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Mastering the Fundamentals of Mathematics

Course Workbook

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Professor James A. Sellers is Professor of Mathematics at The Pennsylvania State University, where he has served as the Director of Undergraduate Mathematics since 2001. From 1992 to 2001, he was a mathematics professor at Cedarville University in Ohio.

Professor Sellers received his B.S. in Mathematics in 1987 from The University of Texas at San Antonio, the city in which he was raised. He then went to Penn State, where he received his Ph.D. in Mathematics in 1992. There, he worked under the direction of his Ph.D. adviser, David Bressoud, and learned a great deal about the beauty of number theory—especially the theory of integer partitions. Professor Sellers has written numerous research articles in the area of partitions and related topics. To date, at least 60 of his papers have appeared in a wide variety of peer-reviewed journals. He is especially fond of coauthoring papers with his undergraduate students; his list of coauthors includes 8 undergraduates he has mentored during his career. Professor Sellers was privileged to spend the spring semester of 2008 as a visiting scholar at the Isaac Newton Institute for Mathematical Sciences at the University of Cambridge, pursuing further studies linking the subjects of partitions and graph theory.

Professor Sellers's teaching reputation is outstanding. He was named the Cedarville University Faculty Scholar of the Year in 1999, a truly distinct honor at the institution. At Penn State, he received the Mary Lister McCammon Award for Distinguished Undergraduate Teaching from his department in 2005. One year later, he received the Mathematical Association of America's (MAA) Allegheny Mountain Section Award for Distinguished Teaching. Since then, he has also received Penn State's Teresa Cohen Service Award in 2007, the MAA's Allegheny Mountain Section Mentor Award in 2009, and Penn State's Donald C. Rung Award for Distinguished Undergraduate Teaching in 2011. Recently named a Fulbright Scholar, Professor Sellers will spend a semester in 2012 teaching and doing research at Johannes Kepler University in Linz, Austria.

Professor Sellers has enjoyed many interactions at the high school and middle school levels. He served as an instructor of middle school students in the TexPREP program in San Antonio, Texas, for 3 summers. He also worked with Saxon Publishers on revisions to a number of high school–level textbooks in the 1990s. As a home educator and father of 5, he has spoken to various home education organizations about mathematics curricula and teaching issues.

Professor Sellers is well known as an entertaining and gifted speaker. He has visited numerous college, university, and high school venues to speak about partitions and combinatorics. He has also spoken at a number of conferences and seminars across the United States, sharing results related to his research as well as his views on teaching and advising undergraduate students. ■

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Mastering the Fundamentals of Mathematics

Scope:

This course is designed to provide the skills that form the groundwork of your understanding of numbers and arithmetic. Mastery of these skills is essential for students preparing to study higher-level mathematics, such as algebra, trigonometry, and calculus. However, this course is also ideal for learners who have found arithmetic frustrating and mysterious in the past or who are looking to refresh their memories and sharpen their skills in a subject that they learned long ago.

This engaging 24-lesson course methodically teaches you the essentials of arithmetic—beginning with the operations of addition, subtraction, multiplication, and division of whole numbers and ending with practical lessons that cover statistics and probability along with a gentle introduction to algebra.

The course is carefully crafted to assist every student to understand and master the basics of arithmetic—to build that crucial foundation necessary to confidently study algebra, trigonometry, and calculus. Following a clear step-by-step approach, this course will guide you through all the major topics of beginning mathematics: arithmetic of whole numbers; equivalent ways to represent fractions; adding, subtracting, multiplying, and dividing fractions; arithmetic with decimal numbers; connections between fractions, decimals, and percents; and practical problems with percentages, including sales tax and tip calculations. The course addresses themes that can sometimes frustrate students: ratios and proportions, exponents and order of operations, and arithmetic with negative and positive integers. The course then expands on these basics by considering other important topics, such as square roots, raising numbers to negative or fractional powers, graphing in the coordinate plane, and an introduction to 2-dimensional geometry. The course concludes by addressing some very important topics in elementary number theory—such as prime numbers and greatest common divisors—along with introductory statistics, probability, and algebra.

Throughout your journey, you will discover the importance of mastering the steps involved in any kind of arithmetic problem, understand why those steps are important, and build an intuition—or "number sense"—about whether an answer is potentially incorrect. Such insights are critical for students who want to move forward in their studies of mathematics. Of course, a mastery of mathematics is also important for students who wish to continue their studies in the sciences, engineering, statistics, and numerous related areas.

This course will help you reach your potential for understanding arithmetic. As you complete each lesson, you will be amazed to find yourself feeling less nervous and more empowered. In no time, you will be ready to move forward in your studies—to algebra and beyond! ■

Addition and Subtraction

Lesson 1

Topics in This Lesson

- Place value.
- Adding whole numbers.
- Subtracting whole numbers.
- Carrying.
- Borrowing.

Summary

This lesson begins with a general overview of the course, what will be taught, and the importance of the material. Next, general mathematical concepts are covered, such as the place value for each digit of a number, how to add and subtract whole numbers, and how to "carry" and to "borrow."

Definitions

borrow: When subtracting numbers, if the number in the top position of a column is smaller than the number below (the one being subtracted), you subtract 1 from the number to the left of the top number, and add 10 to the number from which you are currently subtracting. Because the number to the left has a place value that is 10 times the number to the right, you are "borrowing" 10 from that number.

carry: When adding numbers, if the answer in a column is more than 1 digit in size, then the digit in the tens place of the answer is "carried" to the column to the left to be added.

digit: An individual number from 0 to 9 that can be used to make up a larger number. For example, 786 has 3 digits, and 76,566 has 5 digits.

place value: The value assigned to a digit based on where it is placed in the number.

whole number: Any of the numbers 0, 1, 2, 3, 4, and so on that are not fractions or decimals.

Example from the Lesson

Calculate $842 - 791$.

First, write the numbers vertically, lining up the corresponding digits.

$$\begin{array}{r} 842 \\ -\,791 \\ \hline \end{array}$$

Then, start subtracting from the right-most column and working over to the left. First, 2 − 1 = 1 (no borrowing required), so we just write a 1 under the 1 in 791.

$$\begin{array}{r} 842 \\ -\,791 \\ \hline 1 \end{array}$$

Next, we want 4 − 9. Now, we have to borrow because 4 is less than 9. (It would be like having 4 $10 bills but needing to give someone 9 $10 bills.) So, we change the 8 to a 7 and change the 4 to a 14. Then, we have 14 − 9 = 5, so we write a 5 under the 9.

$$\begin{array}{rrr} {}^{7}\cancel{8} & {}^{1}4 & 2 \\ -\,7 & 9 & 1 \\ \hline & 5 & 1 \end{array}$$

Lastly, we have 7 − 7 = 0. We could write a 0 under the 7 in 791, but this is unnecessary because putting a 0 in front of a number doesn't change the value. We know that 051 is the same as 51, so the final answer is just 51.

$$\begin{array}{rrr} {}^{7}\cancel{8} & {}^{1}4 & 2 \\ -\,7 & 9 & 1 \\ \hline & 5 & 1 \end{array}$$

Additional Example

Add 364 + 984.

The first thing to do is to write the problem vertically, lining up the corresponding digits.

$$\begin{array}{r} 364 \\ +\,984 \\ \hline \end{array}$$

Now, starting from the right (the units column), begin to add, writing the answers below. Because 4 + 4 = 8, write the 8 in the answer section in the units column.

$$\begin{array}{r} 364 \\ +\,984 \\ \hline 8 \end{array}$$

Now, we add the next column to the left, the tens column. We know that 6 + 8 = 14, but we are only able to write 1 digit under each column. So, we carry the 1 from the 14 to the next column to the left and write a 4 under the middle column.

$$\begin{array}{r} \overset{1}{3}64 \\ +\,984 \\ \hline 48 \end{array}$$

Now, we add the final column to the left, not forgetting to add the 1 that we carried: 1 + 3 + 9 = 13. Wait! We can only write 1 number below each column, and there isn't another column to carry our 1 into. You can picture it by just carrying the 1 to the invisible column and adding. However, when it is the farthest column to the left, you simply write both digits under the column.

$$\begin{array}{r} \overset{1}{3}64 \\ +\,984 \\ \hline 1348 \end{array}$$

The final answer is 1348.

Avoiding Common Errors

- When adding, do not forget that only 1 digit can be written in the answer section below a column. The other digit must be carried to the column to the left and added.
- When you borrow, you must remember to decrease the number to the left by 1 and increase the number from which you are subtracting by 10.

Study Tips

- It is very important to keep the columns lined up and write the answer directly below the number being added or subtracted. Consider turning notebook paper sideways and writing 1 number in each column made by the blue lines.
- Writing addition and subtraction problems vertically will help you keep track of which digits should be added or subtracted.
- Estimate what you think an answer might be before you add or subtract. If your estimate is significantly different from the answer you got, see where you might have made a mistake.
- It is sometimes helpful to think of math problems in terms of money. For example, 225 + 50 could be thought of as 2 dollars and 3 quarters (if you think purely in terms of cents).

PROBLEMS

Compute each of the following.

1. $521 + 454$
2. $52 + 428$
3. $835 + 466$
4. $830 + 614$
5. $471 + 353$
6. $944 + 215$
7. $269 + 605$
8. $1951 + 953$
9. $4959 + 160$
10. $4709 + 5503$
11. $544 - 338$
12. $578 - 182$
13. $980 - 352$
14. $322 - 49$
15. $202 - 115$
16. $991 - 258$
17. $5926 - 3661$
18. $8471 - 6154$
19. $3657 - 899$
20. $2030 - 1748$

Multiplication

Lesson 2

Topics in This Lesson

- Determining how addition and multiplication are related.
- Multiplying whole numbers.

Summary

This lesson begins with an explanation of how addition and multiplication are related. Next, it reviews the steps needed for multiplying whole numbers.

Definitions

times: A way of saying "multiplied by." For example, 6 times 7 can be written as 6 × 7.

units digit: The digit to the far right in a number. It is sometimes referred to as the ones digit.

whole number: Any of the numbers 0, 1, 2, 3, 4, and so on that are not fractions or decimals.

Example from the Lesson

Calculate 75 × 14.

First, write the problem vertically, lining up the 5 and 4 (so that both numbers are "right justified" if you want to think of the numbers that way).

$$\begin{array}{r} 75 \\ \times 14 \\ \hline \end{array}$$

Now, we are going to start with the units digit in 14 (that would be the 4), and we are going to multiply that digit with all the digits in 75—1 at a time. So, we first multiply the 4 with the 5: 5 × 4 = 20.

Place a 0 in the answer location, lined up vertically with the 5. Carry the 2 over to just above the 7 in 75.

$$\begin{array}{r} {}^{2} \\ 75 \\ \times 14 \\ \hline 0 \end{array}$$

Next, we multiply the 4 with the 7: 7 × 4 = 28.

Don't forget the number that was carried! It must be added into this amount. That carry was a 2, so we need 28 + 2 = 30. How do we write this? Well, align the 0 in the 30 just under the 7 in 75, and because there are no more digits in 75 that we must multiply with the 4, we can just put the 3 in 30 next to the 0 that we already put in the answer location. So, the answer location at this point should be showing us 300.

$$\begin{array}{r} 75 \\ \times 14 \\ \hline 300 \end{array}$$

We aren't done yet! Now, we must multiply that 1 in 14 with each of the digits in 75. However, a new layer of the process needs to be added. It is straightforward—there's nothing to worry about—but it does require us to be careful. So, let me explain it in a step-by-step fashion.

First, it is important to remember that the 1 in 14 is in the tens digit—not the ones digit. We need to keep track of that somehow in the answer section of the problem. So, here's what we do: Just below the 300, on a second line, we are going to write a 0 in the units digit. Why? Because it is going to keep everything aligned and keep track of the fact that the 1 in 14 is in the tens digit.

$$\begin{array}{r} 75 \\ \times 14 \\ \hline 300 \\ 0 \end{array}$$

Next, we start multiplying that 1 with all the digits in 75, working right to left: 5 × 1 = 5.

So, we write a 5 just to the left of that 0 that we just wrote. There is no carrying here because our answer, 5, is less than 10.

$$\begin{array}{r} 75 \\ \times 14 \\ \hline 300 \\ 50 \end{array}$$

Next, we compute 7 × 1 = 7 and write that to the left of the 5 we just wrote. So, what is in that second line of the answer section? 750. This is correct because 75 × 10 = 750. (Remember: That 1 represents a 10 because the 1 is in the tens digit.)

$$\begin{array}{r} 75 \\ \times 14 \\ \hline 300 \\ 750 \end{array}$$

Now what do we do? Well, believe it or not, we now add! We need to add the 300 and the 750, which are already in vertical form. So, we add—again from right to left.

0 + 0 = 0.
0 + 5 = 5.
3 + 7 = 10.

$$\begin{array}{r} 75 \\ \times 14 \\ \hline 300 \\ 750 \\ \hline 1050 \end{array}$$

The final answer for 75 × 14 is 1050.

Additional Example

Calculate 460 × 38.

Begin by writing the numbers vertically.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline \end{array}$$

Start by multiplying the units digit of the bottom number by each digit of the upper number, and write the answers below each column. Because any number times 0 is 0, 0 × 8 = 0. So, we write a 0 in the units digit of the answer section.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline 0 \end{array}$$

Next, 6 × 8 = 48. Write the 8 below the 3 and carry the 4 above the 4 in the column to the left.

$$\begin{array}{r} \overset{4}{4}60 \\ \times\ 38 \\ \hline 80 \end{array}$$

Next, multiply 4 × 8. We get 32 as our answer, but we need to add the 4 that was carried. So, 32 + 4 = 36. Write this in the answer section—just to the left of the 80 that already appears there.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline 3680 \end{array}$$

Now, add a placeholder 0 in the next line to note that the 3 in 38 is in the tens place.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline 3680 \\ 0 \end{array}$$

We then multiply 3 by each digit in 460. Because 3 × 0 = 0, we insert a 0 to the left of the 0 we already placed in the second line of the answer section.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline 3680 \\ 00 \end{array}$$

Then, 3 × 6 = 18. We need to write the 8 in the answer section and carry the 1.

$$\begin{array}{r} \overset{1}{4}60 \\ \times\ 38 \\ \hline 3680 \\ 800 \end{array}$$

Next, 4 × 3 = 12. Then, add the 1 that was carried to get 12 + 1 = 13. Write this in the answer section.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline 3680 \\ 13800 \end{array}$$

Now, add the columns to get the final answer.

$$\begin{array}{r} 460 \\ \times\ 38 \\ \hline 3680 \\ 13800 \\ \hline 17480 \end{array}$$

The final answer is 17,480.

Avoiding Common Errors

- Do not forget that only 1 digit can be written below a column in the answer section. The other digit must be carried to the column to the left and added after the next 2 numbers are multiplied.
- Do not add the carry amount to the digit before multiplying. Once the 2 digits are multiplied, then you can add the carry to the answer.
- Do not forget to add a placeholder 0 to the answer lines if multiplying 2- or 3-digit numbers or more. An additional placeholder 0 should be added for each row. For example, the first answer row has no 0. The second row has 1 placeholder 0. The third row would have 2 placeholder 0s, and so on.

Study Tips

- It is very important to keep the columns lined up and to write the answer directly below the number being multiplied. Consider turning notebook paper sideways and writing 1 number in each column made by the blue lines.
- Do not become overwhelmed at the beginning of the problem if the numbers involved have many digits. Remember: Just walk through the process step by step and give yourself enough time to finish the problem. Don't try to rush through the problem.

PROBLEMS

Compute each of the following.

1. 26 × 6

2. 73 × 7

3. 15 × 4

4. 44 × 7

5. 72 × 5

6. 38 × 19

7. 82 × 52

8. 39 × 16

9. 54 × 29

10. 62 × 20

11. 48×46

12. 55×34

13. 98×45

14. 99×74

15. 853×34

16. 557×47

17. 105×41

18. 327×57

19. 185×157

20. 477×465

Long Division

Lesson 3

Topics in This Lesson

- The meaning of division.
- Division of whole numbers.
- Long division with a 1-digit divisor.
- Division with remainders.

Summary

This lesson begins with the concept of division and what the term “division” really means. The steps to long division using a 1-digit divisor are then covered.

Definitions

dividend: The number that is being divided into groups. It is the number on the inside of the division bar or the top number if a division problem is written as a fraction.

divisor: A number that divides another number. In number theory, it is a number that divides another number evenly. It is the number on the outside of the division bar—or the bottom number if a division problem is written as a fraction.

quotient: The quantity we obtain by dividing the dividend by the divisor. It’s the answer to a division problem.

remainder: What “remains” after a division problem is finished. It is the part that remains after the last subtraction occurs. It is always less than the divisor.

Example from the Lesson

Calculate 897 ÷ 3.

Begin by rewriting the problem with a division bar.

$$3\overline{)897}$$

First, we ask ourselves: “How many times does 3 go into 8?” Well, you may notice that 3 doesn’t divide 8 evenly—but that doesn’t have to stop us. We just want to know how many 3s we can subtract from 8. Find the number that you can multiply to 3 to get as close to 8 without going over. We choose 2 because 3 × 2 = 6 (which is less than 8) and because 3 × 3 = 9 is too large. Write a 2 above the 8 in the answer section (on the horizontal line segment), and write a 6 below the 8 because 3 × 2 = 6. Then, we subtract to get a 2 and write it below.

$$\begin{array}{r} 2 \\ 3\overline{)897} \\ -6 \\ \hline 2 \end{array}$$

Now, we drop down the next digit in the dividend that we haven’t yet looked at, the 9, so that we have 29 at the bottom of our work.

$$\begin{array}{r} 2 \\ 3\overline{)897} \\ -6 \\ \hline 29 \end{array}$$

Next, we ask the same question: How many 3s are in 29? Well, there are 9 of them because 3 × 9 = 27 (which is less than 29) while 3 × 10 = 30 (which is more than 29). So, write a 9 above the 9 in the dividend, and then write a 27 underneath the 29 at the bottom of our work.

$$\begin{array}{r} 29 \\ 3\overline{)897} \\ -6 \\ \hline 29 \\ -27 \\ \hline \end{array}$$

Subtract these numbers and write the answer, which is 2, at the bottom of the work section. We then drop down the next digit in the dividend, the 7 at the end of 897.

$$\begin{array}{r} 29 \\ 3\overline{)897} \\ -6 \\ \hline 29 \\ -27 \\ \hline 27 \end{array}$$

Ask the same question again: How many 3s are in 27? The answer is 9 because 3 × 9 = 27. So, we write a 9 above the 7 in 897, and because 3 × 9 = 27, we write a 27 underneath the 27 at the bottom of our work.

$$\begin{array}{r} 29 \\ 3\overline{)897} \\ -6 \\ \hline 29 \\ -27 \\ \hline 27 \\ -27 \\ \hline 0 \end{array}$$

When we subtract at the bottom, we get 0. Because there are no more digits in the dividend, we stop—and we have our answer: 897 ÷ 3 = 299. Therefore, if we divided up 897 identical pieces of candy to 3 different children (in as even a way as possible), then each child would receive 299 pieces.

Check that your answer is correct. Multiply 299 by 3. You will find that 299 × 3 = 897 if you do the multiplication work that we completed in the previous lesson.

Additional Example

Calculate 3203 ÷ 5.

Begin by rewriting the problem with a division bar.

$$5\overline{)3203}$$

We start the process by asking: “How many times does 5 go into 3?” Because 5 is larger than 3, the answer is 0 times. So, we have to take the first 2 digits of the dividend and ask: “How many times does 5 go into 32?” The answer is 6 because 5 × 6 = 30, which is close to 30 without going over (5 × 7 = 35, which is larger than 32). Write a 6 above the last digit used in the dividend, which is 2. Write 30 below the 32, and subtract to get 2.

$$\begin{array}{r} 6 \\ 5\overline{)3203} \\ -30 \\ \hline 2 \end{array}$$

Now, drop down the next digit in 3203 that has not been used, the 0.

$$\begin{array}{r} 6 \\ 5\overline{)3203} \\ -30 \\ \hline 20 \end{array}$$

Ask: "How many times does 5 go into 20?" The answer is 4 because 5 × 4 = 20. Write a 4 above the 0 in the dividend, and write 20 under the 20. Then, subtract to get 0.

$$\begin{array}{r} 64 \\ 5\overline{)3203} \\ -30 \\ \hline 20 \\ -20 \\ \hline 0 \end{array}$$

Drop down the next digit in the dividend, which is 3. We now have 03, which is simply 3. Ask: "How many times does 5 go into 3?" The answer is 0 times because 5 × 1 = 5, and 5 is larger than 3. Write a 0 on the answer line, and write a 0 below the 3. Then, subtract.

$$\begin{array}{r} 640 \\ 5\overline{)3203} \\ -30 \\ \hline 20 \\ -20 \\ \hline 03 \\ -0 \\ \hline 3 \end{array}$$

We have a 3 remaining from our division problem, and there are no other digits in the dividend to drop down. So, there is a remainder of 3. Therefore, 3203 ÷ 5 has an answer of 640 R 3, or 640 with a remainder of 3.

Check your work by multiplying the answer, or quotient, by the divisor and adding the remainder: 640 × 5 = 3200 and 3200 + 3 = 3203, so our answer is correct.

Avoiding Common Errors

- Make sure to drop down the next digit in the number before proceeding to divide.
- The divisor always goes on the outside of the division bar. It is the second number when a division problem is written in the following form: number 1 ÷ number 2.
- Be sure to keep your answer work aligned with the correct digits in the dividend. Errors in this alignment can lead to digits in the dividend that are missed or skipped over.

Study Tips

- It is very important to keep the columns lined up. Consider turning notebook paper sideways and writing 1 number in each column made by the blue lines.
- Be sure to check your answer by multiplying the divisor and quotient and adding the remainder.

- If your remainder or the answer after you subtract is larger than the divisor, you have chosen a number that is too small.

- Knowing basic multiplication facts will help immensely with division. Consider using flashcards or a computer flashcard drill program.

PROBLEMS

Compute each of the following.

1. 96 ÷ 4

2. 60 ÷5

3. 288 ÷ 3

4. 516 ÷ 6

5. 352 ÷ 8

6. 3025 ÷ 5

7. 3762 ÷ 6

8. 3682 ÷ 7

9. 471 ÷ 5

10. 407 ÷ 9

11. 3806 ÷ 8

12. 2628 ÷ 7

13. 5412 ÷ 11

14. 5244 ÷ 12

15. 6798 ÷ 11

16. 3000 ÷ 12

17. 8904 ÷ 12

18. 4695 ÷ 15

19. 2694 ÷ 11

20. 7565 ÷ 12

Introduction to Fractions

Lesson 4

Topics in This Lesson

- Definition of fractions.
- Numerators and denominators.
- Equivalent fractions.
- Reducing fractions.
- Improper fractions.
- Mixed numbers.

Summary

This lesson explains what fractions are and defines the parts of a fraction. Next, equivalent fractions are discussed, and 2 ways of determining whether 2 fractions are equivalent are shown. Then, how to reduce fractions is demonstrated. A discussion of mixed numbers, improper fractions, and how to change from one to the other ends this lesson.

Definitions

denominator: The bottom number in a fraction. It tells us how many parts are in the whole.

equivalent fractions: Any 2 fractions that have different numerators and denominators but represent the same amount. If $\frac{a}{b}$ and $\frac{c}{d}$ are 2 equivalent fractions, then $a \times d = b \times c$.

fraction: A part of a whole number. It is usually written as a ratio of 2 whole numbers with 1 number written, a division bar drawn underneath it, and the second number written below.

improper fraction: A fraction whose numerator is larger than the denominator. For example, $\frac{54}{23}$ and $\frac{13}{15}$ are both improper fractions.

mixed number: A whole number that is combined with a fraction. For example, $2\frac{1}{2}$ is a mixed number.

numerator: The top number in a fraction. It tells us how many parts we have of the whole.

reducing: Making the numerator and denominator the lowest numbers they can be with no common factors.

Formulas and Rules

converting from a mixed number to an improper fraction: Multiply the whole number by the denominator and add the numerator.

converting from an improper fraction to a mixed number: The numerator is the dividend, and the denominator is the divisor. Divide, and make the quotient into the whole number part of the mixed number. Make the remainder the numerator of the fraction, and keep the divisor as the denominator.

Example from the Lesson

Determine which is larger: $\frac{7}{8}$ or $\frac{10}{12}$

We need to cross multiply and find the larger amount. Multiply the numerator (the top number) of 1 fraction with the denominator (the bottom number) of the other fraction.

$7 \times 12 = 84.$
$10 \times 8 = 80.$

Which number is larger? Because $84 > 80$, we need to look at 7×12. Which one was the numerator in that case? It was the 7.

So, $\frac{7}{8}$ is larger than $\frac{10}{12}$.

Additional Example

Convert $\frac{45}{7}$ to a mixed number.

We need to figure out how many 7s are in 45 and how much is left over. To do this, we divide 7 into 45. We know that $7 \times 6 = 42$, which is less than 45. We also see that $7 \times 7 = 49$, which is greater than 45. So, we know that 7 divides 45 6 times. Moreover, we know that the remainder in that problem is 3 because $45 - 42 = 3$.

To write our answer as a mixed number, we first write 6 for the whole number. Then, we take our remainder, 3, and write it as the numerator (the top number) and the divisor, 7, as the denominator (the bottom number) because we have 3 parts of 7. Therefore, our final answer is $6\frac{3}{7}$.

Avoiding Common Errors

- When using cross multiplying to determine which fraction is larger, make sure that you multiply the denominator of 1 fraction with the numerator of another. Do not multiply the numerators and denominators of the same fraction.

- When writing a mixed number, make sure to use the divisor (the number outside the division bar) as the bottom number for the fraction. The remainder is the numerator (the top number) of the fraction in the mixed number.

- The numerator of a fraction is not always smaller than the denominator. If the number on the top (the numerator) is larger than the number on the bottom (the denominator), it is an improper fraction and can be written as a mixed number.

Study Tips

- A mixed number and its equivalent improper fraction will always have the same denominator (if both fractions are in reduced terms).
- Learning multiplication and division facts will make the process of reducing fractions much easier.
- The numerator of a fraction is always the number that goes inside the division bar. It is always the dividend. The denominator is the same as the divisor and goes on the outside of the division bar.

PROBLEMS

Write each of the following fractions in reduced terms.

1. $\frac{36}{48}$

2. $\frac{8}{68}$

3. $\frac{56}{96}$

4. $\frac{21}{469}$

5. $\frac{78}{768}$

6. $\frac{126}{574}$

Determine whether the following pairs of fractions are equivalent.

7. $\frac{3}{4}, \frac{24}{32}$

8. $\frac{10}{14}, \frac{25}{35}$

9. $\frac{12}{15}, \frac{11}{14}$

10. $\frac{5}{9}, \frac{34}{63}$

11. $\frac{9}{13}, \frac{36}{52}$

Determine which fraction is larger.

12. $\frac{13}{14}, \frac{50}{56}$

13. $\frac{28}{63}, \frac{3}{7}$

14. $\frac{10}{27}, \frac{20}{52}$

15. $\frac{38}{45}, \frac{8}{9}$

16. $\frac{42}{66}, \frac{15}{22}$

Convert each of the following improper fractions to mixed numbers.

17. $\frac{43}{6}$

18. $\frac{78}{5}$

19. $\frac{613}{10}$

20. $\frac{185}{12}$

Adding and Subtracting Fractions

Lesson 5

Topics in This Lesson

- Adding and subtracting fractions with like denominators.
- Finding lowest common denominators.
- Adding and subtracting fractions with unlike denominators.

Summary

Pizza slices are used to illustrate the concept of adding fractions that have the same denominator. In this lesson, the importance of reducing fractions to their lowest possible terms is explained, and subtracting fractions with like denominators is introduced. Next, students learn what the lowest common denominator is, how to find it, and its importance in adding and subtracting fractions with unlike denominators. Finally, how to add and subtract fractions with unlike denominators is demonstrated.

Definitions

denominator: The bottom number in a fraction. It tells us how many parts are in the whole.

factor: A number that divides another number evenly, leaving no remainder. It is the same as a divisor.

fraction: A part of a whole number. It is usually written as a ratio of 2 whole numbers with 1 number written, a division bar drawn underneath it, and the second number written below.

improper fraction: A fraction whose numerator is larger than the denominator. For example, $\frac{54}{23}$ and $\frac{13}{15}$ are both improper fractions.

least common denominator (LCD): The smallest number that 2 unlike denominators will divide, or the smallest number for which the 2 given denominators are factors. For example, if the denominators were 3 and 4, the least common denominator would be 12.

like denominators: Having the same denominator in 2 or more fractions.

mixed number: A whole number that is combined with a fraction. For example, $2\frac{1}{2}$ is a mixed number.

multiples of a number: The numbers you get when you multiply a given number by 1, then 2, then 3, etc. For example, the multiples of 4 are 4, 8, 12, 16, 20, 24 … .

numerator: The top number in a fraction. It tells us how many parts we have of the whole.

reducing: Making the numerator and denominator the lowest numbers they can be with no common factors.

Example from the Lesson

Calculate $\frac{5}{6}+\frac{3}{8}$.

First, check to see if the denominators are the same or different. Because they are different, we need to go through the process of finding the least common denominator in this problem.

Let's write down the multiples of 6 and then see which one is the smallest that also has 8 as a factor: 6, 12, 18, 24, 30, 36.

So, 8 is definitely not a factor of 6, and it is also not a factor of 12 (8 does not divide 12 evenly or without a remainder). Also, 8 does not divide 18 evenly, but 8 does divide 24 evenly. So, we know that the least common denominator for this problem is 24.

Next, we need to convert $\frac{5}{6}$ and $\frac{3}{8}$ into fractions whose denominators are both 24. What do we need to multiply 6 by to get 24? Well, the answer is 4. So, we multiply both 5 and 6 by 4 to get $\frac{20}{24}$.

That is the fraction with a denominator of 24 that is equivalent to $\frac{5}{6}$. (This fact can be checked with our cross-multiplying technique.)

What about the $\frac{3}{8}$? We need to multiply the 8 by 3 to get 24. So, we multiply both the 3 and 8 in the original fraction by 3 to get $\frac{9}{24}$. That will be our new version of the second fraction.

So, our original problem, $\frac{5}{6}+\frac{3}{8}$, has now been converted to $\frac{20}{24}+\frac{9}{24}$. Now that the denominators are the same, it is simple to add the fractions. The answer, then, is $\frac{29}{24}$ because 20 + 9 = 29.

Additional Example

Calculate $8\frac{3}{4}-3\frac{2}{5}$.

The first thing we will try to do is compute $\frac{3}{4}-\frac{2}{5}$. To do this, we check the denominators. Are they alike? Because they are not, we need to find the lowest common denominator. The factors of 4 are 4, 8, 12, 16, 20, 24, 28, ... The factors of 5 are 5, 10, 15, 20, 25, 30 … . The first factor in common in those 2 lists is 20, so that is the lowest common denominator.

To make $\frac{3}{4}$ into a fraction with 20 as the denominator, we multiply both the top and bottom of the fraction by 5. Therefore, the fraction $\frac{3}{4}$ can be rewritten as $\frac{15}{20}$. Next, to turn $\frac{2}{5}$ into a fraction with 20 as the denominator, we multiply both the top and bottom of the fraction by 4, which will turn $\frac{2}{5}$ into the equivalent fraction $\frac{8}{20}$. Therefore, $\frac{3}{4}-\frac{2}{5}$ is the same as $\frac{15}{20}-\frac{8}{20}$, which is the same as $\frac{7}{20}$. Because we also know that 8 − 3 = 5, we know that our final answer is $5\frac{7}{20}$.

Avoiding Common Errors

- When adding or subtracting fractions (with like denominators), only the numerators are added or subtracted. The denominators stay the same.
- You cannot add or subtract fractions with unlike denominators unless you first change them to having like denominators.
- Remember to multiply both the numerator and denominator of a fraction by the same number when finding like denominators.

Study Tips

- Writing addition and subtraction problems with mixed numbers in a vertical way may help.
- A quick way to find a common denominator is to multiply the 2 denominators together. However, this will not necessarily be the *least* common denominator; in other words, this might make for more reducing of the fraction when the addition or subtraction is completed.

PROBLEMS

Compute each of the following.

1. $\frac{2}{9}+\frac{5}{9}$

2. $\frac{3}{5}+\frac{1}{5}$

3. $\frac{2}{15}+\frac{11}{15}$

4. $\frac{7}{11}-\frac{2}{11}$

5. $\frac{19}{35}-\frac{6}{35}$

6. $\frac{23}{40}-\frac{10}{40}$

7. $\frac{3}{4}+\frac{7}{4}$

8. $\frac{9}{14}+\frac{7}{14}$

9. $\frac{19}{20}-\frac{7}{20}$

10. $\frac{57}{42}-\frac{35}{42}$

11. $\frac{9}{13}+\frac{15}{26}$

12. $\frac{5}{8}+\frac{7}{12}$

13. $\frac{25}{21}+\frac{20}{14}$

14. $\frac{61}{30}+\frac{7}{40}$

15. $\frac{38}{45}+\frac{13}{30}$

16. $\frac{47}{66}-\frac{15}{22}$

17. $\frac{13}{6}-\frac{5}{8}$

18. $\frac{13}{15}-\frac{8}{20}$

19. $\frac{63}{25}-\frac{18}{15}$

20. $\frac{19}{12}-\frac{7}{20}$

Multiplying Fractions

Lesson 6

Topics in This Lesson

- Multiplying fractions.
- Reducing fractions.
- Multiplying whole numbers and fractions.
- Multiplying mixed numbers and improper fractions.

Summary

The basic operation of multiplying 2 fractions is shown with a discussion of how to reduce the product after multiplication has occurred. Cancellation is also introduced. Finally, a demonstration of multiplying whole numbers, mixed numbers, and improper fractions is given.

Definitions

canceling: Taking a common factor out of the numerator of 1 fraction and the denominator of the same fraction or another fraction. This can only be done in multiplication and division—never in addition and subtraction.

denominator: The bottom number in a fraction. It tells us how many parts are in the whole.

improper fraction: A fraction whose numerator is larger than the denominator. For example, $\frac{54}{23}$ and $\frac{13}{15}$ are both improper fractions.

lowest terms: If a fraction has been reduced, it is said to be in lowest terms.

mixed number: A whole number that is combined with a fraction. For example, $2\frac{1}{2}$ is a mixed number.

numerator: The top number in a fraction. It tells us how many parts we have of the whole.

product: The answer to a multiplication problem.

reducing: Making the numerator and denominator the lowest numbers they can be with no common factors.

Example from the Lesson

Calculate $\frac{12}{15}\times\frac{35}{22}$.

Although we can multiply 12 × 35 and 15 × 22, it is not the wisest way to complete this problem. So, let's try some reducing and canceling.

Because 12 and 22 are both even, we can cancel a 2 from each of those, reducing the problem to $\frac{6}{15}\times\frac{35}{11}$.

Next, we see that a 5 can be canceled from the 15 and the 35; $\frac{35}{5} = 7$ and $\frac{15}{5} = 3$, so we can rewrite the problem as $\frac{6}{3}\times\frac{7}{11}$.

Before multiplying out, there is one more thing we can do. Notice that the 6 and the 3 both have a factor of 3 in them, so $\frac{6}{3}$ can be rewritten as $\frac{2}{1}$, which makes our problem $\frac{2}{1}\times\frac{7}{11}$.

There is no more canceling that we can do, so let's multiply the numerators and the denominators. Because 2 × 7 = 14 and 1 × 11 = 11, our answer must be $\frac{14}{11}$. Notice that this fraction is already in reduced terms because the only factor of 11 to consider is 11, and 11 is definitely not a divisor of 14. So, our final answer is $\frac{14}{11}$, or $1\frac{3}{11}$.

Additional Example

Multiply $4\frac{2}{3}\times6\frac{3}{4}$.

We begin by writing the mixed numbers as improper fractions; we do this by multiplying the whole number by the denominator and adding the numerator.

$$4\frac{2}{3}=\frac{14}{3}.$$

$$6\frac{3}{4}=\frac{27}{4}.$$

So the original problem can be rewritten as $\frac{14}{3}\times\frac{27}{4}$.

Next, we see if we can reduce or cancel. The numbers 27 and 3 have a common factor of 3, so we can take a 3 from each of them, leaving us with 9 and 1. The 14 and the 4 are both even, so we can take a 2 from each of them, leaving us with 7 and 2. Therefore, the problem has been reduced to $\frac{7}{1}\times\frac{9}{2}$.

There is no other canceling or reducing to do, so we multiply the numerators with one another and multiply the denominators with one another, which gives us an answer of $\frac{63}{2}$.

Because 63 is not even, we cannot divide it by 2. This improper fraction is in lowest terms. To write our answer as a mixed number, we divide 63 by 2 to get 31 with a remainder of 1, so the improper fraction is $31\frac{1}{2}$.

Avoiding Common Errors

- You can only use cancellation when multiplying and dividing fractions; you cannot use it when adding or subtracting.

- You cannot cancel values from 2 numerators or 2 denominators; you must remove the same factor from a numerator and a denominator.

- Always change mixed numbers to improper fractions before multiplying. If you simply multiply the whole numbers together and then the fractions together, your answer will almost always be incorrect (because you also have to multiply the whole numbers by the fractions).

Study Tips

- Learning basic multiplication and division facts will help reducing and canceling go much more quickly.
- When multiplying mixed numbers, you can estimate the answer by multiplying the 2 whole numbers together. If your answer is significantly different, check your work.

PROBLEMS

Compute each of the following.

1. $\frac{2}{3}\times\frac{4}{5}$
2. $\frac{4}{5}\times\frac{3}{7}$
3. $\frac{7}{10}\times\frac{3}{8}$
4. $\frac{7}{11}\times\frac{2}{7}$
5. $\frac{9}{35}\times\frac{5}{3}$
6. $\frac{21}{40}\times\frac{10}{7}$
7. $\frac{3}{4}\times\frac{20}{21}$
8. $\frac{9}{14}\times\frac{21}{6}$
9. $\frac{25}{20}\times\frac{18}{35}$
10. $\frac{57}{42}\times\frac{35}{19}$
11. $\frac{15}{26}\times 52$

12. $\frac{7}{12}\times 18$

13. $6\times\frac{5}{21}$

14. $20\times\frac{9}{30}$

15. $\frac{8}{45}\times 2\frac{7}{10}$

16. $\frac{35}{66}\times 1\frac{5}{7}$

17. $3\frac{1}{3}\times\frac{5}{8}$

18. $2\frac{13}{15}\times\frac{5}{86}$

19. $2\frac{5}{8}\times 3\frac{1}{7}$

20. $1\frac{7}{12}\times 1\frac{3}{57}$

Dividing Fractions

Lesson 7

Topics in This Lesson

- Dividing fractions.
- Reciprocals of fractions.
- Different notations to show division of fractions.

Summary

This lesson focuses on how to divide fractions. Flipping a fraction to its reciprocal and then multiplying, canceling, reducing, and writing as mixed numbers are all shown.

Definitions

compound fraction: A fraction in which the numerator and/or denominator is also a fraction.

reciprocal: A fraction in which the numerator and denominator switch places.

Example from the Lesson

Calculate $\frac{5}{9} \div \frac{20}{27}$.

We want to replace the division symbol by a multiplication symbol and replace $\frac{20}{27}$ with its reciprocal. Then, the original problem is the same as $\frac{5}{9} \times \frac{27}{20}$.

Before we start multiplying, we should do as much canceling as we can. Note that the 5 and 20 have a 5 in them. So, let's cancel those common divisors of 5, which will leave us with $\frac{1}{9} \times \frac{27}{4}$.

Next, the 9 and 27 both have a 3 in them and a 9 in them. Because 9 is a larger factor, we cancel the 9 from both the numerator and denominator, which will then give us $\frac{1}{1} \times \frac{3}{4}$ because 9 divided by 9 is 1 and 27 divided by 9 is 3.

Because there is no more canceling or reducing to be done, we now multiply.

We know that $1 \times 3 = 3$ and $1 \times 4 = 4$, so our answer is $\frac{3}{4}$.

Additional Example

Calculate $8\frac{1}{3} \div 3\frac{3}{9}$.

Because we have mixed numbers, the first thing to do is to change them to improper fractions.

$$8\frac{1}{3} = \frac{25}{3}.$$

$$3\frac{3}{9} = \frac{30}{9}.$$

Now, we flip the second fraction—changing it to its reciprocal—and multiply: $\frac{25}{3} \times \frac{9}{30}$.

Next, look for common factors to cancel or reduce. Notice that 25 and 30 both have a 5 in common. If we remove a 5 from each, we are left with $\frac{5}{3} \times \frac{9}{6}$.

Next, 9 and 3 have a common factor of 3. Taking out a 3 from each of them leaves us with $\frac{5}{1} \times \frac{3}{6}$.

We could begin multiplying now, but there is still more reducing that we can do. The fraction $\frac{3}{6}$ is the same as $\frac{1}{2}$ because both 3 and 6 have a common factor of 3.

We now have $\frac{5}{1} \times \frac{1}{2}$, which is equal to $\frac{5}{2}$. If we wish to write that as a mixed number, we would write $2\frac{1}{2}$. Therefore, we know that $8\frac{1}{3} \div 3\frac{3}{9} = 2\frac{1}{2}$.

Avoiding Common Errors

- Remember to flip the second fraction before multiplying or doing any canceling.
- Do not flip the first fraction.

Study Tip

- Do not get frustrated with division problems; they are just multiplication problems in disguise.

PROBLEMS

Compute each of the following.

1. $\frac{2}{3} \div \frac{7}{5}$

2. $\frac{4}{5} \div \frac{11}{7}$

3. $\frac{13}{10} \div \frac{8}{3}$

4. $\frac{7}{10} \div \frac{7}{3}$

5. $\frac{7}{15} \div \frac{14}{5}$

6. $\frac{24}{15} \div \frac{8}{21}$

7. $\frac{3}{4} \div \frac{9}{28}$

8. $\frac{6}{21} \div \frac{15}{49}$

9. $\frac{35}{20} \div \frac{14}{30}$

10. $\frac{51}{31} \div \frac{68}{93}$

11. $\frac{15}{26} \div 25$

12. $\frac{8}{5} \div 12$

13. $\frac{5}{21} \div \frac{5}{21}$

14. $\frac{22}{5} \div \frac{11}{5}$

15. $8 \div \frac{4}{5}$

16. $10 \div \frac{25}{27}$

17. $3\frac{1}{3} \div \frac{5}{9}$

18. $3\frac{1}{15} \div \frac{23}{20}$

19. $2\frac{5}{8} \div 5\frac{1}{4}$

20. $1\frac{5}{12} \div 1\frac{11}{40}$

Adding and Subtracting Decimals

Lesson 8

Topics in This Lesson

- An introduction to decimal numbers.
- Place value (what the numbers to the right of the decimal point mean).
- Adding decimal numbers.
- Subtracting decimal numbers.

Summary

This lesson begins with an explanation of decimal numbers and place value. Next, how to add and subtract decimal numbers—and decimal numbers and whole numbers—is discussed.

Definitions

decimal number: A number written in our usual base-10 number system. Often, such a number will contain a decimal point.

decimal point: The dot or period that comes after the ones place and before the tenths place in a decimal number.

hundredths place: The digit second to the right from the decimal point is in the hundredths place.

tenths place: The digit to the immediate right of the decimal point is in the tenths place.

Example from the Lesson

Find 113.24 + 45.607.

We start by writing the 2 numbers vertically, lining up the decimal points.

```
  113.24
+ 45.607
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The 7 seems to be oddly dangling off the end, but it is exactly where it should be. We need to line up the decimal points—not the digits. To help keep everything in its correct place, draw a horizontal line underneath and write the decimal point in the answer line.

$$\begin{array}{r} 113.24 \\ +\ 45.607 \\ \hline . \end{array}$$

Now, we start on the right-hand side of the problem and work our way to the left. So, we see that the 7 is all alone. What do we do? Just treat it as if it were being added to 0. That will give us a 7 in the answer, so write a 7 in the answer line. Next, we have 4 + 0, which is 4, so write in a 4 under the 0. Then, 2 + 6 = 8, so we write in an 8. Next, 3 + 5 = 8, so write in an 8. Then, 1 + 4 = 5, which we write in to the left of the 8. Finally, we have a 1 dangling on the left-hand side. Again, it's just like adding that 1 to a 0—which isn't really there—and we get 1, so we write in a 1 on the far left side of our answer.

$$\begin{array}{r} 113.24 \\ +\ 45.607 \\ \hline 158.847 \end{array}$$

Our final answer is 158.847.

Additional Example

Find 723 − 39.205.

Begin by rewriting the problem vertically, lining up the decimal points. Remember: There is always an understood decimal point at the end of every whole number.

$$\begin{array}{r} 723. \\ -\ 39.205 \\ \hline \end{array}$$

It sometimes helps to put placeholder 0s in when subtracting or adding, so let's add them in and write the decimal in the answer line so that we do not forget it later.

$$\begin{array}{r} 723.000 \\ -\ 39.205 \\ \hline . \end{array}$$

Begin subtracting right to left—just like in every other subtraction problem. We have to borrow to make the final 0 into a 10 so that we can subtract. The 3 changes to a 2, the first 0 to a 9, the second 0 to a 9, and the final 0 to a 10.

$$\begin{array}{r} 7\ \ 2\ {}^{2}\not{3}.\ {}^{9}\not{0}\ {}^{9}\not{0}\ {}^{10}\not{0} \\ -\ 3\ \ 9.\ \ 2\ \ 0\ \ 5 \\ \hline . \end{array}$$

Now that all of the borrowing to the right of the decimal point has taken place, we can begin subtracting. Because $10 - 5 = 5$, $9 - 0 = 0$, and $9 - 2 = 7$, we currently have the following.

$$\begin{array}{r} 7\;2\;{}^{2}\cancel{3}.{}^{9}\cancel{0}\,{}^{9}\cancel{0}\,{}^{10}\cancel{0} \\ -\,3\;9.\;2\;0\;5 \\ \hline .\;7\;9\;5 \end{array}$$

We now need to borrow again. The 2 in the tens place becomes a 1 while the 2 in the ones place becomes a 12. Because $12 - 9 = 3$, we can place a 3 in the answer section just to the left of the decimal point.

$$\begin{array}{r} 7\,{}^{1}\cancel{2}\,{}^{12}\cancel{3}.{}^{9}\cancel{0}\,{}^{9}\cancel{0}\,{}^{10}\cancel{0} \\ -\,3\;9.\;2\;0\;5 \\ \hline 3.\;7\;9\;5 \end{array}$$

Now, we need to borrow once more to make the 7 into a 6 and the 1 into an 11. From there, we can finish the subtracting.

$$\begin{array}{r} {}^{6}\cancel{7}\,{}^{11}\cancel{2}\,{}^{12}\cancel{3}.{}^{9}\cancel{0}\,{}^{9}\cancel{0}\,{}^{10}\cancel{0} \\ -\,3\;9.\;2\;0\;5 \\ \hline 6\;8\;3.\;7\;9\;5 \end{array}$$

Our final answer is 683.795. This can be checked by doing an addition problem to check that $683.795 + 39.205 = 723$.

Avoiding Common Errors

- Line up the decimal points when adding or subtracting decimal numbers.
- Do not forget to write the decimal point in your final answer; 1256 is very different from 12.56.

Study Tips

- Grid paper can help you keep your columns straight when working with decimal numbers. If you do not have any grid paper, turning regular notebook paper 90° will make vertical columns for you to use.
- Rewrite decimal problems vertically to make it easier to line up the decimal points.
- Writing the decimal point in the answer line before beginning any addition or subtraction may help you to not forget to write it in later.
- Estimate the answer to the problem by mentally adding or subtracting only the whole numbers.

PROBLEMS

Compute each of the following.

1. 12.4 + 15.3

2. 97.21 + 41.18

3. 123.45 + 825.77

4. 5454.2 + 1661.4

5. 123.89 + 49.2

6. 943.2 + 82.98

7. 1234.5 + 123.45

8. 6226.21 + 48.076

9. 7800.125 + 942.6767

10. 1267 + 49.35

11. 98.5 − 42.1

12. 903.14 − 801.07

13. 524.45 − 124.37

14. 812.129 − 477.25

15. 145 − 72.8

16. 259.1 − 165.6

17. 9000 − 5678.12

18. 512.75 − 433.285

19. 4321.01 − 3232.85

20. 502.12 − 483.55

Multiplying and Dividing Decimals

Lesson 9

Topics in This Lesson

- Multiplying decimal numbers.
- Dividing decimal numbers.

Summary

We begin this lesson with an explanation of how to multiply 2 decimal numbers. Then, we transition to a discussion of how to divide decimal numbers.

Definitions

decimal number: A number written in our usual base-10 number system. Often, such a number will contain a decimal point.

decimal point: The dot or period that comes after the ones place and before the tenths place in a decimal number.

Example from the Lesson

Multiply 4.02 × 0.65.

We begin by writing the problem vertically, lining up the numbers to the right-hand side. We do not need to line up the decimal points in multiplication problems (even though they happen to be lined up in this problem).

$$\begin{array}{r} 4.02 \\ \times\, 0.65 \\ \hline \end{array}$$

In multiplication problems, we can also ignore the decimal points for now. We will come back to the decimal points near the end of the problem. So, we now start multiplying, moving right to left.

We start to multiply by 5. When we do so—and take care of all of our carries correctly—we are going to get 2010 in the first line of the answer section. Then, we write a 0 in the second line of the answer section to remind us that we are multiplying now by the 6, and then we begin the multiplication by 6 with the 2 in 4.02. Once we have completed that line, we should have 24,120 in the second line.

What about the 0 in 0.65? Well, we really don't need to multiply by that because 0 times any number is still 0. So, we would just end up writing a whole string of 0s in the third line of the answer section, and when we go to add everything up, all of those 0s would be useless.

At this point, our answer section has 2 rows that need to be added.

$$\begin{array}{r} 2010 \\ +\ 24120 \\ \hline \end{array}$$

These columns need to be lined up properly because we are adding, and once we add these correctly, we will have 26,130.

However, we are not done. The answer cannot be that large, given the original numbers in the problem. We finish this problem by going back to the original numbers to see how many digits were to the right of the decimal point. In this case, it was 4 (2 in the first number and 2 in the second number). We now look at 26,130 and count back from the right 4 places. We need to place our decimal point between the 2 and the 6 in 26,130 so that we get the correct answer. Our final answer will then be 2.6130.

Additional Example

Divide 21.44 by 0.8.

The first step is to rewrite the problem by using a division bar. The 0.8 is the divisor (the number outside) and the 21.44 is the dividend (the number inside).

$$0.8\overline{)21.44}$$

We need to move the decimal point 1 place to the right to make the divisor a whole number. Because of this, we also need to move the decimal point 1 place to the right in the dividend. This gives us 214.4 ÷ 8, and the work will look like the following.

$$8\overline{)214.4}$$

Now, place the decimal point in the answer line (at the top of the division bar).

$$\begin{array}{r} . \\ 8\overline{)214.4} \end{array}$$

Next, simply divide, ignoring the decimal point in the dividend.

Does 8 go into 2? No, it does not; so, we give ourselves another digit. Does 8 go into 21? Yes. We know that 8 × 2 = 16 and 8 × 3 = 24, so 8 goes into 21 twice. Write a 2 in the answer section just above the 1 in 214.4 and write a 16 underneath the 21 (because 8 × 2 = 16). Subtract 16 from 21 to get 5.

$$\begin{array}{l} 2. \\ 8\overline{)214.4} \\ \underline{-16} \\ 5 \end{array}$$

Now, we move down another digit. In this case, we drop down the 4, which gives us 54.

$$\begin{array}{r} 2\ . \\ 8\overline{)214.4} \\ -16 \\ \hline 54 \end{array}$$

How many times does 8 go into 54? The answer is 6 because 8 × 6 = 48 (which is smaller than 54) while 8 × 7 = 56 (which is larger than 54). Write a 6 in the answer section above the 4, and write 48 below 54. Subtract to get 6, and then bring down the 4.

$$\begin{array}{r} 26. \\ 8\overline{)214.4} \\ -16 \\ \hline 54 \\ -48 \\ \hline 6\ 4 \end{array}$$

How many times does 8 go into 64? Well, the answer is 8 times because 8 × 8 = 64. Write an 8 above the 4 and a 64 underneath the 64. We subtract and are left with 0.

$$\begin{array}{r} 26.8 \\ 8\overline{)214.4} \\ -16 \\ \hline 54 \\ -48 \\ \hline 64 \\ -64 \\ \hline 0 \end{array}$$

Because there are no other digits in the original dividend to drop down, and because the amount we have left in the work section of the problem is 0, we know that we are done. Our final answer is 26.8.

Avoiding Common Errors

- Count the number of decimal places to the right of the decimal point in both factors when multiplying 2 decimal numbers and add these together to see where the decimal place needs to be placed in the final answer.

- Count from the right of the number to determine how many spaces are needed to move the decimal point. If you count from the left, it is incorrect.

- Make sure that you add the decimal point to your final answer when doing multiplication problems. Just because the multiplication or division is finished, it does not mean that you have the correct answer.
- When you move the decimal point in the divisor of a division problem, the decimal point in the dividend must also be moved the same number of places.

Study Tips

- Write multiplication problems vertically.
- You do not need to line up the decimals when multiplying decimal numbers. Simply line up the digits.
- Estimating your answer before you begin to calculate will help you determine if you made a placement error with the decimal point. Simply round the numbers and do some mental math to give yourself an idea of what the final answer should be.
- When a problem asks you to divide a number by another number, the number after the word "by" is the divisor and goes outside the division bar.
- It is vitally important to keep columns lined up when dividing decimal numbers. Grid paper or turning notebook paper 90° to make vertical rows will help you keep the columns straight.

PROBLEMS

Compute each of the following.

1. 4.37 × 5

2. 8.21 × 3

3. 11 × 93.7

4. 8 × 123.45

5. 4.7 × 2.6

6. 12.73 × 9.21

7. 105.2 × 13.98

8. 1.024 × 1.987

9. 125.61 × 1357.8

10. 1000.25 × 9.102

11. 60.2 ÷ 7

12. 56.8 ÷ 4

13. 147.8 ÷ 10

14. 421.44 ÷ 12

15. 10.236 ÷ 1.2

16. 90.321 ÷ 2.1

17. 185.175 ÷ 1.5

18. 3840.9 ÷ 0.7

19. 15.9486 ÷ 0.03

20. 3.668 ÷ 0.07

Fractions, Decimals, and Percents

Lesson 10

Topics in This Lesson

- Real-world examples of percents.
- Defining the term "percent."
- Converting from percents to fractions.
- Converting from percents to decimals.
- Converting from fractions to decimals.
- Converting from decimals to fractions.

Summary

This lesson starts with some real-world examples that illustrate the importance of percentages. A definition of the term "percent" follows, and then instruction on how to convert from a percent to a fraction is given. Next, how to convert percents to decimals is shown. Converting from fractions to decimals is then explained. The lesson closes with converting decimals to fractions.

Definition

percent: A number with a percent symbol (%) that tells us what part of a hundred is represented. For example, 75% means "75 per hundred."

Formulas and Rules

converting from decimals to fractions: Write any whole numbers by themselves. Next, write what is to the right of the decimal point (leaving off the decimal point) as the numerator (the top of the fraction). Determine which place the last digit of the decimal is in (tenths, hundredths, thousandths, etc.). The denominator (the bottom of the fraction) is the number that corresponds to that place. If the last digit was in the hundredths place, the denominator would be 100. If the last digit was in the tenths place, the denominator would be 10.

converting from fractions to decimals: Perform long division by dividing the denominator (the number on the bottom) into the numerator (the number on the top). Keep using more 0s after the decimal point (as needed) until the answer comes out even or the numbers begin repeating.

converting from percents to decimals: Take the percent amount, move the decimal point 2 places to the left, and remove the percent sign.

converting from percents to fractions:

- Use the percent amount as the numerator (the top number) of the fraction.
- Always make the denominator (the bottom number) 100.
- Simplify the fraction if needed.

Examples from the Lesson

Convert 5% to a fraction.

We build our fraction by using the 5 for the numerator and a 100 for the denominator. Then, we have 5% = $\frac{5}{100}$. We can cancel a 5 from the numerator and denominator, which will leave us with $\frac{1}{20}$. That cannot be reduced any more (because once there's a 1 in the numerator or denominator, reducing is done), so we know that 5% = $\frac{1}{20}$.

Convert 6% to a decimal.

Notice first that 6% = 6.0%. We need to move the decimal point 2 places to the left. Think of the 6 as 06. Move the decimal point once to the left of the 6 and then to the left of the 0. Our final answer is .06. Some people also like to write this as 0.06; either way is correct.

Additional Example

Convert $\frac{7}{8}$ to a decimal.

To convert a fraction to a decimal, we divide the numerator (the top number) by the denominator (the bottom number).

$$8\overline{)7}$$

We put a decimal point after the 7 in the dividend and begin to add 0s.

$$8\overline{)7.000}$$

How many times does 8 go into 70? The answer is 8 times because 8 × 8 = 64. Write an 8 in the answer line and a 64 below the 70. Then, subtract to get 6. Because we have a nonzero remainder, we continue with the problem by dropping down the next number (which is one of the 0s we added to the problem).

$$\begin{array}{r} .8 \\ 8\overline{)7.000} \\ -6\,4 \\ \hline 6\,0 \end{array}$$

How many times does 8 go into 60? The answer is 7 times because 8 × 7 = 56. Write a 7 on the answer line, and write a 56 below the 60. Then, subtract to get 4. Bring down another 0.

$$\begin{array}{r} .87 \\ 8\overline{)7.000} \\ -64 \\ \hline 60 \\ -56 \\ \hline 40 \end{array}$$

How many times does 8 go into 40? The answer is 5 times because 8 × 5 = 40. Write a 5 on the answer line and a 40 under the 40. When we subtract, we have no remainder, so our problem is finished.

$$\begin{array}{r} .875 \\ 8\overline{)7.000} \\ -64 \\ \hline 60 \\ -56 \\ \hline 40 \\ -40 \\ \hline 0 \end{array}$$

Our final answer is .875, or 0.875.

Avoiding Common Errors

- When you convert a fraction to a decimal, make sure that the numerator is the dividend (the number inside the division bar). The numerator of the fraction (the number on the top) may be larger or smaller than the denominator.
- When converting a percent to a decimal, be sure that the decimal point moves 2 places to the left. So, 8% becomes 0.08—not 0.8. Also, 275% becomes 2.75.

Study Tips

- When converting from a decimal to a fraction, count the number of digits to the right of the decimal point. This is the number of 0s after the 1 that you will need in the denominator of the fraction (before any kind of reducing takes place).
- Understanding what fractions, decimals, and percents really are will help you remember how to convert from one to another.
- Knowing your basic arithmetic facts well will help with these conversions.

PROBLEMS

Convert each of the following percents to fractions.

1. 47%
2. 85%
3. 6%
4. 325%
5. 67.5%

Convert each of the following percents to decimals.

6. 39%
7. 83%
8. 4%
9. 625%
10. 57.2%

Convert each of the following fractions to decimals.

11. $\frac{5}{8}$
12. $\frac{14}{5}$
13. $\frac{37}{4}$
14. $\frac{2}{11}$
15. $\frac{5}{6}$

Convert each of the following decimals to fractions.

16. 0.42
17. 7.15
18. 4.375
19. 0.16
20. 0.025

Percent Problems
Lesson 11

Topics in This Lesson

- Real-world examples of percent problems, such as:
    - Calculating tips.
    - Percent reduction.
    - Sales tax.
    - Percent as an amount of a whole.
    - Percent reduction of an item that is already on sale.
    - Percentage increase.

Summary

This lesson deals with real-world examples in which working with percentages is necessary. For example, calculating a tip is an application of percentages. First, finding out how to calculate a sales price when something is x% off is demonstrated. Then, how to determine a certain percent of a whole and how to calculate a percent off with an additional percent off is shown. The lesson closes with a percent that is greater than 100% to show how an increase can be measured.

Definition

percent: A number with a percent symbol (%) that tells us what part of a hundred is represented. For example, 75% means "75 per hundred."

Example from the Lesson

You are considering buying an expensive watch that regularly sells for $600. The watch store is running a 50% off sale. When you go to look at the watch at the store, you find out that the salespeople are willing to take off 30% from that advertised sale price. What is the actual cost of the watch going to be for you?

These kinds of problems—where you have 1 sale on top of another—can often mislead people. It is often believed that you are getting 80% off the original price (because 50 + 30 = 80), but this is not correct.

In order to determine the correct price for the watch, you need to do calculations in stages. First, calculate what the cost would have been after the first 50% was taken. We know that 50% of 600 is the same as 0.5 of 600, or $\frac{1}{2}$ of 600, which is 300. So, we can subtract \$300 from \$600 to determine how much the watch would sell for after the 50% reduction. Because 600 − 300 = 300, the watch would cost \$300 after the discount.

Next, we take 30% from the \$300. (This is why calculating 80% off the original price does not give us the correct amount; the 30% off only comes off of the \$300 price tag—not the original \$600 price tag.) What is 30% of 300? It's 0.30 × 300 = 90, and 300 − 90 = 210. So, the watch costs \$210 after both discounts. Notice that this is very different from \$120, which is what the price would have been if we had reduced the original \$600 price tag by 80%. At least we now know the correct cost—even if it is much higher than we had originally hoped it would be.

In this problem, it turns out that the order in which we took those 2 discounts would not have mattered. If you start with a \$600 watch and take the 30% off first, you will be down to a cost of \$420. Next, take the 50% off of that new sales price: 50% of 420 is the same as 0.50 times 420, which is 210. So, the new price is 420 − 210 = 210, leaving us with the price we found earlier—\$210.

Additional Example

The Mathville Rockets know that this year's attendance at their hockey games is 208% of last year's attendance. If 7250 people came out to the ice rink last season to watch the hockey team play, how many came out this season?

The first thing to notice is that the percent is larger than 100%, which means that our answer will be larger than our original number—7250. Even though the percent is over 100, we proceed in the same way. First, convert 208% to a decimal and multiply. We know that 208% = 2.08, so we need to multiply 7250 by 2.08. Once we do this (using the skills we have learned in previous lessons), we will find that 7250 × 2.08 = 15,080. So, the number of people in attendance this year is 15,080.

This number is bigger than our original number. In fact, it is almost double our original number, which we could have guessed because 100% of the number would have been the same as our original number, and 200% would have been twice the original number. So, we can have confidence that our answer is in the right ballpark—or ice rink, in this case.

Avoiding Common Errors

- When you are finding a percent reduction, make sure you find out what the problem is asking. For example, if the problem says that 36% of a group of 200 people is female, and the problem asks how many are male, you will not have the correct answer if you only find 36% of 200. In this case, you will have determined the number of females in the group of 200. Subtraction is needed to find the number of males.

- Make sure not to add the percentages before calculating a percent off of a percent off. For example, 40% off an item that is already 50% off is not the same as 90% off.

Study Tips

- When you see the word "of" in a mathematical word problem, it usually means that you need to multiply 2 numbers together.
- If you are determining a percentage and the percent is larger than 100%, your answer will be larger than your original number. If it is smaller than 100%, your answer will be smaller than the original number.
- Memorizing common percentages and their fractional and decimal equivalents can save time. For example, 25% is the same as $\frac{1}{4}$, or 0.25.

PROBLEMS

Determine the answers to each of the following.

1. What is an 18% tip on a $25.50 restaurant bill?
2. What is a 20% tip on a $37.79 restaurant bill?
3. What is a 15% tip on a $14.26 restaurant bill?
4. What is an 18% tip on a $92.06 restaurant bill?
5. The original price of a used car was $6000. Then, the dealer decided to give a 12% discount off the price of the car. What is the price of the car now?
6. Last month's electric bill was $327.30. Then, the company decided to give a 5% discount to its customers. How much is the bill now?
7. A new car sells for $25,990, but the dealer is trying to get rid of some of these new cars in order to make room for a newer model, so she is offering an 8% discount to first-time new-car buyers. How much would this car sell for after the 8% discount is given?
8. A restaurant offers a 5% discount to all local students, including your friend Debra. Debra's recent lunch bill should have cost $12.47. What was the actual cost once the 5% discount was taken?
9. The local sales tax is 7%. You want to purchase some clothes for $52.50. What will your actual bill be once the sales tax is added in?
10. The local sales tax is 6.5%. You want to purchase some jewelry for $248. What will your actual bill be once the sales tax is added in?
11. The local income tax is 1.25%. You earned $75,000 last year. How much tax will you have to pay?
12. The state income tax is 3.75%. You earned $52,700 last year. How much tax will you have to pay?

For problems 13–16, use the following information.

In a recent survey, 23,000 people were asked about their pizza preferences. Of those surveyed, 45% of the people stated that they preferred pepperoni as a topping, 30% stated that they preferred vegetables for their topping, and the remaining 25% stated that they simply wanted cheese on their pizzas. (Note that each person could only vote for 1 of the above topping choices.)

13. How many of the people stated that they preferred pepperoni on their pizza?

14. How many of the people stated that they preferred vegetables on their pizza?

15. How many of the people did not want vegetables on their pizza?

16. How many of the people did not want pepperoni on their pizza?

Determine the final cost of an item under the following conditions.

17. The item's original cost was $100. It was then marked down by 20%, and then an additional 10% was taken off that sale price.

18. The item's original cost was $520. It was then marked down by 40%, and then an additional 40% was taken off that sale price.

19. The item's original cost was $57.25. It was then marked down by 15%, and then an additional 25% was taken off that sale price.

20. The item's original cost was $5000. It was then marked down by 60%, and then an additional 20% was taken off that sale price.

Ratios and Proportions

Lesson 12

Topics in This Lesson

- Ratios.
- Proportions.
- Cross multiplying.

Summary

This lesson begins with definitions of the terms "ratio" and "proportion." Many real-world story problems involving proportions and ratios are given. How to check if 2 ratios are equal, how to set up 2 ratios and cross multiply to find an unknown quantity, and how to determine the cost per item are all shown.

Definitions

cross multiplying: A way to determine if 2 ratios are equivalent.

proportion: A statement that 2 ratios are equal. A proportion is a mathematical sentence that contains an equal sign and 2 equal ratios.

ratio: A comparison of 2 amounts. It is typically found by dividing 1 number into another.

Formulas and Rules

how to cross multiply: Write the ratios as fractions. Multiply the numerator of the first fraction with the denominator of the second fraction. Then, multiply the denominator of the first fraction with the numerator of the second fraction. If you drew lines to show which numbers were being multiplied together, it would make an X in the center, or a cross—hence the term "cross multiply."

Example from the Lesson

Constance sold 95 raffle tickets for a total of $142.50. If each ticket costs the same amount of money, how much did each ticket cost?

We can solve this problem by setting up a proportion, which is just an equality statement about 2 ratios. One of the ratios that we can write down is $\frac{142.5}{95}$ (with the dollars on top and the number of tickets on the bottom). Why didn't we use $\frac{95}{142.5}$? We could have done so—as long as we remembered what our ratio compares.

Now, if we use $\frac{142.5}{95}$, then we need to write down another ratio with dollars over number of tickets. We want the cost (in dollars) for 1 ticket, so the other ratio we should write down is $\frac{C}{1}$, where C is the cost of 1 ticket and the 1 stands for 1 ticket. Now, we can write this proportion: $\frac{142.5}{95} = \frac{C}{1}$.

We just want to know C. We know that dividing a number by 1 doesn't do anything, so $\frac{C}{1}$ is just C, which means that our equation is just $\frac{142.5}{95} = \frac{C}{1}$. Now, we need to calculate 142.50 divided by 95.

The first thing to do after writing the long division problem is to move the decimal point up into the answer section—just above the decimal point in the 142.50.

```
       .
95)142.50
```

We notice that 95 doesn't divide 1, nor does 95 divide 14, but 95 does divide 142 once. Write a 1 above the 2 in 142, and subtract 95 from 142 to get 47. Drop down the next digit in 142.50, which is the 5. We now want to know how many times 95 goes into 475.

```
      1.
95)142.50
   −95
     47 5
```

Because 95 × 5 = 475, 95 goes into 475 5 times. Write a 5 in the answer section (just after the decimal point, in this case), and subtract 475 from 475 to get 0. Drop down the next digit from 142.50, which is a 0. We only have 0 from which to divide 95, which means we are done; we can't take any 95s out of 0.

```
      1.5
95)142.50
   −95
     47 5
   −47 5
        0
```

The final answer is 1.5—or $1.50 if we are thinking in terms of money.

So, the proportion $\frac{142.5}{95} = \frac{C}{1}$ just got retranslated into $1.50 = C, which means that the cost of 1 raffle ticket is exactly $1.50.

Additional Example

Totes My Oats cereal sells for $2.75 for a 20-ounce box. Frosted Flax costs $3.00 for a 24-ounce box. Which cereal is cheaper per ounce?

In order to answer this question, we need to make 2 ratios and determine a per-ounce price for each box of cereal. In each case, we should set up our ratio with the cost of the cereal as the first number and the ounces as the second number because we want to compare the cost per ounce, or cost/ounce.

Note that $\frac{2.75}{20}$ is our first ratio, and it equals 0.1375, which means that the cost per ounce of Totes My Oats cereal is 13.75 cents.

The second ratio we want is $\frac{3.00}{24}$, and it equals 0.125, which means that the cost per ounce for Frosted Flax is 12.5 cents.

Now, let's look at our original question. It asks which cereal is cheaper per ounce. The answer is Frosted Flax at 12.5 cents.

Avoiding Common Errors

- Be sure that your ratios represent the same things. If you write a ratio of girls to boys and then another ratio of boys to girls, you cannot cross multiply to see if they are a proportion.
- When you write down a ratio, it is very important that you remember what each of the 2 numbers means. You may want to label the ratio if you are likely to forget, such as 20 cats : 30 dogs.

Study Tips

- Some ratios can tell you 3 things: a part, another part, and the whole amount. If you are told that there is a ratio of 5 men : 7 women, then you also know that there are ratios of 7 women : 12 total people and 5 men : 12 total people.
- Set up your ratios or proportions without numbers first if you are confused—especially when there is an unknown quantity involved. For example, you can write the following.

  $$\frac{\text{miles driven}}{\text{cost of gas}} = \frac{\text{miles driven}}{\text{cost of gas}}$$

  Then, you can go back and fill in the numbers to go along with each of the terms.

PROBLEMS

Determine which of the following ratios are equivalent to one another. For those pairs that are equivalent, write the corresponding proportion statement.

1. 6:8 and 30:40

2. 5:9 and 7:11

3. 14:42 and 17:51

4. 9:10 and 85:100

5. 24:36 and 30:45

6. $\frac{16}{48}$ and $\frac{12}{36}$

7. $\frac{18}{63}$ and $\frac{10}{33}$

8. $\frac{3}{11}$ and $\frac{36}{132}$

9. $\frac{10}{13}$ and $\frac{20}{25}$

10. $\frac{18}{10}$ and $\frac{42}{25}$

Determine the answer to each of the following problems.

11. Jane drove at a constant speed for 3 hours and traveled 186 miles. If she drives at the same speed the next day, how many miles will she travel in 7 hours?

12. Beth read 75 pages of her book in 45 minutes. How many pages will she read in 105 minutes?

13. John always buys 2 pairs of pants every time he buys 3 shirts. So, last year, he had 10 pairs of pants and 15 shirts, but now, he has 24 shirts. How many pairs of pants does John have?

14. If 5 apples cost a total of $1.40, then how much will it cost to buy 9 apples?

15. Linda's company has a policy that for every 6 months a person works, he or she earns 13 vacation days. How many vacation days will Linda earn after she works 4 years?

16. Your favorite team currently has a record of 12 wins and 4 losses. If they continue to win at this pace, how many games will they win if they play 48 games for the season?

17. After looking at the receipt from your latest shopping spree, you noticed that tax on a bill of $32 is exactly $2.24. How much would the tax be on a bill of $60?

18. Jane likes to walk for exercise. She notices that she can walk 2 miles in 25 minutes. How long would it take her to walk 5 miles (assuming that her pace is exactly the same throughout the walk)?

19. A 200-pound person (on Earth) would weigh 75.4 pounds on Mars. How much would a person weigh on Mars if that person weighed 140 pounds on Earth?

20. Currently, 200 Euros is worth 282 U.S. dollars. How many U.S. dollars is 450 Euros worth?

Exponents and Order of Operations

Lesson 13

Topics in This Lesson

- Exponentiation.
- Order of operations.

Summary

This lesson begins with a definition of exponentiation, with several examples and real-world applications given. Order of operations is then discussed. Many examples of how to properly complete a multioperation problem using the order of operations are given.

Definitions

exponent: The small number written high and to the right of a number that tells how many times the number should be multiplied by itself.

exponentiation: Raising a number to a power. In other words, writing the number times itself as many times as the exponent shows. For example, 8^4 is $8 \times 8 \times 8 \times 8$. You could also think of it as repeated multiplication.

operation: A mathematical process. Some operations include addition, subtraction, multiplication, and division.

order of operations: A set of rules that explain in what order operations should be completed in an expression with more than 1 operation.

Formulas and Rules

exponentiation facts:

$a^1 = a$ for any number a, and
as long as $a \neq 0$, $a^0 = 1$.

rules for order of operations:

1. Any operations inside parentheses.
2. Exponents.
3. Multiplication and division from left to right.
4. Addition and subtraction from left to right.

Example from the Lesson

Calculate $15 + 3 \times 4^2$.

We begin by looking over the problem to see what sorts of operations are present. There is addition, multiplication, and exponentiation. We need to follow the order of operations to make sure that we do the problem correctly. The first thing to do is the 4^2.

$$15 + 3 \times 4^2 = 15 + 3 \times 16.$$

Next, we must do the multiplication.

$$15 + 3 \times 4^2 = 15 + 48.$$

Then, we do the addition last.

$$15 + 48 = 63.$$

So, our final answer is 63.

Additional Example

Calculate $75 + 4 / 2 - 5 \times 2^3 + 10$.

Begin by skimming the problem to see what operations are there. All the operations seem to be there, so we simply follow the order of operations to complete the problem correctly.

First, we do exponentiation. $2^3 = 2 \times 2 \times 2 = 8$. We can put this 8 into the original expression and keep working: $75 + 4 / 2 - 5 \times 8 + 10$.

Now, we do multiplication and division from left to right.

$$4 / 2 = 2.$$
$$5 \times 8 = 40.$$

Plug these numbers back into the original expression: $75 + 2 - 40 + 10$.

Now, we are down to addition and subtraction.

$$75 + 2 = 77.$$
$$77 - 40 = 37.$$
$$37 + 10 = 47.$$

Our final answer is 47.

Avoiding Common Errors

- Do not forget to follow the order of operations. If you work all the operations as they occur from left to right, you will sometimes not get the correct answer.
- Do not rely on a calculator to solve multioperation problems; some calculators are programmed to follow the rules for order of operations, but many are not.
- Do not multiply the number and the exponent together to get an answer. The exponent tells you how many times the number should be multiplied by itself. For example, 7^3 is not 7×3—it is $7 \times 7 \times 7$.

Study Tips

- Some people remember the order of operations with the following mnemonic: Please Excuse My Dear Aunt Sally. It reminds us that we first do anything inside parentheses (Please), then exponents (Excuse), then multiplication and division (My Dear), and finally addition and subtraction (Aunt Sally). Although parentheses are not emphasized in this lesson, it is the case that operations within parentheses must be performed before any other operations in the expression.
- Do not try to keep all the steps of a multioperation problem in your head. Substitute your answers back into the original expression as often as possible.
- Using parentheses around the operation you are currently working on will allow you to easily recognize which numbers and operator symbols need to be replaced by the answer after you have worked a step in a multioperation problem.

PROBLEMS

Compute each of the following.

1. 2^7

2. 3^3

3. 5^4

4. 10^3

5. 9^2

Simplify each of the following.

6. $12 + 9 + 8 + 1 + 4 + 3 + 7$

7. $25 + 8 + 25 + 2 + 25 + 15$

8. $12 - 7 + 20 - 18 + 5 + 3$

9. $502 + 413 - 189 - 73 + 43 + 421$

10. $4 \times 7 \times 11 \times 5 \times 5$

11. $2 \times 2 \times 2 \times 5 \times 5 \times 5$

12. $25 \times 3 \div 5 \times 12 \div 4 \div 9$

13. $14 \div 2 \times 8 \times 15 \div 6 \times 7$

14. $103 + 42 - 6 \times 6 + 100 \div 2$

15. $88 \div 4 + 10 - 14 \times 2 - 1$

16. $17 \times 4 + 12 - 8 \times 3 \times 2 + 18 \div 3$

17. $480 - 212 \div 2 \times 3 + 56 \div 7$

18. $123 + 5^3 \times 2 - 207$

19. $2 \times 3^4 - 5 \times 2^3$

20. $1000 \div 2^3 \times 5 + 100 - 15$

Negative and Positive Integers
Lesson 14

Topics in This Lesson

- Signed numbers.
- Absolute value.
- Working with a number line.
- Addition and subtraction of signed numbers.
- Multiplication and division of signed numbers.
- Mixed operations with signed numbers.

Summary

In this lesson, the focus is on signed numbers and how to perform operations with them. A definition of signed numbers is given. A discussion of absolute values and working with number lines follows. Then, how to combine signed numbers using addition and subtraction is presented. Multiplication and division of signed numbers and the rules that govern the sign of the answer is then shown. The lesson concludes with problems that combine all of the operations.

Definitions

absolute value: The distance a number is from 0 on the number line.

integers: The set of signed numbers. All the positive whole numbers (1, 2, 3, …), 0, and all the negative whole numbers (−1, −2, −3, …).

negative number: A number that is less than 0. It has a negative sign (also called a minus sign) in front of it.

number line: A line in which equally spaced dots are labeled with numbers. It usually is written with an arrow on either end to show that the numbers go on infinitely.

signed number: A number with a sign (+ or −) in front of it. Some people use this term to refer to the negative numbers (the numbers with a − sign).

Formulas and Rules

operations with signed numbers:

- If you are subtracting a negative number, the number turns into a positive. For example, $+3 - (-5)$ becomes $+3 + 5$.
- When you are adding a negative number and a positive number, subtract the 2 numbers and take the sign of the number that has a larger absolute value.
- When you are adding 2 numbers that are both negative, add their absolute values together and put a negative sign in front.
- When you multiply or divide 2 numbers that are both positive, the answer is positive.
- When you multiply or divide 2 numbers that are both negative, the answer is positive.
- When you multiply or divide 2 numbers that have different signs, the answer is negative.

Example from the Lesson

Calculate $(-14) / (-2) + (-8) \times (-3)$.

First, we need to do the divisions and multiplications as we read from left to right. So, we need to first find -14 divided by -2. Because both numbers have the same sign, the answer will be positive. So, $(-14) / (-2) = 7$. Also, $(-8) \times (-3) = 24$ for similar reasons. So, we have the following.

$$\begin{aligned} &(-14) / (-2) + (-8) \times (-3) \\ &= 7 + 24 \\ &= 31. \end{aligned}$$

Our final answer is 31.

Additional Example

Calculate $6 - (-2) \times 3^2 - 10 \div 2$.

We need to use what we've learned in the past several lessons to solve this problem. First, skim the problem and see what operations are being performed. Remember that the exponents should be done first.

$$\begin{aligned} &6 - (-2) \times 3^2 - 10 \div 2 \\ &= 6 - (-2) \times 9 - 10 \div 2. \end{aligned}$$

Now, the next thing to do is multiplication and division from left to right.

$$\begin{aligned} &6 - (-2) \times 9 - 10 \div 2 \\ &= 6 - (-18) - 10 \div 2 \\ &= 6 - (-18) - 5. \end{aligned}$$

Now, all that is left to do is to add and subtract from left to right. We see 2 negative signs together. Subtracting a negative number is the same as adding a positive number, so we change the 2 negative signs into 1 positive sign.

$$\begin{aligned} &6 + 18 - 5 \\ &= 24 - 5 \\ &= 19. \end{aligned}$$

Our final answer is 19.

Avoiding Common Errors

- Review and memorize the rules for signed numbers so that you do not make a mistake in what the sign needs to be after the operation is complete.
- Always be sure to follow the order of operations when working with arithmetic problems.

Study Tips

- When you have 2 negative signs next to each other, imagining that those 2 lines come together to form a positive sign will help you to remember the rule for subtracting a negative number.
- When multiplying or dividing, remember that if the signs are the same, it's positive. If the signs are different, it's negative.
- Putting parentheses around negative numbers will help you to keep track of the negative sign.

PROBLEMS

Compute each of the following.

1. $-14 + (-32)$

2. $-26 + 45$

3. $490 + (-187)$

4. $-503 + (-420)$

5. $507 - (-123)$

6. $-8942 - (-917)$

7. $-205 - 478$

8. $1004 - (-432)$

9. $454 + (-273) - (-100)$

10. $-13 - 47 + (-81)$

11. $-99 + 43 - (-63)$

12. $5887 + (-234) - (-198) + 14$

13. $(-4) \times (-15)$

14. $12 \times (-7)$

15. $(-11) \times 14$

16. $(-100) \div 4$

17. $256 \div (-8)$

18. $(-1234) \div (-2)$

19. $(-14) \times 3 \times (-5) \div 6$

20. $27 \div (-3) \times 17 \times (-4) \div (-6)$

Introduction to Square Roots

Lesson 15

Topics in This Lesson

- Defining the term "square root."
- Real-world examples of square roots.
- Finding square roots of perfect squares.
- Finding square roots of numbers that are not perfect squares.
- Multiplying square roots.
- Adding and subtracting square roots.
- Estimating square roots.

Summary

This lesson begins with a definition of square roots and moves on to give real-world examples of when square roots can be used. How to find square roots, both for perfect squares and non–perfect squares, is shown. Next, operations with square roots are shown, such as addition, subtraction, and multiplication. The lesson closes with a discussion of how to estimate square roots.

Definitions

bisection method: A way of approximating square roots by finding the perfect square larger than the number and the perfect square smaller than the number and finding the term in the middle of that number to come close to finding the actual square root.

perfect square: A number whose square root is some whole number times itself. For example, 25 is 5×5, so it is a perfect square, and 7 is 7×1, so it is not a perfect square.

radical: Another name for a square root.

radical symbol: Another name for the square root symbol.

square root: A number that you can multiply by itself to get a certain value. For any real numbers a and b, if $a^2 = b$, then a is a square root of b.

Formulas and Rules

operations with square roots:

You can multiply the numbers under the square root symbols if the 2 are being multiplied. For example, $\sqrt{20}\times\sqrt{5}=\sqrt{100}$.

If there is addition or subtraction under the radical symbol, you cannot break the numbers into separate square roots. For example, $\sqrt{16-4}$ is not the same as $\sqrt{16}-\sqrt{4}$.

You can only add or subtract square roots if the number under the radical symbol is the same.

Example from the Lesson

Simplify $11\sqrt{3}-\sqrt{12}$.

Begin by seeing if the square roots can be rewritten or simplified so that like terms appear.

Notice that $\sqrt{12}=\sqrt{4\times3}=\sqrt{4}\times\sqrt{3}=2\sqrt{3}$.

So, our original problem can be rewritten as $11\sqrt{3}-2\sqrt{3}$.

Now, we have like terms and can do the subtraction.

Because 11 − 2 = 9, we have a final answer of $9\sqrt{3}$.

Additional Example

Simplify $\sqrt{32}\times\sqrt{2}+\sqrt{16-2}$.

Look over this multistep problem to see what can be done. We can simplify $\sqrt{32}$ as $\sqrt{32}=\sqrt{16\times2}=\sqrt{16}\times\sqrt{2}=4\sqrt{2}$. So, our original problem is now the same as $4\sqrt{2}\times\sqrt{2}+\sqrt{16-2}$. Next, we know that $4\sqrt{2}\times\sqrt{2}=4\sqrt{2\times2}=4\sqrt{4}=4\times2=8$. So, the problem has now been simplified to $8+\sqrt{16-2}$, which is the same as $8+\sqrt{14}$. Can we divide 14 into a number that has a perfect square? The factors of 14 are 7 and 2, and neither is a perfect square. So, we have our final answer: $8+\sqrt{14}$.

Avoiding Common Errors

- Do not separate numbers under the radical that are being added or subtracted.
- Do not add square roots that do not have like radicals.
- When doing a multistep problem, do not forget to copy any of the pieces of the answers to the final answer.

Study Tip

- Knowing your basic multiplication facts is extremely important to be able to solve square root problems quickly and correctly.

PROBLEMS

Simplify the following as much as possible.

1. $\sqrt{121}$

2. $\sqrt{48}$

3. $\sqrt{96}$

4. $\sqrt{300}$

5. $\sqrt{128}$

6. $\sqrt{\frac{49}{64}}$

7. $\sqrt{\frac{32}{50}}$

8. $\sqrt{\frac{98}{25}}$

9. $\sqrt{52} \times \sqrt{117}$

10. $\sqrt{8} \times \sqrt{18}$

11. $\sqrt{80} \times \sqrt{64}$

12. $\sqrt{338} \times \sqrt{162}$

13. $5\sqrt{7} + 3\sqrt{7}$

14. $8\sqrt{11} - 7\sqrt{11}$

15. $23\sqrt{15} + 7\sqrt{15} - 12\sqrt{15}$

16. $8\sqrt{98} - 2\sqrt{50}$

17. $14\sqrt{12} + 20\sqrt{75}$

Use the bisection method in the following problems to correctly approximate these square roots to 2 places after the decimal point.

18. $\sqrt{3}$

19. $\sqrt{10}$

20. $\sqrt{22}$

Negative and Fractional Powers
Lesson 16

Topics in This Lesson

- Numbers raised to a fractional power.
- Numbers raised to a negative power.

Summary

After a brief review of squares and square roots, this lesson moves into how to raise a number to a fractional power and gives the formula to follow. Numbers with negative exponents are then addressed, and another formula for simplifying them is given. The final part of the lesson shows how to deal with numbers that are raised to both a fractional and negative exponent.

Definitions

square of a number: The product of a number multiplied by itself.

square root: A number that you can multiply by itself to get a certain value. For any real numbers *a* and *b*, if $a^2 = b$, then *a* is a square root of *b*.

Formulas and Rules

rule of fractional exponents: $a^{b/c} = (a^b)^{1/c}$ and $a^{b/c} = (a^{1/c})^b$.

rule of negative exponents: $a^{-b} = \frac{1}{a^b}$.

Example from the Lesson

Simplify $32^{3/5}$.

Let's write this problem as $(32^{1/5})^3$ using the rule above. Now, let's work through the problem.

$$\begin{aligned} &(32^{1/5})^3 \\ &= 2^3 \\ &= 8. \end{aligned}$$

So, $32^{3/5}$ is just the number 8 written in a disguised form.

Additional Example

Simplify $(-32)^{-2/5}$.

Because we have a negative exponent that is also a fractional exponent, we need to use both of the exponent rules. Let's rewrite the problem using the negative exponent rule first.

$$(-32)^{-2/5} = \frac{1}{(-32)^{2/5}}.$$

Now, we can rewrite the problem again for the fractional exponent.

$$\frac{1}{(-32)^{2/5}} = \frac{1}{\left((-32)^{1/5}\right)^2}.$$

Finally, we can finish our simplifying.

$$\frac{1}{\left((-32)^{1/5}\right)^2} = \frac{1}{(-2)^2} = \frac{1}{4}.$$

Our final answer is $\frac{1}{4}$.

Avoiding Common Errors

- Make sure to remove the negative sign from the exponent when you rewrite the problem in the form $a^{-b} = \frac{1}{a^b}$. Do not remove the negative sign from the number *a* if *a* is negative.
- Make sure that you recopy the problem exactly—not omitting any of the positive or negative signs.
- Be very careful to keep track of signs.

Study Tips

- Write out problems carefully, giving yourself plenty of room to write. Problems like these will take several steps, so do not feel like you have to conserve paper while working them out.
- When you see a negative exponent, you need to make a fraction with a 1 as the numerator and the original problem as the denominator (with the exponent changed to positive).
- When you see a positive and fractional exponent, the numerator of the exponent becomes the exponent on the outside of the parentheses. The denominator tells you how many times the root needs to be multiplied to get the original number.

PROBLEMS

Simplify the following as much as possible.

1. $9^{1/2}$
2. $64^{1/2}$
3. $169^{1/2}$
4. $(-81)^{1/2}$
5. $81^{1/4}$
6. $(-81)^{1/4}$
7. $32^{1/5}$
8. $216^{1/3}$
9. $16^{3/4}$
10. $243^{2/5}$
11. $49^{3/2}$
12. $125^{2/3}$
13. $(-100)^{3/4}$
14. $(-27)^{1/3}$
15. 5^{-2}
16. 3^{-4}
17. 2^{-7}
18. 4^{-3}
19. $32^{-4/5}$
20. $27^{-2/3}$

Graphing in the Coordinate Plane

Lesson 17

Topics in This Lesson

- The coordinate plane.
- Definitions of terms.
- The *x*-axis and *y*-axis.
- Ordered pairs.
- Plotting ordered pairs.
- Graphing lines on the coordinate plane.
- Finding the midpoint of a line segment.

Summary

This lesson begins with a brief history of René Descartes and one of his contributions to the world of mathematics—the Cartesian plane, which is also called the coordinate plane or the *xy*-plane. Many definitions relating to graphing in the coordinate plane are given, including the *x*-axis, *y*-axis, and origin. Quadrants are also introduced. How to write ordered pairs and how to plot points on the coordinate plane are discussed. A discussion of slope and how to calculate it is presented. Finally, the definition of a midpoint of a line segment and a formula for how to find it are given.

Definitions

Cartesian plane: Also called the Cartesian coordinate system, it is another name for the *xy*-plane.

coordinate: The number value assigned to a point on the *x*- or *y*-axis. Each point on the coordinate plane is identified by 2 coordinates.

coordinate plane: The 2-dimensional plane, or the *xy*-plane. A 2-dimensional object formed by 2 number lines—1 horizontal and 1 vertical (so that they intersect at right angles).

midpoint: The middle point of a line segment.

ordered pair: The 2 coordinates used to identify a point on the coordinate plane. Ordered pairs are often written in the form (x, y), with the *x*-axis coordinate first and the *y*-axis coordinate second.

origin: The intersection point of the *x*-axis and *y*-axis.

quadrant: One of the 4 sections of the coordinate plane. Quadrants are labeled counterclockwise from the top as I, II, III, and IV.

slope: The "rise over run," or the change in y over the change in x. It is the quantity of the vertical change in y-values divided by the horizontal change in x-values.

***x*-axis**: The horizontal number line on the coordinate plane.

***y*-axis**: The vertical number line on the coordinate plane.

Formulas and Rules

midpoint of a line segment: Using the coordinates of the 2 endpoints, the midpoint is found by taking $a^{-b} = \frac{x_1 + x_2}{2}$ for the x-coordinate and $\frac{y_1 + y_2}{2}$ for the y-coordinate.

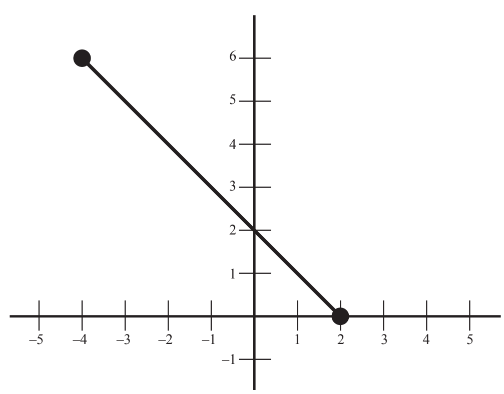
slope of a straight line: The slope of a straight line that goes through the 2 points (x_1, y_1) and (x_2, y_2) is given by the quantity $\frac{y_2 - y_1}{x_2 - x_1}$.

Example from the Lesson

Sketch the graph of the line that goes through the point (−4, 6) and the point (2, 0).

First, draw the pair of axes. Next, plot the point (−4, 6). The −4 tells us to go 4 units to the left of the origin, and the 6 tells us to then go up 6 units. That point is in quadrant II. Next, plot the point (2, 0). The 2 in the x position tells us to go 2 units to the right of the origin. The 0 tells us to go up 0 units, so we don't go up or down at all. The point is going to just sit on the x-axis 2 units away from the origin. That is the correct location of the point (2, 0).

Finally, draw the straight line that connects these 2 points.



Additional Example

Find the slope of the line that passes through the points (−2, 6) and (3, 2).

We use the formula that was mentioned above. It is important that we plug y-values into the numerator and x-values into the denominator of the formula and that we do so in the same order in the numerator as in the denominator. In this case, our slope is given by the following.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2-6}{3-(-2)} = \frac{-4}{3+2} = \frac{-4}{5}.$$

So, our slope is $\frac{-4}{5}$, or $-\frac{4}{5}$.

Avoiding Common Errors

- When solving slope problems, be sure that you do not switch the order in which you enter the information in the numerator and in the denominator (respectively).
- The quadrants are always numbered starting from the upper-right quadrant and moving counterclockwise.
- Pay attention to the signs of the numbers in the ordered pairs when plotting. The ordered pair (−3, 5) is very different from (3, −5).
- Ordered pairs are always written with the x-value first and the y-value second.

Study Tips

- Having graph paper or grid paper will make graphing on the coordinate plane much easier.
- A ruler sometimes helps when drawing straight lines.

PROBLEMS

Plot each of the following points in the Cartesian plane, and state the quadrant in which each point lies.

1. (4, −3)

2. (−7, −1)

3. (−5, 2)

4. (6, 6)

5. (−3, 0)

Plot the line that travels through each of the given pairs of points.

6. (0, 2) and (4, 10)

7. (−3, −1) and (2, 4)

8. (−8, 2) and (−7, 9)

9. (−1, 5) and (2, −7)

10. (1, 3) and (4, −10)

Find the slope of each line that is described below.

11. The line segment described in problem 6.

12. The line segment described in problem 7.

13. The line segment described in problem 8.

14. The line segment described in problem 9.

15. The line segment described in problem 10.

Find the midpoint of each line that is described below.

16. The line segment described in problem 6.

17. The line segment described in problem 7.

18. The line segment described in problem 8.

19. The line segment described in problem 9.

20. The line segment described in problem 10.

Geometry—Triangles and Quadrilaterals

Lesson 18

Topics in This Lesson

- Polygons.
- Triangles.
- Right angles.
- The area and perimeter of triangles.
- Parallelograms.
- The area and perimeter of parallelograms.

Summary

This lesson gives an introduction to polygons with a special focus on triangles and parallelograms. Different types of triangles are discussed, and how to find the area and perimeter of triangles is shown. Right angles are also discussed. Next, different types of parallelograms—such as rectangles and squares—are shown with an explanation of how to find their perimeter and area.

Definitions

equiangular triangle: A triangle that has 3 equal angles. Each of these angles measures 60°.

equilateral triangle: A triangle whose 3 sides are all equal in length.

isosceles triangle: A triangle that has 2 sides, or legs, of equal length.

obtuse triangle: A triangle that has 1 angle that is greater than 90°.

parallelogram: A 4-sided polygon that consists of 2 pairs of parallel line segments.

polygon: A closed figure in the plane whose boundary is a set of at least 3 straight lines that intersect at each vertex, or sharp point, of the figure.

quadrilateral: A polygon with 4 sides, or legs.

rectangle: A parallelogram whose angles are all right angles.

right angle: An angle that measures 90°.

right triangle: A triangle that has 1 angle that measures 90°.

scalene triangle: A triangle whose angles are all less than 90°.

square: A rectangle whose 4 legs are all the same length.

triangle: A polygon that has exactly 3 sides. The sum of all 3 of its angles is 180°.

vertex: The corner-like place where 2 line segments meet in a polygon.

Formulas and Rules

parallelograms:

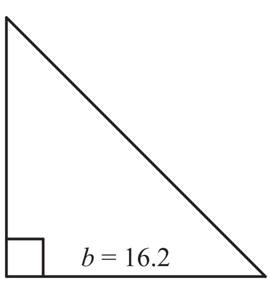
- The sum of the degrees of the angles inside a parallelogram, rectangle, or square is always 360°.
- The area of any parallelogram, rectangle, or square is equal to $b \times h$, where b is the base and h is the height.
- The perimeter of any such object is the sum of the lengths of the 4 legs.

triangles:

- The sum of the degrees of the angles inside a triangle is always 180°.
- The area of a triangle is equal to $\frac{1}{2} \times b \times h$, where b is the base and h is the height of the triangle.
- The perimeter of a triangle is the sum of the lengths of the 3 legs.

Example from the Lesson

Find the area of the isosceles right triangle shown.



We see from the figure that the base is 16.2. The height has not been given to us, but we need it in order to use our area formula. So, what do we do? Well, let's go back to the wording of the problem.

We are told that the triangle is an isosceles right triangle, which actually tells us a few things. First, because it is isosceles, it has 2 legs of equal length. In addition, because it is a right triangle, it has a hypotenuse (which is across from the 90°, or right, angle). Now, let's combine these 2 pieces of information.

We know that there is only 1 hypotenuse in a right triangle because there can only be 1 90° angle—which means that there cannot be another leg that is the same length as the hypotenuse. Because the triangle is isosceles, that means that the other 2 legs in the triangle must have the same length. Now, go back to the picture. We have just determined that the leg that would give us the height of the triangle must have the same length as the base of the triangle. Therefore, the height of the triangle is 16.2 as well.

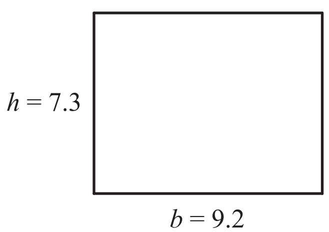
Now, we can plug in all of that information in the area formula for a triangle.

$$\begin{aligned} A &= \frac{1}{2} \times b \times h \\ &= \frac{1}{2} \times 16.2 \times 16.2 \\ &= 8.1 \times 16.2 \\ &= 131.22. \end{aligned}$$

After we do the long multiplication of the 2 decimal numbers, we discover our final answer: 131.22.

Additional Example

Find the perimeter and area of a rectangle with a height of 7.3 and a base of 9.2.



The first thing to do is to make a sketch of the rectangle.

We know that the perimeter of a rectangle is just the measurement of all of its sides added together. We also know that a rectangle has 2 sets of sides of the same length. Therefore, 2 sides measure 7.3, and 2 sides measure 9.2. Lining up our decimal points, we add these numbers to get 33—so the perimeter of the rectangle is 33.

The area is found by multiplying the length by the width: $9.2 \times 7.3 = 67.16$.

So, the area of the rectangle is 67.16.

Avoiding Common Errors

- The height of the triangle does not have to be 1 of the legs of the triangle; it can be the height from the middle of the base to a vertex. If the base of the triangle has been drawn in a horizontal fashion, then the height must be a vertical distance from the bottom of the triangle to the highest point.
- Do not confuse the area and the perimeter. Area is found by multiplying, and perimeter is found by adding.

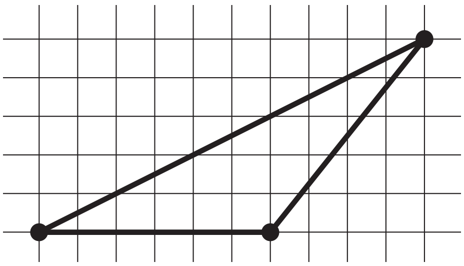
Study Tip

- Take the time to memorize the different terms in this lesson. There are many such vocabulary terms in geometry, and it is very important to know what each one means.

PROBLEMS

Characterize each of the geometric figures shown.

1.



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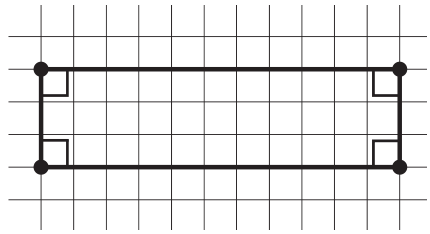

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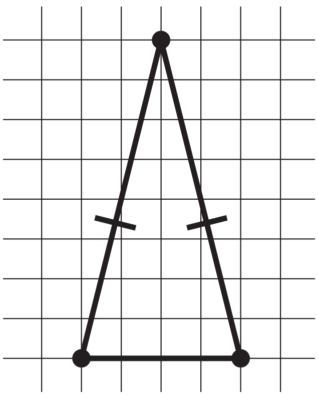

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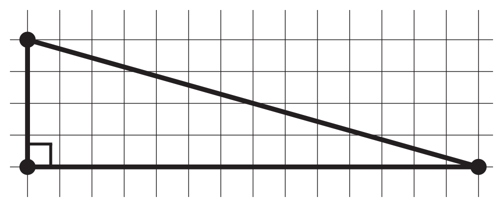

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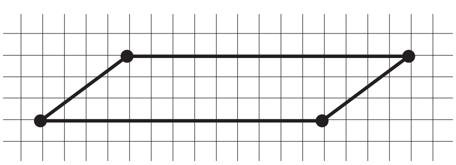

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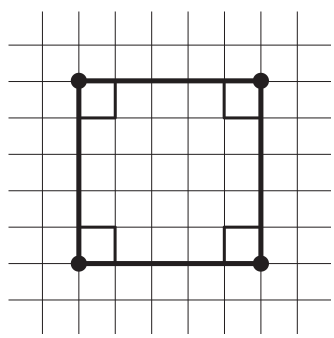

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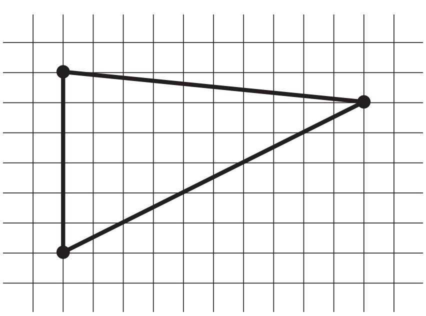

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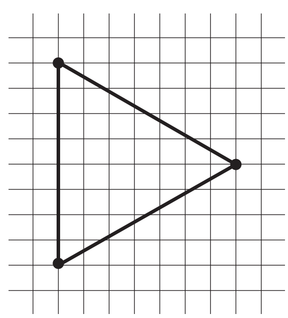
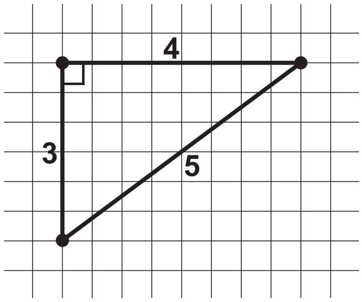


Find the perimeter and area of the following triangles.

9.



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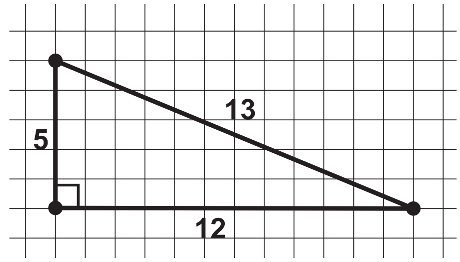

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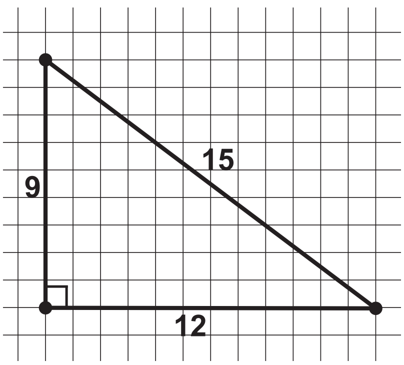
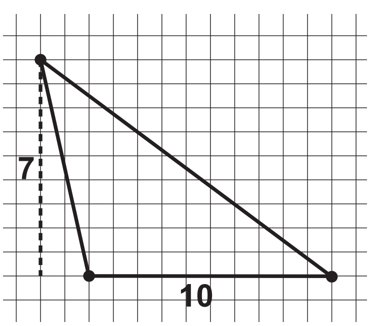


Find the area of the following triangles.

12.



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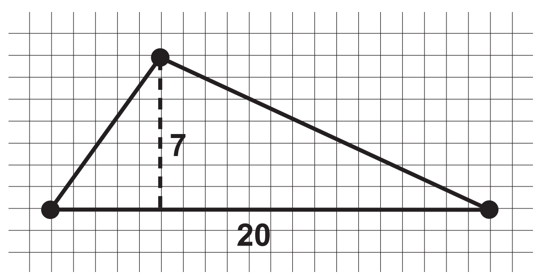

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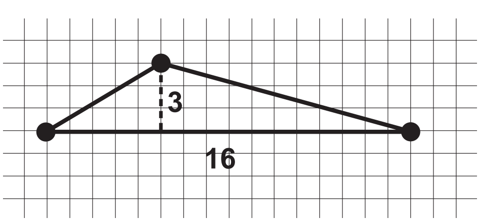
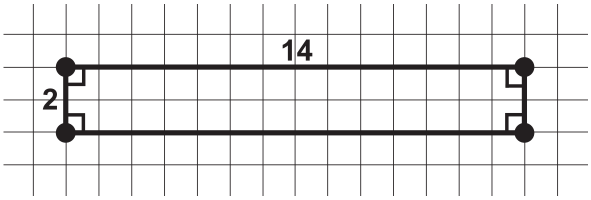


Find the perimeter and area of each of the following.

15.



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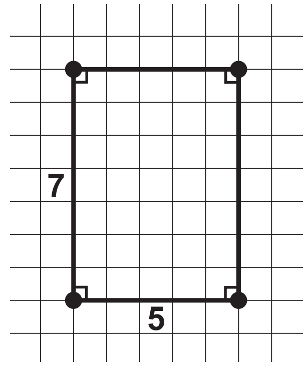

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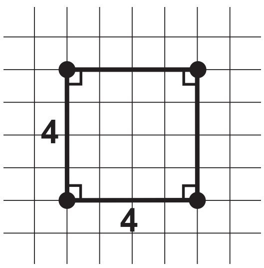
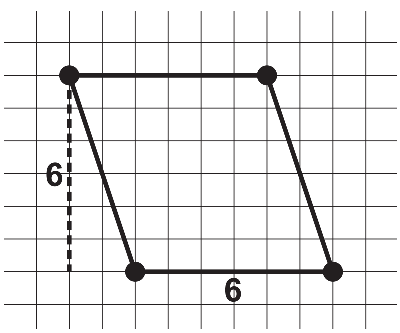


Find the area of the following figures.

18.



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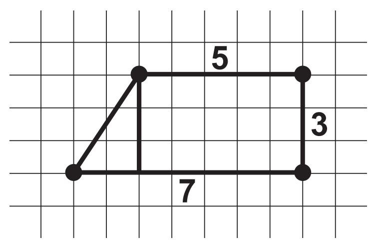

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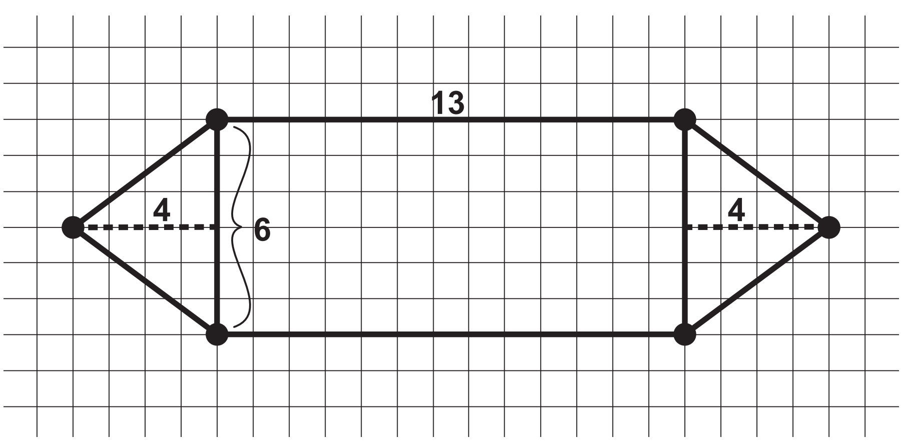


Geometry—Polygons and Circles

Lesson 19

Topics in This Lesson

- Polygons with many sides.
- Regular polygons.
- The sum of angles in a regular polygon.
- Circles.
- The radius and diameter of a circle.
- The circumference and area of a circle.

Summary

This lesson begins with definitions of multisided polygons. How to find the sum of all the angles in a regular polygon is shown. The lesson then moves on to circles and the special number pi (π). How to find the area and circumference of circles is explained.

Definitions

center of the circle: A point that is not actually part of the circle, but it is in the middle of the circle, and each point on the circle is the same distance away from it.

circle: A set of points in the plane that are equidistant (the same distance away) from a fixed point, which we call the center of the circle.

circumference: The distance around a circle.

decagon: A polygon with 10 sides.

diameter: The length of any line segment that connects 2 points on a circle and also goes through the center of the circle. The diameter is always twice the radius distance.

equiangular: When all the angles in a polygon have the same measurement.

equilateral: When all the sides of a polygon have the same length.

heptagon: A polygon with 7 sides.

hexagon: A polygon with 6 sides.

nonagon: A polygon with 9 sides.

octagon: A polygon with 8 sides.

pentagon: A polygon with 5 sides.

pi: The ratio of the perimeter of any circle to the diameter. It is approximately $\frac{22}{7}$ and is often approximated as 3.14. The exact value of pi is a nonrepeating, infinite decimal number. It is given by the symbol π.

radius: The distance from the center of a circle to any point on the circle.

regular polygon: A polygon that is equiangular and equilateral.

Formulas and Rules

area of a circle: $\pi \times r^2$, where r is the radius of the circle.

circumference of a circle: $2 \times \pi \times r$, where r is the radius of the circle.

sum of degrees for a regular polygon: It is exactly 180 times the number that is 2 less than the number of legs. It can be written as $180 \times (L - 2)$, where L is the number of legs (sides) in a polygon.

Example from the Lesson

Mary has a circular pool in her backyard that has a radius of 14 feet. She needs to cover the bottom of the pool with a special fabric that is resistant to ripping. Should she calculate the circumference or area of the pool to figure out how much fabric to buy? How much fabric should she buy? Use $\frac{22}{7}$ as your approximation of π to figure out your answer.

Mary needs to know the area of the bottom of the circular pool in order to know how much fabric to buy. Knowing the circumference only tells her the length of the perimeter of the pool, but that does almost nothing for her in terms of covering the bottom of the pool.

So, let's help Mary approximate the area of the bottom of her pool.

Area = $\pi \times r^2$, and in this case, $r = 14$.

So, the area is approximately $\frac{22}{7} \times 14^2$, using $\frac{22}{7}$ as our approximation of π. That means that we have the following.

$$\begin{aligned} &\frac{22}{7}\times 14\times 14 \\ &= 22\times 2\times 14 \\ &= 44\times 14 \\ &= 616. \end{aligned}$$

So, Mary needs to buy enough fabric to cover 616 square feet at the bottom of her pool.

Additional Example

Find the sum of all the angles in a regular heptagon. What is the measurement of each angle (accurate to 2 decimal places after the decimal point)?

We know from our definitions that a heptagon has 7 sides, and we can find the sum of a regular polygon's angles by taking $180 \times (L - 2)$.

Plugging in 7 for L, we get the following.

$$\begin{aligned} &180\times(7-2) \\ &=180\times 5 \\ &= 900. \end{aligned}$$

So, the sum of all the angles is 900—but what is the sum of 1 angle? We know that a regular polygon is equiangular, which means that each angle has the same measurement. We know that there are 7 angles total, and we know that we need to divide the total by the number of angles to get the approximate measure of 1 angle.

$$900 \div 7 = 128.57.$$

This decimal number keeps going on for a while, but we just need to round it to 2 decimal places.

Avoiding Common Errors

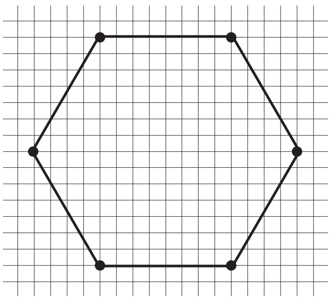
- Do not confuse area and perimeter or area and circumference. Area measures the interior of the polygon; perimeter and circumference are the outside measures.

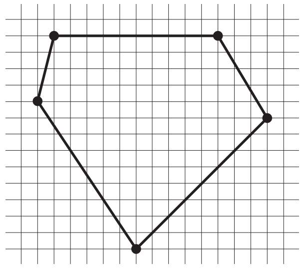
Study Tips

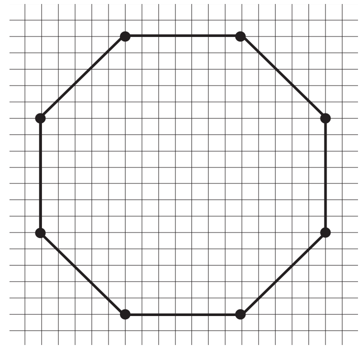
- A way to remember area and perimeter is to think of perimeter as a fence that encloses an area.
- Review basic math skills, such as multiplying with decimal numbers and fractions, as you work with π.

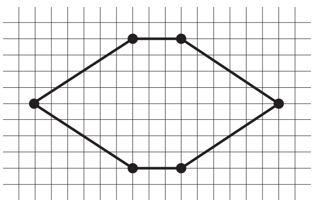
PROBLEMS

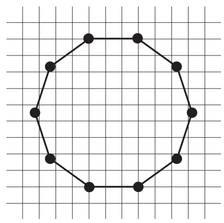
Characterize each of the geometric figures shown—including stating whether the figure is regular.

1.


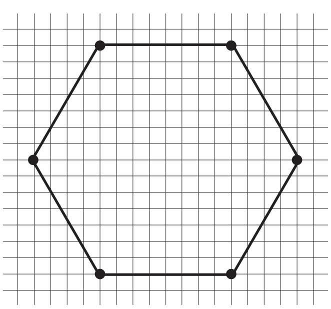
2.


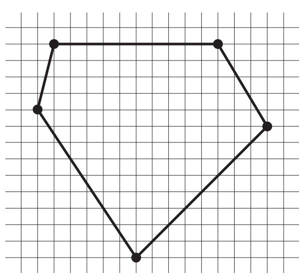
3.


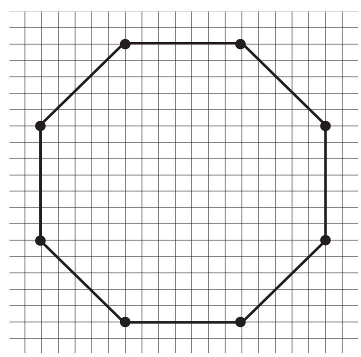
4.


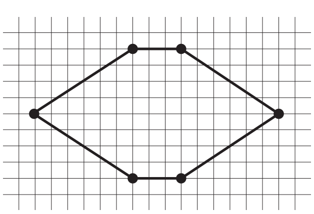
5.


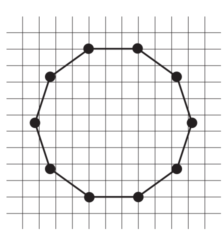
Find the total measure of all the angles in the objects shown.

6.


7.


8.


9.


10.


Approximate the circumference and the area of an object with the following features. Use 3.14 as your approximation of π.

11. A circle with a radius of 6 cm.

12. A circle with a radius of 19 cm.

13. A circle with a radius of 3.2 in.

14. A circle with a radius of 12.5 ft.

15. A circle with a radius of 0.01 ft.

Approximate the area of a circle with the following features. Use $\frac{22}{7}$ as your approximation of π.

16. A circle with a diameter of 14 ft.

17. A circle with a diameter of $16\frac{4}{5}$ in.

18. A circle with a diameter of 2100 cm.

19. A semicircle (or half circle) with a diameter of 84 mm.

20. A semicircle (or half circle) with a diameter of 266 cm.

Number Theory—Prime Numbers and Divisors

Lesson 20

Topics in This Lesson

- Divisors and factors.
- Prime numbers.
- Prime factorization.
- Greatest common divisors.
- Least common factors.
- Least common denominators.

Summary

This lesson begins with a discussion of divisors and factors. Prime numbers are introduced. How to factor a number into primes, which is called prime factorization, is shown. How to use prime factorization to find the greatest common factor and the least common denominator are also explained.

Definitions

divisor: A number that divides another number. In number theory, it is a number that divides another number evenly. It is the number on the outside of the division bar—or the bottom number if a division problem is written as a fraction.

factor: A number that divides another number evenly, leaving no remainder. It is the same as a divisor.

greatest common divisor (GCD): The largest divisor that divides both of the numbers with no remainder. It is also known as the greatest common factor (GCF).

least common denominator (LCD): The smallest number that 2 unlike denominators will divide, or the smallest number for which the 2 given denominators are factors. For example, if the denominators were 3 and 4, the least common denominator would be 12.

least common multiple (LCM): The smallest number for which the 2 given numbers are factors.

number theory: The study of the properties of whole numbers.

prime factorization: Breaking a number down into its prime number factors.

prime number: A number greater than 1 whose divisors are only 1 and the number itself.

Formulas and Rules

finding the greatest common factor: Factor numbers to their prime factorization. Find the smaller occurrence of each prime number in the factors. Multiply these together to obtain the greatest common factor.

finding the least common denominator: Factor numbers to their prime factorization. Find the greatest occurrence of each prime number in the factors. Multiply these together to obtain the least common denominator.

Example from the Lesson

Find the prime factorization of the number 1100.

First, notice that $1100 = 11 \times 100$. So, we start our factor tree with an 11 on one side and a 100 on the other. It might be helpful to circle the 11 because it is prime, so nothing else needs to be done on that side of the factor tree. Next, we focus on the 100, which is the same as 50×2. You can circle the 2 because it is prime. Now, just focus on the 50, which is 25×2. Write 25 and 2 below the 50, and circle the 2. Lastly, $25 = 5 \times 5$, so write 5 and 5 below the 25, and circle each 5 (because 5 is prime).

We now have all that we need to write down the prime factorization of 1100: $1100 = 2^2 \times 5^2 \times 11$, which is the same as $2^2 \times 5^2 \times 11^1$.

Additional Example

Find the least common denominator needed to complete the problem $\frac{1}{36} + \frac{1}{90}$.

If we were to multiply the 2 denominators together, we would have a denominator of $36 \times 90 = 3240$. This would work for a denominator, but reducing the fractions after addition would be much more complicated.

Instead, we should find the least common denominator of 36 and 90. First, we need to find the prime factorization of 36 and 90. We can factor using a factor tree, remembering to circle the prime numbers. The prime factorizations turn out to be $36 = 2^2 \times 3^2$ and $90 = 2^1 \times 3^2 \times 5^1$.

If we find the larger occurrence of each prime number, we get $2^2 \times 3^2 \times 5^1$. Multiplying those numbers together gives us 180. Therefore, the LCD is 180 because 36 and 90 are both factors of this number.

Avoiding Common Errors

- The number 1 is not a prime number.
- The LCD (least common denominator) is not the same as the GCF (greatest common factor). The LCD will be either the same as or larger than both numbers; the GCF will be the same as or smaller than both of the numbers.

Study Tips

- When you write a factor tree, circle the prime numbers to keep track of when a branch of the factor tree comes to its end.
- Allow yourself lots of paper to write out factor trees; don't skimp on space.
- Knowing multiplication and division facts will help greatly when completing prime factorization problems.

PROBLEMS

Determine whether each of the following numbers is prime.

1. 37

2. 79

3. 63

4. 143

5. 157

Find the prime factorization of each of the following.

6. 1250

7. 588

8. 1155

9. 1287

10. 2057

11. 6992

12. 21,000

Compute the following.

13. GCD(72, 54)

14. GCD(242, 264)

15. GCD(400, 693)

16. LCM(72, 54)

17. LCM(30, 60)

18. LCM(100, 36)

Find the least common denominator that would be needed to start each problem below. Do not complete the sum of the fractions; only find the least common denominator.

19. $\frac{1}{20}+\frac{1}{52}$

20. $\frac{1}{18}+\frac{1}{120}$

Number Theory—Divisibility Tricks
Lesson 21

Topics in This Lesson

- Even and odd numbers.
- Parity.
- Divisibility properties.
- Divisibility shortcuts.

Summary

This lesson is all about tricks to find out if a number is divisible by another number. First, basic rules for parity for addition, subtraction, and multiplication are given. Then, rules for finding out if a number is divisible by 2, 3, 4, 5, 8, 9, and 11 are discussed.

Definitions

even number: A number that is divisible by 2.

odd number: A number that has a remainder of 1 when divided by 2.

parity: Whether a number is even or odd.

Formulas and Rules

addition and subtraction parity:

- If you add or subtract 2 numbers that are even, then the result will be even.
- If you add or subtract 2 numbers that are odd, then the result will be even.
- If you add or subtract 2 numbers that have different parity (one of the numbers is even and the other number is odd), then the result will be odd.

divisibility rule for 2: A number is divisible by exactly 2 when its last digit is divisible by 2; in other words, a number is divisible by exactly 2 when the last digit of the number is a 0, 2, 4, 6, or 8.

divisibility rule for 3: A number is divisible by exactly 3 when the sum of all the digits of the number is divisible by 3.

divisibility rule for 4: A number is divisible by exactly 4 when the number created from its last 2 digits is divisible by 4.

divisibility rule for 5: A number is divisible by exactly 5 when its last digit is divisible by 5; in other words, a number is divisible by exactly 5 when the last digit of the number is a 0 or a 5.

divisibility rule for 8: A number is divisible by exactly 8 when the number created from its last 3 digits is divisible by 8.

divisibility rule for 9: A number is divisible by exactly 9 when the sum of all the digits of the number is divisible by 9.

divisibility rule for 11: A number is divisible by exactly 11 when the alternating sum of all the digits of the number is divisible by 11.

multiplication parity:

- If you multiply 2 even numbers together, then the answer you get is an even number.
- If you multiply 2 odd numbers together, then the answer you get is an odd number.
- If you multiply an even number and an odd number, then the answer you get is an even number.

Example from the Lesson

State quickly whether 95,123,473,737,371,820 is divisible by 8.

The rule says we need to check whether 8 divides 820. So, we need to see whether 8 divides 820 with a 0 remainder.

After we divide 820, we see that the remainder obtained is 4 (which is not 0). That means that 8 does not divide 820 evenly, which means that 8 cannot divide the original number 95,123,473,737,371,820.

Additional Example

Quickly determine whether 8,452,872,241 is divisible by 11.

The rule for determining divisibility by 11 is to alternately subtract and add the individual digits of the number. If the final result is divisible by 11, then so is the whole number.

$$\begin{aligned}
&8 - 4 + 5 - 2 + 8 - 7 + 2 - 2 + 4 - 1 \\
&= 4 + 5 - 2 + 8 - 7 + 2 - 2 + 4 - 1 \\
&= 9 - 2 + 8 - 7 + 2 - 2 + 4 - 1 \\
&= 7 + 8 - 7 + 2 - 2 + 4 - 1 \\
&= 15 - 7 + 2 - 2 + 4 - 1
\end{aligned}$$

$= 8 + 2 - 2 + 4 - 1$
$= 10 - 2 + 4 - 1$
$= 8 + 4 - 1$
$= 12 - 1$
$= 11.$

Our final answer is 11, and 11 is certainly divisible by 11. Therefore, 8,452,872,241 is also divisible by 11.

Avoiding Common Errors

- When alternating adding and subtracting to check for divisibility of 11, you need to start with subtraction (because the first digit has a positive sign in front of it, and you need to alternate the operations).

Study Tips

- If a multidigit number ends in an even number, the whole number is even. If a multidigit number ends in an odd number, the whole number is odd.
- Zero is considered to be an even number.
- When checking for divisibility by 11, it might be easier to write the string of $+ - + - +$, leaving room for the digits of the number. Then, write in the digits after the operator symbols are in place.

PROBLEMS

Without actually doing the division completely, answer each of the following questions (using the divisibility tricks discussed in this lesson).

1. Is 120,482,912 divisible by 4?

2. Is 2,394,894,278,230 divisible by 4?

3. Is 9,843,289,237,013,092 divisible by 4?

4. Is 2,309,432,094,294,274 divisible by 4?

5. Is 120,482,912 divisible by 8?

6. Is 9,843,289,237,013,092 divisible by 8?

7. Is 90,240,239,871,230,924,915 divisible by 5?

8. Is 2,398,239,802,309,329,032 divisible by 5?

9. Is 230,949,042 divisible by 3?

10. Is 89,498,249,824 divisible by 3?

11. Is 72,000,004,023 divisible by 3?

12. Is 64,238,792,347 divisible by 3?

13. Is 230,949,042 divisible by 9?

14. Is 72,000,004,023 divisible by 9?

15. Is 83,484,298 divisible by 9?

16. Is 555,555,555 divisible by 9?

17. Is 13,579,709 divisible by 11?

18. Is 72,816,150 divisible by 11?

19. Is 123,123,123 divisible by 11?

20. Is 22,222,222 divisible by 11?

Introduction to Statistics

Lesson 22

Topics in This Lesson

- Mean.
- Median.
- Mode.
- Range.

Summary

The basics of statistics is the focus of this lesson. Definitions of mean, median, mode, and range are given. Many examples of how to organize data to find these values and how to find the values are given.

Definitions

mean: The average of all the numbers in a set of data. Also called the arithmetic mean.

median: The number that is in the exact middle of a set of numbers when they are written from smallest to largest.

mode: The number that occurs most often in a set of data.

range: The difference found by subtracting the largest number in the list from the smallest number in the list of data.

Example from the Lesson

Find the mean, median, mode, and range for the data set 128, 124, 119, 126, 124, 113, 111, 117.

Let's start with the mean. The mean is simply the average of the values in the list. We need to add the values together and then divide by the number of values in the list.

$$\begin{aligned} & 128 + 124 + 119 + 126 + 124 + 113 + 111 + 117 \\ &= 252 + 119 + 126 + 124 + 113 + 111 + 117 \\ &= 371 + 126 + 124 + 113 + 111 + 117 \\ &= 497 + 124 + 113 + 111 + 117 \\ &= 621 + 113 + 111 + 117 \\ &= 734 + 111 + 117 \\ &= 845 + 117 \\ &= 962. \end{aligned}$$

Now, to complete the mean, we need to divide this sum of 962 by the number of values in our original list, which is 8. As with many means, the answer may not be a whole number.

```
     .
8)962.00
```

Let's begin this long division process (which you learned in a previous lesson). The number 8 goes into 9 once, so we put a 1 above the 9 and subtract 8 from 9. That gives us a 1, and then we drop down the 6, so we have 16.

```
  1  .
8)962.00
 −8
  16
```

Next, 8 goes into 16 twice (because $8 \times 2 = 16$), so we write a 2 in the answer section above the 6 and write a 16 below the 16. Then, we subtract to get 0. We then drop down the 2, giving us a 2.

```
  12 .
8)962.00
 −8
  16
 −16
   02
```

Because 8 does not go into 2, we put a 0 in the answer section—just before the decimal point—and then we drop down 1 of the 0s from 962.00.

```
  120.
8)962.00
 −8
  16
 −16
   020
```

How many times does 8 go into 20? Exactly twice because $8 \times 2 = 16$, so we put a 2 in the answer section (just after the decimal point in this case) and then subtract 16 from 20 to get 4. Then, drop the second 0 that appears after the decimal point in 962.00.

```
  120.2
8)962.00
 −8
  16
 −16
   020
   −16
     40
```

Because 8 goes into 40 5 times, we write a 5 in the answer section and subtract 8×5, or 40, from the 40 that is there, which gives us 0. Because the only thing left to drop down would be 0s as well, we know that the division is done. The final answer is 120.25, which is the mean.

Next, find the median. We need to rewrite the list of data from the smallest to the largest value: 111, 113, 117, 119, 124, 124, 126, 128.

Now, we can find the median relatively quickly. Because this list has an even number of values, we need to find the 2 elements that are in the middle of the list—in this case, the 119 and 124. Then, we have to find the average of those 2 numbers, which is $\frac{119+124}{2}$, or 121.5. So, the median is 121.5, which is relatively close to the mean of 120.25.

Next, we need to find the mode, the number that appears most often in the list of data. Because the list is written in order, we can quickly see that the mode is 124.

Finally, we need to find the range of the data set. We subtract the smallest value from the largest value.

$$128 - 111 = 17.$$

So, the range is 17.

Additional Example

Kelly wants to be a starter for the varsity basketball team, but she knows that all the starters on the team score an average of 10 points per game. In her last 5 games, she scored the following.

Game Number	Number of Points
1	5
2	3
3	10
4	14
5	9

What is Kelly's current average? What is the minimum number of points she will need to score in the next game to bring her average up to at least 10 points?

Let's begin by writing the data from least to greatest: 3, 5, 9, 10, 14.

We know that the average of a set of data is the mean, and we find that by adding all the numbers together and dividing by how many numbers there are to start with.

$$3 + 5 + 9 + 10 + 14 = 41.$$

There are 5 numbers in this data set, so we need to divide 41 by 5.

$$41 \div 5 = 8.2.$$

Kelly's current average is 8.2 points per game.

How do we find the points needed to bring this average to 10? We know that the average is found by dividing the sum of the numbers in the set by the total number of items in the set. Kelly played 5 games, so 1 more game means that her total will be 6 games. We know what we want her average to be, which is 10 points, so the only thing we do not know is how many total points she needs to score in the last game. We could write this mathematically as the following.

(Total number of points Kelly needs) ÷ 6 = 10.

We need some number that when divided by 6 will give us 10. If we multiply 6 × 10, it will tell us the number we need.

$6 \times 10 = 60.$

If we put 60 in as the total number of points Kelly needs over the 6 games, the mathematical sentence makes sense.

$60 \div 6 = 10.$

However, we need to find out how many points Kelly needs in her last game to have an average of 10 points. We know that her total in the previous 5 games was 41. If we subtract the total needed from the number of points she already scored, we have the number of points she needs to score.

$60 - 41 = 19.$

Therefore, in order to have an average of 10 points in 6 games, Kelly needs to score 19 points in the sixth game.

Avoiding Common Errors

- It is easy to confuse the terms “mean,” “median,” and “mode.” Make sure that you know the definitions.
- Pay careful attention to what the problem is asking. Simply finding the median or the mean might not be enough to solve the problem.

Study Tips

- A data set will always have 1 median, mean, and range.
- If the number of items in the data set is even, then calculating the median requires taking the average of the 2 numbers in the middle of the list.
- Rewriting the data from the smallest to the largest value will help with many statistical calculations.

PROBLEMS

Find the mean of each of the following data sets.

1. 14, 18, 24, 23, 10, 14, 17, 15, 27
2. 245.3, 187.6, 132, 175.3, 99.7, 172.2, 159, 187.6, 245.3, 187.6
3. 189, 257, 194, 224, 286, 257, 224, 226, 286, 285, 257, 282
4. 86, 84, 82, 94, 94, 94, 90, 83, 81, 82, 94, 97, 88
5. 1.75, 1.87, 2.25, 1.89, 1.87, 2.01, 1.75, 2.22, 2.20, 1.87, 1.99, 1.87

Find the median of each of the following data sets.

6. The data set from problem 1.
7. The data set from problem 2.
8. The data set from problem 3.
9. The data set from problem 4.
10. The data set from problem 5.

Find the mode of each of the following data sets.

11. The data set from problem 1.
12. The data set from problem 2.
13. The data set from problem 3.
14. The data set from problem 4.
15. The data set from problem 5.

Find the range of each of the following data sets.

16. The data set from problem 1.
17. The data set from problem 2.
18. The data set from problem 3.
19. The data set from problem 4.
20. The data set from problem 5.

Introduction to Probability

Lesson 23

Topics in This Lesson

- Historical background of probability.
- Sample space and outcome.
- Multiplication principle.
- Calculating probability.
- Theoretical probability.

Summary

This lesson begins with the historical background of probability. Basic terms relating to probability are given. Then, how to calculate probability is explained with several examples.

Definitions

multiplication principle: If you are trying to count the number of ways to do multiple independent events in succession, then the total will just be the product of the number of ways to do each event separately. In other words, if the events are disconnected, or independent of one another, then you just multiply together the number of ways to do each event, and you have your overall answer.

outcome: A possible result of a probability experiment.

sample space: The set of all possible outcomes for a particular activity or experiment.

Formulas and Rules

theoretical probability: If a sample space has n equally likely outcomes and an event A occurs in m of these outcomes, then the theoretical probability of event A is denoted by $p(A)$ and is given by $p(A) = \frac{m}{n}$.

Example from the Lesson

Ten cards from a traditional deck of playing cards are placed on a table. There are 2 black aces, 2 red aces, 2 black kings, 2 red kings, and 2 black queens in this pile of cards. What is the probability of randomly selecting a black card from this set of cards?

We are not at all worried about which particular card we choose in terms of its face value; all that concerns us in this question is the color of the card.

The denominator of our probability fraction will be 10 because the total number of cards from which we choose is 10. There are exactly 6 black cards in the pile, and we want to choose 1 of those, so the numerator of our probability fraction is 6. Therefore, our probability of randomly selecting a red card from this set of cards is $\frac{6}{10}$, or $\frac{3}{5}$.

Using the same set of cards we just considered, what is the probability of randomly choosing a card that is not a queen?

As with the earlier problem, the size of our sample space is 10, so our denominator will be 10. Next, we want to count how many cards are not queens. The number of queens is 2, so the number of cards that are not queens is 8 (the 4 aces and the 4 kings). Therefore, the numerator of our probability fraction is 8, and that means our probability is $\frac{8}{10}$, or $\frac{4}{5}$.

Additional Example

What is the probability of rolling a numbered cube 4 times and getting a 2 each time?

For this example, we can use the multiplication principle because the result of each roll of the numbered cube does not depend on the previous rolls.

There are 6 possible outcomes for each roll (rolling a 1, 2, 3, 4, 5, or 6), and only 1 of those outcomes is the one we want (getting a 2). The denominator for our fraction is 6, and the numerator is 1. So, the probability of getting a 2 on 1 roll is $\frac{1}{6}$.

However, we want to know the probability for rolling a 2 4 times in a row. The probability for rolling a 2 is $\frac{1}{6}$, so we need to multiply the probability of each instance together to get the result.

$$\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}\times\frac{1}{6}=\frac{1}{1296}.$$

The probability of rolling 4 2s in a row is $\frac{1}{1296}$. This means that if we roll a numbered cube 4 times and record the outcomes, in theory we will obtain 4 2s in a row exactly once in 1296 tries.

Avoiding Common Errors

- Probability is always a fraction. The denominator always tells us the total number of options, and the top number tells us which options we want.
- Be careful to understand what the problem is asking. Is it asking for the chances of choosing a specific color or number—or not choosing that specific color or number?

Study Tip

- Review multiplication of fractions. It will help with probability.

PROBLEMS

For problems 1–5, imagine that you have a container that holds 75 Ping-Pong balls that are labeled for a traditional game of bingo. Namely, balls labeled "B" also have a number from 1 to 15 printed on them, balls marked "I" are numbered 16–30, balls marked "N" are labeled 31–45, balls marked "G" are labeled 46–60, and balls marked "O" are labeled 61–75. So, for example, 1 of the balls is labeled B-7, and another is labeled G-48.

1. Find the probability of randomly selecting 1 ball that is labeled with an N.

2. Find the probability of randomly selecting 1 ball that is labeled with an odd number.

3. Find the probability of randomly selecting 1 ball that is labeled with a vowel.

4. Find the probability of randomly selecting 1 ball that is labeled with a G and that has a numerical label that is less than 20.

5. Find the probability of randomly selecting 1 ball that is labeled with a number that is divisible by 3.

For problems 6–10, imagine that you have a traditional deck of 52 playing cards (with no jokers included).

6. Find the probability of randomly selecting 1 card from the deck that is a black 10.

7. Find the probability of randomly selecting 1 card from the deck that is either red or black.

8. Find the probability of randomly selecting 1 card from the deck that does not have a numerical face value on it.

9. Find the probability of randomly selecting 1 card from the deck that has a numerical face value on it.

10. Find the probability of randomly selecting 2 cards, 1 at a time, both of which are red. (Note that we will count choosing the ace of diamonds followed by the king of hearts as different from choosing the king of hearts followed by the ace of diamonds—so, the order in which we draw them matters. Note also that once the first card is chosen, it is not placed back into the deck before the second card is chosen.)

For problems 11–15, the sample space is the set of all possible 4-digit numbers that can be written using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 (where a 0 in the first position is valid and where repetitions of the digits is allowed).

11. What is the number of elements in this sample space?

12. What is the probability of randomly having a 4-digit number whose digits are all different?

13. What is the probability of randomly having a 4-digit number whose first and last digits are the same?

14. What is the probability of randomly having a 4-digit number whose number is divisible by 5?

15. What is the probability of randomly having a 4-digit number whose third digit is either 7, 8, or 9, and the last digit is larger than the third digit?

16. The mathematics teachers at your school are named Mr. Smith, Ms. Jones, Mr. Jenkins, Ms. Adams, and Mr. Williams. They each teach every math course offered at the school. What is the probability that during your 4 years at the school, you will always have a mathematics teacher whose last name begins with the letter J?

17. How many different combinations of shirts, ties, and pants can be made if you have 8 dress shirts, 5 ties, and 6 pants (and all of these can be worn in combination—there are no clashes between them)?

18. How many times would you expect to see "heads" if you toss a fair coin 3500 times?

19. How many times would you expect to see a number that is a divisor of 6 if you roll a fair numbered cube 1800 times?

20. How many times would you expect to see a number that is prime if you roll a fair numbered cube 240 times?

Introduction to Algebra

Lesson 24

Topics in This Lesson

- Variables and exponents.
- Adding algebraic expressions.
- Multiplying algebraic expressions.
- Dividing algebraic expressions.

Summary

This lesson provides a basic introduction to algebra. Variables and exponents are explained, and some basic operations in adding, multiplying, and dividing algebraic expressions are shown.

Definitions

algebraic expression: A combination of mathematical symbols that might include numbers, variables, and operation symbols.

numerical coefficient: The number in front of a variable. For example, in $8y^3$, the 8 is the numerical coefficient.

variable: A symbol, usually a letter from the alphabet, that represents 1 or more numbers in an expression.

Formulas and Rules

adding expressions with variables: In order to add 2 expressions that involve variables, it must be the case that the expressions are identical except for the numerical coefficient of the expression. (These are often called "like terms.")

dividing expressions with variables: For any nonzero number x and any 2 integers a and b, $x^a/x^b = x^{(a-b)}$. In other words, you subtract the exponents when you divide 2 like variables.

multiplying expressions with variables: For any nonzero number x and any 2 integers a and b, $x^a x^b = x^{(a+b)}$. In other words, you add the exponents when you multiply 2 like variables.

Example from the Lesson

Simplify $(u^4v^7)(t^3u^5v)$.

Notice that we really are just multiplying 5 small terms together: $u^4v^7t^3u^5v$.

Next, let's reorder all the terms to make it easier to see terms that we can combine. Remember, when we multiply several numbers or terms with one another, we can reorder things to our advantage: $t^3u^4u^5v^7v$.

The v on the right-hand side has no exponent, and a variable with no explicitly written exponent has an understood exponent of 1. So, the v is understood to be the same as v^1.

Using our rule for multiplication of variables, we add the exponents of like variables: $t^3u^{(4+5)}v^{(7+1)}$.

Notice that the t^3 doesn't have another term with which it can combine, but that is not a problem.

Final answer: $t^3u^9v^8$.

Additional Example

Simplify $\frac{48t^6x^9y^3}{-6y^3x^3t}$.

Before we begin dividing, we can rewrite the variables in a different order to make it easier to see: $\frac{48t^6x^9y^3}{-6tx^3y^3}$.

We now take care of the numerical coefficients and like terms to simplify this expression.

$$48 \div (-6) = -8$$
$$t^6 \div t^1 = t^{(6-1)} = t^5$$
$$x^9 \div x^3 = x^{(9-3)} = x^6$$
$$y^3 \div y^3 = y^{(3-3)} = y^0.$$

So far, our answer is $-8t^5x^6y^0$.

Any nonzero number that is raised to the power of 0, such as y^0, is the same as 1, so we can write a 1 in place of y^0. Because any number times 1 is that same number, we can forget about that number completely when writing our final answer. Therefore, our final answer is $-8t^5x^6$.

Avoiding Common Errors

- You can only combine the same variables together. For example, x^2y^2 is not xy^4.
- The exponents of variables remain the same when you add. Exponents are added when you multiply variables.

Study Tips

- It may help when multiplying or dividing an expression with many variables to rewrite the expressions vertically, lining up all the like variables. That way, it is easy to see which exponents need to be added or subtracted.
- When you are evaluating an algebraic expression for a specific value, rewrite the problem, leaving a set of parentheses everywhere that variable was. Then, go back and fill in the specific value. For example, $3x^2 + 2x$ is rewritten as $3(\quad)^2 + 2(\quad)$. Then, go back and fill in the value given for x.

PROBLEMS

1. Evaluate the expression $(x - 4)^2 + 3x$ at the value $x = 7$.

2. Evaluate the expression $(2x + 5)^2 - 3x$ at the value $x = -1$.

3. Evaluate the expression $(3t - 1)^3 + 5t^2$ at the value $t = 0$.

4. Evaluate the expression $(3t - 1)^3 + 5t^2$ at the value $t = 2$.

5. Evaluate the expression $(3t - 1)^3 + 5t^2$ at the value $t = -1$.

For problems 6–10, find the area of a triangle using the formula $A = \frac{1}{2}bh$ for each of the following sets of values.

6. $b = 4$, $h = 2$

7. $b = 15$, $h = 14$

8. $b = 41.2$, $h = 29.4$

9. $b = 4\frac{2}{3}$, $h = 8\frac{1}{2}$

10. $b = 65$, $h = 65$

Simplify each of the following.

11. $14x^2y - 9x^2y$

12. $25a^3b^4 + 10a^3b^4$

13. $7t^5v^8 + 15t^5v^8 - 6t^5v^8$

14. $19t^4v^{10} - 15t^4v^{10} - 8t^4v^{10}$

15. $23xy^2 - 22xy^2 + 12xy^3$

16. $20x^3y^2 - 22x^2y^3 + 12x^3y^2$

17. $-20a^7b^2 + 13a^7b^2 + 7a^7b^2$

18. $\dfrac{63x^{10}y^5z^3}{14x^2yz^3}$

19. $\dfrac{25x^4y^5z^3}{5x^2y^3z^{12}}$

20. $\dfrac{-100x^4y^7z^2}{-45x^{-2}z^3}$

Solutions

LESSON 1

1. 975	**2.** 480	**3.** 1301	**4.** 1444	**5.** 824
6. 1159	**7.** 874	**8.** 2904	**9.** 5119	**10.** 10,212
11. 206	**12.** 396	**13.** 628	**14.** 273	**15.** 87
16. 733	**17.** 2265	**18.** 2317	**19.** 2758	**20.** 282

LESSON 2

1. 156	**2.** 511	**3.** 60	**4.** 308	**5.** 360
6. 722	**7.** 4264	**8.** 624	**9.** 1566	**10.** 1240
11. 2208	**12.** 1870	**13.** 4410	**14.** 7326	**15.** 29,002
16. 26,179	**17.** 4305	**18.** 18,639	**19.** 29,045	**20.** 221,805

LESSON 3

1. 24	**2.** 12	**3.** 96	**4.** 86	**5.** 44
6. 605	**7.** 627	**8.** 526	**9.** 94 R 1	**10.** 45 R 2
11. 475 R 6	**12.** 375 R 3	**13.** 492	**14.** 437	**15.** 618
16. 250	**17.** 742	**18.** 313	**19.** 244 R 10	**20.** 630 R 5

LESSON 4

1. $\frac{3}{4}$	**2.** $\frac{2}{17}$	**3.** $\frac{7}{12}$	**4.** $\frac{3}{67}$	**5.** $\frac{13}{128}$
6. $\frac{9}{41}$	**7.** equivalent	**8.** equivalent	**9.** not equivalent	**10.** not equivalent
11. equivalent	**12.** $\frac{13}{14}$	**13.** $\frac{28}{63}$	**14.** $\frac{20}{52}$	**15.** $\frac{8}{9}$
16. $\frac{15}{22}$	**17.** $7\frac{1}{6}$	**18.** $15\frac{3}{5}$	**19.** $61\frac{3}{10}$	**20.** $15\frac{5}{12}$

LESSON 5

1. $\frac{7}{9}$	**2.** $\frac{4}{5}$	**3.** $\frac{13}{15}$	**4.** $\frac{5}{11}$	**5.** $\frac{13}{35}$
6. $\frac{13}{40}$	**7.** $\frac{5}{2}$	**8.** $\frac{8}{7}$	**9.** $\frac{3}{5}$	**10.** $\frac{11}{21}$
11. $\frac{33}{26}$	**12.** $\frac{29}{24}$	**13.** $\frac{55}{21}$	**14.** $\frac{53}{24}$	**15.** $\frac{23}{18}$
16. $\frac{1}{33}$	**17.** $\frac{37}{24}$	**18.** $\frac{7}{15}$	**19.** $\frac{33}{25}$	**20.** $\frac{37}{30}$

LESSON 6

1. $\frac{8}{15}$	**2.** $\frac{12}{35}$	**3.** $\frac{21}{80}$	**4.** $\frac{2}{11}$	**5.** $\frac{3}{7}$
6. $\frac{3}{4}$	**7.** $\frac{5}{7}$	**8.** $\frac{9}{4}$	**9.** $\frac{9}{14}$	**10.** $\frac{5}{2}$
11. 30	**12.** $\frac{21}{2}$	**13.** $\frac{10}{7}$	**14.** 6	**15.** $\frac{12}{25}$
16. $\frac{10}{11}$	**17.** $\frac{25}{12}$ or $2\frac{1}{12}$	**18.** $\frac{1}{6}$	**19.** $\frac{33}{4}$ or $8\frac{1}{4}$	**20.** $\frac{5}{3}$ or $1\frac{2}{3}$

LESSON 7

1. $\frac{10}{21}$	**2.** $\frac{28}{55}$	**3.** $\frac{39}{80}$	**4.** $\frac{3}{10}$	**5.** $\frac{1}{6}$
6. $\frac{21}{5}$	**7.** $\frac{7}{3}$	**8.** $\frac{14}{15}$	**9.** $\frac{15}{4}$	**10.** $\frac{9}{4}$
11. $\frac{3}{130}$	**12.** $\frac{2}{15}$	**13.** 1	**14.** 2	**15.** 10
16. $\frac{14}{5}$	**17.** 6	**18.** $\frac{8}{3}$ or $2\frac{2}{3}$	**19.** $\frac{1}{2}$	**20.** $\frac{10}{9}$ or $1\frac{1}{9}$

LESSON 8

1. 27.7	**2.** 138.39	**3.** 949.22	**4.** 7115.6	**5.** 173.09
6. 1026.18	**7.** 1357.95	**8.** 6274.286	**9.** 8742.8017	**10.** 1316.35
11. 56.4	**12.** 102.07	**13.** 400.08	**14.** 334.879	**15.** 72.2
16. 93.5	**17.** 3321.88	**18.** 79.465	**19.** 1088.16	**20.** 18.57

LESSON 9

1. 21.85	**2.** 24.63	**3.** 1030.7	**4.** 987.60	**5.** 12.22
6. 117.2433	**7.** 1470.696	**8.** 2.034688	**9.** 170,553.258	**10.** 9104.27550
11. 8.6	**12.** 14.2	**13.** 14.78	**14.** 35.12	**15.** 8.53
16. 43.01	**17.** 123.45	**18.** 5487	**19.** 531.62	**20.** 52.4

LESSON 10

1. $\frac{47}{100}$ **2.** $\frac{17}{20}$ **3.** $\frac{3}{50}$ **4.** $\frac{13}{4}$ **5.** $\frac{27}{40}$ **6.** 0.39

7. 0.83 **8.** 0.04 **9.** 6.25 **10.** 0.572 **11.** 0.625 **12.** 2.8

13. 9.25 **14.** 0.18181818181818 … **15.** 0.83333333333333 …

16. $\frac{21}{50}$ **17.** $7\frac{3}{20}$ **18.** $4\frac{3}{8}$ **19.** $\frac{4}{25}$ **20.** $\frac{1}{40}$

LESSON 11

1. \$4.59 **2.** \$7.56 (rounding the numbers up to the nearest cent)

3. \$2.14 (rounding the numbers up to the nearest cent) **4.** \$16.57 **5.** \$5280.00

6. \$310.94 (rounding the numbers up to the nearest cent) **7.** \$23,910.80 **8.** \$11.85

9. \$56.18 **10.** \$264.12 **11.** \$937.50 **12.** \$1976.25 **13.** 10,350 **14.** 6900

15. 16,100 **16.** 12,650 **17.** \$72 **18.** \$187.20 **19.** \$36.50 **20.** \$1600

LESSON 12

1. These ratios are equivalent; one way to write the proportion is $\frac{6}{8} = \frac{30}{40}$.

2. These ratios are not equivalent.

3. These ratios are equivalent; one way to write the proportion is $\frac{14}{42} = \frac{17}{51}$.

4. These ratios are not equivalent.

5. These ratios are equivalent; one way to write the proportion is $\frac{24}{36} = \frac{30}{45}$.

6. These ratios are equivalent; one way to write the proportion is $\frac{16}{48} = \frac{12}{36}$.

7. These ratios are not equivalent.

8. These ratios are equivalent; one way to write the proportion is $\frac{3}{11} = \frac{36}{132}$.

9. These ratios are not equivalent.

10. These ratios are not equivalent.

11. 434 miles	**12.** 175 pages	**13.** 16	**14.** $2.52	**15.** 104
16. 36	**17.** $4.20	**18.** 62.5 minutes	**19.** 52.78 pounds	**20.** $634.50

LESSON 13

1. 128	**2.** 27	**3.** 625	**4.** 1000	**5.** 81
6. 44	**7.** 100	**8.** 15	**9.** 1117	**10.** 7700
11. 1000	**12.** 5	**13.** 980	**14.** 159	**15.** 3
16. 38	**17.** 170	**18.** 166	**19.** 122	**20.** 710

LESSON 14

1. −46	**2.** 19	**3.** 303	**4.** −923	**5.** 630
6. −8025	**7.** −683	**8.** 1436	**9.** 281	**10.** −141
11. 7	**12.** 5865	**13.** 60	**14.** −84	**15.** −154
16. −25	**17.** −32	**18.** 617	**19.** 35	**20.** −102

LESSON 15

1. 11 **2.** $4\sqrt{3}$ **3.** $4\sqrt{6}$ **4.** $10\sqrt{3}$ **5.** $8\sqrt{2}$

6. $\frac{7}{8}$ **7.** $\frac{4}{5}$ **8.** $\frac{7}{5}\sqrt{2}$ **9.** 78 **10.** 12

11. $32\sqrt{5}$ **12.** 234 **13.** $8\sqrt{7}$ **14.** $\sqrt{11}$ **15.** $18\sqrt{15}$

16. $46\sqrt{2}$ **17.** $128\sqrt{3}$ **18.** 1.73 **19.** 3.16 **20.** 4.69

LESSON 16

1. 3 **2.** 8 **3.** 13 **4.** The answer does not exist. **5.** 3

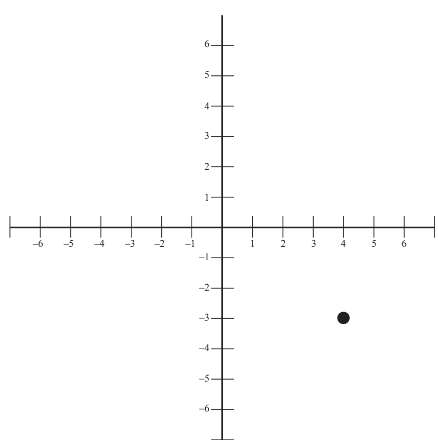
6. The answer does not exist. **7.** 2 **8.** 6 **9.** 8 **10.** 9

11. 343 **12.** 25 **13.** The answer does not exist. **14.** −3 **15.** $\frac{1}{25}$

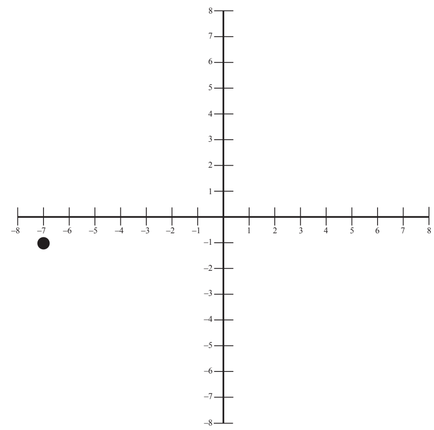
16. $\frac{1}{81}$ **17.** $\frac{1}{128}$ **18.** $\frac{1}{64}$ **19.** $\frac{1}{16}$ **20.** $\frac{1}{9}$

LESSON 17

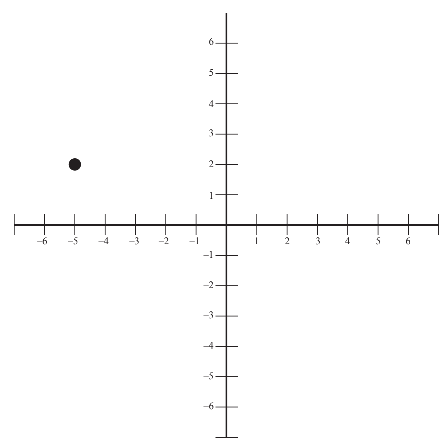
1. This point lies in quadrant IV.



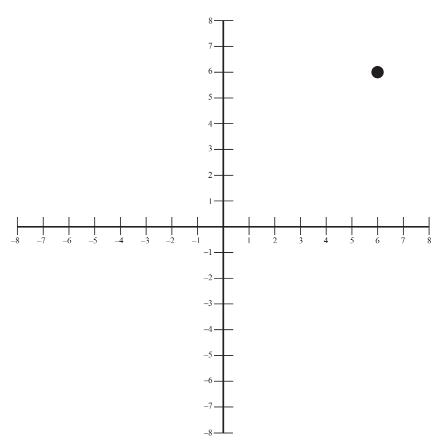
2. This point lies in quadrant III.



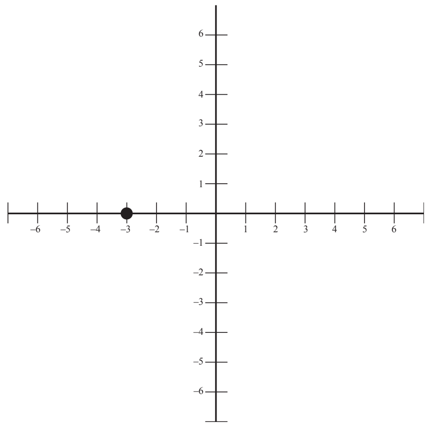
3. This point lies in quadrant II.



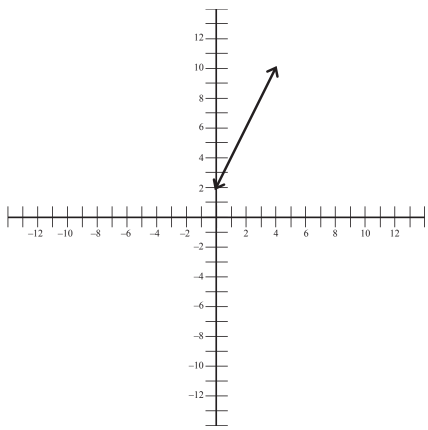
4. This point lies in quadrant I.



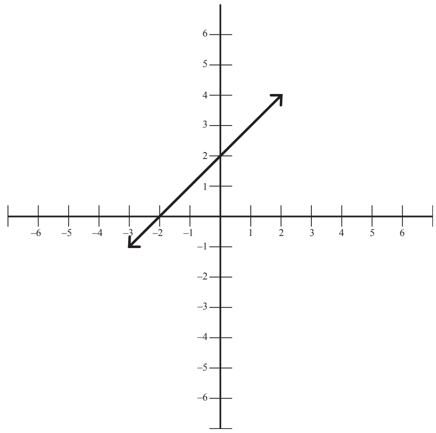
5. This point does not lie in a specific quadrant because it is located on the *x*-axis.



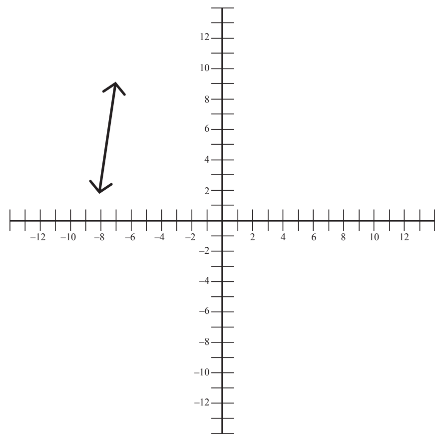
6.



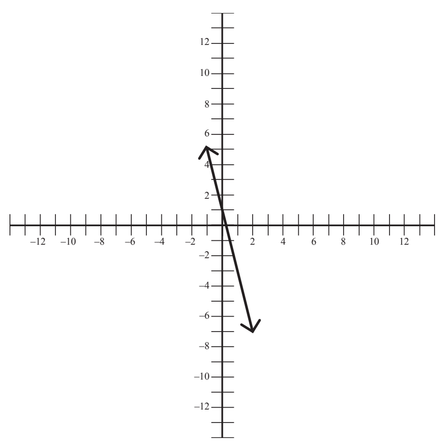
7.



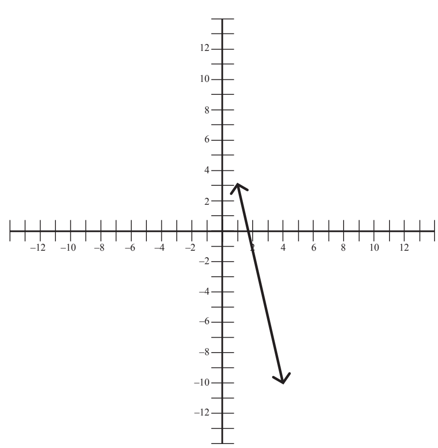
8.



9.



10.



11. 2 **12.** 1 **13.** 7 **14.** −4 **15.** $-\frac{13}{3}$

16. (2, 6) **17.** $\left(-\frac{1}{2}, \frac{3}{2}\right)$ **18.** $\left(-7\frac{1}{2}, 5\frac{1}{2}\right)$ **19.** $\left(\frac{1}{2}, -1\right)$ **20.** $\left(2\frac{1}{2}, -3\frac{1}{2}\right)$

LESSON 18

1. obtuse triangle **2.** rectangle **3.** isosceles triangle **4.** right triangle

5. parallelogram **6.** square **7.** scalene triangle

8. equilateral triangle and equiangular triangle **9.** perimeter = 12; area = 6.

10. perimeter = 30; area = 30. **11.** perimeter = 36; area = 54. **12.** 35

13. 70 **14.** 24 **15.** perimeter = 32; area = 28. **16.** perimeter = 24; area = 35.

17. perimeter = 16; area = 16. **18.** 36 **19.** 18 **20.** 102

LESSON 19

1. hexagon (regular) **2.** pentagon (not regular) **3.** octagon (regular)

4. hexagon (not regular) **5.** decagon (regular) **6.** 720° **7.** 540° **8.** 1080°

9. 720° **10.** 1440° **11.** circumference = 37.68 cm; area = 113.04 cm^2.

12. circumference = 119.32 cm; area = 1133.54 cm^2. **13.** circumference = 20.096 in; area = 32.1536 in^2.

14. circumference = 78.5 ft; area = 490.625 ft^2. **15.** circumference = 0.0628 ft; area = 0.000314 ft^2.

16. 154 ft^2 **17.** $221\frac{19}{25}$ in^2 **18.** 3,465,000 cm^2 **19.** 2772 mm^2 **20.** 27,797 cm^2

LESSON 20

1. prime **2.** prime **3.** not prime **4.** not prime **5.** prime

6. $2^1 \times 5^4$ **7.** $2^2 \times 3 \times 7^2$ **8.** $3 \times 5 \times 7 \times 11$ **9.** $3^2 \times 11 \times 13$ **10.** $11^2 \times 17$

11. $2^4 \times 19 \times 23$ **12.** $2^3 \times 3 \times 5^3 \times 7$ **13.** 18 **14.** 22 **15.** 1

16. 216 **17.** 60 **18.** 900 **19.** 260 **20.** 360

LESSON 21

1. Yes, because the number created from the last 2 digits, 12, is divisible by 4.

2. No, because the number created from the last 2 digits, 30, is not divisible by 4.

3. Yes, because the number created from the last 2 digits, 92, is divisible by 4.

4. No, because the number created from the last 2 digits, 74, is not divisible by 4.

5. Yes, because the number created from the last 3 digits, 912, is divisible by 8.

6. No, because the number created from the last 3 digits, 092, which can just be thought of as 92, is not divisible by 8.

7. Yes, because the last digit of the number, 5, is divisible by 5.

8. No, because the last digit of the number, 2, is not divisible by 5.

9. Yes, because the sum of all the digits of the original number, which is 33, is divisible by 3.

10. No, because the sum of all the digits of the original number, which is 67, is not divisible by 3.

11. Yes, because the sum of all the digits of the original number, which is 18, is divisible by 3.

12. No, because the sum of all the digits of the original number, which is 55, is not divisible by 3.

13. No, because the sum of all the digits of the original number, which is 33, is not divisible by 9.

14. Yes, because the sum of all the digits of the original number, which is 18, is divisible by 9.

15. No, because the sum of all the digits of the original number, which is 46, is not divisible by 9.

16. Yes, because the sum of the digits of the original number, which is 45, is divisible by 9.

17. Yes, because the alternating sum of the digits of the original number, which is 11 (if you start by adding the ones digit, then subtracting the tens digit, and so on), is divisible by 11.

18. Yes, because the alternating sum of the digits of the original number, which is −22 (if you start by adding the ones digit, then subtracting the tens digit, and so on), is divisible by 11.

19. No, because the alternating sum of the digits of the original number, which is 2 (if you start by adding the ones digit, then subtracting the tens digit, and so on), is not divisible by 11.

20. Yes, because the alternating sum of the digits of the original number, which is 0 (if you start by adding the ones digit, then subtracting the tens digit, and so on), is divisible by 11.

LESSON 22

1. 18 **2.** 179.16 **3.** 247.25 **4.** 88.385 (after rounding)

5. 1.962 (after rounding) **6.** 17 **7.** 181.45 **8.** 257 **9.** 88

10. 1.88 **11.** 14 **12.** 187.6 **13.** 257 **14.** 94 **15.** 1.87

16. 17 **17.** 145.6 **18.** 97 **19.** 16 **20.** 0.50

LESSON 23

1. $\frac{1}{5}$ **2.** $\frac{38}{75}$ **3.** $\frac{2}{5}$ **4.** 0 **5.** $\frac{1}{3}$

6. $\frac{1}{26}$ **7.** 1 **8.** $\frac{4}{13}$ **9.** $\frac{9}{13}$ **10.** $\frac{25}{102}$

11. We consider building each element in the sample space 1 digit at a time. There are 10 possibilities for the first digit (any of the numbers from 0 to 9), 10 possibilities for the second digit, 10 possibilities for the third digit, and 10 possibilities for the fourth digit. By the multiplication principle, because none of these choices for the digits impacts the others, we know that the total size of the sample space is 10 × 10 × 10 × 10, or 10,000.

12. The denominator of this probability is the total size of the sample space, which is 10,000. The numerator is the total number of items in the sample space that have no repeated digits. To build such a number, we see that there are 10 choices for the first digit and then only 9 choices for the second digit (because once the first digit is determined, we cannot use that number again). Similarly, there are only 8 choices for the third digit and, finally, 7 choices for the fourth digit. By the multiplication principle, the total number of items in the sample space that have no repeated digits is 10 × 9 × 8 × 7, or 5040. So, our probability fraction is $\frac{5040}{10,000}$, or $\frac{63}{125}$ after we reduce.

13. The denominator of this probability is the total size of the sample space, which is 10,000. The numerator is the total number of items in the sample space that have the same number in the first and last digits. To build such a number, we see that there are 10 choices for the first digit, 10 choices for the second digit, and 10 choices for the third digit. Of course, there is only 1 choice for the fourth digit (because it has to be whatever was chosen for the first digit). By the multiplication principle, the total number of items in the sample space that have the same number in the first and last digits is 10 × 10 × 10 × 1, or 1000. So, our probability fraction is $\frac{1000}{10,000}$, or $\frac{1}{10}$ after we reduce.

14. The denominator of this probability is the total size of the sample space, which is 10,000. The numerator is the total number of items in the sample space that have a 0 or 5 in the last digit. To build such a number, we see that there are 10 choices for the first digit, 10 choices for the second digit, and 10 choices for the third digit. There are 2 choices for the fourth digit (either a 0 or a 5). By the multiplication principle, the total number of items in the sample space that have the same number in the first and last digits is 10 × 10 × 10 × 2, or 2000. So, our probability fraction is $\frac{2000}{10,000}$, or $\frac{1}{5}$ after we reduce.

15. The denominator of this probability is the total size of the sample space, which is 10,000. The numerator is the total number of items in the sample space that satisfy the criteria mentioned in the problem. To build such a number, we see that there are 10 choices for the first digit and 10 choices for the second digit. We now break the problem into some distinct cases. First, the third digit could be a 7. Then, the last digit could be an 8 or a 9. Therefore, in this case, there are 10 × 10 × 1 × 2 possibilities for the numbers, which gives us 200 possibilities. Next, the third digit could be an 8 instead. Then, the last digit must be a 9 in order to satisfy the criteria. So, there are 10 × 10 × 1 × 1 possibilities here, which is 100 more. Lastly, the third digit could be a 9. However, then there are no possibilities for the last digit—because there is no number that we could place into the last digit that would be greater than 9—so this case gives us 0 possibilities. That means that there is a total of 200 + 100 + 0, or 300, numbers in our sample space that fit our criteria. So, our probability fraction is $\frac{300}{10,000}$, or $\frac{3}{100}$ after we reduce.

16. The denominator of our probability fraction is just the number of possible teacher combinations that can occur over your 4 years. You can think of these as a sequence of 4 names (similar to the 4 digits in the earlier problems in this lesson). This number is $5 \times 5 \times 5 \times 5$, or 625 possibilities. The numerator of our probability fraction is the total number of possibilities that satisfy the criterion that all of your teachers' names begin with a J. Because 2 of the teachers' names begin with a J, we see that the total number of possible ways to satisfy the criteria is $2 \times 2 \times 2 \times 2$, or 16. So, the probability fraction is $\frac{16}{625}$.

17. By the multiplication principle, this is simply $8 \times 5 \times 6$, or 240.

18. We expect to see "heads" $\frac{1}{2}$ of the time. So, our answer is $\frac{1}{2} \times 3500$, or 1750 times.

19. The numbers on a numbered cube that are divisors of 6 are 1, 2, 3, and 6. So, we would expect a divisor of 6 to appear with a probability of $\frac{4}{6}$ each time we roll the numbered cube. Thus, if we roll the numbered cube 1800 times, we would expect a divisor of 6 to appear $\frac{4}{6} \times 1800$, or 1200 times.

20. The numbers on a numbered cube that are prime are 2, 3, and 5. So, we would expect a prime to appear with a probability of $\frac{3}{6}$ each time we roll the numbered cube. Thus, if we roll the numbered cube 240 times, we would expect a prime to appear $\frac{3}{6} \times 240$, or 120 times.

LESSON 24

1. 30	**2.** 12	**3.** -1	**4.** 145	**5.** -59
6. 4	**7.** 105	**8.** 605.64	**9.** $\frac{119}{6}$	**10.** 2112.5
11. $5x^2y$	**12.** $35a^3b^4$	**13.** $16t^5v^8$	**14.** $-4t^4v^{10}$	**15.** $xy^2 + 12xy^3$
16. $32x^3y^2 - 22x^2y^3$	**17.** 0	**18.** $\frac{9x^8y^4}{2}$	**19.** $\frac{5x^2y^2}{z^9}$	**20.** $\frac{20x^6y^7}{9z}$

Formulas and Rules

adding expressions with variables: In order to add 2 expressions that involve variables, it must be the case that the expressions are identical except for the numerical coefficient of the expression. (These are often called "like terms.")

addition and subtraction parity:

- If you add or subtract 2 numbers that are even, then the result will be even.
- If you add or subtract 2 numbers that are odd, then the result will be even.
- If you add or subtract 2 numbers that have different parity (one of the numbers is even and the other number is odd), then the result will be odd.

area of a circle: $\pi \times r^2$, where r is the radius of the circle.

circumference of a circle: $2 \times \pi \times r$, where r is the radius of the circle.

converting from a mixed number to an improper fraction: Multiply the whole number by the denominator and add the numerator.

converting from an improper fraction to a mixed number: The numerator is the dividend, and the denominator is the divisor. Divide, and make the quotient into the whole number part of the mixed number. Make the remainder the numerator of the fraction, and keep the divisor as the denominator.

converting from decimals to fractions: Write any whole numbers by themselves. Next, write what is to the right of the decimal point (leaving off the decimal point) as the numerator (the top of the fraction). Determine which place the last digit of the decimal is in (tenths, hundredths, thousandths, etc.). The denominator (the bottom of the fraction) is the number that corresponds to that place. If the last digit was in the hundredths place, the denominator would be 100. If the last digit was in the tenths place, the denominator would be 10.

converting from fractions to decimals: Perform long division by dividing the denominator (the number on the bottom) into the numerator (the number on the top). Keep using more 0s after the decimal point (as needed) until the answer comes out even or the numbers begin repeating.

converting from percents to decimals: Take the percent amount, move the decimal point 2 places to the left, and remove the percent sign.

converting from percents to fractions:

- Use the percent amount as the numerator (the top number) of the fraction.
- Always make the denominator (the bottom number) 100.
- Simplify the fraction if needed.

dividing expressions with variables: For any nonzero number x and any 2 integers a and b, $x^a/x^b = x^{(a-b)}$. In other words, you subtract the exponents when you divide 2 like variables.

divisibility rule for 2: A number is divisible by exactly 2 when its last digit is divisible by 2; in other words, a number is divisible by exactly 2 when the last digit of the number is a 0, 2, 4, 6, or 8.

divisibility rule for 3: A number is divisible by exactly 3 when the sum of all the digits of the number is divisible by 3.

divisibility rule for 4: A number is divisible by exactly 4 when the number created from its last 2 digits is divisible by 4.

divisibility rule for 5: A number is divisible by exactly 5 when its last digit is divisible by 5; in other words, a number is divisible by exactly 5 when the last digit of the number is a 0 or a 5.

divisibility rule for 8: A number is divisible by exactly 8 when the number created from its last 3 digits is divisible by 8.

divisibility rule for 9: A number is divisible by exactly 9 when the sum of all the digits of the number is divisible by 9.

divisibility rule for 11: A number is divisible by exactly 11 when the alternating sum of all the digits of the number is divisible by 11.

exponentiation facts:

> $a^1 = a$ for any number a, and
> as long as $a \neq 0$, $a^0 = 1$.

finding the greatest common factor: Factor numbers to their prime factorization. Find the smaller occurrence of each prime number in the factors. Multiply these together to obtain the greatest common factor.

finding the least common denominator: Factor numbers to their prime factorization. Find the greatest occurrence of each prime number in the factors. Multiply these together to obtain the least common denominator.

how to cross multiply: Write the ratios as fractions. Multiply the numerator of the first fraction with the denominator of the second fraction. Then, multiply the denominator of the first fraction with the numerator of the second fraction. If you drew lines to show which numbers were being multiplied together, it would make an X in the center, or a cross—hence the term "cross multiply."

midpoint of a line segment: Using the coordinates of the 2 endpoints, the midpoint is found by taking $a^{-b} = \frac{x_1 + x_2}{2}$ for the x-coordinate and $\frac{y_1 + y_2}{2}$ for the y-coordinate.

multiplication parity:

- If you multiply 2 even numbers together, then the answer you get is an even number.
- If you multiply 2 odd numbers together, then the answer you get is an odd number.
- If you multiply an even number and an odd number, then the answer you get is an even number.

multiplying expressions with variables: For any nonzero number x and any 2 integers a and b, $x^a x^b = x^{(a + b)}$. In other words, you add the exponents when you multiply 2 like variables.

operations with signed numbers:

- If you are subtracting a negative number, the number turns into a positive. For example, +3 − (−5) becomes +3 + 5.
- When you are adding a negative number and a positive number, subtract the 2 numbers and take the sign of the number that has a larger absolute value.
- When you are adding 2 numbers that are both negative, add their absolute values together and put a negative sign in front.
- When you multiply or divide 2 numbers that are both positive, the answer is positive.
- When you multiply or divide 2 numbers that are both negative, the answer is positive.
- When you multiply or divide 2 numbers that have different signs, the answer is negative.

operations with square roots:

- You can multiply the numbers under the square root symbols if the 2 are being multiplied. For example, $\sqrt{20} \times \sqrt{5} = \sqrt{100}$.
- If there is addition or subtraction under the radical symbol, you cannot break the numbers into separate square roots. For example, $\sqrt{16-4}$ is not the same as $\sqrt{16} - \sqrt{4}$.
- You can only add or subtract square roots if the number under the radical symbol is the same.

parallelograms:

- The sum of the degrees of the angles inside a parallelogram, rectangle, or square is always 360°.
- The area of any parallelogram, rectangle, or square is equal to $b \times h$, where b is the base and h is the height.
- The perimeter of any such object is the sum of the lengths of the 4 legs.

rule of fractional exponents: $a^{b/c} = (a^b)^{1/c}$ and $a^{b/c} = (a^{1/c})^b$.

rule of negative exponents: $a^{-b} = \frac{1}{a^b}$.

rules for order of operations:

1. Any operations inside parentheses.
2. Exponents.
3. Multiplication and division from left to right.
4. Addition and subtraction from left to right.

slope of a straight line: The slope of a straight line that goes through the 2 points (x_1, y_1) and (x_2, y_2) is given by the quantity $\frac{y_2 - y_1}{x_2 - x_1}$.

sum of degrees for a regular polygon: It is exactly 180 times the number that is 2 less than the number of legs. It can be written as $180 \times (L - 2)$, where L is the number of legs (sides) in a polygon.

theoretical probability: If a sample space has n equally likely outcomes and an event A occurs in m of these outcomes, then the theoretical probability of event A is denoted by $p(A)$ and is given by $p(A) = \frac{m}{n}$.

triangles:

- The sum of the degrees of the angles inside a triangle is always 180°.
- The area of a triangle is equal to $\frac{1}{2} \times b \times h$, where b is the base and h is the height of the triangle.
- The perimeter of a triangle is the sum of the lengths of the 3 legs.

Glossary

absolute value: The distance a number is from 0 on the number line.

algebraic expression: A combination of mathematical symbols that might include numbers, variables, and operation symbols.

bisection method: A way of approximating square roots by finding the perfect square larger than the number and the perfect square smaller than the number and finding the term in the middle of that number to come close to finding the actual square root.

borrow: When subtracting numbers, if the number in the top position of a column is smaller than the number below (the one being subtracted), you subtract 1 from the number to the left of the top number, and add 10 to the number from which you are currently subtracting. Because the number to the left has a place value that is 10 times the number to the right, you are "borrowing" 10 from that number.

canceling: Taking a common factor out of the numerator of 1 fraction and the denominator of the same fraction or another fraction. This can only be done in multiplication and division—never in addition and subtraction.

carry: When adding numbers, if the answer in a column is more than 1 digit in size, then the digit in the tens place of the answer is "carried" to the column to the left to be added.

Cartesian plane: Also called the Cartesian coordinate system, it is another name for the *xy*-plane.

center of the circle: A point that is not actually part of the circle, but it is in the middle of the circle, and each point on the circle is the same distance away from it.

circle: A set of points in the plane that are equidistant (the same distance away) from a fixed point, which we call the center of the circle.

circumference: The distance around a circle.

compound fraction: A fraction in which the numerator and/or denominator is also a fraction.

coordinate: The number value assigned to a point on the *x*- or *y*-axis. Each point on the coordinate plane is identified by 2 coordinates.

coordinate plane: The 2-dimensional plane, or the *xy*-plane. A 2-dimensional object formed by 2 number lines—1 horizontal and 1 vertical (so that they intersect at right angles).

cross multiplying: A way to determine if 2 ratios are equivalent.

decagon: A polygon with 10 sides.

decimal number: A number written in our usual base-10 number system. Often, such a number will contain a decimal point.

decimal point: The dot or period that comes after the ones place and before the tenths place in a decimal number.

denominator: The bottom number in a fraction. It tells us how many parts are in the whole.

diameter: The length of any line segment that connects 2 points on a circle and also goes through the center of the circle. The diameter is always twice the radius distance.

digit: An individual number from 0 to 9 that can be used to make up a larger number. For example, 786 has 3 digits, and 76,566 has 5 digits.

dividend: The number that is being divided into groups. It is the number on the inside of the division bar or the top number if a division problem is written as a fraction.

divisor: A number that divides another number. In number theory, it is a number that divides another number evenly. It is the number on the outside of the division bar—or the bottom number if a division problem is written as a fraction.

equiangular: When all the angles in a polygon have the same measurement.

equiangular triangle: A triangle that has 3 equal angles. Each of these angles measures 60°.

equilateral: When all the sides of a polygon have the same length.

equilateral triangle: A triangle whose 3 sides are all equal in length.

equivalent fractions: Any 2 fractions that have different numerators and denominators but represent the same amount. If $\frac{a}{b}$ and $\frac{c}{d}$ are 2 equivalent fractions, then $a \times d = b \times c$.

even number: A number that is divisible by 2.

exponent: The small number written high and to the right of a number that tells how many times the number should be multiplied by itself.

exponentiation: Raising a number to a power. In other words, writing the number times itself as many times as the exponent shows. For example, 8^4 is $8 \times 8 \times 8 \times 8$. You could also think of it as repeated multiplication.

factor: A number that divides another number evenly, leaving no remainder. It is the same as a divisor.

fraction: A part of a whole number. It is usually written as a ratio of 2 whole numbers with 1 number written, a division bar drawn underneath it, and the second number written below.

greatest common divisor (GCD): The largest divisor that divides both of the numbers with no remainder. It is also known as the greatest common factor (GCF).

heptagon: A polygon with 7 sides.

hexagon: A polygon with 6 sides.

hundredths place: The digit second to the right from the decimal point is in the hundredths place.

improper fraction: A fraction whose numerator is larger than the denominator. For example, $\frac{54}{23}$ and $\frac{13}{15}$ are both improper fractions.

integers: The set of signed numbers. All the positive whole numbers (1, 2, 3, …), 0, and all the negative whole numbers (−1, −2, −3, …).

isosceles triangle: A triangle that has 2 sides, or legs, of equal length.

least common denominator (LCD): The smallest number that 2 unlike denominators will divide, or the smallest number for which the 2 given denominators are factors. For example, if the denominators were 3 and 4, the least common denominator would be 12.

least common multiple (LCM): The smallest number for which the 2 given numbers are factors.

like denominators: Having the same denominator in 2 or more fractions.

lowest terms: If a fraction has been reduced, it is said to be in lowest terms.

mean: The average of all the numbers in a set of data. Also called the arithmetic mean.

median: The number that is in the exact middle of a set of numbers when they are written from smallest to largest.

midpoint: The middle point of a line segment.

mixed number: A whole number that is combined with a fraction. For example, $2\frac{1}{2}$ is a mixed number.

mode: The number that occurs most often in a set of data.

multiples of a number: The numbers you get when you multiply a given number by 1, then 2, then 3, etc. For example, the multiples of 4 are 4, 8, 12, 16, 20, 24 … .

multiplication principle: If you are trying to count the number of ways to do multiple independent events in succession, then the total will just be the product of the number of ways to do each event separately. In other words, if the events are disconnected, or independent of one another, then you just multiply together the number of ways to do each event, and you have your overall answer.

negative number: A number that is less than 0. It has a negative sign (also called a minus sign) in front of it.

nonagon: A polygon with 9 sides.

number line: A line in which equally spaced dots are labeled with numbers. It usually is written with an arrow on either end to show that the numbers go on infinitely.

number theory: The study of the properties of whole numbers.

numerator: The top number in a fraction. It tells us how many parts we have of the whole.

numerical coefficient: The number in front of a variable. For example, in $8y^3$, the 8 is the numerical coefficient.

obtuse triangle: A triangle that has 1 angle that is greater than 90°.

octagon: A polygon with 8 sides.

odd number: A number that has a remainder of 1 when divided by 2.

operation: A mathematical process. Some operations include addition, subtraction, multiplication, and division.

order of operations: A set of rules that explain in what order operations should be completed in an expression with more than 1 operation.

ordered pair: The 2 coordinates used to identify a point on the coordinate plane. Ordered pairs are often written in the form (x, y), with the x-axis coordinate first and the y-axis coordinate second.

origin: The intersection point of the x-axis and y-axis.

outcome: A possible result of a probability experiment.

parallelogram: A 4-sided polygon that consists of 2 pairs of parallel line segments.

parity: Whether a number is even or odd.

pentagon: A polygon with 5 sides.

percent: A number with a percent symbol (%) that tells us what part of a hundred is represented. For example, 75% means "75 per hundred."

perfect square: A number whose square root is some whole number times itself. For example, 25 is 5×5, so it is a perfect square, and 7 is 7×1, so it is not a perfect square.

pi: The ratio of the perimeter of any circle to the diameter. It is approximately $\frac{22}{7}$ and is often approximated as 3.14. The exact value of pi is a nonrepeating, infinite decimal number. It is given by the symbol π.

place value: The value assigned to a digit based on where it is placed in the number.

polygon: A closed figure in the plane whose boundary is a set of at least 3 straight lines that intersect at each vertex, or sharp point, of the figure.

prime factorization: Breaking a number down into its prime number factors.

prime number: A number greater than 1 whose divisors are only 1 and the number itself.

product: The answer to a multiplication problem.

proportion: A statement that 2 ratios are equal. A proportion is a mathematical sentence that contains an equal sign and 2 equal ratios.

quadrant: One of the 4 sections of the coordinate plane. Quadrants are labeled counterclockwise from the top as I, II, III, and IV.

quadrilateral: A polygon with 4 sides, or legs.

quotient: The quantity we obtain by dividing the dividend by the divisor. It's the answer to a division problem.

radical: Another name for a square root.

radical symbol: Another name for the square root symbol.

radius: The distance from the center of a circle to any point on the circle.

range: The difference found by subtracting the largest number in the list from the smallest number in the list of data.

ratio: A comparison of 2 amounts. It is typically found by dividing 1 number into another.

reciprocal: A fraction in which the numerator and denominator switch places.

rectangle: A parallelogram whose angles are all right angles.

reducing: Making the numerator and denominator the lowest numbers they can be with no common factors.

regular polygon: A polygon that is equiangular and equilateral.

remainder: What "remains" after a division problem is finished. It is the part that remains after the last subtraction occurs. It is always less than the divisor.

right angle: An angle that measures 90°.

right triangle: A triangle that has 1 angle that measures 90°.

sample space: The set of all possible outcomes for a particular activity or experiment.

scalene triangle: A triangle whose angles are all less than 90°.

signed number: A number with a sign (+ or −) in front of it. Some people use this term to refer to the negative numbers (the numbers with a − sign).

slope: The "rise over run," or the change in *y* over the change in *x*. It is the quantity of the vertical change in *y*-values divided by the horizontal change in *x*-values.

square: A rectangle whose 4 legs are all the same length.

square of a number: The product of a number multiplied by itself.

square root: A number that you can multiply by itself to get a certain value. For any real numbers a and b, if $a^2 = b$, then a is a square root of b.

tenths place: The digit to the immediate right of the decimal point is in the tenths place.

times: A way of saying "multiplied by." For example, 6 times 7 can be written as 6×7.

triangle: A polygon that has exactly 3 sides. The sum of all 3 of its angles is 180°.

units digit: The digit to the far right in a number. It is sometimes referred to as the ones digit.

variable: A symbol, usually a letter from the alphabet, that represents 1 or more numbers in an expression.

vertex: The corner-like place where 2 line segments meet in a polygon.

whole number: Any of the numbers 0, 1, 2, 3, 4, and so on that are not fractions or decimals.

***x*-axis**: The horizontal number line on the coordinate plane.

***y*-axis**: The vertical number line on the coordinate plane.

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