The Real And Complex Number Systems

Integers

1.1 Prove that there is no largest prime.

Proof: Suppose p is the largest prime. Then $p! + 1$ is **NOT** a prime. So, there exists a prime q such that

$$q \,| p! + 1 \Rightarrow q \,| 1$$

which is impossible. So, there is no largest prime.

Remark: There are many and many proofs about it. The proof that we give comes from A**rchimedes 287-212 B. C.** In addition, **Euler Leonhard (1707-1783)** find another method to show it. The method is important since it develops to study the theory of numbers by analytic method. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

1.2 If n is a positive integer, prove the algebraic identity

$$a^n - b^n = (a - b) \sum_{k=0}^{n-1} a^k b^{n-1-k}$$

Proof: It suffices to show that

$$x^n - 1 = (x - 1) \sum_{k=0}^{n-1} x^k.$$

Consider the right hand side, we have

$$\begin{aligned}(x-1)\sum_{k=0}^{n-1}x^k &= \sum_{k=0}^{n-1}x^{k+1}-\sum_{k=0}^{n-1}x^k\\ &= \sum_{k=1}^{n}x^k-\sum_{k=0}^{n-1}x^k\\ &= x^n-1.\end{aligned}$$

1.3 If 2^n-1 is a prime, prove that n is prime. A prime of the form 2^p-1, where p is prime, is called a Mersenne prime.

Proof: If n is not a prime, then say $n=ab$, where $a>1$ and $b>1$. So, we have

$$2^{ab}-1=(2^a-1)\sum_{k=0}^{b-1}(2^a)^k$$

which is not a prime by **Exercise 1.2**. So, n must be a prime.

Remark: The study of **Mersenne prime** is important; it is related with so called **Perfect number**. In addition, there are some **OPEN** problem about it. For example, **is there infinitely many Mersenne nembers?** The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 13-15. (Chinese Version)**

1.4 If 2^n+1 is a prime, prove that n is a power of 2. A prime of the form $2^{2^m}+1$ is called a **Fermat prime.** Hint. Use exercise 1.2.

Proof: If n is a not a power of 2, say $n=ab$, where b is an odd integer. So,

$$2^a+1\,\big|\,2^{ab}+1$$

and $2^a+1<2^{ab}+1$. It implies that 2^n+1 is not a prime. So, n must be a power of 2.

Remark: (1) In the proof, we use the identity

$$x^{2n-1}+1=(x+1)\sum_{k=0}^{2n-2}(-1)^k x^k.$$

Proof: Consider

$$\begin{aligned}(x+1)\sum_{k=0}^{2n-2}(-1)^k x^k &= \sum_{k=0}^{2n-2}(-1)^k x^{k+1} + \sum_{k=0}^{2n-2}(-1)^k x^k \\ &= \sum_{k=1}^{2n-1}(-1)^{k+1} x^k + \sum_{k=0}^{2n-2}(-1)^k x^k \\ &= x^{2n+1}+1.\end{aligned}$$

(2) The study of **Fermat number** is important; for the details the reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 15. (Chinese Version)**

1.5 The Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, \ldots$ are defined by the recursion formula $x_{n+1} = x_n + x_{n-1}$, with $x_1 = x_2 = 1$. Prove that $(x_n, x_{n+1}) = 1$ and that $x_n = (a^n - b^n)/(a-b)$, where a and b are the roots of the quadratic equation $x^2 - x - 1 = 0$.

Proof: Let $d = g.c.d.\,(x_n, x_{n+1})$, then

$$d\,|x_n \text{ and } d\,|x_{n+1} = x_n + x_{n-1}\,.$$

So,

$$d\,|x_{n-1}\,.$$

Continue the process, we finally have

$$d\,|1\,.$$

So, $d = 1$ since d is positive.

Observe that

$$x_{n+1} = x_n + x_{n-1},$$

and thus we consider

$$x^{n+1} = x^n + x^{n-1},$$

i.e., consider

$$x^2 = x + 1 \text{ with two roots, } a \text{ and } b.$$

If we let

$$F_n = (a^n - b^n)/(a-b)\,,$$

then it is clear that

$$F_1 = 1,\ F_2 = 1,\ \text{and } F_{n+1} = F_n + F_{n-1} \text{ for } n > 1.$$

So, $F_n = x_n$ for all n.

Remark: The study of the **Fibonacci numbers** is important; the reader can see the book, **Fibonacci and Lucas Numbers with Applications by Koshy and Thomas.**

1.6 Prove that every nonempty set of positive integers contains a smallest member. This is called the **well–ordering Principle.**

Proof: Given $(\phi \neq) S (\subseteq N)$, we prove that if S contains an integer k, then S contains the smallest member. We prove it by **Mathematical Induction of second form** as follows.

As $k = 1$, it trivially holds. Assume that as $k = 1, 2, ..., m$ holds, consider as $k = m + 1$ as follows. In order to show it, we consider two cases.

(1) If there is a member $s \in S$ such that $s < m + 1$, then by Induction hypothesis, we have proved it.

(2) If every $s \in S$, $s \geq m + 1$, then $m + 1$ is the smallest member.

Hence, by **Mathematical Induction**, we complete it.

Remark: We give a fundamental result to help the reader get more. We will prove the followings are equivalent:

(A. Well–ordering Principle) every nonempty set of positive integers contains a smallest member.

(B. Mathematical Induction of first form) Suppose that $S (\subseteq N)$, if S satisfies that

$$(1).\ 1 \text{ in } S$$
$$(2).\ \text{As } k \in S,\ \text{then } k + 1 \in S.$$

Then $S = N$.

(C. Mathematical Induction of second form) Suppose that $S (\subseteq N)$, if S satisfies that

$$(1).\ 1 \text{ in } S$$
$$(2).\ \text{As } 1, ..., k \in S,\ \text{then } k + 1 \in S.$$

Then $S = N$.

Proof: $(A \Rightarrow B)$: If $S \neq N$, then $N - S \neq \phi$. So, by (**A**), there exists the smallest integer w such that $w \in N - S$. Note that $w > 1$ by (1), so we consider $w - 1$ as follows.

Since $w - 1 \notin N - S$, we know that $w - 1 \in S$. By (2), we know that $w \in S$ which contadicts to $w \in N - S$. Hence, $S = N$.

$(B \Rightarrow C)$: It is obvious.

$(C \Rightarrow A)$: We have proved it by this exercise.

Rational and irrational numbers

1.7 Find the rational number whose decimal expansion is 0.3344444444....

Proof: Let $x = 0.3344444444...$, then

$$\begin{aligned} x &= \frac{3}{10} + \frac{3}{10^2} + \frac{4}{10^3} + ... + \frac{4}{10^n} + .., \text{ where } n \geq 3 \\ &= \frac{33}{10^2} + \frac{4}{10^3}\left(1 + \frac{1}{10} + ... + \frac{1}{10^n} + ..\right) \\ &= \frac{33}{10^2} + \frac{4}{10^3}\left(\frac{1}{1 - \frac{1}{10}}\right) \\ &= \frac{33}{10^2} + \frac{4}{900} \\ &= \frac{301}{900}. \end{aligned}$$

1.8 Prove that the decimal expansion of x will end in zeros (or in nines) if, and only if, x is a rational number whose denominator is of the form $2^n 5^m$, where m and n are nonnegative integers.

Proof: $(\Leftarrow)$Suppose that $x = \frac{k}{2^n 5^m}$, if $n \geq m$, we have

$$\frac{k5^{n-m}}{2^n 5^n} = \frac{5^{n-m}k}{10^n}.$$

So, the decimal expansion of x will end in zeros. Similarly for $m \geq n$.

$(\Rightarrow)$Suppose that the decimal expansion of x will end in zeros (or in nines).

For case $x = a_0.a_1a_2\cdots a_n$. Then

$$x = \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{10^n} = \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n}.$$

For case $x = a_0.a_1a_2\cdots a_n999999\cdots$. Then

$$\begin{aligned} x &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}} + ... + \frac{9}{10^{n+m}} + ... \\ &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{9}{10^{n+1}}\sum_{j=0}^{\infty} 10^{-j} \\ &= \frac{\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n} + \frac{1}{10^n} \\ &= \frac{1+\sum_{k=0}^{n} 10^{n-k}a_k}{2^n5^n}. \end{aligned}$$

So, in both case, we prove that x is a rational number whose denominator is of the form 2^n5^m, where m and n are nonnegative integers.

1.9 Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Proof: If $\sqrt{2}+\sqrt{3}$ is rational, then consider

$$\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{3}-\sqrt{2}\right) = 1$$

which implies that $\sqrt{3}-\sqrt{2}$ is rational. Hence, $\sqrt{3}$ would be rational. It is impossible. So, $\sqrt{2}+\sqrt{3}$ is irrational.

Remark: (1)$\sqrt{p}$ is an irrational if p is a prime.

Proof: If $\sqrt{p} \in Q$, write $\sqrt{p} = \frac{a}{b}$, where $g.c.d.\,(a,b) = 1$. Then

$$b^2p = a^2 \Rightarrow p\left|a^2\right. \Rightarrow p\left|a\right. \qquad (*)$$

Write $a = pq$. So,

$$b^2p = p^2q^2 \Rightarrow b^2 = pq^2 \Rightarrow p\left|b^2\right. \Rightarrow p\left|b\right. . \qquad (*')$$

By (*) and (*'), we get

$$p\left|g.c.d.\,(a,b)\right. = 1$$

which implies that $p = 1$, a contradiction. So, $\sqrt{p}$ is an irrational if p is a prime.

Note: There are many and many methods to prove it. For example, the reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 19-21. (Chinese Version)**

(2) Suppose $a, b \in N$. Prove that $\sqrt{a}+\sqrt{b}$ is rational if and only if, $a = k^2$ and $b = h^2$ for some $h, k \in N$.

Proof: ($\Leftarrow$) It is clear.
($\Rightarrow$) Consider

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right) = a^2 - b^2,$$

then $\sqrt{a} \in Q$ and $\sqrt{b} \in Q$. Then it is clear that $a = h^2$ and $b = h^2$ for some $h, k \in N$.

1.10 If a, b, c, d are rational and if x is irrational, prove that $(ax+b)/(cx+d)$ is usually irrational. When do exceptions occur?

Proof: We claim that $(ax+b)/(cx+d)$ is rational if and only if $ad = bc$.
($\Rightarrow$)If $(ax+b)/(cx+d)$ is rational, say $(ax+b)/(cx+d) = q/p$. We consider two cases as follows.
(i) If $q = 0$, then $ax+b = 0$. If $a \neq 0$, then x would be rational. So, $a = 0$ and $b = 0$. Hence, we have

$$ad = 0 = bc.$$

(ii) If $q \neq 0$, then $(pa - qc)x + (pb - qd) = 0$. If $pa - qc \neq 0$, then x would be rational. So, $pa - qc = 0$ and $pb - qd = 0$. It implies that

$$qcb = qad \Rightarrow ad = bc.$$

($\Leftarrow$)Suppose $ad = bc$. If $a = 0$, then $b = 0$ or $c = 0$. So,

$$\frac{ax+b}{cx+d} = \begin{cases} 0 \text{ if } a = 0 \text{ and } b = 0 \\ \frac{b}{d} \text{ if } a = 0 \text{ and } c = 0 \end{cases}.$$

If $a \neq 0$, then $d = bc/a$. So,

$$\frac{ax+b}{cx+d} = \frac{ax+b}{cx+bc/a} = \frac{a(ax+b)}{c(ax+b)} = \frac{a}{c}.$$

Hence, we proved that if $ad = bc$, then $(ax+b)/(cx+d)$ is rational.

1.11 Given any real $x > 0$, prove that there is an irrational number between 0 and x.

Proof: If $x \in Q^c$, we choose $y = x/2 \in Q^c$. Then $0 < y < x$. If $x \in Q$, we choose $y = x/\sqrt{2} \in Q$, then $0 < y < x$.

Remark: (1) There are many and many proofs about it. We may prove it by the concept of **Perfect set**. The reader can see the book, **Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, pp 41.** Also see the textbook, **Exercise 3.25**.

(2) Given a and $b \in R$ with $a < b$, there exists $r \in Q^c$, and $q \in Q$ such that $a < r < b$ and $a < q < b$.

Proof: We show it by considering four cases. (i) $a \in Q$, $b \in Q$. (ii) $a \in Q$, $b \in Q^c$. (iii) $a \in Q^c$, $b \in Q$. (iv) $a \in Q^c$, $b \in Q^c$.

(i) ($a \in Q$, $b \in Q$) Choose $q = \frac{a+b}{2}$ and $r = \frac{1}{\sqrt{2}}a + \left(1 - \frac{1}{\sqrt{2}}\right) b$.

(ii) ($a \in Q$, $b \in Q^c$) Choose $r = \frac{a+b}{2}$ and let $c = \frac{1}{2^n} < b-a$, then $a+c := q$.

(iii) ($a \in Q^c$, $b \in Q$) Similarly for (iii).

(iv) ($a \in Q^c$, $b \in Q^c$) It suffices to show that there exists a rational number $q \in (a, b)$ by (ii). Write

$$b = b_0.b_1b_2 \cdots b_n \cdots$$

Choose n large enough so that

$$a < q = b_0.b_1b_2 \cdots b_n < b.$$

(It works since $b - q = 0.000..000b_{n+1}... \leq \frac{1}{10^n}$)

1.12 If $a/b < c/d$ with $b > 0$, $d > 0$, prove that $(a + c)/(b + d)$ lies bwtween the two fractions a/b and c/d

Proof: It only needs to conisder the substraction. So, we omit it.

Remark: The result of this exercise is often used, so we suggest the reader keep it in mind.

1.13 Let a and b be positive integers. Prove that $\sqrt{2}$ always lies between the two fractions a/b and $(a + 2b)/(a + b)$. Which fraction is closer to $\sqrt{2}$?

Proof: Suppose $a/b \leq \sqrt{2}$, then $a \leq \sqrt{2}b$. So,

$$\frac{a + 2b}{a + b} - \sqrt{2} = \frac{\left(\sqrt{2} - 1\right)\left(\sqrt{2}b - a\right)}{a + b} \geq 0.$$

In addition,

$$\begin{aligned}\left(\sqrt{2}-\frac{a}{b}\right)-\left(\frac{a+2b}{a+b}-\sqrt{2}\right) &= 2\sqrt{2}-\left(\frac{a}{b}+\frac{a+2b}{a+b}\right)\\ &= 2\sqrt{2}-\frac{a^2+2ab+2b^2}{ab+b^2}\\ &= \frac{1}{ab+b^2}\left[\left(2\sqrt{2}-2\right)ab+\left(2\sqrt{2}-2\right)b^2-a^2\right]\\ &\geq \frac{1}{ab+b^2}\left[\left(2\sqrt{2}-2\right)a\frac{a}{\sqrt{2}}+\left(2\sqrt{2}-2\right)\left(\frac{a}{\sqrt{2}}\right)^2-a^2\right]\\ &= 0.\end{aligned}$$

So, $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$.

Similarly, we also have if $a/b > \sqrt{2}$, then $\frac{a+2b}{a+b} < \sqrt{2}$. Also, $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$ in this case.

Remark: Note that

$$\frac{a}{b} < \sqrt{2} < \frac{a+2b}{a+b} < \frac{2b}{a} \text{ by Exercise 12 and 13.}$$

And we know that $\frac{a+2b}{a+b}$ is closer to $\sqrt{2}$. We can use it to approximate $\sqrt{2}$. Similarly for the case

$$\frac{2b}{a} < \frac{a+2b}{a+b} < \sqrt{2} < \frac{a}{b}.$$

1.14 Prove that $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

Proof: Suppose that $\sqrt{n-1}+\sqrt{n+1}$ is rational, and thus consider

$$\left(\sqrt{n+1}+\sqrt{n-1}\right)\left(\sqrt{n+1}-\sqrt{n-1}\right)=2$$

which implies that $\sqrt{n+1}-\sqrt{n-1}$ is rational. Hence, $\sqrt{n+1}$ and $\sqrt{n-1}$ are rational. So, $n-1=k^2$ and $n+1=h^2$, where k and h are positive integer. It implies that

$$h=\frac{3}{2} \text{ and } k=\frac{1}{2}$$

which is absurb. So, $\sqrt{n-1}+\sqrt{n+1}$ is irrational for every integer $n \geq 1$.

1.15 Given a real x and an integer $N > 1$, prove that there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$. Hint. Consider the $N+1$ numbers $tx - [tx]$ for $t = 0, 1, 2, ..., N$ and show that some pair differs by at most $1/N$.

Proof: Given $N > 1$, and thus consider $tx - [tx]$ for $t = 0, 1, 2, ..., N$ as follows. Since

$$0 \leq tx - [tx] := a_t < 1,$$

so there exists two numbers a_i and a_j where $i \neq j$ such that

$$|a_i - a_j| < \frac{1}{N} \Rightarrow |(i-j)\,x - p| < \frac{1}{N}, \text{ where } p = [jx] - [ix]\,.$$

Hence, there exist integers h and k with $0 < k \leq N$ such that $|kx - h| < 1/N$.

1.16 If x is irrational prove that there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$. Hint. Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and obtain a contradiction by applying Exercise 1.15 with $N > 1/\delta$, where δ is the smallest of the numbers $|x - h_i/k_i|$.

Proof: Assume there are only a finite number $h_1/k_1, ..., h_r/k_r$ and let $\delta = \min_{i=1}^{r} |x - h_i/k_i| > 0$ since x is irrational. Choose $N > 1/\delta$, then by **Exercise 1.15**, we have

$$\frac{1}{N} < \delta \leq \left| x - \frac{h}{k} \right| < \frac{1}{kN}$$

which implies that

$$\frac{1}{N} < \frac{1}{kN}$$

which is impossible. So, there are infinitely many rational numbers h/k with $k > 0$ such that $|x - h/k| < 1/k^2$.

Remark: (1) There is another proof by **continued fractions.** The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 270. (Chinese Version)**

(2) The exercise is useful to help us show the following lemma. $\{ar + b : a \in Z, b \in Z\}$, where $r \in Q^c$ is dense in R. It is equivalent to $\{ar : a \in Z\}$, where $r \in Q^c$ is dense in $[0, 1]$ modulus 1.

Proof: Say $\{ar+b : a \in Z, b \in Z\} = S$, and since $r \in Q^c$, then by **Exercise 1.16**, there are infinitely many rational numbers h/k with $k > 0$ such that $|kr - h| < \frac{1}{k}$. Consider $(x-\delta, x+\delta) := I$, where $\delta > 0$, and thus choosing k_0 large enough so that $1/k_0 < \delta$. Define $L = |k_0 r - h_0|$, then we have $sL \in I$ for some $s \in Z$. So, $sL = (\pm)[(sk_0)r - (sh_0)] \in S$. That is, we have proved that S is dense in R.

1.17 Let x be a positive rational number of the form

$$x = \sum_{k=1}^{n} \frac{a_k}{k!},$$

where each a_k is nonnegative integer with $a_k \leq k-1$ for $k \geq 2$ and $a_n > 0$. Let $[x]$ denote the largest integer in x. Prove that $a_1 = [x]$, that $a_k = [k!x] - k[(k-1)!x]$ for $k = 2, ..., n$, and that n is the smallest integer such that $n!x$ is an integer. Conversely, show that every positive rational number x can be expressed in this form in one and only one way.

Proof: $(\Rightarrow)$First,

$$\begin{aligned}
[x] &= \left[a_1 + \sum_{k=2}^{n} \frac{a_k}{k!}\right] \\
&= a_1 + \left[\sum_{k=2}^{n} \frac{a_k}{k!}\right] \text{ since } a_1 \in N \\
&= a_1 \text{ since } \sum_{k=2}^{n} \frac{a_k}{k!} \leq \sum_{k=2}^{n} \frac{k-1}{k!} = \sum_{k=2}^{n} \frac{1}{(k-1)!} - \frac{1}{k!} = 1 - \frac{1}{n!} < 1.
\end{aligned}$$

Second, fixed k and consider

$$k!x = k! \sum_{j=1}^{n} \frac{a_j}{j!} = k! \sum_{j=1}^{k-1} \frac{a_j}{j!} + a_k + k! \sum_{j=k+1}^{n} \frac{a_j}{j!}$$

and

$$(k-1)!x = (k-1)! \sum_{j=1}^{n} \frac{a_j}{j!} = (k-1)! \sum_{j=1}^{k-1} \frac{a_j}{j!} + (k-1)! \sum_{j=k}^{n} \frac{a_j}{j!}.$$

So,

$$[k!x] = \left[k!\sum_{j=1}^{k-1}\frac{a_j}{j!} + a_k + k!\sum_{j=k+1}^{n}\frac{a_j}{j!}\right]$$
$$= k!\sum_{j=1}^{k-1}\frac{a_j}{j!} + a_k \text{ since } k!\sum_{j=k+1}^{n}\frac{a_j}{j!} < 1$$

and

$$k\left[(k-1)!x\right] = k\left[(k-1)!\sum_{j=1}^{k-1}\frac{a_j}{j!} + (k-1)!\sum_{j=k}^{n}\frac{a_j}{j!}.\right]$$
$$= k\,(k-1)!\sum_{j=1}^{k-1}\frac{a_j}{j!} \text{ since } \ (k-1)!\sum_{j=k}^{n}\frac{a_j}{j!} < 1$$
$$= k!\sum_{j=1}^{k-1}\frac{a_j}{j!}$$

which implies that

$$a_k = [k!x] - k\left[(k-1)!x\right] \text{ for } k = 2, ..., n.$$

Last, in order to show that n is the smallest integer such that $n!x$ is an integer. It is clear that

$$n!x = n!\sum_{k=1}^{n}\frac{a_k}{k!} \in Z.$$

In addition,

$$(n-1)!x = (n-1)!\sum_{k=1}^{n}\frac{a_k}{k!}$$
$$= (n-1)!\sum_{k=1}^{n-1}\frac{a_k}{k!} + \frac{a_n}{n}$$
$$\notin Z \text{ since } \frac{a_n}{n} \notin Z.$$

So, we have proved it.

($\Leftarrow$)It is clear since every a_n is uniquely deermined.

Upper bounds

1.18 Show that the sup and the inf of a set are uniquely determined whenever they exists.

Proof: Given a nonempty set $S\,(\subseteq R)$, and assume $\sup S = a$ and $\sup S = b$, we show $a = b$ as follows. Suppose that $a > b$, and thus choose $\varepsilon = \frac{a-b}{2}$, then there exists a $x \in S$ such that

$$b < \frac{a+b}{2} = a - \varepsilon < x < a$$

which implies that

$$b < x$$

which contradicts to $b = \sup S$. Similarly for $a < b$. Hence, $a = b$.

1.19 Find the sup and inf of each of the following sets of real numbers:

(a) All numbers of the form $2^{-p} + 3^{-q} + 5^{-r}$, where p, q, and r take on all positive integer values.

Proof: Define $S = \{2^{-p} + 3^{-q} + 5^{-r} : p, q, r \in N\}$. Then it is clear that $\sup S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$, and $\inf S = 0$.

(b) $S = \{x : 3x^2 - 10x + 3 < 0\}$

Proof: Since $3x^2 - 10x + 3 = (x-3)(3x-1)$, we know that $S = \left(\frac{1}{3}, 3\right)$. Hence, $\sup S = 3$ and $\inf S = \frac{1}{3}$.

(c) $S = \{x : (x-a)(x-b)(x-c)(x-d) < 0\}$, where $a < b < c < d$.

Proof: It is clear that $S = (a, b) \cup (c, d)$. Hence, $\sup S = d$ and $\inf S = a$.

1.20 Prove the comparison property for suprema (**Theorem 1.16**)

Proof: Since $s \leq t$ for every $s \in S$ and $t \in T$, fixed $t_0 \in T$, then $s \leq t_0$ for all $s \in S$. Hence, by **Axiom 10**, we know that $\sup S$ exists. In addition, it is clear $\sup S \leq \sup T$.

Remark: There is a useful result, we write it as a reference. Let S and T be two nonempty subsets of R. If $S \subseteq T$ and $\sup T$ exists, then $\sup S$ exists and $\sup S \leq \sup T$.

Proof: Since $\sup T$ exists and $S \subseteq T$, we know that for every $s \in S$, we have

$$s \leq \sup T.$$

Hence, by **Axiom 10**, we have proved the existence of $\sup S$. In addition, $\sup S \leq \sup T$ is trivial.

1.21 Let A and B be two sets of positive numbers bounded above, and let $a = \sup A$, $b = \sup B$. Let C be the set of all products of the form xy, where $x \in A$ and $y \in B$. Prove that $ab = \sup C$.

Proof: Given $\varepsilon > 0$, we want to find an element $c \in C$ such that $ab - \varepsilon < c$. If we can show this, we have proved that $\sup C$ exists and equals ab.

Since $\sup A = a > 0$ and $\sup B = b > 0$, we can choose n large enough such that $a - \varepsilon/n > 0$, $b - \varepsilon/n > 0$, and $n > a + b$. So, for this $\varepsilon' = \varepsilon/n$, there exists $a' \in A$ and $b' \in B$ such that

$$a - \varepsilon' < a' \text{ and } b - \varepsilon' < b'$$

which implies that

$$ab - \varepsilon' (a + b - \varepsilon') < a'b' \text{ since } a - \varepsilon' > 0 \text{ and } b - \varepsilon' > 0$$

which implies that

$$ab - \frac{\varepsilon}{n} (a + b) < a'b' := c$$

which implies that

$$ab - \varepsilon < c.$$

1.22 Given $x > 0$, and an integer $k \geq 2$. Let a_0 denote the largest integer $\leq x$ and, assumeing that $a_0, a_1, ..., a_{n-1}$ have been defined, let a_n denote the largest integer such that

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x.$$

Note: When $k = 10$ the integers $a_0, a_1, ...$ are the digits in a decimal representation of x. For general k they provide a representation in the scale of k.

(a) Prove that $0 \leq a_i \leq k - 1$ for each $i = 1, 2, ...$

Proof: Choose $a_0 = [x]$, and thus consider

$$[kx - ka_0] := a_1$$

then

$$0 \leq k(x - a_0) < k \Rightarrow 0 \leq a_1 \leq k - 1$$

and

$$a_0 + \frac{a_1}{k} \leq x \leq a_0 + \frac{a_1}{k} + \frac{1}{k}.$$

Continue the process, we then have

$$0 \leq a_i \leq k - 1 \text{ for each } i = 1, 2, ...$$

and

$$a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} \leq x < a_0 + \frac{a_1}{k} + \frac{a_2}{k^2} + ... + \frac{a_n}{k^n} + \frac{1}{k^n}. \qquad (*)$$

(b) Let $r_n = a_0 + a_1 k^{-1} + a_2 k^{-2} + ... + a_n k^{-n}$ and show that x is the sup of the set of rational numbers $r_1, r_2, ...$

Proof: It is clear by (a)-(*).

Inequality

1.23 Prove **Lagrange's identity** for real numbers:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2.$$

Note that this identity implies that **Cauchy-Schwarz inequality**.

Proof: Consider

$$\left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) = \sum_{1 \leq k,j \leq n} a_k^2 b_j^2 = \sum_{k=j} a_k^2 b_j^2 + \sum_{k \neq j} a_k^2 b_j^2 = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k \neq j} a_k^2 b_j^2$$

and

$$\left(\sum_{k=1}^{n} a_k b_k\right)\left(\sum_{k=1}^{n} a_k b_k\right) = \sum_{1\le k,j\le n} a_k b_k a_j b_j = \sum_{k=1}^{n} a_k^2 b_k^2 + \sum_{k\neq j} a_k b_k a_j b_j$$

So,

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k\right)^2 &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + \sum_{k\neq j} a_k b_k a_j b_j - \sum_{k\neq j} a_k^2 b_j^2 \\ &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + 2\sum_{1\le k<j\le n} a_k b_k a_j b_j - \sum_{1\le k<j\le n} a_k^2 b_j^2 + a_j^2 b_k^2 \\ &= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} \left(a_k b_j - a_j b_k\right)^2.\end{aligned}$$

Remark: (1) The reader may recall the relation with **Cross Product** and **Inner Product**, we then have a fancy formula:

$$\|x\times y\|^2 + |< x, y >|^2 = \|x\|^2\,\|y\|^2,$$

where $x, y \in R^3$.

(2) We often write

$$< a, b >:= \sum_{k=1}^{n} a_k b_k,$$

and the Cauchy-Schwarz inequality becomes

$$|< x, y >| \le \|x\|\,\|y\| \text{ by } \textbf{Remark } (1).$$

1.24 Prove that for arbitrary real a_k, b_k, c_k we have

$$\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 \le \left(\sum_{k=1}^{n} a_k^4\right)\left(\sum_{k=1}^{n} b_k^2\right)^2\left(\sum_{k=1}^{n} c_k^4\right).$$

Proof: Use **Cauchy-Schwarz inequality** twice, we then have

$$\begin{aligned}
\left(\sum_{k=1}^{n} a_k b_k c_k\right)^4 &= \left[\left(\sum_{k=1}^{n} a_k b_k c_k\right)^2\right]^2 \\
&\leq \left(\sum_{k=1}^{n} a_k^2 c_k^2\right)^2 \left(\sum_{k=1}^{n} b_k^2\right)^2 \\
&\leq \left(\sum_{k=1}^{n} a_k^4\right)^2 \left(\sum_{k=1}^{n} c_k^4\right) \left(\sum_{k=1}^{n} b_k^2\right)^2 \\
&= \left(\sum_{k=1}^{n} a_k^4\right) \left(\sum_{k=1}^{n} b_k^2\right)^2 \left(\sum_{k=1}^{n} c_k^4\right).
\end{aligned}$$

1.25 Prove that **Minkowski's inequality**:

$$\left(\sum_{k=1}^{n} (a_k + b_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}.$$

This is the triangle inequality $\|a + b\| \leq \|a\| + \|b\|$ for $n-$dimensional vectors, where $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and

$$\|a\| = \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}.$$

Proof: Consider

$$\begin{aligned}
\sum_{k=1}^{n} (a_k + b_k)^2 &= \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\sum_{k=1}^{n} a_k b_k \\
&\leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 + 2\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2} \quad \text{by Cauchy-Schwarz inequality} \\
&= \left[\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}\right]^2.
\end{aligned}$$

So,

$$\left(\sum_{k=1}^{n}(a_k+b_k)^2\right)^{1/2} \le \left(\sum_{k=1}^{n}a_k^2\right)^{1/2} + \left(\sum_{k=1}^{n}b_k^2\right)^{1/2}.$$

1.26 If $a_1 \ge ... \ge a_n$ and $b_1 \ge ... \ge b_n$, prove that

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right) \le n\left(\sum_{k=1}^{n}a_kb_k\right).$$

Hint. $\sum_{1\le j\le k\le n}(a_k-a_j)(b_k-b_j) \ge 0$.

Proof: Consider

$$0 \le \sum_{1\le j\le k\le n}(a_k-a_j)(b_k-b_j) = \sum_{1\le j\le k\le n}a_kb_k+a_jb_j - \sum_{1\le j\le k\le n}a_kb_j+a_jb_k$$

which implies that

$$\sum_{1\le j\le k\le n}a_kb_j+a_jb_k \le \sum_{1\le j\le k\le n}a_kb_k+a_jb_j. \quad (*)$$

Since

$$\begin{aligned}\sum_{1\le j\le k\le n}a_kb_j+a_jb_k &= \sum_{1\le j<k\le n}a_kb_j+a_jb_k+2\sum_{k=1}^{n}a_kb_k \\ &= \left(\sum_{1\le j<k\le n}a_kb_j+a_jb_k+\sum_{k=1}^{n}a_kb_k\right)+\sum_{k=1}^{n}a_kb_k \\ &= \left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)+\sum_{k=1}^{n}a_kb_k,\end{aligned}$$

we then have, by (*)

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}b_k\right)+\sum_{k=1}^{n}a_kb_k \le \sum_{1\le j\le k\le n}a_kb_k+a_jb_j. \quad (**)$$

In addition,

$$\sum_{1\le j\le k\le n} a_k b_k + a_j b_j$$
$$= \sum_{k=1}^{n} a_k b_k + n a_1 b_1 + \sum_{k=2}^{n} a_k b_k + (n-1)\, a_2 b_2 + ... + \sum_{k=n-1}^{n} a_k b_k + 2a_{n-1} b_{n-1} + \sum_{k=n} a_k b_k$$
$$= n \sum_{k=1}^{n} a_k b_k + a_1 b_1 + a_2 b_2 + ... + a_n b_n$$
$$= (n+1) \sum_{k=1}^{n} a_k b_k$$

which implies that, by (**),

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \le n\left(\sum_{k=1}^{n} a_k b_k\right).$$

Complex numbers

1.27 Express the following complex numbers in the form $a+bi$.

(a) $(1+i)^3$

Solution: $(1+i)^3 = 1+3i+3i^2+i^3 = 1+3i-3-i = -2+2i$.

(b) $(2+3i)\,/\,(3-4i)$

Solution: $\frac{2+3i}{3-4i} = \frac{(2+3i)(3+4i)}{(3-4i)(3+4i)} = \frac{-6+17i}{25} = \frac{-6}{25} + \frac{17}{25}i$.

(c) $i^5 + i^{16}$

Solution: $i^5 + i^{16} = i+1$.

(d) $\frac{1}{2}(1+i)(1+i^{-8})$

Solution: $\frac{1}{2}(1+i)(1+i^{-8}) = 1+i$.

1.28 In each case, determine all real x and y which satisfy the given relation.

(a) $x + iy = |x - iy|$

Proof: Since $|x - iy| \geq 0$, we have

$$x \geq 0 \text{ and } y = 0.$$

(b) $x + iy = (x - iy)^2$

Proof: Since $(x - iy)^2 = x^2 - (2xy)\, i - y^2$, we have

$$x = x^2 - y^2 \text{ and } y = -2xy.$$

We consider tow cases: (i) $y = 0$ and (ii) $y \neq 0$.

(i) As $y = 0 : x = 0$ or 1.

(ii) As $y \neq 0 : x = -1/2$, and $y = \pm \frac{\sqrt{3}}{2}$.

(c) $\sum_{k=0}^{100} i^k = x + iy$

Proof: Since $\sum_{k=0}^{100} i^k = \frac{1-i^{101}}{1-i} = \frac{1-i}{1-i} = 1$, we have $x = 1$ and $y = 0$.

1.29 If $z = x + iy$, x and y real, the complex conjugate of z is the complex number $\bar{z} = x - iy$. Prove that:

(a) Conjugate of $(z_1 + z_2) = \bar{z}_1 + \bar{z}_2$

Proof: Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{(x_1 + x_2) + i\,(y_1 + y_2)} \\
&= (x_1 + x_2) - i\,(y_1 + y_2) \\
&= (x_1 - iy_1) + (x_2 - iy_2) \\
&= \bar{z}_1 + \bar{z}_2.
\end{aligned}$$

(b) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Proof: Write $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{(x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)} \\
&= (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1)
\end{aligned}$$

and

$$\begin{aligned}
\bar{z}_1 \bar{z}_2 &= (x_1 - iy_1)\,(x_2 - iy_2) \\
&= (x_1 x_2 - y_1 y_2) - i\,(x_1 y_2 + x_2 y_1)\,.
\end{aligned}$$

So, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(c) $z\bar{z} = |z|^2$

Proof: Write $z = x + iy$ and thus

$$z\bar{z} = x^2 + y^2 = |z|^2 .$$

(d) $z + \bar{z}$ =twice the real part of z

Proof: Write $z = x + iy$, then

$$z + \bar{z} = 2x,$$

twice the real part of z.

(e) $(z - \bar{z})/i$ =twice the imaginary part of z

Proof: Write $z = x + iy$, then

$$\frac{z - \bar{z}}{i} = 2y,$$

twice the imaginary part of z.

1.30 Describe geometrically the set of complex numbers z which satisfies each of the following conditions:

(a) $|z| = 1$

Solution: The unit circle centered at zero.

(b) $|z| < 1$

Solution: The open unit disk centered at zero.

(c) $|z| \leq 1$

Solution: The closed unit disk centered at zero.

(d) $z + \bar{z} = 1$

Solution: Write $z = x + iy$, then $z + \bar{z} = 1$ means that $x = 1/2$. So, the set is the line $x = 1/2$.

(e) $z - \bar{z} = i$

Proof: Write $z = x + iy$, then $z - \bar{z} = i$ means that $y = 1/2$. So, the set is the line $y = 1/2$.

(f) $z + \bar{z} = |z|^2$

Proof: Write $z = x + iy$, then $2x = x^2 + y^2 \Leftrightarrow (x-1)^2 + y^2 = 1$. So, the set is the unit circle centered at $(1, 0)$.

1.31 Given three complex numbers z_1, z_2, z_3 such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Show that these numbers are vertices of an equilateral triangle inscribed in the unit circle with center at the origin.

Proof: It is clear that three numbers are vertices of triangle inscribed in the unit circle with center at the origin. It remains to show that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$. In addition, it suffices to show that

$$|z_1 - z_2| = |z_2 - z_3|.$$

Note that

$$|2z_1 + z_3| = |2z_3 + z_1| \text{ by } z_1 + z_2 + z_3 = 0$$

which is equivalent to

$$|2z_1 + z_3|^2 = |2z_3 + z_1|^2$$

which is equivalent to

$$(2z_1 + z_3)(2\bar{z}_1 + \bar{z}_3) = (2z_3 + z_1)(2\bar{z}_3 + \bar{z}_1)$$

which is equivalent to

$$|z_1| = |z_3|.$$

1.32 If a and b are complex numbers, prove that:

(a) $|a - b|^2 \leq \left(1 + |a|^2\right)\left(1 + |b|^2\right)$

Proof: Consider

$$\begin{aligned}\left(1 + |a|^2\right)\left(1 + |b|^2\right) - |a - b|^2 &= (1 + \bar{a}a)\left(1 + \bar{b}b\right) - (a - b)\left(\bar{a} - \bar{b}\right) \\ &= (1 + \bar{a}b)\left(1 + a\bar{b}\right) \\ &= |1 + \bar{a}b|^2 \geq 0,\end{aligned}$$

so, $|a-b|^2 \leq \left(1+|a|^2\right)\left(1+|b|^2\right)$

(b) If $a \neq 0$, then $|a+b| = |a| + |b|$ if, and only if, b/a is real and nonnegative.

Proof: ($\Rightarrow$)Since $|a+b| = |a| + |b|$, we have

$$|a+b|^2 = (|a| + |b|)^2$$

which implies that

$$\operatorname{Re}(\bar{a}b) = |a|\,|b| = |\bar{a}|\,|b|$$

which implies that

$$\bar{a}b = |\bar{a}|\,|b|$$

which implies that

$$\frac{b}{a} = \frac{\bar{a}b}{\bar{a}a} = \frac{|\bar{a}|\,|b|}{|a|^2} \geq 0.$$

($\Leftarrow$) Suppose that

$$\frac{b}{a} = k, \text{ where } k \geq 0.$$

Then

$$|a+b| = |a+ka| = (1+k)\,|a| = |a| + k\,|a| = |a| + |b|\,.$$

1.33 If a and b are complex numbers, prove that

$$|a-b| = |1-\bar{a}b|$$

if, and only if, $|a| = 1$ or $|b| = 1$. For which a and b is the inequality $|a-b| < |1-\bar{a}b|$ valid?

Proof: ($\Leftrightarrow$) Since

$$\begin{aligned} |a-b| &= |1-\bar{a}b| \\ &\Leftrightarrow \left(\bar{a}-\bar{b}\right)(a-b) = (1-\bar{a}b)\left(1-a\bar{b}\right) \\ &\Leftrightarrow |a|^2 + |b|^2 = 1 + |a|^2\,|b|^2 \\ &\Leftrightarrow \left(|a|^2-1\right)\left(|b|^2-1\right) = 0 \\ &\Leftrightarrow |a|^2 = 1 \text{ or } |b|^2 = 1. \end{aligned}$$

By the preceding, it is easy to know that

$$|a-b|<|1-\bar{a}b| \Leftrightarrow 0<\left(|a|^2-1\right)\left(|b|^2-1\right).$$

So, $|a-b|<|1-\bar{a}b|$ if, and only if, $|a|>1$ and $|b|>1$. (Or $|a|<1$ and $|b|<1$).

1.34 If a and c are real constant, b complex, show that the equation

$$az\bar{z}+b\bar{z}+\bar{b}z+c=0\ (a\neq 0, z=x+iy)$$

represents a circle in the $x-y$ plane.

Proof: Consider

$$z\bar{z}-\frac{b}{-a}\bar{z}-\frac{\bar{b}}{-a}z+\frac{b}{-a}\left[\overline{\left(\frac{b}{-a}\right)}\right]=\frac{-ac+|b|^2}{a^2},$$

so, we have

$$\left|z-\left(\frac{b}{-a}\right)\right|^2=\frac{-ac+|b|^2}{a^2}.$$

Hence, as $|b|^2-ac>0$, it is a circle. As $\frac{-ac+|b|^2}{a^2}=0$, it is a point. As $\frac{-ac+|b|^2}{a^2}<0$, it is not a circle.

Remark: The idea is easy from the fact

$$|z-q|=r.$$

We square both sides and thus

$$z\bar{z}-q\bar{z}-\bar{q}z+\bar{q}q=r^2.$$

1.35 Recall the definition of the inverse tangent: given a real number t, $\tan^{-1}(t)$ is the unique real number θ which satisfies the two conditions

$$-\frac{\pi}{2}<\theta<+\frac{\pi}{2},\ \tan\theta=t.$$

If $z=x+iy$, show that

(a) $\arg(z)=\tan^{-1}\left(\frac{y}{x}\right)$, if $x>0$

Proof: Note that in this text book, we say $\arg(z)$ is the principal argument of z, denoted by $\theta = \arg z$, where $-\pi < \theta \leq \pi$.

So, as $x > 0$, $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$.

(b) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$, if $x < 0$, $y \geq 0$

Proof: As $x < 0$, and $y \geq 0$. The point (x, y) is lying on $S = \{(x, y) : x < 0,\ y \geq 0\}$. Note that $-\pi < \arg z \leq \pi$, so we have $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) + \pi$.

(c) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right) - \pi$, if $x < 0$, $y < 0$

Proof: Similarly for (b). So, we omit it.

(d) $\arg(z) = \frac{\pi}{2}$ if $x = 0$, $y > 0$; $\arg(z) = -\frac{\pi}{2}$ if $x = 0$, $y < 0$.

Proof: It is obvious.

1.36 Define the folowing "**pseudo-ordering**" of the complex numbers: we say $z_1 < z_2$ if we have either

(i) $|z_1| < |z_2|$ or (ii) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.

Which of Axioms 6,7,8,9 are satisfied by this relation?

Proof: (1) For axiom 6, we prove that it holds as follows. Given $z_1 = r_1 e^{i\arg(z_1)}$, and $r_2 e^{i\arg(z_2)}$, then if $z_1 = z_2$, there is nothing to prove it. If $z_1 \neq z_2$, there are two possibilities: (a) $r_1 \neq r_2$, or (b) $r_1 = r_2$ and $\arg(z_1) \neq \arg(z_2)$. So, it is clear that axiom 6 holds.

(2) For axiom 7, we prove that it does not hold as follows. Given $z_1 = 1$ and $z_2 = -1$, then it is clear that $z_1 < z_2$ since $|z_1| = |z_2| = 1$ and $\arg(z_1) = 0 < \arg(z_2) = \pi$. However, let $z_3 = -i$, we have

$$z_1 + z_3 = 1 - i > z_2 + z_3 = -1 - i$$

since

$$|z_1 + z_3| = |z_2 + z_3| = \sqrt{2}$$

and

$$\arg(z_1 + z_3) = -\frac{\pi}{4} > -\frac{3\pi}{4} = \arg(z_2 + z_3).$$

(3) For axiom 8, we prove that it holds as follows. If $z_1 > 0$ and $z_2 > 0$, then $|z_1| > 0$ and $|z_2| > 0$. Hence, $z_1 z_2 > 0$ by $|z_1 z_2| = |z_1| |z_2| > 0$.

(4) For axiom 9, we prove that it holds as follows. If $z_1 > z_2$ and $z_2 > z_3$, we consider the following cases. Since $z_1 > z_2$, we may have (a) $|z_1| > |z_2|$ or (b) $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$.

As $|z_1| > |z_2|$, it is clear that $|z_1| > |z_3|$. So, $z_1 > z_3$.

As $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$, we have $\arg(z_1) > \arg(z_3)$. So, $z_1 > z_3$.

1.37 Which of Axioms 6,7,8,9 are satisfied if the **pseudo-ordering** is defined as follows? We say $(x_1, y_1) < (x_2, y_2)$ if we have either (i) $x_1 < x_2$ or (ii) $x_1 = x_2$ and $y_1 < y_2$.

Proof: (1) For axiom 6, we prove that it holds as follows. Given $x = (x_1, y_1)$ and $y = (x_2, y_2)$. If $x = y$, there is nothing to prove it. We consider $x \neq y$: As $x \neq y$, we have $x_1 \neq x_2$ or $y_1 \neq y_2$. Both cases imply $x < y$ or $y < x$.

(2) For axiom 7, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x < y$, then there are two possibilities: (a) $x_1 < x_2$ or (b) $x_1 = x_2$ and $y_1 < y_2$.

For case (a), it is clear that $x_1 + z_1 < y_1 + z_1$. So, $x + z < y + z$.

For case (b), it is clear that $x_1 + z_1 = y_1 + z_1$ and $x_2 + z_2 < y_2 + z_2$. So, $x + z < y + z$.

(3) For axiom 8, we prove that it does not hold as follows. Consider $x = (1, 0)$ and $y = (0, 1)$, then it is clear that $x > 0$ and $y > 0$. However, $xy = (0, 0) = 0$.

(4) For axiom 9, we prove that it holds as follows. Given $x = (x_1, y_1)$, $y = (x_2, y_2)$ and $z = (z_1, z_3)$. If $x > y$ and $y > z$, then we consider the following cases. (a) $x_1 > y_1$, or (b) $x_1 = y_1$.

For case (a), it is clear that $x_1 > z_1$. So, $x > z$.

For case (b), it is clear that $x_2 > y_2$. So, $x > z$.

1.38 State and prove a theorem analogous to Theorem 1.48, expressing $\arg(z_1/z_2)$ in terms of $\arg(z_1)$ and $\arg(z_2)$.

Proof: Write $z_1 = r_1 e^{i \arg(z_1)}$ and $z_2 = r_2 e^{i \arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2),$$

where

$$n(z_1, z_2) = \begin{cases} 0 \text{ if } -\pi < \arg(z_1) - \arg(z_2) \leq \pi \\ 1 \text{ if } -2\pi < \arg(z_1) - \arg(z_2) \leq -\pi \\ -1 \text{ if } \pi < \arg(z_1) - \arg(z_2) < 2\pi \end{cases}.$$

1.39 State and prove a theorem analogous to Theorem 1.54, expressing $Log\,(z_1/z_2)$ in terms of $Log\,(z_1)$ and $Log\,(z_2)$.

Proof: Write $z_1 = r_1 e^{i\arg(z_1)}$ and $z_2 = r_2 e^{i\arg(z_2)}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i[\arg(z_1) - \arg(z_2)]}.$$

Hence,

$$\begin{aligned} Log\,(z_1/z_2) &= \log\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right) \\ &= \log|z_1| - \log|z_2| + i\left[\arg(z_1) - \arg(z_2) + 2\pi n(z_1, z_2)\right] \text{ by xercise 1.38} \\ &= Log\,(z_1) - Log\,(z_2) + i2\pi n(z_1, z_2). \end{aligned}$$

1.40 Prove that the nth roots of 1 (also called the nth roots of unity) are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$, and show that the roots $\neq 1$ satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

Proof: By **Theorem 1.51**, we know that the roots of 1 are given by $\alpha, \alpha^2, ..., \alpha^n$, where $\alpha = e^{2\pi i/n}$. In addition, since

$$x^n = 1 \Rightarrow (x-1)\left(1 + x + x^2 + ... + x^{n-1}\right) = 0$$

which implies that

$$1 + x + x^2 + ... + x^{n-1} = 0 \text{ if } x \neq 1.$$

So, all roots except 1 satisfy the equation

$$1 + x + x^2 + ... + x^{n-1} = 0.$$

1.41 (a) Prove that $|z^i| < e^{\pi}$ for all complex $z \neq 0$.

Proof: Since

$$z^i = e^{iLog(z)} = e^{-\arg(z) + i\log|z|},$$

we have

$$\left|z^i\right| = e^{-\arg(z)} < e^{\pi}$$

by $-\pi < \arg(z) \leq \pi$.

(b) Prove that there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Proof: Write $z = x + iy$ and thus,

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

which implies that

$$|\cos x \cosh y| \leq |\cos z|.$$

Let $x = 0$ and y be real, then

$$\frac{e^y}{2} \leq \frac{1}{2}\left|e^y + e^{-y}\right| \leq |\cos z|.$$

So, there is no constant $M > 0$ such that $|\cos z| < M$ for all complex z.

Remark: There is an important theorem related with this exercise. We state it as a reference. (**Liouville's Theorem**) A bounded entire function is constant. The reader can see the book, **Complex Analysis by Joseph Bak, and Donald J. Newman, pp 62-63. Liouville's Theorem** can be used to prove the much important theorem, **Fundamental Theorem of Algebra.**

1.42 If $w = u + iv$ (u, v real), show that

$$z^w = e^{u \log|z| - v \arg(z)} e^{i[v \log|z| + u \arg(z)]}.$$

Proof: Write $z^w = e^{wLog(z)}$, and thus

$$\begin{aligned} wLog(z) &= (u + iv)(\log|z| + i \arg(z)) \\ &= [u \log|z| - v \arg(z)] + i[v \log|z| + u \arg(z)]. \end{aligned}$$

So,

$$z^w = e^{u \log|z| - v \arg(z)} e^{i[v \log|z| + u \arg(z)]}.$$

1.43 (a) Prove that $Log\left(z^{w}\right) = wLog\ z\ + 2\pi in$.

Proof: Write $w = u + iv$, where u and v are real. Then

$$\begin{aligned} Log\left(z^{w}\right) &= \log\left|z^{w}\right| + i\arg\left(z^{w}\right) \\ &= \log\left[e^{u\log|z| - v\arg(z)}\right] + i\left[v\log|z| + u\arg(z)\right] + 2\pi in \text{ by Exercise1.42} \\ &= u\log|z| - v\arg(z) + i\left[v\log|z| + u\arg(z)\right] + 2\pi in. \end{aligned}$$

On the other hand,

$$\begin{aligned} wLogz + 2\pi in &= (u + iv)\left(\log|z| + i\arg(z)\right) + 2\pi in \\ &= u\log|z| - v\arg(z) + i\left[v\log|z| + u\arg(z)\right] + 2\pi in. \end{aligned}$$

Hence, $Log\left(z^{w}\right) = wLog\ z\ + 2\pi in$.

Remark: There is another proof by considering

$$e^{Log(z^{w})} = z^{w} = e^{wLog(z)}$$

which implies that

$$Log\left(z^{w}\right) = wLogz + 2\pi in$$

for some $n \in Z$.

(b) Prove that $\left(z^{w}\right)^{\alpha} = z^{w\alpha}e^{2\pi in\alpha}$, where n is an integer.

Proof: By (a), we have

$$\left(z^{w}\right)^{\alpha} = e^{\alpha Log(z^{w})} = e^{\alpha(wLogz + 2\pi in)} = e^{\alpha wLogz}e^{2\pi in\alpha} = z^{\alpha w}e^{2\pi in\alpha},$$

where n is an integer.

1.44 (i) If θ and a are real numbers, $-\pi < \theta \leq \pi$, prove that

$$\left(\cos\theta + i\sin\theta\right)^{a} = \cos(a\theta) + i\sin(a\theta).$$

Proof: Write $\cos\theta + i\sin\theta = z$, we then have

$$\begin{aligned} \left(\cos\theta + i\sin\theta\right)^{a} = z^{a} = e^{aLogz} = e^{a\left[\log\left|e^{i\theta}\right| + i\arg\left(e^{i\theta}\right)\right]} = e^{ia\theta} \\ = \cos(a\theta) + i\sin(a\theta). \end{aligned}$$

Remark: Compare with the **Exercise 1.43-(b)**.

(ii) Show that, in general, the restriction $-\pi < \theta \leq \pi$ is necessary in (i) by taking $\theta = -\pi$, $a = \frac{1}{2}$.

Proof: As $\theta = -\pi$, and $a = \frac{1}{2}$, we have

$$(-1)^{\frac{1}{2}} = e^{\frac{1}{2}Log(-1)} = e^{\frac{\pi}{2}i} = i \neq -i = \cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right).$$

(iii) If a is an integer, show that the formula in (i) holds without any restriction on θ. In this case it is known as **DeMorvre's theorem**.

Proof: By **Exercise 1.43**, as a is an integer we have

$$(z^w)^a = z^{wa},$$

where $z^w = e^{i\theta}$. Then

$$\left(e^{i\theta}\right)^a = e^{i\theta a} = \cos(a\theta) + i\sin(a\theta).$$

1.45 Use **DeMorvre's theorem** (Exercise 1.44) to derive the trigonometric identities

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

valid for real θ. Are these valid when θ is complex?

Proof: By **Exercise 1.44-(iii)**, we have for any real θ,

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta).$$

By **Binomial Theorem**, we have

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$

and

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$

For complex θ, we show that it holds as follows. Note that $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$ and $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, we have

$$\begin{aligned}
3\cos^2 z\sin z - \sin^3 z &= 3\left(\frac{e^{iz}+e^{-iz}}{2}\right)^2\left(\frac{e^{iz}-e^{-iz}}{2i}\right) - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^3 \\
&= 3\left(\frac{e^{2zi}+e^{-2zi}+2}{4}\right)\left(\frac{e^{iz}-e^{-iz}}{2i}\right) + \frac{e^{3zi}-3e^{iz}+3e^{-iz}-e^{-3zi}}{8i} \\
&= \frac{1}{8i}\left[3\left(e^{2zi}+e^{-2zi}+2\right)\left(e^{zi}-e^{-zi}\right)+\left(e^{3zi}-3e^{iz}+3e^{-iz}-e^{-3zi}\right)\right] \\
&= \frac{1}{8i}\left[\left(3e^{3zi}+3e^{iz}-3e^{-iz}-3e^{-3zi}\right)+\left(e^{3zi}-3e^{iz}+3e^{-iz}-e^{-3zi}\right)\right] \\
&= \frac{4}{8i}\left(e^{3zi}-e^{-3zi}\right) \\
&= \frac{1}{2i}\left(e^{3zi}-e^{-3zi}\right) \\
&= \sin 3z.
\end{aligned}$$

Similarly, we also have

$$\cos^3 z - 3\cos z\sin^2 z = \cos 3z.$$

1.46 Define $\tan z = \sin z/\cos z$ and show that for $z = x+iy$, we have

$$\tan z = \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}.$$

Proof: Since

$$\begin{aligned}
\tan z &= \frac{\sin z}{\cos z} = \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sin x\cosh y + i\cos x\sinh y}{\cos x\cosh y - i\sin x\sinh y} \\
&= \frac{(\sin x\cosh y + i\cos x\sinh y)(\cos x\cosh y + i\sin x\sinh y)}{(\cos x\cosh y - i\sin x\sinh y)(\cos x\cosh y + i\sin x\sinh y)} \\
&= \frac{\left(\sin x\cos x\cosh^2 y - \sin x\cos x\sinh^2 y\right) + i\left(\sin^2 x\cosh y\sinh y + \cos^2 x\cosh y\sinh y\right)}{(\cos x\cosh y)^2 - (i\sin x\sinh y)^2} \\
&= \frac{\sin x\cos x\left(\cosh^2 y - \sinh^2 y\right) + i(\cosh y\sinh y)}{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \text{ since } \sin^2 x + \cos^2 x = 1 \\
&= \frac{(\sin x\cos x) + i(\cosh y\sinh y)}{\cos^2 x + \sinh^2 y} \text{ since } \cosh^2 y = 1 + \sinh^2 y \\
&= \frac{\frac{1}{2}\sin 2x + \frac{i}{2}\sinh 2y}{\cos^2 x + \sinh^2 y} \text{ since } 2\cosh y\sinh y = \sinh 2y \text{ and } 2\sin x\cos x = \sin 2x \\
&= \frac{\sin 2x + i\sinh 2y}{2\cos^2 x + 2\sinh^2 y} \\
&= \frac{\sin 2x + i\sinh 2y}{2\cos^2 x - 1 + 2\sinh^2 y + 1} \\
&= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} \text{ since } \cos 2x = 2\cos^2 x - 1 \text{ and } 2\sinh^2 y + 1 = \cosh 2y.
\end{aligned}$$

1.47 Let w be a given complex number. If $w \neq \pm 1$, show that there exists two values of $z = x + iy$ satisfying the conditions $\cos z = w$ and $-\pi < x \leq \pi$. Find these values when $w = i$ and when $w = 2$.

Proof: Since $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, if we let $e^{iz} = u$, then $\cos z = w$ implies that

$$w = \frac{u^2+1}{2u} \Rightarrow u^2 - 2wu + 1 = 0$$

which implies that

$$(u-w)^2 = w^2 - 1 \neq 0 \text{ since } w \neq \pm 1.$$

So, by **Theorem 1.51**,

$$\begin{aligned}
e^{iz} &= u = w + \left|w^2-1\right|^{1/2} e^{i\phi_k}, \text{ where } \phi_k = \frac{\arg(w^2-1)}{2} + \frac{2\pi k}{2},\ k = 0, 1. \\
&= w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}
\end{aligned}$$

So,

$$ix-y = i\,(x+iy) = iz = \log\left|w \pm \left|w^2-1\right|^{1/2} e^{i\frac{\arg\left(w^2-1\right)}{2}}\right| + i\arg\left(w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}\right)$$

Hence, there exists two values of $z = x+iy$ satisfying the conditions $\cos z = w$ and

$$-\pi < x = \arg\left(w \pm \left|w^2-1\right|^{1/2} e^{i\left(\frac{\arg\left(w^2-1\right)}{2}\right)}\right) \leq \pi.$$

For $w = i$, we have

$$iz = \log\left|\left(1 \pm \sqrt{2}\right) i\right| + i\arg\left(\left(1 \pm \sqrt{2}\right) i\right)$$

which implies that

$$z = \arg\left(\left(1 \pm \sqrt{2}\right) i\right) - i\log\left|\left(1 \pm \sqrt{2}\right) i\right|.$$

For $w = 2$, we have

$$iz = \log\left|2 \pm \sqrt{3}\right| + i\arg\left(2 \pm \sqrt{3}\right)$$

which implies that

$$z = \arg\left(2 \pm \sqrt{3}\right) - i\log\left|2 \pm \sqrt{3}\right|.$$

1.48 Prove Lagrange's identity for complex numbers:

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 = \sum_{k=1}^{n} |a_k|^2 \sum_{k=1}^{n} |b_k|^2 - \sum_{1\leq k<j\leq n} \left(a_k \bar{b}_j - \bar{a}_j b_k\right)^2.$$

Use this to deduce a Cauchy-Schwarz ineqality for complex numbers.

Proof: It is the same as the **Exercise 1.23**; we omit the details.

1.49 (a) By eqating imaginary parts in DeMoivre's formula prove that

$$\sin n\theta = \sin^n\theta\left\{\binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - +...\right\}$$

Proof: By **Exercise 1.44 (i)**, we have

$$\begin{aligned}\sin n\theta &= \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} \sin^{2k-1}\theta \cos^{n-(2k-1)}\theta \\ &= \sin^n\theta \left\{ \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \binom{n}{2k-1} \cot^{n-(2k-1)}\theta \right\} \\ &= \sin^n\theta \left\{ \binom{n}{1}\cot^{n-1}\theta - \binom{n}{3}\cot^{n-3}\theta + \binom{n}{5}\cot^{n-5}\theta - +... \right\}.\end{aligned}$$

(b) If $0 < \theta < \pi/2$, prove that

$$\sin(2m+1)\theta = \sin^{2m+1}\theta P_m\left(\cot^2\theta\right)$$

where P_m is the polynomial of degree m given by

$$P_m(x) = \binom{2m+1}{1}x^m - \binom{2m+1}{3}x^{m-1} + \binom{2m+1}{5}x^{m-2} - +...$$

Use this to show that P_m has zeros at the m distinct points $x_k = \cot^2\{\pi k/(2m+1)\}$ for $k = 1, 2, ..., m$.

Proof: By (a),

$$\begin{aligned}&\sin(2m+1)\theta \\ &= \sin^{2m+1}\theta \left\{ \binom{2m+1}{1}\left(\cot^2\theta\right)^m - \binom{2m+1}{3}\left(\cot^2\theta\right)^{m-1} + \binom{2m+1}{5}\left(\cot^2\theta\right)^{m-2} - +... \right\} \\ &= \sin^{2m+1}\theta P_m\left(\cot^2\theta\right), \text{ where } P_m(x) = \sum_{k=1}^{m+1} \binom{2m+1}{2k-1} x^{m+1-k}. \qquad (*)\end{aligned}$$

In addition, by (*), $\sin(2m+1)\theta = 0$ if, and only if, $P_m(\cot^2\theta) = 0$. Hence, P_m has zeros at the m distinct points $x_k = \cot^2\{\pi k/(2m+1)\}$ for $k = 1, 2, ..., m$.

(c) Show that the sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = \frac{m(2m-1)}{3},$$

and the sum of their squares is given by

$$\sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m+1} = \frac{m\,(2m-1)\,(4m^2+10m-9)}{45}.$$

Note. There identities can be used to prove that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$ and $\sum_{n=1}^{\infty} n^{-4} = \pi^4/90$. (See Exercises 8.46 and 8.47.)

Proof: By (b), we know that sum of the zeros of P_m is given by

$$\sum_{k=1}^{m} x_k = \sum_{k=1}^{m} \cot^2 \frac{\pi k}{2m+1} = -\left(\frac{-\binom{2m+1}{3}}{\binom{2m+1}{1}}\right) = \frac{m\,(2m-1)}{3}.$$

And the sum of their squares is given by

$$\begin{aligned}\sum_{k=1}^{m} x_k^2 &= \sum_{k=1}^{m} \cot^4 \frac{\pi k}{2m+1} \\ &= \left(\sum_{k=1}^{m} x_k\right)^2 - 2\left(\sum_{1\le i<j\le n} x_i x_j\right) \\ &= \left(\frac{m\,(2m-1)}{3}\right)^2 - 2\left(\frac{\binom{2m+1}{5}}{\binom{2m+1}{1}}\right) \\ &= \frac{m\,(2m-1)\,(4m^2+10m-9)}{45}.\end{aligned}$$

1.50 Prove that $z^n - 1 = \prod_{k=1}^{n} \left(z - e^{2\pi ik/n}\right)$ for all complex z. Use this to derive the formula

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

Proof: Since $z^n = 1$ has exactly n distinct roots $e^{2\pi ik/n}$, where $k = 0, ..., n-1$ by **Theorem 1.51.** Hence, $z^n - 1 = \prod_{k=1}^{n} \left(z - e^{2\pi ik/n}\right)$. It implies that

$$z^{n-1} + ... + 1 = \prod_{k=1}^{n-1} \left(z - e^{2\pi ik/n}\right).$$

So, let $z = 1$, we obtain that

$$\begin{aligned}
n &= \prod_{k=1}^{n-1} \left(1 - e^{2\pi ik/n}\right) = \prod_{k=1}^{n-1} \left[\left(1 - \cos\frac{2\pi k}{n}\right) - i\left(\sin\frac{2\pi k}{n}\right)\right] \\
&= \prod_{k=1}^{n-1} \left(2\sin^2\frac{\pi k}{n}\right) - i\left(2\sin\frac{\pi k}{n}\cos\frac{\pi k}{n}\right) \\
&= \prod_{k=1}^{n-1} 2\left(\sin\frac{\pi k}{n}\right)\left(\sin\frac{\pi k}{n} - i\cos\frac{\pi k}{n}\right) \\
&= 2^{n-1}\prod_{k=1}^{n-1} \left(\sin\frac{\pi k}{n}\right)\left(\cos\left(\frac{3\pi}{2} + \frac{\pi k}{n}\right) + i\sin\left(\frac{3\pi}{2} + \frac{\pi k}{n}\right)\right) \\
&= 2^{n-1}\prod_{k=1}^{n-1} \left(\sin\frac{\pi k}{n}\right) e^{i\left(\frac{3\pi}{2} + \frac{\pi k}{n}\right)} \\
&= \left[2^{n-1}\prod_{k=1}^{n-1} \left(\sin\frac{\pi k}{n}\right)\right] e^{\sum_{k=1}^{n-1} \frac{3\pi}{2} + \frac{\pi k}{n}} \\
&= 2^{n-1}\prod_{k=1}^{n-1} \left(\sin\frac{\pi k}{n}\right).
\end{aligned}$$

Some Basic Notations Of Set Theory

References

There are some good books about set theory; we write them down. We wish the reader can get more.

1. Set Theory and Related Topics by Seymour Lipschutz.
2. Set Theory by Charles C. Pinter.
3. Theory of sets by Kamke.
4. Naive set by Halmos.

2.1 Prove Theorem 2.2. Hint. $(a,b) = (c,d)$ means $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$. Now appeal to the definition of set equality.

Proof: $(\Leftarrow)$ It is trivial.

$(\Rightarrow)$ Suppose that $(a,b) = (c,d)$, it means that $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$. It implies that

$$\{a\} \in \{\{c\},\{c,d\}\} \text{ and } \{a,b\} \in \{\{c\},\{c,d\}\}.$$

So, if $a \neq c$, then $\{a\} = \{c,d\}$. It implies that $c \in \{a\}$ which is impossible. Hence, $a = c$. Similarly, we have $b = d$.

2.2 Let S be a relation and let $D(S)$ be its domain. The relation S is said to be

(i) reflexive if $a \in D(S)$ implies $(a,a) \in S$,
(ii) symmetric if $(a,b) \in S$ implies $(b,a) \in S$,
(iii) transitive if $(a,b) \in S$ and $(b,c) \in S$ implies $(a,c) \in S$.

A relation which is symmetric, reflexive, and transitive is called an equivalence relation. Determine which of these properties is possessed by S, if S is the set of all pairs of real numbers (x,y) such that

(a) $x \leq y$

Proof: Write $S = \{(x,y) : x \leq y\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) Since $x \leq x$, $(x, x) \in S$. That is, S is reflexive.

(ii) If $(x, y) \in S$, i.e., $x \leq y$, then $y \leq x$. So, $(y, x) \in S$. That is, S is symmetric.

(iii) If $(x, y) \in S$ and $(y, z) \in S$, i.e., $x \leq y$ and $y \leq z$, then $x \leq z$. So, $(x, z) \in S$. That is, S is transitive.

(b) $x < y$

Proof: Write $S = \{(x, y) : x < y\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) It is clear that for any real x, we cannot have $x < x$. So, S is not reflexive.

(ii) It is clear that for any real x, and y, we cannot have $x < y$ and $y < x$ at the same time. So, S is not symmetric.

(iii) If $(x, y) \in S$ and $(y, z) \in S$, then $x < y$ and $y < z$. So, $x < z$ wich implies $(x, z) \in S$. That is, S is transitive.

(c) $x < |y|$

Proof: Write $S = \{(x, y) : x < |y|\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) Since it is impossible for $0 < |0|$, S is not reflexive.

(ii) Since $(-1, 2) \in S$ but $(2, -1) \notin S$, S is not symmetric.

(iii) Since $(0, -1) \in S$ and $(-1, 0) \in S$, but $(0, 0) \notin S$, S is not transitive.

(d) $x^2 + y^2 = 1$

Proof: Write $S = \{(x, y) : x^2 + y^2 = 1\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = [-1, 1]$, an closed interval with endpoints, -1 and 1.

(i) Since $1 \in D(S)$, and it is impossible for $(1, 1) \in S$ by $1^2 + 1^2 \neq 1$, S is not reflexive.

(ii) If $(x, y) \in S$, then $x^2 + y^2 = 1$. So, $(y, x) \in S$. That is, S is symmetric.

(iii) Since $(1, 0) \in S$ and $(0, 1) \in S$, but $(1, 1) \notin S$, S is not transitive.

(e) $x^2 + y^2 < 0$

Proof: Write $S = \{(x, y) : x^2 + y^2 < 1\} = \phi$, then S automatically satisfies (i) reflexive, (ii) symmetric, and (iii) transitive.

(f) $x^2 + x = y^2 + y$

Proof: Write $S = \{(x, y) : x^2 + x = y^2 + y\} = \{(x, y) : (x - y)(x + y - 1) = 0\}$, then we check that (i) reflexive, (ii) symmetric, and (iii) transitive as follows. It is clear that $D(S) = R$.

(i) If $x \in R$, it is clear that $(x, x) \in S$. So, S is reflexive.

(ii) If $(x, y) \in S$, it is clear that $(y, x) \in S$. So, S is symmetric.

(iii) If $(x, y) \in S$ and $(y, z) \in S$, it is clear that $(x, z) \in S$. So, S is transitive.

2.3 The following functions F and G are defined for all real x by the equations given. In each case where the composite function $G \circ F$ can be formed, give the domain of $G \circ F$ and a formula (or formulas) for $(G \circ F)(x)$.

(a) $F(x) = 1 - x$, $G(x) = x^2 + 2x$

Proof: Write

$$G \circ F(x) = G[F(x)] = G[1 - x] = (1 - x)^2 + 2(1 - x) = x^2 - 4x + 3.$$

It is clear that the domain of $G \circ F(x)$ is R.

(b) $F(x) = x + 5$, $G(x) = |x|/x$ if $x \neq 0$, $G(0) = 0$.

Proof: Write

$$G \circ F(x) = G[F(x)] = \begin{cases} G(x+5) = \frac{|x+5|}{x+5} \text{ if } x \neq -5. \\ 0 \text{ if } x = -5. \end{cases}$$

It is clear that the domain of $G \circ F(x)$ is R.

(c) $F(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{otherwise,} \end{cases}$ $G(x) = \begin{cases} x^2, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$

Proof: Write

$$G \circ F(x) = G[F(x)] = \begin{cases} 4x^2 \text{ if } x \in [0, 1/2] \\ 0 \text{ if } x \in (1/2, 1] \\ 1 \text{ if } x \in R - [0, 1] \end{cases}.$$

It is clear that the domain of $G \circ F(x)$ is R.

Find $F(x)$ if $G(x)$ and $G[F(x)]$ are given as follows:

(d) $G(x) = x^3$, $G[F(x)] = x^3 - 3x^2 + 3x - 1$.

Proof: With help of $(x - 1)^3 = x^3 - 3x^2 + 3x - 1$, it is easy to know that $F(x) = 1 - x$. In addition, there is not other function $H(x)$ such that $G[H(x)] = x^3 - 3x^2 + 3x - 1$ since $G(x) = x^3$ is 1-1.

(e) $G(x) = 3 + x + x^2$, $G[F(x)] = x^2 - 3x + 5$.

Proof: Write $G(x) = \left(x + \frac{1}{2}\right)^2 + \frac{11}{4}$, then

$$G[F(x)] = \left(F(x) + \frac{1}{2}\right)^2 + \frac{11}{4} = x^2 - 3x + 5$$

which implies that

$$(2F(x) + 1)^2 = (2x - 3)^2$$

which implies that

$$F(x) = x - 2 \text{ or } -x + 1.$$

2.4 Given three functions F, G, H, what restrictions must be placed on their domains so that the following four composite functions can be defined?

$$G \circ F,\ H \circ G,\ H \circ (G \circ F),\ (H \circ G) \circ F.$$

Proof: It is clear for answers,

$$R(F) \subseteq D(G) \text{ and } R(G) \subseteq D(H).$$

Assuming that $H \circ (G \circ F)$ and $(H \circ G) \circ F$ can be defined, prove that associative law:

$$H \circ (G \circ F) = (H \circ G) \circ F.$$

Proof: Given any $x \in D(F)$, then

$$\begin{aligned}((H \circ G) \circ F)(x) &= (H \circ G)(F(x)) \\ &= H(G(F(x))) \\ &= H((G \circ F)(x)) \\ &= (H \circ (G \circ F))(x).\end{aligned}$$

So, $H \circ (G \circ F) = (H \circ G) \circ F$.

2.5 Prove the following set-theoretic identities for union and intersection:

(a) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof: For the part, $A \cup (B \cup C) = (A \cup B) \cup C$: Given $x \in A \cup (B \cup C)$, we have $x \in A$ or $x \in B \cup C$. That is, $x \in A$ or $x \in B$ or $x \in C$. Hence, $x \in A \cup B$ or $x \in C$. It implies $x \in (A \cup B) \cup C$. Similarly, if $x \in (A \cup B) \cup C$, then $x \in A \cup (B \cup C)$. Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$.

For the part, $A \cap (B \cap C) = (A \cap B) \cap C$: Given $x \in A \cap (B \cap C)$, we have $x \in A$ and $x \in B \cap C$. That is, $x \in A$ and $x \in B$ and $x \in C$. Hence, $x \in A \cap B$ and $x \in C$. It implies $x \in (A \cap B) \cap C$. Similarly, if $x \in (A \cap B) \cap C$, then $x \in A \cap (B \cap C)$. Therefore, $A \cap (B \cap C) = (A \cap B) \cap C$.

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: Given $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. We consider two cases as follows.

If $x \in B$, then $x \in A \cap B$. So, $x \in (A \cap B) \cup (A \cap C)$.

If $x \in C$, then $x \in A \cap C$. So, $x \in (A \cap B) \cup (A \cap C)$.

So, we have shown that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C). \tag{*}$$

Conversely, given $x \in (A \cap B) \cup (A \cap C)$, then $x \in A \cap B$ or $x \in A \cap C$. We consider two cases as follows.

If $x \in A \cap B$, then $x \in A \cap (B \cup C)$.

If $x \in A \cap C$, then $x \in A \cap (B \cup C)$.

So, we have shown that

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C). \tag{**}$$

By (*) and (**), we have proved it.

(c) $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$

Proof: Given $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$. We consider two cases as follows.

If $x \in A$, then $x \in A \cup (B \cap C)$.

If $x \notin A$, then $x \in B$ and $x \in C$. So, $x \in B \cap C$. It implies that $x \in A \cup (B \cap C)$.

Therefore, we have shown that

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C). \tag{*}$$

Conversely, if $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. We consider two cases as follows.

If $x \in A$, then $x \in (A \cup B) \cap (A \cup C)$.

If $x \in B \cap C$, then $x \in A \cup B$ and $x \in A \cup C$. So, $x \in (A \cup B) \cap (A \cup C)$.

Therefore, we have shown that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C). \qquad (*)$$

By (*) and (**), we have proved it.

(d) $(A \cup B) \cap (B \cup C) \cap (C \cup A) = (A \cap B) \cup (A \cap C) \cup (B \cap C)$

Proof: Given $x \in (A \cup B) \cap (B \cup C) \cap (C \cup A)$, then

$$x \in A \cup B \text{ and } x \in B \cup C \text{ and } x \in C \cup A. \qquad (*)$$

We consider the cases to show $x \in (A \cap B) \cup (A \cap C) \cup (B \cap C)$ as follows.

For the case ($x \in A$):

If $x \in B$, then $x \in A \cap B$.

If $x \notin B$, then by (*), $x \in C$. So, $x \in A \cap C$.

Hence, in this case, we have proved that $x \in (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

For the case ($x \notin A$):

If $x \in B$, then by (*), $x \in C$. So, $x \in B \cap C$.

If $x \notin B$, then by (*), it is impossible.

Hence, in this case, we have proved that $x \in (A \cap B) \cup (A \cap C) \cup (B \cap C)$.

From above,

$$(A \cup B) \cap (B \cup C) \cap (C \cup A) \subseteq (A \cap B) \cup (A \cap C) \cup (B \cap C)$$

Similarly, we also have

$$(A \cap B) \cup (A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap (B \cup C) \cap (C \cup A).$$

So, we have proved it.

Remark: There is another proof, we write it as a reference.

Proof: Consider

$$\begin{aligned}
&(A \cup B) \cap (B \cup C) \cap (C \cup A) \\
&= [(A \cup B) \cap (B \cup C)] \cap (C \cup A) \\
&= [B \cup (A \cap C)] \cap (C \cup A) \\
&= [B \cap (C \cup A)] \cup [(A \cap C) \cap (C \cup A)] \\
&= [(B \cap C) \cup (B \cap A)] \cup (A \cap C) \\
&= (A \cap B) \cup (A \cap C) \cup (B \cap C).
\end{aligned}$$

(e) $A \cap (B - C) = (A \cap B) - (A \cap C)$

Proof: Given $x \in A \cap (B - C)$, then $x \in A$ and $x \in B - C$. So, $x \in A$ and $x \in B$ and $x \notin C$. So, $x \in A \cap B$ and $x \notin C$. Hence,

$$x \in (A \cap B) - C \subseteq (A \cap B) - (A \cap C). \tag{*}$$

Conversely, given $x \in (A \cap B) - (A \cap C)$, then $x \in A \cap B$ and $x \notin A \cap C$. So, $x \in A$ and $x \in B$ and $x \notin C$. So, $x \in A$ and $x \in B - C$. Hence,

$$x \in A \cap (B - C) \tag{**}$$

By (*) and (**), we have proved it.

(f) $(A - C) \cap (B - C) = (A \cap B) - C$

Proof: Given $x \in (A - C) \cap (B - C)$, then $x \in A - C$ and $x \in B - C$. So, $x \in A$ and $x \in B$ and $x \notin C$. So, $x \in (A \cap B) - C$. Hence,

$$(A - C) \cap (B - C) \subseteq (A \cap B) - C. \tag{*}$$

Conversely, given $x \in (A \cap B) - C$, then $x \in A$ and $x \in B$ and $x \notin C$. Hence, $x \in A - C$ and $x \in B - C$. Hence,

$$(A \cap B) - C \subseteq (A - C) \cap (B - C). \tag{**}$$

By (*) and (**), we have proved it.

(g) $(A - B) \cup B = A$ if, and only if, $B \subseteq A$

Proof: ($\Rightarrow$) Suppose that $(A - B) \cup B = A$, then it is clear that $B \subseteq A$.
($\Leftarrow$) Suppose that $B \subseteq A$, then given $x \in A$, we consider two cases.
If $x \in B$, then $x \in (A - B) \cup B$.
If $x \notin B$, then $x \in A - B$. Hence, $x \in (A - B) \cup B$.
From above, we have

$$A \subseteq (A - B) \cup B.$$

In addition, it is obviously $(A - B) \cup B \subseteq A$ since $A - B \subseteq A$ and $B \subseteq A$.

2.6 Let $f : S \to T$ be a function. If A and B are arbitrary subsets of S, prove that

$$f(A \cup B) = f(A) \cup f(B) \text{ and } f(A \cap B) \subseteq f(A) \cap f(B).$$

Generalize to arbitrary unions and intersections.

Proof: First, we prove $f(A \cup B) = f(A) \cup f(B)$ as follows. Let $y \in f(A \cup B)$, then $y = f(a)$ or $y = f(b)$, where $a \in A$ and $b \in B$. Hence, $y \in f(A) \cup f(B)$. That is,

$$f(A \cup B) \subseteq f(A) \cup f(B).$$

Conversely, if $y \in f(A) \cup f(B)$, then $y = f(a)$ or $y = f(b)$, where $a \in A$ and $b \in B$. Hence, $y \in f(A \cup B)$. That is,

$$f(A) \cup f(B) \subseteq f(A \cup B).$$

So, we have proved that $f(A \cup B) = f(A) \cup f(B)$.

For the part $f(A \cap B) \subseteq f(A) \cap f(B)$: Let $y \in f(A \cap B)$, then $y = f(x)$, where $x \in A \cap B$. Hence, $y \in f(A)$ and $y \in f(B)$. That is, $f(A \cap B) \subseteq f(A) \cap f(B)$.

For arbitrary unions and intersections, we have the following facts, and the proof is easy from above. So, we omit the detail.

$$f(\cup_{i \in I} A_i) = \cup_{i \in I} f(A_i), \text{ where } I \text{ is an index set.}$$

And

$$f(\cap_{i \in I} A_i) \subseteq \cap_{i \in I} f(A_i), \text{ where } I \text{ is an index set.}$$

Remark: We should note why the equality does **NOT** hold for the case of intersection. for example, consider $A = \{1, 2\}$ and $B = \{1, 3\}$, where $f(1) = 1$ and $f(2) = 2$ and $f(3) = 2$.

$$f(A \cap B) = f(\{1\}) = \{1\} \subseteq \{1, 2\} \subseteq f(\{1, 2\}) \cap f(\{1, 3\}) = f(A) \cap f(B).$$

2.7 Let $f : S \to T$ be a function. If $Y \subseteq T$, we denote by $f^{-1}(Y)$ the largest subset of S which f maps into Y. That is,

$$f^{-1}(Y) = \{x : x \in S \text{ and } f(x) \in Y\}.$$

The set $f^{-1}(Y)$ is called the inverse image of Y under f. Prove that the following for arbitrary subsets X of S and Y of T.

(a) $X \subseteq f^{-1}[f(X)]$

Proof: Given $x \in X$, then $f(x) \in f(X)$. Hence, $x \in f^{-1}[f(X)]$ by definition of the inverse image of $f(X)$ under f. So, $X \subseteq f^{-1}[f(X)]$.

Remark: The equality may not hold, for example, let $f(x) = x^2$ on R, and let $X = [0, \infty)$, we have

$$f^{-1}[f(X)] = f^{-1}[[0, \infty)] = R.$$

(b) $f(f^{-1}(Y)) \subseteq Y$

Proof: Given $y \in f(f^{-1}(Y))$, then there exists a point $x \in f^{-1}(Y)$ such that $f(x) = y$. Since $x \in f^{-1}(Y)$, we know that $f(x) \in Y$. Hence, $y \in Y$. So, $f(f^{-1}(Y)) \subseteq Y$

Remark: The equality may not hold, for example, let $f(x) = x^2$ on R, and let $Y = R$, we have

$$f\left(f^{-1}(Y)\right) = f(R) = [0, \infty) \subseteq R.$$

(c) $f^{-1}[Y_1 \cup Y_2] = f^{-1}(Y_1) \cup f^{-1}(Y_2)$

Proof: Given $x \in f^{-1}[Y_1 \cup Y_2]$, then $f(x) \in Y_1 \cup Y_2$. We consider two cases as follows.

If $f(x) \in Y_1$, then $x \in f^{-1}(Y_1)$. So, $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

If $f(x) \notin Y_1$, i.e., $f(x) \in Y_2$, then $x \in f^{-1}(Y_2)$. So, $x \in f^{-1}(Y_1) \cup f^{-1}(Y_2)$.

From above, we have proved that

$$f^{-1}[Y_1 \cup Y_2] \subseteq f^{-1}(Y_1) \cup f^{-1}(Y_2). \qquad (*)$$

Conversely, since $f^{-1}(Y_1) \subseteq f^{-1}[Y_1 \cup Y_2]$ and $f^{-1}(Y_2) \subseteq f^{-1}[Y_1 \cup Y_2]$, we have

$$f^{-1}(Y_1) \cup f^{-1}(Y_2) \subseteq f^{-1}[Y_1 \cup Y_2]. \qquad (**)$$

From (*) and (**), we have proved it.

(d) $f^{-1}[Y_1 \cap Y_2] = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

Proof: Given $x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$, then $f(x) \in Y_1$ and $f(x) \in Y_2$. So, $f(x) \in Y_1 \cap Y_2$. Hence, $x \in f^{-1}[Y_1 \cap Y_2]$. That is, we have proved that

$$f^{-1}(Y_1) \cap f^{-1}(Y_2) \subseteq f^{-1}[Y_1 \cap Y_2]. \qquad (*)$$

Conversely, since $f^{-1}[Y_1 \cap Y_2] \subseteq f^{-1}(Y_1)$ and $f^{-1}[Y_1 \cap Y_2] \subseteq f^{-1}(Y_2)$, we have

$$f^{-1}[Y_1 \cap Y_2] \subseteq f^{-1}(Y_1) \cap f^{-1}(Y_2). \qquad (**)$$

From (*) and (**), we have proved it.

(e) $f^{-1}(T - Y) = S - f^{-1}(Y)$

Proof: Given $x \in f^{-1}(T - Y)$, then $f(x) \in T - Y$. So, $f(x) \notin Y$. We want to show that $x \in S - f^{-1}(Y)$. Suppose **NOT**, then $x \in f^{-1}(Y)$ which implies that $f(x) \in Y$. That is impossible. Hence, $x \in S - f^{-1}(Y)$. So, we have

$$f^{-1}(T - Y) \subseteq S - f^{-1}(Y). \qquad (*)$$

Conversely, given $x \in S - f^{-1}(Y)$, then $x \notin f^{-1}(Y)$. So, $f(x) \notin Y$. That is, $f(x) \in T - Y$. Hence, $x \in f^{-1}(T - Y)$. So, we have

$$S - f^{-1}(Y) \subseteq f^{-1}(T - Y). \qquad (**)$$

From (*) and (**), we have proved it.

(f) Generalize (c) and (d) to arbitrary unions and intersections.

Proof: We give the statement without proof since it is the same as (c) and (d). In general, we have

$$f^{-1}(\cup_{i \in I} A_i) = \cup_{i \in I} f^{-1}(A_i).$$

and

$$f^{-1}(\cap_{i \in I} A_i) = \cap_{i \in I} f^{-1}(A_i).$$

Remark: From above sayings and **Exercise 2.6**, we found that the inverse image f^{-1} and the operations of sets, such as intersection and union, can be exchanged. However, for a function, we only have the exchange of f and the operation of union. The reader also see the **Exercise 2.9** to get more.

2.8 Refer to Exercise 2.7. Prove that $f[f^{-1}(Y)] = Y$ for every subset Y of T if, and only if, $T = f(S)$.

Proof: ($\Rightarrow$) It is clear that $f(S) \subseteq T$. In order to show the equality, it suffices to show that $T \subseteq f(S)$. Consider $f^{-1}(T) \subseteq S$, then we have

$$f\left(f^{-1}(T)\right) \subseteq f(S).$$

By hyppothesis, we get $T \subseteq f(S)$.

($\Leftarrow$) Suppose **NOT**, i.e., $f[f^{-1}(Y)]$ is a proper subset of Y for some $Y \subseteq T$ by **Exercise 2.7 (b)**. Hence, there is a $y \in Y$ such that $y \notin f[f^{-1}(Y)]$. Since $Y \subseteq f(S) = T$, $f(x) = y$ for some $x \in S$. It implies that $x \in f^{-1}(Y)$. So, $f(x) \in f[f^{-1}(Y)]$ which is impossible by the choice of y. Hence, $f[f^{-1}(Y)] = Y$ for every subset Y of T.

2.9 Let $f : S \to T$ be a function. Prove that the following statements are equivalent.

(a) f is one-to-one on S.

(b) $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of S.

(c) $f^{-1}[f(A)] = A$ for every subset A of S.

(d) For all disjoint subsets A and B of S, the image $f(A)$ and $f(B)$ are disjoint.

(e) For all subsets A and B of S with $B \subseteq A$, we have

$$f(A - B) = f(A) - f(B).$$

Proof: $(a) \Rightarrow (b)$: Suppose that f is 1-1 on S. By **Exercise 2.6**, we have proved that $f(A \cap B) \subseteq f(A) \cap f(B)$ for all A, B of S. In order to show the equality, it suffices to show that $f(A) \cap f(B) \subseteq f(A \cap B)$.

Given $y \in f(A) \cap f(B)$, then $y = f(a)$ and $y = f(b)$ where $a \in A$ and $b \in B$. Since f is 1-1, we have $a = b$. That is, $y \in f(A \cap B)$. So, $f(A) \cap f(B) \subseteq f(A \cap B)$.

$(b) \Rightarrow (c)$: Suppose that $f(A \cap B) = f(A) \cap f(B)$ for all subsets A, B of S. If $A \neq f^{-1}[f(A)]$ for some A of S, then by **Exercise 2.7 (a),** there is an element $a \notin A$ and $a \in f^{-1}[f(A)]$. Consider

$$\phi = f(A \cap \{a\}) = f(A) \cap f(\{a\}) \text{ by (b)} \qquad (*)$$

Since $a \in f^{-1}[f(A)]$, we have $f(a) \in f(A)$ which contradicts to (*). Hence, no such a exists. That is, $f^{-1}[f(A)] = A$ for every subset A of S.

$(c) \Rightarrow (d)$: Suppose that $f^{-1}[f(A)] = A$ for every subset A of S. If $A \cap B = \phi$, then Consider

$$\begin{aligned} \phi &= A \cap B \\ &= f^{-1}[f(A)] \cap f^{-1}[f(B)] \\ &= f^{-1}(f(A) \cap f(B)) \text{ by } \textbf{Exercise 2.7 (d)} \end{aligned}$$

which implies that $f(A) \cap f(B) = \phi$.

$(d) \Rightarrow (e)$: Suppose that for all disjoint subsets A and B of S, the image $f(A)$ and $f(B)$ are disjoint. If $B \subseteq A$, then since $(A - B) \cap B = \phi$, we have

$$f(A - B) \cap f(B) = \phi$$

which implies that

$$f(A - B) \subseteq f(A) - f(B). \qquad (**)$$

Conversely, we consider if $y \in f(A) - f(B)$, then $y = f(x)$, where $x \in A$ and $x \notin B$. It implies that $x \in A - B$. So, $y = f(x) \in f(A - B)$. That is,

$$f(A) - f(B) \subseteq f(A - B). \qquad (***)$$

By (**) and (***), we have proved it.

$(d) \Rightarrow (a)$: Suppose that $f(A - B) = f(A) - f(B)$ for all subsets A and B of S with $B \subseteq A$. If $f(a) = f(b)$, consider $A = \{a, b\}$ and $B = \{b\}$, we have

$$f(A - B) = \phi$$

which implies that $A = B$. That is, $a = b$. So, f is 1-1.

2.10 Prove that if $A\tilde{}B$ and $B\tilde{}C$, then $A\tilde{}C$.

Proof: Since $A\tilde{}B$ and $B\tilde{}C$, then there exists bijection f and g such that

$$f : A \to B \text{ and } g : B \to C.$$

So, if we consider $g \circ f : A \to C$, then $A\tilde{}C$ since $g \circ f$ is bijection.

2.11 If $\{1, 2, ..., n\} \tilde{} \{1, 2, ..., m\}$, prove that $m = n$.

Proof: Since $\{1, 2, ..., n\} \tilde{} \{1, 2, ..., m\}$, there exists a bijection function

$$f : \{1, 2, ..., n\} \to \{1, 2, ..., m\}.$$

Since f is 1-1, then $n \leq m$. Conversely, consider f^{-1} is 1-1 to get $m \leq n$. So, $m = n$.

2.12 If S is an infinite set, prove that S contains a countably infinite subset. Hint. Choose an element a_1 in S and consider $S - \{a_1\}$.

Proof: Since S is an infinite set, then choose a_1 in S and thus $S - \{a_1\}$ is still infinite. From this, we have $S - \{a_1, .., a_n\}$ is infinite. So, we finally have

$$\{a_1, ..., a_n, ...\} \, (\subseteq S)$$

which is countably infinite subset.

2.13 Prove that every infinite set S contains a proper subset similar to S.

Proof: By Exercise 2.12, we write $S = \tilde{S} \cup \{x_1, ..., x_n, ...\}$, where $\tilde{S} \cap \{x_1, ..., x_n, ...\} = \phi$ and try to show

$$\tilde{S} \cup \{x_2, ..., x_n, ...\} \,\tilde{}\, S$$

as follows. Define

$$f : \tilde{S} \cup \{x_2, ..., x_n, ...\} \to S = \tilde{S} \cup \{x_1, ..., x_n, ...\}$$

by

$$f(x) = \begin{cases} x \text{ if } x \in \tilde{S} \\ x_i \text{ if } x = x_{i+1} \end{cases}.$$

Then it is clear that f is 1-1 and onto. So, we have proved that every infinite set S contains a proper subset similar to S.

Remark: In the proof, we may choose the map

$$f : \tilde{S} \cup \{x_{N+1}, ..., x_n, ...\} \to S = \tilde{S} \cup \{x_1, ..., x_n, ...\}$$

by

$$f(x) = \begin{cases} x \text{ if } x \in \tilde{S} \\ x_i \text{ if } x = x_{i+N} \end{cases}.$$

2.14 If A is a countable set and B an uncountable set, prove that $B - A$ is similar to B.

Proof: In order to show it, we consider some cases as follows. (i) $B \cap A = \phi$ (ii) $B \cap A$ is a finite set, and (iii) $B \cap A$ is an infinite set.

For case (i), $B - A = B$. So, $B - A$ is similar to B.

For case (ii), it follows from the **Remark in Exercise 2.13**.

For case (iii), note that $B \cap A$ is countable, and let $C = B - A$, we have $B = C \cup (B \cap A)$. We want to show that

$$(B - A) \,\tilde{}\, B \Leftrightarrow C \,\tilde{}\, C \cup (B \cap A).$$

By **Exercise 2.12**, we write $C = \tilde{C} \cup D$, where D is countably infinite and $\tilde{C} \cap D = \phi$. Hence,

$$\begin{aligned} C\tilde{\ }C \cup (B \cap A) &\Leftrightarrow \left(\tilde{C} \cup D\right) \tilde{\ } \left[\tilde{C} \cup (D \cup (B \cap A))\right] \\ &\Leftrightarrow \left(\tilde{C} \cup D\right) \tilde{\ } \left(\tilde{C} \cup D'\right) \end{aligned}$$

where $D' = D \cup (B \cap A)$ which is countably infinite. Since $\left(\tilde{C} \cup D\right) \tilde{\ } \left(\tilde{C} \cup D'\right)$ is clear, we have proved it.

2.15 A real number is called **algebraic** if it is a root of an algebraic equation $f(x) = 0$, where $f(x) = a_0 + a_1 x + ... + a_n x^n$ is a polynomial with integer coefficients. Prove that the set of all polynomials with integer coefficients is countable and deduce that the set of algebraic numbers is also countable.

Proof: Given a positive integer $N\,(\geq 2)$, there are only finitely many eqautions with

$$n + \sum_{k=1}^{n} |a_k| = N, \text{ where } a_k \in Z. \tag{*}$$

Let $S_N = \{f : f(x) = a_0 + a_1 x + ... + a_n x^n \text{ satisfies } (*)\}$, then S_N is a finite set. Hence, $\cup_{n=2}^{\infty} S_n$ is countable. Clearly, the set of all polynomials with integer coefficients is a subset of $\cup_{n=2}^{\infty} S_n$. So, the set of all polynomials with integer coefficients is countable. In addition, a polynomial of degree k has at most k roots. Hence, the set of algebraic numbers is also countable.

2.16 Let S be a finite set consisting of n elements and let T be the collection of all subsets of S. Show that T is a finite set and find the number of elements in T.

Proof: Write $S = \{x_1, ..., x_n\}$, then T =the collection of all subsets of S. Clearly, T is a finite set with 2^n elements.

2.17 Let R denote the set of real numbers and let S denote the set of all real-valued functions whose domain in R. Show that S and R are not **equinumrous**. Hint. Assume $S\tilde{\ }R$ and let f be a one-to-one function such that $f(R) = S$. If $a \in R$, let $g_a = f(a)$ be the real-valued function in S which correspouds to real number a. Now define h by the equation $h(x) = 1 + g_x(x)$ if $x \in R$, and show that $h \notin S$.

Proof: Assume $S\tilde{}R$ and let f be a one-to-one function such that $f(R) = S$. If $a \in R$, let $g_a = f(a)$ be the real-valued function in S which correspouds to real number a. Define h by the equation $h(x) = 1 + g_x(x)$ if $x \in R$, then

$$h = f(b) = g_b$$

which implies that

$$h(b) := 1 + g_b(b) = g_b(b)$$

which is impossible. So, S and R are not **equinumrous.**

Remark: There is a similar exercise, we write it as a reference. The cardinal number of $C[a,b]$ is $2^{\aleph_0}$, where $\aleph_0 = \#(N)$.

Proof: First, $\#(R) = 2^{\aleph_0} \leq \#(C[a,b])$ by considering constant function. Second, we consider the map

$$f : C[a,b] \to P(Q \times Q), \text{ the power set of } Q \times Q$$

by

$$f(\varphi) = \{(x,y) \in Q \times Q : x \in [a,b] \text{ and } y \leq \varphi(x)\}.$$

Clearly, f is 1-1. It implies that $\#(C[a,b]) \leq \#(P(Q \times Q)) = 2^{\aleph_0}$.

So, we have proved that $\#(C[a,b]) = 2^{\aleph_0}$.

Note: For notations, the reader can see the textbook, in **Chapter 4, pp 102.** Also, see the book, **Set Theory and Related Topics by Seymour Lipschutz, Chapter 9, pp 157-174. (Chinese Version)**

2.18 Let S be the collection of all sequences whose terms are the integers 0 and 1. Show that S is uncountable.

Proof: Let E be a countable subet of S, and let E consists of the sequences $s_1, .., s_n, ...$ We construct a sequence s as follows. If the nth digit in s_n is 1, we let the nth digit of s be 0, and vice versa. Then the sequence s differes from every member of E in at least one place; hence $s \notin E$. But clearly $s \in S$, so that E is a proper subset of S.

We have shown that every countable subset of S is a proper subset of S. It follows that S is uncountable (for otherwise S would be a proper subset of S, which is absurb).

Remark: In this exercise, we have proved that R, the set of real numbers, is uncountable. It can be regarded as the **Exercise 1.22** for $k = 2$. (**Binary System**).

2.19 Show that the following sets are countable:

(a) the set of circles in the complex plane having the ratiional radii and centers with rational coordinates.

Proof: Write the set of circles in the complex plane having the ratiional radii and centers with rational coordinates, $\{C(x_n; q_n) : x_n \in Q \times Q$ and $q_n \in Q\} := S$. Clearly, S is countable.

(b) any collection of disjoint intervals of positive length.

Proof: Write the collection of disjoint intervals of positive length, $\{I : I$ is an interval of positiv
S. Use the reason in **Exercise 2.21**, we have proved that S is countable.

2.20 Let f be a real-valued function defined for every x in the interval $0 \leq x \leq 1$. Suppose there is a positive number M having the following property: for every choice of a finite number of points $x_1, x_2, ..., x_n$ in the interval $0 \leq x \leq 1$, the sum

$$|f(x_1) + ... + f(x_n)| \leq M.$$

Let S be the set of those x in $0 \leq x \leq 1$ for which $f(x) \neq 0$. Prove that S is countable.

Proof: Let $S_n = \{x \in [0,1] : |f(x)| \geq 1/n\}$, then S_n is a finite set by hypothesis. In addition, $S = \cup_{n=1}^{\infty} S_n$. So, S is countable.

2.21 Find the fallacy in the following "proof" that the set of all intervals of positive length is countable.

Let $\{x_1, x_2, ...\}$ denote the countable set of rational numbers and let I be any interval of positive length. Then I contains infinitely many rational points x_n, but among these there will be one with **smallest index** n. Define a function F by means of the eqaution $F(I) = n$ if x_n is the rational number with smallest index in the interval I. This function establishes a one-to-one correspondence between the set of all intervals and a subset of the positive integers. Hence, the set of all intervals is countable.

Proof: Note that F is not a one-to-one correspondence between the set of all intervals and a subset of the positive integers. So, this is not a proof. In fact, the set of all intervals of positive length is uncountable.

Remark: Compare with **Exercise 2.19,** and the set of all intervals of positive length is uncountable is clear by considering $\{(0, x) : 0 < x < 1\}$.

2.22 Let S denote the collection of all subsets of a given set T. Let $f : S \to R$ be a real-valued function defined on S. The function f is called **additive** if $f(A \cup B) = f(A) + f(B)$ whenever A and B are disjoint subsets of T. If f is additive, prove that for any two subsets A and B we have

$$f(A \cup B) = f(A) + f(B - A)$$

and

$$f(A \cup B) = f(A) + f(B) - f(A \cap B).$$

Proof: Since $A \cap (B - A) = \phi$ and $A \cup B = A \cup (B - A)$, we have

$$f(A \cup B) = f(A \cup (B - A)) = f(A) + f(B - A). \qquad (*)$$

In addition, since$(B - A) \cap (A \cap B) = \phi$ and $B = (B - A) \cup (A \cap B)$, we have

$$f(B) = f((B - A) \cup (A \cap B)) = f(B - A) + f(A \cap B)$$

which implies that

$$f(B - A) = f(B) - f(A \cap B) \qquad (**)$$

By (*) and (**), we have proved that

$$f(A \cup B) = f(A) + f(B) - f(A \cap B).$$

2.23 Refer to Exercise 2.22. Assume f is additive and assume also that the following relations hold for two particular subsets A and B of T :

$$f(A \cup B) = f(A') + f(B') - f(A') f(B')$$

and

$$f(A \cap B) = f(A) f(B)$$

and

$$f(A) + f(B) \neq f(T),$$

where $A' = T - A$, $B' = T - B$. Prove that these relations determine $f(T)$, and compute the value of $f(T)$.

Proof: Write

$$f(T) = f(A) + f(A') = f(B) + f(B'),$$

then

$$\begin{aligned}
[f(T)]^2 &= [f(A) + f(A')][f(B) + f(B')] \\
&= f(A) f(B) + f(A) f(B') + f(A') f(B) + f(A') f(B') \\
&= f(A) f(B) + f(A)[f(T) - f(B)] + [f(T) - f(A)] f(B) + f(A') f(B') \\
&= [f(A) + f(B)] f(T) - f(A) f(B) + f(A') f(B') \\
&= [f(A) + f(B)] f(T) - f(A) f(B) + f(A') + f(B') - f(A \cup B) \\
&= [f(A) + f(B)] f(T) - f(A) f(B) + [f(T) - f(A)] + [f(T) - f(B)] \\
&\quad - [f(A) + f(B) - f(A \cap B)] \\
&= [f(A) + f(B) + 2] f(T) - f(A) f(B) - 2[f(A) + f(B)] + f(A \cap B) \\
&= [f(A) + f(B) + 2] f(T) - 2[f(A) + f(B)]
\end{aligned}$$

which implies that

$$[f(T)]^2 - [f(A) + f(B) + 2] f(T) + 2[f(A) + f(B)] = 0$$

which implies that

$$x^2 - (a+2)x + 2a = 0 \Rightarrow (x-a)(x-2) = 0$$

where $a = f(A) + f(B)$. So, $x = 2$ since $x \neq a$ by $f(A) + f(B) \neq f(T)$.

Charpter 3 Elements of Point set Topology

Open and closed sets in R^1 and R^2

3.1 *Prove that an open interval in R^1 is an open set and that a closed interval is a closed set.*

proof: 1. Let (a,b) be an open interval in R^1, and let $x \in (a,b)$. Consider $\min(x-a, b-x) := L$. Then we have $B(x,L) = (x-L, x+L) \subseteq (a,b)$. That is, x is an interior point of (a,b). Since x is arbitrary, we have every point of (a,b) is interior. So, (a,b) is open in R^1.

2. Let $[a,b]$ be a closed interval in R^1, and let x be an adherent point of $[a,b]$. We want to show $x \in [a,b]$. If $x \notin [a,b]$, then we have $x < a$ or $x > b$. Consider $x < a$, then

$$B(x, \frac{a-x}{2}) \cap [a,b] = (\frac{3x-a}{2}, \frac{x+a}{2}) \cap [a,b] = \phi$$

which contradicts the definition of an adherent point. Similarly for $x > b$.

Therefore, we have $x \in [a,b]$ if x is an adherent point of $[a,b]$. That is, $[a,b]$ contains its all adherent points. It implies that $[a,b]$ is closed in R^1.

3.2 *Determine all the accumulation points of the following sets in R^1 and decide whether the sets are open or closed (or neither).*

(a) All integers.

Solution: Denote the set of all integers by Z. Let $x \in Z$, and consider $(B(x, \frac{x+1}{2}) - \{x\}) \cap S = \phi$. So, Z has no accumulation points.

However, $B(x, \frac{x+1}{2}) \cap S = \{x\} \neq \phi$. So Z contains its all adherent points. It means that Z is closed. Trivially, Z is not open since $B(x,r)$ is not contained in Z for all $r > 0$.

Remark: 1. Definition of an adherent point: Let S be a subset of R^n, and x a point in R^n, x is not necessarily in S. Then x is said to be adherent to S if every n –ball $B(x)$ contains at least one point of S. To be roughly, $B(x) \cap S \neq \phi$.

2. Definition of an accumulation point: Let S be a subset of R^n, and x a point in R^n, then x is called an accumulation point of S if every n –ball $B(x)$ contains at least one point of S distinct from x. To be roughly, $(B(x) - \{x\}) \cap S \neq \phi$. That is, x is an accumulation point if, and only if, x adheres to $S - \{x\}$. Note that in this sense, $(B(x) - \{x\}) \cap S = B(x) \cap (S - \{x\})$.

3. Definition of an isolated point: If $x \in S$, but x is not an accumulation point of S, then x is called an isolated point.

4. Another solution for Z is closed: Since $R - Z = \cup_{n \in Z} (n, n+1)$, we know that $R - Z$ is open. So, Z is closed.

5. In logics, if there does not exist any accumulation point of a set S, then S is automatically a closed set.

(b) *The interval $(a,b]$.*

solution: In order to find all accumulation points of $(a,b]$, we consider 2 cases as follows.

1. $(a,b]$: Let $x \in (a,b]$, then $(B(x,r) - \{x\}) \cap (a,b] \neq \phi$ for any $r > 0$. So, every point of $(a,b]$ is an accumulation point.

2. $R^1 - (a,b] = (-\infty, a] \cup (b, \infty)$: For points in (b,∞) and $(-\infty, a)$, it is easy to know that these points cannot be accumulation points since $x \in (b,\infty)$ or $x \in (-\infty, a)$, there

exists an n –ball $B(x, r_x)$ such that $(B(x, r_x) - \{x\}) \cap (a, b] = \phi$. For the point a, it is easy to know that $(B(a, r) - \{a\}) \cap (a, b] \neq \phi$. That is, in this case, there is only one accumulation point a of $(a, b]$.

So, from 1 and 2, we know that the set of the accumulation points of $(a, b]$ is $[a, b]$.

Since $a \notin (a, b]$, we know that $(a, b]$ cannot contain its all accumulation points. So, $(a, b]$ is not closed.

Since an n –ball $B(b, r)$ is not contained in $(a, b]$ for any $r > 0$, we know that the point b is not interior to $(a, b]$. So, $(a, b]$ is not open.

(c) *All numbers of the form* $1/n$, *(*$n = 1, 2, 3, \ldots$*)*.

Solution: Write the set $\{1/n : n = 1, 2, \ldots\} = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\} := S$. Obviously, 0 is the only one accumulation point of S. So, S is not closed since S does not contain the accumulation point 0. Since $1 \in S$, and $B(1, r)$ is not contained in S for any $r > 0$, S is not open.

Remark: Every point of $\{1/n : n = 1, 2, 3, \ldots\}$ is isolated.

(d) *All rational numbers.*

Solutions: Denote all rational numbers by Q. It is trivially seen that the set of accumulation points is R^1.

So, Q is not closed. Consider $x \in Q$, any n –ball $B(x)$ is not contained in Q. That is, x is not an interior point of Q. (In fact, every point of Q is not an interior point of Q.) So, Q is not open.

(e) *All numbers of the form* $2^{-n} + 5^{-m}$, *(*$m, n = 1, 2, \ldots$*)*.

Solution: Write the set

$$\{2^{-n} + 5^{-m} : m, n = 1, 2, ..\} = \cup_{m=1}^{m=\infty} \{\frac{1}{2} + 5^{-m}, \frac{1}{4} + 5^{-m}, \ldots, \frac{1}{2^n} + 5^{-m}, \ldots\} := S$$

$$= \{\frac{1}{2} + \frac{1}{5}, \frac{1}{2} + \frac{1}{5^2}, \ldots, \frac{1}{2} + \frac{1}{5^m}, \ldots\} \cup \qquad 1$$

$$\{\frac{1}{4} + \frac{1}{5}, \frac{1}{4} + \frac{1}{5^2}, \ldots, \frac{1}{4} + \frac{1}{5^m}, \ldots\} \cup \qquad 2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \cup$$

$$\{\frac{1}{2^n} + \frac{1}{5}, \frac{1}{2^n} + \frac{1}{5^2}, \ldots, \frac{1}{2^n} + \frac{1}{5^m} + \ldots\} \cup \qquad 3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

So, we find that $S' = \{\frac{1}{2^n} : n = 1, 2, \ldots\} \cup \{\frac{1}{5^m} : m = 1, 2, \ldots\} \cup \{0\}$. So, S is not closed since it does not contain 0. Since $\frac{1}{2} \in S$, and $B(\frac{1}{2}, r)$ is not contained in S for any $r > 0$, S is not open.

Remark: By (1)-(3), we can regard them as three sequences

$$\left\{\frac{1}{2} + 5^{-m}\right\}_{m=1}^{m=\infty}, \left\{\frac{1}{4} + 5^{-m}\right\}_{m=1}^{m=\infty} \text{ and } \left\{\frac{1}{2^n} + 5^{-m}\right\}_{m=1}^{m=\infty}, \text{ respectively.}$$

And it means that for (1), the sequence $\{5^{-m}\}_{m=1}^{m=\infty}$ moves $\frac{1}{2}$. Similarly for others. So, it is easy to see why $\frac{1}{2}$ is an accumulation point of $\{\frac{1}{2} + 5^{-m}\}_{m=1}^{m=\infty}$. And thus get the set of all accumulation points of $\{2^{-n} + 5^{-m} : m, n = 1, 2, ..\}$.

(f) *All numbers of the form* $(-1)^n + (1/m)$, *(*$m, n = 1, 2, \ldots$*)*.

Solution: Write the set of all numbers $(-1)^n + (1/m)$, $(m, n = 1, 2, \ldots)$ as

$$\left\{1 + \frac{1}{m}\right\}_{m=1}^{m=\infty} \cup \left\{-1 + \frac{1}{m}\right\}_{m=1}^{m=\infty} := S.$$

And thus by the remark in (e), it is easy to know that $S' = \{1,-1\}$. So, S is not closed since $S' \subsetneq S$. Since $2 \in S$, and $B(2,r)$ is not contained in S for any $r > 0$, S is not open.

(g) *All numbers of the form* $(1/n) + (1/m)$, *$(m,n = 1,2,\ldots)$.*

Solution: Write the set of all numbers $(1/n) + (1/m)$, $(m,n = 1,2,\ldots)$ as

$$\{1 + 1/m\}_{m=1}^{m=\infty} \cup \{1/2 + 1/m\}_{m=1}^{m=\infty} \cup \ldots \cup \{1/n + 1/m\}_{m=1}^{m=\infty} \cup \ldots := S.$$

We find that $S' = \{1/n : n \in N\} \cup \{1/m : m \in N\} \cup \{0\} = \{1/n : n \in N\} \cup \{0\}$. So, S is not closed since $S' \subsetneq S$. Since $1 \in S$, and $B(1,r)$ is not contained in S for any $r > 0$, S is not open.

(h) *All numbers of the form* $(-1)^n/[1 + (1/n)]$, *$(n = 1,2,\ldots)$.*

Soluton: Write the set of all numbers $(-1)^n/[1 + (1/n)]$, $(n = 1,2,\ldots)$ as

$$\left\{\frac{1}{1 + \frac{1}{2k}}\right\}_{k=1}^{k=\infty} \cup \left\{\frac{-1}{1 + \frac{1}{2k-1}}\right\}_{k=1}^{k=\infty} := S.$$

We find that $S' = \{-1,1\}$. So, S is not closed since $S' \subsetneq S$. Since $\frac{-1}{2} \in S$, and $B(\frac{-1}{2},r)$ is not contained in S for any $r > 0$, S is not open.

3.3 The same as Exercise 3.2 for the following sets in R^2.

(a) *All complex z such that* $|z| > 1$.

Solution: Denote $\{z \in C : |z| > 1\}$ by S. It is easy to know that $S' = \{z \in C : |z| \geq 1\}$. So, S is not closed since $S' \subsetneq S$. Let $z \in S$, then $|z| > 1$. Consider $B(z, \frac{|z|-1}{2}) \subseteq S$, so every point of S is interior. That is, S is open.

(b) *All complex z such that* $|z| \geq 1$.

Solution: Denote $\{z \in C : |z| \geq 1\}$ by S. It is easy to know that $S' = \{z \in C : |z| \geq 1\}$. So, S is closed since $S' \subsetneq S$. Since $1 \in S$, and $B(1,r)$ is not contained in S for any $r > 0$, S is not open.

(c) *All complex numbers of the form* $(1/n) + (i/m)$, *$(m,n = 1,2,\ldots)$.*

Solution: Write the set of all complex numbers of the form $(1/n) + (i/m)$, $(m,n = 1,2,\ldots)$ as

$$\left\{1 + \frac{i}{m}\right\}_{m=1}^{m=\infty} \cup \left\{\frac{1}{2} + \frac{i}{m}\right\}_{m=1}^{m=\infty} \cup \ldots \cup \left\{\frac{1}{n} + \frac{i}{m}\right\}_{m=1}^{m=\infty} \cup \ldots := S.$$

We know that $S' = \{1/n : n = 1,2,\ldots\} \cup \{i/m : m = 1,2,\ldots\} \cup \{0\}$. So, S is not closed since $S' \subsetneq S$. Since $1 + i \in S$, and $B(1 + i,r)$ is not contained in S for any $r > 0$, S is not open.

(d) *All points (x,y) such that* $x^2 + y^2 < 1$.

Solution: Denote $\{(x,y) : x^2 + y^2 < 1\}$ by S. We know that $S' = \{(x,y) : x^2 + y^2 \leq 1\}$. So, S is not closed since $S' \subsetneq S$. Let $p = (x,y) \in S$, then $x^2 + y^2 < 1$. It is easy to find that $r > 0$ such that $B(p,r) \subseteq S$. So, S is open.

(e) *All points (x,y) such that* $x > 0$.

Solution: Write all points (x,y) such that $x > 0$ as $\{(x,y) : x > 0\} := S$. It is easy to know that $S' = \{(x,y) : x \geq 0\}$. So, S is not closed since $S' \subsetneq S$. Let $x \in S$, then it is easy to find $r_x > 0$ such that $B(x,r_x) \subseteq S$. So, S is open.

(f) *All points (x,y) such that* $x \geq 0$.

Solution: Write all points (x,y) such that $x \geq 0$ as $\{(x,y) : x \geq 0\} := S$. It is easy to know that $S' = \{(x,y) : x \geq 0\}$. So, S is closed since $S' \subseteq S$. Since $(0,0) \in S$, and $B((0,0),r)$ is not contained in S for any $r > 0$, S is not open.

3.4 *Prove that every nonempty open set S in R^1 contains both rational and irratonal numbers.*

proof: Given a nonempty open set S in R^1. Let $x \in S$, then there exists $r > 0$ such that $B(x,r) \subseteq S$ since S is open. And in R^1, the open ball $B(x,r) = (x-r, x+r)$. Since any interval contains both rational and irrational numbers, we have S contains both rational and irrational numbers.

3.5 *Prove that the only set in R^1 which are both open and closed are the empty set and R^1 itself. Is a similar statement true for R^2?*

Proof: Let S be the set in R^1, and thus consider its complement $T = R^1 - S$. Then we have both S and T are open and closed. Suppose that $S \neq R^1$ and $S \neq \phi$, we will show that it is **impossible** as follows.

Since $S \neq R^1$, and $S \neq \phi$, then $T \neq \phi$ and $T \neq R^1$. Choose $s_0 \in S$ and $t_0 \in T$, then we consider the **new** point $\frac{s_0+t_0}{2}$ which is in S or T since $R = S \cup T$. If $\frac{s_0+t_0}{2} \in S$, we say $\frac{s_0+t_0}{2} = s_1$, otherwise, we say $\frac{s_0+t_0}{2} = t_1$.

Continue these steps, we finally have **two sequences** named $\{s_n\} \subseteq S$ and $\{t_m\} \subseteq T$. In addition, the two sequences are **convergent** to the same point, say p by our construction. So, we get $p \in S$ and $p \in T$ since both S and T are closed.

However, it leads us to get a contradiction since $p \in S \cap T = \phi$. Hence $S = R^1$ or $S = \phi$.

Remark: 1. In the proof, the statement is true for R^n.

2. The construction is not strange for us since the process is called *Bolzano Process*.

3.6 *Prove that every closed set in R^1 is the intersection of a countable collection of open sets.*

proof: Given a closed set S, and consider its complement $R^1 - S$ which is open. If $R^1 - S = \phi$, there is nothing to prove. So, we can assume that $R^1 - S \neq \phi$.

Let $x \in R^1 - S$, then x is an interior point of $R^1 - S$. So, there exists an open interval (a,b) such that $x \in (a,b) \subseteq R^1 - S$. In order to show our statement, we choose a smaller interval (a_x, b_x) so that $x \in (a_x, b_x)$ and $[a_x, b_x] \subseteq (a,b) \subseteq R^1 - S$. Hence, we have

$$R^1 - S = \cup_{x \in R^1 - S} [a_x, b_x]$$

which implies that

$$\begin{aligned} S &= R^1 - \cup_{x \in R^1 - S} [a_x, b_x] \\ &= \cap_{x \in R^1 - S} (R^1 - [a_x, b_x]) \\ &= \cap_{n=1}^{n=\infty} (R^1 - [a_n, b_n]) \text{ (by \textbf{Lindelof Convering Theorem})}. \end{aligned}$$

Remark: 1. There exists another proof by **Representation Theorem for Open Sets on The Real Line.**

2. Note that it is true for that every closed set in R^1 is the intersection of a countable collection of **closed** sets.

3. The proof is suitable for R^n if the statement is that every closed set in R^n is the intersection of a countable collection of open sets. All we need is to change intervals into disks.

3.7 *Prove that a nonempty, bounded closed set S in R^1 is either a closed interval, or that S can be obtained from a closed interval by removing a countable disjoint collection of open intervals whose endpoints belong to S.*

proof: If S is an interval, then it is clear that S is a closed interval. Suppose that S is not an interval. Since $S(\neq \phi)$ is bounded and closed, both $\sup S$ and $\inf S$ are in S. So, $R^1 - S = [\inf S, \sup S] - S$. Denote $[\inf S, \sup S]$ by I. Consider $R^1 - S$ is open, then by **Representation Theorem for Open Sets on The Real Line,** we have

$$R^1 - S = \cup_{m=1}^{m=\infty} I_m$$
$$= I - S$$

which implies that

$$S = I - \cup_{m=1}^{m=\infty} I_m.$$

That is, S can be obtained from a closed interval by removing a countable disjoint collection of open intervals whose endpoints belong to S.

Open and closed sets in R^n

3.8 *Prove that open n −balls and n −dimensional open intervals are open sets in R^n.*

proof: Given an open n −ball $B(x,r)$. Choose $y \in B(x,r)$ and thus consider $B(y,d) \subseteq B(x,r)$, where $d = \min(|x-y|, r - |x-y|)$. Then y is an interior point of $B(x,r)$. Since y is arbitrary, we have all points of $B(x,r)$ are interior. So, the open n −ball $B(x,r)$ is open.

Given an n −dimensional open interval $(a_1,b_1) \times (a_2,b_2) \times \ldots \times (a_n,b_n) := I$. Choose $x = (x_1, x_2, \ldots, x_n) \in I$ and thus consider $r = \min_{i=1}^{i=n}(r_i)$, where $r_i = \min(x_i - a_i, b_i - x_i)$. Then $B(x,r) \subseteq I$. That is, x is an interior point of I. Since x is arbitrary, we have all points of I are interior. So, the n −dimensional open interval I is open.

3.9 *Prove that the interior of a set in R^n is open in R^n.*

Proof: Let $x \in intS$, then there exists $r > 0$ such that $B(x,r) \subseteq S$. Choose any point of $B(x,r)$, say y. Then y is an interior point of $B(x,r)$ since $B(x,r)$ is open. So, there exists $d > 0$ such that $B(y,d) \subseteq B(x,r) \subseteq S$. So y is also an interior point of S. Since y is arbitrary, we find that every point of $B(x,r)$ is interior to S. That is, $B(x,r) \subseteq intS$. Since x is arbitrary, we have all points of $intS$ are interior. So, $intS$ is open.

Remark: 1 It should be noted that S is open if, and only if $S = intS$.

2. $int(intS) = intS$.

3. If $S \subseteq T$, then $intS \subseteq intT$.

3.10 *If $S \subseteq R^n$, prove that intS is the union of all open subsets of R^n which are contained in S. This is described by saying that intS* is **the largest open subset** of S.

proof: It suffices to show that $intS = \cup_{A \subseteq S} A$, where A is open. To show the statement, we consider two steps as follows.

1. ($\subseteq$) Let $x \in intS$, then there exists $r > 0$ such that $B(x,r) \subseteq S$. So, $x \in B(x,r) \subseteq \cup_{A \subseteq S} A$. That is, $intS \subseteq \cup_{A \subseteq S} A$.

2. ($\supseteq$) Let $x \in \cup_{A \subseteq S} A$, then $x \in A$ for some open set $A(\subseteq S)$. Since A is open, x is an interior point of A. There exists $r > 0$ such that $B(x,r) \subseteq A \subseteq S$. So x is an interior point of S, i.e., $x \in intS$. That is, $\cup_{A \subseteq S} A \subseteq intS$.

From 1 and 2, we know that $intS = \cup_{A \subseteq S} A$, where A is open.

Let T be an open subset of S such that $intS \subseteq T$. Since $intS = \cup_{A \subseteq S} A$, where A is open, we have $intS \subseteq T \subseteq \cup_{A \subseteq S} A$ which implies $intS = T$ by $intS = \cup_{A \subseteq S} A$. Hence, $intS$ is the largest open subset of S.

3.11 *If S and T are subsets of* R^n, *prove that*
$(intS) \cap (intT) = int(S \cap T)$ *and* $(intS) \cup (intT) \subseteq int(S \cup T)$.

Proof: For the part $(intS) \cap (intT) = int(S \cap T)$, we consider two steps as follows.
1. $(\subseteq)$ Since $intS \subseteq S$ and $intT \subseteq T$, we have $(intS) \cap (intT) \subseteq S \cap T$ which implies that (Note that $(intS) \cap (intT)$ is open.)

$$(intS) \cap (intT) = int((intS) \cap (intT)) \subseteq int(S \cap T).$$

2. $(\supseteq)$ Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, we have $int(S \cap T) \subseteq intS$ and $int(S \cap T) \subseteq intT$. So,

$$int(S \cap T) \subseteq (intS) \cap (intT).$$

From 1 and 2, we know that $(intS) \cap (intT) = int(S \cap T)$.
For the part $(intS) \cup (intT) \subseteq int(S \cup T)$, we consider $(intS) \subseteq S$ and $(intT) \subseteq T$. So,

$$(intS) \cup (intT) \subseteq S \cup T$$

which implies that (Note that $(intS) \cup (intT)$ is open.)

$$int((intS) \cup (intT)) = (intS) \cup (intT) \subseteq int(S \cup T).$$

Remark: It is not necessary that $(intS) \cup (intT) = int(S \cup T)$. For example, let $S = Q$, and $T = Q^c$, then $intS = \phi$, and $intT = \phi$. However, $int(S \cup T) = intR^1 = R$.

3.12 *Let* S' *denote the derived set and* $\bar{S}$ *the closure of a set S in* R^n. *Prove that*

(a) S' *is closed in* R^n; *that is* $(S')' \subseteq S'$.

proof: Let x be an adherent point of S'. In order to show S' is closed, it suffices to show that x is an accumulation point of S. Assume x is not an accumulation point of S, i.e., there exists $d > 0$ such that

$$(B(x,d) - \{x\}) \cap S = \phi. \quad *$$

Since x adheres to S', then $B(x,d) \cap S' \neq \phi$. So, there exists $y \in B(x,d)$ such that y is an accumulation point of S. Note that $x \neq y$, by assumption. Choose a smaller radius $\tilde{d}$ so that

$$B(y,\tilde{d}) \subseteq B(x,d) - \{x\} \text{ and } B(y,\tilde{d}) \cap S \neq \phi.$$

It implies

$$\phi \neq B(y,\tilde{d}) \cap S \subseteq (B(x,d) - \{x\}) \cap S = \phi \text{ by } (*)$$

which is absurb. So, x is an accumulation point of S. That is, S' contains all its adherent points. Hence S' is closed.

(b) If $S \subseteq T$, *then* $S' \subseteq T'$.

Proof: Let $x \in S'$, then $(B(x,r) - \{x\}) \cap S \neq \phi$ for any $r > 0$. It implies that $(B(x,r) - \{x\}) \cap T \neq \phi$ for any $r > 0$ since $S \subseteq T$. Hence, x is an accumulation point of T. That is, $x \in T'$. So, $S' \subseteq T'$.

(c) $(S \cup T)' = S' \cup T'$

Proof: For the part $(S \cup T)' = S' \cup T'$, we show it by two steps.
1. Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $S' \subseteq (S \cup T)'$ and $T' \subseteq (S \cup T)'$ by (b). So,

$$S' \cup T' \subseteq (S \cup T)'$$

2. Let $x \in (S \cup T)'$, then $(B(x,r) - \{x\}) \cap (S \cup T) \neq \phi$. That is,

$$((B(x,r) - \{x\}) \cap S) \cup ((B(x,r) - \{x\}) \cap T) \neq \phi.$$

So, at least one of $(B(x,r) - \{x\}) \cap S$ and $(B(x,r) - \{x\}) \cap T$ is not empty. If $(B(x,r) - \{x\}) \cap S \neq \phi$, then $x \in S'$. And if $(B(x,r) - \{x\}) \cap T \neq \phi$, then $x \in T'$. So,

$$(S \cup T)' \subseteq S' \cup T'.$$

From 1 and 2, we have $(S \cup T)' = S' \cup T'$.

Remark: Note that since $(S \cup T)' = S' \cup T'$, we have $cl(S \cup T) = cl(S) \cup cl(T)$, where $cl(S)$ is the closure of S.

(d) $(\bar{S})' = S'$.

Proof: Since $\bar{S} = S \cup S'$, then $(\bar{S})' = (S \cup S')' = S' \cup (S')' = S'$ since $(S')' \subseteq S'$ by (a).

(e) $\bar{S}$ *is closed in* R^n.

Proof: Since $(\bar{S})' = S' \subseteq \bar{S}$ by (d), then $\bar{S}$ cantains all its accumulation points. Hence, $\bar{S}$ is closed.

Remark: There is another proof which is like (a). But it is too tedious to write.

(f) $\bar{S}$ *is the intersection of all closed subsets of* R^n *containing* S. *That is,* $\bar{S}$ *is the smallest closed set containing* S.

Proof: It suffices to show that $\bar{S} = \cap_{A \supseteq S} A$, where A is closed. To show the statement, we consider two steps as follows.

1. $(\subseteq)$ Since $\bar{S}$ is closed and $S \subseteq \bar{S}$, then $\cap_{A \supseteq S} A \subseteq \bar{S}$.

2. $(\supseteq)$ Let $x \in \bar{S}$, then $B(x,r) \cap S \neq \phi$ for any $r > 0$. So, if $A \supseteq S$, then $B(x,r) \cap A \neq \phi$ for any $r > 0$. It implies that x is an adherent point of A. Hence if $A \supseteq S$, and A is closed, we have $x \in A$. That is, $x \in \cap_{A \supseteq S} A$. So, $\bar{S} \subseteq \cap_{A \supseteq S} A$.

From 1 and 2, we have $\bar{S} = \cap_{A \supseteq S} A$.

Let $S \subseteq T \subseteq \bar{S}$, where T is closed. Then $\bar{S} = \cap_{A \supseteq S} A \subseteq T$. It leads us to get $T = \bar{S}$. That is, $\bar{S}$ is the smallest closed set containing S.

Remark: In the exercise, there has something to remeber. We list them below.

Remark
1. If $S \subseteq T$, *then* $S' \subseteq T'$.
2. If $S \subseteq T$, *then* $\bar{S} \subseteq \bar{T}$.
3. $\bar{S} = S \cup S'$.
4. S is closed if, and only if $S' \subseteq S$.
5. $\bar{S}$ is closed.
6. $\bar{S}$ is the smallest closed set containing S.

3.13 *Let* S *and* T *be subsets of* R^n. *Prove that* $cl(S \cap T) \subseteq cl(S) \cap cl(T)$ *and that* $S \cap cl(T) \subseteq cl(S \cap T)$ *if* S *is open, where* $cl(S)$ *is the closure of* S.

Proof: Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, then $cl(S \cap T) \subseteq cl(S)$ and, $cl(S \cap T) \subseteq cl(T)$. So, $cl(S \cap T) \subseteq cl(S) \cap cl(T)$.

Given an open set S, and let $x \in S \cap cl(T)$, then we have

1. $x \in S$ and S is open.

$\Rightarrow B(x,d) \subseteq S$ for some $d > 0$.

$$\Rightarrow \begin{array}{l} B(x,r) \cap S \supseteq B(x,r) \text{ if } r \leq d. \\ B(x,r) \cap S \supseteq B(x,d) \text{ if } r > d. \end{array}$$

and

2. $x \in cl(T)$

$\Rightarrow B(x,r) \cap T \neq \phi$ for any $r > 0$.

From 1 and 2, we know

$$B(x,r) \cap (S \cap T) = (B(x,r) \cap S) \cap T = B(x,r) \cap T \neq \phi \text{ if } r \leq d.$$
$$B(x,r) \cap (S \cap T) = (B(x,r) \cap S) \cap T = B(x,d) \cap T \neq \phi \text{ if } r > d.$$

So, it means that x is an adherent point of $S \cap T$. That is, $x \in cl(S \cap T)$. Hence, $S \cap cl(T) \subseteq cl(S \cap T)$.

Remark: It is not necessary that $cl(S \cap T) = cl(S) \cap cl(T)$. For example, $S = Q$ and $T = Q^c$, then $cl(S \cap T) = \phi$ and $cl(S) \cap cl(T) = R^1$.

Note. The statements in Exercises 3.9 through 3.13 are true in any metric space.

3.14 *A set S in R^n is called* ***convex*** *if, for every pair of points x and y in S and every real θ satisfying $0 < \theta < 1$, we have $\theta x + (1-\theta)y \in S$. Interpret this statement geometrically (in R^2 and R^3) and prove that*

(a) Every n −ball in R^n is convex.

Proof: Given an n −ball $B(p,r)$, and let $x, y \in B(p,r)$. Consider $\theta x + (1-\theta)y$, where $0 < \theta < 1$.
Then

$$\begin{aligned} \|\theta x + (1-\theta)y - p\| &= \|\theta(x-p) + (1-\theta)(y-p)\| \\ &\leq \theta\|x-p\| + (1-\theta)\|y-p\| \\ &< \theta r + (1-\theta)r \\ &= r. \end{aligned}$$

So, we have $\theta x + (1-\theta)y \in B(p,r)$ for $0 < \theta < 1$. Hence, by the definition of convex, we know that every n −ball in R^n is convex.

(b) Every n −dimensional open interval is convex.

Proof: Given an n −dimensional open interval $I = (a_1,b_1) \times \ldots \times (a_n,b_n)$. Let $x, y \in I$, and thus write $x = (x_1,x_2,\ldots,x_n)$ and $y = (y_1,y_2,\ldots y_n)$. Consider $\theta x + (1-\theta)y = (\theta x_1 + (1-\theta)y_1, \theta x_2 + (1-\theta)y_2, \ldots, \theta x_n + (1-\theta)y_n)$ where $0 < \theta < 1$.
Then

$$a_i < \theta x_i + (1-\theta)y_i < b_i, \text{ where } i = 1,2,..,n.$$

So, we have $\theta x + (1-\theta)y \in I$ for $0 < \theta < 1$. Hence, by the definition of convex, we know that every n −dimensional open interval is convex.

(c) The interior of a convex is convex.

Proof: Given a convex set S, and let $x, y \in intS$. Then there exists $r > 0$ such that $B(x,r) \subseteq S$, and $B(y,r) \subseteq S$. Consider $\theta x + (1-\theta)y := p \in S$, where $0 < \theta < 1$, since S is convex.

Claim that $B(p,r) \subseteq S$ as follows.

Let $q \in B(p,r)$, We want to find two special points $\tilde{x} \in B(x,r)$, and $\tilde{y} \in B(y,r)$ such that $q = \theta\tilde{x} + (1-\theta)\tilde{y}$.

Since the three n –balls $B(x,r)$, $B(y,r)$, and $B(p,r)$ have the same radius. By parallelogram principle, we let $\tilde{x} = q + (x-p)$, and $\tilde{y} = q + (y-p)$, then $\|x - \tilde{x}\| = \|q-p\| < r$, and $\|\tilde{y} - y\| = \|q - p\| < r$.

It implies that $\tilde{x} \in B(x,r)$, and $\tilde{y} \in B(y,r)$. In addition,

$$\begin{aligned}&\theta\tilde{x} + (1-\theta)\tilde{y}\\ &= \theta(q + (x-p)) + (1-\theta)(q + (y-p))\\ &= q.\end{aligned}$$

Since $\tilde{x}$, $\tilde{y} \in S$, and S is convex, then $q = \theta\tilde{x} + (1-\theta)\tilde{y} \in S$. It implies that $B(p,r) \subseteq S$ since q is arbitrary. So, we have proved the claim. That is, for $0 < \theta < 1$, $\theta x + (1-\theta)y = p \in intS$ if x, $y \in intS$, and S is convex. Hence, by the definition of convex, we know that the interior of a convex is convex.

(d) The closure of a convex is convex.

Proof: Given a convex set S, and let x, $y \in \bar{S}$. Consider $\theta x + (1-\theta)y := p$, where $0 < \theta < 1$, and claim that $p \in \bar{S}$, i.e., we want to show that $B(p,r) \cap S \neq \phi$.

Suppose **NOT,** there exists $r > 0$ such that

$$B(p,r) \cap S = \phi. \qquad *$$

Since x, $y \in \bar{S}$, then $B(x, \frac{r}{2}) \cap S \neq \phi$ and $B(y, \frac{r}{2}) \cap S \neq \phi$. And let $\tilde{x} \in B(x, \frac{r}{2}) \cap S$ and $\tilde{y} \in B(y, \frac{r}{2}) \cap S$. Consider

$$\begin{aligned}\|(\tilde{\theta}\tilde{x} + (1-\tilde{\theta})\tilde{y}) - p\| &= \|(\tilde{\theta}\tilde{x} + (1-\tilde{\theta})\tilde{y}) - (\theta x + (1-\theta)y)\|\\ &\le \|\tilde{\theta}\tilde{x} - \theta x\| + \|(1-\tilde{\theta})\tilde{y} - (1-\theta)y\|\\ &= \begin{array}{l}\|\tilde{\theta}\tilde{x} - \tilde{\theta}x + \tilde{\theta}x - \theta x\| +\\ \|(1-\tilde{\theta})\tilde{y} - (1-\tilde{\theta})y + (1-\tilde{\theta})y - (1-\theta)y\|\end{array}\\ &\le \tilde{\theta}\|\tilde{x} - x\| + (1-\tilde{\theta})\|\tilde{y} - y\| + |\tilde{\theta} - \theta|(\|x\| + \|y\|)\\ &< \frac{r}{2} + |\tilde{\theta} - \theta|(\|x\| + \|y\|)\\ &< r\end{aligned}$$

if we choose a suitable number $\tilde{\theta}$, where $0 < \tilde{\theta} < 1$.

Hence, we have the point $\tilde{\theta}\tilde{x} + (1-\tilde{\theta})\tilde{y} \in B(p,r)$. Note that $\tilde{x}$, $\tilde{y} \in S$ and S is convex, we have $\tilde{\theta}\tilde{x} + (1-\tilde{\theta})\tilde{y} \in S$. It leads us to get a contradiction by (*). Hence, we have proved the claim. That is, for $0 < \theta < 1$, $\theta x + (1-\theta)y = p \in \bar{S}$ if x, $y \in \bar{S}$. Hence, by the definition of convex, we know that the closure of a convex is convex.

3.15 Let F be a collection of sets in R^n, and let $S = \cup_{A \in F} A$ and $T = \cap_{A \in F} A$. For each of the following statements, either give a proof or exhibit a counterexample.

(a) If x is an accumulation point of T, then x is an accumulation point of each set A in F.

Proof: Let x be an accumulation point of T, then $(B(x,r) - \{x\}) \cap T \neq \phi$ for any $r > 0$. Note that for any $A \in F$, we have $T \subseteq A$. Hence $(B(x,r) - \{x\}) \cap A \neq \phi$ for any $r > 0$. That is, x is an accumulation point of A for any $A \in F$.

The conclusion is that If x is an accumulation point of $T = \cap_{A \in F} A$, then x is an accumulation point of each set A in F.

(b) If x is an accumulation point of S, then x is an accumulation point of at least one set A in F.

Proof: No! For example, Let $S = R^n$, and F be the collection of sets consisting of a single point x $(\in R^n)$. Then it is trivially seen that $S = \cup_{A\in F} A$. And if x is an accumulation point of S, then x is not an accumulation point of each set A in F.

3.16 *Prove that the set S of rational numbers in the inerval $(0,1)$ cannot be expressed as the intersection of a countable collection of open sets. Hint: Write $S = \{x_1, x_2, \ldots\}$, assume that $S = \cap_{k=1}^{k=\infty} S_k$, where each S_k is open, and construct a sequence $\{Q_n\}$ of closed intervals such that $Q_{n+1} \subseteq Q_n \subseteq S_n$ and such that $x_n \notin Q_n$. Then use the Cantor intersection theorem to obtain a contradiction.*

Proof: We prove the statement by method of contradiction. Write $S = \{x_1, x_2, \ldots\}$, and assume that $S = \cap_{k=1}^{k=\infty} S_k$, where each S_k is open.

Since $x_1 \in S_1$, there exists a bounded and open interval $I_1 \subseteq S_1$ such that $x_1 \in I_1$. Choose a closed interval $Q_1 \subseteq I_1$ such that $x_1 \notin Q_1$. Since Q_1 is an interval, it contains infinite rationals, call one of these, x_2. Since $x_2 \in S_2$, there exists an open interval $I_2 \subseteq S_2$ and $I_2 \subseteq Q_1$. Choose a closed interval $Q_2 \subseteq I_2$ such that $x_2 \notin Q_2$. Suppose Q_n has been constructed so that

1. Q_n is a closed interval.
2. $Q_n \subseteq Q_{n-1} \subseteq S_{n-1}$.
3. $x_n \notin Q_n$.

Since Q_n is an interval, it contains infinite rationals, call one of these, x_{n+1}. Since $x_{n+1} \in S_{n+1}$, there exists an open interval $I_{n+1} \subseteq S_{n+1}$ and $I_{n+1} \subseteq Q_n$. Choose a closed interval $Q_{n+1} \subseteq I_{n+1}$ such that $x_{n+1} \notin Q_{n+1}$. So, Q_{n+1} satisfies our induction hypothesis, and the construction can process.

Note that

1. For all n, Q_n is not empty.
2. For all n, Q_n is bounded since I_1 is bounded.
3. $Q_{n+1} \subseteq Q_n$.
4. $x_n \notin Q_n$.

Then $\cap_{n=1}^{n=\infty} Q_n \neq \phi$ by **Cantor Intersection Theorem.**

Since $Q_n \subseteq S_n$, $\cap_{n=1}^{n=\infty} Q_n \subseteq \cap_{n=1}^{n=\infty} S_n = S$. So, we have

$$S \cap (\cap_{n=1}^{n=\infty} Q_n) = \cap_{n=1}^{n=\infty} Q_n \neq \phi$$

which is absurb since $S \cap (\cap_{n=1}^{n=\infty} Q_n) = \phi$ by the fact $x_n \notin Q_n$. Hence, we have proved that our assumption does not hold. That is, S the set of rational numbers in the inerval $(0,1)$ cannot be expressed as the intersection of a countable collection of open sets.

Remark: 1. Often, the property is described by saying Q is not an G_δ –set.

2. It should be noted that Q^c is an G_δ –set.

3. For the famous Theorem called **Cantor Intersection Theorem**, the reader should see another classical text book, Principles of Mathematical Analysis written by Walter Rudin, Theorem 3.10 in page 53.

4. For the method of proof, the reader should see another classical text book, Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, in page 41.

Covering theorems in R^n

3.17 If $S \subseteq R^n$, prove that the collection of isolated points of S is countable.

Proof: Denote the collection of isolated points of S by F. Let $x \in F$, there exists an n –ball $(B(x,r_x) - \{x\}) \cap S = \phi$. Write $Q^n = \{x_1, x_2, \ldots\}$, then there are many numbers in Q^n lying on $B(x,r_x) - \{x\}$. We choose the smallest index, say $m = m(x)$, and denote x by x_m.

So, $F = \{x_m : m \in P\}$, where $P(\subseteq N)$, a subset of positive integers. Hence, F is countable.

3.18 Prove that the set of open disks in the xy –plane with center (x,x) and radius $x > 0$, x rational, is a countable covering of the set $\{(x,y) : x > 0, y > 0\}$.

Proof: Denote the set of open disks in the xy –plane with center (x,x) and radius $x > 0$ by S. Choose any point (a,b), where $a > 0$, and $b > 0$. We want to find an 2 –ball $B((x,x),x)$ $(\in S)$ which contains (a,b). It suffices to find $x \in Q$ such that $\|(x,x) - (a,b)\| < x$. Since

$$\|(x,x) - (a,b)\| < x \Leftrightarrow \|(x,x) - (a,b)\|^2 < x^2 \Leftrightarrow x^2 - 2(a+b)x + (a^2+b^2) < 0.$$

Since $x^2 - 2(a+b)x + (a^2+b^2) = [x-(a+b)]^2 - 2ab$, we can choose a suitable rational number x such that $x^2 - 2(a+b)x + (a^2+b^2) < 0$ since $a > 0$, and $b > 0$. Hence, for any point (a,b), where $a > 0$, and $b > 0$, we can find an 2 –ball $B((x,x),x)$ $(\in S)$ which contains (a,b).

That is, S is a countable covering of the set $\{(x,y) : x > 0,\ y > 0\}$.

Remark: The reader should give a geometric appearance or draw a graph.

3.19 The collection Fof open intervals of the form $(1/n, 2/n)$, where $n = 2,3,\ldots$, is an open covering of the open interval $(0,1)$. Prove (without using Theorem 3.31) that no finite subcollection of F covers $(0,1)$.

Proof: Write F as $\{(\frac{1}{2},1),(\frac{1}{3},\frac{2}{3}),\ldots,(\frac{1}{n},\frac{2}{n}),\ldots\}$. Obviously, F is an open covering of $(0,1)$. Assume that there exists a finite subcollection of F covers $(0,1)$, and thus write them as $F' = \{(\frac{1}{n_1},\frac{1}{m_1}),\ldots,(\frac{1}{n_k},\frac{1}{m_k})\}$. Choose $p \in (0,1)$ so that $p < \min_{1\leq i\leq k}(\frac{1}{n_i})$. Then $p \notin (\frac{1}{n_i},\frac{1}{m_i})$, where $1 \leq i \leq k$. It contracdicts the fact F' covers $(0,1)$.

Remark: The reader should be noted that if we use Theorem 3.31, then we cannot get the correct proof. In other words, the author T. M. Apostol mistakes the statement.

3.20 Give an example of a set S which is closed but not bounded and exhibit a coubtable open covering F such that no finite subset of F covers S.

Solution: Let $S = R^1$, then R^1 is closed but not bounded. And let $F = \{(n,n+2) : n \in Z\}$, then F is a countable open covering of S. In additon, it is trivially seen that **no finite subset of F covers S.**

3.21 Given a set S in R^n with the property that for every x in S there is an n –ball $B(x)$ such that $B(x) \cap S$ is coubtable. Prove that S is countable.

Proof: Note that $F = \{B(x) : x \in S\}$ forms an open covering of S. Since $S \subseteq R^n$, then there exists a countable subcover $F'(\subseteq F)$ of S by **Lindelof Covering Theorem.** Write $F' = \{B(x_n) : n \in N\}$. Since

$$S = S \cap (\cup_{n\in N} B(x_n)) = \cup_{n\in N} (S \cap B(x_n)),$$

and

$$S \cap B(x_n) \text{ is countable by hypothesis.}$$

Then S is countable.

Remark: The reader should be noted that exercise 3.21 is equivalent to exercise 3.23.

3.22 Prove that a collection of disjoint open sets in R^n is necessarily countable. Give an example of a collection of disjoint closed sets which is not countable.

Proof: Let F be a collection of disjoint open sets in R^n, and write $Q^n = \{x_1, x_2, \ldots\}$. Choose an open set $S(\neq \phi)$ in F, then there exists an n −ball $B(y,r) \subseteq S$. In this ball, there are infinite numbers in Q^n. We choose the smallest index, say $m = m(y)$. Then we have $F = \{S_m : m \in P \subseteq N\}$ which is countable.

For the example that a collection of disjoint closed sets which is not countable, we give it as follows. Let $G = \{\{x\} : x \in R^n\}$, then we complete it.

3.23 Assume that $S \subseteq R^n$. A point x in R^n is said to be ***condensation*** *point of S if every n −ball $B(x)$ has the property that $B(x) \cap S$ is not countable. Prove that if S is not countable, then there exists a point x in S such that x is a condensation point of S.*

Proof: It is equivalent to exercise 3.21.

Remark: Compare with two definitions on a condensation point and an accumulation point, it is easy to know that a condensation point is an accumulation point. However, am accumulation point is not a condensation point, for example, $S = \{1/n : n \in N\}$. We have 0 is an accumulation point of S, but not a condensation point of S.

3.24 *Assume that $S \subseteq R^n$ and assume that S is not countable. Let T denote the set of condensation points of S. Prove that*

(a) $S - T$ is countable.

Proof: If $S - T$ is uncountable, then there exists a point x in $S - T$ such that *x is a condensation point of $S - T$* by exercise 3.23. Obviously, $x(\in S)$ is also a *condensation point of S*. It implies $x \in T$. So, we have $x \in S \cap T$ which is absurb since $x \in S - T$.

Remark: The reader should regard T as a special part of S, and the advantage of T helps us realize the uncountable set $S(\subseteq R^n)$. Compare with **Cantor-Bendixon Theorem** in exercise 3.25.

(b) $S \cap T$ is not countable.

Proof: Suppose $S \cap T$ is countable, then $S = (S \cap T) \cup (S - T)$ is countable by (a) which is absurb. So, $S \cap T$ is not countable.

(c) T is a closed set.

Proof: Let x be an adherent point of T, then $B(x,r) \cap T \neq \phi$ for any $r > 0$. We want to show $x \in T$. That is to show x is a condensation point of S. Claim that $B(x,r) \cap S$ is uncountable for any $r > 0$.

Suppose **NOT,** then there exists an n −ball $B(x,d) \cap S$ which is countable. Since x is an adherent point of T, then $B(x,d) \cap T \neq \phi$. Choose $y \in B(x,d) \cap T$ so that $B(y,\delta) \subseteq B(x,d)$ and $B(y,\delta) \cap S$ is uncountable. However, we get a contradiction since

$$B(y,\delta) \cap S \left(\text{is uncountable}\right) \subseteq B(x,d) \cap S \left(\text{is countable}\right).$$

Hence, $B(x,r) \cap S$ is uncountable for any $r > 0$. That is, $x \in T$. Since T contains its all adherent points, T is closed.

(d) T contains no isolated points.

Proof: Let $x \in T$, and if x is an isolated point of T, then there exists an n −ball $B(x,d)$ such that $B(x,d) \cap T = \{x\}$. On the other hand, $x \in T$ means that $(B(x,d) - \{x\}) \cap S$ is uncountable. Hence, by exercise 3.23, we know that there exists $y \in (B(x,d) - \{x\}) \cap S$ such that y is a condensation point of $(B(x,d) - \{x\}) \cap S$. So, y is a condensation point of S. It implies $y \in T$. It is impossible since

$$1.\ y(\neq x) \in T.$$
$$2.\ y \in B(x,d).$$
$$3.\ B(x,d) \cap T = \{x\}.$$

Hence, x is not an isolated point of T, if $x \in T$. That is, T contains no isolatd points.

Remark: Use exercise 3.25, by (c) and (d) we know that T is perfect.

Note that Exercise 3.23 is a special case of (b).

3.25 *A set in R^n is called* ***perfect*** *if $S' = S$, that is, if S is a closed set which contains no isolated points. Prove that every uncountable closed set F in R^n can be expressed in the form $F = A \cup B$, where A is perfect and B is countable (**Cantor-Bendixon theorem**).*
Hint. Use Exercise 3.24.

Proof: Let F be a uncountable closed set in R^n. Then by exercise 3.24, $F = (F \cap T) \cup (F - T)$, where T is the set of condensation points of F. Note that since F is closed, $T \subseteq F$ by the fact, a condensation point is an accumulation point. Define $F \cap T = A$ and $F - T = B$, then B is countable and $A(= T)$ is perfect.

Remark: 1. The reader should see another classical text book, Principles of Mathematical Analysis written by Walter Rudin, Theorem 2.43, in page 41. Since the theorem is famous, we list it below.

Theorem 2.43 Let P be a nonempty perfect set in R^k. Then P is uncountable.

Theorem (Modefied 2.43) Let P be a nonempty perfect set in a **complete separable metric space.** Then P is uncountable.

2. Let S has measure zero in R^1. Prove that there is a nonempty perfect set P in R^1 such that $P \cap S = \phi$.

Proof: Since S has measure zero, there exists a collection of open intervals $\{I_k\}$ such that

$$S \subseteq \cup I_k \text{ and } \sum |I_k| < 1.$$

Consider its complement $(\cup I_k)^c$ which is closed with positive measure. Since the complement has a positive measure, we know that it is uncountable. Hence, by Cantor-Bendixon Theorem, we know that

$$(\cup I_k)^c = A \cup B, \text{ where } A \text{ is perfect and } B \text{ is countable.}$$

So, let $A = P$, we have prove it.

Note: From the similar method, we can show that given any set S in R^1 with measure $0 \leq d < \infty$, there is a non-empty perfect set P such that $P \cap S = \phi$. In particular, $S = Q$, S =the set of algebraic numbers, and so on. In addition, even for cases in R^k, it still holds.

Metric Spaces

3.26 In any metric space (M,d) prove that the empty set ϕ and the whole set M are both open and closed.

proof: In order to show the statement, it suffices to show that M is open and closed since $M - M = \phi$. Let $x \in M$, then for any $r > 0$, $B_M(x,r) \subseteq M$. That is, x is an interior point of M. Sinc x is arbitrary, we know that every point of M is interior. So, M is open.

Let x be an adherent point of M, it is clearly $x \in M$ since we consider all points lie in M. Hence, M contains its all adherent points. It implies that M is closed.

Remark: The reader should regard the statement as a common sense.

3.27 Consider the following two metrics in R^n :
$d_1(x,y) = \max_{1 \leq i \leq n}|x_i - y_i|$, $d_2(x,y) = \sum_{i=1}^{i=n}|x_i - y_i|$.
In each of the following metric spaces prove that the ball $B(a;r)$ *has the geometric appearance indicated:*

(a) In (R^2, d_1), *a square with sides parallel to the coordinate axes.*

Solution: It suffices to consider the case $B((0,0),1)$. Let $x = (x_1,x_2) \in B((0,0),1)$, then we have

$$|x_1| < 1, \text{ and } |x_2| < 1.$$

So, it means that the ball $B((0,0),1)$ is a square with sides lying on the coordinate axes. Hence, we know that $B(a;r)$ is a square with sides parallel to the coordinate axes.

(b) In (R^2, d_2), *a square with diagonals parallel to the axes.*

Solution: It suffices to consider the case $B((0,0),1)$. Let $x = (x_1,x_2) \in B((0,0),1)$, then we have

$$|x_1 + x_2| < 1.$$

So, it means that the ball $B((0,0),1)$ is a square with diagonals lying on the coordinate axes. Hence, we know that $B(a;r)$ is a square with diagonals parallel to the coordinate axes.

(c) A cube in (R^3, d_1).

Solution: It suffices to consider the case $B((0,0,0),1)$. Let $x = (x_1,x_2,x_3) \in B((0,0,0),1)$, then we have

$$|x_1| < 1, \ |x_2| < 1, \text{ and } |x_3| < 1.$$

So, it means that the ball $B((0,0,0),1)$ is a cube with length 2. Hence, we know that $B(a;r)$ is a cube with length $2a$.

(d) An octahedron in (R^3, d_2).

Solution: It suffices to consider the case $B((0,0,0),1)$. Let $x = (x_1,x_2,x_3) \in B((0,0,0),1)$, then we have

$$|x_1 + x_2 + x_3| < 1.$$

It means that the ball $B((0,0,0),1)$ is an octahedron. Hence, $B(a;r)$ is an octahedron.

Remark: The exercise tells us one thing that $B(a;r)$ may not be an n −ball if we consider some different matrices.

3.28 Let d_1 *and* d_2 *be the metrics of Exercise 3.27 and let* $\|x - y\|$ *denote the usual Euclidean metric. Prove that the following inequalities for all x and y in* R^n :
$d_1(x,y) \leq \|x-y\| \leq d_2(x,y)$ *and* $d_2(x,y) \leq \sqrt{n}\,\|x-y\| \leq nd_1(x,y)$.

Proof: List the definitions of the three metrics, and compare with them as follows.

1. $d_1(x,y) = \max_{1\le i\le n}|x_i - y_i|$.
2. $\|x-y\| = \left(\sum_{i=1}^{i=n}(x_i-y_i)^2\right)^{1/2}$.
3. $d_2(x,y) = \sum_{i=1}^{i=n}|x_i - y_i|$.

Then we have

(a)

$$d_1(x,y) = \max_{1\le i\le n}|x_i-y_i| = \left(\max_{1\le i\le n}|x_i-y_i|^2\right)^{1/2} \le \left(\sum_{i=1}^{i=n}(x_i-y_i)^2\right)^{1/2} = \|x-y\|.$$

(b)

$$\|x-y\| = \left(\sum_{i=1}^{i=n}(x_i-y_i)^2\right)^{1/2} \le \left[\left(\sum_{i=1}^{i=n}|x_i-y_i|\right)^2\right]^{1/2} = \sum_{i=1}^{i=n}|x_i-y_i| = d_2(x,y).$$

(c)

$$\sqrt{n}\,\|x-y\| = \sqrt{n}\left(\sum_{i=1}^{i=n}(x_i-y_i)^2\right)^{1/2} = \left(n\sum_{i=1}^{i=n}(x_i-y_i)^2\right)^{1/2} \le \left\{n\left(n\left[\max_{1\le i\le n}|x_i-y_i|\right]^2\right)\right\}^{1/2} = n\max_{1\le i\le n}|x_i-y_i| = d_1(x,y).$$

(d)

$$\begin{aligned}[d_2(x,y)]^2 &= \left(\sum_{i=1}^{i=n}|x_i-y_i|\right)^2 = \sum_{i=1}^{i=n}(x_i-y_i)^2 + \sum_{1\le i<j\le n}2|x_i-y_i||x_j-y_j| \\ &\le \sum_{i=1}^{i=n}(x_i-y_i)^2 + (n-1)\sum_{i=1}^{i=n}(x_i-y_i)^2 \text{ by } A.\ P. \ge G.\ P. \\ &= n\sum_{i=1}^{i=n}(x_i-y_i)^2 \\ &= n\|x-y\|^2.\end{aligned}$$

So,

$$d_2(x,y) \le \sqrt{n}\,\|x-y\|.$$

From (a)-(d), we have proved these inequalities.

Remark: 1. Let M be a given set and suppose that (M,d) and $(M,\bar{d})$ are metric spaces. We define the metrics d and $\bar{d}$ are **equivalent** if, and only if, there exist positive constants $\alpha,\ \beta$ such that

$$\alpha d(x,y) \le \bar{d}(x,y) \le \beta d(x,y).$$

The concept is much important for us to consider the same set with different metrics. For example, in this exercise, Since three metrics are equivalent, it is easy to know that (R^k, d_1), (R^k, d_2), and $(R^k, \|.\|)$ are **complete.** (For definition of complete metric space, the reader can see this text book, page 74.)

2. It should be noted that on a finite dimensional vector space X, any two norms are equivalent.

3.29 If (M,d) is a metric space, define $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d' is also a metric for M. Note that $0 \le d'(x,y) < 1$ for all x, y in M.

Proof: In order to show that d' is a metric for M, we consider the following four steps.
(1) For $x \in M$, $d'(x,x) = 0$ since $d(x,x) = 0$.
(2) For $x \neq y$, $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} > 0$ since $d(x,y) > 0$.
(3) For $x,y \in M$, $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$
(4) For $x,y,z \in M$,

$$\begin{aligned}
d'(x,y) &= \frac{d(x,y)}{1+d(x,y)} = 1 - \frac{1}{1+d(x,y)} \\
&\le 1 - \frac{1}{1+d(x,z)+d(z,y)} \text{ since } d(x,y) \le d(x,z)+d(z,y) \\
&= \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\
&\le \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\
&\le \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\
&= d'(x,z) + d'(z,y)
\end{aligned}$$

Hence, from (1)-(4), we know that d' is also a metric for M. Obviously, $0 \le d'(x,y) < 1$ for all x, y in M.

Remark: 1. The exercise tells us how to form a new metric from an old metric. Also, the reader should compare with exercise 3.37. This is another construction.

2. Recall **Discrete metric** d**,** we find that given any set nonempty S, (S,d) is a metric space, and thus use the exercise, we get another metric space (S,d'), and so on. Hence, here is a common sense that given any nonempty set, we can use discrete metric to form many and many metric spaces.

3.30 Prove that every finite subset of a metric space is closed.

Proof: Let x be an adherent point of a finite subet $S = \{x_i : i = 1,2,\ldots,n\}$ of a metric space (M,d). Then for any $r > 0$, $B(x,r) \cap S \neq \phi$. If $x \notin S$, then $B_M(x,\delta) \cap S = \phi$ where $\delta = \min_{1 \le i \neq j \le n} d(x_i, x_j.)$ It is impossible. Hence, $x \in S$. That is, S contains its all adherent points. So, S is closed.

3.31 In a metric space (M,d) the closed ball of radius $r > 0$ about a point a in M is the set $\bar{B}(a;r) = \{x : d(x,a) \le r\}$.

(a) Prove that $\bar{B}(a;r)$ is a closed set.

Proof: Let $x \in M - \bar{B}(a;r)$, then $d(x,a) > r$. Consider $B(x,\delta)$, where $\delta = \frac{d(x,a)-r}{2}$, then if $y \in B(x,\delta)$, we have $d(y,a) \ge d(x,a) - d(x,y) > d(x,a) - \delta = \frac{d(x,a)+r}{2} > r$. Hence, $B(x,\delta) \subseteq M - \bar{B}(a;r)$. That is, every point of $M - \bar{B}(a;r)$ is interior. So, $M - \bar{B}(a;r)$ is open, or equivalently, $\bar{B}(a;r)$ is a closed set.

(b) Give an example of a metric space in which $\bar{B}(a;r)$ is not the closure of the open ball $B(a;r)$.

Solution: Consider discrete metric space M, then we have $\left(\text{let } x \in M\right)$

$$\text{The closure of } B(a;1) = \{a\}$$

and

$$\bar{B}(a;1) = M.$$

Hence, if we let $\{a\}$ is a proper subset of M, then $\bar{B}(a;1)$ is not the closure of the open ball $B(a;1)$.

3.32 In a metric space M, if subsets satisfy $A \subseteq S \subseteq \bar{A}$, where $\bar{A}$ is the closure of A, then A is said to be **dense** *in S. For example, the set Q of rational numbers is dense in R. If A is dense in S and if S is dense in T, prove that A is dense in T.*

Proof: Since A is dense in S and S is dense in T, we have $\bar{A} \supseteq S$ and $\bar{S} \supseteq T$. Then $\bar{A} \supseteq T$. That is, A is dense in T.

3.33 Refer to exercise 3.32. A metric space M is said to be separable if there is a countable subset A which is dense in M. For example, R^1 is separable becasue the set Q of rational numbrs is a countable dense subset. Prove that every Euclidean space R^k is separable.

Proof: Since Q^k is a countable subset of R^k, and $\bar{Q}^k = R^k$, then we know that R^k is separable.

3.34 *Refer to exercise 3.33. Prove that the Lindelof covering theorem (Theorem 3.28) is valid in any separable metric space.*

Proof: Let (M,d) be a separable metric space. Then there exists a countable subset $S = \{x_n : n \in N\} (\subseteq M)$ which is dense in M. Given a set $A \subseteq M$, and an open covering F of A. Write $P = \{B(x_n, r_m) : x_n \in S,\ r_m \in Q\}$.

Claim that if $x \in M$, and G is an open set in M which contains x. Then $x \in B(x_n, r_m) \subseteq G$ for some $B(x_n, r_m) \subseteq P$.

Since $x \in G$, there exists $B(x, r_x) \subseteq G$ for some $r_x > 0$. Note that $x \in cl(S)$ since S is dense in M. Then, $B(x, r_x/2) \cap S \neq \phi$. So, if we choose $x_n \in B(x, r_x/2) \cap S$ and $r_m \in Q$ with $r_x/2 < r_m < r_x/3$, then we have

$$x \in B(x_n, r_m)$$

and

$$B(x_n, r_m) \subseteq B(x, r_x)$$

since if $y \in B(x_n, r_m)$, then

$$\begin{aligned} d(y,x) &\leq d(y,x_n) + d(x_n,x) \\ &< r_m + \frac{r_x}{2} \\ &< \frac{r_x}{3} + \frac{r_x}{2} \\ &< r_x \end{aligned}$$

So, we have prvoed the claim $x \in B(x_n, r_m) \subseteq B(x, r_x) \subseteq G$ or some $B(x_n, r_m) \in P$.

Use the claim to show the statement as follows. Write $A \subseteq \cup_{G \in F} G$, and let $x \in A$, then there is an open set G in F such that $x \in G$. By the claim, there is $B(x_n, r_m) := B_{n+m}$ in P such that $x \in B_{n+m} \subseteq G$. There are, of course, infinitely many such B_{n+m} corresponding to each G, but we choose only one of these, for example, the one of smallest index, say $q = q(x)$. Then we have $x \in B_{q(x)} \subseteq G$.

The set of all $B_{q(x)}$ obtained as x varies over all elements of A is a countable collection of open sets which covers A. To get a countable subcollection of F which covers A, we simply correlate to each set $B_{q(x)}$ one of the sets G of F which contained $B_{q(x)}$. This complete the proof.

3.35 Refer to exercise 3.32. If A is dense in S and B is open in S, prove that $B \subseteq cl(A \cap B)$, where $cl(A \cap B)$ means the closure of $A \cap B$.
Hint. Exercise 3.13.

Proof: Since A is dense in S and B is open in S, $\bar{A} \supseteq S$ and $S \cap B = B$. Then

$$\begin{aligned} B &= S \cap B \\ &\subseteq \bar{A} \cap B, \text{ } B \text{ is open in } S \\ &\subseteq cl(A \cap B) \end{aligned}$$

by exercise 3.13.

3.36 Refer to exercise 3.32. If each of A and B is dense in S and if B is open in S, prove that $A \cap B$ is dense in S.

Proof: Since

$$\begin{aligned} &cl(A \cap B), \text{ } B \text{ is open} \\ &\supseteq cl(A) \cap B \text{ by exercise 3.13} \\ &\supseteq S \cap B \text{ since } A \text{ is dense in } S \\ &= B \text{ since } B \text{ is open in } S \end{aligned}$$

then

$$cl(A \cap B) \supseteq B$$

which implies

$$cl(A \cap B) \supseteq S$$

since B is dense in S.

3.37 Given two metric spaces (S_1,d_1) and (S_2,d_2), a metric ρ for the Cartesian product $S_1 \times S_2$ can be constructed from $d_1 \times d_2$ in may ways. For example, if $x = (x_1,x_2)$ and $y = (y_1,y_2)$ are in $S_1 \times S_2$, let $\rho(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$. Prove that ρ is a metric for $S_1 \times S_2$ and construct further examples.

Proof: In order to show that ρ is a metric for $S_1 \times S_2$, we consider the following four steps.

(1) For $x = (x_1,x_2) \in S_1 \times S_2$, $\rho(x,x) = d_1(x_1,x_1) + d_2(x_2,x_2) = 0 + 0 = 0$.

(2) For $x \neq y$, $\rho(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2) > 0$ since if $\rho(x,y) = 0$, then $x_1 = y_1$ and $x_2 = y_2$.

(3) For $x,y \in S_1 \times S_2$,

$$\begin{aligned} \rho(x,y) &= d_1(x_1,y_1) + d_2(x_2,y_2) \\ &= d_1(y_1,x_1) + d_2(y_2,x_2) \\ &= \rho(y,x). \end{aligned}$$

(4) For $x,y,z \in S_1 \times S_2$,

$$\begin{aligned} \rho(x,y) &= d_1(x_1,y_1) + d_2(x_2,y_2) \\ &\leq d_1(x_1,z_1) + d_1(z_1,y_1) + d_2(x_2,z_2) + d_2(z_2,y_2) \\ &= [d_1(x_1,z_1) + d_2(x_2,z_2)] + [d_1(z_1,y_1) + d_2(z_2,y_2)] \\ &\leq \rho(x,z) + \rho(z,y). \end{aligned}$$

Hence from (1)-(4), we know that ρ is a metric for $S_1 \times S_2$.

For other metrics, we define

$$\rho_1(x,y) := \alpha d_1(x_1,y_1) + \beta d_2(x_2,y_2) \text{ for } \alpha,\beta > 0.$$
$$\rho_2(x,y) := d_1(x_1,y_1) + \frac{d_2(x_2,y_2)}{1 + d_2(x_2,y_2)}$$

and so on. (The proof is similar with us by above exercises.)

Compact subsets of a metric space

3.38 *Assume $S \subseteq T \subseteq M$. Then S is compact in (M,d) if, and only if, S is compact in the metric subspace (T,d).*

Proof: Suppose that S is compact in (M,d). Let $F = \{O_\alpha : O_\alpha \text{ is open in } T\}$ be an open covering of S. Since O_α is open in T, there exists the corresponding G_α which is open in M such that $G_\alpha \cap T = O_\alpha$. It is clear that $\{G_\alpha\}$ forms an open covering of S. So there is a finite subcovering $\{G_1,\ldots,G_n\}$ of S since S is compact in (M,d). That is, $S \subseteq \cup_{k=1}^{k=n} G_k$. It implies that

$$\begin{aligned} S &= T \cap S \\ &\subseteq T \cap (\cup_{k=1}^{k=n} G_k.) \\ &= \cup_{k=1}^{k=n} (T \cap G_k) \\ &= \cup_{k=1}^{k=n} O_k (\in F). \end{aligned}$$

So, we find a fnite subcovering $\{O_1,\ldots,O_n\}$ of S. That is, S is compact in (T,d).

Suppose that S is compact in (T,d). Let $G = \{G_\alpha : G_\alpha \text{ is open in } M\}$ be an open covering of S. Since $G_\alpha \cap T := O_\alpha$ is open in T, the collection $\{O_\alpha\}$ forms an open covering of S. So, there is a finite subcovering $\{O_1,\ldots,O_n\}$ of S since S is compact in (T,d). That is, $S \subseteq \cup_{k=1}^{k=n} O_k$. It implis that

$$S \subseteq \cup_{k=1}^{k=n} O_k \subseteq \cup_{k=1}^{k=n} G_k.$$

So, we find a finite subcovering $\{G_1,\ldots,G_n\}$ of S. That is, S is compact in (M,d).

Remark: The exercise tells us one thing that the property of compact is not changed, but we should note the property of being open may be changed. For example, in the 2 −dimensional Euclidean space, an open interval (a,b) is not open since (a,b) cannot contain any 2 −ball.

3.39 If S is a closed and T is compact, then $S \cap T$ is compact.

Proof: Since T is compact, T is closed. We have $S \cap T$ is closed. Since $S \cap T \subseteq T$, by **Theorem 3.39,** we know that $S \cap T$ is compact.

3.40 The intersection of an arbitrary collection of compact subsets of M is compact.

Proof: Let $F = \{T : T \text{ is compacet in } M\}$, and thus consider $\cap_{T \in F'} T$, where $F' \subseteq F$. We have $\cap_{T \in F'} T$ is closed. Choose $S \in F'$. then we have $\cap_{T \in F'} T \subseteq S$. Hence, by **Theorem 3.39** $\cap_{T \in F'} T$ is compact.

3.41 The union of a finite number of compact subsets of M is cmpact.

Proof: Denote $\{T_k \text{ is a compact subset of } M : k = 1,2,..n\}$ by S. Let F be an open covering of $\cup_{k=1}^{k=n} T_k$. If there does **NOT** exist a finite subcovering of $\cup_{k=1}^{k=n} T_k$, then there does not exist a finite subcovering of T_m for some $T_m \in S$. Since F is also an open covering of T_m, it leads us to get T_m is not compact which is absurb. Hence, if F is an open covering of $\cup_{k=1}^{k=n} T_k$, then there exists a finite subcovering of $\cup_{k=1}^{k=n} T_k$. So, $\cup_{k=1}^{k=n} T_k$ is compact.

3.42 *Consider the metric space Q of rational numbers with the Euclidean metric of R^1. Let S consists of all rational numbers in the open interval (a,b), where a and b are irrational. Then S is a closed and bounded subset of Q which is not compact.*

Proof: Obviously, S is bounded. Let $x \in Q - S$, then $x < a$, or $x > b$. If $x < a$, then $B_Q(x,d) = (x-d,x+d) \cap Q \subseteq Q - S$, where $d = a - x$. Similarly, $x > b$. Hence, x is an interior point of $Q - S$. That is, $Q - S$ is open, or equivalently, S is closed.

Remark: 1. The exercise tells us an counterexample about that in a metric space, a closed and bounded subset is not necessary to be compact.

2. Here is another counterexample. Let M be an infinite set, and thus consider the metric space (M,d) with discrete metric d. Then by the fact $B(x,1/2) = \{x\}$ for any $x \in M$, we know that $F = \{B(x,1/2) : x \in M\}$ forms an open covering of M. It is clear that there does not exist a finite subcovering of M. Hence, M is not compact.

3.In any metric space (M,d), we have three equivalent conditions on compact which list them below. Let $S \subseteq M$.

(a) Given any open covering of S, there exists a finite subcovering of S.

(b) Every infinite subset of S has an accumulation point in S.

(c) S is **totally bounded** and **complete.**

4. It should be note that if we consider the Euclidean space(R^n,d), we have four equivalent conditions on compact which list them below. Let $S \subseteq R^n$.

Remark (a) Given any open covering of S, there exists a finite subcovering of S.

(b) Every infinite subset of S has an accumulation point in S.

(c) S is **totally bounded** and **complete.**

(d) S is bounded and closed.

5. The concept of compact is familar with us since it can be regarded as a extension of **Bolzano – Weierstrass Theorem.**

Miscellaneous properties of the interior and the boundary

If A and B denote arbitrary subsets of a metric space M, prove that:

3.43 $intA = M - cl(M-A)$.

Proof: In order to show the statement, it suffices to show that $M - intA = cl(M-A)$.
1. $(\subseteq)$ Let $x \in M - intA$, we want to show that $x \in cl(M-A)$, i.e., $B(x,r) \cap (M-A) \neq \phi$ for all $r > 0$. Suppose $B(x,d) \cap (M-A) = \phi$ for some $d > 0$. Then $B(x,d) \subseteq A$ which implies that $x \in intA$. It leads us to get a conradiction since $x \in M - intA$. Hence, if $x \in M - intA$, then $x \in cl(M-A)$. That is, $M - intA \subseteq cl(M-A)$.
2. $(\supseteq)$ Let $x \in cl(M-A)$, we want to show that $x \in M - intA$, i.e., x is not an interior point of A. Suppose x is an interior point of A, then $B(x,d) \subseteq A$ for some $d > 0$. However, since $x \in cl(M-A)$, then $B(x,d) \cap (M-A) \neq \phi$. It leads us to get a conradiction since $B(x,d) \subseteq A$. Hence, if $x \in cl(M-A)$, then $x \in M - intA$. That is, $cl(M-A) \supseteq M - intA$.
From 1 and 2, we know that $M - intA = cl(M-A)$, or equvilantly, $intA = M - cl(M-A)$.

3.44 $int(M - A) = M - \bar{A}$.

Proof: Let $B = M - A$, and by exercise 3.33, we know that

$$M - intB = cl(M - B)$$

which implies that

$$intB = M - cl(M - B)$$

which implies that

$$int(M - A) = M - cl(A).$$

3.45 $int(intA) = intA$.

Proof: Since S is open if, and only if, $S = intS$. Hence, Let $S = intA$, we have the equality $int(intA) = intA$.

3.46

(a) $int(\cap_{i=1}^{n} A_i) = \cap_{i=1}^{n} (intA_i)$, *where each* $A_i \subseteq M$.

Proof: We prove the equality by considering two steps.

(1) ($\subseteq$) Since $\cap_{i=1}^{n} A_i \subseteq A_i$ for all $i = 1,2,\ldots,n$, then $int(\cap_{i=1}^{n} A_i) \subseteq intA_i$ for all $i = 1,2,\ldots,n$. Hence, $int(\cap_{i=1}^{n} A_i) \subseteq \cap_{i=1}^{n} (intA_i)$.

(2) ($\supseteq$) Since $intA_i \subseteq A_i$, then $\cap_{i=1}^{n} (intA_i) \subseteq \cap_{i=1}^{n} A_i$. Since $\cap_{i=1}^{n} (intA_i)$ is open, we have

$$\cap_{i=1}^{n} (intA_i) \subseteq int(\cap_{i=1}^{n} A_i).$$

From (1) and (2), we know that $int(\cap_{i=1}^{n} A_i) = \cap_{i=1}^{n} (intA_i)$.

Remark: Note (2), we use the theorem, a finite intersection of an open sets is open. Hence, we ask whether an infinite intersection has the same conclusion or not. Unfortunately, the answer is **NO!** Just see (b) and (c) in this exercise.

(b) $int(\cap_{A \in F} A) \subseteq \cap_{A \in F} (intA)$, *if F is an infinite collection of subsets of M.*

Proof: Since $\cap_{A \in F} A \subseteq A$ for all $A \in F$. Then $int(\cap_{A \in F} A) \subseteq intA$ for all $A \in F$. Hence, $int(\cap_{A \in F} A) \subseteq \cap_{A \in F} (intA)$.

(c) Give an example where eqaulity does not hold in (b).

Proof: Let F=$\{(\frac{-1}{n}, \frac{1}{n}) : n \in N\}$, then $int(\cap_{A \in F} A) = \phi$, and $\cap_{A \in F} (intA) = \{0\}$. So, we can see that in this case, $int(\cap_{A \in F} A)$ is a proper subset of $\cap_{A \in F} (intA)$. Hence, the equality does not hold in (b).

Remark: The key to find the counterexample, it is similar to find an example that an infinite intersection of opens set is not open.

3.47

(a) $\cup_{A \in F} (intA) \subseteq int(\cup_{A \in F} A)$.

Proof: Since $intA \subseteq A$, $\cup_{A \in F} (intA) \subseteq \cup_{A \in F} A$. We have $\cup_{A \in F} (intA) \subseteq int(\cup_{A \in F} A)$ since $\cup_{A \in F} (intA)$ is open.

(b) Give an example of a finite collection F in which equality does not hold in (a).

Solution: Consider $F = \{Q, Q^c\}$, then we have $intQ \cup intQ^c = \phi$ and $int(Q \cup Q^c) = intR^1 = R^1$. Hence, $(intQ) \cup (intQ^c) = \phi$ is a proper subset of $int(Q \cup Q^c) = R^1$. That is, the equality does not hold in (a).

3.48

(a) $int(\partial A) = \phi$ if A is open or if A is closed in M.

Proof: (1) Suppose that A is open. We prove it by the method of contradiction. Assume that $int(\partial A) \neq \phi$, and thus choose

$$\begin{aligned} x &\in int(\partial A) \\ &= int(cl(A) \cap cl(M - A)) \\ &= int(cl(A) \cap (M - A)) \\ &= int(cl(A)) \cap int(M - A) \text{ since } int(S \cap T) = int(S) \cap int(T). \end{aligned}$$

Since

$$x \in int(cl(A)) \Rightarrow B(x, r_1) \subseteq cl(A) = A \cup A'$$

and

$$x \in int(M - A) \Rightarrow B(x, r_2) \subseteq M - A = A^c \quad *$$

we choose $r = \min(r_1, r_2)$, then $B(x,r) \subseteq (A \cup A') \cap A^c = A' \cap A^c$. However,

$$x \in A' \text{ and } x \notin A \Rightarrow B(x,r) \cap A \neq \phi \text{ for this } r. \quad **$$

Hence, we get a contradiction since

$$B(x,r) \cap A = \phi \text{ by (*)}$$

and

$$B(x,r) \cap A \neq \phi \text{ by (**)}.$$

That is, $int(\partial A) = \phi$ if A is open.

(2) Suppose that A is closed, then we have $M - A$ is open. By (1), we have

$$int(\partial(M - A)) = \phi.$$

Note that

$$\begin{aligned} \partial(M - A) &= cl(M - A) \cap cl(M - (M - A)) \\ &= cl(M - A) \cap cl(A) \\ &= \partial A \end{aligned}$$

. Hence, $int(\partial A) = \phi$ if A is closed.

(b) Give an example in which $int(\partial A) = M$.

Solution: Let $M = R^1$, and $A = Q$, then $\partial A = cl(A) \cap cl(M - A) = cl(Q) \cap cl(Q^c) = R^1$. Hence, we have $R^1 = int(\partial A) = M$.

3.49 If $intA = intB = \phi$ and if A is closed in M, then $int(A \cup B) = \phi$.

Proof: Assume that $int(A \cup B) \neq \phi$, then choose $x \in int(A \cup B)$, then there exists $B(x,r) \subseteq A \cup B$ for some $r > 0$. In addition, since $intA = \phi$, we find that $B(x,r) \nsubseteq A$. Hence, $B(x,r) \cap (B - A) \neq \phi$. It implies $B(x,r) \cap (M - A) \neq \phi$. Choose $y \in B(x,r) \cap (M - A)$, then we have

$$y \in B(x,r) \Rightarrow B(y, \varepsilon_1) \subseteq B(x,r), \text{ where } 0 < \varepsilon_1 < r$$

and

$$y \in M - A \Rightarrow B(y, \varepsilon_2) \subseteq M - A, \text{ forsome } \varepsilon_2 > 0.$$

Choose $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, then we have

$$\begin{aligned} B(y, \varepsilon) &\subseteq B(x,r) \cap (M - A) \\ &\subseteq (A \cup B) \cap A^c \\ &\subseteq B. \end{aligned}$$

That is, $intB \neq \phi$ which is absurb. Hence, we have $int(A \cup B) = \phi$.

3.50 Give an example in which $intA = intB = \phi$ *but* $int(A \cup B) = M$.

Solution: Consider the Euclidean sapce $(R^1, |.|)$. Let $A = Q$, and $B = Q^c$, then $intA = intB = \phi$ but $int(A \cup B) = R^1$.

3.51 $\partial A = cl(A) \cap cl(M - A)$ *and* $\partial A = \partial(M - A)$.

Proof: By the definition of the boundary of a set, it is clear that $\partial A = cl(A) \cap cl(M - A)$. In addition, $\partial A = cl(A) \cap cl(M - A)$, and $\partial(M - A) = cl(M - A) \cap cl(M - (M - A)) = cl(M - A) \cap cl(A)$. Hence, we have $\partial A = \partial(M - A)$.

Remark: It had better regard the exercise as a formula.

3.52 If $cl(A) \cap cl(B) = \phi$, *then* $\partial(A \cup B) = \partial A \cup \partial B$.

Proof: We prove it by two steps.

(1) ($\subseteq$) Let $x \in \partial(A \cup B)$, then for all $r > 0$,

$$B(x,r) \cap (A \cup B) \neq \phi \Rightarrow [B(x,r) \cap A] \cup [B(x,r) \cap B] \neq \phi$$

and

$$B(x,r) \cap [(A \cup B)^c] \neq \phi \Rightarrow B(x,r) \cap A^c \cap B^c \neq \phi \qquad *$$

Note that at least one of $[B(x,r) \cap A]$ and $[B(x,r) \cap B]$ is not empty. Without loss of generality, we say $[B(x,r) \cap A] \neq \phi$. Then by (*), we have for all $r > 0$,

$$B(x,r) \cap A \neq \phi, \text{ and } B(x,r) \cap A^c \neq \phi.$$

That is, $x \in \partial A$. Hence, we have proved $\partial(A \cup B) \subseteq \partial A \cup \partial B$.

(2) ($\supseteq$) Let $x \in \partial A \cup \partial B$. Without loss of generality, we let $x \in \partial A$. Then

$$B(x,r) \cap A \neq \phi, \text{ and } B(x,r) \cap A^c \neq \phi.$$

Since $B(x,r) \cap A \neq \phi$, we have

$$B(x,r) \cap (A \cup B) = (B(x,r) \cap A) \cup (B(x,r) \cap B) \neq \phi. \qquad **$$

Claim that $B(x,r) \cap [(A \cup B)^c] = B(x,r) \cap A^c \cap B^c \neq \phi$. Suppsoe **NOT,** it means that $B(x,r) \cap A^c \cap B^c = \phi$. Then we have

$$B(x,r) \subseteq A \Rightarrow B(x,r) \subseteq cl(A)$$

and

$$B(x,r) \subseteq B \Rightarrow B(x,r) \subseteq cl(B).$$

It implies that by hypothesis, $B(x,r) \subseteq cl(A) \cap cl(B) = \phi$ which is absurb. Hence, we have proved the claim. We have proved that

$$B(x,r) \cap (A \cup B) \neq \phi \text{ by(**)}.$$

and

$$B(x,r) \cap [(A \cup B)^c] \neq \phi.$$

That is, $x \in \partial(A \cup B)$. Hence, we have proved $\partial(A \cup B) \supseteq \partial A \cup \partial B$.

From (1) and (2), we have proved that $\partial(A \cup B) = \partial A \cup \partial B$.

Supplement on a separable metric space

Definition (Base) A collection $\{V_\alpha\}$ of open subsets of X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have

$$x \in V_\alpha \subseteq G \text{ for some } \alpha.$$

In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Theorem Every separable metric space has a countable base.

Proof: Let (M,d) be a separable metric space with $S = \{x_1,\ldots,x_n,\ldots\}$ satisfying $cl(S) = M$. Consider a collection $\{B(x_i,\frac{1}{k}) : i,k \in N\}$, then given any $x \in M$ and $x \in G$, where G is open in X, we have $B(x,\delta) \subseteq G$ for some $\delta > 0$. Since S is dense in M, we know that there is a set $B(x_i,\frac{1}{k})$ for some i, k, such that $x \in B(x_i,\frac{1}{k}) \subseteq B(x,\delta) \subseteq G$. So, we know that M has a countable base.

Corollary R^k, where $k \in N$, has a countable base.

Proof: Since R^k is separable, by Theorem 1, we know that R^k has a countable base.

Theorem Every compact metric space is separable.

Proof: Let (K,d) be a compact metric space, and given a radius $1/n$, we have

$$K \subseteq \cup_{i=1}^{p} B\left(x_i^{(n)}, 1/n\right).$$

Let $S = \left\{x_i^{(n)} : i,n \in N\right\}$, then it is clear S is countable. In order to show that S is dense in K, given $x \in K$, we want to show that x is an adherent point of S. Consider $B(x,\delta)$ for any $\delta > 0$, there is a point $x_i^{(n)}$ in S such that $B\left(x_i^{(n)}, 1/n\right) \subseteq B(x,\delta)$ since $1/n \to 0$. Hence, we have shown that $B(x,\delta) \cap S \neq \phi$. That is, $x \in cl(S)$ which implies that $K = cl(S)$. So, we finally have K is separable.

Corollary Every compact metric space has a countable base.

Proof: It is immediately from Theorem 1.

Remark This corallary can be used to show that **Arzela-Ascoli Theorem**.

Limits And Continuity

Limits of sequence

4.1 Prove each of the following statements about sequences in C.

(a) $z^n \to 0$ if $|z| < 1$; $\{z^n\}$ diverges if $|z| > 1$.

Proof: For the part: $z^n \to 0$ if $|z| < 1$. Given $\varepsilon > 0$, we want to find that there exists a positive integer N such that as $n \geq N$, we have

$$|z^n - 0| < \varepsilon.$$

Note that $\log|z| < 0$ since $|z| < 1$, hence if we choose a positive integer $N \geq \left[\log_{|z|}\varepsilon\right] + 1$, then as $n \geq N$, we have

$$|z^n - 0| < \varepsilon.$$

For the part: $\{z^n\}$ diverges if $|z| > 1$. Assume that $\{z^n\}$ converges to L, then given $\varepsilon = 1$, there exists a positive integer N_1 such that as $n \geq N_1$, we have

$$|z^n - L| < 1 (= \varepsilon)$$

$$\Rightarrow |z|^n < 1 + |L|. \quad *$$

However, note that $\log|z| > 0$ since $|z| > 1$, if we choose a positive integer $N \geq \max\left(\left[\log_{|z|} 1 + |L|\right] + 1, N_1\right)$, then we have

$$|z|^N > 1 + |L|$$

which contradicts (*). Hence, $\{z^n\}$ diverges if $|z| > 1$.

Remark: 1. Given any complex number $z \in C - \{0\}$, $\lim_{n\to\infty}|z|^{1/n} = 1$.

2. Keep $\lim_{n\to\infty}(n!)^{1/n} = \infty$ in mind.

3. In fact, $\{z^n\}$ is unbounded if $|z| > 1$. ($\Rightarrow$ $\{z^n\}$ diverges if $|z| > 1$.) Since given $M > 1$, and choose a positive integer $N = \left[\log_{|z|}M\right] + 1$, then $|z|^N \geq M$.

(b) If $z_n \to 0$ and if $\{c_n\}$ is bounded, then $\{c_n z_n\} \to 0$.

Proof: Since $\{c_n\}$ is bounded, say its bound M, i.e., $|c_n| \leq M$ for all $n \in N$. In addition, since $z_n \to 0$, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|z_n - 0| < \varepsilon/M$$

which implies that as $n \geq N$, we have

$$|c_n z_n| \leq M|z_n| < \varepsilon.$$

That is, $\lim_{n\to\infty} c_n z_n = 0$.

(c) $z^n/n! \to 0$ for every complex z.

Proof: Given a complex z, and thus find a positive integer N such that $|z| \leq N/2$. Consider (let $n > N$).

$$\left|\frac{z^n}{n!}\right| = \left|\left(\frac{z^N}{N!}\right)\left(\frac{z^{n-N}}{(N+1)(N+2)\cdots n}\right)\right| \leq \left|\frac{z^N}{N!}\right|\left(\frac{1}{2}\right)^{n-N} \to 0 \text{ as } n \to \infty.$$

Hence, $z^n/n! \to 0$ for every complex z.

Remark: There is another proof by using the fact $\sum_{n=1}^{\infty} a_n$ converges which implies $a_n \to 0$. Since $\sum_{n=1}^{\infty} \frac{z^n}{n!}$ converges by **ratio test** for every complex z, then we have

$z^n/n! \to 0$ for every complex z.

(d) If $a_n = \sqrt{n^2+2} - n$, then $a_n \to 0$.

Proof: Since

$$0 < a_n = \sqrt{n^2+2} - n = \frac{2}{\sqrt{n^2+2}+n} \leq \frac{1}{n} \text{ for all } n \in N,$$

and $\lim_{n\to\infty} 1/n = 0$, we have $a_n \to 0$ as $n \to \infty$ by **Sandwich Theorem.**

4.2 If $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \geq 1$, show that $a_n \to (a_1 + 2a_2)/3$. Hint: $a_{n+2} - a_{n+1} = \frac{1}{2}(a_n - a_{n+1})$.

Proof: Since $a_{n+2} = (a_{n+1} + a_n)/2$ for all $n \geq 1$, we have $b_{n+1} = -b_n/2$, where $b_n = a_{n+1} - a_n$. So, we have

$$b_{n+1} = \left(\frac{-1}{2}\right)^n b_1 \to 0 \text{ as } n \to \infty. \qquad *$$

Consider

$$a_{n+2} - a_2 = \sum_{k=2}^{n+1} b_k = \frac{-1}{2}\sum_{k=1}^{n} b_k = \left(\frac{-1}{2}\right)(a_{n+1} - a_1)$$

which implies that

$$b_n + \left(\frac{3a_{n+1}}{2}\right) = \frac{a_1 + 2a_2}{2}.$$

So we have

$$a_n \to (a_1 + 2a_2)/3 \text{ by } (*).$$

4.3 If $0 < x_1 < 1$ and if $x_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$, prove that $\{x_n\}$ is a decreasing sequence with limit 0. Prove also that $x_{n+1}/x_n \to \frac{1}{2}$.

Proof: Claim that $0 < x_n < 1$ for all $n \in N$. We prove the claim by **Mathematical Induction.** As $n = 1$, there is nothing to prove. Suppose that $n = k$ holds, i.e., $0 < x_k < 1$, then as $n = k+1$, we have

$$0 < x_{k+1} = 1 - \sqrt{1 - x_k} < 1 \text{ by induction hypothesis.}$$

So, by **Mathematical Induction,** we have proved the claim. Use the claim, and then we have

$$x_{n+1} - x_n = (1 - x_n) - \sqrt{1 - x_n} = \frac{x_n(x_n - 1)}{(1-x_n)^2 + (1 - x_n)} < 0 \text{ since } 0 < x_n < 1.$$

So, we know that the sequence $\{x_n\}$ is a decreasing sequence. Since $0 < x_n < 1$ for all $n \in N$, by Completeness of R, (That is, **a monotonic sequence in R which is bounded is a convergent sequence.**) Hence, we have proved that $\{x_n\}$ is a convergent sequence, denoted its limit by x. Note that since

$$x_{n+1} = 1 - \sqrt{1 - x_n} \text{ for all } n \in N,$$

we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 1 - \sqrt{1 - x_n} = 1 - \sqrt{1-x}$ which implies $x(x-1) = 0$. Since $\{x_n\}$ is a decreasing sequence with $0 < x_n < 1$ for all $n \in N$, we finally have $x = 0$.

For proof of $x_{n+1}/x_n \to \frac{1}{2}$. Since

$$\lim_{x\to 0}\frac{1 - \sqrt{1-x}}{x} = \frac{1}{2} \qquad *$$

then we have

$$\frac{x_{n+1}}{x_n} = \frac{1-\sqrt{1-x_n}}{x_n} \to \frac{1}{2}.$$

Remark: In (*), it is the derivative of $1-\sqrt{1-x}$ at the point $x = 0$. Of course, we can prove (*) by **L-Hospital Rule.**

4.4 Two sequences of positive integers $\{a_n\}$ and $\{b_n\}$ are defined recursively by taking $a_1 = b_1 = 1$ and equating rational and irrational parts in the equation

$$a_n + b_n\sqrt{2} = \left(a_{n-1} + b_{n-1}\sqrt{2}\right)^2 \text{ for } n \geq 2.$$

Prove that $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$. Deduce that $a_n/b_n \to \sqrt{2}$ through values $> \sqrt{2}$, and that $2b_n/a_n \to \sqrt{2}$ through values $< \sqrt{2}$.

Proof: Note $a_n + b_n\sqrt{2} = \left(a_{n-1} + b_{n-1}\sqrt{2}\right)^2$ for $n \geq 2$, we have

$$a_n = a_{n-1}^2 + 2b_{n-1}^2 \text{ for } n \geq 2, \text{and} \qquad *$$

$$b_n = 2a_{n-1}b_{n-1} \text{ for } n \geq 2$$

since if $A, B, C,$ and $D \in N$ with $A + B\sqrt{2} = C + D\sqrt{2}$, then $A = C$, and $B = D$.

Claim that $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$. We prove the claim by **Mathematical Induction.** As $n = 2$, we have by (*)
$a_2^2 - 2b_2^2 = (a_1^2 + 2b_1^2)^2 - 2(2a_1b_1)^2 = (1+2)^2 - 2(2)^2 = 1$. Suppose that as $n = k(\geq 2)$ holds, i.e., $a_k^2 - 2b_k^2 = 1$, then as $n = k+1$, we have by (*)

$$\begin{aligned} a_{k+1}^2 - 2b_{k+1}^2 &= (a_k^2 + 2b_k^2)^2 - 2(2a_kb_k)^2 \\ &= a_k^4 + 4b_k^4 - 4a_k^2b_k^2 \\ &= (a_k^2 - 2b_k^2)^2 \\ &= 1 \text{ by induction hypothesis.} \end{aligned}$$

Hence, by **Mathematical Induction,** we have proved the claim. Note that $a_n^2 - 2b_n^2 = 1$ for all $n \geq 2$, we have

$$\left(\frac{a_n}{b_n}\right)^2 = \left(\frac{1}{b_n}\right)^2 + 2 > 2$$

and

$$\left(\frac{2b_n}{a_n}\right)^2 = 2 - \frac{2}{a_n^2} < 2.$$

Hence, $\lim_{n\to\infty}\frac{a_n}{b_n} = \sqrt{2}$ by $\lim_{n\to\infty}\frac{1}{b_n} = 0$ from (*) through values $> \sqrt{2}$, and $\lim_{n\to\infty}\frac{2b_n}{a_n} = \sqrt{2}$ by $\lim_{n\to\infty}\frac{1}{a_n} = 0$ from (*) through values $< \sqrt{2}$.

Remark: From (*), we know that $\{a_n\}$ and $\{b_n\}$ is increasing since $\{a_n\} \subseteq N$ and $\{b_n\} \subseteq N$. That is, we have $\lim_{n\to\infty} a_n = \infty$, and $\lim_{n\to\infty} b_n = \infty$.

4.5 A real sequence $\{x_n\}$ satisfies $7x_{n+1} = x_n^3 + 6$ for $n \geq 1$. If $x_1 = \frac{1}{2}$, prove that the sequence increases and find its limit. What happens if $x_1 = \frac{3}{2}$ or if $x_1 = \frac{5}{2}$?

Proof: Claim that if $x_1 = \frac{1}{2}$, then $0 < x_n < 1$ for all $n \in N$. We prove the claim by **Mathematical Induction**. As $n = 1$, $0 < x_1 = \frac{1}{2} < 1$. Suppose that $n = k$ holds, i.e., $0 < x_k < 1$, then as $n = k+1$, we have

$$0 < x_{k+1} = \frac{x_k^3 + 6}{7} < \frac{1+6}{7} = 1 \text{ by induction hypothesis.}$$

Hence, we have prove the claim by **Mathematical Induction**. Since
$x^3 - 7x + 6 = (x+3)(x-1)(x-2)$, then

$$
\begin{aligned}
x_{n+1} - x_n &= \frac{x_n^3 + 6}{7} - x_n \\
&= \frac{x_n^3 - 7x_n + 6}{7} \\
&> 0 \text{ since } 0 < x_n < 1 \text{ for all } n \in N.
\end{aligned}
$$

It means that the sequence $\{x_n\}$ (strictly) increasing. Since $\{x_n\}$ is bounded, by completeness of R, we know that he sequence $\{x_n\}$ is convergent, denote its limit by x. Since

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^3 + 6}{7} = \frac{x^3 + 6}{7},$$

we find that $x = -3, 1,$ or 2. Since $0 < x_n < 1$ for all $n \in N$, we finally have $x = 1$.

Claim that if $x_1 = \frac{3}{2}$, then $1 < x_n < 2$ for all $n \in N$. We prove the claim by **Mathematical Induction**. As $n = 1$, there is nothing to prove. Suppose $n = k$ holds, i.e., $1 < x_k < 2$, then as $n = k + 1$, we have

$$1 = \frac{1 + 6}{7} < x_{k+1} = \frac{x_k^3 + 6}{7} < \frac{2^3 + 6}{7} = 2.$$

Hence, we have prove the claim by **Mathematical Induction**. Since $x^3 - 7x + 6 = (x + 3)(x - 1)(x - 2)$, then

$$
\begin{aligned}
x_{n+1} - x_n &= \frac{x_n^3 + 6}{7} - x_n \\
&= \frac{x_n^3 - 7x_n + 6}{7} \\
&< 0 \text{ since } 1 < x_n < 2 \text{ for all } n \in N.
\end{aligned}
$$

It means that the sequence $\{x_n\}$ (strictly) decreasing. Since $\{x_n\}$ is bounded, by completeness of R, we know that he sequence $\{x_n\}$ is convergent, denote its limit by x. Since

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n^3 + 6}{7} = \frac{x^3 + 6}{7},$$

we find that $x = -3, 1,$ or 2. Since $1 < x_n < 2$ for all $n \in N$, we finally have $x = 1$.

Claim that if $x_1 = \frac{5}{2}$, then $x_n > \frac{5}{2}$ for all $n \in N$. We prove the claim by **Mathematical Induction**. As $n = 1$, there is nothing to prove. Suppose $n = k$ holds, i.e., $x_k > \frac{5}{2}$, then as $n = k + 1$,

$$x_{k+1} = \frac{x_k^3 + 6}{7} > \frac{\left(\frac{5}{2}\right)^3 + 6}{7} = \frac{173}{56} > 3 > \frac{5}{2}.$$

Hence, we have proved the claim by **Mathematical Induction**. If $\{x_n\}$ was convergent, say its limit x. Then the possibilities for $x = -3, 1,$ or 2. However, $x_n > \frac{5}{2}$ for all $n \in N$. So, $\{x_n\}$ diverges if $x_1 = \frac{5}{2}$.

Remark: Note that in the case $x_1 = 5/2$, we can show that $\{x_n\}$ is increasing by the same method. So, it implies that $\{x_n\}$ is unbounded.

4.6 If $|a_n| < 2$ and $|a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1}^2 - a_n^2|$ for all $n \geq 1$, prove that $\{a_n\}$ converges.

Proof: Let $a_{n+1} - a_n = b_n$, then we have $|b_{n+1}| \leq \frac{1}{8}|b_n||a_{n+1} + a_n| \leq \frac{1}{2}|b_n|$, since $|a_n| < 2$ for all $n \geq 1$. So, we have $|b_{n+1}| \leq \left(\frac{1}{2}\right)^n |b_1|$. Consider (Let $m > n$)

$$
\begin{aligned}
|a_m - a_n| &= |(a_m - a_{m-1}) + (a_{m-1} - a_{m-2}) + \ldots + (a_{n+1} - a_n)| \\
&\leq |b_{m-1}| + \ldots + |b_n| \\
&\leq |b_1| \left[\left(\frac{1}{2}\right)^{m-2} + \ldots + \left(\frac{1}{2}\right)^{n-1} \right],
\end{aligned} \quad *
$$

then $\{a_n\}$ is a Cauchy sequence since $\sum(\frac{1}{2})^k$ converges. Hence, we know that $\{a_n\}$ is a convergent sequence.

Remark: In this exercise, we use the very important theorem, every Cauchy sequence in the Euclidean space R^k is convergent.

4.7 In a metric space (S,d), assume that $x_n \to x$ and $y_n \to y$. Prove that $d(x_n,y_n) \to d(x,y)$.

Proof: Since $x_n \to x$ and $y_n \to y$, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$d(x_n,x) < \varepsilon/2 \text{ and } d(y_n,y) < \varepsilon/2.$$

Hence, as $n \geq N$, we have

$$
\begin{aligned}
|d(x_n,y_n) - d(x,y)| &\leq |d(x_n,x) + d(y_n,y)| \\
&= d(x_n,x) + d(y_n,y) < \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}
$$

So, it means that $d(x_n,y_n) \to d(x,y)$.

4.8 Prove that in a compact meric space (S,d), every sequence in S has a subsequence which converges in S. This property also implies that S is compact but you are not required to prove this. (For a proof see either Reference 4.2 or 4.3.)

Proof: Given a sequence $\{x_n\} \subseteq S$, and let $T = \{x_1,x_2,\ldots\}$. If the range of T is finite, there is nothing to prove. So, we assume that the range of T is infinite. Since S is compact, and $T \subseteq S$, we have T has a accumulation point x in S. So, there exists a point y_n in T such that $B(y_n,x) < \frac{1}{n}$. It implies that $y_n \to x$. Hence, we have proved that every sequence in S has a subsequence which converges in S.

Remark: If every sequence in S has a subsequence which converges in S, then S is compact. We give a proof as follows.

Proof: In order to show S is compact, it suffices to show that every infinite subset of S has an accumulation point in S. Given any infinite subset T of S, and thus we choose $\{x_n\} \subseteq T$ (of course in S). By hypothesis, $\{x_n\}$ has a subsequence $\{x_{k(n)}\}$ which converges in S, say its limit x. From definition of limit of a sequence, we know that x is an accumulation of T. So, S is compact.

4.9 Let A be a subset of a metric space S. If A is complete, prove that A is closed. Prove that the converse also holds if S is complete.

Proof: Let x be an accumulation point of A, then there exists a sequence $\{x_n\}$ such that $x_n \to x$. Since $\{x_n\}$ is convergent, we know that $\{x_n\}$ is a Cauchy sequence. And A is complete, we have $\{x_n\}$ converges to a point $y \in A$. By uniqueness, we know $x = y \in A$. So, A contains its all accumulation points. That is, A is closed.

Suppose that S is complete and A is closed in S. Given any Cauchy sequence $\{x_n\} \subseteq A$, we want to show $\{x_n\}$ is converges to a point in A. Trivially, $\{x_n\}$ is also a Cauchy sequence in S. Since S is complete, we know that $\{x_n\}$ is convergent to a point x in S. By definition of limit of a sequence, it is easy to know that x is an adherent point of A. So, $x \in A$ since A is closed. That is, every Cauchy sequence in A is convergent. So, A is complete.

Supplement

1. Show that the sequence

$$\lim_{n\to+\infty}\frac{(2n)!!}{(2n+1)!!}=0.$$

Proof: Let a and b be positive integers satisfying $a\geq b>1.$ Then we have

$$a!b\leq a!b!\leq(a+b)!\leq(ab)!. \qquad *$$

So, if we let $f(n)=(2n)!,$ then we have, by (*)

$$\frac{(2n)!!}{(2n+1)!!}=\frac{f(n)!}{(f(n)(2n+1))!}\leq\frac{1}{2n+1}\to 0.$$

Hence, we know that $\lim_{n\to+\infty}\frac{(2n)!!}{(2n+1)!!}=0.$

2. Show that

$$a_n=\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{[\sqrt{n}\,]}{n}\right)\to e^{1/2}\text{ as }n\to\infty,$$

where $[x]$ means **Gauss Symbol.**

Proof: Since

$$x-\frac{1}{2}x^2\leq\log(1+x)\leq x,\text{ for all }x\in(-1,1)$$

we have

$$\sum_{k=1}^{k=[\sqrt{n}\,]}\frac{k}{n}-\frac{1}{2}\left(\frac{k}{n}\right)^2\leq\log a_n=\sum_{k=1}^{k=[\sqrt{n}\,]}\log\left(1+\frac{k}{n}\right)\leq\sum_{k=1}^{k=[\sqrt{n}\,]}\frac{k}{n}$$

Consider $i^2<n\leq(i+1)^2,$ then by Sandwish Theorem, we know that

$$\lim_{n\to\infty}\log a_n=1/2$$

which implies that $a_n\to e^{1/2}$ as $n\to\infty.$

3. Show that $(n!)^{1/n}\geq\sqrt{n}$ for all $n\in N.$ ($\Rightarrow (n!)^{1/n}\to\infty$ as $n\to\infty.$)

Proof: We prove it by a special method following Gauss' method. Consider

$$n!=1\cdots\cdots\cdots k\cdots\cdots\cdots n$$
$$=n\cdots(n-k+1)\cdots\cdots 1$$

and thus let $f(k):=k(n-k+1),$ it is easy to show that $f(k)\geq f(1)=n$ for all $k=1,2,\ldots,n.$ So, we have prove that

$$(n!)^2\geq n^n$$

which implies that

$$(n!)^{1/n}\geq\sqrt{n}.$$

Remark: There are many and many method to show $(n!)^{1/n}\to\infty$ as $n\to\infty.$ We do not give a detail proofs about it. But We method it as follows as references.

(a) By $A.P.\geq G.P.,$ we have

$$\frac{\sum_{k=1}^{n}\frac{1}{k}}{n}\geq\left(\frac{1}{n!}\right)^{1/n}$$

and use the fact if $\{a_n\}$ converges to a, then so is $\left\{\frac{\sum_{k=1}^{n} a_k}{n}\right\}$.

(b) Use the fact, by **Mathematical Induction**, $(n!)^{1/n} \geq n/3$ for all n.

(c) Use the fact, $A^n/n! \to 0$ as $n \to \infty$ for any real A.
(d) Consider $p(n) = \left(\frac{n!}{n^n}\right)^{1/n}$, and thus taking $\log p(n)$.
(e) Use the famuos formula, a_n are positive for all n.

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf (a_n)^{1/n} \leq \limsup (a_n)^{1/n} \leq \limsup \frac{a_{n+1}}{a_n}$$

and let $a_n = \left(\frac{n!}{n^n}\right)$.
(f) The radius of the power series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ is ∞.
(g) Ue the fact, $(1+1/n)^n \leq e \leq (1+1/n)^{n+1}$, then $e(n^n e^{-n}) \leq n! \leq e(n^{n+1} e^{-n})$.
(h) More.

Limits of functions

Note. In Exercise 4.10 through 4.28, all functions are real valued.

4.10 Let f be defined on an opne interval (a,b) and assume $x \in (a,b)$. Consider the two statements

(a) $\lim_{h\to 0} |f(x+h) - f(x)| = 0$;
(b) $\lim_{h\to 0} |f(x+h) - f(x-h)| = 0$.

Prove that (a) always implies (b), and give an example in which (b) holds but (a) does not.

Proof: (a) Since

$$\lim_{h\to 0} |f(x+h) - f(x)| = 0 \Leftrightarrow \lim_{h\to 0} |f(x-h) - f(x)| = 0,$$

we consider

$$\begin{aligned} &|f(x+h) - f(x-h)| \\ &= |(f(x+h) - f(x)) + (f(x) - f(x-h))| \\ &\leq |f(x+h) - f(x)| + |f(x) - f(x-h)| \to 0 \text{ as } h \to 0. \end{aligned}$$

So, we have

$$\lim_{h\to 0} |f(x+h) - f(x-h)| = 0.$$

(b) Let

$$f(x) = \begin{cases} |x| \text{ if } x \neq 0, \\ 1 \text{ if } x = 0. \end{cases}$$

Then

$$\lim_{h\to 0} |f(0+h) - f(0-h)| = 0,$$

but

$$\lim_{h\to 0} |f(0+h) - f(0)| = \lim_{h\to 0} ||h| - 1| = 1.$$

So, (b) holds but (a) does not.

Remark: In case (b), there is another example,

$$g(x) = \begin{cases} 1/|x| \text{ if } x \neq 0, \\ 0 \text{ if } x = 0. \end{cases}$$

The difference of two examples is that the limit of $|g(0 + h) - g(0)|$ does not exist as h tends to 0.

4.11 Let f be defined on R^2. If

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

and if the one-dimensional $\lim_{x\to a} f(x,y)$ and $\lim_{y\to b} f(x,y)$ both exist, prove that

$$\lim_{x\to a}\left[\lim_{y\to b} f(x,y)\right] = \lim_{y\to b}\left[\lim_{x\to a} f(x,y)\right] = L.$$

Proof: Since $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $0 < |(x,y) - (a,b)| < \delta$, we have

$$|f(x,y) - L| < \varepsilon/2,$$

which implies

$$\lim_{y\to b}|f(x,y) - L| = \left|\lim_{y\to b} f(x,y) - L\right| \leq \varepsilon/2 \text{ if } 0 < |(x,y) - (a,b)| < \delta$$

which implies

$$\lim_{x\to a}\left|\lim_{y\to b} f(x,y) - L\right| \leq \varepsilon/2 \text{ if } 0 < |(x,y) - (a,b)| < \delta.$$

Hence, we have proved $\lim_{x\to a}|\lim_{y\to b}(x,y) - L| \leq \varepsilon/2 < \varepsilon$. Since ε is arbitrary, we have

$$\lim_{x\to a}\left|\lim_{y\to b} f(x,y) - L\right| = 0$$

which implies that

$$\left|\lim_{x\to a}\lim_{y\to b} f(x,y) - L\right| = 0.$$

So, $\lim_{x\to a}[\lim_{y\to b} f(x,y)] = L$. The proof of $\lim_{y\to b}[\lim_{x\to a} f(x,y)] = L$ is similar.

Remark: 1. The exercise is much important since in mathematics, we would encounter many and many similar questions about the interchange of the order of limits. So, we should keep the exercise in mind.

2. In the proof, we use the concept: $|\lim_{x\to a} f(x)| = 0$ if, and only if $\lim_{x\to a} f(x) = 0$.

3. The hypothesis $f(x,y) \to L$ as $(x,y) \to (a,b)$ tells us that every approach form these points (x,y) to the point (a,b), $f(x,y)$ approaches to L. Use this concept, and consider the special approach from points (x,y) to (x,b) and thus from (x,b) to (a,b). Note that since $\lim_{y\to b} f(x,y)$ exists, it means that we can regrad this special approach as one of approaches from these points (x,y) to the point (a,b). So, it is natural to have the statement.

4. The converse of statement is not necessarily true. For example,

$$f(x,y) = \begin{cases} x + y \text{ if } x = 0 \text{ or } y = 0 \\ 1 \quad \text{otherwise.} \end{cases}$$

Trivially, we have the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$. However,

$$\lim_{y\to 0} f(x,y) = \begin{cases} 0 \text{ if } x = 0, \\ 1 \text{ if } x \neq 0. \end{cases} \quad \text{and} \quad \lim_{x\to 0} f(x,y) = \begin{cases} 0 \text{ if } y = 0, \\ 1 \text{ if } y \neq 0. \end{cases}$$

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 1.$$

In each of the preceding examples, determine whether the following limits exist and evaluate those limits that do exist:

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right]; \ \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]; \ \lim_{(x,y)\to(0,0)} f(x,y).$$

Now consider the functions f defined on R^2 as follows:

(a) $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$.

Proof: 1. Since$(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2} = \begin{cases} \frac{-y^2}{y^2} = -1 \text{ if } y \neq 0, \\ 1 \text{ if } y = 0, \end{cases}$$

we have

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = -1.$$

2. Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2} = \begin{cases} \frac{x^2}{x^2} = 1 \text{ if } x \neq 0, \\ 1 \text{ if } x = 0, \end{cases} \qquad **$$

we have

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 1.$$

3. $((x,y) \neq (0,0))$ Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \\ &= \lim_{r\to 0} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2} \\ &= \cos^2\theta - \sin^2\theta. \end{aligned}$$

So, if we choose $\theta = \pi$ and $\theta = \pi/2$, we find the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$.

Remark: 1. This case shows that

$$1 = \lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \neq \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = -1$$

2. Obviously, the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$. Since if it was, then by (*), (**), and the preceding theorem, we know that

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right]$$

which is absurb.

(b) $f(x,y) = \frac{(xy)^2}{(xy)^2+(x-y)^2}$ if $(x,y) \neq (0,0)$, $f(0,0) = 0$.

Proof: 1. Since $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{(xy)^2}{(xy)^2 + (x-y)^2} = 0 \text{ for all } y,$$

we have

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0.$$

2. Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \lim_{y\to 0} \frac{(xy)^2}{(xy)^2 + (x-y)^2} = 0 \text{ for all } x,$$

we have

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 0.$$

3. $((x,y) \neq (0,0))$ Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$\begin{aligned} f(x,y) &= \frac{(xy)^2}{(xy)^2 + (x-y)^2} \\ &= \frac{r^4\cos^2\theta\sin^2\theta}{r^4\cos^2\theta\sin^2\theta + r^2 - 2r^2\cos\theta\sin\theta} \\ &= \frac{\cos^2\theta\sin^2\theta}{\cos^2\theta\sin^2\theta + \frac{1-2\cos\theta\sin\theta}{r^2}}. \end{aligned}$$

So,

$$f(x,y)\begin{cases} \to 0 \text{ if } r \to 0 \\ 1 \quad\quad \text{if } \theta = \pi/4 \text{ or } \theta = 5\pi/4. \end{cases}$$

Hence, we know that the limit of $f(x,y)$ does not exists as $(x,y) \to (0,0)$.

(c) $f(x,y) = \frac{1}{x}\sin(xy)$ if $x \neq 0$, $f(0,y) = y$.

Proof: 1. Since $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{1}{x}\sin(xy) = y \qquad *$$

we have

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 0.$$

2. Since $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0} \frac{1}{x}\sin(xy) = 0 \text{ if } x \neq 0, \\ \lim_{y\to 0} y = 0 \text{ if } x = 0, \end{cases}$$

we have

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 0.$$

3. $((x,y) \neq (0,0))$ Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$f(x,y) = \begin{cases} \frac{1}{r\cos\theta}\sin(r^2\cos\theta\sin\theta) \text{ if } x = r\cos\theta \neq 0, \\ r\sin\theta \quad \text{if } x = r\cos\theta = 0. \end{cases}$$

$$\to \begin{cases} 0 \text{ if } r \to 0, \\ 0 \text{ if } r \to 0. \end{cases} \qquad **$$

So, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 0.$

Remark: In (*) and (**), we use the famuos limit, that is,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

There are some similar limits, we write them without proofs.

(a) $\lim_{t\to\infty} t\sin(1/t) = 1.$
(b) $\lim_{x\to 0} x\sin(1/x) = 0.$
(c) $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$, if $b \neq 0.$

(d) $f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) \text{ if } x \neq 0 \text{ and } y \neq 0, \\ 0 \quad \text{if } x = 0 \text{ or } y = 0. \end{cases}$

Proof: 1. Since $(x \neq 0)$

$$f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) = x\sin(1/x)\sin(1/y) + y\sin(1/x)\sin(1/y) \text{ if } y \neq 0 \\ 0 \text{ if } y = 0 \end{cases}$$

we have if $y \neq 0$, the limit $f(x,y)$ does not exist as $x \to 0$, and if $y = 0$, $\lim_{x\to 0} f(x,y) = 0.$ Hence, we have $(x \neq 0,\ y \neq 0)$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] \text{ does not exist.}$$

2. Since $(y \neq 0)$

$$f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) = x\sin(1/x)\sin(1/y) + y\sin(1/x)\sin(1/y) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

we have if $x \neq 0$, the limit $f(x,y)$ does not exist as $y \to 0$, and if $x = 0$, $\lim_{y\to 0} f(x,y) = 0.$ Hence, we have $(x \neq 0,\ y \neq 0)$

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] \text{ does not exist.}$$

3. $((x,y) \neq (0,0))$ Consider

$$|f(x,y)| \leq \begin{cases} |x+y| \text{ if } x \neq 0 \text{ and } y \neq 0, \\ 0 \text{ if } x = 0 \text{ or } y = 0. \end{cases}$$

we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

(e) $f(x,y) = \begin{cases} \frac{\sin x - \sin y}{\tan x - \tan y}, \text{ if } \tan x \neq \tan y, \\ \cos^3 x \quad \text{if } \tan x = \tan y. \end{cases}$

Proof: Since we consider the three approaches whose tend to $(0,0)$, we may assume that $x,y \in (-\pi/2, \pi/2)$. and note that in this assumption, $x = y \Leftrightarrow \tan x = \tan y$. Consider

1. $(x \neq 0)$

$$\lim_{x\to 0} f(x,y) = \begin{cases} \lim_{x\to 0} \frac{\sin x - \sin y}{\tan x - \tan y} = \cos y \text{ if } x \neq y. \\ 1 \quad \text{if } x = y. \end{cases}$$

So,

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 1.$$

2. $(y \neq 0)$

$$\lim_{y\to 0} f(x,y) = \begin{cases} \lim_{y\to 0} \frac{\sin x - \sin y}{\tan x - \tan y} = \cos x \text{ if } x \neq y. \\ \cos^3 x \quad \text{if } x = y. \end{cases}$$

So,

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 1.$$

3. Let $x = r\cos\theta$ and $y = r\sin\theta$, where $0 \leq \theta < 2\pi$, and note that $(x,y) \to (0,0) \Leftrightarrow r \to 0$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \begin{cases} \lim_{r\to 0} \frac{\sin(r\cos\theta) - \sin(r\sin\theta)}{\tan(r\cos\theta) - \tan(r\sin\theta)} \text{ if } \cos\theta \neq \sin\theta. \\ \lim_{r\to 0} \cos^3(r\cos\theta) \text{ if } \cos\theta = \sin\theta. \end{cases}$$

$$= \begin{cases} 1 \text{ if } \cos\theta \neq \sin\theta, \text{ by } \textbf{L-Hospital Rule.} \\ 1 \text{ if } \cos\theta = \sin\theta. \end{cases}$$

So, we know that $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

Remark: 1. There is another proof about (e)-(3). Consider

$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

and

$$\tan x - \tan y = \frac{\sin x}{\cos x} - \frac{\sin y}{\cos y},$$

then

$$\frac{\sin x - \sin y}{\tan x - \tan y} = \frac{\cos(\frac{x+y}{2})\cos x \cos y}{\cos(\frac{x-y}{2})}.$$

So,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \begin{cases} \lim_{(x,y)\to(0,0)} \frac{\cos\left(\frac{x+y}{2}\right)\cos x \cos y}{\cos\left(\frac{x-y}{2}\right)} = 1 \text{ if } x \neq y, \\ \lim_{(x,y)\to(0,0)} \cos^3 x = 1 \text{ if } x = y. \end{cases}$$

That is, $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

2. In the process of proof, we use the concept that we write it as follows. Since its proof is easy, we omit it. If

$$\lim_{(x,y)\to(a,b)} f(x,y) = \begin{cases} L \text{ if } x = y \\ L \text{ if } x \neq y \end{cases}$$

or

$$\lim_{(x,y)\to(a,b)} f(x,y) = \begin{cases} L \text{ if } x \neq 0 \text{ and } y \neq 0, \\ L \text{ if } x = 0 \text{ or } y = 0. \end{cases}$$

then we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = L.$$

4.12 If $x \in [0,1]$ prove that the following limit exists,

$$\lim_{m\to\infty}\left[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\right],$$

and that its value is 0 or 1, according to whether x is irrational or rational.

Proof: If x is rational, say $x = q/p$, where $g.c.d.(q,p) = 1$, then $p!x \in N$. So,

$$\lim_{n\to\infty} \cos^{2n}(m!\pi x) = \begin{cases} 1 \text{ if } m \geq p, \\ 0 \text{ if } m < p. \end{cases}$$

Hence,

$$\lim_{m\to\infty}\left[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\right] = 1.$$

If x is irrational, then $m!x \notin N$ for all $m \in N$. So, $\cos^{2n}(m!\pi x) < 1$ for all irrational x. Hence,

$$\lim_{n\to\infty} \cos^{2n}(m!\pi x) = 0 \Rightarrow \lim_{m\to\infty}\left[\lim_{n\to\infty} \cos^{2n}(m!\pi x)\right] = 0.$$

Continuity of real-valued functions

4.13 Let f be continuous on $[a,b]$ and let $f(x) = 0$ when x is rational. Prove that $f(x) = 0$ for every $x \in [a,b]$.

Proof: Given any irrational number x in $[a,b]$, and thus choose a sequence $\{x_n\} \subseteq Q$ such that $x_n \to x$ as $n \to \infty$. Note that $f(x_n) = 0$ for all n. Hence,

$$\begin{aligned} 0 &= \lim_{n\to\infty} 0 \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\left(\lim_{n\to\infty} x_n\right) \text{ by continuity of } f \text{ at } x \\ &= f(x). \end{aligned}$$

Since x is arbitrary, we have shown $f(x) = 0$ for all $x \in [a,b]$. That is, f is constant 0.

Remark: Here is another good exercise, we write it as a reference. Let f be continuous on R, and if $f(x) = f(x^2)$, then f is constant.

Proof: Since $f(-x) = f\left((-x)^2\right) = f(x^2) = f(x)$, we know that f is an even function. So, in order to show f is constant on R, it suffices to show that f is constant on $[0,\infty)$. Given any $x \in (0,\infty)$, since $f(x^2) = f(x)$ for all $x \in R$, we have $f(x^{1/2n}) = f(x)$ for all n. Hence,

$$\begin{aligned} f(x) &= \lim_{n\to\infty} f(x) \\ &= \lim_{n\to\infty} f(x^{1/2n}) \\ &= f\left(\lim_{n\to\infty} x^{1/2n}\right) \text{ by continuity of } f \text{ at } 1 \\ &= f(1) \text{ since } x \neq 0. \end{aligned}$$

So, we have $f(x) = f(1) := c$ for all $x \in (0,\infty)$. In addition, given a sequence $\{x_n\} \subseteq (0,\infty)$ such that $x_n \to 0$, then we have

$$\begin{aligned} c &= \lim_{n\to\infty} c \\ &= \lim_{n\to\infty} f(x_n) \\ &= f\left(\lim_{n\to\infty} x_n\right) \text{ by continuity of } f \text{ at } 0 \\ &= f(0) \end{aligned}$$

From the preceding, we have proved that f is constant.

4.14 Let f be continuous at the point $a = (a_1, a_2, \ldots, a_n) \in R^n$. Keep $a_2, a_3, \ldots, a_n$ fixed and define a new function g of one real variable by the equation

$$g(x) = f(x, a_2, \ldots, a_n).$$

Prove that g is continuous at the point $x = a_1$. (This is sometimes stated as follows**: A continuous function of** n **variables is continuous in each variable separately**.)

Proof: Given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $y \in B(a;\delta) \cap D$, where D is a domain of f, we have

$$|f(y) - f(a)| < \varepsilon. \qquad *$$

So, as $|x - a_1| < \delta$, which implies $|(x, a_2, \ldots, a_n) - (a_1, a_2, \ldots, a_n)| < \delta$, we have

$$|g(x) - g(a_1)| = |f(x, a_2, \ldots, a_n) - f(a_1, a_2, \ldots, a_n)| < \varepsilon.$$

Hence, we have proved g is continuous at $x = a_1$

Remark: Here is an important example like the exercise, we write it as follows. Let $\pi_j : R^n \to R^n$, and $\pi_j : (x_1, x_2, \ldots, x_n) = (0,.., x_j,.., 0)$. Then π_j is continuous on R^n for $1 \leq j \leq n$. Note that π_j is called a **projection**. Note that a projection P is sometimes defined as $P^2 = P$.

Proof: Given any point $a \in R^n$,and given $\varepsilon > 0$, and choose $\delta = \varepsilon$, then as $x \in B(a;\delta)$, we have

$$|\pi_j(x) - \pi_j(a)| = |x_j - a_j| \leq \|x - a\| < \delta = \varepsilon \text{ for each } 1 \leq j \leq n$$

Hence, we prove that $\pi_j(x)$ is continuous on R^n for $1 \leq j \leq n$.

4.15 Show by an example that the converse of statement in Exercise 4.14 is not true in general.

Proof: Let

$$f(x,y) = \begin{cases} x + y \text{ if } x = 0 \text{ or } y = 0 \\ 1 \text{ otherwise.} \end{cases}$$

Define $g_1(x) = f(x,0)$ and $g_2(y) = f(0,y)$, then we have

$$\lim_{x\to 0} g_1(x) = 0 = g_1(0)$$

and

$$\lim_{y\to 0} g_2(y) = 0 = g_2(0).$$

So, $g_1(x)$ and $g_2(y)$ are continuous at 0. However, f is not continuous at $(0,0)$ since

$$\lim_{x\to 0} f(x,x) = 1 \neq 0 = f(0,0).$$

Remark: 1. For continuity, if f is continuous at $x = a$, then it is **NOT** necessary for us to have

$$\lim_{x\to a} f(x) = f(a)$$

this is because a can be an isolated point. However, if a is an accumulation point, we then have

$$f \text{ is continuous at } a \text{ if, and only if, } \lim_{x \to a} f(x) = f(a).$$

4.16 Let f, g, and h be defined on $[0,1]$ as follows:

$f(x) = g(x) = h(x) = 0$, whenever x is irrational;

$f(x) = 1$ and $g(x) = x$, whenever x is rational;

$h(x) = 1/n$, if x is the rational number m/n (in lowest terms);

$h(0) = 1$.

Prove that f is not continuous anywhere in $[0,1]$, that g is continuous only at $x = 0$, and that h is continuous only at the irrational points in $[0,1]$.

Proof: 1. Write

$$f(x) = \begin{cases} 0 \text{ if } x \in (R - Q) \cap [0,1], \\ 1 \text{ if } x \in Q \cap [0,1]. \end{cases}$$

Given any $x \in (R - Q) \cap [0,1]$, and $y \in Q \cap [0,1]$, and thus choose $\{x_n\} \subseteq Q \cap [0,1]$ such that $x_n \to x$, and $\{y_n\} \subseteq (R - Q) \cap [0,1]$ such that $y_n \to y$. If f is continuous at x, then

$$\begin{aligned} 1 &= \lim_{n \to \infty} f(x_n) \\ &= f\left(\lim_{n \to \infty} x_n\right) \text{ by continuity of } f \text{ at } x \\ &= f(x) \\ &= 0 \end{aligned}$$

which is absurb. And if f is continuous at y, then

$$\begin{aligned} 0 &= \lim_{n \to \infty} f(y_n) \\ &= f\left(\lim_{n \to \infty} y_n\right) \text{ by continuity of } f \text{ at } y \\ &= f(y) \\ &= 1 \end{aligned}$$

which is absurb. Hence, f is not continuous on $[0,1]$.

2. Write

$$g(x) = \begin{cases} 0 \text{ if } x \in (R - Q) \cap [0,1], \\ x \text{ if } x \in Q \cap [0,1]. \end{cases}$$

Given any $x \in (R - Q) \cap [0,1]$, and choose $\{x_n\} \subseteq Q \cap [0,1]$ such that $x_n \to x$. Then

$$\begin{aligned} &x \\ &= \lim_{n \to \infty} x_n \\ &= \lim_{n \to \infty} g(x_n) \\ &= \lim g\left(\lim_{n \to \infty} x_n\right) \text{ by continuity of } g \text{ at } x \\ &= g(x) \\ &= 0 \end{aligned}$$

which is absurb since x is irrational. So, f is not continous on $(R - Q) \cap [0,1]$.

Given any $x \in Q \cap [0,1]$, and choose $\{x_n\} \subseteq (R - Q) \cap [0,1]$ such that $x_n \to x$. If g is continuous at x, then

$$\begin{aligned}&0\\&=\lim_{n\to\infty} g(x_n)\\&= g\left(\lim_{n\to\infty} x_n\right) \text{ by continuity of } f \text{ at } x\\&= g(x)\\&= x.\end{aligned}$$

So, the function g may be continuous at 0. In fact, g is continuous at 0 which prove as follows. Given $\varepsilon > 0$, choose $\delta = \varepsilon$, as $|x| < \delta$, we have $|g(x) - g(0)| = |g(x)| \leq |x| < \varepsilon(= \delta)$. So, g is continuous at 0. Hence, from the preceding, we know that g is continuous only at $x = 0$.

3. Write

$$h(x) = \begin{cases} 1 \text{ if } x = 0, \\ 0 \text{ if } x \in (R - Q) \cap [0,1], \\ 1/n \text{ if } x = m/n,\ g.c.d.(m,n) = 1. \end{cases}$$

Consider $a \in (0,1)$ and given $\varepsilon > 0$, there exists the largest positive integer N such that $N \leq 1/\varepsilon$. Let $T = \{x : h(x) \geq \varepsilon\}$, then

$$T = \begin{cases} \{0,1\} \cup \{x : h(x) = 1\} \cup \{x : h(x) = 1/2\}\ldots\cup\{x : h(x) = 1/N\} \text{ if } \varepsilon \leq 1, \\ \phi \text{ if } \varepsilon > 1. \end{cases}$$

Note that T is at most a finite set, and then we can choose a $\delta > 0$ such that $(a - \delta, a + \delta) - \{a\}$contains no points of T and $(a - \delta, a + \delta) \subseteq (0,1)$. So, if $x \in (a - \delta, a + \delta) - \{a\}$, we have $h(x) < \varepsilon$. It menas that

$$\lim_{x\to a} h(x) = 0.$$

Hence, we know that h is continuous at $x \in (0,1) \cap (R - Q)$. For two points $x = 1$, and $y = 0$, it is clear that h is not continuous at $x = 1$, and not continuous at $y = 1$ by the method mentioned in the exercise of part 1 and part 2. Hence, we have proved that h is continuous only at the irrational points in $[0,1]$.

Remark: 1. Sometimes we call f **Dirichlet function.**

2. Here is another proof about g, we write it down to make the reader get more.

Proof: Write

$$g(x) = \begin{cases} 0 \text{ if } x \in (R - Q) \cap [0,1], \\ x \text{ if } x \in Q \cap [0,1]. \end{cases}$$

Given $a \in (0,1]$, and if g is continuous at a, then given $0 < \varepsilon < a$, there exists a $\delta > 0$ such that as $x \in (a - \delta, a + \delta) \subseteq [0,1]$, we have

$$|g(x) - g(a)| < \varepsilon.$$

If $a \in R - Q$, choose $0 < \delta' < \delta$ so that $a + \delta' \in Q$. Then $a + \delta' \in (a - \delta, a + \delta)$ which implies $|g(a + \delta') - g(a)| = |g(a + \delta')| = a + \delta' < \varepsilon < a$. But it is impossible.

If $a \in Q$, choose $0 < \delta' < \delta$ so that $a + \delta' \in R - Q$. $a + \delta' \in (a - \delta, a + \delta)$ which implies $|g(a + \delta') - g(a)| = |-a| = a < \varepsilon < a$. But it is impossible.

If $a = 0$, given $\varepsilon > 0$ and choose $\delta = \varepsilon$, then as $0 \leq x < \delta$, we have $|g(x) - g(0)| = |g(x)| \leq |x| = x < \varepsilon(= \delta)$. It means that g is continuous at 0.

4.17 For each $x \in [0,1]$, let $f(x) = x$ if x is rational, and let $f(x) = 1 - x$ if x is irrational. Prove that:

(a) $f(f(x)) = x$ for all x in $[0,1]$.

Proof: If x is rational, then $f(f(x)) = f(x) = x$. And if x is irrarional, so is $1-x(\in [0,1])$. Then $f(f(x)) = f(1-x) = 1-(1-x) = x$. Hence, $f(f(x)) = x$ for all x in $[0,1]$.

(b) $f(x)+f(1-x) = 1$ for all x in $[0,1]$.

Proof: If x is rational, so is $1-x \in [0,1]$. Then $f(x)+f(1-x) = x+(1-x) = 1$. And if x is irrarional, so is $1-x(\in [0,1])$. Then $f(x)+f(1-x) = (1-x)+1-(1-x) = 1$. Hence, $f(x)+f(1-x) = 1$ for all x in $[0,1]$.

(c) f is continuous only at the point $x = \frac{1}{2}$.

Proof: If f is continuous at x, then choose $\{x_n\} \subseteq Q$ and $\{y_n\} \subseteq Q^c$ such that $x_n \to x$, and $y_n \to x$. Then we have, by continuity of f at x,

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x$$

and

$$f(x) = f\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} 1-y_n = 1-x.$$

So, $x = 1/2$ is the only possibility for f. Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $x \in (1/2-\delta, 1/2+\delta) \subseteq [0,1]$, we have

$$|f(x)-f(1/2)| = |f(x)-1/2| < \varepsilon.$$

Choose $(0 <)\delta < \varepsilon$ so that $(1/2-\delta, 1/2+\delta) \subseteq [0,1]$, then as $x \in (1/2-\delta, 1/2+\delta) \subseteq [0,1]$, we have

$$|f(x)-1/2| = |x-1/2| < \delta < \varepsilon \text{ if } x \in Q,$$
$$|f(x)-1/2| = |(1-x)-1/2| = |1/2-x| < \delta < \varepsilon \text{ if } x \in Q^c.$$

Hence, we have proved that f is continuous at $x = 1/2$.

(d) f assumes every value between 0 and 1.

Proof: Given $a \in [0,1]$, we want to find $x \in [0,1]$ such that $f(x) = a$. If $a \in Q$, then choose $x = a$, we have $f(x = a) = a$. If $a \in R-Q$, then choose $x = 1-a(\in R-Q)$, we have $f(x = 1-a) = 1-(1-a) = a$.

Remark: The range of f on $[0,1]$ is $[0,1]$. In addition, f is an **one-to-one** mapping since if $f(x) = f(y)$, then $x = y$. (The proof is easy, just by definition of 1-1, so we omit it.)

(e) $f(x+y)-f(x)-f(y)$ is rational for all x and y in $[0,1]$.

Proof: We prove it by four steps.

1. If $x \in Q$ and $y \in Q$, then $x+y \in Q$. So,
$$f(x+y)-f(x)-f(y) = x+y-x-y = 0 \in Q.$$
2. If $x \in Q$ and $y \in Q^c$, then $x+y \in Q^c$. So,
$$f(x+y)-f(x)-f(y) = [1-(x+y)]-x-(1-y) = -2x \in Q.$$
3. If $x \in Q^c$ and $y \in Q$, then $x+y \in Q^c$. So,
$$f(x+y)-f(x)-f(y) = [1-(x+y)]-(1-x)-y = -2y \in Q.$$
4. If $x \in Q^c$ and $y \in Q^c$, then $x+y \in Q^c$ or $x+y \in Q$. So,

$$f(x+y)-f(x)-f(y)=\begin{cases}[1-(x+y)]-(1-x)-(1-y)=-1\in Q \text{ if } x+y\in Q^c,\\ (x+y)-(1-x)-(1-y)=-2\in Q \text{ if } x+y\in Q.\end{cases}$$

Remark: Here is an interesting question about functions. Let $f: R-\{0,1\}\to R$. If f satisfies that

$$f(x)+f\left(\frac{x-1}{x}\right)=1+x,$$

then $f(x)=\frac{x^3-x^2-1}{2x(x-1)}$.

Proof: Let $\phi(x)=\frac{x-1}{x}$, then we have $\phi^2(x)=\frac{-1}{x-1}$, and $\phi^3(x)=x$. So,

$$f(x)+f\left(\frac{x-1}{x}\right)=f(x)+f(\phi(x))=1+x \quad *$$

which implies that

$$f(\phi(x))+f(\phi^2(x))=1+\phi(x) \quad **$$

and

$$f(\phi^2(x))+f(\phi^3(x))=f(\phi^2(x))+f(x)=1+\phi^2(x). \quad ***$$

So, by (*), (**), and (***), we finally have

$$\begin{aligned}f(x)&=\frac{1}{2}[1+x-\phi(x)+\phi^2(x)]\\&=\frac{x^3-x^2-1}{2x(x-1)}.\end{aligned}$$

4.18 Let f be defined on R and assume that there exists at least one x_0 in R at which f is continuous. Suppose also that, for every x and y in R, f satisfies the equation

$$f(x+y)=f(x)+f(y).$$

Prove that there exists a constant a such that $f(x)=ax$ for all x.

Proof: Let f be defined on R and assume that there exists at least one x_0 in R at which f is continuous. Suppose also that, for every x and y in R, f satisfies the equation

$$f(x+y)=f(x)+f(y).$$

Prove that there exists a constant a such that $f(x)=ax$ for all x.

Proof: Suppose that f is continuous at x_0, and given any $r\in R$. Since $f(x+y)=f(x)+f(y)$ for all x, then

$$f(x)=f(y-x_0)+f(r), \text{ where } y=x-r+x_0.$$

Note that $y\to x_0 \Leftrightarrow x\to r$, then

$$\begin{aligned}\lim_{x\to r}f(x)&=\lim_{x\to r}f(y-x_0)+f(r)\\&=\lim_{y\to x_0}f(y-x_0)+f(r)\\&=f(r) \text{ since } f \text{ is continuous at } x_0.\end{aligned}$$

So, f is continuous at r. Since r is arbitrary, we have f is continuous on R. Define $f(1)=a$, and then since $f(x+y)=f(x)+f(y)$, we have

$$\begin{aligned}f(1)&=f\left(\frac{1}{m}+..+\frac{1}{m}\right)_{m-\text{times}}\\&=mf\left(\frac{1}{m}\right)\\&\Rightarrow f\left(\frac{1}{m}\right)=\frac{f(1)}{m}\end{aligned} \quad *$$

In addition, since $f(-1) = -f(1)$ by $f(0) = 0,$ we have

$$f(-1) = f\left(\frac{1}{-m} + .. + \frac{1}{-m}\right)_{m-\text{times}}$$
$$= mf\left(\frac{-1}{m}\right)$$
$$\Rightarrow f\left(\frac{1}{-m}\right) = \frac{f(1)}{-m} \qquad *'$$

Thus we have

$$f\left(\frac{n}{m}\right) = f(1/m + \ldots + 1/m)_{n-\text{times}} = nf(1/m) = \frac{n}{m}f(1) \text{ by (*) and (*')} \qquad **$$

So, given any $x \in R,$ and thus choose a sequence $\{x_n\} \subseteq Q$ with $x_n \to x.$ Then

$$f(x) = f\left(\lim_{n\to\infty} x_n\right)$$
$$= \lim_{n\to\infty} f(x_n) \text{ by continuity of } f \text{ on } R$$
$$= \lim_{n\to\infty} x_n f(1) \text{ by (**)}$$
$$= xf(1)$$
$$= ax.$$

Remark: There is a similar statement. Suppose that $f(x+y) = f(x)f(y)$ for all real x and y.

(1) If f is differentiable and non-zero, prove that $f(x) = e^{cx},$ where c is a constant.

Proof: Note that $f(0) = 1$ since $f(x+y) = f(x)f(y)$ and f is non-zero. Since f is differentiable, we define $f'(0) = c.$ Consider

$$\frac{f(x+h) - f(x)}{h} = f(x)\frac{f(h) - f(0)}{h} \to f(x)f'(0) = cf(x) \text{ as } h \to 0,$$

we have for every $x \in R,$ $f'(x) = cf(x).$ Hence,

$$f(x) = Ae^{cx}.$$

Since $f(0) = 1,$ we have $A = 1.$ Hence, $f(x) = e^{cx},$ where c is a constant.

Note: (i) If for every $x \in R,$ $f'(x) = cf(x),$ then $f(x) = Ae^{cx}.$

Proof: Since $f'(x) = cf(x)$ for every $x,$ we have for every $x,$

$$[f'(x) - cf(x)]e^{-cx} = 0 \Rightarrow [e^{-cx}f(x)]' = 0.$$

We note that by **Elementary Calculus**, $e^{-cx}f(x)$ is a constant function $A.$ So, $f(x) = Ae^{cx}$ for all real $x.$

(ii) Suppose that $f(x+y) = f(x)f(y)$ for all real x and $y.$ If $f(x_0) > 0$ for some $x_0,$ then $f(x) > 0$ for all $x.$

Proof: Suppose **NOT**, then $f(a) = 0$ for some $a.$ However,

$$0 < f(x_0) = f(x_0 - a + a) = f(x_0 - a)f(a) = 0.$$

Hence, $f(x) > 0$ for all $x.$

(iii) Suppose that $f(x+y) = f(x)f(y)$ for all real x and $y.$ If f is differentiable at x_0 for some $x_0,$ then f is differentiable for all $x.$ And thus, $f(x) \in C^\infty(R).$

Proof: Since

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x_0+h+x-x_0)-f(x_0+x-x_0)}{h}$$
$$= f(x-x_0)\frac{f(x_0+h)-f(x_0)}{h} \to f(x-x_0)f'(x_0) \text{ as } h \to 0,$$

we have $f'(x)$ is differentiable and $f'(x) = f(x-x_0)f'(x_0)$ for all x. And thus we have $f(x) \in C^\infty(R)$.

(iv) Here is another proof by (iii) and **Taylor Theorem with Remainder term** $R_n(x)$.

Proof: Since f is differentiable, by (iii), we have $f^{(n)}(x) = (f'(0))^n f(x)$ for all x. Consider $x \in [-r,r]$, then by **Taylor Theorem with Remainder term** $R_n(x)$,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_n(x), \text{ where } R_n(x) := \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1},\ \xi \in (0,x) \text{ or } \in (x,0),$$

Then

$$\begin{aligned}|R_n(x)| &= \left|\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}\right| \\ &= \left|\frac{(f'(0))^{n+1}f(\xi)}{(n+1)!}x^{n+1}\right| \\ &\leq \left|\frac{(f'(0)r)^{n+1}}{(n+1)!}\right|M, \text{ where } M = \max_{x\in[-r,r]}|f(x)| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence, we have for every $x \in [-r,r]$

$$\begin{aligned}f(x) &= \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \\ &= f(0)\left(\sum_{k=0}^{\infty}\frac{[f'(0)x]^k}{k!}\right) \\ &= e^{cx}, \text{ where } c := f'(0).\end{aligned}$$

Since r is arbitrary, we have proved that $f(x) = e^{cx}$ for all x.

(2) If f is continuous and non-zero, prove that $f(x) = e^{cx}$, where c is a constant.

Proof: Since $f(x+y) = f(x)f(y)$, we have

$$0 < f(1) = f\left(\frac{1}{n}+\ldots+\frac{1}{n}\right)_{n-\text{times}} = f\left(\frac{1}{n}\right)^n \Rightarrow f\left(\frac{1}{n}\right) = f(1)^{1/n} \qquad *$$

and (note that $f(-1) = f(1)^{-1}$ by $f(0) = 1$,)

$$0 < f(-1) = f\left(\frac{-1}{n}+\ldots+\frac{-1}{n}\right)_{n-\text{times}} = f\left(\frac{-1}{n}\right)^n \Rightarrow f\left(\frac{1}{-n}\right) = f(1)^{-1/n} \qquad *'$$

$$f\left(\frac{m}{n}\right) = f\left(\frac{1}{n}+\ldots+\frac{1}{n}\right)_{m-\text{times}} = f\left(\frac{1}{n}\right)^m = f(1)^{\frac{m}{n}} \text{ by (*) and (*')} \qquad **$$

So, given any $x \in R$, and thus choose a sequence $\{x_n\} \subseteq Q$ with $x_n \to x$. Then

$$
\begin{aligned}
f(x) &= f\left(\lim_{n\to\infty} x_n\right) \\
&= \lim_{n\to\infty} f(x_n) \text{ by continuity of } f \\
&= \lim_{n\to\infty} f(1)^{x_n} \text{ by (**)} \\
&= f(1)^x \\
&= e^{cx}, \text{ where } \log f(1) = c.
\end{aligned}
$$

Note: (i) We can prove (2) by the exercise as follows. Note that $f(x) > 0$ for all x by the remark (1)-(ii) Consider the composite function $g(x) = \log f(x)$, then $g(x+y) = \log f(x+y) = \log f(x)f(y) = \log f(x) + \log f(y) = g(x) + g(y)$. Since log and f are continuous on R, its composite function g is continuous on R. Use the exercise, we have $g(x) = cx$ for some c. Therefore, $f(x) = e^{g(x)} = e^{cx}$.

(ii) We can prove (2) by the remark (1) as follows. It suffices to show that this f is differentiable at 0 by remark (1) and (1)-(iii). Since $f(\frac{m}{n}) = f(1)^{\frac{m}{n}}$ then for every real r, $f(r) = [f(1)]^r$ by continuity of f. Note that $\lim_{r\to 0}\frac{a^r - b^r}{r}$ exists. Given any sequence $\{r_n\}$ with $r_n \to 0$, and thus consider

$$\lim_{r_n\to 0}\frac{f(r_n) - f(0)}{r_n} = \frac{[f(1)]^{r_n} - 1}{r_n} = \frac{[f(1)]^{r_n} - 1^{r_n}}{r_n} \text{ exists,}$$

we have f is differentiable at $x = 0$. So, by remark (1), we have $f(x) = e^{cx}$.

(3) Give an example such that f is not continuous on R.

Solution: Consider $g(x+y) = g(x) + g(y)$ for all x, y. Then we have $g(q) = qg(1)$, where $q \in Q$. By **Zorn's Lemma**, we know that every vector space has a basis $\{v_\alpha : \alpha \in I\}$. Note that $\{v_\alpha : \alpha \in I\}$ is an uncountable set, so there exists a convergent sequence $\{s_n\} \subseteq \{v_\alpha : \alpha \in I\}$. Hence, $S := (\{v_\alpha : \alpha \in I\} - \{s_n\}_{n=1}^{\infty}) \cup \{\frac{s_n}{n}\}_{n=1}^{\infty}$ is a new basis of R over Q. Given $x, y \in R$, and we can find the same N such that

$$x = \sum_{k=1}^{N} q_k v_k \text{ and } y = \sum_{k=1}^{N} p_k v_k, \text{ where } v_k \in S$$

Define the sume

$$x + y := \sum_{k=1}^{N}(p_k + q_k)v_k$$

By uniqueness, we define $g(x)$ to be the sum of coefficients, i.e.,

$$g(x) := \sum_{k=1}^{N} q_k.$$

Note that

$$g\left(\frac{s_n}{n}\right) = 1 \text{ for all } n \Rightarrow \lim_{n\to\infty} g\left(\frac{s_n}{n}\right) = 1$$

and

$$\frac{s_n}{n} \to 0 \text{ as } n \to \infty$$

Hence, g is not continuous at $x = 0$ since if it was, then

$$\begin{aligned} 1 &= \lim_{n\to\infty} g\left(\frac{s_n}{n}\right) \\ &= g\left(\lim_{n\to\infty}\frac{s_n}{n}\right) \text{ by continuity of } g \text{ at } 0 \\ &= g(0) \\ &= 0 \end{aligned}$$

which is absurb. Hence, g is not continuous on R by the exercise. To find such f, it suffices to consider $f(x) = e^{g(x)}$.

Note: Such g (or f) is **not measurable** by **Lusin Theorem**.

4.19 Let f be continuous on $[a,b]$ and define g as follows: $g(a) = f(a)$ and, for $a < x \leq b$, let $g(x)$ be the maximum value of f in the subinterval $[a,x]$. Show that g is continuous on $[a,b]$.

Proof: Define $g(x) = \max\{f(t) : t \in [a,x]\}$, and choose any point $c \in [a,b]$, we want to show that g is continuous at c. Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $x \in (c-\delta, c+\delta) \cap [a,b]$, we have

$$|g(x) - g(c)| < \varepsilon.$$

Since f is continuous at $x = c$, then there exists a $\delta' > 0$ such that as $x \in (c-\delta', c+\delta') \cap [a,b]$, we have

$$f(c) - \varepsilon/2 < f(x) < f(c) + \varepsilon/2. \quad *$$

Consider two cases as follows.

(1) $\max\{f(t) : t \in [a, c+\delta'] \cap [a,b]\} = f(p_1)$, where $p_1 \leq c - \delta'$.
As $x \in (c-\delta', c+\delta') \cap [a,b]$, we have $g(x) = f(p_1)$ and $g(c) = f(p_1)$.
Hence, $|g(x) - g(c)| = 0$.
(2) $\max\{f(t) : t \in [a, c+\delta'] \cap [a,b]\} = f(p_1)$, where $p_1 > c - \delta'$.
As $x \in (c-\delta', c+\delta') \cap [a,b]$, we have by (*) $f(c) - \varepsilon/2 \leq g(x) \leq f(c) + \varepsilon/2$.
Hence, $|g(x) - g(c)| < \varepsilon$.
So, if we choose $\delta = \delta'$, then for $x \in (c-\delta, c+\delta) \cap [a,b]$,

$$|g(x) - g(c)| < \varepsilon \text{ by (1) and (2).}$$

Hence, $g(x)$ is continuous at c. And since c is arbitrary, we have $g(x)$ is continuous on $[a,b]$.

Remark: It is the same result for $\min\{f(t) : t \in [a,x]\}$ by the preceding method.

4.20 Let $f_1, \ldots, f_m$ be m real-valued functions defined on R^n. Assume that each f_k is continuous at the point a of S. Define a new function f as follows: For each x in S, $f(x)$ is the largest of the m numbers $f_1(x), \ldots, f_m(x)$. Discuss the continuity of f at a.

Proof: Assume that each f_k is continuous at the point a of S, then we have $(f_i + f_j)$ and $|f_i - f_j|$ are continuous at a, where $1 \leq i,j \leq m$. Since $\max(a,b) = \frac{(a+b)+|a-b|}{2}$, then $\max(f_1,f_2)$ is continuous at a since both $(f_1 + f_2)$ and $|f_1 - f_2|$ are continuous at a. Define $f(x) = \max(f_1, \ldots f_m)$, use **Mathematical Induction** to show that $f(x)$ is continuous at $x = a$ as follows. As $m = 2$, we have proved it. Suppose $m = k$ holds, i.e., $\max(f_1, \ldots f_k)$ is continuous at $x = a$. Then as $m = k+1$, we have

$$\max(f_1, \ldots f_{k+1}) = \max[\max(f_1, \ldots f_k), f_{k+1}]$$

is continuous at $x = a$ by induction hypothesis. Hence, by **Mathematical Induction**, we have prove that f is continuos at $x = a$.

It is possible that f and g is not continuous on R whihc implies that $\max(f,g)$ is continuous on R. For example, let $f(x) = 0$ if $x \in Q$, and $f(x) = 1$ if $x \in Q^c$ and $g(x) = 1$

if $x \in Q$, and $g(x) = 0$ if $x \in Q^c$.

Remark: It is the same rusult for $\min(f_1,\ldots f_m)$ since $\max(a,b) + \min(a,b) = a + b$.

4.21 Let $f : S \to R$ be continuous on an open set in R^n, assume that $p \in S$, and assume that $f(p) > 0$. Prove that there is an n –ball $B(p;r)$ such that $f(x) > 0$ for every x in the ball.

Proof: Since $(p \in)S$ is an open set in R^n, there exists a $\delta_1 > 0$ such that $B(p,\delta_1) \subseteq S$. Since $f(p) > 0$, given $\varepsilon = \frac{f(p)}{2} > 0$, then there exists an n –ball $B(p;\delta_2)$ such that as $x \in B(p;\delta_2) \cap S$, we have

$$\frac{f(p)}{2} = f(p) - \varepsilon < f(x) < f(p) + \varepsilon = \frac{3f(p)}{2}.$$

Let $\delta = \min(\delta_1,\delta_2)$, then as $x \in B(p;\delta)$, we have

$$f(x) > \frac{f(p)}{2} > 0.$$

Remark: The exercise tells us that under the assumption of continuity at p, we roughly have the same sign in a neighborhood of p, if $f(p) > 0$ $\left(\text{or } f(p) < 0.\right)$

4.22 Let f be defined and continuous on a closed set S in R. Let

$$A = \left\{x : x \in S \text{ and } f(x) = 0\right\}.$$

Prove that A is a closed subset of R.

Proof: Since $A = f^{-1}(\{0\})$, and f is continous on S, we have A is closed in S. And since S is closed in R, we finally have A is closed in R.

Remark: 1. Roughly speaking, the property of being closed has **Transitivity**. That is, in (M,d) let $S \subseteq T \subseteq M$, if S is closed in T, and T is closed in M, then S is closed in M.

Proof: Let x be an adherent point of S in M, then $B_M(x,r) \cap S \neq \phi$ for every $r > 0$. Hence, $B_M(x,r) \cap T \neq \phi$ for every $r > 0$. It means that x is also an adherent point of T in M. Since T is closed in M, we find that $x \in T$. Note that since $B_M(x,r) \cap S \neq \phi$ for every $r > 0$, we have $(S \subseteq T)$

$$B_T(x,r) \cap S = (B_M(x,r) \cap T) \cap S = B_M(x,r) \cap (S \cap T) = B_M(x,r) \cap S \neq \phi.$$

So, we have x is an adherent point of S in T. And since S is closed in T, we have $x \in S$. Hence, we have proved that if x is an adherent point of S in M, then $x \in S$. That is, S is closed in M.

Note: (1) Another proof of remark 1, since S is closed in T, there exists a closed subset U in Msuch that $S = U \cap T$, and since T is closed in M, we have S is closed in M.

(2) There is a similar result, in (M,d) let $S \subseteq T \subseteq M$, if S is open in T, and T is open in M, then S is open in M. (Leave to the reader.)

2. Here is another statement like the exercise, but we should be cautioned. We write it as follows. Let f and g be continuous on (S,d_1) into (T,d_2). Let $A = \{x : f(x) = g(x)\}$, show that A is closed in S.

Proof: Let x be an accumulation point of A, then there exists a sequence $\{x_n\} \subseteq A$ such that $x_n \to x$. So, we have $f(x_n) = g(x_n)$ for all n. Hence, by continuity of f and g, we have

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g\left(\lim_{n\to\infty} x_n\right) = g(x).$$

Hence, $x \in A$. That is, A contains its all adherent point. So, A is closed.

Note: In remark 2, we **CANNOT** use the relation

$$f(x) - g(x)$$

since the difference "$-$" are not necessarily defined on the metric space (T, d_2).

4.23 Given a function $f : R \to R$, define two sets A and B in R^2 as follows:

$$A = \{(x,y) : y < f(x)\},$$
$$B = \{(x,y) : y > f(x)\}.$$

and Prove that f is continuous on R if, and only if, both A and B are open subsets of R^2.

Proof: ($\Rightarrow$) Suppose that f is continuous on R. Let $(a,b) \in A$, then $f(a) > b$. Since f is continuous at a, then given $\varepsilon = \frac{f(a)-b}{2} > 0$, there exists a $(\varepsilon >)\delta > 0$ such that as $|x-a| < \delta$, we have

$$\frac{f(a)+b}{2} = f(a) - \varepsilon < f(x) < f(a) + \varepsilon. \qquad *$$

Consider $(x,y) \in B((a,b);\delta)$, then $|x-a|^2 + |y-b|^2 < \delta^2$ which implies that

1. $|x-a| < \delta \Rightarrow f(x) > \dfrac{f(a)+b}{2}$ by (*) and
2. $|y-b| < \delta \Rightarrow y < b+\delta < b+\varepsilon = \dfrac{f(a)+b}{2}$.

Hence, we have $f(x) > y$. That is, $B((a,b);\delta) \subseteq A$. So, A is open since every point of A is interior. Similarly for B.

($\Leftarrow$) Suppose that A and B are open in R^2. Trivially, $(a, f(a) - \varepsilon/2) := p_1 \in A$, and $(a, f(a) + \varepsilon/2) := p_2 \in B$. Since A and B are open in R^2, there exists a $(\varepsilon/2 >)\delta > 0$ such that

$$B(p_1,\delta) \subseteq A \text{ and } B(p_2,\delta) \subseteq B.$$

Hence, if $(x,y) \in B(p_1,\delta)$, then

$$(x-a)^2 + (y-(f(a)-\varepsilon/2))^2 < \delta^2 \text{ and } y < f(x).$$

So, it implies that

$$|x-a| < \delta,\ |y - f(a) + \varepsilon/2| < \delta, \text{ and } y < f(x).$$

Hence, as $|x-a| < \delta$, we have

$$\begin{aligned} -\delta &< y - f(a) + \varepsilon/2 \\ &\Rightarrow f(a) - \delta - \varepsilon/2 < y < f(x) \\ &\Rightarrow f(a) - \varepsilon < y < f(x) \\ &\Rightarrow f(a) - \varepsilon < f(x). \end{aligned} \qquad **$$

And if $(x,y) \in B(p_2,\delta)$, then

$$(x-a)^2 + (y-(f(a)+\varepsilon/2))^2 < \delta^2 \text{ and } y > f(x).$$

So, it implies that

$$|x-a| < \delta,\ |y - f(a) - \varepsilon/2| < \delta, \text{ and } y > f(x).$$

Hence, as $|x-a| < \delta$, we have

$$f(x) < y < f(a) + \varepsilon/2 + \delta < f(a) + \varepsilon. \qquad ***$$

So, given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|x-a| < \delta$, we have by (**) and (***)

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon.$$

That is, f is continuous at a. Since a is arbitrary, we know that f is continuous on R.

4.24 Let f be defined and bounded on a compact interval S in R. If $T \subseteq S$, the number

$$\Omega_f(T) = \sup\{f(x) - f(y) : x, y \in T\}$$

is called the **oscillation (or span) of** f on T. If $x \in S$, the oscillation of f at x is defined to be the number

$$\omega_f(x) = \lim_{h \to 0^+} \Omega_f(B(x;h) \cap S).$$

Prove that this limit always exists and that $\omega_f(x) = 0$ if, and only if, f is continuous at x.

Proof: 1. Note that since f is bounded, say $|f(x)| \leq M$ for all x, we have $|f(x) - f(y)| \leq 2M$ for all $x, y \in S$. So, $\Omega_f(T)$, the oscillation of f on any subset T of S, exists. In addition, we define $g(h) = \Omega_f(B(x;h) \cap S)$. Note that if $T_1 \subseteq T_2 (\subseteq S)$, we have $\Omega_f(T_1) \leq \Omega_f(T_2)$. Hence, the oscillation of f at x, $\omega_f(x) = \lim_{h \to 0^+} g(h) = g(0+)$ since g is an increasing function. That is, the limit of $\Omega_f(B(x;h) \cap S)$ always exists as $h \to 0^+$.

2. $(\Rightarrow)$ Suppose that $\omega_f(x) = 0$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $h \in (0, \delta)$, we have

$$|g(h)| = g(h) = \Omega_f(B(x;h) \cap S) < \varepsilon/2.$$

That is, as $h \in (0, \delta)$, we have

$$\sup\{f(t) - f(s) : t, s \in B(x;h) \cap S\} < \varepsilon/2$$

which implies that

$$-\varepsilon/2 < f(t) - f(x) < \varepsilon/2 \text{ as } t \in (x - \delta, x + \delta) \cap S.$$

So, as $t \in (x - \delta, x + \delta) \cap S$. we have

$$|f(t) - f(x)| < \varepsilon.$$

That is, f is continuous at x.

$(\Leftarrow)$ Suppose that f is continous at x, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $t \in (x - \delta, x + \delta) \cap S$, we have

$$|f(t) - f(x)| < \varepsilon/3.$$

So, as $t, s \in (x - \delta, x + \delta) \cap S$, we have

$$|f(t) - f(s)| \leq |f(t) - f(x)| + |f(x) - f(s)| < \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3$$

which implies that

$$\sup\{(t) - f(s) : t, s \in (x - \delta, x + \delta) \cap S\} \leq 2\varepsilon/3 < \varepsilon.$$

So, as $h \in (0, \delta)$, we have

$$\Omega_f(B(x;h) \cap S) = \sup\{(t) - f(s) : t, s \in (x - \delta, x + \delta) \cap S\} < \varepsilon.$$

Hence, the oscillation of f at x, $\omega_f(x) = 0$.

Remark: 1. The compactness of S is not used here, we will see the advantage of the oscillation of f in text book, Theorem 7.48, in page 171. (On **Lebesgue's Criterion for Riemann-Integrability**.)

2. One of advantage of the oscillation of f is to show the statement: Let f be defined on $[a, b]$ Prove that a bounded f does **NOT** have the properties:

$$f \text{ is continuous on } Q \cap [a, b], \text{ and discontinuous on } (R - Q) \cap [a, b].$$

Proof: Since $\omega_f(x) = 0$ if, and only if, f is continuous at x, we know that $\omega_f(r) > 0$ for $r \in (R - Q) \cap [a, b]$. Define $J_{1/n} = \{r : \omega_f(r) \geq 1/n\}$, then by hypothesis, we know that $\cup_{n=1}^{\infty} J_{1/n} = (R - Q) \cap [a, b]$. It is easy to show that $J_{1/n}$ is closed in $[a, b]$. Hence, $int[cl(J_{1/n})] = int(J_{1/n}) = \phi$ for all $n \in N$. It means that $J_{1/n}$ is **nowhere dense** for all $n \in N$. Hence,

$$[a,b] = (\cup_{n=1}^{\infty} J_{1/n}) \cup (Q \cap [a,b])$$

is of the **firse category** which is absurb since every complete metric space is of the **second category**. So, this f cannot exist.

Note: 1 The Boundedness of f **can be removed** since we we can accept the concept $\infty > 0$.

2. ($J_{1/n}$ is closed in $[a,b]$) Given an accumulation point x of $J_{1/n}$, if $x \notin J_{1/n}$, we have $\omega_f(x) < 1/n$. So, there exists a 1 –ball $B(x)$ such that $\Omega_f(B(x) \cap [a,b]) < 1/n$. Thus, no points of $B(x)$ can belong to $J_{1/n}$, contradicting that x is an accumulation point of $J_{1/n}$. Hence, $x \in J_{1/n}$ and $J_{1/n}$ is closed.

3. (**Definition of a nowhere dense set)** In a metric space (M,d), let A be a subset of M, we say A is nowhere dense in M if, and only if $\bar{A}$ contains no balls of M, ($\Leftrightarrow int(\bar{A}) = \phi$).

4. (**Definition of a set of the first category and of the second category)** A set A in a metric space M is of the first category if, and only if, A is the union of a countable number of nowhere dense sets. A set B is of the second category if, and only if, B is not of the first category.

5. (**Theorem**) A complete metric space is of the second category.
We write another important theorem about a set of the second category below.
(**Baire Category Theorem**) A nonempty open set in a complete metric space is of the second category.

6. In the notes 3,4 and 5, the reader can see the reference, A First Course in Real Analysis written by M. H. Protter and C. B. Morrey, in pages 375-377.

4.25 Let f be continuous on a compact interval $[a,b]$. Suppose that f has a local maximum at x_1 and a local maximum at x_2. Show that there must be a third point between x_1 and x_2 where f has a local minimum.
Note. To say that f has a local maximum at x_1 means that there is an 1 –ball $B(x_1)$ such that $f(x) \leq f(x_1)$ for all x in $B(x_1) \cap [a,b]$. Local minimum is similarly defined.

Proof: Let $x_2 > x_1$. Suppose **NOT**, i.e., no points on (x_1,x_2) can be a local minimum of f. Since f is continuous on $[x_1,x_2]$, then $\inf\{f(x) : x \in [x_1,x_2]\} = f(x_1)$ or $f(x_2)$ by hypothesis. We consider two cases as follows:
(1) If $\inf\{f(x) : x \in [x_1,x_2]\} = f(x_1)$, then

$$\begin{cases} \text{(i) } f(x) \text{ has a local maximum at } x_1 \text{ and} \\ \text{(ii) } f(x) \geq f(x_1) \text{ for all } x \in [x_1,x_2]. \end{cases}$$

By (i), there exists a $\delta > 0$ such that $x \in [x_1,x_1+\delta) \subseteq [x_1,x_2]$, we have

$$\text{(iii) } f(x) \leq f(x_1).$$

So, by (ii) and (iii), as $x \in [x_1,x_1+\delta)$, we have

$$f(x) = f(x_1)$$

which contradicts the hypothesis that no points on (x_1,x_2) can be a local minimum of f.
(2) If $\inf\{f(x) : x \in [x_1,x_2]\} = f(x_1)$, it is similar, we omit it.
Hence, from (1) and (2), we have there has a third point between x_1 and x_2 where f has a local minimum.

4.26 Let f be a real-valued function, continuous on $[0,1]$, with the following property: For every real y, either there is no x in $[0,1]$ for which $f(x) = y$ or there is exactly one such x. Prove that f is strictly monotonic on $[0,1]$.

Proof: Since the hypothesis says that f is one-to-one, then by Theorem*, we know that f is trictly monotonic on $[0,1]$.

Remark: (**Theorem***) Under assumption of continuity on a compact interval, 1-1 is equivalent to being strictly monotonic. We will prove it in Exercise 4.62.

4.27 Let f be a function defined on $[0,1]$ with the following property: For every real number y, either there is no x in $[0,1]$ for which $f(x) = y$ or there are exactly two values of x in $[0,1]$ for which $f(x) = y$.

(a) Prove that f cannot be continuous on $[0,1]$.

Proof: Assume that f is continuous on $[0,1]$, and thus consider $\max_{x\in[0,1]} f(x)$ and $\min_{x\in[0,1]} f(x)$. Then by hypothesis, there exist exactly two values $a_1 < a_2 \in [0,1]$ such that $f(a_1) = f(a_2) = \max_{x\in[0,1]} f(x)$, and there exist exactly two values $b_1 < b_2 \in [0,1]$ such that $f(b_1) = f(b_2) = \min_{x\in[0,1]} f(x)$.

Claim that $a_1 = 0$ and $a_2 = 1$. Suppose **NOT,** then there exists at least one belonging to $(0,1)$. Without loss of generality, say $a_1 \in (0,1)$. Since f has maximum at $a_1 \in (0,1)$ and $a_2 \in [0,1]$, we can find three points p_1, p_2, and p_3 such that

$$\begin{cases} 1.\ p_1 < a_1 < p_2 < p_3 < a_2, \\ 2.\ f(p_1) < f(a_1),\ f(p_2) < f(a_1), \text{ and } f(p_3) < f(a_2). \end{cases}$$

Since $f(a_1) = f(a_2)$, we choose a real number r so that

$$f(p_1) < r < f(a_1) \Rightarrow r = f(q_1), \text{ where } q_1 \in (p_1, a_1) \text{ by continuity of } f.$$

$$f(p_2) < r < f(a_1) \Rightarrow r = f(q_2), \text{ where } q_2 \in (a_1, p_2) \text{ by continuity of } f.$$

$$f(p_3) < r < f(a_2) \Rightarrow r = f(q_3), \text{ where } q_3 \in (p_3, a_2) \text{ by continuity of } f.$$

which contradicts the hypothesis that for every real number y, there are exactly two values of x in $[0,1]$ for which $f(x) = y$. Hence, we know that $a_1 = 0$ and $a_2 = 1$. Similarly, we also have $b_1 = 0$ and $b_2 = 1$.

So, $\max_{x\in[0,1]} f(x) = \min_{x\in[0,1]} f(x)$ which implies that f is constant. It is impossible. Hence, such f does not exist. That is, f is not continuous on $[0,1]$.

(b) Construct a function f which has the above property.

Proof: Consider $[0,1] = (Q^c \cap [0,1]) \cup (Q \cap [0,1])$, and write $Q \cap [0,1] = \{x_1, x_2, \ldots, x_n, \ldots\}$. Define

$$1.\ f(x_{2n-1}) = f(x_{2n}) = n,$$

$$2.\ f(x) = x \text{ if } x \in (0, 1/2) \cap Q^c,$$

$$3.\ f(x) = 1 - x \text{ if } x \in (1/2, 1) \cap Q^c.$$

Then if $x = y$, then it is clear that $f(x) = f(y)$. That is, f is well-defined. And from construction, we know that the function defined on $[0,1]$ with the following property: For every real number y, either there is no x in $[0,1]$ for which $f(x) = y$ or there are exactly two values of x in $[0,1]$ for which $f(x) = y$.

Remark: $\{x : f \text{ is discontinuous at } x\} = [0,1]$. Given $a \in [0,1]$. Note that since $f(x) \in N$ for all $x \in Q \cap [0,1]$ and Q is dense in R, for any 1-ball $B(a;r) \cap (Q \cap [0,1])$, there is always a rational number $y \in B(a;r) \cap (Q \cap [0,1])$ such that $|f(y) - f(a)| \geq 1$.

(c) Prove that any function with this property has infinite many discontinuities on $[0,1]$.

Proof: In order to make the proof clear, **property** A **of** f means that for every real number y, either there is no x in $[0,1]$ for which $f(x) = y$ or

there are exactly two values of x in $[0,1]$ for which $f(x) = y$

Assume that there exist a finite many numbers of discontinuities of f, say these points $x_1, \ldots, x_n$. By property A, there exists a unique y_i such that $f(x_i) = f(y_i)$ for $1 \leq i \leq n$. Note that the number of the set
$S := \left(\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} \cup \left\{x : f(x) = f(0), \text{ and } f(x) = f(1)\right\}\right)$ is even, say $2m$
We remove these points from S, and thus we have $2m + 1$ subintervals, say I_j, $1 \leq j \leq 2m + 1$. Consider the local extremum in every I_j, $1 \leq j \leq 2m + 1$ and note that every subinterval I_j, $1 \leq j \leq 2m + 1$, has at most finite many numbers of local extremum, say $\#\left(\left\{t \in I_j : f(x) \text{ is the local extremum}\right\} = \left\{t_1^{(j)}, \ldots, t_{p_j}^{(j)}\right\}\right) = p_j$. And by property A, there exists a unique $s_k^{(j)}$ such that $f\left(t_k^{(j)}\right) = f\left(s_k^{(j)}\right)$ for $1 \leq k \leq p_j$. We again remove these points, and thus we have removed even number of points. And odd number of **open** intervals is left, call the odd number $2q - 1$. Note that since the function f is monotonic in every open interval left, R_l, $1 \leq l \leq 2q - 1$, the image of f on these open interval is also an **open** interval. If $R_a \cap R_b \neq \phi$, say $R_a = (a_1, a_1)$ and $R_b = (b_1, b_2)$ with (without loss of generality) $a_1 < b_1 < a_2 < b_2$, then

$$R_a = R_b \text{ by property } A.$$

(Otherwise, b_1 is only point such that $f(x) = f(b_1)$, which contradicts property A.) Note that given any R_a, there must has one and only one R_b such that $R_a = R_b$. However, we have $2q - 1$ **open** intervals is left, it is impossible. Hence, we know that f has infinite many discontinuities on $[0,1]$.

4.28 In each case, give an example of a real-valued function f, continuous on S and such that $f(S) = T$, or else explain why there can be no such f :

(a) $S = (0,1)$, $T = (0,1]$.
Solution: Let

$$f(x) = \begin{cases} 2x \text{ if } x \in (0,1/2], \\ 1 \text{ if } x \in (1/2,1). \end{cases}$$

(b) $S = (0,1)$, $T = (0,1) \cup (1,2)$.

Solution: NO! Since a continuous functions sends a connected set to a connected set. However, in this case, S is connected and T is not connected.

(c) $S = R^1$, $T =$ the set of rational numbers.

Solution: NO! Since a continuous functions sends a connected set to a connected set. However, in this case, S is connected and T is not connected.

(d) $S = [0,1] \cup [2,3]$, $T = \{0,1\}$.

Solution: Let

$$f(x) = \begin{cases} 0 \text{ if } x \in [0,1], \\ 1 \text{ if } x \in [2,3]. \end{cases}$$

(e) $S = [0,1] \times [0,1]$, $T = R^2$.

Solution: NO! Since a continuous functions sends a compact set to a compact set. However, in this case, S is compact and T is not compact.

(f) $S = [0,1] \times [0,1], T = (0,1) \times (0,1)$.

Solution: NO! Since a continuous functions sends a compact set to a compact set. However, in this case, S is compact and T is not compact.

(g) $S = (0,1) \times (0,1), T = R^2$.

Solution: Let

$$f(x,y) = (\cot \pi x, \cot \pi y).$$

Remark: 1. There is some important theorems. We write them as follows.

(**Theorem A**) Let $f : (S, d_s) \to (T, d_T)$ be continuous. If X is a compact subset of S, then $f(X)$ is a compact subset of T.

(**Theorem B**) Let $f : (S, d_s) \to (T, d_T)$ be continuous. If X is a connected subset of S, then $f(X)$ is a connected subset of T.

2. In (g), the key to the example is to find a continuous function $f : (0,1) \to R$ which is onto.

Supplement on Continuity of real valued functions

Exercise Suppose that $f(x) : (0,\infty) \to R$, is continuous with $a \leq f(x) \leq b$ for all $x \in (0,\infty)$, and for any real y, either there is no x in $(0,\infty)$ for which $f(x) = y$ or there are finitely many x in $(0,\infty)$ for which $f(x) = y$. Prove that $\lim_{x\to\infty} f(x)$ exists.

Proof: For convenience, we say property A, it means that for any real y, either there is no x in $(0,\infty)$ for which $f(x) = y$ or there are finitely many x in $(0,\infty)$ for which $f(x) = y$.

We partition $[a,b]$ into n subintervals. Then, by continuity and property A, as x is large enough, $f(x)$ is lying in one and only one subinterval. Given $\varepsilon > 0$, there exists N such that $2/N < \varepsilon$. For this N, we partition $[a,b]$ into N subintervals, then there is a $M > 0$ such that as $x,y \geq M$

$$|f(x) - f(y)| \leq 2/N < \varepsilon.$$

So, $\lim_{x\to\infty} f(x)$ exists.

Exercise Suppose that $f(x) : [0,1] \to R$ is continunous with $f(0) = f(1) = 0$. Prove that

(a) there exist two points x_1 and x_2 such that as $|x_1 - x_2| = 1/n$, we have $f(x_1) = f(x_2) \neq 0$ for all n. In this case, we call $1/n$ the length of horizontal strings.

Proof: Define a new function $g(x) = f(x + \frac{1}{n}) - f(x) : [0, 1 - \frac{1}{n}]$. Claim that there exists $p \in [0, 1 - \frac{1}{n}]$ such that $g(p) = 0$. Suppose **NOT**, by **Intermediate Value Theorem**, without loss of generality, let $g(x) > 0$, then

$$g(0) + g\left(\frac{1}{n}\right) + \ldots + g\left(1 - \frac{1}{n}\right) = f(1) > 0$$

which is absurb. Hence, we know that there exists $p \in [0, 1 - \frac{1}{n}]$ such that $g(p) = 0$. That is,

$$f\left(p + \frac{1}{n}\right) = f(p).$$

So, we have $1/n$ as the length of horizontal strings.

(b) Could you show that there exists 2/3 as the length of horizontal strings?

Proof: The horizontal strings does not exist, for example,

$$f(x) = \begin{cases} x, \text{ if } x \in [0, \frac{1}{4}] \\ -x + \frac{1}{2}, \text{ if } x \in [\frac{1}{4}, \frac{3}{4}] \\ x - 1, \text{ if } x \in [\frac{3}{4}, 1] \end{cases}.$$

Exercise Suppose that $f(x) : [a,b] \to R$ is a continuous and non-constant function. Prove that the function f cannot have any small periods.

Proof: Say f is continuous at $q \in [a,b]$, and by hypothesis that f is non-constant, there is a point $p \in [a,b]$ such that $|f(q) - f(p)| := M > 0$. Since f is continuous at q, then given $\varepsilon = M$, there is a $\delta > 0$ such that as $x \in (q - \delta, q + \delta) \cap [a,b]$, we have

$$|f(x) - f(q)| < M. \qquad *$$

If f has any small periods, then in the set $(q - \delta, q + \delta) \cap [a,b]$, there is a point $r \in (q - \delta, q + \delta) \cap [a,b]$ such that $f(r) = f(p)$. It contradicts to (*). Hence, the function f cannot have any small periods.

Remark 1. There is a function with any small periods.

Solution:The example is Dirichlet function,

$$f(x) = \begin{cases} 0, \text{ if } x \in Q^c \\ 1, \text{ if } x \in Q \end{cases}.$$

Since $f(x + q) = f(x)$, for any rational q, we know that f has any small periods.

2. Prove that there cannot have a non-constant continuous function which has two period p, and q such that q/p is irrational.

Proof: Since q/p is irrational, there is a sequence $\{\frac{q_n}{p_n}\}(\subseteq Q)$ such that

$$\left|\frac{q_n}{p_n} - \frac{q}{p}\right| < \frac{1}{p_n^2} \Rightarrow |pq_n - qp_n| < \left|\frac{p}{p_n}\right| \to 0 \text{ as } n \to \infty.$$

So, f has any small periods, by this exercise, we know that this f cannot a non-constant continuous function.

Note: The inequality is important; the reader should kepp it in mind. There are many ways to prove this inequality, we metion two methods without proofs. The reader can find the proofs in the following references.

(1) An Introduction To The Theory Of Numbers written by G.H. Hardy and E.M. Wright, charpter X, pp 137-138.

(2) In the text book, exercise 1.15 and 1.16, pp 26.

3. Suppose that $f(x)$ is differentiable on R prove that if f has any small periods, then f is constant.

Proof: Given $c \in R$, and consider

$$\frac{f(c + p_n) - f(c)}{p_n} = 0 \text{ for all } n.$$

where p_n is a sequence of periods of function such that $p_n \to 0$. Hence, by differentiability of f, we know that $f'(c) = 0$. Since c is arbitrary, we know that $f'(x) = 0$ on R. Hence, f is constant.

Continuity in metric spaces

In Exercises 4.29 through 4.33, we assume that $f: S \to T$ is a function from one metric space (S, d_S) to another (T, d_T).

4.29 Prove that f is continuous on S if, and only if,

$$f^{-1}(intB) \subseteq int(f^{-1}(B)) \text{ for every subset } B \text{ of } T.$$

Proof: ($\Rightarrow$) Suppose that f is continuous on S, and let B be a subset of T. Since $int(B) \subseteq B$, we have $f^{-1}(intB) \subseteq f^{-1}(B)$. Note that $f^{-1}(intB)$ is open since a pull back of an open set under a continuous function is open. Hence, we have

$$int[f^{-1}(intB)] = f^{-1}(intB) \subseteq int(f^{-1}(B)).$$

That is, $f^{-1}(intB) \subseteq int(f^{-1}(B))$ for every subset B of T.

($\Leftarrow$) Suppose that $f^{-1}(intB) \subseteq int(f^{-1}(B))$ for every subset B of T. Given an open subset $U(\subseteq T)$, i.e., $intU = U$, so we have

$$f^{-1}(U) = f^{-1}(intU) \subseteq int(f^{-1}(U)).$$

In addition, $int(f^{-1}(U)) \subseteq f^{-1}(U)$ by the fact, for any set A, $intA$ is a subset of A. So, f is continuous on S.

4.30 Prove that f is continuous on S if, and only if,

$$f(cl(A)) \subseteq cl(f(A)) \text{ for every subset } A \text{ of } S.$$

Proof: ($\Rightarrow$) Suppose that f is continuous on S, and let A be a subset of S. Since $f(A) \subseteq cl(f(A))$, then $(A \subseteq) f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. Note that $f^{-1}(cl(f(A)))$ is closed since a pull back of a closed set under a continuous function is closed. Hence, we have

$$cl(A) \subseteq cl[f^{-1}(cl(f(A)))] = f^{-1}(cl(f(A)))$$

which implies that

$$f(cl(A)) \subseteq f[f^{-1}(cl(f(A)))] \subseteq cl(f(A)).$$

($\Leftarrow$) Suppose that $f(cl(A)) \subseteq cl(f(A))$ for every subset A of S. Given a closed subset $C(\subseteq T)$, and consider $f^{-1}(C)$ as follows. Define $f^{-1}(C) = A$, then

$$\begin{aligned} f(cl(f^{-1}(C))) &= f(cl(A)) \\ &\subseteq cl(f(A)) = cl(f(f^{-1}(C))) \\ &\subseteq cl(C) = C \text{ since } C \text{ is closed.} \end{aligned}$$

So, we have by $(f(cl(A)) \subseteq C)$

$$cl(A) \subseteq f^{-1}(f(cl(A))) \subseteq f^{-1}(C) = A$$

which implies that $A = f^{-1}(C)$ is closed set. So, f is continuous on S.

4.31 Prove that f is continuous on S if, and only if, f is continuous on every compact subset of S. Hint. If $x_n \to p$ in S, the set $\{p, x_1, x_2, \ldots\}$ is compact.

Proof: ($\Rightarrow$) Suppose that f is continuous on S, then it is clear that f is continuous on every compact subset of S.

($\Leftarrow$) Suppose that f is continuous on every compact subset of S, Given $p \in S$, we consider two cases.

(1) p is an isolated point of S, then f is automatically continuous at p.

(2) p is not an isolated point of S, that is, p is an accumulation point p of S, then there exists a sequence $\{x_n\}(\subseteq)S$ with $x_n \to p$. Note that the set $\{p, x_1, x_2, \ldots\}$ is compact, so we know that f is continuous at p. Since p is arbitrary, we know that f is continuous on S.

Remark: If $x_n \to p$ in S, the set $\{p, x_1, x_2, \ldots\}$ is compact. The fact is immediately from the statement that every infinite subset $\{p, x_1, x_2, \ldots\}$ of has an accumulation point in $\{p, x_1, x_2, \ldots\}$.

4.32 A function $f : S \to T$ is called a closed mapping on S if the image $f(A)$ is closed in T for every closed subset A of S. Prove that f is continuous and closed on S if, and only if,

$$f(cl(A)) = cl(f(A)) \text{ for every subset } A \text{ of } S.$$

Proof: ($\Rightarrow$) Suppose that f is continuous and closed on S, and let A be a subset of S. Since $A \subseteq cl(A)$, we have $f(A) \subseteq f(cl(A))$. So, we have

$$cl(f(A)) \subseteq cl(f(cl(A))) = f(cl(A)) \text{ since } f \text{ is closed.} \quad *$$

In addition, since $f(A) \subseteq cl(f(A))$, we have $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(cl(f(A)))$. Note that $f^{-1}(cl(f(A)))$ is closed since f is continuous. So, we have

$$cl(A) \subseteq cl(f^{-1}(cl(f(A)))) = f^{-1}(cl(f(A)))$$

which implies that

$$f(cl(A)) \subseteq f(f^{-1}(cl(f(A)))) \subseteq cl(f(A)). \quad **$$

From (*) and (**), we know that $f(cl(A)) = cl(f(A))$ for every subset A of S.

($\Leftarrow$) Suppose that $f(cl(A)) = cl(f(A))$ for every subset A of S. Gvien a closed subset C of S, i.e., $cl(C) \subseteq C$, then we have

$$f(C) \supseteq f(cl(C)) = cl(f(C)).$$

So, we have $f(C)$ is closed. That is, f is closed. Given any closed subset B of T, i.e., $cl(B) \subseteq B$, we want to show that $f^{-1}(B)$ is closed. Since $f^{-1}(B) := A \subseteq S$, we have

$$f(cl(f^{-1}(B))) = f(cl(A)) = cl(f(A)) = cl(f(f^{-1}(B))) \subseteq cl(B) \subseteq B$$

which implies that

$$f(cl(f^{-1}(B))) \subseteq B \Rightarrow cl(f^{-1}(B)) \subseteq f^{-1}(f(cl(f^{-1}(B)))) \subseteq f^{-1}(B).$$

That is, we have $cl(f^{-1}(B)) \subseteq f^{-1}(B)$. So, $f^{-1}(B)$ is closed. Hence, f is continuous on S.

4.33 Give an example of a continuous f and a Cauchy sequence $\{x_n\}$ in some metric space S for which $\{f(x_n)\}$ is not a Cauchy sequence in T.

Solution: Let $S = (0,1]$, $x_n = 1/n$ for all $n \in N$, and $f = 1/x : S \to R$. Then it is clear that f is continous on S, and $\{x_n\}$ is a Cauchy sequence on S. In addition, Trivially, $\{f(x_n) = n\}$ is not a Cauchy sequence.

Remark: The reader may compare the exercise with the Exercise 4.54.

4.34 Prove that the interval $(-1,1)$ in R^1 is homeomorphic to R^1. This shows that **neither boundedness nor completeness is a topological property.**

Proof: Since $f(x) = \tan(\frac{\pi x}{2}) : (-1,1) \to R$ is **bijection** and continuous, and its converse function $f^{-1}(x) = \arctan x : R \to (-1,1)$. Hence, we know that f is a Topologic mapping. (Or say f is a homeomorphism). Hence, $(-1,1)$ is homeomorphic to R^1.

Remark: A function f is called a bijection if, and only if, f is 1-1 and onto.

4.35 Section 9.7 contains an example of a function f, continuous on $[0,1]$, with $f([0,1]) = [0,1] \times [0,1]$. Prove that no such f can be one-to-one on $[0,1]$.

Proof: By section 9.7, let $f : [0,1] \to [0,1] \times [0,1]$ be an onto and continuous function. If f is 1-1, then so is its converse function f^{-1}. Note that since f is a 1-1 and continous function defined on a compact set $[0,1]$, then its converse function f^{-1} is also a continous function. Since $f([0,1]) = [0,1] \times [0,1]$, we have the domain of f^{-1} is $[0,1] \times [0,1]$ which is connected. Choose a special point $y \in [0,1] \times [0,1]$ so that $f^{-1}(y) := x \in (0,1)$. Consider a continous function $g = f^{-1}|_{[0,1]\times[0,1]-\{y\}}$, then $g : [0,1] \times [0,1] - \{y\} \to [0,x) \cup (x,1]$ which is continous. However, it is impossible since $[0,1] \times [0,1] - \{y\}$ is connected but $[0,x) \cup (x,1]$ is not connected. So, such f cannot exist.

Connectedness

4.36 Prove that a metric space S is disconnected if, and only if there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S.

Proof: ($\Rightarrow$) Suppose that S is disconnected, then there exist two subset A, B in S such that

$$1.\ A, B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

Note that since A, B are open in S, we have $A = S - B$, $B = S - A$ are closed in S. So, if S is disconnected, then there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S.

($\Leftarrow$) Suppose that there is a nonempty subset A of S, $A \neq S$, which is both open and closed in S. Then we have $S - A := B$ is nonempty and B is open in S. Hence, we have two sets A, B in S such that

$$1.\ A, B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

That is, S is disconnected.

4.37 Prove that a metric space S is connected if, and only if the only subsets of S which are both open and closed in S are empty set and S itself.

Proof: ($\Rightarrow$) Suppose that S is connected. If there exists a subset A of S such that

$$1.\ A \neq \phi,\ 2.\ A \text{ is a proper subset of } S,\ 3.\ A \text{ is open and closed in } S,$$

then let $B = S - A$, we have

$$1.\ A, B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

It is impossible since S is connected. So, this A cannot exist. That is, the only subsets of S which are both open and closed in S are empty set and S itself.

($\Leftarrow$) Suppose that the only subsets of S which are both open and closed in S are empty set and S itself. If S is disconnected, then we have two sets A, B in S such that

$$1.\ A, B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

It contradicts the hypothesis that the only subsets of S which are both open and closed in S are empty set and S itself.

Hence, we have proved that S is connected if, and only if the only subsets of S which are both open and closed in S are empty set and S itself.

4.38 Prove that the only connected subsets of R are
(a) the empty set,
(b) sets consisting of a single point, and
(c) intervals (open, closed, half-open, or infinite).

Proof: Let S be a connected subset of R. Denote the symbol $\#(A)$ to be the number of elements in a set A. We consider three cases as follows. (a) $\#(S) = 0$, (b) $\#(S) = 1$, (c) $\#(S) > 1$.

For case (a), it means that $S = \phi$, and for case (b), it means that S consists of a single point. It remains to consider the case (c). Note that since $\#(S) > 1$, we have $\inf S \neq \sup S$.

Since $S \subseteq R$, we have $S \subseteq [\inf S, \sup S]$. (Note that we accept that $\inf S = -\infty$ or $\sup S = \infty$.) If S is not an interval, then there exists $x \in (\inf S, \sup S)$ such that $x \notin S$. (Otherwise, $(\inf S, \sup S) \subseteq S$ which implies that S is an interval.) Then we have

$$1.\ (-\infty, x) \cap S \ := A \text{ is open in } S$$
$$2.\ (x, +\infty) \cap S \ := B \text{ is open in } S$$
$$3.\ A \cup B = S.$$

Claim that both A and B are not empty. Asume that A is empty, then every $s \in S$, we have $s > x > \inf S$. By the definition of infimum, it is impossible. So, A is not empty. Similarly for B. Hence, we have proved that S is disconnected, a contradiction. That is, S is an interval.

Remark: 1. We note that any interval in R is connected. It is immediate from Exercise 4.44. But we give another proof as follows. Suppose there exists an interval S is not connceted, then there exist two subsets A and B such that

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S.$$

Since $A \neq \phi$ and $B \neq \phi$, we choose $a \in A$ and $b \in B$, and let $a < b$. Consider

$$c := \sup\{A \cap [a,b]\}.$$

Note that $c \in cl(A) = A$ implies that $c \notin B$. Hence, we have $a \leq c < b$. In addition, $c \notin B = cl(B)$, then there exists a $B_S(c;\delta) \cap B = \phi$. Choose $d \in B_S(c;\delta) = (c-\delta, c+\delta) \cap S$ so that

$$1.\ c < d < b \text{ and } 2.\ d \notin B.$$

Then $d \notin A$. (Otherwise, it contradicts $c = \sup\{A \cap [a,b]\}$. Note that $d \in [a,b] \subseteq S = A \cup B$ which implies that $d \in A$ or $d \in B$. We reach a contradiction since $d \notin A$ and $d \notin B$. Hence, we have proved that any interval in R is connected.

2. Here is an application. Is there a continuous function $f : R \to R$ such that $f(Q) \subseteq Q^c$, and $f(Q^c) \subseteq Q$?

Ans: NO! If such f exists, then both $f(Q)$ and $f(Q^c)$ are countable. Hence, $f(R)$ is countable. In addition, $f(R)$ is connected. Since $f(R)$ contains rationals and irrationals, we know $f(R)$ is an interval which implies that $f(R)$ is uncountable, a cotradiction. Hence, such f does not exist.

4.39 Let X be a connected subset of a metric space S. Let Y be a subset of S such that $X \subseteq Y \subseteq cl(X)$, where $cl(X)$ is the closure of X. Prove that Y is also connected. In particular, this shows that $cl(X)$ is connected.

Proof: Given a two valued function f on Y, we know that f is also a two valued function on X. Hence, f is constant on X, (without loss of generality) say $f = 0$ on X. Consider $p \in Y - X$, it ,means that p is an accumulation point of X. Then there exists a dequence $\{x_n\} \subseteq X$ such that $x_n \to p$. Note that $f(x_n) = 0$ for all n. So, we have by continuity of f on Y,

$$f(p) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0.$$

Hence, we have f is constant 0 on Y. That is, Y is conneceted. In particular, $cl(X)$ is connected.

Remark: Of course, we can use definition of a connected set to show the exercise. But, it is too tedious to write. However, it is a good practice to use definition to show it. The reader may give it a try as a challenge.

4.40 If x is a point in a metric space S, let $U(x)$ be the component of S containing x. Prove that $U(x)$ is closed in S.

Proof: Let p be an accumulation point of $U(x)$. Let f be a two valued function defined on $U(x) \cup \{p\}$, then f is a two valud function defined on $U(x)$. Since $U(x)$ is a component of S containing x, then $U(x)$ is connected. That is, f is constant on $U(x)$, (without loss of generality) say $f = 0$ on $U(x)$. And since p is an accumulation point of $U(x)$, there exists a sequence $\{x_n\} \subseteq U(x)$ such that $x_n \to p$. Note that $f(x_n) = 0$ for all n. So, we have by continuity of f on $U(x) \cup \{p\}$,

$$f(p) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0.$$

So, $U(x) \cup \{p\}$ is a connected set containing x. Since $U(x)$ is a component of S containing x, we have $U(x) \cup \{p\} \subseteq U(x)$ which implies that $p \in U(x)$. Hence, $U(x)$ contains its all accumulation point. That is, $U(x)$ is closed in S.

4.41 Let S be an open subset of R. By Theorem 3.11, S is the union of a countable disjoint collection of open intervals in R. Prove that each of these open intervals is a component of the metric subspace S. Explain why this does not contradict Exercise 4.40.

Proof: By Theorem 3.11, $S = \cup_{n=1}^{\infty} I_n$, where I_i is open in R and $I_i \cap I_j = \phi$ if $i \neq j$. Assume that there exists a I_m such that I_m is not a component T of S. Then $T - I_m$ is not empty. So, there exists $x \in T - I_m$ and $x \in I_n$ for some n. Note that the component $U(x)$ is the union of all connected subsets containing x, then we have

$$T \cup I_n \subseteq U(x). \qquad *$$

In addition,

$$U(x) \subseteq T \qquad **$$

since T is a component containing x. Hence, by (*) and (**), we have $I_n \subseteq T$. So, $I_m \cup I_n \subseteq T$. Since T is connected in R^1, T itself is an interval. So, $int(T)$ is still an interval which is open and containing $I_m \cup I_n$. It contradicts the definition of component interval. Hence, each of these open intervals is a component of the metric subspace S.

Since these open intervals is open relative to R, not S, this does not contradict Exercise 4.40.

4.42 Given a compact S in R^m with the following property: For every pair of points a and b in S and for every $\varepsilon > 0$ there exists a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in S with $x_0 = a$ and $x_n = b$ such that

$$\|x_k - x_{k-1}\| < \varepsilon \text{ for } k = 1, 2, .., n.$$

Prove or disprove: S is connected.

Proof: Suppose that S is disconnected, then there exist two subsets A and B such that

$$1.\ A,\ B \text{ are open in } S,\ 2.\ A \neq \phi \text{ and } B \neq \phi,\ 3.\ A \cap B = \phi, \text{ and } 4.\ A \cup B = S. \qquad *$$

Since $A \neq \phi$ and $B \neq \phi$, we choose $a \in A$, and $b \in A$ and thus given $\varepsilon = 1$, then by hypothesis, we can find two points $a_1 \in A$, and $b_1 \in B$ such that $\|a_1 - b_1\| < 1$. For a_1, and b_1, given $\varepsilon = 1/2$, then by hypothesis, we can find two points $a_2 \in A$, and $b_2 \in B$ such that $\|a_2 - b_2\| < 1/2$. Continuous the steps, we finally have two sequence $\{a_n\} \subseteq A$ and $\{b_n\} \subseteq B$ such that $\|a_n - b_n\| < 1/n$ for all n. Since $\{a_n\} \subseteq A$, and $\{b_n\} \subseteq B$, we have $\{a_n\} \subseteq S$ and $\{b_n\} \subseteq S$ by $S = A \cup B$. Hence, there exist two subsequence $\{a_{n_k}\} \subseteq A$ and $\{b_{n_k}\} \subseteq B$ such that $a_{n_k} \to x$, and $b_{n_k} \to y$, where $x, y \in S$ since S is compact. In addition, since A is closed in S, and B is closed in S, we have $x \in A$ and $y \in B$. On the other hand, since $\|a_n - b_n\| < 1/n$ for all n, we have $x = y$. That is,

$A \cap B \neq \phi$ which contradicts (*)-3. Hence, we have prove that S is connected.

Remark: We given another proof by the method of two valued function as follows. Let f be a two valued function defined on S, and choose any two points $a, b \in S$. If we can show that $f(a) = f(b)$, we have proved that f is a constant which implies that S is connected. Since f is a continuous function defined on a compact set S, then f is uniformly on S. Thus, given $1 > \varepsilon > 0$, there exists a $\delta > 0$ such that as $\|x - y\| < \delta$, $x, y \in S$, we have $|f(x) - f(y)| < \varepsilon < 1 \Rightarrow f(x) = f(y)$. Hence, for this δ, there exists a finite set of points $\{x_0, x_1, \ldots, x_n\}$ in S with $x_0 = a$ and $x_n = b$ such that

$$\|x_k - x_{k-1}\| < \delta \text{ for } k = 1, 2, .., n.$$

So, we have $f(a) = f(x_0) = f(x_1) = \ldots = f(x_n) = f(b)$.

4.43 Prove that a metric space S is connected if, and only if, every nonempty proper subset of S has a nonempty boundary.

Proof: ($\Rightarrow$) Suppose that S is connected, and if there exists a nonempty proper subset U of S such that $\partial U = \phi$, then let $B = cl(S - U)$, we have (define $cl(U) = A$)

$$1.\ A \neq \phi.\ B \neq \phi \text{ since } S - U \neq \phi,$$

$$2.\ A \cup B = cl(U) \cup cl(S - U) \supseteq U \cup (S - U) = S$$
$$\Rightarrow S = A \cup B,$$

$$3.\ A \cap B = cl(U) \cap cl(S - U) = \partial U = \phi,$$

and

4. Both A and B are closed in $S \Rightarrow$ Both A and B are open in S.

Hence, S is disconnected. That is, if S is connected, then every nonempty proper subset of S has a nonempty boundary.

($\Leftarrow$) Suppose that every nonempty proper subset of S has a nonempty boundary. If S is disconnected, then there exist two subsets A and B such that

1. A, B are closed in S, 2. $A \neq \phi$ and $B \neq \phi$, 3. $A \cap B = \phi$, and 4. $A \cup B = S$.

Then for this A, A is a nonempty proper subset of S with ($cl(A) = A$, and $cl(B) = B$)

$$\partial A = cl(A) \cap cl(S - A) = cl(A) \cap cl(B) = A \cap B = \phi$$

which contradicts the hypothesis that every nonempty proper subset of S has a nonempty boundary. So, S is connected.

4.44. Prove that every convex subset of R^n is connected.

Proof: Given a convex subset S of R^n, and since for any pair of points a, b, the set $\{(1-\theta)a + \theta b : 0 < \theta < 1\} := T \subseteq S$, i.e., $g : [0,1] \to T$ by $g(\theta) = (1-\theta)a + \theta b$ is a continuous function such that $g(0) = a$, and $g(1) = b$. So, S is path-connected. It implies that S is connected.

Remark: 1. In the exercise, it tells us that every n –ball is connected. (In fact, every n –ball is path-connected.) In particular, as $n = 1$, any interval (open, closed, half-open, or infinite) in R is connected. For $n = 2$, any disk (open, closed, or not) in R^2 is connected.

2. Here is a good exercise on the fact that a path-connected set is connected. Given $[0,1] \times [0,1] := S$, and if T is a countable subset of S. Prove that $S - T$ is connected. (In fact, $S - T$ is path-connected.)

Proof: Given any two points a and b in $S - T$, then consider the vertical line L passing through the middle point $(a + b)/2$. Let $A = \{x : x \in L \cap S\}$, and consider the lines form a to A, and from b to A. Note that A is uncountable, and two such lines (form a to A, and from b to A) are disjoint. So, if every line contains a point of T, then it leads us to get T is uncountable. However, T is countable. So, it has some line (form a to A, and from b to A) is in $S - T$. So, it means that $S - T$ is path-connected. So, $S - T$ is connected.

4.45 Given a function $f : R^n \to R^m$ which is 1-1 and continuous on R^n. If A is open and disconnected in R^n, prove that $f(A)$ is open and disconnected in $f(R^n)$.

Proof: The exercise is **wrong.** There is a counter-example. Let $f : R \to R^2$

$$f = \begin{cases} (\cos(\frac{2\pi x}{1+x} - \frac{\pi}{2}), 1 - \sin(\frac{2\pi x}{1+x} - \frac{\pi}{2})) \text{ if } x \geq 0 \\ (\cos(\frac{2\pi x}{1-x} - \frac{\pi}{2}), -1 + \sin(\frac{2\pi x}{1-x} - \frac{\pi}{2})) \text{ if } x < 0 \end{cases}$$

Remark: If we restrict $n, m = 1$, the conclusion holds. That is, Let $f : R \to R$ be continuous and 1-1. If A is open and disconnected, then so is $f(A)$.

Proof: In order to show this, it suffices to show that f maps an open interval I to another open interval. Since f is continuous on I, and I is connected, $f(I)$ is connected. It implies that $f(I)$ is an interval. Trivially, there is no point x in I such that $f(x)$ equals the endpoints of $f(I)$. Hence, we know that $f(I)$ is an open interval.

Supplement: Here are two exercises on **Homeomorphism** to make the reader get more and feel something.

1. Let $f : E \subseteq R \to R$. If $\{(x, f(x)) : x \in E\}$ is compact, then f is uniformly continuous on E.

Proof: Let $\{(x, f(x)) : x \in E\} = S$, and thus define $g(x) = (I(x) = x, f(x)) : E \to S$. Claim that g is continuous on E. Consider $h : S \to E$ by $h(x, f(x)) = x$. Trivially, h is 1-1, continuous on a compact set S. So, its inverse function g is 1-1 and continuous on a compact set E. The claim has proved.

Since g is continuous on E, we know that f is continuous on a compact set E. Hence, f is uniformly continuous on E.

Note: The question in Supplement 1, there has another proof by the method of contradiction, and use the property of compactness. We omit it.

2. Let $f : (0,1) \to R$. If $\{(x, f(x)) : x \in (0,1)\}$ is path-connected, then f is continuous on $(0,1)$.

Proof: Let $a \in (0,1)$, then there is a compact interval $(a \in)[a_1, a_2] \subseteq (0,1)$. Claim that the set

$$\{(x, f(x)) : x \in [a_1, a_2]\} := S \text{ is compact.}$$

Since S is path-connected, there is a continuous function $g : [0,1] \to S$ such that $g(0) = (a_1, f(a_1))$ and $g(1) = (a_2, f(a_2))$. If we can show $g([0,1]) = S$, we have shown that S is compact. Consider $h : S \to R$ by $h(x, f(x)) = x$; h is clearly continuous on S. So, the composite function $h \circ g : [0,1] \to R$ is also continuous. Note that $h \circ g(0) = a_1$, and $h \circ g(1) = a_2$, and the range of $h \circ g$ is connected. So, $[a_1, a_2] \subseteq h(g([0,1]))$. Hence, $g([0,1]) = S$. We have proved the claim and by Supplement 1, we know that f is continuous at a. Since a is arbitrary, we know that f is continuous on $(0,1)$.

Note: The question in Supplement 2, there has another proof directly by definition of continuity. We omit the proof.

4.46 Let $A = \{(x, y) : 0 < x \leq 1, y = \sin 1/x\}$, $B = \{(x, y) : y = 0, -1 \leq x \leq 0\}$, and let $S = A \cup B$. Prove that S is connected but not arcwise conneceted. (See Fig. 4.5, Section 4.18.)

Proof: Let f be a two valued function defined on S. Since A, and B are connected in S, then we have

$$f(A) = a, \text{ and } f(B) = b, \text{ where } \{a,b\} = \{0,1\}.$$

Given a sequence $\{x_n\}(\subseteq A)$ with $x_n \to (0,0)$, then we have

$$\begin{aligned} a &= \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to 0} x_n\right) \text{ by continuity of } f \text{ at } 0 \\ &= f(0,0) \\ &= b. \end{aligned}$$

So, we have f is a constant. That is, S is connected.

Assume that S is arcwise connected, then there exists a continuous function $g : [0,1] \to S$ such that $g(0) = (0,0)$ and $g(1) = (1,\sin 1)$. Given $\varepsilon = 1/2$, there exists a $\delta > 0$ such that as $|t| < \delta$, we have

$$\|g(t) - g(0)\| = \|g(t)\| < 1/2. \quad *$$

Let N be a positive integer so that $\frac{1}{2N\pi} < \delta$, thus let $(\frac{1}{2N\pi}, 0) := p$ and $\left(\frac{1}{2(N+1)\pi}, 0\right) := q$. Define two subsets U and V as follows:

$$U = \left\{(x,y) : x > \frac{p+q}{2}\right\} \cap g([q,p])$$

$$V = \left\{(x,y) : x < \frac{p+q}{2}\right\} \cap g([q,p])$$

Then we have

(1). $U \cup V = g([q,p])$, (2). $U \neq \phi$, since $p \in U$ and $V \neq \phi$, since $q \in V$,

(3). $U \cap V = \phi$ by the given set A, and (*)

Since $\{(x,y) : x > \frac{p+q}{2}\}$ and $\{(x,y) : x < \frac{p+q}{2}\}$ are open in R^2, then U and V are open in $g([q,p])$. So, we have

(4). U is open in $g([q,p])$ and V is open in $g([q,p])$.

From (1)-(4), we have $g([q,p])$ is disconnected which is absurb since a connected subset under a continuous function is connected. So, such g cannot exist. It means that S is not arcwise connected.

Remark: This exercise gives us an example to say that **connectedness does not imply path-connectedness.** And it is important example which is worth keeping in mind.

4.47 Let $F = \{F_1, F_2, \ldots\}$ be a countable collection of connected compact sets in R^n such that $F_{k+1} \subseteq F_k$ for each $k \geq 1$. Prove that the intersection $\bigcap_{k=1}^{\infty} F_k$ is connected and closed.

Proof: Since F_k is compact for each $k \geq 1$, F_k is closed for each $k \geq 1$. Hence, $\bigcap_{k=1}^{\infty} F_k := F$ is closed. Note that by **Theorem 3.39**, we know that F is compact. Assume that F is not connected. Then there are two subsets A and B with

1.$A \neq \phi$, $B \neq \phi$. 2. $A \cap B = \phi$. 3. $A \cup B = F$. 4. A,B are closed in F.

Note that A, B are closed and disjoint in R^n. By exercise 4.57, there exist U and V which are open and disjoint in R^n such that $A \subseteq U$, and $B \subseteq V$. Claim that there exists F_k such that $F_k \subseteq U \cup V$. Suppose **NOT**, then there exists $x_k \in F_k - (U \cup V.)$ Without loss of generality, we may assume that $x_k \notin F_{k+1}$. So, we have a sequence $\{x_k\} \subseteq F_1$ which implies that there exists a convergent subsequence $\{x_{k(n)}\}$, say $\lim_{k(n)\to\infty} x_{k(n)} = x$. It is clear that $x \in F_k$ for all k since x is an accumulation point of each F_k. So, we have

$$x \in F = \cap_{k=1}^{\infty} F_k = A \cup B \subseteq U \cup V$$

which implies that x is an interior point of $U \cup V$ since U and V are open. So, $B(x;\delta) \subseteq U \cup V$ for some $\delta > 0$, which contradicts to the choice of x_k. Hence, we have proved that there exists F_k such that $F_k \subseteq U \cup V$. Let $C = U \cap F_k$, and $D = V \cap F_k$, then we have

1. $C \neq \phi$ since $A \subseteq U$ and $A \subseteq F_k$, and $D \neq \phi$ since $B \subseteq V$ and $B \subseteq F_k$.
2. $C \cap D = (U \cap F_k) \cap (V \cap F_k) \subseteq U \cap V = \phi$.
3. $C \cup D = (U \cap F_k) \cup (V \cap F_k) = F_k$.
4. C is open in F_k and D is open in F_k by C, D are open in R^n.

Hence, we have F_k is disconnected which is absurb. So, we know that $F = \cap_{k=1}^{\infty} F_k$ is connected.

4.48 Let S be an open connected set in R^n. Let T be a component of $R^n - S$. Prove that $R^n - T$ is connected.

Proof: If S is empty, there is nothing to proved. Hence, we assume that S is nonempty. Write $R^n - S = \cup_{x \in R^n - S} U(x)$, where $U(x)$ is a component of $R^n - S$. So, we have

$$R^n = S \cup (\cup_{x \in R^n - S} U(x)).$$

Say $T = U(p)$, for some p. Then

$$R^n - T = S \cup (\cup_{x \in R^n - S - T} U(x)).$$

Claim that $cl(S) \cap U(x) \neq \phi$ for all $x \in R^n - S - T$. If we can show the claim, given $a, b \in R^n - T$, and a two valued function on $R^n - T$. Note that $cl(S)$ is also connected. We consider three cases. (1) $a \in S$, $b \in U(x)$ for some x. (2) $a, b \in S$. (3) $a \in U(x)$, $b \in U(x')$.

For case (1), let $c \in cl(S) \cap U(x)$, then there are $\{s_n\} \subseteq S$ and $\{u_n\} \subseteq U(x)$ with $s_n \to c$ and $u_n \to c$, then we have

$$f(a) = \lim_{n \to \infty} f(s_n) = f\left(\lim_{n \to \infty} s_n\right) = f(c) = f\left(\lim_{n \to \infty} u_n\right) = \lim_{n \to \infty} f(u_n) = f(b)$$

which implies that $f(a) = f(b)$.

For case (2), it is clear $f(a) = f(b)$ since S itself is connected.

For case (3), we choose $s \in S$, and thus use case (1), we know that

$$f(a) = f(s) = f(b).$$

By case (1)-(3), we have f is constant on $R^n - T$. That is, $R^n - T$ is connected.

It remains to show the claim. To show $cl(S) \cap U(x) \neq \phi$ for all $x \in R^n - S - T$, i.e., to show that for all $x \in R^n - S - T$,

$$\begin{aligned} cl(S) \cap U(x) &= (S \cup S') \cap U(x) \\ &= S' \cap U(x) \\ &\neq \phi. \end{aligned}$$

Suppose **NOT,** i.e., for some x, $S' \cap U(x) = \phi$ which implies that $U(x) \subseteq R^n - cl(S)$ which is open. So, there is a component V of $R^n - cl(S)$ contains $U(x)$, where V is open by **Theorem 4.44.** However, $R^n - cl(S) \subseteq R^n - S$, so we have V is contained in $U(x)$. Therefore, we have $U(x) = V$. Note that $U(x) \subseteq R^n - S$, and $R^n - S$ is closed. So, $cl(U(x)) \subseteq R^n - S$. By definition of component, we have $cl(U(x)) = U(x)$, which is closed. So, we have proved that $U(x) = V$ is open and closed. It implies that $U(x) = R^n$ or ϕ which is absurb. Hence, the claim has proved.

4.49 Let (S, d) be a connected metric space which is not bounded. Prove that for every

a in S and every $r > 0$, the set $\{x : d(x,a) = r\}$ is nonempty.

Proof: Assume that $\{x : d(x,a) = r\}$ is empty. Denote two sets $\{x : d(x,a) < r\}$ by A and $\{x : d(x,a) > r\}$ by B. Then we have

1. $A \neq \phi$ since $a \in A$ and $B \neq \phi$ since S is unbounded,
2. $A \cap B = \phi$,
3. $A \cup B = S$,
4. $A = B(a;r)$ is open in S,

and consider B as follows. Since $\{x : d(x,a) \leq r\}$ is closed in S, $B = S - \{x : d(x,a) \leq r\}$ is open in S. So, we know that S is disconnected which is absurb. Hence, we know that the set $\{x : d(x,a) = r\}$ is nonempty.

Supplement on a connected metric space

Definition Two subsets A and B of a metric space X are said to be separated if both

$$A \cap cl(B) = \phi \text{ and } cl(A) \cap B = \phi.$$

A set $E \subseteq X$ is said to be connected if E is not a union of two nonempty separated sets.

We now prove the definition of connected metric space is **equivalent** to this definiton as follows.

Theorem A set E in a metric space X is connected if, and only if E is not the union of two nonempty disjoint subsets, each of which is open in E.

Proof: ($\Rightarrow$) Suppose that E is the union of two nonempty disjoint subsets, each of which is open in E, denote two sets, U and V. Claim that

$$U \cap cl(V) = cl(U) \cap V = \phi.$$

Suppose **NOT**, i.e., $x \in U \cap cl(V)$. That is, there is a $\delta > 0$ such that

$$B_X(x,\delta) \cap E = B_E(x,\delta) \subseteq U \text{ and } B_X(x,\delta) \cap V \neq \phi$$

which implies that

$$\begin{aligned} B_X(x,\delta) \cap V &= B_X(x,\delta) \cap (V \cap E) \\ &= (B_X(x,\delta) \cap E) \cap V \\ &\subseteq U \cap V = \phi, \end{aligned}$$

a contradiction. So, we have $U \cap cl(V) = \phi$. Similarly for $cl(U) \cap V = \phi$. So, X is disconnected. That is, we have shown that if a set E in a metric space X is connected, then E is not the union of two nonempty disjoint subsets, each of which is open in E.

($\Leftarrow$) Suppose that E is disconnected, then E is a union of two nonempty separated sets, denoted $E = A \cup B$, where $A \cap cl(B) = cl(A) \cap B = \phi$. Claim that A and B are open in E. Suppose **NOT**, it means that there is a point $x (\in A)$ which is not an interior point of A. So, for any ball $B_E(x,r)$, there is a correspounding $x_r \in B$, where $x_r \in B_E(x,r)$. It implies that $x \in cl(B)$ which is absurb with $A \cap cl(B) = \phi$. So, we proved that A is open in E. Similarly, B is open in E. Hence, we have proved that if E is not the union of two nonempty disjoint subsets, each of which is open in E, then E in a metric space X is connected.

Exercise Let A and B be connected sets in a metric space with $A - B$ not connected and suppose $A - B = C_1 \cup C_2$ where $cl(C_1) \cap C_2 = C_1 \cap cl(C_2) = \phi$. Show that $B \cup C_1$ is connected.

Proof: Assume that $B \cup C_1$ is disconnected, and thus we will prove that C_1 is disconnected. Consider, by $cl(C_1) \cap C_2 = C_1 \cap cl(C_2) = \phi$,

$$C_1 \cap cl[C_2 \cup (A \cap B)] = C_1 \cap cl(A \cap B) \ (\subseteq C_1 \cap cl(B)) \qquad *$$

and

$$cl(C_1) \cap [C_2 \cup (A \cap B)] = cl(C_1) \cap (A \cap B) \ (\subseteq cl(C_1) \cap B) \qquad **$$

we know that at least one of (*) and (**) is nonempty by the hypothesis A is connected. In addition, by (*) and (**), we know that at leaset one of

$$C_1 \cap cl(B)$$

and

$$cl(C_1) \cap B$$

is nonempty. So, we know that C_1 is disconnected by the hypothesis B is connected, and the concept of two valued function.

From above sayings and hypothesis, we now have

1. B is connected.
2. C_1 is disconnected.
3. $B \cup C_1$ is disconnected.

Let D be a component of $B \cup C_1$ so that $B \subseteq D;$ we have, let $(B \cup C_1) - D = E(\subseteq C_1)$,

$$D \cap cl(E) = cl(D) \cap E = \phi$$

which implies that

$$cl(E) \cap (A - E) = \phi, \text{ and } cl(A - E) \cap E = \phi.$$

So, we have prove that A is disconnected wich is absurb. Hence, we know that $B \cup C_1$ is connected.

Remark We prove that $cl(A - E) \cap E = cl(E) \cap (A - E) = \phi$ as follows.

Proof: Since

$$D \cap cl(E) = \phi,$$

we obtain that

$$\begin{aligned} & cl(E) \cap (A - E) \\ = \ & cl(E) \cap [(D \cup C_2) \cup (A \cap B)] \\ \subseteq \ & cl(E) \cap [(D \cup C_2) \cup B] \\ = \ & cl(E) \cap (D \cup C_2) \text{ since } B \subseteq D \\ = \ & cl(E) \cap C_2 \text{ since } D \cap cl(E) = \phi \\ \subseteq \ & cl(C_1) \cap C_2 \text{ since } E \subseteq C_1 \\ = \ & \phi. \end{aligned}$$

And since

$$cl(D) \cap E = \phi,$$

we obtain that

$$
\begin{aligned}
& cl(A - E) \cap E \\
=\ & cl[(D \cup C_2) \cup (A \cap B)] \cap E \\
\subseteq\ & cl[(D \cup C_2) \cup B] \cap E \\
=\ & cl(D \cup C_2) \cap E \text{ since } B \subseteq D \\
=\ & cl(C_2) \cap E \text{ since } cl(D) \cap E = \phi \\
\subseteq\ & cl(C_2) \cap C_1 \text{ since } E \subseteq C_1 \\
=\ & \phi.
\end{aligned}
$$

Exercise Prove that every connected metric space with at least two points is uncountable.

Proof: Let X be a connected metric space with two points a and b, where $a \neq b$. Define a set $A_r = \{x : d(x,a) > r\}$ and $B_r = \{x : d(x,a) < r\}$. It is clear that both of sets are open and disjoint. Assume X is countable. Let $r \in \left[\frac{d(a,b)}{4}, \frac{d(a,b)}{2}\right]$, it guarantee that both of sets are non-empty. Since $\left[\frac{d(a,b)}{4}, \frac{d(a,b)}{2}\right]$ is uncountable, we know that there is a $\delta > 0$ such that $A_\delta \cup B_\delta = X$. It implies that X is disconnected. So, we know that such X is countable.

Uniform continuity

4.50 Prove that a function which is uniformly continuous on S is also continuous on S.

Proof: Let f be uniformly continuous on S, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $d(x,y) < \delta$, x and y in S, then we have

$$d(f(x),f(y)) < \varepsilon.$$

Fix y, called a. Then given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $d(x,a) < \delta$, x in S, then we have

$$d(f(x),f(a)) < \varepsilon.$$

That is, f is continuous at a. Since a is arbitrary, we know that f is continuous on S.

4.51 If $f(x) = x^2$ for x in R, prove that f is not uniformly continuous on R.

Proof: Assume that f is uniformly continuous on R, then given $\varepsilon = 1$, there exists a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|f(x) - f(y)| < 1.$$

Choose $x = y + \frac{\delta}{2}$, $(\Rightarrow |x - y| < \delta)$, then we have

$$|f(x) - f(y)| = \left|\delta y + \left(\frac{\delta}{2}\right)^2\right| < 1.$$

When we choose $y = \frac{1}{\delta}$, then

$$\left|1 + \left(\frac{\delta}{2}\right)^2\right| = 1 + \left(\frac{\delta}{2}\right) < 1$$

which is absurb. Hence, we know that f is not uniformly continuous on R.

Remark: There are some similar questions written below.

1. Here is a useful lemma to make sure that a function is uniformly continuous on (a,b), but we need its differentiability.

(**Lemma**) Let $f : (a,b) \subseteq R \to R$ be differentiable and $|f'(x)| \leq M$ for all $x \in (a,b)$. Then f is uniformly continuous on (a,b), where a,b may be $\pm\infty$.

Proof: By **Mean Value Theorem**, we have

$$|f(x) - f(y)| = |f'(z)||x-y|, \text{ where } z \in (x,y) \text{ or } (y,x)$$
$$\leq M|x-y| \text{ by hypothesis.} \quad *$$

Then given $\varepsilon > 0,$ there is a $\delta = \varepsilon/M$ such that as $|x-y| < \delta,\ x,y \in (a,b)$, we have

$$|f(x) - f(y)| < \varepsilon, \text{ by (*)}.$$

Hence, we know that f is uniformly continuous on (a,b).

Note: A standard example is written in Remark 2. But in Remark 2, we still use definition of uniform continuity to practice what it says.

2. $\sin x$ is uniformly continuous on R.

Proof: Given $\varepsilon > 0,$ we want to find a $\delta > 0$ such that as $|x-y| < \delta,$ we have

$$|\sin x - \sin y| < \varepsilon.$$

Since $\sin x - \sin y = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2}),\ |\sin x| \leq |x|,$ and $|\cos x| \leq 1,$ we have

$$|\sin x - \sin y| \leq |x-y|$$

So, if we choose $\delta = \varepsilon,$ then as $|x-y| < \delta,$ it implies that

$$|\sin x - \sin y| < \varepsilon.$$

That is, $\sin x$ is uniformly continuous on R.

Note: $|\sin x - \sin y| \leq |x-y|$ for all $x,y \in R,$ can be proved by **Mean Value Theorem** as follows.

proof: By **Mean Value Theorem,** $\sin x - \sin y = (\sin z)'(x-y)$; it implies that

$$|\sin x - \sin y| \leq |x-y|.$$

3. $\sin(x^2)$ is **NOT** uniformly continuous on R.

Proof: Assume that $\sin(x^2)$ is uniformly continuous on $R.$ Then given $\varepsilon = 1,$ there is a $\delta > 0$ such that as $|x-y| < \delta,$ we have

$$|\sin(x^2) - \sin(y^2)| < 1. \quad *$$

Consider

$$\sqrt{n\pi + \frac{\pi}{2}} - \sqrt{n\pi} = \frac{\frac{\pi}{2}}{\sqrt{n\pi + \frac{\pi}{2}} + \sqrt{n\pi}} < \frac{\pi}{4\sqrt{n\pi}} (\to 0),$$

and thus choose $N = \left[\frac{\pi}{(4\delta)^2}\right] + 1\left(> \frac{\pi}{(4\delta)^2}\right)$ which implies

$$\sqrt{N\pi + \frac{\pi}{2}} - \sqrt{N\pi} < \delta.$$

So, choose $x = \sqrt{N\pi + \frac{\pi}{2}}$ and $y = \sqrt{N\pi}$, then by (*), we have

$$|f(x) - f(y)| = \left|\sin\left(N\pi + \frac{\pi}{2}\right) - \sin(N\pi)\right| = \left|-\sin\frac{\pi}{2}\right| = 1 < 1$$

which is absurb. So, $\sin(x^2)$ is not uniformly continuous on R.

4. $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

Proof: Since $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|}$ for all $x,y \in [0,\infty),$ then given $\varepsilon > 0,$ there exists a $\delta = \varepsilon^2$ such that as $|x-y| < \delta,\ x,y \in [0,\infty),$ we have

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x-y|} < \sqrt{\delta} = \varepsilon.$$

So, we know that $\sqrt{x}$ is uniformly continuous on $[0,\infty)$.

Note: We have the following interesting results:. Prove that, for $x \geq 0,\ y \geq 0,$

$$|x^p - y^p| \leq \begin{cases} |x-y|^p \text{ if } 0 < p < 1, \\ p|x-y|(x^{p-1} + y^{p-1}) \text{ if } 1 \leq p < \infty. \end{cases}$$

Proof: (As $0 < p < 1$) Without loss of generality, let $x \geq y,$ consider $f(x) = (x-y)^p - x^p + y^p,$ then

$$f'(x) = p\left[(x-y)^{p-1} - x^{p-1}\right] \geq 0, \text{ note that } p - 1 < 0.$$

So, we have f is an increasing function defined on $[0,\infty)$ for all given $y \geq 0.$ Hence, we have $f(x) \geq f(0) = 0.$ So,

$$x^p - y^p \leq (x-y)^p \text{ if } x \geq y \geq 0$$

which implies that

$$|x^p - y^p| \leq |x-y|^p$$

for $x \geq 0,\ y \geq 0.$

Ps: The inequality, we can prove the case $p = 1/2$ directly. Thus the inequality is not surprising for us.

(As $1 \leq p < \infty$) Without loss of generality, let $x \geq y,$ consider

$$\begin{aligned} x^p - y^p &= (pz^{p-1})(x-y), \text{ where } z \in (y,x), \text{ by } \textbf{Mean Value Theorem}. \\ &\leq px^{p-1}(x-y), \text{ note that } p-1 \geq 0, \\ &\leq p(x^{p-1} + y^{p-1})(x-y) \end{aligned}$$

which implies

$$|x^p - y^p| \leq p|x-y|(x^{p-1} + y^{p-1})$$

for $x \geq 0,\ y \geq 0.$

5. In general, we have

$$x^r \begin{cases} \text{is uniformly continuous on } [0,\infty), \text{ if } r \in [0,1], \\ \text{is } \textbf{NOT} \text{ uniformly continuous on } [0,\infty), \text{ if } r > 1, \end{cases}$$

and

$$\sin(x^r) = \begin{cases} \text{is uniformly continuous on } [0,\infty), \text{ if } r \in [0,1], \\ \text{is } \textbf{NOT} \text{ uniformly continuous on } [0,\infty), \text{ if } r > 1. \end{cases}$$

Proof: (x^r) As $r = 0,$ it means that x^r is a constant function. So, it is obviuos. As $r \in (0,1],$ then given $\varepsilon > 0,$ there is a $\delta = \varepsilon^{1/r} > 0$ such that as $|x-y| < \delta,\ x,y \in [0,\infty),$ we have

$$\begin{aligned} |x^r - y^r| &\leq |x-y|^r \text{ by note in the exercise} \\ &< \delta^r \\ &= \varepsilon. \end{aligned}$$

So, x^r is uniformly continuous on $[0,\infty),$ if $r \in [0,1].$

As $r > 1,$ assume that x^r is uniformly continuous on $[0,\infty),$ then given $\varepsilon = 1 > 0,$ there exists a $\delta > 0$ such that as $|x-y| < \delta,\ x,y \in [0,\infty),$ we have

$$|x^r - y^r| < 1. \qquad *$$

By **Mean Value Theorem,** we have (let $x = y + \delta/2,\ y > 0$)

$$
\begin{aligned}
x^r - y^r &= rz^{r-1}(x-y) \\
&\geq ry^{r-1}(\delta/2).
\end{aligned}
$$

So, if we choose $y \geq (\frac{2}{r\delta})^{\frac{1}{r-1}}$, then we have

$$x^r - y^r \geq 1$$

which is absurb with (*). Hence, x^r is not uniformly continuous on $[0,\infty)$.

Ps: The reader should try to realize why x^r is not uniformly continuous on $[0,\infty)$, for $r > 1$. The ruin of non-uniform continuity comes from that x is large enough. At the same time, compare it with theorem that a continuous function defined on a compact set K is uniflormly continuous on K.

($\sin x^r$) As $r = 0$, it means that x^r is a constant function. So, it is obviuos. As $r \in (0,1]$, given $\varepsilon > 0$, there is a $\delta = \varepsilon^{1/r} > 0$ such that as $|x-y| < \delta$, $x,y \in [0,\infty)$, we have

$$
\begin{aligned}
|\sin x^r - \sin y^r| &= \left|2\cos\left(\frac{x^r+y^r}{2}\right)\sin\left(\frac{x^r-y^r}{2}\right)\right| \\
&\leq |x^r - y^r| \\
&\leq |x-y|^r \text{ by the note in the Remark 4.} \\
&< \delta^r \\
&= \varepsilon.
\end{aligned}
$$

So, $\sin x^r$ is uniformly continuous on $[0,\infty)$, if $r \in [0,1]$.

As $r > 1$, assume that $\sin x^r$ is uniformly continuous on $[0,\infty)$, then given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x-y| < \delta$, $x,y \in [0,\infty)$, we have

$$|\sin x^r - \sin y^r| < 1. \qquad **$$

Consider a sequence $\left\{(n\pi + \frac{\pi}{2})^{1/r} - (n\pi)^{1/r}\right\}$, it is easy to show that the sequence tends to 0 as $n \to \infty$. So, there exists a positive integer N such that $|x-y| < \delta$, $x = (n\pi + \frac{\pi}{2})^{1/r}$, $y = (n\pi)^{1/r}$. Then

$$\sin x^r - \sin y^r = 1$$

which contradicts (**). So, we know that $\sin x^r$ is not uniformly continuous on $[0,\infty)$.

Ps: For $\left\{(n\pi + \frac{\pi}{2})^{1/r} - (n\pi)^{1/r}\right\} := x_n \to 0$ as $n \to 0$, here is a short proof by using **L-Hospital Rule.**

Proof: Write

$$
\begin{aligned}
x_n &= \left(n\pi + \frac{\pi}{2}\right)^{1/r} - (n\pi)^{1/r} \\
&= (n\pi)^{1/r}\left[\left(1 + \frac{1}{2n}\right)^{1/r} - 1\right] \\
&= \frac{\left[(1 + \frac{1}{2n})^{1/r} - 1\right]}{(n\pi)^{-1/r}}
\end{aligned}
$$

and thus consider the following limit

$$\lim_{x\to\infty} \frac{\left[(1+\frac{1}{2x})^{1/r}-1\right]}{(x\pi)^{-1/r}},\ \left(\frac{0}{0}\right)$$
$$= \lim_{x\to\infty} \frac{\pi^{1/r}}{2} x^{\frac{1}{r}-1}\left(1+\frac{1}{2x}\right)^{\frac{1}{r}-1} \text{ by \textbf{L-Hospital Rule.}}$$
$$= 0.$$

Hence $x_n \to 0$ as $n \to \infty$.

6. Here is a useful criterion for a function which is **NOT** uniformly continuous defined a subset A in a metric space. We say a function f is not uniformly continuous on a subset A in a metric space if, and only if, there exists $\varepsilon_0 > 0$, and two sequences $\{x_n\}$ and $\{y_n\}$ such that as

$$\lim_{n\to\infty} x_n - y_n = 0$$

which implies that

$$|f(x_n) - f(y_n)| \geq \varepsilon_0 \text{ for } n \text{ is large enough.}$$

The criterion is directly from the definition on uniform continuity. So, we omit the proof.

4.52 Assume that f is uniformly continuous on a bounded set S in R^n. Prove that f must be bounded on S.

Proof: Since f is uniformly continuous on a bounded set S in R^n, given $\varepsilon = 1$, then there exists a $\delta > 0$ such that as $\|x-y\| < \delta$, $x,y \in S$, we have

$$d(f(x),f(y)) < 1.$$

Consider the closure of S, $cl(S)$ is closed and bounded. Hence $cl(S)$ is compact. Then for any open covering of $cl(S)$, there is a finite subcover. That is,

$$cl(S) \subseteq \cup_{x\in cl(S)} B(x;\delta/2),$$
$$\Rightarrow cl(S) \subseteq \cup_{k=1}^{k=n} B(x_k;\delta/2), \text{ where } x_k \in cl(S),$$
$$\Rightarrow S \subseteq \cup_{k=1}^{k=n} B(x_k;\delta/2), \text{ where } x_k \in cl(S).$$

Note that if $B(x_k;\delta/2) \cap S = \phi$ for some k, then we remove this ball. So, we choose $y_k \in B(x_k;\delta/2) \cap S$, $1 \leq k \leq n$ and thus we have

$$B(x_k;\delta/2) \subseteq B(y_k;\delta) \text{for } 1 \leq k \leq n,$$

since let $z \in B(x_k;\delta/2)$,

$$\|z-y_k\| \leq \|z-x_k\| + \|x_k-y_k\| < \delta/2 + \delta/2 = \delta.$$

Hence, we have

$$S \subseteq \cup_{k=1}^{k=n} B(y_k;\delta), \text{ where } y_k \in S.$$

Given $x \in S$, then there exists $B(y_k;\delta)$ for some k such that $x \in B(y_k;\delta)$. So,

$$d(f(x),f(x_k)) < 1 \Rightarrow f(x) \in B(f(y_k);1)$$

Note that $\cup_{k=1}^{k=n} B(f(y_k);1)$ is bounded since every $B(f(y_k);1)$ is bounded. So, let B be a bounded ball so that $\cup_{k=1}^{k=n} B(f(y_k);1) \subseteq B$. Hence, we have every $x \in S$, $f(x) \in B$. That is, f is bounded.

Remark: If we know that the codomain is complete, then we can reduce the above proof. See Exercise 4.55.

4.53 Let f be a function defined on a set S in R^n and assume that $f(S) \subseteq R^m$. Let g be defined on $f(S)$ with value in R^k, and let h denote the composite function defined by

$h(x) = g[f(x)]$ if $x \in S$. If f is uniformly continuous on S and if g is uniformly continuous on $f(S)$, show that h is uniformly continuous on S.

Proof: Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $\|x - y\|_{R^n} < \delta$, $x, y \in S$, we have

$$\|h(x) - h(y)\| = \|g(f(x)) - g(f(y))\| < \varepsilon.$$

For the same ε, since g is uniformly continuous on $f(S)$, then there exists a $\delta' > 0$ such that as $\|f(x) - f(y)\|_{R^m} < \delta'$, we have

$$\|g(f(x)) - g(f(y))\| < \varepsilon.$$

For this δ', since f is uniformly continuous on S, then there exists a $\delta > 0$ such that as $\|x - y\|_{R^n} < \delta$, $x, y \in S$, we have

$$\|f(x) - f(y)\|_{R^m} < \delta'.$$

So, given $\varepsilon > 0$, there is a $\delta > 0$ such that as $\|x - y\|_{R^n} < \delta$, $x, y \in S$, we have

$$\|h(x) - h(y)\| < \varepsilon.$$

That is, h is uniformly continuous on S.

Remark: It should be noted that (Assume that all functions written are continuous)

(1) $\left(\text{uniform continuity}\right) \circ \left(\text{uniform continuity}\right) = \text{uniform continuity}$.

(2) (uniform continuity) ∘ (**NOT** uniform continuity) = (a) **NOT** uniform continuity, or (b) uniform continuity.

(3) (**NOT** uniform continuity) ∘ (uniform continuity) = (a) **NOT** uniform continuity, or (b) uniform continuity.

(4) (**NOT** uniform continuity) ∘ (**NOT** uniform continuity) = (a) **NOT** uniform continuity, or (b) uniform continuity.

For (1), it is from the exercise.

For (2), (a) let $f(x) = x$, and $g(x) = x^2$, $x \in R \Rightarrow f(g(x)) = f(x^2) = x^2$.
(b) let $f(x) = \sqrt{x}$, and $g(x) = x^2$, $x \in [0, \infty) \Rightarrow f(g(x)) = f(x^2) = x$.

For (3), (a) let $f(x) = x^2$, and $g(x) = x$, $x \in R \Rightarrow f(g(x)) = f(x) = x^2$.
(b) let $f(x) = x^2$, and $g(x) = \sqrt{x}$, $x \in [0, \infty) \Rightarrow f(g(x)) = f(\sqrt{x}) = x$.

For (4), (a) let $f(x) = x^2$, and $g(x) = x^3$, $x \in R \Rightarrow f(g(x)) = f(x^3) = x^6$.
(b) let $f(x) = 1/x$, and $g(x) = \frac{1}{\sqrt{x}}, x \in (0,1) \Rightarrow f(g(x)) = f\left(\frac{1}{\sqrt{x}}\right) = \sqrt{x}$.

Note. In (4), we have x^r is not uniformly continuous on $(0,1)$, for $r < 0$. Here is a proof.

Proof: Let $r < 0$, and assume that x^r is not uniformly continuous on $(0,1)$. Given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|x^r - y^r| < 1. \qquad *$$

Let $x_n = 2/n$, and $y_n = 1/n$. Then $x_n - y_n = 1/n$. Choose n large enough so that $1/n < \delta$. So, we have

$$|x^r - y^r| = \left|\left(\frac{2}{n}\right)^r - \left(\frac{1}{n}\right)^r\right|$$
$$= \left(\frac{1}{n}\right)^r |2^r - 1| \to \infty, \text{ as } n \to \infty \text{ since } r < 0,$$
which is absurb with (*). Hence, we know that x^r is not uniformly continuous on $(0,1)$, for $r < 0$.

Ps: The reader should try to realize why x^r is not uniformly continuous on $(0,1)$, for $r < 0$. The ruin of non-uniform continuity comes from that x is small enough.

4.54 Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T. (Compare with Exercise 4.33.)

Proof: Given $\varepsilon > 0$, we want to find a positive integer N such that as $n, m \geq N$, we have
$$d(f(x_n), f(x_m)) < \varepsilon.$$
For the same ε, since f is uniformly continuous on S, then there is a $\delta > 0$ such that as $d(x,y) < \delta$, $x,y \in S$, we have
$$d(f(x), f(y)) < \varepsilon.$$
For this δ, since $\{x_n\}$ is a Cauchy sequence in S, then there is a positive integer N such that as $n, m \geq N$, we have
$$d(x_n, x_m) < \delta.$$
Hence, given $\varepsilon > 0$, there is a postive integer N such that as $n, m \geq N$, we have
$$d(f(x_n), f(x_m)) < \varepsilon.$$
That is, $\{f(x_n)\}$ is a Cauchy sequence in T.

Remark: The reader should compare with Exercise 4.33 and Exercise 4.55.

4.55 Let $f : S \to T$ be a function from a metric space S to another metric space T. Assume that f is uniformly continuous on a subset A of S and let T is complete. Prove that there is a unique extension of f to $cl(A)$ which is uniformly continuous on $cl(A)$.

Proof: Since $cl(A) = A \cup A'$, it suffices to consider the case $x \in A' - A$. Since $x \in A' - A$, then there is a sequence $\{x_n\} \subseteq A$ with $x_n \to x$. Note that this sequence is a Cauchy sequence, so we have by Exercise 4.54, $\{f(x_n)\}$ is a Cauchy sequence in T since f is uniformly on A. In addition, since T is complete, we know that $\{f(x_n)\}$ is a convergent sequence, say its limit L. Note that if there is another sequence $\{\tilde{x}_n\} \subseteq A$ with $\tilde{x}_n \to x$, then $\{f(\tilde{x}_n)\}$ is also a convergent sequence, say its limit L'. Note that $\{x_n\} \cup \{\tilde{x}_n\}$ is still a Cauchy sequence. So, we have
$$d(L, L') \leq d(L, f(x_n)) + d(f(x_n), f(\tilde{x}_n)) + d(f(\tilde{x}_n), L') \to 0 \text{ as } n \to \infty.$$
So, $L = L'$. That is, it is well-defined for $g : cl(A) \to T$ by the following
$$g(x) = \begin{cases} f(x) \text{ if } x \in A, \\ \lim_{n\to\infty} f(x_n) \text{ if } x \in A' - A, \text{ where } x_n \to x. \end{cases}$$
So, the function g is a extension of f to $cl(A)$.

Claim that this g is uniformly continuous on $cl(A)$. That is, given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $d(x,y) < \delta$, $x,y \in cl(A)$, we have
$$d(g(x), g(y)) < \varepsilon.$$
Since f is uniformly continuous on A, for $\varepsilon' = \varepsilon/3$, there is a $\delta' > 0$ such that as

$d(x,y) < \delta'$, $x,y \in A$, we have

$$d(f(x),f(y)) < \varepsilon'.$$

Let $x,y \in cl(A)$, and thus we have $\{x_n\} \subseteq A$ with $x_n \to x$, and $\{y_n\} \subseteq A$ with $y_n \to y$. Choose $\delta = \delta'/3$, then we have

$$d(x_n,x) < \delta'/3 \text{ and } d(y_n,y) < \delta'/3 \text{ as } n \geq N_1$$

So, as $d(x,y) < \delta = \delta'/3$, we have $(n \geq N_1)$

$$d(x_n,y_n) \leq d(x_n,x) + d(x,y) + d(y,y_n) < \delta'/3 + \delta'/3 + \delta'/3 = \delta'.$$

Hence, we have as $d(x,y) < \delta$, $(n \geq N_1)$

$$\begin{aligned} d(g(x),g(y)) &\leq d(g(x),f(x_n)) + d(f(x_n),f(y_n)) + d(f(y_n),f(y)) \qquad * \\ &< d(g(x),f(x_n)) + \varepsilon' + d(f(y_n),g(y)) \end{aligned}$$

And since $\lim_{n\to\infty} f(x_n) = g(x)$, and $\lim_{n\to\infty} f(y_n) = g(y)$, we can choose $N \geq N_1$ such that

$$d(g(x),f(x_n)) < \varepsilon' \text{ and}$$
$$d(f(y_n),g(y)) < \varepsilon'.$$

So, as $d(x,y) < \delta$, $(n \geq N)$ we have

$$d(g(x),g(y)) < 3\varepsilon' = \varepsilon \text{ by } (*).$$

That is, g is uniformly on $cl(A)$.

It remains to show that g is a unique extension of f to $cl(A)$ which is uniformly continuous on $cl(A)$. If there is another extension h of f to $cl(A)$ which is uniformly continuous on $cl(A)$, then given $x \in A' - A$, we have, by continuity, (Say $x_n \to x$)

$$h(x) = h\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g\left(\lim_{n\to\infty} x_n\right) = g(x)$$

which implies that $h(x) = g(x)$ for all $x \in A' - A$. Hence, we have $h(x) = g(x)$ for all $x \in cl(A)$. That is, g is a unique extension of f to $cl(A)$ which is uniformly continuous on $cl(A)$.

Remark: 1. We do not require that A is bounded, in fact, A is any non-empty set in a metric space.

2. The exercise is a criterion for us to check that a given function is **NOT** uniformly continuous. For example, let $f : (0,1) \to R$ by $f(x) = 1/x$. Since $f(0+)$ does not exist, we know that f is not uniformly continuous. The reader should feel that a uniformly continuous is sometimes regarded as a **smooth** function. So, it is not surprising for us to know the exercise. Similarly to check $f(x) = x^2, x \in R$, and so on.

3. Here is an exercise to make us know that a uniformly continuous is a **smooth** function. Let $f : R \to R$ be uniformly continuous, then there exist $\alpha, \beta > 0$ such that

$$|f(x)| \leq \alpha|x| + \beta.$$

Proof: Since f is uniformly continuous on R, given $\varepsilon = 1$, there is a $\delta > 0$ such that as $|x - y| < \delta$, we have

$$|f(x) - f(y)| < 1. \qquad *$$

Given any $x \in R$, then there is the positive integer N such that $N\delta > |x| > (N-1)\delta$. If $x > 0$, we consider

$$y_0 = 0,\ y_1 = \delta/2,\ y_2 = \delta, \ldots, y_{2N-1} = N\delta - \frac{\delta}{2}, y_{2N} = x.$$

Then we have

$$|f(x) - f(0)| \leq \sum_{k=1}^{N} |f(y_{2k}) - f(y_{2k-1})| + |f(y_{2k-1}) - f(y_{2k-2})|$$
$$\leq 2N \text{ by } (*)$$
which implies that
$$|f(x)| \leq 2N + |f(0)|$$
$$\leq 2\left(1 + \frac{|x|}{\delta}\right) + |f(0)| \text{ since } |x| > (N-1)\delta$$
$$\leq \frac{2}{\delta}|x| + (2 + |f(0)|).$$
Similarly for $x < 0$. So, we have proved that $|f(x)| \leq \alpha|x| + \beta$ for all x.

4.56 In a metric space (S,d), let A be a nonempty subset of S. Define a function $f_A : S \to R$ by the equation
$$f_A(x) = \inf\{d(x,y) : y \in A\}$$
for each x in S. The number $f_A(x)$ is called the distance from x to A.

(a) Prove that f_A is uniformly continuous on S.

(b) Prove that $cl(A) = \{x : x \in S \text{ and } f_A(x) = 0\}$.

Proof: (a) Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $d(x_1,x_2) < \delta$, $x_1,x_2 \in S$, we have
$$|f_A(x_1) - f_A(x_2)| < \varepsilon.$$
Consider $(x_1,x_2,y \in S)$
$$d(x_1,y) \leq d(x_1,x_2) + d(x_2,y), \text{ and } d(x_2,y) \leq d(x_1,x_2) + d(x_1,y)$$
So,
$$\inf\{d(x_1,y) : y \in A\} \leq d(x_1,x_2) + \inf\{d(x_2,y) : y \in A\} \text{ and}$$
$$\inf\{d(x_2,y) : y \in A\} \leq d(x_1,x_2) + \inf\{d(x_1,y) : y \in A\}$$
which implies that
$$f_A(x_1) - f_A(x_2) \leq d(x_1,x_2) \text{ and } f_A(x_2) - f_A(x_1) \leq d(x_1,x_2)$$
which implies that
$$|f_A(x_1) - f_A(x_2)| \leq d(x_1,x_2).$$
Hence, if we choose $\delta = \varepsilon$, then we have as $d(x_1,x_2) < \delta$, $x_1,x_2 \in S$, we have
$$|f_A(x_1) - f_A(x_2)| < \varepsilon.$$
That is, f_A is uniformly continuous on S.

(b) Define $K = \{x : x \in S \text{ and } f_A(x) = 0\}$, we want to show $cl(A) = K$. We prove it by two steps.

($\subseteq$) Let $x \in cl(A)$, then $B(x;r) \cap A \neq \phi$ for all $r > 0$. Choose $y_k \in B(x;1/k) \cap A$, then we have
$$\inf\{d(x,y) : y \in A\} \leq d(x,y_k) \to 0 \text{ as } k \to \infty.$$
So, we have $f_A(x) = \inf\{d(x,y) : y \in A\} = 0$. So, $cl(A) \subseteq K$.

($\supseteq$) Let $x \in K$, then $f_A(x) = \inf\{d(x,y) : y \in A\} = 0$. That is, given any $\varepsilon > 0$, there is an element $y_\varepsilon \in A$ such that $d(x,y_\varepsilon) < \varepsilon$. That is, $y_\varepsilon \in B(x;\varepsilon) \cap A$. So, x is an adherent point of A. That is, $x \in cl(A)$. So, we have $K \subseteq cl(A)$.

From above saying, we know that $cl(A) = \{x : x \in S \text{ and } f_A(x) = 0\}$.

Remark: 1. The function f_A often appears in Analysis, so it is worth keeping it in mind.

In addition, part (b) comes from intuition. The reader may think it twice about distance 0.

2. Here is a good exercise to pratice. The statement is that suppose that K and F are disjoint subsets in a metric space X, K is compact, F is closed. Prove that there exists a $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$, $q \in F$. Show that the conclusion is may fail for two disjoint closed sets if neither is compact.

Proof: Suppose **NOT**, i.e., for any $\delta > 0$, there exist $p_\delta \in K$, and $q_\delta \in F$ such that $d(p_\delta, q_\delta) \leq \delta$. Let $\delta = 1/n$, then there exist two sequence $\{p_n\} \subseteq K$, and $\{q_n\} \subseteq F$ such that $d(p_n, q_n) \leq 1/n$. Note that $\{p_n\} \subseteq K$, and K is compact, then there exists a subsequence $\{p_{n_k}\}$ with $\lim_{n_k \to \infty} p_{n_k} = p \in K$. Hence, we consider $d(p_{n_k}, q_{n_k}) \leq \frac{1}{n_k}$ to get a contradiction. Since

$$d(p_{n_k}, p) + d(p, q_{n_k}) \leq d(p_{n_k}, q_{n_k}) \leq \frac{1}{n_k},$$

then let $n_k \to \infty$, we have $\lim_{n_k \to \infty} q_{n_k} = p$. That is, p is an accumulation point of F which implies that $p \in F$. So, we get a contradiction since $K \cap F = \phi$. That is, there exists a $\delta > 0$ such that $d(p,q) > \delta$ if $p \in K$, $q \in F$.

We give an example to show that the conclusion does not hold. Let $K = \{(x,0) : x \in R\}$ and $F = \{(x, 1/x) : x > 0\}$, then K and F are closd. It is clear that such δ cannot be found.

Note: Two disjoint closed sets may has the distance 0, however; if one of closed sets is compact, then we have a distance $\delta > 0$. The reader can think of them in R^n, and note that a bounded and closed subsets in R^n is compact. It is why the example is given.

4.57 In a metric space (S,d), let A and B be disjoint closed subsets of S. Prove that there exists disjoint open subsets U and V of S such that $A \subseteq U$ and $B \subseteq V$. Hint. Let $g(x) = f_A(x) - f_B(x)$, in the notation of Exercise 4.56, and consider $g^{-1}(-\infty, 0)$ and $g^{-1}(0, +\infty)$.

Proof: Let $g(x) = f_A(x) - f_B(x)$, then by Exercixe 4.56, we have $g(x)$ is uniformly continuous on S. So, $g(x)$ is continuous on S. Consider $g^{-1}(-\infty, 0)$ and $g^{-1}(0, +\infty)$, and note that A, B are disjoint and closed, then we have by part (b) in Exercise 4.56,

$$g(x) < 0 \text{ if } x \in A \text{ and}$$

$$g(x) > 0 \text{ if } x \in B.$$

So, we have $A \subseteq g^{-1}(-\infty, 0) := U$, and $B \subseteq g^{-1}(0, +\infty) := V$.

Discontinuities

4.58 Locate and classify the discontinuities of the functions f defined on R^1 by the following equations:

(a) $f(x) = \sin x/x$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since $\lim_{x \to 0} \frac{\sin x}{x} = 1$, we know that f has a removable discontinuity at 0.

(b) $f(x) = e^{1/x}$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since $\lim_{x \to 0^+} e^{1/x} = \infty$ and $\lim_{x \to 0^-} e^{1/x} = 0$, we know that f has an irremovable discontinuity at 0.

(c) $f(x) = e^{1/x} + \sin 1/x$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since the limit $f(x)$ does not exist as $x \to 0$, we know that f has a irremovable discontinuity at 0.

(d) $f(x) = 1/(1 - e^{1/x})$ if $x \neq 0$, $f(0) = 0$.

Solution: f is continuous on $R - \{0\}$, and since $\lim_{x \to 0^+} e^{1/x} = \infty$ and $\lim_{x \to 0^-} e^{1/x} = 0$, we know that f has an irremovabel discontinuity at 0. In addition, $f(0+) = 0$ and $f(0-) = 1$, we know that f has the lefthand jump at 0, $f(0) - f(0-) = -1$, and f is continuous from the right at 0.

4.59 Locate the points in R^2 at which each of the functions in Exercise 4.11 is not continuous.

(a) By Exercise 4.11, we know that $f(x,y)$ is discontinuous at $(0,0)$, where

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2} \text{ if } (x,y) \neq (0,0), \text{ and } f(0,0) = 0.$$

Let $g(x,y) = x^2 - y^2$, and $h(x,y) = x^2 + y^2$ both defined on $R^2 - \{(0,0)\}$, we know that g and h are continuous on $R^2 - \{(0,0)\}$. Note that $h \neq 0$ on $R^2 - \{(0,0)\}$. Hence, $f = g/h$ is continuous on $R^2 - \{(0,0)\}$.

(b) By Exercise 4.11, we know that $f(x,y)$ is discontinuous at $(0,0)$, where

$$f(x,y) = \frac{(xy)^2}{(xy)^2 + (x-y)^2} \text{ if } (x,y) \neq (0,0), \text{ and } f(0,0) = 0.$$

Let $g(x,y) = (xy)^2$, and $h(x,y) = (xy)^2 + (x-y)^2$ both defined on $R^2 - \{(0,0)\}$, we know that g and h are continuous on $R^2 - \{(0,0)\}$. Note that $h \neq 0$ on $R^2 - \{(0,0)\}$. Hence, $f = g/h$ is continuous on $R^2 - \{(0,0)\}$.

(c) By Exercise 4.11, we know that $f(x,y)$ is continuous at $(0,0)$, where

$$f(x,y) = \frac{1}{x}\sin(xy) \text{ if } x \neq 0, \text{ and } f(0,y) = y,$$

since $\lim_{(x,y) \to (0,0)} f(x,y) = 0 = f(0,0)$. Let $g(x,y) = 1/x$ and $h(x,y) = \sin(xy)$ both defined on $R^2 - \{(0,0)\}$, we know that g and h are continuous on $R^2 - \{(0,0)\}$. Note that $h \neq 0$ on $R^2 - \{(0,0)\}$. Hence, $f = g/h$ is continuous on $R^2 - \{(0,0)\}$. Hence, f is continuous on R^2.

(d) By Exercise 4.11, we know that $f(x,y)$ is continuous at $(0,0)$, where

$$f(x,y) = \begin{cases} (x+y)\sin(1/x)\sin(1/y) & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0 & \text{if } x = 0 \text{ or } y = 0. \end{cases}$$

since $\lim_{(x,y) \to (0,0)} f(x,y) = 0 = f(0,0)$. It is the same method as in Exercise 4.11, we know that f is discontinuous at $(x,0)$ for $x \neq 0$ and f is discontinuous at $(0,y)$ for $y \neq 0$. And it is clearly that f is continuous at (x,y), where $x \neq 0$ and $y \neq 0$.

(e) By Exercise 4.11, Since

$$f(x,y) = \begin{cases} \frac{\sin x - \sin y}{\tan x - \tan y}, & \text{if } \tan x \neq \tan y, \\ \cos^3 x & \text{if } \tan x = \tan y. \end{cases},$$

we rewrite

$$f(x,y) = \begin{cases} \frac{\cos\left(\frac{x+y}{2}\right)\cos x \cos y}{\cos\left(\frac{x-y}{2}\right)} & \text{if } \tan x \neq \tan y \\ \cos^3 x & \text{if } \tan x = \tan y. \end{cases}.$$

We consider $(x,y) \in (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$, others are similar. Consider two cases (1) $x = y$, and (2) $x \neq y$, we have

(1) ($x = y$) Since $\lim_{(x,y) \to (a,a)} f(x,y) = \cos^3 a = f(a,a)$. Hence, we know that f is

continuous at (a,a).

(2) $(x \neq y)$ Since $x \neq y$, it implies that $\tan x \neq \tan y$. Note that the denominator is not 0 since $(x,y) \in (-\pi/2,\pi/2) \times (-\pi/2,\pi/2)$. So, we know that f is continuous at (a,b), $a \neq b$.

So, we know that f is continuous on $(-\pi/2,\pi/2) \times (-\pi/2,\pi/2)$.

Monotonic functions

4.60 Let f be defined in the open interval (a,b) and assume that for each interior point x of (a,b) there exists a 1 –ball $B(x)$ in which f is increasing. Prove that f is an increasing function throughout (a,b).

Proof: Suppose **NOT**, i.e., there exist p,q with $p < q$ such that $f(p) > f(q)$. Consider $[p,q](\subseteq (a,b))$, and since for each interior point x of (a,b) there exists a 1 –ball $B(x)$ in which f is increasing. Then $[p,q] \subseteq \cup_{x\in[p,q]} B(x;\delta_x)$, (The choice of balls comes from the hypothesis). It implies that $[p,q] \subseteq \cup_{k=1}^{n} B(x_n;\delta_n) := B_n$. Note that if $B_i \subseteq B_j$, we remove such B_i and make one left. Without loss of generality, we assume that $x_1 \leq .. \leq x_n$.

$$f(p) \leq f(x_1) \leq \ldots \leq f(x_n) \leq f(q)$$

which is absurb. So, we know that f is an increasing function throughout (a,b).

4.61 Let f be continuous on a compact interval $[a,b]$ and assume that f does not have a loacal maximum or a local minimum at any interior point. (See the note following Exercise 4.25.) Prove that f must be monotonic on $[a,b]$.

Proof: Since f is continuous on $[a,b]$, we have

$$\max_{x\in[a,b]} f(x) = f(p), \text{ where } p \in [a,b] \text{ and}$$

$$\min_{x\in[a,b]} f(x) = f(q), \text{ where } q \in [a,b].$$

So, we have $\{p,q\} = \{a,b\}$ by hypothesis that f does not have a local maximum or a local minimum at any interior point. Without loss of generality, we assume that $p = a$, and $q = b$. Claim that f is decreasing on $[a,b]$ as follows.

Suppose **NOT**, then there exist $x,y \in [a,b]$ with $x < y$ such that $f(x) < f(y)$. Consider $[x,y]$ and by hyothesis, we know that $f|_{[x,y]}$ has the maximum at y, and $f|_{[a,y]}$ has the minimum at y. Then it implies that there exists $B(y;\delta) \cap [x,y]$ such that f is constant on $B(y;\delta) \cap [x,y]$, which contradicts to the hypothesis. Hence, we have proved that f is decreasing on $[a,b]$.

4.62 If f is one-to one and continuous on $[a,b]$, prove that f must be strictly monotonic on $[a,b]$. That is, prove that **every topological mapping of** $[a,b]$ **onto an interval** $[c,d]$ **must be strictly monotonic.**

Proof: Since f is continuous on $[a,b]$, we have

$$\max_{x\in[a,b]} f(x) = f(p), \text{ where } p \in [a,b] \text{ and}$$

$$\min_{x\in[a,b]} f(x) = f(q), \text{ where } q \in [a,b].$$

Assume that $p \in (a,b)$, then there exists a $\delta > 0$ such that $f(y) \leq f(p)$ for all $y \in (p-\delta,p+\delta) \subseteq [a,b]$. Choose $y_1 \in (x-\delta,x)$ and $y_2 \in (x,x+\delta)$, then we have by 1-1, $f(y_1) < f(x)$ and $f(y_2) < f(x)$. And thus choose r so that

$f(y_1) < r < f(x) \Rightarrow f(z_1) = r$, where $z_1 \in (y_1,x)$ by **Intermediate Value Theorem,**

$f(y_2) < r < f(x) \Rightarrow f(z_2) = r$, where $z_2 \in (x,y_2)$ by **Intermediate Value Theorem,**

which contradicts to 1-1. So, we know that $p \in \{a,b\}$. Similarly, we have $q \in \{a,b\}$.

Without loss of generality, we assume that $p = a$ and $q = b$. Claim that f is strictly decreasing on $[a,b]$.

Suppose **NOT**, then there exist $x,y \in [a,b]$, with $x < y$ such that $f(x) < f(y)$. ("=" does not hold since f is 1-1.) Consider $[x,y]$ and by above method, we know that $f|_{[x,y]}$ has the maximum at y, and $f|_{[a,y]}$ has the minimum at y. Then it implies that there exists $B(y;\delta) \cap [x,y]$ such that f is constant on $B(y;\delta) \cap [x,y]$, which contradicts to 1-1. Hence, for any $x < y (\in [a,b])$, we have $f(x) > f(y)$. ("=" does not hold since f is 1-1.) So, we have proved that f is strictly decreasing on $[a,b]$.

Reamrk: 1. Here is another proof by Exercise 4.61. It suffices to show that 1-1 and continuity imply that f does not have a local maximum or a local minimum at any interior point.

Proof: Suppose **NOT**, it means that f has a local extremum at some interior point x. Without loss of generality, we assume that f has a local minimum at the interior point x. Since x is an interior point of $[a,b]$, then there exists an open interval $(x-\delta, x+\delta) \subseteq [a,b]$ such that $f(y) \geq f(x)$ for all $y \in (x-\delta, x+\delta)$. Note that f is 1-1, so we have $f(y) > f(x)$ for all $y \in (x-\delta, x+\delta) - \{x\}$. Choose $y_1 \in (x-\delta)$ and $y_2 \in (x, x+\delta)$, then we have $f(y_1) > f(x)$ and $f(y_2) > f(x)$. And thus choose r so that

$f(y_1) > r > f(x) \Rightarrow f(p) = r$, where $p \in (y_1, x)$ by **Intermediate Value Theorem,**

$f(y_2) > r > f(x) \Rightarrow f(q) = r$, where $q \in (x, y_2)$ by **Intermediate Value Theorem,**

which contradicts to the hypothesis that f is 1-1. Hence, we have proved that 1-1 and continuity imply that f does not have a local maximum or a local minimum at any interior point.

2. **Under the assumption of continuity on a compact interval, one-to-one is equivalent to being strictly monotonic.**

Proof: By the exercise, we know that an one-to-one and continuous function defined on a compact interval implies that a strictly monotonic function. So, it remains to show that a strictly monotonic function implies that an one-to-one function. Without loss of generality, let f be increasing on $[a,b]$, then as $f(x) = f(y)$, we must have $x = y$ since if $x < y$, then $f(x) < f(y)$ and if $x > y$, then $f(x) > f(y)$. So, we have proved that a strictly monotonic function implies that an one-to-one function. Hence, we get that under the assumption of continuity on a compact interval, one-to-one is equivalent to being strictly monotonic.

4.63 Let f be an increasing function defined on $[a,b]$ and let $x_1, \ldots, x_n$ be n points in the interior such that $a < x_1 < x_2 < \ldots < x_n < b$.

(a) Show that $\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] \leq f(b-) - f(a+)$.

Proof: Let $a = x_0$ and $b = x_{n+1}$; since f is an increasing function defined on $[a,b]$, we know that both $f(x_k+)$ and $f(x_k-)$ exist for $1 \leq k \leq n$. Assume that $y_k \in (x_k, x_{k+1})$, then we have $f(y_k) \geq f(x_k+)$ and $f(x_{k-1}) \geq f(y_{k-1})$. Hence,

$$\begin{aligned}\sum_{k=1}^{n} [f(x_k+) - f(x_k-)] &\leq \sum_{k=1}^{n} [f(y_k) - f(y_{k-1})] \\ &\leq f(y_n) - f(y_0) \\ &\leq f(b-) - f(a+).\end{aligned}$$

(b) Deduce from part (a) that the set of dicontinuities of f is countable.

Proof: Let D denote the set of dicontinuities of f. Consider $D_m = \{x \in [a,b] : f(x+) - f(x-) \geq \frac{1}{m}\}$, then $D = \cup_{m=1}^{\infty} D_m$. Note that $\#(D_m) < \infty$, so we have D is countable. That is, the set of dicontinuities of f is countable.

(c) Prove that f has points of continuity in every open subintervals of $[a,b]$.

Proof: By (b), f has points of continuity in every open subintervals of $[a,b]$, since every open subinterval is uncountable.

Remark: (1) Here is another proof about (b). Denote $Q = \{x_1,\ldots,x_n,\ldots\}$, and let x be a point at which f is not continuous. Then we have $f(x+) - f(x-) > 0$. (If x is the end point, we consider $f(x+) - f(x) > 0$ or $f(x) - f(x-) > 0$) So, we have an open interval I_x such that $I_x \cap f([a,b]) = \{f(x)\}$. The interval I_x contains infinite many rational numbers, we choose the smallest index, say $m = m(x)$. Then the number of the set of discontinuities of f on $[a,b]$ is a subset of N. Hence, the number of the set of discontinuities of f on $[a,b]$ is countable.

(2) There is a similar exercise; we write it as a reference. Let f be a real valued function defined on $[0,1]$. Suppose that there is a positive number M having the following condition: for every choice of a finite number of points $x_1,..,x_n$ in $[0,1]$, we have $-M \leq \sum_{i=1}^{n} x_i \leq M$. Prove that $S : \{x \in [0,1] : f(x) \neq 0\}$ is countable.

Proof: Consider $S_n = \{x \in [0,1] : |f(x)| \geq 1/n\}$, then it is clear that every S_n is countable. Since $S = \cup_{n=1}^{\infty} S_n$, we know that S is countable.

4.64 Give an example of a function f, defined and strictly increasing on a set S in R, such that f^{-1} is not continuous on $f(S)$.

Solution: Let

$$f(x) = \begin{cases} x \text{ if } x \in [0,1), \\ 1 \text{ if } x = 2. \end{cases}$$

Then it is clear that f is strictly increasing on $[0,1]$, so f has the incerse function

$$f^{-1}(x) = \begin{cases} x \text{ if } x \in [0,1), \\ 2 \text{ if } x = 1. \end{cases}$$

which is not continuous on $f(S) = [0,1]$.

Remark: Compare with Exercise 4.65.

4.65 Let f be strictly increasing on a subset S of R. Assume that the image $f(S)$ has one of the following properties: (a) $f(S)$ is open; (b) $f(S)$ is connected; (c) $f(S)$ is closed. Prove that f must be continuous on S.

Proof: (a) Given $a \in S$, then $f(a) \in f(S)$. Given $\varepsilon > 0$, we wan to find a $\delta > 0$ such that as $x \in B(a;\delta) \cap S$, we have $|f(x) - f(a)| < \varepsilon$. Since $f(S)$ is open, then there exists $B(f(a),\varepsilon') \subseteq f(S)$, where $\varepsilon' < \varepsilon$.

Claim that there exists a $\delta > 0$ such that $f(B(a;\delta) \cap S) \subseteq B(f(a),\varepsilon')$. Choose $y_1 = f(a) - \varepsilon'/2$ and $y_2 = f(a) + \varepsilon'/2$, then $y_1 = f(x_1)$ and $y_2 = f(x_2)$, we have $x_1 < a < x_2$ since f is strictly increasing on S. Hence, for $x \in (x_1,x_2) \cap S$, we have $f(x_1) < f(x) < f(x_2)$ since f is strictly increasing on S. So, $f(x) \in B(f(a),\varepsilon')$. Let $\delta = \min(a - x_1, x_2 - a)$, then $B(a;\delta) \cap S = (a-\delta, a+\delta) \cap S \subseteq (x_1,x_2) \cap S$ which implis that $f(B(a;\delta) \cap S) \subseteq B(f(a),\varepsilon')$. $(\subseteq B(f(a),\varepsilon))$

Hence we have prove the claim, and the claim tells us that f is continuous at a. Since a is arbitrary, we know that f is continuous on S.

(b) Note that since $f(S) \subseteq R$, and $f(S)$ is connected, we know that $f(S)$ is an interval I. Given $a \in S$, then $f(a) \in I$. We discuss 2 cases as follows. (1) $f(a)$ is an interior point of I. (2) $f(a)$ is the endpoint of I.

For case (1), it is similar to (a). We omit the proof.

For case (2), it is similar to (a). We omit the proof.

So, we have proved that f is continuous on S.

(c) Given $a \in S$, then $f(a) \in f(S)$. Since $f(S)$ is closed, we consider two cases. (1) $f(a)$ is an isolated point and (2) $f(a)$ is an accumulation point.

For case (1), claim that a is an isolated point. Suppose **NOT**, there is a sequence $\{x_n\} \subseteq S$ with $x_n \to a$. Consider $\{x_n\}_{n=1}^{\infty} = \{x : x_n < a\} \cup \{x : x_n > a\}$, and thus we may assume that $\{x : x_n < a\} := \{a_n\}$ is a infinite subset of $\{x_n\}_{n=1}^{\infty}$. Since f is monotonic, we have $\lim_{n\to\infty} f(x_n) = f(a-)$. Since $f(S)$ is closed, we have $f(a-) \in f(S)$. Therefore, there exists $b \in f(S)$ such that $f(a-) = f(b) \leq f(a)$.

If $f(b) = f(a)$, then $b = a$ since f is strictly increasing. But is contradicts to that $f(a)$ is isolated. On the other hand, if $f(b) < f(a)$, then $b < a$ since f is strictly increasing. In addition, $f(a_n) \leq f(a-) = f(b)$ implies that $a_n \leq b$. But is contradicts to that $a_n \to a$.

Hence, we have proved that a is an isolated point. So, f is sutomatically continuous at a.

For case (2), suppose that $f(a)$ is an accumulation point. Then $B(f(a);\varepsilon) \cap f(S) \neq \phi$ and $B(f(a);\varepsilon)$ has infinite many numbers of points in $f(S)$. Choose $y_1, y_2 \in B(f(a);\varepsilon) \cap f(S)$ with $y_1 < y_2$, then $f(x_1) = y_1$, and $f(x_2) = y_2$. And thus it is similar to (a), we omit the proof.

So, we have proved that f is continuous on S by (1) and (2).

Remark: In (b), when we say f is monotonic on a subset of R, its image is also in R.

Supplement.

It should be noted that the discontinuities of a monotonic function need not be isolated. In fact, given any countable subset E of (a,b), which may even be dense, **we can construct a function f, monotonic on (a,b), discontinuous at every point of E, and at no other point of (a,b).** To show this, let the points of E be arranged in a sequence $\{x_n\}$, $n = 1,2,\ldots$ Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges. Define

$$f(x) = \sum_{x_n < x} c_n \ (a < x < b)$$

Note: The summation is to be understood as follows: Sum over those indices n for whcih $x_n < x$. If there are no points x_n to the left of x, the sum is empty; following the usual convention, we define it to be zero. Since absolute convergence, the order in which the terms are arranged is immaterial.

Then $f(x)$ is desired.

The proof that we omit; the reader should see the book, **Principles of Mathematical Analysis written by Walter Rudin**, pp 97.

Metric space and fixed points

4.66 Let $B(S)$ denote the set of all real-valued functions which are defined and bounded on a nonempty set S. If $f \in B(S)$, let

$$\|f\| = \sup_{x\in S} |f(x)|.$$

The number $\|f\|$ is called the " sup norm " of f.

(a) Provet that the formula $d(f,g) = \|f-g\|$ defines a metric d on $B(S)$.

Proof: We prove that d is a metric on $B(S)$ as follows.

(1) If $d(f,g) = 0$, i.e., $\|f-g\| = \sup_{x\in S}|f(x)-g(x)| = 0 \geq |f(x)-g(x)|$ for all $x \in S$. So, we have $f = g$ on S.

(2) If $f = g$ on S, then $|f(x)-g(x)| = 0$ for all $x \in S$. That is, $\|f-g\| = 0 = d(f,g)$.

(3) Given $f,g \in B(S)$, then

$$\begin{aligned} d(f,g) &= \|f-g\| \\ &= \sup_{x\in S}|f(x)-g(x)| \\ &= \sup_{x\in S}|g(x)-f(x)| \\ &= \|g-f\| \\ &= d(g,f). \end{aligned}$$

(4) Given $f,g,h \in B(S)$, then since

$$|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|,$$

we have

$$\begin{aligned} |f(x)-g(x)| &\leq \left(\sup_{x\in S}|f(x)-h(x)|\right) + \left(\sup_{x\in S}|h(x)-g(x)|\right) \\ &\leq \|f-h\| + \|h-g\| \end{aligned}$$

which implies that

$$\|f-g\| = \sup_{x\in S}|f(x)-g(x)| \leq \|f-h\| + \|h-g\|.$$

So, we have prove that d is a metric on $B(S)$.

(b) Prove that the metric space $(B(S),d)$ is complete. Hint: If $\{f_n\}$ is a Cauchy sequence in $B(S)$, show that $\{f_n(x)\}$ is a Cauchy sequence of real numbers for each x in S.

Proof: Let $\{f_n\}$ be a Cauchy sequence on $(B(S),d)$, That is, given $\varepsilon > 0$, there is a positive integer N such that as $m,n \geq N$, we have

$$d(f,g) = \|f_n - f_m\| = \sup_{x\in S}|f_n(x) - f_m(x)| < \varepsilon. \qquad *$$

So, for every point $x \in S$, the sequence $\{f_n(x)\}(\subseteq R)$ is a Cauchy sequence. Hence, the sequence $\{f_n(x)\}$ is a convergent sequence, say its limit $f(x)$. It is clear that the function $f(x)$ is well-defined. Let $\varepsilon = 1$ in (*), then there is a positive integer N such that as $m,n \geq N$, we have

$$|f_n(x) - f_m(x)| < 1, \text{ for all } x \in S. \qquad **$$

Let $m \to \infty$, and $n = N$, we have by (**)

$$|f_N(x) - f(x)| \leq 1, \text{ for all } x \in S$$

which implies that

$$|f(x)| \leq 1 + |f_N(x)|, \text{ for all } x \in S.$$

Since $|f_N(x)| \in B(S)$, say its bound M, and thus we have

$$|f(x)| \leq 1 + M, \text{ for all } x \in S$$

which implies that $f(x)$ is bounded. That is, $f(x) \in (B(S),d)$. Hence, we have proved that $(B(S),d)$ is a complete metric space.

Remark: 1. We do not require that S is bounded.

2. The boundedness of a function f cannot be remove since sup norm of f is finite.

3. The sup norm of f, often appears and is important; the reader should keep it in mind. And we will encounter it when we discuss on sequences of functions. Also, see Exercise 4.67.

4. Here is an important theorem, the reader can see the definition of uniform convergence in the text book, page 221.

4.67 Refer to Exercise 4.66 and let $C(S)$ denote the subset of $B(S)$ consisting of all funtions **continuous** and bounded on S, where now S is a metric space.

(a) Prove that $C(S)$ is a closed subset of $B(S)$.

Proof: Let f be an adherent point of $C(S)$, then $B(f;r) \cap C(S) \neq \phi$ for all $r > 0$. So, there exists a sequence $\{f_n(x)\}$ such that $f_n \to f$ as $n \to \infty$. So, given $\varepsilon' > 0$, there is a positive integer N such that as $n \geq N$, we have

$$d(f_n,f) = \|f_n - f\| = \sup_{x\in S}|f_n(x) - f(x)| < \varepsilon'.$$

So, we have

$$|f_N(x) - f(x)| < \varepsilon'. \text{ for all } x \in S. \qquad *$$

Given $s \in S$, and note that $f_N(x) \in C(S)$, so for this ε', there exists a $\delta > 0$ such that as $|x - s| < \delta$, $x,s \in S$, we have

$$|f_N(x) - f_N(s)| < \varepsilon'. \qquad **$$

We now prove that f is continuous at s as follows. Given $\varepsilon > 0$, and let $\varepsilon' = \varepsilon/3$, then there is a $\delta > 0$ such that as $|x - s| < \delta$, $x,s \in S$, we have

$$\begin{aligned}|f(x) - f(s)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(s)| + |f_N(s) - f(s)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \text{ by } (*) \text{ and } (**) \\ &= \varepsilon.\end{aligned}$$

Hence, we know that f is continuous at s, and since s is arbitrary, we know that f is continuous on S.

(b) Prove that the metric subspace $C(S)$ is complete.

Proof: By (a), we know that $C(S)$ is complete since a closed subset of a complete metric space is complete.

Remark: 1. In (b), we can see Exercise 4.9.

2. The reader should see the text book in Charpter 9, and note that **Theorem 9.2** and **Theorem 9.3.**

4.68 Refer to the proof of the **fixed points theorem (Theorem 4.48)** for notation.

(a) Prove that $d(p,p_n) \leq d(x,f(x))\alpha^n/(1-\alpha)$.

Proof: The statement is that a contraction f of a complete metric space S has a unique fixed point p. Take any point $x \in S$, and consider the sequence of iterates:

$$x,\ f(x),\ f(f(x)),\ldots$$

That is, define a sequence $\{p_n\}$ inductively as follows:

$$p_0 = x,\ p_{n+1} = f(p_n)\ n = 0,1,2,\ldots$$

We will prove that $\{p_n\}$ converges to a fixed point of f. First we show that $\{p_n\}$ is a Cauchy sequence. Since f is a contraction ($d(f(x),f(y)) \leq \alpha d(x,y)$, $0 < \alpha < 1$ for all $x,y \in S$), we have

$$d(p_{n+1},p_n) = d(f(p_n),f(p_{n-1})) \leq \alpha d(p_n,p_{n-1}),$$

so, by induction, we find

$$d(p_{n+1},p_n) \le \alpha^n d(p_1,p_0) = \alpha^n d(x,f(x)).$$

Use the triangel inequality we find, for $m > n$,

$$\begin{aligned} d(p_m,p_n) &\le \sum_{k=n}^{m-1} d(p_{k+1},p_k) \\ &\le d(x,f(x))\sum_{k=n}^{m-1}\alpha^k \\ &= d(x,f(x))\frac{\alpha^n-\alpha^m}{1-\alpha} \\ &< d(x,f(x))\frac{\alpha^n}{1-\alpha}. \end{aligned} \qquad *$$

Since $\alpha^n \to 0$ as $n \to \infty$, we know that $\{p_n\}$ is a Cauchy sequence. And since S is complete, we have $p_n \to p \in S$. The uniqueness is from the inequality, $d(f(x),f(y)) \le \alpha d(x,y)$.

From (*), we know that (let $m \to \infty$)

$$d(p,p_n) \le d(x,f(x))\frac{\alpha^n}{1-\alpha}.$$

This inequality, which is **useful in numberical work**, provides an estimate for the distance from p_n to the fixed point p. An example is given in (b)

(b) Take $f(x) = \frac{1}{2}(x+2/x)$, $S = [1,+\infty)$. Prove that f is contraction of S with contraction constant $\alpha = 1/2$ and fixed point $p = \sqrt{2}$. Form the sequence $\{p_n\}$ starting wth $x = p_0 = 1$ and show that $\left|p_n - \sqrt{2}\right| \le 2^{-n}$.

Proof: First, $f(x)-f(y) = \frac{1}{2}(x+2/x) - \frac{1}{2}(y+2/y) = \frac{1}{2}[(x-y)+2(\frac{y-x}{xy})]$, then we have

$$\begin{aligned} |f(x)-f(y)| &= \left|\frac{1}{2}\left[(x-y)+2\left(\frac{y-x}{xy}\right)\right]\right| \\ &= \left|\frac{1}{2}(x-y)\left(1-\frac{2}{xy}\right)\right| \\ &\le \frac{1}{2}|x-y| \text{ since } \left|1-\frac{2}{xy}\right| \le 1. \end{aligned}$$

So, f is a contraction of S with contraction constant $\alpha = 1/2$. By **Fixed Point Theorem**, we know that there is a unique p such that $f(p) = p$. That is,

$$\frac{1}{2}\left(p+\frac{2}{p}\right) = p \Rightarrow p = \sqrt{2}. \ (-\sqrt{2} \text{ is not our choice since } S = [1,+\infty).)$$

By (a), it is easy to know that

$$\left|p_n - \sqrt{2}\right| \le 2^{-n}.$$

Remark: Here is a modefied Fixed Point Theorem: Let f be function defined on a complete metric space S. If there exists a N such that $d(f^N(x)-f^N(y)) \le \alpha d(x,y)$ for all $x,y \in S$, where $0 < \alpha < 1$. Then f has a unique fixed point $p \in S$.

Proof: Since f^N is a contraction defined on a complete metric space, with the contraction constant α, with $0 < \alpha < 1$, by Fixed Point Theorem, we know that there exists a unique point $p \in S$, such that

$$\begin{aligned} f^N(p) &= p \\ &\Rightarrow f(f^N(p)) = f(p) \\ &\Rightarrow f^N(f(p)) = f(p). \end{aligned}$$

That is, $f(p)$ is also a fixed point of f^N. By uniqueness, we know that $f(p) = p$. In addition, if there is $p' \in S$ such that $f(p') = p'$. Then we have $f^2(p') = f(p') = p', \ldots, f^N(p') = .. = p'$. Hence, we have $p = p'$. That is, f has a unique fixed point $p \in S$.

4.69 Show by counterexample that the fixed-point theorem for contractions need not hold if either (a) the underlying metric space is not complete, or (b) the contraction constant $\alpha \geq 1$.

Solution: (a) Let $f = \frac{1}{2}(1+x) : (0,1) \to R$, then $|f(x) - f(y)| = \frac{1}{2}|x-y|$. So, f is a contraction on $(0,1)$. However, it has no any fixed point since if it has, say this point p, we get $\frac{1}{2}(1+p) = p \Rightarrow p = 1 \notin (0,1)$.

(b) Let $f = (1+x) : [0,1] \to R$, then $|f(x) - f(y)| = |x-y|$. So, f is a contraction with the contraction constant 1. However, it has no any fixed point since if it has, say this point p, we get $1 + p = p \Rightarrow 1 = 0$, a contradiction.

4.70 Let $f : S \to S$ be a function from a complete metric space (S,d) into itself. Assume there is a real sequence $\{a_n\}$ which converges to 0 such that $d(f^n(x), f^n(y)) \leq \alpha_n d(x,y)$ for all $n \geq 1$ and all x, y in S, where f^n is the nth iterate of f; that is,

$$f^1(x) = f(x),\ f^{n+1}(x) = f(f^n(x)) \text{ for } n \geq 1.$$

Prove that f has a unique point. Hint. Apply the fixed point theorem to f^m for a suitable m.

Proof: Since $a_n \to 0$, given $\varepsilon = 1/2$, then there is a positive integer N such that as $n \geq N$, we have

$$|a_n| < 1/2.$$

Note that $a_n \geq 0$ for all n. Hence, we have

$$d(f^N(x), f^N(y)) \leq \frac{1}{2} d(x,y) \text{ for } x, y \text{ in } S.$$

That is, $f^N(x)$ is a contraction defined on a complete metric space, with the contraction constant 1/2. By Fixed Point Theorem, we know that there exists a unique point $p \in S$, such that

$$\begin{aligned} f^N(p) &= p \\ &\Rightarrow f(f^N(p)) = f(p) \\ &\Rightarrow f^N(f(p)) = f(p). \end{aligned}$$

That is, $f(p)$ is also a fixed point of f^N. By uniqueness, we know that $f(p) = p$. In addition, if there is $p' \in S$ such that $f(p') = p'$. Then we have $f^2(p') = f(p') = p', \ldots, f^N(p') = .. = p'$. Hence, we have $p = p'$. That is, f has a unique fixed point $p \in S$.

4.71 Let $f : S \to S$ be a function from a metric space (S,d) into itself such that

$$d(f(x), f(y)) < d(x,y)$$

where $x \neq y$.

(a) Prove that f has at most one fixed point, and give an example of such an f with no fixed point.

Proof: If p and p' are fixed points of f where $p \neq p'$, then by hypothesis, we have

$$d(p,p') = d(f(p),f(p')) < d(p,p')$$

which is absurb. So, f has at most one fixed point.

Let $f : (0,1/2) \to (0,1/2)$ by $f(x) = x^2$. Then we have

$$|f(x) - f(y)| = |x^2 - y^2| = |x+y||x-y| < |x-y|.$$

However, f has no fixed point since if it had, say its fixed point p, then $p^2 = p \Rightarrow p = 1 \notin (0,1/2)$ or $p = 0 \notin (0,1/2)$.

(b) If S is compact, prove that f has exactly one fixed point. Hint. Show that $g(x) = d(x,f(x))$ attains its minimum on S.

Proof: Let $g = d(x,f(x))$, and thus show that g is continuous on a compact set S as follows. Since

$$\begin{aligned} d(x,f(x)) &\leq d(x,y) + d(y,f(y)) + d(f(y),f(x)) \\ &< d(x,y) + d(y,f(y)) + d(x,y) \\ &= 2d(x,y) + d(y,f(y)) \\ &\Rightarrow d(x,f(x)) - d(y,f(y)) < 2d(x,y) \end{aligned} \quad *$$

and change the roles of x, and y, we have

$$d(y,f(y)) - d(x,f(x)) < 2d(x,y) \quad **$$

Hence, by (*) and (**), we have

$$|g(x) - g(y)| = |d(x,f(x)) - d(y,f(y))| < 2d(x,y) \text{ for all } x,y \in S. \quad ***$$

Given $\varepsilon > 0$, there exists a $\delta = \varepsilon/2$ such that as $d(x,y) < \delta$, $x,y \in S$, we have

$$|g(x) - g(y)| < 2d(x,y) < \varepsilon \text{ by (***)}.$$

So, we have proved that g is uniformly continuous on S.

So, consider $\min_{x \in S} g(x) = g(p)$, $p \in S$. We show that $g(p) = 0 = d(p,f(p))$. Suppose **NOT**, i.e., $f(p) \neq p$. Consider

$$d(f^2(p),f(p)) < d(f(p),p) = g(p)$$

which contradicts to $g(p)$ is the absolute maximum. Hence, $g(p) = 0 \Leftrightarrow p = f(p)$. That is, f has a unique fixed point in S by (a).

(c)Give an example with S compact in which f is not a contraction.

Solution: Let $S = [0,1/2]$, and $f = x^2 : S \to S$. Then we have

$$|x^2 - y^2| = |x+y||x-y| \leq |x-y|.$$

So, this f is not contraction.

Remark: 1. In (b), the Choice of g is natural, since we want to get a fixed point. That is, $f(x) = x$. Hence, we consider the function $g = d(x,f(x))$.

2. Here is a exercise that makes us know more about Remark 1. Let $f : [0,1] \to [0,1]$ be a continuous function, show that there is a point p such that $f(p) = p$.

Proof: Consider $g(x) = f(x) - x$, then g is a continuous function defined on $[0,1]$. Assume that there is no point p such that $g(p) = 0$, that is, no such p so that $f(p) = p$. So, by **Intermediate Value Theorem,** we know that $g(x) > 0$ for all $x \in [0,1]$, or $g(x) < 0$ for all $x \in [0,1]$. Without loss of generality, suppose that $g(x) > 0$ for all $x \in [0,1]$ which is absurb since $g(1) = f(1) - 1 \leq 0$. Hence, we know that there is a point p such that $f(p) = p$.

3. Here is another proof on (b).

Proof: Given any point $x \in S$, and thus consider $\{f^n(x)\} \subseteq S$. Then there is a convergent subsequence $\{f^{n(k)}(x)\}$, say its limit p, since S is compact. Consider

$$\begin{aligned} d(f(p),p) &= d\left(f\left[\lim_{k\to\infty} f^{n(k)}(x)\right], \lim_{k\to\infty} f^{n(k)}(x)\right) \\ &= d\left(\lim_{k\to\infty} f[f^{n(k)}(x)], \lim_{k\to\infty} f^{n(k)}(x)\right) \text{ by continuity of } f \text{ at } p \\ &= \lim_{k\to\infty} d(f^{n(k)+1}(x), f^{n(k)}(x)) \qquad 1 \end{aligned}$$

and

$$d(f^{n(k)+1}(x), f^{n(k)}(x)) \leq \ldots \leq d(f^2[f^{n(k-1)}(x)], f[f^{n(k-1)}(x)]). \qquad 2$$

Note that

$$\begin{aligned} &\lim_{k\to\infty} d(f^2[f^{n(k-1)}(x)], f[f^{n(k-1)}(x)]) \\ &= d\left(\lim_{k\to\infty} f^2[f^{n(k-1)}(x)], \lim_{k\to\infty} f[f^{n(k-1)}(x)]\right) \\ &= d\left(f^2\left[\lim_{k\to\infty} f^{n(k-1)}(x)\right], f\left[\lim_{k\to\infty} f^{n(k-1)}(x)\right]\right) \text{ by continuity of } f^2 \text{ and } f \text{ at } p \\ &= d(f^2(p), f(p)). \qquad 3 \end{aligned}$$

So, by (1)-(3), we know that

$$f(p,f(p)) \leq d(f^2(p), f(p)) \Rightarrow p = f(p)$$

by hypothesis

$$d(f(x), f(y)) < d(x,y)$$

where $x \neq y$. Hence, f has a unique fixed point p by (a) in Exercise.

Note. 1. If $x_n \to x$, and $y_n \to y$, then $d(x_n,y_n) \to d(x,y)$. That is,

$$\lim_{n\to\infty} d(x_n,y_n) = d\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right).$$

Proof: Consider

$$d(x_n,y_n) \leq d(x_n,x) + d(x,y) + d(y,y_n) \text{ and}$$
$$d(x,y) \leq d(x,x_n) + d(x_n,y_n) + d(y_n,y),$$

then

$$|d(x_n,y_n) - d(x,y)| \leq d(x,x_n) + d(y,y_n) \to 0.$$

So, we have prove it.

2. The reader should compare the method with Exercise 4.72.

4.72 Assume that f satisfies the condition in Exercise 4.71. If $x \in S$, let $p_0 = x$, $p_{n+1} = f(p_n)$, and $c_n = d(p_n,p_{n+1})$ for $n \geq 0$.

(a) Prove that $\{c_n\}$ is a decreasing sequence, and let $c = \lim c_n$.

Proof: Consider

$$\begin{aligned} c_{n+1} - c_n &= d(p_{n+1},p_{n+2}) - d(p_n,p_{n+1}) \\ &= d(f(p_n), f(p_{n+1})) - d(p_n,p_{n+1}) \\ &\leq d(p_n,p_{n+1}) - d(p_n,p_{n+1}) \\ &= 0, \end{aligned}$$

so $\{c_n\}$ is a decreasing sequence. And $\{c_n\}$ has a lower bound 0, by **Completeness of** R, we know that $\{c_n\}$ is a convergent sequence, say $c = \lim c_n$.

(b) Assume there is a subsequence $\{p_{k(n)}\}$ which converges to a point q in $S.$ Prove that

$$c = d(q,f(q)) = d(f(q),f[f(q)]).$$

Deduce that q is a fixed point of f and that $p_n \to q.$

Proof: Since $\lim_{n\to\infty} p_{k(n)} = q,$ and $\lim_{n\to\infty} c_n = c,$ we have $\lim_{n\to\infty} c_{k(n)} = c.$ So, we consider

$$\begin{aligned} c &= \lim_{n\to\infty} c_{k(n)} \\ &= \lim_{n\to\infty} d(p_{k(n)},p_{k(n)+1}) \\ &= \lim_{n\to\infty} d(p_{k(n)},f(p_{k(n)})) \\ &= d(q,f(q)) \end{aligned}$$

and

$$d(p_{k(n)},p_{k(n)+1}) \leq d(p_{k(n)-1},p_{k(n)}) \leq \ldots \leq d(f(p_{k(n-1)}),f^2(p_{k(n-1)})),$$

we have

$$c = d(q,f(q)) \leq \lim_{n\to\infty} d(f(p_{k(n-1)}),f^2(p_{k(n-1)})) = d(f(q),f^2(q)). \qquad *$$

So, by (*) and hypoethesis

$$d(f(x),f(y)) < d(x,y)$$

where $x \neq y,$ we know that $q = f(q) \big(\Rightarrow c = 0,$ in fact, this q is a unique fixed point.$\big).$

In order to show that $p_n \to p,$ we consider (let $m \geq k(n)$)

$$d(p_m,q) = d(p_m,f(q)) \leq d(p_{m-1},q) \leq .. \leq d(p_{k(n)},q)$$

So, given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$d(p_{k(n)},q) < \varepsilon.$$

Hence, as $m \geq k(N),$ we have

$$d(p_m,q) < \varepsilon.$$

That is, $p_n \to p.$

Derivatives

Real-valued functions

In each following exercise assume, where mecessary, a knowledge of the formulas for differentiating the elementary trigonometric, exponential, and logarithmic functions.

5.1 Assume that f is said to satisfy a **Lipschitz condition of order** α **at** c if there exists a positive number M (which may depend on c) and 1 −ball $B(c)$ such that

$$|f(x) - f(c)| < M|x - c|^{\alpha}$$

whenever $x \in B(c),\ x \neq c$.

(a) Show that a function which satisfies a Lipschitz condition of order α is continuous at c if $\alpha > 0$, and has a derivative at c if $\alpha > 1$.

Proof: 1. As $\alpha > 0$, given $\varepsilon > 0$, there is a $\delta \leq (\varepsilon/M)^{1/\alpha}$ such that as $x \in (c - \delta, c + \delta) \subseteq B(c)$, we have

$$|f(x) - f(c)| < M|x - c|^{\alpha} \leq M\delta^{\alpha} = \varepsilon.$$

So, we know that f is continuous at c.

2. As $\alpha > 1$, consider $x \in B(c)$, and $x \neq c$, we have

$$\left| \frac{f(x) - f(c)}{x - c} \right| \leq M|x - c|^{\alpha - 1} \to 0 \text{ as } x \to c.$$

So, we know that f has a derivative at c with $f'(c) = 0$.

Remark: It should be note that (a) also holds if we consider the higher dimension.

(b) Given an example of a function satisfying a Lipschitz condition of order 1 at c for which $f'(c)$ does not exist.

Solution: Consider

$$||x| - |c|| \leq |x - c|,$$

we know that $|x|$ is a function satisfying a Lipschitz condition of order 1 at 0 for which $f'(0)$ does not exist.

5.2 In each of the following cases, determine the intervals in which the function f is increasing or decreasing and find the maxima and minima (if any) in the set where each f is defined.

(a) $f(x) = x^3 + ax + b,\ x \in R$.

Solution: Since $f'(x) = 3x^2 + a$ on R, we consider two cases: (i) $a \geq 0$, and (ii) $a < 0$.

(i) As $a \geq 0$, we know that f is increasig on R by $f' \geq 0$ on R. In addition, if f has a local extremum at some point c, then $f'(c) = 0$. It implies that $a = 0$ and $c = 0$. That is, $f(x) = x^3 + b$ has a local extremum at 0. It is impossible since x^3 does not. So, we know that f has no maximum and minimum.

(ii) As $a < 0$, since $f' = 3x^2 + a = 3\left(x - \sqrt{-a/3}\right)\left(x + \sqrt{-a/3}\right)$, we know that

$$f'(x): \begin{array}{ccc} (-\infty, -\sqrt{-a/3}\,] & \left[-\sqrt{-a/3}, \sqrt{-a/3}\,\right] & [\sqrt{-a/3}, +\infty) \\ \geq 0 & \leq 0 & \geq 0 \end{array}$$

which implies that

$$f(x): \begin{array}{ccc} (-\infty, -\sqrt{-a/3}\,] & \left[-\sqrt{-a/3}, \sqrt{-a/3}\,\right] & [\sqrt{-a/3}, +\infty) \\ \nearrow & \searrow & \nearrow \end{array}.$$

*

Hence, f is increasing on $(-\infty, -\sqrt{-a/3}\,]$ and $[\sqrt{-a/3}, +\infty)$, and decreasing on $\left[-\sqrt{-a/3}, \sqrt{-a/3}\,\right]$. In addition, if f has a local extremum at some point c, then $f'(c) = 0$. It implies that $c = \pm\sqrt{-a/3}$. With help of (*), we know that $f(x)$ has a local maximum $f\left(-\sqrt{-a/3}\right)$ and a local minimum $f\left(\sqrt{-a/3}\right)$.

(b) $f(x) = \log(x^2 - 9)$, $|x| > 3$.

Solution: Since $f'(x) = \frac{2x}{x^2-9}$, $|x| > 3$, we know that

$$f'(x) : \begin{array}{ll} (-\infty, -3) & (3, +\infty) \\ < 0 & > 0 \end{array}$$

which implies that

$$f(x) : \begin{array}{ll} (-\infty, -3) & (3, +\infty) \\ \searrow & \nearrow \end{array}.$$

Hence, f is increasing on $(3, +\infty)$, and decreasing on $(-\infty, -3)$. It is clear that f cannot have local extremum.

(c) $f(x) = x^{2/3}(x-1)^4$, $0 \le x \le 1$.

Solution: Since $f'(x) = \frac{2(x-1)^3}{3x^{1/3}}(7x-1)$, $0 \le x \le 1$, we know that

$$f'(x) : \begin{array}{ll} [0, 1/7] & [1/7, 1] \\ \ge 0 & \le 0 \end{array}$$

which implies that

$$f(x) : \begin{array}{ll} [0, 1/7] & [1/7, 1] \\ \nearrow & \searrow \end{array}. \qquad (**)$$

Hence, we know that f is increasing on $[0, 1/7]$, and decreasing on $[1/7, 1]$. In addition, if f has a local extremum at some interior point c, then $f'(c) = 0$. It implies that $c = 1/7$. With help of (**), we know that f has a local maximum $f(1/7)$, and two local minima $f(0)$, and $f(1)$.

Remark: f has the absolute maximum $f(1/7)$, and the absolute minima $f(0) = f(1) = 0$.

(d) $f(x) = (\sin x)/x$ if $x \ne 0$, $f(0) = 1$, $0 \le x \le \pi/2$.

Sulotion: Since $f'(x) = \cos x \frac{x - \tan x}{x^2}$ as $0 < x \le \pi/2$, and $f'_+(0) = 0$, in addition, $f'(x) \to 0$ as $x \to 0^+$ by **L-Hospital Rule**, we know that

$$f'(x) : \begin{array}{l} [0, \pi/2] \\ \le 0 \end{array}$$

which implies that

$$f(x) : \begin{array}{l} [0, \pi/2] \\ \searrow \end{array}. \qquad (***)$$

Hence, we know that f is decreasing on $[0, \pi/2]$. In addition, note that there is no interior point c such that $f'(c) = 0$. With help of (***), we know that f has local maximum $f(0)$, and local minimum $f(\pi/2)$.

Remark: 1. Here is a proof on $f'_+(0)$: Since

$$\lim_{x\to 0^+} \frac{\sin x - 1}{x} = \lim_{x\to 0^+} \frac{-2(\sin x/2)^2}{x} = 0,$$

we know that $f'_+(0) = 0$.

2. f has the absolute maximum $f(0)$, and the absolute minimum $f(\pi/2)$.

5.3 Find a polynomial f of lowest possible degree such that

$$f(x_1) = a_1, f(x_2) = a_2, f'(x_1) = b_1, f'(x_2) = b_2$$

where $x_1 \neq x_2$ and $a_1,\ a_2,\ b_1,\ b_2$ are given real numbers.

Proof: It is easy to know that the lowest degree is at most 3 since there are 4 unknows. The degree is depends on the values of $a_1,\ a_2,\ b_1,\ b_2$.

5.4 Define f as follows: $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$. Show that

(a) f is continuous for all x.

Proof: In order to show f is continuous on R, it suffices to show f is continuous at 0. Since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{1}{e}\right)^{1/x^2} = 0 = f(0),$$

we know that f is continuous at 0.

(b) $f^{(n)}$ is continuous for all x, and $f^{(n)}(0) = 0$, $(n = 1,2,\ldots)$

Proof: In order to show $f^{(n)}$ is continuous on R, it suffices to show $f^{(n)}$ is continuous at 0. Note that

$$\lim_{x\to\pm\infty} \frac{p(x)}{e^x} = 0, \text{ where } p(x) \text{ is any real polynomial.} \quad *$$

Claim that for $x \neq 0$, we have $f^{(n)}(x) = e^{-1/x^2}P_{3n}(1/x)$, where $P_{3n}(t)$ is a real polynomial of degree $3n$ for all $n = 1,2,\ldots$. As $n = 0$, $f^{(0)}(x) = f(x) = e^{-1/x^2} = e^{-1/x^2}P_0(1/x)$, where $P_0(1/x)$ is a constant function 1. So, as $n = 0$, it holds. Suppose that $n = k$ holds, i.e., $f^{(k)}(x) = e^{-1/x^2}P_{3k}(1/x)$, where $P_{3k}(t)$ is a real polynomial of degree $3k$. Consider $n = k+1$, we have

$$\begin{aligned} f^{(k+1)}(x) &= (f^{(k)}(x))' \quad ** \\ &= (e^{-1/x^2}P_{3k}(1/x))' \text{ by induction hypothesis} \\ &= e^{-1/x^2}\left\{\left[2\left(\frac{1}{x}\right)^3 P_{3k}\left(\frac{1}{x}\right)\right] - \left[\left(\frac{1}{x}\right)^2 P'_{3k}\left(\frac{1}{x}\right)\right]\right\}. \end{aligned}$$

Since $[2t^3P_{3k}(t)] - [t^2P'_{3k}(t)]$ is a real polynomial of degree $3k+3$, we define $[2t^3P_{3k}(t)] - [t^2P'_{3k}(t)] = P_{3k+3}(t)$, and thus we have by (**)

$$f^{(k+1)}(x) = e^{-1/x^2}P_{3k+3}(1/x).$$

So, as $n = k+1$, it holds. Therefore, by **Mathematical Induction**, we have proved the claim.

Use the claim to show that $f^{(n)}(0) = 0$, $(n = 1,2,\ldots)$ as follows. As $n = 0$, it is trivial by hypothesis. Suppose that $n = k$ holds, i.e., $f^{(k)}(0) = 0$. Then as $n = k+1$, we have

$$\begin{aligned}\frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} &= \frac{f^{(k)}(x)}{x} \text{ by induction hypothesis}\\ &= \frac{e^{-1/x^2}P_{3k}(1/x)}{x}\\ &= \frac{tP_{3k}(t)}{e^{t^2}} \text{ (let } t = 1/x)\\ &= \left(\frac{tP_{3k}(t)}{e^t}\right)\left(\frac{e^t}{e^{t^2}}\right) \to 0 \text{ as } t \to \pm\infty \ (\Leftrightarrow x \to 0) \text{ by (*).}\end{aligned}$$

Hence, $f^{(k+1)}(0) = 0.$ So, by **Mathematical Induction,** we have proved that $f^{(n)}(0) = 0,$ $(n = 1, 2, \ldots).$

Since

$$\begin{aligned}\lim_{x\to 0} f^{(n)}(x) &= \lim_{x\to 0} e^{-1/x^2}P_{3n}(1/x)\\ &= \lim_{x\to 0}\frac{P_{3n}(1/x)}{e^{1/x^2}}\\ &= \lim_{t\to\pm\infty}\left(\frac{P_{3n}(t)}{e^t}\right)\left(\frac{e^t}{e^{t^2}}\right)\\ &= 0 \text{ by (*)}\\ &= f^{(n)}(0),\end{aligned}$$

we know that $f^{(n)}(x)$ is continuous at 0.

Remark: 1. Here is a proof on (*). Let $P(x)$ be a real polynomial of degree $n,$ and choose an even number $2N > n.$ We consider a **Taylor Expansion with Remainder** as follows. Since for any $x,$ we have

$$e^x = \sum_{k=0}^{2N+1}\frac{1}{k!}x^k + \frac{e^{\xi(x,0)}}{(2N+2)!}x^{2N+2} \geq \sum_{k=0}^{2N+1}\frac{1}{k!}x^k,$$

then

$$0 \leq \left|\frac{P(x)}{e^x}\right| \leq \left|\frac{P(x)}{\sum_{k=0}^{2N+1}\frac{1}{k!}x^k}\right| \to 0 \text{ as } x \to \pm\infty$$

since $\deg(P(x)) = n < \deg\left(\sum_{k=0}^{2N+1}\frac{1}{k!}x^k\right) = 2N+1.$ By **Sandwich Theorem**, we have proved

$$\lim_{x\to\pm\infty}\frac{P(x)}{e^x} = 0.$$

2. Here is another proof on $f^{(n)}(0) = 0,$ $(n = 1, 2, \ldots).$ By Exercise 5.15, it suffices to show that $\lim_{x\to 0} f^{(n)}(x) = 0.$ For the part, we have proved in this exercise. So, we omit the proof. Exercise 5.15 tells us that we need not make sure that the derivative of f at 0. The reader should compare with Exercise 5.15 and Exercise 5.5.

3. In the future, we will encounter the exercise in Charpter 9. The Exercises tells us one important thing that **the Taylor's series about** 0 **generated by** f **converges everywhere on** $R,$ **but it represents** f **only at the origin.**

5.5 Define $f,$ $g,$ and h as follows: $f(0) = g(0) = h(0) = 0$ and, if $x \neq 0,$ $f(x) = \sin(1/x),$ $g(x) = x\sin(1/x),$ $h(x) = x^2\sin(1/x).$ Show that

(a) $f'(x) = -1/x^2\cos(1/x),$ if $x \neq 0;$ $f'(0)$ does not exist.

Proof: Trivially, $f'(x) = -1/x^2 \cos(1/x)$, if $x \neq 0$. Let $\left\{x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}\right\}$, and thus consider

$$\frac{f(x_n) - f(0)}{x_n - 0} = \frac{\sin(1/x_n)}{x_n} = \pi\left(2n + \frac{1}{2}\right) \to \infty \text{ as } n \to \infty.$$

Hence, we know that $f'(0)$ does not exist.

(b) $g'(x) = \sin(1/x) - 1/x \cos(1/x)$, if $x \neq 0$; $g'(0)$ does not exist.

Proof: Trivially, $g'(x) = \sin(1/x) - 1/x \cos(1/x)$, if $x \neq 0$. Let $\left\{x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}\right\}$, and $\{y_n = \frac{1}{2n\pi}\}$, we know that

$$\frac{g(x_n) - g(0)}{x_n - 0} = \sin\left(\frac{1}{x_n}\right) = 1 \text{ for all } n$$

and

$$\frac{g(y_n) - g(0)}{y_n - 0} = \sin\left(\frac{1}{y_n}\right) = 0 \text{ for all } n.$$

Hence, we know that $g'(0)$ does not exist.

(c) $h'(x) = 2x \sin(1/x) - \cos(1/x)$, if $x \neq 0$; $h'(0) = 0$; $\lim_{x \to 0} h'(x)$ does not exist.

Proof: Trivially, $h'(x) = 2x \sin(1/x) - \cos(1/x)$, if $x \neq 0$. Consider

$$\left|\frac{h(x) - h(0)}{x - 0}\right| = |x \sin(1/x)| \leq |x| \to 0 \text{ as } x \to 0,$$

so we know that $h'(0) = 0$. In addition, let $\left\{x_n = \frac{1}{\pi\left(2n+\frac{1}{2}\right)}\right\}$, and $\{y_n = \frac{1}{2n\pi}\}$, we have

$$h'(x_n) = \frac{2}{\pi(2n + \frac{1}{2})} \text{ and } h'(y_n) = -1 \text{ for all } n.$$

Hence, we know that $\lim_{x \to 0} h'(x)$ does not exist.

5.6 Derive Leibnitz's formula for the nth derivative of the product h of two functions f and g :

$$h^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}(x), \text{ where } \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Proof: We prove it by mathematical Induction. As $n = 1$, it is clear since $h' = f'g + g'f$. Suppose that $n = k$ holds, i.e., $h^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}(x)$. Consider $n = k + 1$, we have

$$
\begin{aligned}
h^{(k+1)} &= (h^{(k)})' = \left[\sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}(x) \right]' \\
&= \sum_{j=0}^{k} \binom{k}{j} [f^{(j)} g^{(k-j)}(x)]' \\
&= \sum_{j=0}^{k} \binom{k}{j} \{[f^{(j+1)} g^{(k-j)}] + [f^{(j)} g^{(k-j+1)}]\} \\
&= \begin{array}{l} \sum_{j=0}^{k-1} \binom{k}{j} [f^{(j+1)} g^{(k-j)}] + f^{(k+1)} g^{(0)} \\ + \sum_{j=1}^{k} \binom{k}{j} [f^{(j)} g^{(k-j+1)}] + f^{(0)} g^{(k+1)} \end{array} \\
&= \begin{array}{l} \sum_{j=0}^{k-1} \binom{k}{j} [f^{(j+1)} g^{(k-j)}] + f^{(k+1)} g^{(0)} \\ + \sum_{j=0}^{k-1} \binom{k}{j+1} [f^{(j+1)} g^{(k-j)}] + f^{(0)} g^{(k+1)} \end{array} \\
&= \sum_{j=0}^{k-1} \left[\binom{k}{j} + \binom{k}{j+1} \right] [f^{(j+1)} g^{(k-j)}] + f^{(k+1)} g^{(0)} + f^{(0)} g^{(k+1)} \\
&= \sum_{j=0}^{k-1} \binom{k+1}{j+1} [f^{(j+1)} g^{(k-j)}] + f^{(k+1)} g^{(0)} + f^{(0)} g^{(k+1)} \\
&= \sum_{j=0}^{k+1} \binom{k}{j} [f^{(j+1)} g^{(k-j)}].
\end{aligned}
$$

So, as $n = k + 1,$ it holds. Hence, by **Mathematical Induction**, we have proved the **Leibnitz** formula.

Remark: We use the famous formula called **Pascal Theorem:** $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$, where $0 \leq k < n$.

5.7 Let f and g be two functions defined and having finite third-order derivatives $f'''(x)$ and $g'''(x)$ for all x in R. If $f(x)g(x) = 1$ for all $x,$ show that the relations in (a), (b), (c), and (d) holds at those points where the denominators are not zero:

(a) $f'(x)/f(x) + g'(x)/g(x) = 0.$

Proof: Since $f(x)g(x) = 1$ for all $x,$ we have $f'g + g'f = 0$ for all $x.$ By hypothesis, we have

$$\frac{f'g + g'f}{fg} = 0 \text{ for those points where the denominators are not zero}$$

which implies that

$$f'(x)/f(x) + g'(x)/g(x) = 0.$$

(b) $f''(x)/f'(x) - 2f'(x)/f(x) - g''(x)/g'(x) = 0.$

Proof: Since $f'g + g'f = 0$ for all $x,$ we have $(f'g + g'f)' = f''g + 2f'g' + g''f = 0.$ By hypothesis, we have

$$\begin{aligned} 0 &= \frac{f''g + 2f'g' + g''f}{f'g} \\ &= \frac{f''}{f'} + 2\frac{g'}{g} + \frac{g''}{g\left(\frac{f'}{f}\right)} \\ &= \frac{f''}{f'} - 2\frac{f'}{f} - \frac{g''}{g'} \text{ by (a).} \end{aligned}$$

(c) $\frac{f'''(x)}{f'(x)} - 3\frac{f'(x)g''(x)}{f(x)g'(x)} - 3\frac{f''(x)}{f(x)} - \frac{g'''(x)}{g'(x)} = 0.$

Proof: By (b), we have $(f''g + 2f'g' + g''f)' = 0 = f'''g + 3f''g + 3f'g'' + fg'''$. By hypothesis, we have

$$\begin{aligned} 0 &= \frac{f'''g + 3f''g' + 3f'g'' + fg'''}{f'g} \\ &= \frac{f'''}{f'} + 3\frac{f''g'}{f'g} + 3\frac{g''}{g} + \frac{fg'''}{f'g} \\ &= \frac{f'''}{f'} + 3f''\left(\frac{g'}{f'g}\right) + 3g''\left(\frac{1}{g}\right) + g'''\left(\frac{f}{f'g}\right) \\ &= \frac{f'''}{f'} - 3\frac{f''}{f} - 3\frac{f'g''}{fg'} - \frac{g'''}{g'} \text{ by (a).} \end{aligned}$$

(d) $\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.$

Proof: By (c), we have $\frac{f'''}{f'} - \frac{g'''}{g'} = 3\left(\frac{f''}{f} + \frac{f'g''}{fg'}\right)$. Since

$$\begin{aligned} \frac{f''}{f} + \frac{f'g''}{fg'} &= \left(\frac{f'}{f}\right)\left(\frac{f''}{f'}\right) + \left(\frac{f'}{f}\right)\left(\frac{g''}{g'}\right) \\ &= \left(\frac{f'}{f}\right)\left[\left(\frac{f''}{f'}\right) + \left(\frac{g''}{g'}\right)\right] \\ &= \frac{1}{2}\left[\left(\frac{f''}{f'}\right) - \left(\frac{g''}{g'}\right)\right]\left[\left(\frac{f''}{f'}\right) + \left(\frac{g''}{g'}\right)\right] \text{ by (b)} \\ &= \frac{1}{2}\left[\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right], \end{aligned}$$

we know that $\frac{f'''}{f'} - \frac{g'''}{g'} = \frac{3}{2}\left[\left(\frac{f''}{f'}\right)^2 - \left(\frac{g''}{g'}\right)^2\right]$ which implies that $\frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$.

Note. The expression which appears on the left side of (d) is called the **Schwarzian derivative of *f* at *x*.**

(e) Show that f and g have the same Schwarzian derivative if

$$g(x) = [af(x) + b]/(cf(x) + d), \text{ where } ad - bc \neq 0.$$

Hint. If $c \neq 0$, write $(af + b)/(cf + d) = (a/c) + (bc - ad)/[c(cf + d)]$, and apply part (d).

Proof: If $c = 0$, we have $g = \frac{a}{d}f + \frac{b}{d}$. So, we have

$$\frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$$
$$= \frac{\frac{a}{d}f'''(x)}{\frac{a}{d}f'(x)} - \frac{3}{2}\left(\frac{\frac{a}{d}f''(x)}{\frac{a}{d}f'(x)}\right)^2$$
$$= \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2.$$

So, f and g have the same Schwarzian derivative.

If $c \neq 0$, write $g = (af+b)/(cf+d) = (a/c) + (bc-ad)/[c(cf+d)]$, then
$$(cg-a)\left[\left(\frac{1}{bc-ad}\right)(cf+d)\right] = 1 \text{ since } ad-bc \neq 0.$$

Let $G = cg-a$, and $F = (\frac{1}{bc-ad})(cf+d)$, then $GF = 1$. It implies that by (d),
$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \frac{G'''}{G'} - \frac{3}{2}\left(\frac{G''}{G'}\right)^2$$
which implies that
$$\frac{F'''}{F'} - \frac{3}{2}\left(\frac{F''}{F'}\right)^2 = \frac{(\frac{c}{bc-ad})f'''}{(\frac{c}{bc-ad})f'} - \frac{3}{2}\left[\frac{(\frac{c}{bc-ad})f''}{(\frac{c}{bc-ad})f'}\right]^2$$
$$= \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$
$$= \frac{G'''}{G'} - \frac{3}{2}\left(\frac{G''}{G'}\right)^2$$
$$= \frac{cg'''}{cg'} - \frac{3}{2}\left(\frac{cg''}{cg'}\right)^2$$
$$= \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2.$$

So, f and g have the same Schwarzian derivative.

5.8 Let f_1, f_2, g_1, g_2 be functions having derivatives in (a,b). Define F by means of the determinant
$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix}, \text{ if } x \in (a,b).$$

(a) Show that $F'(x)$ exists for each x in (a,b) and that
$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

Proof: Since $F(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ g_1(x) & g_2(x) \end{vmatrix} = f_1g_2 - f_2g_1$, we have
$$F' = f_1'g_2 + f_1g_2' - f_2'g_1 - f_2g_1'$$
$$= (f_1'g_2 - f_2'g_1) + (f_1g_2' - f_2g_1')$$
$$= \begin{vmatrix} f_1'(x) & f_2'(x) \\ g_1(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) \\ g_1'(x) & g_2'(x) \end{vmatrix}.$$

(b) State and prove a more general result for nth order determinants.

Proof: Claim that if

$$F(x) = \begin{vmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix},$$

then

$$F'(x) = \begin{vmatrix} f'_{11} & f'_{12} & \dots & f'_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix} + \begin{vmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f'_{21} & f'_{22} & \dots & f'_{2n} \\ \dots & \dots & \dots & \dots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{vmatrix} + \dots + \begin{vmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \dots & \dots & \dots & \dots \\ f'_{n1} & f'_{n2} & \dots & f'_{nn} \end{vmatrix}.$$

We prove it by Mahematial Induction. As $n = 2$, it has proved in (a). Suppose that $n = k$ holds, consider $n = k + 1$,

$$\begin{vmatrix} f_{11} & f_{12} & \dots & f_{1k+1} \\ f_{21} & f_{22} & \dots & f_{2k+1} \\ \dots & \dots & \dots & \dots \\ f_{k+11} & f_{k+12} & \dots & f_{k+1k+1} \end{vmatrix}'$$

$$= \left\{ (-1)^{(k+1)+1} f_{k+11} \begin{vmatrix} f_{12} & \dots & f_{1k+1} \\ \dots & \dots & \dots \\ f_{k2} & \dots & f_{kk+1} \end{vmatrix} + \dots + (-1)^{(k+1)+(k+1)} f_{k+1k+1} \begin{vmatrix} f_{11} & \dots & f_{1k} \\ \dots & \dots & \dots \\ f_{k1} & \dots & f_{kk} \end{vmatrix} \right\}'$$

$= \dots$ (The reader can write it down by induction hypothesis).

Hence, by **Mathematical Induction**, we have proved it.

Remark: The reader should keep it in mind since it is useful in Analysis. For example, we have the following Theorem.

(**Theorem**) Suppose that $f, g,$ and h are continuous on $[a, b]$, and differentiable on (a, b). Then there is a $\xi \in (a, b)$ such that

$$\begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof: Let

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix},$$

then it is clear that $F(x)$ is continuous on $[a, b]$ and differentiable on (a, b) since the operations on determinant involving addition, substraction, and multiplication without division. Consider

$$F(a) = F(b) = 0,$$

then by **Rolle's Theorem**, we know that

$$F'(\xi) = 0, \text{ where } \xi \in (a,b),$$

which implies that

$$\begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0. \qquad *$$

(**Application- Generalized Mean Value Theorem**) Suppose that f and g are continuous on $[a,b]$, and differentiable on (a,b). Then there is a $\xi \in (a,b)$ such that

$$[f(b) - f(a)]g'(\xi) = f'(\xi)[g(b) - g(a)].$$

Proof: Let $h(x) = 1$, and thus by (*), we have

$$\begin{vmatrix} f'(\xi) & g'(\xi) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0,$$

which implies that

$$-\begin{vmatrix} f'(\xi) & g'(\xi) \\ f(b) & g(b) \end{vmatrix} + \begin{vmatrix} f'(\xi) & g'(\xi) \\ f(a) & g(a) \end{vmatrix} = 0$$

which implies that

$$[f(b) - f(a)]g'(\xi) = f'(\xi)[g(b) - g(a)].$$

Note: Use the similar method, we can show **Mean Value Theorem** by letting $g(x) = x$, and $h(x) = 1$. And from this viewpoint, we know that **Rolle's Theorem, Mean Value Theorem, and Generalized Mean Value Theorem** are equivalent.

5.9 Given n functions $f_1, \ldots, f_n$, each having nth order derivatives in (a,b). A function W, called the **Wronskian** of $f_1, \ldots, f_n$, is defined as follows: For each x in (a,b), $W(x)$ is the value of the determinant of order n whose element in the kth row and mth column is $f_m^{(k-1)}(x)$, where $k = 1,2,..,n$ and $m = 1,2,\ldots,n$. [The expression $f_m^{(0)}(x)$ is written for $f_m(x)$.]

(a) Show that $W'(x)$ can be obtained by replacing the last row of the determinant defining $W(x)$ by the nth derivatives $f_1^{(n)}(x), \ldots, f_n^{(n)}(x)$.

Proof: Write

$$W(x) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \cdots & \cdots & \cdots & \cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

and note that if any two rows are the same, its determinant is 0; hence, by Exercise 5.8-(b), we know that

$$W'(x) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ \dots & \dots & \dots & \dots \\ f_1^{(n-2)} & f_2^{(n-2)} & \dots & f_n^{(n-2)} \\ f_1^{(n)} & f_2^{(n)} & \dots & f_n^{(n)} \end{vmatrix}.$$

(b) Assuming the existence of n constants $c_1,\dots,c_n$, not all zero, such that $c_1f_1(x)+\dots c_nf_n(x) = 0$ for every x in (a,b), show that $W(x) = 0$ for each x in (a,b).

Proof: Since $c_1f_1(x)+\dots c_nf_n(x) = 0$ for every x in (a,b), where $c_1,\dots,c_n$, not all zero. Without loss of generality, we may assume $c_1 \neq 0$, we know that $c_1f_1^{(k)}(x)+\dots c_nf_n^{(k)}(x) = 0$ for every x in (a,b), where $0 \leq k \leq n$. Hence, we have

$$W(x) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$
$$= 0$$

since the first column is a linear combination of other columns.

Note. A set of functions satisfing such a relation is said to be a l**inearly dependent** set on (a,b).

(c) The vanishing of the Wronskian throughout (a,b) is necessary, but not sufficient. for linear dependence of $f_1,\dots,f_n$. Show that in the case of two functions, if the Wronskian vanishes throughout (a,b) and if one of the functions does not vanish in (a,b), then they form a linearly dependent set in (a,b).

Proof: Let f and g be continuous and differentiable on (a,b). Suppose that $f(x) \neq 0$ for all $x \in (a,b)$. Since the Wronskian of f and g is 0, for all $x \in (a,b)$, we have

$$fg' - f'g = 0 \text{ for all } x \in (a,b). \quad *$$

Since $f(x) \neq 0$ for all $x \in (a,b)$, we have by (*),

$$\frac{fg' - f'g}{f^2} = 0 \Rightarrow \left(\frac{g}{f}\right)' = 0 \text{ for all } x \in (a,b).$$

Hence, there is a constant c such that $g = cf$ for all $x \in (a,b)$. Hence, $\{f,g\}$ forms a linearly dependent set.

Remark: This exercise in (b) is a impotant theorem on **O.D.E**. We often write (b) in other form as follows.

(**Theorem**) Let $f_1,\dots,f_n$ be continuous and differentiable on an interval I. If $W(f_1,\dots,f_n)(t_0) \neq 0$ for some $t_0 \in I$, then $\{f_1,\dots,f_n\}$ is **linearly independent** on I

Note: If $\{f_1,\dots,f_n\}$ is **linearly independent** on I, It is **NOT** necessary that $W(f_1,\dots,f_n)(t_0) \neq 0$ for some $t_0 \in I$. For example, $f(t) = t^2|t|$, and $g(t) = t^3$. It is easy to check $\{f,g\}$ is linearly independent on $(-1,1)$. And $W(f,g)(t) = 0$ for all $t \in (-1,1)$.

Supplement on Chain Rule and Inverse Function Theorem.

The following theorem is called chain rule, it is well-known that let f be defined on an open interval S, let g be defined on $f(S)$, and consider the composite function $g \circ f$ defined on S by the equation

$$g \circ f(x) = g(f(x)).$$

Assume that there is a point c in S such that $f(c)$ is an interior point of $f(S)$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c, and we have

$$g \circ f'(c) = g'(f(c))f'(c).$$

We do not give a proof, in fact, the proof can be found in this text book. We will give another Theorem called The Converse of Chain Rule as follows.

(**The Converse of Chain Rule**) Suppose that f, g and u are related so that $f(x) = g(u(x))$. If $u(x)$ is continuous at x_0, $f'(x_0)$ exists, $g'(u(x_0))$ exists and not zero. Then $u'(x_0)$ is defined and we have

$$f'(x_0) = g'(u(x_0))u'(x_0).$$

Proof: Since $f'(x_0)$ exists, and $g'(u(x_0))$ exists, then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + o(|x - x_0|) \qquad *$$

and

$$g(u(x)) = g(u(x_0)) + g'(u(x_0))(u(x) - u(x_0)) + o(|u(x) - u(x_0)|). \qquad **$$

Since $f(x) = g(u(x))$, and $f(x_0) = g(u(x_0))$, by (*) and (**), we know that

$$u(x) = u(x_0) + \frac{f'(x_0)}{g'(u(x_0))}(x - x_0) + o(|x - x_0|) + o(|u(x) - u(x_0)|). \qquad ***$$

Note that since $u(x)$ is continuous at x_0, we know that $o(|u(x) - u(x_0)|) \to 0$ as $x \to x_0$. So, (***) means that $u'(x_0)$ is defined and we have

$$f'(x_0) = g'(u(x_0))u'(x_0).$$

Remark: The condition that $g'(u(x_0))$ is not zero is essential, for example, $g(x) = 1$ on $(-1,1)$ and $u(x) = |x|$, where $x_0 = 0$.

(**Inverse Function Theroem**) Suppose that f is continuous, strictly monotonic function which has an open interval I for domain and has range J. (It implies that $f(g(x)) = x = g(f(x))$ on its corresponding domain.) Assume that x_0 is a point of J such that $f'(g(x_0))$ is defined and is different from zero. Then $g'(x_0)$ exists, and we have

$$g'(x_0) = \frac{1}{f'(g(x_0))}.$$

Proof: It is a result of **the converse of chain rule** note that

$$f(g(x)) = x.$$

Mean Value Theorem

5.10 Given a function defined and having a finite derivative in (a,b) and such that $\lim_{x \to b^-} f(x) = +\infty$. Prove that $\lim_{x \to b^-} f'(x)$ either fails to exist or is infinite.

Proof: Suppose **NOT**, we have the existence of $\lim_{x \to b^-} f'(x)$, denoted the limit by L. So, given $\varepsilon = 1$, there is a $\delta > 0$ such that as $x \in (b - \delta, b)$ we have

$$|f'(x)| < |L| + 1. \qquad *$$

Consider $x, a \in (b - \delta, b)$ with $x > a$, then we have by (*) and **Mean Value Theorem,**

$$\begin{aligned} |f(x) - f(a)| &= |f'(\xi)(x - a)| \text{ where } \xi \in (a,x) \\ &\leq (|L| + 1)|x - a| \end{aligned}$$

which implies that

$$|f(x)| \le |f(a)| + (|L| + 1)\delta$$

which contradicts to $\lim_{x \to b^-} f(x) = +\infty$.

Hence, $\lim_{x \to b^-} f'(x)$ either fails to exist or is infinite.

5.11 Show that the formula in the **Mean Value Theorem** can be written as follows:

$$\frac{f(x+h) - f(x)}{h} = f'(x + \theta h),$$

where $0 < \theta < 1$.

Proof: (**Mean Value Theorem**) Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $\xi \in (a,b)$ such that $f(b) - f(a) = f'(\xi)(b-a)$. Note that $\xi = a + \theta(b-a)$, where $0 < \theta < 1$. So, we have proved the exercise.

Determine θ as a function of x and h, and keep $x \neq 0$ fixed, and find $\lim_{h \to 0} \theta$ in each case.

(a) $f(x) = x^2$.

Proof: Consider

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = 2x + h = 2(x + \theta h) = f'(x + \theta h)$$

which implies that

$$\theta = 1/2.$$

Hence, we know that $\lim_{h \to 0} \theta = 1/2$.

(b) $f(x) = x^3$.

Proof: Consider

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h} = 3x^2 + 3xh + h^2 = 3(x + \theta h)^2 = f'(x + \theta h)$$

which implies that

$$\theta = \frac{-3x \pm \sqrt{9x^2 + 9xh + 3h^2}}{3h}$$

Since $0 < \theta < 1$, we consider two cases. (i) $x > 0$, (ii) $x < 0$.

(i) As $x > 0$, since

$$0 < \theta = \frac{-3x \pm \sqrt{9x^2 + 9xh + 3h^2}}{3h} < 1,$$

we have

$$\theta = \begin{cases} \frac{-3x + \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h > 0, \text{ and } h \text{ is sufficiently close to } 0, \\ \frac{-3x + \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h < 0, \text{ and } h \text{ is sufficiently close to } 0. \end{cases}$$

Hence, we know that $\lim_{h \to 0} \theta = 1/2$ by **L-Hospital Rule**.

(ii) As $x < 0$, we have

$$\theta = \begin{cases} \frac{-3x - \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h > 0, \text{ and } h \text{ is sufficiently close to } 0, \\ \frac{-3x - \sqrt{9x^2 + 9xh + 3h^2}}{3h} & \text{if } h < 0, \text{ and } h \text{ is sufficiently close to } 0. \end{cases}$$

Hence, we know that $\lim_{h \to 0} \theta = 1/2$ by **L-Hospital Rule.**

From (i) and (ii), we know that as $x \neq 0$, we have $\lim_{h \to 0} \theta = 1/2$.

Remark: For $x = 0$, we can show that $\lim_{h\to 0}\theta = \frac{\sqrt{3}}{3}$ as follows.

Proof: Since

$$0 < \theta = \frac{\pm\sqrt{3h^2}}{3h} < 1,$$

we have

$$\theta = \begin{cases} \frac{\sqrt{3h^2}}{3h} = \frac{\sqrt{3}\,h}{3h} = \frac{\sqrt{3}}{3} \text{ if } h > 0, \\ \frac{-\sqrt{3h^2}}{3h} = \frac{\sqrt{3}\,h}{3h} = \frac{\sqrt{3}}{3} \text{ if } h < 0. \end{cases}$$

Hence, we know that $\lim_{h\to 0}\theta = \frac{\sqrt{3}}{3}$.

(c) $f(x) = e^x$.

Proof: Consider

$$\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^x}{h} = e^{x+\theta h} = f'(x+\theta h)$$

which implies that

$$\theta = \frac{\log\frac{e^h-1}{h}}{h}.$$

Hence, we know that $\lim_{h\to 0}\theta = 1/2$ since

$$\begin{aligned}\lim_{h\to 0}\theta &= \lim_{h\to 0}\frac{\log\frac{e^h-1}{h}}{h} \\ &= \lim_{h\to 0}\frac{e^h h - e^h + 1}{h(e^h-1)} \text{ by L-Hospital Rule.} \\ &\quad \text{Note that } e^h = 1 + h + \frac{h^2}{2} + o(h^2) \\ &= \lim_{h\to 0}\frac{\frac{1}{2} + h + o(1)}{1 + \frac{h}{2} + o(h)} \\ &= 1/2.\end{aligned}$$

(d) $f(x) = \log x$, $x > 0$.

Proof: Consider

$$\frac{f(x+h)-f(x)}{h} = \frac{\log(1+\frac{h}{x})}{h} = \frac{1}{x+\theta h}$$

which implies that

$$\theta = \frac{\frac{h}{x} - \log(1+\frac{h}{x})}{\frac{h}{x}(\log(1+\frac{h}{x}))}.$$

Since $\log(1+t) = t - \frac{t^2}{2} + o(t^2)$, we have

$$\lim_{h\to 0}\theta = \lim_{h\to 0}\frac{\frac{h}{x} - \left(\frac{h}{x} - \frac{1}{2}\left(\frac{h}{x}\right)^2 + o\left(\left(\frac{h}{x}\right)^2\right)\right)}{\frac{h}{x}\left(\frac{h}{x} - \frac{1}{2}\left(\frac{h}{x}\right)^2 + o\left(\left(\frac{h}{x}\right)^2\right)\right)}$$
$$= \lim_{h\to 0}\frac{\frac{1}{2}\left(\frac{h}{x}\right)^2 + o\left(\left(\frac{h}{x}\right)^2\right)}{\left(\frac{h}{x}\right)^2 + \frac{1}{2}\left(\frac{h}{x}\right)^3 + o\left(\left(\frac{h}{x}\right)^3\right)}$$
$$= \lim_{h\to 0}\frac{\frac{1}{2} + o(1)}{1 + \frac{1}{2}\left(\frac{h}{x}\right) + o\left(\frac{h}{x}\right)}$$
$$= 1/2.$$

5.12 Take $f(x) = 3x^4 - 2x^3 - x^2 + 1$ and $g(x) = 4x^3 - 3x^2 - 2x$ in Theorem 5.20. Show that $f'(x)/g'(x)$ is never equal to the quotient $[f(1) - f(0)]/[g(1) - g(0)]$ if $0 < x \leq 1$. How do you reconcile this with the equation

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_1)}{g'(x_1)},\ a < x_1 < b,$$

obtainable from Theorem 5.20 when $n = 1$?

Solution: Note that

$$12x^2 - 6x - 2 = 12\left[x - \left(\frac{1}{4} + \sqrt{\frac{11}{48}}\right)\right]\left[x - \left(\frac{1}{4} - \sqrt{\frac{11}{48}}\right)\right],\text{ where } \left(0 < \frac{1}{4} + \sqrt{\frac{11}{48}} < 1\right).$$

So, when we consider

$$\frac{f(1) - f(0)}{g(1) - g(0)} = 0$$

and

$$f'(x) = 12x^3 - 6x^2 - 2x = xg'(x) = x(12x^2 - 6x - 2),$$

we **CANNOT** write $f'(x)/g'(x) = x$. Otherwise, it leads us to get a contradiction.

Remark: It should be careful when we use **Generalized Mean Value Theorem**, we had better not write the above form unless we know that the denominator is not zero.

5.13 In each of the following special cases of Theorem 5.20, take $n = 1$, $c = a$, $x = b$, and show that $x_1 = (a + b)/2$.

(a) $f(x) = \sin x$, $g(x) = \cos x$;

Proof: Since, by **Theorem 5.20**,

$$\begin{aligned}(\sin a - \sin b)[-\sin(x_1)] &= \left[2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right][-\sin(x_1)] \\ &= (\cos a - \cos b)(\cos x_1) \\ &= \left[-2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)\right](\cos x_1),\end{aligned}$$

we find that if we choose $x_1 = (a + b)/2$, then both are equal.

(b) $f(x) = e^x$, $g(x) = e^{-x}$.

Proof: Since, by **Theorem 5.20**,

$$(e^a - e^b)(-e^{-x_1}) = (e^{-a} - e^{-b})(e^{x_1}),$$

we find that if we choose $x_1 = (a + b)/2$, then both are equal.

Can you find a general class of such pairs of functions f and g for which x_1 will always be $(a + b)/2$ and such that both examples (a) and (b) are in this class?

Proof: Look at the **Generalized Mean Value Theorem**, we try to get something from the equality.

$$[f(a) - f(b)]g'\left(\frac{a+b}{2}\right) = [g(a) - g(b)]f'\left(\frac{a+b}{2}\right), \qquad *$$

if $f(x)$, and $g(x)$ satisfy following two conditions,

$$\text{(i) } f'(x) = g(-x) \text{ and } g'(x) = -f(-x)$$

and

$$\text{(ii) } [f(a) - f(b)]\left[-f\left(-\frac{a+b}{2}\right)\right] = [g(a) - g(b)]\left[g\left(-\frac{a+b}{2}\right)\right],$$

then we have the equality (*).

5.14 Given a function f defined and having a finite derivative f' in the half-open interval $0 < x \leq 1$ and such that $|f'(x)| < 1$. Define $a_n = f(1/n)$ for $n = 1, 2, 3, \ldots,$ and show that $\lim_{n\to\infty} a_n$ exists.

Hint. Cauchy condition.

Proof: Consider $n \geq m$, and by **Mean Value Theorem**,

$$|a_n - a_m| = |f(1/n) - f(1/m)| = |f'(p)|\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n} - \frac{1}{m}\right|$$

then $\{a_n\}$ is a Cauchy sequence since $\{1/n\}$ is a Cauchy sequence. Hence, we know that $\lim_{n\to\infty} a_n$ exists.

5.15 Assume that f has a finite derivative at each point of the open interval (a, b). Assume also that $\lim_{x\to c} f'(x)$ exists and is finite for some interior point c. Prove that the value of this limit must be $f'(c)$.

Proof: It can be proved by Exercise 5.16; we omit it.

5.16 Let f be continuous on (a, b) with a finite derivative f' everywhere in (a, b), expect possibly at c. If $\lim_{x\to c} f'(x)$ exists and has the value A, show that $f'(c)$ must also exist and has the value A.

Proof: Consider, for $x \neq c$,

$$\frac{f(x) - f(c)}{x - c} = f'(\xi) \text{ where } \xi \in (x, c) \text{ or } (c, x) \text{ by } \textbf{Mean Value Theorem,} \qquad *$$

since $\lim_{x\to c} f'(x)$ exists, given $\varepsilon > 0$, there is a $\delta > 0$ such that as $x \in (c - \delta, c + \delta) - \{c\}$, we have

$$A - \varepsilon < f'(x) < A + \varepsilon.$$

So, if we choose $x \in (c - \delta, c + \delta) - \{c\}$ in (*), we then have

$$A - \varepsilon < \frac{f(x) - f(c)}{x - c} = f'(\xi) < A + \varepsilon.$$

That is, $f'(c)$ exists and equals A.

Remark: (1) Here is another proof by **L-Hospital Rule.** Since it is so obvious that we omit the proof.

(2) We should be noted that Exercise 5.16 implies Exercise 5.15. Both methods mentioned in Exercise 5.16 are suitable for Exercise 5.15.

5.17 Let f be continuous on $[0, 1]$, $f(0) = 0$, $f'(x)$ defined for each x in $(0, 1)$. Prove that if f' is an increasing function on $(0, 1)$, then so is too is the function g defined by the equation $g(x) = f(x)/x$.

Proof: Since f' is an increasing function on $(0, 1)$, we know that, for any $x \in (0, 1)$

$$f'(x) - \frac{f(x)}{x} = f'(x) - \frac{f(x) - f(0)}{x - 0} = f'(x) - f'(\xi) \geq 0 \text{ where } \xi \in (0,x). \quad *$$

So, let $x > y$, we have

$$\begin{aligned} g(x) - g(y) &= g'(z)(x-y), \text{ where } y < z < x \\ &= \frac{f'(z)z - f(z)}{z^2}(x-y) \\ &\geq 0 \text{ by } (*) \end{aligned}$$

which implies that g is an increasing function on $(0,1)$.

5.18 Assume f has a finite derivative in (a,b) and is continuous on $[a,b]$ with $f(a) = f(b) = 0$. Prove that for every real λ there is some c in (a,b) such that $f'(c) = \lambda f(c)$.

Hint. Apply Rolle's Theorem to $g(x)f(x)$ for a suitable g depending on λ.

Proof: Consider $g(x) = f(x)e^{-\lambda x}$, then by **Rolle's Theorem**,

$$\begin{aligned} g(a) - g(b) &= g'(c)(a-b), \text{ where } c \in (a,b) \\ &= 0 \end{aligned}$$

which implies that

$$f'(c) = \lambda f(c).$$

Remark: (1) The finding of an auxiliary function usually comes from the equation that we consider. We will give some questions around this to get more.

(2)There are some questions about finding auxiliary functions; we write it as follows.

(i) Show that $e^\pi > \pi^e$.

Proof: (STUDY) Since $\log x$ is a strictly increasing on $(0,\infty)$, in order to show $e^\pi > \pi^e$, it suffices to show that

$$\pi \log e = \log e^\pi > \log \pi^e = e \log \pi$$

which implies that

$$\frac{\log e}{e} > \frac{\log \pi}{\pi}.$$

Consider $f(x) = \frac{\log x}{x} : [e,\infty)$, we have

$$f'(x) = \frac{1 - \log x}{x^2} < 0 \text{ where } x \in (e,\infty).$$

So, we know that $f(x)$ is strictly decreasing on $[e,\infty)$. Hence, $\frac{\log e}{e} > \frac{\log \pi}{\pi}$. That is, $e^\pi > \pi^e$.

(ii) Show that $e^x \geq 1 + x$ for all $x \in R$.

Proof: By **Taylor Theorem with Remainder Term**, we know that

$$e^x = 1 + x + \frac{e^c}{2}x^2, \text{ for some } c.$$

So, we finally have $e^x \geq 1 + x$ for all $x \in R$.

Note: (a) The method in (ii) tells us one thing, we can give a theorem as follows. Let $f \in C^{2n-1}([a,b])$, and $f^{(2n)}(x)$ exists and $f^{(2n)}(x) \geq 0$ on (a,b). Then we have

$$f(x) \geq \sum_{k=0}^{2n-1} \frac{f^{(k)}(a)}{k!}.$$

Proof: By **Generalized Mean Value Theorem**, we complete it.

(b) There are many proofs about that $e^x \geq 1 + x$ for all $x \in R$. We list them as a reference.

(b-1) Let $f(x) = e^x - 1 - x$, and thus consider the extremum.
(b-2) Use **Mean Value Theorem.**
(b-3) Since $e^x - 1 \geq 0$ for $x \geq 0$ and $e^x - 1 \leq 0$ for $x \leq 0$, we then have

$$\int_0^x (e^t - 1)dt \geq 0 \text{ and } \int_x^0 (e^t - 1)dt \leq 0.$$

So, $e^x \geq 1 + x$ for all $x \in R$.

(iii) Let f be continuous function on $[a,b]$, and differentiabel on (a,b). Prove that there exists a $c \in (a,b)$ such that

$$f'(c) = \frac{f(c) - f(a)}{b - c}.$$

Proof: (STUDY) Since $f'(c) = \frac{f(c)-f(a)}{b-c}$, we consider $f'(c)(b-c) - (f(c) - f(a))$. Hence, we choose $g(x) = (f(x) - f(a))(b - x)$, then by **Rolle's Theorem,**

$$g(a) - g(b) = g'(c)(a - b) \text{ where } c \in (a,b)$$

which implies that $f'(c) = \frac{f(c)-f(a)}{b-c}$.

(iv) Let f be a polynomial of degree n, if $f \geq 0$ on R, then we have

$$f + f' + .. + f^{(n)} \geq 0 \text{ on } R.$$

Proof: Let $g(x) = f + f' + .. + f^{(n)}$, then we have

$$g - g' = f \geq 0 \text{ on } R \text{ since } f \text{ is a polynomial of degree } n. \quad *$$

Consider $h(x) = g(x)e^{-x}$, then $h'(x) = (g'(x) - g(x))e^{-x} \leq 0$ on R by (*). It means that h is a decreasing function on R. Since $\lim_{x\to+\infty} h(x) = 0$ by the fact g is still a polynomial, then $h(x) \geq 0$ on R. That is, $g(x) \geq 0$ on R.

(v) Suppose that f is continuous on $[a,b]$, $f(a) = 0 = f(b)$, and $x^2 f''(x) + 4xf'(x) + 2f(x) \geq 0$ for all $x \in (a,b)$. Prove that $f(x) \leq 0$ on $[a,b]$.

Proof: (STUDY) Since $x^2 f''(x) + 4xf'(x) + 2f(x) = [x^2 f(x)]''$ by **Leibnitz Rule,** let $g(x) = x^2 f(x)$, then claim that $g(x) \leq 0$ on $[a,b]$.

Suppose **NOT**, there is a point $p \in (a,b)$ such that $g(p) > 0$. Note that since $f(a) = 0$, and $f(b) = 0$, So, $g(x)$ has an absolute maximum at $c \in (a,b)$. Hence, we have $g'(c) = 0$. By **Taylor Theorem with Remainder term**, we have

$$\begin{aligned} g(x) &= g(c) + g'(c)(x - c) + \frac{g''(\xi)}{2!}(x - c)^2, \text{ where } \xi \in (x,c) \text{ or } (c,x) \\ &\geq g(c) \text{ since } g'(c) = 0, \text{ and } g''(x) \geq 0 \text{ for all } x \in (a,b) \\ &> 0 \text{ since } g(c) \text{ is absolute maximum.} \end{aligned}$$

So,

$$x^2 f(x) \geq c^2 f(c) > 0 \quad **$$

which is absurb since let $x = a$ in (**).

(vi) Suppose that f is continuous and differentiable on $[0,\infty)$, and $\lim_{x\to\infty} f'(x) + f(x) = 0$, show that $\lim_{x\to\infty} f(x) = 0$.

Proof: Since $\lim_{x\to\infty} f'(x) + f(x) = 0$, then given $\varepsilon > 0$, there is $M > 0$ such that as $x \geq M$, we have

$$-\varepsilon < f'(x) + f(x) < \varepsilon.$$

So, as $x \geq M,$ we have

$$\begin{aligned}[-\varepsilon e^x + \varepsilon e^M + \varepsilon e^M f(M)]' &= -\varepsilon e^x \\ &< [e^x f(x)]' \\ &< \varepsilon e^x = [\varepsilon e^x - \varepsilon e^M + \varepsilon e^M f(M)]'.\end{aligned}$$

If we let $-\varepsilon e^x + \varepsilon e^M + \varepsilon e^M f(M) = g(x),$ and $\varepsilon e^x - \varepsilon e^M + \varepsilon e^M f(M) = h(x),$ then we have

$$g'(x) \leq [e^x f(x)]' \leq h'(x)$$

and

$$g(M) = e^M f(M) = h(M).$$

Hence, for $x \geq M,$

$$\begin{aligned}-\varepsilon e^x + \varepsilon e^M + \varepsilon e^M f(M) &= g(x) \\ &\leq e^x f(x) \\ &\leq h(x) = \varepsilon e^x - \varepsilon e^M + \varepsilon e^M f(M)\end{aligned}$$

It implies that, for $x \geq M,$

$$-\varepsilon + e^{-x}[\varepsilon e^M + \varepsilon e^M f(M)] \leq f(x) \leq \varepsilon - e^{-x}[\varepsilon e^M - \varepsilon e^M f(M)]$$

which implies that

$$\lim_{x \to +\infty} f(x) = 0 \text{ since } \varepsilon \text{ is arbitrary.}$$

Note: In the process of proof, we use the result on **Mean Value Theorem**. Let $f, g,$ and h be continuous on $[a,b]$ and differentiable on $(a,b).$ Suppose $f(a) = g(a) = h(a)$ and $f'(x) \leq g'(x) \leq h'(x)$ on $(a,b).$ Show that $f(x) \leq g(x) \leq h(x)$ on $[a,b].$

Proof: By **Mean Value theorem**, we have

$$\begin{aligned}[g(x) - f(x)] - [g(a) - f(a)] &= g(x) - f(x) \\ &= g'(c) - f'(c), \text{ where } c \in (a,x). \\ &\leq 0 \text{ by hypothesis.}\end{aligned}$$

So, $f(x) \leq g(x)$ on $[a,b].$ Similarly for $g(x) \leq h(x)$ on $[a,b].$ Hence, $f(x) \leq g(x) \leq h(x)$ on $[a,b].$

(vii) Let $f(x) = a_1 \sin x + \ldots + a_n \sin nx,$ where a_i are real for $i = 1,2,\ldots n.$ Suppose that $|f(x)| \leq |x|$ for all real $x.$ Prove that $|a_1 + \ldots + na_n| \leq 1.$

Proof: Let $x > 0,$ and by **Mean Value Theorem,** we have

$$\begin{aligned}|f(x) - f(0)| = |f(x)| &= |a_1 \sin x + \ldots + a_n \sin nx| \\ &= |f'(c)x|, \text{ where } c \in (0,x) \\ &= |(a_1 \cos c + \ldots + na_n \cos nc)x| \\ &\leq |x| \text{ by hypothesis.}\end{aligned}$$

So,

$$|a_1 \cos c + \ldots + na_n \cos nc| \leq 1$$

Note that as $x \to 0^+,$ we have $c \to 0^+;$ hence, $|a_1 + \ldots + na_n| \leq 1.$

Note: Here are another type:
(a) $|\sin^2 x - \sin^2 y| \leq |x - y|$ for all $x, y.$
(b) $|\tan x - \tan y| \geq |x - y|$ for all $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2}).$

(viii) Let $f : R \to R$ be differentiable with $f'(x) \geq c$ for all $x,$ where $c > 0.$ Show that there is a point p such that $f(p) = 0$.

Proof: By **Mean Vaule Theorem,** we have

$$f(x) = f(0) + f'(x_1)x \geq f(0) + cx \text{ if } x \geq 0$$
$$= f(0) + f'(x_2)x \leq f(0) + cx \text{ if } x \leq 0.$$

So, as x large enough, we have $f(x) > 0$ and as x is smalle enough, we have $f(x) < 0$. Since f is differentiable on R, it is continuous on R. Hence, by **Intermediate Value Theorem**, we know that there is a point p such that $f(p) = 0$.

(3) Here is another type about integral, but it is worth learning. Compare with (2)-(vii). If

$$c_0 + \frac{c_1}{2} + \ldots + \frac{c_n}{n+1} = 0, \text{ where } c_i \text{ are real constants for } i = 1, 2, ..n.$$

Prove that $c_0 + \ldots + c_n x^n$ has at least one real root between 0 and 1.

Proof: Suppose **NOT,** i.e., (i) $f(x) := c_0 + \ldots + c_n x^n > 0$ for all $x \in [0,1]$ or (ii) $f(x) < 0$ for all $x \in [0,1]$.

In case (i), consider

$$0 < \int_0^1 f(x)dx = c_0 + \frac{c_1}{2} + \ldots + \frac{c_n}{n+1} = 0$$

which is absurb. Similarly for case (ii).

So, we know that $c_0 + \ldots + c_n x^n$ has at least one real root between 0 and 1.

5.19. Assume f is continuous on $[a,b]$ and has a finite second derivative f'' in the open interval (a,b). Assume that the line segment joining the points $A = (a, f(a))$ and $B = (b, f(b))$ intersects the graph of f in a third point P different from A and B. Prove that $f''(c) = 0$ for some c in (a,b).

Proof: Consider a straight line equation, called $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Then $h(x) := f(x) - g(x)$, we knwo that there are three point $x = a$, p and b such that

$$h(a) = h(p) = h(b) = 0.$$

So, by Mean Value Theorem twice, we know that there is a point $c \in (a,b)$ such that

$$h''(c) = 0$$

which implies that $f''(c) = 0$ since g is a polynomial of degree at least 1.

5.20 If f has a finite third derivative f''' in $[a,b]$ and if

$$f(a) = f'(a) = f(b) = f'(b) = 0,$$

prove that $f'''(c) = 0$ for some c in (a,b).

Proof: Since $f(a) = f(b) = 0$, we have $f'(p) = 0$ where $p \in (a,b)$ by **Rolle's Theorem.** Since $f'(a) = f'(p) = 0$, we have $f''(q_1) = 0$ where $q_1 \in (a,p)$ and since $f'(p) = f'(b) = 0$, we have $f''(q_2) = 0$ where $q_2 \in (p,b)$ by **Rolle's Theorem.** Since $f''(q_1) = f''(q_2) = 0$, we have $f'''(c) = 0$ where $c \in (q_1, q_2)$ by **Rolle's Theorem.**

5.21 Assume f is nonnegative and has a finite third derivative f''' in the open interval $(0,1)$. If $f(x) = 0$ for at least two values of x in $(0,1)$, prove that $f'''(c) = 0$ for some c in $(0,1)$.

Proof: Since $f(x) = 0$ for at least two values of x in $(0,1)$, say $f(a) = f(b) = 0$, where $a, b \in (0,1)$. By Rolle's Theorem, we have $f'(p) = 0$ where $p \in (a,b)$. Note that f is nonnegative and differentiable on $(0,1)$, so both $f(a)$ and $f(b)$ are local minima, where a and b are interior to (a,b). Hence, $f'(a) = f'(b) = 0$.

Since $f'(a) = f'(p) = 0,$ we have $f''(q_1) = 0$ where $q_1 \in (a,p)$ and since $f'(p) = f'(b) = 0,$ we have $f''(q_2) = 0$ where $q_2 \in (p,b)$ by **Rolle's Theorem.** Since $f''(q_1) = f''(q_2) = 0,$ we have $f'''(c) = 0$ where $c \in (q_1,q_2)$ by **Rolle's Theorem.**

5.22 Assume f has a finite derivative in some interval $(a,+\infty)$.

(a) If $f(x) \to 1$ and $f'(x) \to c$ as $x \to +\infty,$ prove that $c = 0.$

Proof: Consider $f(x+1) - f(x) = f'(y)$ where $y \in (x,x+1)$ by **Mean Value Theorem,** since

$$\lim_{x\to+\infty} f(x) = 1$$

which implies that

$$\lim_{x\to+\infty}[f(x+1) - f(x)] = 0$$

which implies that $(x \to +\infty \Leftrightarrow y \to +\infty)$

$$\lim_{x\to+\infty} f'(y) = 0 = \lim_{y\to+\infty} f(y)$$

Since $f'(x) \to c$ as $x \to +\infty,$ we know that $c = 0.$

Remark: (i) There is a similar exercise; we write it as follows. If $f(x) \to L$ and $f'(x) \to c$ as $x \to +\infty,$ prove that $c = 0.$

Proof: By the same method metioned in (a), we complete it.

(ii) The exercise tells that the function is smooth; its first derivative is smooth too.

(b) If $f'(x) \to 1$ as $x \to +\infty,$ prove that $f(x)/x \to 1$ as $x \to +\infty.$

Proof: Given $\varepsilon > 0,$ we want to find $M > 0$ such that as $x \geq M$

$$\left|\frac{f(x)}{x} - 1\right| < \varepsilon.$$

Since $f'(x) \to 1$ as $x \to +\infty,$ then given $\varepsilon' = \frac{\varepsilon}{3},$ there is $M' > 0$ such that as $x \geq M',$ we have

$$|f'(x) - 1| < \frac{\varepsilon}{3} \Rightarrow |f'(x)| < 1 + \frac{\varepsilon}{3} \qquad *$$

By **Taylor Theorem with Remainder Term**,

$$\begin{aligned} f(x) &= f(M') + f'(\xi)(x - M') \\ &\Rightarrow f(x) - x = f(M') + (f'(\xi) - 1)x - f'(\xi)M', \end{aligned}$$

then for $x \geq M'$,

$$\begin{aligned} \left|\frac{f(x)}{x} - 1\right| &\leq \left|\frac{f(M')}{x}\right| + |f'(\xi) - 1| + \left|\frac{f'(\xi)M'}{x}\right| \qquad ** \\ &\leq \left|\frac{f(M')}{x}\right| + \frac{\varepsilon}{3} + \left(1 + \frac{\varepsilon}{3}\right)\left|\frac{M'}{x}\right| \text{ by } (*) \end{aligned}$$

Choose $M > 0$ such that as $x \geq M \geq M',$ we have

$$\left|\frac{f(M)}{x}\right| < \frac{\varepsilon}{3} \text{ and } \left|\frac{M'}{x}\right| < \frac{\varepsilon/3}{(1 + \frac{\varepsilon}{3})}. \qquad ***$$

Combine (**) with (***), we have proved that given $\varepsilon > 0,$ there is a $M > 0$ such that as $x \geq M,$ we have

$$\left|\frac{f(x)}{x} - 1\right| < \varepsilon.$$

That is, $\lim_{x\to+\infty} \frac{f(x)}{x} = 1.$

Remark: If we can make sure that $f(x) \to \infty$ as $x \to +\infty$, we can use **L-Hopital Rule**. We give another proof as follows. It suffices to show that $f(x) \to \infty$ as $x \to +\infty$.

Proof: Since $f'(x) \to 1$ as $x \to +\infty$, then given $\varepsilon = 1$, there is $M > 0$ such that as $x \geq M$, we have

$$|f'(x)| < 1 + 1 = 2.$$

Consider

$$f(x) = f(M) + f'(\xi)(x - M)$$

by **Taylor Theorem with Remainder Term**, then

$$\lim_{x \to +\infty} f(x) = +\infty \text{ since } f'(x) \text{ is bounded for } x \geq M.$$

(c) If $f'(x) \to 0$ as $x \to +\infty$, prove that $f(x)/x \to 0$ as $x \to +\infty$.

Proof: The method metioned in (b). We omit the proof.

Remark: (i) There is a similar exercise; we write it as follows. If $f'(x) \to L$ as $x \to +\infty$, prove that $f(x)/x \to L$ as $x \to +\infty$. The proof is mentioned in (b), so we omit it.

(ii) It should be careful that we **CANNOT** use **L-Hospital Rule** since we may not have the fact $f(x) \to \infty$ as $x \to +\infty$. Hence, **L-Hospital Rule** cannot be used here. For example, f is a constant function.

5.23 Let h be a fixed positive number. Show that there is no function f satisfying the following three conditions: $f'(x)$ exists for $x \geq 0$, $f'(0) = 0$, $f'(x) \geq h$ for $x > 0$.

Proof: It is called **Intermediate Value Theorem for Derivatives. (**Sometimes, we also call this theorem **Darboux.)** See the text book in **Theorem 5.16.**

(**Supplement**) 1. Suppose that $a \in R$, and f is a twice-differentiable real function on (a, ∞). Let M_0, M_1, and M_2 are the least upper bound of $|f(x)|$, $|f'(x)|$, and $|f''(x)|$, respectively, on (a, ∞). Prove that $M_1^2 \leq 4M_0M_2$.

Proof: Consider **Taylor's Theorem with Remainder Term**,

$$f(a + 2h) = f(a) + f'(a)(2h) + \frac{f''(\xi)}{2!}(2h)^2, \text{ where } h > 0.$$

then we have

$$f'(a) = \frac{1}{2h}[f(a + 2h) - f(a)] - f''(\xi)h$$

which implies that

$$|f'(a)| \leq \frac{M_0}{h} + hM_2 \Rightarrow M_1 \leq \frac{M_0}{h} + hM_2. \quad *$$

Since $g(h) := \frac{M_0}{h} + hM_2$ has an absolute maximum at $\sqrt{\frac{M_0}{M_2}}$, hence by (*), we know that

$$M_1^2 \leq 4M_0M_2.$$

Remark:

2. Suppose that f is a twice-differentiable real function on $(0, \infty)$, and f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.

Proof: Since $M_1^2 \leq 4M_0M_2$ in **Supplement 1**, we have prove it.

3. Suppose that f is real, three times differentiable on $[-1, 1]$, such that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, and $f'(0) = 0$. Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Proof: Consider **Taylor's Theorem with Remainder Term**,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(c)}{3!}x^3, \text{ where } c \in (x,0) \text{ or } (0,x),$$

Then let $x = \pm 1$, and subtract one from another, we get

$$f^{(3)}(c_1) + f^{(3)}(c_2) \geq 6, \text{ where } c_1 \text{ and } c_2 \text{ in } (-1,1).$$

So, we have prove $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$.

5.24 If $h > 0$ and if $f'(x)$ exists (and is finite) for every x in $(a-h, a+h)$, and if f is continuous on $[a-h, a+h]$, show that we have:

(a)

$$\frac{f(a+h) - f(a-h)}{h} = f'(a+\theta h) + f'(a-\theta h), 0 < \theta < 1;$$

Proof: Let $g(h) = f(a+h) - f(a-h)$, then by **Mean Vaule Theorem**, we have

$$\begin{aligned} g(h) - g(0) &= g(h) \\ &= g'(\theta h)h, \text{ where } 0 < \theta < 1 \\ &= [f'(a+\theta h) + f'(a-\theta h)]h \end{aligned}$$

which implies that

$$\frac{f(a+h) - f(a-h)}{h} = f'(a+\theta h) + f'(a-\theta h), 0 < \theta < 1.$$

(b)

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h} = f'(a+\lambda h) - f'(a-\lambda h), 0 < \lambda < 1.$$

Proof: Let $g(h) = f(a+h) - 2f(a) + f(a-h)$, then by **Mean Vaule Theorem**, we have

$$\begin{aligned} g(h) - g(0) &= g(h) \\ &= g'(\lambda h)h, \text{ where } 0 < \lambda < 1 \\ &= [f'(a+\lambda h) - f'(a-\lambda h)]h \end{aligned}$$

which implies that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h} = f'(a+\lambda h) - f'(a-\lambda h), 0 < \lambda < 1.$$

(c) If $f''(a)$ exists, show that.

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Proof: Since

$$\begin{aligned} &\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \\ &= \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} \text{ by } \textbf{L-Hospital Rule} \\ &= \lim_{h\to 0} \frac{f'(a+h) - f'(a)}{2h} + \frac{f'(a) - f'(a-h)}{2h} \\ &= \frac{1}{2}(2f''(a)) \text{ since } f''(a) \text{ exists.} \\ &= f''(a). \end{aligned}$$

Remark: There is another proof by using **Generalized Mean Value theorem.**

Proof: Let $g_1(h) = f(a+h) - 2f(a) + f(a-h)$ and $g_2(h) = h^2$, then by **Generalized Mean Value theorem,** we have

$$[g_1(h) - g_1(0)]g_2'(\theta h) = g_1'(\theta h)[g_2(h) - g_2(0)]$$

which implies that

$$\frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = \frac{f'(a+\theta h) - f'(a-\theta h)}{2\theta h}$$

Hence,

$$\begin{aligned} &\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \\ &= \lim_{h\to 0} \frac{f'(a+\theta h) - f'(a-\theta h)}{2\theta h} \\ &= f''(a) \text{ since } f''(a) \text{ exists.} \end{aligned}$$

(d) Give an example where the limit of the quotient in (c) exists but where $f''(a)$ does not exist.

Solution: (**STUDY**) Note that in the proof of (c) by using **L-Hospital Rule**. We know that $|x|$ is not differentiable at $x = 0$, and $|x|$ satisfies that

$$\lim_{h\to 0} \frac{|0+h| - |0-h|}{2h} = 0 = \lim_{h\to 0} \frac{f'(0+h) - f'(0-h)}{2h}$$

So, let us try to find a function f so that $f'(x) = |x|$. So, consider its integral, we know that

$$f(x) = \begin{cases} \frac{x^2}{2} \text{ if } x \geq 0 \\ -\frac{x^2}{2} \text{ if } x < 0 \end{cases}$$

Hence, we complete it.

Remark: (i) There is a related statement; we write it as follows. Suppose that f defined on (a,b) and has a derivative at $c \in (a,b)$. If $\{x_n\} \subseteq (a,c)$ and $\{y_n\} \subseteq (c,b)$ with such that $(x_n - y_n) \to 0$ as $n \to \infty$. Then we have

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n}.$$

Proof: Since $f'(c)$ exists, we have

$$f(y_n) = f(c) + f'(c)(y_n - c) + o(y_n - c) \quad *$$

and

$$f(x_n) = f(c) + f'(c)(x_n - c) + o(x_n - c). \quad *'$$

If we combine (*) and (*'), we have

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c) + \frac{o(y_n - c)}{y_n - x_n} + \frac{o(x_n - c)}{y_n - x_n}. \quad **$$

Note that

$$\left|\frac{y_n - c}{y_n - x_n}\right| < 1 \text{ and } \left|\frac{x_n - c}{y_n - x_n}\right| < 1 \text{ for all } n,$$

we have

$$\begin{aligned} &\lim_{n\to\infty}\left[\frac{o(y_n - c)}{y_n - x_n} + \frac{o(x_n - c)}{y_n - x_n}\right] \\ &= \lim_{n\to\infty}\left[\frac{o(y_n - c)}{y_n - c}\frac{y_n - c}{y_n - x_n} + \frac{o(x_n - c)}{x_n - c}\frac{x_n - c}{y_n - x_n}\right] \\ &= 0. \end{aligned}$$

which implies that, by (**)

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n}.$$

(ii) There is a good exercise; we write it as follows. Let $f \in C^2(a,b)$, and $c \in (a,b)$. For small $|h|$ such that $c + h \in (a,b)$, write

$$f(c+h) = f(c) + f'(c + \theta(h)h)h$$

where $0 < \theta < 1$. Show that if $f''(c) \neq 0$, then $\lim_{h\to 0} \theta(h) = 1/2$.

Proof: Since $f \in C^2(a,b)$, by **Taylor Theorem with Remainder Term,** we have

$$\begin{aligned} f(c+h) - f(c) &= f'(c)h + \frac{f''(\xi)}{2!}h^2, \text{ where } \xi \in (c,h) \text{ or } (h,c) \\ &= f'(c + \theta(h)h)h \text{ by hypothesis.} \end{aligned}$$

So,

$$\frac{f'(c+\theta(h)h) - f'(c)}{\theta(h)}\theta(h) = \frac{f''(\xi)}{2!},$$

and let $h \to 0$, we have $\xi \to c$ by continuity of f'' at c. Hence,

$$\lim_{h\to 0} \theta(h) = 1/2 \text{ since } f''(c) \neq 0.$$

Note: We can modify our statement as follows. Let f be defined on (a,b), and $c \in (a,b)$. For small $|h|$ such that $c + h \in (a,b)$, write

$$f(c+h) = f(c) + f'(c + \theta(h)h)h$$

where $0 < \theta < 1$. Show that if $f''(c) \neq 0$, and $\theta(-x) = \theta(x)$ for $x \in (a-h, a+h)$, then $\lim_{h\to 0} \theta(h) = 1/2$.

Proof: Use the exercise (c), we have

$$\begin{aligned} f''(c) &= \lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} \\ &= \lim_{h\to 0} \frac{f'(c+\theta(h)h) - f'(c - \theta(-h)h)}{h} \text{ by hypothesis} \\ &= \lim_{h\to 0} \frac{f'(c+\theta(h)h) - f'(c - \theta(h)h)}{2\theta(h)h} 2\theta(h) \text{ since } \theta(-x) = \theta(x) \text{ for } x \in (a-h, a+h). \end{aligned}$$

Since $f''(c) \neq 0$, we finally have $\lim_{h\to 0} \theta(h) = 1/2$.

5.25 Let f have a finite derivatiive in (a,b) and assume that $c \in (a,b)$. Consider the following condition: For every $\varepsilon > 0$, there exists a $1-$ball $B(c;\delta)$, whose radius δ depends only on ε and not on c, such that if $x \in B(c;\delta)$, and $x \neq c$, then

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \varepsilon.$$

Show that f' is continuous on (a,b) if this condition holds throughout (a,b).

Proof: Given $\varepsilon > 0$, we want to find a $\delta > 0$ such that as $d(x,y) < \delta$, $x,y \in (a,b)$, we have

$$|f'(x) - f'(y)| < \varepsilon.$$

Choose any point $y \in (a,b)$, and thus by hypothesis, given $\varepsilon' = \varepsilon/2$, there is a $1-$ball $B(y;\delta)$, whose radius δ depends only on ε' and not on y, such that if $x \in B(y;\delta)$, and $x \neq y$, then ,

$$\left| \frac{f(x) - f(y)}{x - y} - f'(y) \right| < \varepsilon/2 = \varepsilon'. \quad *$$

Note that $y \in B(x,\delta)$, so, we also have

$$\left|\frac{f(x)-f(y)}{x-y}-f'(x)\right| < \varepsilon/2 = \varepsilon' \qquad *'$$

Combine (*) with (*'), we have

$$|f'(x)-f'(y)| < \varepsilon.$$

Hence, we have proved f' is continuous on (a,b).

Remark: (i) The open interval can be changed into a closed interval; it just need to consider its endpoints. That is, f' is continuous on $[a,b]$ if this condition holds throughout $[a,b]$. The proof is similar, so we omit it.

(ii) The converse of statement in the exercise is alos true. We write it as follows. Let f' be continuous on $[a,b]$, and $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| < \varepsilon$$

whenever $0 < |x-c| < \delta,\ a \leq x, c, \leq b.$

Proof: Given $\varepsilon > 0,$ we want to find a $\delta > 0$ such that

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| < \varepsilon$$

whenever $0 < |x-c| < \delta,\ a \leq x,c \leq b.$ Since f' is continuous on $[a,b],$ we know that f' is uniformly continuous on $[a,b].$ That is, given $\varepsilon' = \varepsilon > 0,$ there is a $\delta > 0$ such that as $d(x,y) < \delta,$ we have

$$|f'(x)-f'(y)| < \varepsilon. \qquad *$$

Consider $d(x,c) < \delta,\ x \in [a,b],$ then by (*), we have

$$\left|\frac{f(x)-f(c)}{x-c}-f'(c)\right| = |f'(x')-f'(c)| < \varepsilon \text{ by } \textbf{Mean Value Theorem}$$

where $d(x',x) < \delta.$ So, we complete it.

Note: This could be expressed by saying that f is uniformly differentiable on $[a,b]$ if f' is continuous on $[a,b]$.

5.26 Assume f has a finite derivative in (a,b) and is continuous on $[a,b],$ with $a \leq f(x) \leq b$ for all x in $[a,b]$ and $|f'(x)| \leq \alpha < 1$ for all x in $(a,b).$ Prove that f has a unique fixed point in $[a,b]$.

Proof: Given any $x,y \in [a,b],$ thus, by **Mean Value Theorem**, we have

$$|f(x)-f(y)| = |f'(z)||x-y| \leq \alpha|x-y| \text{ by hypothesis.}$$

So, we know that f is a contraction on a complete metric space $[a,b].$ So, f has a unique fixed point in $[a,b]$.

5.27 Give an example of a pair of functions f and g having a finite derivatives in $(0,1)$, such that

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0,$$

but such that $\lim_{x\to 0}\frac{f'(x)}{g'(x)}$ does not exist, choosing g so that $g'(x)$ is never zero.

Proof: Let $f(x) = \sin(1/x)$ and $g(x) = 1/x.$ Then it is trivial for that $g'(x)$ is never zero. In addition, we have

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = 0, \text{ and } \lim_{x\to 0}\frac{f'(x)}{g'(x)} \text{ does not exist.}$$

Remark: In this exercise, it tells us that **the converse of L-Hospital Rule** is **NOT** necessary true. Here is a good exercise very like **L-Hospital Rule**, but it does not! We write it as follows.

Suppose that $f'(a)$ and $g'(a)$ exist with $g'(a) \neq 0$, and $f(a) = g(a) = 0$. Prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof: Consider

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{[f(x) - f(a)]/(x - a)}{[g(x) - g(a)]/(x - a)}$$

$$= \frac{f'(a)}{g'(a)} \text{ since } f'(a) \text{ and } g'(a) \text{ exist with } g'(a) \neq 0.$$

Note: (i) It should be noticed that we **CANNOT** use **L-Hospital Rule** since the statement tells that f and g have a derivative at a, we do not make sure of the situation of other points.

(ii) This holds also for complex functions. Let us recall the proof of **L-Hospital Rule,** we need use the order field R; however, C is not an order field. Hence, **L-Hospital Rule** does not hold for C. In fact, no order can be defined in the complex field since $i^2 = -1$.

Supplement on L-Hospital Rule

We do not give a proof about the following fact. The reader may see the book named **A First Course in Real Analysis written by Protter and Morrey**, Charpter 4, pp 88-91.

Theorem $(\frac{0}{0})$ Let f and g be continuous and differentiable on (a,b) with $g' \neq 0$ on (a,b). If

$$\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x) = 0 \text{ and} \qquad *$$

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L.$$

Remark: 1. The size of the interval (a,b) is of no importance; it suffices to have $g' \neq 0$ on $(a, a+\delta)$, for some $\delta > 0$.

2. (*) is a sufficient condition, not a necessary condition. For example, $f(x) = x^2$, and $g(x) = \sin 1/x$ both defined on $(0,1)$.

3. We have some similar results: $x \to a^-$; $x \to a$; $x \to +\infty (\Leftrightarrow 1/x \to 0^+)$; $x \to -\infty (\Leftrightarrow 1/x \to 0^-)$.

Theorem $(\frac{\infty}{\infty})$ Let f and g be continuous and differentiable on (a,b) with $g' \neq 0$ on (a,b). If

$$\lim_{x \to a^+} f(x) = \infty = \lim_{x \to a^+} g(x) = \infty \text{ and} \qquad *$$

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = L.$$

Remark: 1. The proof is skilled, and it needs an algebraic identity.

2. We have some similar results: $x \to a^-$; $x \to a$; $x \to +\infty (\Leftrightarrow 1/x \to 0^+)$; $x \to -\infty (\Leftrightarrow 1/x \to 0^-)$.

3. (*) is a sufficient condition, not a necessary condition. For example, $f(x) = x + \sin x$, and $g(x) = x$.

Theorem (O. Stolz) Suppose that $y_n \to \infty$, and $\{y_n\}$ is increasing. If

$$\lim_{n\to\infty}\frac{x_n - x_{n-1}}{y_n - y_{n-1}} = L, \text{ (or } +\infty)$$

then

$$\lim_{n\to\infty}\frac{x_n}{y_n} = L. \text{ (or } +\infty)$$

Remark: 1. The proof is skilled, and it needs an algebraic identity.

2. The difference between Theorem 2 and Theorem 3 is that x is a **continuous varibale** but x_n is not.

Theorem (Taylor Theorem with Remainder) Suppose that f is a real function defined on $[a,b]$. If $f^n(x)$ is continuous on $[a,b]$, and differentiable on (a,b), then (let $x, c \in [a,b]$, with $x \neq c$) there is a $\tilde{x}$, interior to the interval joining x and c such that

$$f(x) = P_f(x) + \frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x-c)^{n+1},$$

where

$$P_f(x) := \sum_{k=0}^{n}\frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Remark: 1. As $n = 1$, it is exactly **Mean Value Theorem.**

2. The part

$$\frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x-c)^{n+1} := R_n(x)$$

is called the **remainder term.**

3. There are some types about remainder term. (Lagrange, Cauchy, Berstein, etc.)

Lagrange

$$R_n(x) = \frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x-c)^{n+1}.$$

Cauchy

$$R_n(x) = \frac{f^{(n+1)}(c+\theta(x-c))}{n!}[(1-\theta)^n](x-c)^{n+1}, \text{ where } 0 < \theta < 1.$$

Berstein

$$R_n(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$$

5.28 Prove the following theorem:

Let f and g be two functions having finite nth derivatives in (a,b). For some interior point c in (a,b), assume that $f(c) = f'(c) = \ldots = f^{(n-1)}(c) = 0$, and that $g(c) = g'(c) = \ldots = g^{(n-1)}(c) = 0$, but that $g^{(n)}(x)$ is never zero in (a,b). Show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

NOTE. $f^{(n)}$ and $g^{(n)}$ are not assumed to be continuous at c.

Hint. Let

$$F(x) = f(x) - \frac{(x-c)^{n-2} f^{(n)}(c)}{(n-2)!},$$

define G similarly, and apply Theorem 5.20 to the functions F and G.

Proof: Let

$$F(x) = f(x) - \frac{f^{(n)}(c)}{(n-2)!}(x-c)^{n-2}$$

and

$$G(x) = g(x) - \frac{g^{(n)}(c)}{(n-2)!}(x-c)^{n-2}$$

then inductively,

$$F^{(k)}(x) = f^{(k)}(x) - \frac{f^{(n)}(c)}{(n-2-k)!}(x-c)^{n-2-k}$$

and note that

$$F^{(k)}(c) = 0 \text{ for all } k = 0,1,..,n-3, \text{ and } F^{(n-2)}(c) = -f^{(n)}(c).$$

Similarly for G. Hence, by **Theorem 5.20**, we have

$$\left[F(x) - \sum_{k=0}^{n-2} \frac{F^{(k)}}{k!}(x-c)^k\right][G^{(n-1)}(x_1)] = [F^{(n-1)}(x_1)]\left[G(x) - \sum_{k=0}^{n-2} \frac{G^{(k)}}{k!}(x-c)^k\right]$$

where x_1 between x and c, which implies that

$$[f(x)][g^{(n-1)}(x_1)] = [f^{(n-1)}(x_1)][g(x)]. \qquad *$$

Note that since $g^{(n)}$ is never zero on (a,b); it implies that there exists a $\delta > 0$ such that every $g^{(k)}$ is never zero in $(c-\delta, c+\delta) - \{c\}$, where $k = 0,1,2\ldots,n$. Hence, we have, by (*),

$$\begin{aligned}
\lim_{x \to c} \frac{f(x)}{g(x)} &= \lim_{x \to c} \frac{f^{(n-1)}(x_1)}{g^{(n-1)}(x_1)} \\
&= \lim_{x_1 \to c} \frac{f^{(n-1)}(x_1) - f^{(n-1)}(c)}{g^{(n-1)}(x_1) - g^{(n-1)}(c)} \text{ since } x \to c (\Rightarrow x_1 \to c) \\
&= \lim_{x_1 \to c} \frac{(f^{(n-1)}(x_1) - f^{(n-1)}(c))/(x_1 - c)}{(g^{(n-1)}(x_1) - g^{(n-1)}(c))/(x_1 - c)} \\
&= \frac{f^{(n)}(c)}{g^{(n)}(c)} \text{ since } f^{(n)} \text{ exists and } g^{(n)} \text{ exists} (\neq 0) \text{ on } (a,b).
\end{aligned}$$

Remark: (1) The hint is not correct from text book. The reader should find the difference between them.

(2) Here ia another proof by **L-Hospital Rule and Remark in Exercise 5.27.**

Proof: Since $g^{(n)}$ is never zero on (a,b), it implies that there exists a $\delta > 0$ such that every $g^{(k)}$ is never zero in $(c-\delta, c+\delta) - \{c\}$, where $k = 0, 1, 2\ldots, n$. So, we can apply $(n-1)$ **–times L-Hospital Rule methoned in Supplement,** and thus get

$$\begin{aligned}\lim_{x\to c}\frac{f(x)}{g(x)} &= \lim_{x\to c}\frac{f^{(n-1)}(x)}{g^{(n-1)}(x)}\\ &= \lim_{x\to c}\frac{f^{(n-1)}(x)-f^{(n-1)}(c)}{g^{(n-1)}(x)-g^{(n-1)}(c)}\\ &= \lim_{x\to c}\frac{(f^{(n-1)}(x)-f^{(n-1)}(c))/(x-c)}{(g^{(n-1)}(x)-g^{(n-1)}(c))/(x-c)}\\ &= \frac{f^{(n)}(c)}{g^{(n)}(c)} \text{ since } f^{(n)} \text{ exists and } g^{(n)} \text{ exists}(\neq 0) \text{ on } (a,b).\end{aligned}$$

5.29 Show that the formula in Taylor's theorem can also be written as follows:

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{(x-c)(x-x_1)^{n-1}}{(n-1)!}f^{(n)}(x_1),$$

where x_1 is interior to the interval joining x and c. Let $1-\theta = (x-x_1)/(x-c)$. Show that $0 < \theta < 1$ and deduce the following form of the remainder term (due to Cauchy):

$$\frac{(1-\theta)^{n-1}(x-c)^n}{(n-1)!}f^{(n)}[\theta x + (1-\theta)c].$$

Hint. Take $G(t) = t$ in the proof of Theorem 5.20.

Proof: Let

$$F(t) = \sum_{k=0}^{n-1}\frac{f^{(k)}(t)}{k!}(x-t)^k, \text{ and } G(t) = t,$$

and note that

$$F'(t) = \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1}$$

then by **Generalized Mean Value Theorem**, we have

$$[F(x)-F(c)][G'(x_1)] = [G(x)-G(c)][F'(x_1)]$$

which implies that

$$\begin{aligned}f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k &= \frac{f^{(n)}(x_1)}{(n-1)!}(x-x_1)^n\\ &= \frac{f^{(n)}(\theta x + (1-\theta)c)}{(n-1)!}(x-c)^n(1-\theta)^n, \text{ where } x_1 = \theta x + (1-\theta)c.\end{aligned}$$

So, we have prove that

$$f(x) = \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k + R_{n-1}(x),$$

where

$$R_{n-1}(x) = \frac{f^{(n)}(\theta x + (1-\theta)c)}{(n-1)!}(x-c)^n(1-\theta)^n, \text{ where } x_1 = \theta x + (1-\theta)c$$

is called a **Cauchy Remainder.**

Supplement on some questions.

1. Let f be continuous on $[0,1]$ and differentiable on $(0,1)$. Suppose that $f(0) = 0$ and

$|f'(x)| \leq |f(x)|$ for $x \in (0,1)$. Prove that f is constant.

Proof: Given any $x_1 \in (0,1]$, by Mean Value Theorem and hypothesis, we know that

$$|f(x_1) - f(0)| = x_1|f'(x_2)| \leq x_1|f(x_2)|, \text{ where } x_2 \in (0,x_1).$$

So, we have

$$|f(x_1)| \leq x_1 \cdots x_n|f(x_{n+1})| \leq M(x_1 \cdots x_n), \text{ where } x_{n+1} \in (0,x_n), \text{ and } M = \sup_{x\in[a,b]}|f(x)|$$

Since $M(x_1 \cdots x_n) \to 0$, as $n \to \infty$, we finally have $f(x_1) = 0$. Since x_1 is arbirary, we find that $f(x) = 0$ on $[0,1]$.

2. Suppose that g is real function defined on R, with bounded derivative, say $|g'| \leq M$. Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Show that f is 1-1 if ε is small enough. (It implies that f is strictly monotonic.)

Proof: Suppose that $f(x) = f(y)$, i.e., $x + \varepsilon g(x) = y + \varepsilon g(y)$ which implies that

$|y - x| = \varepsilon|g(y) - g(x)| \leq \varepsilon M|y - x|$ by **Mean Value Theorem**, and hypothesis.

So, as ε is small enough, we have $x = y$. That is, f is 1-1.

Supplement on Convex Function.

Definition(Convex Function) Let f be defined on an interval I, and given $0 < \lambda < 1$, we say that f is a convex function if for any two points $x,y \in I$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

For example, x^2 is a convex function on R. Sometimes, the reader may see another weak definition of convex function in case $\lambda = 1/2$. We will show that under continuity, two definitions are equivalent. In addition, it should be noted that a convex function is not necessarily continuous since we may give a jump on a continuous convex function on its boundary points, for example, $f(x) = x$ is a continuous convex function on $[0,1]$, and define a function g as follows:

$$g(x) = x, \text{ if } x \in (0,1) \text{ and } g(1) = g(0) = 2.$$

The function g is not continuous but convex. Note that if $-f$ is convex, we call f is concave, vice versa. Note that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.) It is clear only by definition.

Theorem(Equivalence) Under continuity, two definitions are equivalent.

Proof: It suffices to consider if

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \qquad *$$

then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \text{ for all } 0 < \lambda < 1.$$

Since (*) holds, then by **Mathematical Induction**, it is easy to show that

$$f\left(\frac{x_1+\ldots+x_{2^n}}{2^n}\right) \leq \frac{f(x_1)+\ldots+f(x_{2^n})}{2^n}.$$

Claim that

$$f\left(\frac{x_1+\ldots+x_n}{n}\right) \leq \frac{f(x_1)+\ldots+f(x_n)}{n} \text{ for all } n \in N. \qquad **$$

Using **Reverse Induction**, let $x_n = \frac{x_1+\ldots+x_{n-1}}{n-1}$, then

$$f\left(\frac{x_1+\ldots+x_n}{n}\right) = f\left(\frac{x_1+\ldots+x_{n-1}}{n} + \frac{x_n}{n}\right)$$
$$= f(x_n)$$
$$\leq \frac{f(x_1)+\ldots+f(x_n)}{n} \text{ by induction hypothesis.}$$

So, we have

$$f\left(\frac{x_1+\ldots+x_{n-1}}{n-1}\right) \leq \frac{f(x_1)+\ldots+f(x_{n-1})}{n-1}.$$

Hence, we have proved (**). Given a rational number $m/n \in (0,1)$, where $g.c.d.(m,n) = 1$; we choose $x := x_1 = \ldots = x_m$, and $y := x_{m+1} = \ldots = x_n$, then by (**), we finally have

$$f\left(\frac{mx}{n} + \frac{(n-m)y}{n}\right) \leq \frac{mf(x)}{n} + \frac{(n-m)f(y)}{n} = \frac{m}{n}f(x) + (1-\frac{m}{n})f(y). \qquad ***$$

Given $\lambda \in (0,1)$, then there is a sequence $\{q_n\}(\subseteq Q)$ such that $q_n \to \lambda$ as $n \to \infty$. Then by continuity and (***), we get

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Remark: **The Reverse Induction** is that let $S \subseteq N$ and S has two properties:(1) For every $k \geq 0$, $2^k \in S$ and (2) $k \in S$ and $k-1 \in N$, then $k-1 \in S$. Then $S = N$.

(**Lemma**) Let f be a convex function on $[a,b]$, then f is bounded.

Proof: Let $M = \max(f(a), f(b))$, then every point $z \in I$, write $z = a\lambda + (1-\lambda)b$, we have

$$f(z) = f(a\lambda + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda)f(b) \leq M.$$

In addition, we may write $z = \frac{a+b}{2} - t$, where t is chosen so that z runs through $[a,b]$. So, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}f\left(\frac{a+b}{2} - t\right) + \frac{1}{2}f\left(\frac{a+b}{2} + t\right)$$

which implies that

$$2f\left(\frac{a+b}{2}\right) - f\left(\frac{a+b}{2} + t\right) \leq f\left(\frac{a+b}{2} - t\right) = f(z)$$

which implies that

$$2f\left(\frac{a+b}{2}\right) - M := m \leq f(z).$$

Hence, we have proved that f is bounded above by M and bounded below by m.

(**Theorem**) If $f : I \to R$ is convex, then f satisfies a **Lipschitz condition** on any closed interval $[a,b] \subseteq int(I)$. In addition, f is **absolutely continuous** on $[a,b]$ and continuous on $int(I)$.

Proof: We choose $\varepsilon(> 0)$ so that $[a-\varepsilon, b+\varepsilon](\subseteq int(I))$. By preceding lemma, we know that f is bounded, say $m \leq f(x) \leq M$ on $[a-\varepsilon, b+\varepsilon]$. Given any two points x, and y, with $a \leq x < y \leq b$ We consider an auxiliary point $z = y + \varepsilon$, and a suitable $\lambda = \frac{y-x}{\varepsilon+y-x}$, then $y = \lambda z + (1-\lambda)x$. So,

$$f(y) = f(\lambda z + (1-\lambda)x) \leq \lambda f(z) + (1-\lambda)f(x) = \lambda[f(z) - f(x)] + f(x)$$

which implies that

$$f(y) - f(x) \leq \lambda(M-m) \leq \frac{y-x}{\varepsilon}(M-m).$$

Change roles of x and y, we finally have

$$|f(y) - f(x)| \leq K|y - x|, \text{ where } K = \frac{M - m}{\varepsilon}.$$

That is, f satisfies a **Lipschitz condition** on any closed interval $[a, b]$.

We call that f is absolutely continuous on $[a, b]$ if given any $\varepsilon > 0$, there is a $\delta > 0$ such that for any collection of $\{(a_i, b_i)\}_{i=1}^{n}$ of disjoint open intervals of $[a, b]$ with $\sum_{k=1}^{n} b_i - a_i < \delta$, we have

$$\sum_{k=1}^{n} |f(b_i) - f(a_i)| < \varepsilon.$$

Clearly, the choice $\delta = \varepsilon/K$ meets this requirement. Finally, the continuity of f on $int(I)$ is obvious.

(**Theorem**) Let f be a differentiable real function defined on (a, b). Prove that f is convex if and only if f' is monotonically increasing.

Proof: $(\Rightarrow)$ Suppose f is convex, and given $x < y$, we want to show that $f'(x) \leq f'(y)$. Choose s and t such that $x < u < s < y$, then it is clear that we have

$$\frac{f(u) - f(x)}{u - x} \leq \frac{f(s) - f(u)}{s - u} \leq \frac{f(y) - f(s)}{y - s}. \qquad *$$

Let $s \to y^-$, we have by (*)

$$\frac{f(u) - f(x)}{u - x} \leq f'(y)$$

which implies that, let $u \to x^+$

$$f'(x) \leq f'(y).$$

$(\Leftarrow)$ Suppose that f' is monotonically increasing, it suffices to consider $\lambda = 1/2$, if $x < y$, then

$$\begin{aligned} \frac{f(x) + f(y)}{2} - f\left(\frac{x + y}{2}\right) &= \frac{[f(x) - f(\frac{x+y}{2})] + [f(y) - f(\frac{x+y}{2})]}{2} \\ &= \frac{x - y}{4}[f'(\xi_1) - f'(\xi_2)], \text{ where } \xi_1 \leq \xi_2 \\ &\leq 0. \end{aligned}$$

Similarly for $x > y$, and there is nothing to prove $x = y$. Hence, we know that f is convex.

(**Corollary 1**) Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Proof: $(\Rightarrow)$ Suppose that f is convex, we have shown that f' is monotonically increasing. So, we know that $f''(x) \geq 0$ for all $x \in (a, b)$.

$(\Leftarrow)$ Suppose that $f''(x) \geq 0$ for all $x \in (a, b)$, it implies that f' is monotonically increasing. So, we know that f is convex.

(**Corollary 2**) Let $0 < \beta \leq \alpha$, then we have

$$\left(\frac{|y_1|^\beta + \ldots + |y_n|^\beta}{n}\right)^{1/\beta} \leq \left(\frac{|y_1|^\alpha + \ldots + |y_n|^\alpha}{n}\right)^{1/\alpha}$$

Proof: Let $p \geq 1$, and since $(x^p)'' = p(p-1)x^{p-2} \geq 0$ for all $x > 0$, we know that $f(x) = x^p$ is convex. So, we have (let $p = \frac{\alpha}{\beta}$)

$$\left(\frac{x_1 + \ldots + x_n}{n}\right)^{\alpha/\beta} \leq \frac{x_1^{\alpha/\beta} + \ldots + x_n^{\alpha/\beta}}{n} \qquad *$$

by

$$f\left(\frac{x_1+\ldots+x_n}{n}\right) \le \frac{f(x_1)+\ldots+f(x_n)}{n}.$$

Choose $x_i = |y_i|^\beta$, where $i = 1,2,..,n$. Then by (*), we have

$$\left(\frac{|y_1|^\beta+\ldots+|y_n|^\beta}{n}\right)^{1/\beta} \le \left(\frac{|y_1|^\alpha+\ldots+|y_n|^\alpha}{n}\right)^{1/\alpha}.$$

(**Corollary 3**) Define

$$M_r(y) = \left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)^{1/r}, \text{ where } r > 0.$$

Then $M_r(y)$ is a monotonic function of r on $(0,\infty)$. In particular, we have

$$M_1(y) \le M_2(y),$$

that is,

$$\frac{|y_1|+\ldots+|y_n|}{n} \le \left(\frac{|y_1|^2+\ldots+|y_n|^2}{n}\right)^{1/2}.$$

Proof: It is clear by **Corollary 2.**

(**Corollary 4**) By definition of $M_r(y)$ in **Corollary 3**, we have

$$\lim_{r\to0^+} M_r(y) = (|y_1|\cdots|y_n|)^{1/n} := M_0(y)$$

and

$$\lim_{r\to\infty} M_r(y) = \max(|y_1|,\ldots,|y_n|) := M_\infty(y)$$

Proof: 1. Since $M_r(y) = \left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)^{1/r}$, taking log and thus by **Mean Value Theorem**, we have

$$\frac{\log\left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)-0}{r-0} = \frac{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}\log|y_i|}{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}}, \text{ where } 0 < r' < r.$$

So,

$$\begin{aligned}\lim_{r\to0^+} M_r(y) &= \lim_{r\to0^+} e^{\frac{\log\left(\frac{|y_1|^r+\ldots+|y_n|^r}{n}\right)}{r}} \\ &= \lim_{r\to0^+} e^{\frac{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}\log|y_i|}{(\frac{1}{n})\sum_{i=1}^n |y_i|^{r'}}} \\ &= e^{\frac{\sum_{i=1}^n \log|y_i|}{n}} \\ &= (|y_1|\cdots|y_n|)^{1/n}.\end{aligned}$$

2. As $r > 0$, we have

$$\left\{\frac{[\max(|y_1|,\ldots,|y_n|)]^r}{n}\right\}^{1/r} \le M_r(y) \le \{[\max(|y_1|,\ldots,|y_n|)]^r\}^{1/r}$$

which implies that, by **Sandwich Theorem**,

$$\lim_{r\to\infty} M_r(y) = \max(|y_1|,\ldots,|y_n|)$$

since $\lim_{r\to\infty}(\frac{1}{n})^{1/r} = 1$.

(**Inequality 1**) Let f be convex on $[a,b]$, and let $c \in (a,b)$. Define

$$l(x) = f(a) + \frac{f(c) - f(a)}{c - a}(x - a),$$

then $f(x) \geq l(x)$ for all $x \in [c, b]$.

Proof: Consider $x \in [c, d]$, then $c = \frac{x-c}{x-a}a + \frac{c-a}{x-a}x$, we have

$$f(c) \leq \frac{x - c}{x - a}f(a) + \frac{c - a}{x - a}f(x)$$

which implies that

$$f(x) \geq f(a) + \frac{f(c) - f(a)}{c - a}(x - a) = l(x).$$

(**Inequality 2**) Let f be a convex function defined on (a, b). Let $a < s < t < u < b$, then we have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Proof: By definition of convex, we know that

$$f(x) \leq f(s) + \frac{f(u) - f(s)}{u - s}(x - s),\ x \in [s, u] \qquad *$$

and by **inequality 1**, we know that

$$f(s) + \frac{f(t) - f(s)}{t - s}(x - s) \leq f(x),\ x \in [t, u]. \qquad **$$

So, as $x \in [t, u]$, by (*) and (**), we finally have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}.$$

Similarly, we have

$$\frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Hence, we have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Remark: Using abvoe method, it is easy to verify that if f is a convex function on (a, b), then $f'_-(x)$ and $f'_+(x)$ exist for all $x \in (a, b)$. In addition, if $x < y$, where $x, y \in (a, b)$, then we have

$$f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y).$$

That is, $f'_+(x)$ and $f'_-(x)$ are increasing on (a, b). We omit the proof.

(**Exercise 1**) Let $f(x)$ be convex on (a, b), and assume that f is differentiable at $c \in (a, b)$, we have

$$l(x) = f(c) + f'(c)(x - c) \leq f(x).$$

That is, the equation of tangent line is below $f(x)$ if the equation of tangent line exists.

Proof: Since f is differentiable at $c \in (a, b)$, we write the equation of tangent line at c,

$$l(x) = f(c) + f'(c)(x - c).$$

Define

$$m(s) = \frac{f(s) - f(c)}{s - c} \text{ where } a < s < c \text{ and } m(t) = \frac{f(t) - f(c)}{t - c} \text{ where } b > t > c,$$

then it is clear that

$$m(s) \leq f'(c) \leq m(t)$$

which implies that

$$l(x) = f(c) + f'(c)(x - c) \leq f(x).$$

(**Exercise 2**) Let $f : R \to R$ be convex. If f is bounded above, then f is a constant function.

Proof: Suppose that f is not constant, say $f(a) \neq f(b)$, where $a < b$. If $f(b) > f(a)$, we consider

$$\frac{f(x) - f(b)}{x - b} \geq \frac{f(b) - f(a)}{b - a}, \text{ where } x > b$$

which implies that as $x > b$,

$$f(x) \geq \frac{f(b) - f(a)}{b - a}(x - b) + f(b) \to +\infty \text{ as } x \to +\infty$$

And if $f(b) < f(a)$, we consider

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a}, \text{ where } x < a$$

which implies that as $x < a$,

$$f(x) \geq \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \to +\infty \text{ as } x \to -\infty.$$

So, we obtain that f is not bouded above. So, f must be a constant function.

(**Exercise 3**) Note that e^x is convex on R. Use this to show that $A.P. \geq G.P.$

Proof: Since $(e^x)'' = e^x \geq 0$ on R, we know that e^x is convex. So,

$$e^{\frac{x_1 + \ldots + x_n}{n}} \leq \frac{e^{x_1} + \ldots + e^{x_n}}{n}, \text{ where } x_i \in R,\ i = 1,2,\ldots,n.$$

So, let $e^{x_i} = y_i > 0$, for $i = 1,2,\ldots,n$. Then

$$(y_1 \cdot \cdot \cdot y_n)^{1/n} \leq \frac{y_1 + \ldots + y_n}{n}.$$

Vector-Valued functions

5.30 If a vector valued function f is differentiable at c, prove that

$$f'(c) = \lim_{h \to 0} \frac{1}{h}[f(c + h) - f(c)].$$

Conversely, if this limit exists, prove that f is differentiable at c.

Proof: Write $f = (f_1,\ldots,f_n) : S(\subseteq R) \to R^n$, and let c be an interior point of S. Then if f is differentiable at c, each f_k is differentiable at c. Hence,

$$\begin{aligned}
&\lim_{h \to 0} \frac{1}{h}[f(c + h) - f(c)] \\
&= \lim_{h \to 0}\left(\frac{f_1(c + h) - f_1(c)}{h}, \ldots, \frac{f_n(c + h) - f_n(c)}{h} \right) \\
&= \left(\lim_{h \to 0} \frac{f_1(c + h) - f_1(c)}{h}, \ldots, \lim_{h \to 0} \frac{f_n(c + h) - f_n(c)}{h} \right) \\
&= (f_1'(c), \ldots, f_n'(c)) \\
&= f'(c).
\end{aligned}$$

Conversly, it is obvious by above.

Remark: We give a summary about this. Let f be a vector valued function defined on S. Write $f : S(\subseteq R^n) \to R^m$, c is a interior point.

$f = (f_1,\ldots,f_n)$ is differentiable at $c \Leftrightarrow$ each f_k is differentiable at c, *

and

$$f = (f_1,\ldots,f_n) \text{ is continuous at } c \Leftrightarrow \text{ each } f_k \text{ is continuous at } c.$$

Note: The set S can be a subset in R^n, the definition of differentiation in higher dimensional space makes (*) holds. The reader can see textbook, Charpter 12.

5.31 A vector-valued function f is differentiable at each point of (a,b) and has constant norm $\|f\|$. Prove that $f(t) \cdot f'(t) = 0$ on (a,b).

Proof: Since $\langle f,f\rangle = \|f\|^2$ is constant on (a,b), we have $\langle f,f\rangle' = 0$ on (a,b). It implies that $2\langle f,f'\rangle = 0$ on (a,b). That is, $f(t) \cdot f'(t) = 0$ on (a,b).

Remark: The proof of $\langle f,g\rangle' = \langle f',g\rangle + \langle f,g'\rangle$ is easy from definition of differentiation. So, we omit it.

5.32 A vector-valued function f is never zero and has a derivative f' which exists and is continuous on R. If there is a real function λ such that $f'(t) = \lambda(t)f(t)$ for all t, prove that there is a positive real function u and a constant vector c such that $f(t) = u(t)c$ for all t.

Proof: Since $f'(t) = \lambda(t)f(t)$ for all t, we have

$$(f_1'(t),\ldots,f_n'(t)) = f'(t) = \lambda(t)f(t) = (\lambda(t)f_1(t),\ldots,\lambda(t)f_n(t))$$

which implies that

$$\frac{f_i'(t)}{fi(t)} = \lambda(t) \text{ since } f \text{ is never zero.} \qquad *$$

Note that $\frac{f_i'(t)}{f_i(t)}$ is a continuous function from R to R for each $i = 1,2,\ldots,n$, since f' is continuous on R, we have, by (*)

$$\int_a^x \frac{f_1'(t)}{f_1(t)}dt = \int_a^x \lambda(t)dt \Rightarrow f_i(t) = \frac{f_i(a)}{e^{\lambda(a)}}e^{\lambda(t)} \text{ for } i = 1,2,\ldots,n.$$

So, we finally have

$$\begin{aligned} f(t) &= (f_1(t),\ldots,f_n(t)) \\ &= e^{\lambda(t)}\left(\frac{f_1(a)}{e^{\lambda(a)}},\ldots,\frac{f_n(a)}{e^{\lambda(a)}}\right) \\ &= u(t)c \end{aligned}$$

where $u(t) = e^{\lambda(t)}$ and $c = \left(\frac{f_1(a)}{e^{\lambda(a)}},\ldots,\frac{f_n(a)}{e^{\lambda(a)}}\right)$.

Supplement on Mean Value Theorem in higher dimensional space.

In the future, we will learn so called **Mean Value Theorem in higher dimensional space** from the text book in Charpter 12. We give a similar result as supplement.

Suppose that f is continuous mapping of $[a,b]$ into R^n and f is differentiable in (a,b). Then there exists $x \in (a,b)$ such that

$$\|f(b) - f(a)\| \le (b-a)\|f'(x)\|.$$

Proof: Let $z = f(b) - f(a)$, and define $\phi(x) = f(x) \cdot z$ which is a real valued function defined on (a,b). It is clear that $\phi(x)$ is continuous on $[a,b]$ and differentiable on (a,b). So, by **Mean Value Theorem**, we know that

$$\phi(b) - \phi(a) = \phi'(x)(b-a), \text{ where } x \in (a,b)$$

which implies that

$$\begin{aligned} |\phi(b) - \phi(a)| &= |\phi'(x)(b-a)| \\ &\le \|f(b) - f(a)\|\,\|\phi'(x)\|(b-a) \text{ by Cauchy-Schwarz inequality.} \end{aligned}$$

So, we have

$$\|f(b)-f(a)\| \leq (b-a)\|f'(x)\|.$$

Partial derivatives

5.33 Consider the function f defined on R^2 by the following formulas:

$$f(x,y) = \frac{xy}{x^2+y^2} \text{ if } (x,y) \neq (0,0)\; f(0,0) = 0.$$

Prove that the partial derivatives $D_1f(x,y)$ and $D_2f(x,y)$ exist for every (x,y) in R^2 and evaluate these derivatives explicitly in terms of x and y. Also, show that f is not continuous at $(0,0)$.

Proof: It is clear that for all $(x,y) \neq (0,0)$, we have

$$D_1f(x,y) = y\frac{y^2-x^2}{(x^2+y^2)^2} \text{ and } D_2f(x,y) = x\frac{x^2-y^2}{(x^2+y^2)^2}.$$

For $(x,y) = (0,0)$, we have

$$D_1f(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0.$$

Similarly, we have

$$D_2f(0,0) = 0.$$

In addition, let $y = x$ and $y = 2x$, we have

$$\lim_{x\to 0} f(x,x) = 1/2 \neq \lim_{x\to 0} f(x,2x) = 2/5.$$

Hence, f is not continuous at $(0,0)$.

Remark: The existence of all partial derivatives does not make sure the continuity of f. The trouble with partial derivatives is that they treat a function of several variables as a function of one variable at a time.

5.34 Let f be defined on R^2 as follows:

$$f(x,y) = y\frac{x^2-y^2}{x^2+y^2} \text{ if } (x,,y) \neq (0,0),\; f(0,0) = 0.$$

Compute the first- and second-order partial derivatives of f at the origin, when they exist.

Proof: For $(x,y) \neq (0,0)$, it is clear that we have

$$D_1f(x,y) = \frac{4xy^3}{(x^2+y^2)^2} \text{ and } D_2f(x,y) = \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}$$

and for $(x,y) = (0,0)$, we have

$$D_1f(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = 0,\; D_2f(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y-0} = -1.$$

Hence,

$$D_{1,1}f(0,0) = \lim_{x\to 0}\frac{D_1f(x,0)-D_1f(0,0)}{x-0} = 0,$$

$$D_{1,2}f(0,0) = \lim_{x\to 0}\frac{D_2f(x,0)-D_2f(0,0)}{x-0} = \lim_{x\to 0}\frac{2}{x} \text{ does not exist,}$$

$$D_{2,1}f(0,0) = \lim_{y\to 0}\frac{D_1f(0,y)-D_1f(0,0)}{y-0} = 0,$$

and

$$D_{2,2}f(0,0) = \lim_{y\to 0}\frac{D_2f(0,y) - D_2f(0,0)}{y-0} = \lim_{y\to 0}\frac{0}{y} = 0.$$

Remark: We do not give a detail computation, but here are answers. Leave to the reader as a practice. For $(x,y) \neq (0,0)$, we have

$$D_{1,1}f(x,y) = \frac{4y^3(y^2-3x^2)}{(x^2+y^2)^3},$$

$$D_{1,2}f(x,y) = \frac{4xy^2(3x^2-y^2)}{(x^2+y^2)^3},$$

$$D_{2,1}f(x,y) = \frac{4xy^2(3x^2-y^2)}{(x^2+y^2)^3},$$

and

$$D_{2,2}f(x,y) = \frac{4x^2y(y^2-3x^2)}{(x^2+y^2)^3}.$$

complex-valued functions

5.35 Let S be an open set in C and let S^* be the set of complex conjugates $\bar{z}$, where $z \in S$. If f is defined on S, define g on S^* as follows: $g(\bar{z}) = \bar{f}(z)$, the complex conjugate of $f(z)$. If f is differentiable at c, prove that g is differentiable at $\bar{c}$ and that $g'(\bar{c}) = \bar{f}'(c)$.

Proof: Since $c \in S$, we know that c is an interior point. Thus, it is clear that $\bar{c}$ is also an interior point of S^*. Note that we have

$$\text{the conjugate of }\left(\frac{f(z)-f(c)}{z-c}\right) = \frac{\bar{f}(z)-\bar{f}(c)}{\bar{z}-\bar{c}}$$
$$= \frac{g(\bar{z})-g(\bar{c})}{\bar{z}-\bar{c}} \text{ by } g(\bar{z}) = \bar{f}(z).$$

Note that $z \to c (\Leftrightarrow \bar{z} \to \bar{c})$, so we know that if f is differentiable at c, prove that g is differentiable at $\bar{c}$ and that $g'(\bar{c}) = \bar{f}'(c)$.

5.36 (i) In each of the following examples write $f = u + iv$ and find explicit formulas for $u(x,y)$ and $v(x,y)$: (These functions are to be defined as indicated in Charpter 1.)

(a) $f(z) = \sin z$,

Solution: Since $e^{iz} = \cos z + i\sin z$, we know that

$$\sin z = \frac{1}{2}[(e^y + e^{-y})\sin x + i(e^y - e^{-y})\cos x]$$

from $\sin z = \frac{e^{iz}-e^{-iz}}{2i}$. So, we have

$$u(x,y) = \frac{(e^{-y}+e^y)\sin x}{2}$$

and

$$v(x,y) = \frac{(e^y - e^{-y})\cos x}{2}.$$

(b) $f(z) = \cos z$,

Solution: Since $e^{iz} = \cos z + i\sin z$, we know that

$$\cos z = \frac{1}{2}[(e^{-y}+e^y)\cos x + (e^{-y} - e^y)\sin x]$$

from $\cos z = \frac{e^{iz}+e^{-iz}}{2}$. So, we have

$$u(x,y) = \frac{(e^{-y} + e^{y})\cos x}{2}$$

and

$$v(x,y) = \frac{(e^{-y} - e^{y})\sin x}{2}.$$

(c) $f(z) = |z|$,

Solution: Since $|z| = (x^2 + y^2)^{1/2}$, we know that

$$u(x,y) = (x^2 + y^2)^{1/2}$$

and

$$v(x,y) = 0.$$

(d) $f(z) = \bar{z}$,

Solution: Since $\bar{z} = x - iy$, we know that

$$u(x,y) = x$$

and

$$v(x,y) = -y.$$

(e) $f(z) = \arg z$, $(z \neq 0)$,

Solution: Since $\arg z \in R$, we know that

$$u(x,y) = \arg(x + iy)$$

and

$$v(x,y) = 0.$$

(f) $f(z) = Log\, z$, $(z \neq 0)$,

Solution: Since $Log\, z = \log|z| + i\arg(z)$, we know that

$$u(x,y) = \log(x^2 + y^2)^{1/2}$$

and

$$v(x,y) = \arg(x + iy).$$

(g) $f(z) = e^{z^2}$,

Solution: Since $e^{z^2} = e^{(x^2-y^2)+i(2xy)}$, we know that

$$u(x,y) = e^{x^2-y^2}\cos(2xy)$$

and

$$v(x,y) = e^{x^2-y^2}\sin(2xy).$$

(h) $f(z) = z^\alpha$, (α complex, $z \neq 0$).

Solution: Since $z^\alpha = e^{\alpha Log\, z}$, then we have (let $\alpha = \alpha_1 + i\alpha_2$)

$$\begin{aligned} z^\alpha &= e^{(\alpha_1+i\alpha_2)(\log|z|+i\arg z)} \\ &= e^{(\alpha_1\log|z|-\alpha_2\arg z)+i(\alpha_2\log|z|+\alpha_1\arg z)}. \end{aligned}$$

So, we know that

$$\begin{aligned} u(x,y) &= e^{\alpha_1\log|z|-\alpha_2\arg z}\cos(\alpha_2\log|z| + \alpha_1\arg z) \\ &= e^{\alpha_1\log(x^2+y^2)^{1/2}-\alpha_2\arg(x+iy)}\cos\left(\alpha_2\log(x^2+y^2)^{1/2} + \alpha_1\arg(x+iy)\right) \end{aligned}$$

and

$$v(x,y) = e^{\alpha_1 \log|z| - \alpha_2 \arg z} \sin(\alpha_2 \log|z| + \alpha_1 \arg z)$$
$$= e^{\alpha_1 \log(x^2+y^2)^{1/2} - \alpha_2 \arg(x+iy)} \sin\left(\alpha_2 \log(x^2+y^2)^{1/2} + \alpha_1 \arg(x+iy)\right).$$

(ii) Show that u and v satisfy the Cauchy -Riemanns equation for the following values of z : All z in (a), (b), (g); no z in (c), (d), (e); all z except real $z \leq 0$ in (f), (h).

Proof: (a) $\sin z = u + iv,$ where

$$u(x,y) = \frac{(e^{-y}+e^{y})\sin x}{2} \text{ and } v(x,y) = \frac{(e^{y}-e^{-y})\cos x}{2}.$$

So,

$$u_x = v_y = \frac{(e^{-y}+e^{y})\cos x}{2} \text{ for all } z = x+iy$$

and

$$u_y = -v_x = \frac{(e^{y}-e^{-y})\sin x}{2} \text{ for all } z = x+iy.$$

(b) $\cos z = u + iv,$ where

$$u(x,y) = \frac{(e^{-y}+e^{y})\cos x}{2} \text{ and } v(x,y) = \frac{(e^{-y}-e^{y})\sin x}{2}.$$

So,

$$u_x = v_y = -\frac{(e^{-y}+e^{y})\sin x}{2} \text{ for all } z = x+iy.$$

and

$$u_y = -v_x = \frac{(e^{y}-e^{-y})\cos x}{2} \text{ for all } z = x+iy.$$

(c) $|z| = u + iv,$ where

$$u(x,y) = (x^2+y^2)^{1/2} \text{ and } v(x,y) = 0.$$

So,

$$u_x = x(x^2+y^2)^{-1/2} = v_y = 0 \text{ if } x = 0,\ y \neq 0.$$

and

$$u_y = y(x^2+y^2)^{-1/2} = -v_x = 0 \text{ if } x \neq 0,\ y = 0.$$

So, we know that no z makes Cauchy-Riemann equations hold.

(d) $\bar{z} = u + iv,$ where

$$u(x,y) = x \text{ and } v(x,y) = -y.$$

So,

$$u_x = 1 \neq -1 = v_y.$$

So, we know that no z makes Cauchy-Riemann equations hold.

(e) $\arg z = u + iv,$ where

$$-\pi < u(x,y) = \arg(x^2+y^2)^{1/2} \leq \pi \text{ and } v(x,y) = 0.$$

Note that

$$u(x,y) = \begin{cases} (1)\ \arctan(y/x),\ \text{if } x > 0, y \in R \\ (2)\ \pi/2,\ \text{if } x = 0, y > 0 \\ (3)\ \arctan(y/x) + \pi,\ \text{if } x < 0,\ y \geq 0 \\ (4)\ \arctan(y/x) - \pi,\ \text{if } x < 0,\ y < 0 \\ (5)\ -\pi/2,\ \text{if } x = 0, y < 0. \end{cases}$$

and

$$v_x = v_y = 0.$$

So, we know that by (1)-(5), for $(x,y) \neq (0,0)$

$$u_x = \frac{-y}{x^2+y^2}$$

and for $(x,y) \notin \{(x,y) : x \leq 0, y = 0\}$, we have

$$u_y = \frac{x}{x^2+y^2}.$$

Hence, we know that no z makes Cauchy-Riemann equations hold.

Remark: We can give the conclusion as follows:

$$(\arg z)_x = \frac{-y}{x^2+y^2} \text{ for } (x,y) \neq (0,0)$$

and

$$(\arg z)_y = \frac{x}{x^2+y^2} \text{ for } (x,y) \notin \{(x,y) : x \leq 0, y = 0\}.$$

(f) $Log\, z = u + iv$, where

$$u(x,y) = \log(x^2+y^2)^{1/2} \text{ and } v(x,y) = \arg(x^2+y^2)^{1/2}.$$

Since

$$u_x = \frac{x}{x^2+y^2} \text{ and } u_y = \frac{y}{x^2+y^2}$$

and

$$v_x = \frac{-y}{x^2+y^2} \text{ for } (x,y) \neq (0,0) \text{ and } v_y = \frac{x}{x^2+y^2} \text{ for } (x,y) \notin \{(x,y) : x \leq 0, y = 0\},$$

we know that all z except real $z \leq 0$ make Cauchy-Riemann equations hold.

Remark: $Log\, z$ is differentiable on $C - \{(x,y) : x \leq 0, y = 0\}$ since Cauchy-Riemann equations along with continuity of $u_x + iv_x$, and $u_y + iv_y$.

(g) $e^{z^2} = u + iv$, where

$$u(x,y) = e^{x^2-y^2}\cos(2xy) \text{ and } v(x,y) = e^{x^2-y^2}\sin(2xy).$$

So,

$$u_x = v_y = 2e^{x^2-y^2}[x(\cos 2xy) - y(\sin 2xy)] \text{ for all } z = x+iy.$$

and

$$u_y = -v_x = -2e^{x^2-y^2}[y(\cos 2xy) + x(\sin 2xy)] \text{ for all } z = x+iy.$$

Hence, we know that all z make Cauchy-Riemann equations hold.

(h) Since $z^\alpha = e^{\alpha Log\, z}$, and e^z is differentiable on C, we know that, by the remark of (f), we know that z^α is differentiable for all z except real $z \leq 0$. So, we know that all z except real $z \leq 0$ make Cauchy-Riemann equations hold.

(In part (h), the Cauchy-Riemann equations hold for all z if α is a nonnegative integer, and they hold for all $z \neq 0$ if α is a negative integer.)

Solution: It is clear from definition of differentiability.

(iii) Compute the derivative $f'(z)$ in (a), (b), (f), (g), (h), assuming it exists.

Solution: Since $f'(z) = u_x + iv_x,$ if it exists. So, we know all results by (ii).

5.37 Write $f = u + iv$ and assume that f has a derivative at each point of an open disk D centered at $(0,0)$. If $au^2 + bv^2$ is constant on D for some real a and $b,$ not both 0. Prove that f is constant on D.

Proof: Let $au^2 + bv^2$ be constant on D. We consider three cases as follows.
1. As $a = 0, b \neq 0,$ then we have

$$v^2 \text{ is constant on } D$$

which implies that

$$vv_x = 0.$$

If $v = 0$ on $D,$ it is clear that f is constant.
If $v \neq 0$ on $D,$ that is $v_x = 0$ on $D.$ So, we still have f is contant.
2. As $a \neq 0, b = 0,$ then it is similar. We omit it.
3. As $a \neq 0, b \neq 0,$ Taking partial derivatives we find

$$auu_x + bvv_x = 0 \text{ on } D. \qquad 1$$

and

$$auu_y + bvv_y = 0 \text{ on } D.$$

By **Cauchy-Riemann equations** the second equation can be written as we have

$$-auv_x + bvu_x = 0 \text{ on } D. \qquad 2$$

We consider $(1)(v_x) + (2)(u_x)$ and $(1)(u_x) + (2)(v_x),$ then we have

$$bv(v_x^2 + u_x^2) = 0 \qquad 3$$

and

$$au(v_x^2 + u_x^2) = 0 \qquad 4$$

which imply that

$$(au^2 + bv^2)(v_x^2 + u_x^2) = 0. \qquad 5$$

If $au^2 + bv^2 = c,$ constant on $D,$ where $c \neq 0,$ then $v_x^2 + u_x^2 = 0.$ So, f is constant.
If $au^2 + bv^2 = c,$ constant on $D,$ where $c = 0,$ then if there exists (x,y) such that $v_x^2 + u_x^2 \neq 0,$ then by (3) and (4), $u(x,y) = v(x,y) = 0.$ By continuity of $v_x^2 + u_x^2,$ we know that there exists an open region $S(\subseteq D)$ such that $u = v = 0$ on $S.$ Hence, by **Uniqueness Theorem**, we know that f is constant.

Remark: In complex theory, the Uniqueness theorem is fundamental and important. The reader can see this from the book named **Complex Analysis by Joseph Bak and Donald J. Newman.**

Functions of Bounded Variation and Rectifiable Curves

Functions of bounded variation

6.1 Determine which of the follwoing functions are of bounded variation on $[0,1]$.
(a) $f(x) = x^2 \sin(1/x)$ if $x \neq 0$, $f(0) = 0$.
(b) $f(x) = \sqrt{x} \sin(1/x)$ if $x \neq 0$, $f(0) = 0$.

Proof: (a) Since

$$f'(x) = 2x\sin(1/x) - \cos(1/x) \text{ for } x \in (0,1] \text{ and } f'(0) = 0,$$

we know that $f'(x)$ is bounded on $[0,1]$, in fact, $|f'(x)| \leq 3$ on $[0,1]$. Hence, f is of bounded variation on $[0,1]$.

(b) First, we choose $n+1$ be an even integer so that $\frac{1}{\frac{\pi}{2}(n+1)} < 1$, and thus consider a partition $P = \left\{0 = x_0,\ x_1 = \frac{1}{\frac{\pi}{2}},\ x_2 = \frac{1}{2\frac{\pi}{2}}, \ldots,\ x_n = \frac{1}{n\frac{\pi}{2}},\ x_{n+1} = \frac{1}{(n+1)\frac{\pi}{2}},\ x_{n+2} = 1\right\}$, then we have

$$\sum_{k=1}^{n+2} |\Delta f_k| \geq 2\sqrt{\frac{2}{\pi}}\left(\sum_{k=1}^{n} \sqrt{1/k}\right).$$

Since $\sum \sqrt{1/k}$ diverges to $+\infty$, we know that f is not of bounded variation on $[0,1]$.

6.2 A function f, defined on $[a,b]$, is said to satisfy a uniform Lipschitz condition of order $\alpha > 0$ on $[a,b]$ if there exists a constant $M > 0$ such that $|f(x) - f(y)| < M|x-y|^\alpha$ for all x and y in $[a,b]$. (Compare with Exercise 5.1.)

(a) If f is such a function, show that $\alpha > 1$ implies f is constant on $[a,b]$, whereas $\alpha = 1$ implies f is of bounded variation $[a,b]$.

Proof: As $\alpha > 1$, we consider, for $x \neq y$, where $x, y \in [a,b]$,

$$0 \leq \frac{|f(x) - f(y)|}{|x-y|} < M|x-y|^{\alpha - 1}.$$

Hence, $f'(x)$ exists on $[a,b]$, and we have $f'(x) = 0$ on $[a,b]$. So, we know that f is constant.

As $\alpha = 1$, consider any partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$, we have

$$\sum_{k=1}^{n} |\Delta f_k| \leq M\sum_{k=1}^{n} |x_{k+1} - x_k| = M(b-a).$$

That is, f is of bounded variation on $[a,b]$.

(b) Give an example of a function f satisfying a uniform Lipschitz condition of order $\alpha < 1$ on $[a,b]$ such that f is not of bounded variation on $[a,b]$.

Proof: First, note that x^α satisfies uniform Lipschitz condition of order α, where $0 < \alpha < 1$. Choosing $\beta > 1$ such that $\alpha\beta < 1$ and let $M = \sum_{k=1}^{\infty} \frac{1}{k^\beta}$ since the series converges. So, we have $1 = \frac{1}{M}\sum_{k=1}^{\infty} \frac{1}{k^\beta}$.

Define a function f as follows. We partition $[0,1]$ into infinitely many subsintervals. Consider

$$x_0 = 0,\ x_1 - x_0 = \frac{1}{M}\frac{1}{1^\beta},\ x_2 - x_1 = \frac{1}{M}\frac{1}{2^\beta}, \ldots,\ x_n - x_{n-1} = \frac{1}{M}\frac{1}{n^\beta}, \ldots.$$

And in every subinterval $[x_i, x_{i+1}]$, where $i = 0, 1, \ldots,$ we define

$$f(x) = \left(\left|x - \frac{x_i + x_{i+1}}{2}\right|\right)^\alpha,$$

then f is a continuous function and is not bounded variation on $[0,1]$ since $\sum_{k=1}^{\infty}\left(\frac{1}{2M}\frac{1}{k^\beta}\right)^\alpha$ diverges.

In order to show that f satisfies uniform Lipschitz condition of order α, we consider three cases.

(1) If $x, y \in [x_i, x_{i+1}]$, and $x, y \in [x_i, \frac{x_i+x_{i+1}}{2}]$ or $x, y \in [\frac{x_i+x_{i+1}}{2}, x_{i+1}]$, then

$$|f(x) - f(y)| = |x^\alpha - y^\alpha| \le |x - y|^\alpha.$$

(2) If $x, y \in [x_i, x_{i+1}]$, and $x \in [x_i, \frac{x_i+x_{i+1}}{2}]$ or $y \in [\frac{x_i+x_{i+1}}{2}, x_{i+1}]$, then there is a $z \in [x_i, \frac{x_i+x_{i+1}}{2}]$ such that $f(y) = f(z)$. So,

$$|f(x) - f(y)| = |f(x) - f(z)| \le |x^\alpha - z^\alpha| \le |x - z|^\alpha \le |x - y|^\alpha.$$

(3) If $x \in [x_i, x_{i+1}]$ and $y \in [x_j, x_{j+1}]$, where $i > j$.

If $x \in [x_i, \frac{x_i+x_{i+1}}{2}]$, then there is a $z \in [x_i, \frac{x_i+x_{i+1}}{2}]$ such that $f(y) = f(z)$. So,

$$|f(x) - f(y)| = |f(x) - f(z)| \le |x^\alpha - z^\alpha| \le |x - z|^\alpha \le |x - y|^\alpha.$$

Similarly for $x \in [\frac{x_i+x_{i+1}}{2}, x_{i+1}]$.

Remark: Here is another example. Since it will use **Fourier Theory**, we do not give a proof. We just write it down as a reference.

$$f(t) = \sum_{k=1}^{\infty} \frac{\cos(3^k t)}{3^{k\alpha}}.$$

(c) Give an example of a function f which is of bounded variation on $[a,b]$ but which satisfies no uniform Lipschitz condition on $[a,b]$.

Proof: Since a function satisfies uniform Lipschitz condition of order $\alpha > 0$, it must be continuous. So, we consider

$$f(x) = \begin{cases} x \text{ if } x \in [a,b) \\ b + 1 \text{ if } x = b. \end{cases}$$

Trivially, f is not continuous but increasing. So, the function is desired.

Remark: Here is a good problem, we write it as follows. If f satisfies

$$|f(x) - f(y)| \le K|x - y|^{1/2} \text{ for } x \in [0,1], \text{ where } f(0) = 0.$$

define

$$g(x) = \begin{cases} \frac{f(x)}{x^{1/3}} \text{ if } x \in (0,1] \\ 0 \text{ if } x = 0. \end{cases}$$

Then g satisfies uniform Lipschitz condition of order 1/6.

Proof: Note that if one of x, and y is zero, the result is trivial. So, we may consider $0 < y < x \le 1$ as follows. Consider

$$\begin{aligned} |g(x) - g(y)| &= \left|\frac{f(x)}{x^{1/3}} - \frac{f(y)}{y^{1/3}}\right| \\ &= \left|\frac{f(x)}{x^{1/3}} - \frac{f(y)}{x^{1/3}} + \frac{f(y)}{x^{1/3}} - \frac{f(y)}{y^{1/3}}\right| \\ &\le \left|\frac{f(x)}{x^{1/3}} - \frac{f(y)}{x^{1/3}}\right| + \left|\frac{f(y)}{x^{1/3}} - \frac{f(y)}{y^{1/3}}\right|. \end{aligned} \quad *$$

For the part

$$\begin{aligned}\left|\frac{f(x)}{x^{1/3}}-\frac{f(y)}{x^{1/3}}\right| &= \frac{1}{x^{1/3}}|f(x)-f(y)| \\ &\le \frac{K}{x^{1/3}}|x-y|^{1/2} \text{ by hypothesis} \\ &\le K|x-y|^{1/2}|x-y|^{-1/3} \text{ since } x \ge x-y > 0 \\ &= K|x-y|^{1/6}. \qquad \text{A}\end{aligned}$$

For another part $\left|\frac{f(y)}{x^{1/3}}-\frac{f(y)}{y^{1/3}}\right|$, we consider two cases.

(1) $x \ge 2y$ which implies that $x > x-y \ge y > 0$,

$$\begin{aligned}\left|\frac{f(y)}{x^{1/3}}-\frac{f(y)}{y^{1/3}}\right| &= |f(y)|\left|\frac{x^{1/3}-y^{1/3}}{(xy)^{1/3}}\right| \\ &\le |f(y)|\left|\frac{(x-y)^{1/3}}{(xy)^{1/3}}\right| \text{ since } |x^{1/3}-y^{1/3}| \le |x-y|^{1/3} \text{ for all } x,y \ge 0 \\ &\le |f(y)|\left|\frac{x^{1/3}}{(xy)^{1/3}}\right| \text{ since } (x-y)^{1/3} \le x^{1/3} \\ &\le |f(y)|\left|\frac{1}{y^{1/3}}\right| \\ &\le K\frac{|y|^{1/2}}{|y|^{1/3}} \text{ by hypothesis} \\ &\le K|y|^{1/6} \\ &\le K|x-y|^{1/6} \text{ since } y \le x-y. \qquad \text{B}\end{aligned}$$

(2) $x < 2y$ which implies that $x > y > x-y > 0$,

$$\begin{aligned}\left|\frac{f(y)}{x^{1/3}}-\frac{f(y)}{y^{1/3}}\right| &= |f(y)|\left|\frac{x^{1/3}-y^{1/3}}{(xy)^{1/3}}\right| \\ &\le |f(y)|\left|\frac{(x-y)^{1/3}}{(xy)^{1/3}}\right| \text{ since } |x^{1/3}-y^{1/3}| \le |x-y|^{1/3} \text{ for all } x,y \ge 0 \\ &\le |f(y)|\left|\frac{(x-y)^{1/3}}{y^{2/3}}\right| \text{ since } x > y \\ &\le K|y|^{1/2}\left|\frac{(x-y)^{1/3}}{y^{2/3}}\right| \text{ by hypothesis} \\ &\le K|y|^{-1/6}|x-y|^{1/3} \\ &\le K|x-y|^{-1/6}|x-y|^{1/3} \text{ since } y > x-y \\ &= K|x-y|^{1/6}. \qquad \text{C}\end{aligned}$$

So, by (A)-(C), (*) tells that g satisfies uniform Lipschitz condition of order 1/6.

Note: Here is a general result. Let $0 \le \beta < \alpha < 2\beta$. If f satisfies

$$|f(x)-f(y)| \le K|x-y|^{\alpha} \text{ for } x \in [0,1], \text{ where } f(0) = 0.$$

define

$$g(x) = \begin{cases} \frac{f(x)}{x^\beta} \text{ if } x \in (0,1] \\ 0 \text{ if } x = 0. \end{cases}$$

Then g satisfies uniform Lipschitz condition of order $\alpha - \beta$. The proof is similar, so we omit it.

6.3 Show that a polynomial f is of bounded variation on every compact interval $[a,b]$. Describe a method for finding the total variation of f on $[a,b]$ if the zeros of the derivative f' are known.

Proof: If f is a constant, then the total variation of f on $[a,b]$ is zero. So, we may assume that f is a polynomial of degree $n \geq 1$, and consider $f'(x) = 0$ by two cases as follows.

(1) If there is no point such that $f'(x) = 0$, then by **Intermediate Value Theorem of Differentiability**, we know that $f'(x) > 0$ on $[a,b]$, or $f'(x) < 0$ on $[a,b]$. So, it implies that f is monotonic. Hence, the total variation of f on $[a,b]$ is $|f(b) - f(a)|$.

(2) If there are m points such that $f'(x) = 0$, say $a = x_0 \leq x_1 < x_2 <\ldots< x_m \leq b = x_{m+1}$, where $1 \leq m \leq n$, then we know the monotone property of function f. So, the total variation of f on $[a,b]$ is

$$\sum_{i=1}^{m+1} |f(x_i) - f(x_{i-1})|.$$

Remark: Here is another proof. Let f be a polynomial on $[a,b]$, then we know that f' is bounded on $[a,b]$ since f' is also polynomial which implies that it is continuous. Hence, we know that f is of bounded variation on $[a,b]$.

6.4 A nonempty set S of real-valued functions defined on an interval $[a,b]$ is called a linear space of functions if it has the following two properties:

(a) If $f \in S$, then $cf \in S$ for every real number c.

(b) If $f \in S$ and $g \in S$, then $f + g \in S$.

Theorem 6.9 shows that the set V of all functions of bounded variation on $[a,b]$ is a linear space. If S is any linear space which contains all monotonic functions on $[a,b]$, prove that $V \subseteq S$. This can be described by saying that **the functions of bounded variation form the samllest linear space containing all monotonic functions.**

Proof: It is directlt from Theorem 6.9 and some facts in Linear Algebra. We omit the detail.

6.5 Let f be a real-valued function defined on $[0,1]$ such that $f(0) > 0$, $f(x) \neq x$ for all x, and $f(x) \leq f(y)$ whenever $x \leq y$. Let $A = \{x : f(x) > x\}$. Prove that $\sup A \in A$, and that $f(1) > 1$.

Proof: Note that since $f(0) > 0$, A is not empty. Suppose that $\sup A := a \notin A$, i.e., $f(a) < a$ since $f(x) \neq x$ for all x. So, given any $\varepsilon_n > 0$, then there is a $b_n \in A$ such that

$$a - \varepsilon_n < b_n. \quad *$$

In addition,

$$b_n < f(b_n) \text{ since } b_n \in A. \quad **$$

So, by (*) and (**), we have (let $\varepsilon_n \to 0^+$),

$$a \leq f(a^-)\ (< f(a)) \text{ since } f \text{ is monotonic increasing.}$$

which contradicts to $f(a) < a$. Hence, we know that $\sup A \in A$.

Claim that $1 = \sup A$. Suppose **NOT**, that is, $a < 1$. Then we have

$$a < f(a) < f(1) < 1.$$

Since $a = \sup A$, consider $x \in (a, f(a))$, then

$$f(x) < x$$

which implies that

$$f(a^+) \le a$$

which contradicts to $a < f(a)$. So, we know that $\sup A = 1$. Hence, we have proved that $f(1) > 1$.

Remark: The reader should keep the method in mind if we ask how to show that $f(1) > 1$ directly. The set A is helpful to do this. Or equivalently, let f be strictly increasing on $[0,1]$ with $f(0) > 0$. If $f(1) \le 1$, then there exists a point $x \in [0,1]$ such that $f(x) = x$.

6.6 If f is defined everywhere in R^1, then f is said to be of bounded variation on $(-\infty, +\infty)$ if f is of bounded variation on every finite interval and if there exists a positive number M such that $V_f(a,b) < M$ for all compact interval $[a,b]$. The total variation of f on $(-\infty, +\infty)$ is then defined to be the sup of all numbers $V_f(a,b)$, $-\infty < a < b < +\infty$, and denoted by $V_f(-\infty, +\infty)$. Similar definitions apply to half open infinite intervals $[a, +\infty)$ and $(-\infty, b]$.

(a) State and prove theorems for the inifnite interval $(-\infty, +\infty)$ analogous to the Theorems 6.7, 6.9, 6.10, 6.11, and 6.12.

(**Theorem 6.7***) Let $f : R \to R$ be of bounded variaton, then f is bounded on R.

Proof: Given any $x \in R$, then $x \in [0,a]$ or $x \in [a,0]$. If $x \in [0,a]$, then f is bounded on $[0,a]$ with

$$|f(x)| \le |f(0)| + V_f(0,a) \le |f(0)| + V_f(-\infty, +\infty).$$

Similarly for $x \in [a,0]$.

(**Theorem 6.9***) Assume that f, and g be of bounded variaton on R, then so are thier sum, difference, and product. Also, we have

$$V_{f\pm g}(-\infty, +\infty) \le V_f(-\infty, +\infty) + V_g(-\infty, +\infty)$$

and

$$V_{fg}(-\infty, +\infty) \le AV_f(-\infty, +\infty) + BV_g(-\infty, +\infty),$$

where $A = \sup_{x \in R}|g(x)|$ and $B = \sup_{x \in R}|f(x)|$.

Proof: For sum and difference, given any compact interval $[a,b]$, we have

$$\begin{aligned} V_{f\pm g}(a,b) &\le V_f(a,b) + V_g(a,b), \\ &\le V_f(-\infty, +\infty) + V_g(-\infty, +\infty) \end{aligned}$$

which implies that

$$V_{f\pm g}(-\infty, +\infty) \le V_f(-\infty, +\infty) + V_g(-\infty, +\infty).$$

For product, given any compact interval $[a,b]$, we have (let $A(a,b) = \sup_{x \in [a,b]}|g(x)|$, and $B(a,b) = \sup_{x \in [a,b]}|f(x)|$),

$$\begin{aligned} V_{fg}(a,b) &\le A(a,b)V_f(a,b) + B(a,b)V_g(a,b) \\ &\le AV_f(-\infty, +\infty) + BV_g(-\infty, +\infty) \end{aligned}$$

which implies that

$$V_{fg}(-\infty,+\infty) \le AV_f(-\infty,+\infty) + BV_g(-\infty,+\infty).$$

(**Theorem 6.10***) Let f be of bounded variation on R, and assume that f is bounded away from zero; that is, suppose that there exists a positive number m such that $0 < m \le |f(x)|$ for all $x \in R$. Then $g = 1/f$ is also of bounded variation on R, and

$$V_g(-\infty,+\infty) \le \frac{V_f(-\infty,+\infty)}{m^2}.$$

Proof: Given any compacgt interval $[a,b]$, we have

$$V_g(a,b) \le \frac{V_f(a,b)}{m^2} \le \frac{V_f(-\infty,+\infty)}{m^2}$$

which implies that

$$V_g(-\infty,+\infty) \le \frac{V_f(-\infty,+\infty)}{m^2}.$$

(**Theorem 6.11***) Let f be of bounded variation on R, and assume that $c \in R$. Then f is of bounded variation on $(-\infty,c]$ and on $[c,+\infty)$ and we have

$$V_f(-\infty,+\infty) = V_f(-\infty,c) + V_f(c,+\infty).$$

Proof: Given any a compact interval $[a,b]$ such that $c \in (a,b)$. Then we have

$$V_f(a,b) = V_f(a,c) + V_f(c,b).$$

Since

$$V_f(a,b) \le V_f(-\infty,+\infty)$$

which implies that

$$V_f(a,c) \le V_f(-\infty,+\infty) \text{ and } V_f(c,b) \le V_f(-\infty,+\infty),$$

we know that the existence of $V_f(-\infty,c)$ and $V_f(c,+\infty)$. That is, f is of bounded variation on $(-\infty,c]$ and on $[c,+\infty)$.

Since

$$V_f(a,c) + V_f(c,b) = V_f(a,b) \le V_f(-\infty,+\infty)$$

which implies that

$$V_f(-\infty,c) + V_f(c,+\infty) \le V_f(-\infty,+\infty), \quad *$$

and

$$V_f(a,b) = V_f(a,c) + V_f(c,b) \le V_f(-\infty,c) + V_f(c,+\infty)$$

which implies that

$$V_f(-\infty,+\infty) \le V_f(-\infty,c) + V_f(c,+\infty), \quad **$$

we know that

$$V_f(-\infty,+\infty) = V_f(-\infty,c) + V_f(c,+\infty).$$

(**Theorem 6.12***) Let f be of bounded variation on R. Let $V(x)$ be defined on $(-\infty,x]$ as follows:

$$V(x) = V_f(-\infty,x) \text{ if } x \in R, \text{ and } V(-\infty) = 0.$$

Then (i) V is an increasing function on $(-\infty,+\infty)$ and (ii) $V-f$ is an increasing function on $(-\infty,+\infty)$.

Proof: (i) Let $x < y$, then we have $V(y) - V(x) = V_f(x,y) \ge 0$. So, we know that V is an increasing function on $(-\infty,+\infty)$.

(ii) Let $x < y$, then we have $(V-f)(y) - (V-f)(x) = V_f(x,y) - (f(y) - f(x)) \ge 0$. So, we know that $V - f$ is an increasing function on $(-\infty, +\infty)$.

(b) Show that Theorem 6.5 is true for $(-\infty, +\infty)$ if "monotonic" is replaced by "bounded and monotonic." State and prove a similar modefication of Theorem 6.13.

(**Theorem 6.5***) If f is bounded and monotonic on $(-\infty, +\infty)$, then f is of bounded variation on $(-\infty, +\infty)$.

Proof: Given any compact interval $[a, b]$, then we have $V_f(a, b)$ exists, and we have $V_f(a, b) = |f(b) - f(a)|$, since f is monotonic. In addition, since f is bounded on R, say $|f(x)| \leq M$ for all x, we know that $2M$ is a upper bounded of $V_f(a, b)$ for all a, b. Hence, $V_f(-\infty, +\infty)$ exists. That is, f is of bounded variation on R.

(**Theorem 6.13***) Let f be defined on $(-\infty, +\infty)$, then f is of bounded variation on $(-\infty, +\infty)$ if, and only if, f can be expressed as the difference of two increasing and bounded functions.

Proof: Suppose that f is of bounded variation on $(-\infty, +\infty)$, then by **Theorem 6.12***, we know that

$$f = V - (V - f),$$

where V and $V - f$ are increasing on $(-\infty, +\infty)$. In addition, since f is of bounded variation on R, we know that V and f is bounded on R which implies that $V - f$ is bounded on R. So, we have proved that if f is of bounded variation on $(-\infty, +\infty)$ then f can be expressed as the difference of two increasing and bounded functions.

Suppose that f can be expressed as the difference of two increasing and bounded functions, say $f = f_1 - f_2$, Then by **Theorem 6.9***, and **Theorem 6.5***, we know that f is of bounded variaton on R.

Remark: The representation of a function of bounded variation as a difference of two increasing and bounded functions is by no mean unique. It is clear that **Theorem 6.13*** also holds if "increasing" is replaced by "strictly increasing." For example, $f = (f_1 + g) - (f_2 + g)$, where g is any strictly increasing and bounded function on R. One of such g is $\arctan x$.

6.7 Assume that f is of bounded variation on $[a, b]$ and let

$$P = \{x_0, x_1, \ldots, x_n\} \in \wp[a, b].$$

As usual, write $\Delta f_k = f(x_k) - f(x_{k-1})$, $k = 1, 2, \ldots, n$. Define

$$A(P) = \{k : \Delta f_k > 0\},\ B(P) = \{k : \Delta f_k < 0\}.$$

The numbers

$$p_f(a, b) = \sup\left\{ \sum_{k \in A(P)} \Delta f_k : P \in \wp[a, b] \right\}$$

and

$$n_f(a, b) = \sup\left\{ \sum_{k \in B(P)} |\Delta f_k| : P \in \wp[a, b] \right\}$$

are called respectively, the positive and negative variations of f on $[a, b]$. For each x in $(a, b]$. Let $V(x) = V_f(a, x)$, $p(x) = p_f(a, x)$, $n(x) = n_f(a, x)$, and let $V(a) = p(a) = n(a) = 0$. Show that we have:

(a) $V(x) = p(x) + n(x)$.

Proof: Given a partition P on $[a,x]$, then we have

$$\sum_{k=1}^{n} |\Delta f_k| = \sum_{k \in A(P)} |\Delta f_k| + \sum_{k \in B(P)} |\Delta f_k|$$
$$= \sum_{k \in A(P)} \Delta f_k + \sum_{k \in B(P)} |\Delta f_k|,$$

which implies that (taking supermum)

$$V(x) = p(x) + n(x). \quad *$$

Remark: The existence of $p(x)$ and $q(x)$ is clear, so we know that (*) holds by **Theorem 1.15.**

(b) $0 \le p(x) \le V(x)$ and $0 \le n(x) \le V(x)$.

Proof: Consider $[a,x]$, and since

$$V(x) \ge \sum_{k=1}^{n} |\Delta f_k| \ge \sum_{k \in A(P)} |\Delta f_k|,$$

we know that $0 \le p(x) \le V(x)$. Similarly for $0 \le n(x) \le V(x)$.

(c) p and n are increasing on $[a,b]$.

Proof: Let x,y in $[a,b]$ with $x < y$, and consider $p(y) - p(x)$ as follows. Since

$$p(y) \ge \sum_{k \in A(P),\ [a,y]} \Delta f_k \ge \sum_{k \in A(P),\ [a,x]} \Delta f_k,$$

we know that

$$p(y) \ge p(x).$$

That is, p is increasing on $[a,b]$. Similarly for n.

(d) $f(x) = f(a) + p(x) - n(x)$. Part (d) gives an alternative proof of Theorem 6.13.

Proof: Consider $[a,x]$, and since

$$f(x) - f(a) = \sum_{k=1}^{n} \Delta f_k = \sum_{k \in A(P)} \Delta f_k + \sum_{k \in B(P)} \Delta f_k$$

which implies that

$$f(x) - f(a) + \sum_{k \in B(P)} |\Delta f_k| = \sum_{k \in A(P)} \Delta f_k$$

which implies that $f(x) = f(a) + p(x) - n(x)$.

(e) $2p(x) = V(x) + f(x) - f(a)$, $2n(x) = V(x) - f(x) + f(a)$.

Proof: By (d) and (a), the statement is obvious.

(f) Every point of continuity of f is also a point of continuity of p and of n.

Proof: By (e) and **Theorem 6.14**, the statement is obvious.

Curves

6.8 Let f and g be complex-valued functions defined as follows:

$$f(t) = e^{2\pi i t} \text{ if } t \in [0,1],\ g(t) = e^{2\pi i t} \text{ if } t \in [0,2].$$

(a) Prove that f and g have the same graph but are not equivalent according to defintion in Section 6.12.

Proof: Since $\{f(t) : t \in [0,1]\} = \{g(t) : t \in [0,2]\}$ = the circle of unit disk, we know that f and g have the same graph.

If f and g are equivalent, then there is an 1-1 and onto function $\phi : [0,2] \to [0,1]$ such that

$$f(\phi(t)) = g(t).$$

That is,

$$e^{2\pi i\phi(t)} = \cos 2\pi(\phi(t)) + i\sin 2\pi(\phi(t)) = e^{2\pi it} = \cos 2\pi t + i\sin 2\pi t.$$

In paticular, $\phi(1) := c \in (0,1)$. However,

$$f(c) = \cos 2\pi c + i\sin 2\pi c = g(1) = 1$$

which implies that $c \in Z$, a contradiction.

(b) Prove that the length of g is twice that of f.

Proof: Since

$$\text{the length of } g = \int_0^2 |g'(t)|dt = 4\pi$$

and

$$\text{the length of } f = \int_0^1 |f'(t)|dt = 2\pi,$$

we know that the length of g is twice that of f.

6.9 Let f be rectifiable path of length L defined on $[a,b]$, and assume that f is not constant on any subinterval of $[a,b]$. Let s denote the arc length function given by $s(x) = \Lambda_f(a,x)$ if $a < x \leq b$, $s(a) = 0$.

(a) Prove that s^{-1} exists and is continuous on $[0,L]$.

Proof: By **Theorem 6.19**, we know that $s(x)$ is continuous and strictly increasing on $[0,L]$. So, the inverse function s^{-1} exists since s is an 1-1 and onto function, and by **Theorem 4.29**, we know that s^{-1} is continuous on $[0,L]$.

(b) Define $g(t) = f[s^{-1}(t)]$ if $t \in [0,L]$ and show that g is equivalent to f. Since $f(t) = g[s(t)]$, the function g is said to provide a representation of the graph of f **with arc length as parameter.**

Proof: t is clear by **Theorem 6.20**.

6.10 Let f and g be two real-valued continuous functions of bounded variation defined on $[a,b]$, with $0 < f(x) < g(x)$ for each x in (a,b), $f(a) = g(a)$, $f(b) = g(b)$. Let h be the complex-valued function defined on the interval $[a, 2b-a]$ as follows:

$$\begin{aligned} h(t) &= t + if(t), \text{ if } a \leq t \leq b \\ &= 2b - t + ig(2b-t), \text{ if } b \leq t \leq 2b-a. \end{aligned}$$

(a) Show that h describes a rectifiable curve Γ.

Proof: It is clear that h is continuous on $[a, 2b-a]$. Note that t, f and g are of bounded variation on $[a,b]$, so $\Lambda_h(a, 2b-a)$ exists. That is, h is rectifiable on $[a, 2b-a]$.

(b) Explain, by means of a sketch, the geometric relationship between f, g, and h.

Solution: The reader can give it a draw and see the graph lying on $x-y$ plane is a closed region.

(c) Show that the set of points

$$S = \{(x,y) : a \le x \le b, f(x) \le y \le g(x)\}$$

in a region in R^2 whose boundary is the curve Γ.

Proof: It can be answered by (b), so we omit it.

(d) Let H be the complex-valued function defined on $[a, 2b-a]$ as follows:

$$H(t) = t - \frac{1}{2}i[g(t) - f(t)], \text{ if } a \le t \le b$$
$$= 2b - t + \frac{1}{2}i[g(2b-t) - f(2b-t)], \text{ if } b \le t \le 2b-a.$$

Show that H describes a rectifiable curve Γ_0 which is the boundary of the region

$$S_0 = \{(x,y) : a \le x \le b, f(x) - g(x) \le 2y \le g(x) - f(x)\}.$$

Proof: Let $F(t) = \frac{-1}{2}[g(t) - f(t)]$ and $G(t) = \frac{1}{2}[g(t) - f(t)]$ defined on $[a,b]$. It is clear that $F(t)$ and $G(t)$ are of bounded variation and continuous on $[a,b]$ with $0 < F(x) < G(x)$ for each $x \in (a,b)$, $F(b) = G(b) = 0$, and $F(b) = G(b) = 0$. In addition, we have

$$H(t) = t + iF(t), \text{ if } a \le t \le b$$
$$= 2b - t + iG(2b-t), \text{ if } b \le t \le 2b-a.$$

So, by preceding (a)-(c), we have prove it.

(e) Show that, S_0 has the x –axis as a line of symmetry. (The region S_0 is called the symmetrization of S with respect to x –axis.)

Proof: It is clear since $(x,y) \in S_0 \Leftrightarrow (x,-y) \in S_0$ by the fact

$$f(x) - g(x) \le 2y \le g(x) - f(x).$$

(f) Show that the length of Γ_0 does not exceed the length of Γ.

Proof: By (e), the symmetrization of S with respect to x –axis tells that $\Lambda_H(a,b) = \Lambda_H(b, 2b-a)$. So, it suffices to show that $\Lambda_h(a, 2b-a) \ge 2\Lambda_H(a,b)$. Choosing a partition $P_1 = \{x_0 = a, \ldots, x_n = b\}$ on $[a,b]$ such that

$$\begin{aligned} 2\Lambda_H(a,b) - \varepsilon &< 2\Lambda_H(P_1) \\ &= 2\sum_{i=1}^{n}\left\{(x_i - x_{i-1})^2 + \left[\frac{1}{2}(f-g)(x_i) - \frac{1}{2}(f-g)(x_{i-1})\right]^2\right\}^{1/2} \\ &= \sum_{i=1}^{n}\left\{4(x_i - x_{i-1})^2 + [(f-g)(x_i) - (f-g)(x_{i-1})]^2\right\}^{1/2} \qquad * \end{aligned}$$

and note that $b - a = (2b-a) - b$, we use this P_1 to produce a partition $P_2 = P_1 \cup \{x_n = b, x_{n+1} = b + (x_n - x_{n-1}), \ldots, x_{2n} = 2b-a\}$ on $[a, 2b-a]$. Then we have

$$
\begin{aligned}
\Lambda_h(P_2) &= \sum_{i=1}^{2n} \| h(x_i) - h(x_{i-1}) \| \\
&= \sum_{i=1}^{n} \| h(x_i) - h(x_{i-1}) \| + \sum_{i=n+1}^{2n} \| h(x_i) - h(x_{i-1}) \| \\
&= \sum_{i=1}^{n} \left[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2 \right]^{1/2} + \sum_{i=n+1}^{2n} \left[(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2 \right]^{1/2} \\
&= \sum_{i=1}^{n} \left\{ \left[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2 \right]^{1/2} + \left[(x_i - x_{i-1})^2 + (g(x_i) - g(x_{i-1}))^2 \right]^{1/2} \right\} \qquad **
\end{aligned}
$$

From (*) and (**), we know that

$$2\Lambda_H(a,b) - \varepsilon < 2\Lambda_H(P_1) \le \Lambda_h(P_2) \qquad ***$$

which implies that

$$\Lambda_H(a, 2b-a) = 2\Lambda_H(a,b) \le \Lambda_h(a, 2b-a).$$

So, we know that the length of Γ_0 does not exceed the length of Γ.

Remark: Define $x_i - x_{i-1} = a_i$, $f(x_i) - f(x_{i-1}) = b_i$, and $g(x_i) - g(x_{i-1}) = c_i$, then we have

$$\left(4a_i^2 + (b_i - c_i)^2\right)^{1/2} \le (a_i^2 + b_i^2)^{1/2} + (a_i^2 + c_i^2)^{1/2}.$$

Hence we have the result (***).

Proof: It suffices to square both side. We leave it to the reader.

Absolutely continuous functions

A real-valued function f defined on $[a,b]$ is said to be **absolutely continuous** on $[a,b]$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every n **disjoint** open subintervals (a_k, b_k) of $[a,b]$, $n = 1,2,\ldots,$ the sum of whose lengths $\sum_{k=1}^{n}(b_k - a_k)$ is less than δ.

Absolutely continuous functions occur in the Lebesgue theory of integration and differentiation. The following exercises give some of their elementary properties.

6.11 Prove that every absolutely continuous function on $[a,b]$ is continuous and of bounded variation on $[a,b]$.

Proof: Let f be absolutely continuous on $[a,b]$. Then $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

for every n **disjoint** open subintervals (a_k, b_k) of $[a,b]$, $n = 1,2,\ldots,$ the sum of whose lengths $\sum_{k=1}^{n}(b_k - a_k)$ is less than δ. So, as $|x-y| < \delta$, where $x,y \in [a,b]$, we have

$$|f(x) - f(y)| < \varepsilon.$$

That is, f is uniformly continuous on $[a,b]$. So, f is continuous on $[a,b]$.

In addition, given any $\varepsilon = 1$, there exists a $\delta > 0$ such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals in $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < 1.$$

For this δ, and let K be the smallest positive integer such that $K(\delta/2) \geq b - a$. So, we partition $[a,b]$ into K closed subintervals, i.e.,
$P = \{y_0 = a,\ y_1 = a + \delta/2, \ldots, y_{K-1} = a + (K-1)(\delta/2),\ y_K = b\}$. So, it is clear that f is of bounded variation $[y_i, y_{i+1}]$, where $i = 0, 1, \ldots, K$. It implies that f is of bounded variation on $[a,b]$.

Note: **There exists functions which are continuous and of bounded variation but not absolutely continuous**.

Remark: 1. The standard example is called **Cantor-Lebesgue function**. The reader can see this in the book, **Measure and Integral, An Introduction to Real Analysis by Richard L. Wheeden and Antoni Zygmund**, pp 35 and pp 115.

2. If we wrtie "absolutely continuous" by **ABC**, "continuous" by **C,** and "bounded variation" by **B,** then it is clear that by preceding result, **ABC** implies **B** and **C**, and **B** and **C** do **NOT** imply **ABC**.

6.12 Prove that f is absolutely continuous if it satisfies a uniform Lipschitz condition of order 1 on $[a,b]$. (See Exercise 6.2)

Proof: Let f satisfy a uniform Lipschitz condition of order 1 on $[a,b]$, i.e., $|f(x) - f(y)| \leq M|x - y|$ where $x, y \in [a,b]$. Then given $\varepsilon > 0$, there is a $\delta = \varepsilon/M$ such that as $\sum_{k=1}^{n} (b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open subintervals on $[a,b]$, $k = 1, .., n$, we have

$$\begin{aligned}\sum_{k=1}^{n} |f(b_k) - f(a_k)| &\leq \sum_{k=1}^{n} M|b_k - a_k| \\ &= \sum_{k=1}^{n} M(b_k - a_k) \\ &< M\delta \\ &= \varepsilon.\end{aligned}$$

Hence, f is absolutely continuous on $[a,b]$.

6.13 If f and g are absolutely continunous on $[a,b]$, prove that each of the following is also: $|f|$, cf (c constant), $f + g$, $f \cdot g$; also f/g if g is bounded away from zero.

Proof: (1) ($|f|$ is absolutely continuous on $[a,b]$): Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n} (b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} ||f(b_k)| - |f(a_k)|| < \varepsilon. \qquad 1*$$

Since f is absolutely continunous on $[a,b]$, for this ε, there is a $\delta > 0$ such that as $\sum_{k=1}^{n} (b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon$$

which implies that (1*) holds by the following

$$\sum_{k=1}^{n} ||f(b_k)| - |f(a_k)|| \leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

So, we know that $|f|$ is absolutely continuous on $[a,b]$.

(2) (cf is absolutely continuous on $[a,b]$): If $c = 0$, it is clear. So, we may assume that $c \neq 0$. Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |cf(b_k) - cf(a_k)| < \varepsilon. \qquad 2^*$$

Since f is absolutely continunous on $[a,b]$, for this ε, there is a $\delta > 0$ such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon/|c|$$

which implies that (2*) holds by the following

$$\sum_{k=1}^{n} |cf(b_k) - cf(a_k)| = |c| \sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon.$$

So, we know that cf is absolutely continuous on $[a,b]$.

(3) ($f + g$ is absolutely continuous on $[a,b]$): Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |(f+g)(b_k) - (f+g)(a_k)| < \varepsilon. \qquad 3^*$$

Since f and g are absolutely continunous on $[a,b]$, for this ε, there is a $\delta > 0$ such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \varepsilon/2 \text{ and } \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \varepsilon/2$$

which implies that (3*) holds by the following

$$\begin{aligned}
&\sum_{k=1}^{n} |(f+g)(b_k) - (f+g)(a_k)| \\
&= \sum_{k=1}^{n} |f(b_k) - f(a_k) + g(b_k) - g(a_k)| \\
&\leq \sum_{k=1}^{n} |f(b_k) - f(a_k)| + \sum_{k=1}^{n} |g(b_k) - g(a_k)| \\
&< \varepsilon.
\end{aligned}$$

So, we know that $f + g$ is absolutely continuous on $[a,b]$.

(4) ($f \cdot g$ is absolutely continuous on $[a,b]$.): Let $M_f = \sup_{x \in [a,b]} |f(x)|$ and $M_g = \sup_{x \in [a,b]} |g(x)|$. Given $\varepsilon > 0$, we want to find a $\delta > 0$, such that as $\sum_{k=1}^{n}(b_k - a_k) < \delta$, where $(a_k, b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|(f+g)(b_k)-(f+g)(a_k)|<\varepsilon. \qquad 4^*$$

Since f and g are absolutely continunous on $[a,b]$, for this ε, there is a $\delta>0$ such that as $\sum_{k=1}^{n}(b_k-a_k)<\delta$, where $(a_k,b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|f(b_k)-f(a_k)|<\frac{\varepsilon}{2(M_g+1)} \text{ and } \sum_{k=1}^{n}|g(b_k)-g(a_k)|<\frac{\varepsilon}{2(M_f+1)}$$

which implies that (4*) holds by the following

$$\begin{aligned}
&\sum_{k=1}^{n}|(f\cdot g)(b_k)-(f\cdot g)(a_k)|\\
&=\sum_{k=1}^{n}|f(b_k)(g(b_k)-g(a_k))+g(a_k)(f(b_k)-f(a_k))|\\
&\le M_f\sum_{k=1}^{n}|g(b_k)-g(a_k)|+M_g\sum_{k=1}^{n}|f(b_k)-f(a_k)|\\
&<\frac{\varepsilon M_f}{2(M_f+1)}+\frac{\varepsilon M_g}{2(M_g+1)}\\
&<\varepsilon.
\end{aligned}$$

Remark: The part shows that f^n is absolutely continuous on $[a,b]$, where $n\in N$, if f is absolutely continuous on $[a,b]$.

(5) (f/g is absolutely continuous on $[a,b]$): By (4) it suffices to show that $1/g$ is absolutely continuous on $[a,b]$. Since g is bounded away from zero, say $0<m\le g(x)$ for all $x\in[a,b]$. Given $\varepsilon>0$, we want to find a $\delta>0$, such that as $\sum_{k=1}^{n}(b_k-a_k)<\delta$, where $(a_k,b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|(1/g)(b_k)-(1/g)(a_k)|<\varepsilon. \qquad 5^*$$

Since g is absolutely continunous on $[a,b]$, for this ε, there is a $\delta>0$ such that as $\sum_{k=1}^{n}(b_k-a_k)<\delta$, where $(a_k,b_k)'s$ are disjoint open intervals on $[a,b]$, we have

$$\sum_{k=1}^{n}|g(b_k)-g(a_k)|<m^2\varepsilon$$

which implies that (4*) holds by the following

$$\begin{aligned}
&\sum_{k=1}^{n}|(1/g)(b_k)-(1/g)(a_k)|\\
&=\sum_{k=1}^{n}\left|\frac{g(b_k)-g(a_k)}{g(b_k)g(a_k)}\right|\\
&\le\frac{1}{m^2}\sum_{k=1}^{n}|g(b_k)-g(a_k)|\\
&<\varepsilon.
\end{aligned}$$

Supplement on lim sup and lim inf

Introduction

In order to make us understand the information more on approaches of a given real sequence $\{a_n\}_{n=1}^{\infty}$, we give two definitions, thier names are upper limit and lower limit. It is fundamental but important tools in analysis. We do **NOT** give them proofs. The reader can see the book, **Infinite Series by Chao Wen-Min, pp 84-103. (Chinese Version)**

Definition of limit sup and limit inf

Definition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, we define

$$b_n = \sup\{a_m : m \geq n\}$$

and

$$c_n = \inf\{a_m : m \geq n\}.$$

Example $\{1 + (-1)^n\}_{n=1}^{\infty} = \{0, 2, 0, 2, \ldots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, \ldots\}$, so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1, -2, -3, \ldots\}$, so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n.$$

Proposition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, and thus define b_n and c_n as the same as before.

1. $b_n \neq -\infty$, and $c_n \neq \infty$ $\forall n \in N$.
2. If there is a positive integer p such that $b_p = +\infty$, then $b_n = +\infty$ $\forall n \in N$. If there is a positive integer q such that $c_q = -\infty$, then $c_n = -\infty$ $\forall n \in N$.
3. $\{b_n\}$ is decreasing and $\{c_n\}$ is increasing.

By property 3, we can give definitions on the upper limit and the lower limit of a given sequence as follows.

Definition Given a real sequence $\{a_n\}$ and let b_n and c_n as the same as before.

(1) If every $b_n \in R$, then

$$\inf\{b_n : n \in N\}$$

is called the upper limit of $\{a_n\}$, denoted by

$$\lim_{n\to\infty} \sup a_n.$$

That is,

$$\lim_{n\to\infty} \sup a_n = \inf_n b_n.$$

If every $b_n = +\infty$, then we define

$$\lim_{n\to\infty} \sup a_n = +\infty.$$

(2) If every $c_n \in R$, then

$$\sup\{c_n : n \in N\}$$

is called the lower limit of $\{a_n\}$, denoted by

$$\lim_{n\to\infty} \inf a_n.$$

That is,

$$\lim_{n\to\infty} \inf a_n = \sup_n c_n.$$

If every $c_n = -\infty,$ then we define

$$\lim_{n\to\infty} \inf a_n = -\infty.$$

Remark The concept of lower limit and upper limit first appear in the book (**Analyse Alge'brique**) written by **Cauchy** in 1821. But until 1882, **Paul du Bois-Reymond** gave explanations on them, it becomes well-known.

Example $\{1 + (-1)^n\}_{n=1}^{\infty} = \{0, 2, 0, 2, \dots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n$$

which implies that

$$\lim \sup a_n = 2 \text{ and } \lim \inf a_n = 0.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1, 2, -3, 4, \dots\},$ so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim \sup a_n = +\infty \text{ and } \lim \inf a_n = -\infty.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1, -2, -3, \dots\},$ so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim \sup a_n = -\infty \text{ and } \lim \inf a_n = -\infty.$$

Relations with convergence and divergence for upper (lower) limit

Theorem Let $\{a_n\}$ be a real sequence, then $\{a_n\}$ converges if, and only if, the upper limit and the lower limit are real with

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} a_n.$$

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \sup a_n = +\infty \Leftrightarrow \{a_n\}$ has no upper bound.

(2) $\lim_{n\to\infty} \sup a_n = -\infty \Leftrightarrow$ for any $M > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n \leq -M.$$

(3) $\lim_{n\to\infty} \sup a_n = a$ if, and only if, (a) given any $\varepsilon > 0,$ there are infinite many numbers n such that

$$a - \varepsilon < a_n$$

and (b) given any $\varepsilon > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n < a + \varepsilon.$$

Similarly, we also have

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \inf a_n = -\infty \Leftrightarrow \{a_n\}$ has no lower bound.

(2) $\lim_{n\to\infty} \inf a_n = +\infty \Leftrightarrow$ for any $M > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n \geq M.$$

(3) $\lim_{n\to\infty} \inf a_n = a$ if, and only if, (a) given any $\varepsilon > 0,$ there are infinite many numbers n such that

$$a + \varepsilon > a_n$$

and (b) given any $\varepsilon > 0,$ there is a positive integer n_0 such that as $n \geq n_0,$ we have

$$a_n > a - \varepsilon.$$

From Theorem 2 an Theorem 3, the sequence is divergent, we give the following definitios.

Definition Let $\{a_n\}$ be a real sequence, then we have

(1) If $\lim_{n\to\infty} \sup a_n = -\infty,$ then we call the sequence $\{a_n\}$ diverges to $-\infty$, denoted by

$$\lim_{n\to\infty} a_n = -\infty.$$

(2) If $\lim_{n\to\infty} \inf a_n = +\infty,$ then we call the sequence $\{a_n\}$ diverges to $+\infty$, denoted by

$$\lim_{n\to\infty} a_n = +\infty.$$

Theorem Let $\{a_n\}$ be a real sequence. If a is a limit point of $\{a_n\},$ then we have

$$\lim_{n\to\infty} \inf a_n \leq a \leq \lim_{n\to\infty} \sup a_n.$$

Some useful results

Theorem Let $\{a_n\}$ be a real sequence, then

(1) $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n.$

(2) $\lim_{n\to\infty} \inf(-a_n) = -\lim_{n\to\infty} \sup a_n$ and $\lim_{n\to\infty} \sup(-a_n) = -\lim_{n\to\infty} \inf a_n$

(3) If every $a_n > 0,$ and $0 < \lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n < +\infty,$ then we have

$$\lim_{n\to\infty} \sup \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \inf a_n} \text{ and } \lim_{n\to\infty} \inf \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \sup a_n}.$$

Theorem Let $\{a_n\}$ and $\{b_n\}$be two real sequences.

(1) If there is a positive integer n_0 such that $a_n \leq b_n,$ then we have

$$\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \inf b_n \text{ and } \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup b_n.$$

(2) Suppose that $-\infty < \lim_{n\to\infty} \inf a_n,\ \lim_{n\to\infty} \inf b_n,\ \lim_{n\to\infty} \sup a_n,$ $\lim_{n\to\infty} \sup b_n < +\infty,$ then

$$\begin{aligned}
&\lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \inf b_n \\
&\leq \lim_{n\to\infty} \inf(a_n + b_n) \\
&\leq \lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \sup b_n \text{ (or } \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \inf b_n \text{)} \\
&\leq \lim_{n\to\infty} \sup(a_n + b_n) \\
&\leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n.
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty} \sup(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \sup b_n$$

and

$$\lim_{n\to\infty} \inf(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \inf b_n.$$

(3) Suppose that $-\infty < \lim_{n\to\infty} \inf a_n,\ \lim_{n\to\infty} \inf b_n,\ \lim_{n\to\infty} \sup a_n,$ $\lim_{n\to\infty} \sup b_n < +\infty,$ and $a_n > 0,\ b_n > 0\ \forall n,$ then

$$\begin{aligned}
&\Big(\lim_{n\to\infty} \inf a_n\Big)\Big(\lim_{n\to\infty} \inf b_n\Big) \\
&\le \lim_{n\to\infty} \inf(a_n b_n) \\
&\le \Big(\lim_{n\to\infty} \inf a_n\Big)\Big(\lim_{n\to\infty} \sup b_n\Big) \text{ (or } \Big(\lim_{n\to\infty} \inf b_n\Big)\Big(\lim_{n\to\infty} \sup a_n\Big)) \\
&\le \lim_{n\to\infty} \sup(a_n b_n) \\
&\le \Big(\lim_{n\to\infty} \sup a_n\Big)\Big(\lim_{n\to\infty} \sup b_n\Big).
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty} \sup(a_n b_n) = \Big(\lim_{n\to\infty} a_n\Big) \lim_{n\to\infty} \sup b_n$$

and

$$\lim_{n\to\infty} \inf(a_n + b_n) = \Big(\lim_{n\to\infty} a_n\Big) \lim_{n\to\infty} \inf b_n.$$

Theorem Let $\{a_n\}$ be a **positive** real sequence, then

$$\lim_{n\to\infty} \inf \frac{a_{n+1}}{a_n} \le \lim_{n\to\infty} \inf(a_n)^{1/n} \le \lim_{n\to\infty} \sup(a_n)^{1/n} \le \lim_{n\to\infty} \sup \frac{a_{n+1}}{a_n}.$$

Remark We can use the inequalities to show

$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = 1/e.$$

Theorem Let $\{a_n\}$ be a real sequence, then

$$\lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \inf \frac{a_1 + \ldots + a_n}{n} \le \lim_{n\to\infty} \sup \frac{a_1 + \ldots + a_n}{n} \le \lim_{n\to\infty} \sup a_n.$$

Exercise Let $f : [a, d] \to R$ be a continuous function, and $\{a_n\}$ is a real sequence. If f is increasing and for every $n,$ $\lim_{n\to\infty} \inf a_n,\ \lim_{n\to\infty} \sup a_n \in [a, d],$ then

$$\lim_{n\to\infty} \sup f(a_n) = f\Big(\lim_{n\to\infty} \sup a_n\Big) \text{ and } \lim_{n\to\infty} \inf f(a_n) = f\Big(\lim_{n\to\infty} \inf a_n\Big).$$

Remark: (1) The condition that f is increasing cannot be removed. For example,

$$f(x) = |x|,$$

and

$$a_k = \begin{cases} 1/k \text{ if } k \text{ is even} \\ -1 - 1/k \text{ if } k \text{ is odd.} \end{cases}$$

(2) The proof is easy if we list the definition of limit sup and limit inf. So, we omit it.

Exercise Let $\{a_n\}$ be a real sequence satisfying $a_{n+p} \le a_n + a_p$ for all $n,\ p.$ Show that $\{\frac{a_n}{n}\}$ converges.

Hint: Consider its limit inf.

Remark: The exercise is useful in the theory of **Topological Entorpy**.

Infinite Series And Infinite Products

Sequences

8.1 (a) Given a real-valed sequence $\{a_n\}$ bounded above, let $u_n = \sup\{a_k : k \geq n\}$. Then $u_n \searrow$ and hence $U = \lim_{n\to\infty} u_n$ is either finite or $-\infty$. Prove that
$$U = \lim_{n\to\infty}\sup a_n = \lim_{n\to\infty}(\sup\{a_k : k \geq n\}).$$

Proof: It is clear that $u_n \searrow$ and hence $U = \lim_{n\to\infty} u_n$ is either finite or $-\infty$.

If $U = -\infty$, then given any $M > 0$, there exists a positive integer N such that as $n \geq N$, we have
$$u_n \leq -M$$
which implies that, as $n \geq N$, $a_n \leq -M$. So, $\lim_{n\to\infty} a_n = -\infty$. That is, $\{a_n\}$ is not bounded below. In addition, if $\{a_n\}$ has a finite limit supreior, say a. Then given $\varepsilon > 0$, and given $m > 0$, there exists an integer $n > m$ such that
$$a_n > a - \varepsilon$$
which contradicts to $\lim_{n\to\infty} a_n = -\infty$. From above results, we obtain
$$U = \lim_{n\to\infty}\sup a_n$$
in the case of $U = -\infty$.

If U is finite, then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have
$$U \leq u_n < U + \varepsilon.$$
So, as $n \geq N$, $u_n < U + \varepsilon$ which implies that, as $n \geq N$, $a_n < U + \varepsilon$. In addition, given $\varepsilon' > 0$, and $m > 0$, there exists an integer $n > m$,
$$U - \varepsilon' < a_n$$
by $U \leq u_n = \sup\{a_k : k \geq n\}$ if $n \geq N$. From above results, we obtain
$$U = \lim_{n\to\infty}\sup a_n$$
in the case of U is finite.

(b)Similarly, if $\{a_n\}$ is bounded below, prove that
$$V = \lim_{n\to\infty}\inf a_n = \lim_{n\to\infty}(\inf\{a_k : k \geq n\}).$$

Proof: Since the proof is similar to (a), we omit it.

If U and V are finite, show that:

(c) There exists a subsequence of $\{a_n\}$ which converges to U and a subsequence which converges to V.

Proof: Since $U = \lim\sup_{n\to\infty} a_n$ by (a), then

(i) Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have
$$a_n < U + \varepsilon.$$

(ii) Given $\varepsilon > 0$, and $m > 0$, there exists an integer $P(m) > m$,
$$U - \varepsilon < a_{P(m)}.$$
Hence, $\{a_{P(m)}\}$ is a convergent subsequence of $\{a_n\}$ with limit U.

Similarly for the case of V.

(d) If $U = V,$ every subsequnce of $\{a_n\}$ converges to U.

Proof: By (a) and (b), given $\varepsilon > 0,$ then there exists a positive integer N_1 such that as $n \geq N_1,$ we have

$$a_n < U + \varepsilon$$

and there exists a positive integer N_2 such that as $n \geq N_2,$ we have

$$U - \varepsilon < a_n.$$

Hence, as $n \geq \max(N_1, N_2),$ we have

$$U - \varepsilon < a_n < U + \varepsilon.$$

That is, $\{a_n\}$ is a convergent sequence with limit U. So, every subsequnce of $\{a_n\}$ converges to U.

8.2 Given two real-valed sequence $\{a_n\}$ and $\{b_n\}$ bounded below. Prove hat
(a) $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.

Proof: Note that $\{a_n\}$ and $\{b_n\}$ bounded below, we have $\limsup_{n\to\infty} a_n = +\infty$ or is finite. And $\limsup_{n\to\infty} b_n = +\infty$ or is finite. It is clear if one of these limit superior is $+\infty$, so we may assume that both are finite. Let $a = \limsup_{n\to\infty} a_n$ and $b = \limsup_{n\to\infty} b_n$. Then given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n + b_n < a + b + \varepsilon/2. \qquad *$$

In addition, let $c = \limsup_{n\to\infty}(a_n + b_n),$ where $c < +\infty$ by (*). So, for the same $\varepsilon > 0,$ and given $m = N$ there exists a positive integer K such that as $K \geq N,$ we have

$$c - \varepsilon/2 < a_K + b_K. \qquad **$$

By (*) and (**), we obtain that

$$c - \varepsilon/2 < a_K + b_K < a + b + \varepsilon/2$$

which implies that

$$c \leq a + b$$

since ε is arbitrary. So,

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Remark: (1) The equality may **NOT** hold. For example,

$$a_n = (-1)^n \text{ and } b_n = (-1)^{n+1}.$$

(2) The reader should noted that the finitely many terms does **NOT** change the relation of order. The fact is based on process of proof.

(b) $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$ if $a_n > 0,$ $b_n > 0$ for all $n,$ and if both $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$ are finite or both are infinite.

Proof: Let $\limsup_{n\to\infty} a_n = a$ and $\limsup_{n\to\infty} b_n = b$. It is clear that we may assume that a and b are finite. Given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n b_n < (a + \varepsilon)(b + \varepsilon) = ab + \varepsilon(a + b + \varepsilon). \qquad *$$

In addition, let $c = \limsup_{n\to\infty}(a_n b_n),$ where $c < +\infty$ by (*). So, for the same $\varepsilon > 0,$ and given $m = N$ there exists a positive integer K such that as $K \geq N,$ we have

$$c - \varepsilon < a_K + b_K. \qquad **$$

By (*) and (**), we obtain that

$$c - \varepsilon < a_K + b_K < a + b + \varepsilon(a + b + \varepsilon)$$

which implies that

$$c \le a + b$$

since ε is arbitrary. So,

$$\limsup_{n\to\infty}(a_n b_n) \le \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right).$$

Remark: (1) The equality may **NOT** hold. For example,

$$a_n = 1/n \text{ if } n \text{ is odd and } a_n = 1 \text{ if } n \text{ is even.}$$

and

$$b_n = 1 \text{ if } n \text{ is odd and } b_n = 1/n \text{ if } n \text{ is even.}$$

(2) The reader should noted that the finitely many terms does **NOT** change the relation of order. The fact is based on the process of the proof.

(3) The reader should be noted that if letting $A_n = \log a_n$ and $B_n = \log b_n$, then by (a) and $\log x$ is an increasing function on $(0, +\infty)$, we have proved (b).

8.3 Prove that Theorem 8.3 and 8.4.

(**Theorem 8.3**) Let $\{a_n\}$ be a sequence of real numbers. Then we have:

(a) $\liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n$.

Proof: If $\limsup_{n\to\infty} a_n = +\infty$, then it is clear. We may assume that $\limsup_{n\to\infty} a_n < +\infty$. Hence, $\{a_n\}$ is bounded above. We consider two cases: (i) $\limsup_{n\to\infty} a_n = a$, where a is finite and (ii) $\limsup_{n\to\infty} a_n = -\infty$.

For case (i), if $\liminf_{n\to\infty} a_n = -\infty$, then there is nothing to prove it. We may assume that $\liminf_{n\to\infty} a_n = a'$, where a' is finite. By definition of limit superior and limit inferior, given $\varepsilon > 0$, there exists a positive integer N such that as $n \ge N$, we have

$$a' - \varepsilon/2 < a_n < a + \varepsilon/2$$

which implies that $a' \le a$ since ε is arbitrary.

For case (ii), since $\limsup_{n\to\infty} a_n = -\infty$, we have $\{a_n\}$ is not bounded below. If $\liminf_{n\to\infty} a_n = -\infty$, then there is nothing to prove it. We may assume that $\liminf_{n\to\infty} a_n = a'$, where a' is finite. By definition of limit inferior, given $\varepsilon > 0$, there exists a positive integer N such that as $n \ge N$, we have

$$a' - \varepsilon/2 < a_n$$

which contradicts that $\{a_n\}$ is not bounded below.

So, from above results, we have proved it.

(b) The sequence converges if and only if, $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ are both finite and equal, in which case $\lim_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Proof: ($\Rightarrow$)Given $\{a_n\}$ a convergent sequence with limit a. So, given $\varepsilon > 0$, there exists a positive integer N such that as $n \ge N$, we have

$$a - \varepsilon < a_n < a + \varepsilon.$$

By definition of limit superior and limit inferior, $a = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

($\Leftarrow$)By definition of limit superior, given $\varepsilon > 0$, there exists a positive integer N_1 such that as $n \ge N_1$, we have

$$a_n < a + \varepsilon$$

and by definition of limit superior, given $\varepsilon > 0$, there exists a positive integer N_2 such that as $n \ge N_2$, we have

$$a - \varepsilon < a_n.$$

So, as $n \geq \max(N_1, N_2)$, we have

$$a - \varepsilon < a_n < a + \varepsilon.$$

That is, $\lim_{n\to\infty} a_n = a$.

(c) The sequence diverges to $+\infty$ if and only if, $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = +\infty$.

Proof: ($\Rightarrow$)Given a sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = +\infty$. So, given $M > 0$, there is a positive integer N such that as $n \geq N$, we have

$$M \leq a_n. \qquad *$$

It implies that $\{a_n\}$ is not bounded above. So, $\limsup_{n\to\infty} a_n = +\infty$. In order to show that $\liminf_{n\to\infty} a_n = +\infty$. We first note that $\{a_n\}$ is bounded below. Hence, $\liminf_{n\to\infty} a_n \neq -\infty$. So, it suffices to consider that $\liminf_{n\to\infty} a_n$ is not finite. (So, we have $\liminf_{n\to\infty} a_n = +\infty$.). Assume that $\liminf_{n\to\infty} a_n = a$, where a is finite. Then given $\varepsilon = 1$, and an integer m, there exists a positive $K(m) > m$ such that

$$a_{K(m)} < a + 1$$

which contradicts to (*) if we choose $M = a + 1$. So, $\liminf_{n\to\infty} a_n$ is not finite.

(d) The sequence diverges to $-\infty$ if and only if, $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = -\infty$.

Proof: Note that, $\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n$. So, by (c), we have proved it.

(**Theorem 8.4**)Assume that $a_n \leq b_n$ for each $n = 1, 2, \ldots$. Then we have:

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n \text{ and } \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n.$$

Proof: If $\liminf_{n\to\infty} b_n = +\infty$, there is nothing to prove it. So, we may assume that $\liminf_{n\to\infty} b_n < +\infty$. That is, $\liminf_{n\to\infty} b_n = -\infty$ or b, where b is finite.

For the case, $\liminf_{n\to\infty} b_n = -\infty$, it means that the sequence $\{a_n\}$ is not bounded below. So, $\{b_n\}$ is also not bounded below. Hence, we also have $\liminf_{n\to\infty} a_n = -\infty$.

For the case, $\liminf_{n\to\infty} b_n = b$, where b is finite. We consider three cases as follows.

(i) if $\liminf_{n\to\infty} a_n = -\infty$, then there is nothing to prove it.

(ii) if $\liminf_{n\to\infty} a_n = a$, where a is finite. Given $\varepsilon > 0$, then there exists a positive integer N such that as $n \geq N$

$$a - \varepsilon/2 < a_n \leq b_n < b + \varepsilon/2$$

which implies that $a \leq b$ since ε is arbitrary.

(iii) if $\liminf_{n\to\infty} a_n = +\infty$, then by **Theorem 8.3 (a) and (c)**, we know that $\lim_{n\to\infty} a_n = +\infty$ which implies that $\lim_{n\to\infty} b_n = +\infty$. Also, by **Theorem 8.3 (c)**, we have $\liminf_{n\to\infty} b_n = +\infty$ which is absurb.

So, by above results, we have proved that $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Similarly, we have $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.

8.4 If each $a_n > 0$, prove that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} (a_n)^{1/n} \leq \limsup_{n\to\infty} (a_n)^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Proof: By **Theorem 8.3 (a)**, it suffices to show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} (a_n)^{1/n} \text{ and } \limsup_{n\to\infty} (a_n)^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

We first prove

$$\limsup_{n\to\infty} (a_n)^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

If $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty,$ then it is clear. In addition, since $\frac{a_{n+1}}{a_n}$ is positive, $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} \neq -\infty.$ So, we may assume that $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = a,$ where a is finite.

Given $\varepsilon > 0,$ then there exists a positive integer N such that as $n \geq N,$ we have

$$\frac{a_{n+1}}{a_n} < a + \varepsilon$$

which implies that

$$a_{N+k} < a_N(a+\varepsilon)^k, \text{ where } k = 1,2,\ldots.$$

So,

$$(a_{N+k})^{\frac{1}{N+k}} < (a_N)^{\frac{1}{N+k}}(a+\varepsilon)^{\frac{k}{N+k}}$$

which implies that

$$\begin{aligned}\lim_{k\to\infty}\sup(a_{N+k})^{\frac{1}{N+k}} &\leq \lim_{k\to\infty}\sup(a_N)^{\frac{1}{N+k}}(a+\varepsilon)^{\frac{k}{N+k}} \\ &= a+\varepsilon.\end{aligned}$$

So,

$$\lim_{k\to\infty}\sup(a_{N+k})^{\frac{1}{N+k}} \leq a$$

since ε is arbitrary. Note that the finitely many terms do **NOT** change the value of limit superiror of a given sequence. So, we finally have

$$\lim_{n\to\infty}\sup(a_n)^{1/n} \leq a = \lim_{n\to\infty}\sup\frac{a_{n+1}}{a_n}.$$

Similarly for

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n} \leq \lim_{n\to\infty}\inf(a_n)^{1/n}.$$

Remark: These ineqaulities is much important; we suggest that the reader keep it mind. At the same time, these inequalities tells us that **the root test is more powerful than the ratio test**. We give an example to say this point. Given a series

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$$

where

$$a_{2n-1} = \left(\frac{1}{2}\right)^n, \text{ and } a_{2n} = \left(\frac{1}{3}\right)^n, \; n = 1,2,\ldots$$

with

$$\lim_{n\to\infty}\sup(a_n)^{1/n} = \sqrt{\frac{1}{2}} < 1$$

and

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n} = 0, \; \lim_{n\to\infty}\sup\frac{a_{n+1}}{a_n} = +\infty.$$

8.5 Let $a_n = n^n/n!.$ Show that $\lim_{n\to\infty} a_{n+1}/a_n = e$ and use Exercise 8.4 to deduce that

$$\lim_{n\to\infty}\frac{n}{(n!)^{1/n}} = e.$$

Proof: Since

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}n!}{(n+1)!n^n} = \left(1+\frac{1}{n}\right)^n \to e,$$

by **Exercise 8.4**, we have

$$\lim(a_n)^{1/n} = \lim_{n\to\infty}\frac{n}{(n!)^{1/n}} = e.$$

Remark: There are many methods to show this. We do **NOT** give the detailed proof. But there are hints.

(1) Taking log on $\left(\frac{n!}{n^n}\right)^{1/n}$, and thus consider

$$\frac{1}{n}\left(\log\frac{1}{n}+\ldots+\log\frac{n}{n}\right)\to\int_0^1\log x dx=-1.$$

(2) **Stirling's Formula**:

$$n!=n^n e^{-n}\sqrt{2\pi n}\,e^{\frac{\theta}{12n}},\text{ where }\theta\in(0,1).$$

Note: In general, we have

$$\lim_{x\to+\infty}\frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}}=1,$$

where $\Gamma(x)$ is the **Gamma Function**. The reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 192-195.**

(3) Note that $(1+\frac{1}{x})^x\nearrow e$ and $(1+\frac{1}{x})^{x+1}\searrow e$ on $(0,\infty)$. So,

$$\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1}$$

which implies that

$$e(n^n e^{-n})<n!<e(n^{n+1}e^{-n}).$$

(4) Using **O-Stolz's Theorem:** Let $\lim_{n\to\infty}y_n=+\infty$ and $y_n\nearrow$. If

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=a,\text{ where }a\text{ is finite or }+\infty,$$

then

$$\lim_{n\to\infty}\frac{x_n}{y_n}=a.$$

Let $x_n=\log\frac{1}{n}+\ldots+\log\frac{n}{n}$ and $y_n=n$.

Note: For the proof of **O-Stolz's Theorem,** the reader can see the book, **An Introduction to Mathematical Analysis by Loo-Keng Hua, pp 195. (Chinese Version)**

(5) Note that, if $\{a_n\}$ is a positive sequence with $\lim_{n\to\infty}a_n=a$, then

$$(a_1\cdots a_n)^{1/n}\to a\text{ as }n\to\infty.$$

Taking $a_n=(1+\frac{1}{n})^n$, then

$$(a_1\cdots a_n)^{1/n}=\left(\frac{n^n}{n!}\right)^{1/n}\left(1+\frac{1}{n}\right)\to e.$$

Note: For the proof, it is easy from the **Exercise 8.6**. We give it a proof as follows. Say $\lim_{n\to\infty}a_n=a$. If $a=0$, then by $A.P.\geq G.P.$, we have

$$(a_1\cdots a_n)^{1/n}\leq\frac{a_1+\ldots+a_n}{n}\to 0\text{ by } \textbf{Exercise 8.6}.$$

So, we consider $a\neq 0$ as follows. Note that $\log a_n\to\log a$. So, by **Exercise 8.6,**

$$\frac{\log a_1+\ldots+\log a_n}{n}\to\log a$$

which implies that $(a_1\cdots a_n)^{1/n}\to a$.

8.6 Let $\{a_n\}$ be real-valued sequence and let $\sigma_n=(a_1+\ldots+a_n)/n$. Show that

$$\varliminf_{n\to\infty}a_n\leq\varliminf_{n\to\infty}\sigma_n\leq\varlimsup_{n\to\infty}\sigma_n\leq\varlimsup_{n\to\infty}a_n.$$

Proof: By **Theorem 8.3 (a)**, it suffices to show that

$$\varliminf_{n\to\infty}a_n\leq\varliminf_{n\to\infty}\sigma_n\text{ and }\varlimsup_{n\to\infty}\sigma_n\leq\varlimsup_{n\to\infty}a_n.$$

We first prove

$$\lim_{n\to\infty} \sup \sigma_n \leq \lim_{n\to\infty} \sup a_n.$$

If $\lim\sup_{n\to\infty} a_n = +\infty$, there is nothing to prove it. We may assume that $\lim\sup_{n\to\infty} a_n = -\infty$ or a, where a is finite.

For the case, $\lim\sup_{n\to\infty} a_n = -\infty$, then by **Theorem 8.3 (d)**, we have

$$\lim_{n\to\infty} a_n = -\infty.$$

So, given $M > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$a_n \leq -M. \qquad *$$

Let $n > N$, we have

$$\begin{aligned}\sigma_n &= \frac{(a_1 + \ldots + a_N) + \ldots + a_n}{n} \\ &= \frac{a_1 + \ldots + a_N}{n} + \frac{a_{N+1} + \ldots + a_n}{n} \\ &\leq \frac{a_1 + \ldots + a_N}{n} + \left(\frac{n-N}{n}\right)(-M)\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \sup \sigma_n \leq -M.$$

Since M is arbitrary, we finally have

$$\lim_{n\to\infty} \sup \sigma_n = -\infty.$$

For the case, $\lim\sup_{n\to\infty} a_n = a$, where a is finite. Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$a_n < a + \varepsilon.$$

Let $n > N$, we have

$$\begin{aligned}\sigma_n &= \frac{(a_1 + \ldots + a_N) + \ldots + a_n}{n} \\ &= \frac{a_1 + \ldots + a_N}{n} + \frac{a_{N+1} + \ldots + a_n}{n} \\ &\leq \frac{a_1 + \ldots + a_N}{n} + \left(\frac{n-N}{n}\right)(a + \varepsilon)\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \sup \sigma_n \leq a + \varepsilon$$

which implies that

$$\lim_{n\to\infty} \sup \sigma_n \leq a$$

since ε is arbitrary.

Hence, from above results, we have proved that $\lim\sup_{n\to\infty} \sigma_n \leq \lim\sup_{n\to\infty} a_n$.

Similarly for $\lim\inf_{n\to\infty} a_n \leq \lim\inf_{n\to\infty} \sigma_n$.

Remark: We suggest that the reader keep it in mind since it is the fundamental and useful in the **theory of Fourier Series.**

8.7 Find $\lim\sup_{n\to\infty} a_n$ and $\lim\inf_{n\to\infty} a_n$ if a_n is given by

(a) $\cos n$

Proof: Note that, $\{a + b\pi : a, b \in Z\}$ is dense in R. By $\cos n = \cos(n + 2k\pi)$, we know that

$$\lim_{n\to\infty} \sup \cos n = 1 \text{ and } \lim_{n\to\infty} \inf \cos n = -1.$$

Remark: The reader may give it a try to show that

$$\lim_{n\to\infty}\sup \sin n = 1 \text{ and } \lim_{n\to\infty}\inf \sin n = -1.$$

(b) $(1+\frac{1}{n})\cos n\pi$

Proof: Note that

$$\left(1+\frac{1}{n}\right)\cos n\pi = \begin{cases} 1 \text{ if } n = 2k \\ -1 \text{ if } n = 2k-1 \end{cases}.$$

So, it is clear that

$$\lim_{n\to\infty}\sup\left(1+\frac{1}{n}\right)\cos n\pi = 1 \text{ and } \lim_{n\to\infty}\inf\left(1+\frac{1}{n}\right)\cos n\pi = -1.$$

(c) $n\sin\frac{n\pi}{3}$

Proof: Note that as $n = 1+6k$, $n\sin\frac{n\pi}{3} = (1+6k)\sin\frac{\pi}{3}$, and as $n = 4+6k$, $n = -(4+6k)\sin\frac{\pi}{3}$. So, it is clear that

$$\lim_{n\to\infty}\sup n\sin\frac{n\pi}{3} = +\infty \text{ and } \lim_{n\to\infty}\inf n\sin\frac{n\pi}{3} = -\infty.$$

(d) $\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}$

Proof: Note that $\sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = \frac{1}{2}\sin n\pi = 0$, we have

$$\lim_{n\to\infty}\sup \sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = \lim_{n\to\infty}\inf \sin\frac{n\pi}{2}\cos\frac{n\pi}{2} = 0.$$

(e) $(-1)^n n/(1+n)^n$

Proof: Note that

$$\lim_{n\to\infty}(-1)^n n/(1+n)^n = 0,$$

we know that

$$\lim_{n\to\infty}\sup(-1)^n n/(1+n)^n = \lim_{n\to\infty}\inf(-1)^n n/(1+n)^n = 0.$$

(f) $\frac{n}{3} - [\frac{n}{3}]$

Proof: Note that

$$\frac{n}{3} - \left[\frac{n}{3}\right] = \begin{cases} \frac{1}{3} \text{ if } n = 3k+1 \\ \frac{2}{3} \text{ if } n = 3k+2 \\ 0 \text{ if } n = 3k \end{cases}, \text{ where } k = 0,1,2,\ldots.$$

So, it is clear that

$$\lim_{n\to\infty}\sup \frac{n}{3} - \left[\frac{n}{3}\right] = \frac{2}{3} \text{ and } \lim_{n\to\infty}\inf \frac{n}{3} - \left[\frac{n}{3}\right] = 0.$$

Note. In (f), $[x]$ denoted the largest integer $\leq x$.

8.8 Let $a_n = 2\sqrt{n} - \sum_{k=1}^{n} 1/\sqrt{k}$. Prove that the sequence $\{a_n\}$ converges to a limit p in the interval $1 < p < 2$.

Proof: Consider $\sum_{k=1}^{n} 1/\sqrt{k} := S_n$ and $\int_1^n x^{-1/2}dx := T_n$, then

$$\lim_{n\to\infty} d_n \text{ exists, where } d_n = S_n - T_n$$

by **Integral Test.** We denote the limit by d, then

$$0 \le d < 1 \qquad *$$

by **Theorem 8.23 (i)**. Note that $\{d_n - f(n)\}$ is a positive increasing sequence, so we have

$$d > 0. \qquad **$$

Since

$$T_n = 2\sqrt{n} - 2$$

which implies that

$$\lim_{n\to\infty}\left(2\sqrt{n} - \sum_{k=1}^{n} 1/\sqrt{k}\right) = \lim_{n\to\infty} a_n = 2 - d = p.$$

By (*) and (**), we have proved that $1 < p < 1$.

Remark: (1) The use of **Integral Test** is very useful since we can know the behavior of a given series by integral. However, in many cases, the integrand may be so complicated that it is not easy to calculate. For example: Prove that the convergence of

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \text{ where } p > 1.$$

Of course, it can be checked by **Integral Test**. But there is the Theorem called **Cauchy Condensation Theorem** much powerful than **Integral Test** in this sense. In addition, the reader can think it twice that in fact, **Cauchy condensation Theorem is equivalent to Integral Test.**

(**Cauchy Condensation Theorem**)Let $\{a_n\}$ be a positive decreasing sequence. Then

$$\sum_{n=1}^{\infty} a_n \text{ converges if, and only if, } \sum_{k=0}^{\infty} 2^k a_{2^k} \text{ converges.}$$

Note: (1) The proof is not hard; the reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 61-63.**

(2) There is an extension of **Cauchy Condensation Theorem (Oskar Schlomilch)**: Suppose that $\{a_k\}$ be a positive and decreasing sequence and $\{m_k\}(\subseteq N)$ is a sequence. If there exists a $c > 0$ such that

$$0 < m_{k+2} - m_{k+1} \le c(m_{k+1} - m_k) \text{ for all } k,$$

then

$$\sum_{k=1}^{\infty} a_k \text{ converges if, and only if, } \sum_{k=0}^{\infty} (m_{k+1} - m_k) a_{m_k}.$$

Note: The proof is similar with **Cauchy Condensation Theorem**, so we omit it.

(2) There is a similar Theorem, we write it as a reference. If $t \ge a$, $f(t)$ is a non-negative increasing function, then as $x \ge a$, we have

$$\left|\sum_{a\le n\le x} f(n) - \int_a^x f(t)dt\right| \le f(x).$$

Proof: The proof is easy by drawing a graph. So, we omit it.

P.S.: The theorem is useful when we deal with some sums. For example,

$$f(t) = \log t.$$

Then

$$\left|\sum_{1\le n\le x}\log n - x\log x + x - 1\right| \le \log x.$$

In particular, as $x \in N$, we thus have

$$n\log n - n + 1 - \log n \le \log n! \le n\log n - n + 1 + \log n$$

which implies that

$$n^{n-1}e^{-n+1} \le n! \le n^{n+1}e^{-n+1}.$$

In each of Exercise 8.9. through 8.14, show that the real-valed sequence $\{a_n\}$ is convergent. The given conditions are assumed to hold for all $n \ge 1$. In Exercise 8.10 through 8.14, show that $\{a_n\}$ has the limit L indicated.

8.9 $|a_n| < 2,\ |a_{n+2} - a_{n+1}| \le \frac{1}{8}|a_{n+1}^2 - a_n^2|.$

Proof: Since

$$\begin{aligned}|a_{n+2} - a_{n+1}| &\le \frac{1}{8}|a_{n+1}^2 - a_n^2| \\ &= \frac{1}{8}|a_{n+1} - a_n||a_{n+1} + a_n| \\ &\le \frac{1}{2}|a_{n+1} - a_n| \text{ since } |a_n| < 2\end{aligned}$$

we know that

$$|a_{n+1} - a_n| \le \left(\frac{1}{2}\right)^{n-1}|a_2 - a_1| \le \left(\frac{1}{2}\right)^{n-3}.$$

So,

$$\begin{aligned}|a_{n+k} - a_n| &\le \sum_{j=1}^{k}|a_{n+j} - a_{n+j-1}| \\ &\le \sum_{j=1}^{k}\left(\frac{1}{2}\right)^{n+j-4} \\ &\le \left(\frac{1}{2}\right)^{n-2} \to \infty \text{ as } n \to \infty.\end{aligned}$$

Hence, $\{a_n\}$ is a Cauchy sequence. So, $\{a_n\}$ is a convergent sequence.

Remark: (1) If $|a_{n+1} - a_n| \le b_n$ for all $n \in N$, and $\sum b_n$ converges, then $\sum a_n$ converges.

Proof: Since the proof is similar with the Exercise, we omit it.

(2) In (1), the condition $\sum b_n$ converges **CANNOT** omit. For example,
(i) Let $a_n = \sin\left(\sum_{k=1}^{n}\frac{1}{k}\right)$ Or
(ii) a_n is defined as follows:

$a_1 = 1,\ a_2 = 1/2,\ a_3 = 0,\ a_4 = 1/4,\ a_5 = 1/2,\ a_6 = 3/4,\ a_7 = 1,$ and so on.

8.10 $a_1 \ge 0,\ a_2 \ge 0,\ a_{n+2} = (a_n a_{n+1})^{1/2},\ L = (a_1 a_2^2)^{1/3}.$

Proof: If one of a_1 or a_2 is 0, then $a_n = 0$ for all $n \ge 2$. So, we may assume that $a_1 \ne 0$ and $a_2 \ne 0$. So, we have $a_n \ne 0$ for all n. Let $b_n = \frac{a_{n+1}}{a_n}$, then

$$b_{n+1} = 1/\sqrt{b_n} \text{ for all } n$$

which implies that

$$b_{n+1} = (b_1)^{\left(\frac{-1}{2}\right)^n} \to 1 \text{ as } n \to \infty.$$

Consider

$$\Pi_{j=2}^{n+1} b_j = \Pi_{j=1}^{n} (b_j)^{-1/2}$$

which implies that

$$\left(a_1^{1/2} a_2\right)^{-2/3} a_{n+1} = \left(\frac{1}{b_{n+1}}\right)^{2/3}$$

which implies that

$$\lim_{n\to\infty} a_{n+1} = (a_1 a_2^2)^{1/3}.$$

Remark: There is another proof. We write it as a reference.

Proof: If one of a_1 or a_2 is 0, then $a_n = 0$ for all $n \geq 2$. So, we may assume that $a_1 \neq 0$ and $a_2 \neq 0$. So, we have $a_n \neq 0$ for all n. Let $a_2 \geq a_1$. Since $a_{n+2} = (a_n a_{n+1})^{1/2}$, then inductively, we have

$$a_1 \leq a_3 \leq \ldots \leq a_{2n-1} \leq \ldots \leq a_{2n} \leq \ldots \leq a_4 \leq a_2.$$

So, both of $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge. Say

$$\lim_{n\to\infty} a_{2n} = x \text{ and } \lim_{n\to\infty} a_{2n-1} = y.$$

Note that $a_1 \neq 0$ and $a_2 \neq 0$, so $x \neq 0$, and $y \neq 0$. In addition, $x = y$ by $a_{n+2} = (a_n a_{n+1})^{1/2}$. Hence, $\{a_n\}$ converges to x.

By $a_{n+2} = (a_n a_{n+1})^{1/2}$, and thus

$$\Pi_{j=1}^{n} a_{j+2}^2 = \Pi_{j=1}^{n} a_j a_{j+1} = (a_1 a_2^2 a_{n+1}) \Pi_{j=1}^{n-2} a_{j+2}^2$$

which implies that

$$a_{n+1} a_{n+2}^2 = a_1 a_2^2$$

which implies that

$$\lim_{n\to\infty} a_n = x = (a_1 a_2^2)^{1/3}.$$

8.11 $a_1 = 2,\ a_2 = 8,\ a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1}),\ a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}},\ L = 4.$

Proof: First, we note that

$$a_{2n+1} = \frac{a_{2n} + a_{2n-1}}{2} \geq \sqrt{a_{2n} a_{2n-1}} \text{ by } A.P. \geq G.P. \qquad *$$

for $n \in N$. So, by $a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}}$ and (*),

$$a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}} \leq \sqrt{a_{2n} a_{2n-1}} \leq a_{2n+1} \text{ for all } n \in N.$$

Hence, by **Mathematical Induction**, it is easy to show that

$$a_4 \leq a_6 \leq \ldots \leq a_{2n+2} \leq \ldots \leq a_{2n+1} \leq \ldots \leq a_5 \leq a_3$$

for all $n \in N$. It implies that both of $\{a_{2n}\}$ and $\{a_{2n-1}\}$ converge, say

$$\lim_{n\to\infty} a_{2n} = x \text{ and } \lim_{n\to\infty} a_{2n-1} = y.$$

With help of $a_{2n+1} = \frac{1}{2}(a_{2n} + a_{2n-1})$, we know that $x = y$. In addition, by $a_{2n+2} = \frac{a_{2n} a_{2n-1}}{a_{2n+1}}$, $a_1 = 2$, and $a_2 = 8$, we know that $x = 4$.

8.12 $a_1 = \frac{-3}{2},\ 3a_{n+1} = 2 + a_n^3,\ L = 1.$ Modify a_1 to make $L = -2$.

Proof: By **Mathematical Induction**, it is easy to show that

$$-2 \leq a_n \leq 1 \text{ for all } n. \qquad *$$

So,

$$3(a_{n+1}-a_n) = a_n^3 - 3a_n + 2 \geq 0$$

by (*) and $f(x) = x^3 - 3x + 2 = (x-1)^2(x+2) \geq 0$ on $[-2,1]$. Hence, $\{a_n\}$ is an increasing sequence with a upper bound 1. So, $\{a_n\}$ is a convergent sequence with limit L. So, by $3a_{n+1} = 2 + a_n^3$,

$$L^3 - 3L + 2 = 0$$

which implies that

$$L = 1 \text{ or } -2.$$

So, $L = 1$ sinc $a_n \nearrow$ and $a_1 = -3/2$.

In order to make $L = -2$, it suffices to let $a_1 = -2$, then $a_n = -2$ for all n.

8.13 $a_1 = 3$, $a_{n+1} = \frac{3(1+a_n)}{3+a_n}$, $L = \sqrt{3}$.

Proof: By **Mathematical Induction**, it is easy to show that

$$a_n \geq \sqrt{3} \text{ for all } n. \quad *$$

So,

$$a_{n+1} - a_n = \frac{3-a_n^2}{3+a_n} \leq 0$$

which implies that $\{a_n\}$ is a decreasing sequence. So, $\{a_n\}$ is a convergent sequence with limit L by (*). Hence,

$$L = \frac{3(1+L)}{3+L}$$

which implies that

$$L = \pm\sqrt{3}.$$

So, $L = \sqrt{3}$ since $a_n \geq \sqrt{3}$ for all n.

8.14 $a_n = \frac{b_{n+1}}{b_n}$, where $b_1 = b_2 = 1$, $b_{n+2} = b_n + b_{n+1}$, $L = \frac{1+\sqrt{5}}{2}$.

Hint. Show that $b_{n+2}b_n - b_{n+1}^2 = (-1)^{n+1}$ and deduce that $|a_n - a_{n+1}| < n^{-2}$, if $n > 4$.

Proof: By **Mathematical Induction**, it is easy to show that

$$b_{n+2}b_n - b_{n+1}^2 = (-1)^{n+1} \text{ for all } n$$

and

$$b_n \geq n \text{ if } n > 4$$

Thus, (Note that $b_n \neq 0$ for all n)

$$|a_{n+1} - a_n| = \left|\frac{b_{n+2}}{b_{n+1}} - \frac{b_{n+1}}{b_n}\right| = \left|\frac{(-1)^{n+1}}{b_n b_{n+1}}\right| \leq \frac{1}{n(n+1)} < \frac{1}{n^2} \text{ if } n > 4.$$

So, $\{a_n\}$ is a Cauchy sequence. In other words, $\{a_n\}$ is a convergent sequence, say $\lim_{n\to\infty} b_n = L$. Then by $b_{n+2} = b_n + b_{n+1}$, we have

$$\frac{b_{n+2}}{b_{n+1}} = \frac{b_n}{b_{n+1}} + 1$$

which implies that (Note that $(0 \neq)L \geq 1$ since $a_n \geq 1$ for all n)

$$L = \frac{1}{L} + 1$$

which implies that

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

So, $L = \frac{1+\sqrt{5}}{2}$ since $L \geq 1$.

Remark: (1) The sequence $\{b_n\}$ is the famous sequence named **Fabonacci sequence**. There are many researches around it. Also, it is related with so called **Golden Section**, $\frac{\sqrt{5}-1}{2} = 0.618\ldots$

(2) The reader can see the book, **An Introduction To The Theory Of Numbers by G. H. Hardy and E. M. Wright, Chapter X.** Then it is clear by **continued fractions.**

(3) There is another proof. We write it as a reference.

Proof: (**STUDY**) Since $b_{n+2} = b_n + b_{n+1}$, we may think
$$x^{n+2} = x^n + x^{n+1},$$
and thus consider $x^2 = x + 1$. Say α and β are roots of $x^2 = x + 1$, with $\alpha < \beta$. Then let
$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha},$$
we have
$$F_n = b_n.$$
So, it is easy to show that $L = \frac{1+\sqrt{5}}{2}$. We omit the details.

Note: The reader should be noted that there are many methods to find the formula of **Fabonacci sequence** like F_n. For example, using the concept of **Eigenvalues** if we can find a suitable matrix.

Series

8.15 Test for convergence (p and q denote fixed rela numbers).

(a) $\sum_{n=1}^{\infty} n^3 e^{-n}$

Proof: By **Root Test**, we have
$$\lim_{n\to\infty} \sup \left(\frac{n^3}{e^n} \right)^{1/n} = 1/e < 1.$$
So, the series converges.

(b) $\sum_{n=2}^{\infty} (\log n)^p$

Proof: We consider 2 cases: (i) $p \geq 0$, and (ii) $p < 0$.
For case (i), the series diverges since $(\log n)^p$ does not converge to zero.
For case (ii), the series diverges by **Cauchy Condensation Theorem** (or **Integral Test**.)

(c) $\sum_{n=1}^{\infty} p^n n^p$ $(p > 0)$

Proof: By **Root Test**, we have
$$\lim_{n\to\infty} \sup \left(\frac{p^n}{n^p} \right)^{1/n} = p.$$
So, as $p > 1$, the series diverges, and as $p < 1$, the series converges. For $p = 1$, it is clear that the series $\sum n$ diverges. Hence,
$$\sum_{n=1}^{\infty} p^n n^p \text{ converges if } p \in (0,1)$$
and

$$\sum_{n=1}^{\infty} p^n n^p \text{ diverges if } p \in [1, \infty).$$

(d) $\sum_{n=2}^{\infty} \frac{1}{n^p - n^q}$ $(0 < q < p)$

Proof: Note that $\frac{1}{n^p - n^q} = \frac{1}{n^p} \frac{1}{1-n^{q-p}}$. We consider 2 cases: (i) $p > 1$ and (ii) $p \le 1$. For case (i), by **Limit Comparison Test** with $\frac{1}{n^p}$,

$$\lim_{n\to\infty} \frac{\frac{1}{n^p-n^q}}{\frac{1}{n^p}} = 1,$$

the series converges.

For case (ii), by **Limit Comparison Test** with $\frac{1}{n^p}$,

$$\lim_{n\to\infty} \frac{\frac{1}{n^p-n^q}}{\frac{1}{n^p}} = 1,$$

the series diverges.

(e) $\sum_{n=1}^{\infty} n^{-1-1/n}$

Proof: Since $n^{-1-1/n} \ge n^{-1}$ for all n, the series diverges.

(f) $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$ $(0 < q < p)$

Proof: Note that $\frac{1}{p^n - q^n} = \frac{1}{p^n} \frac{1}{1-\left(\frac{q}{p}\right)^n}$. We consider 2 cases: (i) $p > 1$ and (ii) $p \le 1$. For case (i), by **Limit Comparison Test** with $\frac{1}{p^n}$,

$$\lim_{n\to\infty} \frac{\frac{1}{p^n-q^n}}{\frac{1}{p^n}} = 1,$$

the series converges.

For case (ii), by **Limit Comparison Test** with $\frac{1}{p^n}$,

$$\lim_{n\to\infty} \frac{\frac{1}{p^n-q^n}}{\frac{1}{p^n}} = 1,$$

the series diverges.

(g) $\sum_{n=1}^{\infty} \frac{1}{n\log(1+1/n)}$

Proof: Since

$$\lim_{n\to\infty} \frac{1}{n\log(1+1/n)} = 1,$$

we know that the series diverges.

(h) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$

Proof: Since the identity $a^{\log b} = b^{\log a}$, we have

$$\begin{aligned}(\log n)^{\log n} &= n^{\log\log n} \\ &\ge n^2 \text{ as } n \ge n_0.\end{aligned}$$

So, the series converges.

(i) $\sum_{n=3}^{\infty} \frac{1}{n\log n(\log\log n)^p}$

Proof: We consider 3 cases: (i) $p \le 0$, (ii) $0 < p < 1$ and (iii) $p > 1$.

For case (i), since

$$\frac{1}{n\log n(\log\log n)^p} \geq \frac{1}{n\log n} \text{ for } n \geq 3,$$

we know that the series diverges by the divergence of $\sum_{n=3}^{\infty} \frac{1}{n\log n}$.

For case (ii), we consider (choose n_0 large enough)

$$\sum_{j=n_0}^{\infty} \frac{2^j}{2^j \log 2^j (\log\log 2^j)^p} = \frac{1}{\log 2}\sum_{j=n_0}^{\infty} \frac{1}{j(\log j \log 2)^p}$$

$$\geq \sum_{j=n_0}^{\infty} \frac{1}{j(\log j)^p},$$

then, by **Cauchy Condensation Theorem,** the series diverges since $\sum_{j=n_0}^{\infty} \frac{1}{j(\log j)^p}$ diverges by using **Cauchy Condensation Theorem** again.

For case (iii), we consider (choose n_0 large enough)

$$\sum_{j=n_0}^{\infty} \frac{2^j}{2^j \log 2^j (\log\log 2^j)^p} = \frac{1}{\log 2}\sum_{j=n_0}^{\infty} \frac{1}{j(\log j \log 2)^p}$$

$$\leq 2\sum_{j=n_0}^{\infty} \frac{1}{j(\log j \log 2)^p}$$

$$\leq 4\sum_{j=n_0}^{\infty} \frac{1}{j(\log j)^p},$$

then, by **Cauchy Condensation Theorem,** the series converges since $\sum_{j=n_0}^{\infty} \frac{1}{j(\log j)^p}$ converges by using **Cauchy Condensation Theorem** again.

Remark: There is another proof by **Integral Test**. We write it as a reference.

Proof: It is easy to check that $f(x) = \frac{1}{x\log x(\log\log x)^p}$ is continous, positive, and decreasing to zero on $[a,\infty)$ where $a > 0$ for each fixed p. Consider

$$\int_a^{\infty} \frac{dx}{x\log x(\log\log x)^p} = \int_{\log\log a}^{\infty} \frac{dy}{y^p}$$

which implies that the series converges if $p > 1$ and diverges if $p \leq 1$ by **Integral Test**.

(j) $\sum_{n=3}^{\infty}\left(\frac{1}{\log\log n}\right)^{\log\log n}$

Proof: Let $a_n = \left(\frac{1}{\log\log n}\right)^{\log\log n}$ for $n \geq 3$ and $b_n = 1/n,$ then

$$\frac{a_n}{b_n} = n\left(\frac{1}{\log\log n}\right)^{\log\log n}$$

$$= e^{-(y\log y - e^y)} \to +\infty.$$

So, by **Limit Comparison Test**, the series diverges.

(k) $\sum_{n=1}^{\infty}\left(\sqrt{1+n^2} - n\right)$

Proof: Note that

$$\sqrt{1+n^2} - n = \frac{1}{\sqrt{1+n^2}+n} \geq \frac{1}{\left(1+\sqrt{2}\right)n} \text{ for all } n.$$

So, the series diverges.

(l) $\sum_{n=2}^{\infty} n^p \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right)$

Proof: Note that

$$n^p \left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}} \right) = \frac{1}{n^{\frac{3}{2}-p}} \left(\sqrt{\frac{n}{n-1}} \frac{1}{1 + \sqrt{\frac{n-1}{n}}} \right).$$

So, as $p < 1/2$, the series converges and as $p \geq 1/2$, the series diverges by **Limit Comparison Test.**

(m) $\sum_{n=1}^{\infty} \left((n)^{1/n} - 1 \right)^n$

Proof: With help of **Root Test**,

$$\lim_{n \to \infty} \sup \left[\left((n)^{1/n} - 1 \right)^n \right]^{1/n} = 0 (< 1),$$

the series converges.

(n) $\sum_{n=1}^{\infty} n^p \left(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1} \right)$

Proof: Note that

$$n^p \left(\sqrt{n+1} - 2\sqrt{n} + \sqrt{n-1} \right)$$
$$= \frac{1}{n^{\frac{3}{2}-p}} \left[\frac{n^{\frac{3}{2}}}{\left(\sqrt{n} + \sqrt{n+1} \right) \left(\sqrt{n} + \sqrt{n-1} \right) \left(\sqrt{n-1} + \sqrt{n+1} \right)} \right].$$

So, as $p < 1/2$, the series converges and as $p \geq 1/2$, the series diverges by **Limit Comparison Test.**

8.16 Let $S = \{n_1, n_2, \ldots\}$ denote the collection of those positive integers that do not involve the digit 0 is their decimal representation. (For example, $7 \in S$ but $101 \notin S$.) Show that $\sum_{k=1}^{\infty} 1/n_k$ converges and has a sum less than 90.

Proof: Define $S_j = \{\text{the } j - \text{digit number}\} (\subseteq S)$. Then $\#S_j = 9^j$ and $S = \cup_{j=1}^{\infty} S_j$. Note that

$$\sum_{k \in S_j} 1/n_k < \frac{9^j}{10^{j-1}}.$$

So,

$$\sum_{k=1}^{\infty} 1/n_k \leq \sum_{j=1}^{\infty} \frac{9^j}{10^{j-1}} = 90.$$

In addition, it is easy to know that $\sum_{k=1}^{\infty} 1/n_k \neq 90$. Hence, we have proved that $\sum_{k=1}^{\infty} 1/n_k$ converges and has a sum less than 90.

8.17 Given integers $a_1, a_2, \ldots$ such that $1 \leq a_n \leq n - 1$, $n = 2, 3, \ldots$ Show that the sum of the series $\sum_{n=1}^{\infty} a_n/n!$ is rational if and only if there exists an integer N such that $a_n = n - 1$ for all $n \geq N$. Hint: For sufficency, show that $\sum_{n=2}^{\infty} (n-1)/n!$ is a telescoping series with sum 1.

Proof: ($\Leftarrow$)Assume that there exists an integer N such that $a_n = n - 1$ for all $n \geq N$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a_n}{n!} &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\sum_{n=N}^{\infty}\frac{a_n}{n!}\\ &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\sum_{n=N}^{\infty}\frac{n-1}{n!}\\ &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\sum_{n=N}^{\infty}\frac{1}{(n-1)!}-\frac{1}{n!}\\ &= \sum_{n=1}^{N-1}\frac{a_n}{n!}+\frac{1}{(N-1)!}\in Q.\end{aligned}$$

($\Rightarrow$)Assume that $\sum_{n=1}^{\infty} a_n/n!$ is rational, say $\frac{q}{p}$, where $g.c.d.(p,q) = 1.$ Then

$$p!\sum_{n=1}^{\infty}\frac{a_n}{n!}\in Z.$$

That is, $p!\sum_{n=p+1}^{\infty}\frac{a_n}{n!}\in Z.$ Note that

$$p!\sum_{n=p+1}^{\infty}\frac{a_n}{n!}\leq p!\sum_{n=p+1}^{\infty}\frac{n-1}{n!} = \frac{p!}{p!} = 1 \text{ since } 1\leq a_n\leq n-1.$$

So, $a_n = n-1$ for all $n\geq p+1.$ That is, there exists an integer N such that $a_n = n-1$ for all $n\geq N.$

Remark: From this, we have proved that e is irrational. The reader should be noted that we can use **Theorem 8.16** to show that e is irrational by considering $e^{-1}.$ Since it is easy, we omit the proof.

8.18 Let p and q be fixed integers, $p\geq q\geq 1,$ and let

$$x_n = \sum_{k=qn+1}^{pn}\frac{1}{k},\ s_n = \sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}.$$

(a) Use formula (8) to prove that $\lim_{n\to\infty}x_n = \log(p/q).$

Proof: Since

$$\sum_{k=1}^{n}\frac{1}{k} = \log n + r + O\left(\frac{1}{n}\right),$$

we know that

$$\begin{aligned}x_n &= \sum_{k=1}^{pn}\frac{1}{k}-\sum_{k=1}^{qn}\frac{1}{k}\\ &= \log(p/q)+O\left(\frac{1}{n}\right)\end{aligned}$$

which implies that $\lim_{n\to\infty}x_n = \log(p/q).$

(b) When $q = 1, p = 2,$ show that $s_{2n} = x_n$ and deduce that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \log 2.$$

Proof: We prove it by **Mathematical Induction** as follows. As $n = 1,$ it holds trivially. Assume that $n = m$ holds, i.e.,

$$s_{2m} = \sum_{k=1}^{2m} \frac{(-1)^{k+1}}{k} = \sum_{k=m+1}^{2m} \frac{1}{k} = x_m$$

consider $n = m + 1$ as follows.

$$\begin{aligned} x_{m+1} &= \sum_{k=(m+1)+1}^{2(m+1)} \frac{1}{k} \\ &= x_m - \frac{1}{m+1} + \frac{1}{2m+1} + \frac{1}{2m+2} \\ &= s_{2m} + \frac{1}{2m+1} - \frac{1}{2m+2} \\ &= s_{2(m+1)}. \end{aligned}$$

So, by **Mathematical Induction**, we have proved that $s_{2n} = x_n$ for all n.

By $s_{2n} = x_n$ for all n, we have

$$\lim_{n\to\infty} s_{2n} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \log 2 = \lim_{n\to\infty} x_n.$$

(c) rearrange the series in (b), writing alternately p positive terms followed by q negative terms and use (a) to show that this rearrangement has sum

$$\log 2 + \frac{1}{2}\log(p/q).$$

Proof: We prove it by using **Theorem 8.13.** So, we can consider the new series $\sum_{k=1}^{\infty} a_k$ as follows:

$$a_k = \left[\left(\frac{1}{2(k-1)p+1} + \ldots + \frac{1}{2kp-1} \right) - \left(\frac{1}{2(k-1)q} + \ldots + \frac{1}{2kq} \right) \right]$$

Then

$$\begin{aligned} S_n &= \sum_{k=1}^{n} a_k \\ &= \sum_{k=1}^{2np} \frac{1}{k} - \sum_{k=1}^{np} \frac{1}{2k} - \sum_{k=1}^{nq} \frac{1}{2k} \\ &= \log 2np + \gamma + O\left(\frac{1}{n}\right) - \frac{1}{2}\log np - \frac{\gamma}{2} + O\left(\frac{1}{n}\right) - \frac{1}{2}\log nq - \frac{\gamma}{2} + O\left(\frac{1}{n}\right) \\ &= \log 2np - \log n\sqrt{pq} + O\left(\frac{1}{n}\right) \\ &= \log 2\sqrt{\frac{p}{q}} + O\left(\frac{1}{n}\right). \end{aligned}$$

So,

$$\lim_{n\to\infty} S_n = \log 2 + \frac{1}{2}\log(p/q)$$

by **Theorem 8.13**.

Remark: There is a reference around rearrangement of series. The reader can see the book, **Infinite Series by Chao Wen-Min, pp 216-220. (Chinese Version)**

(d) Find the sum of $\sum_{n=1}^{\infty} (-1)^{n+1} (1/(3n-2) - 1/(3n-1))$.

Proof: Write

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} (-1)^{k+1}\left(\frac{1}{3k-2} - \frac{1}{3k-1}\right) \\
&= \sum_{k=1}^{n} (-1)^{k} \frac{1}{3k-1} + \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{3k-2} \\
&= -\sum_{k=1}^{n} (-1)^{3k-1} \frac{1}{3k-1} - \sum_{k=1}^{n} (-1)^{3k-2} \frac{1}{3k-2} \\
&= -\left[\sum_{k=1}^{n} (-1)^{3k-1} \frac{1}{3k-1} + \sum_{k=1}^{n} (-1)^{3k-2} \frac{1}{3k-2}\right] \\
&= -\left[\sum_{k=1}^{3n} \frac{(-1)^k}{k} - \sum_{k=1}^{n} \frac{(-1)^{3k}}{3k}\right] \\
&= -\left[\sum_{k=1}^{3n} \frac{(-1)^k}{k} - \frac{1}{3}\sum_{k=1}^{n} \frac{(-1)^{k}}{k}\right] \\
&= \sum_{k=1}^{3n} \frac{(-1)^{k+1}}{k} - \frac{1}{3}\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \\
&\to \frac{2}{3}\log 2.
\end{aligned}
$$

So, the series has the sum $\frac{2}{3}\log 2$.

Remark: There is a refernece around rearrangement of series. The reader can see the book, **An Introduction to Mathematical Analysis by Loo-Keng Hua, pp 323-325. (Chinese Version)**

8.19 Let $c_n = a_n + ib_n$, where $a_n = (-1)^n/\sqrt{n}$, $b_n = 1/n^2$. Show that $\sum c_n$ is conditioinally convergent.

Proof: It is clear that $\sum c_n$ converges. Consider

$$\sum |c_n| = \sum \sqrt{\frac{1}{n^2} + \frac{1}{n^4}} = \sum \frac{1}{n}\sqrt{1 + \frac{1}{n^2}} \geq \sum \frac{1}{n}$$

Hence, $\sum |c_n|$ diverges. That is, $\sum c_n$ is conditioinally convergent.

Remark: We say $\sum c_n$ converges if, and only if, the real part $\sum a_n$ converges and the imaginary part $\sum b_n$ converges, where $c_n = a_n + ib_n$.

8.20 Use Theorem 8.23 to derive the following formulas:

(a) $\sum_{k=1}^{n} \frac{\log k}{k} = \frac{1}{2}\log^2 n + A + O\left(\frac{\log n}{n}\right)$ (A is constant)

Proof: Let $f(x) = \frac{\log x}{x}$ define on $[3, \infty)$, then $f'(x) = \frac{1 - \log x}{x^2} < 0$ on $[3, \infty)$. So, it is clear that $f(x)$ is a positive and continuous function on $[3, \infty)$, with

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{1}{x} = 0 \text{ by } \textbf{L-Hospital Rule}.$$

So, by **Theorem 8.23**, we have

$$\sum_{k=3}^{n} \frac{\log k}{k} = \int_3^n \frac{\log x}{x} dx + C + O\left(\frac{\log n}{n}\right), \text{ where } C \text{ is a constant}$$
$$= \frac{1}{2}\log^2 n - \frac{1}{2}\log^2 3 + C + O\left(\frac{\log n}{n}\right), \text{ where } C \text{ is a constant}$$

which implies that

$$\sum_{k=1}^{n} \frac{\log k}{k} = \frac{1}{2}\log^2 n + A + O\left(\frac{\log n}{n}\right),$$

where $A = C + \frac{\log 2}{2} - \frac{1}{2}\log^2 3$ is a constant.

(b) $\sum_{k=2}^{n} \frac{1}{k \log k} = \log(\log n) + B + O\left(\frac{1}{n \log n}\right)$ (B is constant)

Proof: Let $f(x) = \frac{1}{x \log x}$ defined on $[2, \infty),$ then $f'(x) = -\left(\frac{1}{x \log x}\right)^2 (1 + \log x) < 0$ on $[2, \infty).$ So, it is clear that $f(x)$ is a positive and continuous function on $[3, \infty),$ with

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x \log x} = 0.$$

So, by **Theorem 8.23**, we have

$$\sum_{k=2}^{n} \frac{1}{k \log k} = \int_2^n \frac{dx}{x \log x} + C + O\left(\frac{1}{n \log n}\right), \text{ where } C \text{ is a constant}$$
$$= \log \log n + B + O\left(\frac{1}{n \log n}\right), \text{ where } C \text{ is a constant}$$

where $B = C - \log \log 2$ is a constant.

8.21 If $0 < a \leq 1,$ $s > 1,$ define $\zeta(s, a) = \sum_{n=0}^{\infty} (n + a)^{-s}.$

(a) Show that this series converges absolutely for $s > 1$ and prove that

$$\sum_{h=1}^{k} \zeta\left(s, \frac{h}{k}\right) = k^s \zeta(s) \text{ if } k = 1, 2, \ldots$$

where $\zeta(s) = \zeta(s, 1)$ is the Riemann zeta function.

Proof: First, it is clear that $\zeta(s, a)$ converges absolutely for $s > 1.$ Consider

$$\sum_{h=1}^{k} \zeta\left(s, \frac{h}{k}\right) = \sum_{h=1}^{k} \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{h}{k}\right)^s}$$
$$= \sum_{h=1}^{k} \sum_{n=0}^{\infty} \frac{k^s}{(kn + h)^s}$$
$$= \sum_{n=0}^{\infty} \sum_{h=1}^{k} \frac{k^s}{(kn + h)^s}$$
$$= k^s \sum_{n=0}^{\infty} \sum_{h=1}^{k} \frac{1}{(kn + h)^s}$$
$$= k^s \sum_{n=0}^{\infty} \frac{1}{(n + 1)^s}$$
$$= k^s \zeta(s).$$

(b) Prove that $\sum_{n=1}^{\infty}(-1)^{n-1}/n^s = (1-2^{1-s})\zeta(s)$ if $s > 1$.

Proof: Let $\left\{S_n = \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^s}\right\}$, and thus consider its subsequence $\{S_{2n}\}$ as follows:

$$S_{2n} = \sum_{j=1}^{2n} \frac{1}{j^s} - 2\sum_{j=1}^{n} \frac{1}{(2j)^s}$$
$$= \sum_{j=1}^{2n} \frac{1}{j^s} - 2^{1-s}\sum_{j=1}^{n} \frac{1}{j^s}$$

which implies that

$$\lim_{n\to\infty} S_{2n} = (1-2^{1-s})\zeta(s).$$

Since $\{S_n\}$ converges, we know that $\{S_{2n}\}$ also converges and has the same value. Hence,

$$\sum_{n=1}^{\infty}(-1)^{n-1}/n^s = (1-2^{1-s})\zeta(s).$$

8.22 Given a convergent series $\sum a_n$, where each $a_n \geq 0$. Prove that $\sum \sqrt{a_n}\, n^{-p}$ converges if $p > 1/2$. Give a counterexample for $p = 1/2$.

Proof: Since

$$\frac{a_n + n^{-2p}}{2} \geq \sqrt{a_n n^{-2p}} = \sqrt{a_n}\, n^{-p},$$

we have $\sum \sqrt{a_n}\, n^{-p}$ converges if $p > 1/2$ since

$$\sum a_n \text{ converges and } \sum n^{-2p} \text{ converges if } p > 1/2.$$

For $p = 1/2$, we consider $a_n = \frac{1}{n(\log n)^2}$, then

$$\sum a_n \text{ converges by } \textbf{Cauchy Condensation Theorem}$$

and

$$\sum \sqrt{a_n}\, n^{-1/2} = \sum \frac{1}{n\log n} \text{ diverges by } \textbf{Cauchy Condensation Theorem}.$$

8.23 Given that $\sum a_n$ diverges. Prove that $\sum na_n$ also diverges.

Proof: Assume $\sum na_n$ converges, then its partial sum $\sum_{k=1}^{n} ka_k$ is bounded. Then by **Dirichlet Test**, we would obtain

$$\sum (ka_k)\left(\frac{1}{k}\right) = \sum a_k \text{ converges}$$

which contradicts to $\sum a_n$ diverges. Hence, $\sum na_n$ diverges.

8.24 Given that $\sum a_n$ converges, where each $a_n > 0$. Prove that

$$\sum (a_n a_{n+1})^{1/2}$$

also converges. Show that the converse is also true if $\{a_n\}$ is monotonic.

Proof: Since

$$\frac{a_n + a_{n+1}}{2} \geq (a_n a_{n+1})^{1/2},$$

we know that

$$\sum (a_n a_{n+1})^{1/2}$$

converges by $\sum a_n$ converges.

Conversly, since $\{a_n\}$ is monotonic, it must be decreasing since $\sum a_n$ converges. So, $a_n \geq a_{n+1}$ for all n. Hence,

$$(a_n a_{n+1})^{1/2} \geq a_{n+1} \text{ for all } n.$$

So, $\sum a_n$ converges since $\sum (a_n a_{n+1})^{1/2}$ converges.

8.25 Given that $\sum a_n$ converges absolutely. Show that each of the following series also converges absolutely:

(a) $\sum a_n^2$

Proof: Since $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$. So, given $\varepsilon = 1$, there exists a positive integer N such that as $n \geq N$, we have

$$|a_n| < 1$$

which implies that

$$a_n^2 < |a_n| \text{ for } n \geq N.$$

So, $\sum a_n^2$ converges if $\sum |a_n|$ converges. Of course, $\sum a_n^2$ converges absolutely.

(b) $\sum \frac{a_n}{1+a_n}$ (if no $a_n = -1$)

Proof: Since $\sum |a_n|$ converges, we have $\lim_{n\to\infty} a_n = 0$. So, there exists a positive integer N such that as $n \geq N$, we have

$$1/2 < |1 + a_n|.$$

Hence, as $n \geq N$,

$$\left| \frac{a_n}{1 + a_n} \right| < 2|a_n|$$

which implies that $\sum \left| \frac{a_n}{1+a_n} \right|$ converges. So, $\sum \frac{a_n}{1+a_n}$ converges absolutely.

(c) $\sum \frac{a_n^2}{1+a_n^2}$

Proof: It is clear that

$$\frac{a_n^2}{1 + a_n^2} \leq a_n^2.$$

By (a), we have proved that $\sum \frac{a_n^2}{1+a_n^2}$ converges absolutely.

8.26 Determine all real values of x for which the following series converges.

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) \frac{\sin nx}{n}.$$

Proof: Consider its partial sum

$$\sum_{k=1}^{n} \frac{(1 + \frac{1}{2} + \ldots + \frac{1}{k})}{k} \sin kx$$

as follows.

As $x = 2m\pi$, the series converges to zero. So it remains to consider $x \neq 2m\pi$ as follows. Define

$$a_k = \frac{1 + \frac{1}{2} + \ldots + \frac{1}{k}}{k}$$

and

$$b_k = \sin kx,$$

then

$$a_{k+1} - a_k = \frac{1+\frac{1}{2}+\ldots+\frac{1}{k}+\frac{1}{k+1}}{k+1} - \frac{1+\frac{1}{2}+\ldots+\frac{1}{k}}{k}$$
$$= \frac{k(1+\frac{1}{2}+\ldots+\frac{1}{k}+\frac{1}{k+1}) - (k+1)(1+\frac{1}{2}+\ldots+\frac{1}{k})}{k(k+1)}$$
$$= \frac{\frac{k}{k+1} - (1+\frac{1}{2}+\ldots+\frac{1}{k})}{k(k+1)} < 0$$

and

$$\left|\sum_{k=1}^{n} b_k\right| \leq \left|\frac{1}{\sin(\frac{x}{2})}\right|.$$

So, by **Dirichlet Test**, we know that

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} \frac{(1+\frac{1}{2}+\ldots+\frac{1}{k})}{k} \sin kx$$

converges.

From above results, we have shown that the series converges for all $x \in R$.

8.27. Prove that following statements:

(a) $\sum a_n b_n$ converges if $\sum a_n$ converges and if $\sum(b_n - b_{n+1})$ converges absolutely.

Proof: Consider s**ummation by parts**, i.e., **Theorem 8.27**, then

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k).$$

Since $\sum a_n$ converges, then $|A_n| \leq M$ for all $n.$ In addition, by **Theorem 8.10,** $\lim_{n\to\infty} b_n$ exists. So, we obtain that

$$(1).\ \lim_{n\to\infty} A_n b_{n+1} \text{ exists}$$

and

$$(2).\ \sum_{k=1}^{n} |A_k(b_{k+1} - b_k)| \leq M\sum_{k=1}^{n} |b_{k+1} - b_k| \leq M\sum_{k=1}^{\infty} |b_{k+1} - b_k|.$$

(2) implies that

$$(3).\ \sum_{k=1}^{n} A_k(b_{k+1} - b_k) \text{ converges.}$$

By (1) and (3), we have shown that $\sum_{k=1}^{n} a_k b_k$ converges.

Remark: In 1871, **Paul du Bois Reymond (1831-1889)** gave the result.

(b) $\sum a_n b_n$ converges if $\sum a_n$ has bounded partial sums and if $\sum(b_n - b_{n+1})$ converges absolutely, provided that $b_n \to 0$ as $n \to \infty$.

Proof: By **summation by parts**, we have

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k(b_{k+1} - b_k).$$

Since $b_n \to 0$ as $n \to \infty$ and $\sum a_n$ has bounded partial sums, say $|A_n| \leq M$ for all $n.$ Then

$$(1).\ \lim_{n\to\infty} A_n b_{n+1} \text{ exists.}$$

In addition,

$$(2).\ \sum_{k=1}^{n} |A_k(b_{k+1}-b_k)| \le M\sum_{k=1}^{n}|b_{k+1}-b_k| \le M\sum_{k=1}^{\infty}|b_{k+1}-b_k|.$$

(2) implies that

$$(3).\ \sum_{k=1}^{n} A_k(b_{k+1}-b_k) \text{ converges.}$$

By (1) and (3), we have shown that $\sum_{k=1}^{n} a_k b_k$ converges.

Remark: (1) The result is first discovered by **Richard Dedekind (1831-1916).**

(2) There is an exercise by (b), we write it as a reference. Show the convergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k}$.

Proof: Let $a_k = \frac{(-1)^{[\sqrt{k}]}}{k^{2/3}}$ and $b_k = \frac{1}{k^{1/3}}$, then in order to show the convergence of $\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k}$, it suffices to show that $\left\{\sum_{k=1}^{n} a_k := S_n\right\}$ is bounded sequence. Given $n \in N$, there exists $j \in N$ such that $j^2 \le N < (j+1)^2$. Consider

$$S_n = a_1 + a_2 + a_3 + a_4 + \ldots + a_8 + a_9 + \ldots + a_{15} + \ldots + a_{j^2} + \ldots + a_n$$

$$\le \begin{cases} 3a_3 + 5a_4 + 7a_{15} + 9a_{16} + \ldots + (4k-1)a_{(2k)^2-1} + (4k+1)a_{(2k)^2} \text{ if } j = 2k,\ k \ge 2 \\ 3a_3 + 5a_4 + 7a_{15} + 9a_{16} + \ldots + (4k-3)a_{(2k-2)^2} \text{ if } j = 2k-1,\ k \ge 3 \end{cases}$$

then as n large enough,

$$S_n \le \begin{cases} (-3a_4 + 5a_4) + (-7a_{16} + 9a_{16}) + \ldots + \left(-(4k-1)a_{(2k)^2} + (4k+1)a_{(2k)^2}\right) \\ (-3a_4 + 5a_4) + (-7a_{16} + 9a_{16}) + \ldots + \left(-(4k-5)a_{(2k-2)^2} + (4k-3)a_{(2k-2)^2}\right) \end{cases}$$

which implies that as n large enough,

$$S_n \le 2\sum_{j=2}^{\infty} a_{(2j)^2} = 2\sum_{j=2}^{\infty} \frac{1}{(2j)^{4/3}} := M_1 \qquad *$$

Similarly, we have

$$M_2 \le S_n \text{ for all } n \qquad **$$

By (*) and (**), we have shown that $\left\{\sum_{k=1}^{n} a_k := S_n\right\}$ is bounded sequence.

Note: (1) By above method, it is easy to show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{[\sqrt{k}]}}{k^p}$$

converges for $p > 1/2$. For $0 < p \le 1/2$, the series diverges by

$$\frac{1}{(n^2)^p} + \ldots + \frac{1}{(n^2+2n)^p} \ge \frac{2n+1}{(n^2+n)^p} \ge \frac{2n+1}{(n^2+n)^p} \ge \frac{2n+1}{(n+1)^{2p}} \ge \frac{2n+1}{n+1} > 1.$$

(2) There is a similar question, show the divergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^{[\log k]}}{k}$.

Proof: We use **Theorem 8.13** to show it by inserting parentheses as follows. We insert parentheses such that the series $\sum \frac{(-1)^{[\log k]}}{k}$ forms $\sum (-1)^k b_k$. If we can show $\sum (-1)^k b_k$

diverges, then $\sum \frac{(-1)^{[\log k]}}{k}$ also diverges. Consider

$$b_k = \frac{1}{m} + \ldots + \frac{1}{m+r}, \qquad *$$

where

$$\begin{aligned}
&(1).\ [\log m] = N \\
&(2).\ [\log(m-1)] = N-1 \Rightarrow [\log e(m-1)] = N \\
&(3).\ [\log(m+r)] = N \\
&(4).\ [\log(m+r+1)] = N+1 \Rightarrow [\log \tfrac{m+r+1}{e}] = N
\end{aligned}.$$

By (2) and (4),

$$\frac{m+r+1}{e} > m-1 \Rightarrow r+1 \geq m \text{ if } m \text{ is large enough.}$$

By (1) and (3),

$$2m \geq r.$$

So, as k large enough ($\Leftrightarrow m$ is large enough),

$$b_k \geq \frac{r+1}{m+r} \geq \frac{m}{3m} = \frac{1}{3} \text{ by (*).}$$

It implies that $\sum (-1)^k b_k$ diverges since b_k does **NOT** tends to zero as k goes infinity.So, we have proved that the series $\sum \frac{(-1)^{[\log k]}}{k}$ diverges.

(3) There is a good exercise by **summation by parts**, we write it as a reference. Assume that $\sum_{k=1}^{\infty} a_k b_k$ converges and $b_n \nearrow$ with $\lim_{n\to\infty} b_n = \infty$. Show that $b_n \sum_{k=n}^{\infty} a_k$ converges.

Proof: First, we show that the convergence of $\sum_{k=1}^{\infty} a_k$ by **Dirichlet Test** as follows. Since $b_n \nearrow \infty$, there exists a positive integer n_0 such that as $n > n_0$, we have $b_n > 0$. So, we have $\left\{\frac{1}{b_{n+n_0}}\right\}_{n=1}^{\infty}$ is decreasing to zero. So

$$\sum_{k=1}^{\infty} a_{k+n_0} = \sum_{k=1}^{\infty} (a_{k+n_0} b_{k+n_0}) \left(\frac{1}{b_{k+n_0}}\right)$$

converges by **Dirichlet Test**.

For the convergence of $b_n \sum_{k=n}^{\infty} a_k$, let $n > n_0$, then

$$b_n \sum_{k=n}^{\infty} a_k = \sum_{k=n}^{\infty} a_k b_k \frac{b_n}{b_k}$$

and define $c_k = a_k b_k$ and $d_k = \frac{b_n}{b_k}$. Note that $\{d_k\}$ is decreasing to zero. Define $C_k = \sum_{j=1}^{k} c_j$ and thus we have

$$\begin{aligned}
b_n \sum_{k=n}^{m} a_k &= \sum_{k=n}^{m} a_k b_k \frac{b_n}{b_k} \\
&= \sum_{k=n}^{m} (C_k - C_{k-1}) d_k \\
&= \sum_{k=n}^{m-1} C_k (d_k - d_{k+1}) + C_m d_m - C_{n-1} d_n.
\end{aligned}$$

So,

$$
\begin{aligned}
b_n \sum_{k=n}^{\infty} a_k &= \sum_{k=n}^{\infty} a_k b_k \frac{b_n}{b_k} \\
&= \sum_{k=n}^{\infty} C_k (d_k - d_{k+1}) + C_\infty d_\infty - C_{n-1} d_n \\
&= \sum_{k=n}^{\infty} C_k (d_k - d_{k+1}) - C_{n-1} d_n
\end{aligned}
$$

by $C_\infty = \lim_{k\to\infty} C_k$ and $\lim_{k\to\infty} d_k = 0.$ In order to show the existence of $\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k$, it suffices to show the existence of $\lim_{n\to\infty} \sum_{k=n}^{\infty} C_k(d_k - d_{k+1}).$ Since the series $\sum_{k=n}^{\infty} C_k(d_k - d_{k+1})$ exists, $\lim_{n\to\infty} \sum_{k=n}^{\infty} C_k(d_k - d_{k+1}) = 0.$ From above results, we have proved the convergence of $\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k$.

Note: We also show that $\lim_{n\to\infty} b_n \sum_{k=n}^{\infty} a_k = 0$ by preceding sayings.

Supplement on the convergence of series.

(A) Show the divergence of $\sum 1/k.$ We will give some methods listed below. If the proof is easy, we will omit the details.

(1) Use **Cauchy Criterion** for series. Since it is easy, we omit the proof.

(2) Just consider

$$
\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2^n} &\geq 1 + \frac{1}{2} + 2\frac{1}{4} + \ldots + 2^{n-1}\frac{1}{2^n} \\
&= 1 + \frac{n}{2} \to \infty.
\end{aligned}
$$

Remark: We can consider

$$1 + \left(\frac{1}{2} + \ldots + \frac{1}{10}\right) + \left(\frac{1}{11} + \ldots + \frac{1}{100}\right) + \ldots \geq 1 + \frac{9}{10} + \frac{90}{100} + \ldots$$

Note: The proof comes from **Jing Yu**.

(3) Use **Mathematical Induction** to show that

$$\frac{1}{k-1} + \frac{1}{k} + \frac{1}{k+1} \geq \frac{3}{k} \text{ if } k \geq 3.$$

Then

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots . \geq 1 + \frac{3}{3} + \frac{3}{6} + \frac{3}{9} + \ldots$$

Remark: The proof comes from **Bernoulli.**

(4) Use **Integral Test**. Since the proof is easy, we omit it.

(5) Use **Cauchy condensation Theorem**. Since the proof is easy, we omit it.

(6) **Euler Summation Formula**, the reader can give it a try. We omit the proof.

(7) The reader can see the book, **Princilpes of Mathematical Analysis by Walter Rudin, Exercise 11-(b) pp 79.**

Suppose $a_n > 0,\ S_n = a_1 + \ldots + a_n,$ and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

Proof: If $a_n \to 0$ as $n \to \infty,$ then by **Limit Comparison Theorem**, we know that $\sum \frac{a_n}{1+a_n}$ diverges. If $\{a_n\}$ does not tend to zero. Claim that $\frac{a_n}{1+a_n}$ does not tend to zero.

Suppose **NOT**, it means that $\lim_{n\to\infty} \frac{a_n}{1+a_n} = 0.$ That is,

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{a_n}} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$

which contradicts our assumption. So, $\sum \frac{a_n}{1+a_n}$ diverges by claim.

(b) Prove that

$$\frac{a_{N+1}}{S_{N+1}} + \ldots + \frac{a_{N+k}}{S_{N+k}} \geq 1 - \frac{S_N}{S_{N+k}}$$

and deduce that $\sum \frac{a_n}{S_n}$ diverges.

Proof: Consider

$$\frac{a_{N+1}}{S_{N+1}} + \ldots + \frac{a_{N+k}}{S_{N+k}} \geq \frac{a_{N+1} + \ldots + a_{N+k}}{S_{N+k}} = 1 - \frac{S_N}{S_{N+k}}, \qquad *$$

then $\sum \frac{a_n}{S_n}$ diverges by Cauchy Criterion with (*).

Remark: Let $a_n = 1,$ then $\sum \frac{a_n}{S_n} = \sum 1/n$ diverges.

(c) Prove that

$$\frac{a_n}{S_n^2} \leq \frac{1}{S_{n-1}} - \frac{1}{S_n}$$

and deduce that $\sum \frac{a_n}{S_n^2}$ converges.

Proof: Consider

$$\frac{1}{S_{n-1}} - \frac{1}{S_n} = \frac{a_n}{S_{n-1}S_n} \geq \frac{a_n}{S_n^2},$$

and

$$\sum \frac{1}{S_{n-1}} - \frac{1}{S_n} \text{ converges by telescoping series with } \frac{1}{S_n} \to 0.$$

So, $\sum \frac{a_n}{S_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \text{ and } \sum \frac{a_n}{1+n^2a_n}?$$

Proof: For $\sum \frac{a_n}{1+na_n}$: as $a_n = 1$ for all $n,$ the series $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{1+n}$ diverges. As

$$a_n = \begin{matrix} 0 \text{ if } n \neq k^2 \\ 1 \text{ if } n = k^2 \end{matrix} ,$$

the series $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{1+k^2}$ converges.

For $\sum \frac{a_n}{1+n^2a_n}$: Consider

$$\frac{a_n}{1+n^2a_n} = \frac{1}{\frac{1}{a_n}+n^2} \leq \frac{1}{n^2},$$

so $\sum \frac{a_n}{1+n^2a_n}$ converges.

(8) Consider $\sum \sin \frac{1}{n}$ diverges.

Proof: Since

$$\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1,$$

the series $\sum \frac{1}{n}$ diverges by **Limit Comparison Theorem.**

Remark: In order to show the series $\sum \sin \frac{1}{n}$ diverges, we consider Cauchy Criterion as follows.

$$n\sin\left(\frac{1}{2n}\right) \leq \sin\left(\frac{1}{n+1}\right) + \ldots + \sin\left(\frac{1}{n+n}\right)$$

and given $x \in R,$ for $n = 0, 1, 2, \ldots,$ we have

$$|\sin nx| \leq n|\sin x|.$$

So,

$$\sin \frac{1}{2} \leq \sin\left(\frac{1}{n+1}\right) + \ldots + \sin\left(\frac{1}{n+n}\right)$$

for all $n.$ Hence, $\sum \sin \frac{1}{n}$ diverges.

Note: There are many methods to show the divergence of the series $\sum \sin \frac{1}{n}.$ We can use **Cauchy Condensation Theorem** to prove it. Besides, by (11), it also owrks.

(9) **O-Stolz's Theorem.**

Proof: Let $S_n = \sum_{j=1}^{n} \frac{1}{j}$ and $X_n = \log n.$ Then by **O-Stolz's Theorem,** it is easy to see

$$\lim_{n \to \infty} S_n = \infty.$$

(10) Since $\Pi_{k=1}^{n} 1 + \frac{1}{k}$ diverges, the series $\sum 1/k$ diverges by **Theorem 8.52**.

(11) **Lemma:** If $\{a_n\}$ is a decreasing sequence and $\sum a_n$ converges. Then $\lim_{n\to\infty} na_n = 0.$

Proof: Since $a_n \to 0$ and $\{a_n\}$ is a decreasing sequence, we conclude that $a_n \geq 0.$ Since $\sum a_n$ converges, given $\varepsilon > 0,$ there exists a positive integer N such that as $n \geq N,$ we have

$$a_n + \ldots + a_{n+k} < \varepsilon/2 \text{ for all } k \in N$$

which implies that

$$(k+1)a_{n+k} < \varepsilon/2 \text{ since } a_n \searrow.$$

Let $k = n,$ then as $n \geq N,$ we have

$$(n+1)a_{2n} < \varepsilon/2$$

which implies that as $n \geq N$

$$2(n+1)a_{2n} < \varepsilon$$

which implies that

$$\lim_{n\to\infty} 2na_{2n} = 0 \text{ since } \lim_{n\to\infty} a_n = 0. \qquad *$$

Similarly, we can show that

$$\lim_{n\to\infty} (2n+1)a_{2n+1} = 0. \qquad **$$

So, by (*) adn (**), we have proved that $\lim_{n\to\infty} na_n = 0.$

Remark: From this, it is clear that $\sum \frac{1}{n}$ diverges. In addition, we have the convergence of $\sum n(a_n - a_{n+1}).$ We give it a proof as follows.

Proof: Write

$$S_n = \sum_{k=1}^{n} k(a_k - a_{k+1})$$
$$= \sum_{k=1}^{n} a_k - na_{n+1},$$

then

$$\lim_{n\to\infty} S_n \text{ exists}$$

since

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \text{ exists and } \lim_{n\to\infty} na_n = 0.$$

(B) Prove that $\sum \frac{1}{p}$ diverges, where p is a prime.

Proof: Given $N,$ let $p_1, \ldots, p_k$ be the primes that divide at least one integer$\leq N.$ Then

$$\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \ldots\right)$$
$$= \prod_{j=1}^{k} \frac{1}{1 - \frac{1}{p_j}}$$
$$\leq \exp\left(\sum_{j=1}^{k} \frac{2}{p_j}\right)$$

by $(1-x)^{-1} \leq e^{2x}$ if $0 \leq x \leq 1/2.$ Hence, $\sum \frac{1}{p}$ diverges since $\sum \frac{1}{n}$ diverges.

Remark: There are many proofs about it. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

(C) Discuss some series related with $\sum \frac{\sin k}{k}$.

STUDY: (1) We have shown that the series $\sum \sin \frac{1}{k}$ diverges.

(2) The series $\sum \sin(na+b)$ diverges where $a \neq n\pi$ for all $n \in Z$ and $b \in R.$

Proof: Suppose that $\sum \sin(na+b)$ converges, then $\lim_{n\to\infty} \sin(na+b) = 0.$ Hence, $\lim_{n\to\infty} |\sin[(n+1)a+b] - \sin(na+b)| = 0.$ Consider

$$|\sin[(n+1)a+b] - \sin(na+b)|$$
$$= \left|2\cos\left(na+b+\frac{a}{2}\right)\sin\left(\frac{a}{2}\right)\right|$$
$$= \left|2\left[\cos(na+b)\cos\left(\frac{a}{2}\right) - \sin(na+b)\sin\left(\frac{a}{2}\right)\right]\sin\left(\frac{a}{2}\right)\right|$$

which implies that

$$\lim_{n\to\infty}|\sin[(n+1)a+b]-\sin(na+b)|$$
$$= \left|\lim_{n\to\infty}\sin[(n+1)a+b]-\sin(na+b)\right|$$
$$= \left|\lim_{n\to\infty}\sup 2\left[\cos(na+b)\cos\left(\frac{a}{2}\right)-\sin(na+b)\sin\left(\frac{a}{2}\right)\right]\right|\left|\sin\left(\frac{a}{2}\right)\right|$$
$$= \left|\lim_{n\to\infty}\sup 2\left[\cos(na+b)\cos\left(\frac{a}{2}\right)\right]\right|\left|\sin\left(\frac{a}{2}\right)\right|$$
$$= |\sin a| \neq 0$$

which is impossible. So, $\sum\sin(na+b)$ diverges.

Remark: (1) By the same method, we can show the divergence of $\sum\cos(na+b)$ if $a \neq n\pi$ for all $n \in Z$ and $b \in R$.

(2) The reader may give it a try to show that,

$$\sum_{n=0}^{p}\cos(na+b) = \frac{\sin\frac{p+1}{2}b}{\sin\frac{b}{2}}\sin\left(a+\frac{p}{2}b\right) \qquad *$$

and

$$\sum_{n=0}^{p}\sin(na+b) = \frac{\sin\frac{p+1}{2}b}{\sin\frac{b}{2}}\cos\left(a+\frac{p}{2}b\right) \qquad **$$

by considering $\sum_{n=0}^{p}e^{i(na+b)}$. However, it is not easy to show the divergence by (*) and (**).

(3) The series $\sum\frac{\sin k}{k}$ converges conditionally.

Proof: First, it is clear that $\sum\frac{\sin k}{k}$ converges by **Dirichlet's Test** since $|\sum\sin k| \leq \left|\frac{1}{\sin\frac{1}{2}}\right|$. In order to show that the divergence of $\sum\left|\frac{\sin k}{k}\right|$, we consider its partial sums as follows: Since

$$\sum_{k=1}^{3n+3}\left|\frac{\sin k}{k}\right| = \sum_{k=0}^{n}\left|\frac{\sin 3k+1}{3k+1}\right| + \left|\frac{\sin 3k+2}{3k+2}\right| + \left|\frac{\sin 3k+3}{3k+3}\right|$$

and note that there is one value is bigger than 1/2 among three values $|\sin 3k+1|$, $|\sin 3k+2|$, and $|\sin 3k+3|$. So,

$$\sum_{k=1}^{3n+3}\left|\frac{\sin k}{k}\right| \geq \sum_{k=0}^{n}\frac{\frac{1}{2}}{3k+3}$$

which implies the divergence of $\sum\left|\frac{\sin k}{k}\right|$.

Remark: The series is like **Dirichlet Integral** $\int_0^\infty\frac{\sin x}{x}dx$. Also, we know that **Dirichlet Integral** converges conditionally.

(4) The series $\sum\frac{|\sin k|^r}{k}$ diverges for any $r \in R$.

Proof: We prove it by three cases as follows.
(a) As $r \leq 0$, we have

$$\sum\frac{|\sin k|^r}{k} \geq \sum\frac{1}{k}.$$

So, $\sum\frac{|\sin k|^r}{k}$ diverges in this case.
(b) As $0 < r \leq 1$, we have

$$\sum \frac{|\sin k|^r}{k} \geq \sum \frac{|\sin k|}{k}.$$

So, $\sum \frac{|\sin k|^r}{k}$ diverges in this case by (3).

(c) As $r > 1,$ we have

$$\sum_{k=1}^{3n+3} \frac{|\sin k|^r}{k} = \sum_{k=0}^{n} \frac{|\sin 3k+1|^r}{3k+1} + \frac{|\sin 3k+2|^r}{3k+2} + \frac{|\sin 3k+3|^r}{3k+3}$$

$$\geq \sum_{k=0}^{n} \frac{\left(\frac{1}{2}\right)^r}{3k+3}.$$

So, $\sum \frac{|\sin k|^r}{k}$ diverges in this case.

(5) The series $\sum \frac{\sin^{2p-1} k}{k},$ where $p \in N,$ converges.

Proof: We will prove that there is a positive integer $M(p)$ such that

$$\left| \sum_{k=1}^{n} \sin^{2p-1} k \right| \leq M(p) \text{ for all } n. \qquad *$$

So, if we can show (*), then by **Dirichlet's Test**, we have proved it. In order to show (*), we claim that $\sin^{2p-1} k$ can be written as a linear combination of $\sin k,\ \sin 3k, \ldots,$ $\sin(2p-1)k.$ So,

$$\left| \sum_{k=1}^{n} \sin^{2p-1} k \right| = \left| \sum_{k=1}^{n} a_1 \sin k + a_2 \sin 3k + \ldots + a_p \sin(2p-1)k \right|$$

$$\leq |a_1| \left| \sum_{k=1}^{n} \sin k \right| + \ldots + |a_p| \left| \sum_{k=1}^{n} \sin(2p-1)k \right|$$

$$\leq \frac{|a_1|}{\left|\sin \frac{1}{2}\right|} + \ldots + \frac{|a_p|}{\left|\sin \frac{2p-1}{2}\right|} := M(p) \text{ by } \textbf{Theorem 8.30.}$$

We show the claim by **Mathematical Induction** as follows. As $p = 1,$ it trivially holds. Assume that as $p = s$ holds, i.e.,

$$\sin^{2s-1} k = \sum_{j=1}^{s} a_j \sin(2j-1)k$$

then as $p = s+1,$ we have

$$
\begin{aligned}
\sin^{2s+1}k &= \sin^2 k(\sin k)^{2s-1} \\
&= \sin^2 k\left(\sum_{j=1}^{s} a_j \sin(2j-1)k\right) \text{ by induction hypothesis} \\
&= \sum_{j=1}^{s} a_j[\sin^2 k \sin(2j-1)k] \\
&= \sum_{j=1}^{s} a_j\left[\frac{1-\cos 2k}{2}\sin(2j-1)k\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^{s} a_j \sin(2j-1)k - \sum_{j=1}^{s} a_j \cos 2k \sin(2j-1)k\right] \\
&= \frac{1}{2}\left\{\sum_{j=1}^{s} a_j \sin(2j-1)k - \frac{1}{2}\sum_{j=1}^{s} a_j[\sin(2j+1)k + \sin(2j-3)k]\right\}
\end{aligned}
$$

which is a linear combination of $\sin k, \dots, \sin(2s+1)k.$ Hence, we have proved the claim by **Mathematical Induction.**

Remark: By the same argument, the series

$$\sum_{k=1}^{n} \cos^{2p-1} k$$

is also bounded, i.e., there exists a positive number $M(p)$ such that

$$\sum_{k=1}^{n} |\cos^{2p-1} k| \le M(p).$$

(6) Define $\sum_{k=1}^{n} \frac{\sin kx}{k} := F_n(x),$ then $\{F_n(x)\}$ is boundedly convergent on $R.$

Proof: Since $F_n(x)$ is a periodic function with period $2\pi,$ and $F_n(x)$ is an odd function. So, it suffices to consider $F_n(x)$ is defined on $[0,\pi].$ In addition, $F_n(0) = 0$ for all $n.$ Hence, the domain I that we consider is $(0,\pi].$ Note that $\frac{\sin kx}{k} = \int_0^x \cos kt dt.$ So,

$$
\begin{aligned}
F_n(x) &= \sum_{k=1}^{n} \frac{\sin kx}{k} \\
&= \int_0^x \sum_{k=1}^{n} \cos kt dt \\
&= \int_0^x \frac{\sin(n+\frac{1}{2})t - \sin(\frac{1}{2})t}{2\sin(\frac{1}{2})t} dt \\
&= \int_0^x \frac{\sin(n+\frac{1}{2})t}{t} dt + \int_0^x \left(\frac{1}{2\sin\frac{t}{2}} - \frac{1}{t}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right) dt - \frac{x}{2} \\
&= \int_0^{(n+\frac{1}{2})x} \frac{\sin t}{t} dt + \int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right) dt - \frac{x}{2}
\end{aligned}
$$

which implies that

$$|F_n(x)| \le \left|\int_0^{(n+\frac{1}{2})x} \frac{\sin t}{t} dt\right| + \left|\int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right) dt\right| + \frac{\pi}{2}.$$

For the part $\left|\int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt\right|$: Since $\int_0^\infty \frac{\sin t}{t} dt$ converges, there exists a positive M_1 such that

$$\left|\int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt\right| \le M_1 \text{ for all } x \in I \text{ and for all } n.$$

For the part $\left|\int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)(\sin(n+\frac{1}{2})t)dt\right|$: Consider

$$\begin{aligned}
&\left|\int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right)dt\right| \\
&\le \int_0^x \frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} dt \text{ since } t-2\sin\frac{t}{2} > 0 \text{ on } I \\
&\le \int_0^\pi \frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} dt := M_2 \text{ since } \lim_{t\to 0^+} \frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}} = 0.
\end{aligned}$$

Hence,

$$|F_n(x)| \le M_1 + M_2 + \frac{\pi}{2} \text{ for all } x \in I \text{ and for all } n.$$

So, $\{F_n(x)\}$ is uniformly bounded on I. It means that $\{F_n(x)\}$ is uniformly bounded on R.

In addition, since

$$F_n(x) = \int_0^{\left(n+\frac{1}{2}\right)x} \frac{\sin t}{t} dt + \int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right)dt - \frac{x}{2},$$

fixed $x \in I$, we have

$$\int_0^\infty \frac{\sin t}{t} dt \text{ exists.}$$

and by **Riemann-Lebesgue Lemma**, in the text book, **pp 313**,

$$\lim_{n\to\infty} \int_0^x \left(\frac{t-2\sin\frac{t}{2}}{2t\sin\frac{t}{2}}\right)\left(\sin\left(n+\frac{1}{2}\right)t\right)dt = 0.$$

So, we have proved that

$$\lim_{n\to\infty} F_n(x) = \int_0^\infty \frac{\sin t}{t} dt - \frac{x}{2} \text{ where } x \in (0,\pi].$$

Hence, $\{F_n(x)\}$ is pointwise convergent on I. It means that $\{F_n(x)\}$ is pointwise convergent on R.

Remark: (1) For definition of being boundedly convergent on a set S, the reader can see the text book, pp **227**.

(2) In the proof, we also shown the value of **Dirichlet Integral**

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

by letting $x = \pi$.

(3) There is another proof on uniform bound. We write it as a reference.

Proof: The domain that we consider is still $(0,\pi]$. Let $\delta > 0$, and consider two cases as follows.

(a) $x \ge \delta > 0$: Using summation by parts,

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \le \left|\frac{1}{n+1}\sum_{k=1}^{n}\frac{\sin kx}{k}\right| + \left|\sum_{k=1}^{n}\left(\sum_{j=1}^{k}\sin jx\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right|$$
$$\le \frac{1}{n+1}\frac{1}{\sin(\frac{\delta}{2})} + \frac{1}{\sin(\frac{\delta}{2})}\left(1-\frac{1}{n+1}\right)$$
$$= \frac{1}{\sin(\frac{\delta}{2})}.$$

(b) $0 < x \le \delta$: Let $N = [\frac{1}{x}]$, consider two cases as follows.
As $n < N$, then

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \le n|x| < N|x| \le 1 \qquad *$$

and as $n \ge N$, then

$$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right|$$
$$\le \left|\sum_{k=1}^{N-1}\frac{\sin kx}{k}\right| + \left|\sum_{k=N}^{n}\frac{\sin kx}{k}\right|$$
$$\le 1 + \left|\sum_{k=N}^{n}\frac{\sin kx}{k}\right| \text{ by (*)}$$
$$\le 1 + \left|\frac{1}{n+1}\sum_{k=1}^{n}\frac{\sin kx}{k}\right| + \left|\frac{1}{N}\sum_{k=1}^{N-1}\frac{\sin kx}{k}\right| + \left|\sum_{k=N}^{n}\left(\sum_{j=1}^{k}\sin jx\right)\left(\frac{1}{k+1}-\frac{1}{k}\right)\right|$$

by summation by parts

$$\le 1 + \frac{1}{(n+1)\sin\frac{x}{2}} + \frac{1}{N\sin\frac{x}{2}} + \left(\frac{1}{N}-\frac{1}{n+1}\right)\frac{1}{\sin\frac{x}{2}}$$
$$= 1 + \frac{2}{[\frac{1}{x}]\sin\frac{x}{2}}.$$

Note that $\lim_{x\to 0^+}\frac{2}{[\frac{1}{x}]\sin\frac{x}{2}} = 4.$ So, we may choose a $\delta' = \delta$ such that

$$\frac{2}{[\frac{1}{x}]\sin\frac{x}{2}} \le 5 \text{ for all } x \in (0,\delta').$$

By preceding sayings, we have proved that $\{F_n(x)\}$ is uniformly bounded on I. It means that $\{F_n(x)\}$ is uniformly bounded on R.

(D) In 1911, **Otto Toeplitz** proves the following. Let $\{a_n\}$ and $\{x_n\}$ be two sequences such that $a_n > 0$ for all n with $\lim_{n\to\infty}\frac{1}{a_1+\ldots+a_n} = 0$ and $\lim_{n\to\infty}x_n = x$. Then

$$\lim_{n\to\infty}\frac{a_1x_1+\ldots+a_nx_n}{a_1+\ldots+a_n} = x.$$

Proof: Let $S_n = \sum_{k=1}^{n}a_k$ and $T_n = \sum_{k=1}^{n}a_kx_k$, then

$$\lim_{n\to\infty}\frac{T_{n+1}-T_n}{S_{n+1}-S_n} = \lim_{n\to\infty}\frac{a_{n+1}x_{n+1}}{a_{n+1}} = \lim_{n\to\infty}x_{n+1} = x.$$

So, by **O-Stolz's Theorem**, we have prove it.

Remark: (1) Let $a_n = 1$, then it is an extension of **Theorem 8.48.**

(2) Show that

$$\lim_{n\to\infty} \frac{\sin\theta + \ldots + \sin\frac{\theta}{n}}{1 + \ldots + \frac{1}{n}} = \theta.$$

Proof: Write

$$\frac{\sin\theta + \ldots + \sin\frac{\theta}{n}}{1 + \ldots + \frac{1}{n}} = \frac{(\frac{1}{1})1\sin\theta + \ldots + (\frac{1}{n})n\sin\frac{\theta}{n}}{1 + \ldots + \frac{1}{n}},$$

the by **Toeplitz's Theorem,** we have proved it.

(E) **Theorem 8.16** emphasizes the decrease of the sequence $\{a_n\}$, we may ask if we remove the condition of decrease, is it true? The answer is **NOT** necessary. For example, let

$$a_n = \frac{1}{n} + \frac{(-1)^{n+1}}{2n}. (> 0)$$

(F) Some questions on series.

(1) Show the convergence of the series $\sum_{n=1}^{\infty} \log n \sin\frac{1}{n}$.

Proof: Since $n\sin\frac{1}{n} < 1$ for all n, $\log n\sin\frac{1}{n} < 0$ for all n. Hence, we consider the new series

$$\sum_{n=1}^{\infty} -\log n\sin\frac{1}{n} = \sum_{n=1}^{\infty} \log\frac{\sin 1/n}{1/n}$$

as follows. Let $a_n = \log\frac{\sin 1/n}{1/n}$ and $b_n = \log\left(1 + \frac{1}{n^2}\right)$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{6}.$$

In addition,

$$\sum b_n \leq \sum \frac{1}{n^2}$$

by $e^x \geq 1 + x$ for all $x \in R$. From the convergence of $\sum b_n$, we have proved that the convergence of $\sum a_n$ by **Limit Comparison Test.**

(2) Suppose that $a_n \in R$, and the series $\sum_{n=1}^{\infty} a_n^2$ converges. Prove that the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges absolutely.

Proof: By $A.P. \geq G.P.$, we have

$$\frac{a_n^2 + \frac{1}{n^2}}{2} \geq \left|\frac{a_n}{n}\right|$$

which implies that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges absolutely.

Remark: We metion that there is another proof by using **Cauchy-Schwarz inequality**. the difference of two proofs is that one considers a_n, and another considers the partial sums S_n.

Proof: By **Cauchy-Schwarz inequality,**

$$\left(\sum_{k=1}^{n} \frac{|a_n|}{k}\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} \frac{1}{k^2}\right)$$

which implies that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges absolutely.

Double sequences and series

8.28 Investigate the existence of the two iterated limits and the double limit of the double sequence f defined by the followings. **Answer**. Double limit exists in (a), (d), (e), (g). Both iterated limits exists in (a), (b), (h). Only one iterated limit exists in (c), (e). Neither iterated limit exists in (d), (f).

(a) $f(p,q) = \frac{1}{p+q}$

Proof: It is easy to know that the double limit exists with $\lim_{p,q\to\infty} f(p,q) = 0$ by definition. We omit it. In addition, $\lim_{p\to\infty} f(p,q) = 0.$ So, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q)) = 0.$ Similarly, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ Hence, we also have the existence of two iterated limits.

(b) $f(p,q) = \frac{p}{p+q}$

Proof: Let $q = np,$ then $f(p,q) = \frac{1}{n+1}.$ It implies that the double limit does not exist. However, $\lim_{p\to\infty} f(p,q) = 1,$ and $\lim_{q\to\infty} f(p,q) = 0.$ So, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q)) = 1,$ and $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$

(c) $f(p,q) = \frac{(-1)^p p}{p+q}$

Proof: Let $q = np,$ then $f(p,q) = \frac{(-1)^p}{n+1}.$ It implies that the double limit does not exist. In addition, $\lim_{q\to\infty} f(p,q) = 0.$ So, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ However, since $\lim_{p\to\infty} f(p,q)$ does not exist, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ does not exist.

(d) $f(p,q) = (-1)^{p+q}(\frac{1}{p} + \frac{1}{q})$

Proof: It is easy to know $\lim_{p,q\to\infty} f(p,q) = 0.$ However, $\lim_{q\to\infty} f(p,q)$ and $\lim_{p\to\infty} f(p,q)$ do not exist. So, neither iterated limit exists.

(e) $f(p,q) = \frac{(-1)^p}{q}$

Proof: It is easy to know $\lim_{p,q\to\infty} f(p,q) = 0.$ In addition, $\lim_{q\to\infty} f(p,q) = 0.$ So, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ However, since $\lim_{p\to\infty} f(p,q)$ does not exist, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ does not exist.

(f) $f(p,q) = (-1)^{p+q}$

Proof: Let $p = nq,$ then $f(p,q) = (-1)^{(n+1)q}.$ It means that the double limit does not exist. Also, since $\lim_{p\to\infty} f(p,q)$ and $\lim_{q\to\infty} f(p,q)$ do not exist, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ and $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q))$ do not exist.

(g) $f(p,q) = \frac{\cos p}{q}$

Proof: Since $|f(p,q)| \le \frac{1}{q},$ then $\lim_{p,q\to\infty} f(p,q) = 0,$ and $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$ However, since $\{\cos p : p \in N\}$ dense in $[-1,1],$ we know that $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q))$ does not exist.

(h) $f(p,q) = \frac{p}{q^2} \sum_{n=1}^{q} \sin \frac{n}{p}$

Proof: Rewrite

$$f(p,q) = \frac{p \sin \frac{q}{2p} \sin \frac{q+1}{2p}}{q^2 \sin \frac{1}{2p}}$$

and thus let $p = nq,$ $f(p,q) = \frac{\sin \frac{1}{2n} \sin\left(\frac{q+1}{2nq}\right)}{nq \sin \frac{1}{2nq}}.$ It means that the double limit does not exist. However, $\lim_{p\to\infty} f(p,q) = \frac{q+1}{2q}$ since $\sin x \sim x$ as $x \to 0.$ So, $\lim_{q\to\infty}(\lim_{p\to\infty} f(p,q)) = \frac{1}{2}.$

Also, $\lim_{q\to\infty} f(p,q) = \lim_{q\to\infty}\left(p\sin\frac{1}{2p}\right)\left(\frac{\sin\frac{q}{2p}\sin\frac{q+1}{2p}}{q^2}\right) = 0$ since $|\sin x| \leq 1.$ So, $\lim_{p\to\infty}(\lim_{q\to\infty} f(p,q)) = 0.$

8.29 Prove the following statements:

(a) A double series of positive terms converges if, and only if, the set of partial sums is bounded.

Proof: (⇒)Suppose that $\sum_{m,n} f(m,n)$ converges, say $\sum_{m,n} f(m,n) = A_1,$ then it means that $\lim_{p,q\to\infty} s(p,q) = A_1.$ Hence, given $\varepsilon = 1,$ there exists a positive integer N such that as $p,q \geq N,$ we have

$$|s(p,q)| \leq |A_1| + 1.$$

So, let $A_2 = \max\{s(p,q) : 1 \leq p,q < N\},$ we have $|s(p,q)| \leq \max(A_1,A_2)$ for all $p,q.$ Hence, we have proved the set of partial sums is bounded.

(⇐)Suppose that the set of partial sums is bounded by M, i.e., if $S = \{s(p,q) : p,q \in N\},$ then $\sup S := A \leq M.$ Hence, given $\varepsilon > 0,$ then there exists a $s(p_1,q_1) \in S$ such that

$$A - \varepsilon < s(p_1,q_1) \leq A.$$

Choose $N = \max(p_1,q_1),$ then

$$A - \varepsilon < s(p,q) \leq A \text{ for all } p,q \geq N$$

since every term is positive. Hence, we have proved $\lim_{p,q\to\infty} s(p,q) = A.$ That is, $\sum_{m,n} f(m,n)$ converges.

(b) A double series converges if it converges absolutely.

Proof: Let $s_1(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q}|f(m,n)|$ and $s_2(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q} f(m,n),$ we want to show that the existence of $\lim_{p,q\to\infty} s_2(p,q)$ by the existence of $\lim_{p,q\to\infty} s_1(p,q)$ as follows.

Since $\lim_{p,q\to\infty} s_1(p,q)$ exists, say its limit $a.$ Then $\lim_{p\to\infty} s_1(p,p) = a.$ It implies that $\lim_{p\to\infty} s_2(p,p)$ converges, say its limit $b.$ So, given $\varepsilon > 0,$ there exists a positive integer N such that as $p,q \geq N$

$$|s_1(p,p) - s_1(q,q)| < \varepsilon/2$$

and

$$|s_2(N,N) - b| < \varepsilon/2.$$

So, as $p \geq q \geq N,$

$$\begin{aligned} |s_2(p,q) - b| &= |[s_2(N,N) - b] + [s_2(p,q) - s_2(N,N)]| \\ &< \varepsilon/2 + |s_2(p,q) - s_2(N,N)| \\ &< \varepsilon/2 + s_1(p,p) - s_1(N,N) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Similarly for $q \geq p \geq N.$ Hence, we have shown that

$$\lim_{p,q\to\infty} s_2(p,q) = b.$$

That is, we have prove that a double series converges if it converges absolutely.

(c) $\sum_{m,n} e^{-(m^2+n^2)}$ converges.

Proof: Let $f(m,n) = e^{-(m^2+n^2)},$ then by **Theorem 8.44**, we have proved that

$\sum_{m,n} e^{-(m^2+n^2)}$ converges since $\sum_{m,n} e^{-(m^2+n^2)} = \sum_m e^{-m^2} \sum_n e^{-n^2}$.

Remark: $\sum_{m,n=1}^{\infty} e^{-(m^2+n^2)} = \sum_{m=1}^{\infty} e^{-m^2} \sum_{n=1}^{\infty} e^{-n^2} = \left(\frac{e}{e^2-1}\right)^2$.

8.30 Asume that the double series $\sum_{m,n} a(n)x^{mn}$ converges absolutely for $|x| < 1$. Call its sum $S(x)$. Show that each of the following series also converges absolutely for $|x| < 1$ and has sum $S(x)$:

$$\sum_{n=1}^{\infty} a(n)\frac{x^n}{1-x^n}, \ \sum_{n=1}^{\infty} A(n)x^n, \text{ where } A(n) = \sum_{d|n} a(d).$$

Proof: By **Theorem 8.42**,

$$\sum_{m,n} a(n)x^{mn} = \sum_{n=1}^{\infty} a(n)\sum_{m=1}^{\infty} x^{mn} = \sum_{n=1}^{\infty} a(n)\frac{x^n}{1-x^n} \text{ if } |x| < 1.$$

So, $\sum_{n=1}^{\infty} a(n)\frac{x^n}{1-x^n}$ converges absolutely for $|x| < 1$ and has sum $S(x)$.

Since every term in $\sum_{m,n} a(n)x^{mn}$, the term appears once and only once in $\sum_{n=1}^{\infty} A(n)x^n$. The converse also true. So, by **Theorem 8.42** and **Theorem 8.13**, we know that

$$\sum_{n=1}^{\infty} A(n)x^n = \sum_{m,n} a(n)x^{mn} = S(x).$$

8.31 If α is real, show that the double series $\sum_{m,n}(m+in)^{-\alpha}$ converges absolutely if, and only if, $\alpha > 2$. Hint. Let $s(p,q) = \sum_{m=1}^{p}\sum_{n=1}^{q}|m+in|^{-\alpha}$. The set

$$\{m+in : m = 1,2,\ldots p, n = 1,2,\ldots,p\}$$

consists of p^2 complex numbers of which one has absolute value $\sqrt{2}$, three satisfy $|1+2i| \le |m+in| \le 2\sqrt{2}$, five satisfy $|1+3i| \le |m+in| \le 3\sqrt{2}$, etc. Verify this geometricall and deduce the inequlity

$$2^{-\alpha/2}\sum_{n=1}^{p}\frac{2n-1}{n^\alpha} \le s(p,p) \le \sum_{n=1}^{p}\frac{2n-1}{(n^2+1)^{\alpha/2}}.$$

Proof: Since the hint is trivial, we omit the proof of hint. From the hint, we have

$$\sum_{n=1}^{p}\frac{2n-1}{\left(n\sqrt{2}\right)^\alpha} \le s(p,p) = \sum_{m=1}^{p}\sum_{n=1}^{p}|m+in|^{-\alpha} \le \sum_{n=1}^{p}\frac{2n-1}{(1+n^2)^{\alpha/2}}.$$

Thus, it is clear that the double series $\sum_{m,n}(m+in)^{-\alpha}$ converges absolutely if, and only if, $\alpha > 2$.

8.32 (a) Show that the Cauchy product of $\sum_{n=0}^{\infty}(-1)^{n+1}/\sqrt{n+1}$ with itself is a divergent series.

Proof: Since

$$
\begin{aligned}
c_n &= \sum_{k=0}^{n} a_k b_{n-k} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} \\
&= (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}}
\end{aligned}
$$

and let $f(k) = \sqrt{(n-k+1)(k+1)} = \sqrt{-(k-\frac{n}{2})^2 + (\frac{n+2}{2})^2} \leq \frac{n+2}{2}$ for $k = 0, 1, \ldots, n$. Hence,

$$
\begin{aligned}
|c_n| &= \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}} \\
&\geq \frac{2(n+1)}{n+2} \to 2 \text{ as } n \to \infty.
\end{aligned}
$$

That is, the Cauchy product of $\sum_{n=0}^{\infty} (-1)^{n+1}/\sqrt{n+1}$ with itself is a divergent series.

(b) Show that the Cauchy product of $\sum_{n=0}^{\infty} (-1)^{n+1}/(n+1)$ with itself is the series

$$
2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).
$$

Proof: Since

$$
\begin{aligned}
c_n &= \sum_{k=0}^{n} a_k b_{n-k} \\
&= \sum_{k=0}^{n} \frac{(-1)^n}{(n-k+1)(k+1)} \\
&= (-1)^n \sum_{k=0}^{n} \frac{1}{n+2}\left(\frac{1}{k+1} + \frac{1}{n-k+1}\right) \\
&= \frac{2(-1)^n}{n+2} \sum_{k=0}^{n} \frac{1}{k+1},
\end{aligned}
$$

we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} c_n &= \sum_{n=0}^{\infty} \frac{2(-1)^n}{n+2} \sum_{k=0}^{n} \frac{1}{k+1} \\
&= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n+1}\right) \\
&= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right).
\end{aligned}
$$

(c) Does this converge ? Why?

Proof: Yes by the same argument in **Exercise 8.26**.

8.33 Given two absolutely convergent power series, say $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, having sums $A(x)$ and $B(x)$, respectively, show that $\sum_{n=0}^{\infty} c_n x^n = A(x)B(x)$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proof: By **Theorem 8.44** and **Theorem 8.13**, it is clear.

Remark: We can use **Mertens' Theorem**, then it is clear.

8.34 A series of the form $\sum_{n=1}^{\infty} a_n/n^s$ is called a Dirichlet series. Given two absolutely convergent Dirichlet series, say $\sum_{n=1}^{\infty} a_n/n^s$ and $\sum_{n=1}^{\infty} b_n/n^s$, having sums $A(s)$ and $B(s)$, respectively, show that $\sum_{n=1}^{\infty} c_n/n^s = A(s)B(s)$, where $c_n = \sum_{d|n} a_d b_{n/d}$.

Proof: By **Theorem 8.44** and **Theorem 8.13**, we have

$$\left(\sum_{n=1}^{\infty} a_n/n^s\right)\left(\sum_{n=1}^{\infty} b_n/n^s\right) = \left(\sum_{n=1}^{\infty} C_n\right)$$

where

$$\begin{aligned} C_n &= \sum_{d|n} a_d d^{-s} b_{n/d} (n/d)^{-s} \\ &= n^{-s} \sum_{d|n} a_d b_{n/d} \\ &= c_n/n^s. \end{aligned}$$

So, we have proved it.

8.35 $\zeta(s) = \sum_{n=1}^{\infty} 1/n^s,\ s > 1,$ show that $\zeta^2(s) = \sum_{n=1}^{\infty} d(n)/n^s,$ where $d(n)$ is the number of positive divisors of n (including 1 and n).

Proof: It is clear by **Exercise 8.34**. So, we omit the proof.

Ces'aro summability

8.36 Show that each of the following series has $(C,1)$ sum 0 :

(a) $1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 - - + + \cdots$.

Proof: It is clear that $|s_1 + \ldots + s_n| \leq 1$ for all $n,$ where s_n means that the nth partial sum of given series. So,

$$\left|\frac{s_1 + \ldots + s_n}{n}\right| \leq \frac{1}{n}$$

which implies that the given series has $(C,1)$ sum 0.

(b) $\frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2} - 1 + + - \cdots$.

Proof: It is clear that $|s_1 + \ldots + s_n| \leq \frac{1}{2}$ for all $n,$ where s_n means that the nth partial sum of given series. So,

$$\left|\frac{s_1 + \ldots + s_n}{n}\right| \leq \frac{1}{2n}$$

which implies that the given series has $(C,1)$ sum 0.

(c) $\cos x + \cos 3x + \cos 5x + \cdots$ (x real, $x \neq m\pi$).

Proof: Let $s_n = \cos x + \ldots + \cos(2n-1)x,$ then

$$\begin{aligned} s_n &= \sum_{j=1}^{n} \cos(2k-1)x \\ &= \frac{\sin 2nx}{2\sin x}. \end{aligned}$$

So,

$$\begin{aligned} \left| \frac{\sum_{j=1}^{n} s_j}{n} \right| &= \left| \frac{\sum_{j=1}^{n} \sin 2jx}{2n \sin x} \right| \\ &= \left| \frac{\sin nx \sin(n+1)x}{2n \sin x \sin x} \right| \\ &\leq \frac{1}{2n(\sin x)^2} \to 0 \end{aligned}$$

which implies that the given series has $(C,1)$ sum 0.

8.37 Given a series $\sum a_n$, let

$$s_n = \sum_{k=1}^{n} a_k,\ t_n = \sum_{k=1}^{n} k a_k,\ \sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_k.$$

Prove that:

(a) $t_n = (n+1)s_n - n\sigma_n$

Proof: Define $S_0 = 0$, and thus

$$\begin{aligned} t_n &= \sum_{k=1}^{n} k a_k \\ &= \sum_{k=1}^{n} k(s_k - s_{k-1}) \\ &= \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n} k s_{k-1} \\ &= \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n-1} (k+1) s_k \\ &= \sum_{k=1}^{n} k s_k - \sum_{k=1}^{n} (k+1) s_k + (n+1) s_n \\ &= (n+1)s_n - \sum_{k=1}^{n} s_k \\ &= (n+1)s_n - n\sigma_n. \end{aligned}$$

(b) If $\sum a_n$ is $(C,1)$ summable, then $\sum a_n$ converges if, and only if, $t_n = o(n)$ as $n \to \infty$.

Proof: Assume that $\sum a_n$ converges. Then $\lim_{n\to\infty} s_n$ exists, say its limit a. By (a), we have

$$\frac{t_n}{n} = \frac{n+1}{n} s_n - \sigma_n.$$

Then by **Theorem 8.48**, we also have $\lim_{n\to\infty} \sigma_n = a$. Hence,

$$\begin{aligned}\lim_{n\to\infty}\frac{t_n}{n} &= \lim_{n\to\infty}\frac{n+1}{n}s_n - \sigma_n \\ &= \lim_{n\to\infty}\frac{n+1}{n}\lim_{n\to\infty}s_n - \lim_{n\to\infty}\sigma_n \\ &= 1\cdot a - a \\ &= 0\end{aligned}$$

which is $t_n = o(n)$ as $n \to \infty$.

Conversely, assume that $t_n = o(n)$ as $n \to \infty$, then by (a), we have

$$\frac{n}{n+1}\frac{t_n}{n} + \frac{n}{n+1}\sigma_n = s_n$$

which implies that (note that $\lim_{n\to\infty}\sigma_n$ exists by hypothesis)

$$\begin{aligned}\lim_{n\to\infty}s_n &= \lim_{n\to\infty}\frac{n}{n+1}\frac{t_n}{n} + \frac{n}{n+1}\sigma_n \\ &= \lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\frac{t_n}{n} + \lim_{n\to\infty}\frac{n}{n+1}\lim_{n\to\infty}\sigma_n \\ &= 1\cdot 0 + 1\cdot\lim_{n\to\infty}\sigma_n \\ &= \lim_{n\to\infty}\sigma_n\end{aligned}$$

That is, $\sum a_n$ converges.

(c) $\sum a_n$ is $(C,1)$ summable if, and only if, $\sum t_n/n(n+1)$ converges.

Proof: Consider

$$\begin{aligned}\frac{t_n}{n(n+1)} &= \frac{s_n}{n} - \frac{\sigma_n}{n+1} \\ &= \frac{n\sigma_n - (n-1)\sigma_{n-1}}{n} - \frac{\sigma_n}{n+1} \\ &= \frac{n}{n+1}\sigma_n - \frac{n-1}{n}\sigma_{n-1}\end{aligned}$$

which implies that

$$\sum_{k=1}^{n}\frac{t_k}{k(k+1)} = \frac{n}{n+1}\sigma_n. \qquad *$$

($\Rightarrow$)Suppose that $\sum a_n$ is $(C,1)$ summable, i.e., $\lim_{n\to\infty}\sigma_n$ exists. Then $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{t_k}{k(k+1)}$ exists by (*).

($\Leftarrow$)Suppose that $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{t_k}{k(k+1)}$ exists. Then $\lim_{n\to\infty}\sigma_n$ exists by (*). Hence, $\sum a_n$ is $(C,1)$ summable.

8.38 Given a monotonic $\{a_n\}$ of positive terms, such that $\lim_{n\to\infty}a_n = 0$. Let

$$s_n = \sum_{k=1}^{n}a_k,\ u_n = \sum_{k=1}^{n}(-1)^k a_k,\ v_n = \sum_{k=1}^{n}(-1)^k s_k.$$

Prove that:

(a) $v_n = \frac{1}{2}u_n + (-1)^n s_n/2$.

Proof: Define $s_0 = 0$, and thus consider

$$\begin{aligned} u_n &= \sum_{k=1}^{n}(-1)^k a_k \\ &= \sum_{k=1}^{n}(-1)^k (s_k - s_{k-1}) \\ &= \sum_{k=1}^{n}(-1)^k s_k + \sum_{k=1}^{n}(-1)^{k+1} s_{k-1} \\ &= \sum_{k=1}^{n}(-1)^k s_k + \sum_{k=1}^{n}(-1)^k s_k + (-1)^{n+1} s_n \\ &= 2v_n + (-1)^{n+1} s_n \end{aligned}$$

which implies that

$$v_n = \frac{1}{2} u_n + (-1)^n s_n/2.$$

(b) $\sum_{n=1}^{\infty}(-1)^n s_n$ is $(C,1)$ summable and has **Ces'aro sum** $\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n$.

Proof: First, $\lim_{n\to\infty} u_n$ exists since it is an alternating series. In addition, since $\lim_{n\to\infty} a_n = 0,$ we know that $\lim_{n\to\infty} s_n/n = 0$ by **Theorem 8.48**. Hence,

$$\frac{v_n}{n} = \frac{u_n}{2n} + (-1)^n \frac{s_n}{2n} \to 0 \text{ as } n \to \infty.$$

Consider by (a),

$$\begin{aligned} \frac{\sum_{k=1}^{n} v_k}{n} &= \frac{\frac{1}{2}\left(\sum_{k=1}^{n} u_k\right) + \frac{1}{2}\left(\sum_{k=1}^{n}(-1)^k s_k\right)}{n} \\ &= \frac{\sum_{k=1}^{n} u_k}{2n} + \frac{v_n}{2n} \\ &\to \frac{1}{2}\lim_{n\to\infty} u_k \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n a_n \end{aligned}$$

by **Theorem 8.48**.

(c) $\sum_{n=1}^{\infty}(-1)^n(1 + \frac{1}{2} + \ldots + \frac{1}{n}) = -\log\sqrt{2}\ (C,1).$

Proof: By (b) and $\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\log 2$, it is clear.

Infinite products

8.39 Determine whether or not the following infinite products converges. Find the value of each convergent product.

(a) $\prod_{n=2}^{\infty}\left(1 - \frac{2}{n(n+1)}\right)$

Proof: Consider

$$1 - \frac{2}{n(n+1)} = \frac{(n-1)(n+2)}{n(n+1)},$$

we have

$$\prod_{n=2}^{n}\left(1-\frac{2}{k(k+1)}\right) = \prod_{n=2}^{n}\frac{(k-1)(k+2)}{k(k+1)}$$
$$= \frac{1\cdot 4}{2\cdot 3}\frac{2\cdot 5}{3\cdot 4}\frac{3\cdot 6}{4\cdot 5}\cdots\frac{(n-1)(n+2)}{n(n+1)}$$
$$= \frac{n+2}{3n}$$

which implies that

$$\prod_{n=2}^{\infty}\left(1-\frac{2}{n(n+1)}\right) = \frac{1}{3}.$$

(b) $\prod_{n=2}^{\infty}(1-n^{-2})$

Proof: Consider

$$1-n^{-2} = \frac{(n-1)(n+1)}{nn},$$

we have

$$\prod_{k=2}^{n}(1-k^{-2}) = \prod_{k=2}^{n}\frac{(k-1)(k+1)}{kk}$$
$$= \frac{n+1}{2n}$$

which implies that

$$\prod_{n=2}^{\infty}(1-n^{-2}) = 1/2.$$

(c) $\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}$

Proof: Consider

$$\frac{n^3-1}{n^3+1} = \frac{(n-1)(n^2+n+1)}{(n+1)(n^2-n+1)}$$
$$= \frac{(n-1)(n^2+n+1)}{(n+1)\left[(n-1)^2+(n-1)+1\right]}$$

we have (let $f(k) = (k-1)^2+(k-1)+1$),

$$\prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \prod_{k=2}^{n}\frac{(k-1)(k^2+k+1)}{(k+1)\left[(k-1)^2+(k-1)+1\right]}$$
$$= \frac{2}{3}\frac{n^2+n+1}{n(n+1)}$$

which implies that

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1} = \frac{2}{3}.$$

(d) $\prod_{n=0}^{\infty}(1+z^{(2^n)})$ if $|z| < 1$.

Proof: Consider

$$\prod_{k=0}^{n} 1+z^{(2^k)} = (1+z)(1+z^2)\cdots(1+z^{(2^n)})$$

which implies that

$$(1-z)\prod_{k=0}^{n} 1+z^{(2^k)} = 1-z^{(2^{n+1})}$$

which implies that (if $|z| < 1$)

$$\prod_{k=0}^{n} 1+z^{(2^k)} = \frac{1-z^{(2^{n+1})}}{1-z} \to \frac{1}{1-z} \text{ as } n \to \infty.$$

So,

$$\prod_{n=0}^{\infty}(1+z^{(2^n)}) = \frac{1}{1-z}.$$

8.40 If each partial sum s_n of the convergent series $\sum a_n$ is not zero and if the sum itself is not zero, show that the infinite product $a_1 \prod_{n=2}^{\infty}(1-a_n/s_{n-1})$ converges and has the value $\sum_{n=1}^{\infty} a_n$.

Proof: Consider

$$\begin{aligned} a_1 \prod_{k=2}^{n}(1+a_k/s_{k-1}) &= a_1 \prod_{k=2}^{n} \frac{s_{k-1}+a_k}{s_{k-1}} \\ &= a_1 \prod_{k=2}^{n} \frac{s_k}{s_{k-1}} \\ &= s_n \to \sum a_n \neq 0. \end{aligned}$$

So, the infinite product $a_1 \prod_{n=2}^{\infty}(1-a_n/s_{n-1})$ converges and has the value $\sum_{n=1}^{\infty} a_n$.

8.41 Find the values of the following products by establishing the following identities and summing the series:

(a) $\prod_{n=2}^{\infty}(1-\frac{1}{2^n-2}) = 2\sum_{n=1}^{\infty} 2^{-n}$.

Proof: Consider

$$1-\frac{1}{2^n-2} = \frac{2^n-1}{2^n-2} = \frac{1}{2}\frac{2^n-1}{2^{n-1}-1},$$

we have

$$\begin{aligned} \prod_{k=2}^{n}\left(1-\frac{1}{2^k-2}\right) &= \prod_{k=2}^{n}\frac{1}{2}\frac{2^k-1}{2^{k-1}-1} \\ &= 2^{-(n-1)}\prod_{k=2}^{n}\frac{2^k-1}{2^{k-1}-1} \\ &= 2^{-(n-1)}(2^n-1) \\ &= 2^{-(n-1)}(2^{n-1}+\ldots+1) \\ &= 1+\ldots+\frac{1}{2^{n-1}} \\ &= \sum_{k=1}^{n}\frac{1}{2^{k-1}} \\ &= 2\sum_{k=1}^{n}\frac{1}{2^k}. \end{aligned}$$

So,

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{2^n-2}\right) = 2\sum_{n=1}^{\infty} 2^{-n}$$
$$= 2.$$

(b) $\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right) = 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

Proof: Consider

$$1+\frac{1}{n^2-1} = \frac{n^2}{n^2-1} = \frac{nn}{(n-1)(n+1)},$$

we have

$$\prod_{k=2}^{n}\left(1+\frac{1}{k^2-1}\right) = \prod_{k=2}^{n}\frac{kk}{(k-1)(k+1)}$$
$$= 2\frac{n}{n+1}$$
$$= 2\left(1-\frac{1}{n+1}\right)$$
$$= 2\sum_{k=1}^{n}\frac{1}{k(k+1)}.$$

So,

$$\prod_{n=2}^{\infty}\left(1+\frac{1}{n^2-1}\right) = 2\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$$
$$= 2.$$

8.42 Determine all real x for which the product $\prod_{n=1}^{\infty}\cos(x/2^n)$ converges and find the value of the product when it does converge.

Proof: If $x \neq m\pi$, where $m \in Z$, then $\sin\frac{x}{2^n} \neq 0$ for all $n \in N$. Hence,

$$\prod_{k=1}^{n}\cos(x/2^k) = \frac{2^n\sin\frac{x}{2^n}}{2^n\sin\frac{x}{2^n}}\prod_{k=1}^{n}\cos(x/2^k) = \frac{\sin x}{2^n\sin\frac{x}{2^n}} \to \frac{\sin x}{x}.$$

If $x = m\pi$, where $m \in Z$. Then as $m = 0$, it is clear that the product converges to 1. So, we consider $m \neq 0$ as follows. Since $x = m\pi$, choosing n large enough, i.e., as $n \geq N$ so that $\sin\frac{x}{2^n} \neq 0$. Hence,

$$\prod_{k=1}^{n}\cos(x/2^k) = \prod_{k=1}^{N-1}\cos(x/2^k)\prod_{k=N}^{n}\cos(x/2^k)$$
$$= \prod_{k=1}^{N-1}\cos(x/2^k)\frac{\sin(x/2^{N-1})}{2^{n-N+1}\sin(x/2^n)}$$

and note that

$$\lim_{n\to\infty}\frac{\sin(x/2^{N-1})}{2^{n-N+1}\sin(x/2^n)} = \frac{\sin(x/2^{N-1})}{x/2^{N-1}}.$$

Hence,

$$\prod_{k=1}^{\infty}\cos(x/2^k) = \frac{\sin(x/2^{N-1})}{x/2^{N-1}}\prod_{k=1}^{N-1}\cos(x/2^k).$$

So, by above sayings, we have prove that the convergence of the product for all $x \in R$.

8.43 (a) Let $a_n = (-1)^n/\sqrt{n}$ for $n = 1, 2, \ldots$ Show that $\prod(1 + a_n)$ diverges but that $\sum a_n$ converges.

Proof: Clearly, $\sum a_n$ converges since it is alternating series. Consider

$$\begin{aligned}\prod_{k=2}^{2n} 1 + a_k &= \prod_{k=2}^{2n} 1 + \frac{(-1)^k}{\sqrt{k}} \\ &= \left(1 + \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{3}}\right)\left(1 + \frac{1}{\sqrt{4}}\right)\cdots\left(1 - \frac{1}{\sqrt{2n-1}}\right)\left(1 + \frac{1}{\sqrt{2n}}\right) \\ &\leq \left(1 + \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{\sqrt{4}}\right)\left(1 + \frac{1}{\sqrt{4}}\right)\cdots\left(1 - \frac{1}{\sqrt{2n}}\right)\left(1 + \frac{1}{\sqrt{2n}}\right) \\ &= \left(1 + \frac{1}{\sqrt{2}}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2n}\right) \qquad *\end{aligned}$$

and note that

$$\prod_{k=2}^{n} 1 - \frac{1}{2k} := p_n$$

is decreasing. From the divergence of $\sum \frac{1}{2k}$, we know that $p_n \to 0$. So,

$$\prod_{k=2}^{\infty} 1 + a_k = 0.$$

That is, $\prod_{k=2}^{\infty} 1 + a_k$ diverges to zero.

(b) Let $a_{2n-1} = -1/\sqrt{n}$, $a_{2n} = 1/\sqrt{n} + 1/n$ for $n = 1, 2, \ldots$ Show that $\prod(1 + a_n)$ converges but $\sum a_n$ diverges.

Proof: Clearly, $\sum a_n$ diverges. Consider

$$\begin{aligned}\prod_{k=2}^{2n} 1 + a_k &= (1 + a_2)(1 + a_3)(1 + a_4)\cdots(1 + a_{2n}) \\ &= 3(1 + a_3)(1 + a_4)\cdots(1 + a_{2n}) \\ &= 3\left(1 - \frac{1}{2\sqrt{2}}\right)\cdots\left(1 - \frac{1}{n\sqrt{n}}\right) \qquad *\end{aligned}$$

and

$$\begin{aligned}\prod_{k=2}^{2n+1} 1 + a_k &= (1 + a_2)(1 + a_3)(1 + a_4)\cdots(1 + a_{2n})(1 + a_{2n+1}) \\ &= 3\left(1 - \frac{1}{2\sqrt{2}}\right)\cdots\left(1 - \frac{1}{n\sqrt{n}}\right)\left(1 - \frac{1}{\sqrt{n+1}}\right) \qquad **\end{aligned}$$

By (*) and (**), we know that

$$\prod(1 + a_n) \text{ converges}$$

since $\prod_{k=2}^{n}\left(1 - \frac{1}{k\sqrt{k}}\right)$ converges.

8.44 Assume that $a_n \geq 0$ for each $n = 1, 2, \ldots$ Assume further that

$$\frac{a_{2n+2}}{1+a_{2n+2}} < a_{2n+1} < \frac{a_{2n}}{1+a_{2n}} \text{ for } n = 1,2,\ldots$$

Show that $\prod_{k=1}^{\infty}\left(1+(-1)^k a_k\right)$ converges if, and only if, $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

Proof: First, we note that if $\frac{a}{1+a} > b$, then $(1+a)(1-b) > 1$, and if $b > \frac{1+c}{c}$, then $1 > (1-b)(1+c)$. Hence, by hypothesis, we have

$$1 < (1+a_{2n})(1-a_{2n+1}) \qquad *$$

and

$$1 > (1+a_{2n+2})(1-a_{2n+1}). \qquad **$$

($\Leftarrow$)Suppose that $\sum_{k=1}^{\infty}(-1)^k a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Consider **Cauchy Condition for product,**

$$\left|\left(1+(-1)^{p+1}a_{p+1}\right)\left(1+(-1)^{p+2}a_{p+2}\right)\cdots(1+(-1)^{p+q}a_{p+q})-1\right| \text{ for } q = 1,2,3,\ldots.$$

If $p+1 = 2m$, and $q = 2l$, then

$$\begin{aligned}
&\left|\left(1+(-1)^{p+1}a_{p+1}\right)\left(1+(-1)^{p+2}a_{p+2}\right)\cdots(1+(-1)^{p+q}a_{p+q})-1\right| \\
&= |(1+a_{2m})(1-a_{2m+1})\cdots(1+a_{2m+2l})-1| \\
&\le 1+a_{2m}-1 \text{ by (*) and (**)} \\
&= a_{2m} \to 0.
\end{aligned}$$

Similarly for other cases, so we have proved that $\prod_{k=1}^{\infty}\left(1+(-1)^k a_k\right)$ converges by **Cauchy Condition for product.**

($\Rightarrow$)This is a counterexample as follows. Let $a_n = (-1)^n\left[\left(\exp\frac{(-1)^n}{\sqrt{n}}\right)-1\right] \ge 0$ for all n, then it is easy to show that

$$\frac{a_{2n+2}}{1+a_{2n+2}} < a_{2n+1} < \frac{a_{2n}}{1+a_{2n}} \text{ for } n = 1,2,\ldots$$

In addition,

$$\prod_{k=1}^{n}\left(1+(-1)^k a_k\right) = \prod_{k=1}^{n}\exp\frac{(-1)^k}{\sqrt{k}} = \exp\left(\sum_{k=1}^{n}\frac{(-1)^k}{\sqrt{k}}\right) \to \exp(-\log 2) \text{ as } n \to \infty.$$

However, consider

$$\begin{aligned}
&\sum_{k=1}^{n}(a_{2k}-a_{2k-1}) \\
&= \sum_{k=1}^{n}\left[\exp\left(\frac{1}{\sqrt{2k}}\right)-\exp\left(\frac{-1}{\sqrt{2k-1}}\right)\right] \\
&= \sum_{k=1}^{n}\exp(b_k)\left(\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{2k-1}}\right), \text{ where } b_k \in \left(\frac{-1}{\sqrt{2k-1}},\frac{1}{\sqrt{2k}}\right) \\
&\ge \sum_{k=1}^{n}\exp(-1)\left(\frac{1}{\sqrt{2k}}+\frac{1}{\sqrt{2k-1}}\right) \to \infty \text{ as } n \to \infty.
\end{aligned}$$

So, by **Theorem 8.13**, we proved the divergence of $\sum_{k=1}^{\infty}(-1)^k a_k$.

8.45 A complex-valued sequence $\{f(n)\}$ is called **multiplicative** if $f(1) = 1$ and if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. (See Section 1.7) It is called **completely multiplicative** if

$$f(1) = 1 \text{ and if } f(mn) = f(m)f(n) \text{ for all } m \text{ and } n.$$

(a) If $\{f(n)\}$ is **multiplicative** and if the series $\sum f(n)$ converges absolutely, prove that

$$\sum_{n=1}^{\infty} f(n) = \prod_{k=1}^{\infty}\{1 + f(p_k) + f(p_k^2) + \ldots\},$$

where p_k denote the kth prime, the product being absolutely convergent.

Proof: We consider the partial product $P_m = \prod_{k=1}^{m}\{1 + f(p_k) + f(p_k^2) + \ldots\}$ and show that $P_m \to \sum_{n=1}^{\infty} f(n)$ as $m \to \infty$. Writing each factor as a geometric series we have

$$P_m = \prod_{k=1}^{m}\{1 + f(p_k) + f(p_k^2) + \ldots\},$$

a product of a finite number of absolutely convergent series. When we multiple these series together and rearrange the terms such that a typical term of the new absolutely convergent series is

$$f(n) = f(p_1^{a_1}) \bullet\bullet\bullet f(p_m^{a_m}), \text{ where } n = p_1^{a_1} \bullet\bullet\bullet p_m^{a_m},$$

and each $a_i \geq 0$. Therefore, we have

$$P_m = \sum_1 f(n),$$

where $\sum_1$ is summed over those n having all their prime factors $\leq p_m$. By the **unique factorization theorem (Theorem 1.9),** each such n occors once and only once in $\sum_1$. Substracting P_m from $\sum_{n=1}^{\infty} f(n)$, we get

$$\sum_{n=1}^{\infty} f(n) - P_m = \sum_{n=1}^{\infty} f(n) - \sum_1 f(n) = \sum_2 f(n)$$

where $\sum_2$ is summed over those n having at least one prime factor $> p_m$. Since these n occors among the integers $> p_m$, we have

$$\left|\sum_{n=1}^{\infty} f(n) - P_m\right| \leq \sum_{n>p_m} |f(n)|.$$

As $m \to \infty$ the last sum tends to 0 because $\sum_{n=1}^{\infty} f(n)$ converges, so $P_m \to \sum_{n=1}^{\infty} f(n)$.

To prove that the product converges absolutely we use **Theorem 8.52**. The product has the form $\prod(1 + a_k)$, where

$$a_k = f(p_k) + f(p_k^2) + \ldots.$$

The series $\sum|a_k|$ converges since it is dominated by $\sum_{n=1}^{\infty}|f(n)|$. Thereofore, $\prod(1 + a_k)$ also converges absolutely.

Remark: The method comes from **Euler**. By the same method, it also shows that there are infinitely many primes. The reader can see the book, **An Introduction To The Theory Of Numbers by Loo-Keng Hua, pp 91-93. (Chinese Version)**

(b) If, in addition, $\{f(n)\}$ is **completely multiplicative,** prove that the formula in (a) becomes

$$\sum_{n=1}^{\infty} f(n) = \prod_{k=1}^{\infty} \frac{1}{1 - f(p_k)}.$$

Note that Euler's product for $\zeta(s)$ (Theorem 8.56) is the special case in which $f(n) = n^{-s}$.

Proof: By (a), if $\{f(n)\}$ is **completely multiplicative,** then rewrite

$$\begin{aligned} 1 + f(p_k) + f(p_k^2) + \ldots &= \sum_{n=0}^{\infty} [f(p_k)]^n \\ &= \frac{1}{1 - f(p_k)} \end{aligned}$$

since $|f(p_k)| < 1$ for all p_k. (Suppose **NOT,** then $|f(p_k)| \geq 1 \Rightarrow |f(p_k^n)| = |f(p_k)|^n \geq 1$ contradicts to $\lim_{n\to\infty} f(n) = 0.$).

Hence,

$$\sum_{n=1}^{\infty} f(n) = \prod_{k=1}^{\infty} \frac{1}{1 - f(p_k)}.$$

8.46 This exercise outlines a simple proof of the formula $\zeta(2) = \pi^2/6$. Start with the inequality $\sin x < x < \tan x$, valid for $0 < x < \pi/2$, taking recipocals, and square each member to obtain

$$\cot^2 x < \frac{1}{x^2} < 1 + \cot^2 x.$$

Now put $x = k\pi/(2m+1)$, where k and m are integers, with $1 \leq k \leq m$, and sum on k to obtain

$$\sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1} < \frac{(2m+1)^2}{\pi^2} \sum_{k=1}^{m} \frac{1}{k^2} < m + \sum_{k=1}^{m} \cot^2 \frac{k\pi}{2m+1}.$$

Use the formula of **Exercise 1.49(c)** to deduce the ineqaulity

$$\frac{m(2m-1)\pi^2}{3(2m+1)^2} < \sum_{k=1}^{m} \frac{1}{k^2} < \frac{2m(m+1)\pi^2}{3(2m+1)^2}$$

Now let $m \to \infty$ to obtain $\zeta(2) = \pi^2/6$.

Proof: The proof is clear if we follow the hint and **Exercise 1.49 (c)**, so we omit it.

8.47 Use an argument similar to that outlined in **Exercise 8.46** to prove that $\zeta(4) = \pi^4/90$.

Proof: The proof is clear if we follow the **Exercise 8.46** and **Exercise 1.49 (c)**, so we omit it.

Remark: (1) From this, it is easy to compute the value of $\zeta(2s)$, where $s \in \{n : n \in N\}$. In addition, we will learn some new method such as Fourier series and so on, to find the value of **Riemann zeta** function.

(2) Ther is an open problem that $\zeta(2s-1)$, where $s \in \{n \in N : n > 1\}$.

Sequences of Functions

Uniform convergence

9.1 Assume that $f_n \to f$ uniformly on S and that each f_n is bounded on S. Prove that $\{f_n\}$ is uniformly bounded on S.

Proof: Since $f_n \to f$ uniformly on S, then given $\varepsilon = 1$, there exists a positive integer n_0 such that as $n \geq n_0$, we have

$$|f_n(x) - f(x)| \leq 1 \text{ for all } x \in S. \tag{*}$$

Hence, $f(x)$ is bounded on S by the following

$$|f(x)| \leq |f_{n_0}(x)| + 1 \leq M(n_0) + 1 \text{ for all } x \in S. \tag{**}$$

where $|f_{n_0}(x)| \leq M(n_0)$ for all $x \in S$.

Let $|f_1(x)| \leq M(1), ..., |f_{n_0-1}(x)| \leq M(n_0 - 1)$ for all $x \in S$, then by (*) and (**),

$$|f_n(x)| \leq 1 + |f(x)| \leq M(n_0) + 2 \text{ for all } n \geq n_0.$$

So,

$$|f_n(x)| \leq M \text{ for all } x \in S \text{ and for all } n$$

where $M = \max(M(1), ..., M(n_0 - 1), M(n_0) + 2)$.

Remark: (1) In the proof, we also shows that the limit function f is bounded on S.

(2) There is another proof. We give it as a reference.

Proof: Since Since $f_n \to f$ uniformly on S, then given $\varepsilon = 1$, there exists a positive integer n_0 such that as $n \geq n_0$, we have

$$|f_n(x) - f_{n+k}(x)| \leq 1 \text{ for all } x \in S \text{ and } k = 1, 2, ...$$

So, for all $x \in S$, and $k = 1, 2, ...$

$$|f_{n_0+k}(x)| \leq 1 + |f_{n_0}(x)| \leq M(n_0) + 1 \tag{*}$$

where $|f_{n_0}(x)| \leq M(n_0)$ for all $x \in S$.

Let $|f_1(x)| \leq M(1), ..., |f_{n_0-1}(x)| \leq M(n_0 - 1)$ for all $x \in S$, then by (*),

$$|f_n(x)| \leq M \text{ for all } x \in S \text{ and for all } n$$

where $M = \max(M(1), ..., M(n_0 - 1), M(n_0) + 1)$.

9.2 Define two sequences $\{f_n\}$ and $\{g_n\}$ as follows:

$$f_n(x) = x\left(1 + \frac{1}{n}\right) \text{ if } x \in R,\ n = 1, 2, ...,$$

$$g_n(x) = \begin{cases} \frac{1}{n} \text{ if } x = 0 \text{ or if } x \text{ is irrational,} \\ b + \frac{1}{n} \text{ if } x \text{ is rational, say } x = \frac{a}{b},\ b > 0. \end{cases}$$

Let $h_n(x) = f_n(x) g_n(x)$.

(a) Prove that both $\{f_n\}$ and $\{g_n\}$ converges uniformly on every bounded interval.

Proof: Note that it is clear that

$$\lim_{n\to\infty} f_n(x) = f(x) = x,\ \text{for all } x \in R$$

and

$$\lim_{n\to\infty} g_n(x) = g(x) = \begin{cases} 0 \text{ if } x = 0 \text{ or if } x \text{ is irrational,} \\ b \text{ if } x \text{ is ratonal, say } x = \frac{a}{b}, b > 0. \end{cases}$$

In addition, in order to show that $\{f_n\}$ and $\{g_n\}$ converges uniformly on every bounded interval, it suffices to consider the case of any compact interval $[-M, M]$, $M > 0$.

Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$\frac{M}{n} < \varepsilon \text{ and } \frac{1}{n} < \varepsilon.$$

Hence, for this ε, we have as $n \geq N$

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\right| \leq \frac{M}{n} < \varepsilon \text{ for all } x \in [-M, M]$$

and

$$|g_n(x) - g(x)| \leq \frac{1}{n} < \varepsilon \text{ for all } x \in [-M, M].$$

That is, we have proved that $\{f_n\}$ and $\{g_n\}$ converges uniformly on every bounded interval.

Remark: In the proof, we use the easy result directly from definition of uniform convergence as follows. If $f_n \to f$ uniformly on S, then $f_n \to f$ uniformly on T for every subset T of S.

(b) Prove that $h_n(x)$ does not converges uniformly on any bounded interval.

Proof: Write

$$h_n(x) = \begin{cases} \frac{x}{n}\left(1+\frac{1}{n}\right) & \text{if } x = 0 \text{ or } x \text{ is irrational} \\ a + \frac{a}{n}\left(1+\frac{1}{b}+\frac{1}{bn}\right) & \text{if } x \text{ is rational, say } x = \frac{a}{b} \end{cases}.$$

Then

$$\lim_{n\to\infty} h_n(x) = h(x) = \begin{cases} 0 \text{ if } x = 0 \text{ or } x \text{ is irrational} \\ a \text{ if } x \text{ is rational, say } x = \frac{a}{b} \end{cases}.$$

Hence, if $h_n(x)$ converges uniformly on any bounded interval I, then $h_n(x)$ converges uniformly on $[c,d] \subseteq I$. So, given $\varepsilon = \max(|c|,|d|) > 0$, there is a positive integer N such that as $n \geq N$, we have

$$\begin{aligned} \max(|c|,|d|) &> |h_n(x) - h(x)| \\ &= \begin{cases} \left|\frac{x}{n}\left(1+\frac{1}{n}\right)\right| = \frac{|x|}{n}\left|1+\frac{1}{n}\right| & \text{if } x \in Q^c \cap [c,d] \text{ or } x = 0 \\ \left|\frac{a}{n}\left(1+\frac{1}{b}+\frac{1}{bn}\right)\right| & \text{if } x \in Q \cap [c,d],\ x = \frac{a}{b} \end{cases} \end{aligned}$$

which implies that ($x \in [c,d] \cap Q^c$ or $x = 0$)

$$\max(|c|,|d|) > \frac{|x|}{n}\left|1+\frac{1}{n}\right| \geq \frac{|x|}{n} \geq \frac{\max(|c|,|d|)}{n}$$

which is absurb. So, $h_n(x)$ does not converges uniformly on any bounded interval.

9.3

Assume that $f_n \to f$ uniformly on S, $g_n \to f$ uniformly on S.

(a) Prove that $f_n + g_n \to f + g$ uniformly on S.

Proof: Since $f_n \to f$ uniformly on S, and $g_n \to f$ uniformly on S, then given $\varepsilon > 0$, there is a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in S$$

and

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2} \text{ for all } x \in S.$$

Hence, for this ε, we have as $n \geq N$,

$$\begin{aligned} |f_n(x) + g_n(x) - f(x) - g(x)| &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \varepsilon \text{ for all } x \in S. \end{aligned}$$

That is, $f_n + g_n \to f + g$ uniformly on S.

Remark: There is a similar result. We write it as follows. If $f_n \to f$ uniformly on S, then $cf_n \to cf$ uniformly on S for any real c. Since the proof is easy, we omit the proof.

(b) Let $h_n(x) = f_n(x) g_n(x)$, $h(x) = f(x) g(x)$, if $x \in S$. Exercise 9.2 shows that the assertion $h_n \to h$ uniformly on S is, in general, incorrect. Prove that it is correct if each f_n and each g_n is bounded on S.

Proof: Since $f_n \to f$ uniformly on S and each f_n is bounded on S, then f is bounded on S by **Remark (1)** in the **Exercise 9.1.** In addition, since $g_n \to g$ uniformly on S and each g_n is bounded on S, then g_n is uniformly bounded on S by **Exercise 9.1.**

Say $|f(x)| \leq M_1$ for all $x \in S$, and $|g_n(x)| \leq M_2$ for all x and all n. Then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2(M_2 + 1)} \text{ for all } x \in S$$

and

$$|g_n(x) - g(x)| < \frac{\varepsilon}{2(M_1 + 1)} \text{ for all } x \in S$$

which implies that as $n \geq N$, we have

$$\begin{aligned} |h_n(x) - h(x)| &= |f_n(x) g_n(x) - f(x) g(x)| \\ &= |[f_n(x) - f(x)][g_n(x)] + [f(x)][g_n(x) - g(x)]| \\ &\leq |f_n(x) - f(x)| |g_n(x)| + |f(x)| |g_n(x) - g(x)| \\ &< \frac{\varepsilon}{2(M_2 + 1)} M_2 + M_1 \frac{\varepsilon}{2(M_1 + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for all $x \in S$. So, $h_n \to h$ uniformly on S.

9.4 Assume that $f_n \to f$ uniformly on S and suppose there is a constant $M > 0$ such that $|f_n(x)| \leq M$ for all x in S and all n. Let g be continuous on the closure of the disk $B(0; M)$ and define $h_n(x) = g[f_n(x)]$, $h(x) = g[f(x)]$, if $x \in S$. Prove that $h_n \to h$ uniformly on S.

Proof: Since g is continuous on a compact disk $B(0;M)$, g is uniformly continuous on $B(0;M)$. Given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|x-y| < \delta$, where $x, y \in S$, we have

$$|g(x) - g(y)| < \varepsilon. \tag{*}$$

For this $\delta > 0$, since $f_n \to f$ uniformly on S, then there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \delta \text{ for all } x \in S. \tag{**}$$

Hence, by (*) and (**), we conclude that given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|g(f_n(x)) - g(f(x))| < \varepsilon \text{ for all } x \in S.$$

Hence, $h_n \to h$ uniformly on S.

9.5 (a) Let $f_n(x) = 1/(nx+1)$ if $0 < x < 1, n = 1, 2, ...$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $(0,1)$.

Proof: First, it is clear that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1)$. Supppos that $\{f_n\}$ converges uniformly on $(0,1)$. Then given $\varepsilon = 1/2$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| = \left|\frac{1}{1+nx}\right| < 1/2 \text{ for all } x \in (0,1).$$

So, the inequality holds for all $x \in (0,1)$. It leads us to get a contradiction since

$$\frac{1}{1+Nx} < \frac{1}{2} \text{ for all } x \in (0,1) \Rightarrow \lim_{x\to 0^+} \frac{1}{1+Nx} = 1 < 1/2.$$

That is, $\{f_n\}$ converges **NOT** uniformly on $(0,1)$.

(b) Let $g_n(x) = x/(nx+1)$ if $0 < x < 1, n = 1, 2, ...$ Prove that $g_n \to 0$ uniformly on $(0,1)$.

Proof: First, it is clear that $\lim_{n\to\infty} g_n(x) = 0$ for all $x \in (0,1)$. Given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$1/n < \varepsilon$$

which implies that

$$|g_n(x) - g| = \left|\frac{x}{1+nx}\right| = \left|\frac{1}{\frac{1}{x}+n}\right| < \frac{1}{n} < \varepsilon.$$

So, $g_n \to 0$ uniformly on $(0,1)$.

9.6 Let $f_n(x) = x^n$. The sequence $\{f_n(x)\}$ converges pointwise but not uniformly on $[0,1]$. Let g be continuous on $[0,1]$ with $g(1) = 0$. Prove that the sequence $\{g(x)x^n\}$ converges uniformly on $[0,1]$.

Proof: It is clear that $f_n(x) = x^n$ converges **NOT** uniformly on $[0,1]$ since each term of $\{f_n(x)\}$ is continuous on $[0,1]$ and its limit function

$$f = \begin{cases} 0 \text{ if } x \in [0,1) \\ 1 \text{ if } x = 1. \end{cases}$$

is not a continuous function on $[0,1]$ by **Theorem 9.2**.

In order to show $\{g(x)x^n\}$ converges uniformly on $[0,1]$, it suffices to shows that $\{g(x)x^n\}$ converges uniformly on $[0,1)$. Note that

$$\lim_{n\to\infty} g(x)x^n = 0 \text{ for all } x \in [0,1).$$

We partition the interval $[0,1)$ into two subintervals: $[0,1-\delta]$ and $(1-\delta,1)$.

As $x \in [0,1-\delta]$: Let $M = \max_{x\in[0,1]}|g(x)|$, then given $\varepsilon > 0$, there is a positive integer N such that as $n \geq N$, we have

$$M(1-\delta)^n < \varepsilon$$

which implies that for all $x \in [0,1-\delta]$,

$$|g(x)x^n - 0| \leq M|x^n| \leq M(1-\delta)^n < \varepsilon.$$

Hence, $\{g(x)x^n\}$ converges uniformly on $[0,1-\delta]$.

As $x \in (1-\delta,1)$: Since g is continuous at 1, given $\varepsilon > 0$, there exists a $\delta > 0$ such that as $|x-1| < \delta$, where $x \in [0,1]$, we have

$$|g(x) - g(1)| = |g(x) - 0| = |g(x)| < \varepsilon$$

which implies that for all $x \in (1-\delta,1)$,

$$|g(x)x^n - 0| \leq |g(x)| < \varepsilon.$$

Hence, $\{g(x)x^n\}$ converges uniformly on $(1-\delta,1)$.

So, from above sayings, we have proved that the sequence of functions $\{g(x)x^n\}$ converges uniformly on $[0,1]$.

Remark: It is easy to show the followings by definition. So, we omit the proof.

(1) Suppose that for all $x \in S$, the limit function f exists. If $f_n \to f$ uniformly on $S_1 (\subseteq S)$, then $f_n \to f$ uniformly on S, where $\#(S - S_1) < +\infty$.

(2) Suppose that $f_n \to f$ uniformly on S and on T. Then $f_n \to f$ uniformly on $S \cup T$.

9.7 Assume that $f_n \to f$ uniformly on S and each f_n is continuous on S. If $x \in S$, let $\{x_n\}$ be a sequence of points in S such that $x_n \to x$. Prove that $f_n(x_n) \to f(x)$.

Proof: Since $f_n \to f$ uniformly on S and each f_n is continuous on S, by **Theorem 9.2**, the limit function f is also continuous on S. So, given $\varepsilon > 0$, there is a $\delta > 0$ such that as $|y - x| < \delta$, where $y \in S$, we have

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

For this $\delta > 0$, there exists a positive integer N_1 such that as $n \geq N_1$, we have

$$|x_n - x| < \delta.$$

Hence, as $n \geq N_1$, we have

$$|f(x_n) - f(x)| < \frac{\varepsilon}{2}. \qquad (*)$$

In addition, since $f_n \to f$ uniformly on S, given $\varepsilon > 0$, there exists a positive integer $N \geq N_1$ such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2} \text{ for all } x \in S$$

which implies that

$$|f_n(x_n) - f(x_n)| < \frac{\varepsilon}{2}. \qquad (**)$$

By (*) and (**), we obtain that given $\varepsilon > 0$, there exists a positie integer N such that as $n \geq N$, we have

$$\begin{aligned} \left|f_n(x_n) - f(x)\right| &= \left|f_n(x_n) - f(x_n)\right| + \left|f(x_n) - f(x)\right| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

That is, we have proved that $f_n(x_n) \to f(x)$.

9.8 Let $\{f_n\}$ be a seuqnece of continuous functions defined on a compact set S and assume that $\{f_n\}$ converges pointwise on S to a limit function f. Prove that $f_n \to f$ uniformly on S if, and only if, the following two conditions hold.:

(i) The limit function f is continuous on S.

(ii) For every $\varepsilon > 0$, there exists an $m > 0$ and a $\delta > 0$, such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \varepsilon$ for all x in S and all $k = 1, 2, ...$

Hint. To prove the sufficiency of (i) and (ii), show that for each x_0 in S there is a neighborhood of $B(x_0)$ and an integer k (depending on x_0) such that

$$|f_k(x) - f(x)| < \delta \text{ if } x \in B(x_0).$$

By compactness, a finite set of integers, say $A = \{k_1, ..., k_r\}$, has the property that, for each x in S, some k in A satisfies $|f_k(x) - f(x)| < \delta$. Uniform convergence is an easy consequences of this fact.

Proof: ($\Rightarrow$) Suppose that $f_n \to f$ uniformly on S, then by **Theorem 9.2**, the limit function f is continuous on S. In addition, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in S$$

Let $m = N$, and $\delta = \varepsilon$, then (ii) holds.

($\Leftarrow$) Suppose that (i) and (ii) holds. We prove $f_k \to f$ uniformly on S as follows. By (ii), given $\varepsilon > 0$, there exists an $m > 0$ and a $\delta > 0$, such that $n > m$ and $|f_k(x) - f(x)| < \delta$ implies $|f_{k+n}(x) - f(x)| < \varepsilon$ for all x in S and all $k = 1, 2, ...$

Consider $\left|f_{k(x_0)}(x_0) - f(x_0)\right| < \delta$, then there exists a $B(x_0)$ such that as $x \in B(x_0) \cap S$, we have

$$\left|f_{k(x_0)}(x) - f(x)\right| < \delta$$

by continuity of $f_{k(x_0)}(x) - f(x)$. Hence, by (ii) as $n > m$

$$\left|f_{k(x_0)+n}(x) - f(x)\right| < \varepsilon \text{ if } x \in B(x_0) \cap S. \tag{*}$$

Note that S is compact and $S = \cup_{x \in S}(B(x) \cap S)$, then $S = \cup_{k=1}^{p}(B(x_k) \cap S)$. So, let $N = \max_{i=1}^{p}(k(x_p) + m)$, as $n > N$, we have

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } x \in S$$

with help of (*). That is, $f_n \to f$ uniformly on S.

9.9 (a) Use Exercise 9.8 to prove the following theorem of Dini: **If $\{f_n\}$ is a sequence of real-valued continuous functions converginf pointwise to a continuous limit function f on a compact set S, and if $f_n(x) \geq f_{n+1}(x)$ for each x in S and every $n = 1, 2, ...$, then $f_n \to f$ uniformly on S.**

Proof: By **Exercise 9.8**, in order to show that $f_n \to f$ uniformly on S, it suffices to show that (ii) holds. Since $f_n(x) \to f(x)$ and $f_{n+1}(x) \leq f_n(x)$ on S, then fixed $x \in S$, and given $\varepsilon > 0$, there exists a positive integer $N(x) = N$ such that as $n \geq N$, we have

$$0 \leq f_n(x) - f(x) < \varepsilon.$$

Choose $m = 1$ and $\delta = \varepsilon$, then by $f_{n+1}(x) \leq f_n(x)$, then (ii) holds. We complete it.

Remark: (1) **Dini's Theorem** is important in Analysis; we suggest the reader to keep it in mind.

(2) There is another proof by using **Cantor Intersection Theorem**. We give it as follows.

Proof: Let $g_n = f_n - f$, then g_n is continuous on S, $g_n \to 0$ pointwise on S, and $g_n(x) \geq g_{n+1}(x)$ on S. If we can show $g_n \to 0$ uniformly on S, then we have proved that $f_n \to f$ uniformly on S.

Given $\varepsilon > 0$, and consider $S_n := \{x : g_n(x) \geq \varepsilon\}$. Since each $g_n(x)$ is continuous on a compact set S, we obtain that S_n is compact. In addition, $S_{n+1} \subseteq S_n$ since $g_n(x) \geq g_{n+1}(x)$ on S. Then

$$\cap S_n \neq \phi \tag{*}$$

if each S_n is non-empty by **Cantor Intersection Theorem**. However (*) contradicts to $g_n \to 0$ pointwise on S. Hence, we know that there exists a positive integer N such that as $n \geq N$,

$$S_n = \phi.$$

That is, given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|g_n(x) - 0| < \varepsilon.$$

So, $g_n \to 0$ uniformly on S.

(b) Use he sequence in Exercise 9.5(a) to show that compactness of S is essential in Dini's Theorem.

Proof: Let $f_n(x) = \frac{1}{1+nx}$, where $x \in (0,1)$. Then it is clear that each $f_n(x)$ is continuous on $(0,1)$, the limit function $f(x) = 0$ is continuous on $(0,1)$, and $f_{n+1}(x) \leq f_n(x)$ for all $x \in (0,1)$. However, $f_n \to f$ not uniformly on $(0,1)$ by **Exercise 9.5 (a)**. Hence, compactness of S is essential in Dini's Theorem.

9.10 Let $f_n(x) = n^c x (1-x^2)^n$ for x real and $n \geq 1$. Prove that $\{f_n\}$ converges pointwsie on $[0,1]$ for every real c. Determine those c for which the convergence is uniform on $[0,1]$ and those for which term-by-term integration on $[0,1]$ leads to a correct result.

Proof: It is clear that $f_n(0) \to 0$ and $f_n(1) \to 0$. Consider $x \in (0,1)$, then $|1-x^2| := r < 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n^c r^n x = 0 \text{ for any real } c.$$

Hence, $f_n \to 0$ pointwise on $[0,1]$.

Consider

$$f_n'(x) = n^c \left(1-x^2\right)^{n-1} (2n-1) \left(\frac{1}{2n-1} - x^2\right),$$

then each f_n has the absolute maximum at $x_n = \frac{1}{\sqrt{2n-1}}$.

As $c < 1/2$, we obtain that

$$\begin{aligned} |f_n(x)| &\leq |f_n(x_n)| \\ &= \frac{n^c}{\sqrt{2n-1}} \left(1 - \frac{1}{2n-1}\right)^n \\ &= n^{c-\frac{1}{2}} \left[\sqrt{\frac{n}{2n-1}} \left(1-\frac{1}{2n-1}\right)^n\right] \to 0 \text{ as } n \to \infty. \qquad (*) \end{aligned}$$

In addition, as $c \geq 1/2$, if $f_n \to 0$ uniformly on $[0,1]$, then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$|f_n(x)| < \varepsilon \text{ for all } x \in [0,1]$$

which implies that as $n \geq N$,

$$|f_n(x_n)| < \varepsilon$$

which contradicts to

$$\lim_{n\to\infty} f_n(x_n) = \begin{cases} \frac{1}{\sqrt{2e}} \text{ if } c = 1/2 \\ \infty \text{ if } c > 1/2 \end{cases}. \quad (**)$$

From (*) and (**), we conclude that only as $c < 1/2$, the seqences of functions converges uniformly on $[0,1]$.

In order to determine those c for which term-by-term integration on $[0,1]$, we consider

$$\int_0^1 f_n(x)\,dx = \frac{n^c}{2(n+1)}$$

and

$$\int_0^1 f(x)\,dx = \int_0^1 0dx = 0.$$

Hence, only as $c < 1$, we can integrate it term-by-term.

9.11 Prove that $\sum x^n(1-x)$ converges pointwise but not uniformly on $[0,1]$, whereas $\sum (-1)^n x^n (1-x)$ converges uniformly on $[0,1]$. This illustrates that **uniform convergence of $\sum f_n(x)$ along with pointwise convergence of $\sum |f_n(x)|$ does not necessarily imply uniform convergence of $\sum |f_n(x)|$.**

Proof: Let $s_n(x) = \sum_{k=0}^n x^k(1-x) = 1 - x^{n+1}$, then

$$s_n(x) \to \begin{cases} 1 \text{ if } x \in [0,1) \\ 0 \text{ if } x = 1 \end{cases}.$$

Hence, $\sum x^n(1-x)$ converges pointwise but not uniformly on $[0,1]$ by **Theorem 9.2** since each s_n is continuous on $[0,1]$.

Let $g_n(x) = x^n(1-x)$, then it is clear that $g_n(x) \geq g_{n+1}(x)$ for all $x \in [0,1]$, and $g_n(x) \to 0$ uniformly on $[0,1]$ by **Exercise 9.6**. Hence, by **Dirichlet's Test for uniform convergence**, we have proved that $\sum(-1)^n x^n(1-x)$ converges uniformly on $[0,1]$.

9.12 Assume that $g_{n+1}(x) \leq g_n(x)$ for each x in T and each $n = 1, 2, ...,$ and suppose that $g_n \to 0$ uniformly on T. Prove that $\sum (-1)^{n+1} g_n(x)$ converges uniformly on T.

Proof: It is clear by **Dirichlet's Test for uniform convergence.**

9.13 Prove Abel's test for uniform convergence: Let $\{g_n\}$ be a sequence of real-valued functions such that $g_{n+1}(x) \leq g_n(x)$ for each x in T and for every $n = 1, 2, ...$ If $\{g_n\}$ is uniformly bounded on T and if $\sum f_n(x)$ converges uniformly on T, then $\sum f_n(x) g_n(x)$ also converges uniformly on T.

Proof: Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$. Then

$$s_n(x) = \sum_{k=1}^{n} f_k(x) g_k(x) = F_n g_1(x) + \sum_{k=1}^{n} (F_n(x) - F_k(x)) (g_{k+1}(x) - g_k(x))$$

and hence if $n > m$, we can write

$$s_n(x) - s_m(x) = (F_n(x) - F_m(x)) g_{m+1}(x) + \sum_{k=m+1}^{n} (F_n(x) - F_k(x)) (g_{k+1}(x) - g_k(x))$$

Hence, if M is an uniform bound for $\{g_n\}$, we have

$$|s_n(x) - s_m(x)| \leq M |F_n(x) - F_m(x)| + 2M \sum_{k=m+1}^{n} |F_n(x) - F_k(x)|. \quad (*)$$

Since $\sum f_n(x)$ converges uniformly on T, given $\varepsilon > 0$, there exists a positive integer N such that as $n > m \geq N$, we have

$$|F_n(x) - F_m(x)| < \frac{\varepsilon}{M+1} \text{ for all } x \in T \quad (**)$$

By (*) and (**), we have proved that as $n > m \geq N$,

$$|s_n(x) - s_m(x)| < \varepsilon \text{ for all } x \in T.$$

Hence, $\sum f_n(x) g_n(x)$ also converges uniformly on T.

Remark: In the proof, we establish the lemma as follows. We write it as a reference.

(**Lemma**) If $\{a_n\}$ and $\{b_n\}$ are two sequences of complex numbers, define

$$A_n = \sum_{k=1}^{n} a_k.$$

Then we have the identity

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k) \qquad \text{(i)}$$

$$= A_n b_1 + \sum_{k=1}^{n} (A_n - A_k)(b_{k+1} - b_k). \qquad \text{(ii)}$$

Proof: The identity (i) comes from **Theorem 8.27**. In order to show (ii), it suffices to consider

$$b_{n+1} = b_1 + \sum_{k=1}^{n} b_{k+1} - b_k.$$

9.14 Let $f_n(x) = x/(1+nx^2)$ if $x \in R$, $n = 1, 2, ...$ Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

(a) Prove that $f'(x)$ exists for every x but that $f'(0) \neq g(0)$. For what values of x is $f'(x) = g(x)$?

Proof: It is easy to show that the limit function $f = 0$, and by $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$, we have

$$\lim_{n\to\infty} f_n'(x) = g(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}.$$

Hence, $f'(x)$ exists for every x and $f'(0) = 0 \neq g(0) = 1$. In addition, it is clear that as $x \neq 0$, we have $f'(x) = g(x)$.

(b) In what subintervals of R does $f_n \to f$ uniformly?

Proof: Note that

$$\frac{1+nx^2}{2} \geq \sqrt{n}\,|x|$$

by $A.P. \geq G.P.$ for all real x. Hence,

$$\left| \frac{x}{1+nx^2} \right| \leq \frac{1}{2\sqrt{n}}$$

which implies that $f_n \to f$ uniformly on R.

(c) In what subintervals of R does $f_n' \to g$ uniformly?

Proof: Since each $f_n' = \frac{1-nx^2}{(1+nx^2)^2}$ is continuous on R, and the limit function g is continuous on $R - \{0\}$, then by **Theorem 9.2**, the interval I that we consider does not contains 0. Claim that $f_n' \to g$ uniformly on such interval $I = [a, b]$ which does not contain 0 as follows.

Consider

$$\left| \frac{1-nx^2}{(1+nx^2)^2} \right| \leq \frac{1}{1+nx^2} \leq \frac{1}{na^2},$$

so we know that $f_n' \to g$ uniformly on such interval $I = [a, b]$ which does not contain 0.

9.15 Let $f_n(x) = (1/n) e^{-n^2x^2}$ if $x \in R$, $n = 1, 2, \ldots$ Prove that $f_n \to 0$ uniformly on R, that $f_n' \to 0$ pointwise on R, but that the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

Proof: It is clear that $f_n \to 0$ uniformly on R, that $f_n' \to 0$ pointwise on R. Assume that $f_n' \to 0$ uniformly on $[a, b]$ that contains 0. We will prove that it is impossible as follows.

We may assume that $0 \in (a, b)$ since other cases are similar. Given $\varepsilon = \frac{1}{e}$, then there exists a positive integer N' such that as $n \geq \max\left(N', \frac{1}{b}\right) := N$ ($\Rightarrow \frac{1}{N} \leq b$), we have

$$|f_n'(x) - 0| < \frac{1}{e} \text{ for all } x \in [a, b]$$

which implies that

$$\left| 2\frac{Nx}{e^{(Nx)^2}} \right| < \frac{1}{e} \text{ for all } x \in [a, b]$$

which implies that, let $x = \frac{1}{N}$,

$$\frac{2}{e} < \frac{1}{e}$$

which is absurb. So, the convergence of $\{f_n'\}$ is not uniform on any interval containing the origin.

9.16 Let $\{f_n\}$ be a sequence of real-valued continuous functions defined on $[0,1]$ and assume that $f_n \to f$ uniformly on $[0,1]$. Prove or disprove

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

Proof: By **Theorem 9.8**, we have

$$\lim_{n\to\infty} \int_0^{1} f_n(x)\,dx = \int_0^1 f(x)\,dx. \tag{*}$$

Note that $\{f_n\}$ is uniform bound, say $|f_n(x)| \leq M$ for all $x \in [0,1]$ and all n by **Exercise 9.1.** Hence,

$$\left|\int_{1-1/n}^{1} f_n(x)\,dx\right| \leq \frac{M}{n} \to 0. \tag{**}$$

Hence, by (*) and (**), we have

$$\lim_{n\to\infty} \int_0^{1-1/n} f_n(x)\,dx = \int_0^1 f(x)\,dx.$$

9.17 Mathematicinas from Slobbovia decided that the Riemann integral was too complicated so that they replaced it by **Slobbovian integral**, defined as follows: If f is a function defined on the set Q of rational numbers in $[0,1]$, the Slobbovian integral of f, denoted by $S(f)$, is defined to be the limit

$$S(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right),$$

whenever the limit exists. Let $\{f_n\}$ be a sequence of functions such that $S(f_n)$ exists for each n and such that $f_n \to f$ uniformly on Q. Prove that $\{S(f_n)\}$ converges, that $S(f)$ exists, and $S(f_n) \to S(f)$ as $n \to \infty$.

Proof: $f_n \to f$ uniformly on Q, then given $\varepsilon > 0$, there exists a positive integer N such that as $n > m \geq N$, we have

$$|f_n(x) - f(x)| < \varepsilon/3 \tag{1}$$

and

$$\left|f_n(x) - f_m(x)\right| < \varepsilon/2. \tag{2}$$

So, if $n > m \geq N$,

$$\begin{aligned}
\left|S(f_n) - S(f_m)\right| &= \left|\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k}\left(f_n\left(\frac{j}{k}\right) - f_m\left(\frac{j}{k}\right)\right)\right| \\
&= \lim_{k\to\infty} \frac{1}{k}\left|\sum_{j=1}^{k}\left(f_n\left(\frac{j}{k}\right) - f_m\left(\frac{j}{k}\right)\right)\right| \\
&\leq \lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k}\varepsilon/2 \text{ by (2)} \\
&= \varepsilon/2 \\
&< \varepsilon
\end{aligned}$$

which implies that $\{S(f_n)\}$ converges since it is a Cauchy sequence. Say its limit S.

Consider, by (1) as $n \geq N$,

$$\frac{1}{k}\sum_{j=1}^{k}\left[f_n\left(\frac{j}{k}\right) - \varepsilon/3\right] \leq \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq \frac{1}{k}\sum_{j=1}^{k}\left[f_n\left(\frac{j}{k}\right) + \varepsilon/3\right]$$

which implies that

$$\left[\frac{1}{k}\sum_{j=1}^{k} f_n\left(\frac{j}{k}\right)\right] - \varepsilon/3 \leq \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq \left[\frac{1}{k}\sum_{j=1}^{k} f_n\left(\frac{j}{k}\right)\right] + \varepsilon/3$$

which implies that, let $k \to \infty$

$$S(f_n) - \varepsilon/3 \leq \lim_{k\to\infty}\sup \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq S(f_n) + \varepsilon/3 \tag{3}$$

and

$$S(f_n) - \varepsilon/3 \leq \lim_{k\to\infty}\inf \frac{1}{k}\sum_{j=1}^{k} f\left(\frac{j}{k}\right) \leq S(f_n) + \varepsilon/3 \tag{4}$$

which implies that

$$\begin{aligned}
&\left|\lim_{k\to\infty}\sup\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)-\lim_{k\to\infty}\inf\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)\right| \\
&\leq\left|\lim_{k\to\infty}\sup\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)-S\left(f_n\right)\right|+\left|\lim_{k\to\infty}\inf\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{j}{k}\right)-S\left(f_n\right)\right| \\
&\leq\frac{2\varepsilon}{3}\text{ by (3) and (4)} \\
&<\varepsilon. \qquad (5)
\end{aligned}$$

Note that (3)-(5) imply that the existence of $S(f)$. Also, (3) or (4) implies that $S(f)=S$. So, we complete the proof.

9.18 Let $f_n(x)=1/\left(1+n^2x^2\right)$ if $0\leq x\leq 1$, $n=1,2,...$ Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$. Is term-by term integration permissible?

Proof: It is clear that

$$\lim_{n\to\infty}f_n(x)=0$$

for all $x\in[0,1]$. If $\{f_n\}$ converges uniformly on $[0,1]$, then given $\varepsilon=1/3$, there exists a positive integer N such that as $n\geq N$, we have

$$|f_n(x)|<1/3\text{ for all }x\in[0,1]$$

which implies that

$$\left|f_N\left(\frac{1}{N}\right)\right|=\frac{1}{2}<\frac{1}{3}$$

which is impossible. So, $\{f_n\}$ converges pointwise but not uniformly on $[0,1]$.

Since $\{f_n(x)\}$ is clearly uniformly bounded on $[0,1]$, i.e., $|f_n(x)|\leq 1$ for all $x\in[0,1]$ and n. Hence, by **Arzela's Theorem**, we know that the sequence of functions can be integrated term by term.

9.19 Prove that $\sum_{n=1}^{\infty}x/n^{\alpha}\left(1+nx^2\right)$ converges uniformly on every finite interval in R if $\alpha>1/2$. Is the convergence uniform on R?

Proof: By $A.P.\geq G.P.$, we have

$$\left|\frac{x}{n^{\alpha}\left(1+nx^2\right)}\right|\leq\frac{1}{2n^{\alpha+\frac{1}{2}}}\text{ for all }x.$$

So, by **Weierstrass M-test,** we have proved that $\sum_{n=1}^{\infty} x/n^{\alpha}\left(1+nx^2\right)$ converges uniformly on R if $\alpha > 1/2$. Hence, $\sum_{n=1}^{\infty} x/n^{\alpha}\left(1+nx^2\right)$ converges uniformly on every finite interval in R if $\alpha > 1/2$.

9.20 Prove that the series $\sum_{n=1}^{\infty}\left(\left(-1\right)^n/\sqrt{n}\right)\sin\left(1+\left(x/n\right)\right)$ converges uniformly on every compact subset of R.

Proof: It suffices to show that the series $\sum_{n=1}^{\infty}\left(\left(-1\right)^n/\sqrt{n}\right)\sin\left(1+\left(x/n\right)\right)$ converges uniformly on $[0,a]$. Choose n large enough so that $a/n \leq 1/2$, and therefore $\sin\left(1+\left(\frac{x}{n+1}\right)\right) \leq \sin\left(1+\frac{x}{n}\right)$ for all $x \in [0,a]$. So, if we let $f_n(x) = \left(-1\right)^n/\sqrt{n}$ and $g_n(x) = \sin\left(1+\frac{x}{n}\right)$, then by **Abel's test for uniform convergence**, we have proved that the series $\sum_{n=1}^{\infty}\left(\left(-1\right)^n/\sqrt{n}\right)\sin\left(1+\left(x/n\right)\right)$ converges uniformly on $[0,a]$.

Remark: In the proof, we metion something to make the reader get more. (1) since a compact set K is a bounded set, say $K \subseteq [-a,a]$, if we can show the series converges uniformly on $[-a,a]$, then we have proved it. (2) The interval that we consider is $[0,a]$ since $[-a,0]$ is similar. (3) **Abel's test for uniform convergence** holds for $n \geq N$, where N is a fixed positive integer.

9.21 Prove that the series $\sum_{n=0}^{\infty}\left(x^{2n+1}/\left(2n+1\right)-x^{n+1}/\left(2n+2\right)\right)$ converges pointwise but not uniformly on $[0,1]$.

Proof: We show that the series converges pointwise on $[0,1]$ by considering two cases: (1) $x \in [0,1)$ and (2) $x=1$. Hence, it is trivial. Define $f(x) = \sum_{n=0}^{\infty}\left(x^{2n+1}/\left(2n+1\right)-x^{n+1}/\left(2n+2\right)\right)$, if the series converges uniformly on $[0,1]$, then by **Theorem 9.2**, $f(x)$ is continuous on $[0,1]$. However,

$$f(x) = \begin{cases} \frac{1}{2}\log\left(1+x\right) \text{ if } x \in [0,1) \\ \log 2 \text{ if } x = 1 \end{cases}.$$

Hence, the series converges not uniformly on $[0,1]$.

Remark: The function $f(x)$ is found by the following. Given $x \in [0,1)$, then both

$$\sum_{n=0}^{\infty} t^{2n} = \frac{1}{1-t^2} \text{ and } \frac{1}{2}\sum_{n=0}^{\infty} t^n = \frac{1}{2\left(1-t\right)}$$

converges uniformly on $[0,x]$ by **Theorem 9.14.** So, by **Theorem 9.8**, we have

$$\begin{aligned}\int_0^x \sum_{n=0}^{\infty} t^{2n} - \frac{1}{2}\sum_{n=0}^{\infty} t^n &= \int_0^x \frac{1}{1-t^2} - \frac{1}{2(1-t)}dt \\ &= \int_0^x \frac{1}{2}\left(\frac{1}{1-t}+\frac{1}{1+t}\right) - \frac{1}{2}\left(\frac{1}{1-t}\right)dt \\ &= \frac{1}{2}\log(1+x).\end{aligned}$$

And as $x = 1$,

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(x^{2n+1}/(2n+1) - x^{n+1}/(2n+2)\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{2n+1} - \frac{1}{2n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} \text{ by } \textbf{Theorem8.14}. \\ &= \log 2 \text{ by } \textbf{Abel's Limit Theorem}.\end{aligned}$$

9.22 Prove that $\sum a_n \sin nx$ and $\sum b_n \cos nx$ are uniformly convergent on R if $\sum |a_n|$ converges.

Proof: It is trivial by **Weierstrass M-test.**

9.23 Let $\{a_n\}$ be a decreasing sequence of positive terms. Prove that the series $\sum a_n \sin nx$ converges uniformly on R if, and only if, $na_n \to 0$ as $n \to \infty$.

Proof: $(\Rightarrow)$ Suppose that the series $\sum a_n \sin nx$ converges uniformly on R, then given $\varepsilon > 0$, there exists a positive integer N such that as $n \geq N$, we have

$$\left|\sum_{k=n}^{2n-1} a_k \sin kx\right| < \varepsilon. \quad (*)$$

Choose $x = \frac{1}{2n}$, then $\sin\frac{1}{2} \leq \sin kx \leq \sin 1$. Hence, as $n \geq N$, we always have, by (*)

$$
\begin{aligned}
(\varepsilon >) \left|\sum_{k=n}^{2n-1} a_k \sin kx\right| &= \sum_{k=n}^{2n-1} a_k \sin kx \\
&\geq \sum_{k=n}^{2n-1} a_{2n} \sin \frac{1}{2} \text{ since } a_k > 0 \text{ and } a_k \searrow \\
&= \left(\frac{1}{2} \sin \frac{1}{2}\right) (2na_{2n}).
\end{aligned}
$$

That is, we have proved that $2na_{2n} \to 0$ as $n \to \infty$. Similarly, we also have $(2n-1)\, a_{2n-1} \to 0$ as $n \to \infty$. So, we have proved that $na_n \to 0$ as $n \to \infty$.

($\Leftarrow$) Suppose that $na_n \to 0$ as $n \to \infty$, then given $\varepsilon > 0$, there exists a positive integer n_0 such that as $n \geq n_0$, we have

$$
|na_n| = na_n < \frac{\varepsilon}{2(\pi+1)}. \qquad (*)
$$

In order to show the uniform convergence of $\sum_{n=1}^{\infty} a_n \sin nx$ on R, it suffices to show the uniform convergence of $\sum_{n=1}^{\infty} a_n \sin nx$ on $[0, \pi]$. So, if we can show that as $n \geq n_0$

$$
\left|\sum_{k=n+1}^{n+p} a_k \sin kx\right| < \varepsilon \text{ for all } x \in [0, \pi], \text{ and all } p \in N
$$

then we complete it. We consider two cases as follows. $(n \geq n_0)$

As $x \in \left[0, \frac{\pi}{n+p}\right]$, then

$$
\begin{aligned}
\left|\sum_{k=n+1}^{n+p} a_k \sin kx\right| &= \sum_{k=n+1}^{n+p} a_k \sin kx \\
&\leq \sum_{k=n+1}^{n+p} a_k kx \text{ by } \sin kx \leq kx \text{ if } x \geq 0 \\
&= \sum_{k=n+1}^{n+p} (ka_k)\, x \\
&\leq \frac{\varepsilon}{2(\pi+1)} \frac{p\pi}{n+p} \text{ by } (*) \\
&< \varepsilon.
\end{aligned}
$$

And as $x \in \left[\frac{\pi}{n+p}, \pi\right]$, then

$$\begin{aligned}
\left|\sum_{k=n+1}^{n+p} a_k \sin kx\right| &\leq \sum_{k=n+1}^{m} a_k \sin kx + \left|\sum_{k=m+1}^{n+p} a_k \sin kx\right|, \text{ where } m = \left[\frac{\pi}{x}\right] \\
&\leq \sum_{k=n+1}^{m} a_k kx + \frac{2a_{m+1}}{\sin \frac{x}{2}} \text{ by } \textbf{Summation by parts} \\
&\leq \frac{\varepsilon}{2(\pi+1)}(m-n)x + \frac{2a_{m+1}}{\sin \frac{x}{2}} \\
&\leq \frac{\varepsilon}{2(\pi+1)} mx + 2a_{m+1}\frac{\pi}{x} \text{ by } \frac{2x}{\pi} \leq \sin x \text{ if } x \in \left[0, \frac{\pi}{2}\right] \\
&\leq \frac{\varepsilon}{2(\pi+1)}\pi + 2a_{m+1}(m+1) \\
&< \frac{\varepsilon}{2} + 2\frac{\varepsilon}{2(\pi+1)} \\
&< \varepsilon.
\end{aligned}$$

Hence, $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on R.

Remark: (1) In the proof ($\Leftarrow$), if we can make sure that $na_n \searrow 0$, then we can use **the supplement on the convergnce of series in Ch8, (C)-(6)** to show the uniform convergence of $\sum_{n=1}^{\infty} a_n \sin nx = \sum_{n=1}^{\infty} (na_n)\left(\frac{\sin nx}{n}\right)$ by **Dirichlet's test for uniform convergence.**

(2)There are similar results; we write it as references.

(a) Suppose $a_n \searrow 0$, then for each $\alpha \in \left(0, \frac{\pi}{2}\right)$, $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ converges uniformly on $[\alpha, 2\pi - \alpha]$.

Proof: The proof follows from **(12) and (13) in Theorem 8.30 and Dirichlet's test for uniform convergence.** So, we omit it. The reader can see the textbook, **example in pp 231.**

(b) Let $\{a_n\}$ be a decreasing sequence of positive terms. $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly converges on R if and only if $\sum_{n=1}^{\infty} a_n$ converges.

Proof: ($\Rightarrow$) Suppose that $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly converges on R, then let $x = 0$, then we have $\sum_{n=1}^{\infty} a_n$ converges.

($\Leftarrow$) Suppose that $\sum_{n=1}^{\infty} a_n$ converges, then by **Weierstrass M-test,** we have proved that $\sum_{n=1}^{\infty} a_n \cos nx$ uniformly converges on R.

9.24 Given a convergent series $\sum_{n=1}^{\infty} a_n$. Prove that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on the half-infinite interval $0 \leq s < +\infty$. Use this to prove that $\lim_{s \to 0^+} \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n$.

Proof: Let $f_n(s) = \sum_{k=1}^{n} a_k$ and $g_n(s) = n^{-s}$, then by **Abel's test for uniform convergence**, we have proved that the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly on the half-infinite interval $0 \leq s < +\infty$. Then by **Theorem 9.2**, we know that $\lim_{s \to 0^+} \sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n$.

9.25 Prove that the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges uniformly on every half-infinite interval $1 + h \leq s < +\infty$, where $h > 0$. Show that the equation

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

is valid for each $s > 1$ and obtain a similar formula for the kth derivative $\zeta^{(k)}(s)$.

Proof: Since $n^{-s} \leq n^{-(1+h)}$ for all $s \in [1+h, \infty)$, we know that $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges uniformly on every half-infinite interval $1+h \leq s < +\infty$ by **Weierstrass M-test.** Define $T_n(s) = \sum_{k=1}^{n} k^{-s}$, then it is clear that

1. For each n, $T_n(s)$ is differentiable on $[1+h, \infty)$,

2. $\lim_{n \to \infty} T_n(2) = \frac{\pi^2}{6}$.

And

3. $T_n'(s) = -\sum_{k=1}^{n} \frac{\log k}{k^s}$ converges uniformly on $[1+h, \infty)$

by **Weierstrass M-test.** Hence, we have proved that

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$$

by **Theorem 9.13**. By **Mathematical Induction**, we know that

$$\zeta^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{(\log n)^k}{n^s}.$$

0.1 Supplement on some results on Weierstrass M-test.

1. In the textbook, **pp 224-223,** there is a surprising result called **Space-filling curve.** In addition, note the proof is related with **Cantor set** in **exercise 7. 32** in the textbook.

2. There exists a continuous function defined on R which is nowhere differentiable. The reader can see the book, **Principles of Mathematical Analysis by Walter Rudin, pp 154.**

Remark: The first example comes from **Bolzano** in **1834**, however, he did **NOT** give a proof. In fact, he only found the function $f : D \to R$ that he constructed is not differentiable on $D' \, (\subseteq D)$ where D' is countable and dense in D. Although the function f is the example, but he did not find the fact.

In **1861**, **Riemann** gave

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 \pi x)}{n^2}$$

as an example. However, **Reimann** did **NOT** give a proof in his life until **1916**, the proof is given by **G. Hardy.**

In **1860, Weierstrass** gave

$$h(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x), \; b \text{ is odd, } 0 < a < 1, \text{ and } ab > 1 + \frac{3\pi}{2},$$

until **1875,** he gave the proof. The fact surprises the world of **Math**, and produces many examples. There are many researches related with it until now **2003.**

Mean Convergence

9.26 Let $f_n(x) = n^{3/2} x e^{-n^2 x^2}$. Prove that $\{f_n\}$ converges pointwise to 0 on $[-1, 1]$ but that $l.i.m._{n\to\infty} f_n \neq 0$ on $[-1, 1]$.

Proof: It is clear that $\{f_n\}$ converges pointwise to 0 on $[-1, 1]$, so it remains to show that $l.i.m._{n\to\infty}f_n \neq 0$ on $[-1,1]$. Consider

$$\begin{aligned}
\int_{-1}^{1} f_n^2(x)\,dx &= 2\int_0^1 n^3x^2e^{-2n^2x^2}dx \text{ since } f_n^2(x) \text{ is an even function on } [-1,1] \\
&= \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}n} y^2e^{-y^2}dy \text{ by \textbf{Change of Variable}, let } y=\sqrt{2}nx \\
&= \frac{1}{-2\sqrt{2}}\int_0^{\sqrt{2}n} yd\left(e^{-y^2}\right) \\
&= \frac{1}{-2\sqrt{2}}\left[ye^{-y^2}\Big|_0^{\sqrt{2}n} - \int_0^{\sqrt{2}n} e^{-y^2}dy\right] \\
&\to \frac{\sqrt{\pi}}{4\sqrt{2}} \text{ since } \int_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2} \text{ by \textbf{Exercise 7. 19}.}
\end{aligned}$$

So, $l.i.m._{n\to\infty}f_n \neq 0$ on $[-1,1]$.

9.27 Assume that $\{f_n\}$ converges pointwise to f on $[a,b]$ and that $l.i.m._{n\to\infty}f_n = g$ on $[a,b]$. Prove that $f=g$ if both f and g are continuous on $[a,b]$.

Proof: Since $l.i.m._{n\to\infty}f_n = g$ on $[a,b]$, given $\varepsilon_k = \frac{1}{2^k}$, there exists a n_k such that

$$\int_a^b |f_{n_k}(x) - g(x)|^p\,dx \leq \frac{1}{2^k}, \text{ where } p>0$$

Define

$$h_m(x) = \sum_{k=1}^{m}\int_a^x |f_{n_k}(t) - g(t)|^p\,dt,$$

then

a. $h_m(x) \nearrow$ as $x \nearrow$
b. $h_m(x) \leq h_{m+1}(x)$
c. $h_m(x) \leq 1$ for all m and all x.

So, we obtain $h_m(x) \to h(x)$ as $m\to\infty$, $h(x) \nearrow$ as $x\nearrow$, and

$$h(x) - h_m(x) = \sum_{k=m+1}^{\infty}\int_a^x |f_{n_k}(t) - g(t)|^p\,dt \nearrow \text{ as } x \nearrow$$

which implies that

$$\frac{h(x+t)-h(x)}{t} \geq \frac{h_m(x+t)-h_m(x)}{t} \text{ for all } m. \qquad (*)$$

Since h and h_m are increasing, we have h' and h'_m exists a.e. on $[a,b]$. Hence, by (*)

$$h'_m(x) = \sum_{k=1}^{m} |f_{n_k}(t) - g(t)|^p \leq h'(x) \text{ a.e. on } [a,b]$$

which implies that

$$\sum_{k=1}^{\infty} |f_{n_k}(t) - g(t)|^p \text{ exists a.e. on } [a,b].$$

So, $f_{n_k}(t) \to g(t)$ a.e. on $[a,b]$. In addition, $f_n \to f$ on $[a,b]$. Then we conclude that $f = g$ a.e. on $[a,b]$. Since f and g are continuous on $[a,b]$, we have

$$\int_a^b |f-g|\, dx = 0$$

which implies that $f = g$ on $[a,b]$. In particular, as $p = 2$, we have $f = g$.

Remark: (1) A property is said to hold **almost everywhere on a set** S (written: a.e. on S) if it holds everywhere on S except for a set of measurer zero. Also, see the textbook, **pp 254**.

(2) In this proof, we use the theorem which states: A monotonic function h defined on $[a,b]$, then h is differentiable a.e. on $[a,b]$. The reader can see the book, The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 113.**

(3) There is another proof by using **Fatou's lemma**: Let $\{f_k\}$ be a measruable function defined on a measure set E. If $f_k \geq \phi$ a.e. on E and $\phi \in L(E)$, then

$$\int_E \liminf_{k\to\infty} f_k \leq \liminf_{k\to\infty} \int_E f_k.$$

Proof: It suffices to show that $f_{n_k}(t) \to g(t)$ a.e. on $[a,b]$. Since $l.i.m._{n\to\infty} f_n = g$ on $[a,b]$, and given $\varepsilon > 0$, there exists a n_k such that

$$\int_a^b |f_{n_k} - g|^2\, dx < \frac{1}{2^k}$$

which implies that

$$\int_a^b \sum_{k=1}^{m} |f_{n_k} - g|^2 \, dx < \sum_{k=1}^{m} \frac{1}{2^k}$$

which implies that, by **Fatou's lemma,**

$$\int_a^b \liminf_{m\to\infty} \sum_{k=1}^{m} |f_{n_k} - g|^2 \, dx \le \liminf_{m\to\infty} \int_a^b \sum_{k=1}^{m} |f_{n_k} - g|^2 \, dx$$
$$= \sum_{k=1}^{\infty} \int_a^b |f_{n_k} - g|^2 \, dx < 1.$$

That is,

$$\int_a^b \sum_{k=1}^{\infty} |f_{n_k} - g|^2 \, dx < 1$$

which implies that

$$\sum_{k=1}^{\infty} |f_{n_k} - g|^2 < \infty \text{ a.e. on } [a, b]$$

which implies that $f_{n_k} \to g$ a.e. on $[a, b]$.

Note: The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 75.**

(4) There is another proof by using **Egorov's Theorem**: Let $\{f_k\}$ be a measurable functions defined on a finite measurable set E with finite limit function f. Then given $\varepsilon > 0$, there exists a closed set $F\,(\subseteq E)$, where $|E - F| < \varepsilon$ such that

$$f_k \to f \text{ uniformly on } F.$$

Proof: If $f \neq g$ on $[a, b]$, then $h := |f - g| \neq 0$ on $[a, b]$. By continuity of h, there exists a compact subinterval $[c, d]$ such that $|f - g| \neq 0$. So, there exists $m > 0$ such that $h = |f - g| \ge m > 0$ on $[c, d]$. Since

$$\int_a^b |f_n - g|^2 \, dx \to 0 \text{ as } n \to \infty,$$

we have

$$\int_c^d |f_n - g|^2 \, dx \to 0 \text{ as } n \to \infty.$$

then by **Egorov's Theorem,** given $\varepsilon > 0$, there exists a closed susbet F of $[c, d]$, where $|[c, d] - F| < \varepsilon$ such that

$$f_n \to f \text{ uniformly on } F$$

which implies that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int_F |f_n - g|^2 \, dx \\ &= \int_F \lim_{n\to\infty} |f_n - g|^2 \, dx \\ &= \int_F |f - g|^2 \, dx \geq m^2 |F| \end{aligned}$$

which implies that $|F| = 0$. If we choose $\varepsilon < d - c$, then we get a contradiction. Therefore, $f = g$ on $[a, b]$.

Note: The reader can see the book, **Measure and Integral (An Introduction to Real Analysis) written by Richard L. Wheeden and Antoni Zygmund**, **pp 57.**

9.28 Let $f_n(x) = \cos^n x$ if $0 \leq x \leq \pi$.

(a) Prove that $l.i.m._{n\to\infty} f_n = 0$ on $[0, \pi]$ but that $\{f_n(\pi)\}$ does not converge.

Proof: It is clear that $\{f_n(\pi)\}$ does not converge since $f_n(\pi) = (-1)^n$. It remains to show that $l.i.m._{n\to\infty} f_n = 0$ on $[0, \pi]$. Consider $\cos^{2n} x := g_n(x)$ on $[0, \pi]$, then it is clear that $\{g_n(x)\}$ is boundedly convergent with limit function

$$g = \begin{cases} 0 \text{ if } x \in (0, \pi) \\ 1 \text{ if } x = 0 \text{ or } \pi \end{cases}.$$

Hence, by **Arzela's Theorem**,

$$\lim_{n\to\infty} \int_0^\pi \cos^{2n} x dx = \int_0^\pi g(x) \, dx = 0.$$

So, $l.i.m._{n\to\infty} f_n = 0$ on $[0, \pi]$.

(b) Prove that $\{f_n\}$ converges pointwise but not uniformly on $[0, \pi/2]$.

Proof: Note that each $f_n(x)$ is continuous on $[0, \pi/2]$, and the limit function

$$f = \begin{cases} 0 \text{ if } x \in (0, \pi/2] \\ 1 \text{ if } x = 0 \end{cases}.$$

Hence, by **Theorem9.2**, we know that $\{f_n\}$ converges pointwise but not uniformly on $[0, \pi/2]$.

9.29 Let $f_n(x) = 0$ if $0 \leq x \leq 1/n$ or $2/n \leq x \leq 1$, and let $f_n(x) = n$ if $1/n < x < 2/n$. Prove that $\{f_n\}$ converges pointwise to 0 on $[0, 1]$ but that $l.i.m._{n\to\infty} f_n \neq 0$ on $[0, 1]$.

Proof: It is clear that $\{f_n\}$ converges pointwise to 0 on $[0, 1]$. In order to show that $l.i.m._{n\to\infty} f_n \neq 0$ on $[0, 1]$, it suffices to note that

$$\int_0^1 f_n(x)\, dx = 1 \text{ for all } n.$$

Hence, $l.i.m._{n\to\infty} f_n \neq 0$ on $[0, 1]$.

Power series

9.30 If r is the radius of convergence if $\sum a_n (z - z_0)^n$, where each $a_n \neq 0$, show that

$$\lim_{n\to\infty} \inf \left| \frac{a_n}{a_{n+1}} \right| \leq r \leq \lim_{n\to\infty} \sup \left| \frac{a_n}{a_{n+1}} \right|.$$

Proof: By **Exercise 8.4**, we have

$$\frac{1}{\lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right|} \leq r = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}} \leq \frac{1}{\lim_{n\to\infty} \inf \left| \frac{a_{n+1}}{a_n} \right|}.$$

Since

$$\frac{1}{\lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \inf \left| \frac{a_n}{a_{n+1}} \right|$$

and

$$\frac{1}{\lim_{n\to\infty} \inf \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \sup \left| \frac{a_n}{a_{n+1}} \right|,$$

we complete it.

9.31 Given that two power series $\sum a_n z^n$ has radius of convergence 2. Find the radius convergence of each of the following series: In (a) and (b), k is a fixed positive integer.

(a) $\sum_{n=0}^{\infty} a_n^k z^n$

Proof: Since

$$2 = \frac{1}{\lim_{n\to\infty} \sup |a_n|^{1/n}}, \tag{*}$$

we know that the radius of $\sum_{n=0}^{\infty} a_n^k z^n$ is

$$\frac{1}{\lim_{n\to\infty} \sup |a_n^k|^{1/n}} = \frac{1}{\left(\lim_{n\to\infty} \sup |a_n|^{1/n}\right)^k} = 2^k.$$

(b) $\sum_{n=0}^{\infty} a_n z^{kn}$

Proof: Consider

$$\lim_{n\to\infty} \sup \left|a_n z^{kn}\right|^{1/n} = \lim_{n\to\infty} \sup |a_n|^{1/n} |z|^k < 1$$

which implies that

$$|z| < \left(\frac{1}{\lim_{n\to\infty} \sup |a_n|^{1/n}}\right)^{1/k} = 2^{1/k} \text{ by (*)}.$$

So, the radius of $\sum_{n=0}^{\infty} a_n z^{kn}$ is $2^{1/k}$.

(c) $\sum_{n=0}^{\infty} a_n z^{n^2}$

Proof: Consider

$$\lim \sup \left|a_n z^{n^2}\right|^{1/n} = \lim_{n\to\infty} \sup |a_n|^{1/n} |z|^n$$

and claim that the radius of $\sum_{n=0}^{\infty} a_n z^{n^2}$ is 1 as follows.

If $|z| < 1$, it is clearly seen that the series converges. However, if $|z| > 1$,

$$\lim_{n\to\infty} \sup |a_n|^{1/n} \lim_{n\to\infty} \inf |z|^n \leq \lim_{n\to\infty} \sup |a_n|^{1/n} |z|^n$$

which impliest that

$$\lim_{n\to\infty}\sup |a_n|^{1/n} |z|^n = +\infty.$$

so, the series diverges. From above, we have proved the claim.

9.32 Given a power series $\sum a_n x^n$ whose coefficents are related by an equation of the form

$$a_n + Aa_{n-1} + Ba_{n-2} = 0 \ (n = 2, 3, ...).$$

Show that for any x for which the series converges, its sum is

$$\frac{a_0 + (a_1 + Aa_0)\,x}{1 + Ax + Bx^2}.$$

Proof: Consider

$$\sum_{n=2}^{\infty} (a_n + Aa_{n-1} + Ba_{n-2})\,x^n = 0$$

which implies that

$$\sum_{n=2}^{\infty} a_n x^n + Ax \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + Bx^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

which implies that

$$\sum_{n=0}^{\infty} a_n x^n + Ax \sum_{n=0}^{\infty} a_n x^n + Bx^2 \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + Aa_0 x$$

which implies that

$$\sum_{n=0}^{\infty} a_n x^n = \frac{a_0 + (a_1 + Aa_0)\,x}{1 + Ax + Bx^2}.$$

Remark: We prove that for any x for which the series converges, then $1 + Ax + Bx^2 \neq 0$ as follows.

Proof: Consider

$$\left(1 + Ax + Bx^2\right) \sum_{n=0}^{\infty} a_n x^n = a_0 + (a_1 + Aa_0)\,x,$$

if $x = \lambda\,(\neq 0)$ is a root of $1 + Ax + Bx^2$, and $\sum_{n=0}^{\infty} a_n \lambda^n$ exists, we have

$$1 + A\lambda + B\lambda^2 = 0 \text{ and } a_0 + (a_1 + Aa_0)\,\lambda = 0.$$

Note that $a_1 + Aa_0 \neq 0$, otherwise, $a_0 = 0\,(\Rightarrow a_1 = 0)$, and therefore, $a_n = 0$ for all n. Then there is nothing to prove it. So, put $\lambda = \frac{-a_0}{a_1 + Aa_0}$ into $1 + A\lambda + B\lambda^2 = 0$, we then have

$$a_1^2 = a_0 a_2.$$

Note that $a_0 \neq 0$, otherwise, $a_1 = 0$ and $a_2 = 0$. Similarly, $a_1 \neq 0$, otherwise, we will obtain a trivial thing. Hence, we may assume that all $a_n \neq 0$ for all n. So,

$$a_2^2 = a_1 a_3.$$

And it is easy to check that $a_n = a_0 \frac{1}{\lambda^n}$ for all $n \geq N$. Therefore, $\sum a_n \lambda^n = \sum a_0$ diverges. So, for any x for whcih the series converges, we have $1 + Ax + Bx^2 \neq 0$.

9.33 Let $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(0) = 0$.

(a) Show that $f^{(n)}(0)$ exists for all $n \geq 1$.

Proof: By **Exercise 5.4**, we complete it.

(b) Show that the Taylor's series about 0 generated by f converges everywhere on R but that it represents f only at the origin.

Proof: The Taylor's series about 0 generated by f is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} 0 x^n = 0.$$

So, it converges everywhere on R but that it represents f only at the origin.

Remark: It is an important example to tell us that even for functions $f \in C^{\infty}(R)$, the Taylor's series about c generated by f may **NOT** represent f on some open interval. Also see the textbook, **pp 241**.

9.34 Show that the binomial series $(1 + x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ exhibits the following behavior at the points $x = \pm 1$.

(a) If $x = -1$, the series converges for $\alpha \geq 0$ and diverges for $\alpha < 0$.

Proof: If $x=-1$, we consider three cases: (i) $\alpha<0$, (ii) $\alpha=0$, and (iii) $\alpha>0$.

(i) As $\alpha<0$, then

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n=\sum_{n=0}^{\infty}(-1)^n\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!},$$

say $a_n=(-1)^n\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$, then $a_n\geq 0$ for all n, and

$$\frac{a_n}{1/n}=\frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{(n-1)!}\geq-\alpha>0 \text{ for all } n.$$

Hence, $\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n$ diverges.

(ii) As $\alpha=0$, then the series is clearly convergent.

(iii) As $\alpha>0$, define $a_n=n(-1)^n\binom{\alpha}{n}$, then

$$\frac{a_{n+1}}{a_n}=\frac{n-\alpha}{n}\geq 1 \text{ if } n\geq[\alpha]+1. \tag{*}$$

It means that $a_n>0$ for all $n\geq[\alpha]+1$ or $a_n<0$ for all $n\geq[\alpha]+1$. Without loss of generality, we consider $a_n>0$ for all $n\geq[\alpha]+1$ as follows.

Note that (*) tells us that

$$a_n>a_{n+1}>0\Rightarrow\lim_{n\to\infty}a_n \text{ exists.}$$

and

$$a_n-a_{n+1}=\alpha(-1)^n\binom{\alpha}{n}.$$

So,

$$\sum_{n=[\alpha]+1}^{m}(-1)^n\binom{\alpha}{n}=\frac{1}{\alpha}\sum_{n=[\alpha]+1}^{m}(a_n-a_{n+1}).$$

By **Theorem 8.10**, we have proved the convergence of the series $\sum_{n=0}^{\infty}\binom{\alpha}{n}(-1)^n$.

(b) If $x=1$, the series diverges for $\alpha\leq-1$, converges conditionally for α in the interval $-1<\alpha<0$, and converges absolutely for $\alpha\geq 0$.

Proof: If $x=1$, we consider four cases as follows: (i) $\alpha\leq-1$, (ii) $-1<-\alpha<0$, (iii) $\alpha=0$, and (iv) $\alpha>0$:

(i) As $\alpha\leq-1$, say $a_n=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. Then

$$|a_n|=\frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!}\geq 1 \text{ for all } n.$$

So, the series diverges.

(ii) As $-1 < \alpha < 0$, say $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. Then $a_n = (-1)^n b_n$, where

$$b_n = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{n!} > 0.$$

with

$$\frac{b_{n+1}}{b_n} = \frac{n-\alpha}{n} < 1 \text{ since } -1 < -\alpha < 0$$

which implies that $\{b_n\}$ is decreasing with limit L. So, if we can show $L = 0$, then $\sum a_n$ converges by **Theorem 8.16.**

Rewrite

$$b_n = \prod_{k=1}^{n}\left(1 - \frac{\alpha+1}{k}\right)$$

and since $\sum \frac{\alpha+1}{k}$ diverges, then by **Theroem 8.55**, we have proved $L = 0$.

In order to show the convergence is conditionally, it suffices to show the divergence of $\sum b_n$. The fact follows from

$$\frac{b_n}{1/n} = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+n-1)}{(n-1)!} \geq -\alpha > 0.$$

(iii) As $\alpha = 0$, it is clearly that the series converges absolutely.

(iv) As $\alpha > 0$, we consider $\sum \left|\binom{\alpha}{n}\right|$ as follows. Define $a_n = \left|\binom{\alpha}{n}\right|$, then

$$\frac{a_{n+1}}{a_n} = \frac{n-\alpha}{n+1} < 1 \text{ if } n \geq [\alpha] + 1.$$

It implies that $na_n - (n+1)a_n = \alpha a_n$ and $(n+1)a_{n+1} < na_n$. So, by **Theroem 8.10**,

$$\sum a_n = \frac{1}{\alpha}\sum na_n - (n+1)a_n$$

converges since $\lim_{n\to\infty} na_n$ exists. So, we have proved that the series converges absolutely.

9.35 Show that $\sum a_n x^n$ converges uniformly on $[0,1]$ if $\sum a_n$ converges. Use this fact to give another proof of Abel's limit theorem.

Proof: Define $f_n(x) = a_n$ on $[0,1]$, then it is clear that $\sum f_n(x)$ converges uniformly on $[0,1]$. In addition, let $g_n(x) = x^n$, then $g_n(x)$ is unifomrly bouned with $g_{n+1}(x) \leq g_n(x)$. So, by Abel's test for uniform convergence,

$\sum a_n x^n$ converges uniformly on $[0,1]$. Now, we give another proof of **Abel's Limit Theorem** as follows. Note that each term of $\sum a_n x^n$ is continuous on $[0,1]$ and the convergence is uniformly on $[0,1]$, so by **Theorem 9.2,** the power series is continuous on $[0,1]$. That is, we have proved **Abel's Limit Theorem:**

$$\lim_{x\to 1^-} \sum a_n x^n = \sum a_n.$$

9.36 If each $a_n > 0$ and $\sum a_n$ diverges, show that $\sum a_n x^n \to +\infty$ as $x \to 1^-$. (Assume $\sum a_n x^n$ converges for $|x| < 1$.)

Proof: Given $M > 0$, if we can find a y near 1 from the left such that $\sum a_n y^n \geq M$, then for $y \leq x < 1$, we have

$$M \leq \sum a_n y^n \leq \sum a_n x^n.$$

That is, $\lim_{x\to 1^-} \sum a_n x^n = +\infty$.

Since $\sum a_n$ diverges, there is a positive integer p such that

$$\sum_{k=1}^{p} a_k \geq 2M > M. \tag{*}$$

Define $f_n(x) = \sum_{k=1}^{n} a_k x^k$, then by continuity of each f_n, given $0 < \varepsilon\,(< M)$, there exists a $\delta_n > 0$ such that as $x \in [\delta_n, 1)$, we have

$$\sum_{k=1}^{n} a_k - \varepsilon < \sum_{k=1}^{n} a_k x^k < \sum_{k=1}^{n} a_k + \varepsilon \tag{**}$$

By (*) and (**), we proved that as $y = \delta_p$

$$M \leq \sum_{k=1}^{p} a_k - \varepsilon < \sum_{k=1}^{p} a_k y^k.$$

Hence, we have proved it.

9.37 If each $a_n > 0$ and if $\lim_{x\to 1^-} \sum a_n x^n$ exists and equals A, prove that $\sum a_n$ converges and has the sum A. (Compare with Theorem 9.33.)

Proof: By **Exercise 9.36**, we have proved the part, $\sum a_n$ converges. In order to show $\sum a_n = A$, we apply **Abel's Limit Theorem** to complete it.

9.38 For each real t, define $f_t(x) = xe^{xt}/(e^x - 1)$ if $x \in R$, $x \neq 0$, $f_t(0) = 1$.

(a) Show that there is a disk $B(0;\delta)$ in which f_t is represented by a power series in x.

Proof: First, we note that $\frac{e^x-1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} := p(x)$, then $p(0) = 1 \neq 0$. So, by **Theorem 9. 26**, there exists a disk $B(0;\delta)$ in which the reciprocal of p has a power series exapnsion of the form

$$\frac{1}{p(x)} = \sum_{n=0}^{\infty} q_n x^n.$$

So, as $x \in B(0;\delta)$ by **Theorem 9.24.**

$$\begin{aligned} f_t(x) &= xe^{xt}/(e^x - 1) \\ &= \left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}\right) \\ &= \sum_{n=0}^{\infty} r_n(t) x^n. \end{aligned}$$

(b) Define $P_0(t), P_1(t), P_2(t), ...,$ by the equation

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}, \text{ if } x \in B(0;\delta),$$

and use the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$$

to prove that $P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0) t^{n-k}$.

Proof: Since

$$f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \frac{x}{e^x - 1},$$

and

$$f_0(x) = \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} = \frac{x}{e^x - 1}.$$

So, we have the identity

$$\sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}.$$

Use the identity with $e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n$, then we obtain

$$\begin{aligned} \frac{P_n(t)}{n!} &= \sum_{k=0}^{n} \frac{t^{n-k}}{(n-k)!} \frac{P_k(0)}{k!} \\ &= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{n-k} \end{aligned}$$

which implies that

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0)\, t^{n-k}.$$

This shows that each function P_n is a polynomial. There are the **Bernoulli polynomials**. The numbers $B_n = P_n(0)$ $(n = 0, 1, 2, ...)$ are called the **Bernoulli numbers**. Derive the following further properties:

(c) $B_0 = 1,\ B_1 = -\frac{1}{2},\ \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$, if $n = 2, 3, ...$

Proof: Since $1 = \frac{p(x)}{p(x)}$, where $p(x) := \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$, and $\frac{1}{p(x)} := \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!}$. So,

$$\begin{aligned} 1 &= p(x) \frac{1}{p(x)} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} C_n x^n \end{aligned}$$

where

$$C_n = \frac{1}{(n+1)!} \sum_{k=0}^{n} \binom{n+1}{k} P_k(0).$$

So,

$$B_0 = P_0(0) = C_0 = 1,$$

$$B_1 = P_1(0) = \frac{C_1 - P_0(0)}{2} = -\frac{1}{2}, \text{ by } C_1 = 0$$

and note that $C_n = 0$ for all $n \geq 1$, we have

$$\begin{aligned} 0 &= C_{n-1} \\ &= \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} P_k(0) \\ &= \frac{1}{n!} \sum_{k=0}^{n-1} \binom{n}{k} B_k \text{ for all } n \geq 2. \end{aligned}$$

(d) $P_n'(t) = nP_{n-1}(t)$, if $n = 1, 2, ...$

Proof: Since

$$\begin{aligned} P_n'(t) &= \sum_{k=0}^{n} \binom{n}{k} P_k(0)(n-k)t^{n-k-1} \\ &= \sum_{k=0}^{n-1} \binom{n}{k} P_k(0)(n-k)t^{n-k-1} \\ &= \sum_{k=0}^{n-1} \frac{n!(n-k)}{k!(n-k)!} P_k(0)t^{(n-1)-k} \\ &= \sum_{k=0}^{n-1} n\frac{(n-1)!}{k!(n-1-k)!} P_k(0)t^{(n-1)-k} \\ &= n\sum_{k=0}^{n-1} \binom{n-1}{k} P_k(0)t^{(n-1)-k} \\ &= nP_{n-1}(t) \text{ if } n = 1, 2, ... \end{aligned}$$

(e) $P_n(t+1) - P_n(t) = nt^{n-1}$ if $n = 1, 2, ...$

Proof: Consider

$$\begin{aligned} f_{t+1}(x) - f_t(x) &= \sum_{n=0}^{\infty} [P_n(t+1) - P_n(t)] \frac{x^n}{n!} \text{ by (b)} \\ &= xe^{xt} \text{ by } f_t(x) = xe^{xt}/(e^x - 1) \\ &= \sum_{n=0}^{\infty} (n+1) t^n \frac{x^{n+1}}{(n+1)!}, \end{aligned}$$

so as $n = 1, 2, ...$, we have

$$P_n(t+1) - P_n(t) = nt^{n-1}.$$

(f) $P_n(1-t) = (-1)^n P_n(t)$

Proof: Note that

$$f_t(-x) = f_{1-t}(x),$$

so we have

$$\sum_{n=0}^{\infty} (-1)^n P_n(t) \frac{x^n}{n!} = \sum_{n=0}^{\infty} P_n(1-t) \frac{x^n}{n!}.$$

Hence, $P_n(1-t) = (-1)^n P_n(t)$.

(g) $B_{2n+1} = 0$ if $n = 1, 2, ...$

Proof: With help of (e) and (f), let $t = 0$ and $n = 2k+1$, then it is clear that $B_{2k+1} = 0$ if $k = 1, 2, ...$

(h) $1^n + 2^n + ... + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1}$ $(n = 2, 3, ...)$

Proof: With help of (e), we know that

$$\frac{P_{n+1}(t+1) - P_{n+1}(t)}{n+1} = t^n$$

which implies that

$$1^n + 2^n + ... + (k-1)^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1} \quad (n = 2, 3, ...)$$

Remark: (1) The reader can see the book, **Infinite Series by Chao Wen-Min, pp 355-366. (Chinese Version)**

(2) There are some special polynomials worth studying, such as **Legengre Polynomials.** The reader can see the book, **Essentials of Ordinary Differential Equations by Ravi P. Agarwal and Ramesh C. Gupta. pp 305-312.**

(3) The part (h) tells us one formula to calculte the value of the finite series$\sum_{k=1}^{m} k^n$. There is an interesting story from the mail that **Fermat, pierre de (1601-1665)** sent to **Blaise Pascal (1623-1662). Fermat** used the **Mathematical Induction** to show that

$$\sum_{k=1}^{n} k(k+1)\cdots(k+p) = \frac{n(n+1)\cdots(n+p+1)}{p+2}. \qquad (*)$$

In terms of (*), we can obtain another formula on $\sum_{k=1}^{m} k^n$.

Limit sup and limit inf.

Introduction

In order to make us understand the information more on approaches of a given real sequence $\{a_n\}_{n=1}^{\infty}$, we give two definitions, thier names are upper limit and lower limit. It is fundamental but important tools in analysis.

Definition of limit sup and limit inf

Definition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, we define

$$b_n = \sup\{a_m : m \geq n\}$$

and

$$c_n = \inf\{a_m : m \geq n\}.$$

Example $\{1+(-1)^n\}_{n=1}^{\infty} = \{0,2,0,2,\ldots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1,2,-3,4,\ldots\}$, so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1,-2,-3,\ldots\}$, so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n.$$

Proposition Given a real sequence $\{a_n\}_{n=1}^{\infty}$, and thus define b_n and c_n as the same as before.

1 $b_n \neq -\infty$, and $c_n \neq \infty\ \forall n \in N$.
2 If there is a positive integer p such that $b_p = +\infty$, then $b_n = +\infty\ \forall n \in N$.
   If there is a positive integer q such that $c_q = -\infty$, then $c_n = -\infty\ \forall n \in N$.
3 $\{b_n\}$ is decreasing and $\{c_n\}$ is increasing.

By property 3, we can give definitions on the upper limit and the lower limit of a given sequence as follows.

Definition Given a real sequence $\{a_n\}$ and let b_n and c_n as the same as before.

(1) If every $b_n \in R$, then

$$\inf\{b_n : n \in N\}$$

is called the upper limit of $\{a_n\}$, denoted by

$$\limsup_{n\to\infty} a_n.$$

That is,

$$\limsup_{n\to\infty} a_n = \inf_n b_n.$$

If every $b_n = +\infty$, then we define

$$\limsup_{n\to\infty} a_n = +\infty.$$

(2) If every $c_n \in R$, then

$$\sup\{c_n : n \in N\}$$

is called the lower limit of $\{a_n\}$, denoted by

$$\liminf_{n\to\infty} a_n.$$

That is,

$$\lim_{n\to\infty} \inf a_n = \sup_n c_n.$$

If every $c_n = -\infty$, then we define

$$\lim_{n\to\infty} \inf a_n = -\infty.$$

Remark The concept of lower limit and upper limit first appear in the book (**Analyse Alge'brique**) written by **Cauchy** in 1821. But until 1882, **Paul du Bois-Reymond** gave explanations on them, it becomes well-known.

Example $\{1+(-1)^n\}_{n=1}^{\infty} = \{0,2,0,2,\ldots\}$, so we have

$$b_n = 2 \text{ and } c_n = 0 \text{ for all } n$$

which implies that

$$\lim\sup a_n = 2 \text{ and } \lim\inf a_n = 0.$$

Example $\{(-1)^n n\}_{n=1}^{\infty} = \{-1,2,-3,4,\ldots\}$, so we have

$$b_n = +\infty \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim\sup a_n = +\infty \text{ and } \lim\inf a_n = -\infty.$$

Example $\{-n\}_{n=1}^{\infty} = \{-1,-2,-3,\ldots\}$, so we have

$$b_n = -n \text{ and } c_n = -\infty \text{ for all } n$$

which implies that

$$\lim\sup a_n = -\infty \text{ and } \lim\inf a_n = -\infty.$$

Relations with convergence and divergence for upper (lower) limit

Theorem Let $\{a_n\}$ be a real sequence, then $\{a_n\}$ converges if, and only if, the upper limit and the lower limit are real with

$$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} a_n.$$

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \sup a_n = +\infty \Leftrightarrow \{a_n\}$ has no upper bound.

(2) $\lim_{n\to\infty} \sup a_n = -\infty \Leftrightarrow$ for any $M > 0$, there is a positive integer n_0 such that as $n \geq n_0$, we have

$$a_n \leq -M.$$

(3) $\lim_{n\to\infty} \sup a_n = a$ if, and only if, (a) given any $\varepsilon > 0$, there are infinite many numbers n such that

$$a - \varepsilon < a_n$$

and (b) given any $\varepsilon > 0$, there is a positive integer n_0 such that as $n \geq n_0$, we have

$$a_n < a + \varepsilon.$$

Similarly, we also have

Theorem Let $\{a_n\}$ be a real sequence, then we have

(1) $\lim_{n\to\infty} \inf a_n = -\infty \Leftrightarrow \{a_n\}$ has no lower bound.

(2) $\lim_{n\to\infty} \inf a_n = +\infty \Leftrightarrow$ for any $M > 0$, there is a positive integer n_0 such

that as $n \geq n_0$, we have

$$a_n \geq M.$$

(3) $\lim_{n\to\infty} \inf a_n = a$ if, and only if, (a) given any $\varepsilon > 0$, there are infinite many numbers n such that

$$a + \varepsilon > a_n$$

and (b) given any $\varepsilon > 0$, there is a positive integer n_0 such that as $n \geq n_0$, we have

$$a_n > a - \varepsilon.$$

From Theorem 2 an Theorem 3, the sequence is divergent, we give the following definitios.

Definition Let $\{a_n\}$ be a real sequence, then we have

(1) If $\lim_{n\to\infty} \sup a_n = -\infty$, then we call the sequence $\{a_n\}$ diverges to $-\infty$, denoted by

$$\lim_{n\to\infty} a_n = -\infty.$$

(2) If $\lim_{n\to\infty} \inf a_n = +\infty$, then we call the sequence $\{a_n\}$ diverges to $+\infty$, denoted by

$$\lim_{n\to\infty} a_n = +\infty.$$

Theorem Let $\{a_n\}$ be a real sequence. If a is a limit point of $\{a_n\}$, then we have

$$\lim_{n\to\infty} \inf a_n \leq a \leq \lim_{n\to\infty} \sup a_n.$$

Some useful results

Theorem Let $\{a_n\}$ be a real sequence, then

(1) $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$.

(2) $\lim_{n\to\infty} \inf(-a_n) = -\lim_{n\to\infty} \sup a_n$ and $\lim_{n\to\infty} \sup(-a_n) = -\lim_{n\to\infty} \inf a_n$

(3) If every $a_n > 0$, and $0 < \lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n < +\infty$, then we have

$$\lim_{n\to\infty} \sup \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \inf a_n} \text{ and } \lim_{n\to\infty} \inf \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} \sup a_n}.$$

Theorem Let $\{a_n\}$ and $\{b_n\}$be two real sequences.

(1) If there is a positive integer n_0 such that $a_n \leq b_n$, then we have

$$\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \inf b_n \text{ and } \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup b_n.$$

(2) Suppose that $-\infty < \lim_{n\to\infty} \inf a_n$, $\lim_{n\to\infty} \inf b_n$, $\lim_{n\to\infty} \sup a_n$, $\lim_{n\to\infty} \sup b_n < +\infty$, then

$$\begin{aligned}
&\lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \inf b_n \\
&\leq \lim_{n\to\infty} \inf(a_n + b_n) \\
&\leq \lim_{n\to\infty} \inf a_n + \lim_{n\to\infty} \sup b_n \text{ (or } \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \inf b_n \text{)} \\
&\leq \lim_{n\to\infty} \sup(a_n + b_n) \\
&\leq \lim_{n\to\infty} \sup a_n + \lim_{n\to\infty} \sup b_n.
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty} \sup(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} \sup b_n$$

and

$$\lim_{n\to\infty}\inf(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty}\inf b_n.$$

(3) Suppose that $-\infty < \lim_{n\to\infty}\inf a_n,\ \lim_{n\to\infty}\inf b_n,\ \lim_{n\to\infty}\sup a_n,\ \lim_{n\to\infty}\sup b_n < +\infty$, and $a_n > 0,\ b_n > 0\ \forall n$, then

$$\begin{aligned}
&\left(\lim_{n\to\infty}\inf a_n\right)\left(\lim_{n\to\infty}\inf b_n\right)\\
&\le \lim_{n\to\infty}\inf(a_nb_n)\\
&\le \left(\lim_{n\to\infty}\inf a_n\right)\left(\lim_{n\to\infty}\sup b_n\right) \text{ (or } \left(\lim_{n\to\infty}\inf b_n\right)\left(\lim_{n\to\infty}\sup a_n\right))\\
&\le \lim_{n\to\infty}\sup(a_nb_n)\\
&\le \left(\lim_{n\to\infty}\sup a_n\right)\left(\lim_{n\to\infty}\sup b_n\right).
\end{aligned}$$

In particular, if $\{a_n\}$ converges, we have

$$\lim_{n\to\infty}\sup(a_nb_n) = \left(\lim_{n\to\infty} a_n\right)\lim_{n\to\infty}\sup b_n$$

and

$$\lim_{n\to\infty}\inf(a_n+b_n) = \left(\lim_{n\to\infty} a_n\right)\lim_{n\to\infty}\inf b_n.$$

Theorem Let $\{a_n\}$ be a **positive** real sequence, then

$$\lim_{n\to\infty}\inf\frac{a_{n+1}}{a_n} \le \lim_{n\to\infty}\inf(a_n)^{1/n} \le \lim_{n\to\infty}\sup(a_n)^{1/n} \le \lim_{n\to\infty}\sup\frac{a_{n+1}}{a_n}.$$

Remark We can use the inequalities to show

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = 1/e.$$

Theorem Let $\{a_n\}$ be a real sequence, then

$$\lim_{n\to\infty}\inf a_n \le \lim_{n\to\infty}\inf\frac{a_1+\ldots+a_n}{n} \le \lim_{n\to\infty}\sup\frac{a_1+\ldots+a_n}{n} \le \lim_{n\to\infty}\sup a_n.$$

Exercise Let $f : [a,d] \to R$ be a continuous function, and $\{a_n\}$ is a real sequence. If f is increasing and for every n, $\lim_{n\to\infty}\inf a_n,\ \lim_{n\to\infty}\sup a_n \in [a,d]$, then

$$\lim_{n\to\infty}\sup f(a_n) = f\left(\lim_{n\to\infty}\sup a_n\right) \text{ and } \lim_{n\to\infty}\inf f(a_n) = f\left(\lim_{n\to\infty}\inf a_n\right).$$

Remark: (1) The condition that f is increasing cannot be removed. For example,

$$f(x) = |x|,$$

and

$$a_k = \begin{cases} 1/k \text{ if } k \text{ is even} \\ -1-1/k \text{ if } k \text{ is odd.}\end{cases}$$

(2) The proof is easy if we list the definition of limit sup and limit inf. So, we omit it.

Exercise Let $\{a_n\}$ be a real sequence satisfying $a_{n+p} \le a_n + a_p$ for all n, p. Show that $\{\frac{a_n}{n}\}$ converges.

Hint: Consider its limit inf.

Something around the number e

1. Show that the sequence $\{(1+\frac{1}{n})^n\}$ converges, and denote the limit by e.

Proof: Since

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \\ &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2+..+\frac{n(n-1)\cdots 1}{n!}\left(\frac{1}{n}\right)^n \\ &= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\ldots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right) \qquad 1 \\ &\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+..+\frac{1}{2^{(n-1)}}+\ldots \\ &= 3,\end{aligned}$$

and by (1), we know that the sequence is increasing. Hence, the sequence is convergent. We denote its limit e. That is,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Remark: 1. The sequence and e first appear in the mail that **Euler** wrote to **Goldbach.** It is a beautiful formula involving

$$e^{i\pi}+1 = 0.$$

2. Use the exercise, we can show that $\sum_{k=0}^{\infty}\frac{1}{k!} = e$ as follows.

Proof: Let $x_n = (1+\frac{1}{n})^n$, and let $k > n$, we have

$$1+1+\frac{1}{2!}\left(1-\frac{1}{k}\right)+..+\frac{1}{n!}\left(1-\frac{1}{k}\right)\cdot\cdot\left(1-\frac{n-1}{k}\right) \leq x_k$$

which implies that (let $k\to\infty$)

$$y_n := \sum_{i=0}^{n}\frac{1}{i!} \leq e. \qquad 2$$

On the other hand,

$$x_n \leq y_n \qquad 3$$

So, by (2) and (3), we finally have

$$\sum_{k=0}^{\infty}\frac{1}{k!} = e. \qquad 4$$

3. e is an irrational number.

Proof: Assume that e is a rational number, say $e = p/q$, where g.c.d. $(p,q) = 1$. Note that $q > 1$. Consider

$$\begin{aligned}(q!)e &= (q!)\left(\sum_{k=0}^{\infty}\frac{1}{k!}\right) \\ &= (q!)\left(\sum_{k=0}^{q}\frac{1}{k!}\right)+(q!)\left(\sum_{k=q+1}^{\infty}\frac{1}{k!}\right),\end{aligned}$$

and since $(q!)\left(\sum_{k=0}^{q}\frac{1}{k!}\right)$ and $(q!)e$ are integers, we have $(q!)\left(\sum_{k=q+1}^{\infty}\frac{1}{k!}\right)$ is also an integer. However,

$$\begin{aligned}(q!)\left(\sum_{k=q+1}^{\infty}\frac{1}{k!}\right) &= \sum_{k=q+1}^{\infty}\frac{q!}{k!}\\ &= \frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\dots\\ &< \frac{1}{q+1}+\left(\frac{1}{q+1}\right)^2+\dots\\ &= \frac{1}{q}\\ &< 1,\end{aligned}$$

a contradiction. So, we know that e is not a rational number.

4. Here is an estimate about $e = \sum_{k=0}^{n}\frac{1}{k!}+\frac{\theta}{n(n!)}$, where $0 < \theta < 1$. (In fact, we know that $e = 2.71828\ 18284\ 59045\ \dots$.)

Proof: Since $e = \sum_{k=0}^{\infty}\frac{1}{k!}$, we have

$$\begin{aligned}0 < e - x_n &= \sum_{k=n+1}^{\infty}\frac{1}{k!}, \text{ where } x_n = \sum_{k=0}^{n}\frac{1}{k!}\\ &\leq \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\dots\right)\\ &\leq \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\dots\right)\\ &\leq \frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}\\ &\leq \frac{1}{n(n!)} \text{ since } \frac{n+2}{(n+1)^2} < \frac{1}{n}.\end{aligned}$$

So, we finally have

$$e = \sum_{k=0}^{n}\frac{1}{k!}+\frac{\theta}{n(n!)}, \text{ where } 0 < \theta < 1.$$

Note: We can use the estimate dorectly to show e is an irrational number.

2. For continuous variables, we have the samae result as follows. That is,

$$\lim_{x\to\pm\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Proof: (1) Since $(1+\frac{1}{n})^n \to e$ as $n \to \infty$, we know that for any sequence $\{a_n\} \subseteq N$, with $a_n \to \infty$, we have

$$\lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)^{a_n} = e. \tag{5}$$

(2) Given a sequence $\{x_n\}$ with $x_n \to +\infty$, and define $a_n = [x_n]$, then $a_n \leq x_n < a_n + 1$, then we have

$$\left(1+\frac{1}{a_n+1}\right)^{a_n} \leq \left(1+\frac{1}{x_n}\right)^{x_n} \leq \left(1+\frac{1}{a_n}\right)^{a_n+1}.$$

Since

$$\left(1+\frac{1}{a_n+1}\right)^{a_n} \to e \text{ and } \left(1+\frac{1}{a_n}\right)^{a_n+1} \to e \text{ as } x \to +\infty \text{ by (5)}$$

we know that

$$\lim_{n\to+\infty}\left(1+\frac{1}{x_n}\right)^{x_n} = e.$$

Since $\{x_n\}$ is arbitrary chosen so that it goes infinity, we finally obtain that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e. \qquad (6)$$

(3) In order to show $(1+\frac{1}{x})^x \to e$ as $x \to -\infty$, we let $x = -y$, then

$$\begin{aligned}\left(1+\frac{1}{x}\right)^{x} &= \left(1+\frac{1}{-y}\right)^{-y} \\ &= \left(\frac{y}{y-1}\right)^{y} \\ &= \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right).\end{aligned}$$

Note that $x \to -\infty (\Leftrightarrow y \to +\infty)$, by (6), we have shown that

$$\begin{aligned} e &= \lim_{y\to+\infty}\left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right) \\ &= \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x}.\end{aligned}$$

3. Prove that as $x > 0$, we have$(1+\frac{1}{x})^x$ is strictly increasing, and $(1+\frac{1}{x})^{x+1}$ is dstrictly ecreasing.

Proof: Since, by **Mean Value Theorem**

$$\frac{1}{x+1} < \log\left(1+\frac{1}{x}\right) = \log(x+1) - \log(x) = \frac{1}{\xi} < \frac{1}{x} \text{ for all } x > 0,$$

we have

$$\left[x\log\left(1+\frac{1}{x}\right)\right]' = \log\left(1+\frac{1}{x}\right) - \frac{1}{x+1} > 0 \text{ for all } x > 0$$

and

$$\left[(x+1)\log\left(1+\frac{1}{x}\right)\right]' = \log\left(1+\frac{1}{x}\right) - \frac{1}{x} < 0 \text{ for all } x > 0.$$

Hence, we know that

$$x\log\left(1+\frac{1}{x}\right) \text{ is strictly increasing on } (0,\infty)$$

and

$$(x+1)\log\left(1+\frac{1}{x}\right) \text{ is strictly decreasing on } (0,\infty).$$

It implies that

$$\left(1+\frac{1}{x}\right)^{x} \text{ is strictly increasing } (0,\infty), \text{ and } \left(1+\frac{1}{x}\right)^{x+1} \text{ is strictly decreasing on } (0,\infty).$$

Remark: By exercise 2, we know that

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x} = e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x+1}.$$

4. Follow the Exercise 3 to find the smallest a such that $(1+\frac{1}{x})^{x+a} > e$ and strictly decreasing for all $x \in (0,\infty)$.

Proof: Let $f(x) = (1+\frac{1}{x})^{x+a}$, and consider

$$\log f(x) = (x+a)\log\left(1+\frac{1}{x}\right) := g(x),$$

Let us consider

$$
\begin{aligned}
g'(x) &= \log\left(1+\frac{1}{x}\right) - \frac{x+a}{x^2+x} \\
&= -\log(1-y) + [-y+(1-a)y^2]\frac{1}{1-y}, \text{ where } 0 < y = \frac{1}{1+x} < 1 \\
&= \sum_{k=1}^{\infty}\frac{y^k}{k} + [-y+(1-a)y^2]\sum_{k=0}^{\infty} y^k \\
&= \left(\frac{1}{2}-a\right)y^2 + \left(\frac{1}{3}-a\right)y^3 + \ldots + \left(\frac{1}{n}-a\right)y^n + \ldots
\end{aligned}
$$

It is clear that for $a \geq 1/2$, we have $g'(x) < 0$ for all $x \in (0,\infty)$. Note that for $a < 1/2$, if there exists such a so that f is strictly decreasing for all $x \in (0,\infty)$. Then $g'(x) \leq 0$ for all $x \in (0,\infty)$. However, it is impossible since

$$
\begin{aligned}
g'(x) &= \left(\frac{1}{2}-a\right)y^2 + \left(\frac{1}{3}-a\right)y^3 + \ldots + \left(\frac{1}{n}-a\right)y^n + \ldots \\
&\to \frac{1}{2} - a > 0 \text{ as } y \to 1^-.
\end{aligned}
$$

So, we have proved that **the smallest value of a is** 1/2.

Remark: There is another proof to show that $(1+\frac{1}{x})^{x+1/2}$ is strictly decreasing on $(0,\infty)$.

Proof: Consider $h(t) = 1/t$, and two points $(1,1)$ and $\left(1+\frac{1}{x}, \frac{1}{1+\frac{1}{x}}\right)$ lying on the graph From three areas, the idea is that

The area of lower rectangle $<$ The area of the curve $<$ The area of trapezoid

So, we have

$$
\frac{1}{1+x} = \frac{1}{x}\left(\frac{1}{1+\frac{1}{x}}\right) < \log\left(1+\frac{1}{x}\right) < \frac{1}{2x}\left(1+\frac{1}{1+\frac{1}{x}}\right) = \left(x+\frac{1}{2}\right)\left(\frac{1}{x(x+1)}\right). \qquad 7
$$

Consider

$$
\begin{aligned}
\left[\left(1+\frac{1}{x}\right)^{x+1/2}\right]' &= \left[\left(1+\frac{1}{x}\right)^{x+1/2}\right]\left[\log\left(1+\frac{1}{x}\right) - \left(x+\frac{1}{2}\right)\left(\frac{1}{x(x+1)}\right)\right] \\
&< 0 \text{ by (7);}
\end{aligned}
$$

hence, we know that $(1+\frac{1}{x})^{x+1/2}$ is strictly decreasing on $(0,\infty)$.

Note: Use the method of remark, we know that $(1+\frac{1}{x})^x$ is strictly increasing on $(0,\infty)$.