Chapter 4

Maxima and Minima in Several Variables

4.1 Differentials and Taylor's Theorem

In Exercises 1–7 we will first calculate $f(x), f'(x), \ldots, f^{(k)}(x)$ and $f(a), f'(a), \ldots, f^{(k)}(a)$. Then we'll plug into the formula for Taylor's theorem in one variable (Theorem 1.1 in the text):

$$p_k(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

1. Here $a = 0$ and $k = 4$:

$$\begin{aligned} f(x) &= e^{2x} & f(0) &= 1 \\ f^{(n)}(x) &= 2^n e^{2x} & f^{(n)}(0) &= 2^n \end{aligned}$$

so

$$\begin{aligned} p_4(x) &= 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 + \frac{16}{24}x^4 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4. \end{aligned}$$

2. Here $a = 0$ and $k = 3$:

$$\begin{aligned} f(x) &= \ln(1+x) & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\ f''(x) &= -\frac{1}{(1+x)^2} & f''(0) &= -1 \\ f'''(x) &= -2\left(\frac{-1}{(1+x)^3}\right) & f'''(0) &= 2, \end{aligned}$$

so

$$\begin{aligned} p_3(x) &= 0 + x - \frac{1}{2}x^2 + \frac{2}{6}x^3 \\ &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3. \end{aligned}$$

3. Here $a = 1$ and $k = 4$:

$$\begin{aligned} f(x) &= \frac{1}{x^2} & f(1) &= 1 \\ f'(x) &= -\frac{2}{x^3} & f'(1) &= -2 \\ f''(x) &= \frac{6}{x^4} & f''(1) &= 6 \\ f'''(x) &= -\frac{24}{x^5} & f'''(1) &= -24 \\ f''''(x) &= \frac{120}{x^6} & f''''(1) &= 120, \end{aligned}$$

so

$$\begin{aligned} p_4(x) &= 1 - 2(x-1) + \frac{6}{2}(x-1)^2 - \frac{24}{6}(x-1)^3 + \frac{120}{24}(x-1)^4 \\ &= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4. \end{aligned}$$

Students sometimes forget that the Taylor polynomial depends on the choice of a. Some texts include the parameter a in the notation to stress this fact. A nice way to remind your students of this dependence on a is to either assign Exercises 4 and 5 or 6 and 7 together.

We'll do the scratch work for both Exercises 4 and 5 together:

$$\begin{array}{lll} f(x) = \sqrt{x} & f(1) = 1 & f(9) = 3 \\ f'(x) = \dfrac{1}{2\sqrt{x}} & f'(1) = \dfrac{1}{2} & f'(9) = \dfrac{1}{6} \\ f''(x) = \dfrac{-1}{4x^{3/2}} & f''(1) = -\dfrac{1}{4} & f''(9) = -\dfrac{1}{108} \\ f'''(x) = \dfrac{3}{8x^{5/2}} & f'''(1) = \dfrac{3}{8} & f'''(9) = \dfrac{1}{648}. \end{array}$$

4. Here $a = 1$ and $k = 3$ so, using the work above:

$$p_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3.$$

5. Here $a = 9$ and $k = 3$ so, using the work above:

$$p_3(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3.$$

We'll do the scratch work for both Exercises 6 and 7 together:

$$\begin{array}{lll} f(x) = \sin x & f(0) = 0 & f(\pi/2) = 1 \\ f'(x) = \cos x & f'(0) = 1 & f'(\pi/2) = 0 \\ f''(x) = -\sin x & f''(0) = 0 & f''(\pi/2) = -1 \\ f'''(x) = -\cos x & f'''(0) = -1 & f'''(\pi/2) = 0 \\ f''''(x) = \sin x & f''''(0) = 0 & f''''(\pi/2) = 1 \\ f'''''(x) = \cos x & f'''''(0) = 1 & f'''''(\pi/2) = 0. \end{array}$$

6. Here $a = 0$ and $k = 5$ so, using the work above:

$$p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

7. Here $a = \pi/2$ and $k = 5$ so, using the work above:

$$p_5(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24}.$$

Three notes:

- *It makes sense to assign Exercises 8, 9, 16, and 21 together as they explore the same function. Exercise 14 is a higher-dimensional analogue.*
- *In Exercises 8–15, we again do the preliminary calculations and then substitute into the formulas given in Theorem 1.3*

$$p_1(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

and Theorem 1.5

$$p_2(\mathbf{x}) = f(\mathbf{a}) + \sum_{i=1}^{n} f_{x_i}(\mathbf{a})(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j)$$
$$= p_1(\mathbf{x}) + \frac{1}{2}\sum_{i,j=1}^{n} f_{x_i x_j}(\mathbf{a})(x_i - a_i)(x_j - a_j).$$

- *Just as in the one-variable versions of Taylor's theorem, note the lower degree polynomials are contained in the expressions for the higher degree ones.*

We'll do the scratch work for both Exercises 8 and 9 together:

$$\begin{array}{lll}
f(x,y) = \dfrac{1}{x^2+y^2+1} & f(0,0) = 1 & f(1,-1) = 1/3 \\
f_x(x,y) = \dfrac{-2x}{(x^2+y^2+1)^2} & f_x(0,0) = 0 & f_x(1,-1) = -2/9 \\
f_y(x,y) = \dfrac{-2y}{(x^2+y^2+1)^2} & f_y(0,0) = 0 & f_y(1,-1) = 2/9 \\
f_{xx}(x,y) = \dfrac{6x^2-2y^2-2}{(x^2+y^2+1)^3} & f_{xx}(0,0) = -2 & f_{xx}(1,-1) = 2/27 \\
f_{yy}(x,y) = \dfrac{6y^2-2x^2-2}{(x^2+y^2+1)^3} & f_{yy}(0,0) = -2 & f_{yy}(1,-1) = 2/27 \\
f_{xy}(x,y) = \dfrac{8xy}{(x^2+y^2+1)^3} & f_{xy}(0,0) = 0 & f_{xy}(1,-1) = -8/27
\end{array}$$

8. $\mathbf{a} = (0,0)$ so, using the work above:

$$p_1(\mathbf{x}) = f(0,0) + Df(0,0)\mathbf{x} = 1 \quad \text{and}$$
$$p_2(\mathbf{x}) = p_1(\mathbf{x}) + \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$
$$= 1 - x^2 - y^2.$$

9. $\mathbf{a} = (1,-1)$ so, using the work above:

$$p_1(\mathbf{x}) = f(1,-1) + Df(1,-1)(\mathbf{x} - (1,-1)) = \frac{1}{3} + \begin{bmatrix} -\frac{2}{9} & \frac{2}{9} \end{bmatrix}\begin{bmatrix} x-1 \\ y+1 \end{bmatrix}$$
$$= \frac{1}{3} - \frac{2(x-1)}{9} + \frac{2(y+1)}{9} \quad \text{and}$$
$$p_2(\mathbf{x}) = p_1(\mathbf{x}) + \frac{1}{2}(f_{xx}(1,-1)(x-1)^2 + 2f_{xy}(1,-1)(x-1)(y+1) + f_{yy}(1,-1)(y+1)^2)$$
$$= \frac{1}{3} - \frac{2(x-1)}{9} + \frac{2(y+1)}{9} + \frac{(x-1)^2}{27} - \frac{8(x-1)(y+1)}{27} + \frac{(y+1)^2}{27}.$$

10. Here $\mathbf{a} = (0,0)$ and

$$\begin{array}{ll}
f(x,y) = e^{2x+y} & f(0,0) = 1 \\
f_x(x,y) = 2e^{2x+y} & f_x(0,0) = 2 \\
f_y(x,y) = e^{2x+y} & f_y(0,0) = 1 \\
f_{xx}(x,y) = 4e^{2x+y} & f_{xx}(0,0) = 4 \\
f_{yy}(x,y) = e^{2x+y} & f_{yy}(0,0) = 1 \\
f_{xy}(x,y) = 2e^{2x+y} & f_{xy}(0,0) = 2,
\end{array}$$

so

$$\begin{aligned} p_1(\mathbf{x}) &= f(0,0) + Df(0,0)\mathbf{x} = 1 + 2x + y \quad \text{and} \\ p_2(\mathbf{x}) &= 1 + 2x + y + \frac{1}{2}(4x^2 + 2(2)xy + y^2) \\ &= 1 + 2x + y + 2x^2 + 2xy + \frac{y^2}{2} \end{aligned}$$

11. Here $\mathbf{a} = (0, \pi)$ and

$$\begin{aligned} f(x,y) &= e^{2x}\cos 3y & f(0,\pi) &= -1 \\ f_x(x,y) &= 2e^{2x}\cos 3y & f_x(0,\pi) &= -2 \\ f_y(x,y) &= -3e^{2x}\sin 3y & f_y(0,\pi) &= 0 \\ f_{xx}(x,y) &= 4e^{2x}\cos 3y & f_{xx}(0,\pi) &= -4 \\ f_{yy}(x,y) &= -9e^{2x}\cos 3y & f_{yy}(0,\pi) &= 9 \\ f_{xy}(x,y) &= -6e^{2x}\sin 3y & f_{xy}(0,\pi) &= 0, \end{aligned}$$

so

$$\begin{aligned} p_1(\mathbf{x}) &= -1 - 2x \quad \text{and} \\ p_2(\mathbf{x}) &= -1 - 2x + \frac{1}{2}(-4x^2 + 9(y-\pi)^2) \\ &= -1 - 2x - 2x^2 + \frac{9}{2}(y-\pi)^2. \end{aligned}$$

12. Here $\mathbf{a} = (0, 0, 2)$ and

$$\begin{aligned} f(x,y,z) &= ye^{3x} + ze^{2y} & f(0,0,2) &= 2 \\ f_x(x,y,z) &= 3ye^{3x} & f_x(0,0,2) &= 0 \\ f_y(x,y,z) &= e^{3x} + 2ze^{2y} & f_y(0,0,2) &= 5 \\ f_z(x,y,z) &= e^{2y} & f_y(0,0,2) &= 1 \\ f_{xx}(x,y,z) &= 9ye^{3x} & f_{xx}(0,0,2) &= 0 \\ f_{xy}(x,y,z) &= 3e^{3x} & f_{xy}(0,0,2) &= 3 = f_{yx}(0,0,2) \\ f_{xz}(x,y,z) &= 0 & f_{xz}(0,0,2) &= 0 = f_{zx}(0,0,2) \\ f_{yy}(x,y,z) &= 4ze^{2y} & f_{yy}(0,0,2) &= 8 \\ f_{yz}(x,y,z) &= 2e^{2y} & f_{yz}(0,0,2) &= 2 = f_{zy}(0,0,2) \\ f_{zz}(x,y,z) &= 0 & f_{yy}(0,0,2) &= 0, \end{aligned}$$

so

$$\begin{aligned} p_1(\mathbf{x}) &= 2 + 5y + 1(z-2) = 5y + z \quad \text{and} \\ p_2(\mathbf{x}) &= 5y + z + \frac{1}{2}(6xy + 8y^2 + 4y(z-2)) \\ &= y + z + 3xy + 4y^2 + 2yz. \end{aligned}$$

13. Here $\mathbf{a} = (2, -1, 1)$ and

$$\begin{aligned}
f(x,y,z) &= xy - 3y^2 + 2xz & f(2,-1,1) &= -1\\
f_x(x,y,z) &= y + 2z & f_x(2,-1,1) &= 1\\
f_y(x,y,z) &= x - 6y & f_y(2,-1,1) &= 8\\
f_z(x,y,z) &= 2x & f_y(2,-1,1) &= 4\\
f_{xx}(x,y,z) &= 0 & f_{xx}(2,-1,1) &= 0\\
f_{xy}(x,y,z) &= 1 & f_{xy}(2,-1,1) &= 1 = f_{yx}(2,-1,1)\\
f_{xz}(x,y,z) &= 2 & f_{xz}(2,-1,1) &= 2 = f_{zx}(2,-1,1)\\
f_{yy}(x,y,z) &= -6 & f_{yy}(2,-1,1) &= -6\\
f_{yz}(x,y,z) &= 0 & f_{yz}(2,-1,1) &= 0 = f_{zy}(2,-1,1)\\
f_{zz}(x,y,z) &= 0 & f_{yy}(2,-1,1) &= 0,
\end{aligned}$$

so

$$\begin{aligned}
p_1(\mathbf{x}) &= -1 + 1(x-2) + 8(y+1) + 4(z-1) = 1 + x + 8y + 4z \quad \text{and}\\
p_2(\mathbf{x}) &= 1 + x + 8y + 4z + \frac{1}{2}(2(x-2)(y+1) + 4(x-2)(z-1) - 6(y+1)^2)\\
&= xy - 3y^2 + 2xz.
\end{aligned}$$

Note that the second-order polynomial matches the original function exactly. This makes sense, since f is itself a polynomial of degree two.

14. Here $\mathbf{a} = (0, 0, 0)$ and there is quite a bit of symmetry so we'll only calculate:

$$\begin{aligned}
f(x,y,z) &= \frac{1}{x^2+y^2+z^2+1} & f(0,0,0) &= 1\\
f_x(x,y,z) &= \frac{-2x}{(x^2+y^2+z^2+1)^2} & f_x(0,0,0) &= 0 = f_y(0,0,0) = f_z(0,0,0)\\
f_{xx}(x,y,z) &= \frac{6x^2-2y^2-2z^2-2}{(x^2+y^2+z^2+1)^3} & f_{xx}(0,0,0) &= -2 = f_{yy}(0,0,0) = f_{zz}(0,0,0)\\
f_{xy}(x,y) &= \frac{8xy}{(x^2+y^2+z^2+1)^3} & f_{xy}(0,0,0) &= 0 = f_{xz}(0,0,0) = f_{yz}(0,0,0)
\end{aligned}$$

so

$$\begin{aligned}
p_1(\mathbf{x}) &= 1 \quad \text{and}\\
p_2(\mathbf{x}) &= 1 + \frac{1}{2}(-2x^2 - 2y^2 - 2z^2)\\
&= 1 - x^2 - y^2 - z^2.
\end{aligned}$$

15. Again $\mathbf{a} = (0, 0, 0)$ and there is quite a bit of symmetry so we'll only calculate:

$$\begin{aligned}
f(x,y,z) &= \sin xyz & f(0,0,0) &= 0\\
f_x(x,y,z) &= yz\cos xyz & f_x(0,0,0) &= 0 = f_y(0,0,0) = f_z(0,0,0)\\
f_{xx}(x,y,z) &= -y^2z^2\sin xyz & f_{xx}(0,0,0) &= 0 = f_{yy}(0,0,0) = f_{zz}(0,0,0)\\
f_{xy}(x,y) &= z\cos xyz - xyz^2\sin xyz & f_{xy}(0,0,0) &= 0 = f_{xz}(0,0,0) = f_{yz}(0,0,0)
\end{aligned}$$

$$\text{so} \quad p_1(\mathbf{x}) = 0 \quad \text{and} \quad p_2(\mathbf{x}) = 0.$$

16. From Exercise 8 we can read off that the Hessian $Hf(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$.

17. $f(x, y) = \cos x \sin y$

$$\begin{aligned} f_x(x, y) &= -\sin x \sin y & f_y(x, y) &= \cos x \cos y \\ f_{xx}(x, y) &= -\cos x \sin y & f_{yx}(x, y) &= -\sin x \cos y \\ f_{xy}(x, y) &= -\sin x \cos y & f_{yy}(x, y) &= -\cos x \sin y \end{aligned}$$

so

$$Hf\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = \begin{bmatrix} -\frac{\sqrt{6}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} \end{bmatrix}.$$

18. $f(x, y, z) = \dfrac{z}{\sqrt{xy}}$

$$\begin{aligned} f_x(x, y, z) &= -\frac{z}{2x^{3/2}y^{1/2}} & f_y(x, y, z) &= -\frac{z}{2x^{1/2}y^{3/2}} & f_z(x, y, z) &= \frac{1}{\sqrt{xy}} \\ f_{xx}(x, y, z) &= \frac{3z}{4x^{5/2}y^{1/2}} & f_{yx}(x, y, z) &= \frac{z}{4x^{3/2}y^{3/2}} & f_{zx}(x, y, z) &= -\frac{1}{3x^{3/2}y^{1/2}} \\ f_{xy}(x, y, z) &= \frac{z}{4x^{3/2}y^{3/2}} & f_{yy}(x, y, z) &= \frac{3z}{4x^{1/2}y^{5/2}} & f_{zy}(x, y, z) &= -\frac{1}{2x^{1/2}y^{3/2}} \\ f_{xz}(x, y, z) &= -\frac{1}{2x^{3/2}y^{1/2}} & f_{yz}(x, y, z) &= -\frac{1}{2x^{1/2}y^{3/2}} & f_{zz}(x, y, z) &= 0 \end{aligned}$$

so

$$Hf(1, 2, -4) = \begin{bmatrix} -\frac{3}{\sqrt{2}} & -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{3}{4\sqrt{2}} & -\frac{1}{4\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{4\sqrt{2}} & 0 \end{bmatrix}.$$

19. $f(x, y, z) = x^3 + x^2y - yz^2 + 2z^3$

$$\begin{aligned} f_x(x, y, z) &= 3x^2 + 2xy & f_y(x, y, z) &= x^2 - z^2 & f_z(x, y, z) &= -2yz + 6z^2 \\ f_{xx}(x, y, z) &= 6x + 2y & f_{yx}(x, y, z) &= 2x & f_{zx}(x, y, z) &= 0 \\ f_{xy}(x, y, z) &= 2x & f_{yy}(x, y, z) &= 0 & f_{zy}(x, y, z) &= -2z \\ f_{xz}(x, y, z) &= 0 & f_{yz}(x, y, z) &= -2z & f_{zz}(x, y, z) &= -2y + 12z \end{aligned}$$

so

$$Hf(1, 0, 1) = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix}.$$

20. $f(x, y, z) = e^{2x-3y} \sin 5z$

$$\begin{aligned} f_x(x, y, z) &= 2e^{2x-3y} \sin 5z & f_y(x, y, z) &= -3e^{2x-3y} \sin 5z & f_z(x, y, z) &= 5e^{2x-3y} \cos 5z \\ f_{xx}(x, y, z) &= 4e^{2x-3y} \sin 5z & f_{yx}(x, y, z) &= -6e^{2x-3y} \sin 5z & f_{zx}(x, y, z) &= 10e^{2x-3y} \cos 5z \\ f_{xy}(x, y, z) &= -6e^{2x-3y} \sin 5z & f_{yy}(x, y, z) &= 9e^{2x-3y} \sin 5z & f_{zy}(x, y, z) &= -15e^{2x-3y} \cos 5z \\ f_{xz}(x, y, z) &= 10e^{2x-3y} \cos 5z & f_{yz}(x, y, z) &= -15e^{2x-3y} \cos 5z & f_{zz}(x, y, z) &= -25e^{2x-3y} \sin 5z \end{aligned}$$

so

$$Hf(0, 0, 0) = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 0 & -15 \\ 10 & -15 & 0 \end{bmatrix}.$$

For Exercises 21–25 you'll need formula (10): $p_2(\mathbf{x}) = f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + (1/2)\mathbf{h}^T Hf(\mathbf{a})\mathbf{h}$ *where* $\mathbf{h} = \mathbf{x} - \mathbf{a}$.

21. Use the work from Exercises 8 and 16:

$$\begin{aligned} p_2(\mathbf{x}) &= f(0, 0) + Df(0, 0)\mathbf{x} + \frac{1}{2}\mathbf{x}^T \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x} \\ &= 1 + \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

22. Use the work from Exercise 11:

$$p_2(x,y) = f(0,\pi) + Df(0,\pi)\begin{bmatrix} x \\ y-\pi \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y-\pi \end{bmatrix} Hf(0,\pi)\begin{bmatrix} x \\ y-\pi \end{bmatrix}$$
$$= -1 + \begin{bmatrix} -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y-\pi \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y-\pi \end{bmatrix}\begin{bmatrix} -4 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x \\ y-\pi \end{bmatrix}.$$

23. Use the work from Exercise 12:

$$p_2(x,y,z) = f(0,0,2) + Df(0,0,2)\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y & z-2 \end{bmatrix} Hf(0,0,2)\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix}$$
$$= 2 + \begin{bmatrix} 0 & 5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x & y & z-2 \end{bmatrix}\begin{bmatrix} 0 & 3 & 0 \\ 3 & 8 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z-2 \end{bmatrix}.$$

24. Use the work from Exercise 19:

$$p_2(\mathbf{x}) = f(1,0,1) + Df(1,0,1)(\mathbf{x}-(1,0,1)) + \frac{1}{2}(\mathbf{x}-(1,0,1))^T\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}(\mathbf{x}-(1,0,1))$$
$$= 3 + \begin{bmatrix} 3 & 0 & 6 \end{bmatrix}\begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x-1 & y & z-1 \end{bmatrix}\begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 12 \end{bmatrix}\begin{bmatrix} x-1 \\ y \\ z-1 \end{bmatrix}.$$

Exercises 25 and 26 are related and could be assigned together. To make it a cohesive single problem, you may want to tell the students to use the function from Exercise 26 in place of the function given in Exercise 25.

25. The function is $f(x_1, x_2, \ldots, x_n) = e^{x_1+2x_2+\cdots+nx_n}$.

(a) $Df(x_1, x_2, \ldots, x_n) = e^{x_1+2x_2+\cdots+nx_n}\begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}$, and therefore $Df(0,0,\ldots,0) = \begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}$. Taking second derivatives and evaluating at the origin results in:

$$Hf(0,0,\ldots,0) = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}.$$

(c) Since (c) follows immediately from (a) we will skip (b) for a moment.

$$p_2(\mathbf{x}) = f(0,0,\ldots,0) + Df(0,0,\ldots,0)\mathbf{x} + \frac{1}{2}\mathbf{x}^T Hf(0,0,\ldots,0)\mathbf{x}$$
$$= 1 + \begin{bmatrix} 1 & 2 & \cdots & n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \frac{1}{2}\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

(b) Now we can read the answer to (b) right off of our answer to (c).

$$p_1(\mathbf{x}) = 1 + x_1 + 2x_2 + \cdots + nx_n \quad \text{and}$$
$$p_2(\mathbf{x}) = 1 + x_1 + 2x_2 + \cdots + nx_n + \frac{1}{2}\sum_{i,j=1}^{n} ijx_ix_j.$$

26. This is an extension of a special case of Exercise 25. Note that $f_{x_i x_j x_k}(0,0,0) = ijk$ so

$$\begin{aligned} p_3(\mathbf{x}) &= 1 + x + 2y + 3z + \frac{1}{2}(x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz) \\ &\quad + \frac{1}{6}(x^3 + 8y^3 + 27z^3 + 6x^2y + 9x^2z + 12xy^2 + 36y^2z + 27xz^2 + 54yz^2 + 36xyz). \end{aligned}$$

27. $Df(x,y,z) = \begin{bmatrix} 4x^3 + 3x^2y - z^2 + 2xy + 3y & x^3 + 6y^2 + x^2 + 3x & -2xz - 1 \end{bmatrix}$ and

$$Hf(x,y,z) = \begin{bmatrix} 12x^2 + 6xy + 2y & 3x^2 + 2x + 3 & -2z \\ 3x^2 + 2x + 3 & 12y & 0 \\ -2z & 0 & -2x \end{bmatrix}.$$

The only non-zero third derivatives are

$$f_{xxx}(x,y,z) = 24x + 6y \qquad f_{xxy}(x,y,z) = 6x + 2$$
$$f_{xzz}(x,y,z) = -2 \qquad f_{yyy}(x,y,z) = 12$$

and their permutations.

(a) Here $\mathbf{a} = (0,0,0)$ so $f(0,0,0) = 2$, $Df(0,0,0) = \begin{bmatrix} 0 & 0 & -1 \end{bmatrix}$, and $Hf(0,0,0) = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$$\begin{aligned} p_3(\mathbf{x}) &= 2 - z + 3xy + \frac{1}{6}(6x^2y - 6xz^2 + 12y^3) \\ &= 2 - z + 3xy + x^2y - xz^2 + 2y^3. \end{aligned}$$

(b) Here $f(1,-1,0) = -4$, $Df(1,-1,0) = \begin{bmatrix} -4 & 11 & -1 \end{bmatrix}$, and $Hf(1,-1,0) = \begin{bmatrix} 4 & 8 & 0 \\ 8 & -12 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

$$\begin{aligned} p_3(\mathbf{x}) &= -4 - 4(x-1) + 11(y+1) - z \\ &\quad + \frac{1}{2}[4(x-1)^2 + 16(x-1)(y+1) - 12(y+1)^2 - 2z^2] \\ &\quad + \frac{1}{6}[18(x-1)^3 + 3(8)(x-1)^2(y+1) - 3(2)(x-1)z^2 + 12(y+1)^3] \\ &= -4 - 4(x-1) + 11(y+1) - z + 2(x-1)^2 + 8(x-1)(y+1) - 6(y+1)^2 - z^2 \\ &\quad + 3(x-1)^3 + 4(x-1)^2(y+1) - (x-1)z^2 + 2(y+1)^3. \end{aligned}$$

Exercises 28 and 32 are used in Exercise 33 (a) and (b). From Definition 1.4, the total differential of f is $df(\mathbf{a},\mathbf{h}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a})\, dx_i$.

28. $f(x,y) = x^2y^3$ so $df(x,y,\mathbf{h}) = 2xy^3\,dx + 3x^2y^2\,dy$.
29. $f(x,y,z) = x^2 + 3y^2 - 2z^3$ so $df(x,y,z,\mathbf{h}) = 2x\,dx + 6y\,dy - 6z^2\,dz$.
30. $f(x,y,z) = \cos(xyz)$ so $df(x,y,z,\mathbf{h}) = -yz\sin(xyz)\,dx - xz\sin(xyz)\,dy - xy\sin(xyz)\,dz$.
31. $f(x,y,z) = e^x\cos y + e^y \sin z$ so $df(x,y,z,\mathbf{h}) = e^x\cos y\,dx + (-e^x\sin y + e^y\sin z)\,dy + e^y\cos z\,dz$.
32. $f(x,y,z) = 1/\sqrt{xyz}$ so $df(x,y,z,\mathbf{h}) = -\frac{1}{2}(xyz)^{-3/2}(yz\,dx + xz\,dy + xy\,dz)$.
33. **(a)** Use the function from Exercise 28: $f(x,y) = x^2y^3$ with $x = 7$, $y = 2$, $dx = .07$, and $dy = -.02$. So

$$\begin{aligned} (7.07)^2(1.98)^3 \approx 7^2 2^3 + df((7,2),(.07,-.02)) &= 2(7)(2^3)(.07) + 3(7^2)(2^2)(-.02) \\ &= -3.92. \end{aligned}$$

(b) Use the function from Exercise 32: $f(x, y, z) = 1/\sqrt{xyz}$ with $x = 4$, $y = 2$, $z = 2$, $dx = .1$, $dy = -.04$, and $dz = .05$. So

$$\begin{aligned}\frac{1}{\sqrt{(4.1)(1.96)(2.05)}} &\approx \frac{1}{\sqrt{(4)(2)(2)}} - \frac{1}{2}(16)^{-3/2}(4(.1) + 8(-.04) + 8(.05)) \\ &= \frac{1}{4} - \frac{1}{128}(.48) = .24625.\end{aligned}$$

(c) Here the function is $f(x, y, z) = x\cos(yz)$ with $x = 1$, $y = \pi$, $z = 0$, $dx = .1$, $dy = -.03$, and $dz = .12$. So

$$(1.1)\cos((\pi - 0.03)(0.12)) \approx 1 + (\cos 0)(.1) - (\pi \sin 0)(.12) = 1.1.$$

34. $dg(x, y, z, \mathbf{h}) = (3x^2 - 2y + 2xz)\,dx + (-2x)\,dy + (x^2 + 7)\,dz$, so $dg(1, -2, 1, \mathbf{h}) = 9\,dx - 2\,dy + 8\,dz$. This means that changes in x have the most effect.

35. Although students will probably solve this more formally, they should see that, intuitively, changes in the upper left entry are multiplied by the largest number so that is the entry for which the value of the determinant is most sensitive.

36. $r = 2$, $dr = .1$, $h = 3$, and $dh = .05$.

(a) $V = \pi r^2 h$, so $dV = 2\pi rh\,dr + \pi r^2\,dh = 2\pi(2)(3)(.1) + \pi(2^2)(.05) = 1.4\pi$.

(b) $S = 2\pi rh + 2\pi r^2$, so $dS = (2\pi h + 4\pi r)\,dr + 2\pi r\,dh = (2\pi(3) + 4\pi(2))(.1) + 2\pi(2)(.05) = 1.6\pi$.

37. Let x denote the diameter of the can, y the height. Then the volume V is given by

$$V = \pi\left(\frac{x}{2}\right)^2 y = \frac{\pi}{4}x^2 y.$$

The change in volume, ΔV, that occurs when x and y are changed by small amounts dx and dy is given approximately by the differential:

$$\Delta V \approx dV = \frac{\pi}{2}xy\,dx + \frac{\pi}{4}x^2\,dy.$$

When $x = 5$ and $y = 12$ this becomes

$$dV = \pi\left(30\,dx + \frac{25}{4}\,dy\right).$$

If x is decreased by 0.5 cm, so that $dx = -0.5$, then

$$dV = \pi\left(-15 + \frac{25}{4}\,dy\right).$$

For dV to be zero (which represents approximately no change in volume), we see that

$$dy = \frac{60}{25} = 2.4 \text{ cm}.$$

38. **(a)** The area A is given by

$$A = \frac{1}{2}ab\sin\theta, \quad \text{so} \quad dA = \frac{1}{2}b\sin\theta\,da + \frac{1}{2}a\sin\theta\,db + \frac{1}{2}ab\cos\theta\,d\theta.$$

With $a = 3$, $b = 4$, and $\theta = \pi/3$, this becomes

$$dA = \sqrt{3}\,da + \frac{3\sqrt{3}}{4}\,db + 3\,d\theta.$$

Thus, at these values, the area is most sensitive to changes in the angle θ.

(b) We use the differential appearing in part (a):

$$\Delta A \approx dA = \sqrt{3}\,da + \frac{3\sqrt{3}}{4}\,db + 3\,d\theta.$$

If the measurement of a is in error by at most 5%, then

$$|da| \le 0.05(3) = 0.15.$$

Similarly,

$$|db| \le 0.05(4) = 0.2 \quad \text{and} \quad |d\theta| \le 0.02\left(\frac{\pi}{3}\right) = 0.00\overline{6}\pi.$$

Hence the maximum error that results in the calculated value of the area is

$$|dA| \le \sqrt{3}(0.15) + \frac{3\sqrt{3}}{4}(0.2) + 0.02\pi \approx 0.58245\ \text{cm}^2.$$

The percentage error that this represents is calculated as

$$\frac{|dA|}{A} \le \frac{0.58245}{3\sqrt{3}} \approx 0.112,$$

or 11.2%.

39. We are told that $dr = dh$ and know that $V = (1/3)\pi r^2 h$. So $dV = (2/3)\pi rh\,dr + (1/3)\pi r^2 dh = (28\pi/3)\,dr$. Now we want $|dV|$ to be at most .2 so $|dV| = (28\pi/3)|dr| \le .2$ or $|dr| \le .3/(14\pi) \approx .0068209$.

40. $V = xyz$ where $x = 3$, $y = 4$, $z = 2$ and we assume that $dx = dy = dz$. So $dV = (4)(2)\,dx + (3)(2)\,dy + (3)(4)\,dz = 26dx$. We want $|dV| \le .2$ so $|dx| \le .2/26 \approx .00769$. This is a percentage error of $.2/24 = .8333\%$.

41. (a) We do the preliminary calculations:

$$\begin{aligned} f(x,y) &= \cos x \sin y & f(0,\pi/2) &= 1 \\ f_x(x,y) &= -\sin x \sin y & f_x(0,\pi/2) &= 0 \\ f_y(x,y) &= \cos x \cos y & f_y(0,\pi/2) &= 0 \\ f_{xx}(x,y) &= -\cos x \sin y & f_{xx}(0,\pi/2) &= -1 \\ f_{yy}(x,y) &= -\cos x \sin y & f_{yy}(0,\pi/2) &= -1 \\ f_{xy}(x,y) &= -\sin x \cos y & f_{xy}(0,\pi/2) &= 0 \end{aligned}$$

So $p_2(\mathbf{x}) = 1 - x^2/2 - (y - \pi/2)^2/2$.

(b) We'll just follow the estimate in Example 12 in the text: "since all partial derivatives of f will be the product of sines and cosines and hence no larger than 1 in magnitude" and $|h_1|$ and $|h_2|$ are each no more than .3,

$$|R_2(0,\pi/2,h_1,h_2)| \le \frac{1}{6}(|h_1|^3 + 3h_1^2|h_2| + 3|h_1|h_2^2 + |h_2|^3) \le \frac{1}{6}(8\cdot(0.3)^3) = .036.$$

42. (a) We do the preliminary calculations:

$$\begin{aligned} f(x,y) &= e^{x+2y} & f(0,0) &= 1 \\ f_x(x,y) &= e^{x+2y} & f_x(0,0) &= 1 \\ f_y(x,y) &= 2e^{x+2y} & f_y(0,0) &= 2 \\ f_{xx}(x,y) &= e^{x+2y} & f_{xx}(0,0) &= 1 \\ f_{yy}(x,y) &= 4e^{x+2y} & f_{yy}(0,0) &= 4 \\ f_{xy}(x,y) &= 2e^{x+2y} & f_{xy}(0,0) &= 2. \end{aligned}$$

So $p_2(\mathbf{x}) = 1 + x + 2y + x^2/2 + 2xy + 2y^2$.

(b) This time each third derivative has a factor of e^{x+2y} in it. Each derivative with respect to y brings out an additional factor of two. Here $|h_1|$ and $|h_2|$ are no more than .1 and on our set $e^{x+2y} \le e^{.3} < 2$. So

$$|R_2(0,0,h_1,h_2)| \le (2)\frac{1}{6}(|h_1|^3 + 6h_1^2|h_2| + 12|h_1|h_2^2 + 8|h_2|^3) \le \frac{1}{3}(27\cdot(0.1)^3) = .009.$$

43. (a) The preliminary calculations for $f(x,y) = e^{2x}\cos y$ are

$$\begin{aligned} f(x,y) &= e^{2x}\cos y & f(0,\pi/2) &= 0 \\ f_x(x,y) &= 2e^{2x}\cos y & f_x(0,\pi/2) &= 0 \\ f_y(x,y) &= -e^{2x}\sin y & f_y(0,\pi/2) &= -1 \\ f_{xx}(x,y) &= 4e^{2x}\cos y & f_{xx}(0,\pi/2) &= 0 \\ f_{xy}(x,y) &= -2e^{2x}\sin y & f_{xy}(0,\pi/2) &= -2 = f_{yx}(0,\pi/2) \\ f_{yy}(x,y) &= -e^{2x}\cos y & f_{yy}(0,\pi/2) &= 0. \end{aligned}$$

Thus

$$p_2(x, y, z) = -\left(y - \frac{\pi}{2}\right) + \frac{1}{2}\left(-4x\left(y - \frac{\pi}{2}\right)\right) = \frac{\pi}{2} - y - 2x\left(y - \frac{\pi}{2}\right).$$

(b) The eight third-order partial derivatives are:

$$\begin{aligned} f_{xxx}(x, y) &= 8e^{2x}\cos y \\ f_{xxy}(x, y) &= -4e^{2x}\sin y = f_{xyx}(x, y) = f_{yxx}(x, y) \\ f_{xyy}(x, y) &= -2e^{2x}\cos y = f_{yxy}(x, y) = f_{yyx}(x, y) \\ f_{yyy}(x, y) &= e^{2x}\sin y, \end{aligned}$$

Lagrange's form of the remainder tells us that

$$\left|R_2\left(x, y, 0, \frac{\pi}{2}\right)\right| = \frac{1}{3!}\left|\sum_{i,j,k=1}^{2} f_{x_i x_j x_k}(\mathbf{z}) h_i h_j h_k\right|,$$

where $\mathbf{z}$ is a point on the line segment joining $(0, \pi/2)$ and (x, y). Note that the exponential function e^{2x} increases with x and the sine and cosine have maximum values of 1. Thus

$$|f_{xxx}(x, y)| \le 8e^{0.4},$$

and similar results apply to the other third-order partials. Hence

$$\begin{aligned} \left|R_2\left(x, y, 0, \frac{\pi}{2}\right)\right| &\le \frac{1}{6}\left(8e^{0.4}|h_1|^3 + 3 \cdot 4e^{0.4}|h_1|^2|h_2| + 3 \cdot 2e^{0.4}|h_1|\,|h_2|^2 + e^{0.4}|h_2|^3\right) \\ &= \frac{e^{0.4}}{6}\left(8|h_1|^3 + 12|h_1|^2|h_2| + 6|h_1|\,|h_2|^2 + |h_2|^3\right). \end{aligned}$$

If $|h_1| \le 0.2$ and $|h_2| \le 0.1$, then

$$\left|R_2\left(x, y, 0, \frac{\pi}{2}\right)\right| \le \frac{e^{0.4}}{6}\left(8(0.008) + 12(0.004) + 6(0.002) + 0.001\right) \approx 0.03108.$$

4.2 Extrema of Functions

1. $f(x, y) = 4x + 6y - 12 - x^2 - y^2$ so $f_x(x, y) = 4 - 2x$, $f_y(x, y) = 6 - 2y$, $f_{xx}(x, y) = -2$, $f_{xy}(x, y) = 0$, and $f_{yy}(x, y) = -2$.

(a) To find the critical point we will set each of the first partial derivatives equal to 0 and solve: $f_x(x, y) = 0$ when $4-2x = 0$ or when $x = 2$ and $f_y(x, y) = 0$ when $6 - 2y = 0$ or when $y = 3$. So f has a unique critical point at (2, 3).

(b) The increment

$$\begin{aligned} \Delta f &= f(2 + \Delta x, 3 + \Delta y) - f(2, 3) \\ &= 4(2 + \Delta x) + 6(3 + \Delta y) - 12 - (2 + \Delta x)^2 - (3 + \Delta y)^2 \\ &\quad - (4(2) + 6(3) - 12 - 2^2 - 3^2) = -(\Delta x)^2 - (\Delta y)^2. \end{aligned}$$

This tells us that little changes in x and/or y result in a decrease in the value of f. This means that f must have a local maximum at (2, 3).

(c) The Hessian is $Hf(2, 3) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ so $d_1 = -2$ and $d_2 = 4$ so by the second derivative test, f has a local maximum at (2, 3).

2. $g(x, y) = x^2 - 2y^2 + 2x + 3$ so $g_x(x, y) = 2x + 2$, $g_y(x, y) = -4y$, $g_{xx}(x, y) = 2$, $g_{xy}(x, y) = 0$, and $g_{yy}(x, y) = -4$.

(a) To find the critical point we will set each of the first partial derivatives equal to 0 and solve: $g_x(x, y) = 0$ when $2x+2 = 0$ or when $x = -1$ and $g_y(x, y) = 0$ when $-4y = 0$ or when $y = 0$. So g has a unique critical point at $(-1, 0)$.

(b) The increment

$$\begin{aligned}\Delta g &= g(-1+\Delta x, \Delta y) - g(-1,0)\\ &= (-1+\Delta x)^2 - 2(\Delta y)^2 + 2(-1+\Delta x) + 3 - ((-1)^2 + 2(-1) + 3)\\ &= (\Delta x)^2 - 2(\Delta y)^2.\end{aligned}$$

This tells us that any changes in x result in an increase in the value of g and little changes in y result in a decrease in the value of g. This means that f must have a saddle at $(-1,0)$.

(c) The Hessian is $Hg(-1,0) = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ so $d_1 = 2$ and $d_2 = -8$, so by the second derivative test, g has a saddle at $(-1,0)$.

In Exercises 3–20, most of the mistakes will be algebra mistakes made in solving for the critical points. For Exercises 3–14, you are using the familiar rule for the second derivative test at a point $\mathbf{a} = (a,b)$ *where* $f_x(\mathbf{a}) = 0 = f_y(\mathbf{a})$. *The determinant of the Hessian is often referred to as the discriminant:*

$$D(a,b) = |Hf(a,b)| = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

The second derivative test (see Example 5) is then

- *if* $D(a,b) > 0$ *and*

  if $f_{xx}(a,b) > 0$ *then* f *has a local minimum at* (a,b)
  if $f_{xx}(a,b) < 0$ *then* f *has a local maximum at* (a,b)
- *if* $D(a,b) < 0$ *then* f *has a saddle at* (a,b).
- *Otherwise the test tells us nothing.*

In many calculus classes students never see the extension of this test to higher dimensions. In Exercises 15–20, the students will need to use the $\mathbf{R}^3$ *version of the second derivative test.*

3. $f(x,y) = 2xy - 2x^2 - 5y^2 + 4y - 3$, so $f_x(x,y) = 2y - 4x$ and $f_y(x,y) = 2x - 10y + 4$. At a critical point $2y - 4x = 0$ so $y = 2x$. Also $4 = 10y - 2x = 10y - y = 9y$ so $y = 4/9$ and $x = 2/9$. So f has a critical point at (2/9, 4/9).
We easily calculate the Hessian $Hf = \begin{bmatrix} -4 & 2 \\ 2 & -10 \end{bmatrix}$ so $d_1 = -4$ and $d_2 = 36$. So f has a local maximum at (2/9, 4/9).

4. $f(x,y) = \ln(x^2+y^2+1)$, so $f_x(x,y) = \dfrac{2x}{x^2+y^2+1}$ and $f_y(x,y) = \dfrac{2y}{x^2+y^2+1}$. The only critical point of f is at the origin.
The second derivatives are $f_{xx}(x,y) = \dfrac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$, $f_{yy}(x,y) = \dfrac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$, and also $f_{xy}(x,y) = \dfrac{4xy}{(x^2+y^2+1)^2}$. At the origin, the Hessian $Hf(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ so $d_1 = 2$ and $d_2 = 4$. So f has a local minimum at (0, 0).

5. $f(x,y) = x^2 + y^3 - 6xy + 3x + 6y$, so $f_x(x,y) = 2x - 6y + 3$ and $f_y(x,y) = 3y^2 - 6x + 6$. At a critical point for f, $2x = 6y - 3$ and $0 = 3y^2 - 6x + 6$ so $0 = y^2 - 2x + 2$. Substituting, $0 = y^2 - 6y + 5 = (y-1)(y-5)$. We have critical points at (3/2, 1) and (27/2, 5).
The second derivatives are $f_{xx}(x,y) = 2$, $f_{yy}(x,y) = 6y$, and $f_{xy}(x,y) = -6$. $d_1 = 2$ and $d_2 = 12y - 36$. In other words, d_1 is always positive and d_2 is positive when $y = 5$ and negative when $y = 1$ so by the second derivative test f has a saddle point at (3/2, 1) and f has a local minimum at (27/2, 5).

6. $f(x,y) = y^4 - 2xy^2 + x^3 - x$, so $f_x(x,y) = -2y^2 + 3x^2 - 1$ and $f_y(x,y) = 4y^3 - 4xy = 4y(y^2 - x)$. At a critical point for f, $y = 0$ or $y^2 = x$. If $y = 0$ then $x = \pm 1/\sqrt{3}$. If $y^2 = x$ then $0 = 3x^2 - 2x - 1 = (3x+1)(x-1)$. This gives us that $x = 1$ or $x = -1/3$ but x can't be negative. So there are four critical points for f: $(\pm 1/\sqrt{3}, 0)$, and $(1, \pm 1)$.
The second derivatives are $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = 12y^2 - 4x$, and $f_{xy}(x,y) = -4y$. $d_1 = 6x$ and $d_2 = 8(9xy^2 - 2y^2 - 3x^2)$. We'll calculate d_i at each critical point to classify them:

Critical Point	d_1	d_2	Classification
$(1/\sqrt{3}, 0)$	$6/\sqrt{3}$	-8	saddle
$(-1/\sqrt{3}, 0)$	$-6/\sqrt{3}$	-8	saddle
$(1, -1)$	6	32	local minimum
$(1, 1)$	6	32	local minimum

7. $f(x,y) = xy + \dfrac{8}{x} + \dfrac{1}{y}$, so $f_x(x,y) = y - \dfrac{8}{x^2}$ and $f_y(x,y) = x - \dfrac{1}{y^2}$. At a critical point for f, $x = \dfrac{1}{y^2}$ and $y = \dfrac{8}{x^2} = 8y^4$ so $0 = y(8y^3 - 1)$ so either $y = 0$ or $y = 1/2$. Since $y = 0$ is not in the domain of f, the only critical point of f is at (4, 1/2).
The second derivatives are $f_{xx}(x,y) = \dfrac{16}{x^3}$, $f_{yy}(x,y) = \dfrac{2}{y^3}$, and $f_{xy}(x,y) = 1$. $d_1 = \dfrac{16}{x^3}$ and $d_2 = \dfrac{32}{x^3y^3} - 1$. At our critical point both d_1 and d_2 are positive so (4, 1/2) is a local minimum.

8. $f(x,y) = e^x \sin y$ so $f_x(x,y) = e^x \sin y$ and $f_y(x,y) = e^x \cos y$. There are no values of x and y for which both first partials are 0 so there are no critical points.

9. $f(x,y) = e^{-y}(x^2 - y^2)$, so $f_x(x,y) = 2xe^{-y}$ and $f_y(x,y) = -e^{-y}(x^2 - y^2 + 2y)$. At a critical point for f, $x = 0$ and $0 = -y^2 + 2y = -y(y-2)$ so the critical points of f are at (0, 0) and (0, 2).
The second derivatives are $f_{xx}(x,y) = 2e^{-y}$, $f_{yy}(x,y) = e^{-y}(x^2 - y^2 + 4y - 2)$, and $f_{xy}(x,y) = -2xe^{-y}$. $d_1 > 0$ and $d_2(0,y) = -2e^{-2y}(y^2 - 4y + 2)$. In other words, d_1 is always positive and d_2 is negative when $y = 0$ and positive when $y = 2$ so by the second derivative test f has a saddle point at (0, 0) and f has a local minimum at (0, 2).

10. $f(x,y) = x + y - x^2y - xy^2$, so $f_x(x,y) = 1 - 2xy - y^2$ and $f_y(x,y) = 1 - 2xy - x^2$. At a critical point for f, $x^2 = y^2$ so $x = \pm y$. If $x = y$, then $0 = 1 - 2xy - y^2 = 1 - 3x^2$ so $x = y = \pm 1/\sqrt{3}$. If $x = -y$, then $0 = 1 - 2xy - y^2 = 1 + y^2$ for which there are no real solutions. So the critical points for f are $\pm(1/\sqrt{3}, 1/\sqrt{3})$.
The second derivatives are $f_{xx}(x,y) = -2y$, $f_{yy}(x,y) = -2x$, and $f_{xy}(x,y) = -2x - 2y$. $d_1 = -2y$ and $d_2 = -4x^2 - 4xy - 4y^2$. At the critical points d_2 is negative and d_1 is non-zero so f has a saddle point at both $\pm(1/\sqrt{3}, 1/\sqrt{3})$.

11. $f(x,y) = x^2 - y^3 - x^2y + y$, so $f_x(x,y) = 2x - 2xy = 2x(1-y)$ and $f_y(x,y) = -3y^2 - x^2 + 1$. At a critical point for f, either $x = 0$ or $y = 1$. When $x = 0$, y must be $\pm 1/\sqrt{3}$. No solution corresponds to $y = 1$, So the critical points for f are $(0, \pm 1/\sqrt{3})$.
The second derivatives are $f_{xx}(x,y) = 2 - 2y$, $f_{yy}(x,y) = -6y$, and $f_{xy}(x,y) = -2x$. $d_1 = 2 - 2y$ and $d_2 = -12y + 12y^2 - 4x^2$. At $(0, -1/\sqrt{3})$, d_1 is positive and d_2 is positive so f has a local minimum at $(0, -1/\sqrt{3})$. At $(0, 1/\sqrt{3})$, d_1 is positive and d_2 is negative so f has a saddle point at $(0, 1/\sqrt{3})$.

12. $f(x,y) = e^{-x}(x^2 + 3y^2)$, so $f_x(x,y) = (2x - x^2 - 3y^2)e^{-x}$ and $f_y(x,y) = 6ye^{-x}$. From f_y we see that at a critical point for f, we must have $y = 0$. Plugging back into f_x we conclude that there are critical points at (0, 0) and at (2, 0).
The second derivatives are $f_{xx}(x,y) = (2 - 4x + x^2 + 3y^2)e^{-x}$, $f_{yy}(x,y) = -6e^{-x}$, and $f_{xy}(x,y) = -6ye^{-x}$. $d_1 = (2 - 4x + x^2 + 3y^2)e^{-x}$ and $d_2 = 6e^{-2x}(1 - 4x + x^2 - 3y^2)$. At (0, 0), d_1 and d_2 are positive so f has a local minimum at (0, 0). At (2, 0), d_1 and d_2 are negative so f has a saddle point at (2, 0).

13. $f(x,y) = 2x - 3y + \ln xy$, so $f_x(x,y) = 2 + 1/x$ and $f_y(x,y) = -3 + 1/y$. The critical point is $(-1/2, 1/3)$.
The second derivatives are $f_{xx}(x,y) = -1/x^2$, $f_{yy}(x,y) = -1/y^2$, and $f_{xy}(x,y) = 0$. $d_1 = -1/x^2$ and $d_2 = 1/x^2y^2$. At $(-1/2, 1/3)$, d_1 is negative and d_2 is positive so f has a local max at $(-1/2, 1/3)$.

14. $f(x,y) = \cos x \sin y$, so $f_x(x,y) = -\sin x \sin y$ and $f_y(x,y) = \cos x \cos y$. The critical points are of the form $(n\pi, \pi/2 + m\pi)$ and $(\pi/2 + n\pi, m\pi)$ where m and n are integers.
The second derivatives are $f_{xx}(x,y) = -\cos x \sin y$, $f_{yy}(x,y) = -\cos x \sin y$, and $f_{xy}(x,y) = -\sin x \cos y$. $d_1 = -\cos x \sin y$ and $d_2 = \cos^2 x \sin^2 y - \sin^2 x \cos^2 y$. At points of the form $(n\pi, \pi/2 + m\pi)$, d_1 alternates between negative and positive values while d_2 is positive so f has an alternating string of local maxs and mins at such points. At the point $(0, \pi/2)$, for example, f has a local max. At points of the form $(\pi/2 + n\pi, m\pi)$, $d_1 = 0$ and d_2 is negative so such points are saddle points.

15. $f(x,y,z) = x^2 - xy + z^2 - 2xz + 6z$, so $f_x(x,y,z) = 2x - y - 2z$, $f_y(x,y,z) = -x$ and $f_z(x,y,z) = 2z - 2x + 6$. From the second equation, $x = 0$. From the third, then, $z = -3$ and from the first it follows that $y = 6$.
The second derivatives are $f_{xx}(x,y,z) = 2$, $f_{yy}(x,y,z) = 0$, $f_{zz}(x,y,z) = 2$, $f_{xy}(x,y,z) = -1$, $f_{xz}(x,y,z) = -2$ and $f_{yz}(x,y,z) = 0$. $d_1 = 2$, $d_2 = -1$ and $d_3 = -2$ so f has a saddle point at $(0, 6, -3)$.

16. $f(x,y,z) = (x^2 + 2y^2 + 1)\cos z$, so $f_x(x,y,z) = 2x\cos z$, $f_y(x,y,z) = 4y\cos z$ and $f_z(x,y,z) = -(x^2 + 2y^2 + 1)\sin z$. From the third equation, $z = n\pi$. The other two equations imply that x and y both are 0. So the critical points are of the form $(0, 0, n\pi)$.
The second derivatives are $f_{xx}(x,y,z) = 2\cos z$, $f_{yy}(x,y,z) = 4\cos z$, $f_{zz}(x,y,z) = -(x^2 + 2y^2 + 1)\cos z$, $f_{xy}(x,y,z) = 0$, $f_{xz}(x,y,z) = -2x\sin z$ and $f_{yz}(x,y,z) = -4y\sin z$. $d_1 = 2\cos z$ and $d_2 = 8\cos^2 z$. It is easier to calculate d_3 at our critical point. In this case $d_3(0,0,n\pi) = \mp 8$ while $d_1(0,0,n\pi) = \pm 2$, $d_2 = 8$. So f has saddle points at $(0, 0, n\pi)$.

17. $f(x,y,z) = x^2 + y^2 + 2z^2 + xz$ so $f_x(x,y,z) = 2x + z$, $f_y(x,y,z) = 2y$, and $f_z(x,y,z) = 4z + x$. It is easy to see that the only critical point is at the origin.
The Hessian is $Hf = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{bmatrix}$ so $d_1 = 2$, $d_2 = 4$, and $d_3 = 14$. By the second derivative test, f has a local minimum

at (0, 0, 0).

18. $f(x, y, z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$ so $f_x(x, y, z) = 3x^2 + z^2 - 6x$, $f_y(x, y, z) = 2y$, and $f_z(x, y, z) = 2xz + 4z = 2z(x + 2)$. We see immediately that at a critical point of f, $y = 0$ and either $z = 0$ or $x = -2$. If $z = 0$ then $0 = 3x^2 - 6x = 3x(x - 2)$ so $x = 0$ or $x = 2$. If $x = -2$ then $z^2 = -24$ for which there are no real solutions. We conclude that f has critical points at (0, 0, 0) and (2, 0, 0).

The Hessian is $Hf = \begin{bmatrix} 6x-6 & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4 \end{bmatrix}$ so $Hf(x, 0, 0) = \begin{bmatrix} 6x-6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2x+4 \end{bmatrix}$. This makes it easier to calculate $d_1(x, 0, 0) = 6x - 6$, $d_2(x, 0, 0) = 2d_1(x, 0, 0)$, and $d_3(x, 0, 0) = (2x + 4)\, d_2$. At (0, 0, 0) all three d_i's are negative and at (2, 0, 0) all three are positive. By the second derivative test, f has a saddle point at (0, 0, 0) and a local minimum at (2, 0, 0).

19. $f(x, y, z) = xy + xz + 2yz + \dfrac{1}{x}$ so $f_x(x, y, z) = y + z - \dfrac{1}{x^2}$, $f_y(x, y, z) = x + 2z$, and $f_z(x, y, z) = x + 2y$. We see immediately that at a critical point of f, $y = z$ so both $2z = -x$ and $2z = \dfrac{1}{x^2}$ so $-x = \dfrac{1}{x^2}$ so $x = -1$. Therefore, f has a critical point at $(-1, 1/2, 1/2)$.

The Hessian is $Hf = \begin{bmatrix} 2/x^3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ so $d_1(-1, 1/2, 1/2) = -2$, $d_2(-1, 1/2, 1/2) = -1$, and $d_3(-1, 1/2, 1/2) = 12$.

This is the case of the second derivative test where the conditions are valid but neither of the first two cases holds so f has a saddle point at $(-1, 1/2, 1/2)$.

20. $f(x, y, z) = e^x(x^2 - y^2 - 2z^2)$ so $f_x(x, y, z) = e^x(x^2 + 2x - y^2 - 2z^2)$, $f_y(x, y, z) = -2ye^x$, and $f_z(x, y, z) = -4ze^x$. We see immediately that at a critical point of f, $y = z = 0$ and therefore $0 = x^2 + 2x = x(x + 2)$. The two critical points of f are (0, 0, 0) and $(-2, 0, 0)$.

$$\text{The Hessian is } Hf = \begin{bmatrix} e^x(x^2 + 4x + 2 - y^2 - 2z^2) & -2ye^x & -4ze^x \\ -2ye^x & -2e^x & 0 \\ -4ze^x & 0 & -4e^x \end{bmatrix} \text{ so}$$

$$Hf(x, 0, 0) = \begin{bmatrix} e^x(x^2 + 4x + 2) & 0 & 0 \\ 0 & -2e^x & 0 \\ 0 & 0 & -4e^x \end{bmatrix}.$$

For (0, 0, 0), $d_1 > 0$, $d_2 < 0$, and $d_3 < 0$ so f has a saddle at (0, 0, 0). For $(-2, 0, 0)$, $d_1 < 0$, $d_2 > 0$, and $d_3 < 0$ so f has a local maximum at $(-2, 0, 0)$.

21. **(a)** $f(x, y) = \dfrac{2y^3 - 3y^2 - 36y + 2}{1 + 3x^2}$ so $f_x(x, y) = \dfrac{6x(2y^3 - 3y^2 - 36y + 2)}{(1 + 3x^2)}$ and $f_y(x, y) = \dfrac{6(y^2 - y - 6)}{1 + 3x^2}$ $= \dfrac{6(y - 3)(y + 2)}{1 + 3x^2}$. From f_y we see that either $y = 3$ or $y = -2$. Neither of these values makes $f_x = 0$ so $x = 0$. The critical points for f are $(0, -2)$ and $(0, 3)$.

(b)

$$Hf = \begin{bmatrix} \dfrac{6(3x-1)(3x+1)(2y^3 - 3y^2 - 36y + 2)}{(3x^2+1)^3} & -\dfrac{36x(y-3)(y+2)}{(3x^2+1)^2} \\ -\dfrac{36x(y-3)(y+2)}{(3x^2+1)^2} & \dfrac{6(2y-1)}{3x^2+1} \end{bmatrix} \text{ and}$$

$$Hf(0, y) = \begin{bmatrix} -6(2y^3 - 3y^2 - 36y + 2) & 0 \\ 0 & 6(2y - 1) \end{bmatrix}.$$

At $(0, -2)$ we find that $d_1 < 0$ and $d_2 > 0$ so f has a local maximum at $(0, -2)$. At (0, 3) we find that $d_1 > 0$ and $d_2 > 0$ so f has a local minimum at (0, 3).

22. **(a)** $f(x, y) = kx^2 - 2xy + ky^2$ so $f_x(x, y) = 2kx - 2y$ and $f_y(x, y) = -2x + 2ky$. We see that the origin is a critical point for any value of k. The Hessian is $\begin{bmatrix} 2k & -2 \\ -2 & 2k \end{bmatrix}$ so $d_1 = 2k$ and $d_2 = 4k^2 - 4$. For f to have a non-degenerate local maximum or minimum $d_2 > 0$ so $k^2 - 1 > 0$ so either $k > 1$ or $k < -1$. If $k > 1$, then $d_1 > 0$ and the origin is a non-degenerate local minimum. If $k < -1$, then $d_1 < 0$ and the origin is a non-degenerate local maximum.

(b) $g(x, y, z) = kx^2 + kxz - 2yz - y^2 + kz^2/2$ so $g_x(x, y, z) = 2kx + kz, g_y(x, y, z) = -2z - 2y$, and $g_z(x, y, z) = kx - 2y + kz$. The Hessian is $\begin{bmatrix} 2k & 0 & k \\ 0 & -2 & -2 \\ k & -2 & k \end{bmatrix}$. First note that $d_1 = 2k$ and $d_2 = -4k$. These are of opposite signs so a non-degenerate local minimum is not possible. For a non-degenerate local maximum we need $d_1 < 0$ and $d_2 > 0$ so $k < 0$. We also need $d_3 = 2k(-k - 4) < 0$ so $k < -4$. So we have a non-degenerate local maximum when $k < -4$.

23. If you think of this problem geometrically it should be reasonably straightforward. The slices through the origin where only one variable is allowed to change are parabolas. They open up if the coefficient of the term containing that variable is positive and down if it is negative. This tells you that if all of the coefficients are positive then we have a local minimum, if all of the coefficients are negative then we have a local maximum, and if some are positive and some are negative then we have a saddle point.

(a) $f(x, y) = ax^2 + by^2$ so $f_x(x, y) = 2ax$ and $f_y(x, y) = 2by$. Since neither a nor b is 0, the critical point must be the origin. The Hessian is $Hf = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$. The first condition is that $d_2 > 0$ so $4ab > 0$ so a and b are the same sign. Also, $d_1 = 2a$ so when a and b are negative the origin is a local maximum and when a and b are positive the origin is a local minimum.

(b) $f(x, y) = ax^2 + by^2 + cz^2$ so $f_x(x, y, z) = 2ax, f_y(x, y, z) = 2by$ and $f_z(x, y, z) = 2cz$. Since none of a, b and c is 0, the critical point must be the origin. The Hessian is $Hf = \begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix}$. Again, in either case $d_2 > 0$ so $4ab > 0$ so a and b are the same sign. Also, $d_1 = 2a$ and $d_3 = 8abc$. In either case d_1 and d_3 must be the same sign. When a, b and c are negative the origin is a local maximum and when a, b and c are positive the origin is a local minimum.

(c) Really the analysis is no harder, it is just harder to write down. The function is now $f(x_1, x_2, \ldots, x_n) = a_1x_1^2 + a_2x_2^2 + \cdots + a_nx_n^2$. The first derivatives are $f_{x_i}(x_1, x_2, \ldots, x_n) = 2a_ix_i$. Because none of the a_i is zero and all of the first derivatives are 0, we conclude that the only critical point is at the origin. The Hessian is an $n \times n$ matrix with zeros everywhere off of the main diagonal and the entry in position (i, i) is $2a_i$. We easily calculate $d_i = 2^i a_1 a_2 \ldots a_i$. As above, d_2 must be positive so both a_1 and a_2 are of the same sign. We could continue to argue that $d_4 = 4a_3a_4d_2$ so a_3 and a_4 must be of the same sign. In fact, we can continue that reasoning to say for k odd, a_k and a_{k+1} must be of the same sign. For f to have a local maximum $d_1 < 0$ so a_1 and a_2 are both negative. Also, $d_k = 2a_kd_{k-1}$ and for k odd $d_k < 0$ so we can move up through the entries and argue that all of the a_i's must be negative. Similarly, for f to have a local minimum all of the a_i's must be positive.

Note: In Exercises 24–27 we have used a computer algebra system. In fact, I've used Mathematica. In Exercise 24, I've included a list of the relevant commands. These were adapted for each of the exercises.

24. We'll use the following sequence of commands:

- $f[x_-, y_-] = y^4 - 2xy^2 + x^3 - x$
- Solve $[\{D[f[x, y], x] == 0, D[f[x, y], y] == 0\}]$
- $H = \{\{\partial_{x,x}f[x, y], \partial_{x,y}f[x, y]\}, \{\partial_{y,x}f[x, y], \partial_{y,y}f[x, y]\}\}$
- MatrixForm $[H/.\{x \to 1, y \to -1\}]$ (since $(1, -1)$ is the critical point found in the second step)

This is how you define the function, solve $\nabla f = 0$, create the Hessian and display it at the critical points.
In this case we get the following solutions to the simultaneous equations: $(-1/3, \pm i/\sqrt{3})$, $(1, \pm 1)$, and $(\pm 1/\sqrt{3}, 0)$. Let's examine the real-valued solutions.
At $(1, 1)$ the Hessian is $\begin{bmatrix} 6 & -4 \\ -4 & 8 \end{bmatrix}$. This means that $d_1 > 0$ and $d_2 > 0$ so $(1, 1)$ is a local minimum.
At $(1, -1)$ the Hessian is $\begin{bmatrix} 6 & 4 \\ 4 & 8 \end{bmatrix}$. This means that $d_1 > 0$ and $d_2 > 0$ so $(1, -1)$ is a local minimum.
At $(-1/\sqrt{3}, 0)$ the Hessian is $\begin{bmatrix} -2\sqrt{3} & 0 \\ 0 & 4/\sqrt{3} \end{bmatrix}$. This means that $d_1 < 0$ and $d_2 < 0$ so $(-1/\sqrt{3}, 0)$ is a saddle point.
At $(1/\sqrt{3}, 0)$ the Hessian is $\begin{bmatrix} 2\sqrt{3} & 0 \\ 0 & -4/\sqrt{3} \end{bmatrix}$. This means that $d_1 < 0$ and $d_2 < 0$ so $(1/\sqrt{3}, 0)$ is a saddle point.

25. The commands are the same as those outlined in Exercise 24. The critical points are $(0, 0)$, $(\pm\sqrt{3/2}, 0)$, and $\pm(1/\sqrt{2}, -1/\sqrt{2})$.
At $(0, 0)$ the Hessian is $\begin{bmatrix} 0 & -3 \\ -3 & -2 \end{bmatrix}$. This means that $d_1 = 0$ and $d_2 < 0$ so $(0, 0)$ is a saddle point.

At $(\pm\sqrt{3/2}, 0)$ the Hessian is $\begin{bmatrix} 0 & 6 \\ 6 & -2 \end{bmatrix}$. Again, $d_1 = 0$ and $d_2 < 0$ so both $(\sqrt{3/2}, 0)$ and $(-\sqrt{3/2}, 0)$ are saddle points.

At $\pm(1/\sqrt{2}, -1/\sqrt{2})$ the Hessian is $\begin{bmatrix} -6 & 0 \\ 0 & -2 \end{bmatrix}$. This means that $d_1 < 0$ and $d_2 > 0$ so both $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$ are local maxima.

26. We need to slightly alter the commands from the previous two exercises. The command to find the roots specified by the three first partials is now:

Solve $[\{D[f[x,y,z],x] == 0, D[f[x,y,z],y] == 0, D[f[x,y,z],z] == 0\}]$.

We also need to change the specification of the Hessian to:

$$\begin{aligned} H = \{&\{\partial_{x,x} f[x,y,z], \partial_{x,y} f[x,y,z], \partial_{x,z} f[x,y,z]\}, \\ &\{\partial_{y,x} f[x,y,z], \partial_{y,y} f[x,y,z], \partial_{y,z} f[x,y,z]\}, \\ &\{\partial_{z,x} f[x,y,z], \partial_{z,y} f[x,y,z], \partial_{z,z} f[x,y,z]\}\} \end{aligned}$$

Finally, it will be helpful to use the computer to calculate the determinant. For *Mathematica* you type Det[M] where M is the matrix for which you wish to calculate the determinant.

The critical points are at $(1-2\sqrt{2}, -\sqrt{2(4-\sqrt{2})}, -\sqrt{4-\sqrt{2}})$, $(1-2\sqrt{2}, \sqrt{2(4-\sqrt{2})}, \sqrt{4-\sqrt{2}})$, $(1+2\sqrt{2}, -\sqrt{2(4+\sqrt{2})}, \sqrt{4+\sqrt{2}})$, $(1+2\sqrt{2}, \sqrt{2(4+\sqrt{2})}, -\sqrt{4+\sqrt{2}})$, and (0, 0, 0).

At (0, 0, 0) the Hessian is $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -4 \end{bmatrix}$. So $d_1 < 0$, $d_2 > 0$ and $d_3 < 0$ so (0, 0, 0) is a local max.

At $(1-2\sqrt{2}, -\sqrt{2(4-\sqrt{2})}, -\sqrt{4-\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & \sqrt{4-\sqrt{2}} & \sqrt{2(4-\sqrt{2})} \\ \sqrt{4-\sqrt{2}} & -2 & 2\sqrt{2} \\ \sqrt{2(4-\sqrt{2})} & 2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = \sqrt{2} > 0$ and $d_3 = 64 - 16\sqrt{2} > 0$ so $(1-2\sqrt{2}, -\sqrt{2(4-\sqrt{2})}, -\sqrt{4-\sqrt{2}})$ is a saddle point.

At $(1-2\sqrt{2}, \sqrt{2(4-\sqrt{2})}, \sqrt{4-\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & -\sqrt{4-\sqrt{2}} & -\sqrt{2(4-\sqrt{2})} \\ -\sqrt{4-\sqrt{2}} & -2 & 2\sqrt{2} \\ -\sqrt{2(4-\sqrt{2})} & 2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = \sqrt{2} > 0$ and $d_3 = 64 - 16\sqrt{2} > 0$ so $(1-2\sqrt{2}, \sqrt{2(4-\sqrt{2})}, \sqrt{4-\sqrt{2}})$ is a saddle point.

At $(1+2\sqrt{2}, -\sqrt{2(4+\sqrt{2})}, \sqrt{4+\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & -\sqrt{4+\sqrt{2}} & \sqrt{2(4+\sqrt{2})} \\ -\sqrt{4+\sqrt{2}} & -2 & -2\sqrt{2} \\ \sqrt{2(4+\sqrt{2})} & -2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = -\sqrt{2} < 0$ and $d_3 = 64 + 16\sqrt{2} > 0$ so $(1+2\sqrt{2}, -\sqrt{2(4+\sqrt{2})}, \sqrt{4+\sqrt{2}})$ is a saddle point.

At $(1+2\sqrt{2}, \sqrt{2(4+\sqrt{2})}, -\sqrt{4+\sqrt{2}})$ the Hessian is $\begin{bmatrix} -2 & \sqrt{4+\sqrt{2}} & -\sqrt{2(4+\sqrt{2})} \\ \sqrt{4+\sqrt{2}} & -2 & -2\sqrt{2} \\ -\sqrt{2(4+\sqrt{2})} & -2\sqrt{2} & -4 \end{bmatrix}$.

So, $d_1 = -2 < 0$ and $d_2 = -\sqrt{2} < 0$ and $d_3 = 64 + 16\sqrt{2} > 0$ so $(1+2\sqrt{2}, \sqrt{2(4+\sqrt{2})}, -\sqrt{4+\sqrt{2}})$ is a saddle point.

27. The commands are extended as they were in Exercise 26. The critical points are (0, 0, 0, 0), $(-\sqrt{2}, 2\sqrt{2}, 1, -\sqrt{2})$, $(\sqrt{2}, 2\sqrt{2}, -1, -\sqrt{2})$, $(-\sqrt{2}, -2\sqrt{2}, -1, \sqrt{2})$, and $(\sqrt{2}, -2\sqrt{2}, 1, \sqrt{2})$.

At (0, 0, 0, 0) the Hessian is $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = 0$, $d_3 = 0$, and $d_4 = -8 < 0$, so (0, 0, 0, 0) is a saddle point.

At $(-\sqrt{2}, 2\sqrt{2}, 1, -\sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & -1 & -2\sqrt{2} & 0 \\ -1 & 0 & \sqrt{2} & 1 \\ -2\sqrt{2} & \sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(-\sqrt{2}, 2\sqrt{2}, 1, -\sqrt{2})$ is a saddle point.

At $(\sqrt{2}, 2\sqrt{2}, -1, -\sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & 1 & -2\sqrt{2} & 0 \\ 1 & 0 & -\sqrt{2} & 1 \\ -2\sqrt{2} & -\sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(\sqrt{2}, 2\sqrt{2}, -1, -\sqrt{2})$ is a saddle point.

At $(-\sqrt{2}, -2\sqrt{2}, -1, \sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & 1 & 2\sqrt{2} & 0 \\ 1 & 0 & \sqrt{2} & 1 \\ 2\sqrt{2} & \sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(-\sqrt{2}, -2\sqrt{2}, -1, \sqrt{2})$ is a saddle point.

At $(\sqrt{2}, -2\sqrt{2}, 1, \sqrt{2})$ the Hessian is $\begin{bmatrix} -2 & -1 & 2\sqrt{2} & 0 \\ -1 & 0 & -\sqrt{2} & 1 \\ 2\sqrt{2} & -\sqrt{2} & -4 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$. So $d_1 = -2 < 0$, $d_2 = -1 < 0$, $d_3 = 16 > 0$, and $d_4 = 32 > 0$, so $(\sqrt{2}, -2\sqrt{2}, 1, \sqrt{2})$ is a saddle point.

28. We want to maximize $V = xyz$ subject to the constraint $2xy + 2xz + 2yz = c$. Solve the second equation for $z = \dfrac{c - 2xy}{2x + 2y}$ and substitute to get

$$V(x, y) = \frac{cxy - 2x^2y^2}{2x + 2y}.$$

The derivatives are $V_x = -\dfrac{y^2(2x^2 + 4xy - c)}{2(x + y)^2}$ and $V_y = -\dfrac{x^2(2y^2 + 4xy - c)}{2(x + y)^2}$. Since neither x nor y could be zero (we wouldn't have a box), a critical point of f occurs when both $2x^2 + 4xy - c = 0$ and $2y^2 + 4xy - c = 0$. Solving these together we find that $x^2 = y^2$ and since x and y are positive we conclude that $x = y$. Substituting back in, $0 = 2x^2 + 4xy - c = 2x^2 + 4x^2 - c = 6x^2 - c$ so $x = y = \sqrt{c/6}$. $z = \dfrac{c - 2xy}{2x + 2y} = \dfrac{c - (c/3)}{4\sqrt{c/6}} = \sqrt{c/6}$. So our only critical point is when the box is a cube. To conclude that this is a local maximum we see that $d_1 = -\dfrac{y^2(c + 2y^2)}{(x + y)^3} < 0$ and at our critical point $d_2 = -\dfrac{2x^2y^2(2x^2 + 8xy + 2y^2 - 3c)}{(x + y)^4} = -\dfrac{2x^4(12x^2 - 3c)}{(2x)^4} = -\dfrac{2c - 3c}{8} > 0$. So the largest rectangular box with fixed surface area is a cube.

29. We will actually minimize the square of the distance (i.e., the sum of the squares of the differences in each direction): $D(x, y) = x^2 + y^2 + (3x - 4y - 24)^2$ so $D_x(x, y) = 20x - 24y - 144$ and $D_y(x, y) = 34y - 24x + 192$. Set these equal to 0 and solve to get that the point on the plane closest to the origin is $(36/13, -48/13, -12/13)$.

30. Again we will minimize the square of the distance. For points (x, y, z) on the surface we have $z^2 = 4 - xy$, so that the square of the distance $x^2 + y^2 + z^2 = x^2 + y^2 + 4 - xy$; thus we consider the function $D(x, y) = x^2 - xy + y^2 + 4$. We have $D_x(x, y) = 2x - y$ and $D_y(x, y) = 2y - x$. Set the partial derivatives equal to 0 and solve the system

$$\begin{cases} 2x - y = 0 \\ -x + 2y = 0 \end{cases}.$$

The only solution is $(0, 0)$. This solution corresponds to the points $(0, 0, 2)$ and $(0, 0, -2)$ on the surface $xy + z^2 = 4$. To see that these points really do give the minimum distance, we rewrite D as

$$D(x, y) = x^2 - xy + y^2 + 4 = \left(x - \frac{y}{2}\right)^2 + \frac{3y^2}{4} + 4.$$

Thus we see that $D(x, y) \geq 4$ for all (x, y) and $D = 4$ exactly when $x = y = 0$.

31. We solve

$$\begin{cases} R_x(x, y) = 8 - 2x + 2y = 0 \\ R_y(x, y) = 6 - 4y + 2x = 0 \end{cases}.$$

Adding the two equations gives $14 - 2y = 0$ which implies that $y = 7$. Using this in the first equation gives $22 - 2x = 0$ so that $x = 11$. Hence $(11, 7)$ is the unique critical point. A quick check with the Hessian

$$HR(11,7) = \begin{bmatrix} -2 & 2 \\ 2 & -4 \end{bmatrix}$$

reveals that $d_1 = -2$, $d_2 = 8 - 4 = 4$, so this critical point yields a maximum value of R. (Note: we may rewrite the revenue function as $R(x, y) = 8x + 6y - (x - y)^2 - y^2$. From this it is clear that this critical point must be a global maximum.) Thus you should manufacture 1100 units of model X and 700 units of model Y.

Exercises 32–39 force us to check values on the border of our region.

32. $f(x, y) = x^2 + xy + y^2 - 6y$ so $f_x(x, y) = 2x + y$ and $f_y(x, y) = x + 2y - 6$. At a critical point for f, $y = -2x$ so $6 = x + 2y = -3x$. Our only critical point is $(-2, 4)$. We need to check the value of f at the critical point and along the boundary of the region $-3 \le x \le 3, 0 \le y \le 5$.

- $f(-2, 4) = -12$,
- $f(-3, y) = 9 - 9y + y^2$ has a minimum of -11.25 at $y = 4.5$ and a maximum of 9 at $y = 0$,
- $f(3, y) = 9 - 3y + y^2$ has a minimum of 27/4 at $y = 3/2$ and a maximum of 19 at $y = 5$,
- $f(x, 0) = x^2$ which has a minimum of 0 at $x = 0$ and a maximum of 9 at $x = \pm 3$,
- $f(x, 5) = x^2 + 5x - 5$ has a minimum of -11.25 at $x = -5/2$ and a maximum of 19 at $x = 3$.

The absolute maximum is, therefore, 19 at (3, 5) and the absolute minimum is -12 at $(-2, 4)$.

33. $f(x, y, z) = x^2 + xz - y^2 + 2z^2 + xy + 5x$ so $f_x(x, y, z) = 2x + y + z + 5$, $f_y(x, y, z) = x - 2y$, and $f_z(x, y, z) = x + 4z$. At a critical point for f, $x = 2y = -4z$ so $-5 = 2x + y + z = -8z - 2z + z = -9z$. Our only critical point is $(-20/9, -10/9, 5/9)$ which is not within our region. We need to check the value of f along the boundary of the region $-5 \le x \le 0, 0 \le y \le 3, 0 \le z \le 2$. This consists of six two-dimensional faces, twelve one-dimensional edges and eight vertices.

- $f(x, 0, 0) = x^2 + 5x$ has a minimum of -6.25 at $x = -5/2$ and a maximum of 0 at $x = -5$ or 0,
- $f(x, 0, 2) = x^2 + 7x + 8$ has a minimum of -4.25 at $x = -7/2$ and a maximum of 8 at $x = 0$,
- $f(x, 3, 0) = x^2 + 8x - 9$ has a minimum of 25 at $x = -4$ and a maximum of -9 at $x = 0$,
- $f(x, 3, 2) = x^2 + 10x - 1$ has a minimum of -26 at $x = -5$ and a maximum of -1 at $x = 0$,
- $f(-5, y, 0) = -y^2 - 5y$ has a minimum of -24 at $y = 3$ and a maximum of 0 at $y = 0$,
- $f(0, y, 0) = -y^2$ has a minimum of -9 at $y = 3$ and a maximum of 0 at $y = 0$,
- $f(-5, y, 2) = -y^2 - 5y - 2$ has a minimum of -26 at $y = 3$ and a maximum of -2 at $y = 0$,
- $f(0, y, 2) = 8 - y^2$ has a minimum of -1 at $y = 3$ and a maximum of 8 at $y = 0$,
- $f(-5, 0, z) = 2z^2 - 5z$ has a minimum of $-25/8$ at $z = 5/4$ and a maximum of 0 at $z = 0$,
- $f(0, 0, z) = 2z^2$ has a minimum of 0 at $z = 0$ and a maximum of 8 at $z = 2$,
- $f(-5, 3, z) = 2z^2 - 5z - 24$ has a minimum of $-217/8$ at $z = 5/4$ and a maximum of -24 at $z = 0$,
- $f(0, 3, z) = 2z^2 - 9$ has a minimum of -9 at $z = 0$ and a maximum of -1 at $z = 2$.

You also must check for extrema on each face and at each vertex. When you do you find: The absolute maximum is 8 at (0, 0, 2) and the absolute minimum is $-191/7$ at $(-32/7, 3, 8/7)$.

34. In a fit of compassion, the author of the text has not forced Livinia the housefly to walk around the metal plate in search of the hottest and coldest points. The temperature is $T(x, y) = 2x^2 + y^2 - y - 3$ so $T_x(x, y) = 4x$ and $T_y(x, y) = 2y - 1$. We have a critical point for T at (0, 1/2) and $T(0, 1/2) = 2.75$. To check the temperature of the boundary we note that it is a unit disk and so $x = \cos\theta$ and $y = \sin\theta$. We can rewrite $T(\theta) = 2\cos^2\theta + \sin^2\theta - \sin\theta + 3 = \cos^2\theta - \sin\theta + 4$. Then $T_\theta(\theta) = -2\cos\theta\sin\theta - \cos\theta = -\cos\theta(2\sin\theta + 1)$. We, therefore, have critical points on the boundary when $\cos\theta = 0$ (so $\theta = \pi/2$ or $3\pi/2$) and when $\sin\theta = -1/2$ (so $\theta = 7\pi/6$ or $11\pi/6$). Checking the values we see that $T(\pi/2) = 3$, $T(3\pi/2) = 5$ and $T(7\pi/6) = T(11\pi/6) = 21/4$. We conclude that the coldest spot on the plate is at (0, 1/2) where the temperature is 11/4 and the two hottest spots are at $(\pm\sqrt{3}/2, -1/2)$ where the temperature is 21/4.

35. Because the function is "separable", we can analyze it without calculus. The maximum value for f is 1 and the minimum value for f is -1. The absolute maximum is achieved at $(\pi/2, 0)$, $(\pi/2, 2\pi)$, and $(3\pi/2, \pi)$. The absolute minimum is achieved at $(3\pi/2, 0)$, $(3\pi/2, 2\pi)$, and $(\pi/2, \pi)$.

36.

$$\frac{\partial f}{\partial x} = -2\sin x$$
$$\frac{\partial f}{\partial y} = 3\cos y$$

So "ordinary" critical points on $\{(x,y)|0 \le x \le 4,\ 0 \le y \le 3\}$ are at $\left(0, \frac{\pi}{2}\right), \left(\pi, \frac{\pi}{2}\right)$. (In fact, $\left(\pi, \frac{\pi}{2}\right)$ is the only critical point that's actually in the interior of the rectangle.)
Now we look at the boundary of the rectangle:

$$\begin{array}{ll} f_1(x) = f(x,0) = 2\cos x & f_1'(x) = -2\sin x \text{ so critical points at } (0,0), (\pi,0); \\ f_2(x) = f(x,3) = 2\cos x + 3\sin 3 & f_2'(x) = -2\sin x \text{ so critical points at } (0,3), (\pi,3); \\ f_3(y) = f(0,y) = 2 + 3\sin y & f_3'(y) = 3\cos y \text{ so critical point at } \left(0, \frac{\pi}{2}\right); \\ f_4(y) = f(4,y) = 2\cos 4 + 3\sin y & f_4'(y) = 3\cos y \text{ so critical point at } \left(4, \frac{\pi}{2}\right). \end{array}$$

Now we compare values:

(x,y)	$f(x,y) = 2\cos x + 3\sin y$
$\left(0, \frac{\pi}{2}\right)$	5
$\left(\pi, \frac{\pi}{2}\right)$	1
(0, 0)	2
$(\pi, 0)$	-2
(0, 3)	$2 + 3\sin 3 \approx 2.423$
$(\pi, 3)$	$-2 + 3\sin 3 \approx -1.577$
$\left(4, \frac{\pi}{2}\right)$	$2\cos 4 + 3 \approx 1.693$
(4, 0)	$2\cos 4 \approx -1.307$
(4, 3)	$2\cos 4 + 3\sin 3 \approx -0.884$

Thus the absolute minimum occurs at $(\pi, 0)$ and is -2. The absolute maximum occurs at $\left(0, \frac{\pi}{2}\right)$ and is 5.

37. $f(x,y) = 2x^2 - 2xy + y^2 - y + 3$, so $f_x(x,y) = 4x - 2y$ and $f_y(x,y) = -2x + 2y - 1$. At a critical point for f we have $y = 2x$, so $-2x + 4x - 1 = 0$. Thus the only critical point is $(\frac{1}{2}, 1)$.

Now we need to consider the boundary of the region. It consists of three parts: (1) the horizontal line $y = 0$, where $0 \le x \le 2$; (2) the vertical line $x = 0$, where $0 \le y \le 2$; (3) the line $x + y = 2$ (or $y = 2 - x$), where $0 \le x \le 2$. Thus we compare

- $f(\frac{1}{2}, 1) = \frac{5}{2}$,
- $f(x,0) = 2x^2 + 3$ has a minimum of 3 at $x = 0$ and a maximum of 11 at $x = 2$,
- $f(0,y) = y^2 - y + 3$ has a minimum of $\frac{11}{4}$ at $y = \frac{1}{2}$ and a maximum of 5 at $y = 2$,
- $f(x, 2-x) = 5x^2 - 7x + 5$ has a minimum of $\frac{51}{20}$ at $x = \frac{7}{10}$ and a maximum of 11 at $x = 2$

Thus the absolute minimum is $\frac{5}{2}$ occurring at $(\frac{1}{2}, 1)$ and the absolute maximum is 11 occurring at $(2, 0)$.

38. $f(x,y) = x^2 y$ so $f_x(x,y) = 2xy$ and $f_y(x,y) = x^2$. Therefore the only ordinary critical point is $(0,0)$. The boundary of D may be parametrized by $x = 2\cos t$, $y = \sqrt{3}\sin t$ for $0 \le t < 2\pi$. Thus

$$F(t) = f(2\cos t, \sqrt{3}\sin t) = 4\sqrt{3}\cos^2 t \sin t$$

and

$$\begin{aligned} F'(t) &= 4\sqrt{3}\left(-2\cos t \sin^2 t + \cos^3 t\right) \\ &= 4\sqrt{3}\cos t\left(-2(1-\cos^2 t) + \cos^2 t\right) = 4\sqrt{3}\cos t(3\cos^2 t - 2). \end{aligned}$$

We see that $F'(t) = 0$ when either $\cos t = 0$ (in which case $\sin t = \pm 1$) or $\cos t = \pm\sqrt{2/3}$ (in which case $\sin t = \pm 1/\sqrt{3}$). Thus, in addition to $(0,0)$, we need to consider six more points: $(0, \pm\sqrt{3})$, $(\pm 2\sqrt{2/3}, \pm 1)$. From the following table

(x, y)	$f(x, y) = x^2 y$
$(0, 0)$	0
$(0, \pm\sqrt{3})$	0
$\left(\pm\frac{2\sqrt{2}}{\sqrt{3}}, 1\right)$	$\frac{8}{3}$
$\left(\pm\frac{2\sqrt{2}}{\sqrt{3}}, -1\right)$	$-\frac{8}{3}$

we see that absolute minima occur at $(2\sqrt{2/3}, -1)$ and $(-2\sqrt{2/3}, -1)$ and absolute maxima at $(2\sqrt{2/3}, 1)$ and $(-2\sqrt{2/3}, 1)$.

39. The boundary of the closed ball is given by $x^2 + y^2 - 2y + z^2 + 4z = 0$. Completing the square, we find $x^2 + y^2 - 2y + 1 + z^2 + 4z + 4 = 5$ or $x^2 + (y-1)^2 + (z+2)^2 = 5$. (Note also that $x^2 + y^2 - 2y + z^2 + 4z = x^2 + (y-1)^2 + (z+2)^2 - 5$.)
The function $f(x, y, z) = e^{1-x^2-y^2+2y-z^2-4z}$ has

$$\begin{aligned} f_x(x, y, z) &= -2xe^{1-x^2-y^2+2y-z^2-4z} = 0 && \text{when } x = 0 \\ f_y(x, y, z) &= (-2y+2)e^{1-x^2-y^2+2y-z^2-4z} = 0 && \text{when } y = 1 \\ f_z(x, y, z) &= (-2z-4)e^{1-x^2-y^2+2y-z^2-4z} = 0 && \text{when } z = -2 \end{aligned}$$

So $(0, 1, -2)$ is an interior critical point (the only one). Note that on the boundary $x^2 + y^2 - 2y + z^2 + 4z = 0$, we have

$$f(x, y, z) = e^{1-0} = e$$

$$f(0, 1, -2) = e^{1-(-5)} = e^6 \leftarrow \text{so absolute max is at } (0, 1, -2).$$

The absolute minimum of e occurs at *all* points of the boundary. If we set $w = x^2+y^2-2y+z^2+4z$, then $f(x, y, z) = e^{1-w}$, so that it's clear that the minimum must occur when $w = 0$ (since $w \le 0$ defines the domain we are to consider). Likewise, the maximum must occur at the center of the ball.

It's good to take a step back and see that sometimes we can tell what type of critical point we have without using the tools we've developed. In single-variable calculus, when the second derivative test failed to tell us anything we returned either to the first derivative test or to an analysis of the function.

In Exercises 40–45, the exponents are all at least two so (see, for example, Section 2.4, Exercise 27) when the Hessian is evaluated at the origin, all of the entries will be 0. The fact that $Hf(\mathbf{0}) = \mathbf{0}$ means that the Hessian doesn't provide us with any information about the nature of the critical point at the origin. This is part (a) for Exercises 40–45. By a deleted neighborhood of the origin, we will mean a neighborhood of the origin with the origin removed.

40. $f(x, y) = x^2y^2$: in every deleted neighborhood of the origin $f(x, y) > 0$ so $f(0, 0) < f(x, y)$ for every point (x, y) near but not equal to (0, 0) so f has a local minimum at the origin.

41. $f(x, y) = 4 - 3x^2y^2$: in every deleted neighborhood of the origin $x^2y^2 > 0$ so $-3x^2y^2 < 0$ so $f(x, y) < 4$ so $f(0, 0) > f(x, y)$ for every point (x, y) near but not equal to (0, 0) so f has a local maximum at the origin.

42. $f(x, y) = x^3y^3$: in every deleted neighborhood of the origin in quadrants I and III $f(x, y) > 0$ and in quadrants II and IV $f(x, y) < 0$ so f has neither a minimum nor a maximum at the origin.

43. $f(x, y, z) = x^2y^3z^4$: in every deleted neighborhood of the origin where $y > 0$, $f(x, y, z) > 0$; when $y < 0$, $f(x, y, z) < 0$ so f has neither a minimum nor a maximum at the origin.

44. $f(x, y, z) = x^2y^2z^4$: in every deleted neighborhood of the origin $f(x, y, z) > 0$ so $f(0, 0, 0) < f(x, y, z)$ for every point (x, y, z) near but not equal to (0, 0, 0) so f has a local minimum at the origin.

45. $f(x, y, z) = 2 - x^4y^4 - z^4$: in every deleted neighborhood of the origin $x^4y^4 + z^4 > 0$ so $f(x, y, z) < 2$ so $f(0, 0, 0) > f(x, y, z)$ for every point (x, y, z) near but not equal to (0, 0, 0) so f has a local maximum at the origin.

46. $f(x, y) = e^{x^2+5y^2}$. Notice that e^u is a monotone increasing function of u and $x^2 + 5y^2$ has a unique minimum at (0, 0). So f has a local minimum at (0, 0) so $f(0, 0) = 1$ is a global minimum.

47. $f(x, y, z) = e^{2-x^2-2y^2-3x^4}$. Notice that e^u is a monotone increasing function of u and $2 - x^2 - 2y^2 - 3x^4$ has a unique maximum of 2 at (0, 0, 0). So f has a local maximum at (0, 0, 0), so $f(0, 0, 0) = e^2$ is a global maximum.

48. $f(x, y) = x^3 + y^3 - 3xy + 7$.

(a) The first partial derivatives are $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = 3y^2 - 3x$ so we have critical points at (0, 0) and (1, 1). At the origin we have a saddle point. For the behavior at (1, 1), $d_1(1, 1) = 6$ and $d_2(1, 1) = 36 - 9 = 27$. By the second derivative test we have a local minimum.

(b) We know there are no global extrema. Look along the x-axis. The function is $f(x, 0) = x^3 + 7$. As $x \to \infty$ f increases without bound and as $x \to -\infty$ f decreases without bound.

49. There can't be a global maximum because, for example, for fixed y, as $x \to 0+$ the function grows without bound. $f_x(x, y) = y - 1/x$ and $f_y(x, y) = x + 2 - 2/y$ so f has a critical point at (2, 1/2). From the Hessian $\begin{bmatrix} 1/4 & 1 \\ 1 & 8 \end{bmatrix}$ we see that there is a local minimum at (2, 1/2) of $2 + \ln 2$. Note that $f_x(2, y) = y - 1/2$.
We would like to now conclude that f has a unique critical point at (2, 1/2) which is a local minimum and hence it is a global minimum—such a conclusion seems reasonable, but, as Exercise 52 will demonstrate, is not correct. Consider $f_x(x, 1/2) = 1/2 - 1/x$. For $x > 2$ this is positive and so f is increasing along this line. Now look at $f_y(x, y) = x + 2 - 2/y$ for $x \geq 2$. When $y > 1/2$ this is positive and when $0 < y < 1/2$ this is negative. So as we move vertically away from the line $y = 1/2$ for $x \geq 2$ we see that f is increasing. A similar analysis for the remaining regions shows that f has a global minimum at (2, 1/2).

50. First we'll determine the local extrema. We have $f_x(x, y, z) = 3x^2 + 6x - 3z$, $f_y(x, y, z) = 2ye^{y^2+1}$, and $f_z(x, y, z) = 2z - 3x$. Thus the critical points are $(0, 0, 0)$ and $(-1/2, 0, -3/4)$. The Hessian is

$$Hf(x, y, z) = \begin{bmatrix} 6x + 6 & 0 & -3 \\ 0 & (2 + 4y^2)e^{y^2+1} & 0 \\ -3 & 0 & 2 \end{bmatrix}.$$

Thus

$$Hf(0, 0, 0) = \begin{bmatrix} 6 & 0 & -3 \\ 0 & 2e & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

whose sequence of principal minors is $d_1 = 6$, $d_2 = 12e$, $d_3 = 6e$. Thus $(0, 0, 0)$ yields a local minimum. In addition,

$$Hf\left(-\tfrac{1}{2}, 0, -\tfrac{3}{4}\right) = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 2e & 0 \\ -3 & 0 & 2 \end{bmatrix}$$

whose sequence of principal minors is $d_1 = 3$, $d_2 = 6e$, $d_3 = -6e$. Hence this critical point is a saddle point.

There are no global extrema. If we fix y and z both equal to zero, then $f(x, 0, 0) = x^3 + 3x^2 + e$. As $x \to +\infty$, the expression $x^3 + 3x^2 + e$ grows without bound and as $x \to -\infty$, it decreases without bound.

51. (a) We have

$$\frac{\partial f}{\partial x} = -\frac{2}{3}\left[(x-1)(y-2)\right]^{-1/3}(y-2) = -\frac{2(y-2)^{2/3}}{3(x-1)^{1/3}}$$

$$\frac{\partial f}{\partial y} = -\frac{2}{3}\left[(x-1)(y-2)\right]^{-1/3}(x-1) = -\frac{2(x-1)^{2/3}}{3(y-2)^{1/3}}.$$

Note that $\partial f/\partial x$ is undefined when $x = 1$ and zero when $y = 2$ (and $x \neq 1$). Similarly, $\partial f/\partial y$ is undefined when $y = 2$ and zero when $x = 1$ (and $y \neq 2$). Hence the set of critical points consists of all points on the lines $x = 1$ and $y = 2$. Note that these critical points are not isolated.

(b) The domain of f is all of $\mathbf{R}^2$; the expression $[(x-1)(y-2)]^{2/3}$ is always nonnegative and is zero only when either $x = 1$ or $y = 2$. Thus $f(x, y) \leq 3$ for all $(x, y) \in \mathbf{R}^2$ and $f(x, y) = 3$ precisely when either $x = 1$ or $y = 2$. Hence there are (global) maxima of 3 along these lines.

52. (a) Say that f has a local maximum at x_0 and no other critical points. Assume that $f(x_0)$ is not the global maximum. Then there exists a point x_1 such that $f(x_1) > f(x_0)$. By the extreme value theorem, on the closed interval with endpoints x_0 and x_1 there must be a global maximum and a global minimum somewhere on that closed interval. The global minimum could not be at x_0 since it is a local maximum. It could not be at x_1, since $f(x_1) > f(x_0)$. The global minimum must be somewhere on the open interval and it must be at a critical point. This contradicts the assumption that there were no other critical points. If instead the unique critical point of f were a local minimum, then just modify the argument appropriately.

(b) $f(x, y) = 3ye^x - e^{3x} - y^3$ so $f_x(x, y) = 3ye^x - 3e^{3x}$ and $f_y(x, y) = 3e^x - 3y^2$. Solving, $y = 0$ or $y = 1$, but y can't be 0 since $e^x = y^2$. The only critical point for f is at (0, 1) and $f(0, 1) = 1$. Also, $d_1(0, 1) = f_{xx}(0, 1) = -6$ and $d_2(0, 1) = 27$ so at (0, 1) f has a local maximum. Along the y-axis, $f(0, y) = 3y - 1 - y^3$, so as $y \to -\infty$ we see that f increases without bound.

53. (a) Let the local maxima occur at $a < b$. Consider f on [*a, b*]. By the extreme value theorem, f must attain both a maximum and minimum somewhere on [*a, b*]. The minimum cannot occur at a or b since local maxima occur there. Hence there must be some c is the *open* interval (*a, b*) that gives an absolute minimum on [*a, b*]—hence it must be at least a local minimum on **R**.

(b)

$$f_x(x,y) = -2(xy^2 - y - 1)y^2$$
$$f_y(x,y) = -2(xy^2 - y - 1)(2xy - 1) - 4(y^2 - 1)y$$

For $f_x = 0$, either $xy^2 - y - 1 = 0$ or $y^2 = 0$ (so $y = 0$). If $y = 0$, then the $f_y = 0$ equation becomes $-2(-1)(-1) = 0$, which is false. Thus $xy^2 - y - 1 = 0$ and the $f_y = 0$ equation becomes $-4y(y^2 - 1) = 0$. Since $y \neq 0$, we must have $y^2 - 1 = 0$ or $y = \pm 1$. With $y = 1$ in the $f_x = 0$ equation, we have $-2(x - 2) = 0 \Rightarrow x = 2$. With $y = -1$ in the $f_x = 0$ equation, we have $-2(x + 1 - 1) = 0$ so $x = 0$. So we have two critical points: (2, 1) and $(0, -1)$. The Hessian matrix is

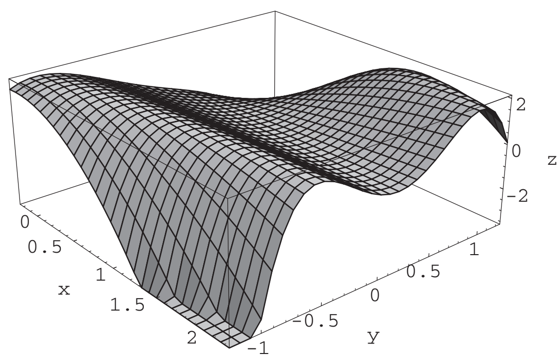
$$Hf(x,y) = \begin{bmatrix} -2y^4 & -2(4xy^3 - 3y^2 - 2y) \\ -2(4xy^3 - 3y^2 - 2y) & -2(6x^2y^2 - 6xy - 2x + 1) - 4(3y^2 - 1) \end{bmatrix}$$

so

$$Hf(2,1) = \begin{bmatrix} -2 & -6 \\ -6 & -26 \end{bmatrix} \text{ sequence of minors is } -2, 16 \Rightarrow \text{local max;}$$

$$Hf(0,-1) = \begin{bmatrix} -2 & 2 \\ 2 & -10 \end{bmatrix} \text{ sequence of minors is } -2, 16 \Rightarrow \text{local max}\,.$$

(c) Best left to a computer. Stay close to the critical points to see the surface details well.



4.3 Lagrange Multipliers

1. The plane is given by $2x - 3y - z = 4$. There will be only one critical point in each case. Geometrically, it cannot be a local maximum because there will always be points nearby which are farther away. There is at least one point on the plane closest to the origin so the single critical point will be at this point. You can also perform the second derivative test.
   (a) We'll minimize the square of the distance: $D(x,y) = x^2 + y^2 + (2x - 3y - 4)^2$. The partials are $D_x(x,y) = 10x - 12y - 16$ and $D_y(x,y) = 20y - 12x + 24$. Set these equal to zero and solve simultaneously to find the critical point $(4/7, -6/7, -2/7)$.
   (b) Minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = 2x - 3y - z = 4$. We solve the system

$$\begin{cases} 2x = 2\lambda \\ 2y = -3\lambda \\ 2z = -\lambda \\ 2x - 3y - z = 4. \end{cases}$$

   We see that $x = \lambda$ so $y = -(3/2)x$ and $z = -x/2$. Substituting into the last equation: $2x + 9x/2 + x/2 = 4$ so $x = 4/7$ and our critical point is $(4/7, -6/7, -2/7)$.
2. The function is $f(x,y) = y$ subject to the constraint $g(x,y) = 2x^2 + y^2 = 4$. We solve the system

$$\begin{cases} 0 = 4\lambda x \\ 1 = 2\lambda y \\ 2x^2 + y^2 = 4. \end{cases}$$

   From the first equation, $\lambda x = 0$, but $\lambda \neq 0$ since $2\lambda y \neq 0$. Hence we must have $x = 0$, so $y^2 = 4$; therefore the critical points are $(0, \pm 2)$.

3. The function is $f(x, y) = 5x + 2y$ subject to the constraint $g(x, y) = 5x^2 + 2y^2 = 14$. We solve the system

$$\begin{cases} 5 = 10\lambda x \\ 2 = 4\lambda y \\ 5x^2 + 2y^2 = 14. \end{cases}$$

By either of the first two equations we see that $\lambda \neq 0$. Together, the first two equations imply that $x = y$ so $7x^2 = 14$ so the critical points are $\pm(\sqrt{2}, \sqrt{2})$.

4. The function is $f(x, y) = xy$ subject to the constraint $g(x, y) = 2x - 3y = 6$. We solve the system

$$\begin{cases} y = 2\lambda \\ x = -3\lambda \\ 2x - 3y = 6. \end{cases}$$

If λ were 0, then both x and y would be 0 which would contradict the third equation. In short, $\lambda \neq 0$. In that case, the first two equations imply that $x = -(3/2)y$ so $-3y - 3y = 6$ or $y = -1$. The critical point is at $(3/2, -1)$.

5. The function is $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 3y + z = 6$. We solve the system

$$\begin{cases} yz = 2\lambda \\ xz = 3\lambda \\ xy = \lambda \\ 2x + 3y + z = 6. \end{cases}$$

One possibility is that two of x, y, and z are zero. In this case the three possible critical points are (3, 0, 0), (0, 2, 0), and (0, 0, 6). If none of x, y, and z is zero then the first two equations imply that $x = (3/2)y$, and the second and third equations together imply that $3y = z$. Hence, $3y + 3y + 3y = 6$, so the final critical point is (1, 2/3, 2).

6. The function is $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = x + y - z = 1$. We solve the system

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ 2z = -\lambda \\ x + y - z = 1. \end{cases}$$

We see immediately that $x = y = -z$, which implies that $x + x + x = 1$. Therefore, the critical point is $(1/3, 1/3, -1/3)$.

7. The function is $f(x, y, z) = 3 - x^2 - 2y^2 - z^2$ subject to the constraint $g(x, y, z) = 2x + y + z = 2$. We solve the system

$$\begin{cases} -2x = 2\lambda \\ -4y = \lambda \\ -2z = \lambda \\ 2x + y + z = 2. \end{cases}$$

Immediately we have $\lambda = -x = -4y = -2z \iff x = 4y = 2z$. Thus $x = 2z$ and $y = z/2$ so that the last equation of the system becomes $4z + z/2 + z = 2 \iff z = 4/11$. Therefore, there is a unique critical point of $\left(\frac{8}{11}, \frac{2}{11}, \frac{4}{11}\right)$.

8. The function is $f(x, y, z) = x^6 + y^6 + z^6$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 6$. We solve the system

$$\begin{cases} 6x^5 = 2\lambda x \\ 6y^5 = 2\lambda y \\ 6z^5 = 2\lambda z \\ x^2 + y^2 + z^2 = 6. \end{cases}$$

The first equation of the system implies either $x = 0$ or $\lambda = 3x^4$. Similarly, the second equation implies either $y = 0$ or $\lambda = 3y^4$ and the third equation implies either $z = 0$ or $\lambda = 3z^4$. No more than two of x, y, or z can be zero, or else the constraint $x^2 + y^2 + z^2 = 6$ cannot be satisfied. Let us suppose that $y = z = 0$. Then $x = \pm\sqrt{6}$ from the constraint. Hence $(\pm\sqrt{6}, 0, 0)$ are two of the critical points. Similarly, if $x = z = 0$, then we obtain $(0, \pm\sqrt{6}, 0)$ as additional critical points, and if $x = y = 0$ we obtain $(0, 0, \pm\sqrt{6})$. If just $z = 0$, then $\lambda = 3x^4 = 3y^4$, so $x = \pm y$ and the constraint $x^2 + y^2 + z^2 = 6$ implies $2x^2 = 6$ or $x = \pm\sqrt{3}$ and there are thus four more critical points $(\pm\sqrt{3}, \pm\sqrt{3}, 0)$. In a similar manner $(\pm\sqrt{3}, 0, \pm\sqrt{3})$ and $(0, \pm\sqrt{3}, \pm\sqrt{3})$ are critical points. Finally, if none of x, y, or z is zero, then $\lambda = 3x^4 = 3y^4 = 3z^4$, which implies $x = \pm y = \pm z$. Hence the last equation of the system implies that $3x^2 = 6$, so $x = \pm\sqrt{2}$. Therefore, there are eight more critical points, namely $(\pm\sqrt{2}, \pm\sqrt{2}, \pm\sqrt{2})$. Thus there are 26 critical points in all.

9. The function is $f(x, y, z) = 2x + y^2 - z^2$ subject to the two constraints $g_1(x, y, z) = x - 2y = 0$ and $g_2(x, y, z) = x + z = 0$. We solve the system

$$\begin{cases} 2 = \lambda + \mu \\ 2y = -2\lambda \\ -2z = \mu \\ x = 2y \\ x = -z. \end{cases}$$

Solving, we see that $2 = \lambda + \mu = -y - 2z = -x/2 + 2x = 3x/2$. So the critical point is $(4/3, 2/3, -4/3)$.

10. The function is $f(x, y, z) = 2x + y^2 + 2z$ subject to the two constraints $g_1(x, y, z) = x^2 - y^2 = 1$ and $g_2(x, y, z) = x + y + z = 2$. We solve the system

$$\begin{cases} 2 = 2\lambda x + \mu \\ 2y = -2\lambda y + \mu \\ 2 = \mu \\ x^2 - y^2 = 1 \\ x + y + z = 2. \end{cases}$$

The third equation of the system implies that the first equation becomes $2\lambda x = 0$. Thus either $\lambda = 0$ or $x = 0$. If $x = 0$, the fourth equation becomes $-y^2 = 1$, which has no solution. If $\lambda = 0$, then the second equation becomes $2y = 2 \iff y = 1$. Hence $x^2 = 2$ in the fourth equation. Using the last equation, we see that $(\sqrt{2}, 1, 1 - \sqrt{2})$ and $(-\sqrt{2}, 1, 1 + \sqrt{2})$ are the only critical points.

11. The function is $f(x, y, z) = xy + yz$ subject to the two constraints $g_1(x, y, z) = x^2 + y^2 = 1$ and $g_2(x, y, z) = yz = 1$. We solve the system

$$\begin{cases} y = 2\lambda x \\ x + z = 2\lambda y + \mu z \\ y = \mu y \\ x^2 + y^2 = 1 \\ yz = 1. \end{cases}$$

The third equation of the system implies that either $\mu = 1$ or $y = 0$. However, y cannot be zero from the last equation. Thus $\mu = 1$ and the second equation reduces to $x = 2\lambda y$, and the first equation becomes $y = 4\lambda^2 y$. Thus either $y = 0$ (which we reject) or $\lambda = \pm 1/2$. This in turn implies that $x = \pm y$, and the fourth equation thus becomes $2x^2 = 1$, so that $x = \pm 1/\sqrt{2}$ and $y = \pm 1/\sqrt{2}$. Now $z = 1/y$ from the last equation, so there are four critical points:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\sqrt{2}\right).$$

12. The function is $f(x, y, z) = x + y + z$ subject to the two constraints $g_1(x, y, z) = y^2 - x^2 = 1$ and $g_2(x, y, z) = x + 2z = 1$. We solve the system

$$\begin{cases} 1 = -2\lambda x + \mu \\ 1 = 2\lambda y \\ 1 = 2\mu \\ y^2 - x^2 = 1 \\ x + 2z = 1. \end{cases}$$

Solving, we see that $\mu = 1/2$ and $2\lambda = 1/y$ so $1/2 = -2\lambda x = -x/y$ or $y = -2x$. This means that $1 = y^2 - x^2 = 3x^2$, so the critical points are $(-1/\sqrt{3}, 2/\sqrt{3}, (3 + \sqrt{3})/6)$ and $(1/\sqrt{3}, -2/\sqrt{3}, (3 - \sqrt{3})/6)$.

13. (a) The function is $f(x, y) = x^2 + y$ subject to the constraint $g(x, y) = x^2 + 2y^2 = 1$. We solve the system

$$\begin{cases} 2x = 2x\lambda \\ 1 = 4y\lambda \\ x^2 + 2y^2 = 1. \end{cases}$$

From the first equation, we see that either $x = 0$ or $\lambda = 1$. If $\lambda = 1$, then $y = 1/4$, so $x = \pm\sqrt{7/8}$. If $x = 0$, then $y = \pm\sqrt{1/2}$. In short, the critical points are $(\pm\sqrt{7/8}, 1/4)$ and $(0, \pm\sqrt{1/2})$.

(b) $L(\lambda; x, y) = x^2 + y - \lambda(x^2 + 2y^2 - 1)$ so

$$H(\lambda; x, y) = \begin{bmatrix} 0 & -2x & -4y \\ -2x & 2 - 2\lambda & 0 \\ -4y & 0 & -4\lambda \end{bmatrix}.$$

So $-d_3 = -16y[x^2 + 1/2 - 2y]$. Substitute the critical points to find that there are local maxima at $(\pm\sqrt{7/8}, 1/4)$ and local minima at $(0, \pm\sqrt{1/2})$.

14. (a) The function is $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the three constraints $g_1(x, y, z, w) = 2x + y + z = 1$, $g_2(x, y, z, w) = x - 2z - w = -2$ and $g_3(x, y, z, w) = 3x + y + 2w = -1$. We solve the system

$$\begin{cases} 2x = 2\lambda + \mu + 3\nu \\ 2y = \lambda + \nu \\ 2z = \lambda - 2\mu \\ 2w = -\mu + 2\nu \\ 2x + y + z = 1 \\ x - 2z - w = -2 \\ 3x + y + 2w = -1. \end{cases}$$

After a great deal of fussing we find that there is a critical point at $\dfrac{1}{68}(-11, 15, 75, -25)$.

(b)

$$HL(\lambda, \mu, \nu, x, y, z, w) = \begin{bmatrix} 0 & 0 & 0 & -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & -3 & -1 & 0 & -2 \\ -2 & -1 & -3 & 2 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 2 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

We calculate $-d_7 = 628$ and conclude that f has a local minimum at the critical point.

Note: For Exercises 15–19 the Mathematica code would be similar to that in Exercise 15.

15. Input the following three lines into *Mathematica* (or the equivalent into your favorite computer algebra system)

$f = 3xy - 4z$

$g = 3x + y - 2xz$

Solve $[\{D[f, x] == \lambda D[g, x], D[f, y] == \lambda D[g, y], D[f, z] == \lambda D[g, z], 3x + y - 2x\, z == 1\}]$
The solutions are

- $\lambda = \sqrt{6}, (x, y, z) = (\sqrt{2/3}, 1/2, (12 - \sqrt{6})/8)$ and
- $\lambda = -\sqrt{6}, (x, y, z) = (-\sqrt{2/3}, 1/2, (12 + \sqrt{6})/8)$.

16. Use the same basic code you used in Exercise 15, allowing for two Lagrange multipliers. The solution is $\lambda_1 = 482/121$, $\lambda_2 = -107/121$, $(x, y, z) = (31, 29, 5)/11$.

17. Many solutions are returned by *Mathematica*. They are

- $(0, -1, 0)$ for $\lambda = -3/2$
- $(0, 1, 0)$ for $\lambda = 3/2$
- $(-2/3, -2/3, -1/3)$ and $(2/3, -2/3, 1/3)$ for $\lambda = -4/3$
- $(-1, 0, 0)$ and $(1, 0, 0)$ for $\lambda = -1$
- $(0, 0, -1)$ and $(0, 0, 1)$ for $\lambda = 0$ and
- $(\sqrt{11/2}/8, -3/8, -3\sqrt{11/2}/8)$ and $(-\sqrt{11/2}/8, -3/8, 3\sqrt{11/2}/8)$ for $\lambda = 1/8$.

18. The solutions given are

- $(1, -1/2, \pm\sqrt{3/2})$ for $\lambda = -1$
- $((-1 - \sqrt{5})/2, (-3 - \sqrt{5})/4, \pm i5^{1/4}/\sqrt{2})$ for $\lambda = (1 + \sqrt{5})/2$.
- $((1 - \sqrt{5})/2, (-3 + \sqrt{5})/4, \pm 5^{1/4}/\sqrt{2})$ for $\lambda = (1 - \sqrt{5})/2$.
- $(-i, i, 0)$ and $(i, -i, 0)$ for $\lambda = -2$, and
- $(-1, -1, 0)$ and $(1, 1, 0)$ for $\lambda = 2$.

Note that several of the solutions are complex and, for the purposes of this discussion, can be discarded.

19. Here there are two solutions:

- $(w, x, y, z) = ((1-\sqrt{2})/2, 1/\sqrt{2}, 1/\sqrt{2}, (1-\sqrt{2})/2)$ for $\lambda_1 = 2 - 1/\sqrt{2}$, $\lambda_2 = 1 - \sqrt{2}$, and $\lambda_3 = 0$, and
- $(w, x, y, z) = ((1+\sqrt{2})/2, -1/\sqrt{2}, -1/\sqrt{2}, (1+\sqrt{2})/2)$ for $\lambda_1 = 2 + 1/\sqrt{2}$, $\lambda_2 = 1 + \sqrt{2}$, and $\lambda_3 = 0$.

20. (a) We need to solve

$$\begin{cases} 3x^2 = \lambda y \\ 6y = \lambda x \\ xy = -4 \end{cases}$$

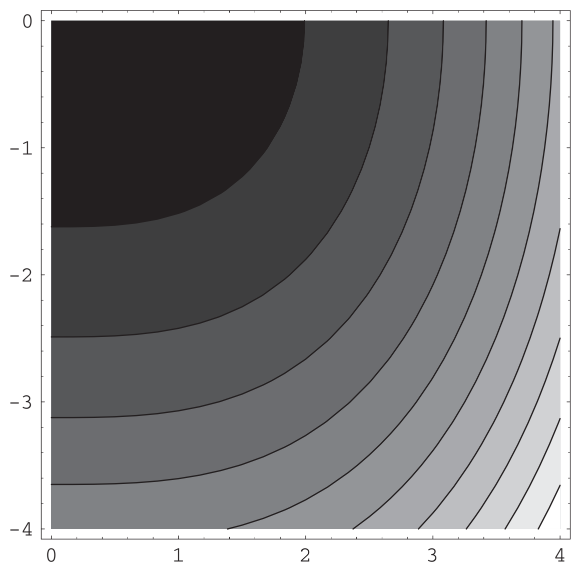
Substitute $y = -4/x$ into the second equation to get $\lambda = -24/x^2$. Substitute both of these into the right side of the first equation to get $x^5 = -32$ or $x = 2$. So $y = -2$ and $\lambda = -6$.

(b) The Hessian in this case is $\begin{bmatrix} 0 & 2 & -2 \\ 2 & 12 & 6 \\ -2 & 6 & 6 \end{bmatrix}$. Following the rule for the second derivative test for constrained local extrema, note that $n = 2$ and $k = 1$ so the only relevant term in sequence (1) is

$$(-1)^1 d_3 = (-1)[(-2)(24) - 2(36)] = 120 > 0.$$

We conclude that there is a constrained local minimum at the point $(2, -2)$.

(c) You can see from the figure below that there is a constrained local minimum at $(2, -2)$ on the curve. This will be the point at which the constraint curve is tangent to one of the level curves.



21. The symmetry of the problem suggests the answer, but we are maximizing $f(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x + y + z = 18$. We solve the system

$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x + y + z = 18. \end{cases}$$

None of the solutions that corresponds to one of x, y, and z being zero is a maximum. The solution we get is $x = y = z$, so $3x = 18$, so the maximum product occurs at the point (6, 6, 6).

22. First, a sphere is a compact surface and the function f is continuous so, by the extreme value theorem, we know that both a minimum and a maximum must be attained. We find the extrema of $f(x, y, z) = x + y - z$ subject to the constraint $x^2 + y^2 + z^2 = 81$. We solve the system

$$\begin{cases} 1 = 2\lambda x \\ 1 = 2\lambda y \\ -1 = 2\lambda z \\ x^2 + y^2 + z^2 = 81. \end{cases}$$

We see that $x = y = -z$, so the critical points are $(3\sqrt{3}, 3\sqrt{3}, -3\sqrt{3})$ and $(-3\sqrt{3}, -3\sqrt{3}, 3\sqrt{3})$. By evaluating at $f(x, y, z) = x + y - z$, we see that the first must yield a maximum of $9\sqrt{3}$, and the second a minimum of $-9\sqrt{3}$.

23. This is a nice problem to assign because by this point some students are only checking boundary values. We are looking for the maximum and minimum values of $f(x, y) = x^2 + xy + y^2$ constrained to be inside the closed disk $g(x, y) = x^2 + y^2 \leq 4$. First we find the critical points without paying attention to the constraint. The partial derivatives are $f_x(x, y) = 2x + y$ and $f_y(x, y) = x + 2y$ so we have a critical point at the origin, and $f(0, 0) = 0$. Next we look for extrema of the function on the boundary of the disk by solving the system

$$\begin{cases} 2x + y = 2\lambda x \\ x + 2y = 2\lambda y \\ x^2 + y^2 = 4. \end{cases}$$

From the first two equations we see that $x^2 = y^2$ so $x = \pm y$ and $x = \pm\sqrt{2}$. Substituting, we find that the minimum is 0 at the origin and the maximum is 6 at $(\sqrt{2}, \sqrt{2})$.

24. We are maximizing $V(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = 2x + 2y + z \leq 108$. In this case, the maximum must occur on the boundary because the only unconstrained critical point requires two of the coordinates to be zero—these points are on the boundary and give the (degenerate) minimum solution of 0. We solve the system

$$\begin{cases} yz = 2\lambda \\ xz = 2\lambda \\ xy = \lambda \\ 2x + 2y + z = 108. \end{cases}$$

Since none of x, y, or z can be zero, we find that $x = y = z/2$, so $3z = 108$ and the critical point is (18, 18, 36). So the dimensions are 18” by 18” by 36”.

25. We are maximizing $f(r, h) = \pi r^2 h$ subject to the constraint that $g(r, h) = 2\pi rh + 2\pi r^2 = c$. We solve the system

$$\begin{cases} 2\pi rh = \lambda(2\pi h + 4\pi r) \\ \pi r^2 = 2\lambda\pi r \\ 2\pi rh + 2\pi r^2 = c. \end{cases}$$

Since $r \neq 0$ the second equation implies that $r = 2\lambda$, so, substituting this into the first equation, we see that $h = 2r$. Hence, the height should equal the diameter.

26. We are minimizing the cost which is $C(r, h) = \pi r^2 + 2(2\pi rh) + 5(2\pi r^2) = 11\pi r^2 + 4\pi rh$ subject to the constraint $g(r, h) = \pi r^2 h + (2/3)\pi r^3 = 900\pi$. We solve the system

$$\begin{cases} 22\pi r + 4\pi h = \pi\lambda(2rh + 2r^2) \\ 4\pi r = \lambda\pi r^2 \\ \pi r^2 h + (2/3)\pi r^3 = 900\pi. \end{cases}$$

As above, we see that $4 = \lambda r$ so $22\pi r + 4\pi h = (4\pi/r)(2rh + 2r^2)$ or $14r = 4h$. Substituting, $900 = (7/2)r^3 + (2/3)r^3 = (25/6)r^3$ so the radius is 6 feet and the height is 21 feet.

27. We wish to minimize $M(x, y, z) = xz - y^2 + 3x + 3$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 9$. We solve the system

$$\begin{cases} z + 3 = 2\lambda x \\ -2y = 2\lambda y \\ x = 2\lambda z \\ x^2 + y^2 + z^2 = 9. \end{cases}$$

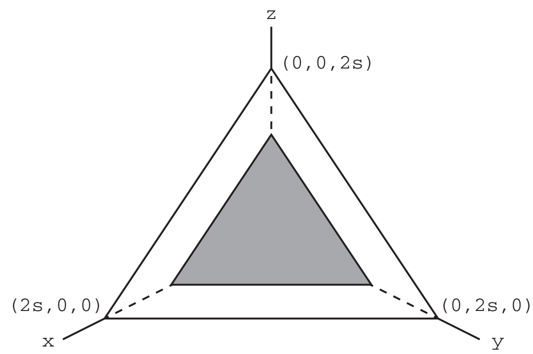
Either $y = 0$ or $\lambda = -1$. If $y = 0$, then $z = -3$ or 3/2 so we get $(0, 0, -3)$ and $(\pm 3\sqrt{3}/2, 0, 3/2)$ as critical points. If $\lambda = -1$, we find the critical points are $(-2, 2, 1)$ and $(-2, -2, 1)$. Comparing values of M, the minimum of -9 is attained at either $(-2, 2, 1)$ or $(-2, -2, 1)$.

28. It’s easier to maximize the *square* of the area $f(x, y, z) = s(s - x)(s - y)(s - z)$ subject to $x + y + z = 2s$ $(= P)$, a constant.

Thus $\nabla f = \lambda \nabla g$ (where $g(x, y, z) = x + y + z$) gives us the system:

$$\begin{cases} -s(s - y)(s - z) = \lambda \\ -s(s - x)(s - z) = \lambda \\ -s(s - x)(s - y) = \lambda \\ \quad x + y + z = 2s \qquad\qquad (0 < x, y, z \leq s) \end{cases}$$

Hence $-s(s-y)(s-z) = -s(s-x)(s-z) = -s(s-x)(s-y)$. The first equality implies $z = s$ or $x = y$. Note that $z = s$ means $f = 0$—so there's zero area which cannot possibly be maximum. Thus $x = y$. From $-s(s-x)(s-z) = -s(s-x)(s-y)$ we similarly conclude that $y = z$. Hence $x = y = z$ $\left(= \frac{2}{3}s\right)$ gives us our critical point and corresponds to having an equilateral triangle. Our constraint looks like a portion of a plane. The dark triangle in the figure below is the part to be considered—it's where f is ≥ 0. Therefore, the point $\left(\frac{2}{3}s, \frac{2}{3}s, \frac{2}{3}s\right)$ yields the maximum.



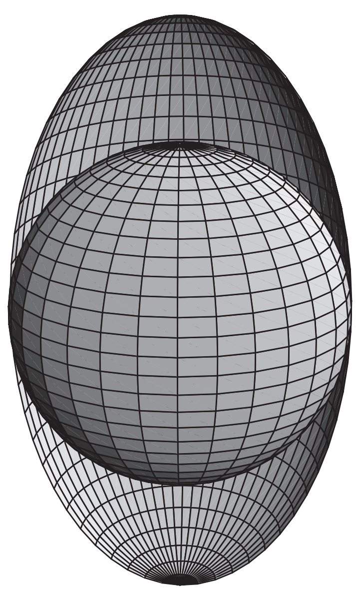
29. A sphere centered at the origin has equation $x^2 + y^2 + z^2 = r^2$. Thus we want to maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 3x^2 + 2y^2 + z^2 = 6$. We can solve this using Lagrange multipliers, but we must make sure we find an *inscribed* sphere. We consider the system

$$\begin{cases} 2x = 6\lambda x & 1^{\text{st}} \text{ equation gives } x = 0 \text{ or } \lambda = 1/3 \\ 2y = 4\lambda y & 2^{\text{nd}} \text{ equation gives } y = 0 \text{ or } \lambda = 1/2 \\ 2z = 2\lambda z & 3^{\text{rd}} \text{ equation gives } z = 0 \text{ or } \lambda = 1 \\ 3x^2 + 2y^2 + z^2 = 6 & (\text{Note that we can't have } x = y = z = 0 \\ & \quad \text{and still satisfy the constraint.}) \end{cases}$$

Thus if $\lambda = 1/3$, $y = z = 0$ and the constraint implies $x = \pm\sqrt{2}$. If $\lambda = 1/2$, $x = z = 0$ and $y = \pm\sqrt{3}$. Finally, if $\lambda = 1$, then $x = y = 0$ and $z = \pm\sqrt{6}$. Comparing values, we have

$$f(\pm\sqrt{2}, 0, 0) = 2, \quad f(0, \pm\sqrt{3}, 0) = 3, \quad f(0, 0, \pm\sqrt{6}) = 6,$$

so that it's tempting to say that the largest sphere has a radius of $\sqrt{6}$. However, such a sphere is not actually inscribed in the ellipsoid. The largest sphere that actually remains inscribed in the ellipsoid has a radius of $\sqrt{2}$.



30. This is just Exercise 1 with two constraints. We are minimizing $f(x, y, z) = x^2 + y^2 + z^2$ with the constraints $g_1(x, y, z) = 2x + y + 3z = 9$ and $g_2(x, y, z) = 3x + 2y + z = 6$. We solve the system

$$\begin{cases} 2x = 2\lambda + 3\mu \\ 2y = \lambda + 2\mu \\ 2z = 3\lambda + \mu \\ 2x + y + 3z = 9 \\ 3x + 2y + z = 6. \end{cases}$$

Eliminate λ and μ and then solve to get a critical point at (1, 2/5, 11/5).

31. This is just Exercise 22 translated by $(2, 5, -1)$. We are minimizing $f(x, y, z) = (x-2)^2 + (y-5)^2 + (z+1)^2$ with the constraints $g_1(x, y, z) = x - 2y + 3z = 8$ and $g_2(x, y, z) = 2z - y = 3$. We solve the system

$$\begin{cases} 2(x-2) = \lambda \\ 2(y-5) = -2\lambda - \mu \\ 2(z+1) = 3\lambda + 2\mu \\ x - 2y + 3z = 8 \\ 2z - y = 3. \end{cases}$$

Eliminate μ by combining the second and third equations and then substitute $2(x-2)$ for λ. Solve to get a critical point at (9/2, 2, 5/2).

32. We want to maximize and minimize the distance function $\sqrt{x^2 + y^2 + z^2}$, but the task is equivalent to finding the extrema of the *square* of the distance. Hence we find the extrema of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints that $g_1(x, y, z) = x + y + z = 4$ and $g_2(x, y, z) = x^2 + y^2 - z = 0$. Note that f is continuous and the ellipse defined by the constraints is compact, so the extreme value theorem guarantees that f has a global maximum and a global minimum on the ellipse. From the Lagrange multiplier equation $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, plus the constraints, we see that we must solve the system

$$\begin{cases} 2x = 2\lambda_1 x + \lambda_2 \\ 2y = 2\lambda_1 y + \lambda_2 \\ 2z = -\lambda_1 + \lambda_2 \\ x + y + z = 4 \\ x^2 + y^2 - z = 0. \end{cases}$$

The first two equations imply $\lambda_2 = 2x - 2\lambda_1 x = 2y - 2\lambda_1 y$, so that $2x(1-\lambda_1) = 2y(1-\lambda_1)$. Hence either $\lambda_1 = 1$ or $x = y$. If $\lambda_1 = 1$, then $\lambda_2 = 0$ and the third equation becomes $2z = -1$, so $z = -1/2$. The last two equations are thus $x + y - 1/2 = 4$ and $x^2 + y^2 + 1/2 = 0$. However, there can be no real solutions to $x^2 + y^2 = -1/2$. Therefore, the case that $\lambda_1 = 1$ leads to no critical points.

If $x = y$, then the last two equations become $2x + z = 4$ and $2x^2 - z = 0$. Hence $z = 4 - 2x$, so that $2x^2 - z = 0$ is equivalent to $2x^2 + 2x - 4 = 0$, which has solutions $x = -2, 1$. Therefore our critical points are $(-2, -2, 8)$ and $(1, 1, 2)$.

Finally, note that $f(-2, -2, 8) = 72 > f(1, 1, 2) = 6$. Hence, in view of the initial observations above, $(1, 1, 2)$ is the point on the ellipse nearest the origin and $(-2, -2, 8)$ the point farthest from the origin.

33. This is the same as Exercise 32 except that we are trying to find extrema for $f(x, y, z) = z$ and the plane has the equation $g_1(x, y, z) = x + y + 2z = 2$. Again, using a computer algebra system we find that the lowest point is at (1/2, 1/2, 1/2) and the highest is at $(-1, -1, 2)$.

34. Minimize $f(x, y, u, v) = (x-u)^2 + (y-v)^2$ subject to the two constraints: $g_1(x, y, u, v) = x^2 + 2y^2 = 1$ and $g_2(x, y, u, v) = u + v = 4$. We solve the system

$$\begin{cases} 2(x-u) = 2\lambda x \\ -2(x-u) = \mu \\ 2(y-v) = 2\lambda y \\ -2(y-v) = \mu \\ x^2 + 2y^2 = 1 \\ u + v = 4. \end{cases}$$

Solving you get two critical points $(x, y, u, v) = (\sqrt{2/3}, \sqrt{1/6}, 2+\sqrt{6}/12, 2-\sqrt{6}/12)$ for which the square of the distance is $35/4 - 2\sqrt{6} \approx 3.85$ and $(x, y, u, v) = (-\sqrt{2/3}, -\sqrt{1/6}, 2-\sqrt{6}/12, 2+\sqrt{6}/12)$ for which the square of the distance is $35/4 + 2\sqrt{6} \approx 13.65$. The minimum distance is $\sqrt{35/4 - 2\sqrt{6}} \approx 1.96$.

35. **(a)** $f(x, y) = x + y$ with the constraint $xy = 6$ so we solve the system

$$\begin{cases} 1 = 2\lambda y \\ 1 = 2\lambda x \\ xy = 6. \end{cases}$$

So $x = y$ and the critical points are at $\pm(\sqrt{6}, \sqrt{6})$.

(b) The constraint curve is not connected. There are two distinct components. Although $(-\sqrt{6}, -\sqrt{6})$ produces a local maximum of $-2\sqrt{6}$ on its component, the value of the function at any point on the other component is greater. Similarly, $(\sqrt{6}, \sqrt{6})$ produces a local minimum of $2\sqrt{6}$ on its component, but the value of the function at any point on the other component is less.

36. We use a Lagrange multiplier to find the maximum value of $f(\alpha, \beta, \gamma) = \sin\alpha \sin\beta \sin\gamma$ subject to the constraint that $\alpha + \beta + \gamma = \pi$. (Note that we also assume that each of α, β, γ must be strictly between 0 and π.) The system of equations to consider is

$$\begin{cases} \cos\alpha \sin\beta \sin\gamma = \lambda \\ \sin\alpha \cos\beta \sin\gamma = \lambda \\ \sin\alpha \sin\beta \cos\gamma = \lambda \\ \alpha + \beta + \gamma = \pi. \end{cases}$$

The first two equations imply that $\cos\alpha \sin\beta \sin\gamma = \sin\alpha \cos\beta \sin\gamma$ This holds if either $\cos\alpha \sin\beta = \sin\alpha \cos\beta$ or $\sin\gamma = 0$. However, if $\sin\gamma = 0$, then γ is 0 or π which we have already ruled out. (Also, f would necessarily be zero and clearly not maximized since any acute triangle will yield a positive value of f.) Hence

$$\cos\alpha \sin\beta = \sin\alpha \cos\beta \iff \sin\alpha \cos\beta - \cos\alpha \sin\beta = 0 \iff \sin(\alpha - \beta) = 0.$$

It follows that $\alpha = \beta$. Similarly, the second and third equations together imply that $\sin\alpha \cos\beta \sin\gamma = \sin\alpha \sin\beta \cos\gamma$ Thus either $\sin\alpha = 0$ (which we reject) or

$$\cos\beta \sin\gamma = \sin\beta \cos\gamma \iff \sin\beta \cos\gamma - \cos\beta \sin\gamma = 0 \iff \sin(\beta - \gamma) = 0.$$

Hence $\beta = \gamma$ and so $\alpha = \beta = \gamma = \pi/3$ using the last equation. Therefore, the maximum value of f is $3\sqrt{3}/8$.

37. Let P have coordinates (x, y, z). The square of the distance from P to the origin is given by the function $f(x, y, z) = x^2 + y^2 + z^2$ and the coordinates of P must satisfy $g(x, y, z) = c$. Thus if f is maximized at P, then, since $\nabla g(x, y, z)$ is given never to vanish, $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ for some λ. If we write this out, we find

$$(2x, 2y, 2z) = \lambda \nabla g(x, y, z).$$

But

$$(2x, 2y, 2z) = 2(x, y, z) = 2\,\overrightarrow{OP},$$

where $\overrightarrow{OP}$ denotes the displacement vector from the origin to P. Therefore,

$$\overrightarrow{OP} = \frac{\lambda}{2} \nabla g(x, y, z);$$

that is, $\overrightarrow{OP}$ is parallel to ∇g. (Note that $\overrightarrow{OP}$ must be nonzero if the distance from the origin to P is to be *maximized*.) Since the gradient vector ∇g at P is known to perpendicular to the level set of g through P, the result follows.

38. This is a non-linear version of Exercise 30. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $g_1(x, y, z) = x^2 + y^2 = 4$ and $g_2(x, y, z) = 2x + 2y + z = 2$. We solve the system

$$\begin{cases} 2x = 2\lambda x + 2\mu \\ 2y = 2\lambda y + 2\mu \\ 2z = \mu \\ x^2 + y^2 = 4 \\ 2x + 2y + z = 2. \end{cases}$$

Solving we see that either $x = y$ or $\lambda = 1$. If $x = y$ then $x = y = \pm\sqrt{2}$ and $z = 2 \mp 4\sqrt{2}$. The farthest point is $(-\sqrt{2}, -\sqrt{2}, 2 + 4\sqrt{2})$. If $\lambda = 1$ then $x = (1 \pm \sqrt{7})/2$, $y = (1 \mp \sqrt{7})/2$, and $z = 0$—these last two are the closest points.

39. We want to find the extreme values of the function $f(x,y) = x^2 + y^2$ (the square of the distance from the point (x,y) to the origin) subject to the constraint $g(x,y) = 3x^2 - 4xy + 3y^2 = 50$. (Note that there will be a global maximum and a global minimum by the extreme value theorem since the ellipse is a compact set in $\mathbf{R}^2$.) We solve the system

$$\begin{cases} 2x = \lambda(6x - 4y) \\ 2y = \lambda(-4x + 6y) \\ 3x^2 - 4xy + 3y^2 = 50. \end{cases}$$

The first two equations together imply

$$\frac{1}{\lambda} = \frac{6x - 4y}{2x} = \frac{-4x + 6y}{2y} \iff 3 - \frac{2y}{x} = 3 - \frac{2x}{y} \iff y^2 = x^2.$$

Thus $y = \pm x$. If $y = x$, then the last equation becomes

$$3x^2 - 4x^2 + 3x^2 = 50 \iff x^2 = 25 \iff x = \pm 5.$$

Thus there are two critical points $(5,5)$ and $(-5,-5)$. If $y = -x$, then the last equation becomes

$$3x^2 + 4x^2 + 3x^2 = 50 \iff x^2 = 5 \iff x = \pm\sqrt{5}.$$

Hence there are two more critical points $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$. Finally, we have

$$f(5,5) = f(-5,-5) = 50 \quad \text{and} \quad f(\sqrt{5}, -\sqrt{5}) = f(-\sqrt{5}, \sqrt{5}) = 10,$$

so that $(5,5)$ and $(-5,-5)$ are the points on the ellipse farthest from the origin and $(\sqrt{5}, -\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$ are the points nearest the origin.

40. (a) This follows immediately from the extreme value theorem. The constraint defines a quarter circle, including the endpoints, which is a compact set in $\mathbf{R}^2$ and the function $f(x,y) = \sqrt{x} + 8\sqrt{y}$ is continuous whenever x and y are both nonnegative.

(b) The system we consider is

$$\begin{cases} \left(\dfrac{1}{2\sqrt{x}}, \dfrac{8}{2\sqrt{y}}\right) = \lambda(2x, 2y) \\ x^2 + y^2 = 17 \end{cases} \qquad \text{or} \qquad \begin{cases} \dfrac{1}{2\sqrt{x}} = 2\lambda x \\ \dfrac{4}{\sqrt{y}} = 2\lambda y \\ x^2 + y^2 = 17. \end{cases}$$

The first two equations of the system together imply that

$$2\lambda = \frac{1}{2x^{3/2}} = \frac{4}{y^{3/2}} \implies y^{3/2} = 8x^{3/2} \implies y = 4x.$$

Using this result in the last equation gives $x^2 + 16x^2 = 17$. Thus $x = 1$ since we only want x (and y) nonnegative. Thus the only critical point we identify in this manner is $(1,4)$.

(c) Note that $\nabla f(x,y)$ is undefined if either x or y is zero. Given the constraint, this means that we should also consider the points $(\sqrt{17}, 0)$ and $(0, \sqrt{17})$. Comparing values, we have

- $f(1,4) = 17$,
- $f(\sqrt{17}, 0) = \sqrt[4]{17}$,
- $f(0, \sqrt{17}) = 8\sqrt[4]{17} \approx 16.24$.

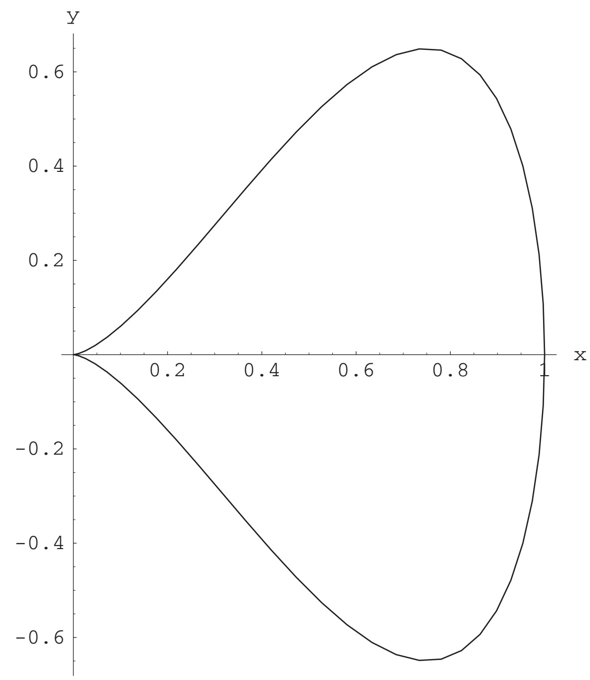
Hence $(1,4)$ yields the global maximum and $(\sqrt{17}, 0)$ the global minimum on the quarter circle.

41. (a) The system is

$$\begin{cases} 1 = \lambda(16x^3 - 12x^2) \\ 0 = 2\lambda y \\ y^2 - 4x^3 + 4x^4 = 0. \end{cases}$$

The second equation implies that either $y = 0$ or $\lambda = 0$. But $\lambda = 0$ cannot satisfy the first equation, so $y = 0$. The last equation implies $4x^3(1 - x) = 0$; thus $x = 0$ or 1. But $x = 0$ cannot satisfy the first equation. Thus the only solution to the system is $(1, 0)$.

(b) The graph of the curve (known as the **piriform**) is shown in the figure below. From it, it's clear that the maximum value of $f(x, y) = x$ occurs at (1, 0) and the minimum value at (0, 0).



(c) Note that $\nabla g(x, y) = (16x^3 - 12x^2, 2y) = (0, 0)$ at (0, 0) (and at (3/4, 0)). (0, 0) is a point on the curve ((3/4, 0) is not). It's the singular point of the piriform and, although not a solution to the Lagrange multiplier system in part (a), it must be considered as a possible site for extrema.

42. (a) The relevant Lagrange multiplier system to solve is

$$\begin{cases} 2x = 0 \\ 2y = 0 \\ 0 = \lambda \\ z = c \end{cases} \qquad \text{The obvious unique solution is } (0, 0, c) \text{ with } \lambda = 0.$$

(b) $L(l; x, y, z) = x^2 + y^2 - l(z - c)$. With c as a constant and $x_1 = x, x_2 = y, x_3 = z$, we have

$$HL(l; x, y, z) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = HL(0; 0, 0, c).$$

The second derivative test asks us to calculate $(-1)^1 d_3$ and $(-1)^1 d_4$ or

$$-d_3 = -\det \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = 0; \quad -d_4 = -\det \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = -(-4) = 4.$$

Thus the second derivative test seems to suggest that we've found a saddle point.

(c) Now we let $x_1 = z, x_2 = y, x_3 = x$ and look at

$$HL(l; z, y, x) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

In this case we find

$$-d_3 = -\det\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = -(-2) = 2 \quad \text{and}$$

$$-d_4 = -\det\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = -(-4) = 4.$$

This time the sound derivative test suggests a local minimum.

(d) Indeed, inspection tells us that the expression $x^2 + y^2$ attains a *global* minimum at $x = y = 0$. So to satisfy the constraint $z = c$, we see that (0, 0, c) yields a global minimum. The difference between the results of (b) and (c) can be explained by looking at $\partial g/\partial x$ vs. $\partial g/\partial z$: $\partial g/\partial x = 0$, but $\partial g/\partial z = 1 \neq 0$.

In part (b), we did not satisfy the hypothesis of the second derivative test that the variables be ordered so that

$$\det\begin{bmatrix} \dfrac{\partial g_1}{\partial x_1}(\mathbf{a}) & \cdots & \dfrac{\partial g_1}{\partial x_k}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k}{\partial x_1}(\mathbf{a}) & \cdots & \dfrac{\partial g_k}{\partial x_k}(\mathbf{a}) \end{bmatrix} \neq 0.$$

(The determinant in this situation is just $\partial g/\partial x$.) In part (c), we did satisfy the hypothesis, since $\partial g/\partial z \neq 0$.

43. (a) In order for $(\boldsymbol{\lambda}, \mathbf{a})$ to be a solution of the constrained problem, $(\boldsymbol{\lambda}, \mathbf{a})$ must solve the system

$$\begin{cases} f_{x_i}(\mathbf{a}) = \displaystyle\sum_{j=1}^{k} \lambda_j (g_j)_{x_i}(\mathbf{a}) & \text{for } 1 \leq i \leq n \\ g_j(\mathbf{a}) = c_j & \text{for } 1 \leq j \leq k. \end{cases}$$

On the other hand, an unconstrained critical point for L must be where all first partials are zero. In other words, we must have

$$L_{l_j} = 0, \quad 1 \leq j \leq k \quad \text{and} \quad L_{x_j} = 0, \quad 1 \leq j \leq n.$$

Upon explicit calculation of the partials these equations are:

$$\begin{cases} f_{l_j}(\mathbf{a}) - (g_j(\mathbf{a}) - c_j) = 0 & \text{for } 1 \leq j \leq k, \text{ and} \\ f_{x_j}(\mathbf{a}) - \sum_{i=1}^{k} \lambda_i (g_i)_{x_j}(\mathbf{a}) = 0 & \text{for } 1 \leq j \leq n. \end{cases}$$

This is the same system as that for the constrained case.

(b) Calculate the Hessian in four blocks. All of the entries in the upper left $k \times k$ block are 0. This is because the entry in position (i, j) is $L_{l_i l_j}$ and the highest power of any l_i appearing in L is 1. The top right block with k rows and n columns gives back the negative first partials of the constraint conditions because the entry in position $(k+i, j)$ is $L_{x_i l_j} = -(g_j - c_j)_{x_i} = -(g_j)_{x_i}$. The lower left block of n rows and k columns is just the transpose of this last block. The lower right $n \times n$ block is such that the entry in position $(k+i, k+j) = L_{x_i x_j} = (f - \sum_{q=1}^{k} l_q g_q)_{x_i x_j}$. When $\boldsymbol{\lambda}$ and $\mathbf{a}$ are substituted for $\mathbf{l}$ and $\mathbf{x}$, the desired matrix is obtained.

44. We find extreme values of $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_i y_i$ subject to the two constraints $g_1(x_1, \ldots, x_n,\ y_1, \ldots,$ $y_n) = {x_1}^2 + \cdots + {x_n}^2 = 1$ and $g_2(x_1, \ldots, x_n, y_1, \ldots, y_n) = {y_1}^2 + \cdots + {y_n}^2 = 1$. Thus we look at $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

together with the constraints to solve

$$\left\{\begin{array}{l} \left.\begin{array}{l} y_1 = 2\lambda_1 x_1 \\ \vdots \\ y_n = 2\lambda_1 x_n \end{array}\right\} \quad \text{The first } n \text{ equations (and the last) imply} \quad \begin{aligned} 1 &= \sum {y_i}^2 = 4{\lambda_1}^2 \sum {x_i}^2 \\ &= 4{\lambda_1}^2 \cdot 1 \\ &\text{so } \lambda_1 = \pm\frac{1}{2}. \end{aligned} \\ \left.\begin{array}{l} x_1 = 2\lambda_2 y_1 \\ \vdots \\ x_n = 2\lambda_2 y_n \end{array}\right\} \quad \text{The next } n \text{ equations (and the next-to-last) imply} \quad \begin{aligned} 1 &= \sum {x_i}^2 = 4{\lambda_2}^2 \sum {y_i}^2 = 4{\lambda_2}^2 \\ &\text{so } \lambda_2 = \pm\frac{1}{2}. \end{aligned} \\ \sum {x_i}^2 = 1 \\ \sum {y_i}^2 = 1 \end{array}\right.$$

Putting all the information together, we find that $\mathbf{x} = \mathbf{y}$ (when $\lambda_1 = \lambda_2 = \frac{1}{2}$) and $\mathbf{x} = -\mathbf{y}$ (when $\lambda_1 = \lambda_2 = -\frac{1}{2}$). When $\mathbf{x} = \mathbf{y}$, $f(\mathbf{x}, \mathbf{y}) = \sum {x_i}^2 = 1$. When $\mathbf{x} = -\mathbf{y}$, $f(\mathbf{x}, -\mathbf{x}) = \sum(-{x_i}^2) = -1$. Though it takes a little bit of argumentation, the hypersphere in $\mathbf{R}^n$ is compact—hence so is the *product* of hyperspheres in $\mathbf{R}^{2n} (= \mathbf{R}^n \times \mathbf{R}^n)$. Thus we find maximum and minimum values of $+1$ and -1, respectively.

45. (a)

$$\sum_{i=1}^{n} {u_i}^2 = {u_1}^2 + \cdots + {u_n}^2 = \frac{{x_1}^2}{(\sqrt{{x_i}^2})^2} + \frac{{x_2}^2}{(\sqrt{{x_i}^2})^2} + \cdots + \frac{{x_n}^2}{(\sqrt{{x_i}^2})^2} = \frac{\sum {x_i}^2}{\sum {x_i}^2} = 1.$$

So **u** is an the unit hypersphere. The case for **v** is identical.

(b) By Exercise 44, we have $-1 \le \sum_{i=1}^{n} u_i v_i \le 1$. Hence

$$\begin{aligned} -1 &\le \sum_i \left(\frac{x_i}{\sqrt{\sum_j x_j^2}} \right) \left(\frac{y_i}{\sqrt{\sum_j y_j^2}} \right) \le 1 \\ \Leftrightarrow -&\sqrt{\sum_j x_j^2}\sqrt{\sum_j y_j^2} \le \sum_i x_i y_i \le \sqrt{\sum_j x_j^2}\sqrt{\sum_j y_i^2} \\ \Leftrightarrow -&\|\mathbf{x}\|\|\mathbf{y}\| \le \mathbf{x}\cdot\mathbf{y} \le \|\mathbf{x}\|\|\mathbf{y}\| \\ \Leftrightarrow \quad & |\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\|\mathbf{y}\|. \end{aligned}$$

4.4 Some Applications of Extrema

1. This problem can be done using calculators or the following table to help with Proposition 4.1:

x_i	y_i	${x_i}^2$	$x_i y_i$
0	2	0	0
1	3	1	3
2	5	4	10
3	3	9	9
4	2	16	8
5	7	25	35
6	7	36	42
21	29	91	107

$$\text{So} \quad m_0 = \frac{7(107) - (21)(29)}{7(91) - (21)^2} = \frac{140}{196} = \frac{35}{49} \approx .71428$$

$$\text{and} \quad b_0 = \frac{(91)(29) - (21)(107)}{7(91) - (21)^2} = \frac{392}{196} = \frac{98}{49} = 2.$$

The equation of the least squares line is $y = (35/49)x + 2$.

2. Again, using Proposition 4.1,

$$m = \frac{2(x_1y_1 + x_2y_2) - (x_1+x_2)(y_1+y_2)}{2(x_1^2+x_2^2) - (x_1+x_2)^2} = \frac{(x_1-x_2)(y_1-y_2)}{(x_1-x_2)^2} = \frac{y_1-y_2}{x_1-x_2}.$$

$$b = \frac{(x_1^2+x_2^2)(y_1+y_2) - (x_1+x_2)(x_1y_1+x_2y_2)}{2(x_1^2+x_2^2) - (x_1+x_2)^2} = \frac{(x_1y_2 - x_2y_1)(x_1-x_2)}{(x_1-x_2)^2} = \frac{x_1y_2 - x_2y_1}{x_1-x_2}.$$

You can check that (x_1, y_1) and (x_2, y_2) are both on the line

$$y = \left(\frac{y_1-y_2}{x_1-x_2}\right)x + \frac{x_1y_2 - x_2y_1}{x_1-x_2}.$$

3. (a) As in the text, the function $D(a, b)$ will be the sum of the squares of the differences between the observed y values and the y values on the curve $y = a/x + b$. This means that

$$D(a,b) = \sum_{i=1}^{n}(y_i - (a/x_i + b))^2.$$

(b) Make the substitution $X_i = 1/x_i$ and then fit the line $y = aX + b$ to this transformed data using Proposition 4.1. We get

$$a = \frac{n\sum X_iy_i - \left(\sum X_i\right)\left(\sum y_i\right)}{n\sum X_i^2 - \left(\sum X_i\right)^2} \quad \text{and} \quad b = \frac{\left(\sum X_i^2\right)\left(\sum y_i\right) - \left(\sum X_i\right)\left(\sum X_iy_i\right)}{n\sum X_i^2 - \left(\sum X_i\right)^2}.$$

Transform the data back, replacing X_i with $1/x_i$, then the curve of the form $y = a/x + b$ that best fits the data has

$$a = \frac{n\sum y_i/x_i - \left(\sum 1/x_i\right)\left(\sum y_i\right)}{n\sum 1/x_i^2 - \left(\sum 1/x_i\right)^2} \quad \text{and} \quad b = \frac{\left(\sum 1/x_i^2\right)\left(\sum y_i\right) - \left(\sum 1/x_i\right)\left(\sum y_i/x_i\right)}{n\sum 1/x_i^2 - \left(\sum 1/x_i\right)^2}.$$

4. We'll use the results of Exercise 3 and organize our sums with the following table:

$1/x_i$	y_i	$1/x_i^2$	y_i/x_i
1	0	1	0
1/2	−1	1/4	−1/2
2	1	4	2
1/3	−1/2	1/9	−1/6
23/6	−1/2	193/36	8/6

$$\text{So} \quad a = \frac{4(8/6) - (23/6)(-1/2)}{4(193/36) - (23/6)^2} = \frac{261}{243} = \frac{29}{27}$$

$$\text{and} \quad b = \frac{(193/36)(-1/2) - (23/6)(8/6)}{4(193/36) - (23/6)^2} = -\frac{561}{486} = -\frac{187}{162}.$$

The equation of the least squares curve of the desired form is $y = 29/(27x) - 187/162$.

5. Again the function $D(a, b, c)$ will be the sum of the squares of the differences between the observed y values and the y values on the curve $y = ax^2 + bx + c$. This means that

$$\begin{aligned} D(a,b,c) &= \sum_{i=1}^{n}(y_i - (ax_i^2 + bx_i + c))^2 \\ &= \sum_{i=1}^{n} y_i^2 + a^2\sum_{i=1}^{n} x_i^4 + b^2\sum_{i=1}^{n} x_i^2 + nc^2 - 2a\sum_{i=1}^{n} x_i^2 y_i - 2b\sum_{i=1}^{n} x_i y_i - 2c\sum_{i=1}^{n} y_i \\ &\quad + 2ab\sum_{i=1}^{n} x_i^3 + 2ac\sum_{i=1}^{n} x_i^2 + 2bc\sum_{i=1}^{n} x_i \quad \text{so} \\ D_a(a,b,c) &= 2a\sum_{i=1}^{n} x_i^4 - 2\sum_{i=1}^{n} x_i^2 y_i + 2b\sum_{i=1}^{n} x_i^3 + 2c\sum_{i=1}^{n} x_i^2, \\ D_b(a,b,c) &= 2b\sum_{i=1}^{n} x_i^2 - 2\sum_{i=1}^{n} x_i y_i + 2a\sum_{i=1}^{n} x_i^3 + 2c\sum_{i=1}^{n} x_i, \quad \text{and} \\ D_c(a,b,c) &= 2cn - 2\sum_{i=1}^{n} y_i + 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i. \end{aligned}$$

Set each of the partial derivatives equal to zero, move the term with coefficient -2 to the other side, and divide by 2 to get the desired equations.

6. You may want to point out to the students that the independent variable x corresponds to hours of sleep because that is what (in theory) Egbert can control.

(a) To get a line $y = ax + b$ we'll need

x_i	y_i	x_i^2	$x_i y_i$
8	85	64	680
8.5	72	72.25	612
9	95	81	855
7	68	49	476
4	52	16	208
8.5	75	72.25	637.5
7.5	90	56.25	675
6	65	36	390
58.5	602	446.75	4533.5

Using the formulas in Proposition 4.1 you'll find that the least squares line is

$$y = (4204/607)x + (14935/607) \approx 6.93x + 24.6.$$

(b) We will need some additional data:

x_i	x_i^3	x_i^4	$x_i^2 y_i$
8	512	4096	5440
8.5	614.125	5220.0625	5202
9	729	6561	7695
7	343	2401	3332
4	64	256	832
8.5	614.125	5220.0625	5418.75
7.5	421.875	3164.0625	5062.5
6	216	1296	2340
58.5	3514.125	28214.1875	35322.25

Use the formulas given in Exercise 5 to obtain the system

$$\begin{cases} 28214.1875a + 3514.125b + 446.75c = 35322.25 \\ 3514.125a + 446.75b + 58.5c = 4533.5 \\ 446.75a + 58.5b + 8c = 602. \end{cases}$$

Solve this system to get the following (approximate) quadratic:

$$y = -.192044054x^2 + 9.42923983x + 17.02314387.$$

(c) Plugging 6.8 into the linear model predicts that Egbert will get 71.7, plugging 6.8 into the quadratic model predicts that Egbert will get 72.26.

7. (a) We are required to show that **F** is a gradient (conservative) vector field. Clearly if $V(x, y) = x^2 + 2xy + 3y^2 + x + 2y$ then $-\nabla V = (-2x - 2y - 1)\mathbf{i} + (-2x - 6y - 2)\mathbf{j} = \mathbf{F}$.

(b) We find equilibrium points of **F** when $\mathbf{F} = \mathbf{0}$. Solve the system of equations

$$\begin{cases} -2x - 2y = 1 \\ -2x - 6y = 2 \end{cases}$$

and find one solution at $(-1/4, -1/4)$. The Hessian is

$$HV = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$$

so both d_1 and d_2 are positive so the equilibrium is stable.

8. Here $V(x, y) = 2x^2 - 8xy - y^2 + 12x - 8y + 12$ so $\nabla V = -\mathbf{F} = (4x - 8y + 12, -8x - 2y - 8)$. This is **0** at $(-11/9, 8/9)$. The Hessian is

$$HV = \begin{bmatrix} 4 & -8 \\ -8 & -2 \end{bmatrix}.$$

Note that $d_1 > 0$ and $d_2 < 0$ so the equilibrium at $(-11/9, 8/9)$ is not stable.

9. Here $V(x, y, z) = 3x^2 + 2xy + z^2 - 2yz + 3x + 5y - 10$ so $\nabla V = -\mathbf{F} = (6x + 2y + 3, 2x - 2z + 5, -2y + 2z)$. This is **0** at $(-1, 3/2, 3/2)$. The Hessian is

$$HV = \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 2 \end{bmatrix}.$$

Note that $d_1 > 0$, $d_2 < 0$, and $d_3 > 0$ so the equilibrium at $(-1, 3/2, 3/2)$ is not stable.

10. (a) Here we are looking for constrained equilibria (as in Example 3 in the text). Our equation is $\mathbf{F} - \nabla V = \lambda \nabla g$ where $g(x, y, z) = 2x^2 + 3y^2 + z^2 = 1$, $\mathbf{F} = -mg\mathbf{k}$, and $V(x, y, z) = 2x$. So our system of equations is

$$\begin{cases} -2 = 4\lambda x \\ 0 = 6\lambda y \\ -mg = 2\lambda z \\ 2x^2 + 3y^2 + z^2 = 1. \end{cases}$$

Note from the first equation that $\lambda \neq 0$ so by the second equation $y = 0$. From the third equation $2\lambda = -mg/z$ so $z = mgx$. Substituting into the equation of the ellipsoid, $2x^2 + m^2g^2x^2 = 1$ so $x = \pm 1/\sqrt{2 + m^2g^2}$. So our two equilibria are at $\pm(1/\sqrt{2 + m^2g^2}, 0, mg/\sqrt{2 + m^2g^2})$.

(b) Note the direction of the force is $(-2, 0, -mg)$ so $-(1/\sqrt{2 + m^2g^2}, 0, mg/\sqrt{2 + m^2g^2})$ is a stable equilibrium.

11. Maximize $R(x, y, z) = xyz^2 - 25000x - 25000y - 25000z$ subject to the constraint $x + y + z = 200000$. Our system of equations is

$$\begin{cases} yz^2 - 25000 = \lambda \\ xz^2 - 25000 = \lambda \\ 2xyz - 25000 = \lambda \\ x + y + z = 200000. \end{cases}$$

The hidden condition is that all of the variables are non-negative. This means that we are finding a maximum on the triangular portion of the plane that lies in the first octant. The maximum revenue will occur at a boundary point or at a critical point. Along the boundary at least one of the variables is 0 and the revenue is at most 0 when at least one of x, y and z is 0. We will see the value of R at the critical point is greater and therefore that it is our global maximum. Assume none of the variables is zero. Then, from the first two equations, since $z \neq 0$ then $x = y$. From the third equation paired with either of the first two we see that $z = 2x = 2y$. Finally, since their sum is 200000 we find the solution (50000, 50000, 100000) is where the maximum revenue occurs.

12. This is similar to Example 4 from the text. We are maximizing $U(x_1, x_2, x_3) = x_1x_2 + 2x_1x_3 + x_1x_2x_3$ subject to the constraint $g(x_1, x_2, x_3) = x_1 + 4x_2 + 2x_3 = 90$. Our system of equations is

$$\begin{cases} x_2 + 2x_3 + x_2x_3 = \lambda \\ x_1 + x_1x_3 = 4\lambda \\ 2x_1 + x_1x_2 = 2\lambda \\ x_1 + 4x_2 + 2x_3 = 90. \end{cases}$$

The only solution of this system with all three of the x_i's non-negative is (33.0149, 6.37314, 15.7463). You can only order integer amounts, so experiment with the different ways of rounding to obtain a maximum at (34, 6, 16).

13. We maximize the function B subject to the constraint $15x + 10y = 500$. Using a Lagrange multiplier, we solve the system

$$\begin{cases} 8x = 15\lambda \\ 2y = 10\lambda \\ 15x + 10y = 500. \end{cases}$$

The first two equations imply that $5\lambda = \frac{8}{3}x = y$. Using this in the constraint equation yields

$$15x + \frac{80}{3}x = 500 \iff x = 12.$$

Thus $(x, y) = (12, 32)$ is our only critical point. We should compare the yield B at this point with that at the boundary values of $\left(\frac{100}{3}, 0\right)$ (all irrigation) and $(0, 50)$ (all fertilizer). We have

$$B(12, 32) = 2200,\ B\left(\tfrac{100}{3}, 0\right) = 5044.\overline{4},\ B(0, 50) = 3100.$$

Thus she should forgo the fertilizer entirely and simply irrigate the field.

14. **(a)** We maximize the given production function f subject to the constraint $8x + 2y = 1000$. Using a Lagrange multiplier, the system we must consider is

$$\begin{cases} 4y - 2 = 8\lambda \\ 4x - 8 = 2\lambda \\ 8x + 2y = 1000. \end{cases}$$

The first two equations of the system imply that

$$4\lambda = 8x - 16 = 2y - 1 \quad \Longrightarrow \quad 8x = 2y + 15.$$

Using this in the last equation we have $4y + 15 = 1000 \iff y = 985/4$. Hence $x = 1015/16$. (Note that in the constraint $8x + 2y = 1000$, we must have $0 \le x \le 125$ and $0 \le y \le 500$. The endpoints $(125, 0)$ and $(0, 500)$ give negative values for f and so $(1015/16, 985/4)$ must yield the maximum value of f on the line segment described by the constraints.) Hence the manufacturer should purchase 63.4375 lb of cashmere and 246.2516 lb of cotton. The ratio of cotton to cashmere is $4\left(\frac{985}{1015}\right) \approx 3.88$.

(b) Most of the essential features of the situation remain unchanged. The constraint equation becomes $8x + 2y = B$, so that the relevant system to solve is

$$\begin{cases} 4y - 2 = 8\lambda \\ 4x - 8 = 2\lambda \\ 8x + 2y = B. \end{cases}$$

As before, $8x = 2y + 15$ and, using this we find that

$$(x, y) = \left(\frac{B + 15}{16}, \frac{B - 15}{4}\right)$$

is the critical point that maximizes f. Thus the ratio of cotton to cashmere should be

$$\frac{(B - 15)/4}{(B + 15)/16} = 4\left(\frac{B - 15}{B + 15}\right).$$

As B becomes very large, we have

$$\lim_{B\to+\infty} 4\left(\frac{B - 15}{B + 15}\right) = \lim_{B\to+\infty} \frac{4\,(1 - 15/B)}{1 + 15/B} = 4,$$

which is the ratio of the cost of cashmere to that of cotton.

15. **(a)** This is an example of the Cobb-Douglas production function with $p = w = 1$ (see Example 5 from the text). The only critical point will be $(K, L) = ((1/3)360000, (2/3)360000) = (120000, 240000)$.

(b) $\partial Q/\partial K = 20(L/K)^{2/3}$ and so at (120000, 240000), $\partial Q/\partial K = 20(2)^{2/3}$. On the other hand, $\partial Q/\partial L = 40(K/L)^{1/3}$ and so at (120000, 240000), $\partial Q/\partial L = 40(1/2)^{1/3}$. These quantities are equal at the critical point.

16. This time we are minimizing $pK + wL = M$ subject to the constraint $Q(K, L) = c$. Our system of equations is

$$\begin{cases} p = \lambda \dfrac{\partial Q}{\partial K} \\ w = \lambda \dfrac{\partial Q}{\partial L}. \end{cases}$$

Since none of p, q, and λ is 0, we can divide the top equation by $p\lambda$, divide the bottom equation by $q\lambda$ and the result is immediate.

True/False Exercises for Chapter 4

1. True.
2. False. (The increment measures the change in the function.)
3. True.
4. True.
5. True.
6. False. ($p_2(x, y) = 1 - 3x + y + 3x^2 + 2xy$.)
7. False. (f is most sensitive to changes in y.)
8. False. (The result is true if f is of class C^2.)
9. False.
10. True.
11. True.
12. False. (The set is not bounded.)
13. False. (Consider the function $f(x, y) = x^2 + y^2$.)
14. True. (This ball is compact.)
15. True.
16. False. (The point **a** might not be a critical point.)
17. False. (The point is not a critical point of the function.)
18. False. (The point (0, 0, 0) gives a local minimum.)
19. True.
20. True.
21. False. (The critical point is a saddle point.)
22. False. (A local extremum can occur where a partial derivative fails to exist.)
23. False. (Extrema may also occur at points where $g = c$ and $\nabla g = \mathbf{0}$.)
24. False. (Solutions to the system only give critical points.)
25. False. (You will have to solve a system of 7 equations in 7 unknowns.)
26. True.
27. True.
28. True.
29. False. (The equilibrium points are the critical points of the potential function.)
30. False. (This is only true at values of labor and capital that maximize the output.)

Miscellaneous Exercises for Chapter 4

1. If $V = \pi r^2 h$ then $dV = 2\pi rh\,dr + \pi r^2 dh$, so in order for V to be equally sensitive to small changes in r and h, we must be at a point (r_0, h_0) where $2\pi r_0 h_0 \approx \pi r_0^2$ so $r_0 = 2h_0$.

2. **(a)** If $f(x_1, x_2, \ldots, x_n) = e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$, then $f_{x_i}(x_1, x_2, \ldots, x_n) = -2x_i e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$ and is 0 only when $x_i = 0$. So the only critical point is at the origin.

(b) If $i \neq j$, then $f_{x_i x_j}(x_1, x_2, \ldots, x_n) = 4x_i x_j e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$, so $f_{x_i x_j}(0, 0, \ldots, 0) = 0$. Also $f_{x_i x_i}(x_1, x_2, \ldots, x_n) = (-2 + 4x_i^2)e^{-x_1^2 - x_2^2 - \cdots - x_n^2}$, so $f_{x_i x_i}(0, 0, \ldots, 0) = -2$. The Hessian is an $n \times n$ diagonal matrix with -2's on the main diagonal and 0's everywhere else. It is easy to calculate $d_i(0, 0, \ldots, 0) = (-2)^i$ and so by the second derivative test, f has a local maximum at the origin.

3. We are asked to maximize the profit $P(x, y) = (x - 2)(80 - 100x + 40y) + (y - 4)(20 + 60x - 35y) = -100x^2 + 40x - 35y^2 + 80y + 100xy - 240$. The partial derivatives are $P_x(x, y) = -200x + 100y + 40$ and $P_y(x, y) = 100x - 70y + 80$. These are both zero at (27/10, 5). You can read the Hessian right off the first derivatives and you see that $d_1 = -200 < 0$ and $d_2 = 4000 > 0$ so profit is maximized when you charge \$2.70 for Mocha and \$5 for Kona.

4. **(a)** Revenue is $R(x, y, z) = 1000x(4 - 0.02x) + 1000y(4.5 - 0.05y) + 1000z(5 - 0.1z) = -20x^2 + 4000x - 50y^2 + 4500y - 100z^2 + 5000z$.

(b) When $(x, y, z) = (6, 5, 4)$, the prices of brands X, Y and Z are, respectively, \$3.88, \$4.25, and \$4.60, and when $(x, y, z) = (1, 3, 3)$, the prices are \$3.98, \$4.35, and \$4.70. The difference is $R(1, 3, 3) - R(6, 5, 4) = 31,130 - 62,930 = -31,800$. The revenue will decline by \$31,800 if the prices are raised.

(c) The partial derivatives are $R_x(x, y, z) = 4000 - 40x$, $R_y(x, y, z) = 4500 - 100y$, and $R_z(x, y, z) = 5000 - 200z$. Thus the critical point is $(100, 45, 25)$ and hence the selling prices should be \$2 for brand X, \$2.25 for brand Y and \$2.50 for brand Z.

5. We note that there must be both a (global) maximum and a minimum value of f because the constraint equation defines the surface of a sphere, which is compact, and f is continuous, so that the extreme value theorem applies.

(a) We find the extrema of $f(x, y, z) = x - \sqrt{3}y$ subject to $g(x, y, z) = x^2 + y^2 + z^2 = 4$. Using the Lagrange multiplier method, we solve

$$\begin{cases} 1 = 2\lambda x \\ -\sqrt{3} = 2\lambda y \\ 0 = 2\lambda z \\ x^2 + y^2 + z^2 = 4. \end{cases}$$

From the first equation, we must have $\lambda \neq 0$, so from the third equation $z = 0$. Then the first two equations imply that $y = -\sqrt{3}x$. Thus, since $x^2 + 3x^2 = 4$, our critical points are $\pm(1, -\sqrt{3}, 0)$. We evaluate f at these points to find that we have a maximum of 4 at $(1, -\sqrt{3}, 0)$ and a minimum of -4 at $(-1, \sqrt{3}, 0)$.

(b) Now we are looking at the function $g(\varphi, \theta) = f(2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi) = 2\sin\varphi\cos\theta - 2\sqrt{3}\sin\varphi\sin\theta$. Thus $g_\varphi(\varphi, \theta) = 2\cos\varphi\cos\theta - 2\sqrt{3}\cos\varphi\sin\theta$ and $g_\theta(\varphi, \theta) = -2\sin\varphi\sin\theta - 2\sqrt{3}\sin\varphi\cos\theta$ so that we should solve

$$\begin{cases} -2\sin\varphi(\sin\theta + \sqrt{3}\cos\theta) = 0 \\ 2\cos\varphi(\cos\theta - \sqrt{3}\sin\theta) = 0. \end{cases}$$

Either $\varphi = 0, \pi$ and $\cos\theta = \sqrt{3}\sin\theta$ (hence $\tan\theta = 1/\sqrt{3}$ so $\theta = \pi/6, 7\pi/6$), or $\varphi = \pi/2, 3\pi/2$ and $\sin\theta = -\sqrt{3}\cos\theta$ (hence $\tan\theta = -\sqrt{3}$ so $\theta = 2\pi/3, 5\pi/3$). Note that these points are (using $(x, y, z) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$):

$$\begin{aligned}
(\varphi, \theta) &= \left(0, \tfrac{\pi}{6}\right) &\iff&& (x, y, z) &= (0, 0, 2) \\
(\varphi, \theta) &= \left(0, \tfrac{7\pi}{6}\right) &\iff&& (x, y, z) &= (0, 0, 2) \\
(\varphi, \theta) &= \left(\pi, \tfrac{\pi}{6}\right) &\iff&& (x, y, z) &= (0, 0, -2) \\
(\varphi, \theta) &= \left(\pi, \tfrac{7\pi}{6}\right) &\iff&& (x, y, z) &= (0, 0, -2) \\
(\varphi, \theta) &= \left(\tfrac{\pi}{2}, \tfrac{2\pi}{3}\right) &\iff&& (x, y, z) &= (-1, \sqrt{3}, 0) \\
(\varphi, \theta) &= \left(\tfrac{\pi}{2}, \tfrac{5\pi}{3}\right) &\iff&& (x, y, z) &= (1, -\sqrt{3}, 0) \\
(\varphi, \theta) &= \left(\tfrac{3\pi}{2}, \tfrac{2\pi}{3}\right) &\iff&& (x, y, z) &= (-1, \sqrt{3}, 0) \\
(\varphi, \theta) &= \left(\tfrac{3\pi}{2}, \tfrac{5\pi}{3}\right) &\iff&& (x, y, z) &= (1, -\sqrt{3}, 0).
\end{aligned}$$

We obtain the same points as in part (a), plus the additional critical points $(0, 0, 2)$ and $(0, 0, -2)$, which are not global extrema, since $f(0, 0, \pm 2) = 0$.

6. **(a)** Here we are maximizing $T(x, y, z) = 200xyz^2$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 1$. Using the

Lagrange multiplier method, we solve

$$\begin{cases} 200yz^2 = 2\lambda x \\ 200xz^2 = 2\lambda y \\ 400xyz = 2\lambda z \\ x^2 + y^2 + z^2 = 1. \end{cases}$$

From the third equation, $\lambda \neq 0$ so $2\lambda = 400xy$ so $2x^2 = 2y^2 = z^2$. From the last equation we see that $4x^2 = 1$ so our critical points are the eight possible combinations of $x = \pm 1/2, y = \pm 1/2$ and $z = \pm 1/\sqrt{2}$. The temperature is a maximum of 25 when the sign of x and y are the same. This is at the four points $\pm(1/2, 1/2, \pm 1/\sqrt{2})$.

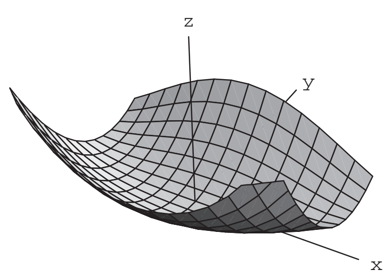
7. (a) $f_x(x,y) = 2x(-3y + 4x^2)$ while $f_y(x,y) = 2y - 3x^2$. From f_x we see that either $x = 0$ or $y = (4/3)x^2$. But from the second equation $y = (3/2)x^2$. So we conclude that the only solution is at (0, 0).

(b) $f_{xx}(x,y) = 6(-y + 4x^2), f_{xy}(x,y) = -6x$, and $f_{yy}(x,y) = 2$. At the origin, the Hessian is $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and so the determinant is 0 and the critical point is degenerate.

(c) If $y = mx$ then the original equation becomes $F(x) = m^2x^2 - 3mx^3 + 2x^4$. We calculate $F'(x) = 2m^2x - 9mx^2 + 8x^3 = 2x(m^2 - 9mx/2 + 4x^2)$. From the second derivative we see that $F''(x) = 2m^2 - 18mx + 24x^2$. This is positive at $x = 0$ for all $m \neq 0$ so there is a minimum for $x = 0$ along any line other than the two axes. When $m = 0, F'(x) = 8x^3$ and so the first derivative test implies that there is a minimum at $x = 0$ when $m = 0$. Finally, consider $G(y) = f(0,y) = y^2$. This clearly has a minimum at $y = 0$. We've shown that along any line through the origin, f has a minimum at (0, 0).

(d) Consider $g(x) = f(x, 3x^2/2) = (-x^2/2)(x^2/2) = -x^4/4$. From the derivative $g'(x) = -x^3$ we see that g has a maximum at $x = 0$ and hence f has a maximum at the origin when constrained to the given parabola. This means that the origin is actually a saddle point for f.

(e) A portion of the surface is shown below.

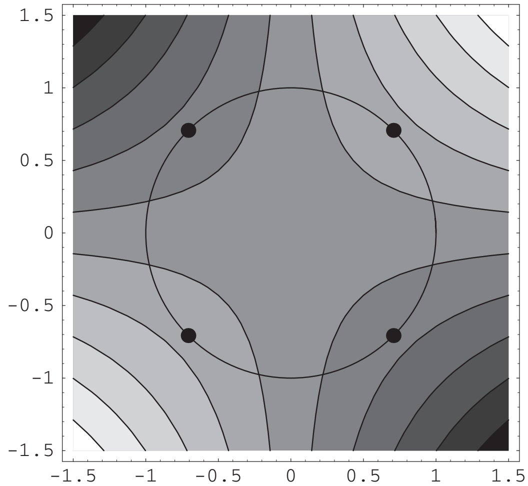


8. (a) Here we are finding the critical points of $f(x,y) = xy$ subject to the constraint $g(x,y) = x^2 + y^2 - 1 = 0$. So taking the partials of $f(x,y) = \lambda g(x,y)$ along with the constraint we get the following system of equations.

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 1 = x^2 + y^2. \end{cases}$$

The solutions correspond to $\lambda = \pm 1/2$ and are the four critical points $(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, 1/\sqrt{2}), (1/\sqrt{2}, -1/\sqrt{2})$, and $(-1/\sqrt{2}, -1/\sqrt{2})$.

(b) Here is a contour plot of $f(x,y) = xy$ along with the constraint curve $x^2 + y^2 = 1$ and the four critical points.



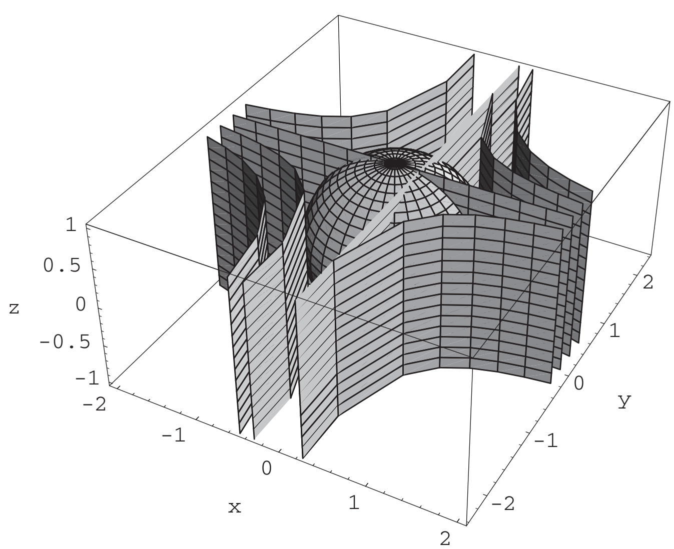
(c) You can see from the figure that f is at its highest value along the constraint curve at two of the critical points and at its lowest at two of the others. In particular, f has a constrained max at $\pm(1/\sqrt{2}, 1/\sqrt{2})$ and has a constrained min at $\pm(1/\sqrt{2}, -1/\sqrt{2})$.

9. (a) Here we are finding the critical points of $f(x, y, z) = xy$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. So taking the partials of $f(x, y, z) = \lambda g(x, y, z)$ along with the constraint we get the following system of equations.

$$\begin{cases} y = 2\lambda x \\ x = 2\lambda y \\ 0 = 2\lambda z \\ 1 = x^2 + y^2 + z^2 \end{cases}$$

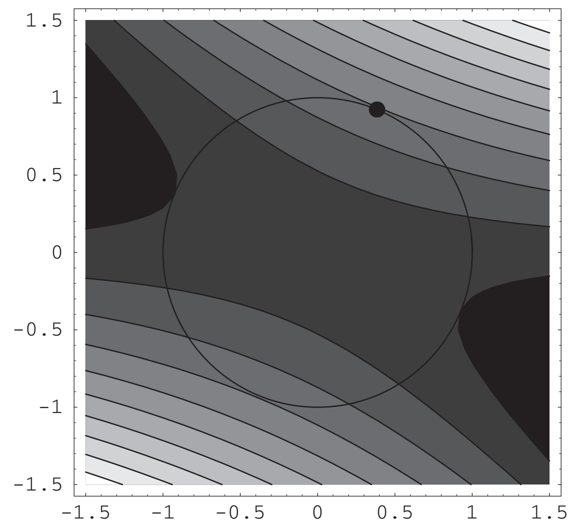
This problem is very similar to Exercise 8 and so it is no surprise that we again get four critical points corresponding to $\lambda = \pm 1/2$. They are $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(-1/\sqrt{2}, 1/\sqrt{2}, 0)$, $(1/\sqrt{2}, -1/\sqrt{2}, 0)$, and $(-1/\sqrt{2}, -1/\sqrt{2}, 0)$. We also get critical points at the two poles corresponding to $\lambda = 0$. These are at $(0, 0, \pm 1)$.

(b) Of course, it is harder to represent this situation than its lower-dimensional counterpart. Here are some level sets, the unit sphere and the critical points.



(c) The arguments that f has a constrained max at $\pm(1/\sqrt{2}, 1/\sqrt{2}, 0)$ and has a constrained min at $\pm(1/\sqrt{2}, -1/\sqrt{2}, 0)$ are the same as in Exercise 8. The two poles must be saddle points. If you travel in a direction where $y = x$, $f(x, y)$ is increasing while if you travel in a direction where $y = -x$, $f(x, y)$ is decreasing. So there are saddle points at $(0, 0, \pm 1)$.

10. From the diagram you can see that we are minimizing $f(x, y) = (x + y)y$ subject to the constraint that $x^2 + y^2 = 1$. Because this is a physical problem, we can assume that $x > 0$ and $y > 0$. A look at the contour plot for f along with the constraint curve lets us see that this solution will be a max.



Our system of equations is

$$\begin{cases} y & = 2\lambda x \\ x + 2y & = 2\lambda y \\ 1 & = x^2 + y^2 \end{cases}$$

Solving gives us one solution for which x and y are positive, namely $x = (\sqrt{2+\sqrt{2}}/2)(\sqrt{2} - 1)$ and $y = (\sqrt{2+\sqrt{2}}/2)$. The area of the rectangle is, therefore, $(\sqrt{2} + 1)/2$.

11. Minimize $f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2$ subject to the constraint $g(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1$ where not all the a_i's are zero. We solve

$$\begin{cases} 2x_i = a_i\lambda & \text{for } 1 \le i \le n \\ a_1x_1 + a_2x_2 + \cdots + a_nx_n = 1. \end{cases}$$

This means that our constrained critical point is at $x_i = a_i\lambda/2$ and $2/(a_1^2 + a_2^2 + \cdots + a_n^2) = \lambda$ so $x_i = a_i/(a_1^2 + a_2^2 + \cdots + a_n^2)$. So our minimum is

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= f\left(\frac{a_1}{a_1^2 + a_2^2 + \cdots + a_n^2}, \frac{a_2}{a_1^2 + a_2^2 + \cdots + a_n^2}, \ldots, \frac{a_n}{a_1^2 + a_2^2 + \cdots + a_n^2}\right) \\ &= \left(\frac{a_1}{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 + \left(\frac{a_2}{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 + \cdots + \left(\frac{a_n}{a_1^2 + a_2^2 + \cdots + a_n^2}\right)^2 \\ &= \frac{1}{a_1^2 + a_2^2 + \cdots + a_n^2}. \end{aligned}$$

12. Minimize the function $f(x_1, x_2, \ldots, x_n) = (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2$ subject to the constraint $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ where not all the a_i's are zero. We solve

$$\begin{cases} 2a_i(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = 2\lambda x_i \ \text{ for } 1 \le i \le n, \text{ and} \\ x_1^2 + x_2^2 + \cdots + x_n^2 = 1. \end{cases}$$

From the first equation, $x_i^2 = (a_i x_i/\lambda)(a_1x_1 + a_2x_2 + \cdots + a_nx_n)$, so

$$\begin{aligned}\lambda &= a_1x_1(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + a_2x_2(a_1x_1 + a_2x_2 + \cdots + a_nx_n) + \cdots \\ &\quad + a_nx_n(a_1x_1 + a_2x_2 + \cdots + a_nx_n) \\ &= (a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2 \quad \text{so} \\ x_i &= \frac{a_i}{a_1x_1 + a_2x_2 + \cdots + a_nx_n} \quad \text{and finally,} \\ \sum_{i=1}^n x_i^2 &= \frac{\sum_{i=1}^n a_i^2}{(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2} = 1.\end{aligned}$$

Now we can substitute back into the original equation:

$$\begin{aligned}f(x_1, x_2, \ldots, x_n) &= f\left(\frac{a_1}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}, \ldots, \frac{a_n}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}\right) \\ &= \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}\right)^2 \\ &= \left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{(a_1x_1 + a_2x_2 + \cdots + a_nx_n)^2}\right)(a_1^2 + a_2^2 + \cdots + a_n^2) \\ &= a_1^2 + a_2^2 + \cdots + a_n^2.\end{aligned}$$

13. Since the faces are parallel to the coordinate planes, we can reduce the problem to maximizing $M(x, y, z) = xyz$ subject to the constraint $g(x, y, z) = x^2 + 2y^2 + 4z^2 = 12$, where x, y, and z are all positive. Here, by the symmetry of the problem, we are maximizing the volume of one eighth of the box and therefore we will have the dimensions of the box itself by doubling x, y, and z. We solve

$$\begin{cases} yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \\ x^2 + 2y^2 + 4z^2 = 12. \end{cases}$$

So $x^2 = 2y^2 = 4z^2$ and $12z^2 = 12$ so a critical point is $(2, \sqrt{2}, 1)$. The dimensions of the box are twice these values so the largest box is $4 \times 2\sqrt{2} \times 2$.

14. We are minimizing the cost of producing a sphere and a cylinder of equal radii with the given constraints. We also need to convert 8000 gallons to $8000/7.480520 \approx 1069.444$ cubic feet. So minimize $V(r, h) = 2\pi rh + 8\pi r^2$ subject to $g(r, h) = \pi r^2 h + (4/3)\pi r^3 = 1069.444$. We solve

$$\begin{cases} 2\pi h + 16\pi r = \lambda(2\pi rh + 4\pi r^2) \\ 2\pi r = \lambda(\pi r^2) \\ \pi r^2 h + (4/3)\pi r^3 = 1069.444. \end{cases}$$

Physically, r cannot be zero, so by the second equation $\lambda = 2/r$ and then by the first $h = 4r$ and so by the third $1069.444 = 4\pi r^3 + (4/3)\pi r^3 = (16\pi/3)r^3$. Therefore, the best dimensions are $r \approx \sqrt[3]{63.8277} \approx 3.9964$ feet and $h \approx 15.9856$ feet.

15. Minimize $M(x, y, z) = x^2 + y^2 + z^2$ subject to $x^2 - (y - z)^2 = 1$. We solve

$$\begin{cases} 2x = 2\lambda x \\ 2y = -2\lambda(y - z) \\ 2z = 2\lambda(y - z) \\ x^2 - (y - z)^2 = 1. \end{cases}$$

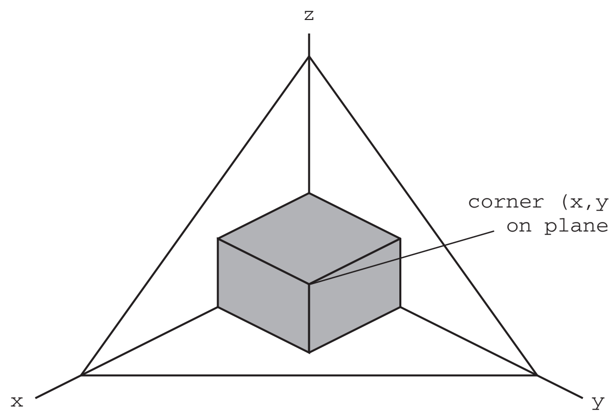
Since the last equation implies that $x \neq 0$, the first equation gives us that $\lambda = 1$, so $y = z = 0$ and thus $x = \pm 1$. The minimum distance is, therefore, 1.

16. Place the vertex of the cone at the North Pole (0, 0, a), with the axis of symmetry of the cone coinciding with the z-axis. The height of the cone is h and the radius is r. We are maximizing $V(r, h) = (1/3)\pi r^2 h$ subject to the constraint $(h-a)^2 + r^2 = a^2$ or $g(h, a) = h^2 - 2ha + r^2 = 0$. We solve

$$\begin{cases} (2/3)\pi r h = 2\lambda r \\ (1/3)\pi r^2 = 2\lambda(h-a) \\ h^2 - 2ha + r^2 = 0. \end{cases}$$

From the first equation we know $\lambda \neq 0$ and $\pi h/3 = \lambda$. So substitute this into the second equation to find that $r^2 = 2h^2 - 2ah$. Solve this with the third equation to find that $h = 4a/3$ and $r = 2\sqrt{2}a/3$.

17. We want to maximize $V = xyz$ subject to $bcx + acy + abz = abc$.



Thus we solve

$$\begin{cases} \nabla V = \lambda \nabla(bcx + acy + abz) \\ bcx + acy + abz = abc \end{cases} \quad \text{or} \quad \begin{cases} yz = \lambda bc \\ xz = \lambda ac \\ xy = \lambda ab \\ \dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1. \end{cases}$$

Hence $\lambda = \dfrac{yz}{bc} = \dfrac{xz}{ac} = \dfrac{xy}{ab}$.

$$\frac{yz}{bc} = \frac{xz}{ac} \Leftrightarrow z = 0 \quad \text{or} \quad y = \frac{bc}{ac}x = \frac{b}{a}x.$$

Now $z = 0$ makes $V = 0$, so this cannot possibly maximize. Thus $y = (b/a)x$. Now $\dfrac{yz}{bc} = \dfrac{xy}{ab} \Leftrightarrow y = 0$ (reject) or $z = \dfrac{bc}{ab}x$ or $z = \dfrac{c}{a}x$. Hence the constraint becomes

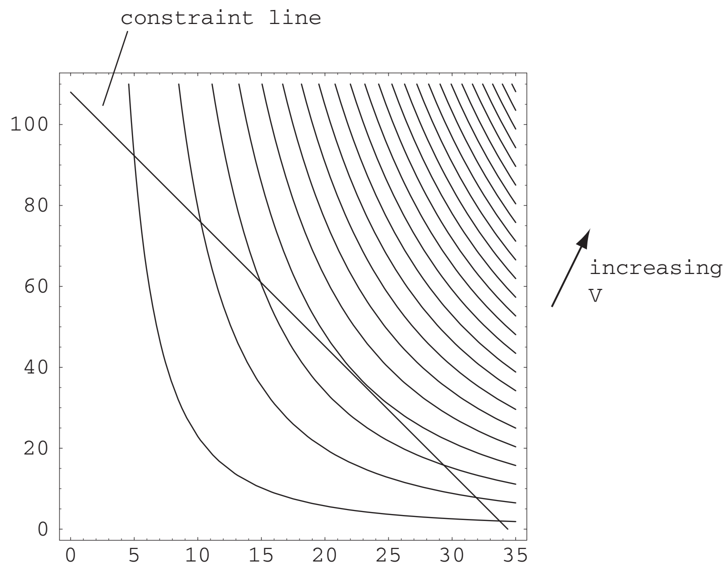
$$\frac{x}{a} + \frac{x}{a} + \frac{x}{a} = 1 \quad \text{so} \quad x = a/3 \Rightarrow y = b/3 \quad z = c/3.$$

18. We have $V(x, y) = \pi \left(\dfrac{x}{2}\right)^2 y = \dfrac{\pi}{4}x^2 y$ with $\pi x + y \leq 108$.

(a) We maximize V subject to $g(x, y) = \pi x + y = 108$. Thus, with a Lagrange multiplier we solve

$$\begin{cases} \dfrac{\pi x y}{2} = \pi\lambda \\ \dfrac{\pi x^2}{4} = \lambda \\ \pi x + y = 108 \end{cases}.$$

The first two equations imply that $\lambda = \dfrac{xy}{2} = \dfrac{\pi x^2}{4}$ so that either $x = 0$ (which we reject) or $\dfrac{y}{2} = \dfrac{\pi x}{4}$, so $y = \dfrac{\pi x}{2}$. Thus, in the constraint we must have $\pi x + \dfrac{\pi x}{2} = 108$ so $x = \dfrac{2 \cdot 108}{3\pi} = \dfrac{72}{\pi}$. Hence the maximizing dimensions are $\dfrac{72}{\pi}''$ diameter, $36''$ length. (That these dimensions really do maximize volume may be seen from the following picture.)



(b) Perhaps this is an easier method: $\pi x + y = 108 \Leftrightarrow y = 108 - \pi x$ so $v(x) = V(x, 108 - \pi x) = \dfrac{\pi x^2}{4}(108 - \pi x)$ defined on $\left[0, \dfrac{108}{\pi}\right]$. Thus $v'(x) = \dfrac{\pi}{4}(216x - 3\pi x^2)$ so critical points are $x = 0, \dfrac{72}{\pi}$.

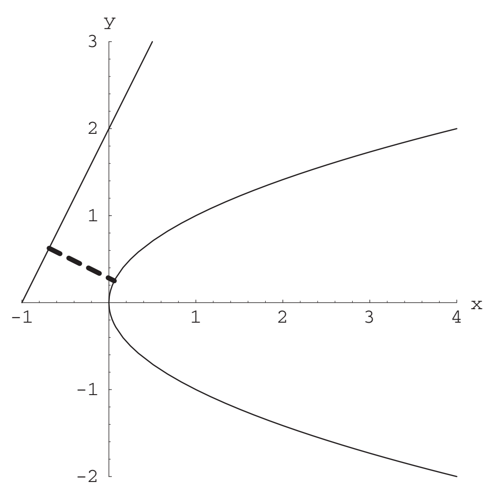
Compare values: $v(0) = 0, \quad v\left(\dfrac{72}{\pi}\right) > 0, \quad v\left(\dfrac{108}{\pi}\right) = 0$, so $x = 72/\pi$ must give the *absolute* maximum.

19. The two equations are $x = y/2 - 1$ and $x = y^2$. We will minimize the square of the distance between a point (x_1, y_1) on the line and a point (x_2, y_2) on the parabola. Maximize $f(y_1, y_2) = (y_1/2 - 1 - y_2^2)^2 + (y_2 - y_1)^2$. Take the first partials:

$$f_{y_1}(y_1, y_2) = \frac{5}{2}y_1 - 1 - y_2^2 - 2y_2 \quad \text{and}$$

$$f_{y_2}(y_1, y_2) = 4y_2^2 - 2y_1y_2 + 6y_2 - 2y_1.$$

Set these equal to zero and solve to find the critical point at $(y_1, y_2) = (5/8, 1/4)$. The minimal distance is therefore $3\sqrt{5}/8$.



20. (a) For each section the time is the distance divided by the rate and the hypotenuse is the altitude divided by the cosine of the angle that is formed by the altitude and the hypotenuse. So

$$T(\theta_1, \theta_2) = \frac{a}{v_1 \cos\theta_1} + \frac{b}{v_2 \cos\theta_2}.$$

(b) We are minimizing time subject to the constraint that the horizontal separation is constant: $a\tan\theta_1 + b\tan\theta_2 = c$. We solve

$$\begin{cases} \dfrac{a\sin\theta_1}{v_1\cos^2\theta_1} = \dfrac{\lambda a}{\cos^2\theta_1} \\ \dfrac{b\sin\theta_2}{v_2\cos^2\theta_2} = \dfrac{\lambda b}{\cos^2\theta_2} \\ a\tan\theta_1 + b\tan\theta_2 = c. \end{cases}$$

The first two equations immediately give the result: $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$.

21. We are minimizing the square of the distance $f(x,y) = (x-x_0)^2 + (y-y_0)^2$ subject to the constraint $ax+by=d$. We solve

$$\begin{cases} 2(x-x_0) = a\lambda \\ 2(y-y_0) = b\lambda \\ ax+by=d. \end{cases}$$

Solving, we see that $x = (a\lambda + 2x_0)/2$ and $y = (b\lambda + 2y_0)/2$ so substituting for x and y in the third equation $(a^2+b^2)\lambda = 2(d - ax_0 - by_0)$. Also substituting for x and y in f we see that

$$\begin{aligned} f(x,y) &= \left(\frac{a\lambda}{2}\right)^2 + \left(\frac{b\lambda}{2}\right)^2 = \left(\frac{a^2+b^2}{4}\right)\lambda^2 = \frac{a^2+b^2}{4}\left(\frac{2(d-ax_0-by_0)}{a^2+b^2}\right)^2 \\ &= \frac{(d-ax_0-by_0)^2}{a^2+b^2} \end{aligned}$$

so the distance D is the square root of this: $D = \dfrac{|ax_0+by_0-d|}{\sqrt{a^2+b^2}}$.

22. This is very similar to Exercise 21. Minimize the square of the distance $f(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$ subject to the constraint $ax+by+cz=d$. We solve

$$\begin{cases} 2(x-x_0) = a\lambda \\ 2(y-y_0) = b\lambda \\ 2(z-z_0) = c\lambda \\ ax+by+cz=d. \end{cases}$$

Solving, we see that $x = (a\lambda+2x_0)/2, y = (b\lambda+2y_0)/2$ and $z = (c\lambda + 2z_0)/2$ so substituting for x, y and z in the fourth equation $(a^2+b^2+c^2)\lambda = 2(d-ax_0-by_0-cz_0)$. Also substituting for x, y and z in f we see that

$$f(x,y,z) = \left(\frac{a^2+b^2+c^2}{4}\right)\lambda^2 = \frac{(d-ax_0-by_0-cz_0)^2}{a^2+b^2}$$

so the distance D is the square root of this: $D = \dfrac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$.

23. (a) We solve

$$\begin{cases} 2xy^2z^2 = 2\lambda x \\ 2x^2yz^2 = 2\lambda y \\ 2x^2y^2z = 2\lambda z \\ x^2+y^2+z^2 = a^2. \end{cases}$$

If $\lambda = 0$, then at least one of x, y and z is 0 and this corresponds to a minimum. If $\lambda \neq 0$, we see that, at a critical point, $x^2 = y^2 = z^2$, so $3x^2 = a^2$ or $x^2 = a^2/3$. Therefore, at a critical point,

$$f(x,y,z) = \left(\frac{a^2}{3}\right)^3 = \frac{a^6}{27}.$$

(b) In part (a) we showed $x^2y^2z^2 \le (a^2/3)^3 = [(x^2+y^2+z^2)/3]^3$ and so this result follows immediately.

(c) We make the appropriate adjustments to parts (a) and (b) and maximize $f(x_1, x_2, \ldots, x_n) = x_1^2 x_2^2 \cdots x_n^2$ subject to $x_1^2 + x_2^2 + \cdots + x_n^2 = a^2$. Because, as in part (a), the case $\lambda = 0$ corresponds to a minimum, we see that at a maximum no x_i is 0. So we solve

$$\begin{cases} 2x_1^2 x_2^2 \cdots x_n^2 / x_i = 2\lambda x_i & \text{for } 1 \le i \le n \\ x_1^2 + x_2^2 + \cdots + x_n^2 = a^2. \end{cases}$$

At a maximum, $x_1^2 = x_2^2 = \cdots = x_n^2$, so $x_i^2 = a^2/n$. Therefore, the maximum of f is $(a^2/n)^n$. So we conclude that $x_1^2 x_2^2 \cdots x_n^2 \le (a^2/n)^n = [(x_1^2 + x_2^2 + \cdots + x_n^2)/n]^n$. The result follows immediately.

(d) We found that f was maximized when $x_1^2 = x_2^2 = \cdots = x_n^2$ so, since here we are assuming $x_i > 0$ for all i, the equality holds when $x_1 = x_2 = \cdots = x_n$.

24. (a)

$$\frac{\partial f}{\partial x_k} = \sum_{j=1}^{n} a_{k_j} x_j + \sum_{i=1}^{n} a_{ik} x_i = \sum_{j=1}^{n} (a_{jk} + a_{kj}) x_j$$

$$\frac{\partial g}{\partial x_k} = 2x_k.$$

Thus the Lagrange multiplier system is

$$\begin{cases} \sum_j (a_{j1} + a_{1j}) x_j = 2\lambda x_1 \\ \vdots \\ \sum_j (a_{jn} + a_{nj}) x_j = 2\lambda x_n \\ x_1^2 + \cdots + x_n^2 = 1. \end{cases}$$

(b) Because A is symmetric, $a_{jk} = a_{kj}$ so the system becomes

$$\begin{cases} \sum_j 2a_{1j} x_j = 2\lambda x_1 \\ \vdots \\ \sum_j 2a_{nj} x_j = 2\lambda x_n \\ x_1^2 + \cdots + x_n^2 = 1. \end{cases}$$

The first n equations come from $\nabla f = \lambda \nabla g$ and simplify to

$$\begin{cases} \sum_j a_{1j} x_j = \lambda x_1 \\ \vdots \\ \sum_j a_{nj} x_j = \lambda x_n. \end{cases}$$

Note that $\sum_j a_{kj} x_j$ is the dot product of the kth row of A with $\mathbf{x}$. So the n equations, taken together, express

$$A\mathbf{x} = \lambda \mathbf{x}.$$

(c)

$$\begin{aligned} f(x_1, \ldots, x_n) &= \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (\lambda \mathbf{x}) \quad (\mathbf{x} \text{ is an eigenvector}) \\ &= \lambda(\mathbf{x}^T \mathbf{x}) = \lambda \mathbf{x} \cdot \mathbf{x} \\ &= \lambda \|\mathbf{x}\|^2 = \lambda \cdot 1, \end{aligned}$$

since $\mathbf{x}$ is assumed to be a unit vector.

25. (a) To set things up using Lagrange multipliers, we solve

$$\begin{cases} 2ax + 2by = 2\lambda x \\ 2bx + 2cy = 2\lambda y \\ x^2 + y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} (a-\lambda)x + by = 0 \\ bx + (c-\lambda)y = 0 \\ x^2 + y^2 = 1. \end{cases}$$

In the last system, multiply the first equation by $\lambda - c$ and the second by b, then add to obtain:

$$((a-\lambda)(\lambda - c) + b^2)x = 0.$$

Now multiply the first equation by b and the second by $\lambda - a$, then add to get:

$$(b^2 + (\lambda - a)(c - \lambda))y = 0.$$

Since $x^2 + y^2 = 1$, we cannot have both x and y equal to 0. Thus

$$b^2 + (\lambda - a)(c - \lambda) = 0 \Leftrightarrow \lambda^2 - (a+c)\lambda + ac - b^2 = 0.$$

Hence

$$\lambda_1, \lambda_2 = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac - b^2)}}{2}.$$

(b) Rewriting, $\lambda_1, \lambda_2 = \dfrac{(a+c) \pm \sqrt{(a-c)^2 + 4b^2}}{2}$. $(a-c)^2 + 4b^2 \geq 0$ so the eigenvalues are always real.

26. (a) $\lambda_1 = \lambda_2 \Leftrightarrow (a-c)^2 + 4b^2 = 0 \Leftrightarrow a = c, b = 0$ so $f(x, y) = a(x^2 + y^2)$.

(b) The eigenvalues are the max and min values of f on the circle. If both are positive, then f has a positive minimum on the circle; hence f must be positive on the entire circle.

(c) If both eigenvalues are negative, then f has a negative maximum on the circle—so f must be negative on the entire circle.

27. (a)

$$f(kx_1, \ldots, kx_n) = \sum_{i,j=1}^{n} a_{ij}(kx_i)(kx_j) = k^2 \sum_{i,j=1}^{n} a_{ij}x_i x_j$$

(b) Let $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|$ when $\mathbf{x} \neq \mathbf{0}$. Then $\mathbf{u}$ is a point on the unit hypersphere. If f has a positive minimum on the hypersphere, then f must be positive on the entire hypersphere. Hence, for $\mathbf{x} \neq \mathbf{0}$:

$$f(\mathbf{x}) = f(k\mathbf{u}) = k^2 f(\mathbf{u}) > 0 \quad (k = \|\mathbf{x}\|).$$

The case where f has a negative maximum on the hypersphere is similar.

(c) Clearly the converses of the results of part (b) hold (i.e., if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then f is positive on the hypersphere ...). From Exercise 24, the minimum value of f is the smallest eigenvalue of A. Thus the quadratic form is positive definite $\Leftrightarrow$ $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ $\Leftrightarrow$ f is positive on the hypersphere $\Leftrightarrow$ the smallest eigenvalue of A is positive $\Leftrightarrow$ all eigenvalues are positive. (The negative definite result is similar.)