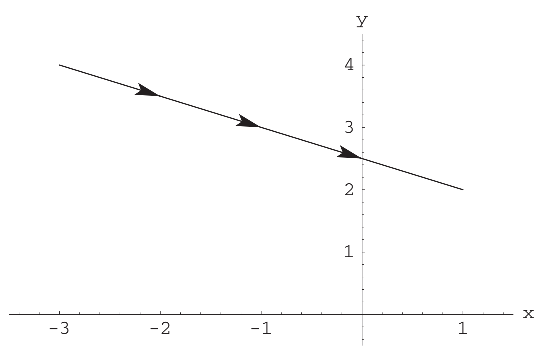
Chapter 3

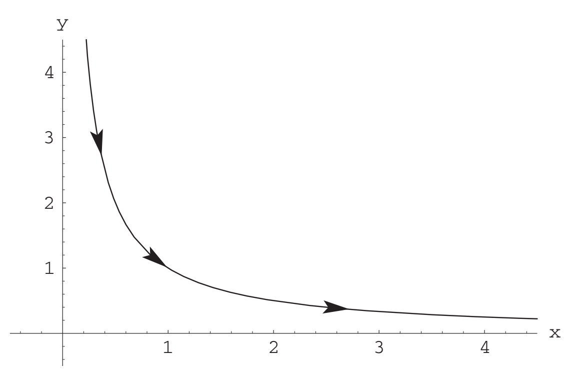
Vector-Valued Functions

3.1 Parametrized Curves and Kepler's Laws

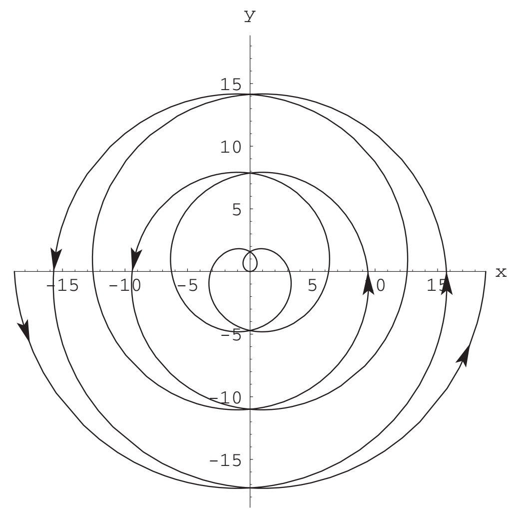
1. The graph is a line segment with slope $-1/2$ and y-intercept 3:



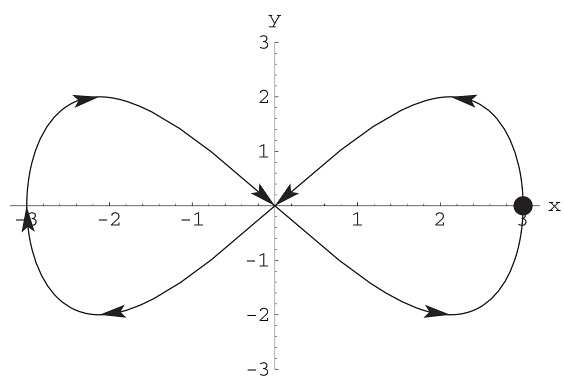
2. In this case $y = 1/x$ and both x and y are positive:



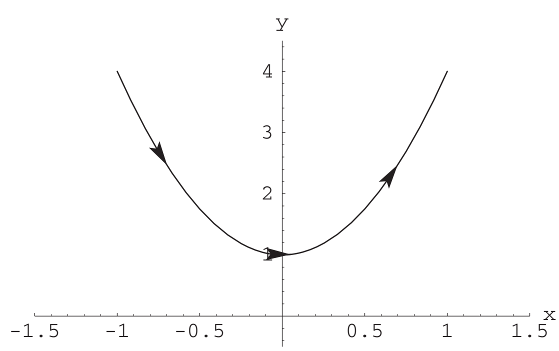
3. This is the spiral $r = \theta$ (note $x = r\cos\theta$ and $y = r\sin\theta$):



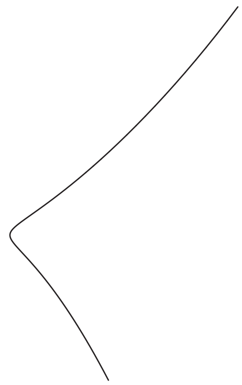
4. This is a lemniscate beginning and ending at the point (3, 0):

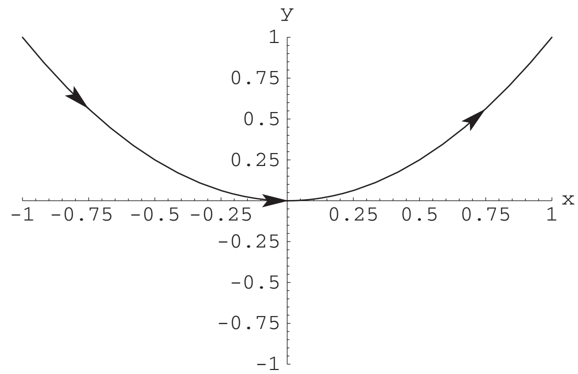


5. Although this is a curve in $\mathbf{R}^3$, because $z \equiv 0$ the curve lives in the xy-plane. It is the parabola $y = 3x^2 + 1$:

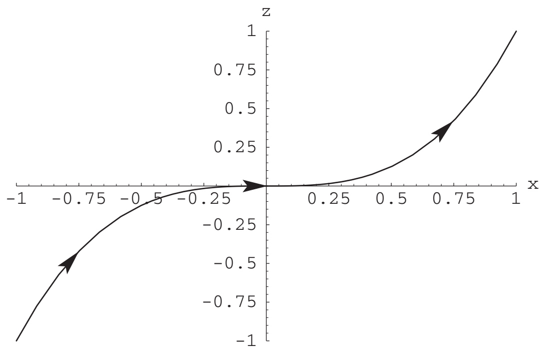


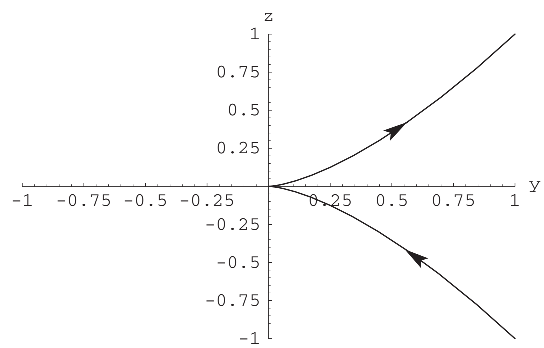
6. It's hard to see what this curve looks like in $\mathbf{R}^3$ (below left):





so I have also projected it onto the three coordinate planes:





For Exercises 7–10, the velocity is the derivative of position, the speed is the length of the velocity vector and the acceleration is the derivative of the velocity vector.

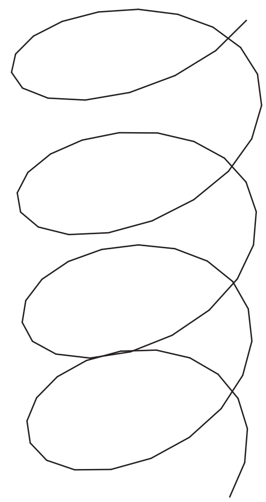
7. $\mathbf{x}(t) = (3t - 5, 2t + 7)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (3, 2)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{3^2 + 2^2} = \sqrt{13}$. Finally, acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (0, 0)$.

8. $\mathbf{x}(t) = (5\cos t, 3\sin t)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (-5\sin t, 3\cos t)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{(-5\sin t)^2 + (3\cos t)^2} = \sqrt{9 + 16\sin^2 t}$. Acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (-5\cos t, -3\sin t) = -\mathbf{x}(t)$.

9. $\mathbf{x}(t) = (t\sin t, t\cos t, t^2)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (\sin t + t\cos t, \cos t - t\sin t, 2t)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t + 4t^2} = \sqrt{1 + 5t^2}$. Finally, acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (2\cos t - t\sin t, -2\sin t - t\cos t, 2)$.

10. $\mathbf{x}(t) = (e^t, e^{2t}, 2e^t)$ so velocity $= \mathbf{v}(t) = \mathbf{x}'(t) = (e^t, 2e^{2t}, 2e^t)$ and speed $= \|\mathbf{v}(t)\| = \sqrt{5e^{2t} + 4e^{4t}} = e^t\sqrt{5 + 4e^{2t}}$. Finally, acceleration $= \mathbf{a}(t) = \mathbf{x}''(t) = (e^t, 4e^{2t}, 2e^t)$.

11. (a)

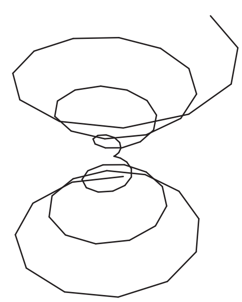


(b) To verify that the curve lies on the surface check that

$$\frac{x^2}{9} + \frac{y^2}{16} = \frac{9\cos^2 \pi t}{9} + \frac{16\sin^2 \pi t}{16} = \cos^2 \pi t + \sin^2 \pi t = 1.$$

The z component just determines the speed traveling up the cylinder.

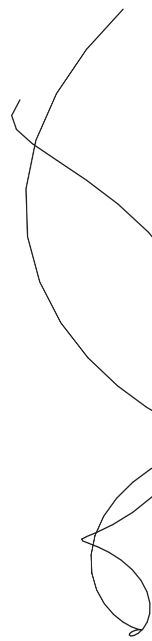
12. (a)



(b) To verify that the curve lies on the surface check that

$$x^2 + y^2 = t^2\cos^2 t + t^2 \sin^2 t = t^2(\cos^2 t + \sin^2 t) = t^2 = z^2.$$

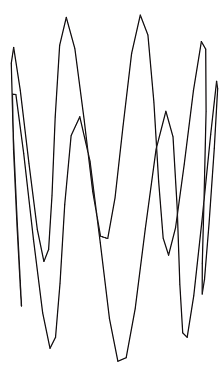
13. (a)



(b) To verify that the curve lies on the surface check that

$$x^2 + y^2 = t^2 \sin^2 2t + t^2 \cos^2 2t = t^2(\sin^2 2t + \cos^2 2t) = t^2 = z.$$

14. (a)



(b) To verify that the curve lies on the surface check that

$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4.$$

In Exercises 15–18 use formulas (2) and (3) from the text. In each case we will need to calculate the position and velocity at the given time.

15. $\mathbf{x}(t) = (te^{-t}, e^{3t})$ so $\mathbf{x}(0) = (0, 1)$ and $\mathbf{x}'(t) = (e^{-t} - te^{-t}, 3e^{3t})$ so $\mathbf{x}'(0) = (1, 3)$. The equation of the tangent line at $t = 0$ is $\mathbf{l}(t) = (0, 1) + (1, 3)t = (t, 1 + 3t)$.

16. $\mathbf{x}(t) = (4\cos t, -3\sin t, 5t)$ so $\mathbf{x}(\pi/3) = (2, -3\sqrt{3}/2, 5\pi/3)$ and $\mathbf{x}'(t) = (-4\sin t, -3\cos t, 5)$ so $\mathbf{x}'(\pi/3) = (-2\sqrt{3}, -3/2, 5)$. The equation of the tangent line at $t = \pi/3$ is

$$\mathbf{l}(t) = (2, -3\sqrt{3}/2, 5\pi/3) + (-2\sqrt{3}, -3/2, 5)(t - \pi/3).$$

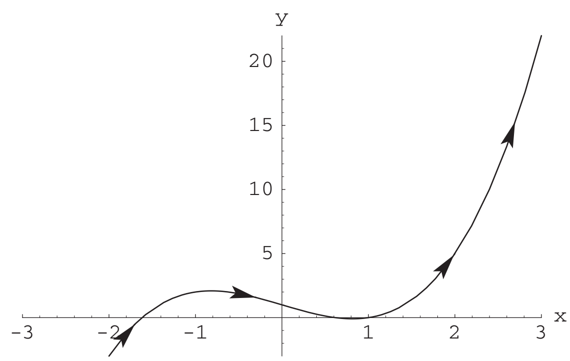
17. $\mathbf{x}(t) = (t^2, t^3, t^5)$ so $\mathbf{x}(2) = (4, 8, 32)$ and $\mathbf{x}'(t) = (2t, 3t^2, 5t^4)$ so $\mathbf{x}'(2) = (4, 12, 80)$. The equation of the tangent line at $t = 2$ is

$$\mathbf{l}(t) = (4, 8, 32) + (4, 12, 80)(t - 2) = (4t - 4, 12t - 16, 80t - 128).$$

18. $\mathbf{x}(t) = (\cos(e^t), 3 - t^2, t)$ so $\mathbf{x}(1) = (\cos e, 2, 1)$ and $\mathbf{x}'(t) = (-e^t \sin(e^t), -2t, 1)$. Therefore, $\mathbf{x}'(1) = (-e \sin e, -2, 1)$. The equation of the tangent line at $t = 1$ is

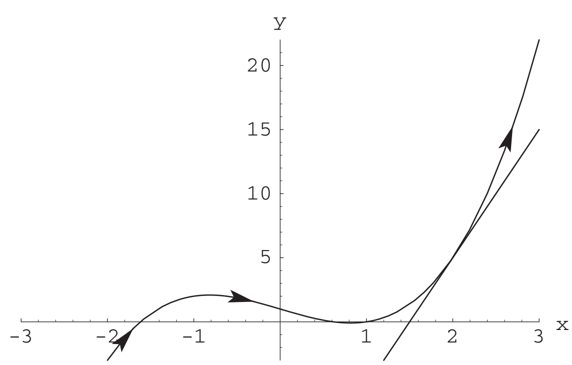
$$\mathbf{l}(t) = (\cos e, 2, 1) + (-e \sin e, -2, 1)(t - 1) = (\cos e + e \sin e - (e \sin e)t, 4 - 2t, t).$$

19. **(a)** The sketch of $\mathbf{x}(t) = (t, t^3 - 2t + 1)$ is:



(b) $\mathbf{x}(2) = (2, 5)$ and since $\mathbf{x}'(t) = (1, 3t^2 - 2)$ we get $\mathbf{x}'(2) = (1, 10)$. The equation of the line is then

$$\mathbf{l}(t) = (2, 5) + (1, 10)(t - 2) = (t, 10t - 15).$$



(c) Since $x = t$ we see that $y = f(x) = x^3 - 2x + 1$.
(d) So the equation of the tangent line at $x = 2$ is $y - f(2) = f'(2)(x - 2)$, where f is as in part (c). Substituting, we get $y - 5 = 10(x - 2)$ or $y = 10x - 15$. This is consistent with our answer for part (b).

20. From the first equation $t = x/(v_0 \cos\theta)$. Substitute this into the second equation to get

$$y = \frac{(v_0 \sin\theta)x}{v_0 \cos\theta} - \frac{1}{2} g \frac{x^2}{(v_0 \cos\theta)^2} = (\tan\theta)x - \frac{g}{2(v_0 \cos\theta)^2} x^2.$$

This is of the form $y = ax^2 + bx$ and the graph is a parabola.

21. We know from the text that Roger is on the ground at $t = 0$ and $t = 2v_0 \sin\theta/g$. By symmetry, Roger is at his maximum height at $t = v_0 \sin\theta/g$. For this exercise this is at time $t = 100 \sin 60^\circ/(32) = 25\sqrt{3}/16$. The maximum height is found by substituting into the equation for y:

$$y = (v_0 \sin\theta)\left(\frac{25\sqrt{3}}{16}\right) - \frac{1}{2}(32)\left(\frac{25\sqrt{3}}{16}\right)^2 = (50\sqrt{3})\left(\frac{25\sqrt{3}}{16}\right) - \frac{(625)(3)}{16} = \frac{(625)(3)}{16}$$

Roger's maximum height is 117.1875 feet.

22. By formula (5) from the text, $x = v_0^2 \sin 2\theta/g$. In this case we can say that $2640 = v_0^2 \sin 120^\circ/32$. Solve this for

$$v_0 = \sqrt{\frac{(2640)(32)}{\sqrt{3}/2}} = 32\sqrt{55\sqrt{3}} \approx 312.329.$$

23. We use the same formula as in Exercise 22 but now solve for θ. So, $x = v_0^2 \sin 2\theta/g$ becomes $1500 = 250^2 \sin 2\theta/32$ or

$$\sin 2\theta = \frac{(1500)(32)}{62500} = \frac{96}{125}.$$

There are two values of θ with $0 \le \theta \le \pi/2$ that satisfy this last equation. One is

$$\theta = (1/2)\sin^{-1}(96/125) \approx 0.43786 \approx 25.088^\circ,$$

and the other is

$$\theta = \pi/2 - (1/2)\sin^{-1}(96/125) \approx 1.13294 \approx 64.913^\circ.$$

24. This is similar to Example 6 from the text. We have the equation:

$$\mathbf{x}(t) = -(1/2)gt^2\mathbf{j} + tv_0 + x_0\mathbf{j}.$$

(a) Here the angle is given as 45° and the initial speed of the water is 7 *m/s*, therefore, $\mathbf{v}_0 = 7(\sqrt{2}/2, \sqrt{2}/2)$. Also x_0 is the initial height of 1 m and gravity is about $-9.8\, m/s^2$. This means that

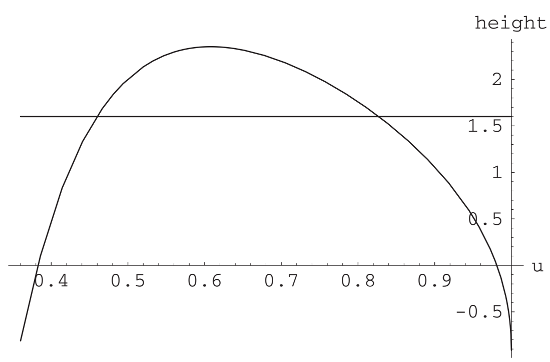
$$\mathbf{x}(t) = -4.9t^2\mathbf{j} + 7(\sqrt{2}/2, \sqrt{2}/2)t + \mathbf{j} = \left(\frac{7\sqrt{2}}{2}t, -4.9t^2 + \frac{7\sqrt{2}}{2}t + 1\right).$$

We want to know the height when the x distance is 5 so first solve $\frac{7\sqrt{2}}{2}t = 5$ for t to get $t = 10/(7\sqrt{2})$. Substitute this into our vertical equation to find that the height would be 1 so the answer is yes, Egbert gets wet.

(b) Here the idea is the same as in part (a). The initial speed of the water is 8 *m/s* and we don't know the direction so $v_0 = 8(u, \sqrt{1-u^2})$ for some u between 0 and 1. So

$$\mathbf{x}(t) = -4.9t^2\mathbf{j} + 8(u, \sqrt{1-u^2})t + \mathbf{j} = (8ut, -4.9t^2 + \sqrt{1-u^2}t + 1).$$

We want the height when the horizontal distance is 5 or when $t = 5/(8u)$. In that case, the height is $-4.9(5/(8u))^2 + 8\sqrt{1-u^2}(5/(8u)) + 1$. For what values of u is this between 0 and 1.6? Consider the figure:



Explore with *Mathematica* or a graphing calculator and you will find the u values in the two intervals which correspond to the correct heights. This gives the two approximate ranges for α as between 11.2° and 34.2° and as between 62.6° and 67.5°.

25. We have $\mathbf{x}(2) = \left(e^4, 8, \frac{3}{2}\right)$ and $\mathbf{x}'(2) = \left(2e^4, 10, \frac{5}{4}\right)$. If the rocket's engines cease when $t = 2$, then the rocket will follow the tangent line path

$$\mathbf{l}(t) = \mathbf{x}(2) + (t-2)\mathbf{x}'(2) = \left(e^4(2t-3), 10t-12, \tfrac{5}{4}t - 1\right).$$

For this path to reach the space station, we must have

$$\left(e^4(2t-3), 10t-12, \frac{5}{4}t - 1\right) = (7e^4, 35, 5).$$

Thus, in particular

$$e^4(2t-3) = 7e^4 \Leftrightarrow 2t - 3 = 7 \Leftrightarrow t = 5.$$

However $\mathbf{l}(5) = \left(7e^4, 38, \frac{21}{4}\right) \ne (7e^4, 35, 5)$. Hence the rocket does *not* reach the repair station.

26. (a) We set $\mathbf{x}(t) = \mathbf{y}(t)$ and solve for t:

$$\left(t^2 - 2, \frac{t^2}{2} - 1\right) = (t, 5 - t^2).$$

Comparing first components, we have $t^2 - 2 = t \Leftrightarrow t^2 - t - 2 = 0 \Leftrightarrow t = -1, 2$. Now $\mathbf{x}(-1) = \left(-1, -\frac{1}{2}\right)$ and $\mathbf{y}(-1) = (-1, 4)$, so this is not a collision point. However, $\mathbf{x}(2) = (2, 1) = \mathbf{y}(2)$. So the balls collide when $t = 2$ at the point (2, 1).

(b) We have $\mathbf{x}'(2) = (4, 2), \mathbf{y}'(2) = (1, -4)$. The angle between the paths is the angle between these tangent vectors, which is

$$\cos^{-1}\left[\frac{\mathbf{x}'(2)\cdot\mathbf{y}'(2)}{\|\mathbf{x}'(2)\|\|\mathbf{y}'(2)\|}\right] = \cos^{-1}\frac{-4}{\sqrt{20}\sqrt{17}} = \cos^{-1}\frac{-2}{\sqrt{5}\sqrt{17}}.$$

27. The calculation is fairly straightforward:

$$\begin{aligned}\frac{d}{dt}(\mathbf{x}\cdot\mathbf{y}) &= \frac{d}{dt}(x_1(t)y_1(t) + x_2(t)y_2(t) + \cdots + x_n(t)y_n(t))\\ &= x_1'(t)y_1(t) + x_1(t)y_1'(t) + x_2'(t)y_2(t) + x_2(t)y_2'(t) + \cdots + x_n'(t)y_n(t) + x_n(t)y_n'(t)\\ &= [x_1'(t)y_1(t) + x_2'(t)y_2(t) + \cdots + x_n'(t)y_n(t)] + [x_1(t)y_1'(t) + x_2(t)y_2'(t) + \cdots + x_n(t)y_n'(t)]\\ &= \mathbf{y}\cdot\frac{d\mathbf{x}}{dt} + \mathbf{x}\cdot\frac{d\mathbf{y}}{dt}.\end{aligned}$$

28. This is similar to Exercise 27:

$$\begin{aligned}\frac{d}{dt}(\mathbf{x}\times\mathbf{y}) &= \frac{d}{dt}[(x_2y_3 - x_3y_2)\mathbf{i} - (x_1y_3 - x_3y_1)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}]\\ &= (x_2'y_3 - x_3'y_2 + x_2y_3' - x_3y_2')\mathbf{i} - (x_1'y_3 - x_3'y_1 + x_1y_3' - x_3y_1')\mathbf{j}\\ &\quad + (x_1'y_2 - x_2'y_1 + x_1y_2' - x_2y_1')\mathbf{k}\\ &= ([x_2'y_3 - x_3'y_2] + [x_2y_3' - x_3y_2'])\mathbf{i} - ([x_1'y_3 - x_3'y_1] + [x_1y_3' - x_3y_1'])\mathbf{j}\\ &\quad + ([x_1'y_2 - x_2'y_1] + [x_1y_2' - x_2y_1'])\mathbf{k}\\ &= \frac{d\mathbf{x}}{dt}\times\mathbf{y} + \mathbf{x}\times\frac{d\mathbf{y}}{dt}.\end{aligned}$$

29. You're asked to show that if $\|\mathbf{x}(t)\|$ is constant, then $\mathbf{x}$ is perpendicular to $d\mathbf{x}/dt$. If $\|\mathbf{x}(t)\|$ is constant, then $\frac{d}{dt}\|\mathbf{x}(t)\| \equiv 0$. So

$$0 = \frac{d}{dt}\|\mathbf{x}(t)\| = \frac{d}{dt}\sqrt{\mathbf{x}\cdot\mathbf{x}} = \left(\frac{1}{2\sqrt{\mathbf{x}\cdot\mathbf{x}}}\right)\left(2\frac{d\mathbf{x}}{dt}\cdot\mathbf{x}\right).$$

This means that $\frac{d\mathbf{x}}{dt}\cdot\mathbf{x} = 0$.

30. (a) $\|\mathbf{x}(t)\|^2 = \cos^2 t + \cos^2 t\sin^2 t + \sin^4 t = \cos^2 t + \sin^2 t(\cos^2 t + \sin^2 t) = 1.$

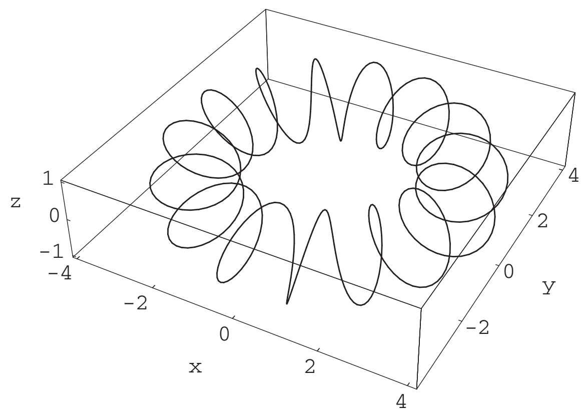
(b) This follows from Proposition 1.7 since $\|\mathbf{x}(t)\| \equiv 1$. The exercise really wants you to calculate the velocity vector: $\mathbf{v} = (-\sin t, -\sin^2 t + \cos^2 t, 2\sin t \cos t)$. Then $\mathbf{v} \cdot \mathbf{x} = 0$.

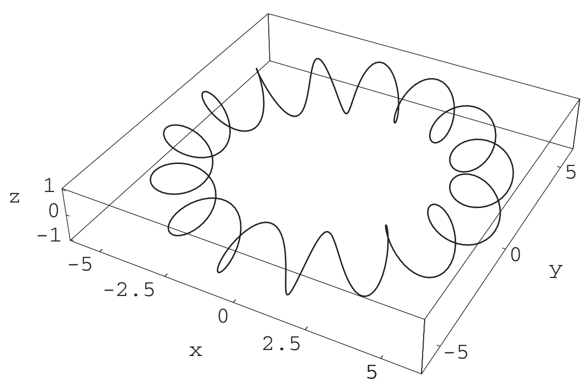
(c) If $\mathbf{x}(t)$ is a path on the unit sphere, $\|\mathbf{x}(t)\| \equiv 1$ so by Proposition 1.7 the position vector is perpendicular to its velocity vector.

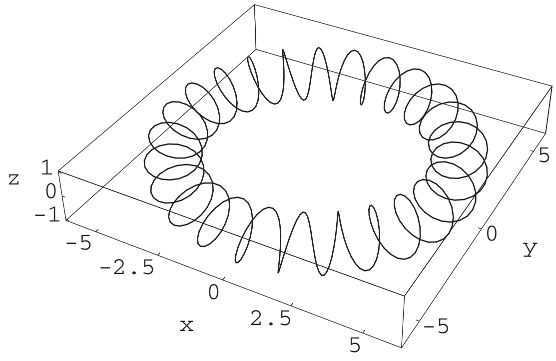
31. (a) Computer graphs are shown for (i), (ii), (iii). The constant a affects the size (radius) of the rings; the constant b affects the size (radius) of the coils; the constant ω affects the number of coils going around the ring.

(b) If $x = (a + b\cos\omega t)\cos t, y = (a + b\cos\omega t)\sin t$, then $x^2 + y^2 = (a + b\cos\omega t)^2$, so that $(\sqrt{x^2+y^2} - a)^2 = (a + b\cos\omega t - a)^2 = b^2\cos^2\ \omega t$. (Note $a > b > 0$.) Hence

$$(\sqrt{x^2+y^2} - a)^2 + z^2 = b^2\cos^2\ \omega t + b^2\sin^2\ \omega t = b^2.$$







32. The angle between $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{x}(t)\cdot\mathbf{x}'(t)}{\|\mathbf{x}(t)\|\,\|\mathbf{x}'(t)\|}\right).$$

Thus we calculate

$$\begin{aligned}
\mathbf{x}'(t) &= \left(e^t(\cos t - \sin t), e^t(\sin t + \cos t)\right);\\
\mathbf{x}(t)\cdot\mathbf{x}'(t) &= e^{2t}\cos t(\cos t - \sin t) + e^{2t}\sin t(\sin t + \cos t) = e^{2t};\\
\|\mathbf{x}(t)\| &= \sqrt{e^{2t}\cos^2 t + e^{2t}\sin^2 t} = e^t;\\
\|\mathbf{x}'(t)\| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2};\\
&= e^t\sqrt{\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\cos t\sin t + \cos^2 t} = \sqrt{2}\,e^t.
\end{aligned}$$

Thus

$$\theta = \cos^{-1}\left(\frac{e^{2t}}{(e^t)(\sqrt{2}e^t)}\right) = \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}.$$

33. **(a)** To have $\mathbf{x}(t_1) = (t_1^2, t_1^3 - t_1) = (t_2^2, t_2^3 - t_2) = \mathbf{x}(t_2)$, we must have $t_1^2 = t_2^2$, so if $t_1 \neq t_2$, then $t_1 = -t_2$. Then, comparing the second components: $t_1^3 - t_1 = -t_1^3 + t_1 \iff 2t_1^3 = 2t_1$. Since $t_1 \neq 0$ (otherwise $t_2 = 0$ as well), we must have $t_1^2 = 1$. Thus $\mathbf{x}(1) = \mathbf{x}(-1) = (1, 0)$.

(b) The velocity vector of the path is $\mathbf{x}'(t) = (2t, 3t^2 - 1)$. Therefore, the corresponding tangent vectors at $t = \pm 1$ are $\mathbf{x}'(-1) = (-2, 2)$ and $\mathbf{x}'(1) = (2, 2)$. Note that $\mathbf{x}'(-1)\cdot\mathbf{x}'(1) = 0$. Since these tangent vectors are parallel to the corresponding tangent lines, we see that the tangent lines must be perpendicular—so the angle they make is $\pi/2$.

34. **(a)** The slope is

$$t = \frac{y - 0}{x - (-1)} = \frac{y}{x+1}.$$

Thus $y = t(x+1)$ is the equation for the line.

(b) Since we have $y = t(x+1)$, we may substitute this expression for y into the equation $x^2 + y^2 = 1$ for the circle. This gives

$$x^2 + t^2(x+1)^2 = 1 \iff (1+t^2)x^2 + 2t^2x + (t^2 - 1) = 0.$$

We may use the quadratic formula with this last equation to solve for x in terms of t:

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(t^2+1)(t^2-1)}}{2(t^2+1)} = \frac{-t^2 \pm 1}{t^2+1}.$$

Hence the two solutions are $x = -1$ (which was to be expected) and

$$x = \frac{-t^2+1}{t^2+1} = \frac{1-t^2}{1+t^2}.$$

(c) From $y = t(x+1)$ in part (a), we see that when $x = -1$, $y = 0$, and when $x = (1-t^2)/(1+t^2)$,

$$y = t\left(\frac{1-t^2}{1+t^2} + 1\right) = t\left(\frac{(1-t^2)+(1+t^2)}{1+t^2}\right) = \frac{2t}{1+t^2}.$$

Hence the parametric equations are

$$x = \frac{1-t^2}{1+t^2},\; y = \frac{2t}{1+t^2}.$$

(d) The parametrization misses the point $(-1, 0)$, since to have $y = 0$ t must be zero, but then $x = 1$, not -1.

35. The distance between a point on the image and the origin is $\|\mathbf{x}(t)\|$ and this is minimized when $t = t_0$. Thus the function

$$f(t) = \|\mathbf{x}(t)\|^2 = \mathbf{x}(t)\cdot\mathbf{x}(t)$$

is also minimized when $t = t_0$. Hence

$$\begin{aligned}
0 = f'(t_0) &= \frac{d}{dt}(\mathbf{x}(t)\cdot\mathbf{x}(t))\Big|_{t=t_0} = \mathbf{x}(t_0)\cdot\mathbf{x}'(t_0) + \mathbf{x}'(t_0)\cdot\mathbf{x}(t_0)\\
&= 2\mathbf{x}(t_0)\cdot\mathbf{x}'(t_0).
\end{aligned}$$

Thus $\mathbf{x}(t_0)$ and $\mathbf{x}'(t_0)$ are orthogonal.

3.2 Arclength and Differential Geometry

In Exercises 1–6 we are using Definition 2.1 to calculate the length of the given paths.

1. $\mathbf{x}(t) = (2t+1, 7-3t)$ so $\mathbf{x}'(t) = (2, -3)$. The length of the path is then

$$L(\mathbf{x}) = \int_{-1}^{2} \|\mathbf{x}'\|\,dt = \int_{-1}^{2} \sqrt{2^2 + (-3)^2}\,dt = \int_{-1}^{2} \sqrt{13}\,dt = \sqrt{13}t\Big|_{-1}^{2} = 3\sqrt{13}.$$

2. $\mathbf{x}(t) = (t^2, 2/3(2t+1)^{3/2})$ so $\mathbf{x}'(t) = (2t, 2(2t+1)^{1/2})$. The length of the path is then

$$\begin{aligned} L(\mathbf{x}) &= \int_0^4 \sqrt{(2t)^2 + 4(2t+1)}\,dt = \int_0^4 \sqrt{4t^2+8t+4}\,dt = 2\int_0^4 |t+1|\,dt \\ &= 2\int_0^4 (t+1)\,dt = 2(t^2/2 + t)\Big|_0^4 = 24. \end{aligned}$$

3. $\mathbf{x}(t) = (\cos 3t, \sin 3t, 2t^{3/2})$ so $\mathbf{x}'(t) = (-3\sin 3t, 3\cos 3t, 3t^{1/2})$. The length of the path is then

$$L(\mathbf{x}) = \int_0^2 \sqrt{9\sin^2 3t + 9\cos^2 3t + 9t}\,dt = 3\int_0^2 \sqrt{1+t}\,dt = 3\int_1^3 \sqrt{u}\,du = 2u^{3/2}\Big|_1^3 = 6\sqrt{3} - 2.$$

4. $\mathbf{x}(t) = (7, t, t^2)$ so $\mathbf{x}'(t) = (0, 1, 2t)$. The length of the path is then

$$\begin{aligned} L(\mathbf{x}) &= \int_1^3 \sqrt{1+4t^2}\,dt = 2\int_1^3 \sqrt{1/4+t^2}\,dt = \left[t\sqrt{1/4+t^2} + (1/4)\ \ln(t + \sqrt{1/4+t^2})\right]\Big|_1^3 \\ &= 3\sqrt{\frac{37}{4}} - \sqrt{\frac{5}{4}} + \left(\frac{1}{4}\right)\left[\ln(3+\sqrt{37/4}) - \ln(1+\sqrt{5/4})\right] \\ &= \frac{3\sqrt{37}-\sqrt{5}}{2} + \left(\frac{1}{4}\right)\left[\ln\left(\frac{6+\sqrt{37}}{2+\sqrt{5}}\right)\right] \approx 8.2681459. \end{aligned}$$

5. $\mathbf{x}(t) = (t^3, 3t^2, 6t)$ so $\mathbf{x}'(t) = (3t^2, 6t, 6)$. The length of the path is then

$$\begin{aligned} L(\mathbf{x}) &= \int_{-1}^{2} \sqrt{9t^4 + 36t^2 + 36}\,dt = \int_{-1}^{2} \sqrt{9(t^2+2)^2}\,dt \\ &= \int_{-1}^{2} 3(t^2+2)\,dt = \left(t^3 + 6t\right)\Big|_{-1}^{2} = 27. \end{aligned}$$

6. $\mathbf{x}(t) = (\ln(\cos t), \cos t, \sin t)$ so $\mathbf{x}'(t) = \left(-\dfrac{\sin t}{\cos t}, \sin t, \cos t\right)$. The length of the path is then

$$\begin{aligned} L(\mathbf{x}) &= \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 t}{\cos^2 t} + \sin^2 t + \cos^2 t}\,dt = \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 t}{\cos^2 t} + 1}\,dt \\ &= \int_{\pi/6}^{\pi/3} \sqrt{\frac{\sin^2 t + \cos^2 t}{\cos^2 t}}\,dt = \int_{\pi/6}^{\pi/3} \frac{1}{\cos t}\,dt = \int_{\pi/6}^{\pi/3} \sec t\,dt \\ &= \ln|\sec t + \tan t|\Big|_{\pi/6}^{\pi/3} = \ln(2+\sqrt{3}) - \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \ln\left(\frac{2\sqrt{3}+3}{3}\right). \end{aligned}$$

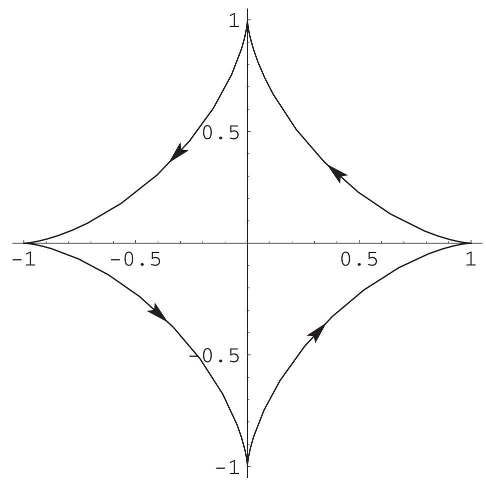
7. $\mathbf{x}(t) = (\ln t, t^2/2, \sqrt{2}t)$ so $\mathbf{x}'(t) = (1/t, t, \sqrt{2})$. The length of the path is then

$$\begin{aligned} L(\mathbf{x}) &= \int_1^4 \sqrt{1/t^2 + t^2 + 2}\,dt = \int_1^4 \sqrt{(1/t+t)^2}\,dt = \int_1^4 (1/t + t)\,dt \\ &= [\ln t + t^2/2]\Big|_1^4 = \ln 4 + 8 - 1/2 = \ln 4 + \tfrac{15}{2}. \end{aligned}$$

8. $\mathbf{x}(t) = (2t\cos t, 2t\sin t, 2\sqrt{2}t^2)$ so $\mathbf{x}'(t) = (2\cos t - 2t\sin t, 2\sin t + 2t\cos t, 4\sqrt{2}t)$. The length of the path is then

$$\begin{aligned}L(\mathbf{x}) &= \int_0^3 \sqrt{4\cos^2 t - 8t\cos t\sin t + 4t^2\sin^2 t + 4\sin^2 t + 8t\sin t\cos t + 4t^2\cos^2 t + 32t^2}\,dt \\ &= \int_0^3 \sqrt{4 + 4t^2 + 32t^2}\,dt = \int_0^3 \sqrt{4 + 36t^2}\,dt \\ &= [t\sqrt{1+9t^2} + \sinh^{-1}(3t)/3]\big|_0^3 = 3\sqrt{82} + \sinh^{-1}(9)/3.\end{aligned}$$

9. A sketch of the curve $\mathbf{x}(t) = (a\cos^3 t, a\sin^3 t)$ for $0 \le t \le 2\pi$ is:



Because of the obvious symmetries we will compute the length of the portion of the curve in the first quadrant and multiply it by 4:

$$\begin{aligned}L(\mathbf{x}) &= 4\int_0^{\pi/2} \|(-3a\cos^2 t\sin t, 3a\sin^2 t\cos t)\|\,dt = 4\int_0^{\pi/2} \sqrt{9a^2(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t)}\,dt \\ &= 4\int_0^{\pi/2} \sqrt{9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)}\,dt = 4\int_0^{\pi/2} 3a\sin t\cos t\,dt = 6a\sin^2 t\big|_0^{\pi/2} \\ &= 6a.\end{aligned}$$

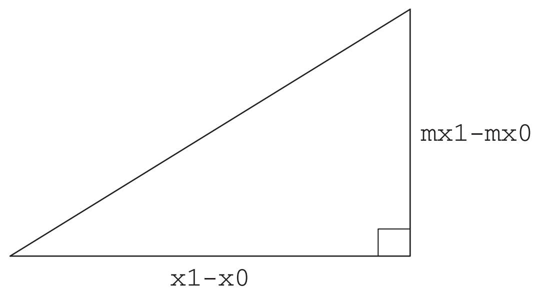
10. If f is a continuously differentiable function then we can calculate the length of the curve $y = f(x)$ between $(a, f(a))$ and $(b, f(b))$ by viewing the curve as the path $\mathbf{y}(x) = (x, f(x))$ so $\mathbf{y}'(x) = (1, f'(x))$, and so by Definition 2.1 the length is

$$L(\mathbf{y}) = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

11. Here $f(x) = mx + b$ and $f'(x) = m$ so by Exercise 10, the length of the curve is

$$L = \int_{x_0}^{x_1} \sqrt{1+m^2}\,dt = (x_1 - x_0)\sqrt{1+m^2}.$$

A quick look at a sketch shows why this should be the case:



If $x_1 > x_0$ then the horizontal distance is $|x_1 - x_0| = x_1 - x_0$ and the vertical distance is $|mx_1 - mx_0| = |m|(x_1 - x_0)$. By the Pythagorean theorem, the length of the hypotenuse is

$$\sqrt{(x_1 - x_0)^2 + (|m|(x_1 - x_0))^2} = \sqrt{(x_1 - x_0)^2(m^2 + 1)} = (x_1 - x_0)\sqrt{m^2 + 1}.$$

12. **(a)** $\mathbf{x}(t) = (a_1 t + b, a_2 t + b)$ so $\mathbf{x}'(t) = (a_1, a_2)$ so

$$L(\mathbf{x}) = \int_{t_0}^{t_1} \sqrt{a_1^2 + a_2^2}\, dt = (t_1 - t_0)\sqrt{a_1^2 + a_2^2}.$$

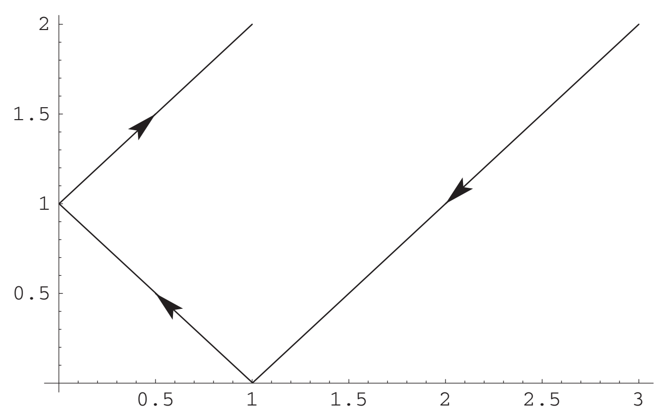
(b) The equation of the line in Exercise 11 could be given as $\mathbf{x}(t) = (t, mt + b)$ in which case part (a) would tell us that the length is $(x_1 - x_0)\sqrt{1 + m^2}$.

(c) $\mathbf{x}(t) = \mathbf{a}t + \mathbf{b}$ so $\mathbf{x}'(t) = \mathbf{a}$. Then

$$L(\mathbf{x}) = \int_{t_0}^{t_1} \|\mathbf{a}\|\, dt = \|\mathbf{a}\|(t_1 - t_0).$$

This, of course, is the same as our answer in part (a).

13. **(a)** A sketch of $\mathbf{x} = |t - 1|\mathbf{i} + |t|\mathbf{j}, -2 \le t \le 2$ is:



(b) Except for two points, the path is smooth (more than C^1). In fact, the path is comprised of three line segments joined end to end. In other words, on the open intervals $-2 \le t < 0, 0 < t < 1$, and $1 < t \le 2$, the path $\mathbf{x}$ is C^1. We say that the path $\mathbf{x}$ is piecewise C^1.

(c) We could figure out the length of each piece and add them together. In the process we will find that we're working too hard.

$$\mathbf{x}(t) = \begin{cases} (1-t)\mathbf{i} - t\mathbf{j} & -2 \le t \le 0 \\ (1-t)\mathbf{i} + t\mathbf{j} & 0 < t \le 1 \\ (t-1)\mathbf{i} + t\mathbf{j} & 1 < t \le 2 \end{cases} \quad \text{so} \quad \mathbf{x}'(t) = \begin{cases} (-1,-1) & -2 \le t \le 0 \\ (-1,1) & 0 < t \le 1 \\ (1,1) & 1 < t \le 2 \end{cases}.$$

So we see that $\|\mathbf{x}'(t)\| \equiv \sqrt{2}$. This means that to calculate the length of the curve we don't have to break up the integral into three pieces:

$$L(\mathbf{x}) = \int_{-2}^{2} \sqrt{2}\, dt = 4\sqrt{2}.$$

14. (a) We have that

$$\|\mathbf{x}(t)\|^2 = e^{-2t}\cos^2 t + e^{-2t}\sin^2 t = e^{-2t}.$$

Thus

$$\lim_{t\to+\infty}\|\mathbf{x}(t)\| = \lim_{t\to+\infty} e^{-t} = 0.$$

Hence $\lim_{t\to+\infty}\mathbf{x}(t) = \mathbf{0}$.

(b) We compute that $\mathbf{x}(t) = (-e^{-t}\cos t - e^{-t}\sin t, e^{-t}\cos t - e^{-t}\sin t)$. Hence

$$\begin{aligned}\|\mathbf{x}'(t)\| &= \sqrt{e^{-2t}(-\cos t - \sin t)^2 + e^{-2t}(\cos t - \sin t)^2}\\ &= e^{-t}\sqrt{\cos^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t\sin t + \sin^2 t}\\ &= \sqrt{2}e^{-t}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_a^\infty \|\mathbf{x}'(t)\|\,dt &= \lim_{t\to\infty}\int_a^t \|\mathbf{x}'(\tau)\|\,d\tau = \lim_{t\to\infty}\int_a^t \sqrt{2}e^{-\tau}\,d\tau\\ &= \lim_{t\to\infty}\left(-\sqrt{2}e^{-t} + \sqrt{2}e^{-a}\right) = \sqrt{2}e^{-a}.\end{aligned}$$

(c) The integral in part (b) represents the length of the path that spirals into $(0,0)$ from the point $\mathbf{x}(a)$. The result of part (b) shows that this arclength is always finite, regardless of a.

15. We use the polar/rectangular conversion equations $x = r\cos\theta$, $y = r\sin\theta$ to define a path $\mathbf{x}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Then

$$\mathbf{x}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta),$$

which implies

$$\begin{aligned}\|\mathbf{x}'(\theta)\| &= \sqrt{(f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 + (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2}\\ &= \sqrt{f'(\theta)^2 + f(\theta)^2},\end{aligned}$$

after expansion and simplification. Hence $L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta$, as desired.

16. (a) We'll use the equation: $s(t) = \int_0^t \|\mathbf{x}'(\tau)\|\,d\tau$. For $\mathbf{x}(\tau) = e^{a\tau}\cos b\tau\mathbf{i} + e^{a\tau}\sin b\tau\mathbf{j} + e^{a\tau}\mathbf{k}$ the derivative is $\mathbf{x}'(\tau) = ae^{a\tau}(\cos b\tau, \sin b\tau, 1) + e^{a\tau}(-b\sin b\tau, b\cos b\tau, 0)$. Therefore the speed is given by $\|\mathbf{x}'(\tau)\| = \sqrt{a^2e^{2a\tau}(2) + b^2e^{2a\tau}} = e^{a\tau}\sqrt{2a^2+b^2}$. This means that

$$s(t) = \int_0^t e^{a\tau}\sqrt{2a^2+b^2}\,d\tau = \frac{\sqrt{2a^2+b^2}}{a}e^{a\tau}\bigg|_0^t = \frac{\sqrt{2a^2+b^2}}{a}(e^{at}-1).$$

(b) Just solve the above for t:

$$t = \left(\ln\left[\frac{as}{\sqrt{2a^2+b^2}} + 1\right]\right)\Big/ a.$$

For Problems 17–20, we'll use

$$\mathbf{T} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}, \quad \mathbf{N} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}, \quad \text{and} \quad \mathbf{B} = \mathbf{T}\times\mathbf{N}.$$

Also

$$\frac{ds}{dt} = \|\mathbf{x}'(t)\|, \quad \kappa(t) = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \left\|\frac{d\mathbf{T}}{ds}\right\|, \quad \text{and} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = -\tau\mathbf{N}.$$

You may want to ask your students to make a guess about τ before they do Exercises 18 and 20. The curves are planar—what might that suggest about τ? Also see Section 3.6, Exercise 28.

17. $\mathbf{x}(t) = (5\cos 3t, 6t, 5\sin 3t)$ so $\mathbf{x}'(t) = (-15\sin 3t, 6, 15\cos 3t)$ and $\|\mathbf{x}'(t)\| = \sqrt{225+36} = \sqrt{261}$.

$$\begin{aligned}\mathbf{T} &= (1/\sqrt{261})(-15\sin 3t, 6, 15\cos 3t)\\ &= (1/\sqrt{29})(-5\sin 3t, 2, 5\cos 3t).\end{aligned}$$

$d\mathbf{T}/dt = (1/\sqrt{29})(-15\cos 3t, 0, -15\sin 3t)$ so

$$\mathbf{N} = (-\cos 3t, 0, -\sin 3t),$$

$$\mathbf{B} = \mathbf{T}\times\mathbf{N} = (1/\sqrt{29})(-2\sin 3t, -5, 2\cos 3t), \text{ and}$$

$$\kappa = \left\| \frac{(1/\sqrt{29})(-15\cos 3t, 0, -15\sin 3t)}{\sqrt{261}} \right\| = \frac{\|(-5\cos 3t, 0, -5\sin 3t)\|}{29} = \frac{5}{29}.$$

Finally,

$$\tau\mathbf{N} = \frac{(1/\sqrt{29})(-6\cos 3t, 0, -6\sin 3t)}{\sqrt{261}} = \frac{(-2\cos 3t, 0, -2\cos 3t)}{29} \quad \text{so}$$

$$\tau = -\frac{2}{29}.$$

18. $\mathbf{x}(t) = (\sin t - t\cos t, \cos t + t\sin t, 2)$ with $t \geq 0$. So $\mathbf{x}'(t) = (t\sin t, t\cos t, 0)$ and $\|\mathbf{x}'(t)\| = |t| = t$.

$$\mathbf{T} = \frac{(t\sin t, t\cos t, 0)}{t} = (\sin t, \cos t, 0), \text{ and}$$

$$\mathbf{N} = (\cos t, -\sin t, 0), \mathbf{B} = (0,0,-1), \text{ and}$$

$$\kappa = \frac{\|(\cos t, -\sin t, 0)\|}{t} = \frac{1}{t}.$$

Finally, $d\mathbf{B}/dt = 0$ so $\tau = 0$.

19. $\mathbf{x}(t) = (t, (1/3)(t+1)^{3/2}, (1/3)(1-t)^{3/2})$ so $\mathbf{x}'(t) = (1, (1/2)(t+1)^{1/2}, -(1/2)(1-t)^{1/2})$, and $\|\mathbf{x}'(t)\| = \sqrt{3/2}$.

$$\mathbf{T} = \sqrt{\frac{2}{3}}\left(1, \tfrac{1}{2}\sqrt{t+1}, -\tfrac{1}{2}\sqrt{1-t}\right), \text{ and}$$

$$\mathbf{N} = \frac{\sqrt{2/3}(0, (1/4)(t+1)^{-1/2}, (1/4)(1-t)^{-1/2})}{\sqrt{(2/3)(1/16)\left(\frac{1}{t+1}+\frac{1}{1-t}\right)}}$$

$$= \frac{1}{\sqrt{2}}(0, \sqrt{1-t}, \sqrt{t+1}), \text{ and}$$

$$\mathbf{B} = \sqrt{\frac{1}{3}}\left(\tfrac{1}{2}(t+1) + \tfrac{1}{2}(1-t), -\sqrt{t+1}, \sqrt{1-t}\right) = \frac{1}{\sqrt{3}}(1, -\sqrt{t+1}, \sqrt{1-t}).$$

Also,

$$\kappa = \frac{\|\sqrt{2/3}(0, (1/4)(t+1)^{-1/2}, (1/4)(1-t)^{-1/2})\|}{\sqrt{3/2}} = \frac{1}{3\sqrt{2(1-t^2)}}.$$

Finally,

$$\frac{d\mathbf{B}}{dt} = \frac{1}{\sqrt{3}}\left(0, -\frac{1}{2\sqrt{t+1}}, -\frac{1}{2\sqrt{1-t}}\right) \text{ so}$$

$$\frac{d\mathbf{B}}{ds} = \frac{1}{\sqrt{3}}\left(0, -\frac{1}{2\sqrt{t+1}}, -\frac{1}{2\sqrt{1-t}}\right)\Big/\sqrt{3/2} = -\tau\mathbf{N}.$$

Solving,

$$\tau = \frac{1}{3\sqrt{(1-t^2)}}.$$

20. $\mathbf{x}(t) = (e^{2t}\sin t, e^{2t}\cos t, 1)$ so $\mathbf{x}'(t) = e^{2t}(2\sin t + \cos t, 2\cos t - \sin t, 0)$, and $\|\mathbf{x}'(t)\| = e^{2t}\sqrt{5}$.

$$\mathbf{T} = \frac{(2\sin t + \cos t, 2\cos t - \sin t, 0)}{\sqrt{5}},$$

$$\mathbf{N} = \frac{(2\cos t - \sin t, -2\sin t - \cos t, 0)}{\sqrt{5}}, \text{ and}$$

$$\mathbf{B} = (0,0,-1).$$

Also,

$$\kappa = \frac{\|(2\cos t - \sin t, -2\sin t - \cos t, 0)\|}{e^{2t}\sqrt{5}} = \frac{1}{e^{2t}\sqrt{5}}.$$

Finally, again we see that $d\mathbf{B}/dt = \mathbf{0}$ so $d\mathbf{B}/ds = \mathbf{0}$ and hence $\tau = 0$.

21. (a) By formula (17): $\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3}$. Let $y = f(x)$ and view the problem as sitting inside of $\mathbf{R}^3$. Then $\mathbf{x} = (x, f(x), 0)$, $\mathbf{x}' = (1, f'(x), 0)$, and $\mathbf{x}'' = (0, f''(x), 0)$. We calculate the cross product $\mathbf{x}' \times \mathbf{x}'' = (0, 0, f''(x))$ so

$$\kappa = \frac{\|(0,0,f''(x))\|}{\|(1, f'(x), 0)\|^3} = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}.$$

(b) If $y = \ln(\sin x)$, then $y' = \cos x/\sin x$ and $y'' = -1/\sin^2 x$. By our results for part (a),

$$\kappa = \frac{|-1/\sin^2 x|}{[1 + (\cos^2 x/\sin^2 x)]^{3/2}} = |\sin x|.$$

22. (a) Formula (17) requires the use of the cross product, so we view this problem as sitting inside of $\mathbf{R}^3$. Let $\mathbf{x} = (x(s), y(s), 0)$. Then $\mathbf{x}' = (x'(s), y'(s), 0)$, and $\mathbf{x}'' = (x''(s), y''(s), 0)$. By formula (17):

$$\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3} = \frac{\|(0, 0, x'y'' - x''y')\|}{\|(x'(s), y'(s), 0)\|^3}.$$

But the curve is parametrized by arclength so $\|(x'(s), y'(s), 0)\| = 1$ so $\kappa = |x'y'' - x''y'|$.

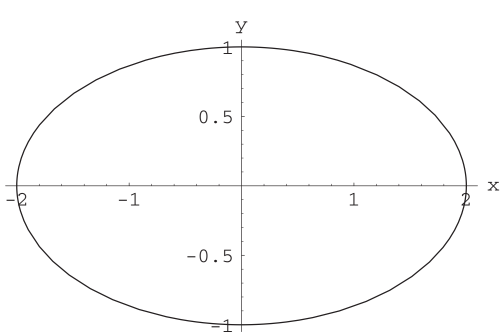
(b) Here $x(s) = (1/2)(1 - s^2)$ and $y(s) = (1/2)(\cos^{-1} s - s\sqrt{1 - s^2})$ so $x'(s) = -s$ and $y'(s) = -\sqrt{1 - s^2}$ so $(x'(s))^2 + (y'(s))^2 = 1$. So the curve is parametrized by arclength. We can then compute its curvature using the formula from part (a):

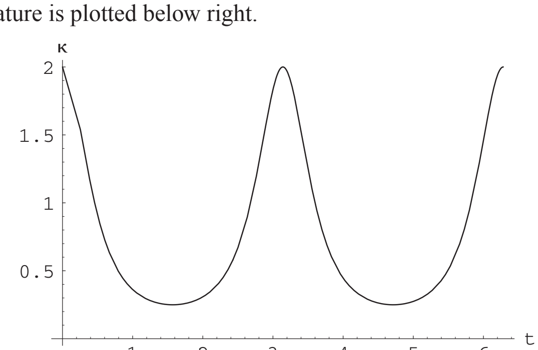
$$\kappa = |x'y'' - x''y'| = \left|(-s)\left(\frac{-s}{\sqrt{1 - s^2}}\right) - (-1)\sqrt{1 - s^2}\right| = \frac{1}{\sqrt{1 - s^2}}.$$

23. (a) The curvature is calculated to be

$$\frac{2}{(\cos^2 t + 4\sin^2 t)^{3/2}}.$$

(b) The path is pictured below left while the corresponding curvature is plotted below right.

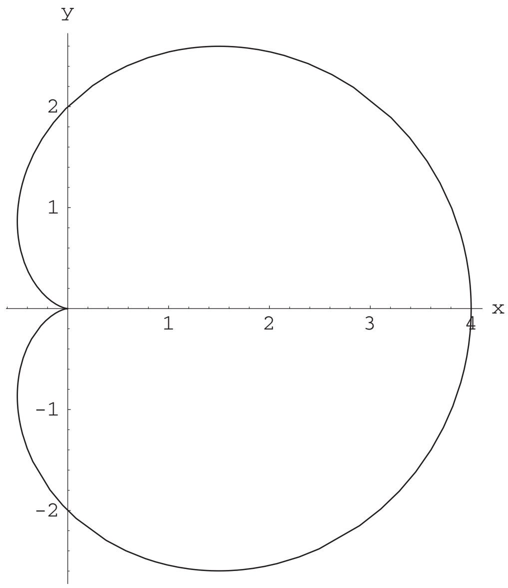


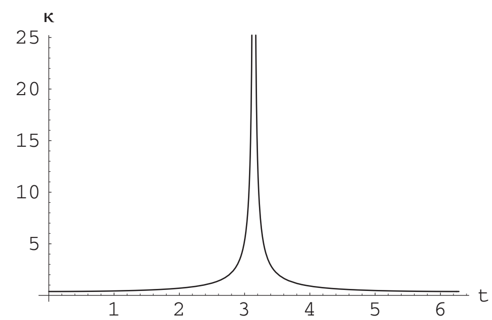


24. (a) The curvature is calculated to be (with some simplification)

$$\frac{3(1 + \cos t)}{16\cos^3(t/2)}.$$

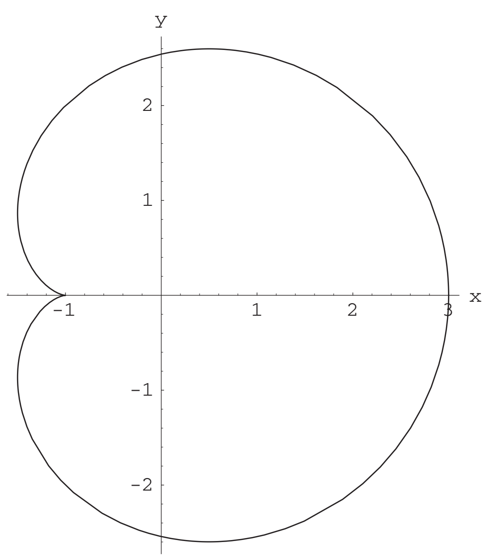
(b) The path is pictured below left while the corresponding curvature is plotted below right.

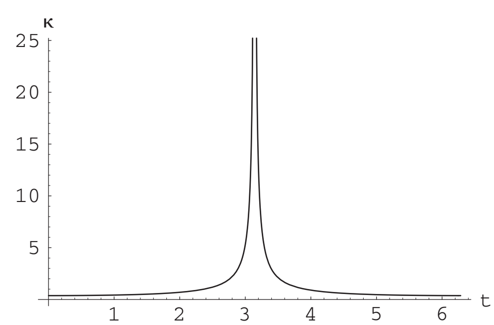




25. (a) The curvature is the same as in Exercise 24.

(b) The path is pictured below left while the corresponding curvature is plotted below right.

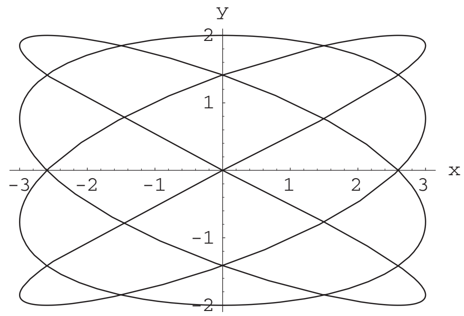


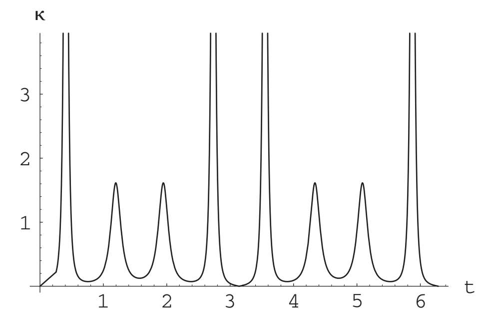


26. (a) The curvature is calculated to be

$$\frac{\sqrt{2}|7\sin t + \sin 7t|}{3(5 + \cos 6t + 4\cos 8t)^{3/2}}.$$

(b) The path is pictured below left while the corresponding curvature is plotted below right.





For Exercises 27–32, calculate the tangential component $\ddot{s}$ and then subtract it from the length of the acceleration to obtain the normal component.

27. $\mathbf{x}(t) = (t^2, t)$ so $\mathbf{x}'(t) = (2t, 1)$ and $\mathbf{x}''(t) = (2, 0)$. The speed is then $\|\mathbf{x}'(t)\| = \sqrt{1+4t^2}$ and so the tangential component of acceleration is $\ddot{s} = 4t/\sqrt{1+4t^2}$. Since $\|\mathbf{a}\| = 2$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 4/(1+4t^2)$, so the normal component of acceleration is $2/\sqrt{1+4t^2}$.

28. $\mathbf{x}(t) = (2t, e^{2t})$ so $\mathbf{x}'(t) = (2, 2e^{2t})$ and $\mathbf{x}''(t) = (0, 4e^{2t})$. The speed is then $\|\mathbf{x}'(t)\| = 2\sqrt{1+e^{4t}}$ and so the tangential component of acceleration is $\ddot{s} = 4e^{4t}/\sqrt{1+e^{4t}}$. Since $\|\mathbf{a}\| = 16e^{4t}$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 16e^{4t}/(1+e^{4t})$, so the normal component of acceleration is $4e^{2t}/\sqrt{1+e^{4t}}$.

29. $\mathbf{x}(t) = (e^t\cos 2t, e^t\sin 2t)$ so $\mathbf{x}'(t) = (e^t(\cos 2t - 2\sin 2t), e^t(\sin 2t + 2\cos 2t))$ and $\mathbf{x}''(t) = (e^t(-3\cos 2t - 4\sin 2t),$ $e^t(4\cos 2t - 3\sin 2t))$. The speed is then $\|\mathbf{x}'(t)\| = e^t\sqrt{5}$ and so the tangential component of acceleration is $\ddot{s} = e^t\sqrt{5}$. Since $\|\mathbf{a}\| = 5e^t$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 25e^{2t} - 5e^{2t}$, so the normal component of acceleration is $2\sqrt{5}e^t$.

30. $\mathbf{x}(t) = (4\cos 5t, 5\sin 4t, 3t)$ so $\mathbf{x}'(t) = (-20\sin 5t, 20\cos 4t, 3)$ and we also have that $\mathbf{x}''(t) = (-100\cos 5t, -80\sin 4t, 0)$. The speed is then $\|\mathbf{x}'(t)\| = \sqrt{400\sin^2 5t + 400\cos^2 4t + 9}$ and so the tangential component of acceleration is

$$\ddot{s} = \frac{(-3200\cos 4t\sin 4t + 4000\cos 5t\sin 5t)}{\sqrt{400\sin^2 5t + 400\cos^2 4t + 9}}.$$

Since $\|\mathbf{a}\| = 20\sqrt{25\cos^2 5t + 16\sin^2 4t}$,

$$\|\mathbf{a}\|^2 - \ddot{s}^2 = 10000\cos^2 5t + 6400\sin^2 4t - \frac{(3200\cos 4t\sin 4t + 4000\cos 5t\sin 5t)^2}{4(400\sin^2 5t + 400\cos^2 4t + 9)},$$

so the normal component of acceleration is the square root of this last quantity.

31. $\mathbf{x}(t) = (t, t, t^2)$ so $\mathbf{x}'(t) = (1, 1, 2t)$ and $\mathbf{x}''(t) = (0, 0, 2)$. The speed is then $\|\mathbf{x}'(t)\| = \sqrt{2+4t^2}$ and so the tangential component of acceleration is $\ddot{s} = 4t/\sqrt{2+4t^2}$. Since $\|\mathbf{a}\| = 2$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 4/(1+2t^2)$, so the normal component of acceleration is $2/\sqrt{1+2t^2}$.

32. $\mathbf{x}(t) = ((3/5)(1-\cos t), \sin t, (4/5)\cos t)$ so $\mathbf{x}'(t) = ((3/5)\sin t, \cos t, (-4/5)\sin t)$ and $\mathbf{x}''(t) =$ $((3/5)\cos t, -\sin t, (-4/5)\cos t)$. The speed is then $\|\mathbf{x}'(t)\| = 1$ and so the tangential component of acceleration is $\ddot{s} = 0$. Since $\|\mathbf{a}\| = 1$, $\|\mathbf{a}\|^2 - \ddot{s}^2 = 1$, so the normal component of acceleration is 1.

33. (a) Tangential component:

$$\ddot{s} = \frac{d\dot{s}}{dt} = \frac{d\|\mathbf{x}'\|}{dt} = \frac{d\sqrt{\mathbf{x}'\cdot\mathbf{x}'}}{dt} = \left(\frac{1}{2\sqrt{\mathbf{x}'\cdot\mathbf{x}'}}\right)(2\mathbf{x}'\cdot\mathbf{x}'') = \frac{\mathbf{x}'\cdot\mathbf{x}''}{\|\mathbf{x}'\|}.$$

Normal component (using formula (17)):

$$\kappa\dot{s}^2 = \left(\frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}\right)\|\mathbf{v}\|^2 = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\|\mathbf{x}'\times\mathbf{x}''\|}{\|\mathbf{x}'\|}.$$

(b) $\mathbf{x}(t) = (t+2, t^2, 3t)$ so $\mathbf{x}'(t) = (1, 2t, 3)$ and $\mathbf{x}''(t) = (0, 2, 0)$. So by part (a), the tangential component of acceleration is $4t/\sqrt{10+4t^2}$, and the normal component of acceleration is $2\sqrt{10}/\sqrt{10+4t^2}$.

34. Here $\mathbf{x} = (x, f(x), 0)$, $\mathbf{x}' = (1, f'(x), 0)$, and $\mathbf{x}'' = (0, f''(x), 0)$. Further, you need to calculate $\|\mathbf{x}'\| = \sqrt{1+[f'(x)]^2}$, $\mathbf{x}'\cdot\mathbf{x}'' = f'(x)f''(x)$, and $\|\mathbf{x}'\times\mathbf{x}''\| = \|(0, 0, f''(x))\| = |f''(x)|$. Substituting into the formulas from Exercise 33 gives us:

$$a_{\text{tang}} = \frac{f'(x)f''(x)}{\sqrt{1+[f'(x)]^2}}, \quad \text{and} \quad a_{\text{norm}} = \frac{|f''(x)|}{\sqrt{1+[f'(x)]^2}}.$$

35. To establish the formula, first note that $\mathbf{v} \times \mathbf{a} = \kappa \dot{s}^3 \mathbf{B}$ (see, for example, the calculation leading up to formula (17)) and $\|\mathbf{v} \times \mathbf{a}\| = \kappa \dot{s}^3 = \kappa \|\mathbf{v}\|^3$. So

$$\frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{\kappa \dot{s}^3 \mathbf{B} \cdot \mathbf{a}'}{\kappa^2 \dot{s}^6} = \frac{\mathbf{B} \cdot \mathbf{a}'}{\kappa \dot{s}^3}.$$

Now, $\mathbf{a}(t) = \ddot{s}\mathbf{T} + \kappa \dot{s}^2 \mathbf{N}$ and by the Frenet equations $\mathbf{N}'(s) = -\kappa \mathbf{T} + \tau \mathbf{B}$. Since we are calculating the dot product of $\mathbf{a}'$ with $\mathbf{B}$, the only piece that will survive is the coefficient of $\mathbf{B}$, so

$$\begin{aligned} \mathbf{a}'(t) &= (\text{something without } \mathbf{B}) + \kappa \dot{s}^2 \mathbf{N}'(s) \frac{ds}{dt} \\ &= (\text{something else without } \mathbf{B}) + \kappa \dot{s}^3 \tau \mathbf{B} \end{aligned}$$

and so, putting it all together,

$$\frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{\kappa \dot{s}^3 \mathbf{B} \cdot \mathbf{a}'}{\kappa^2 \dot{s}^6} = \frac{\mathbf{B} \cdot \mathbf{a}'}{\kappa \dot{s}^3} = \frac{\mathbf{B} \cdot \kappa \dot{s}^3 \tau \mathbf{B}}{\kappa \dot{s}^3} = \tau.$$

36. By equations (11) and (13) we have $\mathbf{T}' = \kappa \mathbf{N}$ and $\mathbf{B}' = -\tau \mathbf{N}$
Hence

$$-\mathbf{T}' \cdot \mathbf{B}' = -(\kappa \mathbf{N}) \cdot (-\tau \mathbf{N}) = \kappa \tau \, \mathbf{N} \cdot \mathbf{N} = \kappa \tau,$$

since $\mathbf{N}$ is a unit vector.

37. From formula (17) $\kappa = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \|\mathbf{x}' \times \mathbf{x}''\|$ since $\mathbf{x}$ must be a unit speed path as it is parametrized by arclength.
By Exercise 35,

$$\tau = \frac{(\mathbf{v} \times \mathbf{a}) \cdot \mathbf{a}'}{\|\mathbf{v} \times \mathbf{a}\|^2} = \frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{\|\mathbf{x}' \times \mathbf{x}''\|^2}$$

Thus

$$\kappa^2 \tau = \|\mathbf{x}' \times \mathbf{x}''\|^2 \cdot \left(\frac{(\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''}{\|\mathbf{x}' \times \mathbf{x}''\|^2} \right) = (\mathbf{x}' \times \mathbf{x}'') \cdot \mathbf{x}'''.$$

38. **(a)** Really, there's nothing much to show in this part—but it really helps you solve part (b). $\mathbf{B}$ is $\mathbf{T} \times \mathbf{N}$ so it is perpendicular to the plane determined by them. In this case, we interpret that as $\mathbf{B}$ is perpendicular to the osculating plane. Make the analagous observations for the other two cases.

(b) Example 9 gives us the formulas for $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$. Using the result of part (a) we can use the perpendicular vector to write down the equation of the plane. First, $\mathbf{B}$ is perpendicular to the osculating plane. So at $t = t_0$ the osculating plane must be of the form $b \sin t_0 (x - a \cos t_0) - b \cos t_0 (y - a \sin t_0) + a(z - bt_0) = 0$. Similarly the rectifying plane can be obtained from $\mathbf{N}$ as $-\cos t_0 (x - a \cos t_0) - \sin t_0 (y - a \sin t_0) = 0$. Finally, the normal plane is obtained from $\mathbf{T}$ as $-a \sin t_0 (x - a \cos t_0) + a \cos t_0 (y - a \sin t_0) + b(z - bt_0) = 0$.

39. We have $\|\mathbf{x} - \mathbf{x}_0\|^2 = (\mathbf{x} - \mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = a^2$. Thus $\|\mathbf{x} - \mathbf{x}_0\| = a$, so $\mathbf{x}(t)$ lies on a sphere of radius a.

40. The normal plane to $\mathbf{x}$ at any point $\mathbf{x}(t)$ is the plane passing through $\mathbf{x}(t)$ and perpendicular to $\mathbf{T}(t)$. Thus the plane has equation $(\mathbf{x} - \mathbf{x}(t)) \cdot \mathbf{T}(t) = 0$ (Here $\mathbf{x}(t)$ and $\mathbf{T}(t)$ are used as "constant" vectors.) Thus, using the product rule,

$$\frac{d}{dt}(\mathbf{x}(t) - \mathbf{x}_0) \cdot (\mathbf{x}(t) - \mathbf{x}_0) = 2(\mathbf{x}(t) - \mathbf{x}_0) \cdot \mathbf{x}'(t).$$

Hence

$$0 = (\mathbf{x}_0 - \mathbf{x}(t)) \cdot \mathbf{T} = -(\mathbf{x}(t) - \mathbf{x}_0) \cdot \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = -\frac{1}{\|\mathbf{x}'(t)\|}(\mathbf{x}(t) - \mathbf{x}_0) \cdot \mathbf{x}'(t)$$

Thus $(\mathbf{x}(t) - \mathbf{x}_0) \cdot \mathbf{x}'(t) = 0$ for all t. Hence $(\mathbf{x}(t) - \mathbf{x}_0) \cdot (\mathbf{x}(t) - \mathbf{x}_0) = \text{constant}$, which implies that we have a sphere curve.

41. We have $\mathbf{T}(t) = \dfrac{(-2 \sin 2t, -2 \cos 2t, -2 \sin t)}{\sqrt{4 + 4 \sin^2 t}}$.

Now we check that $(\mathbf{x}(t) - (1, 0, 0)) \cdot \mathbf{T}(t) = 0$. This equation is

$$\begin{aligned} &(\cos 2t - 1, -\sin 2t, 2 \cos t) \cdot \frac{(-2 \sin 2t, -2 \cos 2t, -2 \sin t)}{\sqrt{4 + 4 \sin^2 t}} \\ &= \frac{1}{\sqrt{4 + 4 \sin^2 t}}(-2 \cos 2t \sin 2t + 2 \sin 2t + 2 \sin 2t \cos 2t + 4 \cos t \sin t) \\ &= 0. \end{aligned}$$

42. By Exercise 27 of §1.4: $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$, so

$$\mathbf{T}\times\mathbf{B} = \mathbf{T}\times(\mathbf{T}\times\mathbf{N}) = -(\mathbf{T}\times\mathbf{N})\times\mathbf{T} = -[(\mathbf{T}\cdot\mathbf{T})\mathbf{N} - (\mathbf{N}\cdot\mathbf{T})\mathbf{T}] = -\mathbf{N}$$

$$\mathbf{N}\times\mathbf{B} = \mathbf{N}\times(\mathbf{T}\times\mathbf{N}) = -(\mathbf{T}\times\mathbf{N})\times\mathbf{N} = -[(\mathbf{T}\cdot\mathbf{N})\mathbf{N} - (\mathbf{N}\cdot\mathbf{N})\mathbf{T}] = \mathbf{T}$$

43.

$$\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w} = (\tau\mathbf{T}+\kappa\mathbf{B})\cdot(\tau\mathbf{T}+\kappa\mathbf{B}) = \tau^2\mathbf{T}\cdot\mathbf{T} + \kappa\tau\mathbf{B}\cdot\mathbf{T} + \kappa\tau\mathbf{T}\cdot\mathbf{B} + \kappa^2\mathbf{B}\cdot\mathbf{B} = \tau^2+\kappa^2$$

44. (a)

$$\begin{aligned}
\mathbf{w}\times\mathbf{T} = (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{T} &= \tau(\mathbf{T}\times\mathbf{T}) + \kappa(\mathbf{B}\times\mathbf{T})\\
&= \kappa(\mathbf{B}\times\mathbf{T}) = \kappa\mathbf{N} \text{ by Exercise 42}\\
&= \mathbf{T}' \text{ by Frenet–Serret}\\
\mathbf{w}\times\mathbf{N} = (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{N} &= \tau(\mathbf{T}\times\mathbf{N}) + \kappa(\mathbf{B}\times\mathbf{N})\\
&= \tau\mathbf{B} - \kappa\mathbf{T} \text{ by Exercise 42}\\
&= \mathbf{N}' \text{ by Frenet–Serret}\\
\mathbf{w}\times\mathbf{B} = (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{B} &= \tau(\mathbf{T}\times\mathbf{B}) = -\tau\mathbf{N} \text{ by Exercise 42}\\
&= \mathbf{B}' \text{ by Frenet–Serret}
\end{aligned}$$

(b) $\mathbf{T}' = \mathbf{w}\times\mathbf{T} = (\tau\mathbf{T}+\kappa\mathbf{B})\times\mathbf{T} = \kappa\mathbf{N}$ by manipulations and Exercise 42. The other equations are similar.

45. $\mathbf{w}$ is a constant vector $\Leftrightarrow \mathbf{w}'(s) = \mathbf{0}$. So

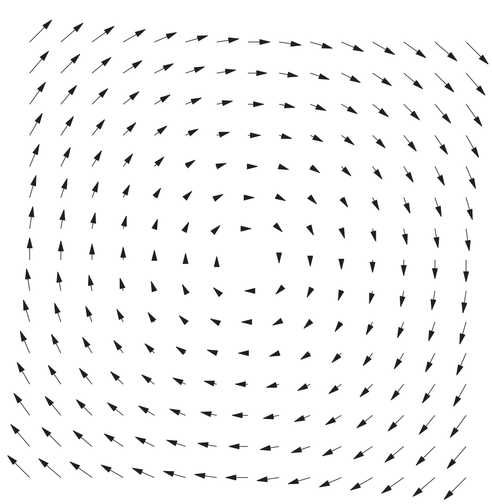
$$\begin{aligned}
\mathbf{0} = \mathbf{w}'(s) &= \tau'\mathbf{T} + \tau\mathbf{T}' + \kappa'\mathbf{B} + \kappa\mathbf{B}'\\
&= \tau'\mathbf{T} + \kappa'\mathbf{B} + \tau\kappa\mathbf{N} - \kappa\tau\mathbf{N} \text{ using Frenet–Serret}\\
&= \tau'\mathbf{T} + \kappa'\mathbf{B}.
\end{aligned}$$

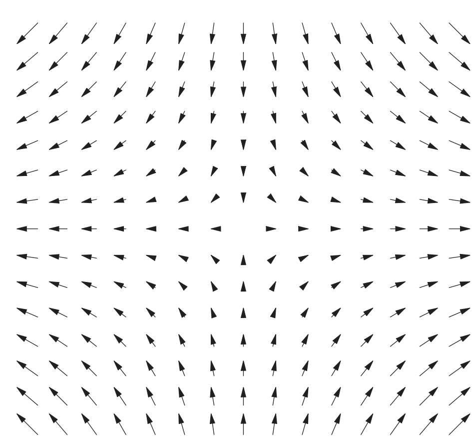
$\mathbf{T}$ and $\mathbf{B}$ are always perpendicular—hence we can never have $\mathbf{T} = c\,\mathbf{B}$ (or vice versa). Thus $\tau' = \kappa' = 0$ so τ, κ are constant and nonzero because $\mathbf{x}'\times\mathbf{x}'' \neq \mathbf{0}$. Thus by Theorem 2.5 the path must be a helix. Conversely, having a helix implies constant τ, κ so $\mathbf{w}' \equiv \mathbf{0}$. Thus $\mathbf{w}$ must be constant.

3.3 Vector Fields: An Introduction

The figures can be generated using Mathematica or Maple. The axes are in the 'usual' positions with the origin at the center. The relative length of the shaft of the arrows corresponds to the length of the vectors. The students should then compare the results in Exercises 1–3 and Exercises 4–6. The differences between the equations for the vector fields should be compared to the differences in the resulting sketches.

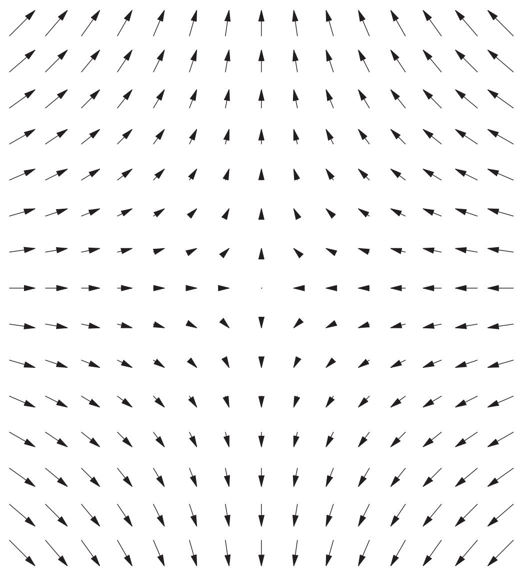
1. $\mathbf{F} = y\mathbf{i} - x\mathbf{j} = (y, -x)$ is shown below left.

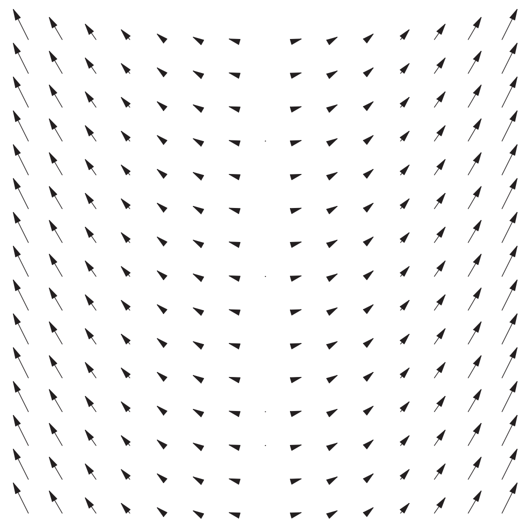




2. $\mathbf{F} = x\mathbf{i} - y\mathbf{j} = (x, -y)$ is shown above right.

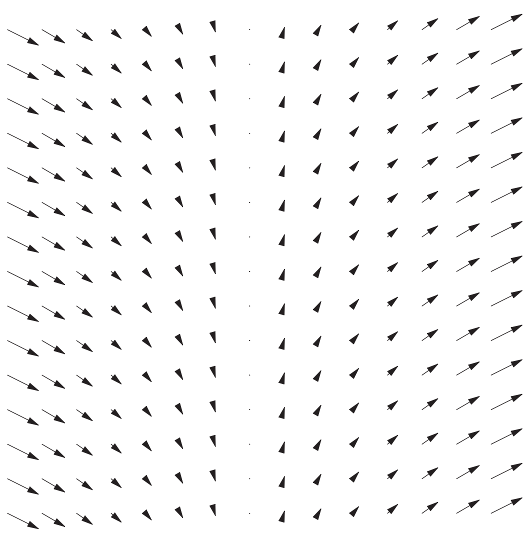
3. $\mathbf{F} = (-x, y)$ is shown below left.

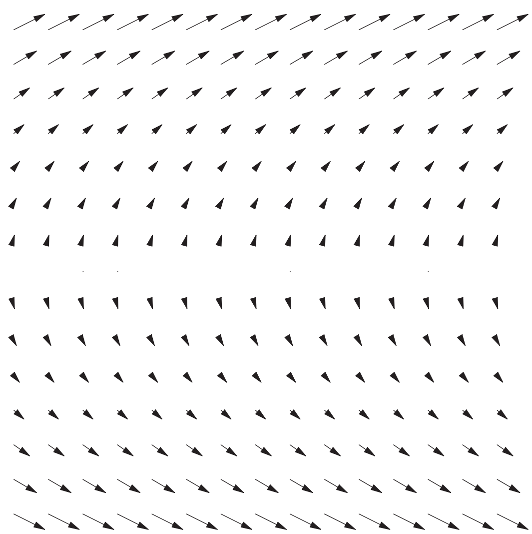




4. $\mathbf{F} = (x, x^2)$ is shown above right.

5. $\mathbf{F} = (x^2, x)$ is shown below left.

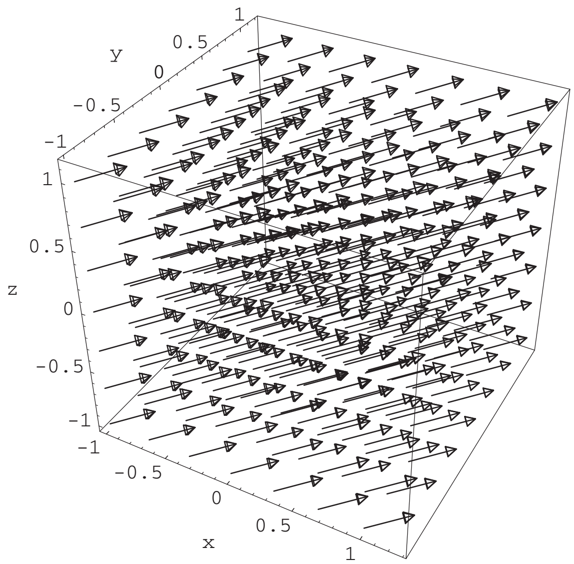


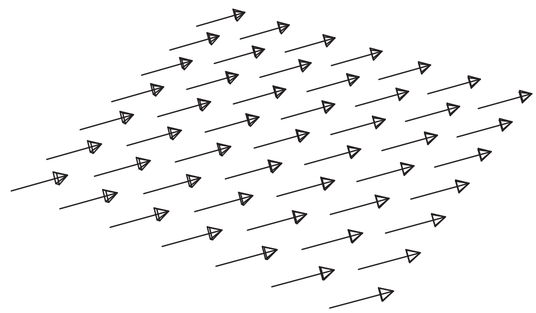


6. $\mathbf{F} = (y^2, y)$ is shown above right.

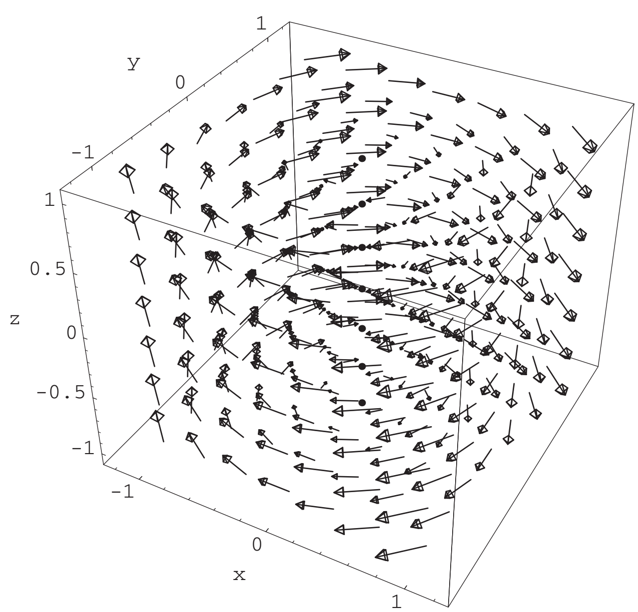
Now we are looking at sketches of vector fields in $\mathbf{R}^3$*. These are harder to see. In most cases, I have also included a sketch of a slice.*

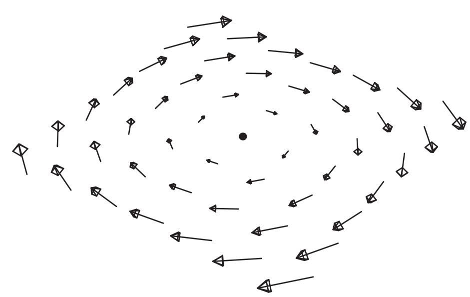
7. $\mathbf{F} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k} = (3, 2, 1)$ is constant. The figure on the right shows the slice in the xy-plane:



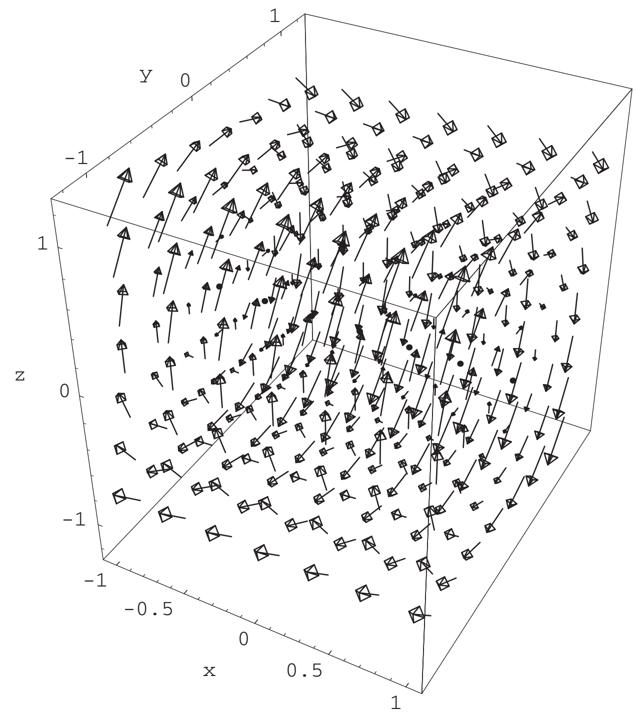


8. $\mathbf{F} = (y, -x, 0)$. The figure on the right shows the slice in the xy-plane—compare this to Exercise 1:

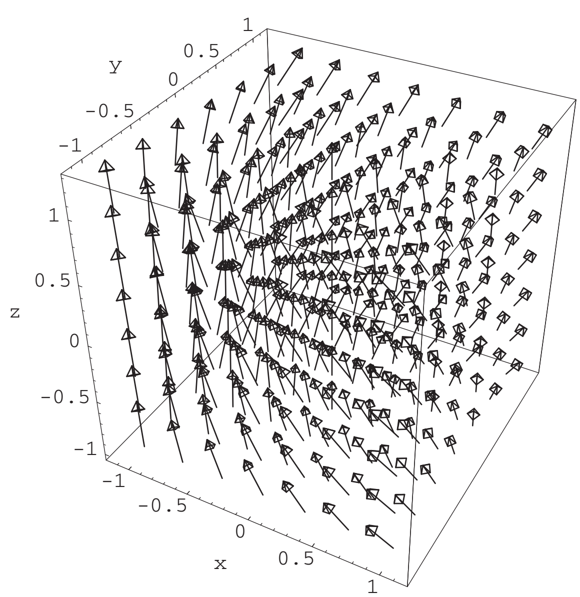


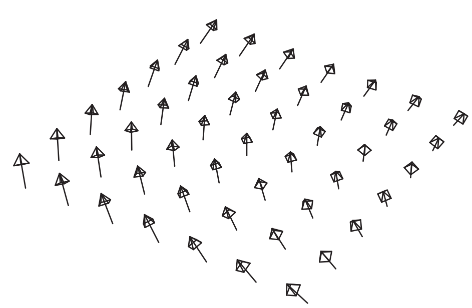


9. $\mathbf{F} = (0, z, -y)$; compare this to Exercise 8:

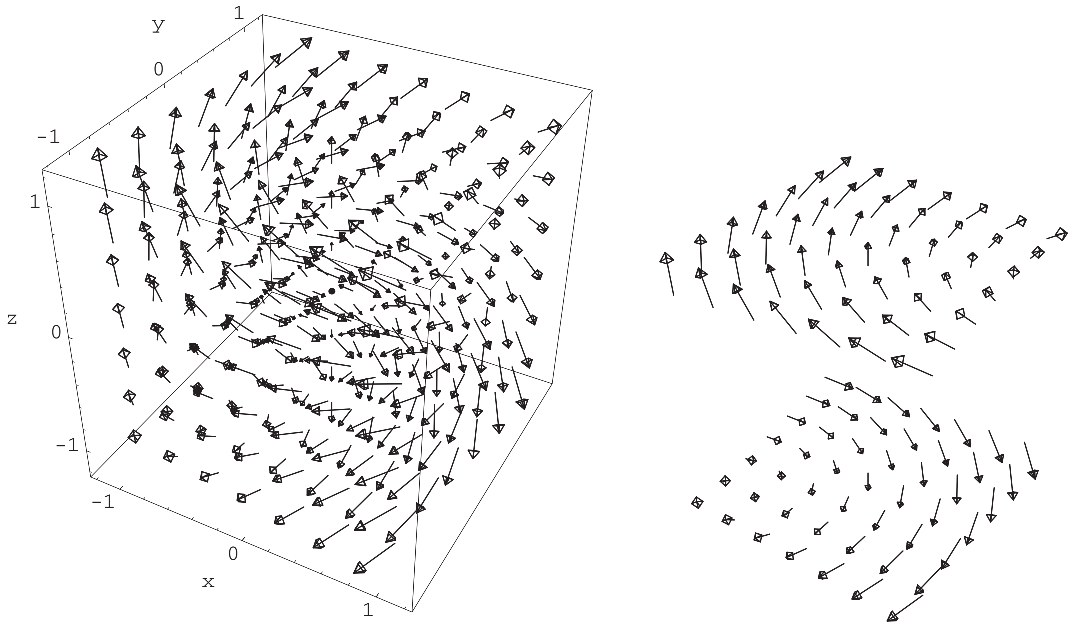


10. $\mathbf{F} = (y, -x, 2)$. The figure on the right shows the slice in the xy-plane—compare this to Exercise 8:

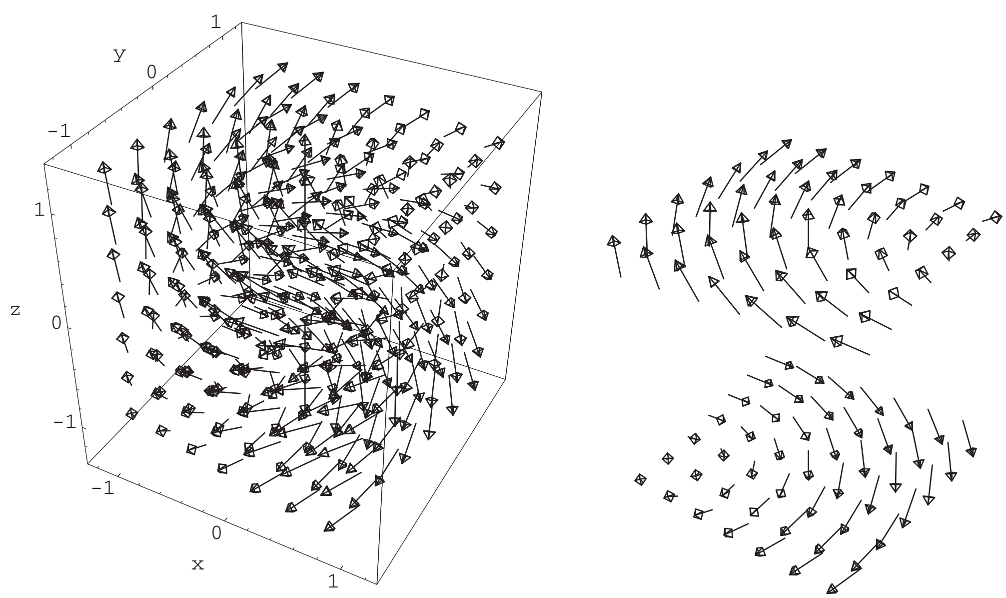




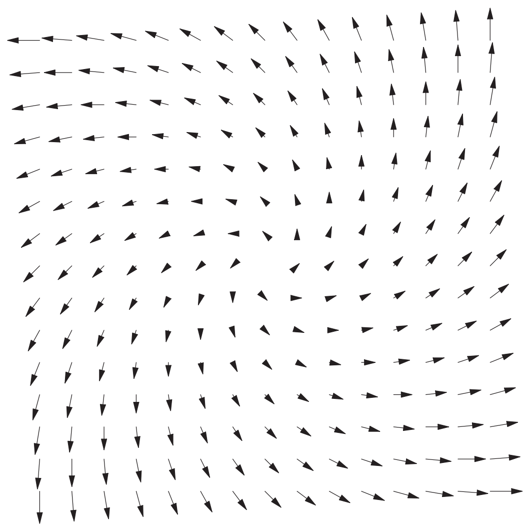
11. $\mathbf{F} = (y, -x, z)$. The figure on the right shows the slices in the $z = 1$ and $z = -1$ planes—compare this to Exercises 8 and 10:

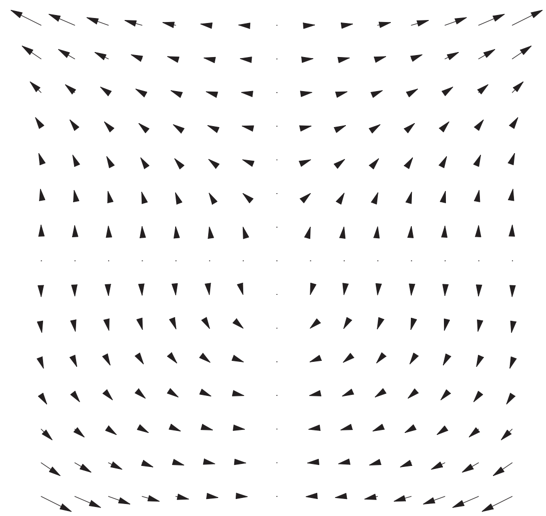


12. $\mathbf{F} = (y, -x, z)/\sqrt{x^2 + y^2 + z^2}$ except at the origin. The figure on the right shows the slices in the $z = 1$ and $z = -1$ planes—compare this to Exercise 11 (they are the same except the vectors in this problem are all unit vectors—they may not look like unit vectors because of the vertical components):



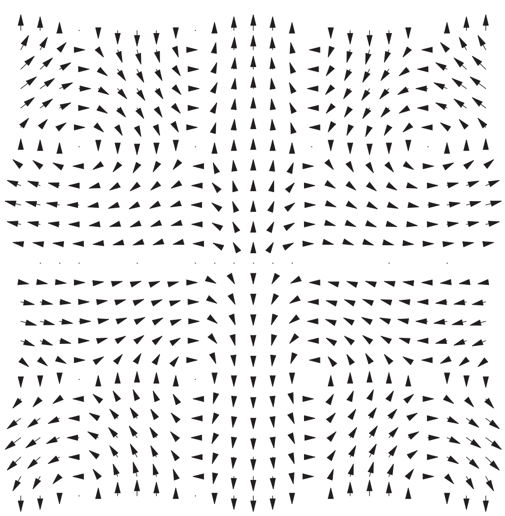
13. The figure is below left.

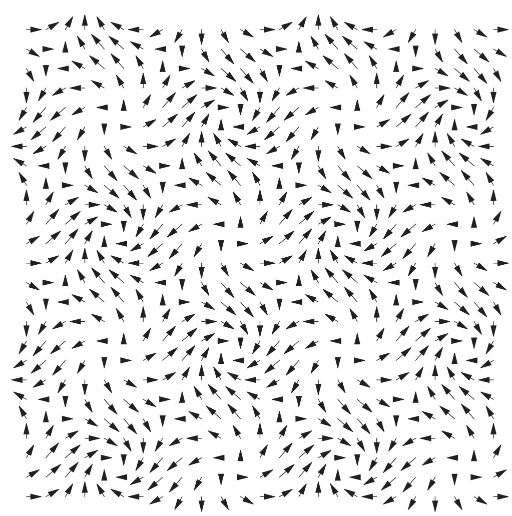




14. The figure is above right.

15. The figure is below left.

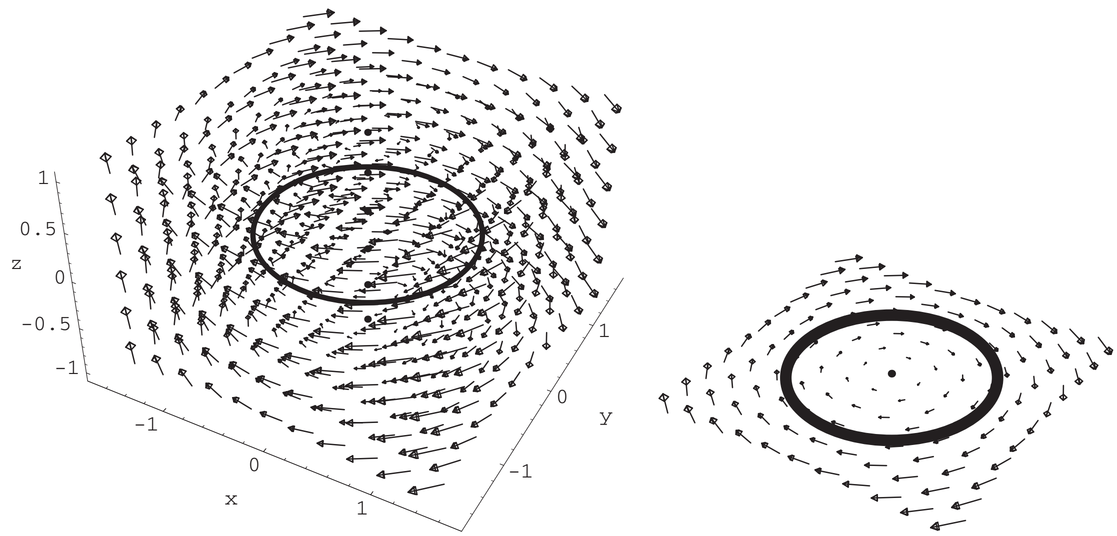




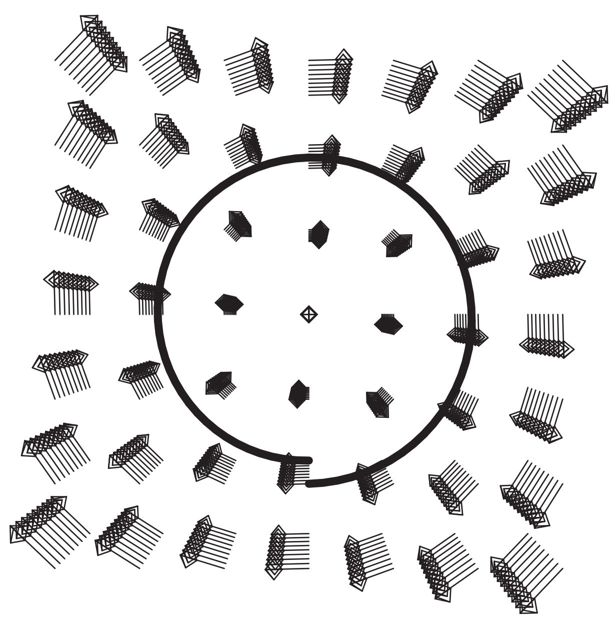
16. The figure is above right.

In Exercises 17–19 we will show that $\mathbf{x}$ *is a flow line of* $\mathbf{F}$ *using Definition 3.2, by showing* $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$.

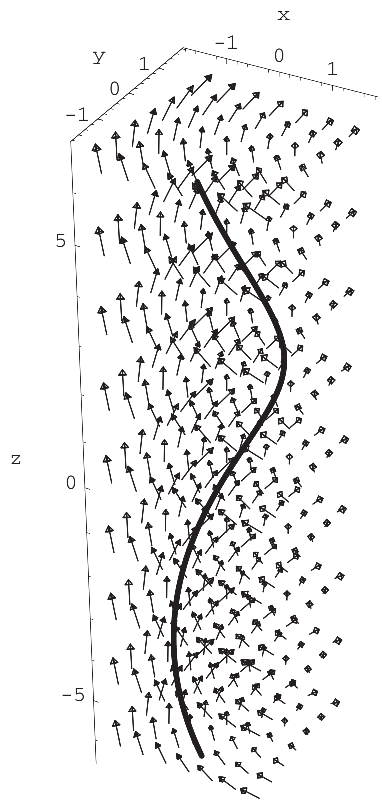
17. $\mathbf{x}(t) = (x, y, z) = (\sin t, \cos t, 0)$ so $\mathbf{x}'(t) = (\cos t, -\sin t, 0) = (y, -x, 0) = \mathbf{F}(\mathbf{x}(t))$. We can see below how the path, in bold, is a flow line for the vector field we saw above in Exercise 8. The figure on the right is the xy-plane slice of the figure on the left.



18. $\mathbf{x}(t) = (x, y, z) = (\sin t, \cos t, 2t)$ so $\mathbf{x}'(t) = (\cos t, -\sin t, 2) = (y, -x, 2) = \mathbf{F}(\mathbf{x}(t))$. Below we see the view from almost directly above one "period" of the path.



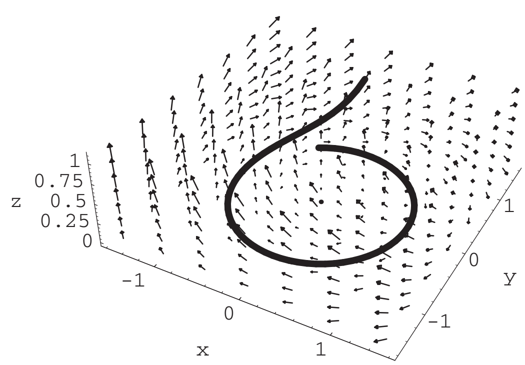
The path, below in bold, is a flow line of the vector field we saw above in Exercise 10.



19. $\mathbf{x}(t) = (x, y, z) = (\sin t, \cos t, e^{2t})$ so

$$\mathbf{x}'(t) = (\cos t, -\sin t, 2e^{2t}) = (y, -x, 2z) = \mathbf{F}(\mathbf{x}(t)).$$

The projection of this path onto the xy-plane is the same as that of the path in Exercise 18. The difference is that the rate at which the path climbs is changing:



20. If $\mathbf{x}(t) = (x, y)$ then $\mathbf{x}'(t) = \mathbf{F}(x, y) = (-x, y)$. Consider x for a moment. This says that $dx/dt = -x$. The solution to this is $x = ce^{-t}$. Our initial condition is that $x(0) = 2$ so $c = 2$. Similarly, $dy/dt = y$ so $y = ke^t$. The initial condition $y(0) = 1$ tells us that $k = 1$. The equation of the flow line is $\mathbf{x}(t) = (2e^{-t}, e^t)$.

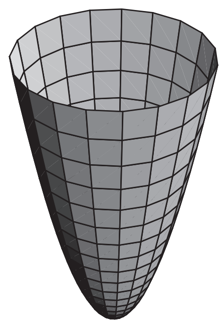
21. If $\mathbf{x}(t) = (x, y)$ then $\mathbf{x}'(t) = \mathbf{F}(x, y) = (x^2, y)$. As in Exercise 20, we know that $y = ke^t$ and $y(1) = e$ tells us that $k = 1$. As for x, $dx/dt = x^2$. This is a separable differential equation $dx/x^2 = dt$. Integrating and solving for x gives us $x = -1/(t + c)$. From the initial condition $x(1) = 1$ we find that $1 = -1/(1 + c)$ or $c = -2$. The equation, therefore, of the flow line is $\mathbf{x}(t) = (1/(2 - t), e^t)$.

22. If $\mathbf{x}(t) = (x, y, z)$ then $\mathbf{x}'(t) = \mathbf{F}(x, y, z) = (2, -3y, z^3)$. We see immediately that the x coordinate function must be linear and of the form $2t + c$. From the initial condition, this constant is 3 so $x = 2t + 3$. As in Exercise 20, we know that $y = ke^{-3t}$

and $y(0) = 5$ tells us that $k = 5$. As for z, $dz/dt = z^3$. This is a separable differential equation $dz/z^3 = dt$. Integrating and solving for z gives us $z = 1/\sqrt{-2(t+c)}$. From the initial condition $z(0) = 7$ we find that $7 = 1/\sqrt{(-2c)}$ or $c = -1/98$. The equation, therefore, of the flow line is $\mathbf{x}(t) = (2t+3, 5e^{-3t}, 7/\sqrt{1-98t})$.

23. **(a)** For the function $f(x, y, z) = 3x - 2y + z$, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is a gradient field.
(b) The equipotiential surfaces are those for which $f(x, y, z)$ is constant. $3x - 2y + z = c$. These are planes with normal vector $(3, -2, 1)$.

24. **(a)** For the function $f(x, y, z) = x^2 + y^2 - 3z$, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is a gradient field.
(b) The equipotiential surfaces are those for which $f(x, y, z)$ is constant. $x^2 + y^2 - 3z = c$ is equivalent to $z = (1/3)(x^2 + y^2 - c)$. These are paraboloids with z intercept $(0, 0, -c/3)$. A typical surface is:



25. Let $\mathbf{x}$ be a flow line of a gradient vector field $\mathbf{F} = \nabla f$ and let $G(t) = f(\mathbf{x}(t))$. We will show that G is an increasing function of t by showing $G'(t) \geq 0$. First, $G'(t) = \nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)$ since $\mathbf{F} = \nabla f$.
Now we use the fact that $\mathbf{x}$ is a flow line of $\mathbf{F}$:

$$\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = \mathbf{x}'(t) \cdot \mathbf{x}'(t) = \|\mathbf{x}'(t)\|^2 \geq 0.$$

For Exercises 26–28, verify that $\frac{\partial}{\partial t}\phi(\mathbf{x}, t) = \mathbf{F}(\phi(\mathbf{x}, t))$ *and* $\phi(\mathbf{x}, 0) = \mathbf{x}$.

26. First we see that $\phi(x, y, 0) = \left(\frac{x+y}{2}e^0 + \frac{x-y}{2}e^0, \frac{x+y}{2}e^0 + \frac{y-x}{2}e^0\right) = (x, y)$. Next,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y, t) &= \left(\frac{x+y}{2}e^t - \frac{x-y}{2}e^{-t}, \frac{x+y}{2}e^t - \frac{y-x}{2}e^{-t}\right)\\ &= \phi(y, x, t) = \mathbf{F}(\phi(x, y, t)).\end{aligned}$$

27. First we see that $\phi(x, y, 0) = (y \sin 0 + x \cos 0, y \cos 0 - x \sin 0) = (x, y)$. Next,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y, t) &= (y \cos t - x \sin t, -y \sin t - x \cos t)\\ &= \phi(y, -x, t) = \mathbf{F}(\phi(x, y, t)).\end{aligned}$$

28. First we see that $\phi(x, y, z, 0) = (x \cos 0 - y \sin 0, y \cos 0 + x \sin 0, ze^0) = (x, y, z)$. Next,

$$\begin{aligned}\frac{\partial}{\partial t}\phi(x, y, z, t) &= (-2x \sin 2t - 2y \cos 2t, -2y \sin 2t + 2x \cos 2t, -ze^{-t})\\ &= \phi(-2y, 2x, -z, t) = \mathbf{F}(\phi(x, y, z, t)).\end{aligned}$$

29. We are assuming that ϕ is a flow of $\mathbf{F}$ and that $\mathbf{x}(t) = \phi(x_0, t)$. Then

$$\mathbf{x}'(t) = \frac{\partial}{\partial t}\phi(\mathbf{x}_0, t) = \mathbf{F}(\phi(\mathbf{x}_0, t)) = \mathbf{F}(\mathbf{x}(t)).$$

The middle equality holds because ϕ is a flow of $\mathbf{F}$.

30. Using the hint, we can apply the results of Exercise 29. If ϕ is a flow of the vector field $\mathbf{F}$ then for any fixed point $\mathbf{x}_0$ in X, the map $\mathbf{x}(t) = \phi(\mathbf{x}_0, t)$ is a flow line of $\mathbf{F}$.

So $\phi(\mathbf{x}_0, s+t)$ is where we are if we flow for $t+s$ seconds while $\phi(\phi(\mathbf{x}_0, t), s)$ is where we are if we first flow for t seconds and then we flow for s seconds. It should be clear that we end up the same place in either case. It is worth checking that your students understand the idea behind the problem—the author of the text has taken great care to make sure that these symbols make some physical sense to them.

31. We know that $\frac{\partial}{\partial t}\phi(\mathbf{x},t) = \mathbf{F}(\phi(\mathbf{x},t))$. So

$$\frac{\partial}{\partial t}D_{\mathbf{x}}\phi(\mathbf{x},t) = D_{\mathbf{x}}\left(\frac{\partial}{\partial t}\phi(\mathbf{x},t)\right) = D_{\mathbf{x}}\mathbf{F}(\phi(\mathbf{x},t)).$$

Now by the chain rule (Theorem 5.3):

$$D_{\mathbf{x}}\mathbf{F}(\phi(\mathbf{x},t)) = D\mathbf{F}(\phi(\mathbf{x},t))D_{\mathbf{x}}\phi(\mathbf{x},t).$$

3.4 Gradient, Divergence, Curl, and The Del Operator

For Exercises 1–6 calculate the divergence of $\mathbf{F}$: *div* $\mathbf{F} = \nabla\cdot\mathbf{F}$.

1. $\mathbf{F} = (x^2, y^2)$, so div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2x + 2y$.

2. $\mathbf{F} = (y^2, x^2)$, so div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 0 + 0 = 0$.

3. $\mathbf{F} = (x+y, y+z, x+z)$, so div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 1 + 1 + 1 = 3$.

4. $\mathbf{F} = (z\cos(e^{y^2}), x\sqrt{z^2+1}, e^{2y}\sin 3x)$, so div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 0 = 0$.

5. $\mathbf{F} = (x_1^2, 2x_2^2, \ldots, nx_n^2)$, so div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n} = 2x_1 + 4x_2 + \cdots + 2nx_n$.

6. $\mathbf{F} = (x_1, 2x_1, \ldots, nx_1)$, so div $\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n} = 1 + 0 + \cdots + 0 = 1$.

For Exercises 7–11 calculate the curl of $\mathbf{F}$: *curl* $\mathbf{F} = \nabla\times\mathbf{F}$.

7. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xe^y & 2xyz \end{vmatrix} = (2xz, -2yz, -e^y)$.

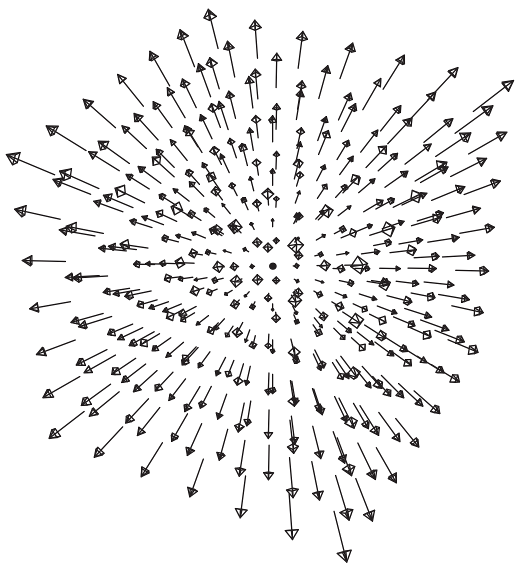
8. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0, 0, 0)$.

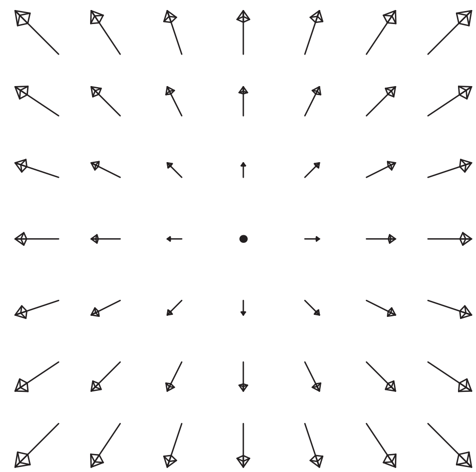
9. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+yz & y+xz & z+xy \end{vmatrix} = (x - x, -y + y, z - z) = (0, 0, 0)$.

10. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos yz - x & \cos xz - y & \cos xy - z \end{vmatrix}$
$= (x(\sin xz - \sin xy), y(\sin xy - \sin yz), z(\sin yz - \sin xz))$.

11. curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z & e^{xyz} & x^2y \end{vmatrix} = (x^2 - xye^{xyz}, y^2 - 2xy, yze^{xyz} - 2yz)$.

12. (a) We denote the vector field from Exercise 8 by $\mathbf{F}_8$ and sketch it below on the left. The figure on the right represents any planar slice through the origin. Every point is being pushed outwards. If you imagine a twig caught in this body of water and you think in terms of spherical coordinates, the change in position is an increase in ρ with no change to φ or θ.





(b) Note that $\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{\mathbf{F}_8}{\sqrt{x^2+y^2+z^2}}$. At each point the direction of $\mathbf{F}$ is the same as that of $\mathbf{F}_8$ but $\mathbf{F}$ is made up of unit vectors. As in part (a) we would argue that the motion of each point is in the direction of increasing ρ and so the curl is again 0.

(c) In Exercise 24 below we'll show that $\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}$. If you don't like citing a future problem, you can follow through the steps in Exercise 24 for this exercise. Note that we know from Exercise 8 that $\nabla \times \mathbf{F}_8 = (0, 0, 0)$.

$$\begin{aligned}
\nabla \times \mathbf{F} &= \nabla \times \left(\frac{1}{\sqrt{x^2+y^2+z^2}}\mathbf{F}_8\right) \\
&= \left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\nabla \times \mathbf{F}_8 + \left[\nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right)\right] \times \mathbf{F}_8 \\
&= (0,0,0) - \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \times \mathbf{F}_8 \\
&= -\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}} \times (x,y,z) = (0,0,0).
\end{aligned}$$

13. (a) At each point "more is moving away than towards" so div $\mathbf{F} > 0$ on all $\mathbf{R}^2$.

(b) At each point "more is moving towards than away" so div $\mathbf{F} < 0$ on all $\mathbf{R}^2$.

(c) Here we have a mixed bag. At each point to the left of the y-axis "more is moving towards than away" and at each point to the right of the y-axis "more is moving away than towards" so div $\mathbf{F} < 0$ for $x < 0$, div $\mathbf{F} > 0$ for $x > 0$, and div $\mathbf{F} = 0$ for $x = 0$.

(d) Again we have a mixed bag. At each point the above the x-axis "more is moving towards than away" and at each point below the x-axis "more is moving away than towards" so div $\mathbf{F} < 0$ for $y > 0$, div $\mathbf{F} > 0$ for $y < 0$, and div $\mathbf{F} = 0$ for $y = 0$.

In Exercises 14 and 15, the student is asked to work examples of the results of Theorems 4.3 and 4.4. Exercise 16 has the student prove Theorem 4.4.

14. $f(x, y, z) = x^2 \sin y + y^2 \cos z$ so $\nabla f = (2x \sin y, x^2 \cos y + 2y \cos z, -y^2 \sin z)$.

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x \sin y & x^2 \cos y + 2y \cos z & -y^2 \sin z \end{vmatrix}$$

$$= (-2y \sin z + 2y \sin z, 0 - 0, 2x \cos y - 2x \cos y) = (0, 0, 0).$$

15. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + e^z \cos x\mathbf{j} + xy^2z^3\mathbf{k}$ so

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & e^z \cos x & xy^2z^3 \end{vmatrix} = (2xyz^3 + e^z \cos x, -y^2z^3 + xy, e^z \sin x - xz).$$

Finally we calculate

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(2xyz^3 + e^z \cos x) + \frac{\partial}{\partial y}(-y^2z^3 + xy) + \frac{\partial}{\partial z}(e^z \sin x - xz)$$
$$= 2yz^3 - e^z \sin x - 2yz^3 + x + e^z \sin x - x = 0.$$

16. We want to show that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$
$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}.$$

Finally, because **F** is of class C^2, the mixed partials are equal and so this last quantity is 0.

17. This is a good warm-up.

$$\nabla r^n = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)^{n/2} = \left(\frac{n}{2}\right)(x^2 + y^2 + z^2)^{(n-2)/2}(2x, 2y, 2z)$$
$$= nr^{n-2}(x, y, z) = nr^{n-2}\mathbf{r}.$$

18. This is similar to Exercise 17 as most of the derivative of the ln pulls out.

$$\nabla(\ln r) = \nabla(\ln(x^2 + y^2 + z^2)^{1/2}) = \tfrac{1}{2}\nabla(\ln(x^2 + y^2 + z^2))$$
$$= \tfrac{1}{2}\left(\frac{1}{x^2 + y^2 + z^2}\right)(2x, 2y, 2z) = \left(\frac{1}{r^2}\right)(x, y, z) = \frac{\mathbf{r}}{r^2}$$

19. In this exercise and the next we'll need to know that $r^n\mathbf{r} = (x^2 + y^2 + z^2)^{n/2}(x, y, z)$.

$$\nabla \cdot (r^n\mathbf{r}) = \frac{\partial}{\partial x}[x(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial y}[y(x^2 + y^2 + z^2)^{n/2}] + \frac{\partial}{\partial z}[z(x^2 + y^2 + z^2)^{n/2}]$$
$$= \left[r^n + x\left(\frac{n}{2}\right)2x(x^2 + y^2 + z^2)^{(n-2)/2}\right] + \left[r^n + y\left(\frac{n}{2}\right)2y(x^2 + y^2 + z^2)^{(n-2)/2}\right]$$
$$+ \left[r^n + z\left(\frac{n}{2}\right)2z(x^2 + y^2 + z^2)^{(n-2)/2}\right]$$
$$= [r^n + nx^2r^{n-2}] + [r^n + ny^2r^{n-2}] + [r^n + nz^2r^{n-2}]$$
$$= 3r^n + n(x^2 + y^2 + z^2)r^{n-2} = 3r^n + nr^2r^{n-2} = 3r^n + nr^n = (n + 3)r^n$$

20. Here

$$\nabla \times (r^n\mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(x^2 + y^2 + z^2)^{n/2} & y(x^2 + y^2 + z^2)^{n/2} & z(x^2 + y^2 + z^2)^{n/2} \end{vmatrix}$$

Let's begin by calculating the coefficient of **i**:

$$\frac{\partial}{\partial y}[z(x^2+y^2+z^2)^{n/2}] - \frac{\partial}{\partial z}[y(x^2+y^2+z^2)^{n/2}]$$
$$= z(x^2+y^2+z^2)^{(n-2)/2}(2y) - y(x^2+y^2+z^2)^{(n-2)/2}(2z) = 0.$$

The calculation is the same for the coefficients of **j** and **k**.

Exercises 21 and 22 follow quickly from properties we explored in Chapter 1. The ∇ seems to distribute over the sum because the derivative of a sum is the sum of the derivatives. Exercises 23–25 are product rules.

21.

$$\nabla \cdot (\mathbf{F}+\mathbf{G}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F_i+G_i) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(F_i) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(G_i) = \nabla \cdot \mathbf{F} + \nabla \cdot \mathbf{G}.$$

22. You can expand the first matrix below and see the result pretty quickly. On the other hand, you can use the result of Exercise 28 from Section 1.6 and the fact that $\frac{d}{dx_i}(F+G) = \frac{d}{dx_i}(F) + \frac{d}{dx_i}(G)$.

$$\nabla \times (\mathbf{F}+\mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1+G_1 & F_2+G_2 & F_3+G_3 \end{vmatrix}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = \nabla \times \mathbf{F} + \nabla \times \mathbf{G}.$$

23.

$$\nabla \cdot (f\mathbf{F}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(fF_i) = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}F_i + f\frac{\partial F_i}{\partial x_i} \right)$$
$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}F_i \right) + \sum_{i=1}^{n} \left(f\frac{\partial F_i}{\partial x_i} \right) = \nabla(f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F})$$
$$= f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f.$$

24.

$$\nabla \times (f\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix}$$
$$= \left[\frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2) \right] \mathbf{i} - \left[\frac{\partial}{\partial x}(fF_3) - \frac{\partial}{\partial z}(fF_1) \right] \mathbf{j} + \left[\frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1) \right] \mathbf{k}$$
$$= \left[\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2 + \frac{\partial F_3}{\partial y}f - \frac{\partial F_2}{\partial z}f \right] \mathbf{i} - \left[\frac{\partial f}{\partial x}F_3 - \frac{\partial f}{\partial z}F_1 + \frac{\partial F_3}{\partial x}f - \frac{\partial F_1}{\partial z}f \right] \mathbf{j}$$
$$+ \left[\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1 + \frac{\partial F_2}{\partial x}f - \frac{\partial F_1}{\partial y}f \right] \mathbf{k} = f\nabla \times \mathbf{F} + \nabla f \times \mathbf{F}.$$

25.

$$\begin{aligned}
\nabla\cdot(\mathbf{F}\times\mathbf{G}) &= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\cdot(F_2G_3-F_3G_2, F_3G_1-F_1G_3, F_1G_2-F_2G_1)\\
&= \frac{\partial F_2}{\partial x}G_3+F_2\frac{\partial G_3}{\partial x}-\frac{\partial F_3}{\partial x}G_2-F_3\frac{\partial G_2}{\partial x}+\frac{\partial F_3}{\partial y}G_1+F_3\frac{\partial G_1}{\partial y}-\frac{\partial F_1}{\partial y}G_3-F_1\frac{\partial G_3}{\partial y}\\
&\quad+\frac{\partial F_1}{\partial z}G_2+F_1\frac{\partial G_2}{\partial z}-\frac{\partial F_2}{\partial z}G_1-F_2\frac{\partial G_1}{\partial z}\\
&= G_1\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-G_2\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+G_3\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)-F_1\left(\frac{\partial G_3}{\partial y}-\frac{\partial G_2}{\partial z}\right)\\
&\quad+F_2\left(\frac{\partial G_3}{\partial x}-\frac{\partial G_1}{\partial z}\right)-F_3\left(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)\\
&= \mathbf{G}\cdot\nabla\times\mathbf{F}-\mathbf{F}\cdot\nabla\times\mathbf{G}.
\end{aligned}$$

26. We will use formulas (6) and (7) from the text. First we establish formula (3):

$$\begin{aligned}
\nabla f &= \frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\\
&= \left(\cos\theta\frac{\partial f}{\partial r}-\frac{\sin\theta}{r}\frac{\partial f}{\partial\theta}\right)\mathbf{i}+\left(\sin\theta\frac{\partial f}{\partial r}+\frac{\cos\theta}{r}\frac{\partial f}{\partial\theta}\right)\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\\
&= \frac{\partial f}{\partial r}(\cos\theta\mathbf{i}+\sin\theta\mathbf{j})+\left(\frac{1}{r}\right)\frac{\partial f}{\partial\theta}(-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})+\frac{\partial f}{\partial z}\mathbf{k}\\
&= \frac{\partial f}{\partial r}\mathbf{e}_r+\left(\frac{1}{r}\right)\frac{\partial f}{\partial\theta}\mathbf{e}_\theta+\frac{\partial f}{\partial z}\mathbf{e}_z.
\end{aligned}$$

Now we establish formula (5). Again we need formulas (6) and (7). First use (6) to obtain: $F_r\mathbf{e}_r+F_\theta\mathbf{e}_\theta=(F_r\cos\theta-F_\theta\sin\theta)\mathbf{i}+(F_r\sin\theta+F_\theta\cos\theta)\mathbf{j}$.

$$\begin{aligned}
\nabla\times\mathbf{F} &= \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ F_x&F_y&F_z\end{vmatrix} = \begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)&\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)&\frac{\partial}{\partial z}\\ (F_r\cos\theta-F_\theta\sin\theta)&(F_r\sin\theta+F_\theta\cos\theta)&F_z\end{vmatrix}\\
&= \left[\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)F_z-\frac{\partial}{\partial z}(F_r\sin\theta+F_\theta\cos\theta)\right]\mathbf{i}\\
&\quad-\left[\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)F_z-\frac{\partial}{\partial z}(F_r\cos\theta-F_\theta\sin\theta)\right]\mathbf{j}\\
&\quad+\left[\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(F_r\sin\theta+F_\theta\cos\theta)\right.\\
&\qquad\left.-\left(\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(F_r\cos\theta-F_\theta\sin\theta)\right]\mathbf{k}\\
&= \left[\frac{1}{r}\frac{\partial}{\partial\theta}F_z-\frac{\partial}{\partial z}F_\theta\right](\cos\theta\mathbf{i}+\sin\theta\mathbf{j})-\left[\frac{\partial}{\partial r}F_z-\frac{\partial}{\partial z}F_r\right](-\sin\theta\mathbf{i}+\cos\theta\mathbf{j})\\
&\quad+\left[\frac{\partial}{\partial r}F_\theta-\left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}F_r\right]\mathbf{k}
\end{aligned}$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial\theta}F_z - \frac{\partial}{\partial z}F_\theta\right]\mathbf{e}_r - \left[\frac{\partial}{\partial r}F_z - \frac{\partial}{\partial z}F_r\right]\mathbf{e}_\theta + \left[\frac{\partial}{\partial r}F_\theta - \left(\frac{1}{r}\right)\frac{\partial}{\partial\theta}F_r\right]\mathbf{e}_z$$

$$= \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \left(\frac{1}{r}\right)\frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & F_\theta & F_z \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.$$

27. We will need formula (9) from Section 1.7:

$$\begin{cases} \mathbf{e}_\rho = \sin\varphi\cos\theta\mathbf{i} + \sin\varphi\sin\theta\mathbf{j} + \cos\varphi\mathbf{k} \\ \mathbf{e}_\varphi = \cos\varphi\cos\theta\mathbf{i} + \cos\varphi\sin\theta\mathbf{j} - \sin\varphi\mathbf{k} \\ \mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \end{cases}$$

From the chain rule, we have the following relations between rectangular and spherical differential operators:

$$\begin{cases} \dfrac{\partial}{\partial\rho} = \sin\varphi\cos\theta\dfrac{\partial}{\partial x} + \sin\varphi\sin\theta\dfrac{\partial}{\partial y} + \cos\varphi\dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial\varphi} = \rho\cos\varphi\cos\theta\dfrac{\partial}{\partial x} + \rho\cos\varphi\sin\theta\dfrac{\partial}{\partial y} - \rho\sin\varphi\dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial\theta} = -\rho\sin\varphi\sin\theta\dfrac{\partial}{\partial x} + \rho\sin\varphi\cos\theta\dfrac{\partial}{\partial y}. \end{cases}$$

Solving for $\partial/\partial x, \partial/\partial y$, and $\partial/\partial z$:

$$\begin{cases} \dfrac{\partial}{\partial x} = \sin\varphi\cos\theta\dfrac{\partial}{\partial\rho} + \dfrac{\cos\varphi\cos\theta}{\rho}\dfrac{\partial}{\partial\varphi} - \dfrac{\sin\theta}{\rho\sin\varphi}\dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial y} = \sin\varphi\sin\theta\dfrac{\partial}{\partial\rho} + \dfrac{\cos\varphi\sin\theta}{\rho}\dfrac{\partial}{\partial\varphi} + \dfrac{\cos\theta}{\rho\sin\varphi}\dfrac{\partial}{\partial\theta} \\ \dfrac{\partial}{\partial z} = \cos\varphi\dfrac{\partial}{\partial\rho} - \dfrac{\sin\varphi}{\rho}\dfrac{\partial}{\partial\varphi}. \end{cases}$$

Now we calculate the gradient:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$= \left(\sin\varphi\cos\theta\frac{\partial f}{\partial\rho} + \frac{\cos\varphi\cos\theta}{\rho}\frac{\partial f}{\partial\varphi} - \frac{\sin\theta}{\rho\sin\varphi}\frac{\partial f}{\partial\theta}\right)\mathbf{i}$$

$$+ \left(\sin\varphi\sin\theta\frac{\partial f}{\partial\rho} + \frac{\cos\varphi\sin\theta}{\rho}\frac{\partial f}{\partial\varphi} + \frac{\cos\theta}{\rho\sin\varphi}\frac{\partial f}{\partial\theta}\right)\mathbf{j} + \left(\cos\varphi\frac{\partial f}{\partial\rho} - \frac{\sin\varphi}{\rho}\frac{\partial f}{\partial\varphi}\right)\mathbf{k}$$

$$= \frac{\partial f}{\partial\rho}(\sin\varphi\cos\theta\mathbf{i} + \sin\varphi\sin\theta\mathbf{j} + \cos\varphi\mathbf{k}) + \left(\frac{1}{\rho}\right)\frac{\partial f}{\partial\varphi}(\cos\varphi\cos\theta\mathbf{i} + \cos\varphi\sin\theta\mathbf{j} - \sin\varphi\mathbf{k})$$

$$+ \left(\frac{1}{\rho\sin\varphi}\right)\frac{\partial f}{\partial\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \frac{\partial f}{\partial\rho}\mathbf{e}_r + \left(\frac{1}{\rho}\right)\frac{\partial f}{\partial\varphi}\mathbf{e}_\varphi + \left(\frac{1}{\rho\sin\varphi}\right)\frac{\partial f}{\partial\theta}\mathbf{e}_\theta.$$

28. $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}.$

(a) $\nabla\cdot\nabla = \left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right)\cdot\left(\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\right) = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}\dfrac{\partial}{\partial y} + \dfrac{\partial}{\partial z}\dfrac{\partial}{\partial z} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} = \nabla^2.$

(b) We use ideas from Exercise 23:

$$\nabla^2(fg) = \nabla\cdot\nabla(fg) = \nabla\cdot((\nabla f)g + f\nabla g) = (\nabla^2 f)g + \nabla f\cdot\nabla g + \nabla f\cdot\nabla g + f\nabla^2 g$$

$$= f\nabla^2 g + g\nabla^2 f + 2(\nabla f\cdot\nabla g).$$

(c) Again we use Exercise 23:

$$\nabla\cdot(f\nabla g - g\nabla f) = \nabla f\cdot\nabla g + f\nabla^2 g - \nabla g\cdot\nabla f - g\nabla^2 f = f\nabla^2 g - g\nabla^2 f.$$

29. $f\nabla f = \left(f\frac{\partial f}{\partial x}, f\frac{\partial f}{\partial y}, f\frac{\partial f}{\partial z}\right)$ hence

$$\begin{aligned}
\nabla\cdot(f\nabla f) &= \frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial f}{\partial z}\right)\\
&= \left(\frac{\partial f}{\partial x}\right)^2+f\frac{\partial^2 f}{\partial x^2}+\left(\frac{\partial f}{\partial y}\right)^2+f\frac{\partial^2 f}{\partial y^2}+\left(\frac{\partial f}{\partial z}\right)^2+f\frac{\partial^2 f}{\partial z^2}\\
&= \left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2+f\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\\
&= \|\nabla f\|^2+f\nabla^2 f.
\end{aligned}$$

30. Write $\mathbf{F} = M\mathbf{i}+N\mathbf{j}+P\mathbf{k}$. Then

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ M & N & P\end{vmatrix} = (P_y-N_z)\mathbf{i}+(M_z-P_x)\mathbf{j}+(N_x-M_y)\mathbf{k}$$

and thus

$$\begin{aligned}
\nabla\times(\nabla\times\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P_y-N_z & M_z-P_x & N_x-M_y\end{vmatrix}\\
&= (N_{xy}-M_{yy}-M_{zz}+P_{xz})\mathbf{i}+(P_{yz}-N_{zz}-N_{xx}+M_{yx})\mathbf{j}\\
&\quad+(M_{zx}-P_{xx}-P_{yy}+N_{zy})\mathbf{k}.
\end{aligned}$$

On the other hand,

$$\nabla(\nabla\cdot\mathbf{F}) = \nabla(M_x+N_y+P_z) = (M_{xx}+N_{yx}+P_{zx})\mathbf{i}+(M_{xy}+N_{yy}+P_{zy})\mathbf{j}+(M_{xz}+N_{yz}+P_{zz})\mathbf{k}$$

and

$$\begin{aligned}
\nabla^2\mathbf{F} = (\nabla\cdot\nabla)\mathbf{F} &= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\mathbf{F}\\
&= (M_{xx}+M_{yy}+M_{zz})\mathbf{i}+(N_{xx}+N_{yy}+N_{zz})\mathbf{j}+(P_{xx}+P_{yy}+P_{zz})\mathbf{k}
\end{aligned}$$

Hence,

$$\nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F} = (N_{yx}+P_{zx}-M_{yy}-M_{zz})\mathbf{i}+(M_{xy}+P_{zy}-N_{xx}-N_{zz})\mathbf{j}+(M_{xz}+N_{yz}-P_{xx}-P_{yy})\mathbf{k}$$

By assumption, $\mathbf{F}$ is of class C^2 so $M_{xy} = M_{yx}$, etc.

Thus we have shown: $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F})-\nabla^2\mathbf{F}$, as desired.

31. (a) Let $\mathbf{G}(t) = \mathbf{F}(\mathbf{a}+t\mathbf{v})$. Then

$$\begin{aligned}
D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = \lim_{t\to 0}\frac{1}{t}(\mathbf{F}(\mathbf{a}+t\mathbf{v})-\mathbf{F}(\mathbf{a})) &= \lim_{t\to 0}\frac{1}{t}(\mathbf{G}(t)-\mathbf{G}(0))\\
&= \mathbf{G}'(0).
\end{aligned}$$

Thus

$$D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = \left.\frac{d}{dt}\mathbf{F}(\mathbf{a}+t\mathbf{v})\right|_{t=0}.$$

(b) $\frac{d}{dt}\mathbf{F}(\mathbf{a}+t\mathbf{v}) = D\mathbf{F}(\mathbf{a}+t\mathbf{v})\frac{d}{dt}(\mathbf{a}+t\mathbf{v}) = D\mathbf{F}(\mathbf{a}+t\mathbf{v})\mathbf{v}$. Now evaluate at $t=0$ to get $D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = D\mathbf{F}(\mathbf{a})\mathbf{v}$.

32. By definition,

$$\begin{aligned} D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) &= \lim_{h\to 0}\frac{1}{h}(\mathbf{F}(\mathbf{a}+h\mathbf{v})-\mathbf{F}(\mathbf{a})) \\ &= \lim_{h\to 0}\frac{1}{h}(F_1(\mathbf{a}+h\mathbf{v})-F_1(\mathbf{a}),\ldots,F_n(\mathbf{a}+h\mathbf{v})-F_n(\mathbf{a})) \\ &= \lim_{h\to 0}\left(\frac{F_1(\mathbf{a}+h\mathbf{v})-F_1(\mathbf{a})}{h},\ldots,\frac{F_n(\mathbf{a}+h\mathbf{v})-F_n(\mathbf{a})}{h}\right) \\ &= \left(\lim_{h\to 0}\frac{F_1(\mathbf{a}+h\mathbf{v})-F_1(\mathbf{a})}{h},\ldots,\lim_{h\to 0}\frac{F_n(\mathbf{a}+h\mathbf{v})-F_n(\mathbf{a})}{h}\right) = (D_{\mathbf{v}}F_1(\mathbf{a}),\ldots,D_{\mathbf{v}}F_n(\mathbf{a})) \end{aligned}$$

using Definition 6.1 of Chapter 2.

33. We use part (b) of Exercise 31 since **F** is evidently differentiable.

$$D\mathbf{F}(x,y,z) = \begin{bmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{bmatrix} \text{ so } D\mathbf{F}(3,2,1) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

$$D_{(\mathbf{i}-\mathbf{j}+\mathbf{k})/\sqrt{3}}\mathbf{F}(3,2,1) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 4/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

34.

$$D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = D\mathbf{F}(\mathbf{a})\mathbf{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix} = \mathbf{v}.$$

In general if $\mathbf{F} = (x_1,\ldots,x_n)$, then $D\mathbf{F}(\mathbf{a}) = I_n$ ($n\times n$ identity matrix), so $D_{\mathbf{v}}\mathbf{F}(\mathbf{a}) = D\mathbf{F}(\mathbf{a})\mathbf{v} = I_n\mathbf{v} = \mathbf{v}$.

True/False Exercises for Chapter 3

1. True.
2. False. (The path has unit speed.)
3. True.
4. False.
5. False. (There should be a negative sign in the second term on the right.)
6. False. ($\kappa = \|d\mathbf{T}/ds\|$, where s is arclength.)
7. True.
8. True.
9. False.
10. False. ($d\mathbf{T}/ds$ must be normalized to give **N**.)
11. True.
12. True.

13. True.
14. False. (It's a vector field.)
15. False. (It's a scalar field.)
16. True.
17. True.
18. False. (It's a scalar field.)
19. False. (It's a meaningless expression.)
20. False. ($\mathbf{F}(\mathbf{x}(t)) \neq \mathbf{x}'(t)$.)
21. True. (Check that $\mathbf{F}(\mathbf{x}(t)) = \mathbf{x}'(t)$.)
22. True. (Verify that $\nabla \cdot \mathbf{F} = 0$.)
23. False. ($\nabla \times \mathbf{F} \neq \mathbf{0}$.)
24. False. (f must be of class C^2.)
25. False. (Consider $\mathbf{F} = y\mathbf{i} + x\mathbf{j}$.)
26. False. (The first term on the right needs a negative sign.)
27. True.
28. False. ($\nabla \times \mathbf{F} \neq \mathbf{0}$.)
29. False. ($\nabla \cdot (\nabla \times \mathbf{F}) \neq 0$.)
30. True.

Miscellaneous Exercises for Chapter 3

1. Here are the answers: (a) D (b) F (c) A
(d) B (e) C (f) E

Here's some explanation: The formulas in (a) and in (f) are the only ones that keep x and y bounded (between -1 and 1), so they must correspond to D and E. Note that in (a) $\mathbf{x}(0) = (0, 0)$, but the graph in E does not pass through the origin. Note that in (c) $x \geq 1$ and the only graph with that property is A. In (b) we see that $\mathbf{x}(-t) = (-t - \sin 5t, t^2 + \cos 6t) = (-x(t), y(t))$. This means that the graph will be symmetric about the y-axis and the only plot that remains with this property is F. What remains is to match the formulas in (d) and (e) with the graphs in B and C. This is easy: in (d) large positive values of t give points in the first quadrant. The graph in C has no points in the first quadrant.

2. *Answers*: (a) E (b) F (c) C (d) B (e) A (f) D

Explanation: (a) Must have z between -1 and 1, but y can be arbitrarily large and positive.

(b) All three coordinates should be bounded (making the only choices B or F). The projection of the curve into the xy-plane should be an astroid—giving choice F.

(c) This is an elliptical helix—so choice C.

(d) All three coordinates are between -1 and 1, so graph B.

(e) Note that $x^2 + y^2 = 4t^2 = \frac{1}{4}z^2$—thus the graph lies on a cone (so A).

(f) Only D remains, but note that we must have $x \geq 1$.

3. First note that $\frac{d}{dt}\|\mathbf{x}'(t)\| = \frac{d}{dt}\sqrt{\mathbf{x}'(t) \cdot \mathbf{x}'(t)} = (\mathbf{x}'(t) \cdot \mathbf{x}''(t))/\|\mathbf{x}'(t)\|$. So $\mathbf{x}$ has constant (non-zero) speed if and only if $\frac{d}{dt}\|\mathbf{x}'(t)\| = 0$ if and only if $\mathbf{x}'(t) \cdot \mathbf{x}''(t) = 0$ (i.e., its velocity and acceleration vectors are perpendicular).

4. **(a)** If we forget about gravity, the glasses travel along the tangent line to $\mathbf{x}$ at $t = 90$. We need the position along this tangent line two seconds after we lose our glasses:

$$\begin{aligned} \mathbf{1}(t) &= \mathbf{x}(90) + 2(\mathbf{x}'(90)) = (-e^{3/2}, 0, 80) + 2(-e^{3/2}/60, -\pi e^{3/2}/30, 0) \\ &= (-31e^{3/2}/30, -\pi e^{3/2}/15, 80). \end{aligned}$$

(b) The only component that changes when we factor in gravity is the height $h(t)$ of the glasses at time t. We know that gravity is $h''(t) = -32\, ft/sec^2$. The initial vertical velocity is zero so $h'(t) = -32t$. We know that when the glasses fall off they are 80 feet off the ground, so $h(t) = -16t^2 + 80$ so $h(2) = 16$ and the position of the glasses two seconds after they fall off is $(-31e^{3/2}/30, -\pi e^{3/2}/15, 16)$.

5. The velocity is $\mathbf{x}'(t) = \left(-\sin(t-1), 3t^2, -\frac{1}{t^2}\right)$ so $\mathbf{x}'(1) = (0, 3, -1)$. At $t = 1$ the position is $\mathbf{x}(1) = (1, 0, -1)$. If we define a surface by the equation $f(x, y, z) = x^3 + y^3 + z^3 - xyz = 0$, then $\nabla f(x, y, z) = (3x^2 - yz, 3y^2 - xz, 3z^2 - xy)$

so $\nabla f(1,0,-1) = (3,1,3)$. In general this vector is normal to the tangent plane at $(1,0,-1)$ and by observation it is also perpendicular to $\mathbf{x}'(1)$ so the curve is tangent to the surface when $t = 1$.

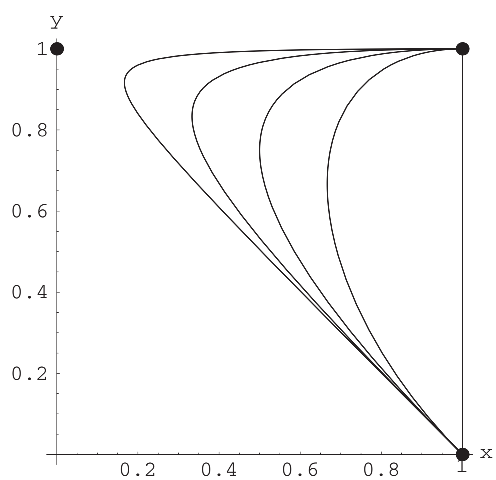
6. **(a)** We'll convert distance to inches and then $r = 240 - 3t$ while $\theta = 4\pi t$.

(b) $x = r\cos\theta$ and $y = r\sin\theta$ so $x(t) = (240-3t)\cos 4\pi t$ and $y(t) = (240-3t)\sin 4\pi t$.

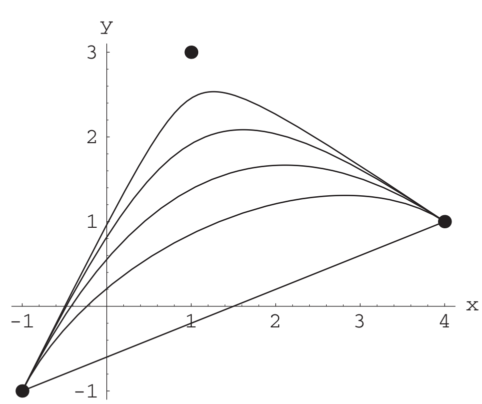
(c) It takes Gregor 80 minutes to reach the center so

$$\begin{aligned}\text{Distance} &= \int_0^{80} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \\ &= \int_0^{80} \sqrt{[-3\cos 4\pi t - 4\pi(240-3t)\sin 4\pi t]^2 + [-3\sin 4\pi t + 4\pi(240-3t)\cos 4\pi t]^2}\, dt \\ &= \int_0^{80} \sqrt{9 + 16\pi^2(240-3t)^2}\, dt = 120638 \text{ inches } \approx 1.90401 \text{ miles.}\end{aligned}$$

7. For $w = 0$ we just get the line segment joining the points x_1 and x_2. As w increases the curve becomes more bent in the direction of the control point.



8. We see the same pattern as in Exercise 7.



9. **(a)** There's nothing much to show. In the equations given in (1), at $t = 0$ all but the first terms in the numerator and denominator disappear and you get $(x(0), y(0)) = (x_1, y_1)$. At $t = 1$ all but the last terms in the numerator and denominator disappear and you get $(x(1), y(1)) = (x_3, y_3)$.

(b) Here we get

$$\begin{aligned}\mathbf{x}(1/2) &= \left(\frac{x_1/4 + wx_2/2 + x_3/4}{(1+w)/2}, \frac{y_1/4 + wy_2/2 + y_3/4}{(1+w)/2}\right)\\ &= \frac{1}{1+w}\left(\frac{x_1+x_3}{2} + wx_2, \frac{y_1+y_3}{2} + wy_2\right)\\ &= \frac{1}{1+w}\left[\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) + (wx_2, wy_2)\right]\\ &= \frac{1}{1+w}\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right) + \frac{w}{1+w}(x_2, y_2).\end{aligned}$$

Note that for $w \geq 0$, both $1/(1+w)$ and $w/(1+w)$ are between 0 and 1 and sum to 1. This tells us that the point **x**(1/2) lies on the line segment joining (x_2, y_2) to the midpoint of the line segment joining (x_1, y_1) to (x_3, y_3).

10. If you're doing this by hand, first simplify the denominator in the expression for $x(t)$ and $y(t)$ by realizing the sum of the first and last terms is 1. In other words, $(1-t)^2 + 2wt(1-t) + t^2 = 2wt(1-t)$. Crank it through and find that $\mathbf{x}'(0) = 2\,w(x_2 - x_1, y_2 - y_1)$ and $\mathbf{x}'(1) = 2\,w(x_3 - x_2, y_3 - y_2)$.

Let l_0 be the tangent line to the curve at **x**(0). Then

$$l_0(t) = \mathbf{x}(0) + t\mathbf{x}'(0) = (x_1, y_1) + 2wt(x_2 - x_1, y_2 - y_1).$$

This cries out for us to check out $t = 1/(2\,w)$. We see that

$$l_0(1/(2\,w)) = (x_1, y_1) + (x_2 - x_1, y_2 - y_1) = (x_2, y_2).$$

Similarly, let l_1 be the tangent line to the curve at **x**(1). Then

$$l_1(t) = \mathbf{x}(1) + t\mathbf{x}'(1) = (x_3, y_3) + 2wt(x_3 - x_2, y_3 - y_2).$$

At $t = -1/(2\,w)$ we see that

$$l_1(-1/(2\,w)) = (x_3, y_3) - (x_3 - x_2, y_3 - y_2) = (x_2, y_2).$$

In other words, the point (x_2, y_2) is on both of the tangent lines.

11. (a) Use part (b) of Exercise 9.

$$\begin{aligned}a = \|\mathbf{x}(1/2) - (x_2, y_2)\| &= \left\|\frac{1}{1+w}\left(\frac{x_1+x_3}{2} + wx_2, \frac{y_1+y_3}{2} + wy_2\right) - (x_2, y_2)\right\|\\ &= \left\|\left(\frac{1}{2(1+w)}\right)(x_1 - 2x_2 + x_3, y_1 - 2y_2 + y_3)\right\|\\ &= \left(\frac{1}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}\end{aligned}$$

(b) This is a similar calculation.

$$\begin{aligned}b &= \left\|\mathbf{x}(1/2) - \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)\right\|\\ &= \left\|\frac{1}{1+w}\left(\frac{x_1+x_3}{2} + wx_2, \frac{y_1+y_3}{2} + wy_2\right) - \left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}\right)\right\|\\ &= \left\|\left(\frac{w}{2(1+w)}\right)(-x_1 + 2x_2 - x_3, -y_1 + 2y_2 - y_3)\right\|\\ &= \left(\frac{w}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}\end{aligned}$$

(c) It's kind of amazing, but

$$\frac{b}{a} = \frac{\left(\frac{w}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}}{\left(\frac{1}{2(1+w)}\right)\sqrt{(x_1 - 2x_2 + x_3)^2 + (y_1 - 2y_2 + y_3)^2}} = w.$$

12. (a) Start with $y' = 2x$. So $y'(-2) = -4$ and $y'(2) = 4$. The two tangent lines $y - 4 = -4(x + 2)$ and $y - 4 = 4(x - 2)$ can be rewritten as $y = -4x - 4$ and $y = 4x - 4$. The point of intersection is $(0, -4)$ and so, by Exercise 10, this is the third control point.

(b) The deal here is that we are actually going to end up with $y = x^2$ between $x = -2$ and $x = 2$. Because **x**(1/2) must be on the line segment connecting the control point we found in (a) to the midpoint of the the line segment connecting the other two control points, it must be on the y-axis. The only point on the parabola that satisfies this is the origin. The constant w is the ratio of the distance between (0, 0) and (0, 4) and the distance between (0, 0) and $(0, -4)$. In this case, $w = 1$.

(c) The Bézier parametrization is

$$\begin{cases} x(t) = -2(1-t)^2 + 2t^2 = 4t - 2 \\ y(t) = 4(1-t)^2 - 4(2t)(1-t) + 4t^2 = (4t-2)^2. \end{cases}$$

13. (a) We have $\mathbf{x}'(t) = \left(\cos t, -\sin t + \frac{1}{2}\cot\frac{t}{2}\sec^2\frac{t}{2}\right)$. Using the double angle formula, we have

$$\tfrac{1}{2}\cot\frac{t}{2}\sec^2\frac{t}{2} = \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}} = \frac{1}{\sin t}.$$

Hence $\mathbf{x}'(t) = \left(\cos t, \frac{1}{\sin t} - \sin t\right)$. Thus, $\mathbf{x}'(t) = (0, 0)$ if and only if $t = \pi/2$.

(b) The tangent line to the tractrix at the point $\mathbf{x}(t_0)$ is given by $\mathbf{l}(s) = \mathbf{x}(t_0) + s\mathbf{x}'(t_0)$. This line crosses the y-axis when $x = 0$ and if we explicitly compute the first component of $\mathbf{l}(s)$, we see that the condition for crossing is

$$\sin t_0 + s\cos t_0 = 0 \Leftrightarrow s = -\tan t_0.$$

The length of the segment we seek is given by

$$\begin{aligned} \|\mathbf{l}(-\tan t_0) - \mathbf{x}(t_0)\| &= \|\mathbf{x}(t_0) - \tan t_0\mathbf{x}'(t_0) - \mathbf{x}(t_0)\| \\ &= |\tan t_0|\|\mathbf{x}'(t_0)\|. \end{aligned}$$

Using the work from part (a), the length of the segment is

$$\begin{aligned} |\tan t_0|\sqrt{\cos^2 t_0 + (\csc\, t_0 - \sin t_0)^2} &= |\tan t_0|\sqrt{\cos^2 t_0 + \csc^2\, t_0 - 2 + \sin^2 t_0} \\ &= |\tan t_0|\sqrt{\csc^2\, t_0 - 1} = |\tan t_0||\cot t_0| = 1. \end{aligned}$$

14. (a) Now we have $\mathbf{y}'(r) = (e^r, \sqrt{1 - e^{2r}})$ by the fundamental theorem of calculus. So the tangent line at the point $\mathbf{y}(r_0)$ is $\mathbf{m}(s) = \mathbf{y}(r_0) + s\mathbf{y}'(r_0)$. This line crosses the y-axis when $x = 0 \Leftrightarrow e^{r_0} + se^{r_0} = 0 \Leftrightarrow s = -1$. As in Exercise 13, we compute $\|\mathbf{m}(-1) - \mathbf{y}(r_0)\| = \|\mathbf{y}(r_0) - \mathbf{y}'(r_0) - \mathbf{y}(r_0)\| = \|\mathbf{y}'(r_0)\| = \sqrt{e^{2r_0} + 1 - e^{2r_0}} = 1$.

(b) Note that, for $\rho < 0$, the integrand $\sqrt{1 - e^{2\rho}}$ is positive. Hence for $r < 0$, the integral $\int_0^r \sqrt{1 - e^{2\rho}}\, d\rho$ is *negative*. Since the exponential e^r varies between 0 and 1 as r varies from $-\infty$ to 0, we see that **y** covers just the bottom half of the tractrix.

15. If $r = f(\theta)$, then we may write $\mathbf{x}(\theta) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Hence $\mathbf{v} = \mathbf{x}'(\theta) = (f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta)$ and $\|\mathbf{v}\| = \sqrt{f'(\theta)^2 + f(\theta)^2} = \sqrt{r'^2 + r^2}$. Also

$$\mathbf{a} = \mathbf{x}''(\theta) = (f''(\theta)\cos\theta - 2f'(\theta)\sin\theta - f(\theta)\cos\theta, f''(\theta)\sin\theta + 2f'(\theta)\cos\theta - f(\theta)\sin\theta).$$

If we calculate $\mathbf{v} \times \mathbf{a}$, we find (after same algebra)

$$\mathbf{v} \times \mathbf{a} = (-f(\theta)f''(\theta) + 2f'(\theta)^2 + f(\theta)^2)\mathbf{k} = (r^2 - rr'' + 2r'^2)\mathbf{k}.$$

Hence, using formula (17), we have

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \frac{|r^2 - rr'' + 2r'^2|}{(r^2 + r'^2)^{3/2}}.$$

16. For the lemniscate $r^2 = \cos 2\theta$, so that, differentiating with respect to θ, we have $2rr' = -2\sin 2\theta$. Hence

$$r' = -\frac{1}{r}\sin 2\theta \quad \text{so } r'^2 = \frac{1}{r^2}\sin^2 2\theta = \frac{\sin^2 2\theta}{\cos 2\theta}.$$

Thus

$$r^2 + r'^2 = \cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta} = \frac{1}{\cos 2\theta}.$$

Now, if we differentiate the equation $rr' = -\sin 2\theta$, we obtain

$$rr'' + r'^2 = -2\cos 2\theta.$$

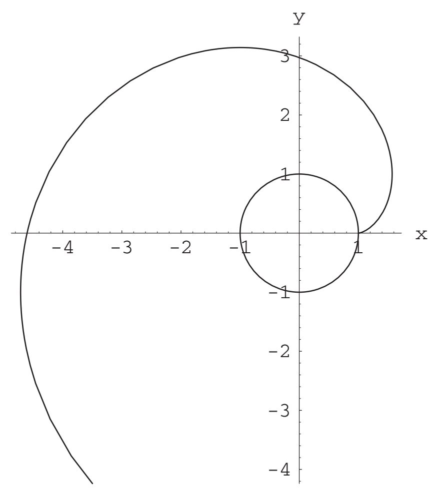
From Exercise 15, we must compute $\kappa = \dfrac{|r^2 - rr'' + 2r'^2|}{(r^2 + r'^2)^{3/2}}$. The denominator is easy; for the numerator, we have

$$\begin{aligned} r^2 - rr'' + 2r'^2 &= r^2 - (rr'' + r'^2) + 3r'^2 \\ &= \cos 2\theta - (-2\cos 2\theta) + \frac{3\sin^2 2\theta}{\cos 2\theta} \\ &= 3\cos 2\theta + \frac{3\sin^2 2\theta}{\cos 2\theta} = \frac{3}{\cos 2\theta}. \end{aligned}$$

Hence $\kappa(\theta) = \dfrac{|3/\cos 2\theta|}{(1/\cos 2\theta)^{3/2}} = 3\sqrt{\cos 2\theta}$.

17. (a) The involute of $\mathbf{x}(t) = (a\cos t, a\sin t)$ is $\mathbf{y}(t) = (a\cos t, a\sin t) - at(-\sin t, \cos t)$.

(b) The circle and involute are shown below.



18. We actually can afford to be a bit sloppy. Look first at $\mathbf{y}'(t) = \mathbf{x}'(t) - s'(t)\mathbf{T}(t) - s(t)\mathbf{T}'(t)$. By the fundamental theorem of calculus, $s'(t) = \|\mathbf{x}'(t)\|$ so $s'(t)\mathbf{T}(t) = \mathbf{x}'(t)$. So we can now say that $\mathbf{y}'(t) = -s(t)\mathbf{T}'(t)$. But by the Frenet–Serret formulas, $\mathbf{T}'(t) = s'(t)\kappa\mathbf{N}$. This means that $\mathbf{y}'(t) = -s(t)s'(t)\kappa\mathbf{N}$. In other words, the tangent vector to the involute is in the opposite direction to the normal vector to the curve, so the unit tangent vector to the involute at t is the opposite of the unit normal vector $\mathbf{N}(t)$ to the original path $\mathbf{x}$.

19. (a) This first conclusion is pretty much by definition. Analytically,

$$\|\mathbf{y}(t) - \mathbf{x}(t)\| = \|\mathbf{x}(t) - s(t)\mathbf{T}(t) - \mathbf{x}(t)\| = \|s(t)\mathbf{T}(t)\| = |s(t)|\|\mathbf{T}(t)\| = |s(t)| = s(t).$$

This last fact follows because $s(t) \geq 0$. Finally we note that $s(t)$ is the distance traveled from $\mathbf{x}(t_0)$ to $\mathbf{x}(t)$ along the underlying curve of $\mathbf{x}$.

(b) We calculated the distance from $\mathbf{x}$ to $\mathbf{y}$ in part (a). We should also observe that this is the distance along the tangent line to $\mathbf{x}$ at time t as it included the point $\mathbf{x}(t)$ and was in the direction $\mathbf{T}(t)$. The conclusion follows—it is as if you are unwinding a taut string from around $\mathbf{x}$: at each point $\mathbf{y}(t)$ is at a point in the direction of the tangent to $\mathbf{x}(t)$ of distance equal to the distance already traveled along $\mathbf{x}$. In other words, the distance is equal to the string already unraveled.

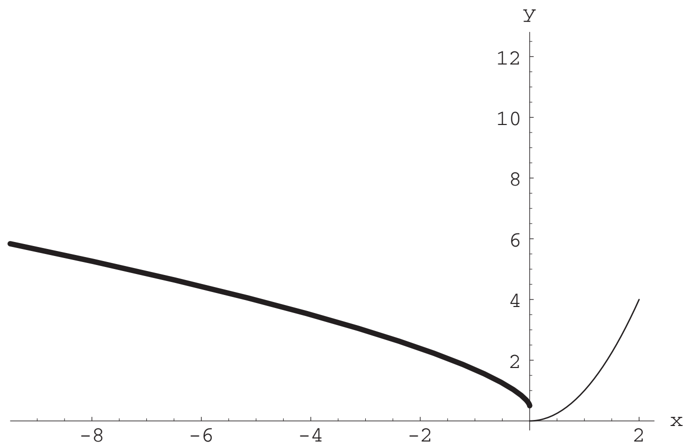
20. (a) The tangent vector is $\mathbf{T} = \mathbf{x}'(t)/\|\mathbf{x}'(t)\| = (1, 2t)/\sqrt{1+4t^2}$. The normal vector is in the xy-plane perpendicular to $\mathbf{T}$, pointing in the direction that $\mathbf{T}$ is changing: $\mathbf{N} = (-2t, 1)/\sqrt{1+4t^2}$. We'll use the formula (from Section 3.2):

$$\kappa = \frac{\|\mathbf{x}' \times \mathbf{x}''\|}{\|\mathbf{x}'\|^3} = \frac{\|(1, 2t, 0) \times (0, 2, 0)\|}{\|(1, 2t, 0)\|^3} = \frac{\|(0, 0, 2)\|}{(1+4t^2)^{3/2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

(b) The formula for the evolute is:

$$\mathbf{y}(t) = \mathbf{x}(t) + \frac{1}{\kappa}\mathbf{N}(t) = (t + (1+4t^2)(-t), t^2 + (1+4t^2)/2) = (-4t^3, 3t^2 + 1/2).$$

(c) In the figure below, the evolute increases in the opposite direction as the parabola. We can see that because the parabola is "straightening out" so the curvature is decreasing so $1/\kappa$ is increasing. The evolute is made up of points distance $1/\kappa$ from the parabola in the normal direction.

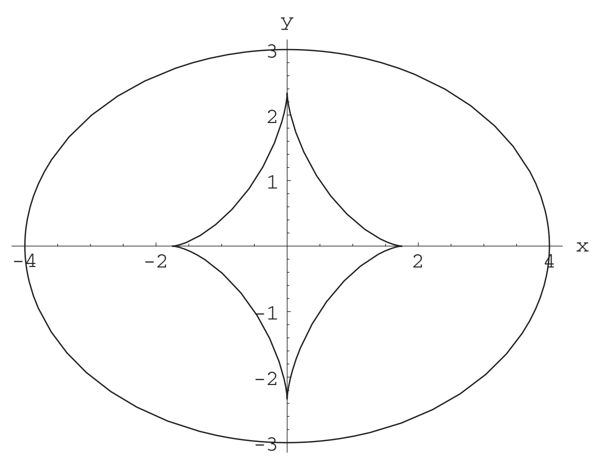


21. The curvature of a circle of radius a is $\kappa = 1/a$. Recall from high school geometry that a tangent to a circle at a given point is perpendicular to a radial line at that point. If the normal vector is oriented inward, then the evolute consists of points of distance equal to the radius of the circle in the direction of the center of the circle. In other words, the evolute of a circle is the center of that circle. Analytically, this is $\mathbf{e}(t) = \mathbf{x}(t) + a\mathbf{N}(t)$.

22. **(a)** Using a computer we get that the evolute of the ellipse $(a \cos t, b \sin t)$ is

$$\left(\cos t\left[a - \frac{b(b^2\cos^2 t + a^2\sin^2 t)}{|ab|}\right], \cos t\left[b - \frac{a(b^2\cos^2 t + a^2\sin^2 t)}{|ab|}\right]\right).$$

(b) An example when $a = 3$ and $b = 4$ is shown below. As a gets close to b the ellipse approaches a circle and the evolute shrinks to a point.



23. You may initially get an ugly looking expression. After some coaxing and explicit help, your computer algebra system should be able to help you to find that $\mathbf{N}(t) = \frac{1}{\sqrt{2}}\left(\frac{\sin t}{\sqrt{1-\cos t}}, -\sqrt{1-\cos t}\right)$ and $\kappa = \frac{1}{2\sqrt{2}a\sqrt{1-\cos t}}$, and to reduce the formula for your evolute of a cycloid to $(at + a\sin t, a\cos t - a)$. This is another cycloid.

24. Punch this into *Mathematica* and you will get

$$\left(2a\cos t(1 + a\cos t) - \frac{2a^3(1 + a^2 + 2a\cos t)(\cos t + a\cos 2t)}{|a^2 + 2a^4 + 3a^3\cos t|},\right.$$
$$\left.2a\sin t(1 + a\cos t) - \frac{a^2(1 + 2a\cos t)(1 + a^2 + 2a\cos t)}{|a^2 + 2a^4 + 3a^3\cos t|}\right).$$

25. Assume that $\mathbf{x}$ is a unit speed curve. To get the direction of the tangent, consider $\mathbf{e}'(t) = \mathbf{x}'(t)+(-1/\kappa^2)\kappa'\mathbf{N}(t)+(1/\kappa)\mathbf{N}'(t)$. By the Frenet–Serret equations, $\mathbf{N}'(t) = -\kappa\mathbf{T}$ since $\mathbf{x}$ is a planar curve. So we see what remains is $\mathbf{e}'(t) = (-1/\kappa^2)\kappa'\mathbf{N}(t)$. This tells us that the unit tangent vector to the evolute is the parallel to the unit normal vector to the original path.

26. First, $\kappa = \|\mathbf{v} \times \mathbf{a}\|/\|\mathbf{v}\|^3$. We know that $\mathbf{v}$ is a unit vector so $\|\mathbf{v}\|^3 = 1$. This means that $\kappa = \|\mathbf{v} \times \mathbf{a}\|$ and also that the tangential component of acceleration is $\frac{d}{dt}\|\mathbf{v}\| = 0$ so $\mathbf{v}$ is perpendicular to $\mathbf{a}$. Finally, this means that $\|\mathbf{v} \times \mathbf{a}\| = \|\mathbf{v}\|\|\mathbf{a}\| = 1$.

27. (a) $[x'(s)]^2 + [y'(s)]^2 = [\cos g(s)]^2 + [\sin g(s)]^2 = 1$.

(b) $\mathbf{v}(s) = (\cos g(s), \sin g(s))$ and $\mathbf{a}(s) = (-g'(s)\sin g(s), g'(s)\cos g(s))$ so

$$\kappa = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|^3} = \|(0, 0, g'(s))\| = |g'(s)|.$$

(c) We use the defining equations with $g'(s) = \kappa(s)$.

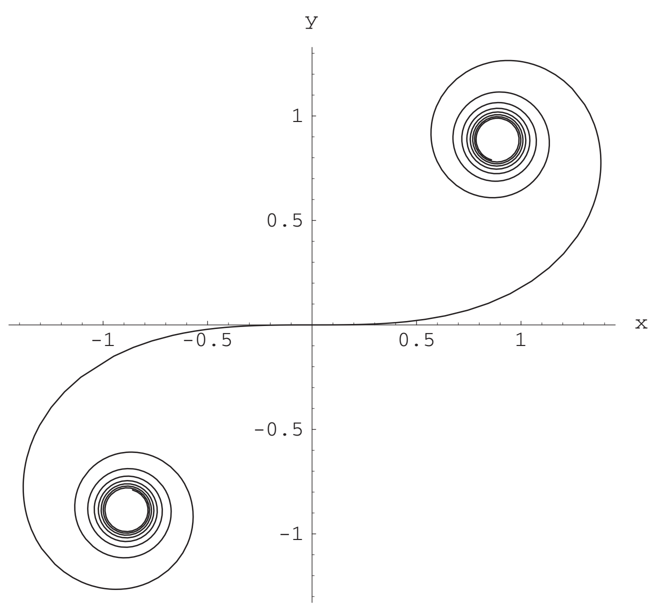
(d) There is more than one solution. For $s \geq 0$ we have $\kappa = s$ therefore, $g(s) = s^2/2$ so

$$x(s) = \int_0^s \cos(t^2/2)\,dt \quad \text{and} \quad y(s) = \int_0^s \sin(t^2/2)\,dt.$$

For $s < 0$, one solution corresponds to $g(s) = -s^2/2$ because for $s < 0, g'(s) = -s = |s|$. By formula (8) in Section 3.2, κ will always be non-negative, so we can also take $g(s) = s^2/2$ for $s < 0$. Because cosine is an even function and sine is an odd function, our two solutions are

$$x(s) = \int_0^s \cos(t^2/2)\,dt \quad \text{and} \quad y(s) = \pm\int_0^s \sin(t^2/2)\,dt.$$

(e) The graph of the clothoid is shown below.



28. (a) $-\tau\mathbf{N} = d\mathbf{B}/ds$ so if $\tau \equiv 0$ then $\mathbf{B}$ is constant.

(b) The velocity $\mathbf{v}(0)$ is in the tangent direction $\mathbf{T}$, so $\mathbf{T}$ lies in the xy-plane. The acceleration $\mathbf{a}(0)$ has components in the direction of $\mathbf{T}$ and $\mathbf{N}$ and since $\mathbf{a}(0)$ and $\mathbf{T}$ are in the xy-plane, so is $\mathbf{N}$. The binormal vector $\mathbf{B}$ must be length one and perpendicular to the plane containing $\mathbf{T}$ and $\mathbf{N}$, so $\mathbf{B} = \pm\mathbf{k}$.

(c) Combining the results from parts (a) and (b), we know that $\mathbf{B} \equiv \mathbf{k}$ or $\mathbf{B} \equiv -\mathbf{k}$. It is always true that $\mathbf{v} \cdot \mathbf{B} = 0$ and $\mathbf{a} \cdot \mathbf{B} = 0$. In this case that is equivalent to $\mathbf{v} \cdot \mathbf{k} = 0$ and $\mathbf{a} \cdot \mathbf{k} = 0$. But $\mathbf{v}(t) \cdot \mathbf{k} = (x'(t), y'(t), z'(t)) \cdot (0, 0, 1) = z'(t)$. We conclude that $z'(t)$ is always zero so $z(t)$ is constant. Since we assumed that $z(0) = 0, z(t) \equiv 0$ and the path remains in the xy-plane.

(d) Look at the plane determined by $\mathbf{v}(0)$ and $\mathbf{a}(0)$. By part (b), $\mathbf{B}$ will be perpendicular to that plane. By part (a), $\mathbf{B}$ will be constant. Part (c) shows that motion will always be orthogonal to the direction of $\mathbf{B}$. It is harder to see in this case, but we can translate the problem so that $\mathbf{x}(0)$ is the origin and rotate so that $\mathbf{a}(0)$ and $\mathbf{v}(0)$ are in the xy-plane, make our conclusions then translate and rotate the solution curve back.

29. Note that we may write $\mathbf{x}(s) = (x(s), y(s), 0)$, where s is the arclength parameter.

(a) $\mathbf{T} = (x'(s), y'(s), 0)$, so $\mathbf{N} = \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \dfrac{1}{\sqrt{x''^2 + y''^2}}(x''(s), y''(s), 0)$. (Hence $\kappa = \sqrt{x''^2 + y''^2}$.) Thus

$$\mathbf{B} = \mathbf{T} \times \mathbf{N} = \left(0, 0, \frac{x'y'' - x''y'}{\kappa}\right) = (0, 0, \pm 1)$$

because $\mathbf{B}$ must be a *unit* vector. Now $\mathbf{B}$ must vary continuously, so either $\mathbf{B} = (0, 0, 1)$ or $(0, 0, -1)$—but in either case, it must be constant.

(b) $\mathbf{B}'(s) = -\tau\mathbf{N}$. $\mathbf{B}$ is constant (by part (a)), so $\mathbf{B}'(s) = \mathbf{0}$. Thus, since $\mathbf{N}$ is *never* zero, we may conclude that $\tau \equiv 0$.

30. We have $0 = \kappa = \left\|\frac{d\mathbf{T}}{ds}\right\|$. Thus $d\mathbf{T}/ds = \mathbf{0}$ so $\mathbf{T}$ must be a constant vector. Since $\mathbf{x}$ is parametrized by arclength, $\mathbf{x}'(s) = \mathbf{T}$ is constant. We may integrate to find:

$$\begin{aligned}\mathbf{x}(s) &= \int_{s_0}^{s} \mathbf{x}'(\sigma)\, d\sigma + \mathbf{x}(s_0) = \int_{s_0}^{s} \mathbf{T}\, d\sigma + \mathbf{x}(s_0) = (s - s_0)\mathbf{T} + \mathbf{x}(s_0)\\ &= s\mathbf{T} + (\mathbf{x}(s_0) - s_0\mathbf{T}),\end{aligned}$$

which is of the form $s\mathbf{a} + \mathbf{b} \Rightarrow$ straight line.

31. (a) The plan is to find the curvature of the strake by finding the curvature of the helix along the pipe. The radius r will be the reciprocal of this curvature (since the curvature of a circle of radius r is $1/r$). The path is $\mathbf{x}(t) = (a\cos\ t, a\sin\ t, ht/2\pi)$ so $\mathbf{x}'(t) = (-a\sin\ t, a\cos\ t, h/2\pi)$ and $\mathbf{x}''(t) = (-a\cos t, -a\sin t, 0)$.

$$\begin{aligned}\kappa &= \frac{\|(-a\sin t, a\cos t, h)/2\pi \times (-a\cos t, -a\sin t, 0)\|}{\|(-a\sin t, a\cos t, h/2\pi)\|^3} = \frac{\|((ah/2\pi)\sin t, -(ah/2\pi)\cos t, a^2)\|}{(a^2 + [h/2\pi]^2)^{3/2}}\\ &= \frac{a\sqrt{h^2/4\pi^2 + a^2}}{(a^2 + [h/2\pi]^2)^{3/2}} = \frac{a}{a^2 + h^2/4\pi^2} \quad \text{so} \quad r = \frac{(a^2 + h^2/4\pi^2)}{a}.\end{aligned}$$

(b) If $a = 3$ and $h = 25$ then $r = (9 + 625/4\pi^2)/3 \approx 8.2771$.

32. Since $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$ is not parametrized by arclength, we may rewrite it as $\mathbf{x}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\right.$ $\left. a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right)$ where $s = \sqrt{a^2 + b^2}\, t$ is arclength (see Example 3 in §3.2). Following Example 9, we have

$$\mathbf{T}(s) = \left(-\frac{a}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right).$$

So the tangent spherical image is a circle of radius $a/\sqrt{a^2 + b^2}$ in the plane $z = b/\sqrt{a^2 + b^2}$.

$\mathbf{N}(s) = \left(-\cos\frac{s}{\sqrt{a^2+b^2}}, -\sin\frac{s}{\sqrt{a^2+b^2}}, 0\right)$; normal spherical image is a unit circle in the xy-plane.

$\mathbf{B}(s) = \left(\frac{b}{\sqrt{a^2+b^2}}\sin\frac{s}{\sqrt{a^2+b^2}}, -\frac{b}{\sqrt{a^2+b^2}}\cos\frac{s}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}\right)$. Thus the binormal spherical image is a circle of radius $b/\sqrt{a^2 + b^2}$ in the plane $z = a/\sqrt{a^2 + b^2}$.

33. By Example 7 of §3.2 and Exercise 30: $\mathbf{x}$ is a straight-line path $\Leftrightarrow \kappa = 0 = \left\|\frac{d\mathbf{T}}{ds}\right\| \Leftrightarrow \mathbf{T}$ is constant.

34. By Exercises 28 and 29, $\mathbf{x}$ is a plane curve $\Leftrightarrow \mathbf{B}$ is constant.

35. $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$ by the Frenet–Serret formula. Now for $\mathbf{N}$ to be defined $\kappa \neq 0$, so if $\tau = 0$, then $\mathbf{N}' = -\kappa\mathbf{T} \neq \mathbf{0}$ (hence $\mathbf{N}$ is not constant). If $\tau \neq 0$, then for $\mathbf{N}'$ to be $\mathbf{0}$, $\mathbf{T}$ and $\mathbf{B}$ would have to be parallel, which they aren't.

36. (a) $\mathbf{x}(t) = r(t)\cos\theta(t)\mathbf{i} + r(t)\sin\theta(t)\mathbf{j} + z(t)\mathbf{k} = r(t)(\cos\theta(t)\mathbf{i} + \sin\theta(t)\mathbf{j}) + z(t)\mathbf{k} = r(t)\mathbf{e}_r + z(t)\mathbf{e}_z$.

(b) We prepare for part (c) by calculating:

$$\begin{aligned}\frac{d\mathbf{e}_r}{dt} &= -\theta'(t)\sin\theta(t)\mathbf{i} + \theta'(t)\cos\theta(t)\mathbf{j} = \theta'(t)\mathbf{e}_\theta,\\ \frac{d\mathbf{e}_\theta}{dt} &= -\theta'(t)\cos\theta(t)\mathbf{i} - \theta'(t)\sin\theta(t)\mathbf{j} = -\theta'(t)\mathbf{e}_r, \text{ and}\\ \frac{d\mathbf{e}_z}{dt} &= \mathbf{0}\end{aligned}$$

(c) We use the results of parts (a) and (b) to calculate:

$$\begin{aligned}\mathbf{v}(t) &= \frac{d}{dt}\mathbf{x}(t) = \frac{d}{dt}[r(t)\mathbf{e}_r + z(t)\mathbf{e}_z]\\ &= r'(t)\mathbf{e}_r + r(t)\frac{d\mathbf{e}_r}{dt} + z'(t)\mathbf{e}_z + z(t)\frac{d\mathbf{e}_z}{dt}\\ &= r'(t)\mathbf{e}_r + r(t)\theta'(t)\mathbf{e}_\theta + z'(t)\mathbf{e}_z, \text{ and}\\ \mathbf{a}(t) &= \frac{d}{dt}[r'(t)\mathbf{e}_r + r(t)\theta'(t)\mathbf{e}_\theta + z'(t)\mathbf{e}_z]\\ &= r''(t)\mathbf{e}_r + r'(t)\theta'(t)\mathbf{e}_\theta + r'(t)\theta'(t)\mathbf{e}_\theta + r(t)\theta''(t)\mathbf{e}_\theta - r(t)[\theta'(t)]^2\mathbf{e}_r + z''(t)\mathbf{e}_z\\ &= (r''(t) - r(t)[\theta'(t)]^2)\mathbf{e}_r + (2r'(t)\theta'(t) + r(t)\theta''(t))\mathbf{e}_\theta + z''(t)\mathbf{e}_z.\end{aligned}$$

37. $\mathbf{x}(t) = (\sin 2t, \sqrt{2}\cos 2t, \sin 2t - 2)$

(a) The loop first closes up when $t = \pi$ so the length of the loop is

$$\text{Length} = \int_0^\pi \sqrt{[2\cos 2t]^2 + [-2\sqrt{2}\sin 2t]^2 + [2\cos 2t]^2}\,dt = \int_0^\pi 2\sqrt{2}\,dt = 2\pi\sqrt{2}.$$

(b) By Definition 3.2, the path is a flow line if $\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}(t))$. Here

$$\mathbf{x}'(t) = (2\cos 2t, -2\sqrt{2}\sin 2t, 2\cos 2t) \quad\text{and}\quad \mathbf{F}(\mathbf{x}(t)) = (\sqrt{2}\cos 2t, -2\sin 2t, \sqrt{2}\cos 2t).$$

So $\mathbf{x}$ is a flow line of the vector field $\sqrt{2}\mathbf{F}(x,y,z) = \sqrt{2}y\mathbf{i} - 2\sqrt{2}x\mathbf{j} + \sqrt{2}y\mathbf{k}$.

38. Poor Livinia, she's been caught in an oven back in Chapter 2, and now here in Chapter 3 she's still looking to get warm.

(a) We saw in Section 2.6 that the gradient is the direction of quickest increase. Livinia should head in a direction parallel to the gradient. In other words, at each point she should travel in the direction $k\nabla T$ so $\mathbf{x}'(t) = k\nabla T$ so $\mathbf{x}$ is a path of $k\nabla T$ for $k \geq 0$.

(b) If $T(x,y,z) = x^2 - 2y^2 + 3z^2$ then $\nabla T = (2x, -4y, 6z)$. We also know that the initial position is $(2,3,-1)$. This means that $x' = 2x$ and $x(0) = 2$ so $x(t) = 2e^{2kt}$. Similarly, $y(t) = 3e^{-4kt}$ and $z(t) = -e^{6kt}$. So the equation of the path is $\mathbf{x}(t) = (2e^{2kt}, 3e^{-4kt}, -e^{6kt})$.

39. $\mathbf{F} = u(x,y)\mathbf{i} - v(x,y)\mathbf{j}$ is an incompressible, irrotational vector field and so $\nabla\cdot\mathbf{F} = 0$ and $\nabla\times\mathbf{F} = \mathbf{0}$.

(a) The Cauchy–Riemann equations follow immediately from the assumptions:

$$\begin{aligned}0 = \nabla\cdot\mathbf{F} &= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}, \text{ so}\\ \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \text{ and}\\ 0 = (\nabla\times\mathbf{F})\cdot\mathbf{k} &= -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \text{ so}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

(b) Take the partial derivative with respect to x of both sides of the equation: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}.$$

An analogous calculation shows the result for v.

40. $\mathbf{F}$ is a gradient field so $\mathbf{F} = \nabla f(\mathbf{x}(t))$. Also $\mathbf{F} = m\mathbf{a}$. From Section 3.3 we know that if $\mathbf{x}$ is a path on an equipotential surface of $\mathbf{F}$ then $f(\mathbf{x}(t))$ is constant so $\frac{d}{dt}f(\mathbf{x}(t)) = 0$. So

$$0 = \frac{d}{dt}f(\mathbf{x}(t)) = \nabla f\cdot\frac{d}{dt}\mathbf{x}(t) = m\mathbf{a}\cdot\mathbf{v}.$$

From Section 3.2 formulas (14) and (16), we see that

$$m\mathbf{a}\cdot\mathbf{v} = m\,\dot{s}\,\ddot{s} = \frac{1}{2}m\frac{d}{dt}(\dot{s}^2).$$

So, since the derivative of $\dot{s}^2 = 0$, we conclude that $\dot{s}^2$ is constant and hence $\dot{s}$ is constant.

41. Using Section 3.1, Exercise 28:

$$\begin{aligned}\frac{d\mathbf{l}}{dt} = \frac{d}{dt}(\mathbf{x} \times m\mathbf{v}) &= \frac{d\mathbf{x}}{dt} \times m\mathbf{v} + \mathbf{x} \times m\frac{d\mathbf{v}}{dt} \\ &= \mathbf{v} \times m\mathbf{v} + \mathbf{x} \times m\mathbf{a} \\ &= \mathbf{0} + \mathbf{x} \times m\mathbf{a} = \mathbf{M}.\end{aligned}$$

42. If $\mathbf{F}$ is a central force, then $\mathbf{F}$ is always parallel to $\mathbf{x}$. Hence $\mathbf{M} = \mathbf{x} \times \mathbf{F} = \mathbf{0}$, By Exercise 41, $\mathbf{M} = \frac{d\mathbf{l}}{dt}$ so $\mathbf{l}$ must be constant.

43. Notice that $\nabla \times \mathbf{F} = (0, 2e^{-x}\cos z, 0) \neq \mathbf{0}$. If $\mathbf{F}$ were a gradient field ∇f of class C^2, then, by Theorem 4.3, $\nabla \times \mathbf{F} = \nabla \times (\nabla f) = \mathbf{0}$.

44. Note that $\nabla \cdot \mathbf{F} = y^2 + 1 + e^z + x^2e^z > 0$ for all $(x, y, z) \in \mathbf{R}^3$. But if $\mathbf{F} = \nabla \times \mathbf{G}$, then $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{G}) \equiv 0$ for any vector field $\mathbf{G}$ of class C^2. Thus $\mathbf{F} \neq \nabla \times \mathbf{G}$.