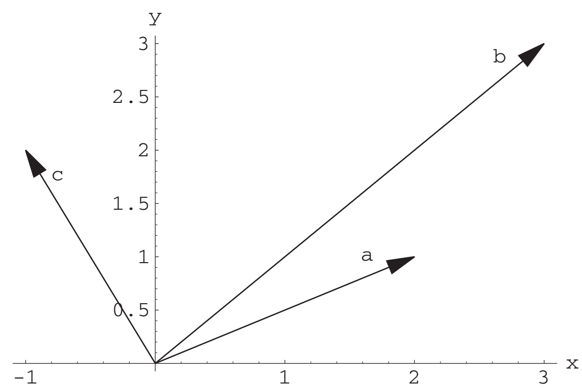
Chapter 1

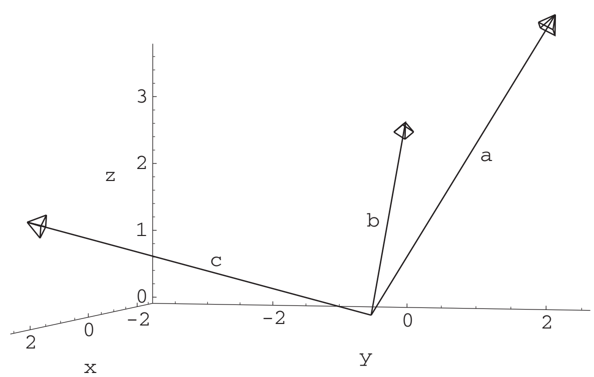
Vectors

1.1 Vectors in Two and Three Dimensions

1. *Here we just connect the point (0, 0) to the points indicated*:



2. *Although more difficult for students to represent this on paper, the figures should look something like the following. Note that the origin is not at a corner of the frame box but is at the tails of the three vectors*.

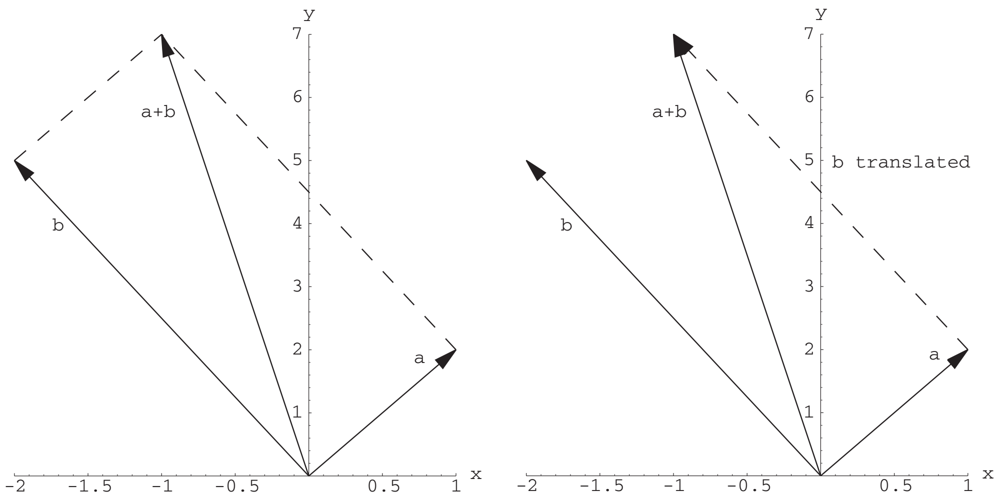


In problems 3 and 4, we supply more detail than is necessary to stress to students what properties are being used:

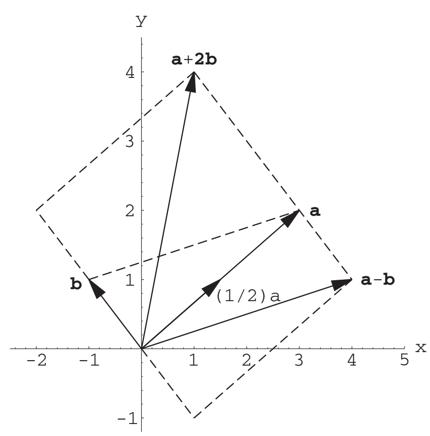
3. **(a)** $(3, 1) + (-1, 7) = (3 + [-1], 1 + 7) = (2, 8)$.
(b) $-2(8, 12) = (-2 \cdot 8, -2 \cdot 12) = (-16, -24)$.
(c) $(8, 9) + 3(-1, 2) = (8 + 3(-1), 9 + 3(2)) = (5, 15)$.
(d) $(1, 1) + 5(2, 6) - 3(10, 2) = (1 + 5 \cdot 2 - 3 \cdot 10, 1 + 5 \cdot 6 - 3 \cdot 2) = (-19, 25)$.
(e) $(8, 10) + 3((8, -2) - 2(4, 5)) = (8 + 3(8 - 2 \cdot 4), 10 + 3(-2 - 2 \cdot 5)) = (8, -26)$.

4. **(a)** $(2, 1, 2) + (-3, 9, 7) = (2 - 3, 1 + 9, 2 + 7) = (-1, 10, 9)$.
(b) $\frac{1}{2}(8, 4, 1) + 2\left(5, -7, \frac{1}{4}\right) = \left(4, 2, \frac{1}{2}\right) + \left(10, -14, \frac{1}{2}\right) = (14, -12, 1)$.
(c) $-2\left((2, 0, 1) - 6\left(\frac{1}{2}, -4, 1\right)\right) = -2((2, 0, 1) - (3, -24, 6)) = -2(-1, 24, -5) = (2, -48, 10)$.

5. We start with the two vectors **a** and **b**. We can complete the parallelogram as in the figure on the left. The vector from the origin to this new vertex is the vector $\mathbf{a} + \mathbf{b}$. In the figure on the right we have translated vector **b** so that its tail is the head of vector **a**. The sum $\mathbf{a} + \mathbf{b}$ is the directed third side of this triangle.



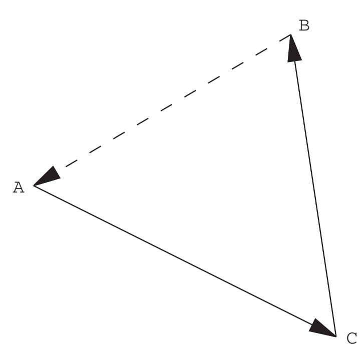
6. $\mathbf{a} = (3, 2)$ $\quad \mathbf{b} = (-1, 1)$
$\mathbf{a} - \mathbf{b} = (3 - (-1), 2 - 1) = (4, 1)$ $\qquad \frac{1}{2}\mathbf{a} = \left(\frac{3}{2}, 1\right)$ $\quad \mathbf{a} + 2\mathbf{b} = (1, 4)$



7. **(a)** $\overrightarrow{AB} = (-3 - 1, 3 - 0, 1 - 2) = (-4, 3, -1)$ $\quad \overrightarrow{BA} = -\overrightarrow{AB} = (4, -3, 1)$

(b) $\overrightarrow{AC} = (2 - 1, 1 - 0, 5 - 2) = (1, 1, 3)$
$\overrightarrow{BC} = (2 - (-3), 1 - 3, 5 - 1) = (5, -2, 4)$
$\overrightarrow{AC} + \overrightarrow{CB} = (1, 1, 3) - (5, -2, 4) = (-4, 3, -1)$

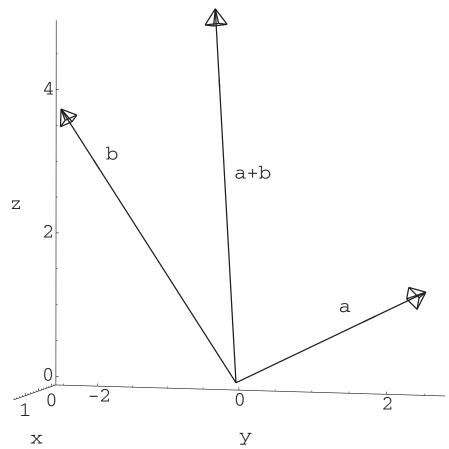
(c) This result is true in general:



Head-to-tail addition demonstrates this.

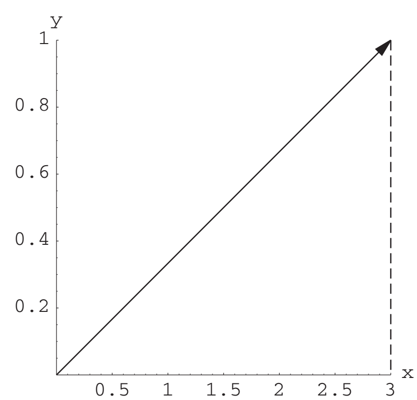
8. The vectors $\mathbf{a} = (1, 2, 1)$, $\mathbf{b} = (0, -2, 3)$ and $\mathbf{a} + \mathbf{b} = (1, 2, 1) + (0, -2, 3) = (1, 0, 4)$ are graphed below. *Again note that the origin is at the tails of the vectors in the figure*.

Also, $-1(1, 2, 1) = (-1, -2, -1)$. This would be pictured by drawing the vector (1, 2, 1) in the opposite direction. Finally, $4(1, 2, 1) = (4, 8, 4)$ which is four times vector **a** and so is vector **a** stretched four times as long in the same direction.

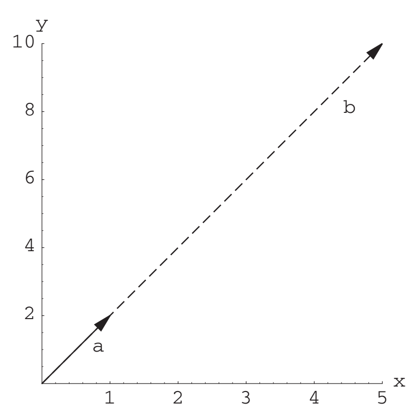


9. *Since the sum on the left must equal the vector on the right componentwise*:
$-12 + x = 2$, $9 + 7 = y$, and $z + -3 = 5$. Therefore, $x = 14$, $y = 16$, and $z = 8$.

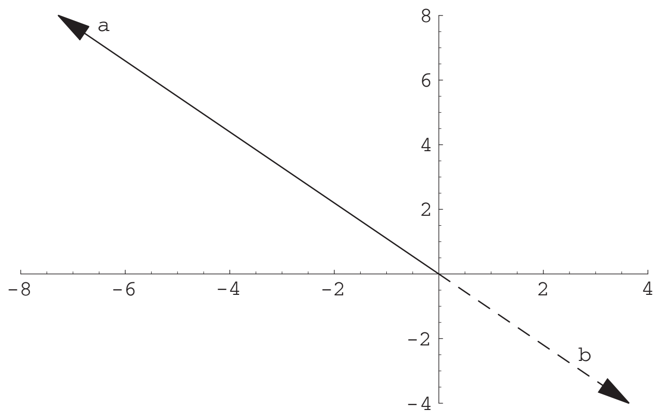
10. If we drop a perpendicular from (3, 1) to the x-axis we see that by the Pythagorean Theorem the length of the vector $(3, 1) = \sqrt{3^2 + 1^2} = \sqrt{10}$.



11. Notice that $\mathbf{b}$ (represented by the dotted line) $= 5\mathbf{a}$ (represented by the solid line).



12. Here the picture has been projected into two dimensions so that you can more clearly see that **a** (represented by the solid line) $= -2\mathbf{b}$ (represented by the dotted line).



13. The natural extension to higher dimensions is that we still add componentwise and that multiplying a scalar by a vector means that we multiply each component of the vector by the scalar. In symbols this means that:

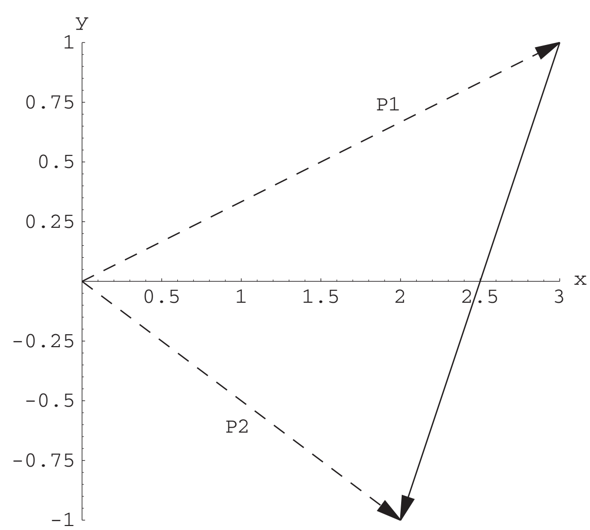
$\mathbf{a}+\mathbf{b} = (a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$ and $k\mathbf{a} = (ka_1, ka_2, \ldots, ka_n)$.

In our particular examples, $(1, 2, 3, 4) + (5, -1, 2, 0) = (6, 1, 5, 4)$, and $2(7, 6, -3, 1) = (14, 12, -6, 2)$.

14. The diagrams for parts (a), (b) and (c) are similar to Figure 1.12 from the text. The displacement vectors are:

(a) $(1, 1, 5)$
(b) $(-1, -2, 3)$
(c) $(1, 2, -3)$
(d) $(-1, -2)$

Note: The displacement vectors for (b) and (c) are the same but in opposite directions (i.e., one is the negative of the other). The displacement vector in the diagram for (d) is represented by the solid line in the figure below:

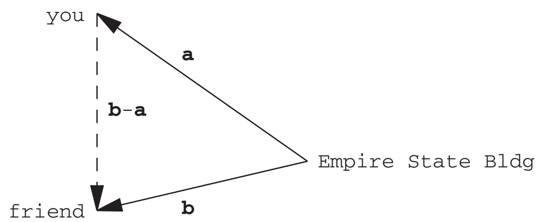


15. In general, we would define the displacement vector from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$ to be $(b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n)$.

In this specific problem the displacement vector from P_1 to P_2 is $(1, -4, -1, 1)$.

16. Let B have coordinates (x, y, z). Then $\overrightarrow{AB} = (x - 2, y - 5, z + 6) = (12, -3, 7)$ so $x = 14$, $y = 2$, $z = 1$ so B has coordinates (14, 2, 1).

17. If $\mathbf{a}$ is your displacement vector from the Empire State Building and $\mathbf{b}$ your friend's, then the displacement vector from you to your friend is $\mathbf{b} - \mathbf{a}$.



18. Property 2 follows immediately from the associative property of the reals:

$$\begin{aligned}(\mathbf{a}+\mathbf{b})+\mathbf{c} &= ((a_1,a_2,a_3)+(b_1,b_2,b_3))+(c_1,c_2,c_3)\\ &= ((a_1+b_1,a_2+b_2,a_3+b_3)+(c_1,c_2,c_3)\\ &= ((a_1+b_1)+c_1,(a_2+b_2)+c_2,(a_3+b_3)+c_3)\\ &= (a_1+(b_1+c_1),a_2+(b_2+c_2),a_3+(b_3+c_3))\\ &= (a_1,a_2,a_3)+((b_1+c_1),(b_2+c_2),(b_3+c_3))\\ &= \mathbf{a}+(\mathbf{b}+\mathbf{c}).\end{aligned}$$

Property 3 also follows from the corresponding componentwise observation:

$$\mathbf{a}+\mathbf{0}=(a_1+0,a_2+0,a_3+0)=(a_1,a_2,a_3)=\mathbf{a}.$$

19. *We provide the proofs for* $\mathbf{R}^3$:

$$\begin{aligned}(1)\ (k+l)\mathbf{a} &= (k+l)(a_1,a_2,a_3)=((k+l)a_1,(k+l)a_2,(k+l)a_3)\\ &= (ka_1+la_1,ka_2+la_2,ka_3+la_3)=k\mathbf{a}+l\mathbf{a}.\\ (2)\ k(\mathbf{a}+\mathbf{b}) &= k((a_1,a_2,a_3)+(b_1,b_2,b_3))=k(a_1+b_1,a_2+b_2,a_3+b_3)\\ &= (k(a_1+b_1),k(a_2+b_2),k(a_3+b_3))=(ka_1+kb_1,ka_2+kb_2,ka_3+kb_3)\\ &= (ka_1,ka_2,ka_3)+(kb_1,kb_2,kb_3)=k\mathbf{a}+k\mathbf{b}.\\ (3)\ k(l\mathbf{a}) &= k(l(a_1,a_2,a_3))=k(la_1,la_2,la_3)\\ &= (kla_1,kla_2,kla_3)=(lka_1,lka_2,lka_3)\\ &= l(ka_1,ka_2,ka_3)=l(k\mathbf{a}).\end{aligned}$$

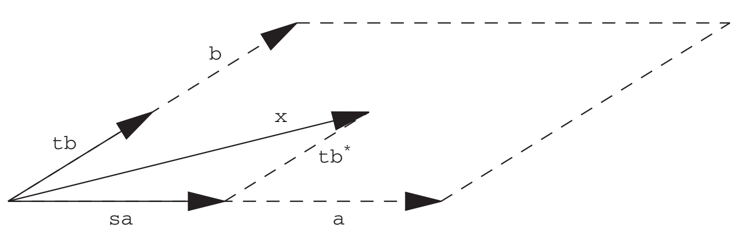
20. (a) $0\mathbf{a}$ is the zero vector. For example, in $\mathbf{R}^3$:

$$0\mathbf{a}=0(a_1,a_2,a_3)=(0\cdot a_1,0\cdot a_2,0\cdot a_3)=(0,0,0).$$

(b) $1\mathbf{a}=\mathbf{a}$. Again in $\mathbf{R}^3$:

$$1\mathbf{a}=1(a_1,a_2,a_3)=(1\cdot a_1,1\cdot a_2,1\cdot a_3)=(a_1,a_2,a_3)=\mathbf{a}.$$

21. (a) The head of the vector $s\mathbf{a}$ is on the x-axis between 0 and 2. Similarly the head of the vector $t\mathbf{b}$ lies somewhere on the vector **b**. Using the head-to-tail method, $s\mathbf{a}+t\mathbf{b}$ is the result of translating the vector $t\mathbf{b}$, in this case, to the right by $2s$ (represented in the figure by $t\mathbf{b}$*****). The result is clearly inside the parallelogram determined by **a** and **b** (and is only on the boundary of the parallelogram if either t or s is 0 or 1.



(b) Again the vectors **a** and **b** will determine a parallelogram (with vertices at the origin, and at the heads of **a**, **b**, and $\mathbf{a} + \mathbf{b}$. The vectors $s\mathbf{a} + t\mathbf{b}$ will be the position vectors for all points in that parallelogram determined by (2, 2, 1) and (0, 3, 2).

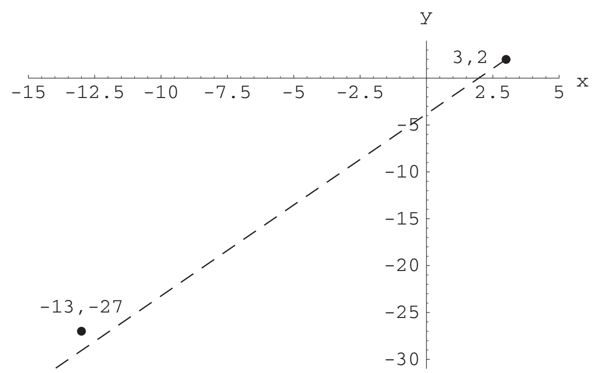
22. Here we are translating the situation in Exercise 21 by the vector $\overrightarrow{OP_0}$. The vectors will all be of the form $\overrightarrow{OP_0} + s\mathbf{a} + t\mathbf{b}$ for $0 \leq s, t \leq 1$.

23. (a) The speed of the flea is the length of the velocity vector $= \sqrt{(-1)^2 + (-2)^2} = \sqrt{5}$ units per minute.

(b) After 3 minutes the flea is at $(3, 2) + 3(-1, -2) = (0, -4)$.

(c) We solve $(3, 2) + t(-1, -2) = (-4, -12)$ for t and get that $t = 7$ minutes. Note that *both* $3-7 = -4$ and $2-14 = -12$.

(d) We can see this algebraically or geometrically: Solving the x part of $(3, 2) + t(-1, -2) = (-13, -27)$ we get that $t = 16$. But when $t = 16$, $y = -30$ not -27. Also in the figure below we see the path taken by the flea will miss the point $(-13, -27)$.



24. (a) The plane is climbing at a rate of 4 miles per hour.

(b) To make sure that the axes are oriented so that the plane passes over the building, the positive x direction is east and the positive y direction is north. Then we are heading east at a rate of 50 miles per hour at the same time we're heading north at a rate of 100 miles per hour. We are directly over the skyscraper in 1/10 of an hour or 6 minutes.

(c) Using our answer in (b), we have traveled for 1/10 of an hour and so we've climbed 4/10 of a mile or 2112 feet. The plane is $2112 - 1250$ or 862 feet about the skyscraper.

25. (a) Adding we get: $\mathbf{F}_1 + \mathbf{F}_2 = (2, 7, -1) + (3, -2, 5) = (5, 5, 4)$.

(b) You need a force of the same magnitude in the opposite direction, so $\mathbf{F}_3 = -(5, 5, 4) = (-5, -5, -4)$.

26. (a) Measuring the force in pounds we get $(0, 0, -50)$.

(b) The z components of the two vectors along the ropes must be equal and their sum must be opposite of the z component in part (a). Their y components must also be opposite each other. Since the vector points in the direction (0, ±2, 1), the y component will be twice the z component. Together this means that the vector in the direction of $(0, -2, 1)$ is $(0, -50, 25)$ and the vector in the direction (0, 2, 1) is (0, 50, 25).

27. The force $\mathbf{F}$ due to gravity on the weight is given by $\mathbf{F} = (0, 0, -10)$. The forces along the ropes are each parallel to the displacement vectors from the weight to the respective anchor points. That is, the tension vectors along the ropes are

$$\mathbf{F}_1 = k((3, 0, 4) - (1, 2, 3)) = k(2, -2, 1)$$
$$\mathbf{F}_2 = l((0, 3, 5) - (1, 2, 3)) = l(-1, 1, 2),$$

where k and l are appropriate scalars. For the weight to remain in equilibrium, we must have $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F} = \mathbf{0}$, or, equivalently, that

$$k(2, -2, 1) + l(-1, 1, 2) + (0, 0, -10) = (0, 0, 0).$$

Taking components, we obtain a system of three equations:

$$\begin{cases} 2k - l = 0 \\ -2k + l = 0 \\ k + 2l = 10. \end{cases}$$

Solving, we find that $k = 2$ and $l = 4$, so that

$$\mathbf{F}_1 = (4, -4, 2) \text{ and } \mathbf{F}_2 = (-4, 4, 8).$$

1.2 More about Vectors

It may be useful to point out that the answers to Exercises 1 and 5 are the "same", but that in Exercise 1, $\mathbf{i} = (1, 0)$ *and in Exercise 5,* $\mathbf{i} = (1, 0, 0)$. *This comes up when going the other direction in Exercises 9 and 10. In other words, it's not always clear whether the exercise "lives" in* $\mathbf{R}^2$ *or* $\mathbf{R}^3$.

1. $(2, 4) = 2(1, 0) + 4(0, 1) = 2\mathbf{i} + 4\mathbf{j}$.

2. $(9, -6) = 9(1, 0) - 6(0, 1) = 9\mathbf{i} - 6\mathbf{j}$.

3. $(3, \pi, -7) = 3(1, 0, 0) + \pi(0, 1, 0) - 7(0, 0, 1) = 3\mathbf{i} + \pi\mathbf{j} - 7\mathbf{k}$.

4. $(-1, 2, 5) = -1(1, 0, 0) + 2(0, 1, 0) + 5(0, 0, 1) = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$.

5. $(2, 4, 0) = 2(1, 0, 0) + 4(0, 1, 0) = 2\mathbf{i} + 4\mathbf{j}$.

6. $\mathbf{i} + \mathbf{j} - 3\mathbf{k} = (1, 0, 0) + (0, 1, 0) - 3(0, 0, 1) = (1, 1, -3)$.

7. $9\mathbf{i} - 2\mathbf{j} + \sqrt{2}\mathbf{k} = 9(1, 0, 0) - 2(0, 1, 0) + \sqrt{2}(0, 0, 1) = (9, -2, \sqrt{2})$.

8. $-3(2\mathbf{i} - 7\mathbf{k}) = -6\mathbf{i} + 21\mathbf{k} = -6(1, 0, 0) + 21(0, 0, 1) = (-6, 0, 21)$.

9. $\pi\mathbf{i} - \mathbf{j} = \pi(1, 0) - (0, 1) = (\pi, -1)$.

10. $\pi\mathbf{i} - \mathbf{j} = \pi(1, 0, 0) - (0, 1, 0) = (\pi, -1, 0)$.

Note: You may want to assign both Exercises 11 and 12 together so that the students may see the difference. You should stress that the reason the results are different has nothing to do with the fact that Exercise 11 is a question about $\mathbf{R}^2$ *while Exercise 12 is a question about* $\mathbf{R}^3$.

11. (a) $(3, 1) = c_1(1, 1) + c_2(1, -1) = (c_1 + c_2, c_1 - c_2)$, so $\begin{cases} c_1 + c_2 = 3, \text{ and} \\ c_1 - c_2 = 1. \end{cases}$

Solving simultaneously (for instance by adding the two equations), we find that $2c_1 = 4$, so $c_1 = 2$ and $c_2 = 1$. So $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$.

(b) Here $c_1 + c_2 = 3$ and $c_1 - c_2 = -5$, so $c_1 = -1$ and $c_2 = 4$. So $\mathbf{b} = -\mathbf{a}_1 + 4\mathbf{a}_2$.

(c) More generally, $(b_1, b_2) = (c_1 + c_2, c_1 - c_2)$, so $\begin{cases} c_1 + c_2 = b_1, \text{ and} \\ c_1 - c_2 = b_2. \end{cases}$

Again solving simultaneously, $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$. So

$$\mathbf{b} = \left(\frac{b_1 + b_2}{2}\right)\mathbf{a}_1 + \left(\frac{b_1 - b_2}{2}\right)\mathbf{a}_2.$$

12. Note that $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, so really we are only working with two (linearly independent) vectors.

(a) $(5, 6, -5) = c_1(1, 0, -1) + c_2(0, 1, 0) + c_3(1, 1, -1)$; this gives us the equations:

$$\begin{cases} 5 = c_1 + c_3 \\ 6 = c_2 + c_3 \\ -5 = -c_1 - c_3. \end{cases}$$

The first and last equations contain the same information and so we have infinitely many solutions. You will quickly see one by letting $c_3 = 0$. Then $c_1 = 5$ and $c_2 = 6$. So we could write $\mathbf{b} = 5\mathbf{a}_1 + 6\mathbf{a}_2$. More generally, you can choose any value for c_1 and then let $c_2 = c_1 + 1$ and $c_3 = 5 - c_1$.

(b) We cannot write (2, 3, 4) as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. Here we get the equations:

$$\begin{cases} c_1 + c_3 = 2 \\ c_2 + c_3 = 3 \\ -c_1 - c_3 = 4. \end{cases}$$

The first and last equations are inconsistent and so the system cannot be solved.

(c) As we saw in part (b), not all vectors in $\mathbf{R}^3$ can be written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. In fact, only vectors of the form $(a, b, -a)$ can be written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. For your students who have had linear algebra, this is because the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are not linearly independent.

Note: As pointed out in the text, the answers for 13–21 are not unique.

13. $\mathbf{r}(t) = (2, -1, 5) + t(1, 3, -6)$ so $\begin{cases} x = 2 + t \\ y = -1 + 3t \\ z = 5 - 6t. \end{cases}$

14. $\mathbf{r}(t) = (12, -2, 0) + t(5, -12, 1)$ so $\begin{cases} x = 12 + 5t \\ y = -2 - 12t \\ z = t. \end{cases}$

15. $\mathbf{r}(t) = (2, -1) + t(1, -7)$ so $\begin{cases} x = 2 + t \\ y = -1 - 7t. \end{cases}$

16. $\mathbf{r}(t) = (2, 1, 2) + t(3 - 2, -1 - 1, 5 - 2)$ so $\begin{cases} x = 2 + t \\ y = 1 - 2t \\ z = 2 + 3t. \end{cases}$

17. $\mathbf{r}(t) = (1, 4, 5) + t(2 - 1, 4 - 4, -1 - 5)$ so $\begin{cases} x = 1 + t \\ y = 4 \\ z = 5 - 6t. \end{cases}$

18. $\mathbf{r}(t) = (8, 5) + t(1 - 8, 7 - 5)$ so $\begin{cases} x = 8 - 7t \\ y = 5 + 2t. \end{cases}$

Note: In higher dimensions, we switch our notation to x_i.

19. $\mathbf{r}(t) = (1, 2, 0, 4) + t(-2, 5, 3, 7)$ so $\begin{cases} x_1 = 1 - 2t \\ x_2 = 2 + 5t \\ x_3 = 3t \\ x_4 = 4 + 7t. \end{cases}$

20. $\mathbf{r}(t) = (9, \pi, -1, 5, 2) + t(-1 - 9, 1 - \pi, \sqrt{2} + 1, 7 - 5, 1 - 2)$ so $\begin{cases} x_1 = 9 - 10t \\ x_2 = \pi + (1 - \pi)t \\ x_3 = -1 + (\sqrt{2} + 1)t \\ x_4 = 5 + 2t \\ x_5 = 2 - t. \end{cases}$

21. (a) $\mathbf{r}(t) = (-1, 7, 3) + t(2, -1, 5)$ so $\begin{cases} x = -1 + 2t \\ y = 7 - t \\ z = 3 + 5t. \end{cases}$

(b) $\mathbf{r}(t) = (5, -3, 4) + t(0 - 5, 1 + 3, 9 - 4)$ so $\begin{cases} x = 5 - 5t \\ y = -3 + 4t \\ z = 4 + 5t. \end{cases}$

(c) Of course, there are infinitely many solutions. For our variation on the answer to (a) we note that a line parallel to the vector $2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$ is also parallel to the vector $-(2\mathbf{i} - \mathbf{j} + 5\mathbf{k})$ so another set of equations for part (a) is:

$$\begin{cases} x = -1 - 2t \\ y = 7 + t \\ z = 3 - 5t. \end{cases}$$

For our variation on the answer to (b) we note that the line passes through both points so we can set up the equation with respect to the other point:

$$\begin{cases} x = -5t \\ y = 1 + 4t \\ z = 9 + 5t. \end{cases}$$

(d) The symmetric forms are:

$$\frac{x+1}{2} = 7 - y = \frac{z-3}{5} \qquad \text{(for (a))}$$

$$\frac{5-x}{5} = \frac{y+3}{4} = \frac{z-4}{5} \qquad \text{(for (b))}$$

$$\frac{x+1}{-2} = y - 7 = \frac{z-3}{-5} \qquad \text{(for the variation of (a))}$$

$$\frac{x}{-5} = \frac{y-1}{4} = \frac{z-9}{5} \qquad \text{(for the variation of (b))}$$

22. Solve for t in each of the parametric equations. Thus

$$t = \frac{x-5}{-2}, t = \frac{y-1}{3}, t = \frac{z+4}{6}$$

and the symmetric form is

$$\frac{x-5}{-2} = \frac{y-1}{3} = \frac{z+4}{6}.$$

23. Solving for t in each of the parametric equations gives $t = x-7$, $t = (y+9)/3$, and $t = (z-6)/(-8)$, so that the symmetric form is

$$\frac{x-7}{1} = \frac{y+9}{3} = \frac{z-6}{-8}.$$

24. Set each piece of the equation equal to t and solve:

$$\frac{x-2}{5} = t \Rightarrow x-2 = 5t \Rightarrow x = 2+5t$$
$$\frac{y-3}{-2} = t \Rightarrow y-3 = -2t \Rightarrow y = 3-2t$$
$$\frac{z+1}{4} = t \Rightarrow z+1 = 4t \Rightarrow z = -1+4t.$$

25. Let $t = (x+5)/3$. Then $x = 3t-5$. In view of the symmetric form, we also have that $t = (y-1)/7$ and $t = (z+10)/(-2)$. Hence a set of parametric equations is $x = 3t-5$, $y = 7t+1$, and $z = -2t-10$.

Note: In Exercises 26–29, we could say for certain that two lines are not the same if the vectors were not multiples of each other. In other words, it takes two pieces of information to specify a line. You either need two points, or a point and a direction (or in the case of $\mathbf{R}^2$*, equivalently, a slope).*

26. The first line is parallel to the vector $\mathbf{a}_1 = (5, -3, 4)$, while the second is parallel to $\mathbf{a}_2 = (10, -5, 8)$. Since $\mathbf{a}_1$ and $\mathbf{a}_2$ are not parallel, the lines cannot be the same.

27. If we multiply each of the pieces in the second symmetric form by -2, we are effectively just traversing the same path at a different speed and with the opposite orientation. So the second set of equations becomes:

$$\frac{x+1}{3} = \frac{y+6}{7} = \frac{z+5}{5}.$$

This looks a lot more like the first set of equations. If we now subtract one from each piece of the second set of equations (as suggested in the text), we are effectively just changing our initial point but we are still on the same line:

$$\frac{x+1}{3} - \frac{3}{3} = \frac{y+6}{7} - \frac{7}{7} = \frac{z+5}{5} - \frac{5}{5}.$$

We have transformed the second set of equations into the first and therefore see that they both represent the same line in $\mathbf{R}^3$.

28. If you first write the equation of the two lines in vector form, we can see immediately that their direction vectors are the same so either they are parallel or they are the same line:

$$\mathbf{r_1}(t) = (-5, 2, 1) + t(2, 3, -6)$$
$$\mathbf{r_2}(t) = (1, 11, -17) - t(2, 3, -6).$$

The first line contains the point $(-5, 2, 1)$. If the second line contains $(-5, 2, 1)$, then the equations represent the same line. Solve just the x component to get that $-5 = 1-2t \Rightarrow t = 3$. Checking we see that $\mathbf{r_2}(3) = (1, 11, -17) - 3(2, 3, -6) = (-5, 2, 1)$ so the lines are the same.

29. Here again the vector forms of the two lines can be written so that we see their headings are the same:

$$\mathbf{r_1}(t) = (2, -7, 1) + t(3, 1, 5)$$
$$\mathbf{r_2}(t) = (-1, -8, -3) + 2t(3, 1, 5).$$

The point $(2, -7, 1)$ is on line one, so we will check to see if it is also on line two. As in Exercise 28 we check the equation for the x component and see that $-1+6t = 2 \Rightarrow t = 1/2$. Checking we see that $\mathbf{r_2}(1/2) = (-1, -8, -3) + (1/2)(2)(3, 1, 5) = (2, -7, 2) \neq (2, -7, 1)$ so the equations do not represent the same lines.

Note: It is a good idea to assign both Exercises 30 and 31 together. Although they look similar, there is a difference that students might miss.

30. If you make the substitution $u = t^3$, the equations become: $\begin{cases} x = 3u + 7, \\ y = -u + 2, \text{ and} \\ z = 5u + 1. \end{cases}$

The map $u = t^3$ is a bijection. The important fact is that u takes on exactly the same values that t does, just at different times. Since u takes on all reals, the parametric equations do determine a line (it's just that the speed along the line is not constant).

31. This time if you make the substitution $u = t^2$, the equations become: $\begin{cases} x = 5u - 1, \\ y = 2u + 3, \text{ and} \\ z = -u + 1. \end{cases}$

The problem is that u cannot take on negative values so these parametric equations are for a ray with endpoint $(-1, 3, 1)$ and heading $(5, 2, -1)$.

32. **(a)** The vector form of the equations is: $\mathbf{r}(t) = (7, -2, 1) + t(2, 1, -3)$. The initial point is then $\mathbf{r}(0) = (7, -2, 1)$, and after 3 minutes the bird is at $\mathbf{r}(3) = (7, -2, 1) + 3(2, 1, -3) = (13, 1, -8)$.

(b) $(2, 1, -3)$

(c) We only need to check one component (say the x): $7 + 2t = 34/3 \Rightarrow t = 13/6$. Checking we see that $\mathbf{r}\left(\frac{13}{6}\right) = (7, -2, 1) + \left(\frac{13}{6}\right)(2, 1, -3) = \left(\frac{34}{3}, \frac{1}{6}, -\frac{11}{2}\right)$.

(d) As in part (c), we'll check the x component and see that $7 + 2t = 17$ when $t = 5$. We then check to see that $\mathbf{r}(5) = (7, -2, 1) + 5(2, 1, -3) = (17, 3, -14) \neq (17, 4, -14)$ so, no, the bird doesn't reach $(17, 4, -14)$.

33. We can substitute the parametric forms of x, y, and z into the equation for the plane and solve for t. So $(3t - 5) + 3(2 - t) - (6t) = 19$ which gives us $t = -3$. Substituting back in the parametric equations, we find that the point of intersection is $(-14, 5, -18)$.

34. Using the same technique as in Exercise 33, $5(1 - 4t) - 2(t - 3/2) + (2t + 1) = 1$ which simplifies to $t = 2/5$. This means the point of intersection is $(-3/5, -11/10, 9/5)$.

35. We will set each of the coordinate equations equal to zero in turn and substitute that value of t into the other two equations.

$$x = 2t - 3 = 0 \Rightarrow t = 3/2. \text{ When } t = 3/2, y = 13/2 \text{ and } z = 7/2.$$

$$y = 3t + 2 = 0 \Rightarrow t = -2/3, \text{ so } x = -13/3 \text{ and } z = 17/3.$$

$$z = 5 - t = 0 \Rightarrow t = 5, \text{ so } x = 7 \text{ and } y = 17.$$

The points are (0, 13/2, 7/2), $(-13/3, 0, 17/3)$, and (7, 17, 0).

36. We could show that two points on the line are also in the plane or that for points on the line: $2x - y + 4z = 2(5 - t) - (2t - 7) + 4(t - 3) = 5$, so they are in the plane.

37. For points on the line we see that $x - 3y + z = (5 - t) - 3(2t - 3) + (7t + 1) = 15$, so the line does not intersect the plane.

38. First we parametrize the line by setting $t = (x - 3)/6$, which gives us $x = 6t + 3$, $y = 3t - 2$, $z = 5t$. Plugging these parametric values into the equation for the plane gives

$$2(6t + 3) - 5(3t - 2) + 3(5t) + 8 = 0 \iff 12t + 24 = 0 \iff t = -2.$$

The parameter value $t = -2$ yields the point $(6(-2) + 3, 3(-2) - 2, 5(-2)) = (-9, -8, -10)$.

39. We find parametric equations for the line by setting $t = (x - 3)/(-2)$, so that $x = 3 - 2t$, $y = t + 5$, $z = 3t - 2$. Plugging these parametric values into the equation for the plane, we find that

$$3(3 - 2t) + 3(t + 5) + (3t - 2) = 9 - 6t + 3t + 15 + 3t - 2 = 22$$

for *all* values of t. Hence the line is contained in the plane.

40. Again we find parametric equations for the line. Set $t = (x + 4)/3$, so that $x = 3t - 4$, $y = 2 - t$, $z = 1 - 9t$. Plugging these parametric values into the equation for the plane, we find that

$$2(3t - 4) - 3(2 - t) + (1 - 9t) = 7 \iff 6t - 8 - 6 + 3t + 1 - 9t = 7 \iff -13 = 7.$$

Hence we have a contradiction; that is, *no* value of t will yield a point on the line that is also on the plane. Thus the line and the plane do *not* intersect.

41. We just plug the parametric expressions for x, y, z into the equation for the surface:

$$\frac{(at+a)^2}{a^2}+\frac{b^2}{b^2}-\frac{(ct+c)^2}{c^2}=\frac{c^2(t+1)^2}{a^2}+1-\frac{c^2(t+1)^2}{c^2}=1$$

for all values of $t \in \mathbf{R}$. Hence all points on the line satisfy the equation for the surface.

42. As explained in the text, we can't just set the two sets of equations equal to each other and solve. If the two lines intersect at a point, we may get to that point at two different times. Let's call these times t_1 and t_2 and solve the equations

$$\begin{cases} 2t_1+3=15-7t_2, \\ 3t_1+3=t_2-2, \text{ and} \\ 2t_1+1=3t_2-7. \end{cases}$$

Eliminate t_1 by subtracting the third equation from the first to get $t_2 = 2$. Substitute back into any of the equations to get $t_1 = -1$. Using either set of equations, you'll find that the point of intersection is $(1, 0, -1)$.

43. The way the problem is phrased tips us off that something is going on. Let's handle this the same way we did in Exercise 42.

$$\begin{cases} 2t_1+1=3t_2+1, \\ -3t_1=t_2+5, \text{ and} \\ t_1-1=7-t_2. \end{cases}$$

Adding the last two equations eliminates t_2 and gives us $t_1 = 13/2$. This corresponds to the point (14, $-39/2$, 11/2). Substituting this value of t_1 into the third equation gives us $t_2 = 3/2$, while substituting this into the first equation gives us $t_2 = 13/3$. This inconsistency tells us that the second line doesn't pass through the point $(14, -39/2, 11/2)$.

44. (a) The distance is $\sqrt{(3t-5+2)^2+(1-t-1)^2+(4t+7-5)^2}=\sqrt{26t^2-2t+13}$.

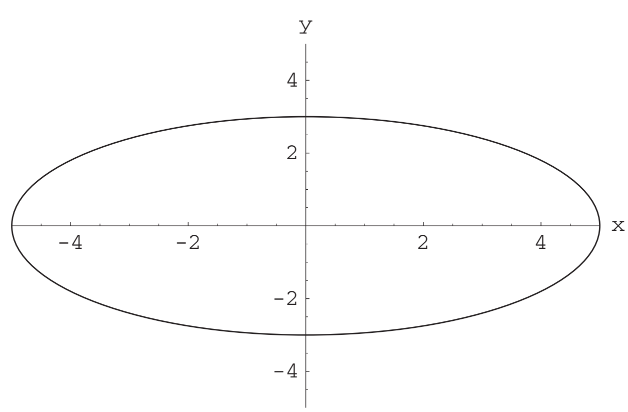
(b) Using a standard first year calculus trick, the distance is minimized when the square of the distance is minimized. So we find $D = 26t^2 - 2t + 13$ is minimized (at the vertex of the parabola) when $t = 1/26$. Substitute back into our answer for (a) to find that the minimal distance is $\sqrt{337/26}$.

45. (a) As in Example 2, this is the equation of a circle of radius 2 centered at the origin. The difference is that you are traveling around it three times as fast. This means that if t varied between 0 and 2π that the circle would be traced three times.

(b) This is just like part (a) except the radius of the circle is 5.

(c) This is just like part (b) except the x and y coordinates have been switched. This is the same as reflecting the circle about the line $y = x$ and so this is also a circle of radius 5. If you care, the circle in (b) was drawn starting at the point (5, 0) counterclockwise while this circle is drawn starting at (0, 5) clockwise.

(d) This is an ellipse with major axis along the x-axis intersecting it at $(\pm 5, 0)$ and minor axis along the y-axis intersecting it at $(0, \pm 3)$: $\frac{x^2}{25}+\frac{y^2}{9}=1$.



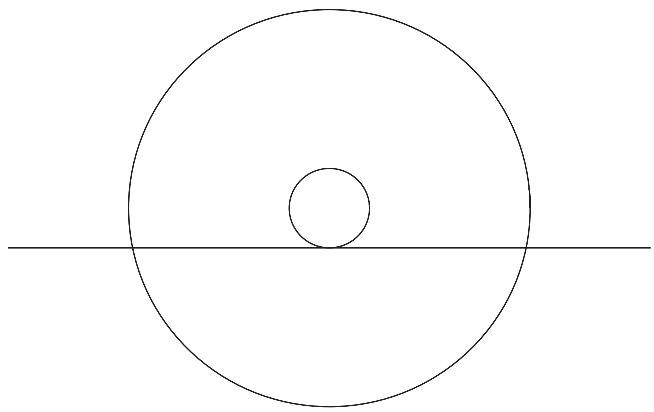
46. The discussion in the text of the cycloid looked at the path traced by a point on the circumference of a circle of radius a as it is rolled without slipping on the x-axis. The vector from the origin to our point P was split into two pieces: $\overrightarrow{OA}$ (the vector from the origin to the center of the circle) and $\overrightarrow{AP}$ (the vector from the center of the circle to P). This split remains the same in our problem.

The center of the circle is always a above the x-axis, and after the wheel has rolled through a central angle of t radians the x coordinate is just *at*. So $\overrightarrow{OA} = (at, a)$. This does not change in our problem.

The vector $\overrightarrow{AP}$ was calculated to be $(-a\sin t, -a\cos t)$. The direction of the vector is still correct but the length is not. If we are b units from the center then $\overrightarrow{AP} = -b(\sin t, \cos t)$.

We conclude then that the parametric equations are $x = at - b\sin t, y = a - b\cos t$. When $a = b$ this is the case of the cycloid described in the text; when $a > b$ we have the curtate cycloid; and when $a < b$ we have the prolate cycloid.

For a picture of how to generate one consider the diagram:



Here the inner circle is rolling along the ground and the prolate cycloid is the path traced by a point on the outer circle. There is a classic toy with a plastic wheel that runs along a handheld track, but your students are too young for that. You could pretend that the big circle is the end of a round roast and the little circle is the end of a skewer. In a regular rotisserie the roast would just rotate on the skewer, but we could imagine rolling the skewer along the edges of the grill. The motion of a point on the outside of the roast would be a prolate cycloid.

47. You are to picture that the circular dispenser stays still so Egbert has to unwind around the dispenser. The direction is $(\cos\theta, \sin\theta)$. The length is the radius of the circle a, plus the amount of tape that's been unwound. The tape that's been unwound is the distance around the circumference of the circle. This is $a\theta$ where θ is again in radians. The equation is therefore $(x, y) = a(1+\theta)(\cos\theta, \sin\theta)$.

1.3 The Dot Product

Exercises 1–16 are just straightforward calculations. For 1–6 use Definition 3.1 and formula (1). For 7–11 use formula (4). For 12–16 use formula (5).

1. $(1,5)\cdot(-2,3) = 1(-2)+5(3) = 13, \quad \|(1,5)\| = \sqrt{1^2+5^2} = \sqrt{26},$
$\|(-2,3)\| = \sqrt{(-2)^2+3^2} = \sqrt{13}.$

2. $(4,-1)\cdot(1/2,2) = 4(1/2)-1(2) = 0, \quad \|(4,-1)\| = \sqrt{4^2+(-1)^2} = \sqrt{17}$
$\|(1/2,2)\| = \sqrt{(1/2)^2+2^2} = \sqrt{17}/2.$

3. $(-1,0,7)\cdot(2,4,-6) = -1(2)+0(4)+7(-6) = -44, \quad \|(-1,0,7)\| = \sqrt{(-1)^2+0^2+7^2} = \sqrt{50} = 5\sqrt{2}$, and $\|(2,4,-6)\| = \sqrt{2^2+4^2+(-6)^2} = \sqrt{56} = 2\sqrt{14}.$

4. $(2,1,0)\cdot(1,-2,3) = 2(1)+1(-2)+0(3) = 0, \quad \|(2,1,0)\| = \sqrt{2^2+1} = \sqrt{5}$, and $\|(1,-2,3)\| = \sqrt{1^2+(-2)^2+3^2} = \sqrt{14}.$

5. $(4\mathbf{i}-3\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 4(1)+-3(1)+1(1) = 2, \quad \|4\mathbf{i}-3\mathbf{j}+\mathbf{k}\| = \sqrt{4^2+3^2+1^2} = \sqrt{26}$, and $\|\mathbf{i}+\mathbf{j}+\mathbf{k}\| = \sqrt{1+1+1} = \sqrt{3}.$

6. $(\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot(-3\mathbf{j}+2\mathbf{k}) = 2(-3)-1(2) = -8, \quad \|\mathbf{i}+2\mathbf{j}-\mathbf{k}\| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$, and $\|-3\mathbf{j}+2\mathbf{k}\| = \sqrt{(-3)^2+2^2} = \sqrt{13}.$

7. $\theta = \cos^{-1}\left(\dfrac{(\sqrt{3}\mathbf{i}+\mathbf{j})\cdot(-\sqrt{3}\mathbf{i}+\mathbf{j})}{\|(\sqrt{3}\mathbf{i}+\mathbf{j})\|\,\|-\sqrt{3}\mathbf{i}+\mathbf{j}\|}\right) = \cos^{-1}\left(\dfrac{-3+1}{(2)(2)}\right) = \cos^{-1}\left(\dfrac{-1}{2}\right) = \dfrac{2\pi}{3}.$

8. $\theta = \cos^{-1}\left(\dfrac{(-1,2)\cdot(3,1)}{\|(-1,2)\|\,\|(3,1)\|}\right) = \cos^{-1}\left(\dfrac{-3+2}{\sqrt{5}\sqrt{10}}\right) = \cos^{-1}\left(-\dfrac{1}{5\sqrt{2}}\right)$.

9. $\theta = \cos^{-1}\left(\dfrac{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\|\mathbf{i}+\mathbf{j}\|\,\|\mathbf{i}+\mathbf{j}+\mathbf{k}\|}\right) = \cos^{-1}\left(\dfrac{1+1}{\sqrt{2}\sqrt{3}}\right) = \cos^{-1}\left(\dfrac{\sqrt{2}}{\sqrt{3}}\right)$.

10. $\theta = \cos^{-1}\left(\dfrac{(\mathbf{i}+\mathbf{j}-\mathbf{k})\cdot(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})}{\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|\,\|-\mathbf{i}+2\mathbf{j}+2\mathbf{k}\|}\right) = \cos^{-1}\left(\dfrac{-1+2-2}{(\sqrt{3})(\sqrt{3})}\right) = \cos^{-1}\left(\dfrac{-1}{3\sqrt{3}}\right)$.

11. $\theta = \cos^{-1}\left(\dfrac{(1,-2,3)\cdot(3,-6,-5)}{\|(1,-2,3)\|\,\|(3,-6,-5)\|}\right) = \cos^{-1}\left(\dfrac{3+12-15}{\sqrt{14}\sqrt{70}}\right) = \cos^{-1}(0) = \dfrac{\pi}{2}$.

Note: The answers to 12 and 13 are the same. You may want to assign both exercises and ask your students why this should be true. You might then want to ask what would happen if vector **a** *was the same but vector* **b** *was divided by* $\sqrt{2}$.

12. $\text{proj}_{\mathbf{i}+\mathbf{j}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \left(\dfrac{(\mathbf{i}+\mathbf{j})\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{(\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}+\mathbf{j})}\right)(\mathbf{i}+\mathbf{j}) = \dfrac{2+3}{1+1}(1,1,0) = \left(\dfrac{5}{2},\dfrac{5}{2},0\right)$.

13. $\text{proj}_{\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}}(2\mathbf{i}+3\mathbf{j}-\mathbf{k}) = \left(\dfrac{\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)\cdot(2\mathbf{i}+3\mathbf{j}-\mathbf{k})}{\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)\cdot\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right)}\right)\left(\frac{\mathbf{i}+\mathbf{j}}{\sqrt{2}}\right) = \dfrac{\frac{1}{\sqrt{2}}(2+3)}{\frac{1+1}{2}}\dfrac{(1,1,0)}{\sqrt{2}} = \left(\dfrac{5}{2},\dfrac{5}{2},0\right)$.

14. $\text{proj}_{5\mathbf{k}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\dfrac{(5\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k})}{(5\mathbf{k})\cdot(5\mathbf{k})}\right)(5\mathbf{k}) = \dfrac{10}{25}(5\mathbf{k}) = 2\mathbf{k}$.

15. $\text{proj}_{-3\mathbf{k}}(\mathbf{i}-\mathbf{j}+2\mathbf{k}) = \left(\dfrac{(-3\mathbf{k})\cdot(\mathbf{i}-\mathbf{j}+2\mathbf{k})}{(-3\mathbf{k})\cdot(-3\mathbf{k})}\right)(-3\mathbf{k}) = \dfrac{-6}{9}(-3\mathbf{k}) = 2\mathbf{k}$.

16. $\text{proj}_{\mathbf{i}+\mathbf{j}+2\mathbf{k}}(2\mathbf{i}-4\mathbf{j}+\mathbf{k}) = \left(\dfrac{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(2\mathbf{i}-4\mathbf{j}+\mathbf{k})}{(\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+2\mathbf{k})}\right)(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = \dfrac{2-4+2}{1+1+4}(1,1,2) = 0$.

17. We just divide the vector by its length: $\dfrac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{||2\mathbf{i}-\mathbf{j}+\mathbf{k}||} = \dfrac{1}{\sqrt{6}}(2,-1,1)$.

18. Here we take the negative of the vector divided by its length: $\dfrac{\mathbf{i}-2\mathbf{k}}{\|\mathbf{i}-2\mathbf{k}\|} = \dfrac{1}{\sqrt{5}}(1,0,-2)$.

19. Same idea as Exercise 17, but multiply by 3: $\dfrac{3(\mathbf{i}+\mathbf{j}-\mathbf{k})}{\|\mathbf{i}+\mathbf{j}-\mathbf{k}\|} = \dfrac{3}{\sqrt{3}}(1,1,-1) = \sqrt{3}(1,1,-1)$.

20. There are a whole plane full of perpendicular vectors. The easiest three to find are when we set the coefficients of the coordinate vectors equal to zero in turn: $\mathbf{i}+\mathbf{j}$, $\mathbf{j}+\mathbf{k}$, and $-\mathbf{i}+\mathbf{k}$.

21. We have two cases to consider.
If either of the projections is zero: $\text{proj}_{\mathbf{a}}\mathbf{b} = \mathbf{0} \Leftrightarrow \mathbf{a}\cdot\mathbf{b} = 0 \Leftrightarrow \text{proj}_{\mathbf{b}}\mathbf{a} = \mathbf{0}$.
If neither of the projections is zero, then the directions must be the same. This means that **a** must be a multiple of **b**. Let $\mathbf{a} = c\mathbf{b}$, then on the one hand

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{c\mathbf{b}}\mathbf{b} = \frac{c\mathbf{b}\cdot\mathbf{b}}{c\mathbf{b}\cdot c\mathbf{b}}c\mathbf{b} = \mathbf{b}.$$

On the other hand

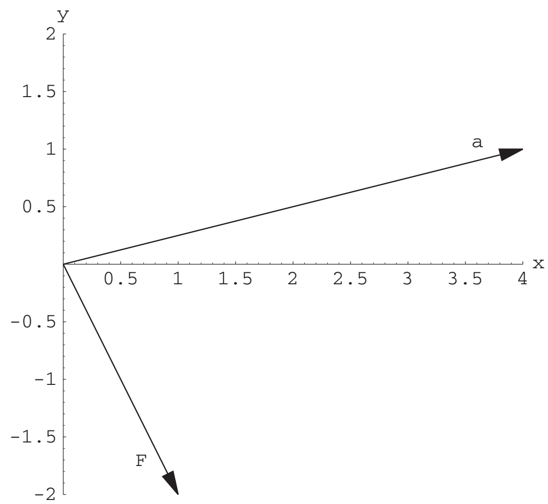
$$\text{proj}_{\mathbf{b}}\mathbf{a} = \text{proj}_{\mathbf{b}}c\mathbf{b} = \frac{\mathbf{b}\cdot c\mathbf{b}}{\mathbf{b}\cdot\mathbf{b}}\mathbf{b} = c\mathbf{b}.$$

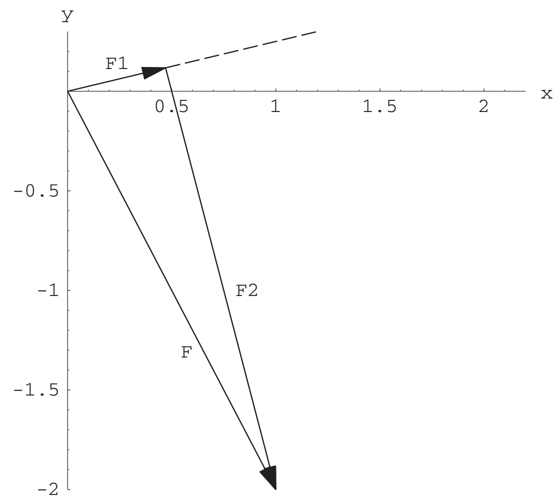
These are equal only when $c = 1$.
In other words, $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$ when $\mathbf{a}\cdot\mathbf{b} = 0$ or when $\mathbf{a} = \mathbf{b}$.

22. Property 2: $\mathbf{a}\cdot\mathbf{b} = (a_1,a_2,a_3)\cdot(b_1,b_2,b_3) = a_1b_1+a_2b_2+a_3b_3 = b_1a_1+b_2a_2+b_3a_3 = \mathbf{b}\cdot\mathbf{a}$.
Property 3: $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = (a_1,a_2,a_3)\cdot((b_1,b_2,b_3)+(c_1,c_2,c_3)) = (a_1,a_2,a_3)\cdot(b_1+c_1,b_2+c_2,b_3+c_3) = a_1(b_1+c_1)+a_2(b_2+c_2)+a_3(b_3+c_3) = (a_1b_1+a_2b_2+a_3b_3)+(a_1c_1+a_2c_2+a_3c_3) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.
Property 4: $(k\mathbf{a})\cdot b = (k(a_1,a_2,a_3))\cdot(b_1,b_2,b_3) = (ka_1,ka_2,ka_3)\cdot(b_1,b_2,b_3) = ka_1b_1+ka_2b_2+ka_3b_3$(*for the* 1^{st} *equality*) $= k(a_1b_1+a_2b_2+a_3b_3) = k(\mathbf{a}\cdot\mathbf{b})$. (*for the* 2^{nd} *equality*) $= a_1\,kb_1+a_2\,kb_2+a_3\,kb_3 = (a_1,a_2,a_3)\cdot(kb_1,kb_2,kb_3) = \mathbf{a}\cdot(k\mathbf{b})$.

23. We have $\|k\mathbf{a}\| = \sqrt{k\mathbf{a}\cdot k\mathbf{a}} = \sqrt{k^2(\mathbf{a}\cdot\mathbf{a})} = \sqrt{k^2}\sqrt{\mathbf{a}\cdot\mathbf{a}} = |k|\,\|\mathbf{a}\|$.

24. The following diagrams might be helpful:





To find $\mathbf{F}_1$, the component of $\mathbf{F}$ in the direction of $\mathbf{a}$, we project $\mathbf{F}$ onto $\mathbf{a}$:

$$\mathbf{F}_1 = \text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{(\mathbf{i}-2\mathbf{j})\cdot(4\mathbf{i}+\mathbf{j})}{(4\mathbf{i}+\mathbf{j})\cdot(4\mathbf{i}+\mathbf{j})}\right)(4\mathbf{i}+\mathbf{j}) = \frac{2}{17}(4,1).$$

To find $\mathbf{F}_2$, the component of $\mathbf{F}$ in the direction perpendicular to $\mathbf{a}$, we can just subtract $\mathbf{F}_1$ from $\mathbf{F}$:

$$\mathbf{F}_2 = (1,-2) - \frac{2}{17}(4,1) = \left(\frac{9}{17}, \frac{-36}{17}\right) = \frac{9}{17}(1,-4).$$

Note that $\mathbf{F}_1$ is a multiple of $\mathbf{a}$ so that $\mathbf{F}_1$ does point in the direction of $\mathbf{a}$ and that $\mathbf{F}_2 \cdot \mathbf{a} = 0$ so $\mathbf{F}_2$ is perpendicular to $\mathbf{a}$.

25. **(a)** The work done by the force is given to be the product of the length of the displacement ($\|\overrightarrow{PQ}\|$) and the component of force in the direction of the displacement ($\pm\|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|$ or in the case pictured in the text, $\|\mathbf{F}\|\cos\theta$). That is,

$$\text{Work} = \|\overrightarrow{PQ}\|\,\|\mathbf{F}\|\cos\theta = \mathbf{F}\cdot\overrightarrow{PQ}$$

using Theorem 3.3.

(b) The displacement vector is $\overrightarrow{PQ} = \mathbf{i}+\mathbf{j}-2\mathbf{k}$ and so, using part (a), we have

$$\text{Work} = \mathbf{F}\cdot\overrightarrow{PQ} = (\mathbf{i}+5\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}-2\mathbf{k}) = 1+5-4 = 2.$$

26. The amount of work is

$$\|\mathbf{F}\|\,\|\overrightarrow{PQ}\|\cos 20^\circ = 60\cdot 12\cdot\cos 20^\circ \approx 676.6 \text{ ft-lb}.$$

27. To move the bananas, one must exert an *upward* force of 500 lb. Such a force makes an angle of 60° with the ramp, and it is the ramp that gives the direction of displacement. Thus the amount of work done is

$$\|\mathbf{F}\|\,\|\overrightarrow{PQ}\|\cos 60^\circ = 500\cdot 40\cdot\tfrac{1}{2} = 10,000 \text{ ft-lb}.$$

28. Note that $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ each point along the positive x-, y-, and z-axes. Therefore, we may use Theorem 3.3 to calculate that

$$\cos\alpha = \frac{(\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot\mathbf{i}}{\|\mathbf{i}+2\mathbf{j}-\mathbf{k}\|(1)} = \frac{1}{\sqrt{6}};$$

$$\cos\beta = \frac{(\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot\mathbf{j}}{\|\mathbf{i}+2\mathbf{j}-\mathbf{k}\|(1)} = \frac{2}{\sqrt{6}};$$

$$\cos\gamma = \frac{(\mathbf{i}+2\mathbf{j}-\mathbf{k})\cdot\mathbf{k}}{\|\mathbf{i}+2\mathbf{j}-\mathbf{k}\|(1)} = -\frac{1}{\sqrt{6}}.$$

29. As in the previous problem, we use $\mathbf{a} = 3\mathbf{i} + 4\mathbf{k}$ to find that

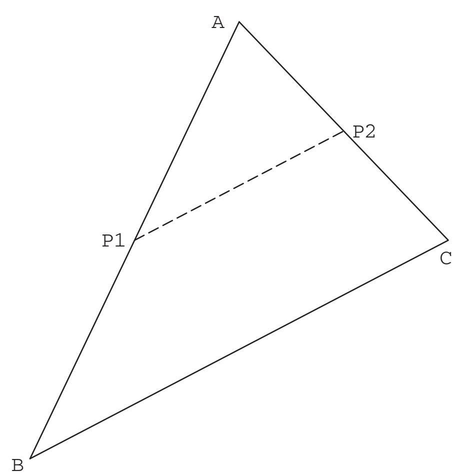
$$\cos\alpha = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{i}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = \frac{3}{5};$$

$$\cos\beta = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{j}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = 0;$$

$$\cos\gamma = \frac{(3\mathbf{i}+4\mathbf{k})\cdot\mathbf{k}}{\|3\mathbf{i}+4\mathbf{k}\|(1)} = \frac{4}{5}.$$

30. You could either use the three right triangles determined by the vector **a** and the three coordinate axes, or you could use Theorem 3.3. By that theorem, $\cos\alpha = \dfrac{\mathbf{a}\cdot\mathbf{i}}{\|\mathbf{a}\|\,\|\mathbf{i}\|} = \dfrac{a_1}{\sqrt{a_1^2+a_2^2+a_3^2}}$. Similarly, $\cos\beta = \dfrac{a_2}{\sqrt{a_1^2+a_2^2+a_3^2}}$ and $\cos\gamma = \dfrac{a_3}{\sqrt{a_1^2+a_2^2+a_3^2}}$.

31. Consider the figure:

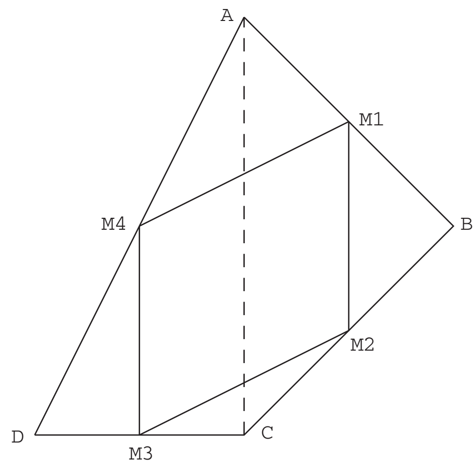


If P_1 is the point on $\overline{AB}$ located r times the distance from A to B, then the vector $\overrightarrow{AP_1} = r\overrightarrow{AB}$. Similarly, since P_2 is the point on $\overline{AC}$ located r times the distance from A to C, then the vector $\overrightarrow{AP_2} = r\overrightarrow{AC}$. So now we can look at the line segment $\overline{P_1P_2}$ using vectors.

$$\overrightarrow{P_1P_2} = \overrightarrow{AP_2} - \overrightarrow{AP_1} = r\overrightarrow{AC} - r\overrightarrow{AB} = r(\overrightarrow{AC} - \overrightarrow{AB}) = r\overrightarrow{BC}.$$

The two conclusions now follow. Because $\overrightarrow{P_1P_2}$ is a scalar multiple of $\overrightarrow{BC}$, they are parallel. Also the positive scalar r pulls out of the norm so $\|\overrightarrow{P_1P_2}\| = \|r\overrightarrow{BC}\| = r\|\overrightarrow{BC}\|$.

32. This now follows immediately from Exercise 31 or Example 6 from the text. Consider first the triangle *ABC*.



If M_1 is the midpoint of $\overline{AB}$ and M_2 is the midpoint of $\overline{BC}$, we've just shown that $\overline{M_1M_2}$ is parallel to $\overline{AC}$ and has half its length. Similarly, consider triangle *DAC* where M_3 is the midpoint of $\overline{CD}$ and M_4 is the midpoint of $\overline{DA}$. We see that $\overline{M_3M_4}$ is parallel to *AC* and has half its length. The first conclusion is that $\overline{M_1M_2}$ and $\overline{M_3M_4}$ have the same length and are parallel. Repeat this process for triangles *ABD* and *CBD* to conclude that $\overline{M_1M_4}$ and $\overline{M_2M_3}$ have the same length and are parallel. We conclude that $M_1M_2M_3M_4$ is a parallelogram. *For kicks—have your students draw the figure for ABCD a non-convex quadrilateral. The argument and the conclusion still hold even though one of the "diagonals" is not inside of the quadrilateral*.

33. In the diagram in the text, the diagonal running from the bottom left to the top right is $\mathbf{a}+\mathbf{b}$ and the diagonal running from the bottom right to the top left is $\mathbf{b}-\mathbf{a}$.

$$\begin{aligned}
\|\mathbf{a}+\mathbf{b}\| &= \|-\mathbf{a}+\mathbf{b}\| && \Leftrightarrow \\
\sqrt{(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})} &= \sqrt{(-\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b})} && \Leftrightarrow \\
\sqrt{\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}+2\mathbf{a}\cdot\mathbf{b}} &= \sqrt{(-1)^2\mathbf{a}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{b}} && \Leftrightarrow \\
\mathbf{a}\cdot\mathbf{b} &= 0
\end{aligned}$$

Since neither **a** nor **b** is zero, they must be orthogonal.

34. Using the same set up as that in Exercise 33, we note first that

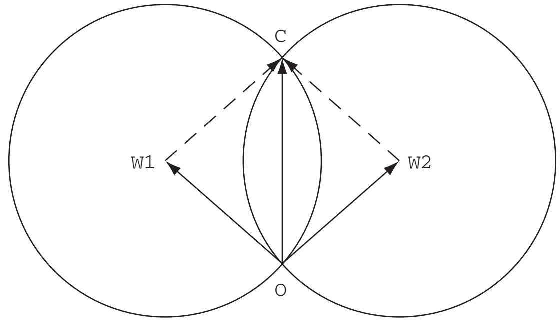
$$(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b}) = \mathbf{a}\cdot(-\mathbf{a})+\mathbf{b}\cdot(-\mathbf{a})+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b} = -\|\mathbf{a}\|^2+\|\mathbf{b}\|^2.$$

It follows immediately that

$$(\mathbf{a}+\mathbf{b})\cdot(-\mathbf{a}+\mathbf{b}) = 0 \Leftrightarrow \|\mathbf{a}\| = \|\mathbf{b}\|.$$

In other words that the diagonals of the parallelogram are perpendicular if and only if the parallelogram is a rhombus.

35. (a) Let's start with the two circles with centers at W_1 and W_2. Assume that in addition to their intersection at point O that they also intersect at point C as shown below.



The polygon OW_1CW_2 is a parallelogram. In fact, because all sides are equal, it is a rhombus. We can, therefore, write the vector $\mathbf{c} = \overrightarrow{OC} = \overrightarrow{OW}_1 + \overrightarrow{OW}_2 = \mathbf{w}_1 + \mathbf{w}_2$. Similarly, we can write $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_3$ and $\mathbf{a} = \mathbf{w}_2 + \mathbf{w}_3$.

(b) Let's use the results of part (a) together with the hint. We need to show that the distance from each of the points A, B, and C to P is r. Let's show, for example, that $\|\overrightarrow{CP}\|$ is r:

$$\|\overrightarrow{CP}\| = \|\overrightarrow{OP} - \overrightarrow{OC}\| = \|(\mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3) - (\mathbf{w}_1 + \mathbf{w}_2)\| = \|\mathbf{w}_3\| = r.$$

The arguments for the other two points are analogous.

(c) What we really need to show is that each of the lines passing through O and one of the points A, B, or C is perpendicular to the line containing the two other points. Using vectors we will show that $\overrightarrow{OA} \perp \overrightarrow{BC}$, $\overrightarrow{OB} \perp \overrightarrow{AC}$, and $\overrightarrow{OC} \perp \overrightarrow{AB}$ by showing their dot products are 0. It's enough to show this for one of them: $\overrightarrow{OA} \cdot \overrightarrow{BC} = (\mathbf{w}_2 + \mathbf{w}_3) \cdot ((\mathbf{w}_1 + \mathbf{w}_2) - (\mathbf{w}_1 + \mathbf{w}_3)) = (\mathbf{w}_2 + \mathbf{w}_3) \cdot (\mathbf{w}_2 - \mathbf{w}_3) = \mathbf{w}_2 \cdot \mathbf{w}_2 + \mathbf{w}_3 \cdot \mathbf{w}_2 - \mathbf{w}_2 \cdot \mathbf{w}_3 - \mathbf{w}_3 \cdot \mathbf{w}_3 = r^2 - r^2 = 0.$

36. (a) This follows immediately from Exercise 34 if you notice that the vectors are the diagonals of the rhombus with two sides $\|\mathbf{b}\|\mathbf{a}$ and $\|\mathbf{a}\|\mathbf{b}$.

Or we can proceed with the calculation: $(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|) \cdot (\|\mathbf{b}\|\mathbf{a} - \|\mathbf{a}\|\mathbf{b})$. The only bit of good news here is that the cross terms clearly cancel each other out and we're left with: $\|\mathbf{b}\|^2(\mathbf{a} \cdot \mathbf{a}) - \|\mathbf{a}\|^2(\mathbf{b} \cdot \mathbf{b}) = \|\mathbf{b}\|^2\|\mathbf{a}\|^2 - \|\mathbf{a}\|^2\|\mathbf{b}\|^2 = 0.$

(b) As in (a), the slicker way is to recall (or reprove geometrically) that the diagonals of a rhombus bisect the vertex angles. Then note that $(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})$ is the diagonal of the rhombus with sides $\|\mathbf{b}\|\mathbf{a}$ and $\|\mathbf{a}\|\mathbf{b}$ and so bisects the angle between them which is the same as the angle between $\mathbf{a}$ and $\mathbf{b}$.

Another way is to let θ_1 be the angle between $\mathbf{a}$ and $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$, and let θ_2 be the angle between $\mathbf{b}$ and $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$. Then

$$\cos^{-1}\theta_1 = \frac{\mathbf{a} \cdot (\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})}{(\|\mathbf{a}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{a}\|^2\|\mathbf{b}\| + \|\mathbf{a}\|\mathbf{a} \cdot \mathbf{b}}{(\|\mathbf{a}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{a}\|\,\|\mathbf{b}\| + \mathbf{a} \cdot \mathbf{b}}{\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|}.$$

Also

$$\cos^{-1}\theta_2 = \frac{\mathbf{b} \cdot (\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})}{(\|\mathbf{b}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\|\mathbf{b}\|\mathbf{b} \cdot \mathbf{a} + \|\mathbf{b}\|^2\|\mathbf{a}\|}{(\|\mathbf{b}\|)\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|} = \frac{\mathbf{b} \cdot \mathbf{a} + \|\mathbf{a}\|\,\|\mathbf{b}\|}{\|(\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b})\|}.$$

So $\|\mathbf{b}\|\mathbf{a} + \|\mathbf{a}\|\mathbf{b}$ bisects the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

1.4 The Cross Product

For Exercises 1–4 use Definition 4.2.

1. $(2)(3) - (4)(1) = 2.$

2. $(0)(6) - (5)(-1) = 5.$

3. $(1)(2)(3) + (3)(7)(-1) + (5)(0)(0) - (5)(2)(-1) - (1)(7)(0) - (3)(0)(3) = -5.$

4. $(-2)(6)(2) + (0)(-1)(4) + (1/2)(3)(-8) - (1/2)(6)(4) - (-2)(-1)(-8) - (0)(3)(2) = -32.$

Note: In Exercises 5–7, the difference between using (2) and (3) really amounts to changing the coefficient of $\mathbf{j}$ *from* $(a_3b_1 - a_1b_3)$ *in formula (2) to* $-(a_1b_3 - a_3b_1)$ *in formula (3). The details are only provided in Exercise 5.*

5. First we'll use formula (2):

$$\begin{aligned}(1,3,-2) \times (-1,5,7) &= [(3)(7) - (-2)(5)]\mathbf{i} + [(-2)(-1) - (1)(7)]\mathbf{j} + [(1)(5) - (3)(-1)]\mathbf{k} \\ &= 31\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = (31, -5, 8).\end{aligned}$$

If instead we use formula (3), we get:

$$\begin{aligned}(1,3,-2) \times (-1,5,7) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -1 & 5 & 7 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -2 \\ 5 & 7 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ -1 & 7 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ -1 & 5 \end{vmatrix}\mathbf{k} \\ &= 31\mathbf{i} - 5\mathbf{j} + 8\mathbf{k} = (31, -5, 8).\end{aligned}$$

6. Just using formula (3):

$$(3\mathbf{i}-2\mathbf{j}+\mathbf{k})\times(\mathbf{i}+\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{k}$$
$$= -3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} = (-3, -2, 5).$$

7. Note that these two vectors form a basis for the xy-plane so the cross product will be a vector parallel to (0, 0, 1). Again, just using formula (3):

$$(\mathbf{i}+\mathbf{j})\times(-3\mathbf{i}+2\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ -3 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ -3 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ -3 & 2 \end{vmatrix}\mathbf{k} = 5\mathbf{k} = (0, 0, 5).$$

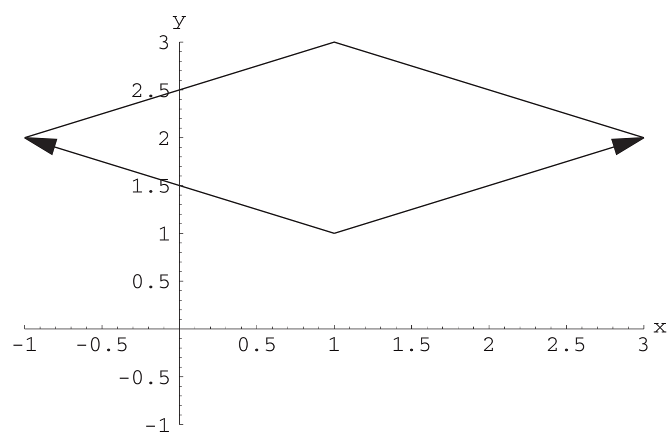
8. By (1) $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}+\mathbf{b})$.
By (2), this $= -\mathbf{c}\times\mathbf{a} + -\mathbf{c}\times\mathbf{b}$.
By (1), this $= \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$.

9. $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = (\mathbf{a}\times\mathbf{a}) + (\mathbf{b}\times\mathbf{a}) - (\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{b})$. The cross product of a vector with itself is $\mathbf{0}$ and also $(\mathbf{b}\times\mathbf{a}) = -(\mathbf{a}\times\mathbf{b})$, so

$$(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = -2(\mathbf{a}\times\mathbf{b}).$$

You may wish to have your students consider what this means about the relationship between the cross product of the sides of a parallelogram and the cross product of its diagonals. In any case, we are given that $\mathbf{a}\times\mathbf{b} = (3, -7, -2)$, so $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = (-6, 14, 4)$.

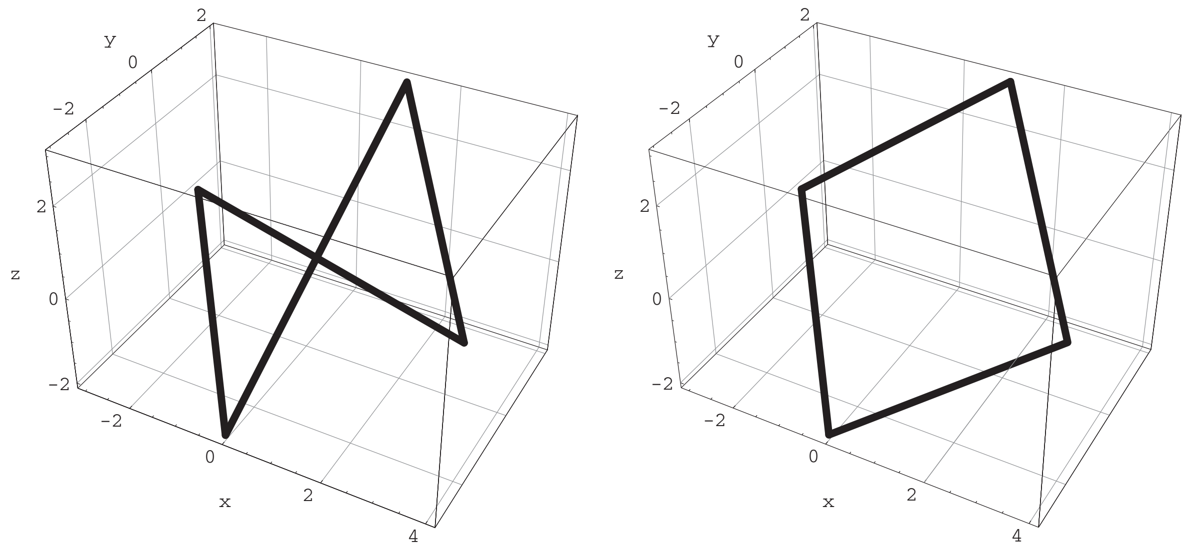
10. If you plot the points you'll see that they are given in a counterclockwise order of the vertices of a parallelogram. To find the area we will view the sides from (1, 1) to (3, 2) and from (1, 1) to $(-1, 2)$ as vectors by calculating the displacement vectors: $(3, 2) - (1, 1)$ and $(-1, 2) - (1, 1)$. We then embed the problem in $\mathbf{R}^3$ and take a cross product. The length of this cross product is the area of the parallelogram.



$$(3-1, 2-1, 0)\times(-1-1, 2-1, 0) = (2, 1, 0)\times(-2, 1, 0) = 4\mathbf{k} = (0, 0, 4).$$

So the area is $\|(0, 0, 4)\| = 4$.

11. This is tricky, as the points are not given in order. The figure on the left shows the sides connected in the order that the points are given.



As the figure on the right shows, if you take the first side to be the side that joins the points (1, 2, 3) and $(4, -2, 1)$ then the next side is the side that joins $(4, -2, 1)$ and $(0, -3, -2)$. We will again calculate the length of the cross product of the displacement vectors. So the area of the parallelogram will be the length of

$$(0-4, -3-(-2), -2-1) \times (1-4, 2-(-2), 3-1) = (-4, -1, -3) \times (-3, 4, 2) = (10, 17, -19).$$

The length of $(10, 17, -19)$ is $\sqrt{10^2 + 17^2 + (-19)^2} = \sqrt{750} = 5\sqrt{30}$.

12. The cross product will give us the right direction; if we then divide this result by its length we will get a unit vector:

$$\frac{(2, 1, -3) \times (1, 0, 1)}{\|(2, 1, -3) \times (1, 0, 1)\|} = \frac{(1, -5, -1)}{\|(1, -5, -1)\|} = \frac{1}{\sqrt{27}}(1, -5, -1).$$

13. For $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ to be zero either

- One or more of the three vectors is $\mathbf{0}$,
- $(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$ which would happen if $\mathbf{a} = k\mathbf{b}$ for some real k, or
- $\mathbf{c}$ is in the plane determined by $\mathbf{a}$ and $\mathbf{b}$.

For Exercises 14–17 we'll just take half of the length of the cross product. Unlike Exercises 10 and 11, in Exercises 16 and 17 we don't have to worry about the ordering of the points. In a triangle, whichever order we choose we are traveling either clockwise or counterclockwise. Just choose any of the vertices as the base for the cross product. Our choices may differ, but the solution won't.

14. $(1/2)\|(1, 1, 0) \times (2, -1, 0)\| = (1/2)\|(0, 0, -3)\| = 3/2$.

15. $(1/2)\|(1, -2, 6) \times (4, 3, -1)\| = (1/2)\|(-16, 25, 11)\| = \sqrt{1002}/2$.

16. $(1/2)\|(-1-1, 2-1, 0) \times (-2-1, -1-1, 0)\| = (1/2)\|(-2, 1, 0) \times (-3, -2, 0)\| =$
$(1/2)\|(0, 0, 7)\| = 7/2$.

17. $(1/2)\|(0-1, 2, 3-1) \times (-1-1, 5, -2-1)\| = (1/2)\|(-1, 2, 2) \times (-2, 5, -3)\| =$
$(1/2)\|(-16, -7, -1)\| = \sqrt{306}/2 = 3\sqrt{34}/2$.

The triple scalar product is used in Exercises 18 and 19 and the equivalent determinant form mentioned in the text is proved in Exercise 20.

Some people write this product as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ *instead of* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. *Exercise 28 shows that these are equivalent.*

18. Here we are given the vectors so we can just use the triple scalar product:

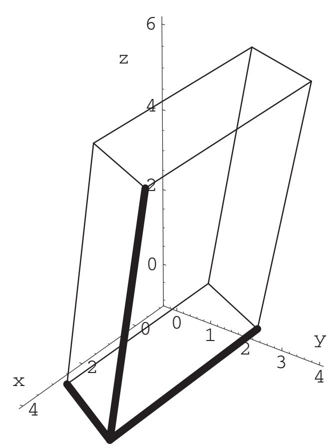
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((3\mathbf{i}-\mathbf{j})\times(-2\mathbf{i}+\mathbf{k}))\cdot(\mathbf{i}-2\mathbf{j}+4\mathbf{k}) = \begin{vmatrix} 3 & -1 & 0 \\ -2 & 0 & 1 \\ 1 & -2 & 4 \end{vmatrix} \\ &= 3\begin{vmatrix} 0 & 1 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 1 \\ 1 & 4 \end{vmatrix} + 0\begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} = 3(2) + (-9) = -3.\end{aligned}$$

Volume $= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = 3$.

19. You need to figure out a useful ordering of the vertices. You can either plot them by hand or use a computer package to help or you can make some observations about them. First look at the z coordinates. Two points have $z = -1$ and two have $z = 0$. These form your bottom face. Of the remaining points two have $z = 5$—these will match up with the bottom points with $z = -1$, and two have $z = 6$—these will match up with the bottom points with $z = 0$. The parallelepiped is shown below.

We'll use the highlighted edges as our three vectors **a**, **b**, and **c**. You could have based the calculation at any vertex. I have chosen $(4, 2, -1)$. The three vectors are:

$$\begin{aligned}\mathbf{a} &= (0,3,0) - (4,2,-1) = (-4,1,1) \\ \mathbf{b} &= (4,3,5) - (4,2,-1) = (0,1,6) \\ \mathbf{c} &= (3,0,-1) - (4,2,-1) = (-1,-2,0)\end{aligned}$$



We can now calculate

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= ((-4,1,1)\times(0,1,6))\cdot(-1,-2,0) = \begin{vmatrix} -4 & 1 & 1 \\ 0 & 1 & 6 \\ -1 & -2 & 0 \end{vmatrix} \\ &= -4\begin{vmatrix} 1 & 6 \\ -2 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & 6 \\ -1 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix} = -4(12) - (6) + (1) = -53.\end{aligned}$$

Finally, Volume $= |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = 53$.

Note: The proofs of Exercises 20 and 28 are easier if you remember that if matrix A is just matrix B with any two rows interchanged then the determinant of A is the negative of the determinant of B. If you don't use this fact (which is explored in exercises later in this chapter), you can prove this with a long computation. That is why the author of the text suggests that a computer algebra system could be helpful—and this would be a great place to use it in a class demonstration.

20. This is not as bad as it might first appear.

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \left(\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right)\cdot(c_1,c_2,c_3)$$

$$= \left(\mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)\cdot(c_1,c_2,c_3)$$

$$= c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

21. $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ by Exercise 20. Similarly, $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$ by Exercise 20. Expand these determinants to see that they are equal.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

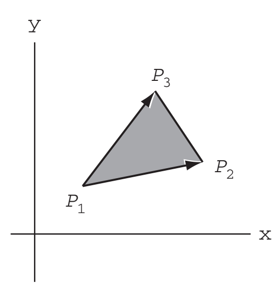
$$\begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = b_1(a_3c_2 - a_2c_3) - b_2(a_3c_1 - a_1c_3) + b_3(a_2c_1 - a_1c_2)$$

22. The value of $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$ is the volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. But so is $|\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c})|$, so the quantities must be equal.

23. **(a)** We have

$$\text{Area} = \tfrac{1}{2}\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\|$$
$$= \tfrac{1}{2}\|(x_2 - x_1, y_2 - y_1, 0)\times(x_3 - x_1, y_3 - y_1, 0)\|$$

$$\text{Now } \overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix}$$
$$= [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\mathbf{k}$$



Hence the area is $\frac{1}{2}|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)|$. On the other hand

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \frac{1}{2}\left(\begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}\right).$$

Expanding and taking absolute value, we obtain

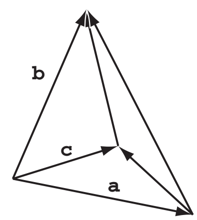
$$\tfrac{1}{2}|x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1|.$$

From here, its easy to see that this agrees with the formula above.

(b) We compute the absolute value of $\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -4 \\ 2 & 3 & -4 \end{vmatrix} = \frac{1}{2}(-8-8+3-4+12+4) = \frac{1}{2}(-1) = -\frac{1}{2}$.

Thus the area is $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

24. Surface area $= \frac{1}{2}(\|\mathbf{a}\times\mathbf{b}\| + \|\mathbf{b}\times\mathbf{c}\| + \|\mathbf{a}\times\mathbf{c}\| + \|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\|)$



25. We assume that **a**, **b**, and **c** are non-zero vectors in $\mathbf{R}^3$.

(a) The cross product $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**.

(b) Scale the cross product to a unit vector by dividing by the length and then multiply by 2 to get $2\left(\dfrac{\mathbf{a}\times\mathbf{b}}{\|\mathbf{a}\times\mathbf{b}\|}\right)$.

(c) $\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$.

(d) Here we divide vector **a** by its length and multiply it by the length of **b** to get $\left(\dfrac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\right)\mathbf{a}$.

(e) The cross product of two vectors is orthogonal to each: $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

(f) A vector perpendicular to $\mathbf{a}\times\mathbf{b}$ will be back in the plane determined by **a** and **b**, so our answer is $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

26. *I love this problem—students tend to go ahead and calculate without thinking through what they're doing first. This would make a great quiz at the beginning of class.*

(a) Vector: The cross product of the vectors **a** and **b** is a vector so you can take its cross product with vector **c**.

(b) Nonsense: The dot product of the vectors **a** and **b** is a scalar so you can't dot it with a vector.

(c) Nonsense: The dot products result in scalars and you can't find the cross product of two scalars.

(d) Scalar: The cross product of the vectors **a** and **b** is a vector so you can take its dot product with vector **c**.

(e) Nonsense: The cross product of the vectors **a** and **b** is a vector so you can take its cross product with vector that is the result of the cross product of **c** and **d**.

(f) Vector: The dot product results in a scalar that is then multiplied by vector **d**. We can evaluate the cross product of vector **a** with this result.

(g) Scalar: We are taking the dot product of two vectors.

(h) Vector: You are subtracting two vectors.

Note: You can have your students use a computer algebra system for these as suggested in the text. I've included worked out solutions for those as old fashioned as I am.

27. Exercise 25(f) shows us that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is in the plane determined by **a** and **b** and so we expect the solution to be of the form $k_1\mathbf{a} + k_2\mathbf{b}$ for scalars k_1 and k_2.

Using formula (3) from the text for $\mathbf{a}\times\mathbf{b}$:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \left(-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_3 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_2\right)\mathbf{i} - \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_3 - \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_1\right)\mathbf{j} \\ &\quad + \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_1\right)\mathbf{k}\end{aligned}$$

Look first at the coefficient of **i**: $-a_1b_3c_3 + a_3b_1c_3 - a_1b_2c_2 + a_2b_1c_2$. If we add and subtract $a_1b_1c_1$ and regroup we have: $b_1(a_1c_1 + a_2c_2 + a_3c_3) - a_1(b_1c_1 + b_2c_2 + b_3c_3) = b_1(\mathbf{a}\cdot\mathbf{c}) - a_1(\mathbf{b}\cdot\mathbf{c})$. Similarly for the coefficient of **j**. Expand then add and subtract $a_2b_2b_3$ and regroup to get $b_2(\mathbf{a}\cdot\mathbf{c}) - a_2(\mathbf{b}\cdot\mathbf{c})$. Finally for the coefficient of **k**, expand then add and subtract $a_3b_3c_3$ and regroup to obtain $b_3(\mathbf{a}\cdot\mathbf{c}) - a_3(\mathbf{b}\cdot\mathbf{c})$. This shows that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}$.

Now here's a version of Exercise 27 worked on *Mathematica*. First you enter the following to define the vectors **a**, **b**, and **c**.

$$a = \{a1, a2, a3\}$$
$$b = \{b1, b2, b3\}$$
$$c = \{c1, c2, c3\}$$

The reply from *Mathematica* is an echo of your input for **c**. Let's begin by calculating the cross product. You can either select the cross product operator from the typesetting palette or you can type the escape key followed by "cross" followed by the escape key. *Mathematica* should reformat this key sequence as $\times$ and you should be able to enter

$$(a \times b) \times c.$$

Mathematica will respond with the calculated cross product

$$\begin{aligned} &\{a2b1c2 - a1b2c2 + a3b1c3 - a1b3c3, \\ &-a2b1c1 + a1b2c1 + a3b2c3 - a2b3c3, \\ &-a3b1c1 + a1b3c1 - a3b2c2 + a2b3c2\}. \end{aligned}$$

Now you can check the other expression. Use a period for the dot in the dot product.

$$(a.c)b - (b.c)\, a$$

Mathematica will immediately respond

$$\begin{aligned} &\{b1(a1c1 + a2c2 + a3c3) - a1(b1c1 + b2c2 + b3c3), \\ &b2(a1c1 + a2c2 + a3c3) - a2(b1c1 + b2c2 + b3c3), \\ &b3(a1c1 + a2c2 + a3c3) - a3(b1c1 + b2c2 + b3c3)\} \end{aligned}$$

This certainly looks different from the previous expression. Before giving up hope, note that this one has been factored and the earlier one has not. You can expand this by using the command

$$\text{Expand}[(a.c)b - (b.c)a]$$

or use *Mathematica*'s command % to refer to the previous entry and just type

$$\text{Expand}[\%].$$

This still might not look familiar. So take a look at

$$(a \times b) \times c - [(a.c)b - (b.c)a].$$

If this *still* isn't what you are looking for, simplify it with the command

$$\text{Simplify}[\%]$$

and *Mathematica* will respond

$$\{0, 0, 0\}.$$

28. The exercise asks us to show that six quantities are equal.

The most important pair is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$. Because of the commutative property of the dot product $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ and so we are showing that $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

$$\begin{aligned} \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}). \end{aligned}$$

The determinants of the 3 by 3 matrices above are equal because we had to interchange two rows twice to get from one to the other. This fact has not yet been presented in the text. This would be an excellent time to use a computer algebra system to show the two determinants are equal. Of course, you could use *Mathematica* or some other such system to do the entire problem.

To show that $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ we use a similar approach:

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}&=\begin{vmatrix} b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\end{vmatrix}\\ &=\begin{vmatrix} c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\\ &=(\mathbf{c}\times\mathbf{a})\cdot\mathbf{b}=\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}).\end{aligned}$$

So we've established that the first three triple scalars are equal.
We get the rest almost for free by noticing that three pairs of equations are trivial:

$$\begin{aligned}&\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b}),\\ &\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=-\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}),\ \text{and}\\ &\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})=-\mathbf{c}\cdot(\mathbf{b}\times\mathbf{a}).\end{aligned}$$

Each of the above pairs are equal by the anticommutativity property of the cross product. If you prefer the matrix approach, this also follows from the fact that interchanging two rows changes the sign of the determinant.

29. By Exercise 28, $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=\mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b}))$.
By anticommutativity, $\mathbf{c}\cdot(\mathbf{d}\times(\mathbf{a}\times\mathbf{b}))=-\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d})$.
By Exercise 27, $-\mathbf{c}\cdot((\mathbf{a}\times\mathbf{b})\times\mathbf{d})=-\mathbf{c}\cdot((\mathbf{a}\cdot\mathbf{d})\mathbf{b}-(\mathbf{b}\cdot\mathbf{d})\mathbf{a})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$.

30. Apply the results of Exercise 27 to each of the three components:

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\times\mathbf{c}+(\mathbf{b}\times\mathbf{c})\times\mathbf{a}+(\mathbf{c}\times\mathbf{a})\times\mathbf{b}&=[(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}]+[(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}]\\ &\quad+[(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}]=\mathbf{0}.\end{aligned}$$

(For example, the $(\mathbf{a}\cdot\mathbf{c})\mathbf{b}$ cancels with the $(\mathbf{c}\cdot\mathbf{a})\mathbf{b}$ because of the commutative property for the dot product.)

31. If your students are using a computer algebra system, they may not notice that this is *exactly* the same problem as Exercise 27. Just replace **c** with $(\mathbf{c}\times\mathbf{d})$ on both sides of the equation in Exercise 27 to obtain the result here.

32. First apply Exercise 29 to the dot product to get

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})=[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})][\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})]-[\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})][\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})].$$

You can either observe that two of these quantities must be 0, or you can apply Exercise 28 to see $\mathbf{a}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{c}\cdot(\mathbf{a}\times\mathbf{a})=0$. Exercise 28 also shows that $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$. The result follows.

33. We did this above in Exercise 29.

34. The amount of torque is the product of the length of the "wrench" and the component of the force perpendicular to the "wrench". In this case, the wrench is the door—so the length is four feet. The 20 lb force is applied perpendicular to the plane of the doorway and the door is open $45°$. So from the text, the amount of torque $=\|\mathbf{a}\|\|\mathbf{F}\|\sin\theta=(4)(20)(\sqrt{2}/2)=40\sqrt{2}$ ft-lb.

35. **(a)** Here the length of **a** is 1 foot, the force $\mathbf{F}=40$ pounds and angle $\theta=120$ degrees. So

$$\textbf{Torque}=(1)(40)\sin 120^\circ=40\left(\frac{\sqrt{3}}{2}\right)=20\sqrt{3}\ \text{foot-pounds.}$$

(b) Here all that has changed is that $\|\mathbf{a}\|$ is 1.5 feet, so

$$\textbf{Torque}=(3/2)(40)\sin 120^\circ=60\left(\frac{\sqrt{3}}{2}\right)=30\sqrt{3}\ \text{foot-pounds.}$$

36. $\mathbf{a} = 2$ in but torque is measured in foot-pounds so $\|\mathbf{a}\| = (1/6)$ ft.

$$\textbf{Torque} = \mathbf{a} \times \mathbf{F} = \left(\frac{1}{6}, 0, 0\right) \times (0, 15, 0) = \left(0, 0, \frac{5}{2}\right).$$

So Egbert is using 5/2 foot-pounds straight up.

37. From the figure

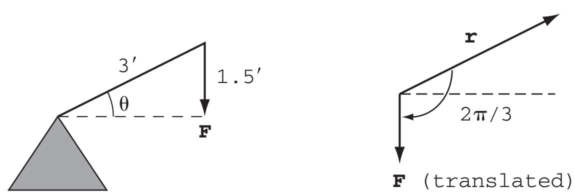
$$\sin\theta = \frac{1.5}{3} = \frac{1}{2}$$
$$\Rightarrow \theta = \pi/6.$$

This is the angle the seesaw makes with horizontal. The angle we want is

$$\pi/6 + \pi/2 = 2\pi/3.$$

Since $\|\mathbf{r}\| = 3$ and $\|\mathbf{F}\| = 50$, the amount of torque is

$$\|\mathbf{T}\| = \|\mathbf{r} \times \mathbf{F}\| = \|\mathbf{r}\|\,\|\mathbf{F}\| \sin\frac{2\pi}{3}$$
$$= 3 \cdot 50 \cdot \frac{\sqrt{3}}{2} = 75\sqrt{3} \text{ ft-lb}$$



38. **(a)** The linear velocity is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ so that

$$\|\mathbf{v}\| = \|\boldsymbol{\omega}\|\,\|\mathbf{r}\| \sin\theta.$$

We have that the angular speed is $\frac{2\pi \text{ radians}}{24 \text{ hrs}} = \frac{\pi}{12}$ radians/hr (this is $\|\boldsymbol{\omega}\|$.) Also $\|\mathbf{r}\| = 3960$, so at $45°$ North latitude, $\|\mathbf{v}\| = \frac{\pi}{12} \cdot 3960 \cdot \sin 45° = \frac{330\pi}{\sqrt{2}} \approx 733.08$ mph.

(b) Here the only change is that $\theta = 90°$. Thus $\|\mathbf{v}\| = \frac{\pi}{2} \cdot 3960 \cdot \sin 90° = 330\pi \approx 1036.73$ mph.

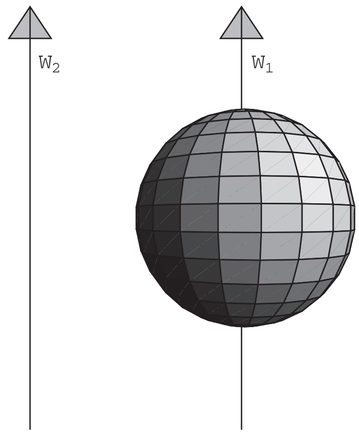
39. Archie's actual experience isn't important in solving this problem; he could have ridden closer to the center. Since we are only interested in comparing Archie's experience with Annie's, it turns out that their difference would be the same so long as the difference in their distance from the center remained at 2 inches. The difference in speed is $(331/3)(2\pi)(6) - (331/3)(2\pi)(4) = (331/3)(2\pi)(2) = 4\pi(331/3) = 1331/3\pi = 400\pi/3$ in/ min.

40. **(a)** $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = (0, 0, 12) \times (2, -1, 3) = (12, 24, 0) = 12\mathbf{i} + 24\mathbf{j}$.

(b) The height of the point doesn't change so we can view this as if it were a problem in $\mathbf{R}^2$. When $x = 2$ and $y = -1$, we can find the central angle by taking $\tan^{-1}(-1/2)$. In one second the angle has moved 12 radians so the new point is

$$(\sqrt{5}\cos(\tan^{-1}(-1/2) + 12), \sqrt{5}\sin(\tan^{-1}(-1/2) + 12), 3) \approx (1.15, -1.92, 3).$$

41. Consider the rotations of a sphere about each of the two parallel axes pictured below.



Assume the two corresponding angular velocity vectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ (denoted w_1 and w_2 in the diagram) are "parallel" and even have the same magnitude. Let them both point straight up (parallel to (0, 0, 1)) with magnitude 2π radians per second. The idea is that as "free vectors" $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are both equal to $(0, 0, 2\pi)$. but that the corresponding rotational motions are very different.

In the case of $\boldsymbol{\omega}_1$, each second every point on the sphere has made a complete orbit around the axis. The corresponding motion is that the sphere is rotating about this axis. (More concretely, take your *Vector Calculus* book and stand it up on its end. Imagine an axis anywhere and spin it around that axis at a constant speed.)

In the case of $\boldsymbol{\omega}_2$, each second every point on the sphere has made a complete orbit around the axis. In this case that means that the corresponding motion is that the sphere is orbiting about this axis. (Hold your *Vector Calculus* book at arms length and you spin around your axis.)

1.5 Equations for Planes; Distance Problems

1. This is a straightforward application of formulas (1) and (2):

$$1(x-3) - (y+1) + 2(z-2) = 0 \quad \Longleftrightarrow \quad x - y + 2z = 8.$$

2. Again we apply formula (2):

$$(x-9) - 2(z+1) = 0 \quad \Longleftrightarrow \quad x - 2z = 11.$$

So what happened to the y term? The equation is independent of y. In the $x - z$ plane draw the line $x - 2z = 11$ and then the plane is generated by "dragging" the line either way in the y direction.

3. We first need to find a vector normal to the plane, so we take the cross product of two displacement vectors:

$$(3-2, -1-0, 2-5) \times (1-2, -2-0, 4-5) = (1, -1, -3) \times (-1, -2, -1) = (-5, 4, -3).$$

Now we can apply formula (2) using any of the three points:

$$-5(x-3) + 4(y+1) - 3(z-2) = 0 \quad \Longleftrightarrow \quad -5x + 4y - 3z = -25.$$

4. We'll again find the cross product of two displacement vectors:

$$(A, -B, 0) \times (0, -B, C) = (-BC, -AC, -AB).$$

Now we apply formula (2):

$$-BC(x-A) - AC(y) - AB(z) = 0 \quad \Longleftrightarrow \quad BCx + ACy + ABz = ABC.$$

5. If the planes are parallel, then a vector normal to one is normal to the other. In this case the normal vector is $\mathbf{n} = (5, -4, 1)$. So using formula (2) we get:

$$5(x-2) - 4(y+1) + (z+2) = 0 \quad \Longleftrightarrow \quad 5x - 4y + z = 12.$$

6. The plane must have a normal vector parallel to the normal $\mathbf{n} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ of the given plane; therefore, the vector $\mathbf{n}$ may also be taken to be the normal to the desired plane. Hence an equation is

$$2(x+1) - 3(y-1) + 1(z-2) = 0 \iff 2x - 3y + z = -3.$$

7. We may take the normal to the plane to be the same as a normal to the given plane; thus we may let $\mathbf{n} = \mathbf{i} - \mathbf{j} + 7\mathbf{k}$. Hence an equation for the desired plane is

$$1(x+2) - 1(y-0) + 7(z-1) = 0 \iff x - y + 7z = 5.$$

8. We may take the normal to the desired plane to be $\mathbf{n} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. Therefore, the equation of the plane must be of the form $2x + 2y + z = D$ for some constant D. For the plane to contain the given line, *every* point on the line must satisfy the equation for the plane. Thus for *all* $t \in \mathbf{R}$ we must have

$$2(2-t) + 2(2t+1) + (3-2t) = D$$

$$\iff 4 - 2t + 4t + 2 + 3 - 2t = D$$
$$\iff 9 = D.$$

Hence the desired equation is $2x + 2y + z = 9$.

9. Any plane parallel to $5x - 3y + 2z = 10$ can be written in the form $5x - 3y + 2z = D$ for some constant D. For this plane to contain the given line, it must be the case that for *all* $t \in \mathbf{R}$ we have

$$5(t+4) - 3(3t-2) + 2(5-2t) = D$$

$$\iff 5t + 20 - 9t + 6 + 10 - 4t = D$$
$$\iff 36 - 8t = D \iff 8t = 36 - D.$$

However, there is no *constant* value for D for which $8t = 36 - D$ for *all* $t \in \mathbf{R}$. Hence the given line will *intersect* each plane parallel to $5x - 3y + 2z = 10$, but it will never be completely contained in any of them.

10. The plane contains the line $\mathbf{r}(t) = (-1, 4, 7) + (2, 3, -1)t$ and the point (2, 5, 0). Choose two points on the line, for example $(-1, 4, 7)$ and (13, 25, 0) and proceed as in Exercises 3 and 4.

$$\begin{aligned}(-1-2, 4-5, 7-0) \times (13-2, 25-5, 0) = (-3, -1, 7) \times (11, 20, 0) &= (-140, 77, -49)\\ &= 7(-20, 11, -7).\end{aligned}$$

We are just looking for the plane perpendicular to this vector so we can ignore the scalar 7.

$$-20(x-2) + 11(y-5) - 7(z) = 0 \iff -20x + 11y - 7z = 15.$$

11. The only relevant information contained in the equation of the line $\mathbf{r}(t) = (-5, 4, 7) + (3, -2, -1)t$ is the vector coefficient of t. This is the normal vector $\mathbf{n} = (3, -2, -1)$.

$$3(x-1) - 2(y+1) - (z-2) = 0 \iff 3x - 2y - z = 3.$$

12. We have two lines given by the vector equations:

$$\mathbf{r}_1(t) = (2, -5, 1) + (1, 3, 5)t$$
$$\mathbf{r}_2(t) = (5, -10, 9) + (-1, 3, -2)t$$

The vector $(1, 3, 5) \times (-1, 3, -2) = (-21, -3, 6) = -3(7, 1, -2)$ is orthogonal to both lines. So the equation of the plane containing both lines is:

$$7(x-2) + y + 5 - 2(z-1) = 0 \iff 7x + y - 2z = 7.$$

13. The line shared by two planes will be orthogonal each of their normal vectors. First, calculate: $(1, 2, -3) \times (5, 5, -1) = (13, -14, -5)$. Now find a point on the line by setting $z = 0$ and solving the two equations

$$\begin{cases} x + 2y = 5 \\ 5x + 5y = 1 \end{cases}$$

to get $x = -23/5$ and $y = 24/5$. The equation of the line is $\mathbf{r}(t) = (-23/5, 24/5, 0) + (13, -14, -5)t$, or in parametric form:

$$\begin{cases} x &= 13t - \frac{23}{5} \\ y &= -14t + \frac{24}{5} \\ z &= -5t. \end{cases}$$

14. The normal to the plane is $\mathbf{n} = (2, -3, 5)$ and the line passes through the point $P = (5, 0, 6)$. The equation of the line

$$\mathbf{r}(t) = P + \mathbf{n}t = (5, 0, 6) + (2, -3, 5)\,t.$$

In parametric form this is:

$$\begin{cases} x = 2t + 5 \\ y = -3t \\ z = 5t + 6. \end{cases}$$

15. The easiest way to solve this is to check that the vector from the coefficients of the first equation $(8, -6, 9A)$ is a multiple of the coefficients of the second equation (A, 1, 2). In this case the first is -6 times the second. This means that $8 = -6A$ or $A = -4/3$. Checking we see this is confirmed by $9A = -6(2)$.

16. For perpendicular planes we check that $0 = (A, -1, 1) \cdot (3A, A, -2)$. This yields the quadratic $0 = 3A^2 - A - 2 = (3A + 2)(A - 1)$. The two solutions are $A = -2/3$ and $A = 1$.

17. This is a direct application of formula (10):

$$\mathbf{x}(s, t) = s\mathbf{a} + t\mathbf{b} + \mathbf{c} = s(2, -3, 1) + t(1, 0, -5) + (-1, 2, 7).$$

In parametric form this is:

$$\begin{cases} x = 2s + t - 1 \\ y = -3s + 2 \\ z = s - 5t + 7 \end{cases}$$

18. Again this follows from formula (10):

$$\mathbf{x}(s, t) = s(-8, 2, 5) + t(3, -4, -2) + (2, 9, -4) \quad \text{or} \quad \begin{cases} x = -8s + 3t + 2 \\ y = 2s - 4t + 9 \\ z = 5s - 2t - 4 \end{cases}$$

19. The plane contains the lines given by the equations:

$$\mathbf{r}_1(t) = (5, -6, 10) + t(2, -3, 4), \text{ and}$$
$$\mathbf{r}_2(t) = (-1, 3, -2) + t(5, 10, 7).$$

So we use formula (10) with the vectors $(2, -3, 4)$ and (5, 10, 7) and either of the two points to get:

$$\mathbf{x}(s, t) = t(2, -3, 4) + s(5, 10, 7) + (-1, 3, -2) \quad \text{or} \quad \begin{cases} x = 2t + 5s - 1 \\ y = -3t + 10s + 3 \\ z = 4t + 7s - 2. \end{cases}$$

20. We need to find two out of the three displacement vectors and use any of the three points:

$$\mathbf{a} = (0, 2, 1) - (7, -1, 5) = (-7, 3, -4) \quad \text{and} \quad \mathbf{b} = (0, 2, 1) - (-1, 3, 0) = (1, -1, 1) \quad \text{so}$$

$$\mathbf{x}(s, t) = s(-7, 3, -4) + t(1, -1, 1) + (0, 2, 1) \quad \text{or} \quad \begin{cases} x = -7s + t \\ y = 3s - t + 2 \\ z = -4s + t + 1. \end{cases}$$

21. The equation of the line $\mathbf{r}(t) = (-5, 10, 9) + t(3, -3, 2)$ immediately gives us one of the two vectors $\mathbf{a} = (3, -3, 2)$. The displacement vector from a point on the line to our given point gives us the vector $\mathbf{b} = (-5, 10, 9) - (-2, 4, 7) = (-3, 6, 2)$. So our equations are:

$$\mathbf{x}(s,t) = s(3, -3, 2) + t(-3, 6, 2) + (-5, 10, 9) \quad \text{or} \quad \begin{cases} x = 3s - 3t - 5 \\ y = -3s + 6t + 10 \\ z = 2s + 2t + 9. \end{cases}$$

22. To convert to the parametric form we will need two vectors orthogonal to the normal direction $\mathbf{n} = (2, -3, 5)$ and a point on the plane. The easiest way to find an orthogonal vector is to let one coordinate be zero and find the other two. For example if the x component is zero then $(2, -3, 5) \cdot (0, y, z) = -3y + 5z$ is solved when $y = 5k$ and $z = 3k$ for any scalar k. In other words, the vectors $\mathbf{a} = (0, 5, 3)$ and $\mathbf{b} = (3, 2, 0)$ are orthogonal to $\mathbf{n}$. For a point in the plane $2x - 3y + 5z = 30$, set any two of x, y, and z to zero. For example (0, 0, 6) is in the plane. Our parametric equations are:

$$\mathbf{x}(s,t) = s(0, 5, 3) + t(3, 2, 0) + (0, 0, 6) \quad \text{or} \quad \begin{cases} x = 3t \\ y = 5s + 2t \\ z = 3s + 6. \end{cases}$$

23. We combine the parametric equations into the single equation:

$$\mathbf{x}(s,t) = s(3, 4, 1) + t(-1, 1, 5) + (2, 0, 3).$$

Use the cross product to find the normal vector to the plane:

$$\mathbf{n} = (3, 4, 1) \times (-1, 1, 5) = (19, -16, 7).$$

So the equation of the plane is:

$$19(x - 2) - 16y + 7(z - 3) = 0 \quad \text{or} \quad 19x - 16y + 7z = 59.$$

24. Using method 1 of Example 7, choose a point B on the line, say $B = (-5, 3, 4)$. Then $\overrightarrow{BP_0} = (-5, 3, 4) - (1, -2, 3) = (-6, 5, 1)$, and $\mathbf{a} = (2, -1, 0)$. So

$$\text{proj}_{\mathbf{a}}\overrightarrow{BP_0} = \left(\frac{\mathbf{a} \cdot \overrightarrow{BP_0}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} = \left(\frac{(2, -1, 0) \cdot (-6, 5, 1)}{(2, -1, 0) \cdot (2, -1, 0)}\right)(2, -1, 0) = \frac{-17}{5}(2, -1, 0).$$

The distance is

$$\|\overrightarrow{BP_0} - \text{proj}_{\mathbf{a}}\overrightarrow{BP_0}\| = \left\|(-6, 5, 1) - \frac{-17}{5}(2, -1, 0)\right\| = (1/5)\|(4, 8, 5)\| = \sqrt{105}/5.$$

25. This time we'll use method 2 of Example 7. Again choose a point B on the line and a vector **a** parallel to the line. The distance is then

$$D = \frac{\|\mathbf{a} \times \overrightarrow{BP_0}\|}{\|\mathbf{a}\|} = \frac{\|(3, 5, 0) \times (7 - 2, -3 + 1, 0)\|}{\|(3, 5, 0)\|} = \frac{31}{\sqrt{34}}.$$

For a method 3, you could have solved for an arbitrary point on the line B such that $\overrightarrow{BP_0} \cdot \mathbf{a} = 0$ and then found the length of $\overrightarrow{BP_0}$. In $\mathbf{R}^2$, the calculation is not too bad.

26. Using method 1 of Example 7, choose a point B on the line, say $B = (5, 3, 8)$. Then $\overrightarrow{BP_0} = (-11, 10, 20) - (5, 3, 8) = (-16, 7, 12)$, and $\mathbf{a} = (-1, 0, 7)$. So

$$\text{proj}_{\mathbf{a}}\overrightarrow{BP_0} = \left(\frac{\mathbf{a} \cdot \overrightarrow{BP_0}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a} = \left(\frac{(-1, 0, 7) \cdot (-16, 7, 12)}{(-1, 0, 7) \cdot (-1, 0, 7)}\right)(-1, 0, 7) = (-2, 0, 14).$$

The distance is

$$\|\overrightarrow{BP_0} - \text{proj}_{\mathbf{a}}\overrightarrow{BP_0}\| = \|(-16, 7, 12) - (-2, 0, 14)\| = \|(-14, 7, -2)\| = \sqrt{249}.$$

27. Use Example 9 and for two points $B_1 = (-1, 3, 5)$ on l_1 and $B_2 = (0, 3, 4)$ on l_2 calculate $\overrightarrow{B_1B_2} = (1, 0, -1)$. To find the vector $\mathbf{n}$, calculate the cross product $\mathbf{n} = (8, -1, 0) \times (0, 3, 1) = (-1, -8, 24)$.

$$\begin{aligned}\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2} &= \left(\frac{\mathbf{n}\cdot\overrightarrow{B_1B_2}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{(-1,-8,24)\cdot(1,0,-1)}{(-1,-8,24)\cdot(-1,-8,24)}\right)(-1,-8,24)\\ &= -\frac{25}{641}(-1,-8,24).\end{aligned}$$

Finally $\left\|-\frac{25}{641}(-1,-8,24)\right\| = \frac{25}{\sqrt{641}}$.

28. Again, use Example 9 and for two points $B_1 = (-7, 1, 3)$ on l_1 and $B_2 = (0, 2, 1)$ on l_2 calculate $\overrightarrow{B_1B_2} = (7, 1, -2)$. To find the vector $\mathbf{n}$, calculate the cross product $\mathbf{n} = (1, 5, -2) \times (4, -1, 8) = (38, -16, -21)$.

$$\begin{aligned}\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2} &= \left(\frac{\mathbf{n}\cdot\overrightarrow{B_1B_2}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{(38,-16,-21)\cdot(7,1,-2)}{(38,-16,-21)\cdot(38,-16,-21)}\right)(38,-16,-21)\\ &= \frac{292}{2141}(38,-16,-21).\end{aligned}$$

Finally $\left\|\frac{292}{2141}(38,-16,-21)\right\| = \frac{292}{\sqrt{2141}}$.

29. **(a)** Again, use Example 9 with the two points $B_1 = (4, 0, 2)$, and $B_2 = (2, 1, 3)$ and normal vector $\mathbf{n} = (3, 1, 2)\times(1, 2, 3) = (-1, -7, 5)$. The displacement vector is $\overrightarrow{B_1B_2} = (-2, 1, 1)$. Note that $\overrightarrow{B_1B_2}$ is orthogonal to $\mathbf{n}$ and so the projection $\text{proj}_{\mathbf{n}}\overrightarrow{B_1B_2} = \mathbf{0}$ (if you'd like, you can go ahead and calculate this) and so the lines are distance 0 apart.

(b) This means that the lines must have a point in common (that they intersect at least once). The lines are not parallel so they have exactly one point in common (i.e., they aren't the same line).

30. **(a)** The shortest distance between a point P_0 and a line l is a straight line that meets P_0 orthogonally. If we have two nonparallel lines then we can use the cross product to find the one direction $\mathbf{n}$ that is orthogonal to each. The shortest segment between two lines will meet each orthogonally, for two skew lines l_1 and l_2 the line that joins them at these closest points will be parallel to $\mathbf{n}$.

If instead l_1 is parallel to l_2 we get a whole plane's worth of orthogonal directions. We have no way of choosing a unique vector $\mathbf{n}$ that is used in the calculation.

O.K., that's why we can't use the method of Example 9. What can we do instead?

(b) Fix a point on l_1, say $P_1 = (2, 0, -4)$. Then as we saw in an earlier exercise, the distance from P_1 to an arbitrary point $P_2 = (1 + t, 3 - t, -5 + 5t)$ on l_2 is

$$\|\overrightarrow{P_1P_2}\| = \sqrt{(t-1)^2 + (3-t)^2 + (-1+5t)^2} = \sqrt{27t^2 - 18t + 11}.$$

$\|\overrightarrow{P_1P_2}\|$ is minimized when $\|\overrightarrow{P_1P_2}\|^2$ is minimized. This is at the vertex of the parabola, when $54t - 18 = 0$ or $t = 1/3$. At this point the distance is

$$\sqrt{27(1/3)^2 - 18(1/3) + 11} = \sqrt{3 - 6 + 11} = \sqrt{8} = 2\sqrt{2}.$$

Note: In Exercises 31–33 we could just cut to the end of Example 8 and realize that the length of $\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \frac{|\mathbf{n}\cdot\overrightarrow{P_1P_2}|}{\|\mathbf{n}\|}$. *Instead we will stay true to the spirit of the examples and follow the argument through.*

31. These planes are parallel so we can use Example 8. The point $P_1 = (1, 0, 0)$ is on plane one and the point $P_2 = (8, 0, 0)$ is on plane two. We project the displacement vector $\overrightarrow{P_1P_2} = (7, 0, 0)$ onto the normal direction $\mathbf{n} = (1, -3, 2)$:

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{(7,0,0)\cdot(1,-3,2)}{(1,-3,2)\cdot(1,-3,2)}\right)(1,-3,2) = \frac{7}{14}(1,-3,2) = \frac{1}{2}(1,-3,2).$$

So the distance is $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \sqrt{14}/2$.

32. These planes are also parallel. We choose point $P_1 = (0, 0, 6)$ on plane one and $P_2 = (0, 0, -2)$ on plane two. The displacement vector is therefore $\overrightarrow{P_1P_2} = (0, 0, -8)$, and the normal vector is $\mathbf{n} = (5, -2, 2)$. So

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{(0,0,-8)\cdot(5,-2,2)}{(5,-2,2)\cdot(5,-2,2)}\right)(5,-2,2) = \frac{-16}{33}(5,-2,2).$$

The distance is $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{16}{\sqrt{33}}$.

33. As in Exercises 27 and 28, we'll choose a point $P_1 = (D_1/A, 0, 0)$ on plane one and $P_2 = (D_2/A, 0, 0)$ on plane two. The displacement vector is

$$\overrightarrow{P_1P_2} = \left(\frac{D_2 - D_1}{A}, 0, 0\right).$$

A vector normal to the plane is $\mathbf{n} = (A, B, C)$.

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{\left(\frac{D_2-D_1}{A}, 0, 0\right)\cdot(A,B,C)}{(A,B,C)\cdot(A,B,C)}\right)(A,B,C) = \frac{D_2 - D_1}{A^2+B^2+C^2}(A,B,C).$$

The distance between the two planes is:

$$\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{|D_2 - D_1|}{A^2+B^2+C^2}\|(A,B,C)\| = \frac{|D_2 - D_1|}{\sqrt{A^2+B^2+C^2}}.$$

34. (a) Plane one is normal to $\mathbf{n_1} = (9, -5, 9) \times (3, -2, 3) = (3, 0, -3)$ while plane two is normal to $\mathbf{n_2} = (-9, 2, -9) \times (-4, 7, -4) = (55, 0, -55)$. So $\mathbf{n_1} = (3/55)\mathbf{n_2}$, i.e. they normal vectors are parallel so the planes are parallel.

(b) We'll use the two points in the given equations to get the displacement vector $\overrightarrow{P_1P_2} = (8, -4, 12)$, and the normal vector $\mathbf{n} = (3, 0, -3)$. So

$$\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2} = \left(\frac{(8,-4,12)\cdot(3,0,-3)}{(3,0,-3)\cdot(3,0,-3)}\right)(3,0,-3) = \frac{-12}{18}(3,0,-3).$$

The distance is $\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_2}\| = \frac{12}{\sqrt{18}} = \frac{12}{3\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$.

35. This exercise follows immediately from Exercise 33 (and can be very difficult without it). Here $A = 1, B = 3, C = -5$ and $D_1 = 2$. The equation in Exercise 33 becomes:

$$3 = \frac{|2 - D_2|}{\sqrt{1^2 + 3^2 + (-5)^2}} = \frac{|2 - D_2|}{\sqrt{35}}.$$

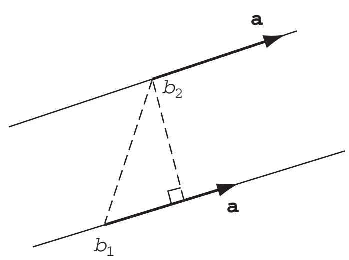
So

$$3\sqrt{35} = |2 - D_2| \quad \text{or} \quad 2 - D_2 = \pm 3\sqrt{35}.$$

So $D_2 = 2 \pm 3\sqrt{35}$ and the equations of the two planes are:

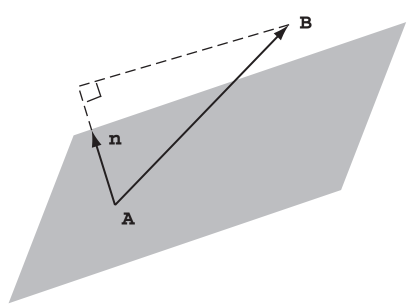
$$x + 3y - 5z = 2 \pm 3\sqrt{35}.$$

36. The lines are parallel, so the distance between them is the same as the distance between any point on one of the lines and the other line. Thus take $\mathbf{b}_2$—the position vector of a point on the second line—and use Example 7. Then $D = \dfrac{\|\mathbf{a} \times (\mathbf{b}_2 - \mathbf{b}_1)\|}{\|\mathbf{a}\|}$.



37. We have $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

$$D = \|\text{proj}_{\mathbf{n}}(\mathbf{b} - \mathbf{a})\| = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|^2}\|\mathbf{n}\|$$
$$= \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|}.$$



(As for the motivation, consider Example 8 with A as P_1, B as P_2.)

38. The parallel planes have equations $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_1) = 0$ and $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_2) = 0$. The desired distance is given by $\|\text{proj}_{\mathbf{n}} \overrightarrow{P_1P_2}\|$ where P_i is the point whose position vector is $\mathbf{x}_i$. Thus $\overrightarrow{P_1P_2} = \mathbf{x}_2 - \mathbf{x}_1$ so

$$\|\text{proj}_{\mathbf{n}} \overrightarrow{P_1P_2}\| = \frac{|\mathbf{n} \cdot (\mathbf{x}_2 - \mathbf{x}_1)|}{\|\mathbf{n}\|^2}\|\mathbf{n}\|$$
$$= \frac{|\mathbf{n} \cdot (\mathbf{x}_2 - \mathbf{x}_1)|}{\|\mathbf{n}\|}.$$

39. By letting $t = 0$ in each vector parametric equation, we obtain $\mathbf{b}_1, \mathbf{b}_2$ as position vectors of points B_1, B_2 on the respective lines. Hence $\overrightarrow{B_1B_2} = \mathbf{b}_2 - \mathbf{b}_1$. A vector $\mathbf{n}$ perpendicular to both lines is given by $\mathbf{n} = \mathbf{a}_1 \times \mathbf{a}_2$. Thus

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{B_1B_2}\| = \frac{|\mathbf{n} \cdot \overrightarrow{B_1B_2}|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{|\mathbf{n} \cdot \overrightarrow{B_1B_2}|}{\|\mathbf{n}\|}$$
$$= \frac{|(\mathbf{a}_1 \times \mathbf{a}_2) \cdot (\mathbf{b}_2 - \mathbf{b}_1)|}{\|\mathbf{a}_1 \times \mathbf{a}_2\|}.$$

1.6 Some n-dimensional Geometry

1. (a) $(1,2,3,\ldots,n) = (1,0,0,\ldots,0) + 2(0,1,0,0,\ldots,0) + \cdots + n(0,0,0,\ldots,0,1) = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 + \cdots + n\mathbf{e}_n$.
(b) $(1,0,-1,1,0,-1,\ldots,1,0,-1) = \mathbf{e}_1 - \mathbf{e}_3 + \mathbf{e}_4 - \mathbf{e}_6 + \mathbf{e}_7 - \mathbf{e}_9 + \cdots + \mathbf{e}_{n-2} - \mathbf{e}_n$.

2. $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n = (1,1,1,\ldots,1)$.

3. $\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 - 4\mathbf{e}_4 + \cdots + (-1)^{n+1}n\mathbf{e}_n = (1,-2,3,-4,\ldots,(-1)^{n+1}n)$.

4. $\mathbf{e}_1 + \mathbf{e}_n = (1,0,0,\ldots,0,1)$.

5. (a) $\mathbf{a}+\mathbf{b} = (1+2, 3-4, 5+6, 7-8, \ldots, 2n-1+(-1)^{n+1}2n) = (3,-1,11,-1,19,-1,\ldots,2n-1+(-1)^{n+1}2n)$.
The n^{th} term is $\begin{cases} 4n-1 & \text{if } n \text{ is odd, and} \\ -1 & \text{if } n \text{ is even.}\end{cases}$
(b) $\mathbf{a}-\mathbf{b} = (1-2, 3+4, 5-6, 7+8, \ldots, 2n-1-(-1)^{n+1}2n) = (-1,7,-1,15,-1,\ldots,2n-1-(-1)^{n+1}2n)$. The n^{th} term is $\begin{cases} 4n-1 & \text{if } n \text{ is even, and} \\ -1 & \text{if } n \text{ is odd.}\end{cases}$
(c) $-3(1,3,5,7,\ldots,2n-1) = (-3,-9,-15,-21,\ldots,-6n+3)$.
(d) $\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{1^2+3^2+5^2+\cdots+(2n-1)^2}$.
(e) $\mathbf{a}\cdot\mathbf{b} = 1(2)+3(-4)+5(6)+\cdots+(2n-1)(-1)^{n+1}2n = 2-12+30-56+\cdots+(-1)^{n+1}2n(2n-1)$.

6. We want to show that $\|\mathbf{a}+\mathbf{b}\| \le \|\mathbf{a}\| + \|\mathbf{b}\|$. Here n is even and **a** and **b** are vectors in $\mathbf{R}^n$,

$$\mathbf{a} = (1,0,1,0,\ldots,0)$$
$$\mathbf{b} = (0,1,0,1,\ldots,1), \text{ and}$$
$$\mathbf{a}+\mathbf{b} = (1,1,1,1,\ldots,1).$$

$$\|\mathbf{a}+\mathbf{b}\| = \underbrace{\sqrt{1^2+1^2+\cdots+1^2}}_{n \text{ times}} = \sqrt{n} = 2\sqrt{n/4} \le 2\sqrt{n/2} = 2\underbrace{\sqrt{1^2+1^2+\cdots+1^2}}_{n/2 \text{ times}} = \|\mathbf{a}\| + \|\mathbf{b}\|.$$

7. First we calculate

$$\|\mathbf{a}\| = \sqrt{1^2+2^2+3^2+\cdots+n^2} = \sqrt{\frac{n(n+1)(2n+1)}{6}}$$
$$\|\mathbf{b}\| = \underbrace{\sqrt{1^2+1^2+\cdots+1^2}}_{n \text{ times}} = \sqrt{n}, \text{ and}$$
$$|\mathbf{a}\cdot\mathbf{b}| = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

So

$$\|\mathbf{a}\|\,\|\mathbf{b}\| = \left(\sqrt{\frac{n(n+1)(2n+1)}{6}}\right)(\sqrt{n}) = \left(\frac{n}{2}\right)\left(\sqrt{\frac{2(n+1)(2n+1)}{3}}\right).$$

For $n=1$, $\sqrt{\frac{2(n+1)(2n+1)}{3}} = 2 = n+1$.
For $n=2$, $\sqrt{\frac{2(n+1)(2n+1)}{3}} = \sqrt{10} \ge 3 = n+1$.
For $n \ge 3$,

$$\left(\frac{n}{2}\right)\left(\sqrt{\frac{2(n+1)(2n+1)}{3}}\right) \ge \left(\frac{n}{2}\right)\frac{2n+1}{\sqrt{3}} \ge \left(\frac{n}{2}\right)(n+1) = |\mathbf{a}\cdot\mathbf{b}|.$$

8. As always,

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \frac{2-5+27-2}{1+1+49+9+4}\mathbf{a} = \frac{22}{64}\mathbf{a}$$
$$= \frac{11}{32}(1,-1,7,3,2) = \left(\frac{11}{32}, \frac{-11}{32}, \frac{77}{32}, \frac{11}{16}\right).$$

9. This is just the triangle inequality:

$$\|\mathbf{a}-\mathbf{b}\| = \|(\mathbf{a}-\mathbf{c})+(\mathbf{c}-\mathbf{b})\| \le \|\mathbf{a}-\mathbf{c}\| + \|\mathbf{c}-\mathbf{b}\|.$$

10. We are given that $\mathbf{c} = \mathbf{a} + \mathbf{b}$ so

$$\|\mathbf{c}\|^2 = \|\mathbf{a} + \mathbf{b}\|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b}.$$

But $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{b} \cdot \mathbf{a}$, so

$$\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} = \|\mathbf{a}\|^2 + \|\mathbf{b}\|^2.$$

This is analogous to the Pythagorean Theorem. Here **a** and **b** are playing the role of the legs. They are orthogonal vectors. The third side of the triangle is $\mathbf{a} + \mathbf{b} = \mathbf{c}$. The theorem in this case says that the sum of the squares of the lengths of the "legs" is the square of the length of the "hypotenuse".

11. We have

$$\|\mathbf{a} + \mathbf{b}\| = \|\mathbf{a} - \mathbf{b}\| \Rightarrow \|\mathbf{a} + \mathbf{b}\|^2 = \|\mathbf{a} - \mathbf{b}\|^2$$
$$\Rightarrow (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}).$$

Expand to find

$$\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$
$$\Rightarrow 4\mathbf{a} \cdot \mathbf{b} = 0,$$

so **a** and **b** are orthogonal.

12. As above, if $\|\mathbf{a} - \mathbf{b}\| > \|\mathbf{a} + \mathbf{b}\|$, then $-2\mathbf{a} \cdot \mathbf{b} > 2\mathbf{a} \cdot \mathbf{b}$ so $-4\mathbf{a} \cdot \mathbf{b} > 0 \Leftrightarrow \mathbf{a} \cdot \mathbf{b} < 0$. Thus $\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \, \|\mathbf{b}\|} < 0$. Hence $\frac{\pi}{2} < 0 \leq \pi$.

13. The equation could also be written in the more suggestive form:

$$(2, 3, -7, 1, -5) \cdot [(x_1, x_2, x_3, x_4, x_5) - (1, -2, 0, 4, -1)] = 0.$$

These are the points in $\mathbf{R}^5$ so that $(x_1, x_2, x_3, x_4, x_5) - (1, -2, 0, 4, -1)$ is orthogonal to the vector $(2, 3, -7, 1, -5)$. This is the four dimensional hyperplane in $\mathbf{R}^5$ orthogonal to $(2, 3, -7, 1, -5)$ containing the point $(1, -2, 0, 4, -1)$.

14. Half of each type of your inventory gives T-shirts in quantities of 10, 15, 12, 10 (in order of lowest to highest selling price). Half of each type of your friend's inventory gives 15, 8, 10, 14 baseball caps. The value of your half of the inventory is

$$(8, 10, 12, 15) \cdot (10, 15, 12, 10) = \$524.$$

The value of your friend's inventory is

$$(8, 10, 12, 15) \cdot (15, 8, 10, 14) = \$530.$$

Thus your friend might be reluctant to accept your offer, unless he's quite a good friend.

15. **(a)** We have

$$\mathbf{p} = (200, 250, 300, 375, 450, 500)$$
$$\text{Total cost} = \mathbf{p} \cdot \mathbf{x} = 200x_1 + 250x_2 + 300x_3 + 375x_4 + 450x_5 + 500x_6$$

(b) With **p** as in part (a), the customer can afford commodity bundles **x** in the set

$$\{\mathbf{x} \in \mathbf{R}^6 | \mathbf{p} \cdot \mathbf{x} \leq 100{,}000\}.$$

The budget hyperplane is $\mathbf{p} \cdot \mathbf{x} = 100{,}000$ or $200x_1 + 250x_2 + 300x_3 + 375x_4 + 450x_5 + 500x_6 = 100{,}000$.

16.

$$3A - 2B = 3\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} -4 & 9 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 6 & 9 \\ -6 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -8 & 18 & 10 \\ 0 & 6 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 11 & -12 & -1 \\ -6 & -6 & 3 \end{bmatrix}$$

17.

$$AC = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 7 \\ 0 & 3 & -2 \end{bmatrix}$$
$$= \begin{bmatrix} 1(1)+2(2)+3(0) & 1(-1)+2(0)+3(3) & 1(0)+2(7)+3(-2) \\ -2(1)+0(2)+1(0) & -2(-1)+0(0)+1(3) & -2(0)+0(7)+1(-2) \end{bmatrix}$$
$$= \begin{bmatrix} 5 & 8 & 8 \\ -2 & 5 & -2 \end{bmatrix}.$$

18.

$$DB = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} -4 & 9 & 5 \\ 0 & 3 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1(-4)+0(0) & 1(9)+0(3) & 1(5)+0(0) \\ 2(-4)-3(0) & 2(9)-3(3) & 2(5)-3(0) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 9 & 5 \\ -8 & 9 & 10 \end{bmatrix}.$$

19.

$$B^T D = \begin{bmatrix} -4 & 0 \\ 9 & 3 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} -4(1)+0(2) & -4(0)+0(-3) \\ 9(1)+3(2) & 9(0)+3(-3) \\ 5(1)+0(2) & 5(0)+0(-3) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 0 \\ 15 & -9 \\ 5 & 0 \end{bmatrix}$$

20. (a)

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) The *ij*th entry of the product of matrices A and B is the product of the ith row of A and the jth column of B. So in case i. we have:

$$(AI_n)_{ij} = [a_{i1} \quad a_{i2} \quad a_{i3} \quad \ldots \quad a_{in}](\mathbf{e}_j)^T = (a_{i1}, a_{i2}, a_{i3}, \ldots, a_{in}) \cdot \mathbf{e}_j = a_{ij}.$$

In case ii. we have:

$$(I_n A)_{ij} = (\mathbf{e}_i) \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ \vdots \\ a_{nj} \end{pmatrix} = \mathbf{e}_i \cdot (a_{1j}, a_{2j}, a_{3j}, \ldots, a_{nj}) = a_{ij}.$$

In both cases we've shown that the *ij*th component of the product is the *ij*th component of matrix A, so $AI_n = A = I_n A$.

21. We'll expand on the first row:

$$\begin{vmatrix} 7 & 0 & -1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & -3 & 0 & 2 \\ 0 & 5 & 1 & -2 \end{vmatrix} = 7 \begin{vmatrix} 0 & 1 & 3 \\ -3 & 0 & 2 \\ 5 & 1 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 0 & 3 \\ 1 & -3 & 2 \\ 0 & 5 & -2 \end{vmatrix}$$
$$= 7\left(-1\begin{vmatrix} -3 & 2 \\ 5 & -2 \end{vmatrix} + 3\begin{vmatrix} -3 & 0 \\ 5 & 1 \end{vmatrix}\right) - \left(2\begin{vmatrix} -3 & 2 \\ 5 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix}\right)$$
$$= 7(-1(-4) + 3(-3)) - (2(-4) + 3(5)) = -42.$$

Note: Exercises 22 and 23 are good exploration problems for students before they've done Exercise 25.

22. Note that if we expand along the first row, only one term survives. If at each step we expand along the first row, the pattern continues. What we are left with is the product of the elements along the diagonal.

$$\begin{vmatrix} 8 & 0 & 0 & 0 \\ 15 & 1 & 0 & 0 \\ -7 & 6 & -1 & 0 \\ 8 & 1 & 9 & 7 \end{vmatrix} = 8 \begin{vmatrix} 1 & 0 & 0 \\ 6 & -1 & 0 \\ 1 & 9 & 7 \end{vmatrix}$$
$$= (8)(1) \begin{vmatrix} -1 & 0 \\ 9 & 7 \end{vmatrix}$$
$$= (8)(1)(-1)(7) = -56.$$

23. This is similar to Exercise 22. Either we could expand along the last row of each matrix at each step or we could expand along the first column at each step. It is easier to keep track of signs if we choose this second approach. We again find that the determinant is the product of the diagonal elements.

$$\begin{vmatrix} 5 & -1 & 0 & 8 & 11 \\ 0 & 2 & 1 & 9 & 7 \\ 0 & 0 & 4 & -3 & 5 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{vmatrix} = 5 \begin{vmatrix} 2 & 1 & 9 & 7 \\ 0 & 4 & -3 & 5 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & -3 \end{vmatrix}$$
$$= (5)(2) \begin{vmatrix} 4 & -3 & 5 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{vmatrix}$$
$$= (5)(2)(4) \begin{vmatrix} 2 & 1 \\ 0 & -3 \end{vmatrix}$$
$$= (5)(2)(4)(2)(-3) = -240.$$

24. There really isn't anything to show. Using the convenient fact provided after Example 8:

- If row i consists of all zeros (i.e., $a_{ij} = 0$ for $1 \le j \le n$) then expand along row i. Using the cofactor notation:

$$\begin{aligned} |A| &= (-1)^{i+1} a_{i1}|A_{i1}| + (-1)^{i+2} a_{i2}|A_{i2}| + \cdots + (-1)^{i+n} a_{in}|A_{in}| \\ &= (-1)^{i+1}(0)|A_{i1}| + (-1)^{i+2}(0)|A_{i2}| + \cdots + (-1)^{i+n}(0)|A_{in}| = 0. \end{aligned}$$

- If column j consists of all zeros (i.e., $a_{ij} = 0$ for all $1 \le i \le n$) then expand along column j. As above we get

$$\begin{aligned} |A| &= (-1)^{1+j} a_{1j}|A_{1j}| + (-1)^{2+j} a_{2j}|A_{2j}| + \cdots + (-1)^{n+j} a_{nj}|A_{nj}| \\ &= (-1)^{1+j}(0)|A_{1j}| + (-1)^{2+j}(0)|A_{2j}| + \cdots + (-1)^{n+j}(0)|A_{nj}| = 0. \end{aligned}$$

25. **(a)** **A lower triangular** matrix is an $n \times n$ matrix whose entries above the main diagonal are all zero. For example the matrix in Exercise 22 is lower triangular.

(b) If we expand the determinant of an upper triangular matrix along its first column we get:

$$\begin{aligned} |A| &= (-1)^{1+1} a_{11}|A_{11}| + (-1)^{2+1} a_{21}|A_{21}| + \cdots + (-1)^{n+1} a_{n1}|A_{n1}| \\ &= (-1)^{1+1}(a_{11})|A_{11}| + (-1)^{2+1}(0)|A_{2j}| + \cdots + (-1)^{n+1}(0)|A_{nj}| = (a_{11})|A_{11}|. \end{aligned}$$

Looking back on what we have found: The determinant of an upper triangular matrix is equal to the term in the upper left position multiplied by the determinant of the matrix that's left when the top most row and left most column are removed. Each time we remove the top row and left column we are left with an upper triangular matrix of one dimension lower. Repeat the process n times and it is clear that

$$|A| = a_{11}|A_{11}| = a_{11}(a_{22}|(A_{11})_{11}|) = \cdots = a_{11}a_{22}a_{33}\cdots a_{nn}.$$

26. (a) **Type I Rule**: If matrix B results from matrix A by exchanging rows i and j then $|A| = -|B|$.

As one example,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \quad \text{while} \quad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1.$$

A more important example is

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = -(b_1a_2 - b_2a_1) = -\begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}.$$

The reason this second example is more important is that you can always expand the determinants of A and B so that you are left with a sum of scalars times the determinants of 2 by 2 matrices involving only the two rows being switched. Since the scalars will be the same in both cases, this second example shows that the effect of switching rows i and j is to switch the sign of every component in the sum and so $|A| = -|B|$.

(b) **Type III Rule**: If matrix B results from matrix A by adding a multiple of row i to row j and leaving row i unchanged then $|A| = |B|$.

As one example,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \quad \text{and also} \quad \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

To see what's going on, let's look at the example

$$\begin{aligned} \begin{vmatrix} a_1 + nb_1 & a_2 + nb_2 \\ b_1 & b_2 \end{vmatrix} &= (a_1 + nb_1)b_2 - (a_2 + nb_2)b_1 = a_1b_2 - a_2b_1 + n(b_1b_2 - b_2b_1) \\ &= a_1b_2 - a_2b_1. \end{aligned}$$

Another way to look at the example above is to see that the determinant splits into two pieces:

$$a_1b_2 - a_2b_1 + n(b_1b_2 - b_2b_1) = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} + n\begin{vmatrix} b_1 & b_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

Note: A more general case of this rule will be proved in Exercise 28.

(c) **Type II Rule**: If matrix B results from matrix A by multiplying the entries in the ith row of A by the scalar c then $|B| = c|A|$.

We will prove this by expanding the determinant for B along the ith row. Because row i is the only one changed, the cofactors B_{ij} are the same as the cofactors A_{ij}.

$$\begin{aligned} |B| &= (-1)^{i+1}b_{i1}|B_{i1}| + (-1)^{i+2}b_{i2}|B_{i2}| + \cdots + (-1)^{i+n}b_{in}|B_{in}| \\ &= (-1)^{i+1}ca_{i1}|A_{i1}| + (-1)^{i+2}ca_{i2}|A_{i2}| + \cdots + (-1)^{i+n}ca_{in}|A_{in}| \\ &= c((-1)^{i+1}a_{i1}|A_{i1}| + (-1)^{i+2}a_{i2}|A_{i2}| + \cdots + (-1)^{i+n}a_{in}|A_{in}|) = c|A|. \end{aligned}$$

27. Here we go: at each step we'll specify what we've done.

$$\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 1 & 0 & 1 & -2 & 4 \\ -1 & 1 & 2 & 3 & -5 \\ 0 & 2 & 3 & 1 & 7 \\ -3 & 2 & -1 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ -1 & 1 & 2 & 3 & -5 \\ 1 & 0 & 1 & -2 & 4 \\ 0 & 2 & 3 & 1 & 7 \\ -3 & 2 & -1 & 0 & 1 \end{vmatrix} \quad \text{switched rows 2 and 3}$$

$$= (-1)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 1 & 0 & 1 & -2 & 4 \\ 0 & 2 & 3 & 1 & 7 \\ 0 & 2 & 2 & -6 & 13 \end{vmatrix} \quad \begin{matrix} \\ \leftarrow \text{row } 2 + \text{row } 3 \\ \\ \\ \leftarrow \text{row } 5 + 3(\text{row } 3) \end{matrix}$$

$$= \left(\frac{-1}{2}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 2 & 0 & 2 & -4 & 8 \\ 0 & 2 & 3 & 1 & 7 \\ 0 & 2 & 2 & -6 & 13 \end{vmatrix} \quad \leftarrow 2(\text{row } 3)$$

$$= \left(\frac{-1}{2}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -1 & 4 & -11 & 0 \\ 0 & 2 & 3 & 1 & 7 \\ 0 & 2 & 2 & -6 & 13 \end{vmatrix} \quad \leftarrow \text{row } 3 - \text{row } 1$$

$$= \left(\frac{-1}{2}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & -3 & -1 & 9 \\ 0 & 0 & -4 & -8 & 15 \end{vmatrix} \quad \begin{matrix} \\ \\ \leftarrow \text{row } 3 + \text{row } 2 \\ \leftarrow \text{row } 4 - 2(\text{row } 2) \\ \leftarrow \text{row } 5 - 2(\text{row } 2) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & -21 & -7 & 63 \\ 0 & 0 & -28 & -56 & 105 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \leftarrow 7(\text{row } 4) \\ \leftarrow 7(\text{row } 5) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & 0 & -37 & 60 \\ 0 & 0 & 0 & -96 & 101 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \leftarrow \text{row } 4 + 3(\text{row } 4) \\ \leftarrow \text{row } 5 + 4(\text{row } 3) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)(-37)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & 0 & -37 & 60 \\ 0 & 0 & 0 & 3552 & -3737 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \\ \leftarrow -37(\text{row } 5) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)(-37)}\right)\begin{vmatrix} 2 & 1 & -2 & 7 & 8 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 7 & -10 & -1 \\ 0 & 0 & 0 & -37 & 60 \\ 0 & 0 & 0 & 0 & 2023 \end{vmatrix} \quad \begin{matrix} \\ \\ \\ \\ \leftarrow \text{row } 5 + 96(\text{row } 4) \end{matrix}$$

$$= \left(\frac{-1}{2(7)(7)(-37)}\right)(2)(1)(7)(-37)(2023) = \frac{2023}{7} = -289.$$

28. **(a)** If you let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $1 = \det(A+B)$ but $\det(A) = \det(B) = 0$. So in general $\det(A+B) \neq \det(A) + \det(B)$.

(b) $\begin{vmatrix} 1 & 2 & 7 \\ 3+2 & 1-1 & 5+1 \\ 0 & -2 & 0 \end{vmatrix} = -58$, while $\begin{vmatrix} 1 & 2 & 7 \\ 3 & 1 & 5 \\ 0 & -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix} = -32 - 26 = -58$. It makes sense that these should be equal; if you imagine expanding on the second row we see that

$$\begin{vmatrix} 1 & 2 & 7 \\ 3+2 & 1-1 & 5+1 \\ 0 & -2 & 0 \end{vmatrix} = (3+2)\begin{vmatrix} 2 & 7 \\ -2 & 0 \end{vmatrix} + (1-1)\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} + (5+1)\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}$$

$$= \left(3\begin{vmatrix} 2 & 7 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} + 5\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}\right) + \left(2\begin{vmatrix} 2 & 7 \\ -2 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix} + 1\begin{vmatrix} 1 & 7 \\ 0 & 0 \end{vmatrix}\right)$$

$$= \begin{vmatrix} 1 & 2 & 7 \\ 3 & 1 & 5 \\ 0 & -2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 7 \\ 2 & -1 & 1 \\ 0 & -2 & 0 \end{vmatrix}.$$

(c) $\begin{vmatrix} 1 & 3 & 2+3 \\ 0 & 4 & -1+5 \\ -1 & 0 & 0-2 \end{vmatrix} = 0$, while $\begin{vmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ -1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 3 & 3 \\ 0 & 4 & 5 \\ -1 & 0 & -2 \end{vmatrix} = -11 + 11 = 0$.

(d) We might characterize the rules for rows as follows:
Let A, B and C be three matrices whose elements are the same except for those in row i where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq j \leq n$. Then $\det(C) = \det(A) + \det(B)$. We prove this by expanding the determinant along row i noting that in that case the cofactors for all three matrices are equal (i.e., $A_{ij} = B_{ij} = C_{ij}$ for $1 \leq j \leq n$):

$$\begin{aligned}
|C| &= (-1)^{i+1}c_{i1}|C_{i1}| + (-1)^{i+2}c_{i2}|C_{i2}| + \cdots (-1)^{i+n}c_{in}|C_{in}| \\
&= (-1)^{i+1}(a_{i1} + b_{i1})|C_{i1}| + (-1)^{i+2}(a_{i2} + b_{i2})|C_{i2}| + \cdots (-1)^{i+n}(a_{in} + b_{in})|C_{in}| \\
&= (-1)^{i+1}(a_{i1})|C_{i1}| + (-1)^{i+2}(a_{i2})|C_{i2}| + \cdots (-1)^{i+n}(a_{in})|C_{in}| \\
&\quad + (-1)^{i+1}(b_{i1})|C_{i1}| + (-1)^{i+2}(b_{i2})|C_{i2}| + \cdots (-1)^{i+n}(b_{in})|C_{in}| \\
&= (-1)^{i+1}(a_{i1})|A_{i1}| + (-1)^{i+2}(a_{i2})|A_{i2}| + \cdots (-1)^{i+n}(a_{in})|A_{in}| \\
&\quad + (-1)^{i+1}(b_{i1})|B_{i1}| + (-1)^{i+2}(b_{i2})|B_{i2}| + \cdots (-1)^{i+n}(b_{in})|B_{in}| \\
&= |A| + |B|.
\end{aligned}$$

The rule for columns is exactly the same:
Let A, B and C be three matrices whose elements are the same except for those in column j where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq n$. Then $\det(C) = \det(A) + \det(B)$. We could prove this by expanding the determinant along column j just as above. Instead note that A^T, B^T, and C^T satisfy the above rule for rows and that the determinant of a matrix is equal to the determinant of its transpose. Our proof is then:

$$|C| = |C^T| = |A^T| + |B^T| = |A| + |B|.$$

29. This is a pretty cool fact. If *AB* and *BA* both exist, these two matrices may not be equal. It doesn't matter. They still have the same determinant. The proof is straightforward: $\det(AB) = (\det\ A)(\det B) = (\det\ B)(\det\ A) = \det\,(BA)$.

30. **(a)** Check the products in both directions . . .

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} (1+0) & (0+0) \\ (1-1) & (0+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+0) & (0+0) \\ (-1+1) & (0+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

(b) Again, the products in both directions yield the identity matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} = \begin{bmatrix} (-40+26+15) & (16-10-6) & (9-6-3) \\ (-80+65+15) & (32-25-6) & (18-15-3) \\ (-40+0+40) & (16+0-16) & (9+0-8) \end{bmatrix}$$
$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} (-40+32+9) & (-80+80+0) & (-120+48+72) \\ (13-10-3) & (26-25+0) & (39-15-24) \\ (5-4-1) & (10-10+0) & (15-6-8) \end{bmatrix}.$$

31. Say the given matrix is A. Then the top left entry in the inverse must be 1/2 because 1 is the top left entry of the product of $A^{-1}A$ and it is twice the top left entry in the inverse matrix.

Looking at the second row of A, in the product AA^{-1} it "picks out" the element in the second row. This means that the second row of A^{-1} is (0, 1, 0). Similarly, the third row of A picks out the opposite of the element in the third row in the product AA^{-1} so the third row of A^{-1} is $(0, 0, -1)$.

The third column of A tells us that the first and third elements of the top row of A^{-1} must be the same. The final element to solve for is the middle element of the top row of A^{-1}. It must be the opposite of the middle element of the third row of A^{-1}. Putting this information together, we have that

$$A^{-1} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

32. Since the first column is 0, the determinant is 0. This means that the matrix could not have an inverse. We'll actually show this in Exercise 35 below. Say, for a minute that you don't accept the results of Exercise 35 and you think you have found an inverse matrix A^{-1} for the given matrix A. Then look at the product $A^{-1}A$. It should be the identity matrix I_3 but the first column of the product will be all 0's. For this reason, no inverse for A could exist.

33. Using the hint, assume that A has two inverses B and C. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

34. We just verify that $B^{-1}A^{-1}$ behaves as an inverse:

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

35. (a) If A is invertible, consider the product $AA^{-1} = I$. By the formula in Exercise 29, $(\det\ A)(\det\ A^{-1}) = \det(AA^{-1}) = \det\ I = 1$. From this we see that $\det\ A \neq 0$. In fact, we see more – the results of part (b) follow immediately.

(b) See part (a).

36. (a)

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I_2$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\cdot\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I_2$$

(b)

$$\begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}^{-1} = \frac{1}{(2\cdot 2-(4)(-1))}\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 \\ 1/8 & 1/4 \end{bmatrix}$$

37. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 3 \end{bmatrix}$, then det $A = 12 + 4 - 2 = 14$, so the formula gives

$$A^{-1} = \frac{1}{14}\begin{bmatrix} \begin{vmatrix} 2 & 4 \\ 0 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 6 & -3 & 2 \\ 4 & 5 & -8 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & -\frac{3}{14} & \frac{1}{7} \\ \frac{2}{7} & \frac{5}{14} & -\frac{4}{7} \\ -\frac{1}{7} & \frac{1}{14} & \frac{2}{7} \end{bmatrix}$$

38. If $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}$, then det $A = 4 + 6 - 18 + 1 = -7$, so the formula gives

$$A^{-1} = -\frac{1}{7}\begin{bmatrix} \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 2 & -2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= -\frac{1}{7}\begin{bmatrix} 2 & 1 & -4 \\ -7 & -7 & 7 \\ -6 & -3 & 5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{7} & -\frac{1}{7} & \frac{4}{7} \\ 1 & 1 & -1 \\ \frac{6}{7} & \frac{3}{7} & -\frac{5}{7} \end{bmatrix}$$

39. We'll transform the cross product into a determinant. To make the determinant easier to calculate we'll replace the fourth row with the sum of the fourth row and five times the second row. Finally we'll expand along the first column.

$$(1, 2, -1, 3) \times (0, 2, -3, 1) \times (-5, 1, 6, 0) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & 1 \\ -5 & 1 & 6 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ 1 & 2 & -1 & 3 \\ 0 & 2 & -3 & 1 \\ 0 & 11 & 1 & 15 \end{vmatrix} \quad \leftarrow \text{row } 4 + 5(\text{row } 2) \qquad = \mathbf{e}_1 \begin{vmatrix} 2 & -1 & 3 \\ 2 & -3 & 1 \\ 11 & 1 & 15 \end{vmatrix} - \begin{vmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 2 & -3 & 1 \\ 11 & 1 & 15 \end{vmatrix}$$

$$= 32\mathbf{e}_1 + 46\mathbf{e}_2 + 19\mathbf{e}_3 - 35\mathbf{e}_4 = (32, 46, 19, -35).$$

40. (a) We use the matrix form to write the cross product as a determinant. We then switch row $i+1$ (the row consisting of $a_{i1}, a_{i2}, \ldots, a_{in}$) with row $j+1$ (the row consisting of $a_{j1}, a_{j2}, \ldots, a_{jn}$) which multiplies the determinant by -1:

$$\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_j \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}$$

$$= (-1)\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{Switch this row } (i+1) \\ \\ \leftarrow \text{with this row } (j+1) \\ \\ \\ \\ \end{matrix}$$

$$= -(\mathbf{a}_1 \times \cdots \times \mathbf{a}_j \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1})$$

(b) Again we will change to the matrix form and then use the rule for the row operation of type II to pull the scalar k out and then rewrite as a cross product.

$$\mathbf{a}_1 \times \cdots \times k\mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{i1} & ka_{i2} & \cdots & ka_{in} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}$$

$$= (k)\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{row } (i+1) \text{ divided by } k \\ \\ \\ \\ \end{matrix}$$

$$= k(\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1})$$

(c) Once again, we will change to the matrix form. This time we will use the rule we developed in Exercise 28 to write this as two determinants. Finally we will convert each back to the cross product form.

$$\mathbf{a}_1 \times \cdots \times (\mathbf{a}_i + \mathbf{b}) \times \cdots \times \mathbf{a}_{n-1} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ (a_{i1}+b_1) & (a_{i2}+b_2) & \dots & (a_{in}+b_n) \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \dots & a_{(n-1)n} \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \dots & a_{(n-1)n} \end{vmatrix} + \begin{vmatrix} l\mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ b_1 & b_2 & \dots & b_n \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \dots & a_{(n-1)n} \end{vmatrix}$$

$$= (\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1}) + (\mathbf{a}_1 \times \cdots \times \mathbf{b} \times \cdots \times \mathbf{a}_{n-1})$$

(d) Expand the determinant along the first row; we'll refer to the cross product matrix as C:

$$\mathbf{b} \cdot |C| = \mathbf{b} \cdot ((\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1}) = \mathbf{b} \cdot \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ a_{11} & a_{12} & \dots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \dots & a_{(n-1)n} \end{vmatrix}$$

$$= \mathbf{b} \cdot (\mathbf{e}_1|C_{11}| - \mathbf{e}_2|C_{12}| + \cdots + (-1)^{1+n}\mathbf{e}_n|C_{1n}|)$$

$$= b_1|C_{11}| - b_2|C_{12}| + \cdots + (-1)^{1+n}b_n|C_{1n}| = \begin{vmatrix} b_1 & b_2 & \cdots & b_n \\ a_{11} & a_{12} & \cdots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}$$

41. This follows immediately from part (d) of Exercise 28. For $1 \le i \le n-1$,

$$\mathbf{a}_i \cdot (\mathbf{a}_1 \times \cdots \times \mathbf{a}_i \times \cdots \times \mathbf{a}_{n-1}) = \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix}.$$

Replace the first row with the difference between row 1 and row $i+1$ and you will get (by Exercise 26) a matrix with the same determinant, namely:

$$\begin{vmatrix} 0 & 0 & \cdots & 0 \\ a_{11} & a_{12} & \cdots & a_{1n} \\ & & \ddots & \\ \vdots & \vdots & & \vdots \\ a_{(n-1)1} & a_{(n-1)2} & \cdots & a_{(n-1)n} \end{vmatrix} = 0.$$

Therefore **b** is orthogonal to $\mathbf{a}_i$ for $1 \le i \le n-1$.

42. To find the normal direction **n** we'll take the cross product of the displacement vectors:

$$\overrightarrow{P_0P_1} = (2,-1,0,0,5) - (1,0,3,0,4) = (1,-1,-3,0,1)$$
$$\overrightarrow{P_0P_2} = (7,0,0,2,0) - (1,0,3,0,4) = (6,0,-3,2,-4)$$
$$\overrightarrow{P_0P_3} = (2,0,3,0,4) - (1,0,3,0,4) = (1,0,0,0,0)$$
$$\overrightarrow{P_0P_4} = (1,-1,3,0,4) - (1,0,3,0,4) = (0,-1,0,0,0)$$

We take the cross product which is the determinant (expand along the fourth row, and then along the last row):

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \\ 1 & -1 & -3 & 0 & 1 \\ 6 & 0 & -3 & 2 & -4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \\ -1 & -3 & 0 & 1 \\ 0 & -3 & 2 & -4 \\ -1 & 0 & 0 & 0 \end{vmatrix} = -\begin{vmatrix} \mathbf{e}_3 & \mathbf{e}_4 & \mathbf{e}_5 \\ -3 & 0 & 1 \\ -3 & 2 & -4 \end{vmatrix}$$
$$= 2\mathbf{e}_3 + 15\mathbf{e}_4 + 6\mathbf{e}_5 = (0,0,2,15,6).$$

We can choose any of the points, say P_0 to find the equation of the hyperplane:

$$2(x_3-3) + 15(x_4) + 6(x_5-4) = 0 \quad \text{or} \quad 2x_3 + 15x_4 + 6x_5 = 30.$$

1.7 New Coordinate Systems

In Exercises 1–3 use equations (1) $x = r\cos\theta$ and $y = r\sin\theta$.

1. $x = \sqrt{2}\cos\pi/4 = (\sqrt{2})(\sqrt{2}/2) = 1$, and $y = \sqrt{2}\sin\pi/4 = 1$. The rectangular coordinates are (1, 1).
2. $x = \sqrt{3}\cos 5\pi/6 = (\sqrt{3})(-\sqrt{3}/2) = -3/2$, and $y = \sqrt{3}\sin 5\pi/6 = (\sqrt{3})(1/2) = \sqrt{3}/2$. The rectangular coordinates are $(-3/2, \sqrt{3}/2)$.
3. $x = 3\cos 0 = 3(1) = 3$, and $y = 3\sin 0 = 0$. The rectangular coordinates are (3, 0).

In Exercises 4–6 use equations (2) $r^2 = x^2 + y^2$, and $\tan\theta = y/x$.

4. $r^2 = (2\sqrt{3})^2 + 2^2 = 16$, so $r = 4$. Also, $\tan\theta = 2/2\sqrt{3} = (1/2)/(\sqrt{3}/2)$. Since we are in the first quadrant the polar coordinates are $(4, \pi/6)$.
5. $r^2 = (-2)^2 + 2^2 = 8$, so $r = 2\sqrt{2}$. Also, $\tan\theta = 2/(-2) = -1$. Since we are in the second quadrant the polar coordinates are $(2\sqrt{2}, 3\pi/4)$.
6. $r^2 = (-1)^2 + (-2)^2 = 5$, so $r = \sqrt{5}$. Also, $\tan\theta = -2/(-1) = 2$. If the point were in the first quadrant, then the angle would be $\tan^{-1} 2$. Since we are in the third quadrant the polar coordinates are $(\sqrt{5}, \pi + \tan^{-1} 2)$.

Exercises 7–9 involve exactly the same idea as Exercises 1–3. Since the z coordinates are the same again we use equations (1) or (3).

7. Here there's nothing to do; the rectangular coordinates are (2 cos 2, 2 sin 2, 2).
8. $x = \pi\cos\pi/2 = (\pi)(0)$, $y = \pi\sin\pi/2 = (\pi)(1)$, and $z = 1$. The rectangular coordinates are $(0, \pi, 1)$.
9. $x = 1\cos 2\pi/3 = -1/2$, $y = 1\sin 2\pi/3 = \sqrt{3}/2$, and $z = -2$. The rectangular coordinates are $(-1/2, \sqrt{3}/2, -2)$.

In Exercises 10–13 use equations (7) $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.

10. $x = 4(\sin\pi/2)(\cos\pi/3) = 4(1)(1/2) = 2$, $y = 4(\sin\pi/2)(\sin\pi/3) = 4(1)(\sqrt{3}/2) = 2\sqrt{3}$, and $z = 4\cos\pi/2 = 4(0) = 0$. So the rectangular coordinates are $(2, 2\sqrt{3}, 0)$.
11. $x = 3(\sin\pi/3)(\cos\pi/2) = 3(\sqrt{3}/2)(0) = 0$, $y = 3(\sin\pi/3)(\sin\pi/2) = 3(\sqrt{3}/2)(1) = 3\sqrt{3}/2$, and $z = 3\cos\pi/3 = 3(1/2) = 3/2$. So the rectangular coordinates are $(0, 3\sqrt{3}/2, 3/2)$.
12. $x = (\sin 3\pi/4)(\cos 2\pi/3) = (\sqrt{2}/2)(-1/2) = -\sqrt{2}/4$, $y = (\sin 3\pi/4)(\sin 2\pi/3) = (\sqrt{2}/2)(\sqrt{3}/2) = \sqrt{6}/4$, and $z = \cos 3\pi/4 = -\sqrt{2}/2$. So the rectangular coordinates are $(-\sqrt{2}/4, \sqrt{6}/4, -\sqrt{2}/2)$. I gave the answer in this form because most students have been told throughout high school that you can never leave a square root in the denominator. They should, of course, feel comfortable leaving the answer as $(-1/\sqrt{8}, \sqrt{3}/\sqrt{8}, -1/\sqrt{2})$, but most won't.
13. $x = 2(\sin\pi)(\cos\pi/4) = 2(0)(\sqrt{2}/2) = 0$, $y = 2(\sin\pi)(\sin\pi/4) = 2(0)(\sqrt{2}/2) = 0$, and $z = 2\cos\pi = 2(-1) = -2$. So the rectangular coordinates are $(0, 0, -2)$.

Exercises 14–16 are basically the same as Exercises 4–6 since the z coordinates are the same in both coordinate systems. Use equations (2) or (4).

14. $r^2 = (-1)^2 + 0^2 = 1$, so $r = 1$. Also, $\tan\theta = 0/(-1) = 0$, so $\theta = \pi$. The cylindrical coordinates are $(1, \pi, 2)$.

15. $r^2 = (-1)^2 + (\sqrt{3})^2$, so $r = 2$. Also, $\tan\theta = \sqrt{3}/(-1) = (\sqrt{3}/2)/(-1/2)$, so $\theta = 2\pi/3$. The cylindrical coordinates are $(2, 2\pi/3, 13)$.

16. $r^2 = 5^2 + 6^2$, so $r = \sqrt{61}$. Also $\tan\theta = 6/5$, so $\theta = \tan^{-1} 6/5$. The cylindrical coordinates are $(\sqrt{61}, \tan^{-1} 6/5, 3)$.

In Exercises 17 and 18 use equations (7) $\rho^2 = x^2 + y^2 + z^2$, $\tan\varphi = \sqrt{x^2 + y^2}/z$, and $\tan\theta = y/x$.

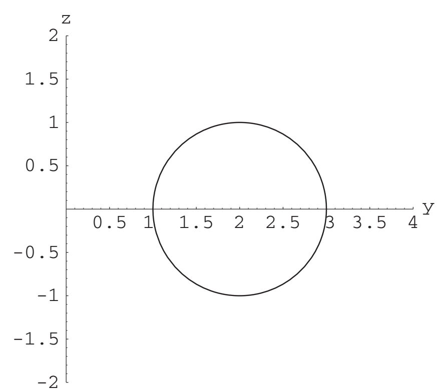
17. $\rho^2 = (1)^2 + (-1)^2 + (\sqrt{6})^2 = 8$, so $\rho = \sqrt{8} = 2\sqrt{2}$. Also, $\tan\varphi = \sqrt{1^2 + (-1)^2}/\sqrt{6} = \sqrt{2}/\sqrt{6} = (1/2)/(\sqrt{3}/2)$, so $\varphi = \pi/6$. Finally, $\tan\theta = -1/1 = -1$, so $\theta = 7\pi/4$ (since the point, when projected onto the xy-plane is in the fourth quadrant). In spherical coordinates the point is $(2\sqrt{2}, \pi/6, 7\pi/4)$.

18. $\rho^2 = 0^2 + (\sqrt{3})^2 + 1^2 = 4$, so $\rho = 2$. Also $\tan\varphi = \sqrt{0^2 + (\sqrt{3})^2}/1 = \sqrt{3}$, so $\varphi = \pi/3$. Finally, when we project the point onto the xy-plane we see that the point is on the positive y-axis so $\theta = \pi/2$. Or, just using the equation $\tan\theta = \sqrt{3}/0$, so $\theta = \pi/2$. In spherical coordinates the point is $(2, \pi/3, \pi/2)$.

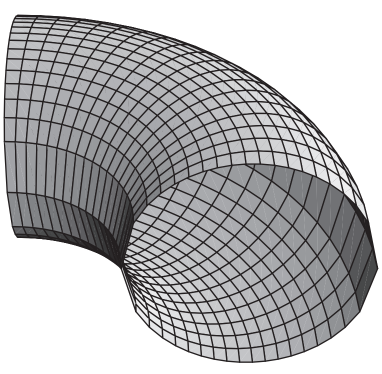
The figures in Exercises 19–21 form a progression. To complete it, the next in line following Exercise 21 would be a sphere.

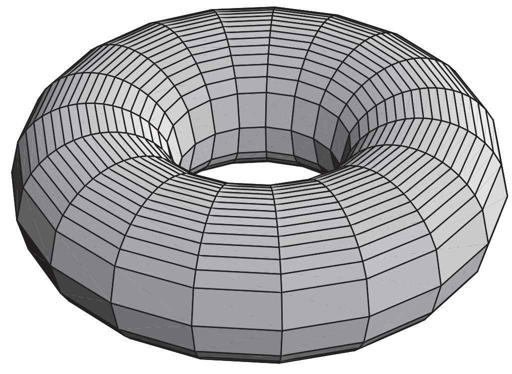
19. As in Example 5, θ does not appear so the surface will be circularly symmetric about the z-axis. Once we have our answer to part (a), we can just rotate it about the z-axis to generate the answer to part (b).

(a) We are slicing in the direction $\pi/2$ which puts us in the yz-plane for positive y. This means that $(r - 2)^2 + z^2 = 1$ becomes $(y - 2)^2 + z^2 = 1$. This is a circle of radius 1 centered at (0, 2, 0).

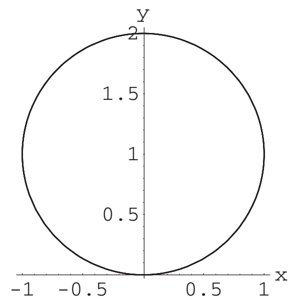


(b) As we start to rotate this about the z-axis, we get a feel for the shape being generated (see below left). In the figure above we see the result of the condition that $r \geq 0$. Without that restriction we would see two circles, each sweeping out a trail like that above. We would end up tracing our surface twice. Rotating this circle (with the restriction on r) about the z-axis, we will end up with a torus (see below right).

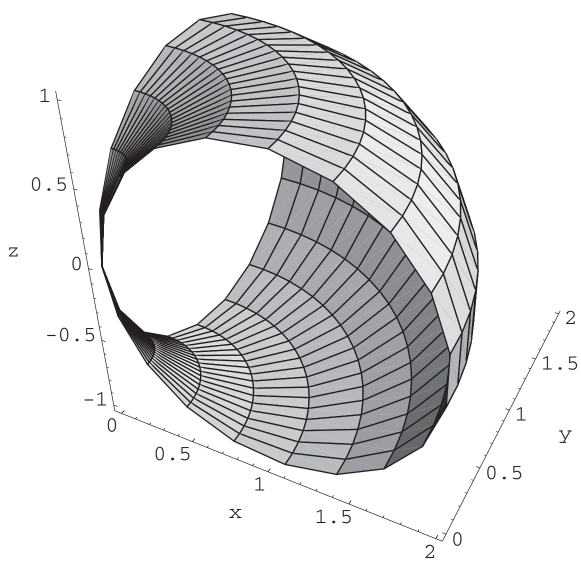


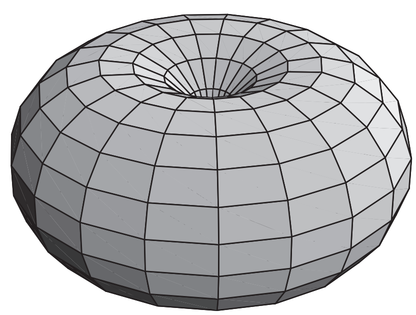


20. (a) As in Example 2, we could reason that our result is a circle that is traced twice (in the figures a is taken to be 1):



(b) When we move to spherical coordinates φ takes on the role of θ from part (a). Note θ does not explicitly appear in this spherical equation. As in the case for cylindrical equations, this means that the surface will be circularly symmetric about the z-axis. As we start to revolve about the z-axis we get the figure on the left.

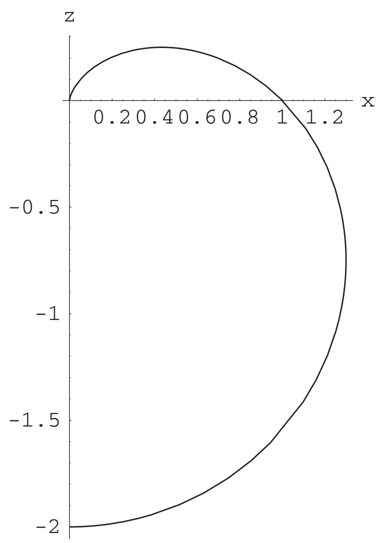


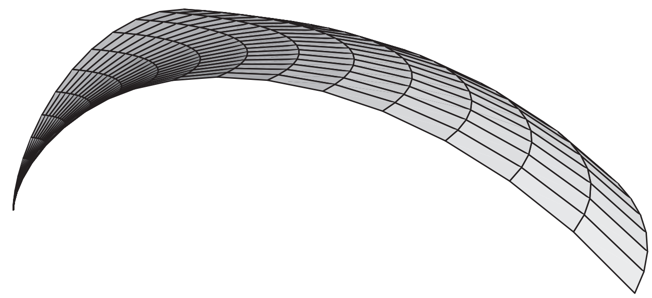


Again, the completed figure is a torus (see above right), but this time the “hole” closes off at the origin.

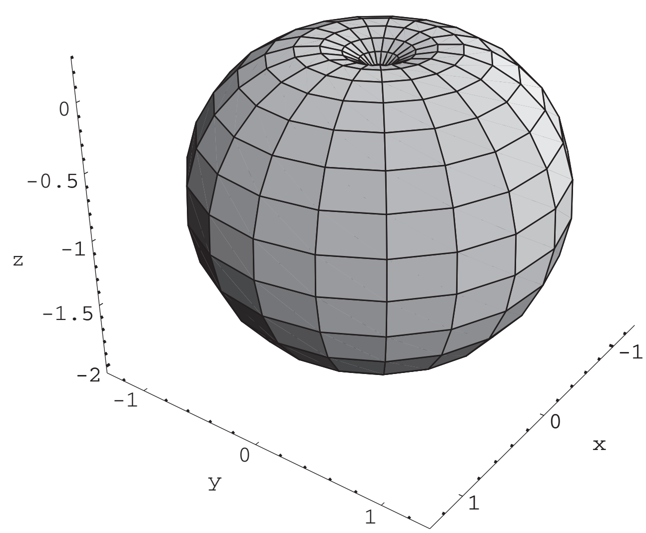
Note: You might want to assign both Exercises 21 and 22. They look so similar and yet the results are very different.

21. As noted above, surface will be circularly symmetric about the z-axis (the equation does not involve θ). In this case we are rotating a piece of the cardioid $1 - \cos\varphi$ shown below left:

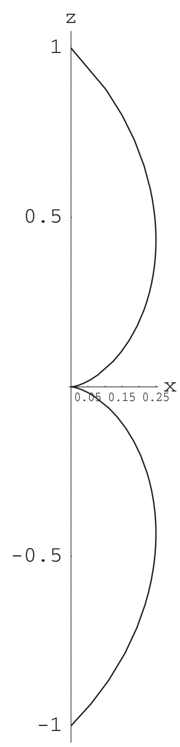


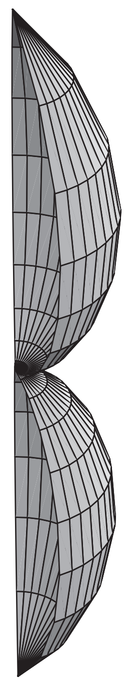


As we start to rotate it we see a “flattened” circle sweeping out the figure pictured above right. The completed figure is like a “dimpled” sphere:

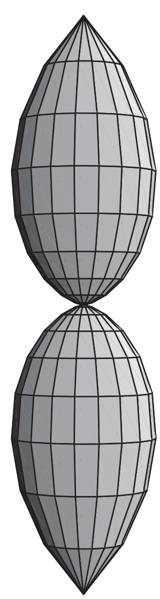


22. Once again, the surface will be circularly symmetric about the z-axis (the equation does not involve θ). In this case we are rotating a piece of the cardioid $1 - \sin\varphi$ shown below left:





As we start to rotate it we see a "double hump" sweeping out the figure pictured above right. The completed figure is shown below:



23. The equation: $\rho \sin\varphi \sin\theta = 2$ is clearly a spherical equation (it involves all three of the spherical coordinates).

- Use equation (7) to convert it to cartesian coordinates: $y = \rho\sin\varphi\sin\theta$ so the cartesian form is simply

$$y = 2.$$

This is a vertical plane parallel to the *xz*-plane.

- Use equation (6) to convert to cylindrical coordinates. $\sin\theta$ stays $\sin\theta$ and $\rho\sin\varphi = r$. So the cylindrical form is

$$r\sin\theta = 2.$$

24. The equation

$$z^2 = 2x^2 + 2y^2$$

is clearly a cartesian equation (it involves all three of the cartesian coordinates).

- Use equation (4) to convert it to cylindrical coordinates: $z^2 = 2(x^2 + y^2) = 2r^2$ so the cylindrical form is simply

$$z^2 = 2r^2.$$

This is a cone which is symmetric about the z-axis, whose vertex is at the origin, one nappe above and one below the xy-plane.

- Use equation (7) to convert to spherical coordinates. $z^2 = 2x^2 + 2y^2$, so $0 = 2(x^2 + y^2 + z^2) - 3z^2$. So the cylindrical form is

$$0 = 2\rho^2 - 3(\rho\cos\varphi)^2 \quad \text{or} \quad \cos\varphi = \pm\sqrt{2/3}.$$

In this final form it is again clear that the surface is a cone.

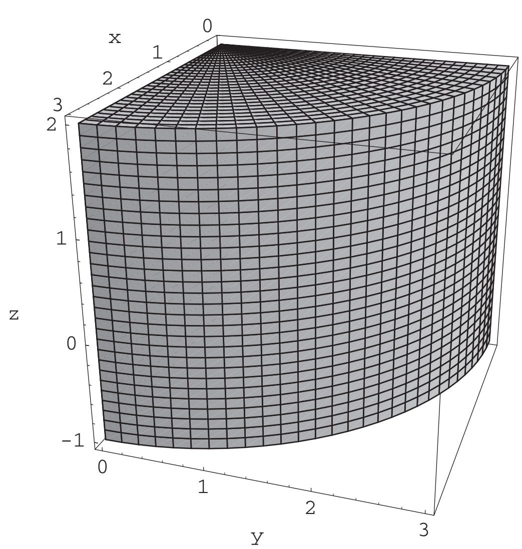
25. $r = 0$ is an equation in cylindrical coordinates. If $r = 0$ then it doesn't matter what θ is and z is free to take on any value. This is the z-axis. In cartesian coordinates this is

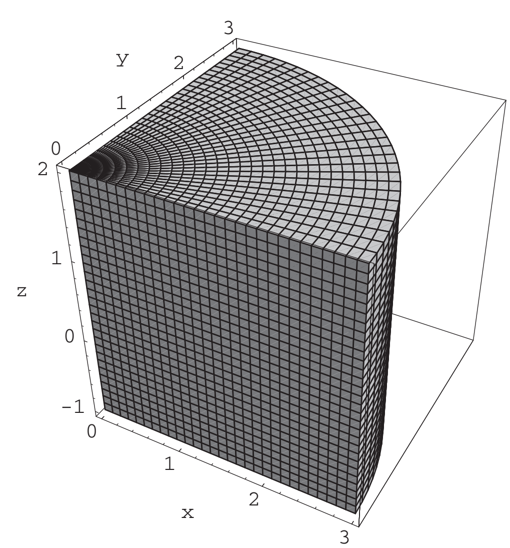
$$x = y = 0,$$

and in spherical coordinates ρ and θ are not constrained but

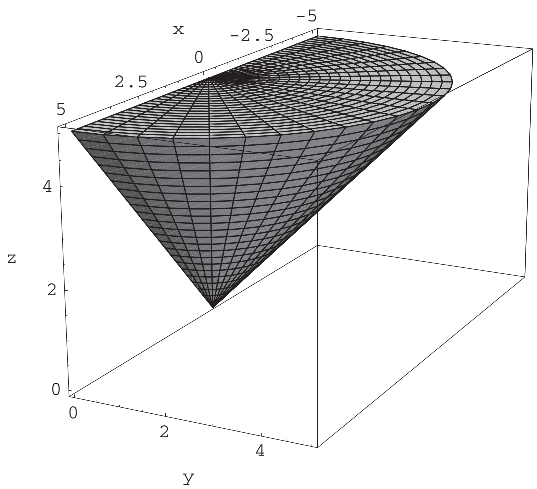
$$\varphi = 0 \quad \text{or} \quad \varphi = \pi.$$

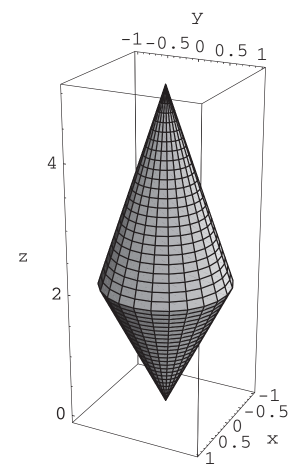
26. You are slicing a wedge out of a cylinder. The result looks like a quarter of a wheel of cheese.





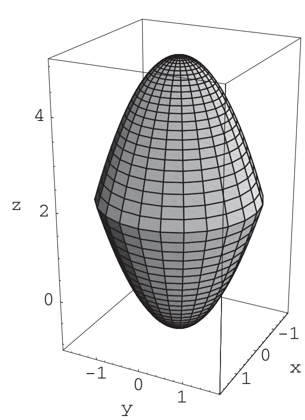
27. Here you are taking the triangular region above the ray $z = r$ and below the ray $z = 5$ in a plane for which θ is fixed (say $\theta = 0$) and rotating it through half a rotation to get half of a cone. The figure is below left.





28. Again we are rotating a triangular region—but this time it is above the line $z = 2r$ and below the line $z = 5 - 3r$. This gives us an image that looks like a diamond spun on a diagonal. The figure is above right.

29. This solid is bound by two paraboloids.



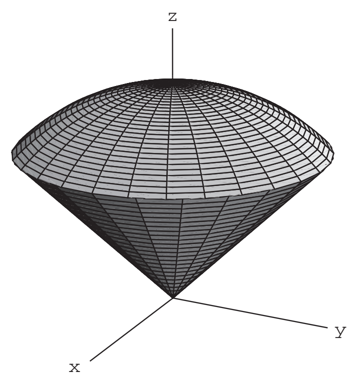
Note: For Exercises 30–32 no sketch is included. I've just roughly described the figure.

30. This is a hollow sphere. The sphere of radius 2 is missing a spherical hole of radius 1.

31. This is the top half of the unit sphere.

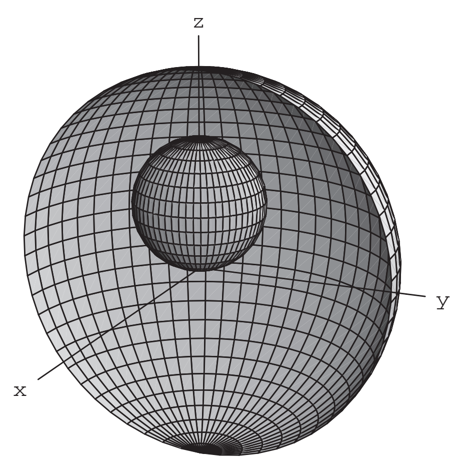
32. This is a quarter of the unit sphere sitting over (and under) the first quadrant.

33. This looks like an ice cream cone:



34. This may look complicated, but it is the cone without the ice cream from the previous problem. The equation $\rho = 2/\cos\varphi$ looks worse than it is. Remember that $z = \rho\cos\varphi$ so this is equivalent to $z = 2$. So we get a flat topped cone with height 2 and tip on the origin.

35. This is a sphere of radius 3 centered at the origin from which we've removed a sphere of radius 1 centered at (0, 0, 1).



36. **(a)** Look for where $(x, y) = (r, \theta)$. We know also that $x = r\cos\theta$, so $r\cos\theta = r$. This implies that $\cos\theta = 1$ so $\theta = 0$. Also $y = r\sin\theta$, but $\sin\theta = 0$ so $y = 0$. So points of the form $(a, 0)$ are the same in both cartesian and polar coordinates.

(b) The only difference between this and part (a) is that a z coordinate has been added to each. So points of the form $(a, 0, b)$ are the same in both rectangular and cylindrical coordinates.

(c) Here (x, y, z) must equal (ρ, φ, θ), where $x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, and $z = \rho\cos\varphi$.
By the first equation $\rho\sin\varphi\cos\theta = \rho$. This implies that $\sin\varphi = 1$ which in turn implies that $\cos\varphi = 0$, $\cos\theta = 1$, and $\sin\theta = 0$. But then $z = \rho\cos\varphi = 0$, and $y = \rho\sin\varphi\sin\theta = \rho(1)(0) = 0$. It looks as if we're headed to solutions on the x-axis again. But wait a minute, if $y = 0$, then $\varphi = 0$, but if $\sin\varphi = 1$ then φ can't be zero. The only point satisfying all of the conditions is the origin (0, 0, 0).

37. **(a)** Picture drawing the graph of the polar equation $r = f(\theta)$ by standing at the origin and turning to angle θ and then walking radially out to $f(\theta)$. You can see that if instead you walked radially out to $-f(\theta)$ you would be heading the same distance in the opposite direction. This tells you that the graph $r = -f(\theta)$ is just the graph $r = f(\theta)$ reflected through the origin.

(b) Although we now have an additional degree of freedom the idea is the same. For each direction specified by φ and θ we would be heading the same distance in the opposite direction. Again this tells you that the graph $\rho = -f(\varphi, \theta)$ is just the graph $\rho = f(\varphi, \theta)$ reflected through the origin.

(c) We're back to the situation in part (a). This time you head in the same direction, you just walk three times as far. So $r = 3f(\theta)$ is as if we expanded the graph $r = f(\theta)$ to three times its original size without changing its shape or orientation.

(d) Analogously, $\rho = 3f(\varphi, \theta)$ is as if we expanded the graph $\rho = f(\varphi, \theta)$ to three times its original size without changing its shape or orientation.

38. Because there is no dependence on θ it means that for each r and the corresponding $z = f(r)$ you have a solution set that corresponds to rotating the point $(r, f(r))$ about the z-axis.

39. **(a)** We need to take six dot products. Each vector dotted with itself must be 1 and each vector dotted with any other must be 0.

$$\mathbf{e}_r \cdot \mathbf{e}_r = (\cos\theta, \sin\theta, 0) \cdot (\cos\theta, \sin\theta, 0) = \cos^2\theta + \sin^2\theta = 1.$$
$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) = \sin^2\theta + \cos^2\theta = 1.$$
$$\mathbf{e}_z \cdot \mathbf{e}_z = (0,0,1) \cdot (0,0,1) = 1.$$
$$\mathbf{e}_r \cdot \mathbf{e}_\theta = (\cos\theta, \sin\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0.$$
$$\mathbf{e}_r \cdot \mathbf{e}_z = (\cos\theta, \sin\theta, 0) \cdot (0,0,1) = 0.$$
$$\mathbf{e}_\theta \cdot \mathbf{e}_z = (-\sin\theta, \cos\theta, 0) \cdot (0,0,1) = 0.$$

(b) We now do the same for the spherical basis vectors.

$$\begin{aligned}\mathbf{e}_\rho \cdot \mathbf{e}_\rho &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) = \sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi \\ &= \sin^2\varphi + \cos^2\varphi = 1.\end{aligned}$$
$$\begin{aligned}\mathbf{e}_\varphi \cdot \mathbf{e}_\varphi &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \cdot (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) = \cos^2\varphi\cos^2\theta \\ &\quad + \cos^2\varphi\sin^2\theta + \sin^2\varphi = \cos^2\varphi + \sin^2\varphi = 1.\end{aligned}$$
$$\mathbf{e}_\theta \cdot \mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0) = \sin^2\theta + \cos^2\theta = 1.$$
$$\begin{aligned}\mathbf{e}_\rho \cdot \mathbf{e}_\varphi &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) = \sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\cos^2\theta \\ &\quad - \sin\varphi\cos\varphi = \sin\varphi\cos\varphi - \sin\varphi\cos\varphi = 0.\end{aligned}$$
$$\mathbf{e}_\rho \cdot \mathbf{e}_\theta = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \cdot (-\sin\theta, \cos\theta, 0) = -\sin\varphi\cos\theta\sin\theta + \sin\varphi\sin\theta\cos\theta = 0.$$
$$\mathbf{e}_\varphi \cdot \mathbf{e}_\theta = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi) \cdot (-\sin\theta, \cos\theta, 0) = -\cos\varphi\cos\theta\sin\theta + \cos\varphi\sin\theta\cos\theta = 0.$$

40. Begin with

$$\begin{aligned}\mathbf{e}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} \\ \mathbf{e}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.\end{aligned}$$

Then

$$\begin{aligned}\sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta &= (\sin\theta\ \cos\theta\,\mathbf{i} + \sin^2\theta\,\mathbf{j}) + (-\cos\theta\ \sin\theta\,\mathbf{i} + \cos^2\theta\,\mathbf{j}) \\ &= \mathbf{j}.\end{aligned}$$

Similarly, $\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta = \mathbf{i}$. Thus, all together

$$\begin{aligned}\mathbf{i} &= \cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta \\ \mathbf{j} &= \sin\theta\,\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta \\ \mathbf{k} &= \mathbf{e}_z.\end{aligned}$$

41. First note that, from (9),

$$\begin{aligned}\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi &= (\sin^2\varphi\cos\theta\,\mathbf{i} + \sin^2\varphi\sin\theta\,\mathbf{j}) + (\cos^2\varphi\cos\theta\,\mathbf{i} + \cos^2\varphi\sin\theta\,\mathbf{j}) \\ &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}.\end{aligned}$$

Hence

$$\begin{aligned}\cos\theta\,(\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi) - \sin\theta\,\mathbf{e}_\theta &= (\cos^2\theta\,\mathbf{i} + \cos\theta\ \sin\theta\,\mathbf{j}) + (\sin^2\theta\,\mathbf{i} - \sin\theta\ \cos\theta\,\mathbf{j}) \\ &= \mathbf{i}.\end{aligned}$$

and, similarly, $\sin\theta\,(\sin\varphi\,\mathbf{e}_\rho + \cos\varphi\,\mathbf{e}_\varphi) + \cos\theta\,\mathbf{e}_\theta = \mathbf{j}$.

Finally, verify that $\cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi = \mathbf{k}$.

So our results are

$$\mathbf{i} = \sin\varphi\cos\theta\,\mathbf{e}_\rho + \cos\varphi\cos\theta\,\mathbf{e}_\varphi - \sin\theta\,\mathbf{e}_\theta$$
$$\mathbf{j} = \sin\varphi\sin\theta\,\mathbf{e}_\rho + \cos\varphi\sin\theta\,\mathbf{e}_\varphi + \cos\theta\,\mathbf{e}_\theta$$
$$\mathbf{k} = \cos\varphi\,\mathbf{e}_\rho - \sin\varphi\,\mathbf{e}_\varphi.$$

42. The exercise is more naturally set up for spherical coordinates.

(a) Here we are inside the portion of the sphere $\rho = 3$ for $|\tan\varphi| \le 1/\sqrt{8}$.

$$\text{Ice cream cone} = \{(\rho,\varphi,\theta) | 0 \le \rho \le 3, 0 \le \varphi \le \tan^{-1}(1/\sqrt{8}), \text{ and } 0 \le \theta < 2\pi\}.$$

(b) Here, z's lower limit is the cone portion so $z \ge \sqrt{8}r$. The upper limit is the portion of the sphere so $z \le \sqrt{3^2 - r^2}$. The variable r is free to be anything between 0 and 1 and θ is free to take on values between 0 and 2π. The cylindrical description is:

$$\{(r,\theta,z) | \sqrt{8}r \le z \le \sqrt{9 - r^2}, 0 \le r \le 1, \text{ and } 0 \le \theta \le 2\pi\}.$$

43. From the formulas in (10) in §1.7, we have that

$$x_1 = \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\cos\varphi_{n-1}$$

and

$$x_2 = \rho\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-2}\sin\varphi_{n-1}.$$

Thus when we take the ratio x_2/x_1, everything cancels to leave us with

$$\frac{x_2}{x_1} = \frac{\sin\varphi_{n-1}}{\cos\varphi_{n-1}} = \tan\varphi_{n-1}.$$

44. (a) Using the formulas in (10), we have that

$$\begin{aligned} x_1^2 + x_2^2 &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}\cos^2\varphi_{n-1} + \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}\sin^2\varphi_{n-1} \\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}\left(\cos^2\varphi_{n-1} + \sin^2\varphi_{n-1}\right) \\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}. \end{aligned}$$

(b) If we assume the restrictions given by the inequalities in (11), then the result in part (a) implies that

$$\begin{aligned} \frac{\sqrt{x_1^2 + x_2^2}}{x_3} &= \frac{\rho\sin\varphi_1\cdots\sin\varphi_{n-3}\sin\varphi_{n-2}}{\rho\sin\varphi_1\cdots\sin\varphi_{n-3}\cos\varphi_{n-2}} \\ &= \frac{\sin\varphi_{n-2}}{\cos\varphi_{n-2}} = \tan\varphi_{n-2}. \end{aligned}$$

45. (a) From part (a) of the previous exercise, we know that $x_1^2 + x_2^2 = \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}$. Thus

$$\begin{aligned} \left(x_1^2 + x_2^2\right) + x_3^2 &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}\sin^2\varphi_{n-2} \\ &\quad + \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}\cos^2\varphi_{n-2} \\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}\left(\sin^2\varphi_{n-2} + \cos^2\varphi_{n-2}\right) \\ &= \rho^2\sin^2\varphi_1\cdots\sin^2\varphi_{n-3}. \end{aligned}$$

(b) Assuming the restrictions given by the inequalities in (11), we obtain

$$\begin{aligned} \frac{\sqrt{x_1^2 + x_2^2 + x_3^2}}{x_4} &= \frac{\rho\sin\varphi_1\cdots\sin\varphi_{n-4}\sin\varphi_{n-3}}{\rho\sin\varphi_1\cdots\sin\varphi_{n-4}\cos\varphi_{n-3}} \\ &= \frac{\sin\varphi_{n-3}}{\cos\varphi_{n-3}} = \tan\varphi_{n-3}. \end{aligned}$$

46. (a) By the work in the previous two exercises, the result holds when $k = 2$ and $k = 3$. To establish the result in general by mathematical induction, we suppose that

$$x_1^2 + \cdots + x_{k-1}^2 = \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-(k-1)}.$$

Then

$$\begin{aligned}\left(x_1^2 + \cdots + x_{k-1}^2\right) + x_k^2 &= \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k} \sin^2 \varphi_{n-k+1} \\ &\quad + \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k} \cos^2 \varphi_{n-k+1} \\ &= \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k} \left(\sin^2 \varphi_{n-k+1} + \cos^2 \varphi_{n-k+1}\right) \\ &= \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-k}.\end{aligned}$$

(b) Assuming the restrictions given by the inequalities in (11), then the result in part (a) implies that

$$\begin{aligned}\frac{\sqrt{x_1^2 + \cdots + x_k^2}}{x_{k+1}} &= \frac{\rho \sin \varphi_1 \cdots \sin \varphi_{n-k-1} \sin \varphi_{n-k}}{\rho \sin \varphi_1 \cdots \sin \varphi_{n-k-1} \cos \varphi_{n-k}} \\ &= \frac{\sin \varphi_{n-k}}{\cos \varphi_{n-k}} = \tan \varphi_{n-k}.\end{aligned}$$

47. By part (a) of the previous exercise with $k = n - 1$, we have

$$x_1^2 + \cdots + x_{n-1}^2 = \rho^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-(n-1)} = \rho^2 \sin^2 \varphi_1.$$

Hence

$$\left(x_1^2 + \cdots + x_{n-1}^2\right) + x_n^2 = \rho^2 \sin^2 \varphi_1 + \rho^2 \cos^2 \varphi_1 = \rho^2.$$

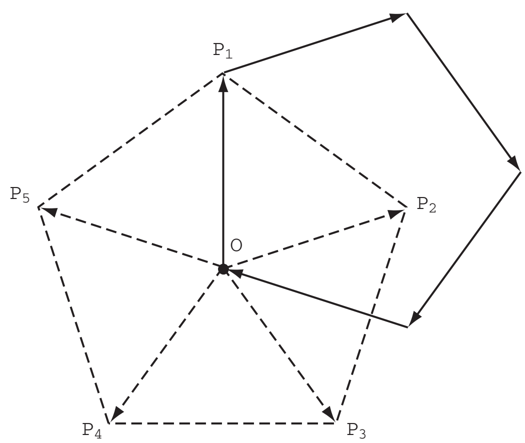
True/False Exercises for Chapter 1

1. False. (The corresponding components must be equal.)
2. True. (Apply two kinds of distributive laws.)
3. False. ($(-4, -3, -3)$ is the displacement vector from P_2 to P_1.)
4. True.
5. False. (Velocity is a vector, but speed is a scalar.)
6. False. (Distance is a scalar, but displacement is a vector.)
7. False. (The particle will be at $(2, -1) + 2(1, 3) = (4, 5)$.)
8. True.
9. False. (From the parametric equations, we may read a vector parallel to the line to be $(-2, 4, 0)$. This vector is not parallel to $(-2, 4, 7)$.)
10. True. (Note that a vector parallel to the line is $(1, 2, 3) - (4, 3, 2) = (-3, -1, 1)$.)
11. False. (The line has symmetric form $\frac{x-2}{-3} = y - 1 = \frac{z+3}{2}$.)
12. True. (Check that the points $(-1, 2, 5)$ and $(2, 1, 7)$ lie on both lines.)
13. False. (The parametric equations describe a *semicircle* because of the restriction on t.)
14. False. (The dot product is the cosine of the angle between the vectors.)
15. False. ($\|k\mathbf{a}\| = |k|\,\|\mathbf{a}\|$.)
16. True.
17. False. (Let $\mathbf{a} = \mathbf{b} = \mathbf{i}$, and $\mathbf{c} = \mathbf{j}$.)
18. True.
19. True.
20. True.
21. True. (Check that each point satisfies the equation.)
22. False. (No values of s and t give the point $(1, 2, 1)$.)
23. False. (The product BA is not defined.)
24. False. (The expression gives the opposite of the determinant.)

25. False. $(\det(2A) = 2^n \det\ A.)$
26. True.
27. False. (The surface with equation $\rho = 4\cos\varphi$ is a sphere.)
28. True. (It's the plane $x = 3$.)
29. True.
30. False. (The spherical equation should be $\varphi = \tan^{-1}\frac{1}{2}$.)

Miscellaneous Exercises for Chapter 1

1. *Solution 1*. We add the vectors head-to-tail by parallel translating $\overrightarrow{OP}_2$ so its tail is at the vertex P_1, translating $\overrightarrow{OP}_3$ so that its tail is at the head of the translated $\overrightarrow{OP}_2$, etc. Since each vector $\overrightarrow{OP}_i$ has the same length and, for $i = 2, \dots, n$, the vector $\overrightarrow{OP}_i$ is rotated $2\pi/n$ from $\overrightarrow{OP}_{i-1}$, the translated vectors will form a closed (regular) n-gon, as the figure below in the case $n = 5$ demonstrates.



Thus, using head-to-tail addition with the closed n-gon, we see that $\sum_{i=1}^{n} \overrightarrow{OP}_i = \mathbf{0}$.

Solution 2. Suppose that $\sum_{i=1}^{n} \overrightarrow{OP}_i = \mathbf{a} \neq \mathbf{0}$. Imagine rotating the entire configuration through an angle of $2\pi/n$ about the center O of the polygon. The vector **a** will have rotated to a different nonzero vector **b**. However, the original polygon will have rotated to an identical polygon (except for the vertex labels), so the new vector sum $\sum_{i=1}^{n} \overrightarrow{OP}_i$ must be unchanged. Hence $\mathbf{a} = \mathbf{b}$, which is a contradiction. Thus $\mathbf{a} = \mathbf{0}$.

2. The line will be $\mathbf{r}(t) = (1, 0, -2) + t(3, -7, 1)$, or $\begin{cases} x = 1 + 3t \\ y = -7t \\ z = -2 + t. \end{cases}$

3. The displacement vector $(3t_0 + 1, 5 - 7t_0, t_0 + 12) - (1, 0, -2) = (3t_0, 5 - 7t_0, t_0 + 14)$ is orthogonal to $(3, -7, 1)$. This means that

$$0 = (3t_0, 5 - 7t_0, t_0 + 14) \cdot (3, -7, 1) = 9t_0 - 35 + 49t_0 + t_0 + 14 = 59t_0 - 21.$$

So $t_0 = 21/59$. The displacement vector gives us the direction of the line:

$$(3t_0, 5 - 7t_0, t_0 + 14) = (1/59)(63, 148, 847).$$

So the equation of the line is

$$\mathbf{r}(t) = (1, 0, -2) + t(63, 148, 847), \quad \text{or} \quad \begin{cases} x = 1 + 63t \\ y = 148t \\ z = -2 + 847t. \end{cases}$$

4. **(a)** If $\mathbf{r}(t) = \overrightarrow{OP}_0 + t\overrightarrow{P_0P_1}$, then $\mathbf{r}(0) = \overrightarrow{OP}_0$ and $\mathbf{r}(1) = \overrightarrow{OP}_0 + \overrightarrow{P_0P_1} = \overrightarrow{OP}_1$.

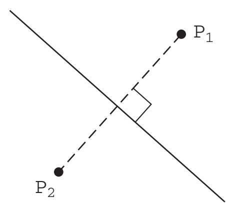
(b) Part (a) set us up for part (b). We know that $\mathbf{r}(0)$ and $\mathbf{r}(1)$ give us the end points of the line segment so $\mathbf{r}(t) = \overrightarrow{OP}_0 + t\overrightarrow{P_0P_1}$, for $0 \le t \le 1$ is the equation of the line segment.

(c) We can just plug into our equation in part (b) to get $\mathbf{r}(t) = (0, 1, 3) + t(2, 4, -10)$ for $0 \le t \le 1$. In parametric form this is

$$\begin{cases} x = 2t \\ y = 1 + 4t \\ z = 3 - 10t \end{cases} \quad \text{for } 0 \le t \le 1.$$

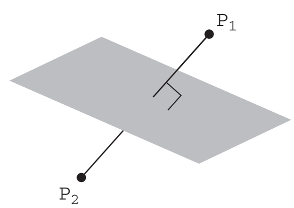
5. (a) The desired line must pass through the midpoint of $\overline{P_1P_2}$, which has coordinates $\left(\frac{-1+5}{2}, \frac{3-7}{2}\right) = (2, -2)$. The line must also be perpendicular to $\overrightarrow{P_1P_2}$. The vector $\overrightarrow{P_1P_2}$ is $(5 + 1, -7 - 3) = (6, -10)$. A vector perpendicular to this must satisfy $(6, -10) \cdot (a_1, a_2) = 0$ so $3a_1 - 5a_2 = 0$ Hence $\mathbf{a} = (5, 3)$ will serve. A vector parametric equation for the line is $\mathbf{l}(t) = (2, -2) + t(5, 3)$, yielding

$$\begin{cases} x = 5t + 2 \\ y = 3t - 2. \end{cases}$$



(b) We generalize part (a). Midpoint of $\overline{P_1P_2}$ is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$. Vector $\overrightarrow{P_1P_2}$ is $(b_1 - a_1, b_2 - a_2)$. A vector $\mathbf{v}$ perpendicular to $\overrightarrow{P_1P_2}$ satisfies $(b_1 - a_1, b_2 - a_2) \cdot \mathbf{v} = 0$ We may therefore take $\mathbf{v}$ to be $\mathbf{v} = (b_2 - a_2, a_1 - b_1)$ so $\mathbf{l}(t) = \left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) + t(b_2 - a_2, a_1 - b_1)$ yielding

$$x = (b_2 - a_2)t + \frac{a_1 + b_1}{2}$$
$$y = (a_1 - b_1)t + \frac{a_2 + b_2}{2}.$$



6. (a) Desired plane passes through midpoint $M(1, 2, -1)$ and has $\overrightarrow{P_1P_2} = (-10, -2, 2)$ as normal vector. So the equation is

$$-10(x - 1) - 2(y - 2) + 2(z + 1) = 0 \iff 5x + y - z = 8.$$

(b) M is $\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2}\right)$; $\overrightarrow{P_1P_2} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.
Equation for plane is

$$(b_1 - a_1)\left(x - \frac{a_1 + b_1}{2}\right) + (b_2 - a_2)\left(y - \frac{a_2 + b_2}{2}\right) + (b_3 - a_3)\left(z - \frac{a_3 + b_3}{2}\right) = 0$$

or

$$(b_1 - a_1)x + (b_2 - a_2)y + (b_3 - a_3)z = \frac{1}{2}(b_1^2 + b_2^2 + b_3^2 - a_1^2 - a_2^2 - a_3^2).$$

7. (a) Midpoint of segment is $\left(\frac{1-3}{2}, \frac{6-2}{2}, \frac{0+4}{2}, \frac{3+1}{2} \frac{-2+0}{2}\right) = (-1, 2, 2, 2, -1)$. Normal to hyperplane is $\overrightarrow{P_1P_2} = (-4, -8, 4, -2, 2)$ so the equation of the hyperplane is $-4(x_1 + 1) - 8(x_2 - 2) + 4(x_3 - 2) - 2(x_4 - 2) + 2(x_5 + 1) = 0$ or $2x_1 + 4x_2 - 2x_3 + x_4 - x_5 = 5$.

(b) Very similar to 6(b). Equation for plane is

$$(b_1 - a_1)\left(x_1 - \frac{a_1 + b_1}{2}\right) + \cdots + (b_n - a_n)\left(x_n - \frac{a_n + b_n}{2}\right) = 0$$

or

$$(b_1 - a_1)x_1 + \cdots + (b_n - a_n)x_n = \frac{1}{2}(b_1^2 + \cdots + b_n^2 - a_1^2 - \cdots - a_n^2).$$

8. We have

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta = \sin\theta, \quad \mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta = \cos\theta,$$

since **a**, **b** are unit vectors. Thus $\|\mathbf{a}\times\mathbf{b}\|^2 + (\mathbf{a}\cdot\mathbf{b})^2 = \sin^2\theta + \cos^2\theta = 1$.

9. (a) No. $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$ just means that the angle between vectors **a** and **b** and the angle between vectors **a** and **b** have the same cosine. If you would prefer, rewrite the equation as $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = 0$ and you can see that what this says is that one of the following is true: vector **a** is orthogonal to the vector $\mathbf{b}-\mathbf{c}$ or $\mathbf{a} = \mathbf{0}$ or $\mathbf{b}-\mathbf{c} = \mathbf{0}$.

(b) No. Use the distributive property of cross products to rewrite the equation as $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = \mathbf{0}$. This could be true if **a** is parallel to $\mathbf{b}-\mathbf{c}$ or if $\mathbf{a} = \mathbf{0}$ or if $\mathbf{b}-\mathbf{c} = \mathbf{0}$.

10. The lines are $\mathbf{r}_1(t) = (-3,1,5) + t(1,-2,2)$ and $\mathbf{r}_2(t) = (4,3,6) + t(-2,4,-4)$. The direction vector for line 2 is -2 times the direction vector for line 1 so either they are parallel or they are the same line. Look at the displacement vector from a point on line 1 to a point on line 2, for example $(4,3,6)-(-3,1,5) = (7,2,1)$. This is not a multiple of the direction vector so they are not the same line. Now, to find the normal direction we'll take

$$(7,2,1)\times(1,-2,2) = (6,-13,-16).$$

The equation of the plane is therefore

$$6(x+3) - 13(y-1) - 16(z-5) = 0 \quad \text{or} \quad 6x - 13y - 16z = -111.$$

11. (a) The angle between the two planes will be the same as the angle between the normal vectors. The normal to $x+y=1$ is $\mathbf{n}_1 = (1,1,0)$, and the normal to $y+z=1$ is $\mathbf{n}_2 = (0,1,1)$.

The angle is then

$$\cos^{-1}\left(\frac{(1,1,0)\cdot(0,1,1)}{\|(1,1,0)\|\,\|(0,1,1)\|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

(b) The line common to both planes must be orthogonal to both $\mathbf{n}_1$ and $\mathbf{n}_2$. We use the cross product to find:

$$\mathbf{n}_1\times\mathbf{n}_2 = (1,1,0)\times(0,1,1) = (1,-1,1).$$

The line must also pass through the point (0, 1, 0). Of course this isn't the only point you could have come up with, but it is the easiest to see. So parametric equations for the line are:

$$\begin{cases} x = t \\ y = 1-t \\ z = t. \end{cases}$$

12. We begin by computing vectors that are parallel to each of the given lines. In particular, we have

$$\begin{aligned} \mathbf{a} &= 4\mathbf{i} - 2\mathbf{j} + 8\mathbf{k} && \text{for line (a),}\\ \mathbf{b} &= -6\mathbf{i} + 3\mathbf{j} - 9\mathbf{k} && \text{for line (b),}\\ \mathbf{c} &= -2\mathbf{i} + \mathbf{j} - 4\mathbf{k} && \text{for line (c),}\\ \mathbf{d} &= 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} && \text{for line (d).} \end{aligned}$$

Note that $\mathbf{a} = -2\mathbf{c}$ and $\mathbf{b} = -3\mathbf{d}$, but **c** and **d** are not scalar multiples of one another. Hence lines (a) and (c) are at least parallel, as are lines (b) and (d), but line (a) is *not* parallel to (b). To see if any of the parallel pairs coincide, note that by letting $t = 0$ in the parametric equations for line (a) we obtain the point (6, 2, 1). This point also lies in line (c): let $t = -2$ in the parametric equations for (c) to obtain it. Hence since we already know that the lines are parallel, this shows that they must in fact be the same. However, if we let $t = 0$ in the parametric equations for (b), we obtain the point (3, 0, 4). This point does *not* lie on line (d) because the only point on (d) with a y-coordinate of 0 is (6, 0, 1). Hence lines (b) and (d) are only parallel.

13. First note that vectors normal to the respective planes are given by:

$$\begin{aligned} \mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} && \text{for plane (a),}\\ \mathbf{b} &= -6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} && \text{for plane (b),}\\ \mathbf{c} &= \mathbf{i} + \mathbf{j} - \mathbf{k} && \text{for plane (c),}\\ \mathbf{d} &= 10\mathbf{i} + 15\mathbf{j} - 5\mathbf{k} && \text{for plane (d),}\\ \mathbf{e} &= 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} && \text{for plane (e).} \end{aligned}$$

It is easy to see that $\mathbf{d} = 5\mathbf{a}$ and $\mathbf{b} = -2\mathbf{e}$ and that $\mathbf{c}$ is not a scalar multiple of any of the other vectors (also that $\mathbf{b}$ and $\mathbf{d}$ are not multiples of one another). Hence planes (a) and (d) must be at least parallel; so must planes (b) and (e). In the case of (b) and (e) note that the equation for (b) may be written as

$$-2(3x - 2y + z) = -2(1).$$

That is, the equation for (b) may be transformed into that for (e) by dividing terms by -2. Hence (b) and (e) are equations for the same plane. In the case of (a) and (d), note that $(0, 0, -3)$ lies on plane (a), but not on (d). Hence (a) and (d) are parallel, but not identical. Finally, it is easy to check that $\mathbf{c} \cdot \mathbf{e} = 3 - 2 - 1 = 0$. Thus the normal vectors to planes (c) and (e) are perpendicular, so that the corresponding planes are perpendicular as well. ($\mathbf{c} \cdot \mathbf{a} = 2 + 3 + 1 = 6 \neq 0$, so plane (c) is perpendicular to neither plane (a) nor (d).)

14. Set up a cube so that one vertex is at the origin and the rest of the bottom face has vertices at (1, 0, 0), (0, 1, 0), and (1, 1, 0). Then the top face will have vertices at (0, 0, 1), (1, 0, 1), (0, 1, 1), and (1, 1, 1).

(a) The angle between the diagonal and one of the edges is

$$\cos^{-1}\left(\frac{(1,1,1)\cdot(1,0,0)}{\|(1,1,1)\|\,\|(1,0,0)\|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

(b) You might be tempted to think that the angle between the diagonal of the cube and the diagonal of one of its faces is (by inspection) half of a right angle. The triangle with the diagonal of the cube as its hypotenuse and the diagonal of one of the faces as one of the legs is a $1 : \sqrt{2} : \sqrt{3}$ right triangle. The cosine of the angle between the diagonal of the cube and the diagonal of a side is $\sqrt{2}/\sqrt{3}$. Using the formula above we also see

$$\cos^{-1}\left(\frac{(1,1,1)\cdot(1,1,0)}{\|(1,1,1)\|\,\|(1,1,0)\|}\right) = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \cos^{-1}\left(\frac{\sqrt{6}}{3}\right).$$

15. The dot product of your two vectors indicates how much you agree with your friend on these five questions. When you both agree or both disagree with an item, the contribution to your dot product is 1. When one of you agrees and the other disagrees the contribution is -1. Your dot product will be an odd number between -5 and 5.

16. **(a)** Following the instructions, we can write $\overrightarrow{BM_1} = \overrightarrow{AM_1} - \overrightarrow{AB} = \frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB}$ because M_1 is the midpoint of $\overline{AC}$. Similarly, $\overrightarrow{CM_2} = \frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}$.

(b) P is on $\overline{BM_1}$ so we can write $\overrightarrow{BP}$ as some multiple of $\overrightarrow{BM_1}$. For definiteness, let's say that $\overrightarrow{BP} = k\overrightarrow{BM_1}$ where $0 < k < 1$. Similarly, $\overrightarrow{CP} = l\overrightarrow{CM_2}$ where $0 < l < 1$. Putting this together with our results from part (a), $\overrightarrow{BP} = k(\frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB})$ and $\overrightarrow{CP} = l(\frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC})$.

(c) First, $\overrightarrow{CB} = \overrightarrow{CP} + \overrightarrow{PB} = \overrightarrow{CP} - \overrightarrow{BP}$. From part (b), this is $l(\frac{1}{2}\overrightarrow{AB} - \overrightarrow{AC}) - k(\frac{1}{2}\overrightarrow{AC} - \overrightarrow{AB}) = (\frac{l}{2} + k)\overrightarrow{AB} - (l + \frac{k}{2})\overrightarrow{AC}$. But, $\overrightarrow{CB}$ also equals $\overrightarrow{CA} + \overrightarrow{AB} = \overrightarrow{AB} - \overrightarrow{AC}$. Equating the coefficients gives us the simultaneous equations $(\frac{l}{2} + k) = 1$ and $(l + \frac{k}{2}) = 1$. This easily gives us $l = k = 2/3$.

(d) Repeat steps (a) through (c) with $\overrightarrow{AM_3}$ and either of the other median vectors. You will again get a point of intersection, say Q. You will show that Q is 2/3 of the way down each median and so must be the same point as P.

17. We are assuming that the plane Π contains the vectors **a, b, c,** and **d**. The vector $\mathbf{n}_1 = \mathbf{a} \times \mathbf{b}$ is orthogonal to Π, and the vector $\mathbf{n}_2 = \mathbf{c} \times \mathbf{d}$ is orthogonal to Π. So the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. This means that $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$.

18. The first two ways that may come to mind to your students each depends on prior knowledge:

Method One: Recall that the area of a triangle is $(1/2)\|\mathbf{a}\|\,\|\mathbf{b}\|\sin C$, where C is the angle between $\mathbf{a}$ and $\mathbf{b}$. So the area is

$$\begin{aligned}\left(\frac{1}{2}\right)\|\mathbf{a}\|\,\|\mathbf{b}\|\sin C &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\sin^2 C}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2(1-\cos^2 C)}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2)\cos^2 C}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2 - (\mathbf{a}\cdot\mathbf{b})^2}.\end{aligned}$$

Method Two: The area of a triangle is 1/2 the area of the parallelogram determined by the same two vectors. The area of the parallelogram is the length of the cross product. So the area is

$$\begin{aligned}\left(\frac{1}{2}\right)\|\mathbf{a}\times\mathbf{b}\| &= \left(\frac{1}{2}\right)\sqrt{(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{b})}\\ \text{(by Section 1.4, Exercise 29)} &= \left(\frac{1}{2}\right)\sqrt{(\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{b})-(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{a})}\\ &= \left(\frac{1}{2}\right)\sqrt{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2-(\mathbf{a}\cdot\mathbf{b})^2}.\end{aligned}$$

19. (a) The vertices are given so that if they are connected in order *ABDC* we will sketch a parallelogram. From Exercise 18 we could say that

$$\begin{aligned}\text{Area} &= \sqrt{\|\overrightarrow{AB}\|^2\,\|\overrightarrow{AC}\|^2-(\overrightarrow{AB}\cdot\overrightarrow{AC})^2}\\ &= [\|(4-1,-1-3,3+1)\|^2\,\|(2-1,5-3,2+1)\|^2\\ &\quad -((4-1,-1-3,3+1)\cdot(2-1,5-3,2+1))^2]^{1/2}\\ &= \sqrt{\|(3,-4,4)\|^2\,\|(1,2,3)\|^2-((3,-4,4)\cdot(1,2,3))^2}\\ &= \sqrt{(41)(14)-(7^2)}=\sqrt{525}=5\sqrt{21}.\end{aligned}$$

(b) When we project the parallelogram in the xy-plane we get the same points with the z coordinate equal to 0. We do the same calculation as in part a with the new vectors:

$$\begin{aligned}\text{Area} &= \sqrt{\|\overrightarrow{AB}\|^2\,\|\overrightarrow{AC}\|^2-(\overrightarrow{AB}\cdot\overrightarrow{AC})^2}\\ &= \sqrt{\|(4-1,-1-3,0)\|^2\,\|(2-1,5-3,0)\|^2-((4-1,-1-3,0)\cdot(2-1,5-3,0))^2}\\ &= \sqrt{\|(3,-4,0)\|^2\,\|(1,2,0)\|^2-((3,-4,0)\cdot(1,2,0))^2}\\ &= \sqrt{(25)(5)-(5^2)}=\sqrt{100}=10.\end{aligned}$$

20. (a) Students raised on the slope-intercept form of a line may be more comfortable once you point out that the slope of the line $ax+by=d$ is $\frac{\Delta y}{\Delta x}=\frac{-a}{b}$. Now the direction that the vector points is clear: $\mathbf{v}=(b,-a)$.

(b) A vector **n** normal to the line l must be orthogonal to the vector **v** you found in part (a). We are also told that the first component of **n** is a. This means that

$$0=\mathbf{n}\cdot\mathbf{v}=(a,?)\cdot(b,-a)=ab-?\,a.$$

So $\mathbf{n}=(a,b)$.

(c) Choose a point P_1 on the line $ax+by=d$. For example, if P_1 has x component zero then $y=d/b$. In other words, choose $P_1=(0,d/b)$. It doesn't matter. We are going to project the displacement vector from the point P_1 to the point $P_0=(x_0,y_0)$ onto **n**.

$$\|\text{proj}_{\mathbf{n}}\overrightarrow{P_0P_1}\|=\left\|\left(\frac{\mathbf{n}\cdot\overrightarrow{P_0P_1}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}\right\|=\frac{|\mathbf{n}\cdot\overrightarrow{P_0P_1}|}{\|\mathbf{n}\|}=\frac{|(a,b)\cdot(x_0,y_0-d/b)|}{\|(a,b)\|}=\frac{|ax_0+by_0-d|}{\sqrt{a^2+b^2}}.$$

(d) We plug into our brand new formula:

$$\text{Distance from }(3,5)\text{ to }l:(3x-5y=2)\text{ is }\frac{|8(3)-5(5)-2|}{\sqrt{8^2+5^2}}=\frac{3}{\sqrt{89}}.$$

21. (a) As should be expected, this is similar to the calculation in Exercise 20. We choose any point P_1 in the plane $\Pi : Ax+By+Cz=D$. For example, let $P_1=(0,0,D/C)$ and $P_0=(x_0,y_0,z_0)$. The normal vector $\mathbf{n}=(A,B,C)$. Again the distance from P_0 to Π is

$$\begin{aligned}\|\text{proj}_{\mathbf{n}}\overrightarrow{P_1P_0}\| &= \left\| \left(\frac{\mathbf{n}\cdot\overrightarrow{P_1P_0}}{\mathbf{n}\cdot\mathbf{n}} \right) \mathbf{n} \right\| \\ &= \frac{|\mathbf{n}\cdot\overrightarrow{P_1P_0}|}{\|\mathbf{n}\|} \\ &= \frac{|(A,B,C)\cdot(x_0,y_0,z_0-D/C)|}{\|(A,B,C)\|} \\ &= \frac{|Ax_0+By_0+Cz_0-D|}{\sqrt{A^2+B^2+C^2}}.\end{aligned}$$

(b) We plug into our formula from part (a):

$$\text{Distance from } (1,5,-3) \text{ to } \Pi\text{: } (x-2y+2z+12=0) \text{ is } \frac{|1(1)-2(5)+2(-3)+12|}{\sqrt{1^2+(-2)^2+2^2}} = \frac{3}{\sqrt{9}} = 1.$$

22. (a) A vector $\mathbf{n}$ normal to Π may be obtained as $\mathbf{n} = \mathbf{b}\times\mathbf{c}$ as both $\mathbf{b} = \overrightarrow{AB}$ and $\mathbf{c} = \overrightarrow{AC}$ are parallel to Π. Thus the distance from P to Π may be found by taking $\|\text{proj}_{\mathbf{n}}\overrightarrow{AP}\| = \|\text{proj}_{\mathbf{n}}\mathbf{p}\|$. Now

$$\text{proj}_{\mathbf{n}}\mathbf{p} = \frac{\mathbf{n}\cdot\mathbf{p}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n} = \frac{\mathbf{n}\cdot\mathbf{p}}{\|\mathbf{n}\|^2}\mathbf{n}.$$

Thus

$$\|\text{proj}_{\mathbf{n}}\mathbf{p}\| = \frac{|\mathbf{n}\cdot\mathbf{p}|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{|\mathbf{n}\cdot\mathbf{p}|}{\|\mathbf{n}\|} = \frac{|(\mathbf{b}\times\mathbf{c})\cdot\mathbf{p}|}{\|\mathbf{n}\|}.$$

(b) We have $\mathbf{b} = (2,-3,1)-(1,2,3) = (1,-5,-2)$, $\mathbf{c} = (2,-1,0)-(1,2,3) = (1,-3,-3)$, and $\mathbf{p} = (1,0,-1)-(1,2,3) = (0,-2,-4)$. Thus

$$\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & -2 \\ 1 & -3 & -3 \end{vmatrix} = (9,1,2).$$

Hence the desired distance is

$$\frac{|(0,-2,-4)\cdot(9,1,2)|}{\|(9,1,2)\|} = \frac{|-10|}{\sqrt{86}} = \frac{10}{\sqrt{86}}.$$

23. (a) The vector $\overrightarrow{AB}\times\overrightarrow{AC} = \mathbf{0}$ if and only if $\overrightarrow{AB}$ is parallel to $\overrightarrow{AC}$. This happens if and only if A, B, and C are collinear.

(b) We note that $\overrightarrow{CD} \neq \mathbf{0}$ since C and D are distinct points. Then $(\overrightarrow{AB}\times\overrightarrow{AC})\cdot\overrightarrow{CD} = 0$ if and only if $\overrightarrow{AB}\times\overrightarrow{AC} = \mathbf{0}$ or $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to $\overrightarrow{CD}$. The first case occurs exactly when A, B, and C are collinear (so A, B, C and D are coplanar). In the second case, $\overrightarrow{AB}\times\overrightarrow{AC}$ is perpendicular to the plane containing A, B, and C and so $\overrightarrow{CD}$ can only be perpendicular to it if and only if D lies in this plane as well.

24. We have the equation that if α is the angle between vectors $\mathbf{x}$ and the vector $\mathbf{k} = (0,0,1)$, then

$$\cos\alpha = \frac{\mathbf{x}\cdot\mathbf{k}}{\|\mathbf{x}\|\,\|\mathbf{k}\|} = \frac{\mathbf{x}\cdot\mathbf{k}}{\|\mathbf{x}\|}.$$

Since we are given that this last quantity $= 1/\sqrt{2}$, $\mathbf{x}$ makes an angle of 45 degrees with the positive z-axis. So the points P satisfying the condition of this exercise sweep out the top nappe of the cone making an angle of 45 degrees with the positive z-axis minus the origin.

25. The equation $\mathbf{a}\times\mathbf{x} = \mathbf{b}$ tells us that $\mathbf{x}$ points in the direction of $\mathbf{b}\times\mathbf{a}$. Now we have to determine the length of $\mathbf{x}$. We can choose any vector in the direction of $\mathbf{x}$. For convenience, let $\mathbf{y}$ be the unit vector in direction of $\mathbf{x}$:

$$\mathbf{y} = \frac{\mathbf{b}\times\mathbf{a}}{\|\mathbf{b}\times\mathbf{a}\|}.$$

The angle between $\mathbf{a}$ and $\mathbf{x}$ is the same as that between $\mathbf{a}$ and $\mathbf{y}$ so

$$\frac{\mathbf{a}\cdot\mathbf{y}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{x}}{\|\mathbf{a}\|\,\|\mathbf{x}\|} = \frac{c}{\|\mathbf{a}\|\,\|\mathbf{x}\|}.$$

So if $c \neq 0$,

$$\|\mathbf{x}\| = \frac{c}{\mathbf{a}\cdot\mathbf{y}}, \quad \text{and,} \quad \mathbf{x} = \left(\frac{c}{\mathbf{a}\cdot\mathbf{y}}\right)\mathbf{y}.$$

If $c = 0$ then $\mathbf{a}$ is orthogonal to $\mathbf{x}$ (and $\mathbf{y}$). Use the fact that

$$\|\mathbf{b}\| = \|\mathbf{a}\times\mathbf{x}\| = \|\mathbf{a}\|\,\|\mathbf{x}\|\sin\theta = \|\mathbf{a}\|\,\|\mathbf{x}\|\sin\pi/2 = \|\mathbf{a}\|\,\|\mathbf{x}\|.$$

So when $c = 0$,

$$\|\mathbf{x}\| = \frac{\|\mathbf{b}\|}{\|\mathbf{a}\|} \quad \text{and} \quad \mathbf{x} = \left(\frac{\|\mathbf{b}\|}{\|\mathbf{a}\|}\right)\mathbf{y}.$$

26. (a) Let $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{c} = \mathbf{j}$. Then

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{i}\times(\mathbf{j}\times\mathbf{j}) = \mathbf{i}\times\mathbf{0} = \mathbf{0},$$

but

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i}.$$

(b) The Jacobi identity states that

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} = \mathbf{0}.$$

This result is equivalent to

$$-(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b}.$$

Since $-(\mathbf{b}\times\mathbf{c})\times\mathbf{a} = \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, we see that we *always* have

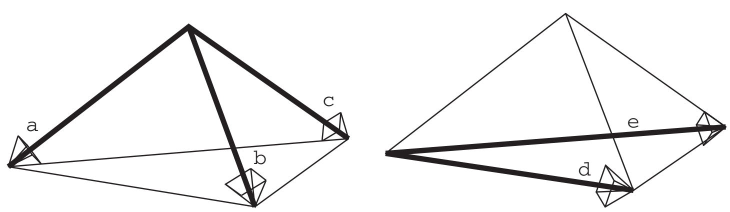
$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b},$$

so that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$$

precisely when $(\mathbf{c}\times\mathbf{a})\times\mathbf{b} = \mathbf{0}$.

27. (a) In the figure below left, the cross product $\mathbf{a}\times\mathbf{b}$ is a vector outwardly normal to the face containing edges $\mathbf{a}$ and $\mathbf{b}$ with length equal to twice the area of the face. To keep the diagram uncluttered, it has been split into two:



So the sum of the four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ asked for in the exercise can be expressed as

$$(1/2)[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) + (\mathbf{e}\times\mathbf{d})].$$

But $\mathbf{d} = \mathbf{b} - \mathbf{a}$ and $\mathbf{e} = \mathbf{c} - \mathbf{a}$ so

$$\begin{aligned}\mathbf{e}\times\mathbf{d} &= (\mathbf{c}-\mathbf{a})\times(\mathbf{b}-\mathbf{a})\\ &= (\mathbf{c}\times\mathbf{b}) - (\mathbf{a}\times\mathbf{b}) - (\mathbf{c}\times\mathbf{a}) + (\mathbf{a}\times\mathbf{a})\\ &= -(\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a}).\end{aligned}$$

We put this together with the above to conclude:

$$\begin{aligned}\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 &= (1/2)[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) + (\mathbf{e}\times\mathbf{d})]\\ &= (1/2)[(\mathbf{a}\times\mathbf{b}) + (\mathbf{b}\times\mathbf{c}) + (\mathbf{c}\times\mathbf{a}) - (\mathbf{a}\times\mathbf{b}) - (\mathbf{b}\times\mathbf{c}) - (\mathbf{c}\times\mathbf{a})]\\ &= \mathbf{0}.\end{aligned}$$

(b) Denote the vectors associated with the first tetrahedron as $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ and the vectors associated with the second tetrahedron as $\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3$, and $\mathbf{v}'_4$. Let the vectors associated with the sides being glued together be $\mathbf{v}_1$ and $\mathbf{v}'_1$.

By construction $\mathbf{v}_1$ and $\mathbf{v}'_1$ have equal lengths and point in opposite directions so $\mathbf{v}_1 + \mathbf{v}'_1 = \mathbf{0}$. From part (a) we know that

$$\mathbf{v}_1 = -(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) \quad \text{and} \quad \mathbf{v}'_1 = -(\mathbf{v}'_2 + \mathbf{v}'_3 + \mathbf{v}'_4).$$

This means that

$$(\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4) + (\mathbf{v}'_2 + \mathbf{v}'_3 + \mathbf{v}'_4) = \mathbf{0}.$$

(c) Just as we can break any polygon into triangles, we can break any polyhedron into tetrahedra. The key to part (b) was that when we glue two tetrahedra together, the vector of the face being hidden is equal to the sum of the three vectors being introduced. In symbols,

$$\mathbf{v}'_1 = \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4.$$

From part (a) we know that for any tetrahedron $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$. So as we build up our polyhedron by gluing tetrahedra together, at each stage (by parts (a) and (b)) the sum of the outward normals with length equal to the area of the face will be zero.

28. We may construct vectors $\mathbf{v}_1, \ldots, \mathbf{v}_4$ outwardly normal to each face of the tetrahedron and with length equal to the area of that face. Using the result of part (a) of Exercise 27, we have that $\mathbf{v}_1 + \cdots + \mathbf{v}_4 = \mathbf{0}$. Hence $\mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$. Let's assume that the vectors are indexed so that $\mathbf{v}_4$ is the vector normal to the face that is opposite to vertex R. Then $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are pairwise perpendicular and thus $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$.

Now we compute

$$\begin{aligned} d^2 &= \|\mathbf{v}_4\|^2 = \|-(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\|^2 = \|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3\|^2 \\ &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \\ &= \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3 + 2\mathbf{v}_1 \cdot \mathbf{v}_2 + 2\mathbf{v}_1 \cdot \mathbf{v}_3 + 2\mathbf{v}_2 \cdot \mathbf{v}_3 \\ &= \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|\mathbf{v}_3\|^2 + 0 + 0 + 0 \\ &= a^2 + b^2 + c^2. \end{aligned}$$

29. (a) Remember, if the adjacent sides of a parallelogram are $\mathbf{a}$ and $\mathbf{b}$, then the diagonals are $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. So the sum of the squares of the lengths of the diagonals are

$$\begin{aligned} \|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) + (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= (\mathbf{a} \cdot \mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}) = 2\|\mathbf{a}\|^2 + 2\|\mathbf{b}\|^2 \end{aligned}$$

which is the sum of the squares of the lengths of the four sides (opposite sides have equal lengths).

(b) $\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2)$.

30. The last line of the proof of the Cauchy–Schwarz inequality in Section 1.6 is

$$\|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \geq (\mathbf{a} \cdot \mathbf{b})^2.$$

Now we only need to notice that

$$\begin{aligned} (\mathbf{a} \cdot \mathbf{b})^2 &= \left[\sum_{i=1}^{n} a_i b_i \right]^2 \\ \|\mathbf{a}\|^2 &= \sum_{i=1}^{n} a_i^2 \\ \|\mathbf{b}\|^2 &= \sum_{i=1}^{n} b_i^2 \end{aligned}$$

and the result follows immediately:

$$\left[\sum_{i=1}^{n} a_i^2 \right] \left[\sum_{i=1}^{n} b_i^2 \right] \geq \left[\sum_{i=1}^{n} a_i b_i \right]^2.$$

31. **(a)**

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

(b) It seems reasonable to guess that

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

(c) We need only show the inductive step:

$$A^{n+1} = AA^n = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix}.$$

32. **(a)** There's nothing much to show. $A^2 = 0$.

(b) You shouldn't need a calculator or computer for this. The diagonal of 1's keeps moving to the left so that

$$A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad A^5 = 0.$$

33. **(a)** The determinants are:

$$|H_2| = \frac{1}{12}, |H_3| = \frac{1}{2160}, |H_4| = \frac{1}{6048000},$$
$$|H_5| = \frac{1}{266716800000}, \text{ and } \quad |H_6| = \frac{1}{186313420339200000}$$

The determinants are going to 0 as n gets larger. As for writing out the matrices, note that H_2 is the upper left two by two matrix in H_{10} in part (b). Similarly, H_3 is the upper left three by three . . . H_6 is the upper left six by six matrix in H_{10}. I would consider deducting points from any student who actually writes these out. They can use a computer algebra system to accomplish this. For *Mathematica* the command for generating H_{10} would be

$$\text{Table}[1/(i+j-1), \{i, 10\}, \{j, 10\}]//\text{MatrixForm}.$$

The command for calculating the determinant would be

$$\text{Det}[\text{Table}[1/(i+j-1), \{i, 10\}, \{j, 10\}]].$$

(b) Using the *Mathematica* commands described in part (a), the determinant

$$|H_{10}| = 1/46206893947914691316295628839036278726983680000000000$$
$$\approx 2.16 \times 10^{-53}.$$

The matrix is

$$H_{10} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{bmatrix}$$

(c) Again, the code examples will be from *Mathematica*. Let's first calculate a numerical approximation A of H_{10} with the command

$$A = \text{N[Table}[1/(i+j-1), \{i, 10\}, \{j, 10\}]].$$

We can then calculate the inverse B and A with the command

$$B = \text{Inverse}[A].$$

You can display these as matrices by appending "//MatrixForm" to the command. Now generate *AB* and *BA* with the commands

$$A.B//\text{MatrixForm} \quad \text{and} \quad B.A//\text{MatrixForm}.$$

You should note that these aren't equal and neither is the 10×10 identity matrix I_{10}.

34. The center of the moving circle is at $(a-b)(\cos t, \sin t)$. Notice that as the moving circle rolls so that its center moves counterclockwise it is turning clockwise relative to its center. When the small circle has traveled completely around the large circle it has rolled over a length of $2\pi(a)$. Its circumference is $2\pi b$ so if it were rolling along a straight line it would have revolved a/b times. The problem is that it is rolling around in a circle and so it has lost a rotation each time the center has traveled completely around. In other words the smaller wheel is turning at a rate of

$$((a/b)-1)t = (a-b)t/b.$$

The position of P relative to the center of the moving circle is

$$b\left(\cos\left(-\frac{(a-b)t}{b}\right), \sin\left(-\frac{(a-b)t}{b}\right)\right) = b\left(\cos\frac{(a-b)t}{b}, -\sin\frac{(a-b)t}{b}\right).$$

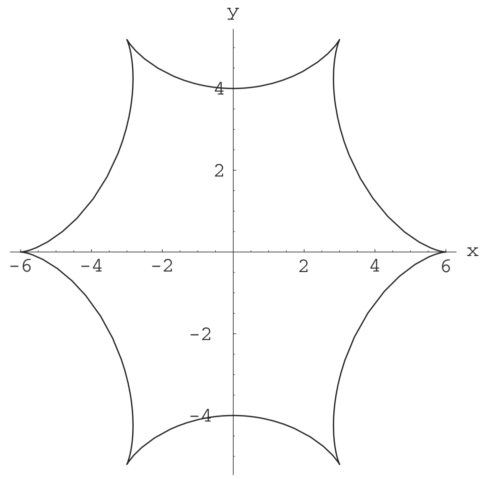
Putting this together, the position of P is the sum of the vector from the origin to the center of the moving circle and the vector from the center of the moving circle to P. This is

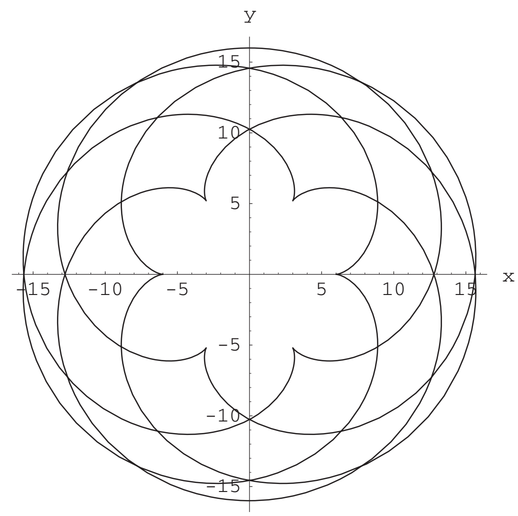
$$(a-b)(\cos t, \sin t) + b\left(\cos\left(\frac{(a-b)t}{b}\right), -\sin\left(\frac{(a-b)t}{b}\right)\right).$$

35. Not much changes here. The center of the moving circle is now at $(a+b)(\cos t, \sin t)$. Now the moving circle gains one revolution each time around the fixed circle and so turns at a rate of $((a/b)+1)t = (a+b)t/b$. Since we are starting P at $(a, 0)$, the initial angle from the center of the moving circle to P is π so the position of P relative to the center of the moving circle is $b\left(\cos\left(\pi + \frac{(a+b)t}{b}\right), \sin\left(\frac{(a-b)t}{b}\right)\right) = -b\left(\cos\frac{(a+b)t}{b}, \sin\frac{(a+b)t}{b}\right)$. As in Exercise 34 we sum the same two vectors to get the expression:

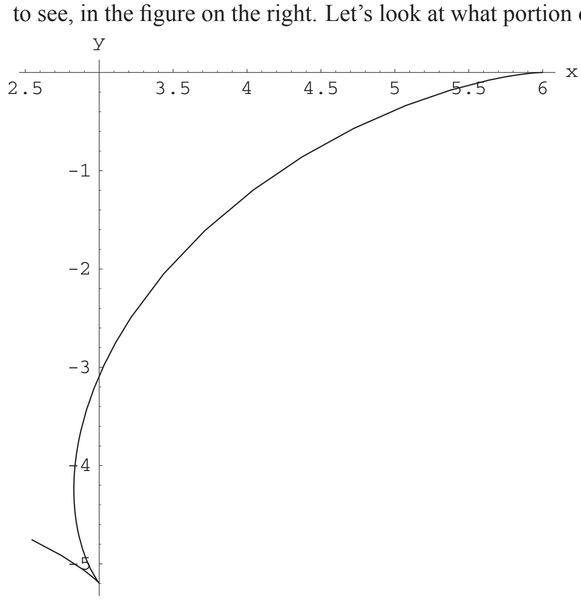
$$(a+b)(\cos t, \sin t) - b\left(\cos\left(\frac{(a+b)t}{b}\right), \sin\left(\frac{(a+b)t}{b}\right)\right).$$

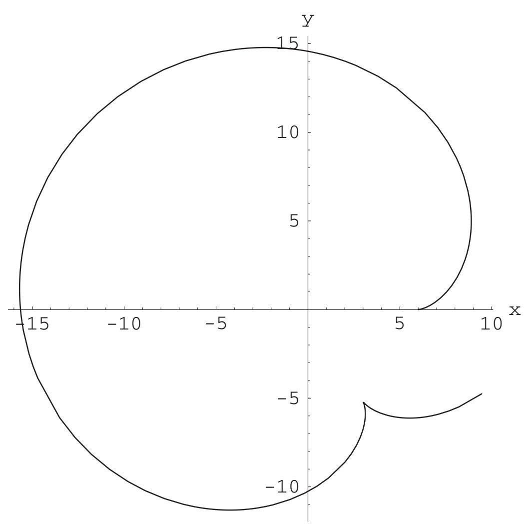
36. **(a)** Let's look at diagrams of hypocycloid (below on the left) and an epicycloid (below on the right) with $a = 6$ and $b = 5$:





What are the roles of a and b? You can see in the figure on the left that there are 6 cusps. This is also true, but harder to see, in the figure on the right. Let's look at what portion of these curves correspond to $0 \leq t \leq 2\pi$.



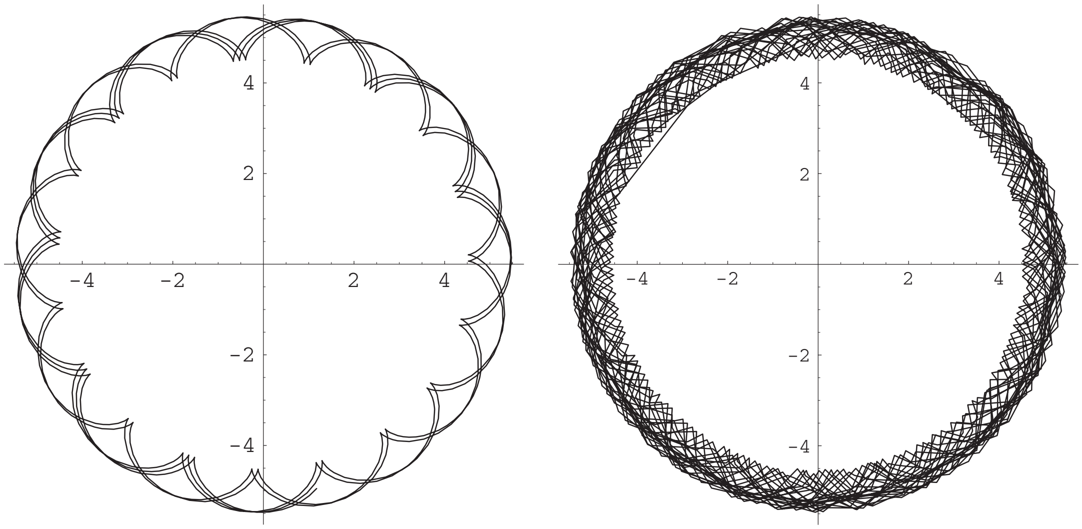


The figure on the left shows that 6/5 of the hypocycloid is covered for $0 \leq t \leq 2\pi$. The figure on the right is the corresponding portion of the epicycloid. Usually what we call the hypocycloid is what we draw until the ends close up. In this case, the hypocycloid is complete when $t = 5(2\pi)$. Again, although it is harder to see, this epicycloid will close up when $t = 5(2\pi)$.

If a and b have no common divisors and are both rational, then the hypocycloid or epicycloid will have a cusps and will close up after $t = b(2\pi)$. If a and b have common divisors then write a/b in lowest terms. The hypocycloid or epicycloid will have as many cusps as the numerator. The same answer holds for epicycloids.

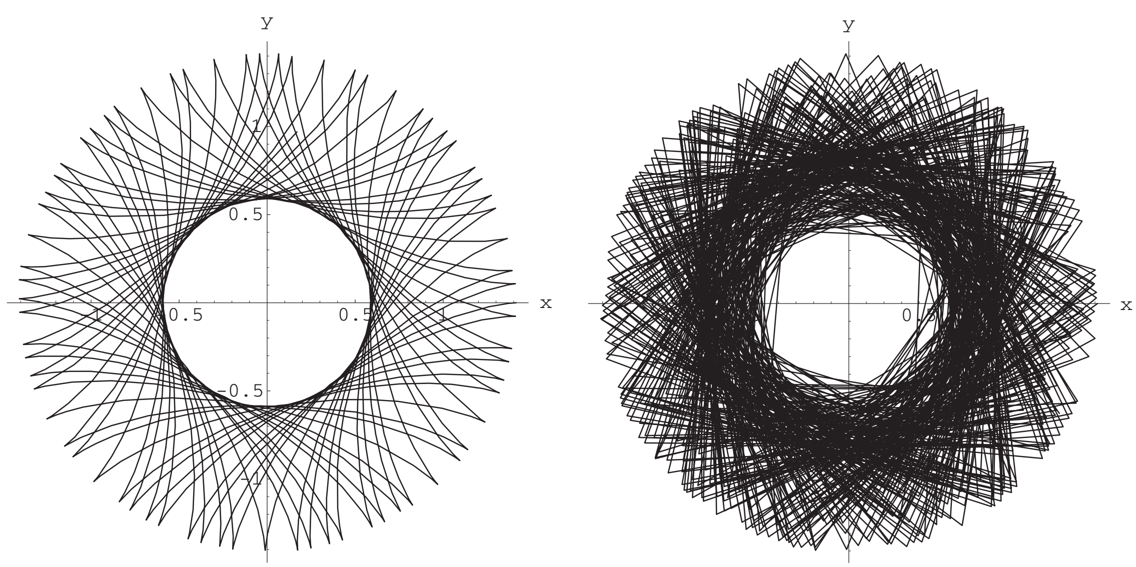
(b) We noted in part (a) that if a/b is rational in lowest terms, the hypocycloid or epicycloid closes up when $t = b(2\pi)$. In the case of the hypocycloid, this is because then $(a-b)(\cos b(2\pi), \sin b(2\pi)) + b\left(\cos\left(\frac{a-b}{b}\right) b(2\pi), -\sin\left(\frac{a-b}{b}\right) b(2\pi)\right) = (a-b)(\cos 0, \sin 0) + b\left(\cos\left(\frac{a-b}{b}\right) 0, -\sin\left(\frac{a-b}{b}\right) 0\right)$. In words, its because the angle is a rational multiple of 2π.

A picture of part of an epicycloid for which a/b is irrational is:



If a/b is irrational then the curve will never close up. It can't. At no time when the center of the moving circle comes back to its original position will P be back in its original position.

A picture of part of a hypocycloid for which a/b is irrational is:



In each case, the figure on the left shows several periods. For the figure on the right we let t get larger. If we let t get arbitrarily large the curve is dense.

37. Look at the second part of the answers in Exercises 34 and 35. The only difference is that we are changing the distance from the center of the moving wheel to P from b to c. The formula for a hypotrochoid is:

$$(a-b)(\cos t, \sin t) + c\left(\cos\left(\frac{(a-b)t}{b}\right), -\sin\left(\frac{(a-b)t}{b}\right)\right).$$

In parametric form, the formulas for a hypotrochoid are:

$$x = (a-b)\cos t + c\cos\left(\frac{(a-b)t}{b}\right),\ y = (a-b)\sin t - c\sin\left(\frac{(a-b)t}{b}\right).$$

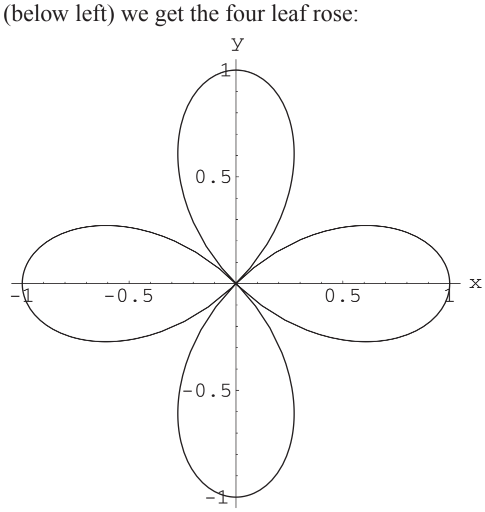
The formula for an epitrochoid is:

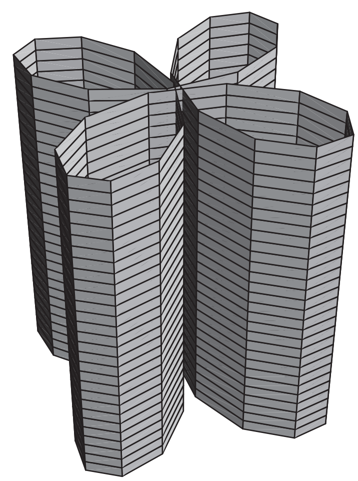
$$(a+b)(\cos t, \sin t) - c\left(\cos\left(\frac{(a+b)t}{b}\right), \sin\left(\frac{(a+b)t}{b}\right)\right).$$

In parametric form, the formulas for an epitrochoid are:

$$x = (a+b)\cos t - c\cos\left(\frac{(a+b)t}{b}\right),\ y = (a+b)\sin t - c\sin\left(\frac{(a+b)t}{b}\right).$$

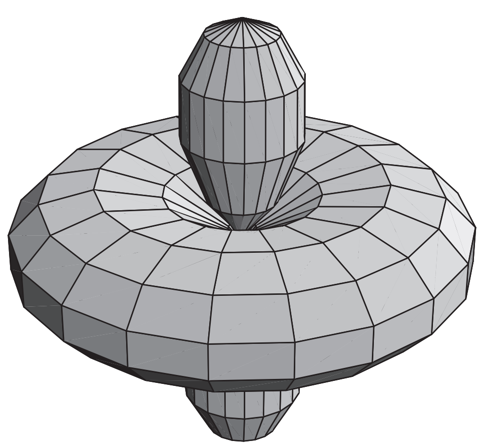
38. (a) Here (below left) we get the four leaf rose:

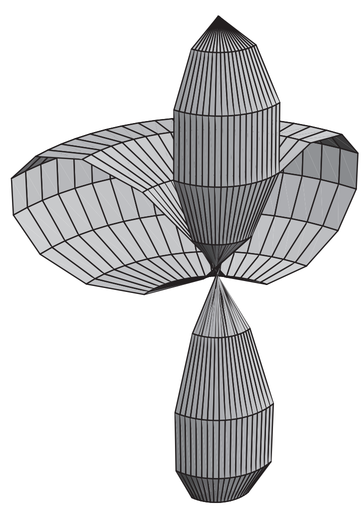




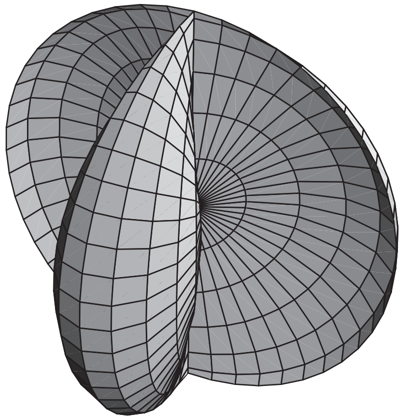
(b) We just erect a cylinder on that base and get the above right image.

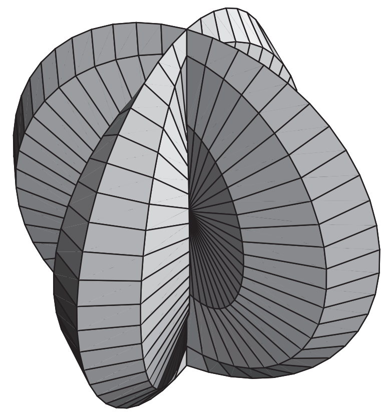
(c) There is no θ explicitly in the equation, so the rose is being rotated about the z-axis (we show both the completed figure and a partial to see how it is formed):



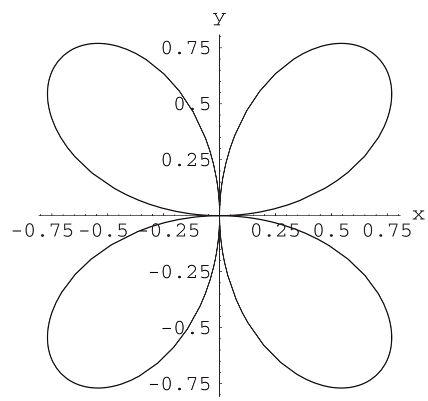


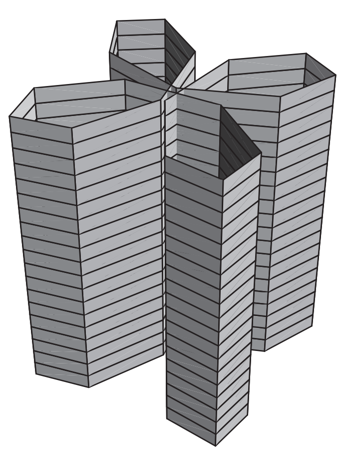
(d) Here we show half of the figure and then the completed figure. From the outside, the figure looks as if the rose has been first rotated about the x-axis and then about the y-axis.

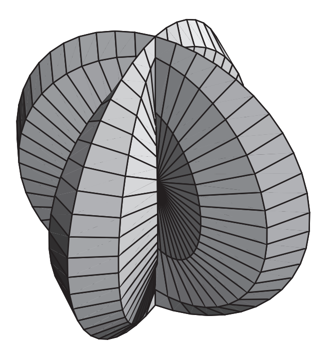




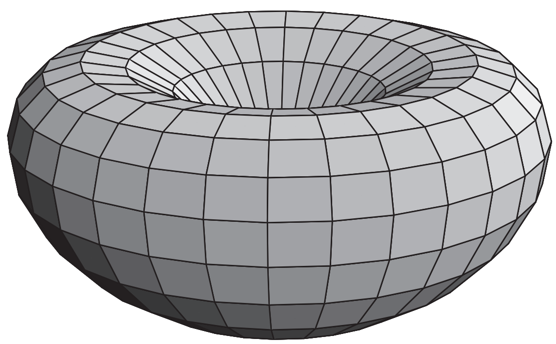
39. Parts (a), (b), and (d) are pictured below top, left, and right. They look very similar to the graphs from the previous exercise.



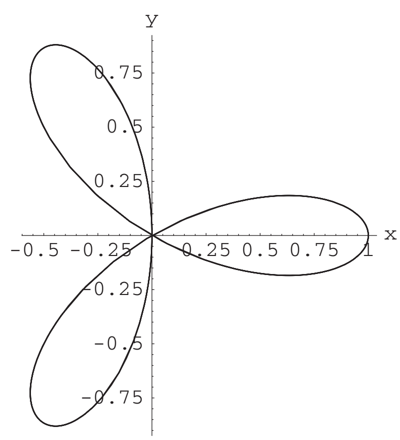


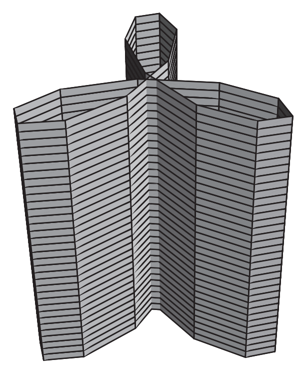


(c) This looks very different from its counterpart for Exercise 38. It looks like a dented sphere.



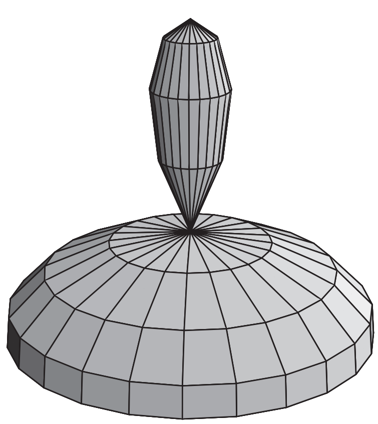
40. (a) We begin with a three leaf rose (the path is traced twice) shown below left.

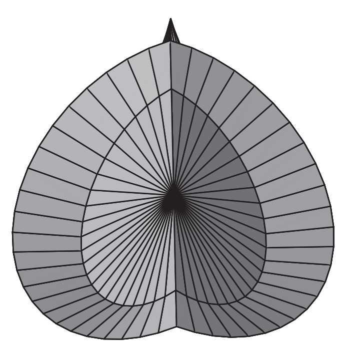




(b) The cylindrical equation again adds nothing. A cylinder is built over the rose. It is shown above right.

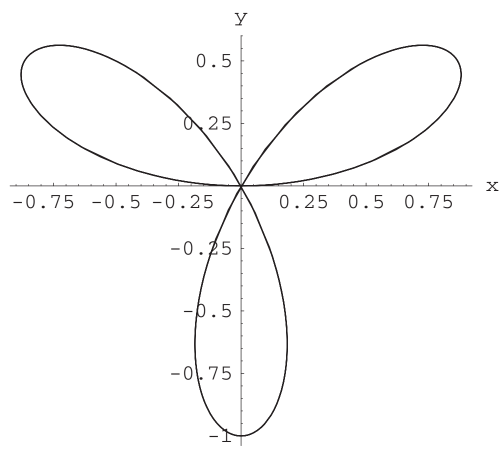
(c) This interesting and different image is shown below left.

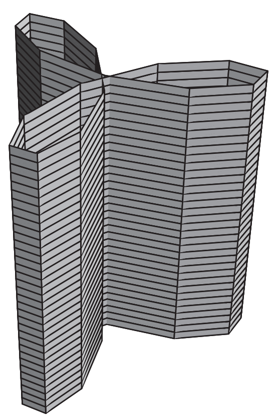


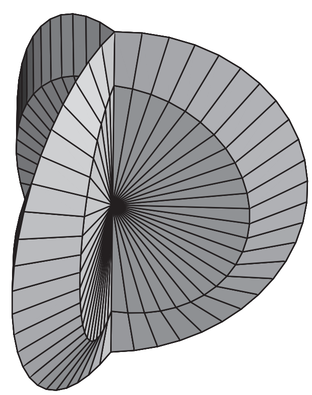


(d) This three leaf version of what we saw in Exercises 38 and 39 is shown above right.

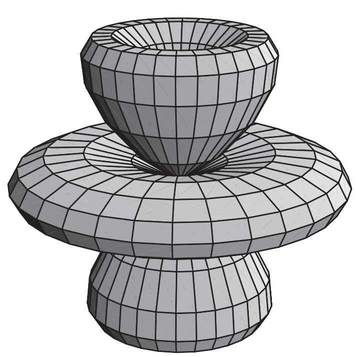
41. The polar plot, cylinder and part (d) are similar to the corresponding solutions for Exercise 40. They are shown below.



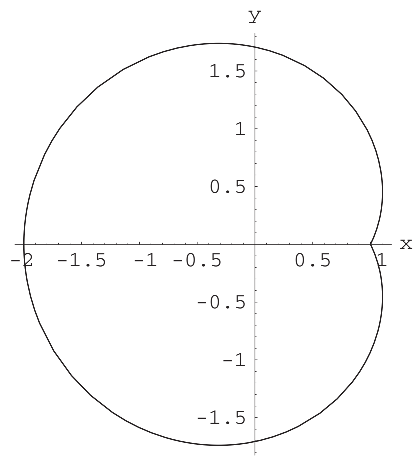


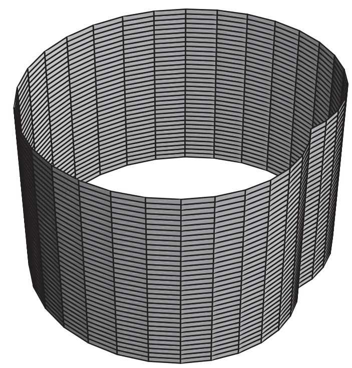


(c) Here in the figure shown below you see a difference in the solid generated by using sine instead of cosine.



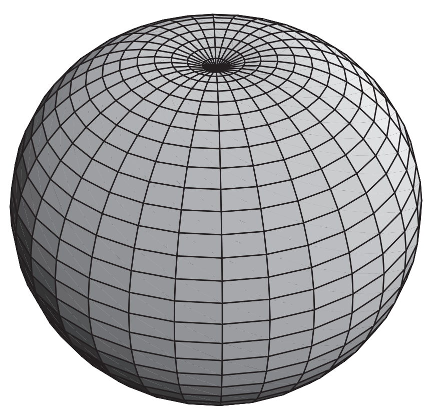
42. **(a)** The nephroid is shown below left.



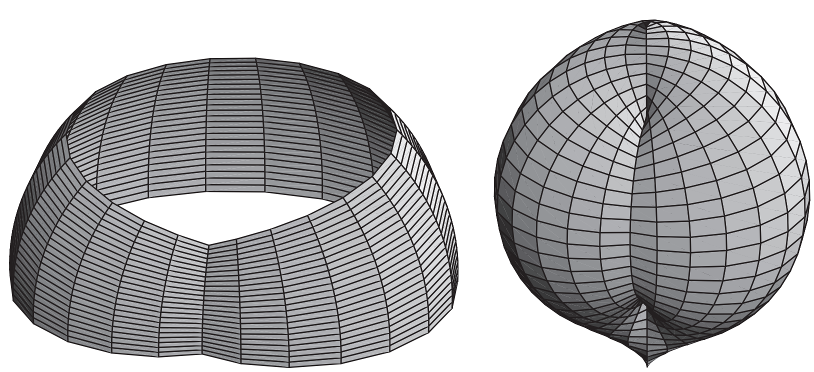


(b) The cylinder based on it is shown above right.

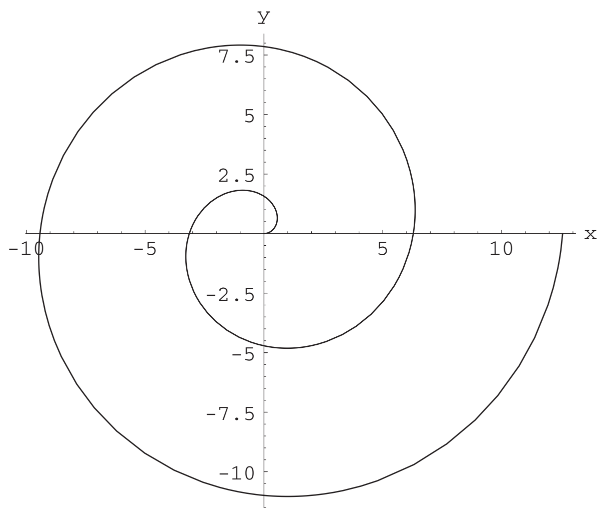
(c) The first spherical graph is a dimpled sphere.

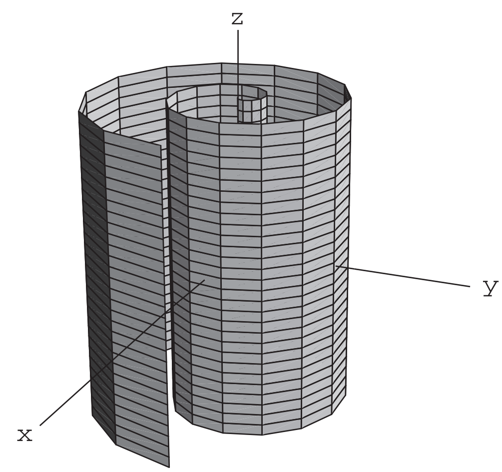


(d) The second spherical graph has a lot of complexity so I have included a partial graph and the completed graph.



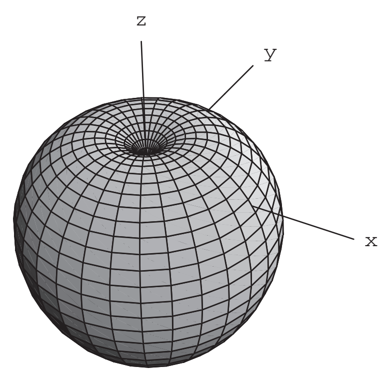
43. (a) The curve is a spiral and is pictured below left.



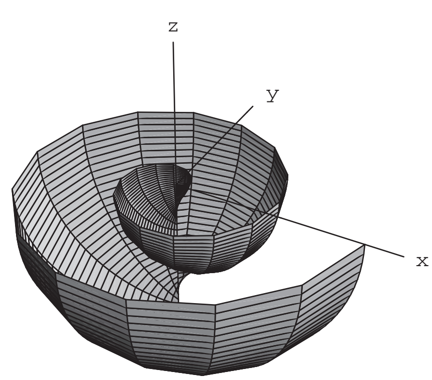


(b) The cylinder based on the spiral in part (a) is shown above right.

(c) Because only part of the spiral is used, the resulting surface is a dimpled ball.



(d) Finally, we see a lovely and intricate shell-like surface.



44. (a) In spherical coordinates the flat top of the hemisphere is the *xy*-plane with spherical equation $\varphi = \pi/2$. The hemispherical bottom has equation $\rho = 5$, but only with $\pi/2 \leq \varphi \leq \pi$. Thus we may describe the object as

$$\{(\rho, \varphi, \theta)|0 \leq \rho \leq 5, \quad \pi/2 \leq \varphi \leq \pi, \quad 0 \leq \theta < 2\pi\}.$$

(b) Now the flat top is described in cylindrical coordinates as $z = 0$ and the bottom hemisphere as $z^2 + r^2 = 25$ with $z \leq 0$, that is, as $z = -\sqrt{25 - r^2}$. Bearing this in mind, the solid object is the set of points

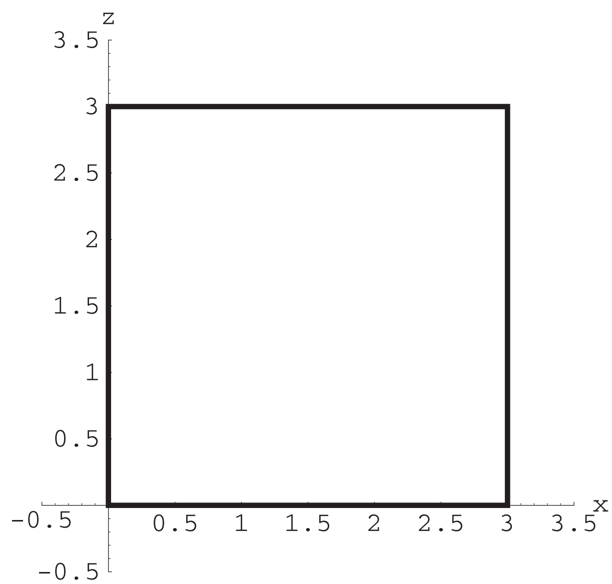
$$\{(r, \theta, z) | -\sqrt{25 - r^2} \le z \le 0, \quad 0 \le r \le 5, \quad 0 \le \theta < 2\pi\}.$$

45. Position the cylinder so that the center of the bottom disk is at the origin and the z-axis is the axis of the cylinder.

(a) In cylindrical coordinates θ is free to take on any values between 0 and 2π. The z-coordinate is bounded by 0 and 3, and $0 \le r \le 3$. To sum up:

$$\{(r, \theta, z) \mid 0 \le r \le 3,\ 0 \le z \le 3,\ 0 \le \theta \le 2\pi\}.$$

(b) Since the solid cylinder is rotationally symmetric about the z-axis, there is no restriction on the θ coordinate, and we may slice the cylinder with the half-plane $\theta =$ constant, in which case we see that the cross section is a filled-in square of side length 3. Consider the cross section by the half-plane $\theta = 0$, pictured below:



The top of the square (which corresponds to the top of the cylinder) has equation $z = 3$, or $\rho \cos \varphi = 3$. Thus the top of the cylinder is the plane $\rho = 3 \sec \varphi$. The bottom is, of course, the plane $z = 0$, which is given by $\rho \cos \varphi = 0$, which implies $\varphi = \pi/2$. The right side of the square, pictured as $x = 3$ in the figure above, corresponds to a cross section of the lateral surface of the cylinder given in cylindrical coordinates as $r = 3$, and thus in spherical coordinates by $\rho \sin \varphi = 3 \iff \rho = 3 \csc \varphi$.

Now fix a value of φ. If this value of φ is between 0 and $\pi/4$, the spherical coordinate ρ must be between 0 and the top of the cylinder $\rho = 3 \sec \varphi$. On the other hand, if this value of φ is between $\pi/4$ and $\pi/2$, the spherical coordinate ρ must be between 0 and the lateral part of the cylinder $\rho = 3 \csc \varphi$. If φ is larger than $\pi/2$, no value of ρ (other than zero) would give a point remaining inside the solid cylinder. To sum up:

$$\{(\rho, \varphi, \theta) \mid 0 \le \rho \le 3 \sec \varphi,\ 0 \le \varphi \le \pi/4,\ 0 \le \theta \le 2\pi\}$$
$$\cup \{(\rho, \varphi, \theta) \mid 0 \le \rho \le 3 \csc \varphi,\ \pi/4 \le \varphi \le \pi/2,\ 0 \le \theta \le 2\pi\}.$$