^q}\,dA = \int_1^b\int_1^a \frac{1}{x^py^q}\,dx\,dy \\ &= \int_1^b \frac{1}{(1-p)y^qx^{p-1}}\Big|_{x=1}^a dy = \frac{1}{1-p}\left(\frac{1}{a^{p-1}} - 1\right)\int_1^b \frac{1}{y^q}\,dy \\ &= \frac{1}{(1-p)(1-q)}\left(\frac{1}{a^{p-1}} - 1\right)\left(\frac{1}{b^{q-1}} - 1\right).\end{aligned}$$

If $p > 1, q > 1$ then as $a, b \to \infty$, $I(a, b) \to \frac{1}{(1-p)(1-q)} = \frac{1}{(p-1)(q-1)}$, so the integral converges in this case. If $p < 1$, then $1/a^{p-1} \to \infty$. Similarly if $q < 1$, $1/b^{p-1} \to \infty$.
If $p = 1, q \ne 1$, then $\int_1^b\int_1^a \frac{1}{xy^q}dx\,dy = \ln a\left(\frac{1}{1-q}\right)\left(\frac{1}{b^{q-1}} - 1\right)$. This becomes infinite as $a, b \to \infty$. Similarly, if $q = 1, p \ne 1$, $I(a, b)$ becomes infinite as $a, b \to \infty$. If $p = q = 1$, then $I(a, b) = \ln a \ln b \to \infty$ as $a, b \to \infty$.
To summarize: the integral converges if and only if $p > 1$ and $q > 1$—in which case the value of the integral is $1/(p - 1)(q - 1)$.

34. (a) We use polar coordinates to make the evaluation. Let

$$\begin{aligned}I(a) &= \iint_{D_a} (1+x^2+y^2)^{-2}\,dA = \int_0^{2\pi}\int_0^a (1+r^2)^{-2}r\,dr\,d\theta \\ &= 2\pi\left(\frac{1}{2}\right)(-(1+r^2)^{-1})\Big|_{r=0}^a = -\pi\left(\frac{1}{1+a^2} - 1\right) \\ &= \pi\left(1 - \frac{1}{1+a^2}\right).\end{aligned}$$

$\lim_{a\to\infty} I(a) = \pi$. Thus the integral converges.

(b) Let $I(a) = \iint_{D_a} (1+x^2+y^2)^p dA = \int_0^{2\pi}\int_0^a (1+r^2)^p r\,dr\,d\theta$. If $p \ne -1$, then

$$I(a) = \frac{\pi}{p+1}((1+a^2)^{p+1} - 1).$$

Now $\lim_{a\to\infty}(1+a^2)^{p+1}$ is finite (and equals 0) just in case $p+1<0$ or $p<-1$. In such case, the integral converges and its value is $-\dfrac{\pi}{p+1}$. If $p=-1$, then $I(-1)=\displaystyle\int_0^{2\pi}\int_0^a \frac{r}{1+r^2}dr\,d\theta = \pi\ln(1+a^2)\to\infty$ as $a\to\infty$. So the integral converges if and only if $p<-1$.

35. Consider

$$I(a)=\iiint_{B_a}\frac{1}{(1+x^2+y^2+z^2)^{3/2}}\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^a\frac{\rho^2\sin\varphi}{(1+\rho^2)^{3/2}}\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^a\int_0^{2\pi}\int_0^{\pi}\frac{\rho^2}{(1+\rho^2)^{3/2}}\sin\varphi\,d\varphi\,d\theta\,d\rho=\int_0^a 4\pi\cdot\frac{\rho^2}{(1+\rho^2)^{3/2}}\,d\rho$$

Now let $\rho=\tan u$ so $d\rho=\sec^2 u\,du$. Then

$$I(a)=4\pi\int_0^{\tan^{-1}a}\frac{\tan^2 u\cdot\sec^2 u\,du}{\sec^3 u}=4\pi\int_0^{\tan^{-1}a}\frac{\sin^2 u}{\cos u}du$$
$$=4\pi\int_0^{\tan^{-1}a}\frac{1-\cos^2 u}{\cos u}\,du=4\pi\int_0^{\tan^{-1}a}(\sec u-\cos u)\,du$$
$$=4\pi(\ln|\sec u+\tan u|-\sin u)\Big|_0^{\tan^{-1}a}=4\pi\left(\ln(\sqrt{a^2+1}+a)-\frac{a}{a^2+1}\right).$$

Since $\lim\limits_{a\to\infty} I(a)=\infty$ the integral does not converge.

36. Consider

$$I(a)=\iiint_{B_a}e^{-\sqrt{x^2+y^2+z^2}}\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^a e^{-\rho}\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$
$$=\int_0^a\int_0^{2\pi}\int_0^{\pi}e^{-\rho}\rho^2\sin\varphi\,d\varphi\,d\theta\,d\rho=4\pi\int_0^a\rho^2e^{-\rho}\,d\rho$$

Now use integration by parts twice: First let $u=\rho^2$ and $dv=e^{-\rho}d\rho$. Then

$$I(a)=4\pi\left(-\rho^2e^{-\rho}\Big|_0^a+2\int_0^a\rho e^{-\rho}d\rho\right)=-4\pi a^2e^{-a}+8\pi\int_0^a\rho e^{-\rho}d\rho.$$

Now let $u=\rho$ and $dv=e^{-\rho}d\rho$ so that

$$I(a)=-4\pi a^2e^{-a}+8\pi\left(-\rho e^{-\rho}|_0^a+\int_0^a e^{-\rho}\,d\rho\right)$$
$$=-4\pi a^2e^{-a}-8\pi ae^{-a}-8\pi e^{-a}+8\pi.$$

$\lim\limits_{a\to\infty} I(a)=8\pi$, so the integral converges and has value 8π.

The importance of Exercise 37 can not be overemphasized. The students have come from a course where they learned one technique of integration after another. They also learned some numerical methods (at least a brief introduction to Riemann sums, the trapezoid rule and Simpson's rule). In a way Exercise 37 is the payoff—it is a chance to mention:

- *Until now they couldn't calculate $\int_{-\infty}^{\infty}e^{-x^2}dx$. The fact that you need the tools of multivariable calculus (or complex analysis) is pretty cool.*
- *They still can't calculate $\int_a^b e^{-x^2}dx$. There is a need for numerical methods to calculate a function as common as the bell curve (with a constant that stretches in the vertical direction and another constant that stretches in the horizontal direction, this is the normal curve). Many will encounter this function in a course on statistics and use the tables; they should know that this is because we can't find the definite integral over a general finite interval.*
- *The technique is pretty and unexpected and is one of the tricks that they should see some time in their mathematical training. The problem is surprisingly straightforward once someone shows you the trick.*

37. (a) $\int_{-1}^{1} e^{-x^2}\,dx$ is finite since e^{-x^2} is bounded on $[-1,1]$. Since $0 \le e^{-x^2} \le 1/x^2$ on both $[1,\infty)$ and $(-\infty,-1]$ and the improper integrals $\int_{1}^{\infty}(1/x^2)\,dx$ and $\int_{-\infty}^{-1}(1/x^2)\,dx$ converge, we see that $\int_{1}^{\infty} e^{-x^2}dx$ and $\int_{-\infty}^{-1} e^{-x^2}dx$ both converge. Hence $\int_{-\infty}^{\infty} e^{-x^2}dx = \int_{-\infty}^{-1} e^{-x^2}dx + \int_{-1}^{1} e^{-x^2}dx + \int_{1}^{\infty} e^{-x^2}dx$ converges.

(b) We have

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy = \iint_{\mathbf{R}^2} e^{-x^2-y^2}\,dA.$$

(c) We'll use polar coordinates.

$$\iint_{D_a} e^{-x^2-y^2}\,dA = \int_0^{2\pi}\int_0^{a} e^{-r^2}\cdot r\,dr\,d\theta = -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)\,d\theta$$
$$= \pi(1-e^{-a^2})$$

(d) Note that, as $a \to \infty$, the disk D_a fills out more and more of $\mathbf{R}^2$. Thus $\lim_{a\to\infty}\iint_{D_a} e^{-x^2-y^2}dA = \iint_{\mathbf{R}^2} e^{-x^2-y^2}dA = I^2$.

(e) Putting everything together:

$$I^2 = \lim_{a\to\infty}\pi(1-e^{-a^2}) = \pi.$$

Thus $I = \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.

38. (a) First, clearly $f(x) \ge 0$. And second, by symmetry,

$$\int_{-\infty}^{\infty} e^{-2|x|}\,dx = 2\int_0^{\infty} e^{-2x}\,dx = -e^{-2x}\Big|_0^{\infty} = 1.$$

(b) We will reduce the calculations in Egbert's problem by recentering.

$$P(250 \le x \le 350) = \int_{250}^{350}\frac{1}{2}e^{-|x-300|}\,dx = 2\int_{300}^{350}\frac{1}{2}e^{-(x-300)}\,dx$$
$$= \int_0^{50} e^{-x}\,dx = -e^{-x}\Big|_0^{50} = 1-e^{-50}.$$

39. (a) Since $\dfrac{2x+y}{140} \ge 0$ on $[0,5]\times[0,4]$, $f(x,y)\ge 0$ for all (x,y). Now

$$\iint_{\mathbf{R}^2} f(x,y) = \int_0^4\int_0^5 \frac{2x+y}{140}\,dx\,dy = \frac{1}{140}\int_0^4(25+5y)\,dy = \frac{1}{140}\left(25y+\frac{5}{2}y^2\right)\Big|_0^4$$
$$= \frac{1}{140}(100+40) = 1.$$

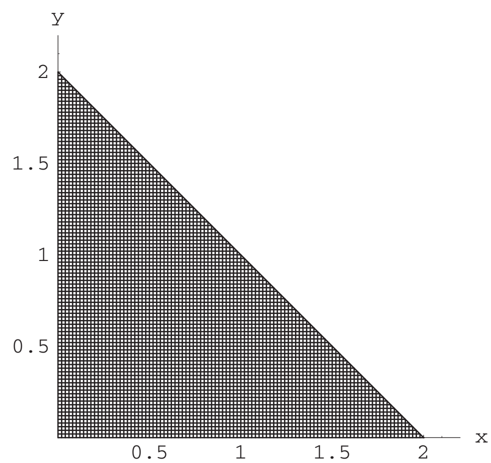
(b) Since $f(x,y)=0$ for $x<0$ or $y<0$,

$$\text{Prob}(x\le 1, y\le 1) = \text{Prob}((x,y)\in[0,1]\times[0,1]) = \int_0^1\int_0^1\frac{2x+y}{140}\,dx\,dy$$
$$= \frac{1}{140}\int_0^1(1+y)\,dy = \frac{1}{140}\left(1+\frac{1}{2}\right) = \frac{3}{280} \approx 0.0107.$$

40. **(a)** Since $ye^{-x-y} \geq 0$ for $y \geq 0$ (note the exponential term is strictly positive), we have that $f(x, y) \geq 0$ for all (x, y). Now we check

$$\iint_{\mathbf{R}^2} ye^{-x-y}\, dA = \int_0^\infty \int_0^\infty ye^{-y}e^{-x}\, dx\, dy$$
$$= \int_0^\infty ye^{-y}\, dy$$
$$= (-ye^{-y} - e^{-y})\Big|_0^\infty = 1.$$

(b) Prob$(x + y \leq 2) = $ Prob$((x, y) \in D)$ where D is the triangular region bounded by $x = 0, y = 0$ and $x + y = 2$.



Thus the desired probability is

$$\int_0^2 \int_0^{2-x} ye^{-y}e^{-x}\, dy\, dx = \int_0^2 (-ye^{-y} - e^{-y})|_0^{2-x} e^{-x}\, dx$$
$$= \int_0^2 ((x-2)e^{x-2} - e^{x-2} + 1)e^{-x}\, dx = \int_0^2 ((x-2)e^{-2} - e^{-2} + e^{-x})\, dx$$
$$= 1 - 5e^{-2}.$$

41. First, we know that $C \geq 0$. Second,

$$1 = \int_{-\infty}^\infty \int_{-\infty}^\infty Ce^{-a|x|-b|y|}\, dx\, dy = 4\int_0^\infty \int_0^\infty Ce^{-ax-by}\, dx\, dy$$
$$= 4C\left[\int_0^\infty e^{-ax}\, dx\right]\left[\int_0^\infty e^{-by}\, dy\right] = 4C\left[-\frac{1}{a}e^{-ax}\Big|_0^\infty\right]\left[-\frac{1}{b}e^{-by}\Big|_0^\infty\right] = \frac{4C}{ab}.$$

So $C = ab/4$.

42. Note that if $C \geq 0$, then $f(x, y) \geq 0$ for all x since a and b are nonnegative. Thus we calculate

$$\iint_{\mathbf{R}^2} f(x, y)\, dA = \int_0^1 \int_0^1 C(ax + by)\, dx\, dy = C\int_0^1 \left(\frac{1}{2}a + by\right)\, dy$$
$$= C\left(\frac{1}{2}a + \frac{1}{2}b\right) = C\left(\frac{a+b}{2}\right).$$

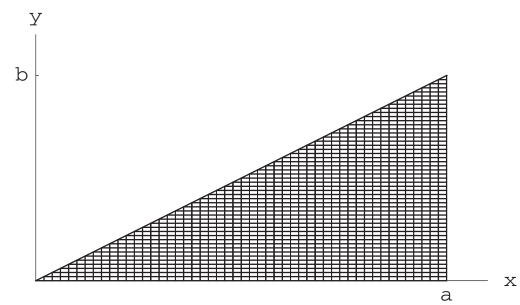
For this to equal 1, we must have $C = \dfrac{2}{a+b}$.

43. (a) If $C \geq 0$, then $f(x, y) \geq 0$ for all (x, y). Thus we calculate

$$\iint_{\mathbf{R}^2} f(x,y)\,dA = \int_0^b \int_0^a C\,xy\,dx\,dy = C\int_0^b \frac{a^2}{2} y\,dy$$
$$= C\frac{a^2b^2}{4}.$$

For this to equal 1 we must have $C = \dfrac{4}{a^2b^2}$.

(b) Prob$(bx - ay \geq 0)$ is the probability that a point (x, y) falls *below* the line $y = \dfrac{b}{a}x$. Since f is zero outside the rectangle $[0, a] \times [0, b]$, we see that the desired probability is the *same* as the probability Prob$((x, y) \in D)$ where D is the triangular region shown below.



This last probability is calculated as

$$\iint_D f dA = \int_0^a \int_0^{(b/a)x} \frac{4}{a^2b^2} xy\,dy\,dx$$
$$= \int_0^a \frac{4}{a^2b^2} x \cdot \frac{1}{2}\left(\frac{b}{a}x\right)^2 dx = \int_0^a \frac{2}{a^4}x^3\,dx$$
$$= \frac{1}{2a^4}x^4\Big|_0^a = \frac{1}{2}.$$

44. We are integrating over the triangle where $0 \leq x + y \leq 60$. The integral is fairly straightforward so the details are omitted:

$$\frac{1}{250}\int_0^{60}\int_0^{60-x} e^{-x/50}e^{-y/5}\,dy\,dx = -\frac{1}{50}\int_0^{60}(e^{-x/50}[e^{-12+x/5} - 1])\,dx$$
$$= 1 - \frac{10}{9}e^{-6/5} + \frac{1}{9}e^{-12} \approx .665340.$$

45. We use polar coordinates:

$$\int_{-1/2}^{1/2}\int_{-\sqrt{(1/4)-x^2}}^{\sqrt{(1/4)-x^2}} \frac{1}{\pi}e^{-x^2-y^2}\,dy\,dx = \int_0^{2\pi}\int_0^{1/2}\left(\frac{1}{\pi}re^{-r^2}\right)dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{1}{\pi}\left(\frac{1}{2}\right)(1 - e^{-1/4})\right)d\theta$$
$$= 1 - e^{-1/4} \approx .22199.$$

46. The joint density function of the components is

$$F(x,y) = f(x)\cdot f(y) = \begin{cases} \dfrac{1}{(2000)^2}e^{-(x+y)/2000} & \text{if } x \geq 0, y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

So we want $\text{Prob}(x \le 2000, y \le 2000) = \int_0^{2000} \int_0^{2000} \frac{1}{2000^2} e^{-x/2000} e^{-y/2000} dx\, dy$

$$= \frac{1}{(2000)^2} \int_0^{2000} -2000e^{-x/2000}\Big|_0^{2000} e^{-y/2000}\, dy$$

$$= \frac{1}{2000} \int_0^{2000} \left(1 - \frac{1}{e}\right) e^{-y/2000}\, dy = \left(1 - \frac{1}{e}\right)^2.$$

47. Formula (8) in Section 5.6 is $I = \iint_W d^2 \delta(x, y, z)\, dV$. If we choose the coordinates so that the center of mass is at the origin, then $\iiint_W x\delta(x, y, z)\, dV = 0$ and $\iiint_W y\delta(x, y, z)\, dV = 0$. We can also choose the coordinates so that $\bar{L}$ is the z-axis. L is a line parallel to the z-axis distance h away, so L is the line corresponding to $x = a$ and $y = b$ where $a^2 + b^2 = h^2$. Then

$$I_{\bar{L}} = \iiint_W (x^2 + y^2)\delta(x, y, z)\, dV$$

and

$$I_L = \iiint_W (x^2 + y^2 + h^2 - 2ax - 2by)\delta(x, y, z)\, dV = \iiint_W (x^2 + y^2 + h^2)\delta(x, y, z)\, dV$$

so

$$I_L - I_{\bar{L}} = \iiint_W h^2 \delta(x, y, z)\, dV = h^2 \iiint_W \delta(x, y, z)\, dV = h^2 M.$$

48. (a) With $\Delta x = (b - a)/m$ and $\Delta y = (d - c)/n$, we have

$$\begin{aligned} T_{m,n} = \frac{\Delta x \Delta y}{4} \Bigg[& f(a, c) + 2 \sum_{i=1}^{m-1} f(x_i, c) + f(b, c) \\ & + 2 \sum_{j=1}^{n-1} f(a, y_j) + 4 \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} f(x_i, y_j) + 2 \sum_{j=1}^{n-1} f(b, y_j) \\ & + f(a, d) + 2 \sum_{i=1}^{m-1} f(x_i, d) + f(b, d) \Bigg]. \end{aligned}$$

Now $f(x, y) = F(x)$, so the formula above becomes

$$\begin{aligned} T_{m,n} = \frac{\Delta x \Delta y}{4} \Bigg[& F(a) + 2 \sum_{i=1}^{m-1} F(x_i) + F(b) \\ & + 2 \sum_{j=1}^{n-1} F(a) + 4 \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} F(x_i) + 2 \sum_{j=1}^{n-1} F(b) \\ & + F(a) + 2 \sum_{i=1}^{m-1} F(x_i) + F(b) \Bigg]. \end{aligned}$$

Note that the terms in $\sum_{j=1}^{n-1} F(a)$ do not depend on j; hence $\sum_{j=1}^{n-1} F(a) = (n - 1)F(a)$. Similarly, we have

$\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} F(x_i) = (n-1)\sum_{i=1}^{m-1} F(x_i)$. Therefore,

$$\begin{aligned} T_{m,n} &= \frac{\Delta x \Delta y}{4}\Bigg[F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b) \\ &\quad + 2(n-1)F(a) + 4(n-1)\sum_{i=1}^{m-1} F(x_i) + 2(n-1)F(b) \\ &\quad + F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b)\Bigg] \\ &= \frac{\Delta x \Delta y}{4}\left[2nF(a) + 4n\sum_{i=1}^{m-1} F(x_i) + 2nF(b)\right] \\ &= \frac{\Delta x \Delta y}{4}(2n)\left[F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b)\right]. \end{aligned}$$

Since $\Delta y = (d-c)/n$, we thus have

$$\begin{aligned} T_{m,n} &= \frac{\Delta x}{2}\left(\frac{d-c}{2n}\right)(2n)\left[F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b)\right] \\ &= (d-c)\frac{\Delta x}{2}\left[F(a) + 2\sum_{i=1}^{m-1} F(x_i) + F(b)\right] \\ &= (d-c)T_m, \end{aligned}$$

using the formula for the trapezoidal rule approximation T_m of the definite integral $\int_a^b F(x)\,dx$.

(b) We proceed in a similar manner. With $\Delta x = (b-a)/(2m)$ and $\Delta y = (d-c)/(2n)$, we have

$$\begin{aligned} &S_{2m,2n} = \\ &\frac{\Delta x \Delta y}{9}\Bigg[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b) \\ &\quad + 2\sum_{j=1}^{n-1} F(a) + 4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} F(x_{2i}) + 8\sum_{j=1}^{n-1}\sum_{i=1}^{m} F(x_{2i-1}) + 2\sum_{j=1}^{n-1} F(b) \\ &\quad + 4\sum_{j=1}^{n} F(a) + 8\sum_{j=1}^{n}\sum_{i=1}^{m-1} F(x_{2i}) + 16\sum_{j=1}^{n}\sum_{i=1}^{m} F(x_{2i-1}) + 4\sum_{j=1}^{n} F(b) \\ &\quad + F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\Bigg]. \end{aligned}$$

Again, we note that the terms in $\sum_{j=1}^{n-1} F(a)$ do not depend on j so that $\sum_{j=1}^{n-1} F(a) = (n-1)F(a)$. In a similar manner, we have $\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} F(x_{2i}) = (n-1)\sum_{i=1}^{m-1} F(x_{2i})$, etc. Thus,

$$\begin{aligned}
&S_{2m,2n} = \\
&\frac{\Delta x \Delta y}{9}\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + F(b)\right. \\
&\quad + 2(n-1)F(a) + 4(n-1)\sum_{i=1}^{m-1} F(x_{2i}) \\
&\quad + 8(n-1)\sum_{i=1}^{m} F(x_{2i-1}) + 2(n-1)F(b) \\
&\quad + 4nF(a) + 8n\sum_{i=1}^{m-1} F(x_{2i}) + 16n\sum_{i=1}^{m} F(x_{2i-1}) + 4nF(b) \\
&\quad \left. + F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right] \\
&= \frac{\Delta x \Delta y}{9}\left[6nF(a) + 12n\sum_{i=1}^{m-1} F(x_{2i}) + 24n\sum_{i=1}^{m} F(x_{2i-1}) + 6nF(b)\right] \\
&= \frac{\Delta x \Delta y}{9}(6n)\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right].
\end{aligned}$$

Since $\Delta y = (d-c)/(2n)$, we have that

$$\begin{aligned}
S_{2m,2n} &= \frac{\Delta x}{3}\left(\frac{d-c}{6n}\right)(6n)\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right] \\
&= (d-c)\frac{\Delta x}{3}\left[F(a) + 2\sum_{i=1}^{m-1} F(x_{2i}) + 4\sum_{i=1}^{m} F(x_{2i-1}) + F(b)\right] \\
&= (d-c)S_{2m},
\end{aligned}$$

using the formula for the Simpson's rule approximation S_{2m} of the definite integral $\int_a^b F(x)\,dx$.

49. With $\Delta x = (b-a)/m$ and $\Delta y = (d-c)/n$, the trapezoidal rule approximation to $\iint_{[a,b]\times[c,d]} f(x)g(y)\,dA =$

$\int_a^b \int_c^d f(x)g(y)\,dy\,dx$ is

$$\begin{aligned}
T_{m,n} &= \frac{\Delta x \Delta y}{4}\Bigg[f(a)g(c) + 2\sum_{i=1}^{m-1} f(x_i)g(c) + f(b)g(c) \\
&\quad + 2\sum_{j=1}^{n-1} f(a)g(y_j) + 4\sum_{j=1}^{n-1}\sum_{i=1}^{m-1} f(x_i)g(y_j) + 2\sum_{j=1}^{n-1} f(b)g(y_j) \\
&\quad + f(a)g(d) + 2\sum_{i=1}^{m-1} f(x_i)g(d) + f(b)g(d)\Bigg] \\
&= \frac{\Delta x \Delta y}{4}\Bigg[g(c)\left(f(a) + 2\sum_{i=1}^{m-1} f(x_i) + f(b)\right) \\
&\quad + 2f(a)\sum_{j=1}^{n-1} g(y_j) + 4\left(\sum_{j=1}^{n-1} g(y_j)\right)\left(\sum_{i=1}^{m-1} f(x_i)\right) + 2f(b)\sum_{j=1}^{n-1} g(y_j) \\
&\quad + g(d)\left(f(a) + 2\sum_{i=1}^{m-1} f(x_i) + f(b)\right)\Bigg] \\
&= \frac{\Delta x \Delta y}{4}\left[\left(f(a) + 2\sum_{i=1}^{m-1} f(x_i) + f(b)\right)\left(g(c) + 2\sum_{j=1}^{n-1} g(y_j) + g(d)\right)\right] \\
&= T_m(f)T_n(g),
\end{aligned}$$

where $T_m(f)$ denotes the trapezoidal rule approximation to $\int_a^b f(x)\,dx$ and $T_n(g)$ the trapezoidal rule approximation to $\int_c^d f(y)\,dy$.