Axioms for the real numbers: Field axioms

(A1) For all $a, b, c \in \mathbb{R}$, $(a+b)+c = a+(b+c)$. (+ associative)

(A2) For all $a, b \in \mathbb{R}$, $a+b = b+a$. (+ commutative)

(A3) There exists $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}$, $a+0 = a$. (Zero)

(A4) For all $a \in \mathbb{R}$, there exists $(-a) \in \mathbb{R}$ such that $a+(-a) = 0$. (Negatives)

(M1) For all $a, b, c \in \mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$. ($\cdot$ associative)

(M2) For all $a, b \in \mathbb{R}$, $a \cdot b = b \cdot a$. ($\cdot$ commutative)

(M3) There exists $1 \in \mathbb{R}$, $1 \neq 0$, s.t. for all $a \in \mathbb{R}$, $a \cdot 1 = a$. (Unit)

(M4) For all $a \neq 0$ in $\mathbb{R}$, there exists $(1/a) \in \mathbb{R}$ such that $a \cdot (1/a) = 1$. (Reciprocals)

(DL) For all $a, b, c \in \mathbb{R}$, $a \cdot (b+c) = a \cdot b + a \cdot c$. (Distributive)

In a nutshell: Arithmetic works.
Axioms still hold if we replace $\mathbb{R}$ with $\mathbb{Q}$, $\mathbb{C}$, integers mod 2 (the set $\{0, 1\}$, taking $1+1=0$).

Axioms for the real numbers: Order axioms

An *ordered field* satisfies axioms (A1)–(A4), (M1)–(M4), and (DL), and also has a relation $\leq$ such that:

(O1) For all $a, b \in \mathbb{R}$, either $a \leq b$ or $b \leq a$.

(O2) For all $a, b \in \mathbb{R}$, if $a \leq b$ and $b \leq a$, then $a = b$.

(O3) For all $a, b, c \in \mathbb{R}$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

(O4) For all $a, b, c \in \mathbb{R}$, if $a \leq b$, then $a + c \leq b + c$.

(O5) For all $a, b, c \in \mathbb{R}$, if $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

Cor: If $c < 0$ and $a \leq b$, then $bc \leq ac$. (Flip!)
Also define:

- $a < b$ means $a \leq b$ and $a \neq b$;
- $a \geq b$ means $b \leq a$;
- $a > b$ means $b < a$.

In a nutshell: Properties of $\leq$, $<$, $\geq$, $>$ are as you (maybe?) learned them in precalculus.
Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields; integers mod 2 and $\mathbb{C}$ are not.

Axioms for the real numbers: Completeness

(C) Every nonempty set of real numbers that has an upper bound also has a **least** upper bound (supremum).

It can be shown that the axioms (A1)–(A4), (M1)–(M4), (DL), (O1)–(O5), and (C) determine $\mathbb{R}$ completely; that is, any other object with the same properties must be essentially the same as $\mathbb{R}$. For the rest of this course, we assume that there exists an object $\mathbb{R}$ that satisfies all of these axioms. All of our results ultimately rely only on these axioms.