Elements of the Linear Theory of Piezoelectricity and the Vibrations Piezoelectric Plates

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To the memory of

Annie Cornman

and

Shirley Klein

PREFACE

This monograph is essentially a compilation of a series of lectures given by the author to the Piezoelectric Crystal Device Department of Bell Telephone Laboratories at the invitation of Dr. W. J. Spencer. The lectures were given, between March 1965 and April 1966, in order to acquaint the people in the department with the very relevant theoretical techniques that had been developed and systematically employed by Prof. R. D. Mindlin of Columbia University during the preceding fifteen years. The lectures appear to have been somewhat successful in that a number of people who attended them and had been unfamiliar with the viewpoints and techniques before have been using them since.

Although most of Prof. Mindlin's work was on the vibrations of elastic plates, this monograph is concerned with the vibrations of piezoelectric plates. In addition, the basic differential equations and boundary conditions governing the behavior of the linear, piezoelectric continuum are developed systematically from fundamental continuum concepts. The field-theoretic viewpoint adopted is quite different from the lumped-circuit attitude prevalent in existing works on this subject. This treatment is more akin to certain treatments of the linear theory of elastic waves and vibrations available in the current literature. During the period in which the lectures were being delivered the monolithic crystal filter was discovered by R. A. Sykes and W. D. Beaver.* As a consequence, the theory presented was applied to the analysis of such a structure at a very early stage in its development.

It is intended that this monograph can be used as a nucleus for a graduate course or a seminar in piezoelectric vibration theory. It is also felt that it can be read, with some effort, by a reasonably competent individual knowing something about the rudiments of elasticity and electromagnetism, and that this effort can enable him to engage in timely research in this field.

The author would like to take this opportunity to thank W. J. Spencer for suggesting that the lectures be given, for constant encouragement, and

* Work on the same type of structure was done by Prof. M. Onoe at the Institute of Industrial Science of the University of Tokyo.

for indispensable help with the preparation of a preliminary version of this monograph; A. H. Meitzler and J. H. Rowen for their support and encouragement throughout; and all those who attended the lectures for their time, effort, and patience. I wish to thank Mrs. E. Jenkins for doing a magnificent typing job on the final version of this manuscript under very trying circumstances. I also wish to thank Prof. J. L. Bleustein of Yale University for some useful comments and for providing a list of corrections to the preliminary version of this monograph. The author, of course, takes full responsibility for any errors and inadequacies that remain.

H.F.T.

Murray Hill, New Jersey
July 1968

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PART III — Approximation Techniques and Applications

INTRODUCTION

The small vibrations of piezoelectric bodies are governed by the equations of the linear theory of piezoelectricity. In piezoelectricity the *quasi-static* electric field is coupled to the *dynamic* mechanical motion. To be more specific, the equations of linear elasticity are coupled to the charge equation of electrostatics by means of the piezoelectric constants. The equations of piezoelectricity have been available and used since the days of Voigt ([1]). Although there are a reasonable number of books on piezoelectricity currently available ([2,3]), they are all written from different practical points of view, and use the theory sporadically without a systematic development. In fact, a systematic derivation of the equations and relevant boundary conditions appears to be virtually nonexistent in the open literature, with the exception of the work of Mindlin ([4]), which is very brief. Moreover, all the existing texts discuss the piezoelectric vibrations of bodies only in the simplest cases of the one-wave thickness vibrations and the extremely low-frequency extensional and flexural vibrations of thin rods. In addition, considerations of large piezoelectric coupling are completely absent in texts on this subject except for discussions of the simplest cases of elementary thickness vibrations. However, during the past two decades techniques for the solution of more complicated piezoelectric plate vibration problems have been developed and employed, and appear in the current research literature, although not in the texts on the subject. Although there are a number of such methods, the technique that has been most used and the one that seems most fruitful to this author is the one resulting from the investigations of Prof. R. D. Mindlin of Columbia University and some of his students. Even though Prof. Mindlin has written a very valuable monograph ([5]) on the vibrations of elastic plates which appeared as a Signal Corps report, it was never published as a book. As a consequence, it is, unfortunately, not readily available, and some acquaintance with much of its contents is essential to an understanding of much of the current literature on this subject. Hence it seems advisable at this time to prepare for publication the present rather brief monograph, which originated as a series of lectures given by the author within Bell Telephone Laboratories at the request of W. J. Spencer, and which includes some very relevant material (and ideas) that appear (in greater detail) in the Signal Corps monograph

by Mindlin. However, it should be noted that the material presented here includes the *piezoelectric interaction*, whereas Mindlin's monograph was devoted to the vibrations of *purely elastic* plates. In addition, linear piezoelectric theory is developed in some detail and some space is devoted to a discussion of approximation techniques other than the ones considered by Mindlin in his monograph.

The material presented can broadly be separated into four categories:

1. The development of the three-dimensional linear differential equations and appropriate boundary condition.
2. Solution of pertinent three-dimensional standing wave problems, which can be solved and serve as the basis of the approximation techniques.
3. The approximation techniques.
4. Applications to practical problems.

In developing the three-dimensional theory, the mechanical concepts of infinitesimal deformation and stress are presented in some detail, but the electrical discussion proceeds from Maxwell's equations, which are assumed to be known. However, the quasistatic electric field equations, which are used in linear piezoelectric theory, are obtained from Maxwell's equations, and the attendant assumptions and limitations are carefully delineated. The degenerate form of the Poynting vector for the quasistatic electric field and the conditions for its validity appear naturally in the course of the derivation. At this point it should be noted that the importance of the word *linear* appearing in the title of this monograph cannot be overemphasized. The assumption of linearity is far-reaching and obscures much interesting physical detail which is of no importance for the type of small vibrations being described. As a consequence of this assumption, consideration of such things as electric body forces and couples and the distinction between the final and initial position are omitted, along with a number of other related things. Almost all of the equations presented rest, in one way or another, on this assumption of linearity, and if a situation is to be described in which nonlinear effects are present, the entire description presented in this monograph must be abandoned and the appropriate invariant, nonlinear description derived. Important results concerning the nonlinear theory have been presented by Toupin ([6]), but are not needed or included in this monograph. A derivation of Hamilton's variational principle for the linear piezoelectric continuum is included, since it is needed for a number of the approximate techniques.

The three-dimensional problems that are presented, which are extremely important for the approximate techniques of vibration analysis developed

later, are the thickness vibrations of piezoelectric plates and two-dimensional standing eigenwaves in plates, which, fortunately, are problems that can be solved exactly in the framework of the three-dimensional theory. If all the vibration problems could be solved exactly within the framework of the three-dimensional theory, there would be no need for approximation technique. However, this is not the case, and approximation techniques are almost always required in any real physical problem.

The approximation techniques that are developed are not perturbation techniques in any sense. Actually, three techniques are discussed. Two of these take modal solutions of the three-dimensional equations just mentioned; these solutions satisfy the differential equations and boundary conditions on the major surfaces of the plate exactly, and a number of such solutions are summed so as to satisfy the remaining boundary conditions on the minor surfaces approximately, using either the variational formulation or the method of least squares. The third technique, which is considered to be most fruitful and is concentrated on most heavily, is somewhat different in philosophy from the other two, in that the variational formulation is used to construct a system of approximate two-dimensional plate equations, which may then be solved exactly in some instances in which the three-dimensional equations cannot. This latter procedure originated with Poisson, Cauchy, and Kirchhoff in the development of the classical theory of elastic plates, and has been developed most extensively by Mindlin. A great deal of present research in the area of piezoelectric plate vibrations uses these approximate equations. In this monograph a further simplification in these equations is introduced, and the more tractable *thickness-shear approximation* is obtained along with the appropriate edge conditions in a manner exhibiting the natural limitations inherent in the approximation. However, these latter equations turn out to be extremely accurate in the frequency range of most practical interest, i.e., in the vicinity of the thickness-shear frequency.

These equations are applied in the description of the steady-state forced vibrations of a monolithic crystal structure. Structures of this type show great promise of significantly furthering the crystal filter art, and, indeed, have been fabricated and used and are already available commercially. Another area of application of these approximate procedures is in the design of AT cut quartz resonators and thickness-shear resonators using high coupling materials.

Chapter 1

ELEMENTS OF VECTOR AND TENSOR NOTATION

1. CARTESIAN COORDINATE TRANSFORMATIONS

Consider the orthogonal coordinate system with axes x_1, x_2, and x_3 shown in Fig. 1. The base vectors are $\mathbf{e}_i$ ($i = 1, 2, 3$), where $|\mathbf{e}_i| = 1$. Clearly, the vector $\mathbf{r}$ may be written

$$\mathbf{r} = \mathbf{e}_i x_i = \mathbf{e}_1 x_1 + \mathbf{e}_2 x_2 + \mathbf{e}_3 x_3 , \tag{1.1}$$

and since the coordinate system is orthogonal,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} , \tag{1.2}$$

where δ_{ij} is the Kronecker delta defined by

$$\begin{aligned} \delta_{ij} &= 1 \quad \text{if} \quad i = j \\ &= 0 \quad \text{if} \quad i \neq j , \end{aligned} \tag{1.3}$$

and we have introduced the summation convention ([7]) for repeated indices.

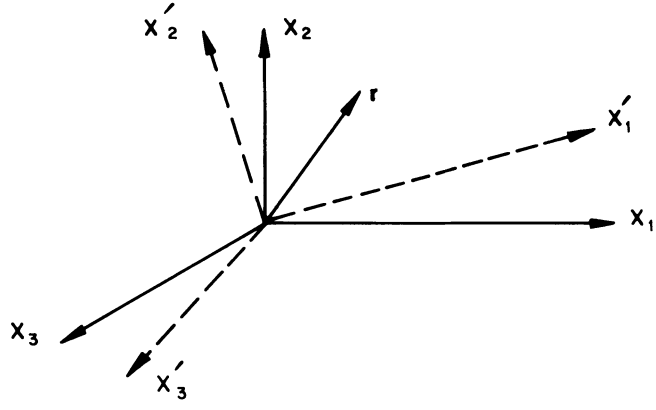

Fig. 1. Rectangular Cartesian coordinate axes.

A vector is an invariant quantity, i.e., it is independent of the frame of reference in which its components are measured; this is why it is useful for the description of physical phenomena. Consider another orthogonal coordinate system x_l' (shown dotted in Fig. 1). Then we may write

$$\mathbf{r} = \mathbf{e}_l' x_l' = \mathbf{e}_i x_i \cdot \tag{1.4}$$

Remembering that $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ and $\mathbf{e}_l' \cdot \mathbf{e}_{m'} = \delta_{lm}$, upon dotting $\mathbf{e}_{m'}$ into both sides of (1.4) we obtain $x_{m'} = \mathbf{e}_{m'} \cdot \mathbf{e}_i x_i$. Let $a_{mi} = \mathbf{e}_{m'} \cdot \mathbf{e}_i$; then

$$x_{m'} = a_{mi} x_i , \tag{1.5}$$

which is the transformation law for the components of a first rank tensor. Similarly, we have

$$x_j = a_{lj} x_l' , \tag{1.6}$$

which is the transformation law going the other way. Note the placement of the indices in (1.5) and (1.6). Since $\mathbf{r} = (\mathbf{r} \cdot \mathbf{e}_i)\mathbf{e}_i$, and $\mathbf{r}$ can be $\mathbf{e}_l'$, we have

$$\mathbf{e}_l' = (\mathbf{e}_l' \cdot \mathbf{e}_i)\mathbf{e}_i ,$$

from which we obtain

$$\mathbf{e}_{m'} \cdot \mathbf{e}_l' = (\mathbf{e}_l' \cdot \mathbf{e}_i)\mathbf{e}_{m'} \cdot \mathbf{e}_i ,$$

which with the definition of a_{mi}, (1.2), and (1.4) yields

$$a_{li} a_{mi} = \delta_{lm} . \tag{1.7}$$

Similarly, we obtain

$$a_{li} a_{lj} = \delta_{ij} . \tag{1.8}$$

Equations (1.7) and (1.8) are the orthogonality relations, which tell us that the transformation from one orthogonal coordinate system to another is orthogonal.

Since det $\delta_{ij} = 1$ and the determinant of a matrix product is equal to the product of the determinants, we must have

$$\det a_{li} \equiv |a_{li}| = \pm 1 . \tag{1.9}$$

Proper rotations have det $a_{li} = +1$, and improper rotations -1. Inversion and reflection operations give improper rotations.

2. VECTORIAL OPERATIONS AND INTEGRAL THEOREMS

According to any text on vector and/or tensor analysis, the scalar product of any two vectors **A** and **B** may be written

$$\mathbf{A}\cdot\mathbf{B} = A_i\mathbf{e}_i\cdot B_j\mathbf{e}_j = A_iB_i. \tag{1.10}$$

Similarly, the vector product may be written

$$\mathbf{A}\times\mathbf{B} = A_i\mathbf{e}_i\times B_j\mathbf{e}_j = A_iB_j\mathbf{e}_i\times\mathbf{e}_j, \tag{1.11}$$

and we may write

$$\mathbf{e}_i\times\mathbf{e}_j = e_{ijk}\mathbf{e}_k, \tag{1.12}$$

where e_{ijk} is the three-dimensional skew-symmetric tensor or Levi-Civita symbol defined by

$$\begin{aligned} e_{ijk} &= +1 && \text{if } ijk \text{ is a cyclic permutation of 1, 2, 3} \\ &= \ \ 0 && \text{if any two indices are equal} \\ &= -1 && \text{if } ijk \text{ is an anticyclic permutation,} \end{aligned} \tag{1.13}$$

in any coordinate system; hence e_{ijk} does not transform as a tensor under improper transformations. Substituting from (1.12) into (1.11), we obtain

$$\mathbf{A}\times\mathbf{B} = e_{ijk}A_iB_j\mathbf{e}_k = e_{kij}A_iB_j\mathbf{e}_k, \tag{1.14}$$

which shows that $\mathbf{A}\times\mathbf{B}$ is not a true (polar) vector. In other words, (1.14) shows that $\mathbf{A}\times\mathbf{B}$ behaves as a vector under proper rotations, but not under improper rotations. It is the vectorial representation of an antisymmetric second rank tensor in three-dimensional space. It is very useful in 3-space, but not in spaces of higher dimensions. It is called a dual, oriented, or relative quantity by mathematicians, and either an axial or pseudo vector by physicists.

Texts on vector and tensor analysis define the spatial differential operator ∇ by

$$\nabla \equiv \mathbf{e}_i\frac{\partial}{\partial x_i} = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}. \tag{1.15}$$

Then we may write the gradient of a scalar φ in the form

$$\nabla\varphi \equiv \mathbf{e}_i\,\partial\varphi/\partial x_i \equiv \mathbf{e}_i\varphi_{,i}, \tag{1.16}$$

where the comma followed by an index denotes differentiation with respect to the space coordinate x_i. Clearly, we may write the divergence and curl of a vector $\mathbf{A}$ in the forms

$$\nabla \cdot \mathbf{A} \equiv \partial A_i / \partial x_i \equiv A_{i,i}, \tag{1.17}$$

$$\nabla \times \mathbf{A} \equiv (\mathbf{e}_j \, \partial / \partial x_j) \times \mathbf{e}_k A_k \equiv \mathbf{e}_j \times \mathbf{e}_k A_{k,j} = e_{ijk} \mathbf{e}_i A_{k,j}. \tag{1.18}$$

There exist the following integral theorems, which we write in both vector and Cartesian tensor notation

$$\int_S \mathbf{n} \cdot \mathbf{A} \, ds = \int_V \nabla \cdot \mathbf{A} \, dV, \tag{1.19a}$$

$$\int_S n_i A_i \, ds = \int_V A_{i,i} \, dV, \tag{1.19b}$$

$$\oint_c \mathbf{A} \cdot d\mathbf{r} = \int_s \mathbf{n} \cdot (\nabla \times \mathbf{A}) \, ds, \tag{1.20a}$$

$$\oint_c A_i \, dx_i = \int_s n_i e_{ijk} A_{k,j} \, ds, \tag{1.20b}$$

where S denotes a closed surface enclosing a volume V, and c a closed curve enclosing an open area s.

3. DYADICS AND HIGHER RANK TENSORS

Let $\mathbf{A} = A_i \mathbf{e}_i$ and $\mathbf{B} = B_j \mathbf{e}_j$. Form the outer product $\mathbf{P} = \mathbf{AB}$ (no dot, no cross), i.e.,

$$\mathbf{P} = \mathbf{AB} = A_i \mathbf{e}_i B_j \mathbf{e}_j = \mathbf{e}_i A_i B_j \mathbf{e}_j = \mathbf{e}_i P_{ij} \mathbf{e}_j. \tag{1.21}$$

The symbol $\mathbf{P}$ represents a dyadic or second rank tensor, and P_{ij} are its nine components in a rectangular Cartesian coordinate system. A dyadic (second rank tensor) is an invariant just like a vector (first rank tensor) and scalar (zero rank tensor). Consequently, from (1.21) we have

$$\mathbf{P} = \mathbf{e}_i P_{ij} \mathbf{e}_j = \mathbf{e}_l' P'_{lm} \mathbf{e}_m', \tag{1.22}$$

from which we may obtain the transformation law for the components of second rank tensors by successively dotting both sides of (1.22) with primed base vectors as we did in obtaining (1.5) for the transformation of the components of vectors. The resulting transformation laws for second rank tensors corresponding to (1.5) and (1.6) for vectors are

$$P'_{lm} = a_{li}a_{mj}P_{ij}\,, \tag{1.23}$$

$$P_{ij} = a_{li}a_{mj}P'_{lm}. \tag{1.24}$$

In a similar way we may form triads or third rank tensors and tetrads or fourth rank tensors, and so on. The transformation law always turns out to be of the form

$$H'_{lmn\alpha\gamma\beta\ldots} = a_{li}a_{mj}a_{nk}a_{\alpha t}a_{\gamma s}a_{\beta r}\ldots H_{ijktsr\ldots}. \tag{1.25}$$

In Cartesian tensor notation we simply write the components and ignore the base vectors.

4. PROPERTIES OF ANTISYMMETRIC SECOND RANK TENSORS

We will now show that there is a one-to-one correspondence between an antisymmetric second rank polar tensor and an axial vector. Consider $\omega_{ij}(\boldsymbol{\omega} = \mathbf{e}_i\omega_{ij}\mathbf{e}_j)$, where

$$\omega_{ij} = -\omega_{ji}(\boldsymbol{\omega} = -\boldsymbol{\omega}^T). \tag{1.26}$$

Clearly there are three independent components, i.e., the number of independent components possessed by a vector in 3-space. Form the vector $\boldsymbol{\Omega}$ from the tensor $\boldsymbol{\omega}$ by

$$\Omega_i = \tfrac{1}{2}e_{ijk}\omega_{jk}\,, \tag{1.27}$$

and $\boldsymbol{\Omega} = \mathbf{e}_i\Omega_i$ is an axial vector, whereas $\boldsymbol{\omega}$ is a polar tensor (dyadic). Note that

$$\omega_{lm} = e_{ilm}\Omega_i = \tfrac{1}{2}e_{ilm}e_{ijk}\omega_{jk}\,, \tag{1.28}$$

since we have the well-known tensor identity between the Levi-Civita and Kronecker symbols

$$e_{ilm}e_{ijk} = \delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj}. \tag{1.29}$$

Consider the polar vector $\mathbf{v} = \boldsymbol{\omega}\cdot\mathbf{r}$. Then

$$v_i = \omega_{ij}x_j = e_{kij}\Omega_k x_j = e_{ijk}x_j\Omega_k\,, \tag{1.30a}$$

and

$$\mathbf{v} = \boldsymbol{\omega}\cdot\mathbf{r} = \mathbf{r}\times\boldsymbol{\Omega}. \tag{1.30b}$$

5. PROPERTIES OF SYMMETRIC SECOND RANK TENSORS

Let us now determine the eigenvalues and eigenvectors of a symmetric matrix. Consider $S_{ij}(\mathbf{S} = \mathbf{e}_i S_{ij} \mathbf{e}_j)$, where

$$S_{ij} = S_{ji}(\mathbf{S} = \mathbf{S}^T). \tag{1.31}$$

Consider $\mathbf{S}$ operating on $\mathbf{r}$ to form $\mathbf{v}$, where $\mathbf{v}$ is any other vector. Then we have

$$\mathbf{v} = \mathbf{S}\cdot\mathbf{r} \qquad v_i = S_{ij} x_j. \tag{1.32}$$

Now ask whether there are directions whereby $\mathbf{v} = \mathbf{S}\cdot\mathbf{r}$ is in the direction $\mathbf{r}$. This may be stated mathematically by writing $\mathbf{v} = \lambda\mathbf{r}$, where λ is a scalar multiplier. Then from (1.32) we have

$$S_{ij} x_j = \lambda x_i, \tag{1.33}$$

and

$$(S_{ij} - \lambda\delta_{ij}) x_j = 0, \tag{1.34}$$

which can have nontrivial solutions only if the determinant of the coefficients of x_j vanishes, i.e., if

$$|S_{ij} - \lambda\delta_{ij}| = 0. \tag{1.35}$$

Equation (1.35) is a cubic equation and yields 3 λ's. Each $\lambda^{(n)}$ $(n = 1, 2, 3)$ determines amplitude ratios $x_j^{(n)}$. Let $\tilde{\lambda}^{(n)}$ denote the complex conjugate of $\lambda^{(n)}$ and $\tilde{x}_j^{(n)}$ denote the complex conjugate of $x_j^{(n)}$. Taking the complex conjugate of (1.33) for $\lambda^{(n)}$, i.e., of $S_{ij} x_j^{(n)} = \lambda^{(n)} x_i^{(n)}$, we get

$$\widetilde{S_{ij} x_j^{(n)}} = \widetilde{\lambda^{(n)} x_i^{(n)}}, \tag{1.36}$$

or

$$\tilde{S}_{ij} \tilde{x}_j^{(n)} = \tilde{\lambda}^{(n)} \tilde{x}_i^{(n)}. \tag{1.37}$$

Form

$$\tilde{x}_i^{(n)} S_{ij} x_j^{(n)} - x_i^{(n)} \tilde{S}_{ij} \tilde{x}_j^{(n)} = (\lambda^{(n)} - \tilde{\lambda}^{(n)}) x_i^{(n)} \tilde{x}_i^{(n)},$$

and

$$\tilde{x}_i^{(n)} S_{ij} x_j^{(n)} - \tilde{x}_i^{(n)} S_{ij} x_j^{(n)} = (\lambda^{(n)} - \tilde{\lambda}^{(n)}) x_i^{(n)} \tilde{x}_i^{(n)} = 0, \tag{1.38}$$

where we have used (1.31), (1.33), and (1.37) in obtaining (1.38). Since $x_i^{(n)} \tilde{x}_i^{(n)}$ is real, $\tilde{\lambda}^{(n)} = \lambda^{(n)}$, and $\lambda^{(n)}$ is real. Thus $x_j^{(n)}$ is real, since it is

determined from linear equations which contain only real quantities. Now assume all $\lambda^{(n)}$ distinct (no degeneracy). Normalize $x_i^{(n)}$ so that $\bar{x}_i^{(n)}\bar{x}_i^{(n)} = 1$, i.e., $x_i^{(n)}x_i^{(n)} = N_{(n)}^2$ and $\bar{x}_i^{(n)} = x_i^{(n)}/N_{(n)}$. The $\mathbf{e}_j\bar{x}_j^{(n)}$ are eigenvectors, one eigenvector for each value of n. Thus there are three eigenvectors. We can write the $\bar{x}_j^{(n)}$ $(n = 1, 2, 3;\ j = 1, 2, 3)$ as $\bar{x}_{nj}$ and consider them as a 3×3 array which is a transformation. We will now show that for distinct $\lambda^{(n)}$ the eigenvectors $\mathbf{e}_j\bar{x}_j^{(n)}$ are mutually orthogonal, or what is the same thing, that the transformation given by the $\bar{x}_{nj}$ is orthogonal. To this end, we write the eigenvalue equation for two distinct values of $n = l,m$,

$$S_{ij}\bar{x}_j^{(l)} = \lambda^{(l)}\bar{x}_i^{(l)}, \tag{1.39a}$$

$$S_{ij}\bar{x}_j^{(m)} = \lambda^{(m)}\bar{x}_i^{(m)}, \tag{1.39b}$$

and form

$$\bar{x}_i^{(m)}S_{ij}\bar{x}_j^{(l)} - \bar{x}_i^{(l)}S_{ij}\bar{x}_j^{(m)} = (\lambda^{(l)} - \lambda^{(m)})\bar{x}_i^{(l)}\bar{x}_i^{(m)},$$

from which with (1.31) we have

$$(\lambda^{(l)} - \lambda^{(m)})\bar{x}_i^{(l)}\bar{x}_i^{(m)} = 0. \tag{1.40}$$

Since $\lambda^{(l)} \neq \lambda^{(m)}$ and $\bar{x}_i^{(n)}\bar{x}_i^{(n)} = 1$, from (1.40) we have $\bar{x}_i^{(l)}\bar{x}_i^{(m)} = \delta_{lm}$, and the eigenvectors are orthogonal; thus $\bar{x}_{li}\bar{x}_{mi} = \delta_{lm}$, and the transformation is orthogonal. Let us now transform the S_{ij} to the particular orthogonal coordinate system which is composed of the aforementioned eigenvectors. It is clear that we may do this by writing

$$S'_{lm} = \bar{x}_{li}\bar{x}_{mj}S_{ij};$$

but from (1.39b) we have

$$\bar{x}_{mj}S_{ij} = S_{ij}\bar{x}_j^{(m)} = \lambda^{(m)}\bar{x}_i^{(m)}$$

so that

$$S'_{lm} = \bar{x}_{li}\lambda^{(m)}\bar{x}_i^{(m)} = \bar{x}_i^{(l)}\lambda^{(m)}\bar{x}_i^{(m)} = \lambda^{(m)}\delta_{lm}. \tag{1.41}$$

From (1.41) it is clear that in the coordinate system composed of the eigenvectors the **S** matrix is diagonal, and the diagonal values are equal to the eigenvalues. In the case of a degeneracy it turns out that the eigenvectors are not unique but may be selected to be orthogonal. For a discussion of the degenerate case when two or even all three of the eigenvalues are equal see ([8]), Section 72.

6. LAGRANGIAN UNDETERMINED MULTIPLIERS

We will now consider the stationary value of a function $F(x_k) \equiv F(x_1, x_2, x_3)$. For a stationary value of F at P we have

$$dF = (\partial F/\partial x_k)\,|_P\, dx_k = 0. \tag{1.42}$$

Since all dx_k are arbitrary, we have the three independent conditions

$$(\partial F/\partial x_k)\,|_P = 0. \tag{1.43}$$

However, suppose we want F to be stationary but the variables x_k are not independent and satisfy an additional relation of the form

$$g(x_k) = c. \tag{1.44}$$

Now we cannot have $(\partial F/\partial x_k)\,|_P = 0$, since all the dx_k are no longer arbitrary. Nevertheless, we can solve (1.44) for any one of the x_i in terms of the other two, and then make $F(x_\alpha)$ stationary with respect to the other two. However, this is a cumbersome and unnecessarily selective and asymmetric procedure. However, by taking the total differential of (1.44) and solving for any one of the dx_k in terms of the other two and substituting in (1.42), it can be shown that we can proceed in a more symmetric and less cumbersome manner by introducing an unknown scalar multiplier λ and forming

$$dF - \lambda\, dg = \left(\frac{\partial F}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i}\right) dx_i = 0. \tag{1.45}$$

This procedure can also be viewed as selecting λ so that any *one* of the terms in parentheses vanishes. Then the other two dx_α are independent, and the coefficients can be equated to zero independently. We can find the additional unknown λ from the condition (1.44). Thus we can obtain a stationary value of F, subject to (1.44), by making

$$H = F - \lambda(g - c) \tag{1.46}$$

stationary and allowing all dx_i to be arbitrary.

PART I

ELEMENTS OF THE LINEAR THEORY OF PIEZOELECTRICITY

Chapter 2

MECHANICAL CONSIDERATIONS

1. DEFINITION OF THE FIELD VARIABLES AND THE CONSERVATION EQUATIONS

In a mechanical continuum two distinct types of forces act, body forces **f** and contact forces **t**. The body forces arise as a result of some distant action. They are long-range forces. The contact forces arise as a result of the contact of adjacent elements of a body. Macroscopically speaking, they are surface forces. Microscopically, they are caused by very short-range near-neighbor interactions between adjacent microscopic elements. The body force exerted by electric or magnetic fields will be neglected below. We will also neglect body-couples and surface-couples as well. The quantities being neglected can be shown to be very small in the linear theory with which we will be concerned.

The traction vector $\mathbf{t}(\mathbf{n})$ is defined as the force per unit area acting on a surface, usually exerted by a neighboring surface, and is shown in Fig. 2. The body force **f** is applied and has the units of force per unit volume. An example is gravity. The mechanical linear momentum of a continuum is given by $\varrho\mathbf{v}$, where ϱ is the mass density and **v** denotes the velocity of a point. Both **f** and $\varrho\mathbf{v}$ are volumetric quantities, and are shown applied to an arbitrary point of a continuum in Fig. 2.

The motion of an arbitrary portion of a continuum is governed by the following conservation equations:

Mass

$$(d/dt)\int_V \varrho\, dV = 0. \tag{2.1}$$

Linear momentum

$$\int_S \mathbf{t}(\mathbf{n})\, dS + \int_V \mathbf{f}\, dV = (d/dt)\int_V \varrho\mathbf{v}\, dV. \tag{2.2}$$

Angular momentum

$$\int_S \mathbf{r}\times\mathbf{t}(\mathbf{n})\, dS + \int_V \mathbf{r}\times\mathbf{f}\, dV = (d/dt)\int \mathbf{r}\times\varrho\mathbf{v}\, dV. \tag{2.3}$$

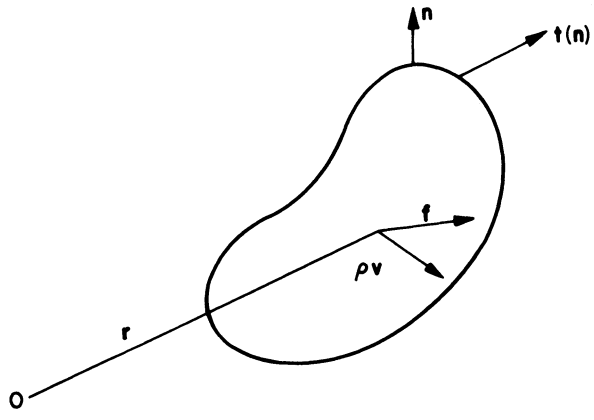

Fig. 2. An arbitrary material volume element in motion.

In (2.1)–(2.3) d/dt denotes the material derivative, but as we shall see shortly, such considerations are of no importance in the linear theory.

2. THE STRESS TENSOR

Applying (2.2) to the elementary tetrahedron shown in Fig. 3 and taking the limit as the volume approaches zero, i.e., as $h \to 0$, we obtain

$$\mathbf{t}(\mathbf{n})\, \Delta S_n + \sum_{i=1}^{3} \mathbf{t}(-\mathbf{e}_i)\, \Delta S_i + h\left(\mathbf{f} - \varrho \frac{d}{dt}\mathbf{v}\right) \Delta S_n = 0.$$

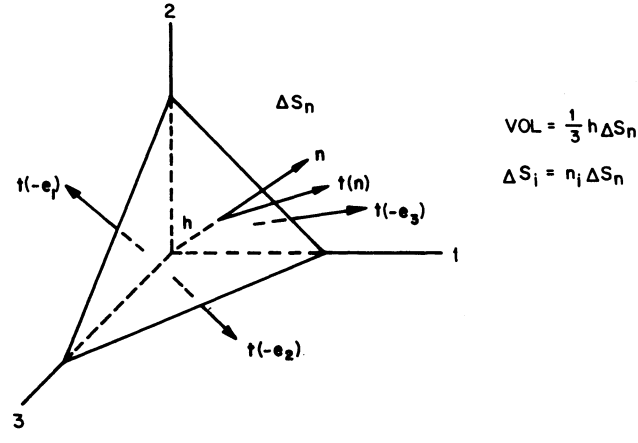

Fig. 3. Elementary tetrahedron.

Then substituting $\Delta S_i = n_i \, \Delta S_n$ (Fig. 3) and taking the limit as $h \to 0$, we find

$$\mathbf{t}(\mathbf{n}) + \sum_{i=1}^{3} \mathbf{t}(-\mathbf{e}_i) n_i = 0. \tag{2.4}$$

Considering $\mathbf{n} = (1, 0, 0)$ as a limit of (2.4), we find

$$\mathbf{t}(\mathbf{e}_1) + \mathbf{t}(-\mathbf{e}_1) = 0, \qquad \mathbf{t}(\mathbf{e}_1) = -\mathbf{t}(-\mathbf{e}_1).$$

If we consider the other two perpendicular surfaces, we get

$$\mathbf{t}(\mathbf{e}_j) = -\mathbf{t}(-\mathbf{e}_j). \tag{2.5}$$

Substituting from (2.5) into (2.4), we find

$$\mathbf{t}(\mathbf{n}) = \sum_{i=1}^{3} n_i \mathbf{t}(\mathbf{e}_i) = n_i \mathbf{t}(\mathbf{e}_i) = n_j \mathbf{e}_j \cdot \mathbf{e}_i \mathbf{t}(\mathbf{e}_i) = \mathbf{n} \cdot \mathbf{e}_i \mathbf{t}(\mathbf{e}_i). \tag{2.6}$$

As we have seen, the quantity $\mathbf{e}_i \mathbf{t}(\mathbf{e}_i)$ is a dyadic or second rank tensor, and hereafter will be denoted by $\boldsymbol{\tau}$ or $\mathbf{T}$, i.e.,

$$\boldsymbol{\tau} \equiv \mathbf{e}_i \mathbf{t}(\mathbf{e}_i), \tag{2.7}$$

and from (2.7) and (2.6)

$$\mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot \boldsymbol{\tau}. \tag{2.8}$$

From Fig. 3 it is clear that the component representation of the vector $\mathbf{t}$ may be written

$$\mathbf{t}(\mathbf{n}) = \tau_{nj} \mathbf{e}_j. \tag{2.9}$$

Consequently,

$$\mathbf{t}(\mathbf{e}_i) = \tau_{ij} \mathbf{e}_j, \tag{2.10}$$

where the subscript i indicates the surface on which $\mathbf{t}$ acts and j indicates the direction in which $\mathbf{t}$ acts. Finally,

$$\boldsymbol{\tau} \equiv \mathbf{e}_i \mathbf{t}(\mathbf{e}_i) = \mathbf{e}_i \tau_{ij} \mathbf{e}_j. \tag{2.11}$$

We may also write the vector $\mathbf{t}(\mathbf{n})$ in the form

$$\mathbf{t}(\mathbf{n}) = t_j(\mathbf{n}) \mathbf{e}_j \equiv t_j \mathbf{e}_j. \tag{2.12}$$

Clearly, from (2.9) and (2.12) we have

$$t_j = t_j(\mathbf{n}) \equiv \tau_{nj}, \tag{2.13}$$

and from (2.8), (2.11), and (2.12) we have

$$t_j \mathbf{e}_j = n_l \mathbf{e}_l \cdot \mathbf{e}_i \tau_{ij} \mathbf{e}_j = n_i \tau_{ij} \mathbf{e}_j , \tag{2.14}$$

so that in Cartesian tensor component form we have

$$t_j = n_i \tau_{ij} . \tag{2.15}$$

From (1.23) and (1.24) we know that the components of the tensor $\boldsymbol{\tau}$ transform according to the formulae

$$\tau'_{kl} = a_{ki} a_{lj} \tau_{ij} \qquad \text{and} \qquad \tau_{ij} = a_{ki} a_{lj} \tau'_{kl} . \tag{2.16}$$

3. THE STRESS EQUATIONS OF MOTION

Substituting from (2.8) into the equation of the conservation of linear momentum (2.2) and applying the divergence theorem, we obtain

$$\int_V \boldsymbol{\nabla} \cdot \boldsymbol{\tau} \, dV + \int_V \mathbf{f} \, dV = \int_V \varrho (d\mathbf{v}/dt) \, dV ,$$

from which, since V is arbitrary, we find

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \mathbf{f} = \varrho \, d\mathbf{v}/dt , \tag{2.17a}$$

which are the stress equations of motion. In Cartesian component form we have

$$\tau_{ij,i} + f_j = \varrho \, dv_j/dt . \tag{2.17b}$$

Rewriting (2.3) in component form and substituting from (2.15), we obtain

$$\int_S e_{klj} x_l n_i \tau_{ij} \, dS + \int_V e_{klj} [x_l f_j - x_l \varrho (dv_j/dt) - \varrho v_l v_j] \, dV = 0 . \tag{2.18}$$

Using the divergence theorem in (2.18), we find

$$\int_V e_{klj} x_l \left(\tau_{ij,i} + f_j - \varrho \frac{dv_j}{dt} \right) dV + \int_V e_{kjl} x_{l,i} \tau_{ij} \, dV = 0 .$$

Since $x_{l,i} = \delta_{li}$ and V is arbitrary and we have (2.17*b*), we may write

$$e_{kji} \tau_{ij} = 0 , \tag{2.19}$$

from which we may conclude that

$$\tau_{ij} = \tau_{ji} , \tag{2.20}$$

and the stress tensor is symmetric.

We now introduce the first linearizing approximation,

$$\frac{dv_j}{dt} = \frac{\partial v_j}{\partial t} = \frac{\partial^2 u_j}{\partial t^2} = \ddot{u}_j, \tag{2.21}$$

where $\mathbf{u}$ is the mechanical displacement of a material point. The stress equations of motion (2.17) now take the component form

$$\tau_{ij,i} + f_j = \varrho \ddot{u}_j \qquad \begin{pmatrix} 3 \text{ equations} \\ 9 \text{ variables} \end{pmatrix}. \tag{2.22}$$

From (2.11) we obtain

$$\tau_{kl} = \mathbf{e}_k \cdot \boldsymbol{\tau} \cdot \mathbf{e}_l, \tag{2.23}$$

and in particular

$$\tau_{(k)(k)} = \mathbf{e}_k \cdot \boldsymbol{\tau} \cdot \mathbf{e}_k, \qquad k \text{ no sum} \tag{2.24}$$

(the parentheses mean no sum), where $\mathbf{e}_k$ and $\mathbf{e}_l$ denote mutually orthogonal unit vectors referred to a particular Cartesian coordinate system.

Let $\mathbf{n}$ denote a unit vector normal to a surface (any surface). Then from (2.24)

$$\tau_{(n)(n)} = \mathbf{n} \cdot \boldsymbol{\tau} \cdot \mathbf{n} = n_i \tau_{ij} n_j, \tag{2.25}$$

and $\tau_{(n)(n)}$ represents the component of the traction vector on that surface, normal to that surface.

4. THE NATURE OF THE STRESS TENSOR

It is natural to ask the question: For a given stress tensor referred to a specific Cartesian coordinate system, are there surfaces with normal $\mathbf{n}$ on which the normal component of stress $\tau_{(n)(n)}$ is locally a maximum or a minimum, and if so, where are they? To answer this, we try to find orientations $\mathbf{n}$ for which $\tau_{(n)(n)}$ is stationary, subject to the constraint $\mathbf{n} \cdot \mathbf{n} = 1$. From our discussion of Lagrangian multipliers it is clear that we proceed by forming the function

$$f = n_i \tau_{ij} n_j - \lambda(n_i n_i - 1), \tag{2.26}$$

and making f stationary for arbitrary δn_k, i.e.,

$$\delta f = (\partial f / \partial n_k)\, \delta n_k = 0, \tag{2.27}$$

so that

$$\partial f / \partial n_k = 0 \tag{2.28}$$

are the conditions. Now, since we must regard the n_k as independent, we have

$$\partial n_i/\partial n_k = \delta_{ik}. \tag{2.29}$$

Hence from (2.26), (2.28), and (2.29) we find

$$\begin{aligned} \delta_{ik}\tau_{ij}n_j + n_i\tau_{ij}\delta_{jk} - \lambda(n_i\delta_{ik} + \delta_{ik}n_i) = 0 \\ \tau_{kj}n_j + n_i\tau_{ik} - 2\lambda n_k = 0\,, \end{aligned} \tag{2.30}$$

and from (2.20) and (2.30) we obtain

$$\tau_{ki}n_i = \lambda n_k; \tag{2.31}$$

but (2.31) is just the eigenvalue equation for a symmetric matrix, and we already know all the results. There are three real eigenvalues, and three mutually orthogonal eigenvectors which form a triad, and the symmetric tensor is diagonal and is equal to the eigenvalues when referred to this triad. Therefore we can immediately conclude that there are three mutually orthogonal planes on which the normal stress is locally a maximum or minimum, and all shear stresses are zero on these planes. Moreover, the eigenvalues of this problem are the locally extreme values of the stresses.

Chapter 3

INFINITESIMAL DEFORMATION THEORY

1. THE INFINITESIMAL DISPLACEMENT FIELD

Consider a line element $d\mathring{\mathbf{r}}$ which moves in such a manner that it rotates and extends (or contracts) to $d\mathbf{r}$ as shown in Fig. 4. From Fig. 4 it is clear that

$$d\mathring{\mathbf{r}} = \mathring{\mathbf{r}}^2 - \mathring{\mathbf{r}}^1, \qquad d\mathbf{r} = \mathbf{r}^2 - \mathbf{r}^1, \tag{3.1}$$

$$\mathbf{r}^1 - \mathring{\mathbf{r}}^1 = \mathbf{u}^1, \qquad \mathbf{r}^2 - \mathring{\mathbf{r}}^2 = \mathbf{u}^2, \tag{3.2}$$

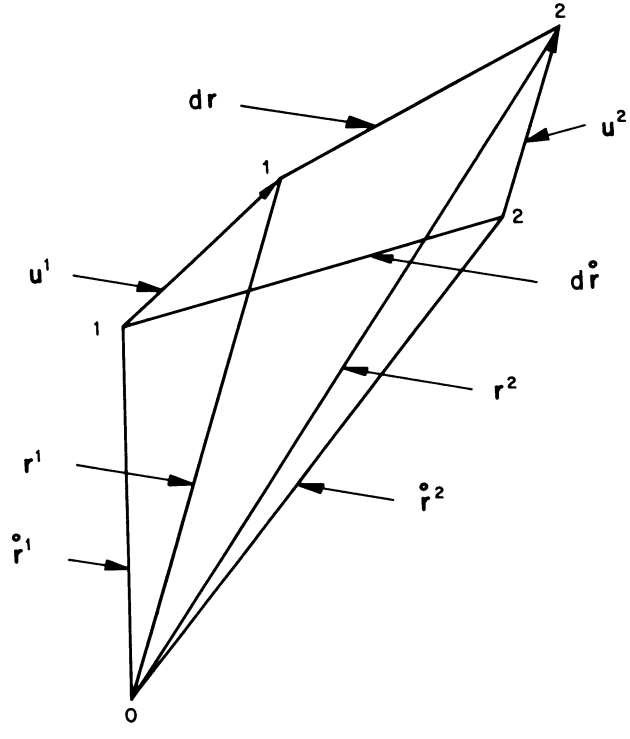

Fig. 4. Deformation of an arbitrary line element.

and from (3.1) and (3.2) we have

$$d\mathbf{r} - d\mathring{\mathbf{r}} = \mathbf{u}^2 - \mathbf{u}^1. \tag{3.3}$$

Let $\mathbf{u}^2 - \mathbf{u}^1 = d\mathbf{u}$, where $\mathbf{u}$ is the displacement field. Then from (3.3) we have

$$d\mathbf{u} = d\mathbf{r} - d\mathring{\mathbf{r}},$$

or

$$d\mathbf{r} = d\mathring{\mathbf{r}} + d\mathbf{u}. \tag{3.4}$$

When the deformation is infinitesimal we may assume that

$$\mathbf{r} \approx \mathring{\mathbf{r}}, \tag{3.5}$$

$$|\partial u_i/\partial x_j| \ll 1, \tag{3.6}$$

which means that $\mathbf{u}$ is very small and that products of components of the displacement gradients are negligible compared to the components of the displacement gradients themselves.

Expand $\mathbf{u}$ in a Taylor's series:

$$u_i^2 = u_i^1 + \frac{\partial u_i^1}{\partial x_j} dx_j + \text{h.o.t.} \tag{3.7a}$$

where the higher order terms (h.o.t.) are negligible, or, vectorially,

$$\mathbf{u}^2 = \mathbf{u}^1 + d\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{u} = \mathbf{u}^1 + d\mathring{\mathbf{r}}\cdot\mathring{\boldsymbol{\nabla}}\mathbf{u}, \tag{3.7b}$$

where $\boldsymbol{\nabla} = \mathring{\boldsymbol{\nabla}}$ because of the assumption given by (3.5), i.e., because Eulerian and Lagrangian coordinates are equivalent. Thus, since the infinitesimal strain assumptions have already been made, we can no longer distinguish between the final and the initial positions. This is a far-reaching assumption and obscures much important detail, but it is perfectly all right for the type of linear theory in which we are interested. Almost all of the simplified treatment which will be presented rests on this assumption. If this assumption is removed, the mathematics becomes much more involved.

From (3.7*b*) and the definition of $d\mathbf{u}$ we have

$$du_i = (\partial u_i/\partial x_j)\, dx_j, \tag{3.8a}$$

or

$$d\mathbf{u} = d\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{u}. \tag{3.8b}$$

Since $\partial u_i/\partial x_j$ is a second rank tensor with nine terms, we can decompose it into its symmetric and antisymmetric parts. Thus let

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \mathbf{S} = \tfrac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\boldsymbol{\nabla}), \tag{3.9}$$

where $\mathbf{u}\boldsymbol{\nabla} \equiv (\boldsymbol{\nabla}\mathbf{u})^{\text{transpose}}$, and

$$\omega_{ij} = \tfrac{1}{2}(u_{j,i} - u_{i,j}), \qquad \boldsymbol{\omega} = \tfrac{1}{2}(\boldsymbol{\nabla}\mathbf{u} - \mathbf{u}\boldsymbol{\nabla}). \tag{3.10}$$

Then, adding (3.9) and (3.10), we obtain

$$u_{i,j} = S_{ij} + \omega_{ji}, \tag{3.11a}$$

or

$$\boldsymbol{\nabla}\mathbf{u} = \mathbf{S} + \boldsymbol{\omega}, \tag{3.11b}$$

and

$$d\mathbf{u} = d\mathbf{r}\cdot\mathbf{S} + d\mathbf{r}\cdot\boldsymbol{\omega},$$

$$du_i = S_{ij}\,dx_j + \omega_{ji}\,dx_j.$$

Since ω_{ij} is an antisymmetric second rank tensor, according to (1.28) it may be written

$$\omega_{ij} = +\,e_{kij}\Omega_k, \tag{3.12}$$

where $\boldsymbol{\Omega}$ is an axial vector, and according to (1.27) we have

$$\Omega_l = +\,\tfrac{1}{2}e_{lij}\omega_{ij}. \tag{3.13}$$

Moreover, from (3.10) and (3.13) we have

$$\Omega_l = \tfrac{1}{2}e_{lij}u_{j,i}, \tag{3.14a}$$

or

$$\boldsymbol{\Omega} = \tfrac{1}{2}\boldsymbol{\nabla}\times\mathbf{u}. \tag{3.14b}$$

2. THE NATURE OF AN INFINITESIMAL DEFORMATION

From (3.4) and (3.8b) we have

$$d\mathbf{r} = d\mathring{\mathbf{r}} + d\mathring{\mathbf{r}}\cdot\boldsymbol{\nabla}\mathbf{u} = d\mathring{\mathbf{r}}\cdot(\mathbf{I} + \boldsymbol{\nabla}\mathbf{u}), \tag{3.15a}$$

where $\mathbf{I} = \mathbf{e}_i\delta_{ij}\mathbf{e}_j = \mathbf{e}_i\mathbf{e}_i$, or

$$dx_i = d\mathring{x}_j(\delta_{ij} + u_{i,j}), \tag{3.15b}$$

and from (3.11*a*) and (3.15*b*) we obtain

$$dx_i = d\mathring{x}_j\,(\delta_{ij} + S_{ij} - \omega_{ij}). \tag{3.16}$$

Let

$$T_{ij} = \delta_{ij} + S_{ij} - \omega_{ij}, \tag{3.17}$$

and regard T_{ij} as a linear transformation which transforms $d\mathring{\mathbf{r}}$ to $d\mathbf{r}$. Further, let

$$R_{ik} = \delta_{ik} - \omega_{ik} \tag{3.18}$$

and

$$L_{ik} = \delta_{ik} + S_{ik}. \tag{3.19}$$

Then from (3.17)–(3.19)

$$T_{ij} = R_{ik}L_{jk}$$

since by (3.9), (3.10), and (3.6) $\boldsymbol{\omega}$ and $\mathbf{S}$ are infinitesimal, and products of components are negligible. Thus we now have two infinitesimal transformations $\mathbf{R}$ and $\mathbf{L}$ operating successively in transforming $d\mathring{\mathbf{r}}$ to $d\mathbf{r}$.

Now, the square of the length of any arbitrary line element before the deformation is given by

$$l_0^2 = d\mathring{x}_j\,d\mathring{x}_j,$$

and the square of the length of the corresponding element after the deformation is given by

$$l^2 = dx_i\,dx_i.$$

Let us compute l^2 using (3.16). Thus

$$\begin{aligned} dx_i\,dx_i &= d\mathring{x}_j\,(\delta_{ij} + S_{ij} - \omega_{ij})\,d\mathring{x}_k\,(\delta_{ik} + S_{ik} - \omega_{ik}) \\ &= d\mathring{x}_j\,d\mathring{x}_k\,(\delta_{jk} + 2S_{jk}), \end{aligned} \tag{3.20}$$

since $\mathbf{S}$ and $\boldsymbol{\omega}$ are infinitesimal and $\boldsymbol{\omega}$ is antisymmetric.

Consider a deformation in which the $S_{jk} \equiv 0$. Then from (3.20) we have

$$dx_i\,dx_i = d\mathring{x}_j\,d\mathring{x}_j,$$

or $l^2 = l_0^2$, and the length of *any* element does not change. Therefore, since the length of any element does not change under a deformation consisting of an $\boldsymbol{\omega}$ only ($\mathbf{S} \equiv 0$), $\boldsymbol{\omega}$ must correspond to a pure infinitesimal rotation.

It is interesting to note that from (3.4), (3.16), and (3.12) we may write

$$du_i = dx_i - d\mathring{x}_i = S_{ij}\, d\mathring{x}_j + e_{ikj}\Omega_k\, d\mathring{x}_j\,, \tag{3.21a}$$

or

$$d\mathbf{u} = d\mathbf{r} - d\mathring{\mathbf{r}} = \mathbf{S}\cdot d\mathring{\mathbf{r}} + \boldsymbol{\Omega}\times d\mathring{\mathbf{r}}\,, \tag{3.21b}$$

where $\boldsymbol{\Omega}$ is given by (3.14*b*). Therefore the infinitesimal change in displacement between the ends of any differential line element consists of a portion which produces a change in length and another portion which does not produce any change in length, and corresponds to a pure rotation. It is important to note that whereas the rotational deformation $\boldsymbol{\omega}$ (or $\boldsymbol{\Omega}$) cannot change the length of *any* element, the extensional deformation $\mathbf{S}$ can (and generally does) rotate almost all differential line elements $d\mathring{\mathbf{r}}$ issuing from a point. The difference is that the rotation $\boldsymbol{\omega}$ rotates every element by the same amount, i.e., rigidly, while the strain $\mathbf{S}$ rotates different elements different amounts, i.e., nonrigidly. The off-diagonal terms of $\mathbf{S}$ do the rotating.

Let us ask if for a given deformation $(u_{i,j})$ referred to a specific Cartesian coordinate system, out of all differential elements issuing from a point, having the same initial length l_0, are there any elements which locally have a maximum or minimum final length, and if so, where are they? To answer this question, we maximize the final length (or the square) $dx_i\, dx_i$, subject to the constraint that the initial length (or rather the square) is equal to a constant, i.e., $d\mathring{x}_j\, d\mathring{x}_j = l_0^2$. We know that we may proceed by forming the function

$$G = dx_i\, dx_i - \lambda(d\mathring{x}_j\, d\mathring{x}_j - l_0^2)\,, \tag{3.22}$$

and making G stationary for arbitrary $d\mathring{x}_k$, i.e.,

$$\delta G = \frac{\partial G}{\partial(d\mathring{x}_m)}\,\delta(d\mathring{x}_m) = 0\,,$$

so that

$$\partial G/\partial(d\mathring{x}_m) = 0 \tag{3.23}$$

are the conditions. Substituting from (3.20) into (3.22), we obtain

$$G = (\delta_{jk} + 2S_{jk})\, d\mathring{x}_j\, d\mathring{x}_k - \lambda(d\mathring{x}_j\, d\mathring{x}_j - l_0^2). \tag{3.24}$$

Since we must regard the $d\mathring{x}_k$ as independent, we have

$$\partial(d\mathring{x}_j)/\partial(d\mathring{x}_m) = \delta_{jm}\,, \tag{3.25}$$

and from (3.23) and (3.24), with (3.25), we obtain

$$\frac{\partial G}{\partial(d\mathring{x}_m)} = (\delta_{jk} + 2S_{jk})(\delta_{jm}\, d\mathring{x}_k + \delta_{km}\, d\mathring{x}_j) - 2\lambda\delta_{jm}\, d\mathring{x}_j = 0$$

$$= (\delta_{mk} + 2S_{mk})\, d\mathring{x}_k + (\delta_{jm} + 2S_{jm})\, d\mathring{x}_j - 2\lambda\, d\mathring{x}_m = 0; \quad (3.26)$$

but $\delta_{mj} + 2S_{mj} = \delta_{jm} + 2S_{jm}$, and

$$(\delta_{mk} + 2S_{mk})\, d\mathring{x}_k = (\delta_{mj} + 2S_{mj})\, d\mathring{x}_j. \quad (3.27)$$

Hence from (3.26) and (3.27) we have

$$(\delta_{mj} + 2S_{mj})\, d\mathring{x}_j = \lambda\, d\mathring{x}_m,$$

or

$$S_{mj}\, d\mathring{x}_j = \tfrac{1}{2}(\lambda - 1)\, d\mathring{x}_m \equiv \lambda^1\, d\mathring{x}_m. \quad (3.28)$$

Again, we see that this is just the eigenvalue equation for a symmetric matrix S_{mj}, and we know all the results.

Therefore we can immediately conclude that there are three mutually orthogonal directions $d\mathring{x}_m^{(n)}$ along which the change in length is locally a maximum or minimum, and consequently, the extensional strain is locally a maximum or minimum along these same mutually orthogonal directions. Moreover, from our knowledge of the solution of the eigenvalue problem it is clear that when the **S** tensor is referred to the eigenvector triad, the off-diagonal (or shear) strains vanish. Thus in this coordinate system we may write

$$S_{ij} = S_{(i)}\, \delta_{ij}, \quad (3.29)$$

where $S_{(i)} \equiv \lambda^{1(i)}$, and from (3.16) and (3.29) we have

$$\begin{aligned} dx_i &= d\mathring{x}_j\, (\delta_{ij} + S_{(i)}\, \delta_{ij} - \omega_{ij}) \\ dx_i &= (1 + S_{(i)})\, d\mathring{x}_i - \omega_{ij}\, d\mathring{x}_j. \end{aligned} \quad (3.30)$$

Now we consider only the three line elements which coincide with the axes of the eigenvector triad. In order to keep track of the three elements, let us denote which element we are considering by an m, i.e., instead of $d\mathring{x}_j$ we have $d\mathring{x}_j^{(m)}$. Then substituting from (3.12) into (3.30), we obtain

$$dx_i^{(m)} = (1 + S_{(i)})\, d\mathring{x}_i^{(m)} + e_{ikj}\Omega_k\, d\mathring{x}_j^{(m)}. \quad (3.31)$$

Since we are considering only those elements lying along the axes of the

eigenvector triad, we may write

$$d\mathring{x}_j^{(m)} = \delta_{mj} l_0 , \tag{3.32}$$

and substituting from (3.32) into (3.31), we have

$$\begin{aligned} dx_i^{(m)} &= (1 + S_{(i)})\delta_{mi} l_0 + e_{ikj}\Omega_k \delta_{mj} l_0 \\ dx_i^{(m)} &= [(1 + S_{(m)})\delta_{mi} + e_{ikm}\Omega_k] l_0 . \end{aligned} \tag{3.33}$$

Multiplying (3.33) by $\mathbf{e}_i$ and summing over i, we obtain

$$\begin{aligned} d\mathbf{r}^{(m)} &= [(1 + S_{(m)})\mathbf{e}_m + \mathbf{e}_i e_{ikm}\Omega_k] l_0 , \\ d\mathbf{r}^{(m)} &= [(1 + S_{(m)})\mathbf{e}_m + \mathbf{e}_k \Omega_k \times \mathbf{e}_m] l_0 . \end{aligned} \tag{3.34}$$

Since $\mathbf{e}_m l_0 = d\mathring{\mathbf{r}}^{(m)}$, from (3.34) we find

$$d\mathbf{r}^{(m)} = (1 + S_{(m)})\, d\mathring{\mathbf{r}}^{(m)} + \boldsymbol{\Omega} \times d\mathring{\mathbf{r}}^{(m)} , \tag{3.35}$$

from which it is clear that the strain deformation $\mathbf{S}$ does not rotate the eigenvectors of the strain tensor, it simply elongates (or contracts) them, and that each eigenvector is rotated through the same infinitesimal angle $\boldsymbol{\Omega}$. Since the eigenvectors are orthogonal before the deformation and rotate through the same angle, they remain orthogonal during the deformation. Thus it is clear that an arbitrary infinitesimal deformation consists of a small translation $\mathbf{u}$ of a point, a rotation of three mutually perpendicular lines through the point, and an extension or contraction of these lines.

3. INFINITESIMAL VOLUME CHANGE

We will now determine the expression for an infinitesimal change in volume. The most expedient way of doing this is to consider the mutually orthogonal eigenvectors of the strain tensor $\mathbf{S}$. These vectors are orthogonal before the deformation and remain orthogonal during the deformation. They are simply rotated rigidly by $\boldsymbol{\Omega}$ and extended (or contracted) by $\mathbf{S}$. The rectangular volume enclosed by these vectors before the deformation is [(8), Section 19]

$$dV_0 = d\mathring{\mathbf{r}}^{(1)} \cdot d\mathring{\mathbf{r}}^{(2)} \times d\mathring{\mathbf{r}}^{(3)} . \tag{3.36}$$

The corresponding volume enclosed by the same vectors after the deformation is

$$dV = d\mathbf{r}^{(1)} \cdot d\mathbf{r}^{(2)} \times d\mathbf{r}^{(3)} . \tag{3.37}$$

Substituting from (3.35) into (3.37), we obtain

$$dV = (1 + S_{(1)})\, d\mathring{\mathbf{r}}^{(1)} \cdot (1 + S_{(2)})\, d\mathring{\mathbf{r}}^{(2)} \times (1 + S_{(3)})\, d\mathring{\mathbf{r}}^{(3)}, \tag{3.38}$$

since $\boldsymbol{\Omega}$ and $S_{(m)}$ are infinitesimal and

$$d\mathring{\mathbf{r}}^{(1)} \cdot d\mathring{\mathbf{r}}^{(2)} \times (\boldsymbol{\Omega} \times d\mathring{\mathbf{r}}^{(3)}) + d\mathring{\mathbf{r}}^{(1)} \cdot (\boldsymbol{\Omega} \times d\mathring{\mathbf{r}}^{(2)}) \times d\mathring{\mathbf{r}}^{(3)} + (\boldsymbol{\Omega} \times d\mathring{\mathbf{r}}^{(1)}) \cdot d\mathring{\mathbf{r}}^{(2)} \times d\mathring{\mathbf{r}}^{(3)} = 0,$$

where each term vanishes separately, since $d\mathring{\mathbf{r}}^{(1)}$, $d\mathring{\mathbf{r}}^{(2)}$ and $d\mathring{\mathbf{r}}^{(3)}$ are mutually orthogonal. Therefore from (3.36) and (3.38) we obtain

$$dV - dV_0 = [(1 + S_{(1)})(1 + S_{(2)})(1 + S_{(3)}) - 1]\, d\mathring{\mathbf{r}}^{(1)} \cdot d\mathring{\mathbf{r}}^{(2)} \times d\mathring{\mathbf{r}}^{(3)}, \tag{3.39}$$

and from (3.39) and (3.36) we find

$$(dV - dV_0)/dV_0 = S_{(1)} + S_{(2)} + S_{(3)}, \tag{3.40}$$

since the $S_{(m)}$ are infinitesimal.

The quantity $(dV - dV_0)/dV_0$ is called the dilatation and is represented by the symbol Δ. Thus from (3.40) we have

$$\Delta = S_{(1)} + S_{(2)} + S_{(3)}, \tag{3.41}$$

that is to say, the dilatation is equal to the sum of the eigenvalues of the strain tensor. But the sum of the diagonal components of a second rank tensor is an invariant, and we have

$$\Delta = S_{ii}, \tag{3.42}$$

in any coordinate system, and since $|S_{ij}| \ll 1$, $\Delta \ll 1$. Moreover, since from (3.40) and (3.41) $dV = (1 + \Delta)\, dV_0$ and $\Delta \ll 1$, $dV \approx dV_0$; and since $\varrho\, dV = \varrho_0\, dV_0$, $\varrho = (1 - \Delta)\varrho_0$, $\varrho \approx \varrho_0$. Thus in infinitesimal deformation theory, although Δ represents the change in volume per unit volume, the final volume and mass density may be taken to be equal to the initial volume and mass density, respectively. Note that

$$\frac{\partial}{\partial t}\, dV = \dot{\Delta}\, dV_0, \tag{3.43}$$

and that

$$\Delta = S_{ii} = u_{i,i} = \boldsymbol{\nabla} \cdot \mathbf{u}. \tag{3.44}$$

Chapter 4

ELECTROMAGNETIC CONSIDERATIONS

1. THE FIELD VECTORS AND THE FIELD EQUATIONS

Maxwell's equations in Gaussian units are ([9])

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \tag{4.1}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{4.2}$$

where $\mathbf{H}$ is the magnetic field intensity, $\mathbf{E}$ is the electric field intensity, $\mathbf{D}$ is the electric displacement vector, and $\mathbf{B}$ is the magnetic flux vector. These vector fields are related by the equations

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \tag{4.3}$$

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \tag{4.4}$$

where $\mathbf{P}$ is the polarization vector and $\mathbf{M}$ is the magnetization vector, with $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{P}$ polar vectors, while $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$ are axial vectors. Along with the six Maxwell equations, (4.1) and (4.2), we have the auxiliary equations

$$\nabla \cdot \mathbf{B} = 0, \tag{4.5}$$

$$\nabla \cdot \mathbf{D} = 4\pi\varrho_e. \tag{4.6}$$

Equation (4.5) is satisfied automatically, and (4.6) simply defines ϱ_e such that the equation of the conservation of electric charge

$$\partial\varrho_e/\partial t + \nabla \cdot \mathbf{J} = 0 \tag{4.7}$$

is satisfied.

Boundary conditions are determined from the integral form of Maxwell's equations. Maxwell's differential equations may be determined from the integral forms when suitable differentiability conditions are assumed. The integral forms are:

Ampere's law:

$$\oint_l \mathbf{H}\cdot d\mathbf{r} = (1/c)\int_s \mathbf{n}\cdot\dot{\mathbf{D}}\, ds + (4\pi/c)\int_s \mathbf{n}\cdot\mathbf{J}\, ds, \tag{4.8}$$

Faraday's law:

$$\oint_l \mathbf{E}\cdot d\mathbf{r} = -(1/c)\int_s \mathbf{n}\cdot\dot{\mathbf{B}}\, ds, \tag{4.9}$$

$$\int_S \mathbf{n}\cdot\mathbf{B}\, ds = 0, \tag{4.10}$$

$$\int_S \mathbf{n}\cdot\mathbf{D}\, ds = 4\pi\int_V \varrho_e\, dV, \tag{4.11}$$

where l in (4.8) and (4.9) denotes an arbitrary closed circuit and s an arbitrary open surface contained within l, and S in (4.10) and (4.11) denotes an arbitrary closed surface. All surfaces and circuits are stationary with respect to an inertial reference frame. The integral form of the conservation of electric charge takes the form

$$\int_S \mathbf{n}\cdot\mathbf{J}\, ds = -\int_V \dot{\varrho}_e\, dV, \tag{4.12}$$

where S is a closed surface.

In indicial notation Maxwell's equations (4.1) and (4.2) and (4.5) and (4.6) take the form, respectively,

$$e_{ijk}H_{k,j} = (1/c)\dot{D}_i + (4\pi/c)J_i, \tag{4.13}$$

$$e_{ijk}E_{k,j} = -(1/c)\dot{B}_i, \tag{4.14}$$

$$B_{i,i} = 0, \tag{4.15}$$

$$D_{i,i} = 4\pi\varrho_e, \tag{4.16}$$

2. THE ELECTROMAGNETIC POTENTIALS

These equations may be reformulated in terms of the vector and scalar potentials $\mathbf{A}$ and φ:

$$B_i = e_{ijk}A_{k,j}, \tag{4.17}$$

$$e_{ijk}\left(E_k + \frac{1}{c}\dot{A}_k\right)_{,j} = 0, \tag{4.18}$$

$$E_k + (1/c)\dot{A}_k = -\varphi_{,k}. \tag{4.19}$$

Note that $A_{k,k}$ is arbitrary. Different selections of $A_{k,k}$ determine different gauges. The vector potential **A** is not unique because any other $\bar{\mathbf{A}}$ defined by

$$\bar{\mathbf{A}} = \mathbf{A} + \boldsymbol{\nabla}\Psi, \tag{4.20}$$

where Ψ is any scalar, could also have been used as the vector potential, since

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\Psi \equiv 0.$$

When written in terms of the vector and scalar potential Maxwell's equations become (4.13), (4.17), and (4.19), which we rewrite here as

$$e_{ijk}H_{k,j} = \frac{1}{c}\dot{D}_i + \frac{4\pi}{c}J_i, \tag{4.21}$$

$$B_i = e_{ijk}A_{k,j}, \tag{4.22}$$

$$E_k = -\varphi_{,k} - \frac{1}{c}\dot{A}_k, \tag{4.23}$$

along with the definitions (4.3) and (4.4), which we rewrite here as

$$D_i = E_i + 4\pi P_i, \qquad H_i = B_i - 4\pi M_i, \tag{4.24}$$

and the auxiliary equation (4.16), which we rewrite here as

$$D_{i,i} = 4\pi\varrho_e. \tag{4.25}$$

Although **A** and hence φ are not unique, **E**, **H**, **B**, **D**, and **J** are unique.

3. POYNTING'S THEOREM AND ELECTROMAGNETIC ENERGY FLUX

From (4.13) and (4.14) we form

$$E_i e_{ijk}H_{k,j} = (1/c)E_i\dot{D}_i + (4\pi/c)E_iJ_i \tag{4.26}$$

$$H_i e_{ijk}E_{k,j} = -(1/c)H_i\dot{B}_i, \tag{4.27}$$

and subtracting, we obtain

$$e_{ijk}E_iH_{k,j} - e_{ijk}H_iE_{k,j} = \frac{1}{c}E_i\dot{D}_i + \frac{1}{c}H_i\dot{B}_i + \frac{4\pi}{c}E_iJ_i$$

$$-(e_{jik}E_iH_k)_{,j} = \frac{1}{c}(E_i\dot{D}_i + H_i\dot{B}_i) + \frac{4\pi}{c}E_iJ_i. \tag{4.28}$$

Integrating (4.28) over an arbitrary volume and using the divergence theorem, we find

$$-(c/4\pi)\int_S \mathbf{n}\cdot\mathbf{E}\times\mathbf{H}\,dS = \int_V (1/4\pi)(\mathbf{E}\cdot\dot{\mathbf{D}}+\mathbf{H}\cdot\dot{\mathbf{B}})\,dV + \int_V \mathbf{E}\cdot\mathbf{J}\,dV. \quad (4.29)$$

Equation (4.29) is Poynting's theorem. So far it is nothing more than a mathematical consequence of Maxwell's equations. We will now try to interpret the terms physically for the purely electromagnetic case. When we do this we will have an energy equation for the purely electromagnetic case only. It is not an energy equation when there is *any* interaction present, such as, e.g., thermal or mechanical. However, in any event it is always a mathematical identity which is compatible with Maxwell's equations.

Let us define

$$\mathbf{h} = (c/4\pi)\,\mathbf{E}\times\mathbf{H}, \quad (4.30)$$

and identify $(\mathbf{E}\cdot\dot{\mathbf{D}} + \mathbf{H}\cdot\dot{\mathbf{B}})/4\pi$ with the rate of change of electromagnetic energy, $\dot{U}$. This last identification is questionable in general, but adequate for our purposes. Moreover, questions of this nature, which are closely related to the electromagnetic body force question, will, like and for the same reasons as the body force and finite deformation questions, be entirely ignored. Then in this purely electromagnetic case Poynting's theorem (4.29) gives us

$$\frac{\partial}{\partial t}\int_V U\,dV = -\int_S \mathbf{n}\cdot\mathbf{h}\,dS - \int_V \mathbf{E}\cdot\mathbf{J}\,dV. \quad (4.31)$$

Under the interpretation we are using, the $\mathbf{E}\cdot\mathbf{J}$ term is the usual Joule heat term. The energy equation (4.31) for the purely electromagnetic case now asserts that the time rate of change of internal energy in an arbitrary volume is equal to minus the rate at which electromagnetic energy flows out of the surface enclosing that volume minus the rate at which electric energy is dissipated inside the volume by thermal means.

The previous considerations have told us the important fact that the vector $\mathbf{h}$ defined in (4.30) represents the energy flux vector across a surface, and it is taken to be positive when outwardly directed across a closed surface. It is called the Poynting vector, and it has the units of energy per unit area per unit time or power per unit area.

The electromagnetic potentials $\mathbf{A}$ and φ enable us to put Poynting's theorem in a form which is particularly useful for our purposes. Consider the term in Poynting's differential equation, (4.28), which is responsible for the Poynting energy flux vector, $\mathbf{h}$, in (4.31). If (4.28) is multiplied

throughout by $c/4\pi$, the term may be written

$$-h_{i,i} = -(c/4\pi)(e_{ijk}E_jH_k)_{,i},$$

and substituting from (4.23), we have

$$\begin{aligned} -h_{i,i} &= +\frac{c}{4\pi}\left[e_{ijk}\left(\varphi_{,j} + \frac{1}{c}\dot{A}_j\right)H_k\right]_{,i} \\ &= \frac{c}{4\pi}e_{ijk}\left[\varphi_{,j}H_{k,i} + \frac{1}{c}(\dot{A}_jH_k)_{,i}\right], \end{aligned}$$

and substituting from (4.21), we find

$$\begin{aligned} -h_{i,i} &= -\frac{c}{4\pi}\varphi_{,j}\left(\frac{1}{c}\dot{D}_j + \frac{4\pi}{c}J_j\right) + \frac{1}{4\pi}(e_{ijk}\dot{A}_jH_k)_{,i} \\ &= -\frac{1}{4\pi}[(\varphi\dot{D}_j)_{,j} - \varphi\dot{D}_{j,j}] - (\varphi J_j)_{,j} + \varphi J_{j,j} + \frac{1}{4\pi}(e_{ijk}\dot{A}_jH_k)_{,i}, \end{aligned}$$

which, with the divergence of (4.21), yields

$$-h_{i,i} = -\left[\varphi\left(\frac{\dot{D}_i}{4\pi} + J_i\right) - \frac{1}{4\pi}e_{ijk}\dot{A}_jH_k\right]_{,i}. \tag{4.32}$$

Substituting from (4.32) into (4.28) and integrating over an arbitrary volume and applying the divergence theorem, we get Poynting's theorem, (4.29), in the form

$$\begin{aligned} &-\int_S n_i\left[\varphi\left(\frac{\dot{D}_i}{4\pi} + J_i\right) - \frac{1}{4\pi}e_{ijk}\dot{A}_jH_k\right]dS \\ &\quad = \int_V \frac{1}{4\pi}(E_i\dot{D}_i + H_i\dot{B}_i)\,dV + \int_V E_iJ_i\,dV, \end{aligned} \tag{4.33}$$

and as a consequence we obtain from (4.32) and (4.33) for the purely electromagnetic case the previous energy equation, (4.31), with

$$h_i = \varphi\left(\frac{\dot{D}_i}{4\pi} + J_i\right) - \frac{1}{4\pi}e_{ijk}\dot{A}_jH_k. \tag{4.34}$$

Now, whereas $h_i = (c/4\pi)e_{ijk}E_jH_k$ is unique, the present h_i in (4.34) is not, because φ and A_j are not unique. Nevertheless, in the present case we know that $\int_S n_ih_i\,dS$, where S is any closed surface, is unique, and that is all that counts. Moreover, the unique form of $\mathbf{h}$ can have added to it the $\nabla \times \mathbf{V}$, where $\mathbf{V}$ is any vector, without violating Poynting's theorem,

because $\int_S \mathbf{n}\cdot\boldsymbol{\nabla}\times\mathbf{V}\,dS = 0$, where S is an arbitrary closed surface. That is to say, the nonunique $\mathbf{h}$'s all differ by the curl of some vector, and that doesn't matter because the surface integral vanishes.

4. THE QUASISTATIC ELECTRIC FIELD

When we consider piezoelectricity or biased electrostriction, which, to a good approximation, gives the same equations, we will consider polarizable (but not magnetizable) dielectrics only. Consequently, we may set

$$\varrho_e = J_i = M_i \equiv 0 \tag{4.35}$$

in all the equations. It should be noted that under these circumstances from (4.35) and (4.24)

$$H_i \equiv B_i, \tag{4.36}$$

and H_i (and consequently A_i) cannot be zero, because from (4.21) and (4.35)

$$\boldsymbol{\nabla}\times\mathbf{H} = (1/c)\dot{\mathbf{D}}, \tag{4.37}$$

and

$$\dot{\mathbf{D}} \not\equiv 0. \tag{4.38}$$

Under these circumstances Maxwell's equations (4.21)–(4.23) take the form

$$e_{ijk}H_{k,j} = (1/c)\dot{D}_i, \tag{4.39}$$

$$H_k = e_{klm}A_{m,l}, \tag{4.40}$$

$$E_i = -\varphi_{,i} - (\dot{A}_i/c), \tag{4.41}$$

and from (4.24)

$$D_i = E_i + 4\pi P_i, \tag{4.42}$$

and from (4.25) and (4.35) we have the auxiliary equation

$$D_{i,i} = 0. \tag{4.43}$$

The basic simplifying assumption which is used to get the piezoelectric equations is that each component of $\varphi_{,i}$ that enters in a problem satisfies

$$|\dot{A}_i/c| \ll |\varphi_{,i}|. \tag{4.44}$$

This assumption obviously is well borne out experimentally. It basically

is valid when the electromagnetic waves essentially uncouple from the elastic waves, and when we are considering wavelengths near the elastic wave, which are much shorter than the electromagnetic wavelength of the same frequency. The condition for the validity of (4.44) is

$$\omega/c \ll | k_i | , \tag{4.45}$$

where k_i represents a component of the wave number for a wavelike solution of (4.39)–(4.41). It may readily be seen that (4.45) is the condition for the validity of (4.44) by considering a wavelike solution of (4.39)–(4.41) in the form of a successive approximation about $\dot{A}_i/c = 0$. When $\dot{A}_i/c$ is neglected in (4.41), the auxiliary equation (4.43) may be used to find φ. When φ is thus determined, A_i may be determined by using (4.39) and (4.40) with the now known $\dot{D}_i$ on the r.h.s. of (4.39), with the result that each

$$| A_i | \ll | \varphi | , \tag{4.46}$$

when (4.45) holds, thereby showing that (4.44) is indeed satisfied whenever (4.45) holds. As a matter of fact we could continue with this successive approximation procedure with the now known nonzero A_i and determine a second approximation for φ, and so on. However, as a consequence of (4.45), the first approximation (or the theory of the quasistatic electric field) is certainly as accurate as necessary. Moreover, when (4.45) is valid, it is clear from (4.39) and (4.46) that the magnetic portion of the Poynting energy flux in (4.34) is negligible compared to the electric portion, and by virtue of this fact and (4.35), the degenerate form of the Poynting energy flux in this quasistatic electric case may be written from (4.34) in the form

$$h_i = \varphi \dot{D}_i/4\pi . \tag{4.47}$$

In addition, since (4.45) must hold in the vacuum (or air) immediately outside a piezoelectric body as well as inside—in order to satisfy the electric boundary conditions at the interface—(4.44), (4.46), and (4.47) also hold in the same region and, consequently, the electric equation in the vacuum in the immediate vicinity of the piezoelectric body is

$$\varphi_{,kk} = 0 . \tag{4.48}$$

We are now in a position to write the conservation of energy for a linear piezoelectric continuum. However, before proceeding it should be noted that in this chapter we have used Gaussian units exclusively, primarily

because we wished to make the quasistatic electric field approximation and $\mathbf{A}$ and φ have the same Gaussian units, thereby enabling the approximation to be made more easily in those units. Nevertheless, once the quasistatic approximation has been made, we are free to use any electrical units we wish; and we employ MKS units exclusively throughout the remainder of this monograph because they are more convenient.

Chapter 5

THE LINEAR THEORY OF PIEZOELECTRICITY

1. ENERGY CONSIDERATIONS

The principle of conservation of energy for a piezoelectric medium states that in any volume V bounded by a surface S with unit outward normal $\mathbf{n}$ the rate of increase of energy (kinetic plus internal) is equal to the rate at which work is done by the surface tractions acting across S less the flux of electric energy outward across S. Thus we have

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\varrho \dot{u}_j \dot{u}_j + U\right) dV = \int_S (t_j \dot{u}_j - n_j \varphi \dot{D}_j)\, dS\,, \tag{5.1}$$

as the equation of the conservation of energy. Basically, this equation postulates the existence of the internal energy function U.

From our previous work we have the following equations: The stress equations of motion, (2.22), with $f_j = 0$,

$$\tau_{ij,i} = \varrho \ddot{u}_j\,, \tag{5.2}$$

(where $\tau_{ij} = \tau_{ji}$). The charge equation of electrostatics, (4.43),

$$D_{i,i} = 0\,. \tag{5.3}$$

The electric field–electric potential relations, (4.41), with (4.44),

$$E_k = -\varphi_{,k}\,. \tag{5.4}$$

The strain–mechanical displacement relations, (3.9),

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i})\,. \tag{5.5}$$

All these will be needed in what follows. Previously discussed linearizing approximations have already been included, such as $d/dt \approx \partial/\partial t$, $\mathbf{v} \approx \dot{\mathbf{u}}$, infinitesimal strain, the absence of the electric body force and couple, and

the fact that we cannot distinguish between the initial and the final positions. We also have the relation

$$dV = dV_0(1 + \Delta) \approx dV_0, \tag{5.6}$$

since

$$\Delta = u_{k,k} \ll 1, \tag{5.7}$$

and hence

$$\varrho = \varrho_0(1 - \Delta) \approx \varrho_0, \tag{5.8}$$

and, (3.43),

$$\frac{\partial}{\partial t} dV \approx \dot{\Delta}\, dV_0 \approx \dot{\Delta}\, dV. \tag{5.9}$$

Substituting from (2.15) into (5.1), applying the divergence theorem, and utilizing the fact that the resulting equation is valid for an arbitrary volume V, we obtain

$$\varrho \ddot{u}_j \dot{u}_j + \dot{U} = (\tau_{ij}\dot{u}_j)_{,i} - (\varphi \dot{D}_i)_{,i}$$

$$\dot{U} = (\tau_{ij,i} - \varrho \ddot{u}_j)\dot{u}_j + \tau_{ij}\dot{u}_{j,i} - \varphi \dot{D}_{i,i} - \varphi_{,i}\dot{D}_i,$$

which with (5.2)–(5.5) yields

$$\dot{U} = \tau_{ij}\dot{S}_{ij} + E_i\dot{D}_i. \tag{5.10}$$

Equation (5.10) is called the first law of thermodynamics for the piezoelectric medium.

2. PIEZOELECTRIC CONSTITUTIVE EQUATIONS

Let us define the electric enthalpy H by

$$H = U - E_iD_i. \tag{5.11}$$

Then differentiating (5.11) with respect to time, we obtain

$$\dot{H} = \dot{U} - E_i\dot{D}_i - \dot{E}_iD_i,$$

which with (5.10) yields

$$\dot{H} = \tau_{ij}\dot{S}_{ij} - D_i\dot{E}_i. \tag{5.12}$$

Equation (5.12) implies that

$$H = H(\mathbf{S}, \mathbf{E}), \tag{5.13}$$

and differentiating (5.13) with respect to time, we find

$$\dot{H} = \frac{\partial H}{\partial S_{ij}} \dot{S}_{ij} + \frac{\partial H}{\partial E_i} \dot{E}_i ,$$

and substituting from (5.12), we obtain

$$\left(\tau_{ij} - \frac{\partial H}{\partial S_{ij}}\right) \dot{S}_{ij} - \left(D_i + \frac{\partial H}{\partial E_i}\right) \dot{E}_i = 0. \tag{5.14}$$

Since Eq. (5.14) is an identity which must hold for *arbitrary* $\dot{S}_{ij}$ and $\dot{E}_i$ which are consistent with the condition $\dot{S}_{ij} = \dot{S}_{ji}$, we obtain

$$\tau_{ij} = \frac{1}{2}\left(\frac{\partial H}{\partial S_{ij}} + \frac{\partial H}{\partial S_{ji}}\right), \tag{5.15}$$

$$D_i = -\partial H/\partial E_i. \tag{5.16}$$

If we further agree to construct H so that

$$\frac{\partial H}{\partial S_{ij}} = \frac{\partial H}{\partial S_{ji}},$$

we may write in place of (5.15)

$$\tau_{ij} = \partial H/\partial S_{ij}. \tag{5.17}$$

Since we are interested in a linear theory only, we construct a homogeneous quadratic form for H

$$H = \tfrac{1}{2} c^E_{ijkl} S_{ij} S_{kl} - e_{ijk} E_i S_{jk} - \tfrac{1}{2} \varepsilon^S_{ij} E_i E_j , \tag{5.18}$$

where

$$c_{ijkl} = c_{ijlk} = c_{jikl} = c_{klij} ,$$

$$e_{ijk} = e_{ikj} ,$$

$$\varepsilon_{ij} = \varepsilon_{ji} ,$$

and here and below we have dropped the superscript E on the elastic constants and the superscript S on the dielectric constants because all other constants occurring in this monograph will be defined in terms of these constants and e_{ijk}. Thus we have 21 independent elastic constants, 18 independent piezoelectric constants, and six independent dielectric constants in the most general case ([10]) (triclinic crystal without center of symmetry). Since **e** is an odd rank polar tensor, it cannot exist in any material that has a center of symmetry.

From (5.16)–(5.18) we obtain the linear piezoelectric constitutive equations

$$\tau_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \tag{5.19}$$

$$D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k. \tag{5.20}$$

3. THE DIFFERENTIAL EQUATIONS OF PIEZOELECTRICITY

We now have a determinate system of equations, i.e., the number of equations is equal to the number of variables, as is clear from Table I. This system of 22 equations in 22 variables can readily be reduced by simple substitutions to four equations in the four variables u_j and φ. The equations are

$$c_{ijkl}u_{k,li} + e_{kij}\varphi_{,ki} = \varrho\ddot{u}_j, \tag{5.21}$$

$$e_{ikl}u_{k,li} - \varepsilon_{ik}\varphi_{,ki} = 0. \tag{5.22}$$

We must solve this system of equations subject to boundary conditions which we have yet to determine. To determine the boundary conditions, we will obtain conditions sufficient for a unique solution analogous to those of Neumann for the purely elastic case [([11]), Section 124]. This will give us a uniqueness theorem.

Before we obtain the uniqueness theorem we should note that the internal energy function U must be a positive-definite quadratic form in order to secure the stability of the system. From (5.18), (5.11), and (5.20) it is readily seen that

$$U = \tfrac{1}{2}c_{ijkl}S_{ij}S_{kl} + \tfrac{1}{2}\varepsilon_{ij}E_iE_j. \tag{5.23}$$

TABLE I

	Number of equations	Additional variables
$\tau_{ij,i} = \varrho\ddot{u}_j$	3	9
$D_{i,i} = 0$	1	3
$\tau_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k$	6	9
$D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k$	3	0
$S_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$	6	0
$E_k = -\varphi_{,k}$	3	1
Totals	22	22

4. UNIQUENESS OF SOLUTION

Consider two solutions of the 22 equations

$$\tau_{ij,i} = \varrho \ddot{u}_j , \tag{5.24}$$

$$D_{i,i} = 0 , \tag{5.25}$$

$$\tau_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k , \tag{5.26}$$

$$D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k , \tag{5.27}$$

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}) , \tag{5.28}$$

$$E_k = - \varphi_{,k} . \tag{5.29}$$

Since every equation is linear, the difference of the two solutions is also a solution of the same system of equations. Let τ_{ij}^*, u_j^*, D_i^*, S_{kl}^*, E_k^*, and φ^* denote the difference solution, where $\tau_{ij}^* = \tau_{ij}^{(2)} - \tau_{ij}^{(1)}$, etc. Now, consider the dependent variables in the above equations to be starred (i.e., the difference variables), and drop the star.

From (5.24) form the scalar

$$(\tau_{ij,i} - \varrho \ddot{u}_j)\dot{u}_j = 0 ,$$

and integrate over the volume of the region being considered

$$\int_V (\tau_{ij,i}\dot{u}_j - \varrho \ddot{u}_j \dot{u}_j)\, dV = 0$$

$$= \int_V [(\tau_{ij}\dot{u}_j)_{,i} - \tau_{ij}\dot{u}_{j,i} - \tfrac{1}{2}\dot{\varrho}\dot{u}_j\dot{u}_j]\, dV . \tag{5.30}$$

Applying the divergence theorem to (5.30) and employing the symmetry of the stress tensor and (5.28), we obtain

$$\int_S n_i \tau_{ij}\dot{u}_j\, dS = \int_V [\tau_{ij}\dot{S}_{ji} + \dot{T}]\, dV , \tag{5.31}$$

where

$$T = \tfrac{1}{2}\varrho \dot{u}_j \dot{u}_j . \tag{5.32}$$

The substitution of (5.26) into (5.31) yields

$$\int_S n_i \tau_{ij}\dot{u}_j\, dS = \int_V [c_{ijkl} S_{kl}\dot{S}_{ij} - e_{kij} E_k \dot{S}_{ij} + \dot{T}]\, dV ,$$

and the further substitution of (5.27) yields

$$\int_S n_i \tau_{ij} \dot{u}_j \, dS = \int_V [c_{ijkl} S_{kl} \dot{S}_{ij} - E_k(\dot{D}_k - \varepsilon_{ki} \dot{E}_k) + \dot{T}] \, dV,$$

which with (5.29) and (5.25) yields

$$\int_S n_i \tau_{ij} \dot{u}_j \, dS = \int_V [c_{ijkl} S_{kl} \dot{S}_{ij} + \varepsilon_{ik} E_i \dot{E}_k + (\varphi \dot{D}_k)_{,k} + \dot{T}] \, dV. \quad (5.33)$$

Applying the divergence theorem to (5.33), we obtain

$$\int_V [\dot{T} + c_{ijkl} S_{kl} \dot{S}_{ij} + \varepsilon_{ik} E_i \dot{E}_k] \, dV = \int_S [n_i \tau_{ij} \dot{u}_j - n_i \varphi \dot{D}_i] \, dS. \quad (5.34)$$

At this point it should be remembered that this equation is for the difference solution, i.e., every dependent variable should have a star. Let us define the internal energy U^* of the difference system as

$$U^* = \tfrac{1}{2} c_{ijkl} S_{ij}^* S_{kl}^* + \tfrac{1}{2} \varepsilon_{ik} E_i^* E_k^*. \quad (5.35)$$

Then differentiating (5.35) with respect to time and utilizing the symmetries of all the quantities, we find

$$\dot{U}^* = c_{ijkl} S_{kl}^* \dot{S}_{ij}^* + \varepsilon_{ik} E_i^* \dot{E}_k^*,$$

which enables us to write (5.34) in the form

$$\int_V [\dot{T}^* + \dot{U}^*] \, dV = \int_S [n_i \tau_{ij}^* \dot{u}_j^* - n_i \varphi^* \dot{D}_i^*] \, dS. \quad (5.36)$$

Integrating (5.36) with respect to time, we have

$$\int_{t_0}^t dt \int_V (\dot{T}^* + \dot{U}^*) \, dV = \int_{t_0}^t dt \int_S (n_i \tau_{ij}^* \dot{u}_j^* - n_i \varphi^* \dot{D}_i^*) \, dS,$$

and hence

$$\mathscr{T}^* + \mathscr{U}^* = \mathscr{T}_0^* + \mathscr{U}_0^* + \int_{t_0}^t dt \int_S (t_j^* \dot{u}_j^* - n_i \varphi^* \dot{D}_i^*) \, dS, \quad (5.37)$$

where

$$\mathscr{T}^* = \int_V T^* \, dV, \qquad \mathscr{U}^* = \int_V U^* \, dV, \quad (5.38)$$

$$t_j^* = n_i \tau_{ij}^*. \quad (5.39)$$

Since T^* and U^* are positive-definite homogeneous quadratic functions ([12])

of $\dot{u}_j^*$, S_{ij}^* and E_i^*, making the right-hand side zero (and hence the left-hand-side zero) is sufficient to ensure that

$$\dot{u}_j^* = S_{ij}^* = E_i^* = 0\,,$$

and hence that the two solutions are identical to within a static rigid body displacement and a constant potential [see ([11]), Section 18]. Therefore from the right-hand side of (5.37) we can read off initial and boundary conditions sufficient for a unique solution.

Initial conditions may be the specification of $\dot{u}_j$, u_j, and E_i at each point of the body at t_0, in which case the aforementioned static rigid body displacement of the difference system vanishes. Boundary conditions consist of the specification at all t and at each point of the surface of the body any combination of conditions which make the surface integral in (5.37) vanish. For example: (t_1, t_2, t_3, φ); $(t_1, t_2, t_3, n_i D_i)$; (t_1, u_2, u_3, φ); $(t_1, u_2, t_3, n_i D_i)$; etc.

The above is the piezoelectric generalization of Neumann's theorem in linear elasticity [see ([11]), p. 176]. The uniqueness theorem which has been presented is subject to a number of restrictions which are not especially important for our purposes. Some of these restrictions may be removed [see Mindlin's monograph ([5]), pp. 1.17–1.18].

Chapter 6

HAMILTON'S PRINCIPLE

1. THE PROCESS OF VARIATION

Consider a function $f(x)$, $a < x < b$. Change the function at any (or every) point $x = x_0$ to $\bar{f}(x)$, while holding $x = x_0$ fixed. Form the difference

$$\bar{f}(x_0) - f(x_0) \equiv \delta f, \tag{6.1}$$

which is called the variation of f. Note that δf differs fundamentally from df, which is defined as the difference in the value of the function f at two neighboring points x_1 and x_0. In taking the variation, the functional form is varied and the position is held fixed. In taking the differential, the position is varied and the functional form is held fixed.
Now

$$\frac{d}{dx}f = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \qquad x_1 = x_0 + \Delta x, \tag{6.2}$$

and

$$\begin{aligned}
\delta\left(\frac{df}{dx}\right) &= \frac{\overline{df}}{dx} - \frac{df}{dx} = \lim_{\Delta x \to 0}\left[\frac{\overline{f(x+\Delta x) - f(x)}}{\Delta x} - \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0}\left[\frac{\bar{f}(x+\Delta x) - \bar{f}(x)}{\Delta x} - \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\bar{f}(x+\Delta x) - f(x+\Delta x) - \bar{f}(x) + f(x)\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\delta f(x+\Delta x) - \delta f(x)\right] = \frac{d}{dx}(\delta f).
\end{aligned} \tag{6.3}$$

Thus the δ-process (variation) commutes with d/dx (differentiation). Consider $\delta \int_a^b F(x)\,dx$; then by definition

$$\begin{aligned}
\delta \int_a^b F(x)\,dx &= \overline{\int_a^b F(x)\,dx} - \int_a^b F(x)\,dx = \int_a^b \bar{F}(x)\,dx - \int_a^b F(x)\,dx \\
&= \int_a^b \left[\bar{F}(x) - F(x)\right] dx = \int_a^b \delta F(x)\,dx.
\end{aligned} \tag{6.4}$$

This shows that the variation of a definite integral is equal to the definite integral of the variation.

In practice we are not interested in the simple cases we have just discussed, in which F is a *prescribed* function of the *independent* variable x. Instead we are interested in the situation in which F is a prescribed function of a *dependent* variable, say, y, and its first derivative, y', which in turn depends on the independent variable x. Moreover, we are especially concerned when F appears as the integrand of a *definite* integral. Under these circumstances we have

$$G = \int_a^b F(y, y', x)\, dx\,, \tag{6.5}$$

and we wish to make the functional G in (6.5) stationary. In Eq. (6.5) the dependence of F on y, y', and x does not change during the variation. However, the dependent variable y (and, of course, y') is changed during the variation in accordance with (6.1). Clearly then, taking the variation of the functional G in (6.5), we obtain

$$\delta G = \int_a^b \delta F(y, y', x)\, dx = 0\,, \tag{6.6}$$

since the limits a and b remain fixed. Since the dependence of F on y and y' does not change as y (and, hence, y') are varied, from (6.1) we obtain

$$\delta F = F(\bar{y}, \bar{y}', x) - F(y, y', x)\,, \tag{6.7}$$

where $\bar{y} = y + \delta y$ and $\bar{y}' = y' + \delta y'$. Expanding $F(\bar{y}, \bar{y}', x)$ in a Taylor series about y and y' and neglecting terms in δy and $\delta y'$ higher than the first since δy and $\delta y'$ approach zero, from (6.7) we obtain

$$\delta F = \frac{\partial F}{\partial \bar{y}}\bigg|_{\delta y=0} \delta y + \frac{\partial F}{\partial \bar{y}'}\bigg|_{\delta y'=0} \delta y'\,,$$

which may be written

$$\delta F = \frac{\partial F}{\partial y}\, \delta y + \frac{\partial F}{\partial y'}\, \delta y'\,. \tag{6.8}$$

Substituting from (6.8) into (6.6), we find

$$\delta G = \int_a^b \left[\frac{\partial F}{\partial y}\, \delta y + \frac{\partial F}{\partial y'}\, \delta y'\right] dx = 0\,,$$

which may be written

$$\delta G = \int_a^b \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right] \delta y \, dx + \left[\frac{\partial F}{\partial y'} \, \delta y\right]_a^b = 0, \qquad (6.9)$$

which must hold for *arbitrary* variations δy. In particular, if δy vanishes at a and b, it is easy to show [([13]), Chapter II, Section 10] that we must have

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0, \qquad (6.10)$$

which is the Euler equation for the variational problem. If, on the other hand, δy does not vanish at one end, say b, we must have

$$\frac{\partial F}{\partial y'} = 0 \qquad \text{at} \quad x = b, \qquad (6.11)$$

and (6.10) is satisfied.

2. HAMILTONIAN MECHANICS

In classical mechanics there is a function L of all the pertinent variables (the independent coordinates and velocities) which is given by [([13]), Chapter V, Section 1]

$$L = T(\dot{q}_k) - V(q_k, t) = L(q_k, \dot{q}_k, t), \qquad (6.12)$$

where T is the kinetic and V the potential energy. There is a principle which states that for a conservative holonomic system

$$\delta \int_{t_0}^{t} L \, dt = 0 \qquad (6.13)$$

for all variations δq_k of the position coordinates q_k which are consistent with the holonomic [([13]), Chapter I, Section 6] constraints and which vanish at t_0 and t [([13]), Chapter V, Section 1]. The principle is called Hamilton's principle, and all of classical mechanics (subject to the restrictions mentioned) may be shown to be a consequence of that single statement. If the constraints are not holonomic, the principle can be generalized by introducing Lagrangian undetermined multipliers, but we are not interested in this. If the force system is not conservative, the principle can be generalized directly simply by calculating the virtual work δW done by the nonconservative forces in a virtual displacement consistent with the constraints, and reformulating the principle as follows:

$$\delta \int_{t_0}^{t} L \, dt + \int_{t_0}^{t} \delta W \, dt = 0. \qquad (6.14)$$

This is the form of Hamilton's principle which we will use. There is a Hamilton's principle for the electromagnetic field equations in free space [see ([9]), Chapter 24].

It is commonly agreed that it is both interesting and useful to have a variational principle which reproduces an entire system of equations which were obtained previously in the alternative manner of defining field variables and applying conservation theorems. It is interesting (from the standpoint of having a single statement embodying the entire theory in a manner which exhibits energies of interaction) and useful for the purpose of obtaining approximate solutions of the equations or of generalizing the theory.

We are interested in obtaining approximate solutions of the equations which we have already derived.

3. THE VARIATIONAL PRINCIPLE FOR A LINEAR PIEZOELECTRIC CONTINUUM*

Consider a piezoelectric body subject to prescribed surface tractions, $\mathbf{t}$, and surface charge per unit area σ. The given surface tractions $\mathbf{t}$ and surface charge σ may, of course, be zero at any part. The virtual work per unit area done by the prescribed surface tractions in a small virtual displacement of the surface is $t_k\,\delta u_k$. The electrical analog of the virtual work per unit area done by the prescribed surface charge σ in a small variation $\delta\varphi$ of electrical potential φ is $-\sigma\,\delta\varphi$. The minus sign occurs because in the variational principle for our electromechanical medium it turns out that the electric enthalpy $H = U - E_i D_i$ takes the place of the internal energy function U in the Lagrange density, i.e., the effective electrical energy content of H is opposite in sign to that of U. In any event, we will show that the variational principle presented here yields the differential equations and boundary conditions previously derived.

The Lagrangian for this bounded piezoelectric medium is defined by

$$L = \int_V [\tfrac{1}{2}\varrho\dot{u}_j\dot{u}_j - H(S_{kl}, E_k)]\,dV, \tag{6.15}$$

and

$$\delta W = \int_S (\bar{t}_k\,\delta u_k - \bar{\sigma}\,\delta\varphi)\,dS. \tag{6.16}$$

* All linearizing approximations will be already built in.

Hence, from (6.14)–(6.16) the variational principle takes the form

$$\delta \int_{t_0}^{t} dt \int_V [\tfrac{1}{2}\varrho \dot{u}_j \dot{u}_j - H(S_{kl}, E_k)]\, dV + \int_{t_0}^{t} dt \int_S (\bar{t}_k\, \delta u_k - \bar{\sigma}\, \delta\varphi)\, dS = 0, \tag{6.17}$$

where $\bar{t}_k$ and $\bar{\sigma}$ are prescribed and all variations vanish at t_0 and t. Remember that we have the relations (5.5), (5.4), (5.17), and (5.16) and the approximation for the material derivative:

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \qquad E_k = -\varphi_{,k},$$

$$\tau_{kl} = \frac{\partial H}{\partial S_{kl}}, \qquad D_k = -\frac{\partial H}{\partial E_k}, \qquad \frac{\partial}{\partial t} \approx \frac{d}{dt},$$

which we will need in the derivation.

Consider the expression for the variational principle term by term.

First term:

$$\begin{aligned}\delta \int_{t_0}^{t} dt \int_V \tfrac{1}{2}\varrho \dot{u}_j \dot{u}_j\, dV &= \int_{t_0}^{t} dt \int_V \varrho \dot{u}_j\, \delta \dot{u}_j\, dV \\ &= \int_{t_0}^{t} dt \int_V \left[\frac{\partial}{\partial t}(\varrho \dot{u}_j\, \delta u_j\, dV) - \varrho \ddot{u}_j\, \delta u_j\, dV\right] \\ &= \int_V \left[\varrho \dot{u}_j\, \delta u_j\right]_{t_0}^{t} dV - \int_{t_0}^{t} dt \int_V \varrho \ddot{u}_j\, \delta u_j\, dV = -\int_{t_0}^{t} dt \int_V \varrho \ddot{u}_j\, \delta u_j\, dV,\end{aligned}$$

since δu_j vanishes at t_0, t.

Second term:

$$\delta \int_{t_0}^{t} dt \int_V H(S_{kl}, E_k)\, dV = \int_{t_0}^{t} dt \int_V \left[\frac{\partial H}{\partial S_{kl}}\, \delta S_{kl} + \frac{\partial H}{\partial E_k}\, \delta E_k\right] dV.$$

Now,

$$\delta S_{kl} = \tfrac{1}{2}\, \delta(u_{k,l} + u_{l,k}) = \tfrac{1}{2}[(\delta u_k)_{,l} + (\delta u_l)_{,k}], \tag{6.18}$$

$$\delta E_k = -\delta\varphi_{,k} = -(\delta\varphi)_{,k}. \tag{6.19}$$

Hence because of the symmetry of τ_{kl}

$$\begin{aligned}\delta \int_{t_0}^{t} dt \int_V H\, dV &= \int_{t_0}^{t} dt \int_V [\tau_{kl}(\delta u_l)_{,k} + D_k(\delta\varphi)_{,k}]\, dV \\ &= \int_{t_0}^{t} dt \int_S [n_k \tau_{kl}\, \delta u_l + n_k D_k\, \delta\varphi]\, dS - \int_{t_0}^{t} dt \int_V [\tau_{kl,k}\, \delta u_l + D_{k,k}\, \delta\varphi]\, dV,\end{aligned}$$

from the divergence theorem.

Thus after substituting the above expressions for the first two terms back into the variational principle, (6.17), and rearranging terms we have

$$\delta \int_{t_0}^{t} dt \int_V [\tfrac{1}{2}\varrho \dot{u}_j \dot{u}_j - H(S_{kl}, E_k)]\, dV + \int_{t_0}^{t} dt \int_S (\bar{t}_k\, \delta u_k - \bar{\sigma}\, \delta\varphi)\, dS$$
$$= \int_{t_0}^{t} dt \left[\int_V (\tau_{kl,k} - \varrho \ddot{u}_l)\, \delta u_l\, dV + \int_V D_{k,k}\, \delta\varphi\, dV \right.$$
$$\left. + \int_S (\bar{t}_l - n_k \tau_{kl})\, \delta u_l\, dS - \int_S (\bar{\sigma} + n_k D_k)\, \delta\varphi\, dS \right] = 0. \tag{6.20}$$

Since the variations δu_l and $\delta\varphi$ are arbitrary inside the volume V, we have:

Stress equations of motion

$$\tau_{kl,k} - \varrho \ddot{u}_l = 0. \tag{6.21}$$

Charge equation of electrostatics

$$D_{k,k} = 0. \tag{6.22}$$

Now, on the surface S:

(α) Either δu_l is arbitrary and $\bar{t}_l - n_k \tau_{kl} = 0$, or u_l is prescribed and $\delta u_l = 0$.

(β) Either $\delta\varphi$ is arbitrary and $\bar{\sigma} + n_k D_k = 0$, or φ is prescribed and $\delta\varphi$ is zero.

Thus we see that we have obtained both the differential equations and the boundary conditions from this single variational principle.

Note that $\bar{t}_l = 0$ for traction-free boundary conditions. The surface charge $\bar{\sigma}$ exists in general at all interfaces in this formalism. It may be taken as zero at a dielectric–dielectric interface if the appropriate dielectric constant on the side of interest is sufficiently greater than the corresponding dielectric constant on the other side. This will usually be the case, and for our purposes it will be nonzero only when solved for at the end of a problem. It will be taken as zero on all surfaces on which it must be prescribed. In introducing $\bar{\sigma}$ and ignoring the electromagnetic field on the vacuum (or air) side of the dielectric–vacuum (or air) interface we have made a restrictive but usually valid assumption. This assumption may be removed, but we will not do so here.

We are interested in the last variational form, (6.20), of the variational principle. This is the variational equation from which we will obtain approximate solutions.

4. A MODIFICATION OF HAMILTON'S PRINCIPLE

As we shall see, Eq. (6.20), which we obtained from Hamilton's principle, turns out to be useful in certain instances and not in others. More specifically, Eq. (6.20) is particularly useful, within the theoretical framework we will espouse, when the boundary conditions which are to be satisfied approximately are traction free and/or charge free, but not when the mechanical displacement and/or the electrical potential vanishes. The reason for this limitation is that in the variational principle the variations of u_k and φ are constrained to vanish on those portions of the boundary on which they are prescribed, and any approximating functions used with this variational principle must satisfy the variational constraints contained in the principle. On the other hand, the traction and/or charge boundary conditions arise when the variations of u_k and φ are unconstrained. In addition, for similar reasons Eq. (6.20) is not useful when there is an internal surface of discontinuity present as shown in Fig. 5. However, it turns out that all these difficulties present in (6.20) may be removed if we modify ([14]) Hamilton's principle slightly.

It is clear from the discussion in the preceding paragraph that the limitations of the utility of (6.20) arise as a consequence of the constraints on the variations imposed in the principle. Thus it is clear that if we can remove the constraints, we can remove the limitation. Now, it is well known in variational calculus [([15]), Chapter IV] (and in stationary problems in general) that a constraint on a variation may be removed by adding to the Lagrangian each constraint as a zero times a Lagrange multiplier and then treating all variations of the field variables as unconstrained [([15]),

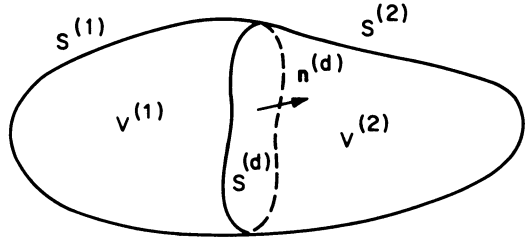

Fig. 5. Diagram of a bounded region containing an internal surface of discontinuity.

Chapter IV, Section 9.1]. We now proceed to modify Eq. (6.17) accordingly, i.e., in place of (6.17) we write

$$\delta \int_{t_0}^{t} dt \sum_{m=1}^{2} \Big[\int_{V^{(m)}} [\tfrac{1}{2}\varrho_j^{(m)} \dot{u}^{(m)} \dot{u}_j^{(m)} - H^{(m)}(S_{kl}^{(m)}, E_k^{(m)})]\, dV$$
$$+ \int_{S_N^{(m)}} (\bar{t}_k^{(m)} u_k^{(m)} - \bar{\sigma}^{(m)} \varphi^{(m)})\, dS + \int_{S_C^{(m)}} \lambda_k^{(m)} (u_k^{(m)} - \bar{u}_k^{(m)})\, dS$$
$$- \int_{S_C^{(m)}} l^{(m)} (\varphi^{(m)} - \bar{\varphi}^{(m)})\, dS \Big]$$
$$+ \delta \int_{t_0}^{t} dt \int_{S^{(d)}} \lambda_k^{(d)} (u_k^{(2)} - u_k^{(1)})\, dS - \delta \int_{t_0}^{t} dt \int_{S^{(d)}} l^{(d)} (\varphi^{(2)} - \varphi^{(1)})\, dS = 0\,, \tag{6.23}$$

where $S_N^{(m)}$, $S_C^{(m)}$, and $S^{(d)}$, respectively, stand for the portion of the mth surface on which the traction $\bar{t}_k^{(m)}$ and/or the charge $\bar{\sigma}^{(m)}$ are prescribed, the portion of the mth surface on which the mechanical displacement $u_k^{(m)}$ and/or the electrical potential $\varphi^{(m)}$ are prescribed, and the surface of discontinuity shown in Fig. 5. In taking the variation of the expression on the left-hand side in (6.23), all the $\delta u_k^{(m)}$ and $\delta\varphi^{(m)}$ are unconstrained everywhere except at t and t_0,* where they are constrained as in the usual version of Hamilton principle, and the Lagrangian multipliers $\lambda_k^{(m)}$, $l^{(m)}$, $\lambda_k^{(d)}$, and $l^{(d)}$ are to be varied freely [(15), Chapter IV, Section 9.2]. Carrying out the variations in (6.23), as we did in going from (6.17) to (6.20), and taking into consideration the fact that the divergence theorem is valid only up to and not across the surface of discontinuity $S^{(d)}$, we obtain

$$\int_{t_0}^{t} dt \sum_{m=1}^{2} \Big[\int_{V^{(m)}} [(\tau_{kl,k}^{(m)} - \varrho^{(m)} \ddot{u}_l^{(m)})\, \delta u_l^{(m)} + D_{k,k}^{(m)}\, \delta\varphi^{(m)}]\, dV$$
$$+ \int_{S_N^{(m)}} [(\bar{t}_l^{(m)} - n_k^{(m)} \tau_{kl}^{(m)})\, \delta u_l^{(m)} - (\bar{\sigma}^{(m)} + n_k^{(m)} D_k^{(m)})\, \delta\varphi^{(m)}]\, dS$$
$$+ \int_{S_C^{(m)}} [(\lambda_l^{(m)} - n_k^{(m)} \tau_{kl}^{(m)})\, \delta u_l^{(m)} - (l^{(m)} + n_k^{(m)} D_k^{(m)})\, \delta\varphi^{(m)}$$
$$+ \delta\lambda_l^{(m)} (u_l^{(m)} - \bar{u}_l^{(m)}) - \delta l^{(m)} (\varphi^{(m)} - \bar{\varphi}^{(m)})]\, dS \Big]$$
$$+ \int_{t_0}^{t} dt \int_{S^{(d)}} [-(\lambda_l^{(d)} + n_k^{(d)} \tau_{lk}^{(1)})\, \delta u_l^{(1)} + (\lambda_l^{(d)} + n_k^{(d)} \tau_{lk}^{(2)})\, \delta u_l^{(2)}$$
$$+ (l^{(d)} - n_k^{(d)} D_k^{(1)})\, \delta\varphi^{(1)} - (l^{(d)} - n_k^{(d)} D_k^{(2)})\, \delta\varphi^{(2)}$$
$$+ \delta\lambda_k^{(d)} (u_k^{(2)} - u_k^{(1)}) + \delta l^{(d)} (\varphi^{(2)} - \varphi^{(1)})]\, dS = 0\,, \tag{6.24}$$

* This constraint can be removed also. [See (14)].

where $n_l^{(d)}$ denotes the components of the unit normal to the surface of discontinuity $S^{(d)}$ directed from $V^{(1)}$ to $V^{(2)}$. Since the volumetric variations in (6.24) are arbitrary, we have (6.21) and (6.22) in both $V^{(1)}$ and $V^{(2)}$. Since all the surface variations $\delta u_k^{(m)}$, $\delta\varphi^{(m)}$, $\delta\lambda_k^{(m)}$, $\delta l^{(m)}$, $\delta\lambda_k^{(d)}$, and $\delta l^{(d)}$ are independent (unconstrained), we have

$$\bar{t}_l^{(m)} - n_k^{(m)}\tau_{kl}^{(m)} = 0 \quad \text{on} \quad S_N^{(m)}, \tag{6.25}$$

$$\bar{\sigma}^{(m)} + n_k^{(m)}D_k^{(m)} = 0 \quad \text{on} \quad S_N^{(m)}, \tag{6.26}$$

$$\lambda_l^{(m)} - n_k^{(m)}\tau_{kl}^{(m)} = 0 \quad \text{on} \quad S_C^{(m)}, \tag{6.27}$$

$$l^{(m)} + n_k^{(m)}D_k^{(m)} = 0 \quad \text{on} \quad S_C^{(m)}, \tag{6.28}$$

$$u_l^{(m)} - \bar{u}_l^{(m)} = 0 \quad \text{on} \quad S_C^{(m)}, \tag{6.29}$$

$$\varphi^{(m)} - \bar{\varphi}^{(m)} = 0 \quad \text{on} \quad S_C^{(m)}, \tag{6.30}$$

$$\lambda_l^{(d)} + n_k^{(d)}\tau_{lk}^{(1)} = 0 \quad \text{on} \quad S^{(d)}, \tag{6.31}$$

$$\lambda_l^{(d)} + n_k^{(d)}\tau_{lk}^{(2)} = 0 \quad \text{on} \quad S^{(d)}, \tag{6.32}$$

$$l^{(d)} - n_k^{(d)}D_k^{(1)} = 0 \quad \text{on} \quad S^{(d)}, \tag{6.33}$$

$$l^{(d)} - n_k^{(d)}D_k^{(2)} = 0 \quad \text{on} \quad S^{(d)}, \tag{6.34}$$

$$u_k^{(2)} - u_k^{(1)} = 0 \quad \text{on} \quad S^{(d)}, \tag{6.35}$$

$$\varphi^{(2)} - \varphi^{(1)} = 0 \quad \text{on} \quad S^{(d)}. \tag{6.36}$$

From (6.27) and (6.28) we obtain the Lagrangian multipliers $\lambda_l^{(m)}$ and $l^{(m)}$ in the form

$$\lambda_l^{(m)} = n_k^{(m)}\tau_{kl}^{(m)}, \tag{6.37}$$

$$l^{(m)} = -n_k^{(m)}D_k^{(m)}. \tag{6.38}$$

Note that the subtraction of (6.32) from (6.31) yields

$$n_k^{(d)}(\tau_{lk}^{(2)} - \tau_{lk}^{(1)}) = 0, \tag{6.39}$$

and the subtraction of (6.34) from (6.33) yields

$$n_k^{(d)}(D_k^{(2)} - D_k^{(1)}) = 0. \tag{6.40}$$

Equations (6.39) and (6.40), respectively, tell us that the traction vector and the normal component of electric displacement are continuous across $S^{(d)}$. At this point it is clear that our variational formalism (6.23) yields

the differential equations (6.21) and (6.22) in the mth region and the boundary conditions (6.25), (6.26), (6.29), (6.30), 6.35), (6.36), (6.39), and (6.40) of the linear theory of piezoelectricity for the problem at hand.

In order to find *the most appropriate forms* of (6.23) and (6.24) to be used in obtaining approximate solutions to boundary value problems, add (6.31) to (6.32) and (6.33) to (6.34) to obtain, respectively,

$$\lambda_l^{(d)} = -\tfrac{1}{2}n_k^{(d)}(\tau_{lk}^{(1)} + \tau_{lk}^{(2)}), \tag{6.41}$$

$$I^{(d)} = \tfrac{1}{2}n_k^{(d)}(D_k^{(1)} + D_k^{(2)}), \tag{6.42}$$

and then substitute from (6.37), (6.38), (6.41), and (6.42) into (6.23) and (6.24) to obtain, respectively,

$$\begin{aligned}
&\delta\int_{t_0}^{t} dt \sum_{m=1}^{2}\Big[\int_{V^{(m)}}(\tfrac{1}{2}\varrho^{(m)}\ddot{u}_j^{(m)}\dot{u}_j^{(m)} - H^{(m)})\,dV + \int_{S_N^{(m)}}(\bar{t}_k^{(m)}u_k^{(m)} - \bar{\sigma}^{(m)}\varphi^{m})\,dS \\
&\qquad + \int_{S_C^{(m)}} n_l^{(m)}\tau_{kl}^{(m)}(u_k^{(m)} - \bar{u}_k^{(m)})\,dS + \int_{S_C^{(m)}} n_k^{(m)}D_k^{(m)}(\varphi^{(m)} - \bar{\varphi}^{(m)})dS\Big] \\
&-\delta\int_{t_0}^{t} dt \int_{S^{(d)}} \tfrac{1}{2}n_k^{(d)}[(\tau_{lk}^{(1)}+\tau_{lk}^{(2)})(u_l^{(2)}-u_l^{(1)})+(D_k^{(1)}+D_k^{(2)})(\varphi^{(2)}-\varphi^{(1)})]dS=0,
\end{aligned} \tag{6.43}$$

$$\begin{aligned}
&\int_{t_0}^{t} dt \sum_{m=1}^{2}\Big[\int_{V^{(m)}}[(\tau_{kl,k}^{(m)} - \varrho^{(m)}\ddot{u}_l^{(m)})\,\delta u_l^{(m)} + D_{k,k}^{(m)}\,\delta\varphi^{(m)}]\,dV \\
&\qquad + \int_{S_N^{(m)}}[(\bar{t}_l^{(m)} - n_k^{(m)}\tau_{kl}^{(m)})\,\delta u_l^{(m)} - (\bar{\sigma}^{(m)} + n_k{}^{(m)}D_k^{(m)})\,\delta\varphi_k^{(m)}]\,dS \\
&\qquad + \int_{S_C^{(m)}} n_k^{(m)}[(u_l^{(m)} - \bar{u}_l^{(m)})\,\delta\tau_{kl}^{(m)} + (\varphi^{(m)} - \bar{\varphi}^{(m)})\,\delta D_k^{(m)}]\,dS\Big] \\
&\quad + \int_{t_0}^{t} dt \int_{S^{(d)}} n_k^{(d)}\,\tfrac{1}{2}[(\tau_{kl}^{(2)} - \tau_{kl}^{(1)})(\delta u_l^{(1)} + \delta u_l^{(2)}) \\
&\qquad\qquad + (u_l^{(1)} - u_l^{(2)})(\delta\tau_{lk}^{(1)} + \delta\tau_{lk}^{(2)}) \\
&\qquad\qquad + (D_k^{(2)} - D_k^{(1)})(\delta\varphi^{(1)} + \delta\varphi^{(2)}) \\
&\qquad\qquad + (\varphi^{(2)} - \varphi^{(1)})(\delta D_k^{(1)} + \delta D_k^{(2)})]\,dS = 0.
\end{aligned} \tag{6.44}$$

Equation (6.44) is a form that is very useful for approximation. The integrals over $S_C^{(m)}$ in (6.43) and (6.44) were first presented by Eer Nisse ([16]) and Holland and Eer Nisse ([17]) but without a complete derivation. A derivation was provided in ([14]), in which the integrals over $S^{(d)}$ first appear.

Chapter 7

MATERIAL SYMMETRY CONSIDERATIONS

1. COMPRESSED NOTATION AND MATRIX ARRAYS

In order to determine the solution of piezoelectric (or elastic) vibration problems, we will have to know the arrays of material coefficients for the particular symmetry of the material we are considering. The book by Nye ([10]) is very useful in this context. The compressed matrix notation turns out to be more useful than the extended tensor notation *when discussing symmetry*. This matrix notation consists of replacing ij or kl by p or q, where i, j, k, and l take the values 1, 2, and 3, and p and q take the values 1, 2, 3, 4, 5, and 6 according to the prescription in Table II.
Furthermore

$$c_{ijkl} = c_{pq}, \qquad e_{ikl} = e_{ip}, \qquad \tau_{ij} \equiv T_{ij} = T_p \equiv \tau_p.$$

By virtue of the above identifications and the fact that we wish the constitutive relations (5.19) and (5.20) to be written

$$T_p = c_{pq}^E S_q - e_{kp} E_k, \tag{7.1}$$

$$D_i = e_{iq} S_q + \varepsilon_{ik}^S E_k, \tag{7.2}$$

TABLE II

ij or kl	p or q
11	1
22	2
33	3
23 or 32	4
31 or 13	5
12 or 21	6

we find that

$$\begin{aligned} S_{ij} &= S_p \qquad \text{when} \quad i = j,\ p = 1, 2, 3, \\ 2S_{ij} &= S_p \qquad \text{when} \quad i \neq j,\ p = 4, 5, 6. \end{aligned} \tag{7.3}$$

We may now write the elastic and piezoelectric constants as well as the dielectric constants as matrices, since they all are described by two indices. The arrays for an arbitrarily anisotropic (triclinic) material without a center of symmetry are

$$c_{pq}^{E} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix}, \tag{7.4}$$

$$e_{ip} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & e_{25} & e_{26} \\ e_{31} & e_{32} & e_{33} & e_{34} & e_{35} & e_{36} \end{pmatrix}, \tag{7.5}$$

$$\varepsilon_{ij}^{S} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}. \tag{7.6}$$

Here we see the $21 + 18 + 6 = 45$ independent constants exhibited. The arrays for a material with monoclinic symmetry, with x_1 the digonal axis (which by the International Symbol is class 2, or by the Schoenflies Symbol is C_2), are

$$c_{pq}^{E} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0 \\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0 \\ c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & c_{56} \\ 0 & 0 & 0 & 0 & c_{56} & c_{66} \end{pmatrix}, \tag{7.7}$$

$$e_{ip} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & e_{25} & e_{26} \\ 0 & 0 & 0 & 0 & e_{35} & e_{36} \end{pmatrix}, \tag{7.8}$$

$$\varepsilon_{ij}^{S} = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & \varepsilon_{23} \\ 0 & \varepsilon_{23} & \varepsilon_{33} \end{pmatrix}. \tag{7.9}$$

These arrays exhibit the symmetry of rotated Y-cut quartz when the equations are referred to rectangular axes in and normal to the plane of the plate. For a monoclinic crystal all $13 + 8 + 4 = 25$ constants are independent, whereas for rotated Y-cut quartz the 25 constants are not independent but are derived from a smaller number of independent constants, which are the independent constants of a trigonal crystal with x_3 the trigonal axis and x_1 a diagonal axis. The arrays for such a crystal (32, D_3) are

$$c_{pq}^{E} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{pmatrix}, \tag{7.10}$$

$$c_{66} = \tfrac{1}{2}(c_{11} - c_{12}),$$

$$e_{ip} = \begin{pmatrix} e_{11} & -e_{11} & 0 & e_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & -e_{14} & -e_{11} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.11}$$

$$\varepsilon_{ij}^{S} = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \tag{7.12}$$

Thus we see that a material with this type of trigonal symmetry is described by $6 + 2 + 2 = 10$ independent material constants. Clearly, the 25 constants for the previously mentioned rotated Y-cut quartz may be expressed in terms of the 10 independent material constants for a crystal in class D_3 by using the appropriate tensor transformation laws.

Lithium tantalate and lithium niobate are two new crystalline materials which have higher piezoelectric coupling than quartz, and are currently being investigated for potential resonator and transducer applications. Both of these materials are trigonal, but they are a different type of trigonal crystal than quartz. Although x_3 is the trigonal axis in both quartz and these crystals, x_1 is normal to a mirror plane in these crystals instead of being a twofold rotation axis as it is in quartz. For these two crystals the crystal class is $C_{3v} = 3m$. The arrays of the elastic and dielectric constants are the same as for quartz when referred to the principal axes, and are given in (7.10) and (7.12). However, the array of piezoelectric constants is quite different, and is given by

$$e_{ip} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & -e_{22} \\ -e_{22} & e_{22} & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}. \tag{7.13}$$

When a rotated Y-cut is formed by rotating about x_1, as shown in Fig. 6, and the arrays are referred to rectangular axes in and normal to the plane of the plate the arrays have m-monoclinic symmetry with x_1 normal to a mirrorplane. The arrays of the elastic and dielectric constants are the same as for 2-monoclinic symmetry and are given in (7.7) and (7.9). The array of piezoelectric constants is given by

$$e_{ip} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & e_{16} \\ e_{21} & e_{22} & e_{23} & e_{24} & 0 & 0 \\ e_{31} & e_{32} & e_{33} & e_{34} & 0 & 0 \end{pmatrix}. \tag{7.14}$$

Thus it is clear that a material with m-monoclinic symmetry has $13 + 10 + 4 = 27$ independent constants. However, if the arrays (7.7), (7.9), and (7.14) are for rotated Y-cut lithium niobate referred to axes in and normal to the plane of the plate, the 27 constants are not independent, but are expressible in terms of the $6 + 4 + 2 = 12$ independent constants of a material with $3m$ symmetry by using the appropriate tensor transformation laws.

The polarized ferroelectric ceramics (with x_3 in the poling direction) effectively have the symmetry of a hexagonal crystal in class $C_{6v} = 6mm$.

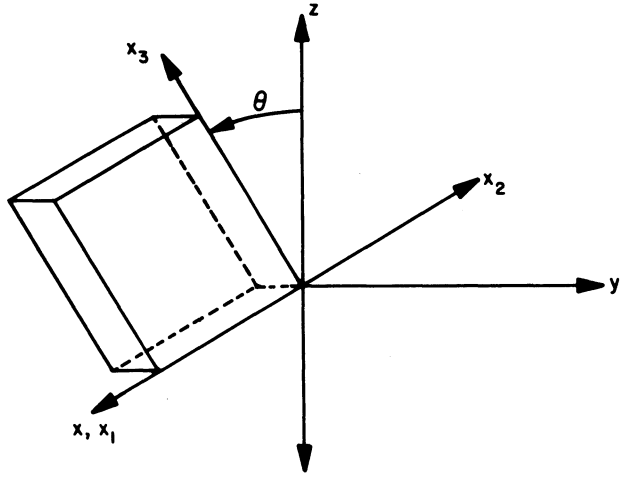

Fig. 6. Diagram of a rotated Y-cut of quartz.

The arrays for a material possessing this symmetry are

$$c_{pq}^{E} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}, \tag{7.15}$$

$$c_{66} = \tfrac{1}{2}(c_{11} - c_{12}),$$

$$e_{ip} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}, \tag{7.16}$$

$$\varepsilon_{ij}^{S} = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}. \tag{7.17}$$

Thus it is clear that a material with this type of symmetry is described by $5 + 3 + 2 = 10$ independent material constants. Materials with this type of symmetry are important because the polarized ceramics have high piezoelectric coupling. An isotropic material has arrays which are similar to the arrays for class C_{6v} except that there are some additional relations among the coefficients. For one thing, all the e_{ip} vanish, and there is no piezoelectric coupling. In addition to that, there are the relations

$$c_{12} = c_{13}, \qquad c_{11} = c_{33}, \qquad c_{44} = c_{66}, \qquad \varepsilon_{11} = \varepsilon_{33}. \tag{7.18}$$

Hence there are two independent elastic constants and one independent dielectric constant. The relationships between the isotropic c_{pq} elastic constants and the Lamé constants λ and μ are

$$c_{11} = \lambda + 2\mu, \qquad c_{12} = \lambda, \qquad c_{44} = \mu. \tag{7.19}$$

2. EQUATIONS FOR DIFFERENT SYMMETRIES

When the arrays for the material with monoclinic symmetry (class 2, C_2) are substituted in the constitutive equations (7.1) and (7.2) we obtain

$$T_{11} = c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{11}\varphi_{,1},$$
$$T_{22} = c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{12}\varphi_{,1},$$

$$
\begin{aligned}
T_{33} &= c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{13}\varphi_{,1}\,,\\
T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{14}\varphi_{,1}\,,\\
T_{31} &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{25}\varphi_{,2} + e_{35}\varphi_{,3}\,,\\
T_{12} &= c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{26}\varphi_{,2} + e_{36}\varphi_{,3}\,,\\
D_1 &= e_{11}u_{1,1} + e_{12}u_{2,2} + e_{13}u_{3,3} + e_{14}(u_{2,3} + u_{3,2}) - \varepsilon_{11}\varphi_{,1}\,,\\
D_2 &= e_{25}(u_{3,1} + u_{1,3}) + e_{26}(u_{1,2} + u_{2,1}) - \varepsilon_{22}\varphi_{,2} - \varepsilon_{23}\varphi_{,3}\,,\\
D_3 &= e_{35}(u_{3,1} + u_{1,3}) + e_{36}(u_{1,2} + u_{2,1}) - e_{23}\varphi_{,2} - \varepsilon_{33}\varphi_{,3}\,,
\end{aligned}
\tag{7.20}
$$

When these constitutive relations are substituted in the stress equations of motion

$$T_{ij,i} = \varrho\ddot{u}_j\,, \tag{7.21}$$

and the charge equation of electrostatics

$$D_{i,i} = 0\,, \tag{7.22}$$

and terms are combined we obtain

$$
\begin{aligned}
&c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{55})u_{3,13} + (c_{14} + c_{56})u_{2,13}\\
&\quad + (c_{14} + c_{56})u_{3,12} + 2c_{56}u_{1,23} + c_{66}u_{1,22} + c_{55}u_{1,33}\\
&\quad + e_{11}\varphi_{,11} + e_{26}\varphi_{,22} + (e_{36} + e_{25})\varphi_{,23} + e_{35}\varphi_{,33} = \varrho\ddot{u}_1\,,\\
&c_{56}u_{3,11} + (c_{56} + c_{14})u_{1,13} + (c_{66} + c_{12})u_{1,12} + c_{66}u_{2,11}\\
&\quad + c_{22}u_{2,22} + (c_{23} + c_{44})u_{3,23} + 2c_{24}u_{2,23} + c_{24}u_{3,22}\\
&\quad + c_{34}u_{3,33} + c_{44}u_{2,33} + (e_{26} + e_{12})\varphi_{,12} + (e_{36} + e_{14})\varphi_{,13} = \varrho\ddot{u}_2\,,\\
&c_{55}u_{3,11} + (c_{55} + c_{13})u_{1,13} + (c_{56} + c_{14})u_{1,12} + c_{56}u_{2,11}\\
&\quad + c_{24}u_{2,22} + 2c_{34}u_{3,23} + (c_{44} + c_{23})u_{2,23} + c_{44}u_{3,22}\\
&\quad + c_{33}u_{3,33} + c_{34}u_{2,33} + (e_{25} + e_{14})\varphi_{,12} + (e_{35} + e_{13})\varphi_{,13} = \varrho\ddot{u}_3\,,\\
&e_{11}u_{1,11} + (e_{12} + e_{26})u_{2,12} + (e_{13} + e_{35})u_{3,13} + (e_{14} + e_{36})u_{2,13}\\
&\quad + (e_{14} + e_{25})u_{3,12} + (e_{25} + e_{36})u_{1,23} + e_{26}u_{1,22}\\
&\quad + e_{35}u_{1,33} - \varepsilon_{11}\varphi_{,11} - \varepsilon_{22}\varphi_{,22} - 2\varepsilon_{23}\varphi_{,23} - \varepsilon_{33}\varphi_{,33} = 0\,.
\end{aligned}
\tag{7.23}
$$

These are the equations for rotated Y-cut quartz referred to axes (x_1, x_2, x_3) in and normal to the plane of the plate, with x_2 normal to the plane of the plate and x_1 the digonal axis in the plane of the plate. The coordinate axes for the rotated Y-cut are related to the principal axes of trigonal quartz as shown in Fig. 6. In Fig. 6, θ is the angle of the cut. The equations for quartz (trigonal, class D_3) when referred to the principal axes (x, y, z),

which are then denoted, respectively, (x_1, x_2, x_3), are given by (7.20) and (7.23) provided

$$\begin{aligned}
&c_{22} = c_{11}, \quad c_{23} = c_{13}, \quad c_{24} = -c_{14}, \quad c_{34} = 0,\\
&c_{55} = c_{44}, \quad c_{56} = c_{14}, \quad c_{66} = \tfrac{1}{2}(c_{11} - c_{12}),\\
&e_{12} = -e_{11}, \quad e_{13} = 0, \quad e_{25} = -e_{14}, \quad e_{26} = -e_{11},\\
&e_{35} = 0, \quad e_{36} = 0, \quad \varepsilon_{22} = \varepsilon_{11}, \quad \varepsilon_{23} = 0.
\end{aligned} \tag{7.24}$$

When the arrays for a material with m-monoclinic symmetry are substituted in the constitutive equations (7.1) and (7.2) we obtain

$$\begin{aligned}
T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + c_{14}(u_{2,3} + u_{3,2}) + e_{21}\varphi_{,2} + e_{31}\varphi_{,3}\\
T_{22} &= c_{12}u_{1,1} + c_{22}u_{2,2} + c_{23}u_{3,3} + c_{24}(u_{2,3} + u_{3,2}) + e_{22}\varphi_{,2} + e_{32}\varphi_{,3}\\
T_{33} &= c_{13}u_{1,1} + c_{23}u_{2,2} + c_{33}u_{3,3} + c_{34}(u_{2,3} + u_{3,2}) + e_{23}\varphi_{,2} + e_{33}\varphi_{,3}\\
T_{23} &= c_{14}u_{1,1} + c_{24}u_{2,2} + c_{34}u_{3,3} + c_{44}(u_{2,3} + u_{3,2}) + e_{24}\varphi_{,2} + e_{34}\varphi_{,3}\\
T_{31} &= c_{55}(u_{3,1} + u_{1,3}) + c_{56}(u_{1,2} + u_{2,1}) + e_{15}\varphi_{,1}\\
T_{12} &= c_{56}(u_{3,1} + u_{1,3}) + c_{66}(u_{1,2} + u_{2,1}) + e_{16}\varphi_{,1}\\
D_1 &= e_{15}(u_{1,3} + u_{3,1}) + e_{16}(u_{1,2} + u_{2,1}) - \varepsilon_{11}\varphi_{,1}\\
D_2 &= e_{21}u_{1,1} + e_{22}u_{2,2} + e_{23}u_{3,3} + e_{24}(u_{2,3} + u_{3,2}) - \varepsilon_{22}\varphi_{,2} - \varepsilon_{23}\varphi_{,3}\\
D_3 &= e_{31}u_{1,1} + e_{32}u_{2,2} + e_{33}u_{3,3} + e_{34}(u_{2,3} + u_{3,2}) - \varepsilon_{23}\varphi_{,2} - \varepsilon_{33}\varphi_{,3}
\end{aligned} \tag{7.25}$$

When these constitutive equations are substituted in (7.21) and (7.22) we obtain

$$\begin{aligned}
&c_{11}u_{1,11} + (c_{12}+c_{66})u_{2,12} + (c_{13}+c_{55})u_{3,13} + (c_{14}+c_{56})u_{2,13} + (c_{14}+c_{56})u_{3,12}\\
&\quad + 2c_{56}u_{1,23} + c_{66}u_{1,22} + c_{55}u_{1,33} + (e_{21}+e_{16})\varphi_{,12} + (e_{31}+e_{15})\varphi_{,13} = \varrho\ddot{u}_1,\\
&c_{56}u_{3,11} + (c_{56}+c_{14})u_{1,13} + (c_{66}+c_{12})u_{1,12} + c_{66}u_{2,11} + c_{22}u_{2,22}\\
&\quad + (c_{23}+c_{44})u_{3,23} + 2c_{44}u_{2,23} + c_{24}u_{3,22} + c_{34}u_{3,33} + c_{44}u_{2,33}\\
&\quad + e_{16}\varphi_{,11} + e_{22}\varphi_{,22} + (e_{32}+e_{24})\varphi_{,23} + e_{34}\varphi_{,33} = \varrho\ddot{u}_2,\\
&c_{55}u_{3,11} + (c_{55}+c_{13})u_{1,13} + (c_{56}+c_{14})u_{1,12} + c_{56}u_{2,11} + c_{24}u_{2,22}\\
&\quad + 2c_{34}u_{3,23} + (c_{44}+c_{23})u_{2,23} + c_{44}u_{3,22} + c_{33}u_{3,33} + c_{34}u_{2,33}\\
&\quad + e_{15}\varphi_{,11} + e_{24}\varphi_{,22} + (e_{34}+e_{23})\varphi_{,23} + e_{33}\varphi_{,33} = \varrho\ddot{u}_3,\\
&(e_{15}+e_{31})u_{1,31} + e_{15}u_{3,11} + (e_{16}+e_{21})u_{1,12} + e_{16}u_{2,11} + e_{22}u_{2,22}\\
&\quad + (e_{23}+e_{34})u_{3,23} + (e_{24}+e_{32})u_{2,23} + e_{24}u_{3,22} + e_{33}u_{3,33}\\
&\quad + e_{34}u_{2,33} - \varepsilon_{11}\varphi_{,11} - \varepsilon_{22}\varphi_{,22} - 2\varepsilon_{23}\varphi_{,23} - \varepsilon_{33}\varphi_{,33} = 0.
\end{aligned} \tag{7.26}$$

These are the equations for rotated Y-cut lithium niobate or lithium tan-

talate referred to axes (x_1, x_2, x_3) in and normal to the plane of the plate. The equations for these two materials (trigonal, class $3m = C_{3v}$) when referred to the principal axes (x, y, z), which are then denoted, respectively, (x_1, x_2, x_3), are given by (7.25) and (7.26) provided

$$\begin{aligned}
&c_{22} = c_{11}, \quad c_{23} = c_{13}, \quad c_{24} = -c_{14}, \quad c_{34} = 0,\\
&c_{55} = c_{44}, \quad c_{56} = c_{14}, \quad c_{66} = \tfrac{1}{2}(c_{11} - c_{12}),\\
&e_{24} = e_{15}, \quad e_{16} = -e_{22}, \quad e_{21} = -e_{22}, \quad e_{23} = 0,\\
&e_{32} = e_{31}, \quad e_{34} = 0, \quad \varepsilon_{22} = \varepsilon_{11}, \quad \varepsilon_{23} = 0.
\end{aligned} \tag{7.27}$$

When the arrays for the material in class C_{6v} are substituted in the constitutive equations (7.1) and (7.2) we obtain

$$\begin{aligned}
T_{11} &= c_{11}u_{1,1} + c_{12}u_{2,2} + c_{13}u_{3,3} + e_{31}\varphi_{,3},\\
T_{22} &= c_{12}u_{1,1} + c_{11}u_{2,2} + c_{13}u_{3,3} + e_{31}\varphi_{,3},\\
T_{33} &= c_{13}u_{1,1} + c_{13}u_{2,2} + c_{33}u_{3,3} + e_{33}\varphi_{,3},\\
T_{23} &= c_{44}(u_{3,2} + u_{2,3}) + e_{15}\varphi_{,2},\\
T_{31} &= c_{44}(u_{3,1} + u_{1,3}) + e_{15}\varphi_{,1},\\
T_{12} &= c_{66}(u_{1,2} + u_{2,1}),\\
D_1 &= e_{15}u_{3,1} + e_{15}u_{1,3} - \varepsilon_{11}\varphi_{,1},\\
D_2 &= e_{15}(u_{3,2} + u_{2,3}) - \varepsilon_{11}\varphi_{,2},\\
D_3 &= e_{31}u_{1,1} + e_{31}u_{2,2} + e_{33}u_{3,3} - \varepsilon_{33}\varphi_{,3}.
\end{aligned} \tag{7.28}$$

When these constitutive equations are substituted in the stress equations of motion (7.21) and the charge equation of electrostatics (7.22) we obtain

$$\begin{aligned}
&c_{11}u_{1,11} + (c_{12} + c_{66})u_{2,12} + (c_{13} + c_{44})u_{3,13} + c_{66}u_{1,22}\\
&\quad + c_{44}u_{1,33} + (e_{31} + e_{15})\varphi_{,13} = \varrho\ddot{u}_1,\\
&c_{66}u_{2,11} + (c_{66} + c_{12})u_{1,12} + c_{11}u_{2,22} + (c_{13} + c_{44})u_{3,23}\\
&\quad + c_{44}u_{2,33} + (e_{31} + e_{15})\varphi_{,23} = \varrho\ddot{u}_2,\\
&c_{44}u_{3,11} + (c_{44} + c_{13})u_{1,31} + c_{44}u_{3,22} + (c_{44} + c_{13})u_{2,23}\\
&\quad + c_{33}u_{3,33} + e_{15}\varphi_{,11} + e_{15}\varphi_{,22} + e_{33}\varphi_{,33} = \varrho\ddot{u}_3,\\
&e_{15}u_{3,11} + (e_{15} + e_{31})u_{1,13} + e_{15}u_{3,22} + (e_{15} + e_{31})u_{2,32}\\
&\quad + e_{33}u_{3,33} - \varepsilon_{11}\varphi_{,11} - \varepsilon_{11}\varphi_{,22} - \varepsilon_{33}\varphi_{,33} = 0.
\end{aligned} \tag{7.29}$$

The constitutive equations for an isotropic material may be written suc-

cinctly in indicial notation and in vector notation in terms of the Lamé constants λ and μ, since, for the isotropic array shown

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{7.30}$$

Substituting in the elastic constitutive relations

$$T_{ij} = c_{ijkl}u_{k,l}, \tag{7.31}$$

we obtain

$$T_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i}). \tag{7.32}$$

Substituting this into the stress equations of motion, (7.21), we obtain

$$(\lambda + \mu)u_{k,kj} + \mu u_{j,kk} = \varrho\ddot{u}_j. \tag{7.33}$$

In invariant vector notation we have

$$\mathbf{T} = \lambda\boldsymbol{\nabla}\cdot\mathbf{u}\mathbf{I} + \mu(\boldsymbol{\nabla}\mathbf{u} + \mathbf{u}\boldsymbol{\nabla}), \tag{7.32'}$$

for the constitutive equations and

$$(\lambda + \mu)\,\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{u} + \mu\,\nabla^2\mathbf{u} = \varrho\ddot{\mathbf{u}}, \tag{7.33'}$$

for the displacement equations of motion for an isotropic material.

3. MATERIAL CONSTANTS

Values for the material constants for left-hand quartz have been determined by Bechmann ([18]). When referred to the crystal axes in accordance with the notation in this chapter, for left-hand quartz the constants have the values

$$\begin{aligned}
&c_{11} = c_{22} = 86.74\times10^9\ \mathrm{N/m^2}, \quad c_{14} = -c_{24} = c_{56} = -17.91, \quad c_{12} = 6.99,\\
&c_{34} = 0, \quad c_{13} = c_{23} = 11.91, \quad c_{44} = c_{55} = 57.94, \quad c_{33} = 107.2,\\
&c_{66} = \tfrac{1}{2}(c_{11} - c_{12}) = 39.88;\\
&e_{11} = -e_{12} = -e_{26} = 0.171\ \mathrm{C/m^2}, \quad e_{14} = -e_{25} = -0.0406,\\
&e_{13} = e_{35} = e_{36} = 0;\\
&\varepsilon_{11} = \varepsilon_{22} = 39.21\times10^{-12}\ \mathrm{C/V\text{-}m}, \quad \varepsilon_{33} = 41.03, \quad \varepsilon_{23} = 0.
\end{aligned} \tag{7.34}$$

The mass density of quartz is

$$\varrho = 2649 \quad \text{kg/m}^3. \tag{7.35}$$

When the constants are referred to the (x_1, x_2, x_3) axes shown in Fig. 6 and $\theta = 35.25°$ the rotated Y-cut is an AT-cut and the constants for the AT-cut determined from Bechmann's constants given in (7.34) are

$$\begin{aligned}
&c_{11} = 86.74, \quad c_{22} = 129.77, \quad c_{33} = 102.83, \quad c_{12} = -\,8.25, \quad c_{13} = 27.15,\\
&c_{14} = -\,3.66, \quad c_{23} = -\,7.42, \quad c_{24} = 5.7, \quad c_{34} = 9.92, \quad c_{44} = 38.61,\\
&c_{55} = 68.81, \quad c_{66} = 29.01, \quad c_{56} = 2.53;\\
&e_{11} = 0.171, \quad e_{12} = -\,0.152, \quad e_{13} = -\,0.0187, \quad e_{14} = 0.067, \quad e_{25} = 0.108,\\
&e_{26} = -\,0.095, \quad e_{35} = -\,0.0761, \quad e_{36} = 0.067;\\
&\varepsilon_{11} = 39.21, \quad \varepsilon_{22} = 39.82, \quad \varepsilon_{33} = 40.42, \quad \varepsilon_{23} = 0.86.
\end{aligned} \tag{7.36}$$

Values for the material constants for lithium tantalate and lithium niobate have been determined by Warner, Onoe, and Coquin ([19]). When referred to the crystal axes in accordance with the notation of this chapter the constants for lithium tantalate have the values

$$\begin{aligned}
&c_{11} = c_{22} = 2.33 \times 10^{11}\ \text{N/m}^2, \quad c_{14} = -\,c_{24} = c_{56} = -\,0.11, \quad c_{12} = 0.47,\\
&c_{34} = 0, \quad c_{13} = c_{23} = 0.80, \quad c_{44} = c_{55} = 0.94, \quad c_{33} = 2.75,\\
&c_{66} = \tfrac{1}{2}(c_{11} - c_{12}) = 0.93;\\
&e_{15} = e_{24} = 2.6\ \text{C/m}^2, \quad e_{22} = -\,e_{16} = -\,e_{21} = 1.6\\
&e_{31} = e_{32} = 0.0, \quad e_{33} = 1.9, \quad e_{23} = e_{34} = 0;\\
&\varepsilon_{11} = \varepsilon_{22} = 36.3 \times 10^{-11}\ \text{C/V-m}, \quad \varepsilon_{33} = 38.2, \quad \varepsilon_{23} = 0.
\end{aligned} \tag{7.37}$$

The mass density of lithium tantalate is

$$\varrho = 7450 \quad \text{kg/m}^3. \tag{7.38}$$

The constants for lithium niobate have the values

$$\begin{aligned}
&c_{11} = c_{22} = 2.03 \times 10^{11}\ \text{N/m}^2, \quad c_{14} = -\,c_{24} = c_{56} = 0.09, \quad c_{12} = 0.53,\\
&c_{34} = 0, \quad c_{13} = c_{23} = 0.75, \quad c_{44} = c_{55} = 0.60, \quad c_{33} = 2.45,\\
&c_{66} = \tfrac{1}{2}(c_{11} - c_{12}) = 0.75;\\
&e_{15} = e_{24} = 3.7\ \text{C/m}^2, \quad e_{22} = -\,e_{16} = -\,e_{21} = 2.5,\\
&e_{31} = e_{32} = 0.2, \quad e_{33} = 1.3, \quad e_{23} = e_{34} = 0;\\
&\varepsilon_{11} = \varepsilon_{22} = 38.9 \times 10^{-11}\ \text{C/V-m}, \quad \varepsilon_{33} = 25.7, \quad \varepsilon_{23} = 0.
\end{aligned} \tag{7.39}$$

The mass density of lithium niobate is

$$\varrho = 4700 \quad \text{kg/m}^3. \tag{7.40}$$

There are so many different high coupling polarized ceramics which have the effective symmetry of a material in class C_{6v} that there seems to be no point in reproducing the constants for all or many of these ceramics here. Constants for many of these ceramics are given in the excellent article by Berlincourt *et al.* ([20]).

PART II

FUNDAMENTAL STANDING WAVE SOLUTIONS

Chapter 8

SOME ASPECTS OF THE THEORY OF WAVES AND VIBRATIONS

1. THE INHOMOGENEOUS SCALAR WAVE EQUATION

Before considering piezoelectric (or even elastic) vibrations—which are vectorial—it is enlightening to consider the mathematics of scalar vibration theory in some detail because the basic ideas of the two theories are the same and vectorial vibration theory is sufficiently complex and cumbersome to obscure the basic ideas if one is not already aware of them. On the other hand, scalar vibration theory is sufficiently simple and straightforward so as not to obscure the basic ideas. Of course, in either case we are considering linear vibration theory only.

Consider the scalar wave equation in one space variable and time and containing a source term:

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = \varrho(x, t), \tag{8.1}$$

where ϱ denotes the source term, which is the inhomogeneous term of the differential equation. It must be prescribed. The scalar φ is the dependent variable. Any term containing φ or any of its partial derivatives is called a homogeneous term. The differential equation is defined in a region of space-time defined by

$$a < x < b, \qquad t > 0. \tag{8.2}$$

If $a = -\infty$ and $b = +\infty$ the region is unbounded and we do not have to consider the boundary conditions associated with the differential equation. If a is finite and $b = +\infty$ the region is unbounded on one side, i.e., a half space, and we do have to consider the boundary conditions on the bounded side. If both a and b are finite, the body is bounded and we have to consider the boundary conditions at both a and b. The problem of the unbounded and semibounded region are similar and have many things in common, in particular, the existence of a *continuous* distribution of eigen-

states. On the other hand, the problem of the fully bounded body is quite different in character from the other two, namely, in the existence of a *discrete* distribution of eigenstates only. We are here concerned with the fully bounded body which possesses a discrete distribution of eigenstates, and will discuss the other problem only in passing. The differential equation we are discussing could be describing the transverse vibrations of a string, the longitudinal oscillations of a rod, as well as other things.

The boundary conditions at each end consist of the specification of either φ or $\partial\varphi/\partial x$ or any combination thereof. These conditions could be obtained from a uniqueness theorem as in three-dimensional piezoelectricity. The most general condition at a boundary point then is of the form

$$c\varphi + e\,\partial\varphi/\partial x = f(t), \tag{8.3}$$

where $f(t)$ denotes a prescribed inhomogeneous term and c and e could be functions of time. Only those situations in which c and e are constants are of interest to us and will be considered here. Now our problem consists of the differential equation (8.1) and the boundary conditions

$$\begin{aligned} c_1\varphi + e_1\,\partial\varphi/\partial x &= f_1(t) \quad \text{at} \quad x = a \text{ and } t > 0, \\ c_2\varphi + e_2\,\partial\varphi/\partial x &= f_2(t) \quad \text{at} \quad x = b \text{ and } t > 0, \end{aligned} \tag{8.4}$$

and the initial conditions

$$\begin{aligned} \varphi &= g(x) \quad \text{at} \quad t = 0,\ a < x < b, \\ \partial\varphi/\partial t &= h(x) \quad \text{at} \quad t = 0,\ a < x < b. \end{aligned} \tag{8.5}$$

The quantities ϱ, f_1, f_2, g, and h are the inhomogeneous terms which must be prescribed and which force the system into oscillation. The initial conditions g and/or h produce so-called free vibrations, while the boundary and interior forcing terms f_1, f_2, and ϱ are responsible for the steady-state forced vibrations. It should be noted that we are not interested in the free vibrations here, but only in the steady forced vibrations. Consequently, we take $g = h = 0$ and assume that any free-vibrational term which is generated by f_1, f_2, or ϱ is sufficiently damped to be ignored. We further assume that the damping is sufficiently small to be ignored in the steady-state forced vibrational solution. Both of these assumptions are essentially satisfied after a sufficient number of cycles. Thus we are obtaining the solution after a sufficient number of cycles and we call this solution the steady-state solution. In brief, we have eliminated any consideration of the initial conditions and all transients associated therewith.

2. HOMOGENEOUS SOLUTIONS

When f_1, f_2, and ϱ vanish we have what is called a homogeneous boundary value problem (or eigenvalue problem). It is important to note that the solution to the inhomogeneous (forced) problem, which exists when any one of f_1, f_2, and ϱ are nonzero, frequently may be composed of solutions of the associated homogeneous ($\varrho = f_1 = f_2 = 0$) problem. Consequently, the solution of the homogeneous problem is of fundamental interest.

Let us first consider the homogeneous problem in the infinite medium (line). Then the boundary conditions may be left out of account. Our differential equation is

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0. \tag{8.6}$$

We may verify that a traveling wave solution in the $+ x$ direction is given by

$$\varphi = A \cos(\eta x - \omega t), \tag{8.7}$$

provided

$$\eta^2 = \omega^2/c^2, \tag{8.8a}$$

or

$$\omega = c\eta, \tag{8.8b}$$

since we restrict ω to be positive and real always and η to be positive in this instance. If we plot an ω *vs.* η diagram from (8.8*b*), we obtain the straight line shown in Fig. 7, i.e., a straight line with slope c. If instead of writing (8.7) we write

$$\varphi = A \cos \eta(x - vt), \tag{8.9}$$

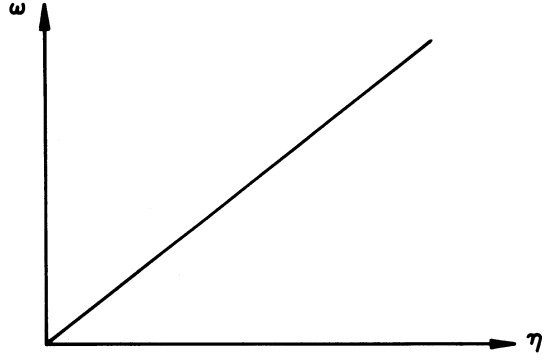

Fig. 7. Frequency *vs.* wave number diagram for a nondispersive string.

where v is the velocity of an infinite train of waves, then

$$\omega = \eta v , \tag{8.10}$$

and the slope c of the above dispersion curve is identical with the phase velocity v of the infinite train of waves in this simple case.

If we had been obtaining a plate wave solution of a more complicated equation the dispersion curve associated with the solution might not have been a straight line as above, but might have been curved as shown in Fig. 8, or any of a number of other shapes for that matter. As a matter of fact, there might be portions of the dispersion curves for which η is imaginary or even complex, but ω is always restricted to be real and positive. In any event, when η is real the slope of the radius vector from the origin to the curve denotes the phase velocity v.

If in the present wave equation we had considered a traveling sine wave, we would, of course, have obtained the same dispersion curve. Moreover, if we had considered standing waves of the forms

$$\begin{aligned} \varphi &= B \cos \eta x \cos \omega t , \\ \varphi &= B \sin \eta x \cos \omega t , \end{aligned} \tag{8.11}$$

we would have obtained the same straight-line dispersion relation shown in Fig. 7. It is precisely these latter standing waves which will be of importance to us in obtaining solutions to vibration problems.

Let us now consider the homogeneous problem of, say, a string of

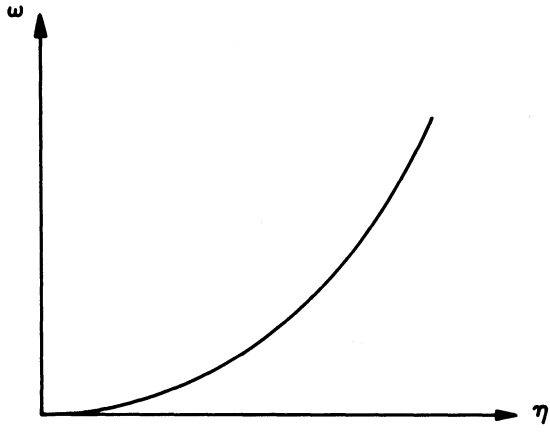

Fig. 8. Frequency *vs.* wave number diagram for a hypothetical dispersive medium.

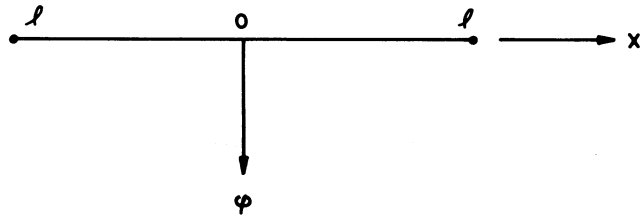

Fig. 9. Stretched string of length $2l$.

length $2l$ fixed at both ends as shown in Fig. 9. For convenience, let us place the origin of coordinates in the center of the string; we could place it anywhere. Here φ is the displacement. The equation and boundary conditions are

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \qquad -l < x < l, \tag{8.12}$$

$$\varphi = 0 \qquad \text{at} \quad x = \pm l. \tag{8.13}$$

We will now obtain two independent sets of solutions, which correspond, respectively, to the symmetryc and antisymmetric motions (or modes). For the symmetric modes we have

$$\varphi = A \cos \eta x \cos \omega t, \tag{8.14}$$

and for the antisymmetric modes

$$\varphi = B \sin \eta x \cos \omega t. \tag{8.15}$$

In either case, from (8.12) we obtain the ω *vs.* η (or dispersion) relation

$$\omega = c\eta, \tag{8.16}$$

and A and B are arbitrary. Substituting in the boundary conditions (8.13), we find in the symmetric case

$$A \cos \eta l \cos \omega t = 0. \tag{8.17}$$

For a nontrivial solution to exist, i.e., A nonzero, we must have

$$\cos \eta l = 0. \tag{8.18}$$

Hence

$$\eta l = n\pi/2, \qquad n = 1, 3, 5, 7, \ldots, \tag{8.19}$$

and from (8.19) and (8.16) we obtain

$$\omega = nc\pi/2l, \qquad n = 1, 3, 5, 7, \ldots . \tag{8.20}$$

These values of ω in (8.20) are the natural (or eigen) frequencies of the symmetric modes. From the boundary conditions (8.13), in the antisymmetric case we find

$$B \sin \eta l \cos \omega t = 0, \tag{8.21}$$

and for a nontrivial solution we must have

$$\sin \eta l = 0. \tag{8.22}$$

Equations of this nature are called transcendental frequency equations. We will be concerned with equations of this nature—but much more complicated ones of course—throughout.

From (8.22) we have

$$\eta l = n\pi/2, \qquad n = 2, 4, 6, 8, \ldots , \tag{8.23}$$

and from (8.23) and (8.16) we obtain

$$\omega = nc\pi/2l, \qquad n = 2, 4, 6, 8, \ldots . \tag{8.24}$$

These are the eigenfrequencies of the antisymmetric modes.

Suppose we consider the same string, but have the slope vanish at both ends instead of the displacement. The boundary conditions now become

$$\partial\varphi/\partial x = 0 \qquad \text{at} \quad x = \pm l. \tag{8.25}$$

Then for the symmetric motions we have

$$\varphi = A \cos \eta x \cos \omega t, \tag{8.26}$$

and from (8.25) we obtain

$$\sin \eta l = 0, \tag{8.27}$$

so that from (8.27) and (8.16) we have

$$\omega = nc\pi/2l, \qquad n = 2, 4, 6, 8, \ldots ; \tag{8.28}$$

and for the antisymmetric motions we have

$$\varphi = B \sin \eta x \cos \omega t, \tag{8.29}$$

and from (8.25) we obtain

$$\cos \eta l = 0, \tag{8.30}$$

so that from (8.30) and (8.16) we have

$$\omega = nc\pi/2l, \qquad n = 1, 3, 5, 7, \ldots . \tag{8.31}$$

From this we see that the boundary conditions have a strong influence on the natural frequencies of the system. In fact, in the two systems the natural frequencies of the symmetric and antisymmetric modes have been interchanged.

Suppose we consider the same string, but have the slope vanish at one end and the displacement at the other. The boundary conditions become

$$\partial\varphi/\partial x = 0 \qquad \text{at} \quad x = -l, \tag{8.32}$$

$$\varphi = 0 \qquad \text{at} \quad x = l. \tag{8.33}$$

Now, we will not have separate, uncoupled symmetric and antisymmetric motions because the system is no longer symmetric. From the differential equations we have the independent solutions

$$\begin{aligned} \varphi &= A \cos \eta x \cos \omega t, \\ \varphi &= B \sin \eta x \cos \omega t, \end{aligned} \tag{8.34}$$

where in both solutions

$$\eta = \omega/c, \tag{8.35}$$

and A and B are arbitrary. Hence we have the sum solution

$$\varphi = (A \cos \eta x + B \sin \eta x) \cos \omega t, \tag{8.36}$$

since the system is linear. Substituting from (8.36) into the boundary conditions (8.32) and (8.33), we obtain

$$[-A\eta \sin \eta(-l) + B\eta \cos \eta(-l)] \cos \omega t = 0, \tag{8.37}$$

$$[A \cos \eta l + B \sin \eta l] \cos \omega t = 0, \tag{8.38}$$

or

$$A\eta \sin \eta l + B\eta \cos \eta l = 0, \tag{8.37'}$$

$$A \cos \eta l + B \sin \eta l = 0. \tag{8.38'}$$

For a nontrivial solution to exist, i.e., for A and B nonzero, the determinant of the coefficients of A and B must vanish, and we obtain

$$\begin{vmatrix} \eta \sin \eta l & \eta \cos \eta l \\ \cos \eta l & \sin \eta l \end{vmatrix} = 0. \tag{8.39}$$

Expanding (8.39), we find

$$\eta \sin^2 \eta l - \eta \cos^2 \eta l = 0. \tag{8.40}$$

This is a more complicated transcendental frequency equation than obtained previously. Since $\eta \neq 0$, we have

$$\sin^2 \eta l - \cos^2 \eta l = 0, \tag{8.41}$$

or

$$\cos 2\eta l = 0, \tag{8.42}$$

and

$$2\eta l = n\pi/2, \qquad n = 1, 3, 5, \ldots, \tag{8.43}$$

and from (8.43) and (8.35) we find

$$\omega = nc\pi/4l, \qquad n = 1, 3, 5, \ldots . \tag{8.44}$$

We could go on to consider more complicated homogeneous boundary conditions, but no new ideas would be introduced. However, the transcendental frequency equations would become even more complicated.

3. STEADY-STATE FORCED VIBRATIONS. I

Let us now consider an inhomogeneous (forced) vibration problem with an interior forcing term. The boundary conditions are homogeneous —say, zero displacement conditions (8.13)—and the interior forcing term in (8.1) is taken to be

$$\varrho = K \cos \omega t. \tag{8.45}$$

The classical procedure consists of first obtaining the solutions to the associated homogeneous eigenvibration problem, which we have already done, then expanding K in the complete set of eigenfunctions of the homogeneous solution, expanding the solution to the forced problem in the same set of functions, and evaluating the coefficients of each term of the solution. Let us briefly run through this procedure. We will then run through

another very useful procedure, which has applications when the classical procedure doesn't. Both forms of solution are useful in their own way.

In the solution let

$$\varphi(x, t) = \varphi(x) \cos \omega t. \tag{8.46}$$

Then substituting (8.45) and (8.46) into (8.1), we obtain

$$\frac{d^2\varphi}{dx^2} + \frac{\omega^2}{c^2}\varphi = K, \qquad -l < x < l, \tag{8.47}$$

and from (8.13) we have

$$\varphi = 0 \qquad \text{at} \quad x = \pm l. \tag{8.48}$$

We have already obtained the eigensolutions to the associated homogeneous problem. They are

$$\varphi_n = A_n \cos \eta_n x, \qquad \eta_n = n\pi/2l, \quad n = 1, 3, 5, \ldots, \tag{8.49}$$

for the symmetric modes and

$$\varphi_n = B_n \sin \eta_n x, \qquad \eta_n = n\pi/2l, \quad n = 2, 4, 6, \ldots, \tag{8.50}$$

for the antisymmetric modes. Expanding K in a series of the eigenfunctions, we have

$$K = \sum_{n=1,3}^{\infty} c_n \cos \eta_n x + \sum_{n=2,4}^{\infty} d_n \sin \eta_n x, \tag{8.51}$$

where by virtue of the orthogonality of the eigenfunctions we find

$$c_n = (K/l) \int_{-l}^{l} \cos \eta_n x \, dx = (2K/\eta_n l) \sin \eta_n l, \; d_n = 0. \tag{8.52}$$

Expanding φ in a series of the eigenfunctions, we have

$$\varphi = \sum_{n=1,3}^{\infty} A_n \cos \eta_n x + \sum_{n=2,4}^{\infty} B_n \sin \eta_n x, \tag{8.53}$$

and substituting in the differential equation (8.47), we find

$$\begin{aligned} &- \sum_{n=1,3}^{\infty} \eta_n^2 A_n \cos \eta_n x - \sum_{n=2,4}^{\infty} \eta_n^2 B_n \sin \eta_n x \\ &\quad + \frac{\omega^2}{c^2} \sum_{n=1,3}^{\infty} A_n \cos \eta_n x + \frac{\omega^2}{c^2} \sum_{n=2,4}^{\infty} B_n \sin \eta_n x = \sum_{n=1,3}^{\infty} c_n \cos \eta_n x. \end{aligned} \tag{8.54}$$

Now from the eigensolutions we know that

$$\eta_n^2 = \omega_n^2/c^2. \tag{8.55}$$

Hence, substituting from (8.55) into (8.54), we obtain

$$\sum_{n=1,3}^{\infty} [A_n(\omega^2 - \omega_n^2) - c^2 c_n] \cos \eta_n x + \sum_{n=2,4}^{\infty} B_n(\omega^2 - \omega_n^2) \sin \eta_n x = 0, \tag{8.56}$$

which, by means of the orthogonality [([21]), Chapter 1] of the eigensolutions, yields

$$B_n = 0, \qquad A_n = c^2 c_n/(\omega^2 - \omega_n^2). \tag{8.57}$$

Hence, substituting from (8.57) into (8.53), we see that our solution may be written

$$\varphi = c^2 \sum_{n=1,3}^{\infty} \frac{c_n}{\omega^2 - \omega_n^2} \cos \eta_n x, \tag{8.58}$$

from which it is clear that a symmetric (in this case constant) forcing function cannot force the antisymmetric motions, which was clear on intuitive grounds. From the form of the solution (8.58), i.e., the fact that each term contains a resonance denominator, it is clear that when the driving frequency ω is very near a particular natural frequency ω_i the ith term dominates the entire series and the solution may be written

$$\varphi = \frac{c^2 c_i}{\omega^2 - \omega_i^2} \cos \eta_i x, \tag{8.59}$$

and when $\omega = \omega_i$ the steady-state solution blows up.

4. ORTHOGONALITY OF THE EIGENSOLUTIONS

It is clear from the foregoing that the orthogonality of the eigensolutions was crucial to obtaining our result. Although the eigensolutions turned out to be trigonometric functions in this simple case, and it is well known that the trigonometric functions are orthogonal, the orthogonality properties are more far reaching (fundamental), and in fact are a direct consequence of our differential eigensystem. That is, any solution of the homogeneous differential equation and homogeneous boundary conditions can be shown to be orthogonal without actually obtaining the solution. This is an important fact, since when we have such a general proof of orthogonality we don't have to examine the orthogonality of complicated

solutions, which is very desireable with complicated functions—especially complicated vectorial functions.

To this end consider two eigensolutions to the homogeneous differential equation (8.47) subject to the same homogeneous boundary conditions

$$\frac{d^2\varphi_m}{dx^2} + \frac{\omega_m^2}{c^2}\varphi_m = 0, \tag{8.60}$$

$$\frac{d^2\varphi_n}{dx^2} + \frac{\omega_n^2}{c^2}\varphi_n = 0. \tag{8.61}$$

From (8.60) and (8.61) form the equation

$$\varphi_n \frac{d^2\varphi_m}{dx^2} - \varphi_m \frac{d^2\varphi_n}{dx^2} + \frac{\omega_m^2 - \omega_n^2}{c^2}\varphi_m\varphi_n = 0, \tag{8.62}$$

which with an integration from $-l$ to $+l$ can be written

$$[\varphi_n\varphi_m' - \varphi_m\varphi_n']_{-l}^{l} = \frac{\omega_n^2 - \omega_m^2}{c^2}\int_{-l}^{l} \varphi_m\varphi_n\, dx, \tag{8.63}$$

from which it is clear that for *any* homogeneous boundary condition [Eq. (8.3) with $f(t) = 0$] the left-hand side vanishes. Hence if $\omega_m \neq \omega_n$

$$\int_{-l}^{l} \varphi_m\varphi_n\, dx = 0. \tag{8.64}$$

We will now obtain the solution to the previous forced vibration problem in the second manner, which will be a particularly useful procedure to us in future problems.

5. STEADY-STATE FORCED VIBRATIONS. II

The second manner of obtaining the steady-state solution of (8.47) subject to (8.48) consists of writing

$$\varphi = \chi + \psi, \tag{8.65}$$

and substituting in (8.47) and (8.48) to obtain

$$\frac{d^2\chi}{dx^2} + \frac{\omega^2}{c^2}\chi + \frac{d^2\psi}{dx^2} + \frac{\omega^2}{c^2}\psi = K, \qquad -l < x < l, \tag{8.66}$$

$$\chi + \psi = 0 \qquad \text{at} \quad x = \pm l. \tag{8.67}$$

We now select χ so that ψ satisfies a homogeneous equation. Hence

$$\chi = c^2K/\omega^2, \tag{8.68}$$

$$\frac{d^2\psi}{dx^2} + \frac{\omega^2}{c^2}\psi = 0, \qquad -l < x < l, \tag{8.69}$$

$$\psi = -c^2K/\omega^2 \qquad \text{at} \quad x = \pm l. \tag{8.70}$$

Clearly, the differential equation (8.69) is satisfied by

$$\psi = A\cos\eta x + B\sin\eta x, \tag{8.71}$$

where

$$\eta = \omega/c. \tag{8.72}$$

Substituting in the boundary conditions (8.70), we find

$$\begin{aligned} A\cos\eta l - B\sin\eta l &= -c^2K/\omega^2, \\ A\cos\eta l + B\sin\eta l &= -c^2K/\omega^2. \end{aligned} \tag{8.73}$$

Adding, we obtain

$$A\cos\eta l = -c^2K/\omega^2, \tag{8.74}$$

and subtracting, we obtain

$$B\sin\eta l = 0. \tag{8.75}$$

Equation (8.74) is inhomogeneous, and shows that symmetric motions are forced by a uniform forcing field; whereas (8.75) is homogeneous, and shows that antisymmetric motions cannot be forced by a uniform forcing field. More precisely, from (8.75) we have

$$B = 0, \tag{8.76}$$

and from (8.74)

$$A = -c^2K/(\omega^2\cos\eta l). \tag{8.77}$$

Resonance occurs when the amplitude of the forced oscillation goes to ∞, i.e., when $A = \infty$. From (8.77) we see that this occurs when

$$\cos\eta l = 0. \tag{8.78}$$

Note that

$$\varphi = \frac{c^2K}{\omega^2}\left(1 - \frac{\cos\eta x}{\cos\eta l}\right)\cos\omega t, \qquad \eta = \omega/c, \tag{8.79}$$

is the steady-state solution for any driving frequency ω off resonance, and simply diverges at resonance.

6. BOUNDARY FORCING

Now let us consider the problem of a boundary (here an edge) forcing term. We have the differential equation (the time factor $\cos \omega t$ has been removed)

$$\frac{d^2\varphi}{dx^2} + \frac{\omega^2}{c^2}\varphi = 0, \qquad -l < x < l, \tag{8.80}$$

and the boundary conditions

$$\varphi = 0 \qquad \text{at } x = -l, \tag{8.81}$$

$$\varphi = H \qquad \text{at } x = l \qquad (\text{really } H\cos \omega t). \tag{8.82}$$

If we wish to use the classical (Fourier) procedure, we write

$$\varphi = \chi + \psi, \tag{8.83}$$

and, substituting from (8.83) into (8.80)–(8.82), obtain

$$\frac{d^2\chi}{dx^2} + \frac{\omega^2}{c^2}\chi + \frac{d^2\psi}{dx^2} + \frac{\omega^2}{c^2}\psi = 0, \tag{8.84}$$

$$\chi + \psi = 0 \qquad \text{at } x = -l, \tag{8.85}$$

$$\chi + \psi = H \qquad \text{at } x = l. \tag{8.86}$$

We now select χ so that the boundary conditions on ψ are homogeneous. That is, we wish χ to satisfy the conditions

$$\chi = 0 \qquad \text{at } x = -l, \tag{8.87}$$

$$\chi = H \qquad \text{at } x = l. \tag{8.88}$$

This is accomplished by the selection

$$\chi = \frac{H}{2}\left(1 + \frac{x}{l}\right). \tag{8.89}$$

Substituting from (8.89) into (8.84)–(8.86), we find

$$\frac{d^2\psi}{dx^2} + \frac{\omega^2}{c^2}\psi = -\frac{\omega^2}{c^2}\frac{H}{2}\left(1 + \frac{x}{l}\right), \qquad -l < x < l, \tag{8.90}$$

$$\psi = 0 \qquad \text{at} \quad x = \pm l. \tag{8.91}$$

We now proceed as before, i.e., we obtain the solutions to the homogeneous eigenvalue problem defined by

$$\frac{d^2\psi}{dx^2} + \frac{\omega^2}{c^2}\psi = 0, \qquad -l < x < l, \tag{8.92}$$

$$\psi = 0 \qquad \text{at} \quad x = \pm l, \tag{8.93}$$

and then expand the forcing term

$$-\frac{\omega^2 H}{2c^2}\left(1 + \frac{x}{l}\right), \tag{8.94}$$

and the function ψ as an infinite sum of all the eigensolutions ψ_n of (8.92)–(8.93), and then evaluate the amplitudes (coefficients) of each ψ_n by means of the orthogonality of the ψ_n. There is no point in repeating this procedure here, since it is exactly the same as we followed in Section 3 except for the fact that the forcing term is different. From the form of the forcing term it is clear that a motion will be neither symmetric nor antisymmetric in general; but a resonant mode will be either symmetric or antisymmetric.

Now the second manner of obtaining a solution of (8.80)–(8.82) is perfectly straightforward. Clearly, we may write

$$\varphi = A\cos\eta x + B\sin\eta x, \qquad \eta = \omega/c. \tag{8.95}$$

Substituting from (8.95) into the boundary conditions (8.81)–(8.82), we find

$$\begin{aligned} A\cos\eta l - B\sin\eta l &= 0, \\ A\cos\eta l + B\sin\eta l &= H. \end{aligned} \tag{8.96}$$

At this point we could simplify the problem by adding and subtracting the two equations in (8.96), but it is more instructive to proceed in the following way because it is more typical of what has to be done in more complicated problems. We solve the inhomogeneous simultaneous equations in (8.96) for A and B to obtain

$$\begin{aligned} A &= \begin{vmatrix} 0 & -\sin\eta l \\ H & \sin\eta l \end{vmatrix} \Big/ \begin{vmatrix} \cos\eta l & -\sin\eta l \\ \cos\eta l & \sin\eta l \end{vmatrix}, \\ B &= \begin{vmatrix} \cos\eta l & 0 \\ \cos\eta l & H \end{vmatrix} \Big/ \begin{vmatrix} \cos\eta l & -\sin\eta l \\ \cos\eta l & \sin\eta l \end{vmatrix}. \end{aligned} \tag{8.97}$$

Expanding (8.97), we obtain

$$\begin{aligned} A &= (H \sin \eta l)/(2 \sin \eta l \cos \eta l) = H/(2 \cos \eta l), \\ B &= (H \cos \eta l)/(2 \sin \eta l \cos \eta l) = H/(2 \sin \eta l). \end{aligned} \tag{8.98}$$

These are two inhomogeneous equations for A and B. These equations show that in general a motion will be neither symmetric nor antisymmetric. However, since the resonance denominator factored into two parts, a resonant mode will be either symmetric or antisymmetric. This is clear from the above equations since there are two separate conditions for resonance, i.e., resonance occurs when either A or B go to infinity. In this case they go to infinity separately (and not together), and that is why a resonant mode is either symmetric or antisymmetric even though the solution is asymmetric. Note that

$$\varphi = \frac{H}{2}\left(\frac{\cos \eta x}{\cos \eta l} + \frac{\sin \eta x}{\sin \eta l}\right) \cos \omega t, \qquad \eta = \omega/c, \tag{8.99}$$

is the steady-state solution for any driving frequency ω off resonance, and simply diverges at a symmetric or antisymmetric resonance.

7. ORTHOGONALITY OF PIEZOELECTRIC VIBRATIONS

Consider two eigensolutions of the homogeneous piezoelectric equations

$$\tau_{ij,i} = \varrho \ddot{u}_j, \tag{8.100}$$

$$D_{i,i} = 0, \tag{8.101}$$

$$\tau_{ij} = c_{ijkl} S_{kl} - e_{kij} E_k, \tag{8.102}$$

$$D_i = e_{ikl} S_{kl} + \varepsilon_{ik} E_k, \tag{8.103}$$

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \tag{8.104}$$

$$E_k = -\varphi_{,k}, \tag{8.105}$$

one solution at frequency ω_m and the other at ω_n. The solution at frequency ω_m satisfies the equations

$$\tau^m_{ij,i} + \varrho \omega_m{}^2 u_j{}^m = 0, \tag{8.106}$$

$$D^m_{i,i} = 0, \tag{8.107}$$

and the solution at ω_n the equations

$$\tau^n_{ij,i} + \varrho\omega_n{}^2 u_j{}^n = 0\,, \tag{8.108}$$

$$D^n_{i,i} = 0. \tag{8.109}$$

From (8.106) and (8.108) form the equation

$$u_j{}^n\tau^m_{ij,i} - u_j{}^m\tau^n_{ij,i} + \varrho(\omega_m{}^2 - \omega_n{}^2)u_j{}^m u_j{}^n = 0.$$

Now, we have

$$\begin{aligned} u_j{}^n\tau^m_{ij,i} - u_j{}^m\tau^n_{ij,i} &= (u_j{}^n\tau^m_{ij})_{,i} - S^n_{ij}\tau^m_{ij} - (u_j{}^m\tau^n_{ij})_{,i} + S^m_{ij}\tau^n_{ij} \\ &= (u_j{}^n\tau^m_{ij} - u_j{}^m\tau^n_{ij})_{,i} - S^n_{ij}\tau^m_{ij} + S^m_{ij}\tau^n_{ij}\,, \end{aligned} \tag{8.110}$$

and from (8.101)–(8.103), and the symmetry of the elastic and dielectric coefficients we have

$$\begin{aligned} S^m_{ij}\tau^n_{ij} - S^n_{ij}\tau^m_{ij} &= S^m_{ij}(c_{ijkl}S^n_{kl} - e_{kij}E_k{}^n) - S^n_{ij}(c_{ijkl}S^m_{kl} - e_{kij}E_k{}^m) \\ &= e_{kij}S^n_{ij}E_k{}^m - e_{kij}S^m_{ij}E_k{}^n \\ &= (D_k{}^n - \varepsilon_{kl}E_l{}^n)E_k{}^m - (D_k{}^m - \varepsilon_{kl}E_l{}^m)E_k{}^n = D_k{}^nE_k{}^m - D_k{}^mE_k{}^n \\ &= -D_k{}^n\varphi^m_{,k} + D_k{}^m\varphi^n_{,k} = (D_k{}^m\varphi^n)_{,k} - (D_k{}^n\varphi^m)_{,k}. \end{aligned} \tag{8.111}$$

Substituting from (8.111) into (8.110), we obtain

$$(u_j{}^n\tau^m_{ij} - u_j{}^m\tau^n_{ij} + D_i{}^m\varphi^n - D_i{}^n\varphi^m)_{,i} + \varrho(\omega^2{}_m - \omega^2{}_n)u_j{}^m u_j{}^n = 0. \tag{8.112}$$

Integrating (8.112) throughout the volume V, we find

$$\begin{aligned} &\int_S (n_i\tau^m_{ij}u_j{}^n - n_i\tau^n_{ij}u_j{}^m + n_iD_i{}^m\varphi^n - n_iD_i{}^n\varphi^m)\,dS \\ &\quad = \varrho(\omega_m{}^2 - \omega_n{}^2)\int_V u_j{}^m u_j{}^n\,dV. \end{aligned} \tag{8.113}$$

From (8.113) it is clear that for any homogeneous boundary conditions the left-hand side vanishes. Hence, if $\omega_m \neq \omega_n$, for homogeneous boundary conditions we have

$$\int_V u_j{}^m u_j{}^n\,dV = 0 \qquad \text{if} \quad m \neq n\,, \tag{8.114}$$

which is the orthogonality condition for piezoelectric vibrations.

Chapter 9

THICKNESS VIBRATIONS OF PLATES

1. FREE ELASTIC THICKNESS VIBRATIONS

The plate is infinite in extent and bounded by two parallel planes located at, say, $x_2 = \pm h$, as shown in Fig. 10. For our purposes, both faces of the plate may (or may not) be completely coated with electrodes which are infinitesimally thin. Since the electrodes are infinitesimally thin, all possible mechanical effects may be ignored. When the electrodes are there we shall assume that an alternating potential difference is applied to them.

Let us first consider a few purely elastic eigenvibration problems so that later on we can note the influence of the piezoelectric coupling. This approach will also be instructive, since the purely elastic problems are less cumbersome. The mechanical boundary conditions are

$$T_{2j} = 0 \qquad \text{at} \quad x_2 = \pm h. \tag{9.1}$$

Referring to Fig. 10, thickness vibrations correspond to solutions which depend on the x_2 spatial coordinate only, and are independent of x_1 and x_3.

For the thickness eigenvibration problem for an isotropic plate the differential equations become

$$\begin{aligned} \mu u_{1,22} &= \varrho \ddot{u}_1 , \\ (\lambda + 2\mu) u_{2,22} &= \varrho \ddot{u}_2 , \\ \mu u_{3,22} &= \varrho \ddot{u}_3 , \end{aligned} \tag{9.2}$$

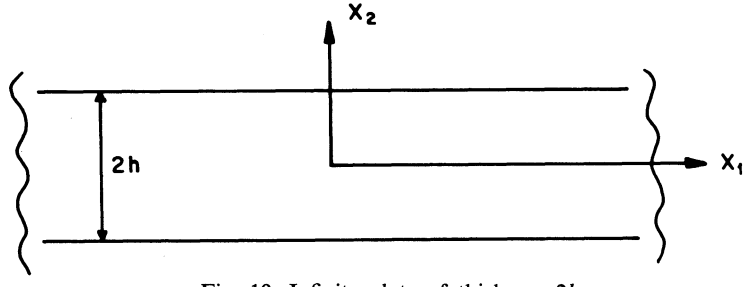

Fig. 10. Infinite plate of thickness $2h$.

and the boundary conditions become

$$\begin{aligned} T_{21} &= \mu u_{1,2} = 0 && \text{at} \quad x_2 = \pm h, \\ T_{22} &= (\lambda + 2\mu) u_{2,2} = 0 && \text{at} \quad x_2 = \pm h, \\ T_{23} &= \mu u_{3,2} = 0 && \text{at} \quad x_2 = \pm h, \end{aligned} \tag{9.3}$$

from which it is clear that the three displacements u_1, u_2, and u_3 uncouple, and we have three independent scalar wave equations, which have already been solved in Chapter 8. The only additional information which we now have is that

$$c = (\mu/\varrho)^{1/2} \tag{9.4}$$

for shear vibrations and

$$c = [(\lambda + 2\mu)/\varrho]^{1/2} \tag{9.5}$$

for extensional vibrations. Exactly the same sort of thing happens for the material with hexagonal symmetry, for the symmetry axis either in the plane of the plate or perpendicular to the plane of the plate.

For the material with monoclinic symmetry, with x_1 the digonal axis in the plane of the plate and x_2 perpendicular to the plane of the plate, the differential equations and boundary conditions, respectively, become

$$\begin{aligned} c_{66} u_{1,22} &= \varrho \ddot{u}_1, \\ c_{22} u_{2,22} + c_{24} u_{3,22} &= \varrho \ddot{u}_2, \\ c_{24} u_{2,22} + c_{44} u_{3,22} &= \varrho \ddot{u}_3, \end{aligned} \tag{9.6}$$

$$\begin{aligned} T_{21} &= c_{66} u_{1,2} = 0 && \text{at} \quad x_2 = \pm h, \\ T_{22} &= c_{22} u_{2,2} + c_{24} u_{3,2} = 0 && \text{at} \quad x_2 = \pm h, \\ T_{23} &= c_{24} u_{2,2} + c_{44} u_{3,2} = 0 && \text{at} \quad x_2 = \pm h. \end{aligned} \tag{9.7}$$

Thus it is clear that the u_1 displacement is uncoupled from u_2 and u_3, but u_2 and u_3 remain coupled even in a thickness solution. The thickness vibrations depending on the u_1 displacement are governed by a scalar wave equation as before with

$$c = (c_{66}/\varrho)^{1/2}, \tag{9.8}$$

and we have already obtained the solution. The thickness vibrations depending on u_2 and u_3 are a little more complicated than the previous ones. Let us obtain the solutions. Consider as a solution of the differential equations (9.6)

$$\begin{aligned} u_1 &= 0, \\ u_2 &= (A_2 \cos \eta x_2 + B_2 \sin \eta x_2) \cos \omega t, \\ u_3 &= (A_3 \cos \eta x_2 + B_3 \sin \eta x_2) \cos \omega t, \end{aligned} \tag{9.9}$$

which satisfies (9.6) provided

$$\begin{aligned} A_2(c_{22}\eta^2 - \varrho\omega^2) + A_3 c_{24}\eta^2 &= 0, \\ A_2 c_{24}\eta^2 + A_3(c_{44}\eta^2 - \varrho\omega^2) &= 0, \end{aligned} \tag{9.10}$$

$$\begin{aligned} B_2(c_{22}\eta^2 - \varrho\omega^2) + B_3 c_{24}\eta^2 &= 0, \\ B_2 c_{24}\eta^2 + B_3(c_{44}\eta^2 - \varrho\omega^2) &= 0, \end{aligned} \tag{9.11}$$

which shows that the symmetric and antisymmetric motions are not coupled by the differential equations. In other words, we can have a nontrivial solution with either $A_\alpha = 0$ or $B_\alpha = 0$ ($\alpha = 2,3$). Dividing Eqs. (9.10) and (9.11) by η^2 and defining λ as $\varrho\omega^2/\eta^2$, and noting that for a nontrivial symmetric solution $B_\alpha = 0$ and the determinant of the coefficients of the A_α must vanish, i.e.,

$$\begin{vmatrix} c_{22} - \lambda & c_{24} \\ c_{24} & c_{44} - \lambda \end{vmatrix} = 0, \tag{9.12}$$

we obtain the two values of λ

$$\lambda^{\pm} = \tfrac{1}{2}(c_{22} + c_{44}) \pm \tfrac{1}{2}[(c_{22} - c_{44})^2 + 4c_{24}^2]^{1/2}. \tag{9.13}$$

For each value of λ we obtain amplitude ratios

$$[A_2^{\pm}:A_3^{\pm}] = [c_{24}:(\lambda^{\pm} - c_{22})], \tag{9.14}$$

and a dispersion curve as shown in Fig. 11. Thus we have two dispersion curves in this case. For a nontrivial antisymmetric solution $A_\alpha = 0$, and we obtain the same $\lambda^{\pm}$ and amplitude ratios for the $B_\alpha{}^{\pm}$ as we did for the $A_\alpha{}^{\pm}$ in the other case. We now have the four boundary conditions (two on each surface) given in (9.7) remaining to be satisfied. It is instructive—because it is at times necessary—to proceed in the following general way. Take u_α as a sum of all four solutions—two symmetric and two antisymmetric—of the differential equations. Thus

$$\begin{aligned} u_2 &= [C^+A_2^+ \cos \eta^+ x_2 + C^-A_2^- \cos \eta^- x_2 + D^+B_2^+ \sin \eta^+ x_2 \\ &\quad + D^-B_2^- \sin \eta^- x_2] \cos \omega t, \\ u_3 &= [C^+A_3^+ \cos \eta^+ x_2 + C^-A_3^- \cos \eta^- x_2 + D^+B_3^+ \sin \eta^+ x_2 \\ &\quad + D^-B_3^- \sin \eta^- x_2] \cos \omega t. \end{aligned} \tag{9.15}$$

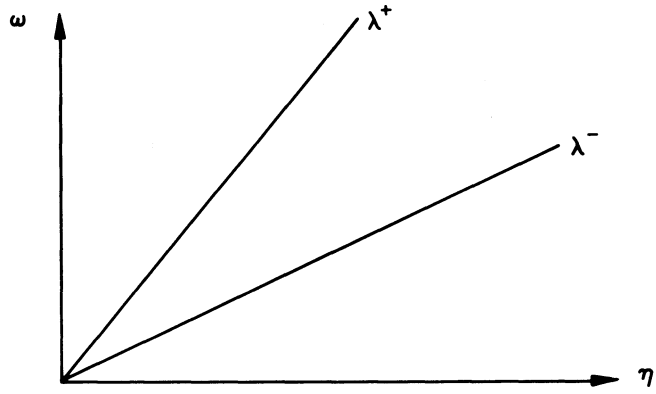

Fig. 11. Frequency *vs.* wave number diagram for elastic thickness vibrations in a rotated *Y*-cut quartz plate.

Substituting from (9.15) into the boundary conditions (9.7), we obtain

$$\begin{aligned}
&\mp C^+(c_{22}A_2{}^+ + c_{24}A_3{}^+)\eta^+(\sin \eta^+h) \mp C^-(c_{22}A_2{}^- + c_{24}A_3{}^-)\eta^-(\sin \eta^-h)\,, \\
&\quad + D^+(c_{22}B_2{}^+ + c_{24}B_3{}^+)\eta^+(\cos \eta^+h) + D^-(c_{22}B_2{}^- + c_{24}B_3{}^-)\eta^-(\cos \eta^-h) = 0, \\
&\mp C^+(c_{24}A_2{}^+ + c_{44}A_3{}^+)\eta^+(\sin \eta^+h) \mp C^-(c_{24}A_2{}^- + c_{44}A_3{}^-)\eta^-(\sin \eta^-h) \\
&\quad + D^+(c_{24}B_2{}^+ + c_{44}B_3{}^+)\eta^+(\cos \eta^+h) + D^-(c_{24}B_2{}^- + c_{44}B_3{}^-)\eta^-(\cos \eta^-h) = 0,
\end{aligned} \tag{9.16}$$

in which the upper sign refers to the boundary conditions at the upper surface and the lower sign at the lower surface. Adding, we obtain

$$\begin{aligned}
&D^+(c_{22}B_2{}^+ + c_{24}B_3{}^+)\eta^+(\cos \eta^+h) + D^-(c_{22}B_2{}^- + c_{24}B_3{}^-)\eta^-(\cos \eta^-h) = 0\,, \\
&D^+(c_{24}B_2{}^+ + c_{44}B_3{}^+)\eta^+(\cos \eta^+h) + D^-(c_{24}B_2{}^- + c_{44}B_3{}^-)\eta^-(\cos \eta^-h) = 0\,,
\end{aligned} \tag{9.17}$$

and subtracting, we obtain

$$\begin{aligned}
&C^+(c_{22}A_2{}^+ + c_{24}A_3{}^+)\eta^+(\sin \eta^+h) + C^-(c_{22}A_2{}^- + c_{24}A_3{}^-)\eta^-(\sin \eta^-h) = 0\,, \\
&C^+(c_{24}A_2{}^+ + c_{44}A_3{}^+)\eta^+(\sin \eta^+h) + C^-(c_{24}A_2{}^- + c_{44}A_3{}^-)\eta^-(\sin \eta^-h) = 0\,,
\end{aligned} \tag{9.18}$$

which shows that the symmetric and antisymmetric solutions of the differential equations are not coupled by the boundary conditions. For a nontrivial antisymmetric solution $C^+ = C^- = 0$, and the determinant of the coefficients of D^+ and D^- must vanish, i.e.,

$$\begin{vmatrix} (c_{22}B_2{}^+ + c_{24}B_3{}^+)\eta^+ \cos \eta^+ h & (c_{22}B_2{}^- + c_{24}B_3{}^-)\eta^- \cos \eta^- h \\ (c_{24}B_2{}^+ + c_{44}B_3{}^+)\eta^+ \cos \eta^+ h & (c_{24}B_2{}^- + c_{44}B_3{}^-)\eta^- \cos \eta^- h \end{vmatrix} = 0, \quad (9.19)$$

and from (9.19) we obtain the two independent transcendental frequency equations

$$\cos \eta^+ h = 0, \qquad \cos \eta^- h = 0. \quad (9.20)$$

Consequently, either $D^- = 0$ or $D^+ = 0$, and the two independent antisymmetric solutions of the differential equations are not coupled by the boundary conditions in this case. From the transcendental frequency equations (9.20) we have

$$\eta^{\pm} h = n\pi/2, \qquad n = 1, 3, 5, \ldots, \quad (9.21)$$

and the eigenfrequencies are given by

$$\omega^{\pm} = (\lambda^{\pm}/\varrho)^{1/2} n\pi/2h, \qquad n = 1, 3, 5, \ldots . \quad (9.22)$$

Exactly the same sort of thing happens for the symmetric modes, when $D^+ = D^- = 0$; and the eigenfrequencies are given by Eq. (9.22), but with $n = 2, 4, 6, \ldots$.

Now, let us consider an arbitrarily anisotropic material. For thickness vibrations we have the differential equations

$$c_{2jk2}u_{k,22} = \varrho \ddot{u}_j, \quad (9.23)$$

and boundary conditions

$$T_{2j} = c_{2jk2}u_{k,2} = 0 \qquad \text{at} \quad x_2 = \pm h. \quad (9.24)$$

Let us see if we can determine purely antisymmetric solutions. Consider as a solution of the differential equations

$$u_j = A_j \sin \eta x_2 \cos \omega t, \qquad j = 1, 2, 3. \quad (9.25)$$

This is a solution of (9.23) provided

$$(c_{2jk2} - \lambda \delta_{jk})A_k = 0. \quad (9.26)$$

Equation (9.26) is a system of linear, homogeneous, algebraic equations in the A_k, and for a nontrivial solution to exist the determinant of the coefficients of the A_k must vanish, i.e.,

$$| c_{2jk2} - \lambda \delta_{jk} | = 0. \quad (9.27)$$

This is a cubic equation in λ and yields $3\lambda^{(\alpha)}$, which, for our purposes, are all assumed distinct. Since

$$c_{2jk2} = c_{k22j} = c_{2kj2}, \tag{9.28}$$

this is an algebraic eigenvalue problem for a symmetric matrix, and this problem has already been covered in great detail by us, from which we can assert that the $3\lambda^{(\alpha)}$ (here distinct) are all real and that the eigenvectors $\mathbf{A}^{\alpha}$ of the three independent solutions are mutually orthogonal, so that the displacements $\mathbf{u}^{\alpha}$ are, of course, mutually orthogonal also. From the foregoing it is clear that there are three dispersion curves which are all straight lines, as shown in Fig. 12. In order to satisfy the six boundary conditions in (9.24) (three on each surface), we will try a sum of the three independent antisymmetric solutions of the differential equations. If we were to require six solutions, we would need the three independent symmetric solutions as well. Thus we take

$$u_j = \sum_{\alpha=1}^{3} C^{(\alpha)} A_j^{(\alpha)} (\sin \eta^{(\alpha)} x_2) \cos \omega t \tag{9.29}$$

as the solution. Substituting from (9.29) into the boundary conditions (9.24), we obtain

$$\sum_{\alpha=1}^{3} C^{(\alpha)} c_{2jk2} A_k^{(\alpha)} \eta^{(\alpha)} \cos \eta^{(\alpha)} h = 0 \tag{9.30}$$

on each surface. Consequently, the antisymmetric solutions are adequate, and we do not need the symmetric solutions. The equations are linear,

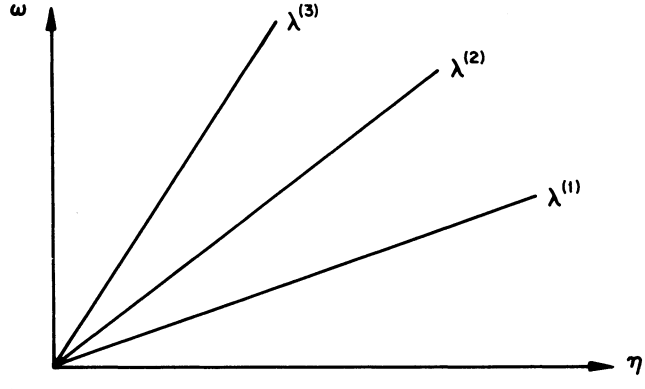

Fig. 12. Frequency *vs.* wave number diagram for elastic thickness vibrations in an arbitrarily anisotropic plate.

homogeneous, algebraic equations in the $C^{(\alpha)}$, and for a nontrivial solution to exist the determinant of the coefficients of the $C^{(\alpha)}$ must vanish, i.e.,

$$| c_{2jk2} A_k^{(\alpha)} \eta^{(\alpha)} \cos \eta^{(\alpha)} h | = 0, \tag{9.31}$$

and we obtain the three independent transcendental frequency equations

$$\cos \eta^{(\alpha)} h = 0, \qquad \alpha = 1, 2, 3. \tag{9.32}$$

Consequently, for each eigensolution two of the three $C^{(\alpha)} = 0$, and the three independent solutions of the differential equations are not coupled by the boundary conditions. Therefore from the previous discussion we know what the antisymmetric eigenmodes look like. The eigenfrequencies of the antisymmetric modes are

$$\omega^{(\alpha)} = (\lambda^{(\alpha)}/\varrho)^{1/2} n\pi/2h; \qquad \alpha = 1, 2, 3; \quad n = 1, 3, 5, \ldots . \tag{9.33}$$

This result is due to Koga ([22]). From the foregoing the symmetric eigensolutions are obvious. The implications for propagating elastic waves in anisotropic crystals are obvious.

2. FORCED PIEZOELECTRIC THICKNESS VIBRATIONS

Let us now consider a forced piezoelectric thickness vibration problem. We will consider a material with monoclinic symmetry (rotated Y-cut quartz), since the material with hexagonal symmetry is somewhat simpler. The differential equations are

$$\begin{aligned} c_{66} u_{1,22} + e_{26} \varphi_{,22} &= \varrho \ddot{u}_1, \\ c_{22} u_{2,22} + c_{24} u_{3,22} &= \varrho \ddot{u}_2, \\ c_{24} u_{2,22} + c_{44} u_{3,22} &= \varrho \ddot{u}_3, \\ e_{26} u_{1,22} - \varepsilon_{22} \varphi_{,22} &= 0; \end{aligned} \tag{9.34}$$

and the boundary conditions are

$$\begin{aligned} T_{21} &= c_{66} u_{1,2} + e_{26} \varphi_{,2} = 0, && \text{at} \quad x_2 = \pm h, \\ T_{22} &= c_{22} u_{2,2} + c_{24} u_{3,2} = 0, && \text{at} \quad x_2 = \pm h, \\ T_{23} &= c_{24} u_{2,2} + c_{44} u_{3,2} = 0, && \text{at} \quad x_2 = \pm h, \\ \varphi &= \pm \varphi_0 \cos \omega t, && \text{at} \quad x_2 = \pm h. \end{aligned} \tag{9.35}$$

From the differential equations (9.34) and the boundary conditions (9.35)

it is obvious that only the u_1 displacement is coupled to the electric potential. Hence we take

$$\begin{aligned} u_2 &= u_3 = 0, \\ u_1(x_2, t) &= u_1(x_2) \cos \omega t, \\ \varphi(x_2, t) &= \varphi(x_2) \cos \omega t, \end{aligned} \tag{9.36}$$

and substitute in the differential equations (9.34), which are then manipulated to obtain

$$[c_{66} + (e_{26}^2/\varepsilon_{22})]u_{1,22} + \varrho\omega^2 u_1 = 0, \tag{9.37}$$

$$\varphi = (e_{26}/\varepsilon_{22})u_1 + L_1 x_2 + L_2. \tag{9.38}$$

Substituting from (9.36) and (9.38) into the nontrivial boundary conditions in (9.35), we obtain

$$\begin{aligned} &[c_{66} + (e_{26}^2/\varepsilon_{22})]u_{1,2} + e_{26}L_1 = 0 \quad \text{at} \quad x_2 = \pm h, \\ &(e_{26}/\varepsilon_{22})u_1 \pm L_1 h + L_2 = \pm \varphi_0, \quad \text{at} \quad x_2 = \pm h. \end{aligned} \tag{9.39}$$

Consider the antisymmetric function

$$u_1 = A \sin \eta x_2 \tag{9.40}$$

as a solution of the differential equation (9.37). This satisfies (9.37) provided

$$\bar{c}_{66}\eta^2 = \varrho\omega^2, \tag{9.41}$$

where

$$\bar{c}_{66} = c_{66} + (e_{26}^2/\varepsilon_{22}) \tag{9.42}$$

is the piezoelectrically stiffened elastic constant. If we were to consider the symmetric solution, we would find that it could not be forced. Substituting from (9.40) into the four remaining boundary conditions (9.39), we obtain

$$\bar{c}_{66}\eta A \cos \eta h + e_{26}L_1 = 0 \quad \text{at} \quad x_2 = \pm h, \tag{9.43}$$

$$\pm (e_{26}/\varepsilon_{22})A \sin \eta h \pm L_1 h + L_2 = \pm \varphi_0 \quad \text{at} \quad x_2 = \pm h. \tag{9.44}$$

Hence solving (9.44) for L_1 and L_2, we find

$$\begin{aligned} L_2 &= 0, \\ L_1 &= (\varphi_0/h) - (e_{26}A/\varepsilon_{22}h) \sin \eta h, \end{aligned} \tag{9.45}$$

and substituting from (9.45) into (9.43), we obtain

$$A\left(\bar{c}_{66}\eta \cos \eta h - \frac{e_{26}^2}{\varepsilon_{22}} \frac{\sin \eta h}{h}\right) = - e_{26} \frac{\varphi_0}{h}. \tag{9.46}$$

Equation (9.46) is an *inhomogeneous* linear algebraic equation in A. Consequently, the antisymmetric modes are forced by the application of an alternating voltage to the surface electrodes. Resonance occurs when the coefficient of A in (9.46) vanishes, i.e., when

$$\tan \eta h = (\varepsilon_{22}\bar{c}_{66}/e_{26}^2)\, \eta h = \eta h / k_{26}^2. \tag{9.47}$$

Thus for this piezoelectric plate the wavelengths of overtone resonances are not integral fractions of the fundamental; as a consequence the resonant frequencies of overtone modes are not integral multiples of the fundamental. The deviation from the integral multiple relationship depends on the electromechanical coupling factor k_{26} only. Consequently, the coupling factor can be determined from simple resonance measurements (fundamental plus at least one overtone) only, and antiresonance measurements are not required. Tables for doing this exist in the literature ([23]).

Now let us consider an arbitrarily anisotropic piezoelectric plate ([24]). Let the index ν, instead of the index 2, denote the direction normal to the surface of the plate. We will adopt the convention that there will be no sum over repeated Greek indices, although we will continue to sum over repeated Latin indices. The differential equations are

$$\begin{aligned} c_{\nu jk\nu} u_{k,\nu\nu} + e_{\nu\nu j}\varphi_{,\nu\nu} &= \varrho \ddot{u}_j, \\ e_{\nu k\nu} u_{k,\nu\nu} - \varepsilon_{\nu\nu}\varphi_{,\nu\nu} &= 0, \end{aligned} \tag{9.48}$$

and the boundary conditions are

$$\begin{aligned} T_{\nu j} = c_{\nu jk\nu} u_{k,\nu} + e_{\nu\nu j}\varphi_{,\nu} = 0 \qquad &\text{at} \quad x_\nu = \pm h, \\ \varphi = \pm \varphi_0 \cos \omega t \qquad &\text{at} \quad x_\nu = \pm h. \end{aligned} \tag{9.49}$$

Let

$$\begin{aligned} u_j(x_\nu, t) &= u_j(x_\nu) \cos \omega t, \\ \varphi(x_\nu, t) &= \varphi(x_\nu) \cos \omega t, \end{aligned} \tag{9.50}$$

and substitute in the differential equations (9.48), which are then manipulated to obtain

$$\bar{c}_{\nu jk\nu} u_{k,\nu\nu} + \varrho \omega^2 u_j = 0, \tag{9.51}$$

$$\varphi = (e_{\nu\nu k}/\varepsilon_{\nu\nu})u_k + L_1 x_\nu + L_2\,, \tag{9.52}$$

where

$$\bar{c}_{\nu jk\nu} = c_{\nu jk\nu} + (e_{\nu\nu j}e_{\nu\nu k}/\varepsilon_{\nu\nu})\,, \tag{9.53}$$

and L_1 and L_2 are integration constants. Substituting from (9.50) into the boundary conditions (9.49), we obtain

$$\bar{c}_{\nu jk\nu}u_{k,\nu} + e_{\nu\nu j}L_1 = 0 \qquad \text{at} \quad x_\nu = \pm h, \tag{9.54}$$

$$\frac{e_{\nu\nu k}}{\varepsilon_{\nu\nu}}\,u_k(\pm h) \pm L_1 h + L_2 = \pm\varphi_0 \qquad \text{at} \quad x_\nu = \pm h. \tag{9.55}$$

Consider the antisymmetric functions

$$u_j = A_j \sin \eta x_\nu \tag{9.56}$$

as a solution of the three coupled ordinary differential equations (9.51). This satisfies (9.51) provided

$$(\bar{c}_{\nu jk\nu} - \bar{C}\delta_{jk})A_k = 0\,, \tag{9.57}$$

where

$$\bar{C} = \varrho\omega^2/\eta^2. \tag{9.58}$$

For a nontrivial solution to (9.57) the determinant of the coefficients of the A_k must vanish:

$$|\,\bar{c}_{\nu jk\nu} - \bar{C}\delta_{jk}\,| = 0. \tag{9.59}$$

Equation (9.59) is a cubic equation in $\bar{C}$ and yields three real positive roots, $\bar{C}^{(1)}$, $\bar{C}^{(2)}$, $\bar{C}^{(3)}$. Hence for a given ω there are three real $\eta^{(n)}$, one for each $\bar{C}^{(n)}$. Each $\bar{C}^{(n)}$ then yields an independent solution, and amplitude ratios are found from the linear algebraic equations (9.57). The amplitude ratios will be denoted by

$$(A_1^{(n)}{:}A_2^{(n)}{:}A_3^{(n)}) = (\beta_1^{(n)}{:}\beta_2^{(n)}{:}\beta_3^{(n)}).$$

Thus we have three antisymmetric solutions given by

$$u_j^{(n)} = \beta_j^{(n)} \sin \eta^{(n)} x_\nu\,, \qquad n = 1, 2, 3; \tag{9.60}$$

and from (9.52) for each solution there exists an electric potential $\varphi^{(n)}$ given by

$$\varphi^{(n)} = \frac{e_{\nu\nu k}}{\varepsilon_{\nu\nu}}\,\beta_k^{(n)}\,(\sin \eta^{(n)} x_\nu) + L_1^{(n)} x_\nu + L_2^{(n)}. \tag{9.61}$$

Similarly, there are three symmetric solutions given by

$$u_j^{(m)} = \beta_j^{(m)} \cos \eta^{(m)} x_\nu, \qquad m = 1, 2, 3, \tag{9.62}$$

where the $\eta^{(m)}$ and $\beta_j^{(m)}$ are exactly the same as in the antisymmetric solutions. One solution of the differential equations is insufficient to satisfy the eight boundary conditions. All six solutions seem to be required. Six solutions will certainly be enough, since the two integration constants L_1 and L_2 already appear in the boundary conditions, and we require at most eight linear algebraic equations in eight unknowns which will consist of L_1, L_2, and the amplitudes of the six solutions. Hence we take

$$u_j = \sum_{n=1}^{3} P^{(n)} \beta_j^{(n)} \sin \eta^{(n)} x_\nu + \sum_{m=1}^{3} Q^{(m)} \beta_j^{(m)} \cos \eta^{(m)} x_\nu \tag{9.63}$$

as the solution of the problem. Clearly, from (9.61) we have

$$\varphi = \sum_{n=1}^{3} P^{(n)} \frac{e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \beta_k^{(n)} (\sin \eta^{(n)} x_\nu) + \sum_{m=1}^{3} Q^{(m)} \frac{e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \beta_k^{(m)} (\cos \eta^{(m)} x_\nu) + L_1 x_\nu + L_2. \tag{9.64}$$

From the two boundary conditions on the electric potential (9.55) we obtain

$$\begin{aligned} L_1 &= \frac{\varphi_0}{h} - \frac{1}{h} \sum_{n=1}^{3} P^{(n)} \frac{e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \beta_k^{(n)} \sin \eta^{(n)} h, \\ L_2 &= - \sum_{m=1}^{3} Q^{(m)} \frac{e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \beta_k^{(m)} \cos \eta^{(m)} h. \end{aligned} \tag{9.65}$$

Substituting from (9.63)–(9.65) into the remaining six boundary conditions (9.54), we obtain

$$\begin{aligned} &\sum_{n=1}^{3} P^{(n)} \bar{c}_{\nu jk\nu} \beta_k^{(n)} \eta^{(n)} \cos \eta^{(n)} h \mp \sum_{m=1}^{3} Q^{(m)} \bar{c}_{\nu jk\nu} \beta_k^{(m)} \eta^{(m)} \sin \eta^{(m)} h \\ &- \frac{1}{h} \sum_{n=1}^{3} P^{(n)} \frac{e_{\nu\nu j} e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \beta_k^{(n)} (\sin \eta^{(n)} h) + e_{\nu\nu j} \frac{\varphi_0}{h} = 0 \\ &\qquad \text{at} \quad x_\nu = \pm h. \end{aligned} \tag{9.66}$$

Adding the equations at $x_\nu = \pm h$ and multiplying by h, we obtain

$$\sum_{n=1}^{3} P^{(n)} \beta_k^{(n)} \left[\bar{c}_{\nu jk\nu} \eta^{(n)} h (\cos \eta^{(n)} h) - \frac{e_{\nu\nu j} e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \sin \eta^{(n)} h \right] = - e_{\nu\nu j} \varphi_0, \tag{9.67}$$

while subtracting the equations at $x_\nu = \pm h$, we obtain

$$\bar{c}_{\nu jk\nu} \sum_{m=1}^{3} Q^{(m)} \beta_k^{(m)} \eta^{(m)} \sin \eta^{(m)} h = 0, \tag{9.68}$$

which shows that the symmetric and antisymmetric solutions of the equations are not coupled by the boundary conditions, and, moreover, that the symmetric solutions are not driven by the application of an alternating voltage to the surface electrodes. For the antisymmetric modes which are driven by an alternating voltage resonance occurs when the determinant of the coefficients of the $P^{(n)}$ vanishes, i.e., when

$$\left| \beta_k^{(n)} \left[\bar{c}_{\nu jk\nu} \eta^{(n)} h \cos \eta^{(n)} h - \frac{e_{\nu\nu j} e_{\nu\nu k}}{\varepsilon_{\nu\nu}} \sin \eta^{(n)} h \right] \right| = 0. \tag{9.69}$$

This is the transcendental frequency equation, from which the resonant frequencies may be determined. It cannot be factored, and shows that the three fundamental antisymmetric solutions of the differential equations are coupled by the boundary conditions. It is a very complicated transcendental equation indeed. Each term in the 3×3 determinant consists of a sum of three terms with a cosine function and three terms with a sine function. It is fortunate that in practical cases there are usually enough zeros so that the determinant simplifies considerably, as it does for rotated Y-cut quartz plates, polarized ceramic plates with surfaces parallel to principal axes, and many other instances. However, it does not simplify very much for certain principal cuts of a material in trigonal class C_{3v} (lithium niobate) or C_3 or monoclinic class m. One may calculate resonances from the complicated determinant by first noting that

$$\eta^{(n)} h = [\varrho / \bar{C}^{(n)}]^{1/2} \omega h, \tag{9.70}$$

substituting the constants $\bar{c}_{\nu jk\nu}$ in the determinant (9.59), and then calculating the three $\bar{C}^{(n)}$ from the cubic, and the $\beta_i^{(n)}$ from the linear algebraic equations (9.57) that led to the cubic. Then everything is known in the complicated determinant (9.69) except ωh, which are then determined as the roots of the determinant. Note that when the piezoelectric constants vanish the determinant factors and reduces to

$$\cos \eta^{(n)} h = 0, \qquad n = 1, 2, 3, \tag{9.71}$$

which is the same as (9.32) in the purely elastic case. It should, of course,

be clear that every thickness vibration problem of a piezoelectric (or elastic) plate is a special case of the foregoing.

Equation (9.69) can be written in a more compact form, which turns out to be more illuminating for certain purposes. To obtain this form construct the matrix

$$F^{mn} = \beta_j^{(m)} G_j^{(n)}, \tag{9.72}$$

where

$$G_j^{(n)} = \beta_k^{(n)} \left[\bar{c}_{\nu jk\nu} \eta^{(n)} (\cos \eta^{(n)} h) - \frac{e_{\nu\nu j} e_{\nu\nu k}}{\varepsilon_{\nu\nu}} (\sin \eta^{(n)} h) \right], \tag{9.73}$$

and the $\beta_j^{(n)}$ are assumed to be normalized. Now since the determinant of a matrix product is equal to the product of the separate determinants and $\det \beta_j^{(n)}$ is nonzero (it equals one), we have

$$| F^{mn} | = | \beta_j^{(m)} G_j^{(n)} | = | \beta_j^{(m)} | \, | G_j^{(n)} | = 0, \tag{9.74}$$

which, with (9.57) and $\beta_j^{(n)} \beta_j^{(m)} = \delta_{mn}$, enables us to write

$$\eta^{(1)} \eta^{(2)} \eta^{(3)} h^3 \cos \eta^{(1)} h \cos \eta^{(2)} h \cos \eta^{(3)} h \left| \delta_{mn} - k^{(m)} k^{(n)} \frac{\sin \eta^{(n)} h}{\eta^{(n)} h \cos \eta^{(n)} h} \right| = 0, \tag{9.75}$$

where

$$(k^{(n)})^2 = (\beta_j^{(n)} e_{\nu\nu j})^2 / \bar{C}^{(n)} \varepsilon_{\nu\nu}. \tag{9.76}$$

Expansion of the determinant in (9.75) yields

$$\eta^{(1)} \eta^{(2)} \eta^{(3)} \cos \eta^{(1)} h \cos \eta^{(2)} h \cos \eta^{(3)} h \left[\sum_{n=1}^{3} (k^{(n)})^2 \frac{\tan \eta^{(n)} h}{\eta^{(n)} h} - 1 \right] = 0. \tag{9.77}$$

Equation (9.77) was first obtained by G. A. Coquin of Bell Telephone Laboratories, who communicated the result to the author.

Chapter 10

TWO-DIMENSIONAL STANDING WAVES IN ELASTIC PLATES

1. SOLUTION FOR POLARIZED CERAMIC

Since the solutions of problems in the theory of the vibrations of *bounded* plates may be composed of sums of solutions of the appropriate problems of two-dimensional standing waves in unbounded plates, the solution of the problem of two-dimensional standing waves in an infinte plate will be discussed in considerable detail. In fact, one cannot even attempt to obtain a solution to a vibration problem of a bounded plate by the methods we will adopt until one is thoroughly familiar with the details of the solution to the corresponding problem of the unbounded plate. The exact reasons for this will become increasingly clear as we proceed. Since waves in purely elastic plates are simpler than waves in piezoelectric plates—and more is known about the details of the solution—and we wish to see the influence of the piezoelectric coupling on the solution, we will consider two-dimensional standing waves in purely elastic plates before proceeding to a consideration of the equivalent problem for piezoelectric plates. To this end, we will consider the polarized ceramic material —excluding piezoelectric coupling—first, since it is simpler than rotated *Y*-cut quartz ([25]). Let the surfaces of the unbounded plate be normal to the polarization axis, which is in the x_3 direction, and let us consider standing waves which depend on x_1 and x_3 only. Note that the solution will be similar to that for plate-waves propagating in the x_1 direction. Consider as a solution of the differential equations (7.29), with the $e_{ijk} = 0$,

$$\begin{aligned} u_1 &= A_1 \cos \eta x_3 \cos \xi x_1 \cos \omega t\,, \\ u_2 &= 0\,, \\ u_3 &= A_3 \sin \eta x_3 \cos \xi x_1 \cos \omega t\,, \end{aligned} \tag{10.1}$$

which satisfies (7.29) with the $e_{ijk} = 0$, provided

$$\begin{aligned} (c_{11}\xi^2 + c_{44}\eta^2 - \varrho\omega^2)A_1 + (c_{13} + c_{44})\xi\eta A_3 &= 0\,, \\ (c_{44} + c_{13})\xi\eta A_1 + (c_{44}\xi^2 + c_{33}\eta^2 - \varrho\omega^2)A_3 &= 0\,. \end{aligned} \tag{10.2}$$

This system of linear homogeneous equations in A_1 and A_3 yields nontrivial solutions when the determinant of the coefficients of A_1 and A_3 vanishes, i.e., when

$$\begin{vmatrix} (c_{11}\xi^2 + c_{44}\eta^2 - \varrho\omega^2) & (c_{13} + c_{44})\,\xi\eta \\ (c_{13} + c_{44})\,\xi\eta & (c_{44}\xi^2 + c_{33}\eta^2 - \varrho\omega^2) \end{vmatrix} = 0. \tag{10.3}$$

This equation is quadratic in ω^2, ξ^2, and η^2. Therefore for a given ω and ξ there are two η^2, $[(\eta^{(1)})^2, (\eta^{(2)})^2]$. The frequency ω must be *real*, but ξ can be real, imaginary, or even complex. For real or imaginary ξ an η^2 may be either real-positive, real-negative, or they may occur in complex conjugate pairs. If an η^2 is real-positive, the corresponding η is real. Although both $+\eta$ and $-\eta$ result, only the $+\eta$ need be considered, since the $-\eta$ gives the same solution as the $+\eta$ because of the form of the solution (10.1). If an η^2 is real-negative, the corresponding η is imaginary. Again, and for the same reason, only the + imaginary η need be considered. If the two η^2 occur in complex conjugate pairs, the resulting four η consist of two complex conjugates and the negatives of them. Again, because of the form of the solution (10.1) the negative pair give the same solution as the positive pair, and hence do not have to be considered. Each solution yields amplitude ratios when substituted in the linear equations (10.2) in A_1 and A_3. The amplitude ratios are given by

$$\begin{aligned} A_1^{(n)} &= -\,(c_{13} + c_{44})\xi\eta_{(n)}\,, \\ A_3^{(n)} &= c_{11}\xi^2 + c_{44}\eta_{(n)}^2 - \varrho\omega^2. \end{aligned} \tag{10.4}$$

The boundary conditions at each surface of the plate are

$$T_{31} = T_{32} = T_{33} = 0 \qquad \text{at} \quad x_3 = \pm h. \tag{10.5}$$

Since $u_2 \equiv 0$ and u_1 and u_3 are independent of x_2, by virtue of (7.28) with the $e_{ijk} = 0$, $T_{32} \equiv 0$. Thus we have two remaining boundary conditions to satisfy at each surface of the plate. One solution of the differential equations is insufficient to satisfy the boundary conditions; both solutions are required. Hence we take

$$\begin{aligned} u_1 &= \sin\,\xi x_1 \cos\omega t \sum_{n=1}^{2} C^{(n)} A_1^{(n)} \cos\,\eta_{(n)} x_3 \\ u_3 &= \cos\,\xi x_1 \cos\omega t \sum_{n=1}^{2} C^{(n)} A_3^{(n)} \sin\,\eta_{(n)} x_3 \end{aligned} \tag{10.6}$$

as the solution of the problem. Substituting from (10.6) into the boundary conditions (10.5), with the aid of (7.28) with $e_{jik} = 0$ we obtain

$$\sum_{n=1}^{2} C^{(n)} c_{44}(A_3^{(n)}\xi + A_1^{(n)}\eta_{(n)}) \sin \eta_{(n)}h = 0$$
$$\sum_{n=1}^{2} C^{(n)}(c_{15}A^{(n)}\xi + c_{23}A_3^{(n)}\eta_{(n)}) \cos \eta_{(n)}h = 0. \tag{10.7}$$

Equations (10.7) constitute a system of linear homogeneous algebraic equations in the $C^{(n)}$. This system yields nontrivial solutions when the determinant of the coefficients of $C^{(1)}$ and $C^{(2)}$ vanishes, i.e., when

$$\begin{vmatrix} c_{44}(A_3^{(1)}\xi + A_1^{(1)}\eta_{(1)}) \sin \eta_{(1)}h & c_{44}(A_3^{(2)}\xi + A_1^{(2)}\eta_{(2)}) \sin \eta_{(2)}h \\ (c_{13}A_1^{(1)}\xi + c_{33}A_3^{(1)}\eta_{(1)}) \cos \eta_{(1)}h & (c_{13}A_1^{(2)}\xi + c_{33}A_3^{(2)}\eta_{(2)}) \cos \eta_{(2)}h \end{vmatrix} = 0. \tag{10.8}$$

Equation (10.8) is a transcendental equation, the roots of which determine the ω *vs.* ξ relation for this plate. For a fixed ω and ξ—and hence $\eta^{(1)}$, $\eta^{(2)}$, $A_3^{(1)}$, $A_1^{(1)}$, $A_3^{(2)}$, and $A_1^{(2)}$—this equation contains an infinite number of roots h_m, each of which determines a point on the ωh *vs.* ξh relation, and yields amplitude relations ($C^{(1)}$:$C^{(2)}$) when substituted in the linear algebraic equations (10.7) in $C^{(1)}$ and $C^{(2)}$.

The foregoing has implicitly assumed both η real. However, when an η is imaginary, it may be carried through the entire calculation as an imaginary quantity. In fact, some of the modern computing machines do complex arithmetic directly and even enable the direct treatment of some functions of a complex variable. Also, when $\eta_{(1)}$ and $\eta_{(2)}$ are complex conjugate they may be carried through the calculation as complex quantities in the same way. Of course, when you obtain the final solution, it will be real—either because each term is real or because the complex terms occur in conjugate pairs. Until now we have considered ξ either pure real or pure imaginary. However, when ξ is complex it may be carried through the entire calculation as a complex quantity. Of course, when ξ is complex $\eta_{(1)}$ and $\eta_{(2)}$ are complex, and they are not complex conjugates. Consequently, all successive quantities occurring in the solution are complex, and are not complex conjugate. Therefore when ξ is complex the frequency determinat is complex and both real and imaginary parts must vanish simultaneously. However, when ξ is pure real or pure imaginary the frequency determinant is either pure real or pure imaginary, so that only one quantity has to vanish. If ξ (complex) is a point on the ω *vs.* ξ curve, then $-\xi$ and $\bar{\xi}$ (complex conjugate) are also points on the ω *vs.* ξ curve because of the form of the solution. When a final complex solution is obtained it will be complex. However, combinations of complex conjugate (ξ and $\bar{\xi}$) solutions will be real.

For purposes of calculation, it is convenient to write the pertinent equations involved in the calculation in dimensionless form. To this end, we define

$$\gamma = 2\xi h/\pi, \qquad \alpha = 2\eta h/\pi, \qquad \bar{c}_{pq} = c_{pq}/c_{44}, \qquad \Omega = \omega/\bar{\omega}, \tag{10.9}$$

where

$$\bar{\omega} = (\pi/2h)(c_{44}/\varrho)^{1/2} \tag{10.10}$$

is the lowest thickness shear frequency, and substitute in Eqs. (10.3), (10.4), and (10.8) to obtain

$$\begin{vmatrix} (\bar{c}_{11}\gamma^2 + \alpha^2 - \Omega^2) & (\bar{c}_{13} + 1)\gamma\alpha \\ (\bar{c}_{13} + 1)\gamma\alpha & (\gamma^2 + \bar{c}_{33}\alpha^2 - \Omega^2) \end{vmatrix} = 0, \tag{10.11}$$

$$\begin{aligned} \bar{A}_1^{(n)} &= -(\bar{c}_{13} + 1)\gamma\alpha_{(n)}, \\ \bar{A}_3^{(n)} &= \bar{c}_{11}\gamma^2 + \alpha_{(n)}^2 - \Omega^2, \end{aligned} \tag{10.12}$$

$$\begin{vmatrix} (\bar{A}_3^{(1)}\gamma + \bar{A}_1^{(1)}\alpha_{(1)}) \sin \tfrac{1}{2}\pi\alpha_{(1)} & (\bar{A}_3^{(2)}\gamma + \bar{A}_1^{(2)}\alpha_{(2)}) \sin \tfrac{1}{2}\pi\alpha_{(2)} \\ (\bar{c}_{13}\bar{A}_1^{(1)}\gamma + \bar{c}_{33}\bar{A}_3^{(1)}\alpha_{(1)}) \cos \tfrac{1}{2}\pi\alpha_{(1)} & (\bar{c}_{13}\bar{A}_1^{(2)}\gamma + \bar{c}_{33}\bar{A}_3^{(2)}\alpha_{(2)}) \cos \tfrac{1}{2}\pi\alpha_{(2)} \end{vmatrix} = 0. \tag{10.13}$$

The relation between Ω and γ is wanted from Eqs. (10.11)–(10.13). A straightforward procedure for calculating this relation begins by selecting a value of γ. The further choice of Ω permits $\alpha^{(1)}$ and $\alpha^{(2)}$ to be determined from the biquadratic equation in α, (10.11). After this $\bar{A}_1^{(1)}$, $\bar{A}_3^{(1)}$, $\bar{A}_1^{(2)}$, and $\bar{A}_3^{(2)}$ may be evaluated from (10.12). If the values thus determined satisfy the transcendental frequency equation (10.13), the selected values of γ and Ω consitute a point on the Ω *vs.* γ (dispersion) curve. If the transcendental frequency equation (10.13) is not satisfied, repeat the calculation for different values of Ω (and/or γ) until the frequency equation is satisfied and a root has been obtained. When a sufficient number of values of γ and Ω satisfying the frequency equation have been obtained the dispersion curves may be plotted.

2. DISPERSION RELATIONS FOR ELASTIC WAVES WITH REAL PROPAGATION WAVE-NUMBERS

One could now proceed randomly and do a tremendous amount of calculation and determine the curves. One could also proceed by determining certain critical points analytically and obtain the curves with much

less calculation by starting at these points. We will, of course, discuss the latter procedure. Furthermore, these equations must be studied exhaustively in the Ω *vs.* γ region of interest because it is essential that all branches of the dispersion curves in the region of interest be determined before proceeding with vibration problems of the bounded plate by means of the procedures discussed in this monograph.

Before proceeding with a discussion of the tracing of the dispersion curves we should note that the solution and dispersion curves we have been discussing completely determine the longitudinal modes for this elastic plate. Another independent solution can be obtained by interchanging $\cos \eta x_3$ and $\sin \eta x_3$ in the solution functions (10.1) and (10.6) and carrying through the solution in the same manner. This latter solution yields dispersion curves which are independent of the others and determine the flexural modes for the plate.

In determining critical (or starting) points on the dispersion curves, we examine the solution when $\gamma \equiv 0$ and $\Omega \not\equiv 0$. The solution degenerates and the frequency determinant (10.13) takes the form

$$\begin{vmatrix} \bar{A}_1^{(1)}\alpha_{(1)} \sin \tfrac{1}{2}\pi\alpha_{(1)} & 0 \\ 0 & \bar{c}_{33}\bar{A}_3^{(2)}\alpha_{(2)} \cos \tfrac{1}{2}\pi\alpha_{(2)} \end{vmatrix} = 0, \tag{10.14}$$

which yields the two transcendental equations

$$\sin \tfrac{1}{2}\pi\alpha_{(1)} = 0, \qquad \cos \tfrac{1}{2}\pi\alpha_{(2)} = 0, \tag{10.15}$$

the first of which, as we know, corresponds to the symmetric thickness shear modes, and the second to the antisymmetric thickness stretch modes. Thus we have

$$\begin{aligned} \alpha_{(1)} &= n, \qquad n = 2, 4, 6, \ldots, \\ \alpha_{(2)} &= m, \qquad m = 1, 3, 5, \ldots, \end{aligned} \tag{10.16}$$

and from (10.11), we have

$$\alpha_{(1)} = \Omega, \qquad \alpha_{(2)} = \Omega/\bar{c}_{33}^{1/2}. \tag{10.17}$$

Hence at $\xi = 0$ we have the points

$$\begin{aligned} \Omega &= n, \qquad n = 2, 4, 6, \ldots, \\ \Omega &= \bar{c}_{33}^{1/2} m, \qquad m = 1, 3, 5, \ldots. \end{aligned} \tag{10.18}$$

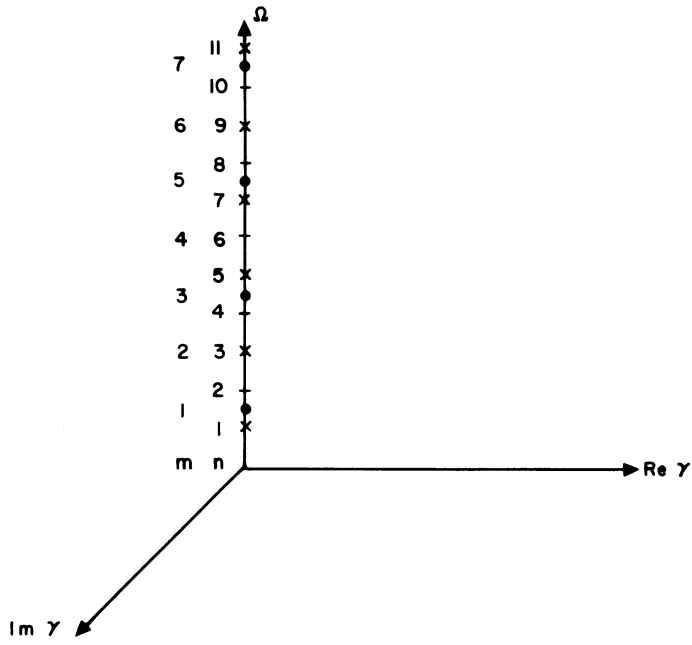

Fig. 13. Location of thickness frequencies on dispersion spectrum for polarized ceramic plates.

If $\bar{c}_{33}^{1/2} = 1.5$, then the points of the upper set and the points of the lower set in (10.18) take the respective positions shown on the dispersion spectrum in Fig. 13. If we had obtained the antisymmetric solution, we would have found the points m even and n odd. These results for the antisymmetric modes are also shown in Fig. 13. We have to determine the dispersion spectrum in some region of this space shown in Fig. 13.

If the material were isotropic, the basic characteristics of the dispersion spectrum would be essentially the same as in this anisotropic case, and would differ only in detail. Furthermore, when the elastic constants satisfy the isotropic relations, the algebra and, consequently, some of the associated discussion simplifies considerably. Moreover, for a first discussion the isotropic case is complicated enough, and a knowledge of the isotropic case turns out to be a useful—if not an essential—prerequisite for a discussion of the algebraically more complicated anisotropic case. Hence we will now introduce the assumption of elastic isotropy into our problem. When a material is elastically isotropic we have

$$c_{11} = c_{33} = \lambda + 2\mu, \quad c_{13} = \lambda, \quad c_{44} = \tfrac{1}{2}(c_{11} - c_{13}) = \mu. \quad (10.19)$$

Expansion of the algebraic determinant (10.11) yields

$$\alpha^4 + \left[\left(\bar{c}_{11} - \frac{\bar{c}_{13}^2}{\bar{c}_{33}} - \frac{2\bar{c}_{13}}{\bar{c}_{33}}\right)\gamma^2 - \left(\frac{1}{\bar{c}_{33}} + 1\right)\Omega^2\right]\alpha^2 + \frac{\bar{c}_{11}}{\bar{c}_{33}}\gamma^4 - \left(\frac{\bar{c}_{11}}{\bar{c}_{33}} + \frac{1}{\bar{c}_{33}}\right)\gamma^2\Omega^2 + \frac{\Omega^4}{\bar{c}_{33}} = 0. \tag{10.20}$$

Introducing the isotropy relations, (10.20) may be written

$$\alpha^4 + \left[2\gamma^2 - \left(1 + \frac{V_2^2}{V_1^2}\right)\Omega^2\right]\alpha^2 + \gamma^4 - \left(1 + \frac{V_2^2}{V_1^2}\right)\gamma^2\Omega^2 + \frac{V_2^2}{V_1^2}\Omega^4 = 0, \tag{10.21}$$

where

$$V_2^2 = \mu/\varrho, \qquad V_1^2 = (\lambda + 2\mu)/\varrho. \tag{10.22}$$

The roots of the biquadratic in α in (10.21) are given by

$$\alpha_\pm^2 = \frac{1}{2}\left(1 + \frac{V_2^2}{V_1^2}\right)\Omega^2 - \gamma^2 \pm \frac{1}{2}\Omega^2\left(1 - \frac{V_2^2}{V_1^2}\right),$$

from which we obtain

$$\alpha_+^2 \equiv \alpha_2^2 = \Omega^2 - \gamma^2, \tag{10.23}$$

$$\alpha_-^2 \equiv \alpha_1^2 = (V_2^2/V_1^2)\Omega^2 - \gamma^2, \tag{10.24}$$

which shows the major simplification which results from the assumed isotropy—the biquadratic equation in α in (10.21) factors into the two parts (10.23) and (10.24). This algebraic simplification does not occur in the anisotropic case.

Substituting from (10.23) and (10.24) into (10.12) for the amplitude ratios, we obtain for the coefficients of the trigonometric functions in the transcendental determinant

$$\begin{aligned}
\bar{A}_3^{(1)}\gamma + \bar{A}_1^{(1)}\alpha_{(1)} &= 2\gamma\alpha_{(1)}^2\left(1 - \frac{V_1^2}{V_2^2}\right),\\
\bar{A}_3^{(2)}\gamma + \bar{A}_1^{(2)}\alpha_{(2)} &= \gamma(\alpha_{(2)}^2 - \gamma^2)\left(1 - \frac{V_1^2}{V_2^2}\right),\\
\bar{c}_{13}\bar{A}_1^{(1)}\gamma + \bar{c}_{33}\bar{A}_3^{(1)}\alpha_{(1)} &= \mu\alpha_{(1)}(\alpha_{(2)}^2 - \gamma^2)\left(1 - \frac{V_1^2}{V_2^2}\right),\\
\bar{c}_{13}\bar{A}_1^{(2)}\gamma + \bar{c}_{33}\bar{A}_3^{(2)}\alpha_{(2)} &= -2\mu\gamma^2\alpha_{(2)}\left(1 - \frac{V_1^2}{V_2^2}\right).
\end{aligned} \tag{10.25}$$

Substituting from (10.25) into the transcendental determinant (10.13), we obtain

$$\mu\gamma\left(1-\frac{V_1^2}{V_2^2}\right)^2\alpha_{(1)}\begin{vmatrix} 2\alpha_{(1)}\sin\tfrac{1}{2}\pi\alpha_{(1)} & (\alpha_{(2)}^2-\gamma^2)\sin\tfrac{1}{2}\pi\alpha_{(2)} \\ (\alpha_{(2)}^2-\gamma^2)\cos\tfrac{1}{2}\pi\alpha_{(1)} & -2\gamma^2\alpha_{(2)}\cos\tfrac{1}{2}\pi\alpha_{(2)}\end{vmatrix}=0. \tag{10.26}$$

Expanding (10.26), we find

$$\tan\frac{\pi}{2}\alpha_{(1)}=-\frac{(\gamma^2-\alpha_{(2)}^2)^2}{4\gamma^2\alpha_{(1)}\alpha_{(2)}}\tan\frac{\pi}{2}\alpha_{(2)}, \tag{10.27}$$

which is the Rayleigh ([26]) frequency equation governing the symmetric motions of an isotropic elastic plate. If in the basic solution of the problem we interchange $\cos\eta x_3$ and $\sin\eta x_3$, we obtain

$$\tan\frac{\pi}{2}\alpha_{(1)}=-\frac{4\gamma^2\alpha_{(1)}\alpha_{(2)}}{(\gamma^2-\alpha_{(2)}^2)^2}\tan\frac{\pi}{2}\alpha_{(2)}, \tag{10.28}$$

which is the Rayleigh ([26]) frequency equation governing the antisymmetric motions of an isotropic elastic plate. The frequency–propagation wave number–thickness wave number relations, (10.23)–(10.24), are the same as for the symmetric motions.

In attempting to sketch the dispersion spectrum we first focus our attention on γ real, and note that the Ω–Re γ plane breaks up into three distinct regions ([5,27]) depending on the character of $\alpha_{(1)}$ and $\alpha_{(2)}$. When

$$\begin{aligned} &\Omega/\gamma > V_1/V_2 && \alpha_{(1)} \text{ and } \alpha_{(2)} \text{ are real,} \\ &V_1/V_2 > \Omega/\gamma > 1 && \alpha_{(2)} \text{ real, } \alpha_{(1)} \text{ imaginary,} \\ &1 > \Omega/\gamma && \alpha_{(1)} \text{ and } \alpha_{(2)} \text{ are imaginary.} \end{aligned} \tag{10.29}$$

The character of the solutions and the mode shapes are, of course, quite different in the three regions. Schematically, we have the regions shown in Fig. 14. Besides the thickness solutions which we have already discussed, there are other simple solutions at which the frequency equation degenerates, and which are helpful in the approximate sketching of the spectrum. One such set of simple solutions are the Lamé solutions, which are given by

$$\alpha_{(2)}^2=\gamma^2, \qquad \cos\tfrac{1}{2}\pi\alpha_{(2)}=0, \qquad \alpha_{(2)}=1,\,3,\,5,\ldots, \tag{10.30}$$

for the symmetric motions and

$$\alpha_{(2)}^2=\gamma^2, \qquad \sin\tfrac{1}{2}\pi\alpha_{(2)}=0, \qquad \alpha_{(2)}=2,\,4,\,6,\ldots, \tag{10.31}$$

for the antisymmetric motions. Clearly, these solutions intersect the line $\Omega=\gamma\sqrt{2}$. Schematically, we have the Lamé frequencies shown in Fig. 15.

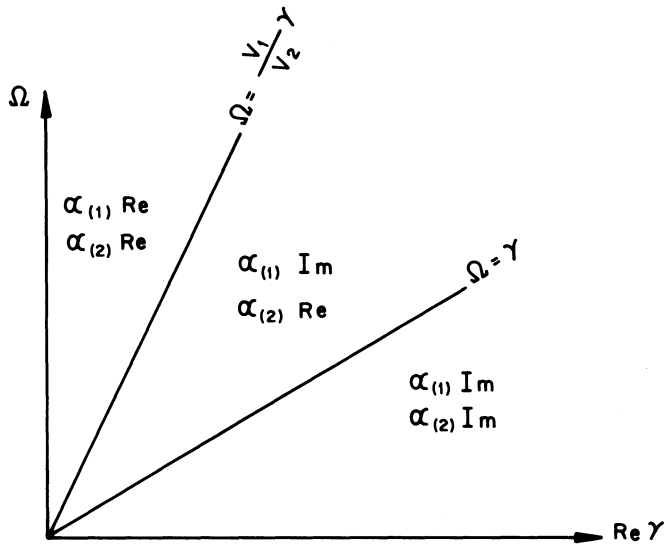

Fig. 14. Regions of real and imaginary thickness wave numbers in the Ω *vs.* Re γ plane for an infinite isotropic plate.

Another set of simple degenerate solutions are those along the line $\Omega = V_1\gamma/V_2$ when $\alpha_1 = 0$. We will not bother to discuss this set.

In the lowest range of Ω/γ, $\alpha_{(1)}$ and $\alpha_{(2)}$ are pure imaginary. An imaginary $\alpha_{(1)}$ or $\alpha_{(2)}$ will be denoted by

$$\alpha_{(1)} = i\bar{\alpha}_{(1)}, \qquad \alpha_{(2)} = i\bar{\alpha}_{(2)}. \tag{10.32}$$

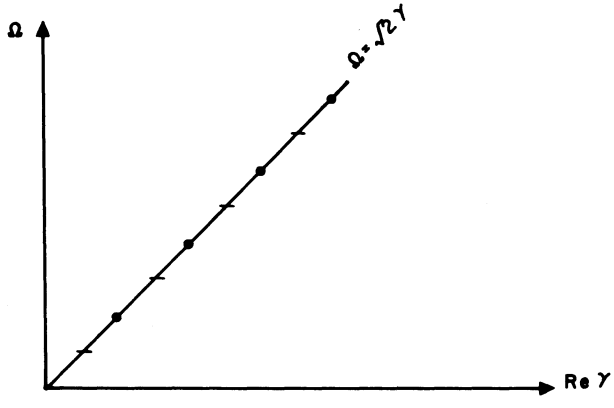

Fig. 15. Location of the Lamé modes on the dispersion spectrum of an infinite isotropic plate.

It should be noted at this point that although the $\alpha_{(n)}$ are either pure real or pure imaginary for real γ in the isotropic case, in the anisotropic case there are frequently ranges in which they are complex. This is one of the algebraic complications which are avoided in the isotropic case. In the lowest range of Ω/γ the equation for the symmetric modes (10.27) becomes

$$\frac{\tanh(\pi\bar{\alpha}_{(2)}/2)}{\tanh(\pi\bar{\alpha}_{(1)}/2)} = \frac{4\gamma^2\bar{\alpha}_{(1)}\bar{\alpha}_{(2)}}{(\gamma^2+\bar{\alpha}_{(2)}^2)^2} \tag{10.33}$$

and (10.28) for the antisymmetric modes becomes

$$\frac{\tanh(\pi\bar{\alpha}_{(2)}/2)}{\tanh(\pi\bar{\alpha}_{(1)}/2)} = \frac{(\gamma^2+\bar{\alpha}_{(2)}^2)^2}{4\gamma^2\bar{\alpha}_{(1)}\bar{\alpha}_{(2)}}. \tag{10.34}$$

For γ very large (wavelengths very short compared to h), we obtain from both (10.33) and (10.34) for both the symmetric and antisymmetric motions

$$(\gamma^2+\bar{\alpha}_{(2)}^2)^2 = 4\gamma^2\bar{\alpha}_{(1)}\bar{\alpha}_{(2)}. \tag{10.35}$$

Substituting from (10.23) and (10.24) into (10.35), we obtain

$$\frac{\Omega^2}{\gamma^2}\left[\frac{\Omega^6}{\gamma^6} - 8\frac{\Omega^4}{\gamma^4} + 8\left(3 - 2\frac{V_2^2}{V_1^2}\right)\frac{\Omega^2}{\gamma^2} - 16\left(1 - \frac{V_2^2}{V_1^2}\right)\right] = 0, \tag{10.36}$$

which is the equation governing the velocity (Ω/γ) of Rayleigh ([28]) surface waves in an isotropic solid. This equation, which is valid only in the region $\Omega/\gamma < 1$, has only one root in this range. For $V_1/V_2 = 1.5$ that root is given by

$$\Omega/\gamma = 0.8935, \tag{10.37}$$

which corresponds to a straight line slightly below the $\Omega = \gamma$ line on the dispersion spectrum. Clearly, this line will be an asymptote for at least one curve for the symmetric motions and one curve for the antisymmetric motions.

Still in the lowest Ω/γ range, we have for γ very small (wavelengths very long compared to h) $\Omega \to 0$, $\bar{\alpha}_{(2)} \to 0$, and $\bar{\alpha}_{(1)} \to 0$. In this limit the equation for the symmetric modes (10.33) becomes

$$(\gamma^2+\bar{\alpha}_{(2)}^2)^2 = 4\gamma^2\bar{\alpha}_{(1)}^2, \tag{10.38}$$

from which, with the aid of (10.23) and (10.24), we obtain

$$\Omega = 2\gamma[1 - (V_2^2/V_1^2)]^{1/2}, \tag{10.39}$$

which does not lie in the range $\Omega/\gamma < 1$ since $V_2^2/V_1^2 < \frac{3}{4}$, and hence is invalid. In the same limit (γ very small) for the antisymmetric modes we must use the first two terms in the expansion for the tanh x, i.e., tanh $x = x - \frac{1}{3}x^3$. Using this expression, the transcendental equation (10.34) for the antisymmetric modes becomes

$$4\gamma^2\bar{\alpha}_{(2)}^2 - \tfrac{1}{3}\pi^2\gamma^2\bar{\alpha}_{(2)}^4 = (\gamma^2 + \bar{\alpha}_{(2)}^2)^2(1 - \tfrac{1}{12}\pi^2\bar{\alpha}_{(1)}^2), \tag{10.40}$$

from which, remembering that $\Omega \to 0$ and $\gamma \to 0$ and using (10.23) and (10.24), we obtain

$$\Omega = \pi\gamma^2\left[\frac{1}{3}\left(1 - \frac{V_2^2}{V_1^2}\right)\right]^{1/2}, \tag{10.41}$$

which is an unfamiliar form of the dispersion equation in the classical theory of the flexure of thin plates.

In the intermediate range of Ω/γ the transcendental equation (10.27) for the symmetric modes takes the form

$$\frac{\tan(\pi\alpha_{(2)}/2)}{\tanh(\pi\bar{\alpha}_{(1)}/2)} = \frac{4\gamma^2\bar{\alpha}_{(1)}\alpha_{(2)}}{(\gamma^2 - \alpha_{(2)}^2)^2}, \tag{10.42}$$

and (10.28) for the antisymmetric modes

$$\frac{\tan(\pi\alpha_{(2)}/2)}{\tanh(\pi\bar{\alpha}_{(1)}/2)} = -\frac{(\gamma^2 - \alpha_{(2)}^2)^2}{4\gamma^2\bar{\alpha}_{(1)}\alpha_{(2)}}. \tag{10.43}$$

For γ very small $\Omega \to 0$, $\bar{\alpha}_{(1)} \to 0$, and $\alpha_{(2)} \to 0$. In this limit Eq. (10.42) for the symmetric modes becomes

$$(\gamma^2 - \alpha_{(2)}^2)^2 = 4\gamma^2\bar{\alpha}_{(1)}^2, \tag{10.44}$$

from which we obtain

$$\Omega = 2\gamma[1 - (V_2^2/V_1^2)]^{1/2}, \tag{10.45}$$

which is a straight line intersecting the origin, and does lie in the intermediate range $1 < \Omega/\gamma < V_1/V_2$ since $0 < (V_2^2/V_1^2) < \frac{3}{4}$. Equation (10.45) is an unfamiliar form of the dispersion equation in the classical theory of the extension of thin plates. In the same limit the transcendental equation (10.43) for the antisymmetric modes takes the form

$$-4\gamma^2\alpha_{(2)}^2\left(1 + \frac{1}{12}\pi^2\alpha_{(2)}^2\right) = (\gamma^2 - \alpha_{(2)}^2)\left(1 - \frac{\pi^2}{12}\bar{\alpha}_{(1)}^2\right), \tag{10.46}$$

which yields the same dispersion relation, (10.41), as for the classical theory of flexure, which lies below the intermediate range, and hence is invalid.

The lowest antisymmetric curve is confined to the lowest Ω/γ region and is asymptotic to the Rayleigh surface-wave line from below. The lowest symmetric curve crosses the line $\Omega = \gamma$ separating the lowest from the intermediate region. The crossover point is found by taking the limit of the transcendental equation (10.42) for the symmetric modes as $\alpha_{(2)} \to 0$, to obtain

$$4\bar{\alpha}_{(1)} \tanh \tfrac{1}{2}\pi\bar{\alpha}_{(1)} = \tfrac{1}{2}\pi\gamma^2,$$

from which with the aid of (10.24) we find

$$4\left(1 - \frac{V_2^2}{V_1^2}\right) \tanh \frac{\pi}{2}\bar{\alpha}_{(1)} = \frac{\pi}{2}\bar{\alpha}_{(1)}. \tag{10.47}$$

Then the location of the crossover point on the dispersion spectrum is given by

$$\Omega = \gamma = \bar{\alpha}_{(1)}[1 - (V_2^2/V_1^2)]^{-1/2}, \tag{10.48}$$

where $\bar{\alpha}_{(1)}$ is the root of the previous transcendental equation. For $V_1/V_2 = 1.5$, $\bar{\alpha}_{(1)} = 2.22$ and $\gamma = 2.97$. This lowest curve is then asymptotic to the Rayleigh surface-wave line from above.

Thus so far in addition to the thickness frequencies and the Lamé modes we also have, in the two lowest Ω/γ ranges, the curves shown in Fig. 16. In Fig. 16 the upper heavy line is the lowest longitudinal mode

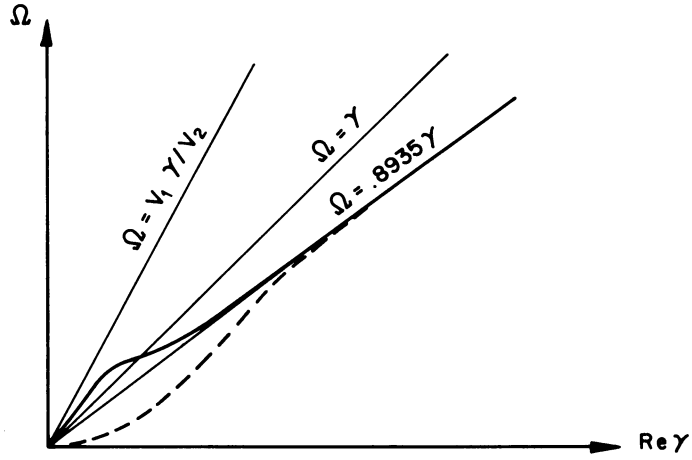

Fig. 16. Dispersion curves for the lowest extensional and flexural modes in an infinite isotropic plate.

and the lower dotted line is the lowest flexural mode. As already mentioned, the behavior of these curves as $\Omega \to 0$ and $\gamma \to 0$ coincides with that predicted by the classical theories of the extensional and flexural motions of thin plates. Note that these are the only curves in the lowest Ω/γ region shown in Fig. 14.

In the highest Ω/γ region in Fig. 14 the transcendental equation (10.27) for the symmetric modes degenerates at certain points, at which it has the special roots

$$\sin \tfrac{1}{2}\pi\alpha_{(1)} = 0, \qquad \sin \tfrac{1}{2}\pi\alpha_{(2)} = 0, \tag{10.49}$$

and the amplitude ratio determined from (10.7) is given by

$$C^{(1)}/C^{(2)} = \pm\, 2\gamma^2\alpha_{(2)}/(\alpha_{(2)}^2 - \gamma^2)\alpha_{(1)}. \tag{10.50}$$

Under these circumstances

$$\alpha_{(1)} = n = 2, 4, 6, \ldots, \qquad \alpha_{(2)} = m = 2, 4, 6, \ldots. \tag{10.51}$$

When $\alpha_{(1)} = n$ and $\alpha_{(2)} = m$, from (10.23) and (10.24) we have

$$\Omega^2 - \gamma^2 = n^2, \tag{10.52}$$

$$\Omega^2 - (V_1^2/V_2^2)\gamma^2 = (V_1^2/V_2^2)m^2. \tag{10.53}$$

Equation (10.52) represents a series of hyperbolas on the Ω *vs.* γ diagram which intersect the Ω axis at certain of the thickness frequencies and are asymptotic to the line $\Omega = \gamma$ as $\gamma \to \infty$. Equation (10.53) also represents hyperbolas which intersect the Ω axis at thickness frequencies, but which are asymptotic to the line $\Omega = V_1\gamma/V_2$ as $\gamma \to \infty$.

The intersections of these two sets of hyperbolas determine points on the dispersion curves. The transcendental equation (10.27) for the symmetric modes degenerates at another sequence of points, at which it has the special roots

$$\cos \tfrac{1}{2}\pi\alpha_{(1)} = 0, \qquad \cos \tfrac{1}{2}\pi\alpha_{(2)} = 0, \tag{10.54}$$

and at which the amplitude ratio determined from (10.7) is given by

$$C^{(1)}/C^{(2)} = \pm\, (\gamma^2 - \alpha_{(2)}^2)/2\alpha_{(1)}^2. \tag{10.55}$$

Under these circumstances

$$\alpha_{(1)} = n = 1, 3, 5, \ldots, \qquad \alpha_{(2)} = m = 1, 3, 5, \ldots, \tag{10.56}$$

and the two sets of hyperbolas discussed previously may be drawn for n-odd, m-odd as well as for n-even, m-even. Thus it is clear that the intersections of the hyperbolas for m-even, n-even and for m-odd, n-odd determine points on the dispersion curves for the symmetric motions, whereas mixed intersections do not. In an entirely similar way it may be shown that the same set of intersection points determine points on the dispersion curves for the antisymmetric motions. It is also clear that the curves m-odd, n-odd and m-even, n-even form bounds for the dispersion curves, i.e., the dispersion curves can cross the m, n curves only at the aforementioned intersection points.

3. IMAGINARY PROPAGATION WAVE NUMBER

When γ is pure imaginary, i.e.,

$$\gamma = i\bar{\gamma}, \tag{10.57}$$

where $\bar{\gamma}$ is a real number, from (10.23) and (10.24) we have

$$\alpha_{(2)}^2 = \Omega^2 + \bar{\gamma}^2, \tag{10.58}$$

$$\alpha_{(1)}^2 = (V_2^2/V_1^2)\Omega^2 + \bar{\gamma}^2. \tag{10.59}$$

Therefore $\alpha_{(1)}$ and $\alpha_{(2)}$ are always real when γ is pure imaginary. Consequently, the previous degenerate roots corresponding to the intersections of the hyperbolas for m-odd, n-odd and m-even, n-even when γ is real and $\Omega/\gamma > V_1/V_2$ are also roots whenever γ is pure imaginary. The only difference is that the curves are no longer hyperbolas. Now the set of curves from (10.58) are circles and the other set from (10.59) are ellipses. Thus at this stage we can exhibit the set of dispersion curves shown in Fig. 17 for the isotropic plate. Actually, to really draw the curves in as much detail as shown, we would have to determine the slopes and curvatures at all critical points. We will not bother with these steps because we intend to use a calculator to determine the precise detail. Note that for real γ all curves except the lowest two are asymptotic to the line $\Omega = \gamma$ as $\gamma \to \infty$. The lowest two are asymptotic to the Rayleigh surface-wave line $\Omega = 0.8935\gamma$.

4. COMPLEX PROPAGATION WAVE NUMBER

This brings us to a discussion of the complex branches ([29]). The first thing to do is to determine the points of intersection of the complex branches with the three coordinate planes $-\Omega = 0$, $\operatorname{Im}\gamma = 0$, and $\operatorname{Re}\gamma = 0$.

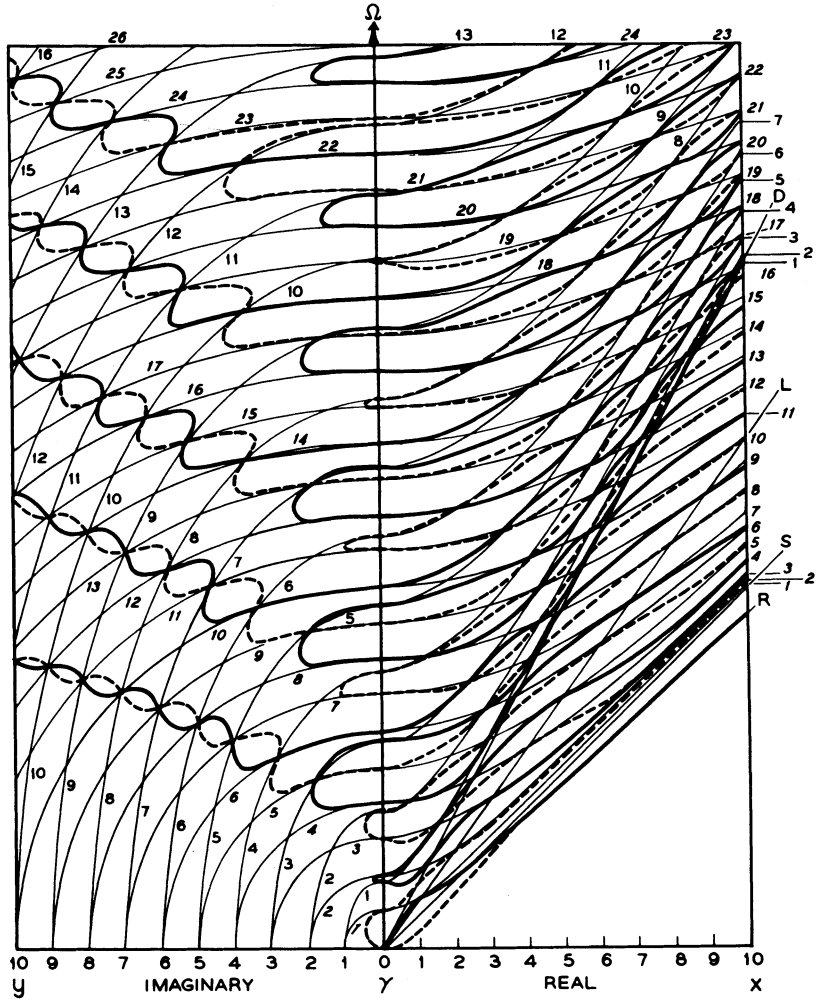

Fig. 17. Frequency spectrum of an infinite isotropic plate for real and imaginary wave numbers ($\nu = 0.31$). After Mindlin [33].

When γ is complex, i.e.,

$$\gamma = x + iy, \tag{10.60}$$

$\alpha_{(1)}$ and $\alpha_{(2)}$ become complex. The points of intersection of the complex branches with the plane $\Omega = 0$ cannot be found simply by setting $\Omega = 0$ in the equations

$$\alpha_{(2)}^2 = \Omega^2 - \gamma^2, \qquad \alpha_{(1)}^2 = (V_2^2\Omega^2/V_1^2) - \gamma^2, \tag{10.61}$$

because then $\alpha_{(1)} = \alpha_{(2)}$ and the solution evaporates, and the Rayleigh frequency equation for the symmetric modes

$$\tan \tfrac{1}{2}\pi\alpha_{(1)}/\tan \tfrac{1}{2}\pi\alpha_{(2)} = -(\gamma^2 - \alpha_{(2)}^2)^2/4\gamma^2\alpha_{(1)}\alpha_{(2)}, \tag{10.62}$$

degenerates to an identity which permits any value of γ. In other words, in this isotropic case we cannot first select the plane $\Omega = 0$ and then ask where the intersections of the branches with the plane $\Omega = 0$ are. We must first get on a branch at some small value of $\Omega > 0$ and then ask for the limiting position of the branch as $\Omega \to 0$. This complication does *not* exist in an anisotropic case, in which one may proceed in the straightforward manner of first selecting the plane $\Omega = 0$ and then determining the intersection with that plane.

In proceeding with the isotropic (and anisotropic) case, it is expedient to note that the transcendental frequency equation (10.62) and the two quadratic relations (10.61) actually amount to an implicit equation of the form

$$F(\gamma, \Omega) = 0. \tag{10.63}$$

In the isotropic case we now make a Taylor expansion of (10.63) at some $\gamma = \gamma_0$, which is not required to be the intersection of a branch with the $\Omega = 0$ plane, in powers of Ω, and retain only the first term in Ω to obtain

$$F(\gamma_0, \Omega) = F(\gamma_0, 0) + \Omega(\partial F/\partial\Omega)_{\Omega=0,\gamma=\gamma_0} = 0. \tag{10.64}$$

Since $F(\gamma_0, 0) \equiv 0$ (as already noted, it gives no information in the isotropic case), and since from (10.61) and (10.62) $\partial F/\partial\Omega$ is linearly proportional to Ω and $\Omega \neq 0$, from (10.64) we must have

$$\Omega^{-1}(\partial F/\partial\Omega)_{\Omega=0} = 0 \tag{10.65}$$

as the condition for the intersection of the branches with the plane $\Omega = 0$. Carrying out the operations

$$\frac{1}{\Omega}\frac{\partial F}{\partial\Omega} = \frac{1}{\Omega}\frac{\partial F}{\partial\alpha_{(1)}}\frac{\partial\alpha_{(1)}}{\partial\Omega} + \frac{1}{\Omega}\frac{\partial F}{\partial\alpha_{(2)}}\frac{\partial\alpha_{(2)}}{\partial\Omega} = 0$$

by means of (10.61) and (10.62) and noting that $\alpha_{(1)} = \alpha_{(2)} = i\gamma$ when $\Omega = 0$, we obtain after some manipulation

$$(\sinh \pi\gamma) + \pi\gamma = 0 \tag{10.66}$$

for the symmetric modes. In a similar way but replacing (10.62) by (10.28), we obtain

$$(\sinh \pi\gamma) - \pi\gamma = 0 \tag{10.67}$$

for the antisymmetric modes. Equations (10.66) and (10.67) are complex equations, and equating real and imaginary parts of (10.66) we obtain for the symmetric modes

$$\begin{aligned} (\sinh \pi x)(\cos \pi y) + \pi x &= 0\,, \\ (\cosh \pi x)(\sin \pi y) + \pi y &= 0\,, \end{aligned} \tag{10.68}$$

with similar equations for the antisymmetric modes. The two equations in (10.68) determine curves in the $\Omega = 0$ plane. The intersections of these families of curves determine the intersections of the complex branches with the plane $\Omega = 0$. We can determine the character of the complex branches in the vicinity of the plane $\Omega = 0$ by differentiating (10.63) totally with respect to Ω. Thus

$$\frac{\partial F}{\partial \gamma}\,\frac{\partial \gamma}{\partial \Omega} + \frac{\partial F}{\partial \Omega} = 0\,,$$

and since, from (10.61) and (10.62) $\partial F/\partial\gamma$ is not zero, with (10.65) we have

$$\Omega^{-1}\,\partial\gamma/\partial\Omega = 0 \qquad \text{as} \quad \Omega \to 0\,. \tag{10.69}$$

We have derived (10.69) for the isotropic case only. The derivation for the anisotropic case would proceed by setting $\Omega = 0$ in (10.11)–(10.13) and finding the γ_0 at $\Omega = 0$, and then evaluating $\Omega^{-1}\,\partial\gamma/\partial\Omega$ at γ_0 and $\Omega = 0$ from (10.11)–(10.13).

The behavior of the complex branches in the vicinity of the plane $\operatorname{Im}\gamma = 0$ can be investigated by making a Taylor expansion at some real $\gamma = x_0$ and $\Omega = \Omega_0$, which is not required to be a root of (10.63), in powers of iy and retaining only the first term in iy to obtain

$$F(\gamma, \Omega_0) = F(x_0\,, \Omega_0) + iy(\partial F/\partial\gamma)_{\gamma=x_0} = 0\,.$$

Since for analytic functions of a complex variable the derivative is independent of the direction in the complex plane, we have $\partial F/\partial\gamma = \partial F/\partial x = \partial F/\partial iy$. Consequently,

$$F(\gamma, \Omega_0) = F(x_0\,, \Omega_0) + iy(\partial F/\partial x)_{x=x_0} = 0\,,$$

and separating real and imaginary parts, we obtain the equations

$$F(x_0, \Omega_0) = 0, \qquad y(\partial F/\partial x)_{x=x_0} = 0,$$

governing the roots in the complex space near the plane $\operatorname{Im}\gamma = 0$. These equations yield the usual real roots

$$F(x_0, \Omega_0) = 0, \qquad y = 0, \tag{10.70}$$

and also the roots

$$F(x_0, \Omega_0) = 0, \qquad y \neq 0, \qquad \partial F/\partial x|_{x=x_0} = 0, \tag{10.71}$$

which govern the complex branches in the vicinity of the plane $\operatorname{Im}\gamma = 0$. Differentiating (10.63) with $\gamma = x$ totally with respect to x, we obtain

$$\frac{\partial F}{\partial \Omega}\frac{\partial \Omega}{\partial x} + \frac{\partial F}{\partial x} = 0. \tag{10.72}$$

Since, as is usually the case, $\partial F/\partial \Omega \neq 0$, from (10.71) and (10.72) we have

$$\partial \Omega/\partial x = 0, \tag{10.73}$$

thereby clearly showing that the complex branches intersect the real branches at points of zero slope of the real branches. Exceptional cases where $\partial F/\partial \Omega = 0$ occur, but they will not be discussed here. These cases are exceptional in other ways which we have also ignored here. The exceptional cases are discussed by Mindlin ([5,29]). The theorem we have just proved is referred to by Mindlin as Onoe's theorem, and is true for any analytic function of one real and one complex variable, as noted by Kaul and Mindlin ([30]). Equation (10.73) is valid in the anisotropic as well as the isotropic case, since the isotropic relations were in no way used in obtaining (10.73).

A similar investigation of the behavior of the roots in the vicinity of the plane $\operatorname{Re}\gamma = 0$ shows again that the complex branches intersect the imaginary branches at points of zero slope of the imaginary branches.

When the end points of a complex branch have been determined the entire branch can be calculated starting at the known points. Thus any desired portion of the frequency spectrum can be calculated.

5. CALCULATION OF THE DISPERSION CURVES

When the material is somewhat anisotropic—cubic, C_{6v} (not quartz)—the same curves exist but are shifted somewhat ([31]). The amount of numerical work does not change much, but much of the algebra that we went through

for the isotropic case becomes extremely cumbersome in an anisotropic case. Nevertheless, it should be clear that all the information we obtained in the isotropic case is not really necessary if a machine calculation is to be made. The really crucial information consists of the following:

1. The thickness frequencies, which have already been determined in general.
2. The behavior of the real branches in the vicinity of zero frequency and zero wave number. I will provide a simple procedure which works in every case.
3. The location of the Rayleigh surface-wave line, which is located by solving the appropriate surface wave problem. This will be discussed later.
4. The location of the lowest of the three wave velocity lines for the infinite medium. This is straightforward. In the isotropic case this was simply the shear velocity line.
5. The location of the intersection of the complex branches with the plane $\Omega = 0$. This has already been discussed.

After the aforementioned have been determined the order of the machine calculation should be:

1. The calculation of the real branches.
2. The calculation of the imaginary branches.
3. The calculation of the complex branches.

Of course, step 3 must proceed *after* steps 1 and 2 because the real and imaginary branches are needed in order to locate the intersections of the complex branches with the real and imaginary planes (points of zero slope in the respective planes).

The exact procedure is, of course, a computational decision. At this point we again note that a vibration problem can be *started* using the techniques presented in this monograph only after the frequency spectrum has been plotted in detail in some defined region in Ω *vs.* γ space.

6. LOW-FREQUENCY EXTENSIONAL PLATE EQUATIONS

The aforementioned procedure for determining the behavior of the lowest branches in the real plane ($\Omega \to 0$ *vs.* $\gamma \to 0$) begins with the governing equations (7.28) and (7.29), which with $u_2 = 0$ and no x_2 dependence yield the nontrivial equations

$$T_{11} = c_{11}u_{1,1} + c_{13}u_{3,3}, \quad T_{33} = c_{13}u_{1,1} + c_{33}u_{3,3}, \quad T_{13} = c_{44}(u_{1,3} + u_{3,1}); \tag{10.74}$$

$$T_{11,1} + T_{31,1} + \varrho\omega^2 u_1 = 0, \tag{10.75}$$

$$T_{13,1} + T_{33,3} + \varrho\omega^2 u_3 = 0. \tag{10.76}$$

We also have the boundary conditions

$$T_{31} = T_{33} = 0 \qquad \text{at} \quad x_3 = \pm h. \tag{10.77}$$

We now look for the lowest longitudinal branch. To this end, we make the approximations and the expansions directly in the equations. The simplest way of describing the approximation consists of assuming that

$$T_{33} = T_{31} = 0 \qquad \text{for all} \quad x_1, x_3, \tag{10.78}$$

along with

$$\varrho\omega^2 u_3 \approx 0, \tag{10.79}$$

so that the boundary conditions (10.77) and the differential equation (10.76) are satisfied identically. The one remaining differential equation (10.75) with the aid of (10.78) and (10.74) yields

$$\left(c_{11} - \frac{c_{13}^2}{c_{33}}\right) u_{1,11} + \varrho\omega^2 u_1 = 0, \tag{10.80}$$

from which we can see that the lowest longitudinal branch approaches $\Omega = 0$, $\gamma = 0$ as a straight line with the slope

$$\frac{\Omega}{\gamma} = \left(\frac{c_{11}}{c_{44}} - \frac{c_{13}^2}{c_{33}c_{44}}\right)^{1/2}. \tag{10.81}$$

For the isotropic case this becomes

$$\Omega/\gamma = 2[(\lambda + \mu)/(\lambda + 2\mu)]^{1/2} = 2[1 - (V_2^2/V_1^2)]^{1/2}, \tag{10.82}$$

as we have already shown in (10.45).

7. LOW-FREQUENCY FLEXURAL PLATE EQUATIONS

We now look for the lowest flexural branch by means of an essentially analogous procedure. To this end we assume that

$$T_{33} = 0, \qquad \text{for all} \quad x_1, x_3 \tag{10.83}$$

and that

$$u_1 = x_3\psi(x_1)\,, \tag{10.84}$$

$$u_3 = w(x_1)\,, \tag{10.85}$$

and perform the following operation on (10.75):

$$\int_{-h}^{h} x_3(T_{11,1} + T_{31,3} + \varrho\omega^2 u_1)\, dx_3 = 0\,,$$

which yields

$$M_{,1} - V + I\omega^2\psi = 0\,, \tag{10.86}$$

where

$$M = \int_{-h}^{h} x_3 T_{11}\, dx_3\,, \qquad V = \int_{-h}^{h} T_{31}\, dx_3\,, \qquad I = \varrho \int_{-h}^{h} x_3^2\, dx_3\,. \tag{10.87}$$

We now suppose that

$$I\omega^2\psi \approx 0\,, \tag{10.88}$$

which means that we are neglecting what is commonly termed rotatory inertia; and from (10.86) and (10.88) we have

$$V = M_{,1}\,. \tag{10.89}$$

Substituting from (10.83), (10.85), (10.87), and (10.89) in the integrated form of (10.76), we obtain

$$M_{,11} + 2h\varrho\omega^2 w = 0\,. \tag{10.90}$$

From (10.83) and the second of (10.74) we further have

$$u_{3,3} = -\,(c_{13}/c_{33})u_{1,1}\,, \tag{10.91}$$

and substituting from (10.91) and (10.84) into the first of (10.74) and integrating through the thickness and using (10.87), we obtain

$$M = \left(c_{11} - \frac{c_{13}^2}{c_{33}}\right)\frac{2h^3}{3}\,\psi_{,1}\,. \tag{10.92}$$

Finally, we assume that the shearing strain S_{13} is negligible, and obtain

$$\psi = -\,w_{,1}\,. \tag{10.93}$$

Substituting from (10.92) and (10.93) into (10.90), we find

$$-\left(c_{11}-\frac{c_{13}^2}{c_{33}}\right)\frac{h^2}{3}w_{,1111}+\varrho\omega^2 w=0\,, \tag{10.94}$$

from which it is clear that the dispersion relation for the lowest antisymmetric (flexural) branch is a parabola as $\Omega\to 0$, $\gamma\to 0$, which is given by the equation

$$\Omega=\frac{\pi\gamma^2}{2}\left[\frac{1}{3}\left(\frac{c_{11}}{c_{44}}-\frac{c_{13}^2}{c_{33}c_{44}}\right)\right]^{1/2}. \tag{10.95}$$

For the isotropic case Eq. (10.95) becomes

$$\Omega=\pi\gamma^2\left[\frac{\lambda+\mu}{3(\lambda+2\mu)}\right]^{1/2}=\pi\gamma^2\left[\frac{1}{3}\left(1-\frac{V_2^2}{V_1^2}\right)\right]^{1/2}, \tag{10.96}$$

in accordance with (10.41), as we obtained previously.

I have gone through these last operations in accord with the elementary theories of the extension and flexure of thin plates. This may be regarded as a preliminary introduction to the principles underlying Mindlin's theory of plates ([5]).

8. ELASTIC SURFACE WAVES

We will now go through the procedure for determining the surface wave velocity. We begin with Eqs. (10.74)–(10.76), but we take as solutions of the equations the following:

$$\begin{aligned} u_1&=A_1(\exp-\eta x_3)\sin\xi x_1\,,\\ u_3&=A_3(\exp-\eta x_3)\cos\xi x_1\,, \end{aligned} \tag{10.97}$$

which satisfy the equations provided

$$\begin{aligned} (c_{11}-c_{44}k^2-\varrho v^2)A_1-(c_{13}+c_{44})kA_3&=0\,,\\ (c_{13}+c_{44})kA_1+(c_{44}-c_{33}k^2-\varrho v^2)A_3&=0\,, \end{aligned} \tag{10.98}$$

where $v=\omega/\xi$ and $k=\eta/\xi$. This system of linear homogeneous equations, (10.98), in A_1 and A_3 yields nontrivial solutions when the determinant of the coefficients of A_1 and A_3 vanishes, i.e., when

$$\begin{vmatrix} (c_{11}-c_{44}k^2-\varrho v^2) & -(c_{13}+c_{44})k\\ (c_{13}+c_{44})k & (c_{44}-c_{33}k^2-\varrho v^2)\end{vmatrix}=0. \tag{10.99}$$

This equation is quadratic in k^2 and v^2. Hence for a given v there are two k^2 (k_1^2, k_2^2). For a physically meaningful solution v must be real, and k^2 must be positive if it is real. However, the k^2 may occur in complex conjugate pairs and the solution is still physically meaningful. In any event, each k^2 yields two solutions, one the negative of the other, and the one corresponding to a negative real part is discarded since it cannot correspond to a physically meaningful solution. Each solution, of course, yields amplitude ratios when substituted in the linear equations (10.98) in A_1 and A_3,

$$\begin{aligned} A_1^{(n)} &= (c_{13} + c_{44})k_{(n)}\,, \\ A_3^{(n)} &= (c_{11} - c_{44}k_{(n)}^2 - \varrho v^2)\,. \end{aligned} \tag{10.100}$$

The boundary conditions at the surface of the semiinfinite body are

$$T_{31} = T_{32} = T_{33} = 0 \qquad \text{at} \quad x_3 = 0\,, \tag{10.101}$$

and everything vanishes as $x_3 \to \infty$. Since $T_{32} \equiv 0$, one condition is satisfied identically, and two remain. Hence we take

$$\begin{aligned} u_1 &= (\sin \xi x_1) \sum_{n=1}^{2} C^{(n)} A_1^{(n)} \exp(-\eta_{(n)} x_3)\,, \\ u_3 &= (\cos \xi x_1) \sum_{n=1}^{2} C^{(n)} A_3^{(n)} \exp(-\eta_{(n)} x_3)\,, \end{aligned} \tag{10.102}$$

as the solution of the problem. Substituting from (10.74) and (10.102) into the nontrivial boundary conditions (10.101), we obtain

$$\begin{aligned} &\sum_{n=1}^{2} C^{(n)}(A_3^{(n)} + k_n A_1^{(n)}) = 0\,, \\ &\sum_{n=1}^{2} C^{(n)}(c_{13}A_1^{(n)} - c_{33}k_{(n)}A_3^{(n)}) = 0\,. \end{aligned} \tag{10.103}$$

Equations (10.103) constitute a system of linear homogeneous algebraic equations in the $C^{(n)}$. This system yields nontrivial solutions when the determinant of the coefficients of $C^{(1)}$ and $C^{(2)}$ vanish, i.e., when

$$\begin{vmatrix} (A_3^{(1)} + k_1 A_1^{(1)}) & (A_3^{(2)} + k_2 A_1^{(2)}) \\ (c_{13}A_1^{(1)} - c_{33}k_1 A_3^{(1)}) & (c_{13}A_1^{(2)} - c_{33}k_2 A_3^{(2)}) \end{vmatrix} = 0\,. \tag{10.104}$$

This equation may be regarded as an algebraic equation in k_1, k_2, and v, or as a higher-order algebraic equation in v alone. A number of values of v satisfy the system. However, only one value of v turns out to yield a

physically permissible solution, i.e., a solution with real v and such that all variables vanish at infinity. To my knowledge, this has not been proven in general, but it usually works out this way. It has been proven for the isotropic case, and Synge has shown that at times there may be no physically meaningful solution. The v that yields the physically meaningful solution is called the Rayleigh surface wave velocity.

The surface wave velocity v may now be calculated by selecting a value of v and calculating the corresponding k_1 and k_2 from (10.99). After this the $A_1^{(n)}$ and $A_3^{(n)}$ may be calculated from (10.100). All quantities appearing in the determinant (10.104) are now known, and it either does or does not vanish. If it vanishes, the v which has been selected is a root, and if it does not vanish, change v and repeat the calculation until the determinant does vanish. When you have a root such that v is real, then k_1 and k_2 must be checked to assure that the solution vanishes as $x_3 \to \infty$. The one physically acceptable solution, when it exists, corresponds to a v_R which is almost always less than the lowest plane wave velocity in the infinite medium in the propagation (here x_1) direction. The surface wave velocity V_R will usually be in the vicinity of 0.9 of the lowest plane wave velocity in that direction. At any rate, that value provides a good starting point for the calculation.

Chapter 11

TWO-DIMENSIONAL STANDING WAVES IN PIEZOELECTRIC PLATES

1. SOLUTION FOR POLARIZED CERAMIC MATERIAL

We now consider the solution for two-dimensional standing waves in a piezoelectric plate ([32]) of the same material considered in Chapter 10. The surfaces of the plate are coated with infinitesimally thin electrodes which are shorted. Consider the following as a solution of the differential equations:

$$\begin{aligned} u_1 &= A_1 \cos \eta x_3 \sin \xi x_1 \cos \omega t\,, \\ u_2 &= 0\,, \\ u_3 &= A_3 \sin \eta x_3 \cos \xi x_1 \cos \omega t\,, \\ \varphi &= B \sin \eta x_3 \cos \xi x_1 \cos \omega t\,, \end{aligned} \tag{11.1}$$

which satisfies the differential equations (7.29) provided

$$\begin{aligned} &(c_{11}\xi^2 + c_{44}\eta^2 - \varrho\omega^2)A_1 + (c_{13} + c_{44})\xi\eta A_3 + (e_{31} + e_{15})\xi\eta B = 0\,, \\ &(c_{44} + c_{13})\xi\eta A_1 + (c_{44}\xi^2 + c_{33}\eta^2 - \varrho\omega^2)A_3 + (e_{15}\xi^2 + e_{33}\eta^2)B = 0\,, \\ &(e_{15} + e_{31})\xi\eta A_1 + (e_{15}\xi^2 + e_{33}\eta^2)A_3 - (\varepsilon_{11}\xi^2 + \varepsilon_{33}\eta^2)B = 0\,. \end{aligned} \tag{11.2}$$

This system of linear homogeneous equations in A_1, A_3, and B yields nontrivial solutions when the determinant of the coefficients vanishes, i.e., when

$$\begin{vmatrix} (c_{11}\xi^2 + c_{44}\eta^2 - \varrho\omega^2) & (c_{13} + c_{44})\xi\eta & (e_{31} + e_{15})\xi\eta \\ (c_{44} + c_{13})\xi\eta & (c_{44}\xi^2 + c_{33}\eta^2 - \varrho\omega^2) & (e_{15}\xi^2 + e_{33}\eta^2) \\ (e_{15} + e_{31})\xi\eta & (e_{15}\xi^2 + e_{33}\eta^2) & -(\varepsilon_{11}\xi^2 + \varepsilon_{33}\eta^2) \end{vmatrix} = 0\,. \tag{11.3}$$

Equation (11.3) is quadratic in ω^2, but cubic in ξ^2 and η^2. Hence for a given ω and ξ there are three η (η_1, η_2, η_3), each of which yields amplitude ratios when substituted in any two of the three linear algebraic equations

(11.2) in A_1, A_3, and B. The amplitude ratios will be designated by

$$(A_1^{(n)}:A_3^{(n)}:B^{(n)}) = (\beta_1^{(n)}:\beta_3^{(n)}:\beta_2^{(n)}), \tag{11.4}$$

and are readily determined so that *there are no denominators* in the expressions. The above determinantal equation (11.3) clearly shows that the piezoelectric constants couple the quasistatic electric solution to the dynamic mechanical solution. These two solutions uncouple only if all the e_{ip} vanish. It is precisely the coupling of these two solutions that is ignored in purely mechanical treatments of this problem. This coupling of a quasistatic phenomenon to a dynamic phenomenon is the underlying reason that although in this instance a given wave normal in the x_1–x_3 plane results in only two phase velocities, a given frequency and propagation wave number results in three thickness wave numbers. The existence of these three thickness wave numbers is what enables the solution of the problem. The boundary conditions at each surface of the plate are

$$T_{31} = T_{32} = T_{33} = \varphi = 0 \qquad \text{at} \quad x_3 = \pm h. \tag{11.5}$$

Since according to (7.28) and (11.1) $T_{32} \equiv 0$, we have three boundary conditions to satisfy at each surface of the plate. Clearly, all three solutions of the differential equations are required in order to satisfy the remaining conditions. Hence we take

$$\begin{aligned} u_1 &= \cos \omega t \sin \xi x_1 \sum_{n=1}^{3} C^{(n)} \beta_1^{(n)} \cos \eta_n x_3, \\ u_3 &= \cos \omega t \cos \xi x_1 \sum_{n=1}^{3} C^{(n)} \beta_3^{(n)} \sin \eta_n x_3, \\ \varphi &= \cos \omega t \cos \xi x_1 \sum_{n=1}^{3} C^{(n)} \beta_2^{(n)} \sin \eta_n x_3, \end{aligned} \tag{11.6}$$

as the solution of the problem. Substituting from (11.6) into (7.28) and from (7.28) into the boundary conditions (11.5), we obtain

$$\begin{aligned} \sum_{n=1}^{3} C^{(n)} L_1^{(n)} \sin \eta_n h &= 0, \\ \sum_{n=1}^{3} C^{(n)} L_2^{(n)} \cos \eta_n h &= 0, \\ \sum_{n=1}^{3} C^{(n)} \beta_2^{(n)} \sin \eta_n h &= 0, \end{aligned} \tag{11.7}$$

where

$$\begin{aligned} L_1^{(n)} &= c_{44}(\beta_3^{(n)}\xi + \beta_1^{(n)}\eta_n) + e_{15}\beta_2^{(n)}\xi\,, \\ L_2^{(n)} &= c_{13}\beta_1^{(n)}\xi + c_{33}\beta_3^{(n)}\eta_n + e_{33}\beta_2^{(n)}\eta_n\,. \end{aligned} \tag{11.8}$$

The above system of homogeneous linear algebraic equations, (11.7), in $C^{(1)}$, $C^{(2)}$, and $C^{(3)}$ yields nontrivial solutions when the determinant of the coefficients of the $C^{(n)}$ vanishes, i.e., when

$$\begin{vmatrix} L_1^{(1)} \sin \eta_1 h & L_1^{(2)} \sin \eta_2 h & L_1^{(3)} \sin \eta_3 h \\ L_2^{(1)} \cos \eta_1 h & L_2^{(2)} \cos \eta_2 h & L_2^{(3)} \cos \eta_3 h \\ \beta_2^{(1)} \sin \eta_1 h & \beta_2^{(2)} \sin \eta_2 h & \beta_2^{(3)} \sin \eta_3 h \end{vmatrix} = 0\,. \tag{11.9}$$

Equation (11.9) is a transcendental equation, the roots of which enable the determination of the ω *vs.* ξ relation for this piezoelectric plate. The procedure for making a calculation is similar to, but more complicated than, that in the purely mechanical case, and should be obvious. However, it should be noted that in putting the equations in dimensionless form for a calculation, the dimensionless constants are most conveniently defined by

$$\bar{c}_{pq} = c_{pq}/c_{44}\,, \qquad \bar{e}_{ip} = e_{ip}/(c_{44}\varepsilon_{33})^{1/2}, \qquad \bar{\varepsilon}_{ij} = \varepsilon_{ij}/\varepsilon_{33}\,. \tag{11.10}$$

In addition, note that the transcendental determinant (11.9) can be written in the more familiar form

$$\sum_{n=1}^{3} M_n \cot \eta_n h = 0\,, \tag{11.11}$$

where the M_n are composed of combinations of $L_1^{(n)}$, $L_2^{(n)}$, and $\beta_2^{(n)}$. The determinantal form (11.9) is better suited to a calculation, however, since it does not contain terms which sometimes diverge.

2. LIMITING ROOTS

No complete calculation for the determination of the roots of (11.9) has been performed to date. However, we shall determine the important critical information for starting a calculation and note some of the similarities and differences between this and the purely elastic case. First we determine the thickness frequencies. This solution at infinite wavelength can be obtained directly from the previous solution simply by setting $\xi = 0$. However, the procedure is not quite as straightforward as in the purely elastic case. Consequently, we will present the details. When ξ is set equal to zero the set of algebraic equations, (11.2), in A_1, A_3, and B and the

corresponding determinant (11.3) become, respectively,

$$\begin{aligned}(c_{44}\eta^2 - \varrho\omega^2)A_1 &= 0,\\ (c_{33}\eta^2 - \varrho\omega^2)A_3 + e_{33}\eta^2 B &= 0,\\ e_{33}\eta^2 A_3 - \varepsilon_{33}\eta^2 B &= 0,\end{aligned} \tag{11.12}$$

$$\begin{vmatrix} (c_{44}\eta^2 - \varrho\omega^2) & 0 & 0 \\ 0 & (c_{33}\eta^2 - \varrho\omega^2) & e_{33}\eta^2 \\ 0 & e_{33}\eta^2 & -\varepsilon_{33}\eta^2 \end{vmatrix} = 0. \tag{11.13}$$

The determinant (11.13) readily yields the three η_n, which are given by

$$\eta_1 = \omega/(c_{44}/\varrho)^{1/2}, \qquad \eta_2 = 0, \qquad \eta_3 = \omega/(\bar{c}_{33}/\varrho)^{1/2}, \tag{11.14}$$

where

$$\bar{c}_{33} = c_{33} + (e_{33}^2/\varepsilon_{33}). \tag{11.15}$$

The substitution of the three η_n in (11.14) successively into the algebraic equations (11.12) yields

$$\beta_j^{(n)} = \left\| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & e_{33}/\varepsilon_{33} \\ 0 & 0 & 1 \end{matrix} \right\|, \tag{11.16}$$

where the column refers to the η_n and the row to the u_j or φ. The linear equations (11.7) in the $C^{(n)}$ now become

$$\begin{aligned} C^{(1)}c_{44}\eta_1 \sin \eta_1 h &= 0,\\ C^{(2)}e_{33}\eta_2 \cos \eta_2 h + C^{(3)}[c_{33} + (e_{33}^2/\varepsilon_{33})]\eta_3 \cos \eta_3 h &= 0,\\ C^{(2)} \sin \eta_2 h + C^{(3)}(e_{33}/\varepsilon_{33}) \sin \eta_3 h &= 0.\end{aligned} \tag{11.17}$$

The last two equations in (11.17) show that as $\eta_2 \to 0$, $C^{(2)} \to \infty$, since $\sin \eta_3 h$ and $\cos \eta_3 h$ cannot be equal to zero simultaneously. However, in (11.17) $C^{(2)}$ and η_2 always occur as an indeterminate product. Hence Eqs. (11.17) may be written

$$\begin{aligned} C^{(1)}c_{44}\eta_1 \sin \eta_1 h &= 0,\\ Ke_{33} + C^{(3)}\bar{c}_{33}\eta_3 \cos \eta_3 h &= 0,\\ Kh + C^{(3)}(e_{33}/\varepsilon_{33}) \sin \eta_3 h &= 0,\end{aligned} \tag{11.18}$$

where

$$K \equiv C^{(2)}\eta_2. \tag{11.19}$$

Equations (11.18) yield a nontrivial solution when

$$\begin{vmatrix} c_{44}\eta_1 \sin \eta_1 h & 0 & 0 \\ 0 & e_{33} & \bar{c}_{33}\eta_3 \cos \eta_3 h \\ 0 & h & (e_{33}/\varepsilon_{33}) \sin \eta_3 h \end{vmatrix} = 0. \tag{11.20}$$

This transcendental determinant (11.20) yields the two transcendental equations

$$\sin \eta_1 h = 0, \qquad \tan \eta_3 h = \eta_3 h \varepsilon_{33} \bar{c}_{33}/e_{33}^2 = \eta_3 h/k_t^2. \tag{11.21}$$

Thus we see that the first transcendental equation yields thickness frequencies and modes which are identical with the purely elastic, symmetric, thickness-shear frequencies and modes. We also see that the second transcendental equation yields thickness frequencies and modes which are identical with those of the piezoelectric, antisymmetric, thickness-stretch solution which we have obtained previously. These two sets of roots determine the starting points of the dispersion curves on the Ω axis ($\gamma = 0$ line).

The three wave velocities for propagation in any direction in an arbitrarily anisotropic crystal have been obtained previously. Hence the lowest velocity for our direction may be obtained easily. The determination of the surface wave velocity is straightforward and, by virtue of Chapter 10, Section 8, the procedure should be obvious. The procedure for determining the intersections of the complex branches with the plane $\Omega = 0$ is the same as in the anisotropic, purely elastic case and is straightforward. Onoe's theorem concerning the intersection of the complex branches with the real ($\operatorname{Im} \gamma = 0$) and the imaginary ($\operatorname{Re} \gamma = 0$) planes remains valid, since the proof covers this case also. We have still to determine the behavior of the lowest branches in the real plane as $\gamma \to 0$ ($\Omega \to 0$ as $\gamma \to 0$). Although we can proceed with the approximation in an essentially similar (*but somewhat different*) manner in this piezoelectric case as in the elastic case, we will not do so now because the analysis is a little complicated, and, more importantly, we will obtain the approximate equations automatically a little later on and to obtain them now also seems a waste of time.

3. PLATE OF INFINITESIMAL WIDTH

In the foregoing we have considered two-dimensional standing waves in *infinitely wide* as well as infinitely long plates. We will now show that the solution we have presented is also applicable in the case of standing

waves in an *infinitesimally wide* (i.e., very narrow) and infinitely long plate, provided some changes are made in the elastic, piezoelectric, and dielectric constants.

We now take the coordinate system in the same way as before, as shown in Fig. 18, so that the additional faces of the plate are at $x_2 = \pm w$. The boundary conditions on the additional faces of the plate are, assuming a high enough dielectric constant,

$$T_2 = T_4 = T_6 = D_2 = 0 \qquad \text{at} \quad x_2 = \pm w. \tag{11.22}$$

The proper assumptions for such a plate in which $w/h \ll 1$ are

$$T_2 = T_4 = T_6 = D_2 = 0, \tag{11.23}$$

for all x_i, along with

$$\varrho\omega^2 u_2 \approx 0. \tag{11.24}$$

Introducing the assumptions (11.23) into the constitutive equations (7.28), we obtain after elimination of certain terms

$$\begin{aligned}
T_1 &= c_{11}^P u_{1,1} + c_{13}^P u_{3,3} + e_{31}^P \varphi_{,3}, \\
T_3 &= c_{13}^P u_{1,1} + c_{33}^P u_{3,3} + e_{33}^P \varphi_{,3}, \\
T_5 &= c_{44} u_{3,1} + c_{44} u_{1,3} + e_{15} \varphi_{,1}, \\
D_1 &= e_{15} u_{3,1} + e_{15} u_{1,3} - \varepsilon_{11} \varphi_{,1}, \\
D_3 &= e_{31}^P u_{1,1} + e_{33}^P u_{3,3} - \varepsilon_{33}^P \varphi_{,3}, \\
T_2 &= T_4 = T_6 = D_2 = 0,
\end{aligned} \tag{11.25}$$

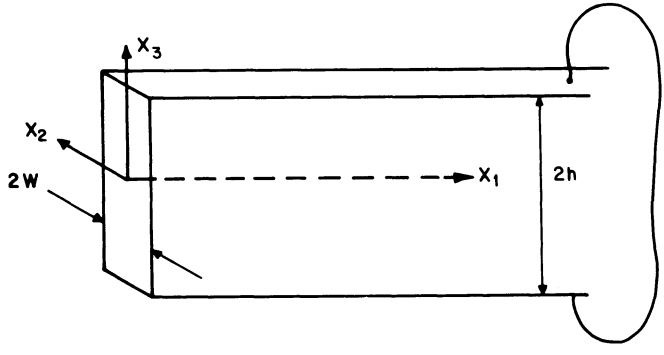

Fig. 18. Plate of infinitesimal width and infinite length between shorted electrodes.

where

$$\begin{aligned} c_{11}^P &= [c_{11} - (c_{12}^2/c_{11})], & c_{13}^P &= [c_{13} - (c_{12}c_{13}/c_{11})], \\ e_{31}^P &= [e_{31} - (c_{12}e_{31}/c_{11})], & c_{33}^P &= [c_{33} - (c_{13}^2/c_{11})], \\ e_{33}^P &= [e_{33} - (c_{13}e_{31}/c_{11})], & \varepsilon_{33}^P &= [\varepsilon_{33} + (e_{31}^2/c_{11})], \end{aligned} \tag{11.26}$$

and the constants without the superscript remain unchanged. The substitution of the constitutive equations (11.25) along with the unused assumption (11.24) into the equations

$$\begin{aligned} T_{1,1} + T_{6,2} + T_{5,3} + \varrho\omega^2 u_1 &= 0, \\ T_{6,1} + T_{2,2} + T_{4,3} + \varrho\omega^2 u_2 &= 0, \\ T_{5,1} + T_{4,2} + T_{3,3} + \varrho\omega^2 u_3 &= 0, \\ D_{1,1} + D_{2,2} + D_{3,3} &= 0, \end{aligned} \tag{11.27}$$

yields a system of equations which are identical, except for the modification of the constants, with the equations we used before in the case of the infinitely wide ($2w = \infty$) plate, i.e., Eqs. (7.28) and (7.29) with $u_2 = 0$ and no x_2 dependence. The nontrivial boundary conditions on the electroded surfaces $x_3 = \pm h$ remain the same, and are given in (11.5). The boundary conditions (11.22) on the additional surfaces are, of course, satisfied identically by virtue of the assumptions. Thus, since the nontrivial equations and boundary conditions are identical with those used in the case of the infinitely wide plate, that analysis applies without change to the thin (narrow) plate provided the c_{pq}, e_{ip}, and ε_{ij} are replaced, respectively, by the c_{pq}^P, e_{ip}^P, and ε_{ij}^P which we have defined. It should be noted that for the analysis of the thin plate which we have presented to be applicable the frequency must be well below the lowest width frequency because of the assumptions we have made. It should be noted further that these *plate* equations which we have presented are the piezoelectric counterparts of the (so-called) plane-stress equations of the theory of elasticity.

Since the entire solution for the infinitely wide plate is valid in the case of the infinitesimally wide plate if the constants are changed, the thickness solution must, of course, also be valid. Thus we obtain the transcendental equation

$$\tan \eta_3 h = \frac{\eta_3 h \varepsilon_{33}^P \bar{c}_{33}^P}{(e_{33}^P)^2} = \frac{\eta_3 h}{(k_{33}^P)^2}, \tag{11.28}$$

where

$$\eta_3 = \omega/(\bar{c}_{33}^P/\varrho)^{\frac{1}{2}}, \tag{11.29}$$

and

$$\bar{c}_{33}^{P} = c_{33}^{P} + [(e_{33}^{P})^2/\epsilon_{33}^{P}]. \tag{11.30}$$

The transcendental equation (11.28) governs the piezoelectric thickness-stretch modes for a plate which is narrow in one direction and very long in another. These modes can, of course, be driven by the application of an alternating voltage to the surface electrodes. It is evident that we have obtained a piezoelectric coupling factor k_{33}^{P} which is neither the one for the infinite medium nor the one for the thin rod.

If we were to go further and introduce additional boundaries at $x_1 = \pm l$ and make the assumption that $l/h \ll 1$ as well as $w/h \ll 1$, we would introduce the conditions

$$T_1 = T_5 = T_6 = D_1 = 0,$$

and obtain the equations, resonant frequencies, and coupling factor for the thin rod with a high dielectric constant. The procedure is obvious.

PART III

APPROXIMATION TECHNIQUES AND APPLICATIONS

Chapter 12

EXPANSION IN PLATE EIGENSOLUTIONS

1. METHOD OF LEAST SQUARES

The method of least squares [([21]), Section 6.C] is a very powerful technique for obtaining approximate solutions to boundary value problems with a high degree of accuracy. Before proceeding with a problem of interest we will introduce the general idea by applying the procedure to Fourier series.

Supposing we wish to approximate $f(x)$ in the range $-l < x < l$ by a trigonometric series in the form

$$S_N(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos(n\pi x/l) + \sum_{n=1}^{N} B_n \sin(n\pi x/l). \tag{12.1}$$

To proceed with the method of least squares, we form the error ε_N,

$$\varepsilon_N = f(x) - S_N(x), \tag{12.2}$$

and following Gauss we calculate the mean-square error

$$M = (1/2l) \int_{-l}^{l} \varepsilon_N{}^2 \, dx, \tag{12.3}$$

and reduce M to a minimum through the choice of the only quantities left to us, i.e., the A_n and B_n. Note that although ε_N can be either positive or negative, $\varepsilon_N{}^2$ must always be positive so that there can be no cancellations from different Δx regions. That is why the minimization of M yields an excellent approximation. The mean-square error M will be a minimum if

$$\delta M = (1/l) \int_{-l}^{l} \varepsilon_N \, \delta\varepsilon_N \, dx = 0, \tag{12.4}$$

where

$$\delta\varepsilon_N = -\delta S_N = -\frac{\partial S_N}{\partial A_0}\delta A_0 - \sum_{m=1}^{N} \frac{\partial S_N}{\partial A_m}\delta A_m - \sum_{m=1}^{N} \frac{\partial S_N}{\partial B_m}\delta B_m, \tag{12.5}$$

and

$$\partial S_N/\partial A_0 = \tfrac{1}{2}, \quad \partial S_N/\partial A_m = \cos(m\pi x/l), \quad \partial S_N/\partial B_m = \sin(m\pi x/l). \tag{12.6}$$

Substituting from (12.5) and (12.6) into (12.4), we obtain

$$-\frac{1}{l}\int_{-l}^{l}\left[\tfrac{1}{2}\varepsilon_N\,\delta A_0 + \varepsilon_N \sum_{m=1}^{N} \cos\frac{m\pi x}{l}\,\delta A_m + \varepsilon_N \sum_{m=1}^{N} \sin\frac{m\pi x}{l}\,\delta B_m\right] dx = 0, \tag{12.7}$$

from which, since all the A_k and B_k are independent, we obtain the equations

$$\begin{aligned} (1/l)\int_{-l}^{l} (S_N - f)\cos(m\pi x/l)\,dx &= 0, \qquad m = 0, 1, 2, \ldots N, \\ (1/l)\int_{-l}^{l} (S_N - f)\sin(m\pi x/l)\,dx &= 0, \qquad m = 1, 2, \ldots N. \end{aligned} \tag{12.8}$$

Equations (12.8) constitute $2N + 1$ linear algebraic equations for the determination of the $2N + 1$ unknowns A_k, B_k. In this particular application of the method of least squares a tremendous simplification of the algebra occurs because of the orthogonality of the trigonometric functions, which enables each A_k and B_k to be determined separately by an independent equation. In many situations—as in the ones we will be concerned with—this simplification does not occur, but the procedure is still applicable and the algebra simply becomes more cumbersome.

For the trigonometric functions the orthogonality relations are

$$\begin{aligned} \int_{-l}^{l} \cos(m\pi x/l)\sin(n\pi x/l)\,dx &= 0, \\ \int_{-l}^{l} \cos(m\pi x/l)\cos(n\pi x/l)\,dx &= l\,\delta_{mn}, \\ \int_{-l}^{l} \sin(m\pi x/l)\sin(n\pi x/l)\,dx &= l\,\delta_{mn}. \end{aligned} \tag{12.9}$$

Hence from (12.8) and (12.9) for the coefficients of the trigonometric series in (12.1) we obtain

$$A_m = \frac{1}{l}\int_{-l}^{l} f(x)\cos(m\pi x/l)\,dx, \qquad B_m = \frac{1}{l}\int_{-l}^{l} f(x)\sin(m\pi x/l)\,dx. \tag{12.10}$$

We could go on and show that the series is complete [([15]), Chapter II,

Section 1.3] since $M \to 0$ as $N \to \infty$, but mathematical considerations of this nature are not of interest to us here.

Now suppose we wish to approximate a function in the range $-l<x<l$ by a trigonometric series when we are given the function $f(x)$ for $-l<x<a$ and the slope $f'(x) \equiv g(x)$ for $a < x < l$. The series is the same as before, but the error term changes. We now must define two error functions $\varepsilon_N^{\mathrm{I}}$ and $\varepsilon_N^{\mathrm{II}}$ such that

$$\begin{aligned} \varepsilon_N^{\mathrm{I}} &= f(x) - S_N(x) && -l < x < a, \\ \varepsilon_N^{\mathrm{II}} &= [g(x) - S_N'(x)]a && a < x < l, \end{aligned} \tag{12.11}$$

where we have introduced the a for dimensional purposes, and

$$S_N' = -\sum_{n=1}^{N} \frac{n\pi}{l} A_n \sin\frac{n\pi x}{l} + \sum_{n=1}^{N} \frac{n\pi}{l} B_n \cos\frac{n\pi x}{l}. \tag{12.12}$$

We now form the mean-square error

$$M = \frac{1}{l+a}\int_{-l}^{a} (\varepsilon_N^{\mathrm{I}})^2\,dx + \frac{1}{l-a}\int_{a}^{l} (\varepsilon_N^{\mathrm{II}})^2\,dx, \tag{12.13}$$

proceed as before, and obtain

$$\frac{1}{l+a}\int_{-l}^{a}(S_N - f)\cos\frac{m\pi x}{l}\,dx - \frac{a^2}{l-a}\int_{a}^{l}(S_N' - g)\frac{m\pi}{l}\sin\frac{m\pi x}{l}\,dx = 0 \tag{12.14}$$

$$\frac{1}{l+a}\int_{-l}^{a}(S_N - f)\sin\frac{m\pi x}{l}\,dx + \frac{a^2}{l-a}\int_{a}^{l}(S_N' - g)\frac{m\pi}{l}\cos\frac{m\pi x}{l}\,dx = 0 \tag{12.15}$$

$$m = 0, 1, 2, \ldots, N.$$

Equations (12.14)–(12.15) constitute a system of $2N + 1$ linear algebraic equations for the $2N + 1$ unknowns A_k, B_k. However, in this case the orthogonality relations are of no use to us because of the limits on the integrals, and the algebraic equations have to be inverted in the usual manner to find the A_k and B_k. Note that each time the number N is changed all the A_k and B_k change. However, the solution (S_N) will usually converge, and the solution will not change much after a certain $N = N_0$. We are here bordering on some complicated questions relating to the convergence of series, which questions are answered by concepts from modern functional analysis relating to the theory of the Lebesgue integral ([33]). We shall not concern ourselves with questions of this nature.

2. FORCED VIBRATIONS OF A BOUNDED PIEZOELECTRIC PLATE BY THE METHOD OF LEAST SQUARES

Consider the plate shown in Fig. 19, and let the top and bottom surfaces be completely coated with infinitesimally thin electrodes, while the right and left surfaces are exposed. The plate may be either infinitely wide in the direction out of the paper or infinitesimally narrow with the additional faces free and without electrodes. As we have already shown, the only difference between the two cases is in the specific values of the constants in the equations. The plate is driven into vibration by the application of an alternating potential difference to the surface electrodes. For definiteness we will be considering the plate which is infinite in the direction out of the paper.

From Eqs. (7.28) and (7.29) the pertinent nontrivial equations are

$$T_{1,1} + T_{5,3} = \varrho \ddot{u}_1, \quad T_{5,1} + T_{3,3} = \varrho \ddot{u}_3, \quad D_{1,1} + D_{3,3} = 0, \tag{12.16}$$

and the constitutive equations are

$$\begin{aligned} T_1 &= c_{11}u_{1,1} + c_{13}u_{3,3} + e_{31}\varphi_{,3}, & T_3 &= c_{13}u_{1,1} + c_{33}u_{3,3} + e_{33}\varphi_{,3}, \\ T_5 &= c_{44}(u_{1,3} + u_{3,1}) + e_{15}\varphi_{,1}, & & \\ D_1 &= e_{15}(u_{1,3} + u_{3,1}) - \varepsilon_{11}\varphi_{,1}, & D_3 &= e_{31}u_{1,1} + e_{33}u_{3,3} - \varepsilon_{33}\varphi_{,3}, \end{aligned} \tag{12.17}$$

and simple substitution of (12.17) in (12.16) yields three differential equations in the three variables u_1, u_3, and φ. The boundary conditions are

$$\begin{aligned} &T_1 = T_5 = 0, \quad \varphi = \pm \varphi_0 \cos \omega t && \text{at} \quad x_1 = \pm h, \\ &T_3 = T_5 = D_3 = 0 && \text{at} \quad x_3 = \pm l. \end{aligned} \tag{12.18}$$

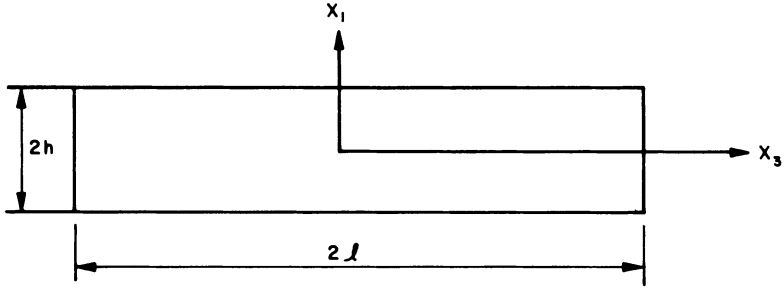

Fig. 19. A bounded plate of length $2l$ and thickness $2h$.

Our previous work has shown that the thickness solution satisfies the conditions

$$T_1 = T_5 = 0, \qquad \varphi = \pm \varphi_0 \cos \omega t \qquad \text{at} \quad x_1 = \pm h, \tag{12.19}$$

and the two-dimensional standing waves satisfy the conditions

$$T_1 = T_5 = \varphi = 0 \qquad \text{at} \quad x_1 = \pm h. \tag{12.20}$$

Hence the thickness solution plus any number of the two-dimensional solutions satisfy, at the driving frequency ω, the prescribed conditions on $x_1 = \pm h$. Our procedure then consists of taking a sum of the aforementioned solutions with undetermined coefficients and satisfying the boundary conditions on $x_3 = \pm l$ approximately by the method of least squares.

The thickness solution is given by

$$\begin{gathered} u_1 = u_2 = 0, \qquad u_3 = A \sin \eta_0 x_1, \\ \varphi = B (\sin \eta_0 x_1) + C x_1, \end{gathered} \tag{12.21}$$

where we have ignored the time factor $\cos \omega t$, and

$$\begin{gathered} \eta_0 = \omega/(\bar{c}_{44}/\varrho)^{1/2}, \qquad \bar{c}_{44} = c_{44} + [(e_{15})^2/\varepsilon_{11}], \\ B = e_{15} A/\varepsilon_{11}, \qquad C = (\varphi_0/h) - (e_{15} A/\varepsilon_{11} h) \sin \eta_0 h, \\ A[\bar{c}_{44} \eta_0 h \cos \eta_0 h - (e_{15}^2/\varepsilon_{11}) \sin \eta_0 h] = - e_{15} \varphi_0. \end{gathered} \tag{12.22}$$

Thus, given an ω and a φ_0, the η_0, A, B, and C are known in terms of ω and φ_0 unless ω is a thickness frequency.

We are interested in the two-dimensional solutions in which φ is symmetric in x_3, since the driving field, and hence the thickness solution, are symmetric in x_3. These two-dimensional solutions take the form

$$\begin{aligned} u_1 &= \sin \xi x_3 \sum_{n=1}^{3} H^{(n)} \beta_1^{(n)} \cos \eta_{(n)} x_1, \\ u_2 &= 0, \\ u_3 &= \cos \xi x_3 \sum_{n=1}^{3} H^{(n)} \beta_3^{(n)} \sin \eta_{(n)} x_1, \\ \varphi &= \cos \xi x_3 \sum_{n=1}^{3} H^{(n)} \beta_2^{(n)} \sin \eta_{(n)} x_1, \end{aligned} \tag{12.23}$$

where for a given ξ and ω the $\eta_{(n)}$ are the three roots of

$$\begin{vmatrix} (c_{11}\eta^2 + c_{44}\xi^2 - \varrho\omega^2) & (c_{13} + c_{44})\eta\xi & (e_{31} + e_{15})\eta\xi \\ (c_{44} + c_{13})\eta\xi & (c_{44}\eta^2 + c_{33}\xi^2 - \varrho\omega^2) & (e_{15}\eta^2 + e_{33}\xi^2) \\ (e_{15} + e_{31})\eta\xi & (e_{15}\eta^2 + e_{33}\xi^2) & -(\varepsilon_{11}\eta^2 + \varepsilon_{33}\xi^2) \end{vmatrix} = 0. \tag{12.24}$$

The amplitude ratios $\beta_1^{(n)}$, $\beta_2^{(n)}$, and $\beta_3^{(n)}$ are determined in the usual way such that there are no denominators. For a given ω the correct ξ must satisfy

$$\begin{vmatrix} L_1^{(1)} \sin \eta_{(1)}h & L_1^{(2)} \sin \eta_{(2)}h & L_1^{(3)} \sin \eta_{(3)}h \\ L_2^{(1)} \cos \eta_{(1)}h & L_2^{(2)} \cos \eta_{(2)}h & L_2^{(3)} \cos \eta_{(3)}h \\ \beta_2^{(1)} \sin \eta_{(1)}h & \beta_2^{(2)} \sin \eta_{(2)}h & \beta_2^{(3)} \sin \eta_{(3)}h \end{vmatrix} = 0, \tag{12.25}$$

where

$$\begin{aligned} L_1^{(n)} &= c_{11}\beta_1^{(n)}\eta_{(n)} + c_{13}\beta_3^{(n)}\xi + e_{31}\beta_2^{(n)}\xi, \\ L_2^{(n)} &= c_{44}\beta_3^{(n)}\eta_{(n)} + c_{44}\beta_1^{(n)}\xi + e_{15}\beta_2^{(n)}\eta^{(n)}. \end{aligned} \tag{12.26}$$

The amplitude ratios $H^{(1)}$, $H^{(2)}$, and $H^{(3)}$ are determined in the usual way such that there are no denominators.

After the dispersion curves have been determined we must decide in how many two-dimensional eigensolutions we will expand in order to obtain an accurate solution. This approximation is crucial and cannot be made until *after all* the dispersion curves have been determined in a given ω–ξ region. Let us suppose that in the ω–ξ region of interest to us the dispersion curves look as shown in Fig. 20. We are interested in $\omega < A$. Other branches in the region $\omega < A$ — and there are an infinite number of such branches — have a very large Im ξ, so large that their influence on the frequency spectrum is negligible and they may be ignored. In particular, the complex branch emanating from B is assumed to have an Im ξ value well in excess of that of the essentially vertical branch, shown in the Im ξ plane, for all $\omega \leq A$. If this were not so, it would be ridiculous to include the vertical branch without also including the *two* complex-conjugate branches emanating from B. The essentially vertical branch is a consequence of the piezoelectric coupling and will be included in the analysis. It can subsequently be ignored simply by setting its amplitude equal to zero throughout. Obviously this could be done with the complex-conjugate branches emanating from B also, or any other branches for that matter. A branch shown in the position of the essentially vertical branch will probably be negligible, but we will include it at this point anyway.

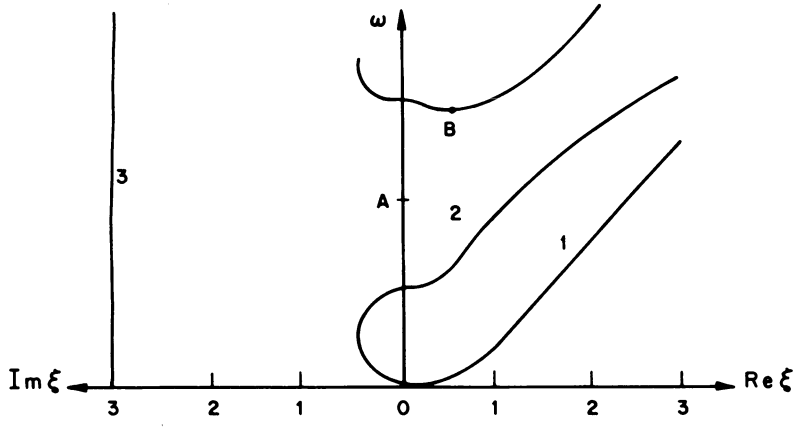

Fig. 20. Dispersion curves for the low antisymmetric modes of an infinite, polarized ceramic plate for real and imaginary wave numbers including the influence of piezoelectricity.

Thus, as far as we are concerned, there are three branches (or solutions) which we have to consider for any $\omega \leq A$ in order to obtain an accurate solution. The three branches are labeled 1, 2, 3 in the diagram. We now take the approximate solution in the form

$$u_1 = \sum_{m=1}^{3} K^{(m)} u_1^{(m)}, \qquad u_3 = \sum_{m=1}^{3} K^{(m)} u_3^{(m)} + A \sin \eta_0 x_1,$$
$$\varphi = \sum_{m=1}^{3} K^{(m)} \varphi^{(m)} + (e_{15}/\varepsilon_{11}) A(\sin \eta_0 x_1) + C x_1, \tag{12.27}$$

where

$$u_1^{(m)} = \sin \xi_{(m)} x_3 \sum_{n=1}^{3} H^{(nm)} \beta_1^{(nm)} \cos \eta_{(nm)} x_1,$$
$$u_3^{(m)} = \cos \xi_{(m)} x_3 \sum_{n=1}^{3} H^{(nm)} \beta_3^{(nm)} \sin \eta_{(nm)} x_1, \tag{12.28}$$
$$\varphi^{(m)} = \cos \xi_{(m)} x_3 \sum_{n=1}^{3} H^{(nm)} \beta_2^{(nm)} \sin \eta_{(nm)} x_1,$$

and η_0, A, and C are known in terms of ω and φ_0 from the thickness solution. We now express the quantities appearing in the remaining boundary conditions on $x_3 = \pm l$ in terms of the u_1, u_3, and φ shown above, form the mean-square error M, and minimize M by the appropriate selection of

the $K^{(m)}$. To this end, we write the expressions

$$\begin{aligned} T_3 &= c_{13}\sum_{m=1}^{3} K^{(m)}u_{1,1}^{(m)} + c_{33}\sum_{m=1}^{3} K^{(m)}u_{3,3}^{(m)} + e_{33}\sum_{m=1}^{3} K^{(m)}\varphi_{,3}^{(m)}, \\ T_5 &= c_{44}\sum_{m=1}^{3} K^{(m)}u_{1,3}^{(m)} + c_{44}\left(\sum_{m=1}^{3} K^{(m)}u_{3,1}^{(m)} + A\eta_0 \cos \eta_0 x_1\right) \\ &\quad + e_{15}\left[\sum_{m=1}^{3} K^{(m)}\varphi_{,1}^{(m)} + (e_{15}/\varepsilon_{11})\, A\eta_0(\cos \eta_0 x_1) + C\right], \\ D_3 &= e_{31}\sum_{m=1}^{3} K^{(m)}u_{1,1}^{(m)} + e_{33}\sum_{m=1}^{3} K^{(m)}u_{3,3}^{(m)} - \varepsilon_{33}\sum_{m=1}^{3} K^{(m)}\varphi_{,3}^{(m)}, \end{aligned} \tag{12.29}$$

and after the differentiations are performed we set $x_3 = l$ in the above and form

$$M = \frac{1}{2h}\int_{-h}^{h}\left[(T_3)^2 + (T_5)^2 + \left(\frac{c_{44}}{e_{15}}\, D_3\right)^2\right] dx_1, \tag{12.30}$$

where the factor c_{44}/e_{15} has been introduced for dimensional reasons. This factor is not unique; other factors with the same dimensionality could have been chosen. This is an undesirable feature of this procedure. The mean-square error M in (12.30) will be a minimum if

$$\delta M = \frac{1}{h}\int_{-h}^{h}\sum_{p=1}^{3}\left[T_3\frac{\partial T_3}{\partial K^{(p)}} + T_5\frac{\partial T_5}{\partial K^{(p)}} + \frac{c_{44}^2}{e_{15}^2}\, D_3\frac{\partial D_3}{\partial K^{(p)}}\right]\delta K^{(p)}\, dx_1 = 0. \tag{12.31}$$

Since all the $K^{(p)}$ are independent, we obtain

$$\int_{-h}^{h}\left[T_3\frac{\partial T_3}{\partial K^{(p)}} + T_5\frac{\partial T_5}{\partial K^{(p)}} + \frac{c_{44}^2}{e_{15}^2}\, D_3\frac{\partial D_3}{\partial K^{(p)}}\right] dx_1 = 0, \qquad p = 1, 2, 3, \tag{12.32}$$

which yields three linear inhomogeneous algebraic equations in the three $K^{(p)}$. This system can be solved for the $K^{(p)}$ unless we have resonance, in which case the $K^{(p)}$ diverge. The resonance condition occurs when the determinant of the coefficients of the $K^{(p)}$ vanishes. The vanishing of said determinant yields a transcendental frequency equation, the roots of which determine ω *vs.* l/h. The procedure for making a calculation should be obvious. The determination of the algebraic equations in the $K^{(p)}$ requires some straightforward if tedious effort. There is no point in presenting the final algebraic equations here, since they are quite lengthy, and by themselves provide no additional insight.

3. SMALL PIEZOELECTRIC COUPLING

If we had assumed small coupling, the problem considered in Section 2 of this chapter would simplify somewhat, in that we would have had

$$E_1 = -(\varphi_0/h)\cos\omega t, \qquad E_3 = 0, \tag{12.33}$$

so that a pure elasticity problem would remain with E_1 as the driving term. The solution would proceed in exactly the same way except that the electric differential equations and boundary conditions would be ignored, so that the thickness vibration and two-dimensional standing wave solutions would be purely elastic. Under these circumstances the purely imaginary vertical branch on the dispersion curves would not exist. The sums in the two-dimensional mode equations would be over 1 and 2, and both determinants would have the third row and column eliminated, and all piezoelectric constants would be eliminated from any equation pertaining to the two-dimensional standing waves. In the final solution equations the sum on m would be over 1 and 2, and φ would not appear. The expression for M would not contain D_3, and we would obtain two linear inhomogeneous algebraic equations in $K^{(1)}$ and $K^{(2)}$ in place of the three in the piezoelectric case.

It should be noted that in either case if an off-resonant steady-state solution is obtained, the charge on (and of course, the current through) the crystal may be obtained from the relation

$$Q = -b\int_{-l}^{l} D_1\Big]_{x_1=h} dx_3, \tag{12.34}$$

where b is the length into the paper, and the resonances do not depend on b. The time derivative of the above equation gives the relation between the voltage across and the current through the crystal. This is the relation needed to put the crystal in a circuit.

4. USE OF VARIATIONAL TECHNIQUES

In Chapter 6 [Eq. (6.20)] we derived the variational equation

$$\int_{t_0}^{t} dt\Big[\int_V (\tau_{kl,k} - \varrho\ddot{u}_l)\,\delta u_l\,dV + \int_V D_{k,k}\,\delta\varphi\,dV + \int_S (\bar{t}_l - n_k\tau_{kl})\,\delta u_l\,dS - \int_S (\bar{\sigma} + n_k D_k)\,\delta\varphi\,dS\Big] = 0. \tag{12.35}$$

We may employ this principle instead of the method of least squares to obtain an approximate solution to the problem treated in Section 2 of this

chapter. The variational procedure is very similar to the previous one except for the last step.

Exactly as in Section 2 we expand the solution in a sum of the thickness solution and the two-dimensional standing waves. Then all that remains in (12.35) is

$$- b \int_{-h}^{h} (T_{31}\, \delta u_1 + T_{33}\, \delta u_3)\, dx_1 - b \int_{-h}^{h} D_3\, \delta\varphi\, dx_1 = 0 \quad \text{at} \quad x_3 = \pm l, \tag{12.36}$$

since the plate eigenmodes, which satisfy the differential equations and boundary conditions at $x_1 = \pm h$, are such that all the other terms vanish. We then substitute in the above equation instead of (12.31) and equate the coefficients of each $\delta K^{(m)}$ in (12.36) to zero in order to obtain the three equations in the $K^{(m)}$.

Note that the technique presented in this section does not have the unfortunate nonuniqueness of the least-squares technique. However, if the boundary at, say, $x_3 = l$ of the plate in Fig. 19, is held rigidly, the variational equation (12.35) is not applicable when the solution consists of an expansion in plate eigenmodes because under such circumstances the solution functions u_k must vanish at $x_3 = l$ and the plate eigensolutions cannot. On the other hand, the least squares technique can be used when the aforementioned boundary cannot move, because with the use of least squares the solution functions u_k need not satisfy any conditions at $x_3 = l$. Nevertheless, in the boundary value problem we are now considering the least squares technique suffers from the additional dimensional difficulty that the mechanical displacement and stress terms appearing in the error equation are not naturally dimensionally compatible. These terms can be made dimensionally compatible by introducing a nonunique geometric factor (l or h?) and some nonunique combination of elastic constants (c_{44} or c_{33}?). However, the difficulties inherent in the variational equation (12.35) and the least squares technique are both eliminated if the modification of Hamilton's principle presented in Section 4 of Chapter 6 is employed. In this modification the variational equation that replaces (12.35) is (6.44). Since there is no surface of discontinuity in the problem being considered here, the integral over $S^{(d)}$ in (6.44) does not exist and the superscript (m) and attendant summation sign can be eliminated. Since the plate eigenmodes satisfy the differential equations and boundary conditions at $x_1 = \pm h$, all that remains of (6.44) is

$$\begin{aligned} &+ b \int_{-h}^{h} [T_{31}\, \delta u_1 + T_{33}\, \delta u_3]_{x_3=-l}\, dx_1 + b \int_{-h}^{h} [D_3\, \delta\varphi]_{x_3=-l}\, dx_1 \\ &+ b \int_{-h}^{h} [u_1\, \delta T_{31} + u_3\, \delta T_{33}]_{x_3=l}\, dx_1 - b \int_{-h}^{h} [D_3\, \delta\varphi]_{x_3=l}\, dx_1 = 0. \end{aligned} \tag{12.37}$$

At this point it should be noted that since the boundary conditions in the problem we are considering now are asymmetric with respect to x_3, the plate eigenmodes in which φ is symmetric in x_3 are no longer adequate and the eigensolutions in which φ is antisymmetric in x_3 are also required. We may now take both the symmetric and antisymmetric (in x_3) plate eigensolutions with amplitude coefficients $K^{(m)}$ and substitute in (12.37) and then equate the coefficients of each $\delta K^{(m)}$ in (12.37) to zero in order to obtain the same number of equations as of $K^{(m)}$.

If a surface of discontinuity were present in Fig. 19, we would expand the solution in two sets of plate eigensolutions, one for each region separated by the surface of discontinuity. The amplitude coefficients $K^{(m)}$ for each term of each set would be independent, and we would employ (6.44) with the integral over $S^{(d)}$ included, and, as in the immediately preceding discussion, obtain the requisite number of linear algebraic equations for the $K^{(m)}$.

It should be noted that variational formulations such as (6.17) or (6.20) and (6.43) or (6.44) can be used to obtain approximate solutions without expanding in plate eigenmodes. In other words, the expansion functions selected need not even satisfy the differential equations and boundary conditions on the major surfaces in addition to not satisfying the boundary conditions on the minor surfaces. Under such circumstances none of the integrals appearing in (6.20) and (6.44) can be eliminated. Although this latter approach can yield very accurate and useful information, this writer feels that it does not yield as much understanding and insight into the nature of plate vibrations as does the expansion in eigenmodes. Consequently, this latter approach will not be treated at all in this monograph. In recent years this approach has been fruitfully exploited by Eer Nisse and Holland ([34])* in treating the vibrations of complicated geometric structures.

* Also ([16,17]) and a number of other publications cited in these references.

Chapter 13

TWO-DIMENSIONAL PIEZOELECTRIC PLATE EQUATIONS

1. GENERAL PLATE EQUATIONS FROM A POWER SERIES EXPANSION

Since we are interested here in obtaining plate differential equations only, and not in establishing three-dimensional plate boundary conditions, we consider the volumetric portion of the variational principle (6.20) and ignore the surface portion. If we were to include the surface portion, we would obtain no additional terms in the resulting plate equations. The volumetric portion of the variational principle is

$$\int_{t_0}^{t} dt \int_V [(T_{ij,i} - \varrho \ddot{u}_j)\, \delta u_j + D_{i,i}\, \delta\varphi]\, dV = 0. \tag{13.1}$$

In addition to the variational principle we have the strain–mechanical displacement and electric field–electric potential relations, (5.5) and (5.4),

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{13.2}$$

$$E_i = -\varphi_{,i}, \tag{13.3}$$

respectively, and the linear piezoelectric constitutive relations, (5.19) and (5.20),

$$T_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k, \tag{13.4}$$

$$D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k. \tag{13.5}$$

The faces of the plate are at $x_2 = \pm h$ and the remaining boundary is a cylindrical surface with generators perpendicular to the faces as shown in Fig. 21. We now expand the mechanical displacement ([5,35,36]) u_j and electric displacement ([37]) D_i in a series of powers of the thickness coordinate x_2. Thus

$$u_j = \sum_{n=0}^{g} x_2^n u_j^{(n)}, \tag{13.6}$$

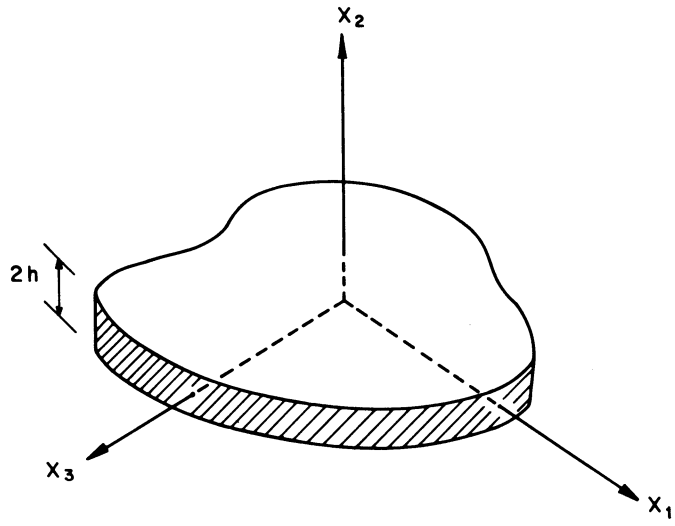

Fig. 21. Orientation of the coordinate system with respect to a bounded plate.

$$D_j = \sum_{n=0}^{g} x_2^n D_j^{(n)} , \tag{13.7}$$

where g is a positive integer and the variables $u_j^{(n)}$ and $D_j^{(n)}$ are independent of x_2 but are functions of x_1 and x_3, and $u_j^{(n)}$ and $D_j^{(n)}$ vanish for $n > g$. Successively higher values of g determine successively higher-order theories. When the series expansions (13.6) and (13.7) are substituted in the variational principle (13.1) and the integrations with respect to x_2 are performed we obtain

$$\int_{t_0}^{t} dt \int_A \sum_{n=0}^{g} \left[\left(T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} - \varrho \sum_{m=0}^{g} H_{mn} \ddot{u}_j^{(m)} \right) \delta u_j^{(n)} + (D_{i,i}^{(n)} + (n+1) D_2^{(n+1)}) \, \delta\varphi^{(n)} \right] dA = 0 , \tag{13.8}$$

where A is the area of the plate and we have introduced the definitions

$$T_{ij}^{(n)} \equiv \int_{-h}^{h} x_2^n T_{ij} \, dx_2 , \tag{13.9}$$

$$F_j^{(n)} \equiv [x_2^n T_{2j}]_{-h}^{h} , \tag{13.10}$$

$$\varphi^{(n)} \equiv \int_{-h}^{h} x_2^n \varphi \, dx_2 , \tag{13.11}$$

for the nth order component of stress resultant, the nth order component

of applied plate force per unit area, and the nth order potential resultant, respectively; and

$$H_{mn} \equiv \int_{-h}^{h} x_2{}^m x_2{}^n \, dx_2 = \frac{2h^{m+n+1}}{m+n+1}, \qquad m+n \text{ even}, \qquad (13.12)$$
$$= 0, \qquad m+n \text{ odd}.$$

Since the variations $\delta u_j^{(n)}$ and $\delta\varphi^{(n)}$ in (13.8) are arbitrary, we obtain

$$T_{ij,i}^{(n)} - nT_{2j}^{(n-1)} + F_j^{(n)} = \varrho \sum_{m=0}^{g} H_{mn} \ddot{u}_j^{(m)}, \qquad (13.13)$$

$$D_{i,i}^{(n)} + (n+1)D_2^{(n+1)} = 0, \qquad (13.14)$$

as the two-dimensional equations of motion and of electrostatics of order n. Note that the substitution of the power series expansion for the electric displacement D_i given in (13.7) into the three-dimensional equation of electrostatics, $D_{i,i} = 0$, yields

$$\sum_{n=0}^{g} x_2{}^n [D_{i,i}^{(n)} + (n+1)D_2^{(n+1)}] = 0,$$

which is satisfied identically by virtue of the form of the two-dimensional equations of electrostatics (13.14).

Substituting the power series expansion (13.6) for u_j in the three-dimensional expression (13.2) for the strain and rearranging terms, we obtain

$$S_{ij} = \sum_{n=0}^{g} x_2{}^n S_{ij}^{(n)}, \qquad (13.15)$$

where we have introduced the definition

$$S_{ij}^{(n)} \equiv \tfrac{1}{2}[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{2j} u_i^{(n+1)} + \delta_{2i} u_j^{(n+1)})], \qquad (13.16)$$

for the nth order component of strain. Now, we define the nth order component of electric field resultant by forming

$$E_i^{(n)} \equiv \int_{-h}^{h} x_2{}^n E_i \, dx_2, \qquad (13.17)$$

and we substitute from (13.3) into (13.17) to find

$$E_i^{(n)} \equiv -\varphi_{,i}^{(n)} - \delta_{2i}(\Phi^{(n)} - n\varphi^{(n-1)}), \qquad (13.18)$$

where we have introduced the definition

$$\Phi^{(n)} = [x_2{}^n\varphi]_{-h}^{h} \tag{13.19}$$

for the nth order component of applied voltage. All that remains now is the determination of the two-dimensional constitutive equations. To this end, we substitute the power series expansions (13.7) and (13.15) in the constitutive equations (13.4) and (13.5), multiply by $x_2{}^m\,dx_2$, integrate over the thickness, and employ (13.9) and (13.17), with the result

$$T_{ij}^{(m)} = c_{ijkl}\sum_{n=0}^{g} H_{mn}S_{kl}^{(n)} - e_{kij}E_k^{(m)}, \tag{13.20}$$

$$\sum_{n=0}^{g} H_{mn}(D_i^{(n)} - e_{ikl}S_{kl}^{(n)}) = \varepsilon_{ik}E_k^{(m)}, \tag{13.21}$$

which, with (13.12), shows that the nth order stress resultants and electric-field resultants depend on all the strains and electric displacements of even order if n is even and odd order if n is odd. If we define

$$H_{mn}^{-1} = (\text{cof } H_{mn})/|H_{mn}|, \tag{13.22}$$

$$\sum_{m=0}^{g} H_{mr}^{-1}H_{mn} = \delta_{rn}, \tag{13.23}$$

(13.21) may be written in the more useful form

$$D_i^{(n)} = e_{ikl}S_{kl}^{(n)} + \sum_{m=0}^{g} H_{mn}^{-1}\varepsilon_{ik}E_k^{(m)}. \tag{13.24}$$

Thus we have obtained a complete system of two-dimensional equations from the three-dimensional equations.

At this point it should be noted that the stress at a point cannot be determined from a solution of the plate equations because the equations are written in terms of the stress-resultants $T_{ij}^{(n)}$ and not the stresses T_{ij}. Similarly, we cannot obtain the electric potential at a point, but only the electric-potential resultants $\varphi^{(n)}$. In addition, the actual mode shapes cannot be determined from a solution because the x_2 dependence of u_j and D_j has been *assumed*. Thus it should be clear that we have given up all hope of obtaining a solution of the three-dimensional equations, and have replaced them by a larger system of two-dimensional equations which we can hope to solve in some cases. It is not surprising then that we are unable to obtain the aforementioned three-dimensional information from a solution of the two-dimensional equations. It is, of course, implicit that the information which we cannot obtain is not pertinent to the information we wish

to obtain. This assumption is valid in certain instances—such as the determination of frequency spectra—but not in all instances. It should be clear that we have replaced the philosophy of obtaining an approximate solution to an exact system of equations discussed in Chapter 12 by that of obtaining an exact solution to an approximate system of equations. Note that the $F_j^{(n)}$ are the inhomogeneous mechanical forcing terms in the equations and the $\Phi^{(n)}$ are the inhomogeneous electrical forcing terms. Hence the $\Phi^{(n)}$ are important even in the small coupling case, since they are the electrical driving terms which force the elastic solution. Now we define two-dimensional kinetic energy, internal energy, and electric enthalpy densities by substituting the appropriate series expansions in the respective volumetric densities and integrating through the thickness. Thus we obtain

$$\mathscr{K} \equiv \int_{-h}^{h} \tfrac{1}{2}\varrho \dot{u}_j \dot{u}_j \, dx_2 = \tfrac{1}{2}\varrho \sum_{m=0}^{g} \sum_{n=0}^{g} H_{mn} \dot{u}_j^{(m)} \dot{u}_j^{(n)}, \tag{13.25}$$

$$\mathscr{U} = \int_{-h}^{h} \tfrac{1}{2}(T_{ij}S_{ij} + E_iD_i) \, dx_2 = \tfrac{1}{2} \sum_{n=0}^{g} (T_{ij}^{(n)}S_{ij}^{(n)} + E_i^{(n)}D_i^{(n)}), \tag{13.26}$$

$$\mathscr{H} = \int_{-h}^{h} \tfrac{1}{2}(T_{ij}S_{ij} - E_iD_i) \, dx_2 = \tfrac{1}{2} \sum_{n=0}^{g} (T_{ij}^{(n)}S_{ij}^{(n)} - E_i^{(n)}D_i^{(n)}). \tag{13.27}$$

2. TRUNCATION OF SERIES FOR A SPECIFIC APPROXIMATION

The equations in Section 1 are not particularly useful unless they are truncated to form a finite system. Unfortunately, the truncation is not quite as straightforward as the expansion because it entails an approximation which is always a bit difficult because of its nature. In this section we will truncate in order to obtain a system that includes the lowest-order coupled extensional and flexural motion of the plate. However, before proceeding with the truncation it is enlightening to study the character of certain of the equations in Section 1 in some detail.

An examination of Eq. (13.8) reveals that in making an approximation we have the choice of either taking a $\delta u_j^{(n)} = 0$ and ignoring that nth order equation (13.13) or letting $\delta u_j^{(n)}$ be arbitrary and satisfying the nth order equation (13.13). Clearly, from (13.8) a similar relationship exists between a particular $\delta\varphi^{(n)}$ and the nth order equation (13.14). The specific choice in a given instance depends on some knowledge of the two-dimensional standing wave solutions discussed in Chapter 10 in the frequency-wave number range of interest. It is also worth noting that the nth order equations in (13.13) are coupled to the $(n-1)$th order equations through the $T_{2j}^{(n-1)}$.

Moreover if n is even, the nth order equation in (13.13) is coupled to all the even order equations for which $\delta u_j^{(n)}$ does not vanish by virtue of (13.12), the inertia terms on the right-hand side of Eqs. (13.13), and the strains of even order appearing on the right-hand side of Eqs. (13.20) for m even. Similar statements hold, of course, for m and n odd. In a similar way the nth order electrical equations are coupled together for all n's for which $\delta\varphi^{(n)}$ does not vanish by virtue of (13.24), (13.12), and (13.18). Thus it is clear that in making a truncation we will have the choice, with the highest order equations considered, of either including the equation and letting the associated $\delta u_j^{(n)}$ (or $\delta\varphi^{(n)}$) be arbitrary, or ignoring the equation and letting the associated $\delta u_j^{(n)}$ (or $\delta\varphi^{(n)}$) vanish. The choice will be made so as to optimize the approximation for the size of the system of equations obtained by an appeal to the character of the solutions discussed in Chapter 10.

We begin the truncation in which we are interested, and which includes coupled extensional and flexural motions, by first setting $g = 2$ in the equations of Section 1. At this stage we have three sets of Eqs. (13.13) and (13.14) corresponding to $n = 0, 1, 2$. This seems simple enough, but the truncation is not yet complete because this is far from an optimum approximation for such a large number (twelve) of dynamic equations. Clearly, we have

$$u_j^{(n)} = 0 \qquad \text{for} \quad n > 2\,, \tag{13.28}$$

$$D_j^{(n)} = 0 \qquad \text{for} \quad n > 2\,, \tag{13.29}$$

at this stage of the truncation. We now wish to eliminate the differential equations of order two from our approximation. For the mechanical equations (13.13) this can readily be accomplished by setting $\delta u_j^{(2)} = 0$. But such an approach will not yield an accurate description of the flexural motion because it prevents contraction during the extension accompanying flexure, which is undesirable. We would obtain a much more accurate description of the flexural motion by permitting the contraction during the extension accompanying flexure to take place *freely*. This can be accomplished by letting $\delta u_2^{(2)}$ be arbitrary and setting

$$T_{22}^{(1)} = 0\,, \tag{13.30}$$

and assuming $\ddot{u}_2^{(2)}$ to be negligible in the equations of motion (13.13). The other two mechanical displacement conditions

$$\delta u_1^{(2)} = 0, \qquad \delta u_3^{(2)} = 0\,, \tag{13.31}$$

are retained. Since the three-dimensional electric potential φ is most likely essentially antisymmetric in x_2 because of (13.17) and (13.3), we take

$$E_1^{(2)} = 0, \qquad E_3^{(2)} = 0, \tag{13.32}$$

and retain $D_a^{(2)}$, $a = 1,3$. This consideration will undoubtedly yield different conditions for the even and odd order truncations. From (13.32) and (13.18) it is clear that we have

$$\delta\varphi^{(2)} \equiv 0 \tag{13.33}$$

in this approximation, and we have thus eliminated the equation (13.14) for $n = 2$. At this stage from (13.13) and (13.14) we have a total of eight differential equations. From (13.20), (13.21), (13.28), and (13.29) we also have the constitutive equations

$$\begin{aligned} T_p^{(0)} &= 2hc_{pq}S_q^{(0)} - e_{ip}E_i^{(0)}, \\ T_p^{(1)} &= \tfrac{2}{3}h^3c_{pq}S_q^{(1)} - e_{ip}E_i^{(1)}, \end{aligned} \tag{13.34}$$

$$\begin{aligned} \tfrac{2}{3}h^3D_i^{(1)} &= \tfrac{2}{3}h^3e_{iq}S_q^{(1)} + \varepsilon_{ik}E_k^{(1)}, \\ 2hD_i^{(0)} + \tfrac{2}{3}h^3D_i^{(2)} &= 2he_{iq}S_q^{(0)} + \varepsilon_{ik}E_k^{(0)}, \\ \tfrac{2}{3}h^3D_i^{(0)} + \tfrac{2}{5}h^5D_i^{(2)} &= \tfrac{2}{3}h^3e_{iq}S_q^{(0)} + \varepsilon_{i2}E_2^{(2)}, \end{aligned} \tag{13.35}$$

where we have introduced the abbreviated indicial notation. The last two constitutive equations in (13.35) may be solved simultaneously for $D_j^{(0)}$ and $D_j^{(2)}$ to give

$$\begin{aligned} D_i^{(0)} &= e_{iq}S_q^{(0)} + \frac{9}{8h}\,\varepsilon_{ik}E_k^{(0)} - \frac{15}{8h^3}\,\varepsilon_{i2}E_2^{(2)}, \\ D_i^{(2)} &= \frac{15}{8h^3}\left(\frac{3}{h^2}\,\varepsilon_{i2}E_2^{(2)} - \varepsilon_{ik}E_k^{(0)}\right). \end{aligned} \tag{13.36}$$

This system of equations contains, among other things, the lowest thickness-stretch resonance, which Mindlin has shown [([36]); ([5]), Sections 5.01, 6.02; ([38])] can be eliminated without appreciably altering the range of validity of the approximation. This unnecessary thickness resonance can be eliminated by setting

$$T_{21}^{(1)} = 0, \qquad T_{23}^{(1)} = 0, \tag{13.37}$$

$$T_{22}^{(0)} = 0, \tag{13.38}$$

and neglecting $\ddot{u}_2^{(1)}$ in the equations of motion (13.13). This essentially completes the truncation.

The condition (13.38) permits the elimination of $S_2^{(0)}$ from the constitutive equations (13.34) and (13.35) with the result

$$T_p^{(0)} = 2hc_{pq}^* S_q^{(0)} - e_{ip}^* E_i^{(0)}, \tag{13.39}$$

$$D_i^{(0)} = e_{iq}^* S_q^{(0)} + (1/2h)\varepsilon_{ij}^* E_j^{(0)} - (15/8h^3)\varepsilon_{i2} E_2^{(2)}, \tag{13.40}$$

where

$$\begin{aligned} c_{pq}^* &= c_{pq} - (c_{p2}c_{2q}/c_{22}), \\ e_{ip}^* &= e_{ip} - (e_{i2}c_{p2}/c_{22}), \\ \varepsilon_{ij}^* &= \tfrac{9}{4}\varepsilon_{ij} + (e_{i2}e_{j2}/c_{22}). \end{aligned} \tag{13.41}$$

The conditions (13.30) and (13.37) permit the elimination of the $S_{2j}^{(1)}$ from (13.34) and (13.35) with the result

$$T_r^{(1)} = \tfrac{2}{3}h^3\gamma_{rs}S_s^{(1)} - \psi_{ir}E_i^{(1)}, \tag{13.42}$$

$$D_i^{(1)} = \psi_{ir}S_r^{(1)} + (3/2h^3)\zeta_{ij}E_j^{(1)}, \tag{13.43}$$

where with $i,j = 1, 2, 3$; $r,s = 1, 3, 5$; $v,w = 2, 4, 6$;

$$\begin{aligned} \gamma_{rs} &= c_{rs} - c_{rw}c_{vs}(c_{vw})^{-1}, \\ \psi_{ir} &= e_{ir} - e_{iv}c_{rw}(c_{vw})^{-1}, \\ \zeta_{ij} &= \varepsilon_{ij} + e_{iv}e_{jw}(c_{vw})^{-1}. \end{aligned} \tag{13.44}$$

Thus we now have the constitutive equations for this intermediate approximation.

At this stage our two-dimensional electric enthalpy density takes the form

$$\begin{aligned} \mathscr{H} &= \tfrac{1}{2}\,(T_p^{(0)}S_p^{(0)} + T_p^{(1)}S_p^{(1)} - E_i^{(0)}D_i^{(0)} - E_i^{(1)}D_i^{(1)} - E_2^{(2)}D_2^{(2)}) \\ &= \tfrac{1}{2}\,(2hc_{pq}^* S_p^{(0)}S_q^{(0)} - 2e_{ip}^* S_p^{(0)}E_i^{(0)} + \tfrac{2}{3}\,h^3\gamma_{rs}S_r^{(1)}S_s^{(1)} \\ &\quad - 2\psi_{ir}E_i^{(1)}S_r^{(1)} - (1/2h)\varepsilon_{ij}^* E_i^{(0)}E_j^{(0)} - (3/2h^3)\zeta_{ij}E_i^{(1)}E_j^{(1)} \\ &\quad + (15/4h^3)\varepsilon_{2k}E_2^{(2)}E_k^{(0)} - (45/8h^5)\varepsilon_{22}E_2^{(2)}E_2^{(2)}), \end{aligned} \tag{13.45}$$

and an analogous expression may be written for the internal energy density.

Even before this point we could have substituted the strain–mechanical displacement and electric field–electric potential relations (13.16) and (13.18) in the constitutive equations (13.39), (13.40), (13.42), and (13.43) and substituted the latter in the equations of motion (13.13) and (13.14) for $n = 0,1$ to obtain seven differential equations in the seven variables $u_j^{(0)}$, $u_a^{(1)}$, $\varphi^{(0)}$, and $\varphi^{(1)}$. However, if we had done this and specialized the resulting equa-

tions to rotated Y-cut quartz (or even an isotropic material for that matter) and obtained the thickness solution we would have obtained a thickness-shear frequency that was too high. This sort of thing happens because we have assumed a thickness dependence of the displacement which is linear, whereas the actual one is trigonometric. In order to compensate for this (incorrect) assumption without altering the description of the fundamental modes at long wavelengths, we follow Mindlin ([36]) and introduce correction factors $\varkappa_4$ and $\varkappa_6$ by replacing $S_4^{(0)}$ and $S_6^{(0)}$ in $\mathscr{H}$ and $\mathscr{U}$ with $\varkappa_4 S_4^{(0)}$ and $\varkappa_6 S_6^{(0)}$, respectively. This replacement produces important alterations in the zero-order constitutive equations. The values of $\varkappa_4$ and $\varkappa_6$ are to be determined so that the two lowest thickness-shear frequencies predicted by these approximate equations agree with the exact values predicted by the three-dimensional equations in Chapters 10 and 11. The correction factors may conveniently be inserted in the appropriate position by replacing c_{pq}^* and e_{ip}^* in $\mathscr{H}$ and $\mathscr{U}$ by

$$c_{pq}^{**} = \varkappa_p^\alpha \varkappa_q^\beta c_{pq}^*, \quad e_{iq}^{**} = \varkappa_q^\beta e_{iq}^* \quad \text{(no sum)}, \tag{13.46}$$

where α and β are the powers $\alpha = \cos^2(p\pi/2)$, $\beta = \cos^2(q\pi/2)$. Thus $\varkappa_p^\alpha$ (or $\varkappa_q^\beta$) is equal to $\varkappa_4$, $\varkappa_6$, or 1 depending on whether p (or q) in c_{pq}^{**} and e_{ip}^{**} is 4, 6, or neither, respectively. The same definitions hold for c_{ijkl}^{**}, e_{ikl}^{**}, $\varkappa_{ij}^\alpha$, and $\varkappa_{kl}^\beta$ if p and q are replaced by ij and kl.

At this point it should be noted that piezoelectric correction factors should be introduced in addition to the aforementioned elastic correction factors in the fully coupled piezoelectric case. Furthermore, it is believed that these piezoelectric correction factors should be introduced by replacing $E_1^{(1)}$ and $E_3^{(1)}$ by corrected terms in $\mathscr{H}$ and $\mathscr{U}$ because of the incorrect assumption of the thickness dependence of D_1 and D_3. However, nothing has been done in connection with the determination of such piezoelectric correction factors, mainly because the dispersion curves in a fully coupled piezoelectric case have never been determined. Such correction factors are, of course, of no importance in a small coupling case (such as quartz). Consequently, they are being neglected in this development.

To recapitulate, after the truncation and adjustments we have the following:

The electric enthalpy density $\mathscr{H}$ is given by the previous expression with c_{pq}^* and e_{ip}^* replaced by c_{pq}^{**} and e_{ip}^{**}.

The internal energy density $\mathscr{U}$ is given by

$$\mathscr{U} = \mathscr{H} + E_i^{(0)} D_i^{(0)} + E_i^{(1)} D_i^{(1)} + E_2^{(2)} D_2^{(2)}. \tag{13.47}$$

The kinetic energy density $\mathscr{K}$ is given by

$$\mathscr{K} = \varrho h(\dot{u}_i^{(0)}\dot{u}_i^{(0)} + (h^2/3)\dot{u}_a^{(1)}\dot{u}_a^{(1)}), \tag{13.48}$$

where here, and hereafter, indices a, b, c, d range over 1, 3.

The constitutive relations are given by

$$\begin{aligned} T_p^{(0)} &= \partial\mathscr{H}/\partial S_p^{(0)} = 2hc_{pq}^{**}S_q^{(0)} - e_{kp}^{**}E_k^{(0)}, \\ T_r^{(1)} &= \partial\mathscr{H}/\partial S_r^{(1)} = \tfrac{2}{3}h^3\gamma_{rs}S_s^{(1)} - \psi_{ir}E_i^{(1)}, \end{aligned} \tag{13.49}$$

$$\begin{aligned} D_i^{(0)} &= -\partial\mathscr{H}/\partial E_i^{(0)} = e_{iq}^{**}S_q^{(0)} + (1/2h)\varepsilon_{ij}^{*}E_j^{(0)} - (15/8h^3)\varepsilon_{i2}E_2^{(2)}, \\ D_i^{(1)} &= -\partial\mathscr{H}/\partial E_i^{(1)} = \psi_{ir}S_r^{(1)} + (3/2h^3)\zeta_{ij}E_j^{(1)}, \\ D_2^{(2)} &= -\partial\mathscr{H}/\partial E_2^{(2)} = (15/8h^5)(3\varepsilon_{22}E_2^{(2)} - h^2\varepsilon_{2k}E_k^{(0)}). \end{aligned} \tag{13.50}$$

The equations of motion take the form:

$$T_{aj,a}^{(0)} + F_j^{(0)} = 2\varrho h\ddot{u}_j^{(0)}, \qquad T_{ab,a}^{(1)} - T_{2b}^{(0)} + F_b^{(1)} = \tfrac{2}{3}\varrho h^3\ddot{u}_b^{(1)}. \tag{13.51}$$

The equations of electrostatics take the form

$$D_{a,a}^{(0)} + D_2^{(1)} = 0, \qquad D_{a,a}^{(1)} + 2D_2^{(2)} = 0. \tag{13.52}$$

The strain-displacement relations are given by

$$\begin{aligned} S_{ij}^{(0)} &= \tfrac{1}{2}(u_{i,j}^{(0)} + u_{j,i}^{(0)} + \delta_{2j}u_i^{(1)} + \delta_{2i}u_j^{(1)}), \\ S_{ab}^{(1)} &= \tfrac{1}{2}(u_{a,b}^{(1)} + u_{b,a}^{(1)}). \end{aligned} \tag{13.53}$$

The electric field–potential relations are given by

$$\begin{aligned} E_i^{(0)} &= -\varphi_{,i}^{(0)} - \delta_{2i}\Phi^{(0)}, \\ E_i^{(1)} &= -\varphi_{,i}^{(1)} + \delta_{2i}(\varphi^{(0)} - \Phi^{(1)}), \\ E_2^{(2)} &= 2\varphi^{(1)} - \Phi^{(2)}, \end{aligned} \tag{13.54}$$

and we note that $\Phi^{(2)} = h^2\Phi^{(0)}$.

The foregoing equations (13.49)–(13.54) comprise 37 equations in the 37 dependent variables: five each of $T_p^{(0)}$ and $S_p^{(0)}$; three each of $T_r^{(1)}$, $S_r^{(1)}$, $D_i^{(0)}$, $E_i^{(0)}$, $D_i^{(1)}$, $E_i^{(1)}$, and $u_i^{(0)}$; two $u_a^{(1)}$; and $D_2^{(2)}$, $E_2^{(2)}$, $\varphi^{(0)}$, and $\varphi^{(1)}$. The 37 equations may readily be reduced to seven in seven variables by first substituting from (13.53) and (13.54) into (13.49) and (13.50) with the result

$$T_{ij}^{(0)} = 2hc_{ijkl}^{**}(u_{k,l}^{(0)} + \delta_{2l}\, u_k^{(1)}) + e_{kij}^{**}\varphi_{,k}^{(0)} + e_{2ij}^{**}\Phi^{(0)},$$
$$T_{ab}^{(1)} = \tfrac{2}{3}\, h^3\gamma_{abcd}u_{c,d}^{(1)} - \psi_{2ab}\varphi^{(0)} + \psi_{iab}\varphi_{,i}^{(1)} + \psi_{2ab}\Phi^{(1)}, \tag{13.55}$$

$$D_i^{(0)} = e_{ikl}^{**}(u_{k,l}^{(0)} + \delta_{2l}\, u_k^{(1)}) - (1/2h)\varepsilon_{ij}^{*}\varphi_{,j}^{(0)} - (15/4h^3)\varepsilon_{i2}\varphi^{(1)} - (1/8h)\varepsilon_{i2}'\Phi^{(0)},$$
$$D_i^{(1)} = \psi_{iab}u_{a,b}^{(1)} + (3/2h^3)\zeta_{i2}\varphi^{(0)} - (3/2h^3)\zeta_{ij}\varphi_{,j}^{(1)} - (3/2h^3)\zeta_{i2}\Phi^{(1)}, \tag{13.56}$$
$$D_2^{(2)} = (15/8h^5)(h^2\varepsilon_{2k}\varphi_{,k}^{(0)} + 6\varepsilon_{22}\varphi^{(1)} - 2h^2\varepsilon_{22}\Phi^{(0)}),$$

where $\varepsilon_{i2}' = 4\varepsilon_{i2}^{*} - 15\varepsilon_{i2}$; and then substituting from (13.55) and (13.56) into (13.51) and (13.52), respectively, to obtain

$$2hc_{ijkl}^{**}(u_{k,li}^{(0)} + \delta_{2k}u_{l,i}^{(1)}) + e_{kij}^{**}\varphi_{,ki}^{(0)} + F_j^{(0)} + e_{2ij}^{**}\Phi_{,i}^{(0)} = 2\varrho h\ddot{u}_j^{(0)}$$
$$\tfrac{2}{3}\, h^3\gamma_{abcd}u_{c,da}^{(1)} - 2hc_{2bkl}^{**}(u_{k,l}^{(0)} + \delta_{2k}u_l^{(1)}) - e_{i2b}^{**}\varphi_{,i}^{(0)} \tag{13.57}$$
$$+ \psi_{iab}(\varphi_{,ia}^{(1)} - \delta_{2i}\varphi_{,a}^{(0)}) + F_b^{(1)} - e_{22b}^{**}\Phi^{(0)} + \psi_{2ab}\Phi_{,a}^{(1)} = \tfrac{2}{3}\,\varrho h^3\ddot{u}_b^{(1)},$$

$$\tfrac{2}{3}\, h^3\psi_{2kl}u_{k,l}^{(1)} + \tfrac{2}{3}\, h^3e_{ikl}^{**}(u_{k,li}^{(0)} + \delta_{2k}u_{l,i}^{(1)}) - \tfrac{1}{3}\, h^2\varepsilon_{ij}^{*}\varphi_{,ij}^{(0)}$$
$$- \zeta_{2j}(\varphi_{,j}^{(1)} - \delta_{2j}\varphi^{(0)}) - \tfrac{5}{2}\,\varepsilon_{k2}\varphi_{,k}^{(1)} - (h^2/12)\varepsilon_{i2}'\Phi_{,i}^{(0)} - \zeta_{22}\Phi^{(1)} = 0, \tag{13.58}$$
$$\tfrac{2}{3}\, h^3\psi_{iab}u_{a,bi}^{(1)} - \zeta_{ij}(\varphi_{,ij}^{(1)} - \delta_{2j}\varphi_{,i}^{(0)}) + \tfrac{5}{2}\,\varepsilon_{2k}\varphi_{,k}^{(0)} + (15/h^2)\varepsilon_{22}\varphi^{(1)}$$
$$- 5\varepsilon_{22}\Phi^{(0)} - \zeta_{i2}\Phi_{,i}^{(1)} = 0.$$

Equations (13.57) and (13.58) are seven second-order differential equations in the seven dependent variables $u_i^{(0)}$, $u_a^{(1)}$, $\varphi^{(0)}$, and $\varphi^{(1)}$ with seven inhomogeneous (forcing) terms; the five surface tractions $F_j^{(0)}$, $F_b^{(1)}$ and the two surface potentials $\Phi^{(0)}$, $\Phi^{(1)}$. Note that there are three independent variables, x_1, x_3, and t, and that the major surfaces are already included in the case of the infinite medium (plate). Note that the inhomogeneous potentials $\Phi^{(0)}$, $\Phi^{(1)}$ are important even in the small coupling case, since they are the electrical terms that drive the motion.

3. UNIQUENESS OF SOLUTION OF THE TRUNCATED SYSTEM

We have still to establish edge (boundary) conditions at the bounding surface of a finite plate. To this end, we will establish a theorem of uniqueness of solution of the aforementioned 37 equations in the classical Neumann manner. We consider two sets of the 37 variables, each set satisfying the system of equations, and form a system comprising the 37 differences between corresponding variables in the two sets. Since each set satisfies the equations and the equations are linear, the difference set also satisfies the equations. In terms of the difference quantities, we form the equation

$$\int_{t_0}^{t} dt \int_A [(\bar{T}_{ij,i}^{(0)} + \bar{F}_j^{(0)} - 2\varrho h\ddot{\bar{u}}_j^{(0)})\dot{\bar{u}}_j^{(0)} + (\bar{T}_{ab,a} - \bar{T}_{2b}^{(0)} + \bar{F}_b^{(1)} - \tfrac{2}{3}\varrho h^3\ddot{\bar{u}}_b^{(1)})\dot{\bar{u}}_b^{(1)}]\, dA = 0, \tag{13.59}$$

where the bar denotes the difference. After much tedious manipulation and the use of all other difference equations along with many applications of the divergence theorem the above equation may be brought to the form

$$\int_A [\bar{\mathscr{U}} + \bar{\mathscr{K}}]_{t_0}^{t} dA = \int_{t_0}^{t} dt \oint_C n_a(\bar{T}_{aj}^{(0)}\dot{\bar{u}}_j^{(0)} + \bar{T}_{ab}^{(1)}\dot{\bar{u}}_b^{(1)} - \bar{\varphi}^{(0)}\dot{\bar{D}}_{(a)}^{(0)} - \bar{\varphi}^{(1)}\dot{\bar{D}}_a^{(1)})\, ds + \int_{t_0}^{t} dt \int_A [\bar{F}_j^{(0)}\dot{\bar{u}}_j^{(0)} + \bar{F}_b^{(1)}\dot{\bar{u}}_b^{(1)} - \bar{\Phi}^{(0)}(\dot{\bar{D}}_2^{(0)} + h^2\dot{\bar{D}}_2^{(2)}) - \bar{\Phi}^{(1)}\dot{\bar{D}}_2^{(1)}]\, dA, \tag{13.60}$$

where $\bar{\mathscr{U}}$ and $\bar{\mathscr{K}}$ denote the internal and kinetic energy densities, respectively, of the difference system; C denotes the edge of the plate; and the n_a are the components of the outward unit normal to the edge of the plate in the plane of the plate. From the above equation we see by the usual arguments based on the positive-definiteness of $\bar{\mathscr{U}}$ and $\bar{\mathscr{K}}$ that there are seven conditions to be specified at each point of the interior and at each point on the edge of the plate in addition to the initial values of $u_i^{(0)}$, $u_a^{(1)}$, $\varphi^{(0)}$, $\dot{u}_i^{(0)}$, and $\dot{u}_a^{(1)}$. Referred to orthogonal coordinates n, s, x_2, the edge conditions are one member of each of the seven products

$$T_{nn}^{(0)}u_n^{(0)},\ T_{ns}^{(0)}u_s^{(0)},\ T_{n2}^{(0)}u_2^{(0)},\ T_{nn}^{(1)}u_n^{(1)},\ T_{ns}^{(1)}u_s^{(1)},\ \varphi^{(0)}D_n^{(0)},\ \varphi^{(1)}D_n^{(1)}. \tag{13.61}$$

In terms of the orthogonal coordinates, α, β, x_2, the interior conditions are one member of each of the seven products

$$F_2^{(0)}u_2^{(0)},\ F_\alpha^{(0)}u_\alpha^{(0)},\ F_\beta^{(0)}u_\beta^{(0)},\ F_\alpha^{(1)}u_\alpha^{(1)},\ F_\beta^{(1)}u_\beta^{(1)},\ \Phi^{(0)}(D_2^{(0)}+h^2D_2^{(2)}),\ \Phi^{(1)}D_2^{(1)}. \tag{13.62}$$

The terms $D_2^{(0)} + h^2D_2^{(2)}$ and $D_2^{(1)}$ in the interior conditions (13.62) can be interpreted by recalling that, to our approximation,

$$D_2 = D_2^{(0)} + x_2D_2^{(1)} + x_2^2D_2^{(2)}. \tag{13.63}$$

Hence from (13.63) we have

$$\begin{aligned} D_2^{(0)} + h^2D_2^{(2)} &= \tfrac{1}{2}[D_2(h) + D_2(-h)], \\ D_2^{(1)} &= (1/2h)[D_2(h) - D_2(-h)]. \end{aligned} \tag{13.64}$$

The above interior conditions (13.64) and the interpretation indicates that these equations, which have been derived for the situation where the major surfaces of the plate are completely coated with electrodes, may be applied

even when the major surfaces are without electrodes provided the dielectric constant ε_{22} of the plate is sufficiently larger than the dielectric constant of the surrounding region simply by setting

$$D_2^{(0)} + h^2 D_2^{(2)} = 0, \qquad D_2^{(1)} = 0, \tag{13.65}$$

at all interior points. Under these circumstances $\Phi^{(0)}$ and $\Phi^{(1)}$ become unknown variables. However, the above two conditions in (13.65) permit the elimination of $\Phi^{(0)}$ and $\Phi^{(1)}$ from the equations (13.49)–(13.54). When a solution is obtained the unknown quantities $\Phi^{(0)}$ and $\Phi^{(1)}$ can, of course, be obtained from the solution.

4. ORTHOGONALITY OF SOLUTION OF THE TRUNCATED SYSTEM

In solving a forced vibration problem it is sometimes useful to have orthogonality conditions. To this end we consider two solutions

$$\begin{aligned}(u_j^{(0)}, u_b^{(1)}, \varphi^{(0)}, \varphi^{(1)}) &= (u_j^{(0)\mu}, u_b^{(1)\mu}, \varphi^{(0)\mu}, \varphi^{(1)\mu}) e^{i\omega_\mu t}, \\ (u_j^{(0)}, u_b^{(1)}, \varphi^{(0)}, \varphi^{(1)}) &= (u_j^{(0)\nu}, u_b^{(1)\nu}, \varphi^{(0)\nu}, \varphi^{(1)\nu}) e^{i\omega_\nu t},\end{aligned} \tag{13.66}$$

of the equations of motion (with $F_j^{(0)} = F_b^{(1)} = 0$) and electrostatics. The substitution of the first of (13.66) in the equations of motion (13.51) yields

$$\begin{aligned}2\varrho h\omega_\mu{}^2 u_j^{(0)\mu} &= -T_{ij,i}^{(0)\mu}, \\ \tfrac{2}{3}\varrho h^3 \omega_\mu{}^2 u_b^{(1)\mu} &= -T_{ab,a}^{(1)\mu} + T_{2b}^{(0)\mu},\end{aligned} \tag{13.67}$$

and the second yields

$$\begin{aligned}2\varrho h\omega_\nu{}^2 u_j^{(0)\nu} &= -T_{ij,i}^{(0)\nu}, \\ \tfrac{2}{3}\varrho h^3 \omega_\nu{}^2 u_b^{(1)\nu} &= -T_{ab,a}^{(1)\nu} + T_{2b}^{(0)\nu}.\end{aligned} \tag{13.68}$$

From (13.67) and (13.68) we now form the equation

$$\begin{aligned}&2h\varrho(\omega_\mu{}^2 - \omega_\nu{}^2) \int_A (u_j^{(0)\mu} u_j^{(0)\nu} + \tfrac{1}{3}h^2 u_b^{(1)\mu} u_b^{(1)\nu})\, dA \\ &= \int_A (T_{ij,i}^{(0)\nu} u_j^{(0)\mu} - T_{ij,i}^{(0)\mu} u_j^{(0)\nu} + T_{ab,a}^{(1)\nu} u_b^{(1)\mu} - T_{ab,a}^{(1)\mu} u_b^{(1)\nu} \\ &\quad + T_{2b}^{(0)\mu} u_b^{(1)\nu} - T_{2b}^{(0)\nu} u_b^{(1)\mu})\, dA.\end{aligned} \tag{13.69}$$

After much tedious manipulation and the use of the constitutive equations (13.49) and (13.50), the equations of electrostatics (13.52), the relations

(13.53) and (13.54), the symmetries of the material constants, and the divergence theorem, Eq. (13.69) may be brought to the form

$$
\begin{aligned}
2\varrho h(\omega_\mu{}^2 - \omega_\nu{}^2) \int_A & (u_j^{(0)\mu} u_j^{(0)\nu} + \tfrac{1}{3}h^2 u_b^{(1)\mu} u_b^{(1)\nu})\, dA \\
&= \int_A [\Phi^{(0)\mu}(D_2^{(0)\nu} + h^2 D_2^{(2)\nu}) - \Phi^{(0)\nu}(D_2^{(0)\mu} + h^2 D_2^{(2)\mu}) \\
&\quad + \Phi^{(1)\mu} D_2^{(1)\nu} - \Phi^{(1)\nu} D_2^{(1)\mu}]\, dA \\
&\quad + \oint_C n_a (T_{aj}^{(0)\nu} u_j^{(0)\mu} - T_{aj}^{(0)\mu} u_j^{(0)\nu} + T_{ab}^{(1)\nu} u_b^{(1)\mu} - T_{ab}^{(1)\mu} u_b^{(1)\nu} \\
&\quad + D_a^{(0)\nu} \varphi^{(0)\mu} - D_a^{(0)\mu} \varphi^{(0)\nu} + D_a^{(1)\nu} \varphi^{(1)\mu} - D_a^{(1)\mu} \varphi^{(1)\nu})\, ds. \qquad (13.70)
\end{aligned}
$$

The area integral on the right-hand side of (13.70) vanishes for either zero electric potential or zero electric displacement on both major surfaces of the plate, and the edge integrals vanish for homogeneous boundary conditions on the edge. Thus for homogeneous conditions on the major surfaces and edge we find

$$
\int_A (u_j^{(0)\mu} u_j^{(0)\nu} + \tfrac{1}{3}h^2 u_b^{(1)\mu} u_b^{(1)\nu})\, dA = N_{(\mu)} \delta_{\mu\nu}, \qquad (13.71)
$$

where $N_{(\mu)}$ is a normalization factor. These are the orthogonality conditions for the approximation.

5. STEADY-STATE FORCED VIBRATIONS. I

Let us consider the problem of steady vibrations forced by an ac voltage applied to electrodes on the major surfaces of a plate with its edge free. Under these circumstances the inhomogeneous terms $F_j^{(0)}$, $F_b^{(1)}$, and $\Phi^{(1)}$ vanish and only the inhomogeneous term $\Phi^{(0)}$ remains. The boundary conditions on the edge are

$$
n_a T_{aj}^{(0)} = n_a T_{ab}^{(1)} = n_a D_a^{(0)} = n_a D_a^{(1)} = 0, \qquad (13.72)
$$

since we have assumed the dielectric constants of the plate to be large compared with those of the surrounding region. Although these edge conditions appear to be homogeneous, they are actually inhomogeneous due to the presence of the prescribed potential $\Phi^{(0)}$ in the constitutive equations (13.49) and (13.50) by virtue of (13.54). It will be convenient for our purposes here to transfer all inhomogeneous terms from the boundary conditions to the differential equations. It should be noted that at times it might be better to transfer the inhomogeneous terms from the differential equations to the boundary conditions, as would be the case in the presence of dissipation.

Nonetheless, we shall proceed as specified. To this end, we shall choose the auxiliary function so that it removes the inhomogeneity not only from the boundary conditions (13.72), but also from the electrostatic equations (13.58), thereby leaving a residual problem of inhomogeneous equations of motion, homogeneous electrostatic equations, and homogeneous boundary conditions. Let

$$(u_j^{(0)}, u_b^{(1)}, \varphi^{(0)}, \varphi^{(1)}) = (u_j^{(0)A}, u_b^{(1)A}, \varphi^{(0)A}, \varphi^{(1)A})\, e^{i\omega t} + (u_j^{(0)R}, u_b^{(1)R}, \varphi^{(0)R}, \varphi^{(1)R})\, e^{i\omega t}, \tag{13.73}$$

where the superscript A identifies the auxiliary functions and R the residual solution. Inserting this solution in the constitutive relations (13.49) and (13.50) and cancelling the factor $e^{i\omega t}$, we obtain expressions of the type

$$T_{ij}^{(0)} = T_{ij}^{(0)R} + T_{ij}^{(0)A} + e_{2ij}^{**}\Phi^{(0)}, \tag{13.74}$$

where

$$T_{ij}^{(0)R} = 2hc_{ijkl}^{**}(u_{k,l}^{(0)R} + \delta_{2l}u_k^{(1)R}) + e_{kij}^{**}\varphi_{,k}^{(0)R},$$
$$T_{ij}^{(0)A} = 2hc_{ijkl}^{**}(u_{k,l}^{(0)A} + \delta_{2l}u_k^{(1)A}) + e_{kij}^{**}\varphi_{,k}^{(0)A}.$$

With resolutions like (13.74) for all the constitutive equations, the differential equations (13.51) and (13.52) take the form

$$T_{ij,i}^{(0)R} + 2\varrho h\omega^2 u_j^{(0)R} + T_{ij,i}^{(0)A} + 2\varrho h\omega^2 u_j^{(0)A} + e_{2ij}^{**}\Phi_{,i}^{(0)} = 0, \tag{13.75}$$
$$T_{ab,a}^{(1)R} - T_{2b}^{(0)R} + \tfrac{2}{3}\varrho h^3\omega^2 u_b^{(1)R} + T_{ab,a}^{(1)A} - T_{2b}^{(0)A} + \tfrac{2}{3}\varrho h^3\omega^2 u_b^{(1)A} - e_{22b}^{**}\Phi^{(0)} = 0,$$

$$D_{i,i}^{(0)R} + D_2^{(1)R} + D_{i,i}^{(0)A} + D_2^{(1)A} - (1/8h)\varepsilon_{i2}'\Phi_{,i}^{(0)} = 0, \tag{13.76}$$
$$D_{i,i}^{(1)R} + 2D_2^{(2)R} + D_{i,i}^{(1)A} + 2D_2^{(2)A} - (15/2h^3)\varepsilon_{22}\Phi^{(0)} = 0,$$

and the boundary conditions (13.72) take the form

$$n_a T_{aj}^{(0)R} + n_a T_{aj}^{(0)A} + n_a e_{2aj}^{**}\Phi^{(0)} = 0,$$
$$n_a T_{ab}^{(1)R} + n_a T_{ab}^{(1)A} = 0, \tag{13.77}$$
$$n_a D_a^{(0)R} + n_a D_a^{(0)A} - (1/8h)n_a\varepsilon_{a2}'\Phi^{(0)} = 0,$$
$$n_a D_a^{(1)R} + n_a D_a^{(1)A} = 0.$$

We now choose the auxiliary functions $u_i^{(0)A}$, $u_a^{(1)A}$, $\varphi^{(0)A}$, and $\varphi^{(1)A}$ to be particular solutions of the differential equations

$$D_{i,i}^{(0)A} + D_2^{(1)A} = (\varepsilon_{i2}'/8h)\Phi_{,i}^{(0)},$$
$$D_{i,i}^{(1)A} + 2D_2^{(2)A} = (15/2h^3)\varepsilon_{22}^{s}\Phi^{(0)}, \tag{13.78}$$

and to satisfy the boundary conditions

$$\begin{aligned} &n_a T_{aj}^{(0)A} + n_a e_{2aj}^{**}\Phi^{(0)} = 0\,, \\ &n_a T_{ab}^{(1)A} = 0\,, \\ &n_a D_a^{(0)A} - n_a(\varepsilon'_{a2}/8h)\Phi^{(0)} = 0, \qquad n_a D_a^{(1)A} = 0\,. \end{aligned} \tag{13.79}$$

Then, inserting (13.78) and (13.79) in (13.75), (13.76), and (13.77), respectively, we find the residual problem governed by the differential equations

$$T_{ij,i}^{(0)R} + 2\varrho h\omega^2 u_j^{(0)R} + 2\varrho h\omega^2 u_j^{(0)A} + G_j^{(0)} = 0\,, \tag{13.80}$$

$$T_{ab,a}^{(1)R} - T_{2b}^{(0)R} + \tfrac{2}{3}\varrho h^3\omega^2 u_b^{(1)R} + \tfrac{2}{3}\varrho h^3\omega^2 u_b^{(1)A} + \tfrac{1}{3}h^2 G_b^{(1)} = 0\,, \tag{13.81}$$

$$D_{i,i}^{(0)R} + D_2^{(1)R} = 0\,, \tag{13.82}$$

$$D_{i,i}^{(1)R} + 2D_2^{(2)R} = 0\,, \tag{13.83}$$

and the boundary conditions

$$n_a T_{aj}^{(0)R} = n_a T_{ab}^{(1)R} = n_a D_a^{(0)R} = n_a D_a^{(1)R} = 0\,, \tag{13.84}$$

where the inhomogeneous forcing terms are given by

$$G_j^{(0)} = T_{ij,i}^{(0)A} + e_{2ij}^{**}\Phi_{,i}^{(0)}, \quad G_b^{(1)} = (3/h^2)(T_{ab,a}^{(1)A} - T_{2b}^{(0)A} - e_{22b}^{**}\Phi^{(0)}).$$

The solution of the residual problem may be expressed as an infinite series

$$(u_j^{(0)R}, u_b^{(1)R}, \varphi^{(0)R}, \varphi^{(1)R}) = \sum_{\mu} A_\mu (u_j^{(0)\mu}, u_b^{(1)\mu}, \varphi^{(0)\mu}, \varphi^{(1)\mu})\,, \tag{13.85}$$

where $u_j^{(0)\mu}$, $u_b^{(1)\mu}$, $\varphi^{(0)\mu}$, and $\varphi^{(1)\mu}$ are the orthogonal solutions of the associated homogeneous ($\Phi^{(0)} = 0$) eigensystem,

$$\begin{aligned} &T_{ij,i}^{(0)\mu} + 2\varrho h\omega_\mu{}^2 u_j^{(0)\mu} = 0\,, \\ &T_{ab,a}^{(1)\mu} - T_{2b}^{(0)\mu} + \tfrac{2}{3}\varrho h^3\omega_\mu{}^2 u_b^{(1)\mu} = 0\,, \end{aligned} \tag{13.86}$$

$$\begin{aligned} &D_{i,i}^{(0)\mu} + D_2^{(1)\mu} = 0\,, \\ &D_{i,i}^{(1)\mu} + 2D_2^{(2)\mu} = 0\,, \end{aligned} \tag{13.87}$$

$$n_a T_{aj}^{(0)\mu} = n_a T_{ab}^{(1)\mu} = n_a D_a^{(0)\mu} = n_a D_a^{(1)\mu} = 0 \qquad \text{on} \qquad C. \tag{13.88}$$

Noting, from (13.85), (13.55), and (13.56) (with $\Phi^{(0)} = 0$), that

$$(T_{ij}^{(0)R}, T_{ab}^{(1)R}, D_i^{(0)R}, D_i^{(1)R}, D_2^{(2)R}) = \sum_{\mu} A_\mu (T_{ij}^{(0)\mu}, T_{ab}^{(1)\mu}, D_i^{(0)\mu}, D_i^{(1)\mu}, D_2^{(2)\mu})\,, \tag{13.89}$$

we see from (13.87) that the residual electrostatic equations (13.82) and (13.83) are satisfied identically, since they are satisfied by each term of the sum in (13.85) separately. In order to find the A_μ which will make the series solution (13.85) satisfy the residual equations of motion (13.80) and (13.81), we must first express the *known* inhomogeneous terms, in (13.80) and (13.81) in series form:

$$G_j^{(0)} = \sum_\mu B_\mu u_j^{(0)\mu}, \qquad u_j^{(0)A} = \sum_\mu C_\mu u_j^{(0)\mu}, \tag{13.90}$$

$$G_b^{(1)} = \sum_\mu B_\mu u_b^{(1)\mu}, \qquad u_b^{(1)A} = \sum_\mu C_\mu u_b^{(1)\mu}. \tag{13.91}$$

Multiplying (13.90) by $u_j^{(0)\nu}$, (13.91) by $\frac{1}{3}h^2 u_b^{(1)\nu}$, adding, integrating over the area of the plate, and using the orthogonality relations (13.71), we find

$$\begin{aligned} B_\mu &= N_{(\mu)}^{-1} \int_A (G_j^{(0)} u_j^{(0)\mu} + \tfrac{1}{3}h^2 G_b^{(1)} u_b^{(1)\mu})\, dA, \\ C_\mu &= N_{(\mu)}^{-1} \int_A (u_j^{(0)A} u_j^{(0)\mu} + \tfrac{1}{3}h^2 u_b^{(1)A} u_b^{(1)\mu})\, dA. \end{aligned} \tag{13.92}$$

Now substituting (13.85) and (13.89)–(13.91) in (13.80) and (13.81), using (13.86), multiplying (13.80) by $u_j^{(0)\nu}$ and (13.81) by $u_b^{(1)\nu}$, adding, integrating over the area, and then using (13.71), we find

$$A_\mu = (B_\mu + 2\varrho h\omega^2 C_\mu)/2\varrho h(\omega_\mu{}^2 - \omega^2), \tag{13.93}$$

thereby showing that each A_μ has a resonance denominator. Thus we have

$$\begin{aligned} (u_j^{(0)}, u_b^{(1)}, \varphi^{(0)}, \varphi^{(1)}) &= (u_j^{(0)A}, u_b^{(1)A}, \varphi^{(0)A}, \varphi^{(1)A}) e^{i\omega t} \\ &\quad + \sum_\mu A_\mu (u_j^{(0)\mu}, u_b^{(1)\mu}, \varphi^{(0)\mu}, \varphi^{(1)\mu}) e^{i\omega t}, \end{aligned} \tag{13.94}$$

as the complete steady-state solution of (13.57) and (13.58) with boundary conditions (13.72). Obviously, very near a resonance—say ω_ν—one term of the series expansion in (13.94) dominates all others and the quasistatic auxiliary solution as well, so that our solution (13.94) becomes

$$(u_j^{(0)}, u_b^{(1)}, \varphi^{(0)}, \varphi^{(1)}) = A_\nu (u_j^{(0)\nu}, u_b^{(1)\nu}, \varphi^{(0)\nu}, \varphi^{(1)\nu}) e^{i\omega t}. \tag{13.95}$$

The solution, of course, blows up at $\omega = \omega_\nu$ because A_ν diverges.

6. DETERMINATION OF SURFACE CHARGE

Since the ac voltage is applied uniformly over traction-free faces by means of a thin perfectly conducting film, we have

$$\Phi^{(0)} = Ve^{i\omega t}, \tag{13.96}$$

where V is the constant voltage drop across the thickness. The current through the plate is equal to the time derivative of the integrated surface charge, Q, over the whole area of a face, where

$$Q = \int_A D_2 \Big]_{x_2=-h} dA. \tag{13.97}$$

Substituting from (13.63) into (13.97) and using the two-dimensional equations of electrostatics (13.52), the divergence theorem, and the electric boundary conditions on C in (13.72), we find

$$Q = \int_A D_2^{(0)}\, dA. \tag{13.98}$$

Then substituting (13.94) in the first of (13.56) and the resulting expression for $D_2^{(0)}$ in (13.98), we have for the total surface charge

$$Q = V\left[L - (\varepsilon'_{22}/8h)A + \sum_\mu A_\mu Y_\mu\right], \tag{13.99}$$

where A is the area of the plate and the A_μ are given in (13.93) and

$$Y_\mu = \int_A [e^{**}_{2kl}(u^{(0)\mu}_{k,l} + \delta_{2l}u^{(1)\mu}_k) - (\varepsilon^*_{2j}/2h)\varphi^{(0)\mu}_{,j} - (15/4h^3)\varepsilon^s_{22}\varphi^{(1)\mu}]\, dA, \tag{13.100}$$

$$L = \int_A [e^{**}_{2kl}(u^{(0)A}_{k,l} + \delta_{2l}u^{(1)A}_k) - (\varepsilon^*_{2j}/2h)\varphi^{(0)A}_{,j} - (15/4h^3)\varepsilon^s_{22}\varphi^{(1)A}]\, dA, \tag{13.101}$$

with the constant V factored out. The formula (13.99) for the surface charge Q takes account of the action of the crystal both as a capacitor and as a charge generator, including in both cases the distortion of the field due to the finite dimensions of the plate. When in the vicinity of a resonant frequency $\omega \approx \omega_\nu$ one term in the sum in (13.99) dominates all others, and the formula for surface charge takes the form

$$Q = V[L - (\varepsilon'_{22}/8h)A + A_\nu Y_\nu]. \tag{13.102}$$

At resonance ($\omega = \omega_\nu$) A_ν and hence Q approach infinity. Right near the resonant frequency there is the so-called antiresonant frequency at which the quasistatic terms in (13.102) cancel the resonant term, so that Q vanishes.

7. STEADY-STATE FORCED VIBRATIONS. II

In the solution to the nondissipative forced vibration problem which we have obtained we have transferred the inhomogeneities from the boundary conditions and the electrostatic equations to the equations of motion.

This procedure enabled the solution to be obtained as an expansion in the eigensolutions of the associated homogeneous problem. As we already know, there is another procedure whereby we transfer the inhomogeneities from the differential equations to the boundary conditions. Moreover, this latter procedure is equally applicable when dissipation is present. We may obtain the solution based on this latter procedure simply by defining our auxiliary solution in such a way that the residual differential equations are homogeneous, rather than in the way we did before. To this end, we choose the auxiliary functions $u_i^{(0)A}$, $u_b^{(1)A}$, $\varphi^{(0)A}$, and $\varphi^{(1)A}$ to be particular solutions of the equations

$$\begin{aligned}
&T_{ij,i}^{(0)A} + 2\varrho h\omega^2 u_j^{(0)A} + e_{2ij}^{**}\Phi_{,i}^{(0)} = 0\,,\\
&T_{ab,a}^{(1)A} - T_{2b}^{(0)A} + \tfrac{2}{3}\varrho h^3\omega^2 u_b^{(1)A} - e_{22b}^{**}\Phi^{(0)} = 0\,,\\
&D_{i,i}^{(0)A} + D_2^{(1)A} - (\varepsilon'_{i2}/8h)\Phi_{,i}^{(0)} = 0\,,\\
&D_{i,i}^{(1)A} + 2D_2^{(2)A} - (15/2h^3)\varepsilon_{22}^{s}\Phi^{(0)} = 0\,,
\end{aligned} \tag{13.103}$$

without regard to boundary conditions. Then inserting (13.103) in the full equations (13.75)–(13.77), we find the residual problem governed by the *homogeneous equations*

$$\begin{aligned}
&T_{ij,i}^{(0)R} + 2\varrho h\omega^2 u_j^{(0)R} = 0\,,\\
&T_{ab,a}^{(1)R} - T_{2b}^{(0)R} + \tfrac{2}{3}\varrho h^3\omega^2 u_b^{(1)R} = 0\,,\\
&D_{i,i}^{(0)R} + D_2^{(1)R} = 0, \qquad D_{i,i}^{(1)R} + 2D_2^{(2)R} = 0\,,
\end{aligned} \tag{13.104}$$

and the *inhomogeneous boundary conditions*

$$\begin{aligned}
&n_a T_{aj}^{(0)R} = -\,n_a T_{aj}^{(0)A} - n_a e_{2aj}^{**}\Phi^{(0)}, \qquad n_a T_{ab}^{(1)R} = -\,n_a T_{ab}^{(1)A}\,,\\
&n_a D_a^{(0)R} = -\,n_a D_a^{(0)A} + n_a(\varepsilon'_{a2}/8h)\Phi^{(0)}, \qquad n_a D_a^{(1)R} = -\,n_a D_a^{(1)A}\,,
\end{aligned} \tag{13.105}$$

wherein the entire right-hand sides of (13.105) are known from the aforementioned particular auxiliary solution. We must now obtain an appropriate number of independent solutions of the residual homogeneous equations (13.104) at any given frequency, and take a sum of those independent solutions in order to satisfy the seven inhomogeneous boundary conditions (13.105) of the residual problem. The boundary condition equations (13.105) yield a number of inhomogeneous algebraic equations in the same number of amplitudes of the independent solutions of the residual homogeneous equations. Hence the amplitudes may be solved for in terms of the driving terms and the resultant solution thereby obtained. The above may be

accomplished, of course, without further approximation for a highly limited number of geometries only. The series method of solution naturally suffers from the same limitation. When the present solution is obtained the surface charge Q may be obtained from the equation

$$Q = \int_A (D_2^{(0)A} + D_2^{(0)R})\, dA\,, \tag{13.106}$$

where $D_2^{(0)A}$ and $D_2^{(0)R}$ are determined from the constitutive relation for $D_2^{(0)}$. The present solution may be generalized formally to account for a small amount of dissipation simply by replacing the real elastic constants by complex quantities, as is usually done in linear viscoelasticity theory. The complex quantities will, of course, be functions of frequency. When the complex quantities are introduced in place of the elastic constants the resonances may be defined as the frequencies at which Q is a maximum for a given voltage. When the system is left nondissipative the solution blows up (Q becomes infinite) at resonance and cannot be used for a calculation.

8. TRUNCATED PLATE EQUATIONS FOR ROTATED Y-CUT QUARTZ

When the arrays of elastic piezoelectric and dielectric constants for a crystal with 2-monoclinic symmetry given in (7.7)–(7.9) are substituted in (13.49) and (13.50) we obtain the constitutive equations

$$T_1^{(0)} = 2h[c_{11}^* u_{1,1}^{(0)} + c_{14}^* \varkappa_4 u_{2,3}^{(0)} + c_{14}^* \varkappa_4 u_3^{(1)} + c_{13}^* u_{3,3}^{(0)}] + e_{11}^* \varphi_{,1}^{(0)}\,,$$

$$T_6^{(0)} = 2h\varkappa_6[c_{66}^* \varkappa_6 u_{2,1}^{(0)} + c_{66}^* \varkappa_6 u_1^{(1)} + c_{56}^* u_{1,3}^{(0)} + c_{56}^* u_{3,1}^{(0)}] + e_{26}^* \varkappa_6 \Phi^{(0)} + e_{36}^* \varkappa_6 \varphi_{,3}^{(0)}\,,$$

$$T_5^{(0)} = 2h[c_{56}^* \varkappa_6 u_{2,1}^{(0)} + c_{56}^* \varkappa_6 u_1^{(1)} + c_{55}^* u_{1,3}^{(0)} + c_{55}^* u_{3,1}^{(0)}] + e_{25}^* \Phi^{(0)} + e_{35}^* \varphi_{,3}^{(0)}\,,$$

$$T_4^{(0)} = 2h\varkappa_4[c_{14}^* u_{1,1}^{(0)} + c_{44}^* \varkappa_4 u_{2,3}^{(0)} + c_{44}^* \varkappa_4 u_3^{(1)} + c_{34}^* u_{3,3}^{(0)}] + e_{14}^* \varkappa_4 \varphi_{,1}^{(0)}\,,$$

$$T_3^{(0)} = 2h[c_{13}^* u_{1,1}^{(0)} + c_{34}^* \varkappa_4 u_{2,3}^{(0)} + c_{34}^* \varkappa_4 u_3^{(1)} + c_{33}^* u_{3,3}^{(0)}] + e_{13}^* \varphi_{,1}^{(0)}\,,$$

$$D_1^{(0)} = e_{11}^* u_{1,1}^{(0)} + e_{14}^* \varkappa_4 u_{2,3}^{(0)} + e_{14}^* \varkappa_4 u_3^{(1)} + e_{13}^* u_{3,3}^{(0)} - (1/2h)\varepsilon_{11}^* \varphi_{,1}^{(0)}\,,$$

$$D_2^{(0)} = e_{26}^* \varkappa_6 u_{2,1}^{(0)} + e_{26}^* \varkappa_6 u_1^{(1)} + e_{25}^* u_{1,3}^{(0)} + e_{25}^* u_{3,1}^{(0)} - (1/2h)[\varepsilon_{22}^* \Phi^{(0)} + \varepsilon_{23}^* \varphi_{,3}^{(0)}] - (15/4h^3)\varepsilon_{22}^s \varphi^{(1)} + (15/8h)\varepsilon_{22}^s \Phi^{(0)}\,,$$

$$D_3^{(0)} = e_{36}^* \varkappa_6 u_{2,1}^{(0)} + e_{36}^* \varkappa_6 u_1^{(1)} + e_{35}^* u_{1,3}^{(0)} + e_{35}^* u_{3,1}^{(0)} - (1/2h)[\varepsilon_{23}^* \Phi^{(0)} + \varepsilon_{33}^* \varphi_{,3}^{(0)}] - (15/4h^3)\varepsilon_{23}^s \varphi^{(1)} + (15/8h)\varepsilon_{23}^s \Phi^{(0)}\,,$$

$$T_1^{(1)} = (2h^3/3)[\gamma_{11} u_{1,1}^{(1)} + \gamma_{13} u_{3,3}^{(1)}] + \psi_{11} \varphi_{,1}^{(1)}\,,$$

$$T_5^{(1)} = (2h^3/3)[\gamma_{55} u_{1,3}^{(1)} + \gamma_{55} u_{3,1}^{(1)}] + \psi_{25} \Phi^{(1)} - \psi_{25} \varphi^{(0)} + \psi_{35} \varphi_{,3}^{(1)}\,,$$

$$
\begin{aligned}
T_3^{(1)} &= (2h^3/3)[\gamma_{13}u_{1,1}^{(1)} + \gamma_{33}u_{3,3}^{(1)}] + \psi_{13}\varphi_{,1}^{(1)},\\
D_1^{(1)} &= \psi_{11}u_{1,1}^{(1)} + \psi_{13}u_{3,3}^{(1)} - (3/2h^3)\zeta_{11}\varphi_{,1}^{(1)},\\
D_2^{(1)} &= \psi_{25}u_{1,3}^{(1)} + \psi_{25}u_{3,1}^{(1)} - (3/2h^3)[\zeta_{22}\Phi^{(1)} - \zeta_{22}\varphi^{(0)} + \zeta_{23}\varphi_{,3}^{(1)}],\\
D_3^{(1)} &= \psi_{35}u_{1,3}^{(1)} + \psi_{35}u_{3,1}^{(1)} - (3/2h^3)[\zeta_{23}\Phi^{(1)} - \zeta_{23}\varphi^{(0)} + \zeta_{33}\varphi_{,3}^{(1)}],\\
D_2^{(2)} &= (15/8h^3)[\varepsilon_{22}^s\Phi^{(0)} + \varepsilon_{23}^s\varphi_{,3}^{(0)}] - (45/8h^3)\varepsilon_{22}^s\Phi^{(0)} + (45/4h^5)\varepsilon_{22}^s\varphi^{(1)}.
\end{aligned}
\tag{13.107}
$$

Constitutive equations for rotated Y-cut quartz in this approximation have been presented by Beaver ([39]). However, there are discrepancies between his Eqs. (9) and our (13.107) above.

When the constitutive equations (13.107) are substituted in the plate stress equations of motion (13.51) and the plate charge equations of electrostatics (13.52) we obtain the differential equations

$$
\begin{aligned}
&c_{11}^*u_{1,11}^{(0)} + c_{55}^*u_{1,33}^{(0)} + (c_{14}^*\varkappa_4 + c_{56}^*\varkappa_6)u_{2,13}^{(0)} + (c_{13}^* + c_{55}^*)u_{3,13}^{(0)} + c_{56}^*\varkappa_6u_{1,3}^{(1)}\\
&\quad + c_{14}^*\varkappa_4u_{3,1}^{(1)} + \frac{e_{11}^*}{2h}\varphi_{,11}^{(0)} + \frac{e_{35}^*}{2h}\varphi_{33}^{(0)} + \frac{e_{25}^*}{2h}\Phi_{,3}^{(0)} + \frac{F_1^{(0)}}{2h} = \varrho\ddot{u}_1^{(0)},
\end{aligned}
$$

$$
\begin{aligned}
&(c_{14}^*\varkappa_4 + \varkappa_6c_{56}^*)u_{1,13}^{(0)} + c_{66}^*\varkappa_6{}^2u_{2,11}^{(0)} + c_{44}^*\varkappa_4{}^2u_{2,33}^{(0)} + \varkappa_6c_{56}^*u_{3,11}^{(0)} + c_{34}^*\varkappa_4u_{3,33}^{(0)}\\
&\quad + c_{66}^*\varkappa_6{}^2u_{1,1}^{(1)} + c_{44}^*\varkappa_4{}^2u_{3,3}^{(1)} + \left(\frac{e_{14}^*\varkappa_4 + e_{36}^*\varkappa_6}{2h}\right)\varphi_{,13}^{(0)} + \frac{e_{26}^*}{2h}\varkappa_6\Phi_{,1}^{(0)} + \frac{F_2^{(0)}}{2h} = \varrho\ddot{u}_2^{(0)},
\end{aligned}
$$

$$
\begin{aligned}
&(c_{13}^* + c_{55}^*)u_{1,13}^{(0)} + c_{56}^*\varkappa_6u_{2,11}^{(0)} + c_{34}^*\varkappa_4u_{2,33}^{(0)} + c_{55}^*u_{3,11}^{(0)} + c_{33}^*u_{3,33}^{(0)} + c_{56}^*\varkappa_6u_{1,1}^{(1)}\\
&\quad + c_{34}^*\varkappa_4u_{3,3}^{(1)} + \left(\frac{e_{13}^* + e_{35}^*}{2h}\right)\varphi_{,13}^{(0)} + \frac{e_{25}^*}{2h}\Phi_{,1}^{(0)} + \frac{F_3^{(0)}}{2h} = \varrho\ddot{u}_3^{(0)},
\end{aligned}
$$

$$
\begin{aligned}
&e_{11}^*u_{1,11}^{(0)} + e_{35}^*u_{1,33}^{(0)} + (e_{14}^*\varkappa_4 + e_{36}^*\varkappa_6)u_{2,13}^{(0)} + (e_{13}^* + e_{35}^*)u_{3,13}^{(0)} + (e_{36}^*\varkappa_6 + \psi_{25})u_{1,3}^{(1)}\\
&\quad + (e_{14}^*\varkappa_4 + \psi_{25})u_{3,1}^{(1)} - \frac{\varepsilon_{11}^*}{2h}\varphi_{,11}^{(0)} - \frac{\varepsilon_{33}^*}{2h}\varphi_{,33}^{(0)} + \frac{3}{2h^3}\zeta_{22}\varphi^{(0)}\\
&\quad - \frac{3}{2h^3}\left(\frac{5}{2}\varepsilon_{23}^s + \zeta_{23}\right)\varphi_{,3}^{(1)} - \frac{\varepsilon_{23}^*}{2h}\Phi_{,3}^{(0)} - \frac{3}{2h^3}\zeta_{22}\Phi^{(1)} + \frac{15}{8h}\varepsilon_{23}^s\Phi_{,3}^{(0)} = 0,
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{h^2}c_{56}^*\varkappa_6u_{1,3}^{(0)} - \frac{3}{h^2}c_{66}^*\varkappa_6{}^2u_{2,1}^{(0)} - \frac{3}{h^2}c_{56}^*\varkappa_6u_{3,1}^{(0)} - \frac{3}{h^2}c_{66}^*\varkappa_6{}^2u_1^{(1)} + \gamma_{11}u_{1,11}^{(1)}\\
&\quad + \gamma_{55}u_{1,33}^{(1)} + (\gamma_{13} + \gamma_{55})u_{3,13}^{(1)} - \frac{3}{2h^3}(\varkappa_6e_{36}^* + \psi_{25})\varphi_{,3}^{(0)} + \frac{3}{2h^3}\psi_{11}\varphi_{,11}^{(1)}\\
&\quad + \frac{3}{2h^3}\psi_{35}\varphi_{,33}^{(1)} - \frac{3}{2h^3}e_{26}^*\varkappa_6\Phi^{(0)} + \frac{3}{2h^3}\psi_{25}\Phi_{,3}^{(1)} + \frac{3}{2h^3}F_1^{(1)} = \varrho\ddot{u}_1^{(1)},
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{h^2} c_{14}^* \varkappa_4 u_{1,1}^{(0)} - \frac{3}{h^2} c_{44}^* \varkappa_4^2 u_{2,3}^{(0)} - \frac{3}{h^2} c_{34}^* \varkappa_4 u_{3,3}^{(0)} - \frac{3}{h^2} \varkappa_4^2 c_{44}^* u_3^{(1)} \\
&\quad + (\gamma_{13} + \gamma_{55}) u_{1,13}^{(1)} + \gamma_{55} u_{3,11}^{(1)} + \gamma_{33} u_{3,33}^{(1)} - \frac{3}{2h^3} (e_{14}^* \varkappa_4 + \psi_{25}) \varphi_{,1}^{(0)} \\
&\quad + \frac{3}{2h^3} (\psi_{13} + \psi_{35}) \varphi_{,13}^{(1)} + \frac{3}{2h^3} \psi_{25} \Phi_{,1}^{(1)} + \frac{3}{2h^3} F_3^{(1)} = \varrho \ddot{u}_3^{(1)}, \\
&\psi_{11} u_{1,11}^{(1)} + \psi_{35} u_{1,33}^{(1)} + (\psi_{13} + \psi_{35}) u_{3,13}^{(1)} + \frac{3}{2h^3} \left(\zeta_{23} + \frac{5}{2} \varepsilon_{23}^s \right) \varphi_{,3}^{(0)} \\
&\quad + \frac{45}{2h^5} \varepsilon_{22}^s \varphi^{(1)} - \frac{3}{2h^3} \zeta_{11} \varphi_{,11}^{(1)} - \frac{3}{2h^3} \zeta_{33} \varphi_{,33}^{(1)} + \frac{15}{4h^3} \varepsilon_{22}^s \Phi^{(0)} \\
&\quad - \frac{3}{2h^3} \zeta_{23} \Phi_{,3}^{(1)} - \frac{45}{4h^3} \varepsilon_{22}^s \Phi^{(0)} = 0.
\end{aligned} \tag{13.108}
$$

9. AN APPLICATION TO A ROTATED Y-CUT QUARTZ PLATE

As an example of the use of the equations obtained let us consider a rotated Y-cut quartz plate with edges at $x_1 = \pm l$ and unbounded in the x_3 direction, as shown in Fig. 22. We will obtain the solution using both of the aforementioned procedures. Let us first consider the series expansion procedure discussed in Section 5. Clearly the problem, and hence the solution, is independent of x_3. An examination of Eqs. (13.108), which are applicable to rotated Y-cut quartz, when they are independent of x_3 reveals that we may take

$$u_1^{(0)} = u_3^{(1)} = \varphi^{(0)} = 0, \tag{13.109}$$

while $u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$, and $\varphi^{(1)}$ remain coupled to the driving voltage. Then from (13.108) the governing differential equations are

$$
\begin{aligned}
&c_{66} \varkappa_6^2 (u_{2,11}^{(0)} + u_{1,1}^{(1)}) + c_{56} \varkappa_6 u_{3,11}^{(0)} + \varrho \omega^2 u_2^{(0)} = 0, \\
&c_{56} \varkappa_6 (u_{2,11}^{(0)} + u_{1,1}^{(1)}) + c_{55} u_{3,11}^{(0)} + \varrho \omega^2 u_3^{(0)} = 0, \\
&\tfrac{2}{3} h^3 \gamma_{11} u_{1,11}^{(1)} - 2h \varkappa_6^2 c_{66} (u_{2,1}^{(0)} + u_1^{(1)}) - 2h \varkappa_6 c_{56} u_{3,1}^{(0)} + \psi_{11} \varphi_{,11}^{(1)} \\
&\qquad + \tfrac{2}{3} h^3 \varrho \omega^2 u_1^{(1)} - \varkappa_6 e_{26} V = 0, \\
&\tfrac{2}{3} h^3 \psi_{11} u_{1,11}^{(1)} - \zeta_{11} \varphi_{,11}^{(1)} + (15/h^2) \varepsilon_{22}^s \varphi^{(1)} - 5 \varepsilon_{22}^s V = 0.
\end{aligned} \tag{13.110}
$$

Note that the constant e_{26} couples the thickness-shear deformation with the applied voltage, and ψ_{11} couples the flexural deformation with the induced electric field.

In accordance with Fig. 22, the plate is bounded by $x_1 = \pm l$ and $x_3 = \pm w$. The appropriate boundary conditions on $x_1 = \pm l$ are

$$T_{12}^{(0)} = T_{13}^{(0)} = T_{11}^{(1)} = D_1^{(1)} = 0, \tag{13.111}$$

and we ignore the boundary conditions on $x_3 = \pm w$. Substituting from the constitutive equations (13.107) into the boundary conditions (13.111) on $x_1 = \pm l$ and taking account of (13.109), we obtain

$$\begin{aligned} 2h\varkappa_6 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + 2hc_{56}u_{3,1}^{(0)} + e_{26}V &= 0, \\ 2h\varkappa_6 c_{56}(u_{2,1}^{(0)} + u_1^{(1)}) + 2hc_{55}u_{3,1}^{(0)} + e_{25}V &= 0, \\ \tfrac{2}{3}h^3\gamma_{11}u_{1,1}^{(1)} + \psi_{11}\varphi_{,1}^{(1)} &= 0, \\ \tfrac{2}{3}h^3\psi_{11}u_{1,1}^{(1)} - \zeta_{11}\varphi_{,1}^{(1)} &= 0. \end{aligned} \tag{13.112}$$

A set of auxiliary functions which may be used to remove the inhomogeneous terms, V, from the electrostatic equation, the fourth of (13.110), and the boundary conditions (13.112), is

$$\begin{gathered} u_2^{(0)A} = (1/2h)VP_2x_1, \quad u_3^{(0)A} = (1/2h)VP_3x_1, \quad u_1^{(1)A} = 0 \\ \varphi^{(1)A} = \tfrac{1}{3}h^2V, \end{gathered} \tag{13.113}$$

where

$$P_2 = \frac{e_{25}c_{56} - e_{26}c_{55}}{\varkappa_6(c_{55}c_{66} - c_{56}^2)}, \qquad P_3 = \frac{e_{26}c_{56} - e_{25}c_{66}}{(c_{55}c_{66} - c_{56}^2)}.$$

Then $G_j^{(0)} = G_b^{(1)} = 0$ and the residual inhomogeneous differential equations are

$$\begin{aligned} &c_{66}\varkappa_6^2(u_{2,11}^{(0)R} + u_{1,1}^{(1)R}) + c_{56}\varkappa_6 u_{3,11}^{(0)R} + \varrho\omega^2 u_2^{(0)R} + (1/2h)\varrho\omega^2 VP_2x_1 = 0, \\ &c_{56}\varkappa_6(u_{2,11}^{(0)R} + u_{1,1}^{(1)R}) + c_{55}\varkappa_6 u_{3,11}^{(0)R} + \varrho\omega^2 u_3^{(0)R} + (1/2h)\varrho\omega^2 VP_3x_1 = 0, \\ &\tfrac{2}{3}h^3\gamma_{11}u_{1,11}^{(1)R} - 2h\varkappa_6^2 c_{66}(u_{2,1}^{(0)R} + u_1^{(1)R}) - 2h\varkappa_6 c_{56}u_{3,1}^{(0)R} + \psi_{11}\varphi_{,11}^{(1)R} \\ &\quad + \tfrac{2}{3}\varrho h^3\omega^2 u_1^{(1)R} = 0, \\ &\tfrac{2}{3}h^3\psi_{11}u_{1,11}^{(1)R} - \zeta_{11}\varphi_{,11}^{(1)R} + (15/h^2)\varepsilon_{22}^s\varphi^{(1)R} = 0. \end{aligned} \tag{13.114}$$

In this case we see that the third residual equation in (13.114) has also become homogeneous. The boundary conditions to be satisfied by the residual solution are

$$\begin{aligned} 2h\varkappa_6 c_{66}(u_{2,1}^{(0)R} + u_1^{(1)R}) + 2hc_{56}u_{3,1}^{(0)R} &= 0, \\ 2h\varkappa_6 c_{56}(u_{2,1}^{(0)R} + u_1^{(1)R}) + 2hc_{55}u_{3,1}^{(0)R} &= 0, \\ \tfrac{2}{3}h^3\gamma_{11}u_{1,1}^{(1)R} + \psi_{11}\varphi_{,1}^{(1)R} = 0, \qquad \tfrac{2}{3}h^3\psi_{11}u_{1,1}^{(1)R} - \zeta_{11}\varphi_{,1}^{(1)R} &= 0. \end{aligned} \tag{13.115}$$

We now find the orthogonal functions for the series solution of the residual problem, i.e., we find the solutions of the above systems (13.114) and (13.115) with $V = 0$. Consider

$$(u_2^{(0)}, u_3^{(0)}) = (A, B) \sin \xi x_1, \quad (u_1^{(1)}, \varphi^{(1)}) = (C, D) \cos \xi x_1. \tag{13.116}$$

These satisfy (13.114) with $V = 0$ provided

$$\begin{aligned}
&A(\varkappa_6^2 c_{66}\xi^2 - \varrho\omega^2) + B\varkappa_6 c_{56}\xi^2 + C\varkappa_6^2 c_{66}\xi = 0,\\
&A\varkappa_6 c_{56}\xi^2 + B(c_{55}\xi^2 - \varrho\omega^2) + C\varkappa_6 c_{56}\xi = 0,\\
&A\varkappa_6^2 c_{66}\xi + B\varkappa_6 c_{56}\xi + C(\varkappa_6^2 c_{66} + \tfrac{1}{3}h^2\gamma_{11}\xi^2 - \tfrac{1}{3}\varrho h^2\omega^2) + D(\psi_{11}/2h)\xi^2 = 0,\\
&C\psi_{11}\xi^2 - D(3/2h^3)[(15/h^2)\varepsilon_{22}^s + \zeta_{11}\xi^2] = 0.
\end{aligned} \tag{13.117}$$

For a nontrivial solution the determinant of the coefficients of A, B, C, and D in (13.117) must vanish, leading to a quartic equation in ξ^2, as against a cubic when the piezoelectric constant ψ_{11} vanishes. The additional root is imaginary for all ω, i.e., it produces a nonpropagating mode. Thus there will be no additional resonances due to the piezoelectric effect, i.e., there will be a modification of the existing resonances only. Since ψ_{11}^2 is small in comparison with $\zeta_{11}\gamma_{11}$ and $\varepsilon_{22}^s\gamma_{11}$, the remaining three roots are almost the same as with $\psi_{11} = 0$. Hence there will be little error in the wave numbers ξ if they are computed as the roots of the aforementioned vanishing determinant with $\psi_{11} = 0$. Under these circumstances we have ξ_n, $n = 1, 2, 3$, as functions of ω^2 from the portion of the determinant from the first three of the linear equations (13.117) in A, B, and C, and

$$\xi_4^2 = -15\varepsilon_{22}^s/h^2\zeta_{11}, \tag{13.118}$$

independent of ω, from the fourth. From the three-dimensional solution presented in Chapter 11 it is seen that (13.118) should be more like

$$\xi_4^2 = -\pi^2\varepsilon_{22}^s/h^2\varepsilon_{11}^s. \tag{13.119}$$

An examination of the equations reveals that an electrical correction factor $\varkappa_e$ should be introduced, and the most probable position should be alongside $E_1^{(1)}$ in the expression for the plate electric enthalpy density $\mathscr{H}$. Nothing has been done along these lines, and it really doesn't matter for small coupling materials such as quartz. Corresponding to ξ_n, $n = 1, 2, 3$, we may determine amplitude ratios

$$(p_n{:}q_n{:}s_n) = (A_n{:}B_n{:}C_n) \tag{13.120}$$

from the first two of the linear equations (13.117) in A, B, and C, and

$$r_n = \frac{\varepsilon_{22}^s}{\psi_{11}} D_n = \frac{2}{3} \frac{h^3 \varepsilon_{22}^s \xi_n^2 s_n}{(15\varepsilon_{22}^s h^{-2} + \zeta_{11}\xi_n^2)}, \qquad n = 1, 2, 3, \quad (13.121)$$

from the fourth. In accordance with our approximation $p_4 = q_4 = s_4 = 0$, $r_4 = 1$. Consequently, we may write (13.116) as

$$\begin{aligned} (u_2^{(0)}, u_3^{(0)}) &= \sum_{n=1}^{3} M_n(p_n, q_n) \sin \xi_n x_1 , \\ u_1^{(1)} &= \sum_{n=1}^{3} M_n s_n \cos \xi_n x_1 , \\ \varphi^{(1)} &= M_4 \cosh \xi_4' x_1 + (\psi_{11}/\varepsilon_{22}^s) \sum_{n=1}^{3} M_n r_n \cos \xi_n x_1 , \end{aligned} \quad (13.122)$$

as the solution of the problem. In (13.122) $\xi_4' \equiv i\xi_4$. Substituting (13.122) into the boundary conditions (13.115) for the residual solution, we obtain

$$\begin{aligned} &\sum_{n=1}^{3} M_n[\varkappa_6 c_{66}(p_n\xi_n + s_n) + c_{56} q_n \xi_n] \cos \xi_n l = 0 , \\ &\sum_{n=1}^{3} M_n[\varkappa_6 c_{56}(p_n\xi_n + s_n) + c_{55} q_n \xi_n] \cos \xi_n l = 0 , \\ &M_4 \psi_{11} \varepsilon_{22}^s \xi_4' \sinh(\xi_4' l) - \sum_{n=1}^{3} M_n(\tfrac{2}{3} h^3 \gamma_{11} \varepsilon_{22}^s s_n + \psi_{11}^2 r_n)\xi_n \sin \xi_n l = 0 , \\ &M_4 \zeta_{11} \varepsilon_{22}^s \xi_4' \sinh(\xi_4' l) + \psi_{11} \sum_{n=1}^{3} M_n(\tfrac{2}{3} h^3 \varepsilon_{22}^s s_n - \zeta_{11} r_n)\xi_n \sin \xi_n l = 0 . \end{aligned} \quad (13.123)$$

Equations (13.123) constitute a system of linear homogeneous algebraic equations in M_1, M_2, M_3, and M_4. This system yields nontrivial solutions when the determinant of the coefficients of the M_n vanishes. As usual, the vanishing of said determinant results in a transcendental frequency equation which ultimately relates the l/h ratio to the eigenfrequencies. For a given frequency ω_μ the equation yields an infinite number of l/h ratios (or *vice-versa*), to each of which there correspond amplitude ratios

$$(M_1 : M_2 : M_3 : M_4) \quad (13.124)$$

from three of the four linear equations in the M_n, and the amplitude ratios

$$(p_n : q_n : s_n : r_n) \quad (13.125)$$

from three of the four linear equations in A, B, C, and D, as already discus-

sed. Thus we obtain the expressions

$$(u_2^{(0)\mu}, u_3^{(0)\mu}) = \sum_{n=1}^{3} M_n{}^{\mu}(p_n{}^{\mu}, q_n{}^{\mu}) \sin \xi_n{}^{\mu} x_1 ,$$

$$u_1^{(1)\mu} = \sum_{n=1}^{3} M_n{}^{\mu} s_n{}^{\mu} \cos \xi_n{}^{\mu} x_1 , \tag{13.126}$$

$$\varphi^{(1)\mu} = M_4{}^{\mu} \cosh \xi_4{}' x_1 + (\psi_{11}/\varepsilon_{22}^{s}) \sum_{n=1}^{3} M_n{}^{\mu} r_n{}^{\mu} \cos \xi_n{}^{\mu} x_1 ,$$

for the orthogonal eigenfunctions for the series solution of the residual problem.

Little error in a computation of frequencies and amplitude ratios will result if the frequency equation is obtained from the first three of (13.123) with $M_4 = 0$. The frequency equation would then be of the same form as in the purely elastic case ([38]), but the coefficients of the transcendental functions would have slightly different values due to the presence of the piezoelectric constants. In such an approximation the amplitude ratios $M_1{}^{\mu}$, $M_2{}^{\mu}$, and $M_3{}^{\mu}$ would be obtained from the first two of (13.123) and $M_4{}^{\mu}$ from the fourth.

The normalization factor $N_{(\mu)}$ is found by inserting the eigen-solution (13.126) in (13.71), remembering that $u_1^{(0)\mu} = u_3^{(1)\mu} = 0$, with the result

$$N_{(\mu)} = \tfrac{1}{2} A \sum_{m=1}^{3} \sum_{n=1}^{3} M_m{}^{\mu} M_n{}^{\mu} [(p_m{}^{\mu} p_n{}^{\mu} + q_m{}^{\mu} q_n{}^{\mu} + \tfrac{1}{3} h^2 s_m{}^{\mu} s_n{}^{\mu}) S_{mn}^{-}$$
$$- (p_m{}^{\mu} p_n{}^{\mu} + q_m{}^{\mu} q_n{}^{\mu} - \tfrac{1}{3} h^2 s_m{}^{\mu} s_n{}^{\mu}) S_{mn}^{+}] , \tag{13.127}$$

where

$$S_{mn}^{\pm} = [\sin(\xi_m{}^{\mu} l \pm \xi_n{}^{\mu} l)]/(\xi_m{}^{\mu} l \pm \xi_n{}^{\mu} l).$$

Now, from (13.92) we have

$$B_{\mu} = 0 ,$$

$$C_{\mu} = \frac{A}{2N_{(\mu)}} \sum_{n=1}^{3} \frac{M_n{}^{\mu}}{\xi_n{}^{\mu} h} (P_2 p_n{}^{\mu} + P_3 q_n{}^{\mu}) \left(\frac{\sin \xi_n{}^{\mu} l}{\xi_n{}^{\mu} l} - \cos \xi_n{}^{\mu} l \right) , \tag{13.128}$$

since $G_j^{(0)} = G_b^{(1)} = 0$ and $u_1^{(0)A} = u_1^{(1)A} = u_3^{(1)A} = 0$.

Finally, from (13.93) we have

$$A_{\mu} = C_{\mu} \omega^2 / (\omega_{\mu}{}^2 - \omega^2). \tag{13.129}$$

Note that for small damping, dissipation may be introduced simply by allowing some of the elastic constants in the associated eigenproblem to be

complex, thereby introducing a small amount of viscoelastic behavior, which contributes a time decay portion to the eigensolution of the free-vibration problem. This is admittedly a very approximate procedure, and one which we have not justified. However, it is probably very accurate for the small damping which occurs in vibrating piezoelectric crystals. In other words, it is very doubtful that a more accurate analysis is warranted. This procedure permits the relation of the time decay factor of a free vibration to the Q of the crystal, which should be related to the steady-state spatially-decaying mode rather than the transient time-decaying mode. Nevertheless, this procedure affords a useful and essentially valid simplification. Introducing the usual expression for small damping into (13.129), we obtain

$$A_\mu = \frac{C_\mu \omega^2}{\omega_\mu{}^2 + i(\omega_\mu{}^2/Q) - \omega^2}. \tag{13.130}$$

This latter expression prevents the solution from blowing up at resonance.

The solution is completed except for the expression for the correction factor $\varkappa_6$. This is determined by equating the thickness frequency predicted by the third of (13.110) with the thickness-shear frequency for rotated Y-cut quartz plates, which we have obtained previously from the appropriate solution of the three-dimensional equations and is given in Chapter 9 as the lowest root of (9.47) along with (9.41) and (9.42). The result is

$$\varkappa_6{}^2 = \frac{1}{3}\alpha_0{}^2\left(1 + \frac{e_{26}^2}{\varepsilon_{22}^S c_{66}^E}\right), \tag{13.131}$$

where α_0 is the lowest positive root of

$$\tan\alpha = \alpha\left(1 + \frac{\varepsilon_{22}^S c_{66}^E}{e_{26}^2}\right). \tag{13.132}$$

The formula for surface charge, (13.99), requires expressions for Y_μ and L in addition to A_μ. These are obtained by substituting (13.126) and (13.113) in (13.100) and (13.101), respectively, with the results

$$Y_\mu = A\sum_{n=1}^{3}\frac{M_n{}^\mu}{\xi_n{}^\mu l}\left(\varkappa_6 e_{26}(p_n{}^\mu \xi_n{}^\mu + s_n{}^\mu) + e_{25}q_n{}^\mu \xi_n{}^\mu - \frac{15}{4h^3}\psi_{11}r_n{}^\mu\right)\sin\xi_n{}^\mu l$$
$$- A\frac{15}{4h^3}\varepsilon_{22}^s M_4{}^\mu \frac{\sinh\xi_4{}' l}{\xi_4{}' l},$$
$$L = (A/2h)(\varkappa_6 e_{26}P_2 + e_{25}P_3 - \tfrac{5}{2}\varepsilon_{22}^s). \tag{13.133}$$

As noted previously, since the expression (13.129) for A_μ has a reson-

ance denominator, near a resonance only one term in the infinite series in the formula for the surface charge Q need be computed. In the foregoing it has been suggested that in computing the roots of the equations obtained by setting determinants of the coefficients in (13.117) and (13.123) equal to zero certain of the terms having ψ_{11} as a factor be neglected because of the smallness of ψ_{11}. By following this suggestion the coupling between the applied voltage V and the strain would be taken into account through e_{26}, as would the generation of the induced electric field by the strain gradient through the amplitude ratios r_n, but a part of the small counter-effect of the induced electric field on the strain would be neglected. This approximation need not be made if adequate computing facilities are available for computing the roots of the 4×4 algebraic determinant obtained from (13.117), which yields a quartic equation, and the 4×4 transcendental determinant which would replace the present transcendental determinant obtained from (13.123).

As noted previously, the solution to a problem including dissipation may be obtained more properly by means of the other method of solution, which transfers the inhomogeneities from the differential equations to the boundary conditions, thereby leaving a residual problem of homogeneous equations and inhomogeneous boundary conditions. For completeness we will indicate this second method of solution of the rotated Y-cut quartz plate problem, which solution we have just obtained by the first method. To this end, we take the auxiliary solution in the form

$$u_2^{(0)A} = u_3^{(0)A} = 0, \quad u_1^{(1)A} = R_1 V, \quad \varphi^{(1)A} = R_2 V, \tag{13.134}$$

where

$$R_1 = \varkappa_6 e_{26}/2h(\tfrac{1}{3}h^2\varrho\omega^2 - \varkappa_6{}^2 c_{66}), \qquad R_2 = h^2/3. \tag{13.135}$$

Substituting these expressions for the auxiliary solution, (13.134) and (13.135), into (13.110) and (13.112), it is clear that the residual problem consists of homogeneous differential equations with inhomogeneous boundary conditions, and all inhomogeneities are proportional to V. If the problem is nondissipative, it is clear that we obtain exactly the same resonance condition as before, and an off-resonance solution is not a series. If the problem is dissipative, the solution exists at all driving frequencies and we may obtain the surface charge Q (or any other quantity) in terms of V and the driving frequency ω. Resonance may then be defined as that ω at which Q is a maximum for a given V. This procedure involves much more calculation than the first, but it is exact, whereas the first is an approximation, although one which undoubtedly is very accurate in most practical instances.

Chapter 14

MECHANICAL EFFECT OF ELECTRODE PLATING

1. EQUATIONS FOR THE CRYSTAL PLATE

We are here interested in determining the effect of the elastic stiffness and inertia of the electrodes shown in Fig. 22 on the truncated form of the two-dimensional equations of motion we have obtained previously in Chapter 13. The equations for the unplated crystal (13.51) and (13.52) are reproduced here

$$T^{(0)}_{aj,a} + F^{(0)}_j = 2\varrho h \ddot{u}^{(0)}_j, \qquad T^{(1)}_{ab,a} - T^{(0)}_{2b} + F^{(1)}_b = \tfrac{2}{3}\varrho h^3 \ddot{u}^{(1)}_b, \tag{14.1}$$

$$D^{(0)}_{a,a} + D^{(1)}_2 = 0, \qquad D^{(1)}_{a,a} + 2D^{(2)}_2 = 0, \tag{14.2}$$

and the constitutive equations (13.49) and (13.50) are also reproduced here as

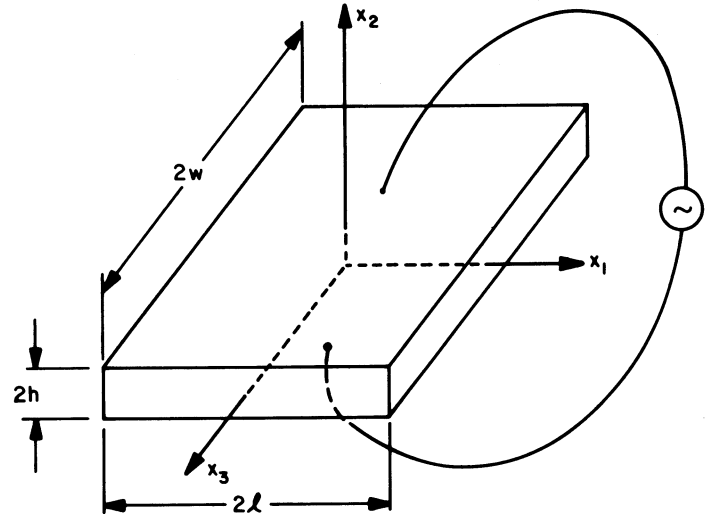

Fig. 22. A rectangular plate with electroded surfaces connected to a driving circuit.

$$
\begin{aligned}
T_{ij}^{(0)} &= 2hc_{ijkl}^{**}S_{kl}^{(0)} - e_{kij}^{**}E_k^{(0)}, \\
T_{ab}^{(1)} &= \tfrac{2}{3}h^3\gamma_{abcd}S_{cd}^{(1)} - \psi_{iab}E_i^{(1)},
\end{aligned}
\tag{14.3}
$$

$$
\begin{aligned}
D_i^{(0)} &= e_{iq}^{**}S_q^{(0)} + (1/2h)\varepsilon_{ij}^{*}E_j^{(0)} - (15/8h^3)\varepsilon_{i2}E_2^{(2)}, \\
D_i^{(1)} &= \psi_{iab}S_{ab}^{(1)} + (3/2h^3)\zeta_{ij}E_j^{(1)}, \\
D_2^{(2)} &= (15/8h^5)(3\varepsilon_{22}E_2^{(2)} - h^2\varepsilon_{2k}E_k^{(0)}),
\end{aligned}
\tag{14.4}
$$

and all the quantities have been defined previously in Chapter 13.

Now, before proceeding to the platings we must discuss a few points in a little greater detail than we have previously; namely, the significance of the quantities $T_{ij}^{(0)}$, $T_{ab}^{(1)}$, $F_j^{(0)}$, and $F_b^{(1)}$. First consider the stress resultants $T_{ij}^{(0)}$, which are defined by

$$
T_{ij}^{(0)} = \int_{-h}^{h} T_{ij}\, dx_2, \tag{14.5}
$$

from which we can see that $T_{11}^{(0)}$ and $T_{33}^{(0)}$ represent extensional forces per unit length in the plane of the plate. Similarly, we see that $T_{13}^{(0)}$ represents a shearing force per unit length in the plane of the plate, and that $T_{12}^{(0)}$ and $T_{32}^{(0)}$ represent shearing forces per unit length acting in the direction normal to the plane of the plate. The stress resultants $T_{ab}^{(1)}$ are defined by

$$
T_{ab}^{(1)} = \int_{-h}^{h} x_2 T_{ab}\, dx_2, \tag{14.6}
$$

from which it is clear that $T_{11}^{(1)}$ and $T_{33}^{(1)}$ represent bending moments and $T_{13}^{(1)}$ represents a twisting moment. Note that $T_{11}^{(0)}$, $T_{33}^{(0)}$, and $T_{13}^{(0)}$ are forces per unit length which occur in the elementary theory of the extension of thin plates and $T_{12}^{(0)}$, $T_{32}^{(0)}$, $T_{11}^{(1)}$, $T_{33}^{(1)}$, and $T_{13}^{(1)}$ are forces and moments per unit length which occur in the elementary theory of the flexure of thin plates ([40]). The surface loadings $F_j^{(0)}$ and $F_b^{(1)}$ are defined by

$$
F_j^{(0)} \equiv [T_{2j}]_{-h}^{h} = T_{2j}(h) - T_{2j}(-h), \tag{14.7}
$$

$$
F_b^{(1)} \equiv [x_2 T_{2b}]_{-h}^{h} = hT_{2b}(h) + hT_{2b}(-h), \tag{14.8}
$$

and will be of the utmost importance in obtaining the equations for the electroded crystal.

2. EQUATIONS FOR THE PLATINGS

The electrodes are assumed to be perfectly conducting, and are extremely thin compared with the thickness of the crystal plate (Fig. 23). Consequently, the purely mechanical portions of Eqs. (14.1) and (14.3),

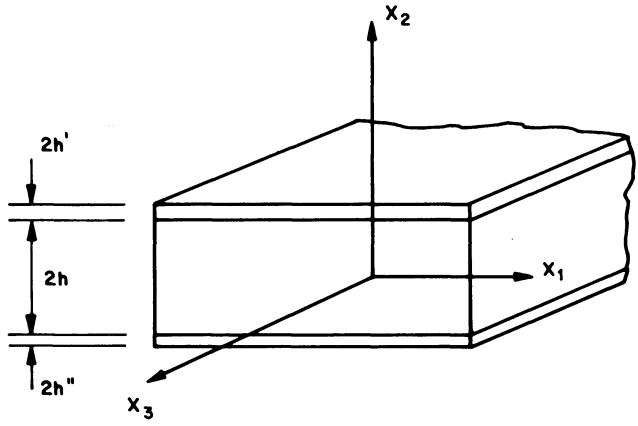

Fig. 23. Diagram of a plated crystal plate showing thicknesses of platings.

i.e., excluding all electrical terms, for the unplated crystal are applicable to each electrode with the additional simplification that all forces, $T_{2b}^{(0)}$, and moments, $T_{ab}^{(1)}$, per unit length associated with the elementary theory of flexure of thin plates may be neglected, since the plate is so thin that only its extensional resistance need be considered. Nevertheless, both its extensional and *transverse* inertia must be included, but rotatory inertia, $2\varrho h^3 \ddot{u}_b^{(1)}/3$, may be neglected. Under these circumstances all that remains of (14.1) and (14.3) are

$$\delta_{jb} T_{ab,a}^{(0)\prime} + F_j^{(0)\prime} = 2h' \varrho' \ddot{u}_j^{(0)\prime}, \tag{14.9}$$

$$T_{ab}^{(0)\prime} = 2h' \gamma'_{abcd} S_{cd}^{(0)\prime}, \tag{14.10}$$

where $\delta_{jb} = 0$ for $j = 3$ and

$$F_j^{(0)\prime} = \left[T'_{2j} \right]_{-h'}^{h'} = T'_{2j}(h') - T'_{2j}(-h'). \tag{14.11}$$

Equations (14.9)–(14.11) are for the upper electrode, and we have an identical set of equations for the lower electrode but with the primes replaced by double primes.

3. EQUATIONS FOR THE PLATED CRYSTAL PLATE

Since the electrodes are attached to the crystal plate, we must impose the conditions of continuity of surface tractions and mechanical displacements across the interfaces as did Mindlin ([41]). However, since we are

already limited to wave numbers which are not too large ($\xi h < 2$), we may ignore the free thickness-stretch motion ($u_2^{(1)\prime}$) which accompanies the extensional motion of the plating and write

$$T'_{2j}(-h') = T_{2j}(h), \qquad T''_{2j}(h'') = T_{2j}(-h), \tag{14.12}$$

$$u_i^{(0)\prime} = u_i^{(0)} + h\delta_{ia}u_a^{(1)}, \qquad u_i^{(0)\prime\prime} = u_i^{(0)} - h\delta_{ia}u_a^{(1)}. \tag{14.13}$$

Using (14.12), we may express the surface loadings $F_j^{(0)}$ and $F_a^{(1)}$ of the unelectroded crystal in terms of the surface loadings $F_j^{(0)\prime}$ and $F_j^{(0)\prime\prime}$ of the electrodes. Then the equations of motion of the electrodes, (14.9), may be incorporated in the stress equations of motion of the unplated crystal, (14.1), thus forming five stress equations of motion of the electroded crystal plate; and $u_i^{(0)\prime}$ and $u_i^{(0)\prime\prime}$ may be eliminated from the five equations as well as from all the constitutive relations by means of (14.13). In essence, this procedure will leave us with five stress equations of motion of the electroded crystal plus two electrostatic equations, for a total of seven equations in the seven dependent variables $u_j^{(0)}$, $u_a^{(0)}$, $\varphi^{(0)}$, and $\varphi^{(1)}$. To carry out this process from (14.7), (14.8) and (14.12) we note that

$$F_j^{(0)} = T'_{2j}(-h') - T''_{2j}(h''), \quad F_b^{(1)} = hT'_{2b}(-h') + hT''_{2b}(h''). \tag{14.14}$$

The successive addition and subtraction of (14.11) and its double primed counterpart, respectively, yield

$$\begin{aligned} T'_{2j}(-h') - T''_{2j}(h'') &= \mathscr{T}_j^{(0)} - F_j^{(0)\prime} - F_j^{(0)\prime\prime}, \\ hT'_{2b}(-h') + hT''_{2b}(h'') &= \mathscr{T}_b^{(1)} - h(F_b^{(0)\prime} - F_b^{(0)\prime\prime}), \end{aligned} \tag{14.15}$$

where

$$\mathscr{T}_j^{(0)} \equiv T'_{2j}(h') - T''_{2j}(-h''), \qquad \mathscr{T}_b^{(1)} \equiv hT'_{2b}(h') + hT''_{2b}(-h''). \tag{14.16}$$

Thus from (14.14) and (14.15) we obtain

$$F_j^{(0)} = \mathscr{T}_j^{(0)} - (F_j^{(0)\prime} + F_j^{(0)\prime\prime}), \qquad F_b^{(1)} = \mathscr{T}_b^{(1)} - h(F_b^{(0)\prime} - F_b^{(0)\prime\prime}). \tag{14.17}$$

We further obtain from the sum and difference of (14.9) and its double primed counterpart the equations

$$\begin{aligned} F_j^{(0)\prime} + F_j^{(0)\prime\prime} &= -\delta_{jb}(T_{ab}^{(0)\prime} + T_{ab}^{(0)\prime\prime})_{,a} + 2h'\varrho' \ddot{u}_j^{(0)\prime} + 2h''\varrho'' \ddot{u}_j^{(0)\prime\prime}, \\ F_b^{(0)\prime} - F_b^{(0)\prime\prime} &= -(T_{ab}^{(0)\prime} - T_{ab}^{(0)\prime\prime})_{,a} + 2h'\varrho' \ddot{u}_b^{(0)\prime} - 2h''\varrho'' \ddot{u}_b^{(0)\prime\prime}. \end{aligned} \tag{14.18}$$

Hence substituting from (14.18) into (14.17) and using the continuity of mechanical displacement conditions (14.13), we find

$$\begin{aligned} F_j^{(0)} &= \mathscr{F}_j^{(0)} + \delta_{jb}(T_{ab}^{(0)\prime} + T_{ab}^{(0)\prime\prime})_{,a} - 2h\varrho R_S \ddot{u}_j^{(0)} - 2h^2\varrho R_D \delta_{ja}\ddot{u}_a^{(1)}, \\ F_b^{(1)} &= \mathscr{F}_b^{(1)} + h(T_{ab}^{(0)\prime} - T_{ab}^{(0)\prime\prime})_{,a} - 2h^3\varrho R_S \ddot{u}_b^{(1)} - 2h^2\varrho R_D \ddot{u}_b^{(0)}, \end{aligned} \tag{14.19}$$

where

$$R_S = (\varrho' h' + \varrho'' h'')/\varrho h, \tag{14.20}$$

$$R_D = (\varrho' h' - \varrho'' h'')/\varrho h, \tag{14.21}$$

so that R_S and R_D are the ratios of the sums and differences of the weights of the electrodes to the weight of the portion of the crystal located between the electrode. Finally, substituting from (14.19) into (14.1), we obtain

$$\begin{aligned} \tau_{ij,i}^{(0)} + \mathscr{F}_j^{(0)} &= 2h\varrho(1 + R_S)\ddot{u}_j^{(0)} + 2h^2\varrho R_D \delta_{ja}\ddot{u}_a^{(1)}, \\ \tau_{ab,a}^{(1)} - \tau_{2b}^{(0)} + \mathscr{F}_b^{(1)} &= \tfrac{2}{3}h^3\varrho(1 + 3R_S)\ddot{u}_b^{(1)} + 2h^2\varrho R_D \ddot{u}_b^{(0)}, \end{aligned} \tag{14.22}$$

where

$$\begin{aligned} \tau_{ij}^{(0)} &\equiv T_{ij}^{(0)} + \delta_{ia}\delta_{jb}(T_{ab}^{(0)\prime} + T_{ab}^{(0)\prime\prime}), \\ \tau_{ab}^{(1)} &\equiv T_{ab}^{(1)} + h(T_{ab}^{(0)\prime} - T_{ab}^{(0)\prime\prime}). \end{aligned} \tag{14.23}$$

Noting from the conditions of continuity of mechanical displacement at the interface (14.13) that

$$S_{cd}^{(0)\prime} = S_{cd}^{(0)} + hS_{cd}^{(1)}, \qquad S_{cd}^{(0)\prime\prime} = S_{cd}^{(0)} - hS_{cd}^{(1)}, \tag{14.24}$$

and substituting from the constitutive equations for the unelectroded crystal (14.3) and the electrodes (14.10) and its double primed counterpart, and using (14.24), we obtain the stress constitutive equations for the electroded crystal in the form

$$\begin{aligned} \tau_{ij}^{(0)} &= 2h(c_{ijkl}^{**} + \delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld}\gamma_{abcd}^{S})S_{kl}^{(0)} + 2h^2\delta_{ia}\delta_{jb}\gamma_{abcd}^{D}S_{cd}^{(1)} + e_{kij}^{**}E_k^{(0)}, \\ \tau_{ab}^{(1)} &= \tfrac{2}{3}h^3(\gamma_{abcd} + 3\gamma_{abcd}^{S})S_{cd}^{(1)} + 2h^2\gamma_{abcd}^{D}S_{cd}^{(0)} - \Psi_{iab}E_i^{(1)}, \end{aligned} \tag{14.25}$$

where

$$\gamma_{abcd}^{S} \equiv (h'\gamma'_{abcd} + h''\gamma''_{abcd})/h, \tag{14.26}$$

$$\gamma_{abcd}^{D} \equiv (h'\gamma'_{abcd} - h''\gamma''_{abcd})/h, \tag{14.27}$$

and the electric-displacement constitutive equations for the electroded

crystal remain the same as for the unelectroded crystal, (14.4). Thus we now have the full system of equations for the electroded crystal, which may readily be reduced, by straightforward substitution of the constitutive equations (14.25) and the kinematical and potential relations (13.53) and (13.54) in the five stress equations of motion (14.22) and the two equations of electrostatics (14.2), to seven equations in the seven dependent variables $u_j^{(0)}$, $u_a^{(1)}$, $\varphi^{(0)}$, and $\varphi^{(1)}$. The equations are similar to but a bit more complicated than the equations for the unelectroded crystal. At this point we could proceed in the usual way to obtain initial, edge, and interior conditions sufficient for a unique solution ([41]). However, in view of our previous work in Chapter 13, the results are obvious, and we need not present the uniqueness theorem here. We could also obtain the orthogonality conditions for the electroded crystal in the usual straightforward manner, but we shall not bother to do so, even though the results are slightly different for the electroded crystal. Nevertheless, the orthogonality relation for homogeneous conditions in the interior and on the edge probably takes the form

$$\int_A [(1 + R_S)u_j^{(0)\mu}u_j^{(0)\nu} + \tfrac{1}{3}h^2(1 + 3R_S)u_b^{(1)\mu}u_b^{(1)\nu} + hR_D(u_b^{(1)\mu}u_b^{(0)\nu} + u_b^{(0)\mu}u_b^{(1)\nu})]\,dA = N_{(\mu)}\delta_{\mu\nu}, \tag{14.28}$$

where $N_{(\mu)}$ is a normalization factor.

Of particular importance to us is the situation where the upper and lower electrodes are identical and isotropic. Under these circumstances from (14.21) and (14.27) we have $\gamma^D_{abcd} = R_D = 0$, and from (14.26), (7.30), and (11.26)

$$\gamma^S_{abcd} = [\lambda^0\delta_{ab}\delta_{cd} + \mu(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})]2h'/h, \tag{14.29}$$

where $\lambda^0 = 2\mu\lambda/(\lambda + 2\mu)$ is the plate Lamé constant of the electrode and λ and μ are the Lamé constants for the electrode material. Then the stress equations of motion (14.22) for the symmetrically electroded crystal take the form

$$\begin{aligned} \tau^{(0)}_{ij,i} + \mathscr{T}^{(0)}_j &= 2h\varrho(1 + R_S)\ddot{u}_j^{(0)}, \\ \tau^{(1)}_{ab,a} - \tau^{(0)}_{2b} + \mathscr{T}^{(1)}_b &= \tfrac{2}{3}h^3\varrho(1 + 3R_S)\ddot{u}_b^{(1)}, \end{aligned} \tag{14.30}$$

and the stress constitutive equations (14.25) take the form

$$\begin{aligned} \tau^{(0)}_{ij} &= 2hc^{***}_{ijkl}S^{(0)}_{kl} - e^{**}_{kij}E^{(0)}_k, \\ \tau^{(1)}_{ab} &= \tfrac{2}{3}h^3\gamma^*_{abcd}S^{(1)}_{cd} - \Psi_{iab}E^{(1)}_i, \end{aligned} \tag{14.31}$$

where

$$c^{***}_{ijkl} = c^{**}_{ijkl} + (2h'/h)[\lambda^0\delta_{ib}\delta_{jb}\delta_{kd}\delta_{ld} + \mu(\delta_{ic}\delta_{jd}\delta_{ke}\delta_{ld} + \delta_{id}\delta_{jc}\delta_{kc}\delta_{ld})],$$
$$\gamma^*_{abcd} = \gamma_{abcd} + (6h'/h)[\lambda^0\delta_{ab}\delta_{cd} + \mu(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})]. \tag{14.32}$$

Thus it is clear that in the case of symmetric isotropic electrodes the equations are exactly the same as in the case of the unelectroded crystal except that the elastic constants and translational and rotational inertial densities are different.

4. FURTHER USES OF THE PLATING EQUATIONS

Before proceeding further I would like to indicate how the mechanical effect of the electrode plating may be included in the least squares or variational methods discussed in Chapter 12 for solving problems in the vibration of bounded plates. In either of these methods we superpose solutions of the three-dimensional equations which exactly satisfy the differential equations and the boundary conditions on the major surfaces in order to satisfy the conditions on the remaining surfaces approximately. The major difference is that in the previous method of using the two-dimensional equations the mechanical effect of the electrode served to change the differential equations and the edge conditions, while in the least squares and variational methods it changes the boundary conditions on the major surfaces of the plate and the edge conditions. That is to say, we proceed exactly as we did before with either the least squares or variational method except that we first change the mechanical boundary conditions on the major surfaces of the plate so as to account for the mechanical effect of the electrode on those surfaces, and then extend the integration over the minor surfaces so as to include the electrode plating. Without the electrode the mechanical boundary conditions for a plate with major surfaces at $x_2 = \pm h$ are

$$T_{2j} = 0 \qquad \text{at} \quad x_2 = \pm h. \tag{14.33}$$

We are here going to consider the case of symmetric electrodes only. The electrodes are sufficiently thin compared to the thickness of the plate that the mechanical equations for the upper electrode are the same equations as before, i.e.,

$$\delta_{jb}T^{(0)\prime}_{ab,a} + F^{(0)\prime}_j = 2h'\varrho'\ddot{u}^{(0)\prime}_j, \tag{14.34}$$

$$F^{(0)\prime}_j \equiv [T'_{2j}]^{h'}_{-h'} \equiv T'_{2j}(h') - T'_{2j}(-h'), \tag{14.35}$$

$$T^{(0)\prime}_{ab} = 2h'\gamma'_{abcd}\,S^{(0)\prime}_{cd}. \tag{14.36}$$

Since the electrodes are attached to the crystal and the wave numbers of interest are not too large ($\xi h < 5$), we may ignore the free thickness-stretch motion ($u_2^{(1)\prime}$) which accompanies the extensional motion of the plating and write

$$T'_{2j}(-h') = T_{2j}(h), \qquad u_j^{(0)\prime} = u_j(h). \tag{14.37}$$

Moreover, since the outside of the electrode is traction free, we have

$$T'_{2j}(h') = 0. \tag{14.38}$$

From the mechanical displacement continuity conditions in (14.13) we have

$$S_{cd}^{(0)\prime} = S_{cd}(h), \tag{14.39}$$

and from (5.19) we also have the stress constitutive equations

$$T_{ij} = c_{ijkl}u_{k,l} + e_{kij}\varphi_{,k}. \tag{14.40}$$

Substituting (14.36) in (14.34) and then (14.34) and (14.40) in the first of (14.37), we obtain the mechanical boundary conditions

$$c_{2jkl}u_{k,l} + e_{k2j}\varphi_{,k} = \delta_{jb}2h'\gamma'_{abcd}u_{c,ad} - 2h'\varrho'\ddot{u}_j \tag{14.41}$$

at $x_2 = +h$. We obtain a negative right-hand side at $x_2 = -h$, since the electrode on the lower surface is identical with the electrode on the upper surface and x_2 points up. Moreover, since the electrode is perfectly conducting, the potential within the electrode must be constant, and

$$\varphi_{,a} = 0,$$

so that the boundary conditions become

$$c_{2jkl}u_{k,l} + e_{22j}\varphi_{,2} \mp 2h'\delta_{jb}\gamma'_{abcd}u_{c,da} \pm 2h'\varrho'\ddot{u}_j = 0 \quad \text{at } x_2 = \pm h, \tag{14.42}$$

and since the electrodes are isotropic we have

$$\gamma'_{abcd} = \lambda'\delta_{ab}\delta_{cd} + \mu(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}). \tag{14.43}$$

It can readily be shown in the usual way that these boundary conditions, along with the usual electrical condition, are sufficient for a unique solution to the steady-state problem of the infinite plate. The electrical condition is, of course,

$$\varphi = \pm\varphi_0 \cos\omega t \qquad \text{at } x_2 = \pm h. \tag{14.44}$$

The homogeneous form ($\varphi_0 = 0$) of (14.44) and (14.42) determines the two-dimensional standing wave solutions and corresponding dispersion curves for the infinite plate including the mechanical effect of the electrode plating ([42]). Note that the new mechanical boundary conditions (14.42) on the major surfaces are homogeneous and linear. Consequently, the same procedure of superposing the thickness solution and an appropriate number of two-dimensional standing wave solutions of the homogeneous problem (i.e., with $\varphi = 0$ at $x_2 = \pm h$) in order to satisfy the remaining boundary conditions approximately by either the method of least squares or a variational technique is applicable. Note from (14.42) that even the thickness solution changes because of the inertia of the electrode. Note further that in integrating either the least-squares conditions or the variational conditions in the thickness direction along the minor surfaces of the plate the contribution of the electrode plating must be included. For example, suppose in a symmetric, purely elastic problem that the boundary conditions on each of the minor surfaces are, say,

$$T_{1j} = 0 \qquad \text{at} \quad x_1 = \pm l. \tag{14.45}$$

Then, according to the method of least squares discussed in Chapter 12, we must minimize

$$M = \frac{1}{2(h + 2h')} \left[\int_{-h}^{h} T_{1j} T_{1j}\, dx_2 + 2 \int_{h}^{h+2h'} T_{1b} T_{1b}\, dx_2 \right] \text{ at } x_1 = l, \tag{14.46}$$

since, as already discussed, T_{12} is negligible in the electrode plating. Now

$$T_{1b}^{(0)\prime} = \int_{h}^{h+2h'} T_{1b}\, dx_2, \tag{14.47}$$

and since T_{1b} is essentially constant in the electrode plating, we have

$$T_{1b} = T_{1b}^{(0)\prime} / 2h'. \tag{14.48}$$

Hence we must minimize

$$M = \frac{1}{2(h + 2h')} \left[\int_{-h}^{h} T_{1j} T_{1j}\, dx_2 + \frac{1}{h'} T_{1b}^{(0)\prime} T_{1b}^{(0)\prime} \right] \quad \text{at } x_1 = l. \tag{14.49}$$

In the same problem, using the variational technique discussed in Section 4 of Chapter 12, we have the variational condition

$$- \int_{-h}^{h} T_{1j}\, \delta u_j\, dx_2 - 2 \int_{h}^{h+2h'} T_{1b}\, \delta u_b\, dx_2 = 0 \qquad \text{at} \quad x_1 = l, \tag{14.50}$$

since T_{12} is negligible in the electrode plating. Since u_b is essentially constant in the electrode plating, as is T_{1b}, we have from (14.37), (14.48), and (14.50)

$$-\int_{-h}^{h} T_{1j}\,\delta u_j\,dx_2 - 2T_{1b}^{(0)\prime}\,\delta u_b^{(0)\prime} = 0 \qquad \text{at} \quad x_1 = l \tag{14.51}$$

as the variational condition for the approximation.

Chapter 15

SOME ELECTRICAL CIRCUIT CONSIDERATIONS

1. ELECTRICAL ADMITTANCE

In Section 6 of Chapter 13 we found the current through the crystal for a prescribed driving voltage across the crystal. In electrical engineering terminology we have found the admittance of the crystal [([43]), Chapter 3, Section 18)]. However, before we can introduce these terms properly we must discuss a sign convention associated with the terminology. The sign convention we will employ is contained in the diagram in Fig. 24, which indicates that by convention the current is considered positive if it flows from active to passive + to + [([43]), Chapter 9, Sections 1, 2]. The passive element can be represented by an admittance Y and eliminated from the diagram by means of the relation

$$I = YV, \tag{15.1}$$

so that we may consider the diagram in Fig. 25 instead of the one in Fig. 24. Thus it is clear that our formula for surface charge has effectively determined the admittance of the crystal, which, in that case, was a passive element.

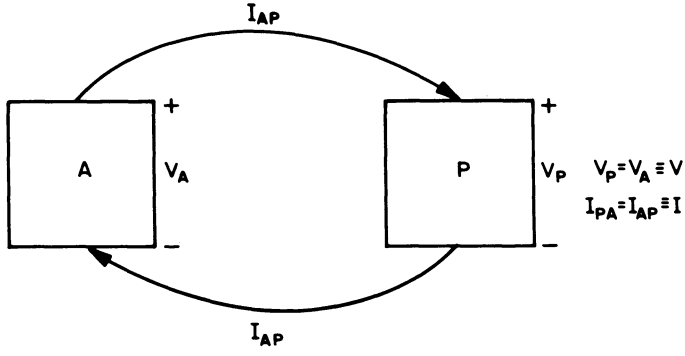

Fig. 24. An electrical circuit with a passive and an active element.

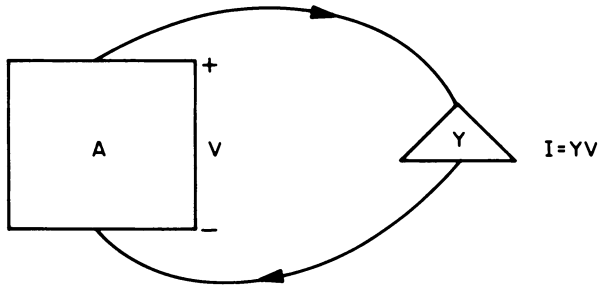

Fig. 25. An electrical circuit with the passive element represented by an admittance.

However, we must find out if I is $+\dot{Q}$ or $-\dot{Q}$ according to our convention. Since the x_2 axis has been taken to be positive in the up direction and according to our electrical convention the current flows down in this passive element, we have

$$I = -\int_A \dot{D}_2(-h)\,dA = -\dot{Q}, \tag{15.2}$$

and in our particular case from (13.102) and (15.2) we find near a resonance ($\omega \approx \omega_\nu$)

$$I = -i\omega V\left(L - \frac{\varepsilon'_{22}}{8h}A + \frac{Y_\nu C_\nu \omega^2}{\omega_\nu^2 - \omega^2}\right). \tag{15.3}$$

Hence the admittance of the crystal is given by

$$Y = -i\omega\left(L - \frac{\varepsilon'_{22}A}{8h} + \frac{Y_\nu C_\nu \omega^2}{\omega_\nu^2 - \omega^2}\right), \tag{15.4}$$

and is purely susceptive [([43]), Chapter 3, Section 18] when there is no damping. If damping is introduced in the usual manner, a conductive ([43]) component enters (15.4) by virtue of (13.130). We can now use the above formula (15.4) to put the crystal in a circuit with an active element and determine the characteristics of the entire circuit as a function of frequency if the frequency is confined in the neighborhood of ω_ν.

However, of prime interest to us here is the situation where the crystal is the active element, because in that case the problem cannot be completely formulated field-theoretically without determining the analytical expression for the admittance relation for the remaining passive portion of the circuit. We are again concerned about whether a sign is plus or minus according to our convention. Since the x_2 axis has been taken to be positive in the up

direction and according to our electrical convention the current flows up in an active element, we have

$$I = + \int_A \dot{D}_2(-h)\, dA = + \dot{Q}. \tag{15.5}$$

Thus from (15.1) and (15.5) our required expression for the admittance relation for the remaining passive portion of the circuit is

$$\int_A \dot{D}_2(-h)\, dA = YV. \tag{15.6}$$

As we shall see, this expression is crucial to the solution of a crystal vibration problem when the crystal is an active element.

2. COMPLEX NOTATION

Note that in Section 1 we introduced the usual electrical engineering convention [([43]), Chapter 3, Sections 13–15] of introducing complex quantities and meaning that the real part should be taken after the complete expression is written in terms of real quantities including the time dependence. For example, in Section 1

$$Y = Y^R + iY^I, \qquad V = (V_0{}^R + iV_0{}^I)e^{i\omega t},$$

and according to the convention, by YV we mean

$$\begin{aligned} YV &= \mathrm{Re}(Y^R + iY^I)(V_0{}^R + iV_0{}^I)(\cos\omega t + i\sin\omega t), \\ YV &= (Y^R V_0{}^R - Y^I V_0{}^I)\cos\omega t - (Y^R V_0{}^I + Y^I V_0{}^R)\sin\omega t. \end{aligned} \tag{15.7}$$

We shall adhere to this convention where convenient. Moreover, when we have a standing wave function as a solution we shall write, say,

$$U_1 = A \sin\eta x_2 \cos\xi x_1\, e^{i\omega t} \tag{15.8}$$

where A is complex and $\sin\eta x_2 \cos\xi x_1$ is real, so that in (15.8) we mean

$$U_1 = \sin\eta x_2 \cos\xi x_1\, (A^R \cos\omega t - A^I \sin\omega t). \tag{15.9}$$

When we have a traveling wave function as a solution we shall write, say,

$$U_1 = A \cosh\eta x_2 \exp - i(\xi x_1 - \omega t), \tag{15.10}$$

where A is complex and $\cosh \eta x_2$ is real, so that according to our convention, by (15.10) we mean

$$U_1 = \cosh \eta x_2 \, [A^R \cos(\xi x_1 - \omega t) + A^I \sin(\xi x_1 - \omega t)]. \quad (15.11)$$

This convention reduces the amount of writing considerably.

Chapter 16

APPLICATION TO A MONOLITHIC STRUCTURE

1. COUPLED THICKNESS-SHEAR AND FLEXURE

Consider the rotated Y-cut quartz plate shown in Fig. 26 with five sections which are denoted 0, 1, 2, 3, and 4. The 1 and 3 sections contain electrodes and the others do not. The 1 section is driven by the application of an alternating voltage to the surface electrodes and the energy is detected in the 3 section by a passive detecting circuit of admittance Y. As usual, the dimension out of the paper will be ignored. The solution actually requires the full system of two-dimensional equations (13.110) which we derived for x_1 dependence in rotated Y-cut quartz, and which contain coupled thickness-shear, flexure, and face-shear motions. However, we shall make an approximation which results in a tremendous simplification and which yields very accurate results if we confine the frequency to a certain very narrow range of great practical importance. Nevertheless, before we make the simplification we will discuss the solution of the more general problem in some detail, including the full piezoelectric coupling. When we make the approximation we will, of course, assume that the piezoelectric coupling is small.

The differential equations in all four sections are of the same form, but the voltage must be eliminated in sections 0, 2, and 4 by means of the zero surface charge conditions (13.65). As we know from Chapter 14, on account of the plating the mechanical material constants (elastic and inertial) in sections 1 and 3 are different than those in sections 0, 2, and 4. The first thing we do in all five sections is introduce auxiliary solutions in

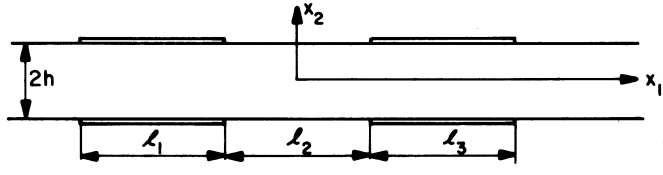

Fig. 26. A monolithic structure.

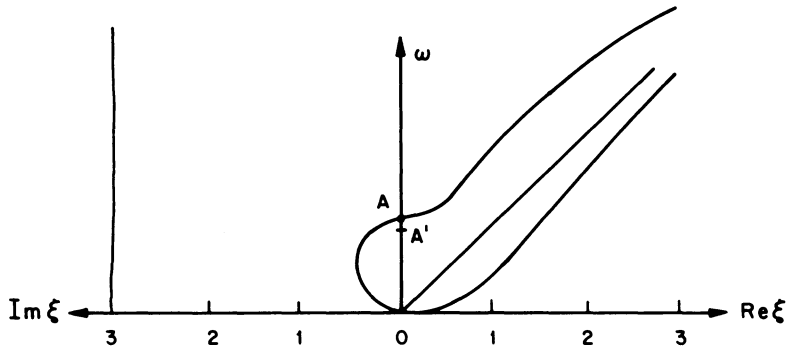

Fig. 27. Dispersion curves for the lowest antisymmetric modes of an infinite rotated *Y*-cut quartz plate for real and imaginary wave numbers including the influence of piezoelectricity.

(13.110) which leave residual homogeneous differential equations and inhomogeneous boundary conditions. We know that the standing and traveling wave solutions of these differential equations yield the dispersion curves shown in Fig. 27. The curves shown are for the unplated regions 0, 2, and 4. The curves for the plated regions 1 and 3 are of the same shape, but have the important difference that the nonzero intersection point A with the ω axis appears lower, say at the point A', on the diagram shown. Real values of the wave number ξ correspond to propagating waves in regions 0 and 4 and trigonometric standing waves in regions 1, 2, and 3. Imaginary values of ξ correspond to decaying (or growing) standing waves in regions 0 and 4 and hyperbolic standing waves in regions 1, 2, and 3. In regions 0 and 4 we can tolerate only decaying standing and outward traveling waves. Thus at any frequency we have four complex solutions in each of these two regions. In regions 1, 2, and 3 we can have both sin and cos (or sinh and cosh) for the spatial dependence of each variable, thereby giving eight complex solutions at any one frequency in each of regions 1, 2, and 3. Thus so far we have 32 independent solutions, and hence 32 complex unknowns. The voltage V in Section 3 is an additional complex unknown, and we have a total of 33 complex unknowns. The boundary conditions are the continuity of

$$T_{12}^{(0)},\quad T_{13}^{(0)},\quad T_{11}^{(1)},\quad D_1^{(1)},\quad u_2^{(0)},\quad u_3^{(0)},\quad u_1^{(1)},\quad \varphi^{(1)} \tag{16.1}$$

at each point of connection of one region with the adjacent one. There are four such points of connection of the five regions. Hence the continuity of the eight quantities in (16.1) at each of the four connection points gives us 32 complex equations in the 33 complex unknowns. However, we have

the admittance condition in Section 3, which takes the form

$$\int_A [\dot{D}_3^{(0)} + h^2 \dot{D}_3^{(2)}]\, dA = Y V_3, \tag{16.2}$$

in this approximation. This gives us a total of 33 complex linear algebraic equations in 33 complex unknowns, which may be solved numerically.

2. THE THICKNESS-SHEAR APPROXIMATION [46]

Numerical results from the aforementioned system of equations discussed in Section 1 in simpler cases ([44,45]) indicate that for frequencies in the vicinity of the thickness-shear frequency (say $A' < \omega < A$) and for small wave numbers ($|\xi h| \ll 1$) the thickness-shear mode dominates and the amplitudes of all other modes are extremely small. Consequently, we will formulate the problem discussed in Section 1 considering only that one mode and assuming small piezoelectric coupling. The significant characteristics of this essentially thickness-shear approximation are:

1. The frequency is very close to—both above and below—the thickness-shear frequency in each region.
2. The wave number ξ is sufficiently small that $|\xi^2 h^2| \ll 1$.

This means that our approximation will contain two parameters of smallness, (1) the ratio (deviation from the thickness-shear frequency): (thickness-shear frequency) and (2) the dimensionless quantity ξh. We now proceed to make the approximation in the two-dimensional equations. The equations are the four differential equations (13.110) in the dependent variables $u_2^{(0)}$, $u_3^{(0)}$, $u_1^{(1)}$, and $\varphi^{(1)}$, which we used when we considered the vibrations of a rotated Y-cut quartz plate in Chapter 13. We are at this stage considering the typical equations which are valid in any section, with different sections having different constants. The pertinent constitutive equations may be obtained from (13.107) when x_3 dependence is excluded, and are

$$\begin{aligned}
T_{12}^{(0)} &= 2h\varkappa_6^2 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + 2h\varkappa_6 c_{56} u_{3,1}^{(0)} + \varkappa_6 e_{26} V, \\
T_{13}^{(0)} &= 2h\varkappa_6 c_{56}(u_{2,1}^{(0)} + u_1^{(1)}) + 2h c_{55} u_{3,1}^{(0)} + e_{25} V, \\
T_{11}^{(1)} &= \tfrac{2}{3} h^3 \gamma_{11} u_{1,1}^{(1)} + \psi_{11} \varphi_{,1}^{(1)}, \\
D_1^{(1)} &= \psi_{11} u_{1,1}^{(1)} - (3/2h^3)\zeta_{11} \varphi_{,1}^{(1)}, \\
D_2^{(0)} &= \varkappa_6 e_{26}(u_{2,1}^{(0)} + u_1^{(1)}) + e_{25} u_{3,1}^{(0)} - (15/4h^3)\varepsilon_{22}\varphi^{(1)} + (3/4h)\varepsilon_{22} V, \\
D_2^{(2)} &= (15/4h^5)\varepsilon_{22}(3\varphi^{(1)} - h^2 V).
\end{aligned} \tag{16.3}$$

Since the piezoelectric coupling is small, for purposes of obtaining a solution all electrical quantities can be ignored in the unelectroded sections 0, 2, and 4, the electrostatic equation can be ignored in the driving section 1, and the stress equation and electrostatic equations can be solved separately in the detection Section 3, and all electrical edge conditions can be omitted in the analysis. However, if when the solution is obtained, we wish to calculate the current through the driving Section 1, we must first obtain the induced electrical-potential resultant $\varphi^{(1)}_1$ from the electrostatic equation in Section 1 using the mechanical functions known from the solution. Thus under the assumption of small piezoelectric coupling it is perfectly permissible to make the approximation, in the three stress equations of motion in (13.110), of ignoring the induced electrical potential resultant $\varphi^{(1)}$.

We begin the approximation by noting that $c_{56} \ll c_{55}$, and neglecting c_{56}. Then since we are not interested in the resonances associated with $u_3^{(0)}$ we may ignore the second differential equation and $T_{13}^{(0)}$. At this stage the two remaining mechanical equations in (13.110) take the form

$$c_{66}\varkappa_6^2(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + \varrho\omega^2 u_2^{(0)} = 0\,, \tag{16.4}$$

$$(2h^3/3)\gamma_{11}u_{1,11}^{(1)} - 2h\varkappa_6^2 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + \tfrac{2}{3}h^3\varrho\omega^2 u_1^{(1)} - \varkappa_6 e_{26}V = 0\,, \tag{16.5}$$

and the remaining constitutive equations (16.3) take the form

$$\begin{aligned}
T_{12}^{(0)} &= 2h\varkappa_6^2 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + \varkappa_6 e_{26}V\,,\\
T_{11}^{(1)} &= \tfrac{2}{3}\,h^3\gamma_{11}u_{1,1}^{(1)}\,,\\
D_1^{(1)} &= \psi_{11}u_{1,1}^{(1)} - (3/2h^3)\zeta_{11}\varphi_{,1}^{(1)}\,,\\
D_2^{(0)} &= \varkappa_6 e_{26}(u_{2,1}^{(0)} + u_1^{(1)}) - (15/4h^3)\varepsilon_{22}\varphi^{(1)} + (3/4h)\varepsilon_{22}V\,,\\
D_2^{(2)} &= (15/4h^5)\varepsilon_{22}(3\varphi^{(1)} - h^2V).
\end{aligned} \tag{16.6}$$

Note from (16.4)–(16.6) that the two remaining mechanical differential equations (16.4) and (16.5) may be written

$$T_{12,1}^{(0)} + \varrho\omega u_2^{(0)} = 0\,, \tag{16.7}$$

$$T_{11,1}^{(1)} - T_{12}^{(0)} + \tfrac{2}{3}h^3\varrho\omega^2 u_1^{(1)} = 0. \tag{16.8}$$

Ignoring the mechanical effect of the electrode, we take

$$u_2^{(0)} = A\sin\xi x_1\,, \qquad u_1^{(1)} = B\cos\xi x_1\,, \tag{16.9}$$

such that

$$\omega = (\pi/2h)(c_{66}/\varrho)^{1/2} + \varepsilon = \omega_0 + \varepsilon, \quad |\xi^2 h^2| \ll 1, \quad |\varepsilon| \ll \omega_0, \tag{16.10}$$

and substitute (16.9) in the homogeneous form ($V = 0$) of (16.4) and (16.5), to obtain

$$A = (h^2\xi/3)B, \tag{16.11}$$

$$-\tfrac{2}{3}h\gamma_{11}\xi^2h^2B - (\pi^2/6)c_{66}\xi hA + \tfrac{2}{3}h^2\pi(c_{66}\varrho)^{1/2}\varepsilon B = 0. \tag{16.12}$$

Substituting from (16.11) into (16.12), we obtain

$$\tfrac{2}{3}h[\gamma_{11} + (\pi^2/12)c_{66}]\xi^2h^2B - \tfrac{2}{3}h^2\pi(c_{66}\varrho)^{1/2}\varepsilon B = 0. \tag{16.13}$$

The implications of the foregoing are that (16.4) serves to enable us to solve for $u_2^{(0)}$ in terms of $u_1^{(1)}$ in the form

$$u_2^{(0)} = -(h^2/3)\, u_{1,1}^{(1)}, \tag{16.14}$$

which may then be substituted in (16.5) to yield

$$\tfrac{2}{3}h^3(\gamma_{11} + \varkappa_6{}^2c_{66})u_{1,11}^{(1)} - 2h\varkappa_6{}^2c_{66}u_1^{(1)} + \tfrac{2}{3}h^3\varrho\omega^2u_1^{(1)} - \varkappa_6e_{26}V = 0. \tag{16.15}$$

The above approximation was for the unelectroded portion of the crystal. A similar approximation may, of course, be performed for the electroded portion. However, since the mass loading of the electrodes is very small, the effect of small piezoelectric coupling may be comparable with the effect of the mass loading. Consequently, before proceeding further we must obtain the thickness solution of a piezoelectric plate including the mass loading of the electrode.

3. THICKNESS VIBRATIONS OF ELECTRODED ROTATED Y-CUT QUARTZ INCLUDING THE MASS LOADING OF THE ELECTRODE

We begin this section by writing the equations for the thickness-shear vibrations of a rotated Y-cut crystal with surface electrodes in the form

$$c_{66}u_{1,22} + e_{26}\varphi_{,22} = \varrho\ddot{u}_1, \qquad e_{26}u_{1,22} - \varepsilon_{22}\varphi_{,22} = 0, \tag{16.16}$$

$$c_{66}u_{1,2} + e_{26}\varphi_{,2} \pm 2h'\varrho'\ddot{u}_1 = 0 \qquad \text{at} \quad x_2 = \pm h, \tag{16.17}$$

$$\varphi = \pm\varphi_0 \cos\omega t \qquad \text{at} \quad x_2 = \pm h. \tag{16.18}$$

Equations (16.16) are the nontrivial equations of (9.34), (16.18) is the last of (9.35), and (16.17) is the form taken by (14.42) in this case. Ignoring the time factor $\cos \omega t$, the steady-state solution satisfying the differential equations (16.16) and the electrical boundary conditions (16.18) is given by

$$u_1 = A \sin \eta x_2, \qquad \left(c_{66} + \frac{e_{26}^2}{\varepsilon_{22}}\right)\eta^2 = \varrho\omega^2,$$

$$\varphi = \frac{e_{26}}{\varepsilon_{22}} A \sin \eta x_2 - \frac{e_{26} A \sin \eta h}{\varepsilon_{22} h} x_2 + \frac{\varphi_0}{h} x_2. \tag{16.19}$$

Substituting from (16.19) into the mechanical boundary conditions (16.17), we obtain

$$A(\eta h \cos \eta h - k_{26}^2 \sin \eta h - R\eta^2 h^2 \sin \eta h) = - e_{26}\varphi_0/\bar{c}_{66}, \tag{16.20}$$

where

$$k_{26}^2 = e_{26}^2/\bar{c}_{66}\varepsilon_{22}, \qquad R = 2\varrho' h'/\varrho h, \tag{16.21}$$

and

$$\bar{c}_{66} = c_{66} + (e_{26}^2/\varepsilon_{22}). \tag{16.22}$$

Resonance occurs when the coefficient of A in (16.20) vanishes, i.e., when

$$\eta h \cos \eta h - k^2 \sin \eta h - R\eta^2 h^2 \sin \eta h = 0,$$

or

$$\tan \eta h = \eta h/(k^2 + R\eta^2 h^2), \tag{16.23}$$

where we have dropped the subscripts 26 on k. Since k^2 and R are very small, the first root ηh of (16.23) will differ from $\pi/2$ by a small quantity, say Δ. Then we have

$$\eta h = (\pi/2) - \Delta, \tag{16.24}$$

and substituting (16.24) in (16.23), we find

$$\frac{\cos \Delta}{\sin \Delta} = \frac{(\pi/2) - \Delta}{k^2 + R(\frac{1}{4}\pi^2 - \pi\Delta + \Delta^2)}. \tag{16.25}$$

Expanding the trigonometric functions in (16.25) as power series in Δ, recalling that $R \ll 1$ and $k^2 \ll 1$, and retaining up to quadratic terms in Δ, we obtain

$$\Delta^2 - \tfrac{1}{2}\pi\Delta + k^2 + \tfrac{1}{4}R\pi^2 = 0, \tag{16.26}$$

which has the pertinent root

$$\Delta = (2k^2/\pi) + \tfrac{1}{2}\pi R. \tag{16.27}$$

Hence from (16.27) and (16.24) we have

$$\eta h = \frac{\pi}{2}\left(1 - R - \frac{4k^2}{\pi^2}\right), \tag{16.28}$$

and the piezoelectric effect of the electrode is negligible compared to the mass loading effect if $4k^2/\pi^2 \ll R$. Thus from (16.28), the second of (16.19), and (16.22), the thickness frequency in the electroded portion is given by

$$\bar{\omega}_s = \frac{\pi}{2h}\left(1 - R - \frac{4k^2}{\pi^2}\right)\left(\frac{\bar{c}_{66}}{\varrho}\right)^{1/2}. \tag{16.29}$$

It can readily be shown that the thickness frequency in the unelectroded region is given by

$$\omega_N = (\pi/2h)(\bar{c}_{66}/\varrho)^{1/2}. \tag{16.30}$$

The thickness frequency which we used in Section 2 of this chapter and appears in (16.10) is

$$\omega_0 = (\pi/2h)(c_{66}/\varrho)^{1/2}. \tag{16.31}$$

Now, from the first of (16.21) and (16.22) we have

$$c_{66} = \bar{c}_{66}(1 - k^2), \tag{16.32}$$

and since $k^2 \ll 1$, we have

$$\bar{c}_{66} = c_{66}(1 + k^2). \tag{16.33}$$

The quantity which is really of interest is $\bar{\omega}_s/\omega_N$, and from (16.29) and (16.30) this is given by

$$\bar{\omega}_s/\omega_N = 1 - R - (4k^2/\pi^2). \tag{16.34}$$

Now from (16.33) we have

$$(\bar{c}_{66})^{1/2} = [c_{66}(1 + k^2)]^{1/2} = (c_{66})^{1/2}(1 + \tfrac{1}{2}k^2). \tag{16.35}$$

Hence from (16.30), (16.31), and (16.35) we have $\omega_N/\omega_0 = 1 + \frac{1}{2}k^2$, and if we wish to continue to use the thickness-shear frequency ω_0 from our previous work in Section 2 in the small piezoelectric coupling approxi-

mation we must define a fictitious piezoelectrically unstiffened thickness-shear frequency $\bar{\omega}_0$ by

$$\bar{\omega}_0 = \bar{\omega}_s(1 - \tfrac{1}{2}k^2). \tag{16.36}$$

Then from (16.36), (16.29), (16.31), (16.35), and (16.34) we have

$$\bar{\omega}_0/\omega_0 = \bar{\omega}_s/\omega_N = 1 - R - (4k^2/\pi^2),$$

and the important relationship (16.34) is adhered to. From (16.29), (16.31) and (16.35) note that

$$\begin{aligned}\frac{\bar{\omega}_s}{\omega_0} &= \left(1 - R - \frac{4k^2}{\pi^2}\right)\left(1 + \frac{k^2}{2}\right) = 1 + \frac{k^2}{2}\left(1 - \frac{8}{\pi^2}\right) - R \\ &\approx 1 + 0.1k^2 - R\end{aligned} \tag{16.37}$$

is of no physical significance here, since the piezoelectric constants cannot be varied. However, for electrically biased electrostriction the effective piezoelectric constants could be varied, and then (16.37) would have physical significance. Note from (16.37) that for small piezoelectric coupling the combination of piezoelectric stiffening and the electric boundary condition at a driven electrode serves to raise—not lower—the thickness frequency.

4. THE THICKNESS-SHEAR APPROXIMATION IN THE ELECTRODED REGION

We may now make the approximation for the electroded portion. In doing so we will naturally introduce the defined thickness-shear frequency $\bar{\omega}_0$ given in (16.36).

From (14.30)–(14.32) we find that the pertinent equations for the electroded portion analogous to Eqs. (16.4) and (16.5) for the unelectroded portion are

$$c_{66}\bar{\varkappa}_6^2(u_{2,11}^{(0)} + u_{1,1}^{(1)}) + \varrho(1 + R)\omega^2 u_2^{(0)} = 0, \tag{16.38}$$

$$\tfrac{2}{3}h^3\bar{\gamma}_{11}u_{1,11}^{(1)} - 2h\bar{\varkappa}_6^2 c_{66}(u_{2,1}^{(0)} + u_1^{(1)}) + \tfrac{2}{3}h^3\varrho(1 + 3R)\omega^2 u_1^{(1)} - \bar{\varkappa}_6 e_{26}V = 0. \tag{16.39}$$

where the barred coefficients are for the plated region. The thickness-shear frequency predicted by (16.39) is given by

$$\bar{\omega}_t = \frac{\bar{\varkappa}_6}{h}\left[\frac{3c_{66}}{(1 + 3R)\varrho}\right]^{1/2}, \tag{16.40}$$

and is to be compared with $\bar{\omega}_0$ defined in (16.36), which from (16.36), (16.29), and (16.35) is given by

$$\bar{\omega}_0 = \frac{\pi}{2h}\left(1 - R - \frac{4k^2}{\pi^2}\right)\left(\frac{c_{66}}{\varrho}\right)^{1/2}, \tag{16.41}$$

instead of $\bar{\omega}_s$, because we used ω_0 instead of ω_N in the unelectroded region, as noted in Section 3. Hence from (16.40) and (16.41) we have

$$\bar{\varkappa}_6^2 = \frac{\pi^2}{12}\left(1 - R - \frac{4k^2}{\pi^2}\right)^2 (1 + 3R),$$

which for $R \ll 1$ and $k^2 \ll 1$ may be written

$$\bar{\kappa}_6^2 = \frac{\pi^2}{12}\left(1 + R - \frac{8k^2}{\pi^2}\right). \tag{16.42}$$

Now, proceeding as in Section 2 for $\omega = \bar{\omega}_0 + \bar{\varepsilon}$, $|\bar{\varepsilon}| \ll \bar{\omega}_0$, and $|\xi^2 h^2| \ll 1$, from (16.38), (16.41), and (16.42) we find

$$u_2^{(0)} = -\frac{h^2}{3}\,\frac{1 + R - (8k^2/\pi^2)}{1 - R - (8k^2/\pi^2)}\,u_{1,1}^{(1)},$$

or since $R \ll 1$ and $k^2 \ll 1$, we have

$$u_2^{(0)} = -\tfrac{1}{3}h^2(1 + 2R)u_{1,1}^{(1)}, \tag{16.43}$$

and substituting (16.43) in (16.39), we obtain

$$\tfrac{2}{3}h^3[\bar{\gamma}_{11} + \bar{\varkappa}_6^2(1 + 2R)c_{66}]\,u_{1,11}^{(1)} - 2h\bar{\varkappa}_6^2 c_{66}u_1^{(1)} + \tfrac{2}{3}h^3\varrho(1 + 3R)\,\omega^2 u_1^{(1)} - \bar{\varkappa}_6 e_{26}V = 0, \tag{16.44}$$

which is a differential equation of the same form as (16.15) but with slightly different coefficients.

5. THE EDGE CONDITIONS FOR THE THICKNESS-SHEAR APPROXIMATION

The edge conditions in the thickness-shear approximation [Eqs. (16.15) and (16.44)] are extremely complicated indeed, as a result of the fact that one branch—the flexural—has been eliminated and only one—the thickness-shear—remains. The approximation itself is valid in any one region

only if the frequency is very near the thickness-shear frequency. In any steady-state vibration problem in which there are two or more different sections the frequency must be the same in each section. Consequently, for Eqs. (16.15) and (16.44) to be valid the thickness-shear frequencies must be very nearly the same in adjacent sections. I cannot emphasize too much the need to note carefully that Eq. (16.15) [or (16.44)] is an approximation to the equations of coupled thickness-shear and flexure, and it is valid in a highly limited range only. Now, the equations of coupled thickness-shear and flexure have four continuity conditions at an edge connecting one region with another. The four conditions are the continuity of

$$T_{12}^{(0)}, \quad T_{11}^{(1)}, \quad u_2^{(0)}, \quad u_1^{(1)}; \tag{16.45}$$

and two branches in each section are required in order to satisfy the four conditions; however, the present approximation no longer has two branches, it has only one. The first step in resolving this dilemma is to recall that to the present approximation we have in the unlectroded region

$$T_{12}^{(0)} = -\tfrac{2}{3}h^3\varkappa_6{}^2c_{66}u_{1,11}^{(1)} + 2h\varkappa_6{}^2c_{66}u_1^{(1)}, \tag{16.46}$$

$$T_{11}^{(1)} = \tfrac{2}{3}h^3\gamma_{11}u_{1,1}^{(1)}, \tag{16.47}$$

$$u_2^{(0)} = -\tfrac{1}{3}h^2u_{1,1}^{(1)}. \tag{16.48}$$

Since in this approximation $|\xi^2h^2| \ll 1$, we may write $T_{12}^{(0)}$ in the form

$$T_{12}^{(0)} = 2h\varkappa_6{}^2c_{66}u_1^{(1)}, \tag{16.49}$$

in place of (16.46). Note from (16.47)–(16.49) that of the four continuity conditions in (16.45) $T_{12}^{(0)}$ and $u_1^{(1)}$ are large and $T_{11}^{(1)}$ and $u_2^{(0)}$ are small, since $|\xi h| \ll 1$, and the large ones and small ones are separately proportional. Note further that the product terms

$$T_{12}^{(0)}\dot{u}_2^{(0)}, \qquad T_{11}^{(1)}\dot{u}_1^{(1)}, \tag{16.50}$$

which appear in Mindlin's ([36]) uniqueness theorem for the equations of coupled thickness-shear and flexure, may, by virtue of (16.47)–(16.49), be written

$$-\tfrac{2}{3}h^3\varkappa_6{}^2c_{66}u_1^{(1)}\dot{u}_{1,1}^{(1)}, \qquad \tfrac{2}{3}h^3\gamma_{11}u_{1,1}^{(1)}\dot{u}_1^{(1)}, \tag{16.51}$$

from which it is clear that both terms are of the same order of magnitude.

Consequently, it is impossible to eliminate the flexure in favor of the shear, or *viceversa*, at this stage. Hence at a prescribed homogeneous edge condition (free, fixed, or mixed) we must satisfy two conditions approximately with one branch, and at a junction connecting two adjacent regions we must satisfy all four conditions approximately with one branch in each section. In the prescribed homogeneous cases we can always eliminate one of the two conditions, since, as noted previously, two of the four quantities specifying the conditions are small compared to the other two and the large ones and small ones are separately proportional. Hence we have simply to satisfy the large condition and ignore the small negligible one when it exists. Thus when the edge is either completely free ($T_{12}^{(0)} = T_{11}^{(1)} = 0$) or completely fixed ($u_2^{(0)} = u_1^{(1)} = 0$), we have to satisfy the edge condition

$$u_1^{(1)} = 0. \tag{16.52}$$

Clearly we must satisfy this same condition when the edge is supported in such a way that $T_{12}^{(0)} = u_1^{(1)} = 0$. Only when the edge is supported in such a way that $T_{11}^{(1)} = u_2^{(0)} = 0$ do we have to satisfy

$$u_{1,1}^{(1)} = 0. \tag{16.53}$$

In the case of a junction connecting two adjacent regions we can satisfy all four conditions approximately only when two of the four conditions are approximately equal to the other two. This will be the case when adjacent sections have approximately the same thickness and flexural and shear properties, i.e., when h, γ_{11}, and $\varkappa_6{}^2 c_{66}$ have very nearly the same values in adjacent sections, as in the problems with which we are concerned. Under these circumstances the four continuity conditions, (16.45), at a junction may be replaced by the two continuity conditions

$$u_1^{(1)}, \quad u_{1,1}^{(1)}, \tag{16.54}$$

or any of the other equivalent combinations of two conditions for that matter. Thus the *thickness-shear approximation* is sensible only if the thickness and shear and flexural stiffnesses are approximately the same on adjacent sides of a junction, and under these circumstances all four continuity conditions are satisfied approximately by the one remaining branch associated with each section. These considerations imply that if the shear and/or flexural characteristics in two adjacent sections differ considerably, the *thickness-shear approximation* would be useless and the fully coupled equations of thickness-shear and flexure would have to be used.

6. APPLICATION OF THE THICKNESS-SHEAR APPROXIMATION TO A MONOLITHIC STRUCTURE

We now have the equations and boundary conditions to solve the problem discussed in Section 1 using the *thickness-shear approximation.* In regions 0, 2, and 4 the equation is (16.15), which, for convenience, we rewrite here as

$$\tfrac{2}{3}h^3(\gamma_{11} + \varkappa_6{}^2 c_{66})\, u^{(1)}_{n\,1,11} - 2h\varkappa_6{}^2 c_{66} u^{(1)}_{n\,1} + \tfrac{2}{3}h^3\varrho\omega^2 u^{(1)}_{n\,1} = 0. \tag{16.55}$$

In region 1 the equation is (16.44), which, for convenience, we rewrite here as

$$\tfrac{2}{3}h^3[\bar{\gamma}_{11} + \bar{\varkappa}_6{}^2(1 + 2R)c_{66}]\, u^{(1)}_{1\,1,11} - 2h\bar{\varkappa}_6{}^2 c_{66}\, u^{(1)}_{1\,1} + \tfrac{2}{3}h^3\varrho(1 + 3R)\omega^2\, u^{(1)}_{1\,1} - \bar{\varkappa}_6 e_{26} V_1 = 0. \tag{16.56}$$

In region 3 we have the same equation as (16.56) but with a lower script 3 instead of a 1, and from the last of (13.110) the electrostatic equation

$$\tfrac{2}{3}h^3\psi_{11}\, u^{(1)}_{3\,1,11} - \zeta_{11}\, \varphi^{(1)}_{3,11} + (15/h^2)\varepsilon_{22}\, \varphi^{(1)}_{3} - 5\varepsilon_{22} V_3 = 0, \tag{16.57}$$

which, because of small piezoelectric coupling, may be solved separately from the mechanical equation analogous to (16.56) for region 3. In region 3 from (15.6), (13.63), and the fact that $D^{(1)}_2 = 0$ according to (13.107) we also have the admittance condition

$$\int_A \left[\dot{D}^{(0)}_{3\,2} + h^2 \dot{D}^{(2)}_{3\,2}\right] dA = Y V_3, \tag{16.58}$$

where from (13.107) and (16.43) we have

$$D^{(0)}_{3\,2} = -\bar{\varkappa}_6 e_{26} \frac{h^2}{3}(1 + 2R)\, u^{(1)}_{3\,1,11} + \bar{\varkappa}_6 e_{26}\, u^{(1)}_{3\,1} - \frac{15}{4h^3}\varepsilon_{22}\varphi^{(1)}_3 + \frac{3}{4h}\varepsilon_{22} V_3,$$

$$D^{(2)}_{3\,2} = (15/4h^5)\,\varepsilon_{22}(3\varphi^{(1)}_3 - h^2 V_3). \tag{16.59}$$

At each point of connection of one region with the adjacent one we have according to (16.54)

$$u^{(1)}_{n\,1} = u^{(1)}_{n+1\,1} \qquad n = 0, 1, 2, 3, \tag{16.60}$$

$$u^{(1)}_{n\,1,1} = u^{(1)}_{n+1\,1,1} \qquad n = 0, 1, 2, 3. \tag{16.61}$$

Since the frequencies must be confined to be in the neighborhood of the thickness-shear resonances, we may simplify the equations somewhat by recalling from (16.10) that

$$\omega = \omega_0 + \varepsilon = \frac{\pi}{2h}\left(\frac{c_{66}}{\varrho}\right)^{1/2} + \varepsilon, \tag{16.62}$$

and then defining $\bar{\varepsilon}$ so that

$$\omega = \bar{\omega}_0 + \bar{\varepsilon} = \frac{\pi}{2h}\left(1 - R - \frac{4k^2}{\pi^2}\right)\left(\frac{c_{66}}{\varrho}\right)^{1/2} + \bar{\varepsilon}, \tag{16.63}$$

and then substituting (16.62) and (16.63) in (16.55) and (16.56) to obtain

$$\tfrac{2}{3}h^3(\gamma_{11} + \varkappa_6{}^2 c_{66})\, \underset{n}{u}{}^{(1)}_{1,11} + \tfrac{2}{3}\pi h^2 (c_{66}\varrho)^{1/2}\varepsilon\, \underset{n}{u}{}^{(1)}_{1} = 0, \tag{16.64}$$

for $n = 0, 2, 4$, and

$$\tfrac{2}{3}h^3[\bar{\gamma}_{11} + \bar{\varkappa}_6^2(1 + 2R)c_{66}]\, \underset{n}{u}{}^{(1)}_{1,11} + \tfrac{2}{3}\pi h^2\left(1 + 2R - \frac{4k^2}{\pi^2}\right)(c_{66}\varrho)^{1/2}\bar{\varepsilon}\, \underset{n}{u}{}^{(1)}_{1} - \bar{\varkappa}_6 e_{26}\, \underset{n}{V} = 0, \tag{16.65}$$

for $n = 1, 3$. Clearly, from (16.62) and (16.63) we have

$$\bar{\varepsilon} = \varepsilon + \omega_0 - \bar{\omega}_0 = \varepsilon + \frac{\pi}{2h}\left(\frac{c_{66}}{\varrho}\right)^{1/2}\left(R + \frac{4k^2}{\pi^2}\right). \tag{16.66}$$

In regions 1 and 3 we now express the solution as a sum of an auxiliary and a residual part, the auxiliary part selected so that the residual part satisfies homogeneous differential equations. Clearly, from (16.65) the auxiliary part is given by

$$\underset{n}{\overset{A}{u}}{}^{(1)}_{1} = \frac{3}{2h^2\pi}\,\frac{\bar{\varkappa}_6 e_{26}}{[1 + 2R - (4k^2/\pi^2)](c_{66}\varrho)^{1/2}\,\bar{\varepsilon}}\,\underset{n}{V},$$

or since $R \ll 1$ and $k^2 \ll 1$

$$\underset{n}{\overset{A}{u}}{}^{(1)}_{1} = \frac{\sqrt{3}}{4h^2}\,\frac{(1 - \frac{3}{2}R)}{(c_{66}\varrho)^{1/2}\,\bar{\varepsilon}}\,e_{26}\,\underset{n}{V}, \tag{16.67}$$

and the residual solution satisfies the homogeneous mechanical equation

$$\tfrac{2}{3}h^3[\bar{\gamma}_{11} + \bar{\varkappa}_6{}^2(1 + 2R)c_{66}]\, \underset{n}{\overset{R}{u}}{}^{(1)}_{1,11} + \tfrac{2}{3}\pi h^2[1 + 2R - (4k^2/\pi^2)](c_{66}\varrho)^{1/2}\,\bar{\varepsilon}\, \underset{n}{\overset{R}{u}}{}^{(1)}_{1} = 0 \tag{16.68}$$

for $n = 1, 3$, and in these regions

$$\underset{n}{u_1^{(1)}} = \overset{A}{\underset{n}{u_1^{(1)}}} + \overset{R}{\underset{n}{u_1^{(1)}}}. \tag{16.69}$$

Moreover, in region 3 the electrostatic equation (16.57) remains essentially unchanged and may be written

$$\tfrac{2}{3}h^3\psi_{11}\overset{R}{\underset{3}{u_{1,11}^{(1)}}} - \zeta_{11}\underset{3}{\varphi_{,11}^{(1)}} + (15/h^2)\varepsilon_{22}\underset{3}{\varphi^{(1)}} - 5\varepsilon_{22}\underset{3}{V} = 0. \tag{16.70}$$

When expressed in terms of the residual variables in regions 1 and 3 the mechanical continuity conditions (16.60) and (16.61) take the form

$$\underset{n}{u_1^{(1)}} = \frac{\sqrt{3}}{4h^2}\frac{(1-\frac{3}{2}R)}{(c_{66}\varrho)^{1/2}\bar{\varepsilon}}e_{26}\underset{n-1}{\overset{n+1}{V}} + \underset{n-1}{\overset{n+1}{\overset{R}{u_1^{(1)}}}} \qquad \begin{matrix} n=0,2, \\ n=2,4. \end{matrix} \tag{16.71}$$

$$\underset{n}{u_{1,1}^{(1)}} = \underset{n-1}{\overset{n+1}{\overset{R}{u_{1,1}^{(1)}}}} \qquad \begin{matrix} n=0,2, \\ n=2,4. \end{matrix} \tag{16.72}$$

Thus at this stage we have five homogeneous mechanical equations, (16.64) and (16.68), eight mechanical boundary conditions, (16.71) and (16.72), one electrostatic equation, (16.70), and one admittance condition, (16.58). We also have a prescribed voltage $\underset{1}{V}$. In each section a typical solution

$$\underset{n}{u_1^{(1)}} = \underset{n}{A}\cos\underset{n}{\xi}x_1,$$

yields a dispersion relation as shown in Fig. 28, and the dispersion relation for the unelectroded region lies above the dispersion relation for the electroded region, and is shown dotted. We will confine ourselves to frequencies

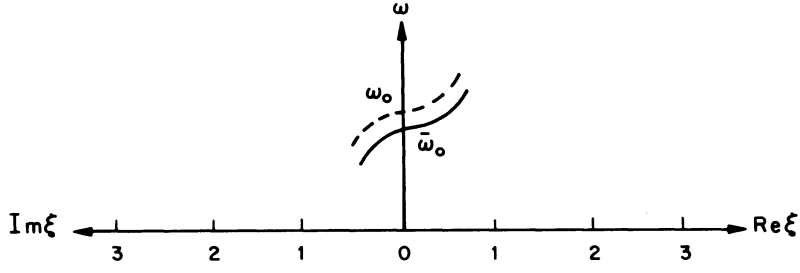

Fig. 28. Thickness-shear dispersion curves for plated and unplated piezoelectric plates for real and imaginary wave numbers.

ω, where $\bar{\omega}_0 < \omega < \omega_0$. Then ε is always negative. We may then write the mechanical portion of the solution in the form

$$
\begin{aligned}
u_1^{(1)} &= A_0 \exp -[i\xi_0(x_1 + l_1 + \tfrac{1}{2}l_2)], \\
u_1^{R(1)} &= A_1 \cos \xi_1[x_1 + \tfrac{1}{2}(l_1 + l_2)] + B_1 \sin \xi_1[x_1 + \tfrac{1}{2}(l_1 + l_2)], \\
u_1^{(1)} &= A_2 \cosh i\xi_2 x_1 - B_2 \sinh i\xi_2 x_1, \\
u_1^{R(1)} &= A_3 \cos \xi_3[x_1 - \tfrac{1}{2}(l_2 + l_3)] + B_3 \sin \xi_3[x_1 - \tfrac{1}{2}(l_2 + l_3)], \\
u_1^{(1)} &= A_4 \exp[i\xi_4(x_1 - l_1 - \tfrac{1}{2}l_2)],
\end{aligned}
\tag{16.73}
$$

where the time factor is understood to be $e^{i\omega t}$ and the A_n and B_n are complex, and $\xi_0 = \xi_2 = \xi_4$ and $\xi_1 = \xi_3$, and which are found, respectively, from the relations

$$
\xi_0^2 = \frac{(c_{66}\varrho)^{1/2}}{\gamma_{11} + \varkappa_6^2 c_{66}} \frac{\pi\varepsilon}{h}, \tag{16.74}
$$

$$
\xi_1^2 = \frac{1 + 2R - (4k^2/\pi^2)}{\bar{\gamma}_{11} + \bar{\varkappa}_6^2(1 + 2R)c_{66}} (c_{66}\varrho)^{1/2} \frac{\pi\bar{\varepsilon}}{h}, \tag{16.75}
$$

which are found by substituting from (16.73) into (16.64) and (16.68), respectively. Substituting from (16.73) into the eight mechanical continuity conditions given in (16.71) and (16.72) yields eight complex, linear, algebraic equations in the nine complex unknowns $A_0, A_1, A_2, A_3, A_4, B_1, B_2, B_3$, and V_3.

We must now satisfy the electrostatic equation (16.70) in region 3. To this end, we substitute $u_1^{R(1)}$ from (16.73) and write $\varphi_3^{(1)}$ as the sum of an auxiliary and residual part, so that

$$
\varphi_3^{(1)} = \overset{A}{\varphi}{}_3^{(1)} + \overset{R}{\varphi}{}_3^{(1)}, \tag{16.76}
$$

where

$$
\overset{A}{\varphi}{}_3^{(1)} = (h^2 V_3)/3, \tag{16.77}
$$

so that from (16.76), (16.77), and (16.70) we see that $\overset{R}{\varphi}{}_3^{(1)}$ satisfies

$$-\tfrac{2}{3}h^3\psi_{11}\underset{3}{\xi^2}\overset{R}{\underset{3}{u_1^{(1)}}} - \zeta_{11}\overset{R}{\underset{3}{\varphi^{(1)}_{,11}}} + (15/h^2)\varepsilon_{22}\overset{R}{\underset{3}{\varphi^{(1)}}} = 0. \tag{16.78}$$

Since we have assumed small piezoelectric coupling and ignored all electrical edge conditions, we have already ignored the essentially vertical piezoelectric branch shown in Fig. 27. Inasmuch as the homogeneous solution of (16.78) for $\overset{R}{\underset{3}{\varphi^{(1)}}}$ gives rise to this essentially vertical branch which has already been ignored, this homogeneous (complementary) solution can be—indeed has been—ignored and only the inhomogeneous (particular) solution of (16.78) need be considered. This latter solution takes the form

$$\overset{R}{\underset{3}{\varphi^{(1)}}} = \frac{2}{45}h^3\frac{\psi_{11}}{\varepsilon_{22}}h^2\underset{3}{\xi^2}\overset{R}{\underset{3}{u_1^{(1)}}}. \tag{16.79}$$

Moreover, as we shall see this term turns out to be negligible when inserted in the admittance relation (16.58). From (16.76), (16.77), and (16.79) we now have

$$\underset{3}{\varphi^{(1)}} = \frac{h^2}{3}\underset{3}{V} + \frac{2}{45}h^3\frac{\psi_{11}}{\varepsilon_{22}}h^2\underset{3}{\xi^2}\overset{R}{\underset{3}{u_1^{(1)}}}. \tag{16.80}$$

Substituting from (16.73) and (16.80) into the constitutive equations (16.59), which must be integrated over the area of the electrode in region 3, in the admittance relation (16.58), we find

$$\begin{aligned}\underset{3}{D_2^{(0)}} &= \tfrac{1}{3}\bar{\varkappa}_6 e_{26}(1+2R)\underset{1}{\xi^2}h^2\overset{R}{\underset{3}{u_1^{(1)}}} + \bar{\varkappa}_6\frac{\sqrt{3}\,(1-\frac{3}{2}R)}{4h^2(c_{66}\varrho)^{1/2}\bar{\varepsilon}}e_{26}^2\underset{3}{V}\\ &\quad + \bar{\varkappa}_6 e_{26}\overset{R}{\underset{3}{u_1^{(1)}}} - \frac{5}{4h}\varepsilon_{22}\underset{3}{V} - \frac{1}{6}\psi_{11}h^2\underset{1}{\xi^2}\overset{R}{\underset{3}{u_1^{(1)}}} + \frac{3}{4h}\varepsilon_{22}\underset{3}{V}, \qquad (16.81)\\ \underset{3}{D_2^{(2)}} &= (1/2h^2)\psi_{11}h^2\underset{1}{\xi^2}\overset{R}{\underset{3}{u_1^{(1)}}}.\end{aligned}$$

Since ψ_{11} is of the order of e_{26} and $\underset{1}{\xi^2}h^2 \ll 1$, from (16.81), we obtain

$$\begin{aligned}\underset{3}{D_2^{(0)}} + h^2\underset{3}{D_2^{(2)}} &= \bar{\varkappa}_6\frac{\sqrt{3}\,(1-\frac{3}{2}R)e_{26}^2}{4h^2(c_{66}\varrho)^{1/2}\bar{\varepsilon}}\underset{3}{V} - \frac{\varepsilon_{22}}{2h}\underset{3}{V}\\ &\quad + \bar{\varkappa}_6 e_{26}\{\underset{3}{A}\cos\underset{1}{\xi}[x_1 - \tfrac{1}{2}(l_2+l_3)] + \underset{3}{B}\sin\underset{1}{\xi}[x_1 - \tfrac{1}{2}(l_2+l_3)]\}. \qquad (16.82)\end{aligned}$$

Substituting (16.82) into the admittance condition (16.58) and integrating

over the area of the electrode in region 3, we obtain

$$-i\omega b l_3 \frac{\varepsilon_{22}}{2h}\left[1-\frac{\bar{\varkappa}_6\sqrt{3}\,(1-\frac{3}{2}R)e_{26}^2}{2h(c_{66}\varrho)^{1/2}\bar{\varepsilon}\varepsilon_{22}}\right]V_3+i\omega 2b\frac{\bar{\varkappa}_6 e_{26}}{\xi_1}A_3\sin\xi_1\frac{l_3}{2}=YV_3, \tag{16.83}$$

where b is the length of the electrode out of the paper. Equation (16.83) may be written

$$\left[bl_3\frac{\varepsilon_{22}}{2h}\left(1-\frac{\bar{\varkappa}_6\sqrt{3}\,(1-\frac{3}{2}R)e_{26}^2}{2h(c_{66}\varrho)^{1/2}\bar{\varepsilon}\varepsilon_{22}}\right)-\frac{iY}{\omega}\right]V_3=\frac{2b\bar{\varkappa}_6 e_{26}}{\xi_1}A_3\sin\xi_1\frac{l_3}{2}. \tag{16.84}$$

Equation (16.84) is the admittance relation, which provides the ninth complex linear algebraic equation. As already stated, the other eight equations come from the eight mechanical continuity conditions in (16.71) and (16.72). We have not bothered to write them down because they are extremely straightforward and simply require a great amount of writing. Thus we now have nine complex, linear, algebraic equations in nine complex unknowns. The value of the admittance Y depends, of course, on the external circuitry associated with region 3.

Before making a calculation the equations should, of course, be put in dimensionless form wherever possible. This may be accomplished by writing the pertinent equations so that the dimensionless quantities ξh, l_n/h, ε/ω_0, and $\bar{\varepsilon}/\omega_0$ occur. A calculation would then proceed by selecting a value of ω/ω_0, which would determine values of ε/ω_0 and $\bar{\varepsilon}/\omega_0$. Then $\xi_0 h$ and $\xi_1 h$ could be determined from (16.74) and (16.75), respectively, after which the nine complex equations may be solved simultaneously to give the nine complex unknowns in terms of the applied voltage V_1. Thus V_3/V_1 may be determined. In making a calculation, some obvious numerical simplifications can be made because of the smallness of certain quantities. That is, since the electrodes and piezoelectric coupling are such that

$$\bar{\gamma}_{11}\approx\gamma_{11}, \qquad \bar{\varkappa}_6\approx\varkappa_6, \qquad R\ll 1, \qquad 4k^2/\pi^2\ll 1, \tag{16.85}$$

we may replace the continuity relations (16.71) and (16.72) by the simpler ones

$$u_{1\,n}^{(1)}=\frac{\sqrt{3}\,e_{26}}{2\pi h c_{66}\bar{\delta}}\,V_{n+1-1}+u_{1\,n+1\;n-1}^{R\,(1)} \qquad \begin{matrix} n=0,2 \\ n=2,4, \end{matrix} \tag{16.86}$$

$$u^{(1)}_{1,1\ n\ n} = u^{R\ (1)}_{1,1\ n+1\ n-1} \qquad \begin{matrix} n=0,2 \\ n=2,4, \end{matrix} \tag{16.87}$$

and the dimensionless form of the dispersion relations (16.74) and (16.75) by the simpler ones

$$\xi^2_0 h^2 = \frac{\pi^2 \delta}{2(\tilde{\gamma}_{11} + \varkappa_6^2)}, \qquad \xi^2_1 h^2 = \frac{\pi^2 \bar{\delta}}{2(\tilde{\gamma}_{11} + \varkappa_6^2)}, \tag{16.88}$$

where

$$\delta = \varepsilon/\omega_0, \qquad \bar{\delta} = \bar{\varepsilon}/\omega_0, \qquad \tilde{\gamma}_{11} = \gamma_{11}/c_{66}. \tag{16.89}$$

Similarly, the admittance condition (16.84) may be written in the form

$$\left[\frac{\varepsilon_{22}}{2}\frac{l_3}{h}\left(1 - \frac{\varkappa_6\sqrt{3}\,k_{26}^2}{\pi\bar{\delta}}\right) - \frac{iY}{b\omega_0}\right]\frac{V}{h}_3 = \frac{2\varkappa_6 e_{26}}{\xi h}_1 A_3 \sin \xi_1 \frac{hl_3}{2h}. \tag{16.90}$$

Thus it is clear that the entire effect is contained in the relation

$$\bar{\delta} - \delta = R + (4k^2/\pi^2), \tag{16.91}$$

and

$$\delta = (\omega/\omega_0) - 1, \qquad \bar{\omega}_0 < \omega < \omega_0. \tag{16.92}$$

Once the solution has been obtained, the driving current through region 1 may be calculated. However, as already stated the electric-potential resultant $\varphi^{(1)}_1$ must first be determined as a solution of the electrostatic equation in region 1. The solution for $\varphi^{(1)}_1$ in region 1 proceeds in exactly the same manner as the solution for $\varphi^{(1)}_3$ in region 3, and with the same terms being negligible. In fact, the expression for $\varphi^{(1)}_1$ may be obtained from the expression for $\varphi^{(1)}_3$ in (16.80) merely by replacing all lower scripts 3 with lower scripts 1. In this way the expressions for $D^{(0)}_{2\,1}$ and $D^{(2)}_{2\,1}$ may be obtained from the expressions for $D^{(0)}_{2\,3}$ and $D^{(2)}_{2\,3}$ in (16.81). Moreover, since region 1 is passive in relation to the driving circuit, we have

$$I_1 = -\int_A \left(\dot{D}^{(0)}_{2\,1} + h^2 \dot{D}^{(2)}_{2\,1}\right) dA = -i\omega \int_A \left(D^{(0)}_{2\,1} + h^2 D^{(2)}_{2\,1}\right) dA. \tag{16.93}$$

Substituting in (16.93) from the expressions for $D_2^{(0)}_1$ and $D_2^{(2)}_1$, we obtain

$$\begin{aligned} I_1 = &- i\omega b l_1 \frac{\varepsilon_{22}}{2h}\left[\frac{\bar{\varkappa}_6\sqrt{3}\,(1-\tfrac{3}{2}R)e_{26}^2}{2h(c_{66}\varrho)^{1/2}\bar{\varepsilon}\varepsilon_{22}} - 1\right] V_1 \\ &- i\omega 2b \frac{\bar{\varkappa}_6 e_{26}}{\xi_1} A_1 \sin\xi_1 \frac{l_1}{2}, \end{aligned} \tag{16.94}$$

which, after the introduction of the approximations in (16.85), may be written in the form

$$I_1 = -\frac{i\omega b l_1 \varepsilon_{22}}{2}\left[\left(\frac{\varkappa_6\sqrt{3}\,k_{26}^2}{\pi\bar{\delta}} - 1\right)\frac{V_1}{h} + \frac{4e_{26}}{\varepsilon_{22}}\varkappa_6\frac{h}{l_1}\frac{A_1}{\xi_1 h}\sin\frac{\xi_1 h}{2}\frac{l_1}{h}\right]. \tag{16.95}$$

Since $A_1 e_{26}/\varepsilon_{22}$ is known in terms of V_1/h from the solution, Eq. (16.95) determines the admittance of the entire structure as seen by the driving circuit.

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