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Alex Talbot Coram

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Alex Talbot Coram
Professor in Political Science
University of Western Australia

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To Joan, Andrew, Alicia and to the memory of my father

Contents

Preface

Purpose of the book

Societies generally, and in the long run, require some sort of mechanism to prevent chronic and widespread internal violence and to provide for an orderly process of collective decision making. I say 'generally and in the long run' because such problems may not be solved, or only partially solved, for considerable periods of time. Questions about the nature of these problems might be thought of as questions about the foundations of political-economic systems.

The purpose of this book is to provide social scientists with an introduction to some of the applications of strategic choice, or game, theory to some of these questions. It also deals with some aspects of collective decision making.

The focus on questions about the foundations of political-economic systems places this work within an old, and continuing, tradition in political-economic theory. The aim of this tradition is to understand political-economic systems as broadly as possible by starting at the most basic level. This is the level that requires the smallest number of assumptions about the existence of institutions, such as the state and markets, and questions the preconditions for their existence. That is, rather than simply assume the existence of order and property rights and some systematic process of choice, it is asked, what would happen in the absence of these institutions? Why are they needed?

This focus is a departure from much of the more familiar literature in economics and public choice. In this literature the tendency is to start by assuming the existence of a market without analysing the preconditions required for trade and exchange. The problem here is that the structure of the system as a whole may sometimes be forgotten. The preconditions required for trade to take place and for security of property tend to be overlooked. One result of this is that government and collective choice mechanisms tend to be treated as a residual category that compensate for market failure. At best this is a distorted framework. It may be more useful to think of the market as a means of compensating for collective choice failures.

The techniques of game theory provide the most appropriate formal approach to the analysis of problems of security and collective choice because these problems arise as the result of strategic interaction. They do not arise as the result of individuals taking actions that have no consequences for anyone else. Moreover, the game theory approach has the advantage of consistency. This is because it applies the assumption that individuals are optimizing in a consistent way and allows individuals to make any moves that are optimal, rather than those that are somehow nice. Among the examples of fully optimal moves are the use of violence and the organization of force.

An immediate consequence of taking optimization seriously is that the state, or some institution for ensuring order, has to be built into any general theory from the beginning. It is not consistent to ignore the necessity of having an institution to enforce rules in order to attack problems of general equilibrium in a pure market environment. If rules are required to make transactions possible, they must be explicitly dealt with in the analysis.

The applications of game theory are explored in a sequence that follows the logic of the problems being dealt with. In tracking these problems, the game theory can be set out so that the discussion of the techniques in game theory progresses with the analysis. This provides a natural progression from simpler to more complex techniques. One advantage of this is that it is possible to provide some substantive content for the applications of the theory and to build on previous solutions. Another is that a reader, interested in a particular type of problem, can turn immediately to the appropriate techniques.

Since the book is organized around problems it differs from books that attempt to provide a systematic introduction to game theory. These are usually organized around the type of mathematical technique used or the characteristics of the particular class of game. Excellent examples of this type of book at the basic level are Luce and Raiffa's classic *Games and Decisions* and Binmore's *Fun and Games*. Many others are included in the bibliography.

Although not intended as a systematic introduction, this book provides all the basic definitions and theorems needed. Nearly all the material is accessible to anyone who remembers their high school calculus, or is prepared to brush up as the study progresses.

This book will serve as a text for courses in political economy, or on the theory of the state and collective decisions for students in economics, political science, public choice and related disciplines. Most of the material in the early part of each chapter is accessible to undergraduates, especially if accompanied by lectures and some additional reading. I have found it easy to make this material comprehensible to students with a traditional arts and humanities background. This includes students who have an active dislike of mathematics.

Outline

The book starts with some basic questions about the problem of order and then progresses to more complex questions about property rights, markets for protection and collective choice mechanisms. Each stage introduces additional game theory techniques and shows why they are needed to solve the problems at hand. The study is able to progress fairly systematically because some of the most simple techniques provide a reasonably adequate basis for analysing some of the most basic problems of social life.

Chapter 1 outlines the general problem of analyzing the place of the state and collective decision processes in political-economic systems. It also outlines some of the substantive implications of the study. In particular, it looks at the debate between the view that social order is a political, and hence collective, construct and the view that markets are natural and part of a spontaneous order. It also provides an introduction to the techniques to be used.

Chapters 2, 3 and 4 look at the question of whether it is desirable to enforce collective decisions or whether enforcement is necessary to make such decisions binding. This is done in the traditional manner by examining the outcome in a situation where there are no rules to protect life or property.

In Chapter 5 the question of whether markets could emerge without a state to enforce rules and whether it would be possible to buy rules on a market is considered. This question is interesting for an assessment of the political-economic and pure economic approaches. If it were possible for the market to provide the rules it needs, it would be possible to construct a closed economic theory.

Chapter 6 looks at the notion of spontaneous evolution and evolutionary stable strategies. This deals with the important question of what we can learn about outcomes if the notion of global optimization is dropped. In this case, individuals are assumed to base their choices on imperfect information and trial and error tactics.

Chapter 7 examines the criteria that might be used to make collective decisions. What economists call collective goods are dealt with in this section. It will be seen that these goods are simply a subset of strategic action and do not present any new problems. From a strategic choice approach, it is the market which is the special case.

In Chapter 8 I examine problems of collective decision making and welfare maximization. If there is a state to enforce collective decisions would it be possible to design a mechanism to allow the utilities of individuals to be aggregated? One problem is that individuals might lie in order to manipulate the outcome.

In chapter nine I look at voting from the viewpoint of strategic manipulation.

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1
State, Anarchy and Game Theory

1.1 Introduction

Social scientists currently give a great deal of attention to questions about the nature of collective decision making and about the extent to which a centralized authority is necessary to enforce such decisions. These questions have been around, in their modern form, at least since the time of Hobbes's *Leviathan* and Locke's *Two Treatises of Government*. They are central to much of modern formal theory as well as to popular debates between proponents of various choice processes, such as those that leave final outcomes to markets and those that determine final outcomes directly. They also form the basis of what is now commonly studied under the headings of social choice theory, public choice, and modern political economy. They include the following: Why is it necessary to have a state, or any other mechanism, to enforce rules over things like protection for life and property? Why do not individuals simply co-operate to do what is best for the society? If rules are needed, why do not individuals spontaneously provide these, or buy them on the market?[1] Why are collective decisions necessary? How is it possible to ascertain which choices are best? Does society have a preference in any coherent sense? How should collective decisions be made? Would it be possible to find out what individuals want by asking them?

The purpose of this book is to consider some of the applications of strategic choice, or game, theory to questions of this type. It is essentially an introduction to applied game theory.

The words 'state' and 'anarchy' are used in the title to emphasize the links between this study and the traditional concerns of political economists with enforceability, authority and questions of violence. The analysis of these questions gives the study a somewhat broader focus than the more usual collective choice theory. I follow the standard practice of dealing with questions about the outcome of individual decisions as they arise in a situation without a state or government, or what is usually referred to as a state of nature. No attempt is made to address the sufficient conditions for the state, or any other enforcement mechanism, to be justified.

It must be noted that there is a distinction between the question, 'how did enforcement mechanisms for collective choice come into being?' and the questions, 'why do we need them?, what do they do?' This book is primarily concerned with the second type of question. The first question is historical. The answer most likely has something to do with treachery, bloodshed and violence.

The applications of game theory are developed by starting with the most basic problems of security and moving progressively to more advanced problems of property and decision making. This trajectory is partly justified by the argument that it is more defensible to start with problems that require less, rather than more, assumptions. It is also partly justified by the primacy of security in social life. This point is taken up in 1.3.

As the main task of this study is to provide an introduction to the game theory approach, no attempt is made to argue a single thesis. In so far as there is an argument, it is implicit in the sequence of development outlined. The argument is that the focus on problems of strategic action make this sequence appear to be a natural trajectory for any attempt to develop a general analysis of political economic phenomena. That is, there seems to be some inherent logic in the order of the approach to the problems. This does not mean that a different sequence could not be used. It does mean that it is more likely to involve more messy assumptions.

Apart from the emphasis placed on the sequence of the analysis, the position adopted is that the study of social phenomena is best approached by treating them as a series of problems. These are simply dealt with on an individual basis. It is almost certainly true that everything is connected. I do not think, however, that we are in much of a position to deal with these connections at the present stage.

To begin, it might be useful to provide a few notes on the approach to be adopted. I shall also take this opportunity to expand the previous comments and to give the subsequent game theoretical analysis some general context.

1.2 The game theory approach

The main advantage of a game theory approach is that it provides a framework for analysing a broad range of social institutions in a formal and rigorous way. This stems from the fact that it analyses social interaction or interdependence. More precisely, it analyses the case in which the decision of one player affects others. That is, each individual, *i*, chooses the strategy that will give the optimum pay-off bearing in mind that each other player will choose whatever strategy will maximize their return given that *i* is choosing an optimum strategy. Since the actions of one individual may impose costs on another, or prevent others achieving some desirable outcome, it is necessary from the outset to consider questions about the nature of these costs and the related questions of authority, collective decisions and rules. For example, the fact that each individual is attempting to maximize in isolation means that their choices may not result in an outcome that provides a collective optimum.

The difference between non-strategic and strategic action can be illustrated by considering the difference between transactions in a market with an infinite number of traders and the decisions of opposing generals in a war. The market is essentially a single agent model in which the actions of one individual do not influence others. Although the bread price faced by any one individual will depend on the decisions of all others, the decisions of any one individual will not affect this price or the pay-offs of others. For analytical purposes the decisions can be treated as independent. The decision of one general cannot be treated as analytically separate from the decisions of the opposing general. The pay-offs of each general will, in most cases, depend on the decisions of the other.

Since the game theory approach provides the framework for the simultaneous analysis of phenomena that are often divided into political and economic, it might also be considered the basis of a formal political economy. It might be speculated that one reason for the separation of the analysis of political phenomena and economic phenomena into political science and economics shortly after the turn of the century was the inability of the available mathematical techniques to deal with strategic, or political, interaction. The early attempts to study interpendence, such as Cournot's model of duopoly, were restricted to a small class of cases. Hence the study of political phenomena tended to remain largely historical and descriptive. Von Neumann and Morgenstern's *Theory of Games and Economic Behaviour* (1944) provided the first systematic development of the techniques needed for a formal attack on political and economic phenomena together. The single agent model of agents in markets becomes a special case of the multi-agent model.

The fact that strategic action generates suboptimalities was familiar to political philosophers such as Hobbes, Locke and Hume. Something of the notion also underlies some of Rousseau's work. Rousseau's belief that society is necessary for people to be good but corrupts them, or that the general will can only be expressed in small communities, are examples. Part of the argument for small communities is that, where communities exceed a certain size, individuals may no longer identify the common good with their individual good. This gives rise to strategic action problems.

Somewhat intriguingly, the problem of strategic action turns out to be that individuals have too much choice, or that too many worthwhile strategies are available. This may seem a somewhat strange idea for those brought up in the liberal-democratic tradition. This tell us that choice is a good thing, and that an expansion of options increases the welfare of individuals. This is only true, however, if there is no interdependence. It turns out that, with interdependence, increasing the range of worthwhile choices does not always improve the outcome.

To illustrate the argument, imagine a rock concert. If the only choice is to remain seated, all can see. If the range of choice is increased so allowing people to stand or move to the front, no more can see and many may be injured.

The point just covered might be called the paradox of strategic interaction. This is that an increase in the range of actions may cause individuals to move to an inferior position.

The characteristic feature of the game theory, or political-economic, approach is that it simplifies problems as much as possible by studying a system made up of optimizing individuals.[2] It is assumed that these individuals are trying to maximize such things as their security, or material welfare or whatever is relevant to the issue being analysed. The questions that will be asked concern the outcome of these choices under different rules. What will be the outcome if there are no rules? What would happen if a collective choice procedure of type A or B were in operation?

The emphasis on optimization means that the analysis abstracts from considerations of norms and values in so far as they directly affect the choice of strategies. It could be thought of as a three billion stooges view of social life since it gives a very sparse view of human thought processes. It is justified in that it gives a useful starting point for thinking about more complex environments. In the simple systems studied, the choices made depend on nothing more than the available strategies and the endowments and preferences of individuals.

The analysis also abstracts from all question of the moral worth of different arrangements. It does not, for example, have anything to say about the body of claims that runs through Plato and Rousseau to the modern Hegelians that the state and political life is an end in itself. This study is not intended to criticize these claims by omission. It seems quite plausible to argue that collective action is a good in itself and provides a means to both express and develop good intentions.[3] These views are simply not discussed.

A more systematic idea of the problems generated by strategic interaction can be gained by observing that there are three general cases that are of concern when considering the consequences of individual and collective choice. These are characterized according to the type of outcomes that individual actions will produce. The possibilities are as follows:

[Ai] Spontaneous selection of the strategies that produce a collectively optimal outcome in some sense. This means that it will pay all individuals to follow the strategy that leads to the collectively optimal strategy. Any deviation will make the individual who deviates worse off.

[Aii] Voluntary agreement to adhere to some rule that produces an optimal outcome. Imagine, for example, that all players meet and agree to do x in order to get some desired outcome, but there is no punishment for deviating. It will be in the interest of all individuals to keep this promise only if the pay-offs for keeping it are greater than the pay-offs for breaking it. Examples would be an agreement to meet at a certain place for mutual advantage, or a trade that makes all parties better off.

[Aiii] Individuals only keep an agreement that leads to an optimal outcome if there is an enforcement mechanism. In this case there is an optimal outcome that all individuals want but neither [Ai] nor [Aii] apply. The only way to get the agreement is to set up a mechanism to enforce the appropriate action.

Each of these possibilities is generated by interactions within a different structure. That is, a game that produces spontaneous co-operation has a different structure than one in which co-operation has to be enforced.

1.3 The problems considered

The problems to be dealt with fall naturally into two sections.

1.3.1 Section one. Security

The first set of problems concern the question of whether individuals could spontaneously co-operate to provide things such as physical security and rules of property. These are essentially questions about the structure of interaction that applies to each case. It is obvious that, if all interactions were of types [Ai] neither social choice nor enforcement would be desirable in the first instance. There would not even be an argument that a government is useful to provide defence against external aggressors. This is because those who were organized and had a structure of authority would not necessarily have any military advantage. Spontaneous peoples' armies would undertake scientific research, automatically form fighting units and so on. Alternatively, if once security were provided interactions were of type [Ai] and [Aii], then only a minimal security-providing state might be required.

More specifically, I shall concentrate on the following questions.

[Qi] Security problems created by individuals pursuing optimal strategies without any enforceable rules. Would it be possible for individuals to provide physical security without a mechanism to enforce collective decisions?

The question of security and of the enforcement of collective decisions has been placed first because security is usually taken to be the prerequisite for any social order and because its analysis throws some light on the nature of collective decisions and why they are needed in the first place.[4] The problem of order is commonly known as the Hobbesian problem. The problem is that, where there are no rules, individuals who desire peace and safety for life and limb may not be able to attain it through their own efforts without some form of authority structure. Whether this is a justified description of the state of nature is taken up in Chapter 2. What is obvious, however, is that the security problem has some claim to primacy in thinking about the foundations of welfare. Without security there is 'no place for industry because the fruit thereof is uncertain: and consequently no culture of the earth; no navigation and use of commodities that may be imported by sea ... no arts, no letters, no society' (Hobbes, 1968, p. 186).

The analysis of security problems, and of the conditions that make it difficult to arrive at collectively optimal outcomes as the result of spontaneous action, throws some light on collective decisions because it helps us to understand the need for such decisions. It also helps in understanding when decisions need to be enforced as well as the type of mechanisms that are required.

This analysis has implications for a range of secondary issues not directly addressed in the body of this study.

[Qi.a] The question of what would happen if there were not an institution to enforce collective decisions explores some of the necessary conditions for a state or government to be desirable. These conditions are necessary in the sense that the effort to sustain some co-operative organization would not be voluntarily forthcoming unless it helped solve some problems that would otherwise exist. It is assumed that individuals would not give some institution a near monopoly of violence, for example, unless it made them better off. Similarly, unless it is assumed that individuals are naturally subordinate to authority, or have an innate tendency to desire traffic lights, an explanation of why they would create such arrangements is needed. One way to get an insight into structures of authority, markets, or the rules of the road, is to ask what sorts of problem they solve, and for whom.

[Qi.b] If the state is required to provide security, then it can be understood why it requires a preponderance of violence in some area. Without the ability to uphold its claims to be the enforcement agency, it could not prevent other institutions making the same claim. Where there is more than one such claim, the different groups making the claims have no authority over them to constrain their actions. Hence, the conditions that generated the Hobbesian problem would remain.

[Qi.c] If a state is in place to provide security, then the solution to subsequent problems of collective choice have a different form. This is because all subsequent questions take the form of, what should be done with an already exiting mechanism? That is, once the security problem is solved, the setting-up costs of the state have already been incurred. The difference is somewhat like the difference between the decision to purchase a car for work, and the decision of whether to use it to go to the store for a can of beer.

Point [Qi.c] is often misunderstood. Elster, for example, believes that for all collective goods 'decentralized solutions are more fundamental than centralized ones, since compliance with centralized directives is itself a collective action problem', (Elster, 1989, p. 17). This is only true for the problem of establishing the state in the first instance. The problem of getting an agreement to accept an authoritative mechanism does not have to be solved again for every subsequent issue.

The focus is solely on the security problem as it arises between individuals. Security problems between states are not of concern. In some cases the players could be taken as groups without affecting the interpretation of the models. In other cases, however, the analysis of groups presents problems that have to be dealt with in co-operative game theory. This is because groups involve consideration of internal co-ordination and bargaining.

This point is illustrated by comparing the problems facing states in an international system with those facing individuals. Bull (1977, p. 46) argues that there is order between states. There are several reasons why this does not allow us to assume that there will be a similar order between individuals. First, the number of states that can threaten the security of any other state is relatively small. As will be seen, this generates a different dynamic than games with a large number of players. Second, states have a collective entity such that the losses from conflict may be much less than the losses to an individual. Bull notes similarly that 'states are not vulnerable to violent attack to the same degree that individuals are' (1977, p. 49). Third, the assumption of order between states is not justified. If states are treated as having a corporate life-span, the number of major conflicts during a life is significant.

From a formal perspective, the security problem for groups may look more like the problem of struggles over material possession faced by individuals than the security problem for individuals. Because groups do not necessarily face annihilation, struggles between groups may be thought of in terms of players trying to minimize the costs of the conflict, rather than in terms of avoiding physical destruction.

[Qii] Struggles over material possessions. Is the state necessary to enforce rules of property?

The problem of physical security is not the same as the problem of providing security for property. One of the weaknesses of the Hobbesian approach is that it makes the mistake of treating the question of property as if it were subsumed by the problem of order. Some attempt is made to disaggregate these problems in the Lockean tradition, but this is often done by making arbitrary assumptions about rights. It is easy to see, however, that these problems can be separated. It would logically be possible to have a state that protected life and limb but did not protect property. Imagine, for example, a state that solved the Hobbesian problem by protecting life and privatizing all other functions of the police force. Individuals would then have to decide whether it was worthwhile renting police to protect property. Something like this already happens with private security firms.

[Qiii] Markets for security. Could the market provide its own rules?

This question has some interesting implications for political economy. If the market could provide its own rules, it could operate as a closed system that produced its inputs as outputs, once physical safety was assured. If this were the case, we might be able to get close to a theory of markets that did not require that the state be treated endogenously.

[Qiv] Evolutionary stability. Could equilibria evolve?

This is the problem of equilibria when the assumption that individuals are global optimizers is relaxed. It looks at the possibility of spontaneous order emerging for trial and error type strategies.

1.3.2 Section two. Collective choice

The questions in this section concern the criteria or mechanisms which might be used to choose between the various possible alternatives. It is assumed that enforcement of collective decisions is no longer a problem. In this case, the primary question thus becomes, how are social choices to be made?

The most obvious distinction is between a mechanism that depends on collective choice over ends and a mechanism that accepts the unintended consequence of individual decisions under specified rules. The conditions under which each mechanism is most likely to be appropriate partly depend on whether the pay-offs for one individual are related to the decisions of others, and how they are related. The goods that can be provided optimally by individual decisions are those that do not have a strategic component. Some goods have a strategic component that cannot be eliminated by properly designed rules. Among these may be the rules themselves. In these cases the problem is irreducibly one of collective choice.

[Qi] What criteria might be used for co-operative decision-making and for sharing the gains from co-operation?

To be worthwhile for all individuals, the condition for co-operation is that there is some mechanism for sharing the joint gains or the surplus from co-operation that makes no-one worse off. It follows that there must be some choice mechanism that has this characteristic for all co-operative decisions. Note that this type of mechanism preserves the optimal outcome for each individual. This is not necessarily the case with the welfare maximizing mechanism discussed below.

[Qii] Would it be possible for a planner or a benevolent despot to maximize welfare?

The simplest approach to collective choice would be to select whatever outcome maximizes welfare, in some sense. It might be imagined that individuals collectively accept this as an appropriate decision criterion even if it does not provide the optimal outcome for each individual. The planner, or benevolent despot, could determine the outcome that maximizes welfare, if it were possible to get sufficient information on what each individual would prefer. Maybe the planner could simply ask the population. The problem here is that some of the considerations that make a collective choice mechanism necessary in the first place may operate to subvert the collective choice. For example, suppose that each individual is asked to reveal the value that is placed on some good that all desire, and the reported value determines the share of the cost. In this case each individual may understate this value, thus distorting the choice process.

When the assumption that collective choice is made by a benevolent dictator is dropped, some mechanism must be found to aggregate the preferences of the population. This is often done by choice among a menu of outcomes through some scoring or voting procedure.

[Qiii] Is it possible to make preference aggregation procedures strategy proof?

This question concerns the mechanisms that might be put into place to allow individuals to choose the optimum outcome from some menu of possible outcomes. There are a large number of possibilities here from choice by lottery to various sorts of voting systems. Once more, if individuals vote strategically, problem of collective sub-optimalities may arise. In considering this question, the study will only focus on solutions to choice problems that have a strategic content.

1.4 Some implications of the approach

Starting with the assumption that individuals are unrestrained optimizing agents and allowing all strategies, including violence, produces an emphasis on security and rules that is much less evident in many neo-classical market, or individual agent, based approaches. It also means that many special problems such as collective goods, externalities and free riding become subsumed in the more general stategic action framework.

The possibility that there is something missing from the majority of neoclassical market based approaches has also been observed by Usher in *The Welfare Economics of Markets, Voting and Predation*. He notes the ‘remarkable fact about welfare economics’ is that ‘there is no role for violence at all’.[5] The difference between

Usher's work and this study is in the approach. The point made here is that the omission of violence may not be accidental or an oversight that can be rectified within the existing framework. It may be the product of a non-strategic approach.

In many ways the trajectory developed in this study is closer to that of the earlier classical political economy and political science. Some of the possible consequences of this are briefly dealt with here. These are speculative and no attempt is made to argue the case fully.

The most important consequence is that the institutions and rules supported by government are important in sustaining social life because spontaneous equilibria over things like the rules of property cannot easily be obtained. Hence it is necessary to consider the possibility that some form of rule enforcement mechanism may be required for such things as large-scale markets. If so, markets are largely created and held together by an enforcement mechanism, rather than being self-sustaining. That is, collective choice and government are not fully explained as institutions that are required for the purpose of correcting market failures. They logically precede and construct the market.

The statement that political decisions are important in creating and maintaining much of the basis of social life is not particularly profound. The point is that the implications of this are often obscured in approaches that do not take strategic interaction as the starting point. The consequence is that rules and enforcement are pushed into the background. Not only is the state sometimes explained in terms of market failure, but it is often assumed that the market is somehow natural or pre-political, and the state is artificial. Related to this is the observation that concepts such as redistribution may be misleading. If rules emerge as the result of agreement, then there is no natural distribution to be redistributed. There is only a choice between different distributions.

In *Politics and Vision*, Wolin develops some of the historical aspects of the way in which perception of the relation between the state and society have developed. Wolin argues that there are two broad positions on this relation. The first, starting with Hobbes sees the state as necessary to provide the framework within which material production and other forms of activity take place. The second follows from Locke and is traced through modern libertarians such as Nozick. In this second approach there is some natural condition, or spontaneous equilibrium which precedes the state and allows for a substantial amount of welfare.

The qualification that welfare must be substantial is important. It is not simply the existence of equilibrium that matters. Unless Hollywood is not to be believed, 'kill them all' is often an equilibrium strategy. If a spontaneous equilibrium with a substantial level of welfare exists, the state is not essential, or even necessary, for a commodious existence. It is an add-on. It is seen as 'something like a better set of accommodation for those who already were home owners, rather than a shelter erected in desperation by the shelterless' (Wolin, 1960, p. 306).

The consequences of these two positions have considerable normative interest. If there were a set of holdings that precede the state it would be possible to argue that these should not be altered by collective choice. For such a set of holdings to exist there must be a spontaneous and reproducible order. This order must be such that it requires no enforcement mechanism for its support.

The claim that there is a natural order gets some of its underpinnings from Smith's *The Wealth of Nations*. Smith argued that a system of uncontrolled exchange amongst optimizing agents could bring about an outcome in which the system could continue to operate and the demand for goods and services was satisfied. The individual attempting to maximize his returns 'intends only his own gain' but 'he is in this, as in many other cases led by an invisible hand to promote an end which was no part of his intention' (Smith, 1937, p. 423). This intuition was later systematized in Walrasian general equilibrium theory. What this demonstrated is that a perfectly free market could, under the proper conditions, reach an equilibrium in which all prices and quantities were determined. In addition, at equilibrium, it is not possible to improve on the welfare of any one individual without reducing that of another.

Supporters of the notion of a natural or spontaneous order have sought an analogue of market equilibrium in a wider class of social phenomena. This idea has been pursued by many recent writers. It has been expanded at length by Hayek in *Law, Legislation and Liberty*. It is taken up from an anarchist perspective by Taylor in *Community, Anarchy and Liberty* and from a libertarian perspective by Nozick in *Anarchy, State, and Utopia*. Spontaneous order arguments are also explored by Elster in *The Cement of Society* and Sugden in *The Economics of Rights, Co-operation, and Welfare*. These are less sweeping in their claims than Hayek or Nozick.

Hayek's and Nozick's arguments depend on special assumptions that are outside a game theory analysis. Some of Taylor's arguments are considered in chapters 2 and 3.

Sugden and Elster's arguments raise the question of the degree to which the notion of spontaneity applies to the sort of large scale problems that are of concern when considering the need for enforceable rules. It is true that individuals can, through conventions, mutual coercion, or other devices, solve many co-operation problems. The examples that are usually given of this are, however, small scale problems such as who does the washing up, or of reciprocity in sending Christmas cards, or who goes through a door first (Sugden, 1986, p. 51–4). What is important, however, is whether spontaneous order holds for problems such as safety, security of property against theft, or the provision of collective goods such as an acceptable environment.

In its most simplistic form, much of the argument of the spontaneous rules approach depends on the observation that 'a competitive society with no externality-producing activities will inevitably allocate its resources efficiently. Thus it will have no role for government' (Auster and Silver, 1979, p. 8). This is true by definition. No externalities means no strategic interaction. The obvious question is, 'why will rational individuals not act strategically?'

1.5 Brief notes on some selected literature

Arrow's *Social Choice and Individual Values*, the papers in Tullock's *Studies in the Theory of Anarchy* and the work by Buchanan and Tullock, *The Calculus of Consent* are among the earliest of the distinctly modern attempts to give choice an axiomatic foundation and to extend the Hobbesian and Lockean traditions within a contemporary political-economic framework. The question of rules and social order has been pursued by Buchanan in his writings on constitutional political economy. Brennan and Buchanan have recognized the importance of a rational choice explanation of the origins of rules in their *The Reason of Rules*.

Bush and Mayer, and more recently, Grossman and Kim, Hirshleifer, Skaperdas and others have explored the question of the allocation of resources between productive and unproductive activities in their studies of struggles over material goods, organized crime and theft. This group of studies throws light on the property problem and questions about the need for rules of property. It also helps in understanding the relation between the state and other wielders of violence. Some of these questions are considered in chapters 3 and 4 of this book. Usher also considers violence and the evolution of forms of state in his *The Welfare Economics* mentioned in the previous section. He departs from many standard texts in welfare economics by developing the argument, in a systematic manner, that theft and predation are the major barriers to efficiency.

There have also been several recent works dealing with the problem of the necessity of the state in general terms. The most interesting are those of Taylor and Nozick, mentioned in the previous section. This study applies game theory to a much broader range of questions than the work of Taylor and Nozick.[6] It attempts to give an appreciation of the broader applications of the game theory approach, and of different ways of interpreting each question.

Some writers have attempted to apply notions of monopoly trading and transaction costs from neoclassical economics to the state.[7] In these approaches the state is analysed by looking at various aspects of externalities and economies of scale. Similar ideas are found scattered through much of the literature on public choice and public policy. One example of such work is Auster and Silver's *The State as a Firm*. Auster and Silver rightly believe that the theory of the firm has many applications to an understanding of the state. A similar point is made by North in *Structure and Change in Economic History* (1981, p. 21). This belief is, however, probably more true of some of the recent work on the foundations of the firm than the earlier style of analysis referred to by Auster and Silver.[8]

One problem with Auster and Silver's approach is that it is confined to an analysis of marginal rates of return. The state's role in providing security, for example, is analysed by treating the state as a firm which equates the marginal returns from resources devoted to enforcement with the marginal costs. This is reasonable if there is a state already in place. In this case the problem is to analyse the benefits of more expenditure on such things as reducing violence or the externalities from pollution. This approach does not explain the necessity of states and enforcement activities, however. Nor does it tell us what is essential about collective choice mechanisms. It cannot simply be assumed that the relations between the state and citizens are the same as between customers and a producer of a good or service. For example, security, justice and welfare do not seem to be like corn flakes or cans of beer in all respects.

North also attempts to develop a neoclassical theory of the state. He essentially sees the state as a producer of security. This is traded with the population in exchange for revenue. The state attempts to maximize revenue and is constrained by the potential of other would be states to offer a better deal (North, 1981, p. 23). This also tends to make too close an analogy with the firm and to treat the state as if it were attempting to maximize profits or revenues in a competitive market. This may make more sense as a proto-model of party competition within democratic systems.

Axelrod's books, *The Evolution of Cooperation* and *The Complexity of Cooperation*, are probably the best known and thorough attempts to analyse the prisoner's dilemma problem. Of particular note is the extensive use of simulations and agent-based models in the second of these books. These models allow experimental research into different forms of interaction and the problem of the evolution of strategies to be addressed through the use of genetic algorithms. This is outside the analytical approach of this book.

On the social choice side, much of the emphasis has been on voting as a scoring method which allows choice between alternatives. This trajectory runs through such writers as de Borda, Condorcet, Dodgson, Arrow and modern social choice theorists. Revelation problems and voting problems are dealt with extensively in the literature of social choice and its offshoot, public choice theory. The work by Buchanan and Tulloch is a classic in the field. Two excellent studies in social choice are those of Moulin, *The Strategy of Social Choice* and *Axioms of Cooperative Decision Making*. A fairly standard, and less technical, reference to public choice is Mueller *Public Choice II*.

More recent work on social choice has developed a topological approach. Pattanaik and Salles's *Social Choice and Welfare* gives an introduction to this work. Although interesting, the topological approach requires a good deal of mathematical sophistication and is beyond the scope of this book.

This study will only deal with a small part of the social choice problem. In keeping with the emphasis on game theory it will focus on strategic aspects of preference revelation and choice mechanisms.

1.6 Some game theoretical concepts

1.6.1 The elements of a game

The elements of a game are the players, the available strategies, the payoffs and the information available to each player.

[1] The players are the individuals involved in the game. This set is written N. Player i is an element of N. This is written $i \in N$. I prefer 'players' to individuals or actors because it serves as a reminder of the level of abstraction being used. I shall refer to players as 'it' or 'them' throughout to avoid the ugly him/her or variants.

[2] Players have preferences that are complete and transitive. Completeness means that an individual has a preference ordering over all possible outcomes. The symbol $\geq$ will be used in its usual sense of greater than or equal to and also to stand for preferred or equivalent to. This is an abuse of the notation, but the meaning will be clear in each context. Transitive means that $a \geq b$ and $b \geq c$ then $a \geq c$. This is a consistency requirement. To see the point let $a = 10, b = 5$ and $c = 1$.

[3] Pay-offs can be measured in utilities or in the value of an outcome. Let $v(\cdot)$ be the pay-off function, or the value, assigned to an outcome. Then if $a \geq b$, $v(a) \geq v(b)$. Observe that nothing has been said about the substance of a and b. a may be a rat in a brown paper bag and b may be a million dollars.

[4] It is assumed that players always attempt to maximize their pay-offs. An optimizing individual will sometimes be referred to as rational.[9]

[5] S is the set of strategies that are available. A strategy available to i is $s_i \in S_i$. Hence $S = \times_{i \in N} S_i$. For all players other than i a strategy is written s_{-i}, or s_j.

The connection between pay-offs and strategies is specified by letting V be the set of values or pay-offs. Each strategy gives a pay-off depending on the strategy of the opponent in all non-trivial cases. A strategy choice s_i^* for individual i gives a pay-off

$$v_i : (s_i^*, s_{-i}) \rightarrow x \in \mathbf{R}$$

for each $s_i \in S_i$. This says that the pay-off function provides a mapping from the strategy of player i and the strategy of all other players to a pay-off given as real number.

[6] Information is given by the set I. It is assumed that each player has complete information unless otherwise specified.

These elements give a game

$$\gamma = [S,\ V,\ N,\ I]$$

The game theory interpretation of questions about non-co-operative behaviour and situations where co-operation is possible is captured in the following distinction.

[1] Non-co-operative games are those in which all strategies are completely specified. In other words each player always plays whatever strategy maximizes its returns given the available pay-offs. It would be possible in such a game, for example, for players to keep promises, but it would be necessary to prove that this was the optimal strategy for each individual given the rewards and penalties. Every aspect of the relevant institutions and rules, and the moves available to all the players would have to be specified.

These games may be one shot simultaneous move game or repeated games. Assume that the game is one shot unless it is specified otherwize and that players do not communicate unless they are specifically allowed to send signals.

[2] Co-operative games are those in which players are allowed to keep promises or to make commitments to certain strategies before the play of any particular game. These games start with assumptions about the sociology of individual and group behaviour. These assumptions are not derived from a more primitive model of optimization. Since players can make commitments, they can pursue strategies that are collectively optimal. In a fully co-operative game with transferable utilities and side-payments, a move to the Pareto frontier is always possible because the gains can be distributed among the players by prior agreement.

1.6.2 The solution to a game

The solution to a game is the set of strategies that it would be optimal for each of the players to choose. Let the solution be $s^* \in S$, where

$$s^* = (s_1^*, s_2^*, \ldots, s_n^*).$$

If a solution exists it can be thought of as being produced by some mechanism that gives a mapping from the specifications of the game to an outcome. This is written:

$$\varphi :\ \gamma \rightarrow\ s^* \tag{1.i}$$

where $s_i^* \in S$ solves

$$\text{maximize}_{si}\ v_i\ =\ v_i(s_i,\ s_{-i}) \tag{1.ii}$$

When such an s^* exists for all players the game has an equilibrium.

A mistake that is often made in trying to determine s^* is that of confusing what is collectively reasonable and what is individually optimal. This mistake is common in the analysis of the problem of security and other problems of co-operation. It is illustrated by de Jasay's belief that game theoretical problems of security are the result of the assumption that individuals in the state of nature are 'myopic simpletons clad in animal skins clubbing each other on the head' (de Jasay, p. 3). In other words, they are too stupid to see that violence produces a sub-optimal solution.

This type of mistake may stem from the assumption that it is always rational to do what is reasonable. The chain of thought probably goes something like the following: it seems reasonable for individuals not to engage in destructive behaviour, therefore it is rational not to do so. It follows that anyone who does engage in such behaviour is stupid.

In game theory terms, this confuses the non-co-operative question of the optimal strategies for individuals with the co-operative question of what set of bargains individuals might make if promises could be kept and contracts enforced. To find out what is rational it is necessary to analyse the structure of the game. Myopic simpletons might, for example, attain security more easily than individuals trained in mathematical optimization theory. Incompetent generals might lose fewer troops in their campaigns than competent generals because they are unable to locate their enemy.

The problem to be solved by the mechanism φ is that of finding all the solutions to equation (1.ii) that have the property that no player has an incentive to change its strategy, given the strategies chosen by others. In other words, a solution exists if there is no move that a player could make that would make it better off. In some cases a player will have a strategy that solves (1.ii) regardless of the strategies pursued by the other players. This can be seen as a dominant strategy. In other cases the selection of strategies will be more difficult.

The problem of finding a solution to a game with two players is illustrated with the following example. To simplify the illustration assume that a strategy is the choice of a point on some closed interval, say $[a, b]$. This might occur for example where the strategy is some amount of a resource to devote to an activity and the resource can be treated as if it is infinitely divisible. In this case s_i^k is a choice of some amount, x^k, where $x^k \in [a, b]$. Let the strategy sets for players i and j be the intervals I and J respectively.

The intuition is that player i will play a strategy that maximizes its pay-off against a strategy by j. This allows us to draw a graph by plotting the strategies available to i on the horizontal and those available to j on the vertical. $r_i = r_i(s_j)$ can be thought of as a best response correspondence. It is the best reply of i to a strategy s_j played by j.

The great insight developed by Nash was that the strategies of all players will be in equilibrium when no player has an incentive to deviate or change its reply in

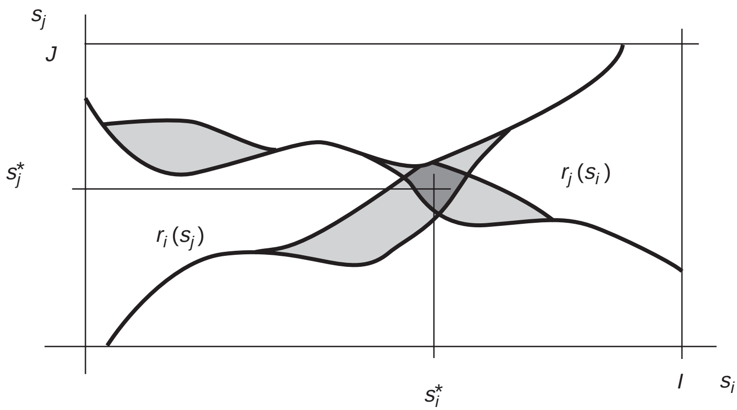


Figure 1.1 Reaction functions for two players

response to the best strategy of any other player. This means that φ selects a s^* such that each player has picked the s that is the best response to the other player's best response. This idea is set out in Figure 1.1.

Any point within the heavily shaded area is an equilibrium for both players. The proof that such a point exists for the specified class of games depends on Brouwer's beautiful fixed point theorem. It can be seen intuitively, however, that if player *i* and player *j* pick strategies that intersect in the shaded area neither has an incentive to change its choice.

An interesting question is, how do the players get to the equilibrium? There are three main possibilities.

[1] A process of calculation. The players solve the game or calculate the optimum strategies directly. This may assume a higher degree of computational capacity than is reasonable.

[2] A process of evolution. In this process each generation of players gets differential rewards according to the pay-offs from its strategies and reproduction rates reflect these payoffs. The requirement here is a repeated game with a large number of players. It is often argued that animals arrive at optimal strategies in this way.[10] Alternatively, it might be thought that strategies evolve through some sort of observation and learning process.

[3] An updating strategy. Players update their strategies based on observations of their success against the strategies of other players. There are a large number of trial and error updating schemes that might be used. The simplest is the tâtonnement process. This process is not very well specified in that it assumes that players never attempt to anticipate responses. It is, however, useful for illustrative purposes. I shall discuss it in

order to illustrate the general idea of stability of equilibria and some of the problems this entails. I deal with evolutionary stable strategies in chapter 6.

Definition: A tâtonnement process occurs when player *i* picks an s_i at time *t* and *j* chooses the best reply $s_j(s_i)$ at $t+1$, assuming that s_i remains unchanged. *i* now selects the best reply at $t+2$ to *j*'s strategy at $t+1$, and so on. ■

Now, consider when *i* will stop changing strategies. This will occur when the set of strategies *i* has chosen are also the best reply to what *j* has chosen. If the strategies that *j* has chosen are the best reply to what *i* has chosen, neither will have an incentive to deviate. In this case $r_i(s_j^*) = s_i^*$ and $r_j(s_i^*) = s_j^*$. Hence the strategies will be an equilibrium and

$$s^* = (s_i^*, s_j^*)$$

The equilibria for a game may be stable or unstable relative to whatever updating process is used. Roughly a stable equilibrium can be thought of as one towards which successive strategies move, or around which the strategy paths would orbit. An equilibrium is unstable if the paths move away. This will be determined by the properties of the reaction correspondences and the process chosen.[11] It is also possible that an equilibrium is stable for starting points within some small radius of the equilibrium but not for regions further away.

Figure 1.2 illustrates the case where there is an equilibrium but the tâtonnement process does not lead to it for a starting point some distance away. This does not

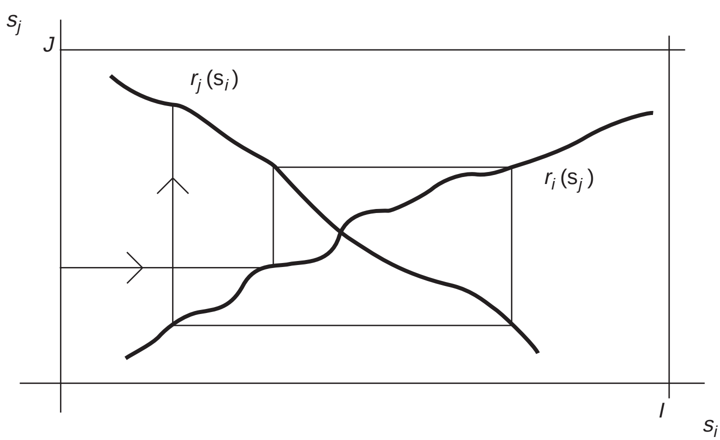


Figure 1.2 Equilibrium without convergence

mean, however, that there is not some other iterative process that would produce converge.

1.6.3 Expected pay-offs where there is uncertainty

Where players are uncertain about outcomes, it is assumed that they assign a value to the probability of an outcome and attempt to maximize the expected value of this outcome. For simplicity attitudes to risk are ignored. Individuals that prefer more money to less will assign a higher value to a bet that gives them one chance in ten of winning a hundred dollars to a bet that gives them one chance in five of winning forty dollars.

To calculate the expected value, let the result of a strategy s^o be a set of outcomes, where outcome i occurs with probability p_i and has value v^i. $p_i \in p, i = 1, \ldots, n$ where p is a probability measure.[12] Then the expected value of s^o is

$$E[s^o \mid p] = \sum p_i v^i$$

1.6.4 Sub-game perfectness and credible threats

Players might issue threats to each other in the form of retaliation for a previous action. For example, one player might make the threat that if some other player steals its resources it will retaliate by fighting. One of the lessons of game theory, and life, is that talk is cheap. In order to deal with threats, we need a notion of credibility. This is covered by the concept of sub-game perfect equilibria.

Sub-game perfection refers to the common sense idea that, once a round of the game has been played, a response is only optimal if it maximizes the returns given the game from that point forward. In economics this is covered under the heading of sunk costs. This point is illustrated in Figure 1.3. There are two players and player 1 moves first and player 2 moves second.

Imagine that player two has indicated that it will retaliate if player 1 plays $s_1 = a$ by playing $s_2 = a$ to give a pay-off $(-1, 0)$. Player 1 plays a. What does player 2 do in the sub-game starting on branch a? From the assumption that players are pay-off maximizing, it must play b. Hence the threat is not credible because the strategy (a, a) cannot be sustained in the second move of the game.

1.6.5 Strategic form and extensive form

There are two ways to represent the strategic interaction problem in non-co-operative games. One is to construct a diagram with branches that connect every strategic choice that a player might make to the strategic choices of all other players. The outcome of each sequence of strategies is given as a pay-off. In this case the game would be in extensive form. Figure 1.3 gives an example. The other

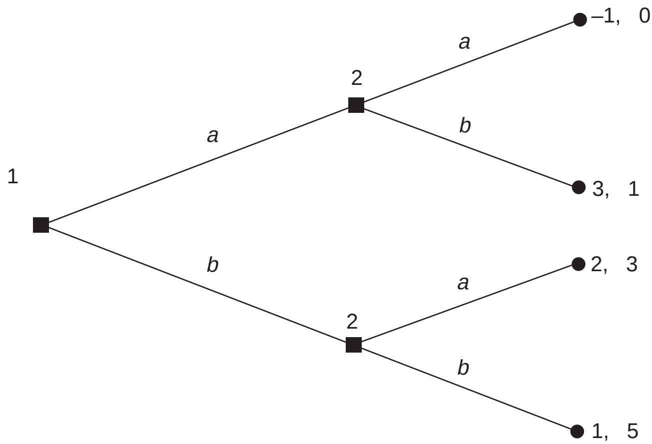


Figure 1.3 Extensive form and subgame perfect equilibria

would be to write a strategy as the choice of a sequence of strategies. Let a game have m nodes and s_i^k be the kth strategy for player i. Then

$$s_i^k = (s_i^{1k}, \ldots, s_i^{mk})$$

is the kth strategy move at each node of a game. This means that, in the above diagram, each of the paths that leads to a pay-off would be represented by a single strategy. In this case the game is said to be in strategic form.

Since the number of possible strategies and responses is often large, the extensive form poses some problems of tractability. I will not deal with games in extensive form in what follows. This may involve some loss of information. In some cases the sequence of moves and the dynamics of the path is important. I think that this problem can only be satisfactorily solved by a dynamic theory of non-co-operative games.

1.7 The measure of welfare and Pareto superiority

The terms welfare, and Pareto superiority are used throughout this study. These terms and some of the assumptions they require are dealt with below.

Aggregate welfare is some measure of the welfare derived by all individuals from some state of affairs. For example the welfare from some state of affairs a may be measured by adding the utilities derived by each individual

$$\omega(a) = v_1(a) + v_2(a) + \ldots + v_3(a)$$

Situations can also be compared using the concept of Pareto superiority. All this requires is ordinal information, that is the rankings that individuals assign to different states of affairs. It does not require any measure that makes interpersonal comparison possible. Let $a = (a_1, \ldots, a_n)$ be a vector that gives outcome a_i to individual i. Then

Definition: a is Pareto-superior to b if all individuals are at least as well off with a as with b, and one individual is strictly better off. That is $v_i(a) \geq v_i(b)$ for all $i \in N$ and $v_j(a) > v_j(b)$ for at least one $j \in N$. ■

A state of affairs where no further Pareto superior moves could be made is said to be on the Pareto frontier.

Definition: The Pareto frontier is the set of points where one individual cannot be made better off without making some other individual worse off. ■

Note that a programme that maximizes welfare will be on the Pareto frontier since everything will be distributed. A welfare maximizing move will not necessarily be Pareto superior, however. Think of taking a good from a person for whom it has very little utility and giving it to another person for whom it has a great deal of utility. An example would be a dollar from a millionaire to a starving person.

The idea of Pareto superiority is found in Hobbes, for example, when he claims that the move from the state of nature to a situation with an enforcement mechanism is an improvement for all. Locke has a version of Pareto superiority in mind when he argues that property rights make everybody better off. He compares the best holding of material goods without property and accumulation with the worst holding with accumulation. He claims that 'a king of a large and fruitful territory there feeds, lodges, and is clad worse than a day labourer in England' (Locke, 1963, p. 339). In this case he is making a claim for what might be thought of as super-Pareto superiority.

Pareto-superior moves and the Pareto frontier are illustrated in Figure 1.4. Imagine that there are two individuals and the pay-offs for individual one and two are written v_1 and v_2. Assume that there is some good that is infinitely divisible and that the pay-offs can be represented by a continuous function. It might be though that the good is the benefit from buying and distributing a pizza or from some joint venture. Let the amount they can get in the original position be c_1 for individual 1 and c_2 for individual 2. It will be observed that any move to the right of c_1 makes player 1 better off and any move above c_2 makes player 2 better off. In the area a, b, c both players can be made better off. Any move from c into this region is Pareto superior.

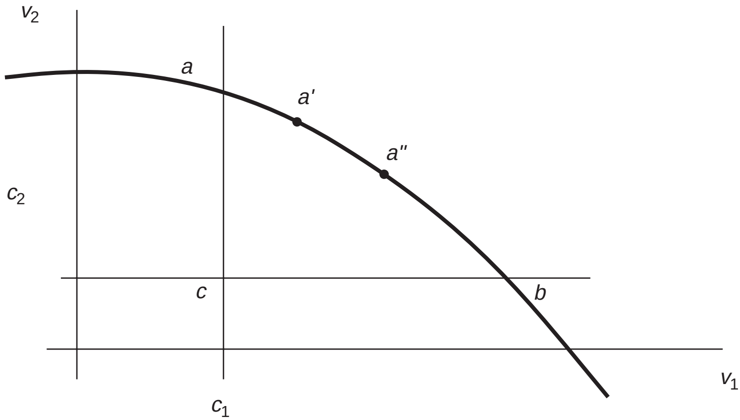


Figure 1.4 The Pareto frontier for two players in the positive quadrant

From the assumption that more utilities are preferred to less, individuals will move to this frontier. Note that the location on the frontier is not determined. It may be some point like a' or a''.

1.8 Note on the general problem of optimal outcomes

It was previously claimed that the equilibrium strategies of individuals may not always provide a Pareto optimal outcome in non-co-operative games and outcomes of type [Ai] in chapter 1 may not be common. This claim will be examined in the subsequent chapters. Before doing this, however, it may be worthwhile considering the case for games in general. The point that Pareto efficient outcomes are not, in general, to be expected can be illustrated with a fairly simple example. This example is somewhat abstract. It is included here because it is a rather neat way to make the point independently of the particular arguments and it sums much of the detailed analysis of the succeeding chapters. It can be skipped without any loss of comprehension in what follows.

Return to the general form of the problem given in equation (1.ii) above and assume that the problem for each player is to decide how much to spend on some activity, such as buying security, or undertaking research, or fishing. A strategy is the amount spent.

Assume that the pay-offs are continuously differentiable with respect to the strategies at all points in the domain of v, and that the Pareto frontier is strictly concave. Strictly concave means no horizontal or vertical sections. These assumptions could be dropped but it would make the proof tedious. Let s_i be a strategy of spending some amount of a resource and treat the strategy space as a closed interval, $s_i \in [0, a]$.

1.8.1 Nash equilibrium

Suppose that there is some strategy s_i that is a non-co-operative equilibrium for player *i*. A Nash equilibrium is defined more fully in chapter 2. The first order condition is that a strategy, s^*, is an equilibrium if a player cannot be made better off by changing. This means

$$(\partial v_i (s^*)/\partial s_i)\, ds_i \leq 0 \tag{1.iii}$$

for all $i = 1, \ldots, n$.

1.8.2 Pareto optimality

The definition of Pareto-optimality says that *s* is on the Pareto frontier if a gain for any one player causes a loss to some other player. To see this consider Figure 1.4 with $v(s)$ some point on the frontier. That is, for $(\partial v_i(s)/\partial s_i)ds_i > 0, (\partial v_j(s)/\partial s_i)ds_i < 0, j \neq i$. This means that an outcome is Pareto optimal if, for some $\alpha > 0$, it satisfies the equation

$$\sum_{j=1}^{n} \alpha_j(\partial v_j (s)/\partial s_i)ds_i = 0 \tag{1.iv}$$

1.8.3 Comparison

The solutions to equation (1.iv) will exist for any *s* such that $v(s)$ is on the Pareto frontier. However, from equation (1.iii) the individual equilibrium for a player, *i*, tells us nothing about

$$\partial v_j(s^*)/\partial s_i$$

Therefore the solution to equation (1.iii) is not, in general, a solution to equation (1.iv). It follows that Pareto optimal solutions are not, in general, non-co-operative equilibrium solutions.

It might be thought that agreement can solve the problem. Suppose that players agree to make a move to a Pareto optimal outcome. The question is, why should any player keep the agreement? From equation (1.iv) and the assumption that the Pareto frontier is strictly concave, a general condition for an outcome to be on this frontier is that there exists a change in the strategy of player *i* such that

$(\partial v_i(s)/\partial s_i)ds_i > 0$. In this case, unless there are constraints, it will pay every player to choose a different strategy.

Taking the same problem from the other direction, it may be a Pareto superior move for an agent to change its strategy from s^* to s. But, from equation (1.iii), $(\partial v_i(s)/\partial s_i)ds_i \leq 0$ at the optimum. Hence $\Delta v_i < 0$ for any infinitesimal change in s_i. Hence, the change is not an optimal move for the agents acting individually.

This also reveals an interesting connection between strategic interaction and the Pareto frontier. The conditions $(\partial v_i(s^*)/\partial s_i)ds_i \leq 0$ and equation (1.iv) is satisfied for an internal solution when

$$\partial v_i(s^*)/\partial s_i = 0 \text{ and } \partial v_j(s^*)/\partial s_i = 0$$

for all j. Define strategic dependence as a situation where the pay-offs for i depend on the strategies of j. Then

$$\partial v_j(s)/\partial s_i \neq 0.$$

This means that the non-co-operative equilibrium will always meet the conditions required for the Pareto frontier where there is no strategic dependence.

2

Security in the Hobbesian State of Nature and Related Non-Co-operative Games

2.1 Introduction

States are necessary to enforce collective decisions over security for life and limb, it is argued in the Hobbesian tradition, because, without enforceable rules, the optimal choices of individuals create a situation where there is no security or any of the other preconditions for material production or civilized existence. Hobbes argues, for example, that the desire for security will lead to continual preparation for war and a condition in which there is 'continual fear and danger of violent death, and the life of man solitary, poor, nasty, brutish and short' (1968, p. 186). This argument is a specific case of the more general problem of individual actions leading to a Pareto inferior outcome outlined in Chapter 1. Note that the argument does not depend on the assumption that individuals want this outcome. If security would emerge through the working of some spontaneous mechanism there would be no need to submit to a centralized authority in order to get protection and to solve problems of conflict. If, on the other hand, security requires an enforcement mechanism to provide the rules necessary for non-violence to be an equilibrium outcome, the argument for centralized control of force is more plausible.[1] It must be stressed that the assumptions made about such things as the nature of the pay-offs in the absence of authority, and the extent to which the game can be repeated, are critical when thinking about the problem of security. It is not at all clear, for example, that the case where the same game can be repeated a large number of times is appropriate for the security problem.

This chapter introduces basic models in static non-co-operative games to analyse the security problem and some related questions. It also looks at the structure of these games when repetition is allowed.

It will be noted that the structure of the Hobbesian problem is analogous to the structure of collective goods problems more generally. From this angle, the problem of security is a variant of the more general collective goods problem.

One of the themes that emerges from the chapter is that, for the type of game analysed, spontaneous co-operation is neither easy to come by nor particularly robust. Many of the interactions produce outcomes of type [Aiii].

Since it has received a great deal of attention in the literature, some brief consideration is given in the last section of this chapter to the question of whether individuals could agree to a structure of authority.

2.2 The simple Hobbesian problem

The Hobbesian argument is set up by starting with the methodological device of a fictitious state of nature in which there are no constraints on action. The argument depends on the assumption that being dominated is the worst of all possible outcomes, and that individuals can only choose between arming themselves for battle or being unarmed and dominated by an armed player. The state of war includes both battle and preparing for offensive and defensive activities. In seeking security, individuals are engaged in a quest for power. Power is relative in the sense that one individual's power can only increase at the expense of others in the relevant group. Hence the quest for security creates a 'perpetual and restless desire for power after power that ceaseth only in death (Hobbes 1968, p. 161).

Such problems have been interpreted in many ways by political theorists.[2] It is not the purpose of this study to make an argument in favour of any particular interpretation. All that needs to be emphasized is that it is the structure of the game which produces the Hobbesian problem, not a desire for aggression.

This point is often misunderstood. Paglia says that the point of Hobbes's argument is that 'aggression comes from nature' (1990, p. 2). This is simply wrong in any non-trivial sense of the argument.

The justification for concentrating on the strategies of arming or not arming is that other strategies are not of interest for the analysis of the Hobbesian state of nature. Similarly, costs of fighting are a secondary consideration. It is always better in this state of nature to fight at any cost than to be dominated. It might be argued that this assumption is too strong. This would amount to claiming that costs of fighting have to be taken into account, and that submitting might be preferable to fighting. This possibility is dealt with in the subsequent chapters.

In order to analyse this situation consider a simple game with a dominant sub-optimal strategy. This game is well known under the unfortunate title of the prisoner's dilemma. This is analysed by treating it as a one-shot interaction between players. This means that the game is played once and that both players move simultaneously.

2.2.1 The prisoner's dilemma and dominant strategies

The prisoner's dilemma represents the Hobbesian problem, and problems of collective action more generally, in their most simple form. This game also gives a simple model of a range of other problems and is seen by many as ubiquitous in social interaction.[3] It has probably been more widely applied than it deserves, and is often used to draw conclusions that are more general than seem justified. Many of its striking predictions arise from the restrictive assumptions of the one-shot game. None the less, it is a useful starting point and is of particular interest for the problem of security. It also gives rise to a number of perversities in interaction that are instructive for an introduction to game theory.[4]

The set up is that there are two players and each has two strategies. This might also be thought of as a game between player 1 taken at random and one, or any number, of other individuals given by player 2. The strategies are a choice between arming and not arming, or cheating and not cheating, depending on the precise problem. These strategies are usually called co-operate and defect.

Let S_i be the strategy set of individual i. Let

$$S_1 = (s_1^c, s_1^d) \text{ and } S_2 = (s_2^c, s_2^d)$$

where the superscripts c and d stand for not arm and arm, or co-operate and defect, respectively. The pay-off is security. Hence to dominate is better than being equal and to being dominated. To arm against domination is better than being dominated. This is because the possession of arms gives a non-zero probability of avoiding domination. That is, for player,

$$v_1(s_1^d, s_2^c) > v_1(s_1^c, s_2^c) \text{ and } v_1(s_1^d, s_2^d) > v_1(s_1^c, s_2^d).$$

The general structure of the game is given in strategic form in Figure 2.1(a).

The game is symmetric for all players so that the pay-off for player 1 against player 2 using strategy s_2° is the same as the pay-off for 2 against 1 playing s_1°. In the Hobbesian state of nature $c_1 > a_1$ and $d_1 > b_1$. For ease of reference these inequalities can be represented by any order preserving numbers. That is any number x can be assigned to c_1 and y to a_1, provided that $x > y$. An example is given in Figure 2.1(b).

In order to analyse this game the following definition is useful.

Definition: Dominant strategies. A strategy s_i^* will be said to strictly dominate s_i when

$$v_i({s_i}^*, s_{-i}) > v_i(s_i, s_{-i})$$

That is when the pay-offs for s_i^* are strictly greater than the pay-offs for any other strategy against all the strategies used by other players. If

$$v_i(s_i^*, s_{-i}) \geq v_i(s_i, s_{-i})$$

s_i^* weakly dominates s_i. ■

	s_2^c	s_2^d
s_1^c	a_1, a_2	b_1, b_2
s_1^d	c_1, c_2	d_1, d_2

(a)

	s_2^c	s_2^d
s_1^c	2, 2	–1, 4
s_1^d	4, –1	0, 0

(b)

Figure 2.1 The prisoner's dilemma

Theorem 2.1: A player never uses a dominated strategy.
Proof: Obvious ■

It follows immediately from the pay-offs and the definition that the game has a dominant strategy and the solution is

$$\varphi = (s_1^d, s_2^d)$$

Remark [i]: The dominant strategy is played for all strategies of the opponent and hence these do not have to be considered. The force of this point is often missed. One provision that is commonly made in presenting the prisoner's dilemma is that players are not allowed to communicate.[5] It will be seen, however, that communication will not alter the outcome. This is because there is no alternative equilibrium that players could converge on which gives both a greater pay-off than defecting. An agreement to co-operate is not an equilibrium because the value of defecting is greater than the value of co-operating against any strategy chosen by an opponent.

Remark [ii]: This game illustrates the more general point in section 1.8 that equilibria need not be Pareto optimal. Although it is one of a class of such games, it has been taken as the prime example of a particularly perverse or unreasonable outcome. This is probably because of its simplicity and the fact that the players get what they were most trying to avoid.

Remark [iii]: It is sometimes suggested that players would evolve norms of co-operation or some other pattern of behaviour in order to avoid such a sub-optimal outcome (Ullman-Margalit, 1977). This may be true in some cases. What must be stressed, however, is that the notion of players having a change of heart and developing the right sort of values is, in no sense, a solution to the prisoner's dilemma. If the norms are so altered that a player prefers, say, co-operation with a high probability of being dead to non-co-operation, the game is no longer a prisoner's dilemma. This is because the pay-off structure is altered.

What would be needed, then, if the Hobbesian argument were to be avoided is a solution to the game that can be generated within the logic of non-co-operative strategic interaction. This leads to a study of repeated games.

2.3 Repeated interactions

Axelrod has examined some of the conditions under which repetition of the prisoner's dilemma makes co-operation possible in *The Evolution of Co-operation* and *The Complexity of Co-operation*. The first contains the celebrated tit-for-tat strategy. In the second he develops a simulation that allows strategies to evolve as the result of a selection process that rewards those that are more successful. Taylor and others have suggested that the argument underlying the Hobbesian problem may not be valid if the prisoner's dilemma game is repeated; de Jasay agrees with this claim (1985, p. 44). I shall briefly consider these claims before looking at the characteristics of the repeated prisoner's dilemma.

The argument that co-operation is possible when the game is repeated a large number of times is correct. It would not be correct, however, to extend this to the proposition that repetition can solve the Hobbesian problem. This is because the conditions required for punishment to be effective may not hold in many cases. Most importantly, a game of security might only be played once (Zupan, 1991). It is then possible for the winner to write the rules for subsequent engagements. If so, a strategy of co-operate on the first encounter may prove fatal. In Axelrod's work on the evolution of strategies each round of the evolutionary game consisted of 151 plays with each opponent (1997, p. 20).

More generally repetition requires special circumstances that do not always hold in games where there are large pay-offs from defecting if another player co-operates. This problem is most acute in large groups. Even if the same game were played several times it may only be played repeatedly with the same individuals in special conditions. Transactions with large pay-offs, or large potential losses, such as buying a house or investing savings, for example, are not often repeated under circumstances that allow punishment for defection. This is why it is plausible that small communities, where all individuals know each other, can more readily enforce co-operation than large open groups of individuals.[6]

The feature that is common to all repeated games is that the structure of the game changes when players are concerned about future interactions. A strategy of co-operation can be enforced by making the strategy chosen at time t dependent on the strategies of opponents at times $(0, 1, \ldots, t-1)$. Thus the strategy set for the repeated game is not the same as the one shot game because it is possible to punish opponents who do not co-operate by playing strategies that reduce their future pay-offs. For example, if player 2 defects on any round player 1 could punish by defecting on a number of subsequent round and co-operating when player 2 has been sufficiently co-operative.

One way to think of a repeated game is to imagine that players can announce a strategy which includes their proposed response to the strategies of other players in the previous rounds of the game. Such a game would give a self-enforcing optimal collective outcome if it is optimal for other players to respond in a way that supports this outcome.

Definition: A repeated game is a series of rounds of the same one-shot stage game with the same pay-offs at each round. In this case, the game has to be extended to take into account the information that each player has received from previous moves. This gives

$$\gamma = [S, V, N, \kappa]$$

where κ is the information that the play has received up to round $t-1$ of the game. ■

It will be noted that, unlike the dominant strategy case, the optimum response of each player now depends on what it thinks that the other players will do. The solution requires the idea of the Nash equilibrium in Figure 1.1. Specifically:

Definition: Nash equilibria. A strategy s^* is a Nash equilibrium if for all players

$$v_i({s_i}^*, {s_{-i}}^*) \geq v_i(s_i, {s_{-i}}^*)$$

for all $s_i \in S_i$. ■

If all other players play ${s_{-i}}^*$, then a strategy $s_i \neq s_i^*$ cannot give a higher pay-off for player i than s_i^*.

One problem that is faced in the analysis of repeated games is that of making the pay-offs meaningful. Let w_i^t be the pay-off for player i in round t. It is obvious that where $t = 1 \ldots n$, and n tends to infinity

$$v_i = \sum w_i^t$$

also tends to infinity.

Since we are dealing with pay-offs stretching into the future, however, it is appropriate to consider that a pay-off at some future time may be worth less than the same pay-off now. This is dealt with by discounting. It follows that the value of the pay-off streams would decline in some fashion. This assumption is justified by noting that interest rates are usually positive and life is uncertain. One way to do this is to multiply future pay-offs by a discount factor $\delta < 1$. Note that the lower the value of δ the more the future is discounted. In this case the pay-off stream would be $w_i + \delta w_i + \delta w_i + \ldots + \delta^n w_i$ or

$$v_i = \sum_{t=1}^{n} \delta^{t-1} w_i^t$$

Since $\delta < 1$ the pay-off converges because $\delta^t \to 0$ as $n \to \infty$.

2.4 Analysis of repeated games

The argument that infinite repetitions of the prisoner's dilemma may allow co-operation is most easily seen in what is called the folk theorem. This says that in infinitely repeated games any outcome is possible. An infinitely repeated game and a game without a known termination date have similar characteristics as long as a sufficiently low probability is attached to termination at any one game. At each stage each player knows what moves every other player has made previously. This outcome may be a string of co-operate moves, or of defect moves, or any permutation of a string of co-operates and defects.

Theorem 2.2: (Folk theorem). For an infinitely repeated game there is a discount rate such that there is a Nash equilibrium for every feasible strategy vector s and pay-off vector v, where $v_i \geq v_i^m$. v_i^m is the minimum to which all other players can hold i. ■

The idea can be grasped without a formal proof. What the theorem says is that, if the discount rate is sufficiently low, any pattern of co-operation can be enforced by players who are prepared to punish a non-co-operator. Thus, any combination of pay-offs can result.

This says that co-operation is sometimes possible with infinite plays of the game, but says nothing about the strategies that might be employed or their results. Consider some specific strategies under the best assumptions for the argument that spontaneous co-operation is possible.

2.4.1 Example one: trigger strategy or grim strategy with a large number of players

The grim, or trigger, strategy is particularly interesting because, if it is an equilibrium, it enforces co-operation throughout the game. Moreover, it would enforce co-operation without any repeated interaction between individual players. All that would be required is a communication by all players that they intended to use this strategy and the belief that this is a truthful message.

This strategy punishes any player who deviates from s^c at t^* by a response of s^d forever from that point by all other players. To see the set-up of the argument, approximate the discounted stream of pay-offs by a smooth curve over time and consider the pay-off stream in Figure 2.2.

Assume that the discount rate δ is sufficiently high that the area

$$[a] = [v_i(s_i^d, s_{-i}^c) - v_i(s_i^c, s_{-i}^c)]\Delta t$$

at t^* is less than the area between $v(s_i^c, s_{-i}^c)$ and $v_i(s_i^d, s_{-i}^d)$ from $t^* + 1$. If the threat is credible player i does not defect. Hence $\varphi = (s_i^c, s_{-i}^c)$.

The first point to consider in thinking about the grim strategy is the role of the discount factor. This can be examined by using the pay-offs in Figure 2.1.

Let $v_i(s_i^c, s_{-i}^c) = a$, $v_i(s_i^d, s_{-i}^c) = c$, and $v_i(s_i^d, s_{-i}^d) = d$. For player i to co-operate on all rounds it must be the case that the pay-off is greater than defecting on any round $t + 1$. Hence

$$(1 + \delta + \ldots)\, a > (1 + \delta + \ldots + \delta^t)\, a + \delta^{t+1} c + (\delta^{t+2} + \ldots) d$$

Subtracting the a terms and dividing both sides through by $(\delta^{t+1} + \ldots)$, and noting that $\delta^{t+1}/(\delta^{t+1} + \ldots) = (1 - \delta)$ gives

$$\delta > (c - a)/(c - d) \qquad (2.i)$$

This gives $\delta > 0$ from the assumption that c is the largest pay-off in the game.

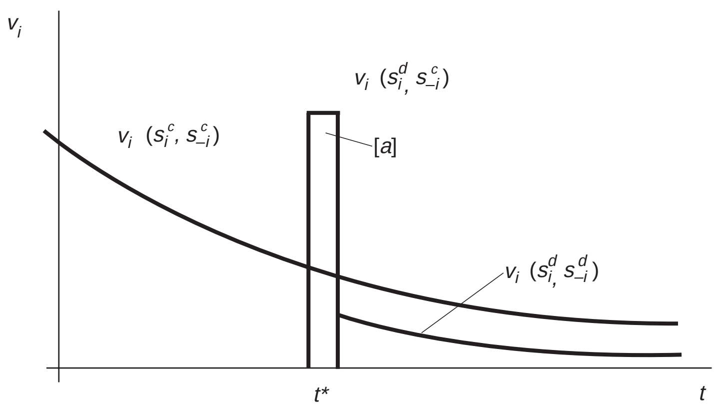


Figure 2.2 Pay-off stream with discounting

Since $c - a$ is the difference between the pay-off for defecting if everyone else co-operates and co-operating if everyone else co-operates δ needs to increase as this difference increases. That is, the future has to be less heavily discounted as this difference increases.

Note that for $(c - a) > (c - d)$ players never co-operate since this requires $\delta > 1$. This means that for large differences between the co-operation and the defection pay-off, or sufficiently large pay-off for all players defecting, co-operation is impossible.

An interesting implication of this is for the case where players have some reason to assume that some other player in the game will be tempted to defect and shift all strategies to s^d. This assumption may arise from uncertainty about discount factors or from the possibility that players make mistakes. As the probability of a defection increases the value of the discount factor is reduced. This reduces the likelihood that δ can meet the condition in equation (2.i). It follows that uncertainty makes the trigger strategy equilibrium precarious.

The second point to consider is that the threat of defecting forever is not always credible. It is sub-game perfect in the sense that, if every other player defects the non-co-operative equilibrium for i is to defect also. To see this consider the two player game. Observe that (s_1^d, s_2^d), is a Nash equilibrium for each individual game. However, this depends on both players staying with their defect strategy. It is not clear that this threat would be enforced.

The way out of this problem is to invoke an additional restriction on equilibrium for dynamic games known as Pareto perfection. The idea here is that players will not use a dominated equilibrium in any sub-game.

Consider the two player game. Observe that in the grim strategy players are allowed to announce a threat. If a threat can be announced before move one, it can be re-announced or renegotiated at move t. Suppose player i defects at some time and then announces that it wishes to co-operate for all future games. Since

$$v_j(s_i^c, s_j^c) > v_j(s_i^d, s_j^d)$$

player j can do better by returning to a co-operative strategy.

Similarly, in the n player case where the pay-off is a positive and increasing function of the number of contributors it may be possible to sustain a number of permanent defectors. Let the number who contribute be m and $v_i = f(m)$ with $f(0) = 0$. Imagine player i defects on round t^* and says that it will not co-operate in any future game. Now each of the other players is faced with the choice of

$$v_j(s_i^d, s_j^c) = f(m - 1)$$

where $j \neq i$ for all future rounds or

$$v_j(s_i^d, s_j^d) = f(m-1)$$

where $f(0) < f(m-1), m > 1$. Hence it is not optimal to defect forever even if some players never co-operate.

2.4.2 Example two: tit-for-tat with *n* players

The idea behind the tit-for-tat strategy is that co-operation can be induced in a repeated prisoner's dilemma by a direct eye-for-an-eye response (Axelrod, 1984). Player i starts by playing s_i^c and then plays whatever the other player used on the last round. A move of s_{-i}^d is punished on the next round by s_i^d. It is easy to see that tit-for-tat is a Nash equilibrium and is sub-game perfect where the game is played repeatedly, the discount factor is sufficiently large and the pay-off matrix meets a condition called condition ζ.

To define ζ use the pay-offs from Figure 2.1(a). Then $v_i(s_i^c, s-_i^c) = a$, $v_i(s_i^d, s_{-i}^c) = c$, and $v_i(s_i^d, s_{-i}^d) = d.v_i(s_i^c, s_{-i}^d) = b$.

The matrix game 2.1(a) meets condition ζ if and only if $c + b < 2a$.

Under these conditions tit-for-tat is a Nash equilibrium because it is a best reply to itself. It gives pay-offs at each round of $v_i(s_1^c, s_2^c)$. No strategy does better since gains from defecting at t are lost at $t + 1$. It is sub-game perfect because a threat of (s_i^d, s_{-i}^d) is a Nash equilibrium for player i in any sub-game.

The first thing to notice is that the tit-for-tat strategy requires special conditions to produce an equilibrium. It will not work for games that give an opponent an advantage in future rounds or that fail to meet condition ζ. This is the same as saying it will not work for games that have a large penalty for losing.

This point is illustrated by using Axelrod's computer tournament with the iterated prisoner's dilemma (Axelrod, 1997, p. 16). This meets condition ζ since $a = 3, c = 5, b = 0$. Hence $c + b < 2a$.

Suppose now that the pay-offs for defecting while the other player co-operates increase to $a = 10$. Then

$$c + b = 10 > 2a.$$

Let players i and $-i$ play $s_i^* = s_{-i}^*$ = tit-for-tat and $v_i = \sum v_i^t$ where t is a round of the game and discounting is ignored. Then there exists an s_i such that

$$v_i(s_i{}^*, s_{-i}{}^*) < v_i(s_i, s_{-i}{}^*)$$

hence tit-for-tat is not Nash. In particular let

$$s_i = s_i^d, s_i^c, s_i^d, s_i^c, s_i^d \ldots.$$

This is a sort of a reverse tit-for-tat. It meets each co-operation with a defect and each defect with a co-operation. This gives pay-offs $10 + 0 + 10 + 0 + \ldots > 3 + 3 + 3 + \ldots$.

To avoid this consequence the strategy would have to be altered to play *m* defections for each defection by an opponent. *m* must be such that the pay-offs meet a new condition ζ'.

The matrix game 2.1(a) meets conditions ζ' if and only if $c + mb > (m + 1)a$

Similarly, tit-for-tat will not work where there are a large number of players, unless player *i* plays *j* an indefinite and large number of times, or every player is decisive.[7] That is to say, in an *n* player game without repeated interactions, it must be necessary for $n - 1$ to co-operate in order to make it worthwhile for all others to co-operate. It will be noted that condition ζ' puts much heavier weight on interaction than condition ζ in that a much more lengthy run of interactions is needed to make retaliation effective.

In addition certain restrictions on discounting are required. If the game is played once a year, for example, and *i* discounts next year heavily retaliation may not induce co-operation on round one.

Repetition generates an interesting problem where there are many players, repetition is not guaranteed, and players are not certain about the type of their opponent. With what probability would the game have to be repeated for tit-for-tat to be a worthwhile strategy? This problem is now analysed. Since it is not possible to consider every case, consider the following example.

Assume that the initial pairing of players occurs at random. The players only have a choice between tit-for-tat and a strategy of defect, and the probability that the game will continue to the next round is given by $\sigma \in [0, 1]$. Let the good be order and the pay-offs $v_i(s_i^c, s_{-i}^c) = a, v_i \ (s_i^d, s_{-i}^c) = c$, and $v_i(s_i^d, s_{-i}^d) = 0$. $v_i(s_1^c, s_{-i}^d) = b$, and $a, c > 0$ and $b < 0$.

In this game a tit-for-tat player will get a pay-off of

$$\mathrm{a}\,(1 + \sigma + \sigma^2 \ldots + \sigma^n)$$

which is the same as $a/(1 - \sigma)$ if the other player plays tit-for-tat and $b + 0 + \ldots 0$ if the other player defects (Figure 2.3).

Let the probability that the opponent is a tit-for-tat player be $p, p \in [0, 1] \ldots$. Then to play tit-for-tat requires that

	s_2^c	s_2^d
s_1^c	$a/(1-\sigma)$	b
s_1^d	c	0

Figure 2.3 Pay-offs with tit-for-tat and defecting players

$$p[a/(1-\sigma)] + (1-p)b \geq pc,$$

which requires

$$p \geq -b(1-\sigma)/[a-(b+c)(1-\sigma)] \quad (2.ii)$$

Inequality (2.ii) has some interesting features. Observe that it requires $a \geq c(1-\sigma)$ for $p \leq 1$. By definition $c > a$. This means that the value of σ cannot be too small.

As noted, the usual assumption in the Hobbesian game is that domination is the best outcome and being dominated is the worst. For example, let $c = 2, a = 1$ and $b = -3$ and the probability that the other player is tit-for-tat be $\frac{1}{2}$. For tit-for-tat to be the best first move for player one, the probability that the game will be played again for every round must be given by

$$\tfrac{1}{2} \geq -b(1-\sigma)/[a-(b+c)(1-\sigma)]$$

.

This gives

$$\sigma \geq 1 - a/(c-b)$$

which is 0.8.

2.4.3 Example three: finite number of plays

The case where the game has a known termination date produces the backward induction paradox (Luce and Raiffa, 1985).[8] It gets the label 'backward' because the game unravels from the end point.

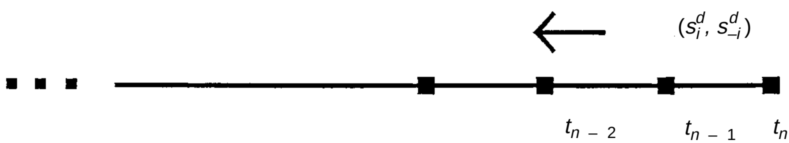
To see how the backward induction works, suppose there are two players and each player knows that the game terminates at some time t_n. Hence

$$\varphi_n = (s_1^d, s_2^d).$$

At t_{n-1} both players know that (s_1^d, s_2^d) will be played on the next move and no further punishment is possible. Hence

$$\varphi_{n-1} = (s_1^d, s_2^d).$$

In this way the game unravels so that (s_1^d, s_2^d) is played on every move.



This argument has worried many writers since Luce and Raiffa suggested that it seemed unreasonable in the 1950s (Luce and Raiffa 1985, p. 101). For example $v_1(s_1^d, s_2^c) - v_1(s_1^c, s_2^c)$ might only be one cent whereas $v_1(s_1^c, s_2^c) - v_1(s_1^d, s_2^d)$ might be a million dollars. If there are r rounds of the game, the players sacrifice a chance of $r(10^6 - 0.01)$ dollars to gain one cent.

To see this outcome as paradoxical is to miss the point made previously that intuitive understandings of what is optimal for each individual player may often be wrong. It simply illustrates the point made in Chapter 1 that the equilibrium strategies for individuals may not be on the Pareto frontier.

2.5 Pay-offs depend on contributions

The analysis of security games can be extended by returning to the one shot simultaneous moves case. Consider the possibility that individuals attempt to provide security through some voluntary agreement to contribute towards the costs of order. In this case, pay-offs depend on contributions. This example falls short of the full security market that is analysed subsequently. It is interesting here because it is conceivable that such a game may not be a prisoner's dilemma at all levels of contribution. In addition, Taylor has argued that co-operation may be self-enforcing under a voluntary contributions scheme.

	s_{-i}^{c}	s_{-i}^{d}
s_i^c	$f(m) - c$	$f(1) - c$
s_i^d	$f(m - 1)$	0

Figure 2.4 Continuous pay-offs with conditional co-operation

2.5.1 Example one: pay-offs depend on contributions

This can be analysed as follows. The value of security will be treated as a continuous function $f(m)$ where there are n players and m is the number of contributors. The cost is c where

$$f(1) < c < f(m) \text{ for some } m.$$

Let the case with no security have value zero. Pay-offs are for player i. In this case the pay-off matrix for i against all others is set out in Figure 2.4.

For s^d not to be a dominant strategy,

$$f(m) - c > f(m - 1)$$

at some level of contribution. That is each additional contributor must produce more additional pay-off in security than the cost of the contribution.

2.5.1a Additional security not greater than the cost

In the case $f(m) - c < f(m - 1)$ any player can announce permanent defection in order to get

$$v_i(s_i^d, s_{-i}^c) = f(m - 1).$$

The threat on the behalf of other players to retaliate is sub-game perfect because (s_i^d, s_{-i}^d) is a Nash equilibrium for the game. As with the previous analysis of repeated games, this threat is not Pareto perfect. It reduces the pay-off of all other players to zero forever.

2.5.1b Additional security greater than the cost at some level of contribution

Suppose that, for some level of m, the contribution to security of an additional player produces a return greater than the cost. It might be the case, for example, that the production function has increasing economies of scale over some of its range. At very low levels of contribution, a low return is expected. This return then begins to increase as contributions increase.

To make the analysis simple assume that the number of players n is sufficiently large that it can be approximated by a dense set in some interval on $\boldsymbol{R}$ and the pay-off function can be represented by a continuous curve.[9] The pay-offs for this game can then be represented in Figure 2.5.

The value $f(m^*)$ on this figure is the point where

$$f(m) - c = f(m-1).$$

To see what is happening imagine a similar curve for $f(m-1)$ on the same graph. For $m = 1$,

$$f(1) - c < f(0) = 0.$$

As the function f gets steeper the returns for each contribution increase. Where the return for an additional contribution is c the two curves would intersect. The first intersection is the point $m = m^{**}$ where $df/dm = c$. Think of starting with $f(m^{**})$ and taking away a contribution. $f(m^{**} - 1)$ is now equal to $f(m^{**}) - c$. At $m = m^*$ the curves intersect again at $df/dm = c$. For $m > m^*$,

$$f(m) - c < f(m-1).$$

We can ignore points around m^{**} for this argument.

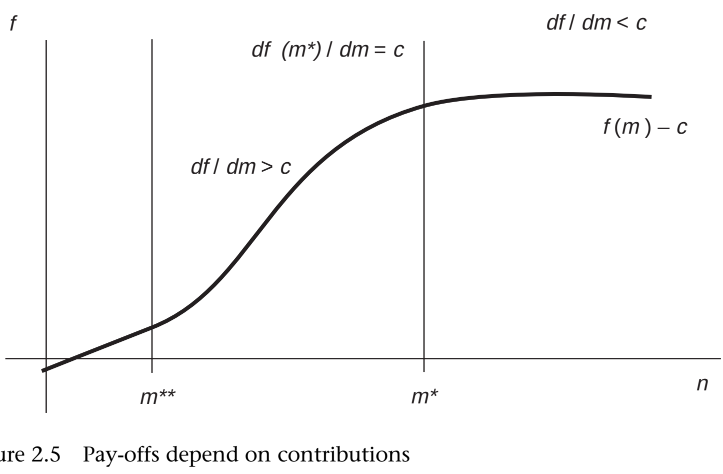


Figure 2.5 Pay-offs depend on contributions

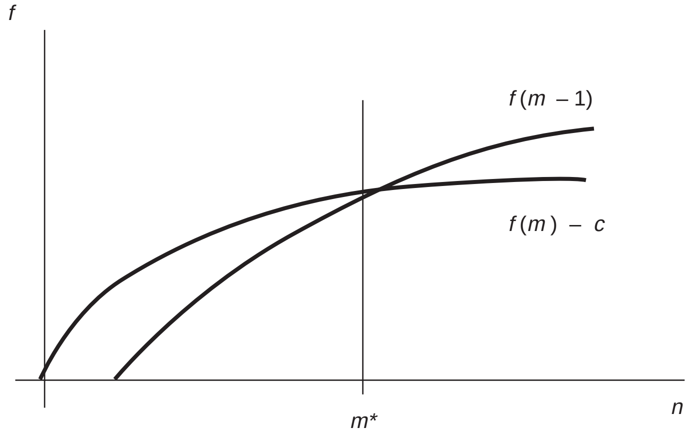


Figure 2.6 Continuous pay-off function that violates $v_i(s_i^d, s_{-i}^d) > v_i(s_i^c, s_{-i}^d)$.

It is easy to see that one Nash equilibrium for this game is for m^* of the population to contribute to providing the good. Any additional individual gets a smaller pay-off than the no contribute pay-off, and any member of m^* gets less if she withdraws. If less than m^* have contributed it pays any non-contributing member to contribute.

It will be noted that, for $m > m^*$ the pay-off for any player from being in the set M with m^* members where M contributes is less than being in the set that does not contribute. Hence if i knows that m^* others would contribute, i is better off not contributing. If m others intend to contribute for $m < m^*$, then i is better off contributing.

A consequence of this is that, if the number of players $n \leq m^*$, then the game has an equilibrium. If, however, the number of players increase such that $n > m^*$ every player's optimal strategy is s_i^d. In other words the game has an equilibrium in co-operative strategies for n low but switches to an equilibrium in which no player contributes at some critical level $n = m^* + 1$.

To see the reasoning here imagine a game between i and $n > m^*$ other players with the matrix of pay-offs in Figure 2.4. In this case

$$v_i(s_i^d, s_{-i}^c) > v_i(s_i^c, s_{-i}^c) \text{ and } v_i(s_i^d, s_{-i}^d) > v_i(s_i^c, s_{-i}^d).$$

Note that the pay-off function had to be drawn in a particular way for $v_i(s_i^d, s_{-i}^d) > v_i(s_i^c, s_{-i}^d)$ to apply for $m = 1$. Consider the pay-off function in Figure 2.6.

The difference here is that

$$v_i(s_i^d, s_i^d) < v_i(s_i^c, s_i^d) \text{ for } m = 1.$$

If all other players do not co-operate then it pays player 1 to co-operate unilaterally since $f(1) - c > f(0)$. The analysis of this game is more complicated and is taken up in Chapter 5.

An important variant of this type of problem is one where the players are prepared to offer conditional co-operation.

2.5.2 Example two: the problem of conditional co-operation

A strategy of conditional co-operation has the form of a number of players each offering to co-operate if and only if $a - 1$ for $a \leq n$ other players co-operate. Taylor makes the claim that the strategy of conditional co-operation may produce an equilibrium in which collective goods, such as security, are provided in the n person prisoner's dilemma (1987, pp. 88–92, 137). This claim only holds under very strong assumptions. Moreover, it violates the provision that threats have to be credible as required by sub-game perfect equilibria. I omit the discussion of the conditions required for discount factors to simplify the notation.

The essence of Taylor's argument can be expressed in our notation as follows. The pay-off function for the collective good is $f(m)$ and the pay-off for any contributor is $f(m) - c$. $f(m)$ is an increasing function of m and $f(m) - c < f(m-1)$. If none of the good is provided the pay-off is zero. The total number of players is n and there are two types of players conditional co-operators and defectors. Conditional co-operators co-operate provided that m' others co-operate, and defectors always defect. There are r conditional co-operators

It is obvious in this case that, if the appropriate assumptions are made, then there is an equilibrium in which some conditional co-operators co-operate. The assumptions that are required are restrictive, however, and it is not at all obvious when they could be met. There are several different permutations possible.

Consider the example that Taylor discusses of a group of conditional co-operators that assign different values to m. Let a group be R_i with r_i members. $r = r_1 + r_2 + r_3$.

For a group R_i with $m' = a - 1$ to co-operate the following conditions are necessary.

$$\text{[Ci]} \quad f_i(m') - c > f_i(0)$$

$$\text{[Cii]} \quad r - \sum r_j = a$$

$$\text{[Ciii]} \quad f_i(m' - 1) - c \leq f_i(0)$$

Condition [Ci] says that the expected pay-off to each player in R_i when a players contribute is greater than the pay-off when no-one contributes. Condition [Cii] says that the players will only co-operate if $r_i = a$. That is, after players in the first other two groups have withdrawn it must be the case that defection by one player in group R_i would cause conditional co-operation to collapse.

To see the necessity of condition [Cii], suppose that $r_i > a$. In this case the optimal strategy for each player in group R_i is to defect. Similarly if $r_i < a$ no member of group R_i finds it worthwhile to co-operate.

It follows immediately that, for different values of m', only one group of conditional co-operators co-operate.

Condition [Ciii] says that for every member of a group of conditional co-operators with some value of m' it must be the case that if one player withdraws each of the other conditional co-operators is indifferent between co-operating and defecting or is made worse off by co-operating. Unless this were true, the threat of conditionality is not sub-game perfect. Suppose that each player in R_j sets $m = m'$. Suppose j announces permanent defection. This leaves each i with a pay-off $f_i(m' - 1) - c$. Let $f_i(m' - 1) - c > f_i(0)$. In this case the threat of permanent defection by the remaining players is not credible.

2.6 A note on agreement to form a state

The problem of whether individuals in the Hobbesian state of nature could enter into an agreement to set up a some form of authority structure is now briefly considered. The problem is simpler than is sometimes thought because the game faced by players attempting to leave the state of nature is not a prisoner's dilemma.[10] This problem is dealt with here in its simplest form without reference to any particular authority structure. It involves no new game theoretical ideas and can be dealt with using one-shot and repeated games.

2.6.1a Stage one

Start with some individual z who offers to act as a rule enforcer and invites other individuals to form a protective coalition A. The cost of joining A is some payment ε where ε can be infinitesimally small. This is retained by player z. The pay-off to z increases with the size of A. A member of A is obliged to accept the instructions of z or be punished by the other members of A. Write the number of members of a coalition A as a. A coalition A defeats a coalition A' if $a > a'$.

Let a strategy of acting as an enforcer be s_z^e and of accepting the offer be s_i^c and rejecting the offer be s_i^d. Let the state of war pay-off be $v_i(h)$. The pay-off for a member of a coalition A is

$$v_i(s_A^e, s_i^c) - \varepsilon > s_i^d = v_i(h)$$

since the coalition always defeats an individual. For the case where $A = z$ the coalition $A = z \cup i$ does better against any random opponent than i does alone. Hence i does better in the coalition than outside it.

Proposition: (a) There is some z such that $s_z(s_i^c) = s_z^e$; (b) $\varphi = (s_z^e, s_i^c)$.

Proof: (a) Suppose z does not offer to act as an enforcer. Then z gets the state of war pay-off $v_z(h)$. If z acts as an enforcer it gets $v(s_z^e) + (a-1)\varepsilon \geq v_z(h)$. (b) Immediate from the preceding. ■

2.6.1b Stage two

It is optimal for any member of A to follow an instruction by z to reduce its arms provided all other members follow the same instruction. Since instruction following is enforced by A it is optimal for all members of A to reduce their arms.

A more sophisticated variant of this game would be to allow players two moves in the first stage. Without formalities it is sketched as follows. In the first stage they choose an enforcer. It is tempting to say that this would allow a selection between enforcers that offer the best deal, but such a deal is not binding in the state of nature. The choice is a statement of preference that serves a signalling device to each player about the intention of others. In the second stage the potential leader with the most support forms a coalition A and asks non-supporters to join. Since A is the largest coalition it is optimal to join.

Some obvious refinements of this argument are needed for an analysis of actual states. Although outside our immediate interest they are worth a brief mention.

One of the implications of the assumption about the size of coalition A is that there should only be a single state. Why are there multiple states? Clearly coalition size does not map monotonically into coalition strength. An obvious consideration is the internal costs to a coalition of accommodating individuals with different pay-off functions. These are usually summed under things like culture and national identity.

Assume, for example, that these differences increased monotonically from some central point. The coalition cost function would then be discontinuous at a barrier to military activity, say a river or a mountain pass. This would start to produce a division into multiple coalitions.

2.7 Note on the strategic goods problem in general

The strategic goods, or what is commonly known as collective goods, problem is that where individuals can get some benefits from the contributions of others they will contribute less to provision of a good than is collectively optimal. Some examples of such goods are clean air, conserving fisheries and wilderness, and contributing to preventing greenhouse emissions. Many of the essential features of such problems have the same structure as the prisoner's dilemma. In other words, security can be seen as a strategic good. For completeness I sketch out here the pay-off matrix for the n player case where n is large.

	s_{-i}^c	s_{-i}^d
s_i^c	$g(n) - c$	$\varepsilon - c$
s_i^d	$g(n) - \varepsilon$	0

Figure 2.7 An *n* person collective goods problem

The pay-off for the good is g where g is a function of the number of contributors. $g(n-1) = g(n) - \varepsilon$ where ε is small. Each player contributes $c > \varepsilon$. This gives the pay-off matrix in figure 2.7. $g(1) = \varepsilon$. All the previous analysis follows.

3
Security and Material Possessions in a Lockean State of Nature – Non-Co-operative Games

3.1 Introduction

States also provide protection for material possessions by enforcing rules of ownership and exchange, and this raises questions about the security of property and enforcement analogous to those in Chapter 2. This chapter makes a start on these questions. This will be done by extending the analysis of conflict in the state of nature to the case where there are no rules of possession and there are conflicts over material goods. In order to deal with this question it must be assumed that the Hobbesian problem is solved and that individuals have some level of material wealth. This is referred to as a Lockean state of nature. Without this assumption it is difficult to see how material possessions can be treated as a greater concern than domination and fear of violent death. The Hobbesian problem might be avoided by assuming that there is already an ultra-minimal state that protects life. The absence of rules of property means that individuals are involved in struggles where there are material pay-offs. For these interactions to be interesting it is also necessary to assume that strategies are costly. It turns out that the problem of studying struggles over possessions is more complicated than the analysis of struggles over domination. This is because the single dominant strategy of the Hobbesian problem is lost and the equilibrium of one-shot games may not be uniquely determined.

This chapter continues the previous analysis of non-co-operative games with two moves by allowing players to have a choice between strategies of being aggressive and attempting to steal and protect their own property, or being non-aggressive. It also introduces games in which strategies are a choice of how much resource to devote to each activity.

One of the consequence of concentrating on struggles over material possessions is that the costs of fighting might exceed the gains. Despite the additional complications, the general result that state of nature games may not produce collectively desirable outcomes is carried through.

3.2 A security problem with mixed strategies: hawk–dove games

The feature of struggles over material possessions that differentiates it from the prisoner's dilemma is that it is possible to imagine that a player might be better off not fighting an aggressor. Assume that the conflict is costly to both the winning player and the losing player. This is because players spend resources in preparing to fight and in the fight itself. Perhaps it would be possible to flee, for example. In this case, the costs of fighting may be greater than the value of winning. In such cases immediate surrender or flight would be the optimum strategy.

The obvious example of such games is conflicts which may result in death, but where the consequences of losing are preferable to being dead. The conflict may be over material possessions such as territory, or a stock of food, or other goods. At some point the costs of defending, or attempting to take the good, may exceed the material value of the object of the struggle.

Since the addition of costs of fighting means that the dominant equilibria associated with the prisoner's dilemma is lost, each player has to take into account the strategies of other players in the one-shot game. This may lead to multiple equilibria and the possibility that players may randomize between the pure strategies. That is a strategy might consist of a probability distribution between the available pure strategies.

To consider this problem start with one-shot games and assume that strategy sets are finite.

Definition: A mixed strategy is a random selection from among the pure strategies available to a player. ■

A mixed strategy is given by a probability distribution

$$\sigma = \sum \sigma^j$$

summed over all the pure strategies $s_i^1 \ldots s_i^n \in S_i$ with $j = 1, \ldots, n$. It assigns a probability to every pure strategy, $\sigma^j \geq 0$. For a pure strategy that would not be played $\sigma^j = 0$. Since at least one strategy must be used,

$$\sum \sigma^j = 1.$$

A pure strategy is simply a degenerate case of a mixed strategy. In this case probability zero is assigned to every other strategy than the one played. $\sigma^j = 1$ and $\sigma^i = 0$, for all $i \neq j$. Hence i plays s_i^j with probability one.

The possibility that players may use mixed strategies makes the following theorem possible.

Theorem: Nash (1950). Every finite strategic form game has an equilibrium in mixed strategies. ■

This theorem is proven in any of the standard texts, for example Fudenberg and Tirole (1992, p. 29) or Binmore (1992, p. 322). Its importance is that it gives an assurance that a solution exists for this type of game. It still leaves the problem of finding the solutions.

The most well known example of a finite strategy game where the costs of fighting may exceed the costs of losing is called a hawk–dove or chicken game.

3.2.1 Hawk–dove game

The hawk–dove game can be understood by thinking of two players in dispute about some material good. Assume that, if player 2 capitulates or acts peacefully, player 1 is better off claiming the good. If player 2 does not capitulate, then player 1 is better off capitulating if the costs of fighting are greater than the value of the prize in dispute.

The logic of this game can be understood from the game of chicken which gives this structure its alternative name. This is the game of two players driving cars towards each other and getting most points for not swerving if the opponent swerves. In the hawk–dove game the players have only two strategies, or are one of two types. The hawk strategy is to not swerve, or to fight, whenever confronted with an opponent, the dove strategy is to flee. If a hawk meets a dove the hawk gets the higher pay-off. If two hawks meet they damage each other. A hawk gets a lesser pay-off against another hawk than a dove gets against a hawk. Doves do better against each other than hawks against each other

This game is set out in its general form in Figure 3.1(a) with pay-offs for player 1. Let hawk be s^h and dove be s^d. $c_1 > a_1$ and $b_1 > d_1$. For simplicity some values are given in Figure 3.1(b).

This game has two Nash equilibria in pure strategies.

	s_2^d	s_2^h
s_1^d	a_1	b_1
s_1^h	c_1	d_1

(a)

	s_2^d	s_2^h
s_1^d	1, 1	0, 3
s_1^h	3, 0	–1, –1

(b)

Figure 3.1 The hawk–dove game

$$\varphi_1 = (s_1^h, s_2^d) \text{ and } \varphi_2 = (s_1^d, s_2^h).$$

It will be observed that player one prefers φ_1 and player two prefers φ_2.

Communication did not solve the prisoner's dilemma. In this game, if players were able to communicate, the messages they send would be part of their strategy set. Player 1 may send a signal that it intends to use s_1^h, in an attempt to bring about φ_1. One obvious difficulty here is that player 2 may attempt to send the same signal. This leads to the analysis of the strategies the players may adopt to ensure that their signals are credible. This problem will be pursued in the analysis of repeated games and war of attrition games.

This game also has a solution in mixed strategies. A mixed strategy may be thought of as a player choosing to act like a dove with a probability x and like a hawk with a probability $(1 - x)$ with $x \in [0, 1]$. This means that

$$s_1 = (xs_1^d, (1 - x)s_1^h).$$

Another interpretation of mixed strategies for a game with several players is that some percentage of the population play hawk and some percentage play dove. In this case the mixed strategy gives the distribution of the population between the two types. This interpretation has some applications to a study of the different strategies that players might adopt in a struggle over property. It is interesting to note that it has also received considerable attention in the analysis of the evolution of animal behaviour.[1]

The best reply strategy can be thought of in terms of evolutionary stability. If a stable Nash equilibrium exists it is the pattern that would emerge as the result of random maximizing behaviour with selection in favour of the strategies with the best pay-offs. It will be observed that neither a population of hawks nor a population of doves is stable. A strategy of playing hawk is not a best reply to a strategy in which everyone else plays hawk, for example. The population of hawks gets

$$v(s_1^h, s_2^h) = -1$$

and can be invaded by a dove which gets

$$v_1(s_1^d, s_2^h) = 0.$$

A population of doves can always be invaded by a hawk which gets

$$v_1(s_1^h, s_2^d) = 3.$$

It follows that the only pattern that would be stable must contain a mix of hawks and doves.

A simple way to calculate the mixed strategy is to reason as follows.[2] Since φ is a mixed strategy, neither hawk nor dove can be dominant. Hence, from theorem 2.1

$$v_1(s_1^h) \leq v_1(s_1^d) \text{ and } v_1(s_1^d) \leq v_1(s_1^h).$$

Therefore $v_1(s_1^h) = v_1(s_1^d)$.

Let player 2 use the mixed strategy $s_2 = [ys_2^d, (1-y)s_2^h]$. From the pay-off matrix in Figure 3.1 this means,

$$ya_1 + (1-y)b_1 = yc_1 + (1-y)d_1$$

Hence

$$y = (d_1 - b_1)/(a_1 + d_1 - b_1 - c_1)$$

Substituting the numerical pay-offs $y = \frac{1}{3}$. Since the game is the same for all players

$$s_1^* = s_2^* = \left(\tfrac{1}{3}, \tfrac{2}{3}\right).$$

The expected value of the game for player 1 and player 2 is

$$E(v_1) = xy(1) + x(1-y)(0) + (1-x)y(3) + (1-x)(1-y)(-1)$$

This gives

$$E(v_1) = E(v_2) = \tfrac{1}{3}.$$

Since the players do not always fight in equilibrium the pay-off for each player is not as bad as the Hobbesian game in which everyone fights and the value is given by

$$v(s_1^h, s_2^h) = -1.$$

It is, however, worse than the co-operative outcome

$$v(s_1^d, s_2^d) = 1.$$

Remark: The mixed strategy for player 1 was calculated using the pay-offs for player 2. This observation holds for all games with two players and a finite strategy set with mixed strategy equilibria. This means that changing the pay-offs available to a player will not change its behaviour. To see the importance of this observation, consider the following example.

Suppose that the game is a struggle over material goods and the strategies are steal and not steal for player 1 and guard and not guard for player 2. Player 2 could be the police, for example. The police may not wish to guard continuously. If player 1 does not wish to get caught stealing when player 2 is guarding the game will have the hawk–dove structure just analysed. It follows that increasing the penalties for stealing by player 2 will not decrease the amount of stealing.

This is a strong result. It is for the social scientist to judge whether it holds for the situation being analysed, or whether it is merely an artefact of the simple model which is used. For example, would the police play the mixed strategy equilibrium, or would they play hawk with probability $(1 - y)$ where $(1 - y)$ is set such that the dominant strategy for the thief is s_i^d?

3.2.2 Security in mixed strategies

The difficulty with the mixed strategy is that, compared with playing safe and being a dove, it is risky. This raises the question of security levels, or the safety of a move, and whether mixed strategies or pure strategies are safer.

The secure pay-off of a player is defined as the best pay-off against an opponent who inflicts the maximum damage. In the case of games of absolute conflict, such as zero sum games, this level is the best that a player can expect to get against a rational opponent. In other cases it would take a degree of paranoia to assume that all opponents will use the strategy that inflicts maximum losses regardless of cost to themselves. None the less, where the stakes are high the security of any move is a reasonable consideration.

Definition: The security pay-off is the best that can be obtained against an opponent that inflicts maximum losses. This can be defined as

$$v_i^s = \max{}_{si} \min{}_{s-i}(s_i, s_{-i}) = \min{}_{s-i} \max{}_{si}(s_i, s_{-i})$$

In other words, it is the best that i can do against an opponent if i maximizes the minimum pay-off or $-i$ minimizes the maximum pay-off. ■

The concept of a security strategy is important in applied and pure game theory and is supported by the well-known von Neumann minimax theorem. It can be understood by introducing the notion of a saddle point.

Imagine a saddle shape. This has two transverse curves, one open up and the other open down, like this ∪∩. The first is on top of the second and they touch at their minimum and maximum points. Player $-i$ controls the first curve and so holds it to the minimum. Player i controls the second curve and pushes it to its maximum.

Formally, a pay-off matrix has a saddle point if there is some pay-off that is the largest for its column and the smallest for its row. That is

$$v_i(s_i^o, s_{-i}) \geq v_i(s_i^o, s_{-i}^o) \geq v_i(s_i, s_{-i}^o)$$

where s_{-i}^o, s_i^o is the saddle point. This give the following theorem.

Theorem 3.1: The security strategy s_i^o gives the security value $\max_{si}\min_{s-i} = \min_{s-i}\max_{si}$ of a game with a saddle point.

Proof:: The strategy s_i^o gives a pay-off such that

$$v_i(s_i^o, s_{-i}) \geq v_i(s_i^o, s_{-i}^o).$$

Hence player i must get at least $v_i(s_i^o, s_{-i}^o)$ which is $\max_{si}$ if the opponent picks the column that minimizes the pay-off in each row $= \max_{si}\min_{s-i}$. $v_i(s_i^o, s_{-i}^o) \geq v_i(s_i, s_{-i}^o)$. Hence i cannot get more than $v_i(s_i^o, s_{-i}^o)$ because this is the maximum of the column the opponent selects to minimize the maximum pay-off $= \min_{s-i}\max_{si}$. ■

This general theorem can be illustrated by returning to the hawk–dove game. Here there is a saddle point in the pay-offs for player one with pay-off $v_1 = 0$. Observe that there is no mixed strategy that can do better than s_1^d against s_2^h.

There is always a security strategy in games with a finite number of pure strategies in which $\max_{si}\min_{s-i}(s_i, s_{-i}) = \min\max$. In general, if the game does not have a saddle point then the security strategy may be a mixed strategy.

3.3 Repeated hawk–dove games

The repeated hawk–dove game will now be considered. As with the analysis of Hobbesian problems, a question that arises is whether the repeated hawk–dove game is more likely than the repeated prisoner's dilemma to produce a stable co-operative equilibrium, in which there is security for property.

The difficulty with the repeated hawk–dove game is that a retaliation strategy like tit-for-tat is not sub-game perfect as it was in the prisoner's dilemma. This is because, if player j chooses hawk on every round and never deviates, it is not a Nash equilibrium for player i to punish by choosing hawk in every round. That is

$$v_i(s_i^d, s_j^h) > v_i(s_i^h, s_j^h)$$

Hence retaliation is not optimal against a determined hawk player.

Myerson suggests a strategy for the hawk–dove game called the q positional strategy (1991, p. 329). He argues that it is plausible to assume that the player who has played hawk in the past might be more likely to play hawk in the future. One motivation for this is that hawkish behaviour might be an attribute of a type of player. Alternatively a player might have to make an investment in being a hawk, perhaps by acquiring greater technologies of violence.

Myerson's suggested strategy is that in a contest between i and j, i chooses hawk if i has chosen hawk strictly more times than j, and dove if it has chosen hawk strictly less times. If both players have chosen hawk an equal number of times they choose hawk with a probability of q and dove with a probability of $(1 - q)$. This strategy is a Nash equilibrium and also a sub-game perfect equilibrium for some value of q.

The problem is that there are pay-offs from establishing a reputation for being tough, and this may lead each player to follow a strategy of investing in being tough now in an attempt to get the benefits of an equilibrium in which it has strategy $s_i^h, i = 1, 2$. This gives a variant of the problem of credible signals noted in the analysis of the one shot game. In this case, what may emerge is a strategy in which each player tries to out-spend the other.

Myerson shows that, for a low future discount which gives a discount factor approaching one, the value of q will also approach one. This means that the game will tend to converge to a hawk, hawk equilibrium. If so, there is no reason for believing that the repeated hawk–dove game holds out a greater hope for a spontaneous Pareto-efficient equilibrium than the repeated prisoner's dilemma.

To analyse the question of which strategies are likely to emerge in repeated games the notion of the resistance is introduced.

3.3.1 Resistance in repeated games

The question of whether the repeated hawk–dove game is more likely to produce a Pareto-efficient equilibrium than the prisoner's dilemma can be studied by constructing an index of the relative stability of a retaliatory strategy in the two games. This can be done by exploring the idea of resistance.[3] The resistance of a strategy s^* against some other strategy s indicates the capacity of s^* to sustain itself when other players are using a different strategy. Specifically, the resistance of s^* against s is the maximum proportion of s players that could be put into a population of players using s^* before the pay-off for the s players is greater than for the s^* players (Myerson, 1991, p. 119). Obviously, the higher the proportion of players that are required the greater the resistance of s^* against s, and hence the more probable a strategy can be sustained in a repeated game.

In order to analyse resistance a more tractable measure of the pay-offs for games without a finite stopping point is needed than the discounted pay-offs dealt with previously. This is done by defining the discounted average of a sequence of pay-offs as

$$v_i = (1 - \delta) \sum_{k=1}^{\infty} \delta^{k-1} w_i^k(s_i, s_{-i}) \tag{3.i}$$

Each player's objective is now to maximize the discounted average pay-off. Observe that multiplying the game by $(1 - \delta)$ does not alter the equilibrium strategies. This is often called a time averaging process (Fudenberg and Tirole, 1992, p. 149).

For an example of how the averaging process works, imagine that the pay-off is w_i for every round. Use the result

$$\sum_{n=0}^{\infty} \delta^n = 1/(1-\delta).$$

It will be seen that

$$v_i = (1-\delta)[(1+\delta+\ldots+\delta^n)]w_i = w_i.$$

It is now possible to solve the problem of finding the resistance of strategy s^* against s. Let the proportion of s players required for $v(s) \geq v(s^*)$ be α. The resistance is calculated by finding the largest value of $\alpha \in [0, 1]$ such that

$$v_i[s_i^*, (\alpha s_{-i} + (1-\alpha)s_{-i}^*)] \geq v_i[s_i, (\alpha s_{-i} + (1-\alpha)s_{-i}^*)] \tag{3.ii}$$

The resistance of the tit-for-tat strategy against the always defect strategy for the prisoner's dilemma game can be calculated by using the pay-offs in Figure 2.1(a). Let s^t stand for tit-for-tat and s^d for always defect. Then

$$v_i(s_i^t, s_{-i}^t) = a_i, \text{ and } v_i(s_i^d, s_{-i}^d) = d_i$$

If the second player is not tit-for-tat, then tit-for-tat gives b_i on the first round and d_i on every other round.[4]

$$v_i(s_i^t, s_{-i}^d) = (1-\delta)b_i + \delta d_i$$

Similarly

$$v_i(s_i^d, s_{-i}^t) = (1-\delta)c_i + \delta d_i.$$

Substituting these values into equation (3.ii).

$$\alpha \leq [a_i - c_i(1-\delta) - \delta d_i]/[a_i + d_i - b_i(1-\delta) - c_i(1-\delta) - 2\delta d_i] \tag{3.iii}$$

Hence

$$\alpha \leq x$$

where x tends to $a/a = 1$ as δ tends to 1.

It follows that, for a negligible value of the discount factor tit-for-tat can survive unless the proportion of permanent defectors is high. Observe that the discount factor cannot be 1. If $\delta = 1$ the pay-off function is zero. Also note that for $\delta < 1, \alpha$ may be small or negative since $c > a$.

Since $v_i(s_i^h, s_{-i}^h)$ is not sub-game perfect for the hawk–dove game the same calulation does not make sense. To get some idea of the resistance of tit-for-tat against the aggressive strategy of always playing hawk we could use the q positional strategy.

The resistance of tit-for-tat against the q positional strategy is calculated using the pay-offs in Figures 3.1(a) and 3.1(b). The expected values of this strategy will involve permutations of the probabilities. For example

$$E[v_i(s_i^q, s_j^q)] = (1-\delta)[qqd_i + (1-q)[c_i(\delta+\delta^2+\ldots)] + (1-q)q[b_i(\delta+\delta^2+\ldots)]$$

$$+(1-q)(1-q)a_i + qqqqd + qqq(1-q)[c_i(\delta^2+\delta^3+\ldots)] + \ldots]$$

We can get some idea what is happening if these are simplified by disregarding high powers of q and letting $\delta \to 1$ as before. For $q(1-q)$ not too small this gives approximate values[5]

$$v_i(s_i^q, s_{-i}^q) = (b_i + c_i)q(1-q),\ v_i(s_i^q, s_{-i}^t) = v_i(s_i^t, s_{-i}^q) = qd_i + (1-q)a_i,\ v_i(s_i^t, s_{-i}^t) = a_i.$$

Substituting these gives the approximation and using α' for the resistance

$$\alpha' \leq [q(a_i - d_i)]/[a_i(2q-1) + (b_i + c_i)q(1-q) - 2q\,d_i]$$

This gives

$$\alpha' \leq x'$$

where $x' = 2q/[q(7-3q) - 1]$

It follows that it may be more difficult for tit-for-tat to be sustained in the hawk–dove game than in the prisoner's dilemma for $q(1-q)$ not too small. For example if $q = 0.5$ then $x' \doteqdot 0.57$.

3.4 A pirate retaliator game with different pay-offs

Struggles over property might sometimes involve individuals of different types such as settlers and nomads or farmers and raiders. The mixed strategy equilibrium can be used to analyse this situation by interpreting it as an equilibrium between different types of players, rather than between players of the same type using different strategies. If players are thought of as different types, then it is possible to extend the analysis beyond simple hawk–dove games to study conflicts where different players may get different utilities from some course of action. This also adds the possibility that players may be uncertain about the type of their opponent.

Consider a variant of the one shot simultaneous move hawk–dove problem in which there are two types of individuals, pirates and retaliators. Pirates enjoy blood and danger and will always get a greater pay-off from fighting, no matter what the opponent does. In the hawk–dove game the optimum strategy for a non-pirate would be to play dove in a game with a pirate. To make the analysis more interesting, assume that non- pirates are retaliators. Retaliators will get greater pay-offs from not fighting if an opponent chooses to not fight. In response to aggression, they will get a greater pay-off from fighting.[6]

The motivation behind these assumptions is less clear than that behind the assumption in the Hobbesian game that all individuals are the same. It might be justified as a way of testing the model if this assumption is relaxed. Alternatively, it might be justified as a claim about the propensity of individuals to have different degrees of passivity or aggressivity through causes that are not modelled, or about the value of reputation for future games. This analysis could be adapted to the prisoner's dilemma game.

Assume that player 1 is a retaliator. A player does not know whether its opponent is a pirate or a retaliator. It is also assumed that uncertainty causes some retaliators to act out of fear that their opponent is a pirate. The probability that player 2 is a pirate is p and the probability that player 2 is not a pirate is $(1-p)$. The probability that player 2 will act out of fear of player 1 and fight is q. This gives an aggregate probability of

$$(p+q-pq)$$

that player 2 fights.[7] Let to fight be s^h and not to fight be s^d.

Since the pirate gets a greater pay-off from fighting

$$v_2(s_2^h, s_1) \geq v_2(s_2^d, s_1).$$

The optimum strategies for the retaliator are to fight if the opponent fights and to not fight if the opponent does not fight, that is

$$v_1(s_1^d, s_2^d) \geq v_1(s_1^h, s_2^d) \text{ and } v_1(s_1^h, s_2^h) \geq v_1(s_1^d, s_2^h).$$

The pay-offs are those given in Figure 3.1(a). Note that $a_1 > c_1$ and $d_1 > b_1$.

The expected pay-off for a retaliator in this game is

$$E[v_1(s_1^d)] = (p + q - pq)(b_1) + (1 - p - q + pq)(a_1)$$

$$E[v_1(s_1^h)] = (p + q - pq)(d_1) + (1 - p + q - pq)(c_1)$$

A strategy of fighting dominates a strategy of not fighting whenever

$$E[v_1(s_1^h)] > E[v_1(s_1^d)].$$

Let the expected value of fighting be $E[r_1(s_1^h)]$ and define this as

$$E[r_1(s_1^h)] = E[v_1(s_1^h)] - E[v_1(s_1^d)]$$

Calculating this gives

$$E[r_1(s_1^h)] = (p + q - pq)(d_1 + a_1 - b_1 - c_1) + c_1 - a_1$$

Since the optimum strategy for the pirates is given, the equilibrium in the model is determined by the optimum strategies for retaliators. Clearly if all players were retaliators, then the outcome would be the Pareto efficient equilibrium

$$\varphi = (s_1^d, s_2^d).$$

What is interesting is how the optimum strategies for retaliators change as the percentage of pirates in the game increases for different values of q. Writing $E[r_1(s_1^h)] = r_1$, a typical relationship is illustrated Figure 3.2. $k^* = q(d_1 + a_1 - b_1 - c_1) + c_1 - a_1$.

When $r_1 > 0$, player one plays s_1^h. The straight line gives the value of r_1 for variations in p with $q = 0$. The curved line gives r_1 for $q > 0$. Observe that since

$$(p + q - pq) \geq p$$

r_1 on the curved line is always greater than r_1 on the straight line. $r_1(q > 0)$ intersects the line $r_1 = 0$ at a lower value of p than $r_1(q = 0)$.

The probability of acting out of fear is q. Thus imperfect information increases the probability that $\varphi = (s_1^h, s_2^h)$.

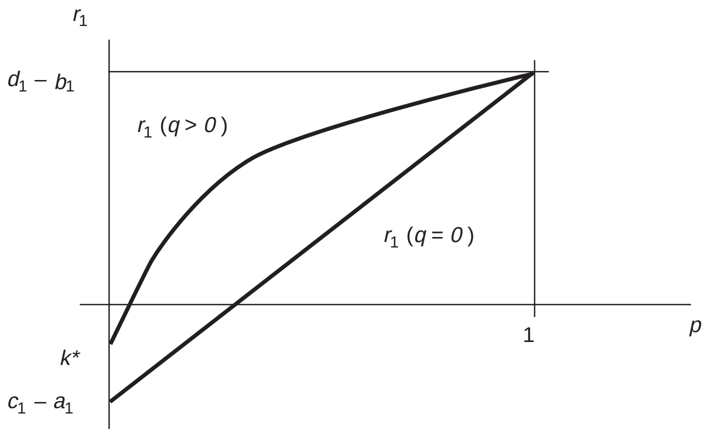


Figure 3.2 A pirate retaliator game

3.5 War of attrition games with continuous strategies and a fixed treasure

The analysis of struggles over material possessions also raises the possibility that victory might go to the player who spends the most on military or fighting capacity. Wars and struggles between settlers and raiders are obvious examples. Another example is the case discussed previously where players are concerned about reputation. This introduced the possibility that each player might try to out-spend the others in order to establish an equilibrium in future games in which it plays hawk. This type of game is known as a war of attrition. The strategy of a player is an amount to spend on aggression or hawkish behaviour. Each player wishes to out-spend the other while spending less than the value of the prize for winning. Hence the strategy is to wear down the other player as in military conflicts.

War of attrition games were originally developed by evolutionary biologists such as John Maynard Smith to help explain animal behaviour. The story is set out in J. M. Smith's *Evolution and the Theory of Games*. In this case a stable Nash equilibrium refers literally to the evolutionary stable strategy that the animals would develop through a selection process that eliminated sub-optimal strategies. For present purposes it is the Nash equilibrium in military conflicts that are of concern. Evolutionary stable strategies are dealt with in more detail in Chapter 6.

3.5.1 Description of the game and informal analysis

The game is a contest between n players over a prize which has the same value for each player. This might be thought of as some treasure, or some land, or a mine or a pile of goods. It is assumes that the player that has spent most wins the conflict. The cost of the conflict is given by whatever its opponent has spent. This captures the idea that the harder the opponent fights, the greater the costs to the winning side. Each player loses the entire value of its military expenditure if it is defeated. This is plausible and provides a 'bigger they come the harder they fall penalty'. It would not make any substantial difference to the model if these proportions were altered. For example, it could be imagined that only half the military expenditure is lost, or a third of the wealth captured.

A player's expenditure might represent such things as the amount of time or money spent on military hardware, or on defence, or on buying support. If it is assumed that each player's strategy is continuous. A strategy is represented by $s_i^i = x_i \in [a, b] \in \mathbf{R}$.[8]

Remark i: One difficulty in games with continuous strategies is that any point on an interval has zero probability of selection. For this reason probabilities are thought of as applying to some measurable distance, or interval, in $[a, b]$.

Remark ii: It has already been seen that a game with a finite strategy space has a Nash equilibrium in mixed strategies. Similar result can be proven for games with infinite strategy spaces and pay-off functions that are not too badly behaved (Fudenberg, 1992, pp. 487–9).

Consider the two player case. $s_1^i = x_i \in [0, \infty)$ and $s_2^j = y_j \in [0, \infty)$. Let the value of the prize be m. This gives the following pay-off:

$$v_1(x_i,\ y_j) = \left| \begin{array}{l} m - y_j,\ if\ x_i > y_j \\ m/2 - x_i,\ if\ x_i = y_j \\ -\ x_i,\ if\ x_i < y_j \end{array} \right.$$

In other words, if player 1 spends more than player 2 on arms or military hardware, the return is m less the cost of the fight. This is determined by what the opponent spends and is given by $m - y_j$. This may be greater or less than zero. If player 1 spends less on military hardware than player 2, the value of x_i is lost.

The pay-offs for this game are illustrated by setting $v_1(x_i, y)$ on the vertical axis and looking at the pay-offs for two values of y_j. These are $y_j^1 < m, y_j^2 > m$. The value of x is on the horizontal axis. This is given in Figure 3.3.

To get a feel for the model consider what happens when military expenditure starts to escalate. Imagine that expenditure has gone up to the point where every player has spent $k > m$. In this case the value of the pay-offs to the winner is $v_i = m - k$, which is less than zero. The loser gets $-k$. This means that the hawks would exterminate each other.

It is clear from the diagram that there is no strategy in this game that is a best reply to itself. Suppose both players use some strategy z. For player one this gives

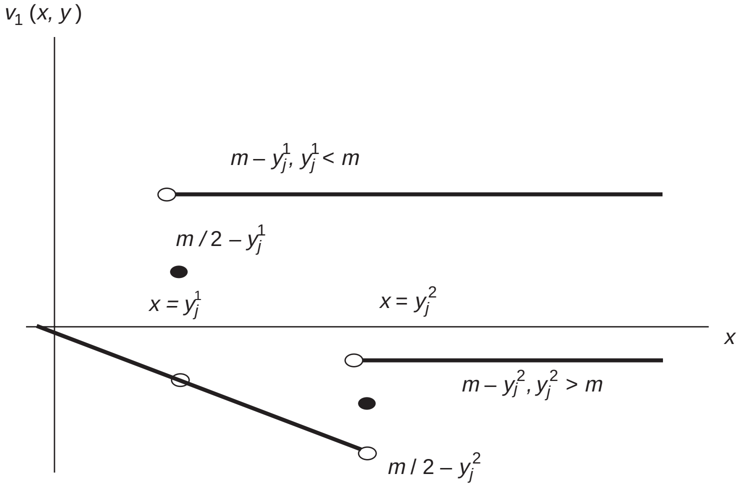


Figure 3.3 Pay-offs for two values of *y* in a war of attrition

$$v(z, z) = m/2 - z$$

Let $z < m$. Then

$$v(z, z) < v(z + \varepsilon, z) = m - z.$$

Let $z = m$. Then

$$v(z, z) < v(z + \varepsilon, z) = v(0, z) = 0.$$

Thus it pays players to increase expenditure. Let $z > m$. Then

$$v(z, z) < v(0, z).$$

Similarly for the *n* player case, there is no pure strategy for this game. If all, or a large number of players were spending more than *m*, it would pay any one player to set *x* at zero and thus get a greater pay-off over a series of rounds of the game. It might be imagined, for example, that when the pirates are coming this group simply abandons its holdings of wealth and flees with its current consumption goods.

3.5.2 Formal analysis

The mixed strategy equilibrium is calculated as follows. Since the probability that any player spends exactly some amount, say *x*, is zero, are defined probabilities over a range.

This is done by letting $p(x)$ be the probability that any player has stopped increasing military expenditure at some point k.[9] For k very large, for example, the probability $p(x)$ would be close to one since it would not be very likely that a player was still spending. The probability that expenditure is between a and k is $p(k) - p(a)$.

The integral of the derivative of this function is often convenient for calculation purposes. The derivative is $p'(x)$. The probability that expenditure is between a and k is given by

$$\int_a^k p'(x)dx$$

which is $p(k) - p(a)$ as required. This idea is used to get the probability that expenditure has stopped between x and $x + \delta$. This is given by $p'(x)/\delta$.

Write the strategy of the player using $p(x)$ as p^*. It follows that the pay-off for any player using a pure strategy k against p^* is a summation of the expected pay-offs

$$E[v(k, p^*)] = \int_o^k (m - x)p'(x)dx - \int_k^\infty kp'(x)dx \tag{3.iv}$$

Recall that the calculation of mixed strategies says that the pay-off for each of the pure strategies against a mixed strategy is equal. Hence if $k + \delta$ is some pure strategy it must have the same pay-off as k. Therefore, $E[v(k + \delta, p^*)] - E[v(k, p^*)] = 0$. This gives

$$\int_o^{k+\delta} (m - x)p'(x)dx - \int_{-k+\delta}^\infty (k + \delta)p'(x)dx - \int_o^k (m - x)p'(x)dx + \int_k^\infty kp'(x)dx = 0$$

Summing gives

$$\int_k^{k+\delta} (m - x + k)p'(x)dx - \delta \int_{k+\delta}^\infty p'(x)dx = 0$$

If δ is small $x \to k$ and $\int(m - x + k)p'(x)dx$ tends to $m\int p'(x)dx$ evaluated at k over the interval δ which tends to $\delta mp'(k)$. $\int p'(x)dx$ evaluated between k, ∞ is $1 - p(k)$. Hence

$$p'/(1 - p) = 1/m$$

Note that $p'/(1-p) = -d[1n(1-p)]/dx$. Taking integrals

$$1n(1-p) = -\int 1/m = -x/m + c$$

for c some arbitrary constant. Therefore

$$p(x) = 1 - e^{-x/m} \tag{3.v}$$

using the fact that $p(0) = 0$.

What equation (3.v) tells us is that, for any fixed value of material wealth, the probability that a player selected at random has stopped spending on military hardware increases as the level of expenditure increases. This can be interpreted as follows. Each player would choose some function for its military expenditure. If there are some players spending less than their optimum, it will pay at least one player to increase its expenditure. If the other players are spending too much, it will pay at least one player to decrease its expenditure.

The returns to each player will, of course, be the same in any equilibrium strategy. In this game the returns are zero. To see this notice that, if any player gets a positive return, there must be at least one other player getting a negative return. If so, it pays that player to alter its expenditure.

The probability density function is illustrated in Figure 3.4. What the curve shows is that as x increases the probability that a player has ceased spending on military hardware approaches one. For a two player game the optimal strategy for each player is the mixed strategy that says stop spending in any interval with probability p. For a n player game, with n sufficiently large to be treated as a dense interval, an optimum strategy for a player is a location on this curve.

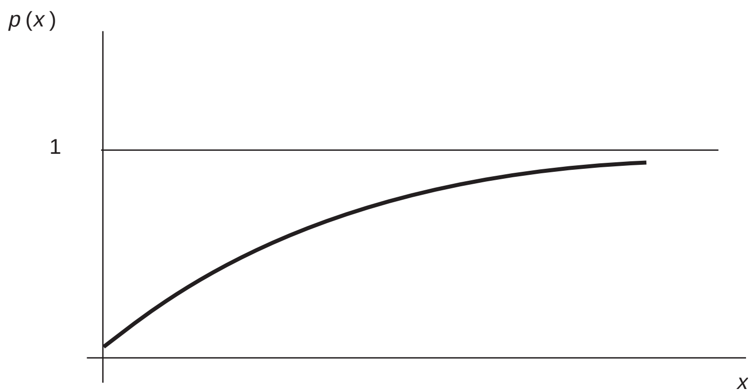


Figure 3.4 Probability that a player has stopped spending for any x

It might be asked whether this equilibrium is stable. What happens, for example, if the prize is worth more to some players than to others? Perhaps it has more strategic value to some players, or they are more desperate, or the struggle is over the wealth of the players and information is imperfect.

Some aspects of this problem will be considered.

3.6 War of attrition with unequal prizes

The analysis of the game with unequal prizes will be simplified by considering the case where there are only two types of players. Type 1 players give the prize a higher value than type 2 players. Type 1 occurs with probability q and type 2 with probability $(1 - q)$. This gives a probability distribution

$$p = qp_1 + (1 - q)p_2$$

To analyse the effect of differences in the value of the prize observe that differentiating equation (3.v) gives

$$\partial p/\partial m_i = -(e^{x/m})/m^2 < 0$$

This means that, for a given x, the probability distribution function for type 1 is below that of type 2.

What needs to be ascertained is whether these two distribution functions overlap. If they overlap, p_1 would be below p_2 for some x. This would say that the probability that type 1 had stopped spending for any value of x is less than that of type 2, but type 2 may have spent more. If they do not overlap then type one would always spend more.

To rule out overlap means showing that a value of k cannot have more than one solution. An outcome like that in Figure 3.5 is not possible.[10] This can be shown in the following proposition adopted from Smith (1986, pp. 194–6).

Proposition: Suppose $k \in p_1$. Then $k \notin p_2$.

Proof: Let $s = (p_1, p_2)$. s^* is the Nash equilibrium $p^* \in s^*$. The expected value for s against s^* is

$$E[(s, s^*)] = q[v_1 f(k) - g(k)] + (1 - q)h \qquad (3.vi)$$

In the first half of the left-hand side of equation (3.iv) f is the probability that k wins against p^*, and g is the expected cost of choosing k against p^*. The second half of equation (3.iv) gives h as the pay-off of a type two player using s against s^*.

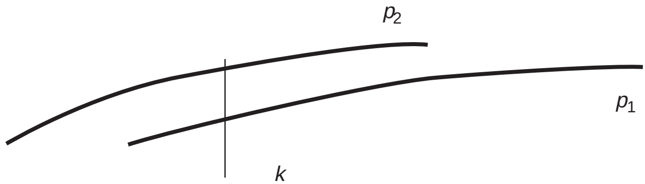


Figure 3.5 Overlapping distribution functions

If $k \in p_1$ then the pay-offs must be the same for all k giving

$$v_1 f(k) - g(k) = a \quad \text{(3.vii)}$$

where a is some constant.

Since f is the probability that k wins against p^* and g is the cost of using k, the values of f and g do not depend on whether the player is type 1 or type 2 if $k \in p_2$. Therefore

$$v_2 f(k) - g(k) = b \quad \text{(3.viii)}$$

where b is some constant.

Subtracting equation (3.viii) from equation (3.vii) gives

$$(v_1 - v_2)f = a - b \quad \text{(3.vii)}$$

f is the probability that k defeats p, and is monotonically increasing with k. Hence the solution to (3.vii) must be unique. But v_1 and v_2 are constant because the cost of playing k are in the $g(k)$ term. Thus

$$(v_1 - v_2)f = a - b$$

is constant. Contradiction. Therefore p_1 and p_2 do not overlap. ■

With some additional proof that the curves cannot have gaps or atoms of probability the strategies in the two-type game are given by probability distribution functions like those in Figure 3.6.

It can be shown that where there are n players that place different values on the prize, there are n non-overlapping distribution functions with no gaps between them.[11] The player that places the higher value on the prize always spends more than a player with the lower value.

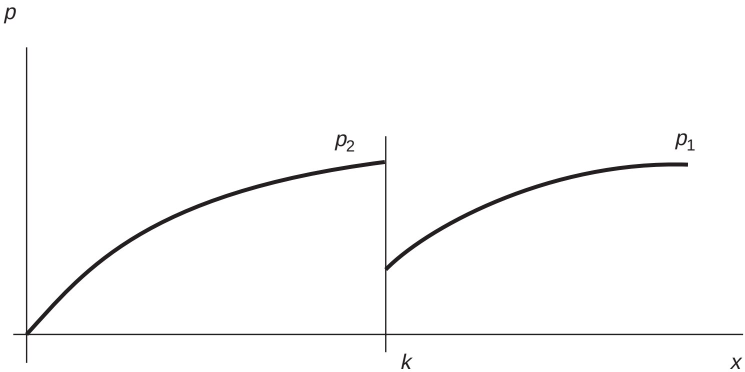


Figure 3.6 Evolutionary stable strategies with different prizes

4

Struggles over Property with Stealing, Production and Guarding – Non-Co-Operative Games

4.1 Introduction

The problem of struggles over material possessions is analysed in more detail in this chapter. The addition of material wealth to the state of nature opens up the possibility that the strategies of individuals might go beyond simply stealing and not stealing. For example, since individuals are concerned with material wealth, it is natural to imagine that they will undertake wealth improving activities other than stealing. Production and guarding whatever is produced are obvious possibilities. Once these complications are allowed, a new set of questions emerges. These are related to Friedman's question, 'what would happen if there were no systematic restraints on theft?' (Friedman, 1973, p. 207). For example, would such a system be stable and produce orderly anarchy, as writers such as Taylor (1982) claim? If there is stability, at what level? If stability is at a sufficiently high level of production it might not be worthwhile extending the state to protect property. Alternatively, would it be characterized by most efforts going to piracy and little to production?

In order to deal with these questions *n*-person games with continuous strategies and continuous pay-offs will be analysed. I shall also look at the case of a dynamic game in which players strategies change over time.

4.2 Struggles over material possessions with continuous strategies

The problem of struggles over material goods with continuous strategies can most easily be thought of as an extension of the hawk–dove and the war of attrition games. The war of attrition framework is retained in that the return that players get is a function of their expenditure and the expenditure of others. The additional feature is that the strategies are now continuous and each player wants to maximize over some function which depends on how much other players are spending, but also on the costs in terms of alternative uses of whatever resource is being spent. For example, whatever is spent on stealing will affect the returns from stealing, and the returns from any alternative use of resources that are available. This type of game is analysed by applying the following generalization of the Nash equilibrium theorem for continuous pay-offs.

Theorem 4.1: (Debreu, Glicksberg and Fan) (Fudenberg and Tirole, 1992, p. 34): A strategic form game whose strategy spaces S are non-empty compact convex subsets of Euclidean space and pay-offs v_i are continuous in s and quasi- concave in s_i, has a pure strategy Nash equilibrium.

This can be translated roughly as follows. Euclidean space has the sort of measure properties we are familiar with on a daily basis. Compact means that the strategy set has some upper and lower limits and includes its end points. The requirement that the pay-offs are quasi-concave for a continuous function means that the function can monotonically increase, can monotonically decrease and can increase and decrease like a familiar concave function. What it cannot do is decrease then increase.

Discussion of proof: The proof here is the same as the finite game case. Essentially it is to look for a strategy that is a best response to itself. If all players use this, then there are no gains from deviating. This is the same as showing that an equilibrium set of strategies exist such as that in Figure 1.1. In other words there must be some strategy that is the best response to itself and $r_i(s^*_{-i}) = s^*_i$ and $r_{-i}(s^*_i) = s^*_{-i}$. ■

4.3 The stealing and leisure game: Bush and Mayer

Bush and Mayer (1974) attempted to analyse what would happen in a state of nature in which each player had the same amount of some material good and a choice between stealing and leisure. They assume that the pay-off for players will be a function of the expenditure of their energies on stealing and of the amount of energy that others expend on stealing. The analysis is simplified by assuming that holdings of wealth are fixed. In this case the problem is one of allocating energies to stealing. They show that in this case there is an equilibrium, but, as would be expected, it is at a much lower level of welfare than with some co-operative arrangement. Their analysis is summarized as follows.

4.3.1 The model

Bush and Mayer assume that each day every individual is allocated one unit of some all purpose good x. There are no barriers to stealing. The value of the game for any individual i will be given by v_i where v_i is quasi-concave and continuous in all its arguments, $i = 1, \ldots, n$. Let e_i represent the level of effort that can be used to generate income by theft. v_i is concave in x_i and e_i. f_i is the technology of stealing and is the ability to take x from others. c_i is the amount that i loses to theft.

$$\begin{aligned} v_i &= v_i(x_i, e_i) \\ x_i &= 1 + f_i e_i - c_i \end{aligned} \tag{4.i}$$

Bush and Mayer also assume that stealing is uniform so that each individual loses the same amount of wealth given by

$$c = (\sum_{i=1}^{n} f_j e_j)/(n-1)$$

for $i \neq j$. Income is now given by

$$x_i = 1 + f_i e_i - c.$$

This assumption is not necessary in the proof of the existence of equilibrium below.

Since the good is desired and effort is costly the pay-off increases in x and decreases in e. That is $\partial v_i/\partial x_i > 0$ and $\partial v_i/\partial e_i < 0$. The players will set effort so that utility is maximised. This means that for an internal solution

$$dv_i/de_i = \partial v_i/\partial e_i + (\partial v_i/\partial x_i)(\partial x_i/\partial e_i) = 0.$$

This gives

$$f_i \partial v_i/\partial x_i = -\partial v_i/\partial e_i \tag{4.ii}$$

That is, to optimize a player sets the marginal rate at which utilities are lost through effort equal to the marginal rate at which utilities are increased from the income that effort produces.

4.3.2 Analysis of the stealing and leisure game

The first problem to be considered is whether there is an equilibrium for this game. This can be dealt with more easily than in Bush and Mayer's paper by using the previous theorem.

Proposition 4.a: The stealing game has an equilibrium. Bush and Mayer call this the natural equilibrium.

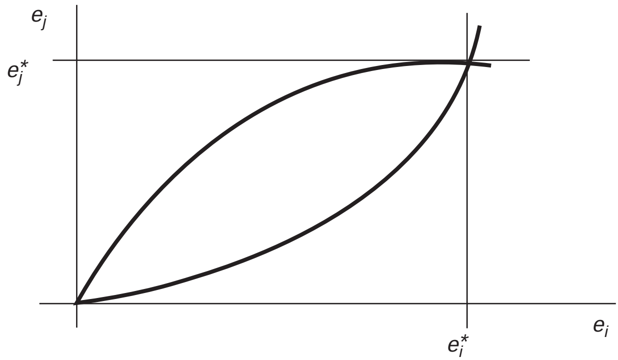


Figure 4.1 Stealing equilibrium

Proof: The proof that there is a Nash equilibrium solution follows immediately from the Debreu, Glicksberg, Fan theorem and the fact that the pay-off functions are continuous and quasi-concave in a player's own strategy. ■

The nature of the equilibrium can be demonstrated by noting that as e_j increases x_i will decrease. In order to restore the equilibrium

$$\partial v_i/\partial x_i = -(\partial v_i/\partial e_i)/f_i,$$

e_i must increase. The idea of reaction curves is again useful. These will have the shape in Figure 4.1.

These curves show the desired level of effort for player 1 for each level of effort for player 2. A point not on the intersection of the curves is not a Nash equilibrium since $r_i(e_j) \neq r_j(e_j)$ and both players have an incentive to change their strategies. It will also be noted that under a simple tâtonnement scheme the equilibrium is stable and there would be convergence.

Assume that effort is not so unpleasant, or so ineffective, that a player would never steal. It follows that the level of v_i is less than it would be if co-operation were possible and there is an allocation that must make all players better off than the best they can do without rules of property. Bush and Mayer call this the orderly anarchistic allocation. To see this observe that there is some level of income for $e_i = 0$ written $(x_i^o, 0)$ that i would prefer to (x_i^*, e_i^*), for $e_i^* > 0$ and $x_i^o \leq x_i^*$ where $\sum x_i^* = n$ since each player starts with one unit of the resource. Hence any final distribution $((b_1, 0), \ldots, (b_n, 0))$ is an orderly anarchy for $b_i \leq x_i^o$ and $\sum b_i \geq n$.

It must be noted, however, that the existence of a collectively optimal orderly anarchistic allocation does not mean that this allocation would be reached by players optimizing their returns. Would this be expected?

Bush and Mayer call this orderly anarchistic allocation a 'fragile equilibrium.' (1974, p. 410). What they actually mean is that players would not reach this equilibrium if they used their Nash equilibrium strategies, even though it is on the Pareto frontier. Hence we have the problem set out in section 1.9. The term 'orderly allocation' is rather misleading. This problem is dealt with in more detail when the theory of the core is analysed in chapter six.

This is a fairly restricted model because the only strategy that players have is to trade leisure for stealing. What happens when production is taken into account and when players are allowed to steal and guard? This question is considered below.

4.4 The breakdown of anarchy: Hirshleifer

Hirshleifer (1995) analyses a model in which players can produce and fight over resources. The idea that drives the model is that the resources available to the players are determined by a continuous struggle. The amount produced depends on the stock of resources and the stock of resources depends on the effort players put into fighting. Effort put into fighting reduces the amount of effort available for producing. These assumptions are used to consider the stability of an anarchistic system.

The set up of the model, and two of its findings, is presented below. To avoid the problems of solving a continuous game the players make a single strategy choice at the beginning of the game. This is known as a steady state or open loop equilibrium.

4.4.1 The model

The resource available to player *i* is w_i and this is divided between productive effort and fighting. Transformation of these resources into the different types of activities has a conversion cost. Let x_i be productive effort and y_i fighting effort with conversion costs a_i and b_i respectively. Then

$$w_i = a_i x_i + b_i y_i \tag{4.iii}$$

It is also possible to deal with the intensities of productive and fighting effort by defining $e_i = x_i/w_i$ and $f_i = y_i/w_i$. This gives

$$a_i e_i + b_i f_i = 1$$

Hirshleifer assumes that players wish to optimize the amount of the good they produce for consumption and that production depends on the amount of effort and the resource. Let production be

$$m_i = (x_i)^h = (e_i w_i)^h \tag{4.iv}$$

Resource control depends on the probability that fighting is successful and is defined by $w_i = p_i w$ where p_i is the probability of success and w is the total resource. The success function for player 1 against player 2 is

$$p_1 = y_1{}^{\alpha}/(y_1{}^{\alpha} + y_2{}^{\alpha})$$

with the analogous expression for p_2.

This gives

$$p_1/p_2 = (y_1/y_2)\alpha$$

where α is an index of the decisiveness of conflict. The higher the value of α more effective is an attack. In the First World War, for example, trench strategies gave α a low value.

4.4.2 The sustainability of the system

Consider the problem of the sustainability of a two-player anarchistic system. From the previous

$$w_1/w_2 = (y_1/y_2)^{\alpha} = (f_1 w_1/f_2 w_2)^{\alpha}$$

So

$$f_1^{\alpha} w_1^{\alpha-1} = f_2^{\alpha} w_2^{\alpha-1}$$

and

$$p_1/p_2 \;=\; (f_1/f_2)^{\alpha/(1-\alpha)} \tag{4.v}$$

from the identity

$$f^{\alpha} f^{\alpha\alpha} f^{\alpha\alpha\alpha} \ldots = f^{\alpha+\alpha\alpha+\alpha\alpha\alpha+\ldots} = f^{\alpha/(1-\alpha)}$$

This gives the following straightforward proposition.

Proposition 4.b: The necessary conditions for an anarchistic system to be stable are (a) the decisiveness parameter $\alpha < 1$ and (b) a high value of m_i.

Proof: Immediate. ■

Condition (b) says that the players must be able to produce enough to stay alive.

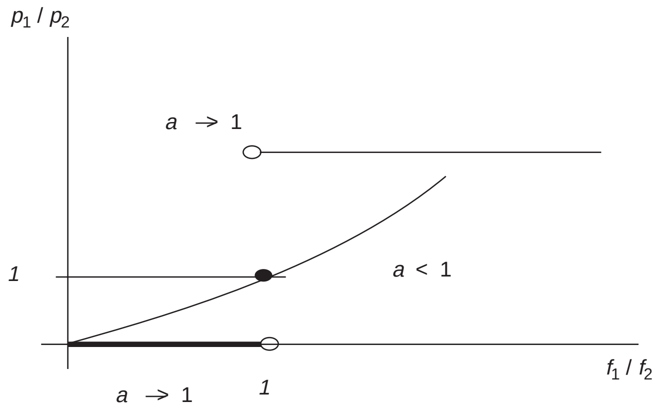


Figure 4.2 Success functions for different values of decisiveness

Condition (a) is more interesting. It can be understood as follows.

Examine equation (4.v). Note that for $\alpha \to 1, \alpha/(1-\alpha) \to \infty$. This means for

$$f_1/f_2 > 1, (f_1/f_2)^{\alpha/(1-\alpha)} \to \infty$$

and for

$$f_1/f_2 < 1, (f_1/f_2)^{\alpha/(1-\alpha)} \to 0.$$

This is illustrated for $\alpha < 1$ and $\alpha \to 1$ in Figure 4.2.

Observe that, at the limit p_1/p_2 jumps at $f_1/f_2 = 1$. This means that player 1 wants $f_1/f_2 > 1$ and player 2 wants $f_1/f_2 < 1$. Hence the system is not stable for $\alpha \to 1$.

4.4.3 Optimization and equilibrium

The optimum strategies for the players in this system are those that maximize m_i. From the set up of the model it is obvious that there is a trade-off between fighting for more of the resource and putting effort into production. In order to solve this problem we need to solve

$$\begin{aligned} &\max m_i = \max(e_i p_i w)^h = [(e_1 w f_1^\eta)/(f_1^\eta + f_2^\eta)]^h \\ &\text{subject to } a_i e_i + b_i f_i = 1 \end{aligned} \qquad (4.\text{vi})$$

where $\eta = \alpha/(1-\alpha)$.

This is straightforward optimization problem and is solved by constructing the Lagrangian. This is a standard method for constrained problems and is found in any undergraduate mathematics text. The Lagrangian is written

$$L = [(e_1 w f_1^{\eta})/(f_1^{\eta} + f_2^{\eta})]^h + \lambda(1 - a_i e_i - b_i f_i)$$

Solving for the first order conditions

$$\partial L/\partial f_i = 0,$$

$$\partial L/\partial e_i = 0$$

for $i = 1, 2$ and dividing $\partial L/\partial f_i$ by $\partial L/\partial e_i$ eliminates the λ term. a_i is eliminated using

$$a_i = (1 - b_i f_i)/e_i$$

from the constraint term. The details are in Hirshleifer (1995, pp. 34–5).

It is now possible to write the optimum strategy of player 1 in terms of the strategy of player 2 from the solution to equation (4.vi). It is

$$f_1^{\eta}/f_2^{\eta} = \eta/(b_1 f_1) - (\eta + 1)$$

For the simple case where $a_i = a$ and $b_i = b$ for $i = 1, 2$ the strategies for both players are the same and this reduces to

$$f_1 = f_2 = \alpha/b(2 - \alpha) \qquad (4.vii)$$

This gives the following:

Proposition 4.c: The larger the value of the decisiveness parameter, α, the more resources devoted to fighting. The lower the cost of diverting resources to fighting, b, the greater the resources devoted to fighting.

Proof: Immediate, from equation (4.vii). ■

4.5 Swords into plowshares: Grossman and Kim

Grossman and Kim (1995) analyse a model in which players can allocate resources to defence, production and offence. They are concerned with the conditions under which a non-aggressive equilibrium is possible. This is an equilibrium in which no resources are allocated to offensive weapons. In this case resources are allocated to defence and production.

4.5.1 The model

The model is for two players and the game is set out in two stages. In stage one an allocation of resources to defence is chosen. In stage two resources are allocated to production and to offence. The optimum allocation of resources in stage one depends on the response in stage two. Hence the game is solved for stage one by backward induction from stage two. Since the mathematical details of the argument can be obtained from the original I shall only provide a sketch of some of its features.

Total resources for player *i* are a_i. This is allocated to meet the condition

$$a_i = x_i + y_i + z_i$$

where x is resources allocated to production, y is resources allocated to offence and z is resources allocated to defence. Since resources are allocated to defence in stage one,

$$a_i = z_i = x_i + y_i$$

The production function is αx_i. The security for property depends on allocation to defence and offence. The fraction of the total endowment retained by an agent is

$$p_i = z_i/(z_i - \theta y_j)$$

where θ indicates the effectiveness of offence against defence.

It is assumed that fighting over resources is destructive. Hence agent *i* gains the fraction of agent *j*'s endowmnent given by $(1-p_j)(1-\beta)$, where β measures the losses through fighting.

The total pay-off to player *i* is

$$v_i = \alpha x_i + p_i a_i + (1 - p_j)\,(1 - \beta) a_j \qquad (4.\text{viii})$$

which can easily be seen to be quasi-concave as required by theorem 4.1.

Since the model is in two steps it has the advantage of allowing y_i to be calculated in stage two under the assumption that a_i and z_i are fixed from stage one. Thus the partial derivative $\partial v_i/\partial y_i$ does not have to take the $\partial z_i/\partial y_i$ into account.

4.5.2 Security of claims to property

The security of claims to property is calculated by differentiating equation (4.viii) with respect to y_i in stage two and then using this to calculate the optimum level of

z_i in stage one. The derivatives are

$$dv_i/dy_i = -\alpha - (1-\beta)\,(\partial p_j/\partial y_j)\,a_j$$

and

$$dv_j/dz_j = [\partial p_i/z_j + (\partial p_j/dy_j)(\partial y_i/\partial z_i)]\,a_j - \alpha$$

Substitution from the equilibrium z_i into the equilibrium y_i from these two equations gives

$$y_i = \begin{vmatrix} [1 - 1/(2(1-\beta)\theta)]a_j/2\alpha\theta & \text{for } 2(1-\beta)\theta > 1 \\ 0 & \text{for } 2(1-\beta)\theta \leq 1 \end{vmatrix} \qquad (4.\text{vix})$$

This says that when $2(1-\beta)\theta \leq 1$ the optimum level of defence is such that there is no fighting at all over property. It will be noted that this is a function of the parameter that gives the effectiveness of offence against defence and the parameter that gives the losses due to fighting. It is not a function of wealth endowments.

An interesting question is the relationship between the equilibrium allocation of resources to offensive activities and the effectiveness of offence. It might be expected that resources devoted to offence would increase. To test this, use equation (4.vix) to get

$$\partial y_i/\partial\theta = a_j[1 - (1-\beta)\theta]/2\theta^3\alpha(1-\beta)$$

This tells us that, in this model, resources devoted to offence begin to decrease at some sufficiently high level of development of the relative capacity of offensive technology. An example is given in Figure 4.3.

In other words at some high level of effectiveness of offence the players are better off shifting the smaller amount of resources that remain after predation into other activities such as production.

It might be thought that the two stage characteristic of this model imposes too strong an assumption and that it is more reasonable to assume that players allocate all resources simultaneously. Let us now consider this case.

4.6 Stealing production and guarding

Suppose that we have a system in which players attempt to maximize their holdings of material goods by allocating resources between production, stealing and guarding simultaneously. Stealing is anachronistic, but it has a stylistic advantage.[2] The main problem in trying to analyse such a system is that, with three strategies the decision

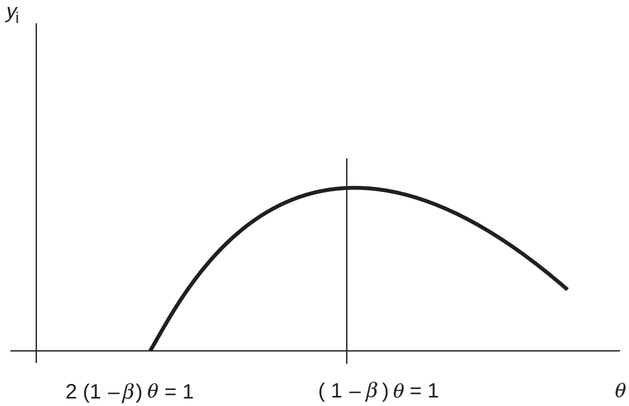


Figure 4.3 Resources devoted to offence

to devote resources to any one activity does not necessarily determine the resources devoted to the remaining two. For example, if production goes up there would be more to steal. Does stealing or guarding go up? This indeterminacy is itself interesting because it is probably a feature of many real world outcomes.

4.6.1 The model

The model is constructed by imagining that there are two players and that each player controls one units of some resource. This might be thought of as labour time. A strategy for $i, i = 1, 2$ is a choice of the amount of this resource to devote to production, stealing and guarding. These amounts are designated x_i, y_i and z_i respectively.

The goods that are produced and stolen can be thought of as a pile of possessions which is to be guarded. Unlike the Hirshleifer (1995) and Grossman and Kim (1995) models, it is assumed that theft is limited to produced goods rather than all resources. That is, the players raid, but they do not struggle over territory. Players derive monotone increasing utility from increases in the amount of material goods held.

The game is played repeatedly with an infinite time horizon. Players have perfect information at the start of the game and use an open loop strategy.

Returns to a player for stealing and guarding will depend on the amount of resources devoted to stealing by player i and the amount of resource devoted to guarding by player j, $i \neq j$. They will also depend on the amount there is to steal. Returns to guarding will depend on the amount of resources devoted to this activity and the amount of resources devoted to stealing by the other player, as well as the amount available to steal. Returns to production depend on the resources devoted to production.

The production, stealing and guarding functions are written respectively

$$m_i\,(x_i)$$
$$h_i\,(y_i, x_i, z_i, x_j, y_j, z_j)$$
$$g_i\,(z_i, y_j)$$

where $i \neq j$. These functions preclude manna-from-heaven technologies where goods are free.

The function h_i takes into account the fact that what is stolen from player j depends on j's produced goods and what j has stolen from i. I shall say more about this later. Assume that these functions are continuously differentiable in their domains up to the required order and concave in x_i, y_i, z_i respectively. Define $g \in [0, 1]$.

The problem for player i is to maximize the amount of goods it holds. Since it was assumed that each player guards the stock of goods that it has produced and stolen, the problem can be written as follows.

$$\begin{aligned} &\gamma = \text{maximize}_{si}\ v_i = g_i(m_i + h_i) \\ &\text{subject to } x_i + y_i + z_i = 1 \end{aligned} \tag{4.x}$$

In what follows I will drop the subscripts to ease the notation where the meaning is clear.

4.6.1.1 *Equilibrium properties of the general model*

The equilibrium properties are again considered to see whether the game is stable, or there is at least one player that is always better off changing its strategy, no matter what strategies every other player has chosen. This means that the system is unstable and probably cannot continue in its current form.

The conditions required for an internal equilibrium are:

$$\partial v/\partial x = \partial v/\partial y = \partial v/\partial z \tag{4.xi}$$

where $\partial v/\partial x = g\partial(m+h)/\partial x$ and $\partial v/\partial y = g\partial h/\partial y$. $\partial v/\partial z = (\partial v/\partial g)\,(\partial g/\partial z)(m+h) + g\dfrac{\partial n}{\partial z}$. It is also assumed that $\partial v/\partial z$ is concave.

Proposition 4d: (Equilibrium) (a) The game has a Nash equilibrium in pure strategies for sufficiently small values of k where $0 < \partial^2 h/\partial x \partial y < k$. (b) If g is non linear, and at least one of m and h is non-linear then $s_i(s_j)$ is a one to one function.

Proof: (a) The proof depends on the Debreu, Glicksberg, Fan theorem in section 4.2. Continuity is obvious. Quasi-concavity is established from the properties of the

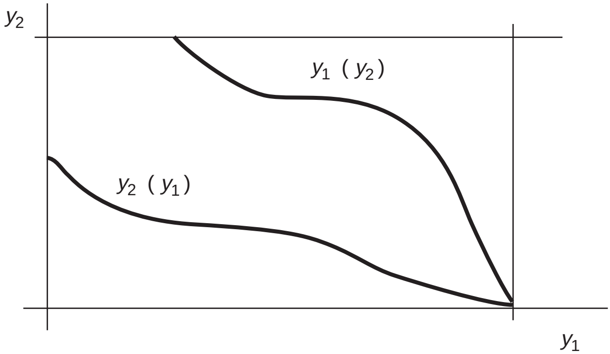


Figure 4.4 Example of reaction functions with a specialized predator

bordered Hessian matrix for v.[3] It is straightforward to show that the determinants of this matrix will have the appropriate signs for $\partial^2 h/\partial x \partial y$ sufficiently small.[4] (b) See Appendix 4.8.1 below. ■

Proposition 4d says that the system will only be stable if the production and/or the stealing functions are constrained in some way. Where the constraints are not met stability cannot be guaranteed.

The proof that the best response function is one to one in part (b) is important because it narrows the range of possible responses to a single best reply.

It is easy to pick an example that violates concavity where best reply functions are not unique. Consider v with increasing returns to scale from guarding.

The rest of the analysis concentrates on games that meet the conditions required for an equilibrium. What is not clear is whether production and stealing both have to be constrained, or only one of these. What is the nature of these constraints?

It is important to note the game analysed so far is consistent with a range of equilibria, including the possibility that one player is a specialized predator. Consider the best response functions given in Figure 4.4 for the case without an internal equilibrium. The reaction for player 1 is always greater than, or equal to, that of player 2. A trivial example of specialized predation is given by the case where the technologies are such that $\partial v_1/\partial x_1 < \partial v_1/\partial y_1$ and $\partial v_2/\partial x_2 > \partial v_2/\partial y_2$ for the domain of the problem.

4.6.1.2 Change in the technology of production

The forces of production have been seen as one of the main factors in historical development by many social scientists, and in particular by Marxian political economists. One way of thinking about this would be to consider the case where a general change in technology increases productive capacity in the long run, but roughly balances stealing and guarding capacity. Wheeled vehicles can be used as chariots to increase raiding, but also to help construct stockades to protect goods. This case can be approximated by considering a change in productive technology with stealing and raiding technology constant.

For changes in the technology of production, with the technology of stealing constant, we might expect the allocation of resources to change with perhaps more devoted to production. A surprise here is that this may not happen.

Define a separable equation as one that can be written

$$\psi(kx) = l(k)\psi(x)$$

for ψ and l continuous injective functions. Examples are $\psi = x, x^2, \sqrt{x}$. This is a weaker condition than homogeneity, for example.

Let an increase in productive technology be represented by an increase in the parameter k.

Proposition 4e: (Constant proportionality of stealing). The equilibrium strategies are invariant to any increase in the productive technology that can be represented by a parameter, k, where $m = m(kx)$ and m is separable in k.

Proof: Immediate from the fact that strategies are invariant under multiplication of pay-offs. Since stealing depends on production by player j the pay-off function is $v = g(l(k) + l(k)h) = l(k)v$. ■

Proposition 4e tells us that, for a specified class of production and stealing functions, the strategies of the players will not alter with changes in the technology. This can be illustrated by considering a situation where there are a number of bands with an inefficient technology of production. Perhaps they can only gather nuts and berries. These bands allocate their resources among production, stealing and guarding. Suppose the technology of production increased to the level of modern societies. Despite this change in production, the relative resource allocation remains unchanged.

The prediction that a constant proportion of resources is devoted to stealing and guarding throughout history is intriguing. There is a fairly simple intuition underlying this result. It is that, as productivity increases the gains from stealing and guarding both increase. These gains offset the tendency to shift resources out of either of these activities.

The extent which the conditions assumed for the model hold for actual social systems could, in principle, be empirically tested. Since the notion of stealing may be too narrow for comparison across different societies it would be best to think of it as non-productive acquisition. Such a test would also raise some interesting questions in measurement and classification. Should some of the activities of lawyers in modern societies be treated as resources devoted to non-productive acquisition, for example?

So far the model has been analysed in very general terms. Let us consider a specific example.

4.6.2 A two player example with symmetric players and concave pay-off functions

The example to be analysed is a symmetric game in which each player has the same initial resources and pay-off functions. The assumption of symmetry is interesting in itself and is used by Hobbes and Locke. It also allows us to avoid the analytical difficulties of explicitly analysing a system in which three resources are allocated simultaneously.

Since the game is symmetrical we know that there must be at least one Nash equilibrium where both players use the same strategy. I will concentrate on this equilibrium in what follows. I do not have a proof that this equilibrium is unique although there is an argument that it is a focal point for the game.[5]

Stealing from produced goods is written in full as

$$h_1 = f_1[m_2 + f_2(m_1 + f_1(m_2 + \ldots)).$$

For m separable this gives the separability that was required in proposition 4e. Hence the game in equation (4.x) has pay-offs

$$v_i = g_i m_i + f_i m_j \tag{4.xii}$$

at the symmetrical equilibrium.

Let $a, b \in \boldsymbol{R}$, $a \geq 0, b \geq 0$. Write the production stealing and guarding functions as

$$\begin{aligned} m_i &= x_i^b \\ f_i &= 2a\sqrt{y_i}/(1 + z_j) \\ g_i &= 1 - 2a\sqrt{y_j}/(1 + z_i) \end{aligned}$$

Note that the production function has been constructed so that it is non-separable to give some new results.

This gives the conditions for a Nash equilibrium as $g_i m_i{}' = f_i{}' m_j = g_i{}' m_i$ where $(.)'$ is the derivative with respect to its own control. This means

$$[1 - 2a\sqrt{y_j}/(1 + z_i)]bx_i^{b-1} = ax_j^b/\sqrt{y_i}(1 + z_j) = 2ax_j^b\sqrt{y_j}/(1 + z_i)^2$$

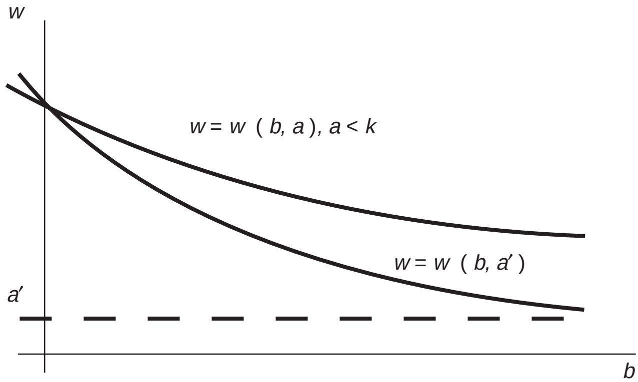


Figure 4.5 Relation between technology of production and resources devoted to theft

From $g_i m_i{}' = g_i{}' m_i$ and $f_i{}' m_j = g_i{}' m_i$ and the assumption of symmetry we get

$$\begin{aligned} 2y &= 1 + z \\ x &= 2b\,[y^2/(a\sqrt{y}) - y] \end{aligned} \tag{4.xiii}$$

Consider changes in the technology of production.

4.6.2.1 Changes in the technology of production

The questions here are the effect of a change in the technology of production on the amount of resources devoted to theft and the effect of these parameters on the equilibrium of the system. From proposition 4d we are particularly concerned with the question of whether it is constraints on the technology of production or on the technology of stealing that is central to the existence of equilibria. Since the consequence of a change in production depends on the technology of stealing these have to be considered together.

The resource constraint $x + y + z = 1$ and equation (4.xiii) gives the equilibrium condition

$$b = (2 - 3w^2)2w^2\,(w/a - 1) \tag{4.xiv}$$

where $\sqrt{y} = w$.

Letting ψ represent a continuous function, equation (4.xiv) can be expressed as

$$b = \psi(w, a)$$

ψ is injective and has a continuous inverse within the domain of acceptable values of w for this problem, a is treated as a parameter. Hence

$$w = \psi^{-1}(b, a) = w(b, a).$$

Write $w(b, a') = w(b)'$. This is analysed to establish the following proposition.

Proposition 4f: (a) There is a $k \in \boldsymbol{R}$ such that, for $a \geq k$ an equilibrium only exists for $b = 0$. This equilibrium is unique. (b) For a $< k$ the amount of resources devoted to stealing decreases as b increases for a constant. (c) For $a > a'$, $w(b) > w(b)'$.

Proof: Appendix 4.8.2. ■

Proposition 4f is illustrated in Figure 4.5. It gives us the following additional bits of information about the system.

4.6.2.1a The technology of stealing trumps the technology of production. A solution exists for all b for $a < k$. For $a > k$, however, the only solution is the trivial one where $b = 0$. This is a manna from heaven technology and there is no production. For any situation where the amount of goods available is altered by the resources devoted to production, there is no equilibrium and the system is unstable.

4.6.2.1b The resources devoted to theft will decrease as the technology of production improves for a given technology of stealing less than some critical value k. This tells us that, at least for the symmetric game with a non-separable technology of production, it is the absolute value of the stealing technology that is important and not the ratio between stealing and production. This is because the rate of decrease in stealing for an increase in production will be reduced as the technology of stealing improves. At the limit an increase in the technology of production will not reduce resources devoted to stealing.

This might be compared with the constancy of stealing hypothesis in proposition 4e. Although the resources devoted to stealing are not constant where the technology of production is not separable, they cannot fall below some level that increases with the technology of stealing. Hence we have a weaker hypothesis of a minimal level of resources devoted to stealing for all levels of production.

4.6.2.2 Changes in the technology of stealing

The technology of stealing is now taken as a variable parameter. To analyse this we get, from equation (4.xiii), and the resource constraint

$$a = 2bw^3/[2 - w^2(3 - 2b)] \tag{4.xv}$$

This gives

$$w = \psi^{-1}(a, b) = w(a),$$

where w is continuous and injective for

$$2 - w^2(3 - 2b) \neq 0.$$

It is immediate that the sign of $\partial w/\partial a$ remains the same for all b. From proposition 4g we also know that the resources devoted to stealing at each value of the stealing parameter are reduced as the technology of production increases. That is, for any $b > b'$ we have $w(a) < w(a)'$.

Consider the way in which the system responds to changes in the stealing technology with b constant.

An analysis of equation (4.xv) establishes the following proposition.

Proposition 4g: (a) Resources devoted to stealing increase for an increase in the technology of stealing; (b) the total amount stolen begins to decrease for some value of a; (c) an improvement in the technology of stealing makes both players worse off.

Proof: Appendix 4.8.3. ■

Proposition 4g (b) is illustrated in Figure 4.6. It is simple to show that as technology of production increases the value of a at which the returns from stealing start to reduce increases.

Part (a) of proposition 4g is what would be expected. Together with the second and third part it shows that, even though an increase in stealing makes both players worse off, the optimal strategy for each player is to attempt to steal more.

4.6.3 Summary of results

The main findings from the analysis of this model are summarized below.

4.6.3.1 Systems in which stealing production and guarding are possible will only have an equilibrium if the level of stealing technology is low. This is consistent with Hirshleifer's (1995) findings on the break down of anarchy if stealing technology is treated as decisiveness of conflict. This was shown to be true in both the case of the general game and the symmetric game. In addition it was shown for the symmetric game that increases in the technology of stealing could not be offset by increases in the technology of production.

4.6.3.2 Resources devoted to stealing and guarding may be constant over a large domain of change in the technology of production. This constancy hypothesis should, in principle, be testable.

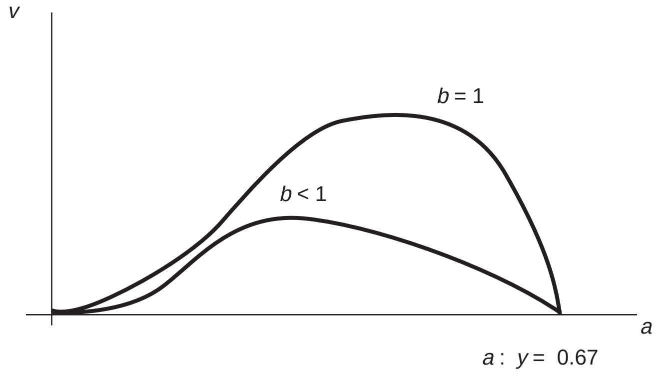


Figure 4.6 Returns for an increase in the parameter of stealing

4.6.3.3 For the symmetrical game, increases in the technology of production reduce resources devoted to stealing and increases in the technology of stealing increase resources devoted to stealing. In the long run stealing overwhelms production.

4.6.3.4 For the symmetrical game the returns for stealing first increase and then decrease.

4.6.3.5 Everyone is worse off in the symmetrical game for an increase in stealing technology. Conversely, everyone is better off for an increase in guarding technology.

4.7 Comment on the models

What the models tend to show is that stability is not guaranteed in the quasi-anarchistic state of nature. It may be necessary to impose strong restrictions on the efficacy of theft or offence in order to produce any equilibrium at all. Even where the state of nature is stable, resources will be wasted on stealing, fighting and/or guarding activities.

One implication of this is that there may be gains from a central authority or protection agency that reduces these deadweight costs. The extent of these gains would depend on what assumptions were made about the guarding technology and the level of losses as the result of stealing and guarding.

4.8 Appendices

Appendix 4.8.1 Proof of Proposition 4d(b)

Let $s_j = s_j^o$ and $s_i = (x_i, y_i, z_i)$ and $s_i^{\#} = (x_i^{\#}, y_i^{\#}, z_i^{\#})$ be best response strategies to s_j^o. For h, m, and g non-linear equilibrium requires from (ii) that $\partial(m+h)/\partial x = \partial h/\partial y = (\partial v/\partial z)/g$. Suppose $x_i^{\#} > x_i$. Since h, m and g are concave $y_i^{\#} > y_i$ and $z_i^{\#} < z_i$. This means $\partial v_i/\partial x_i$ is strictly decreasing and $(\partial v_i/\partial z_i)/g$ is increasing. Contradiction. Now let m_i be linear. $\partial v_i/\partial x_i = k$. Hence $\partial h_i/\partial y_i = k$ and y_i is fixed. If x_i increases z_i must decrease and $(\partial v/\partial z)/g$ increases which violates the equality. Similarly if x_i decreases. Contradiction. The argument is similar for h_i linear.

Appendix 4.8.2 Proof of proposition 4f

Assume that $w/a - 1 > 0$ for all w in the relevant domain for an equilibrium. The derivative of equation (iv) gives

$$\partial b/\partial w = -[6_w\,(2_w^2\,(w/a-1) + (2-2_w^2)\,(6_w^2/a - 4_w)]/2_w^2\,(w/a-1)^2$$

The sign of $\partial w/\partial b$ = sgn $(\partial b/\partial w)^{-1}$ = sgn $-[6w(2w^2(w/a-1)) + (2-3w^2)$ $(6w^2/a - 4w)]$.

(a) There is a $k > 0$ such that for a $< k, \partial w/\partial b < 0$ and for $a > k \partial w/\partial b > 0$. Numerical analysis gives $k' \approx 0.83$ for $b = 0$. At this value of b, production is a constant $x^o = 1$ for all x. The solution for the two move stealing and guarding game is $y = 2/3$. Hence $w = 0.816$. For $k > k'$ we have $\partial w/\partial b > 0$. This is not possible since $2y = 1 + z$ and the resource constraint is violated.

(b) For $k < k'$ we have $\partial w/\partial b < 0$.

(c) From (iv) $b \to \infty$ as $w/a \to 1$. Hence lim inf w is $w = a$. For $a > a'$ lim inf $w(b) > w(b)'$ and $w(b) > w(b)'$. for all b.

The assumption that $w/a - 1 > 0$ is true if $w > a$ for $w \leq 0.816$. This follows from equation (vi).

Appendix 4.8.3 Proof of proposition 4g

(a) The inverse is defined so $\partial w/\partial a = (\partial a/\partial w)^{-1} = (2-w^2)/2w^2(6-\,w^2) > 0$ for all admissible w. $f = a/w$. This gives $\partial f/\partial a = (w - a\partial w/\partial a)\,/w^2$. The sign of $\partial f/\partial a$ = sgn $2w^3(6-w^2) - a(2-w^2)$. $\partial f/\partial a > 0$ for $2w^3\,(6-w^2)/(2-w^2) > a = 2w^2/(2-w^2)$.

(b) The total amount stolen is given by $fx = (a/\sqrt{y})\,[2(y^2/a\sqrt{y} - y)] = 2(w^2 - aw)$. This gives $\partial fx/\partial a = 2(2w\partial w/\partial a - w -\ a\partial w/\partial a)$. Substituting for $\partial w/\partial a$ gives sgn $\partial fx/\partial a$ = sgn $(2w-a)\ (2-w^2)^2 - 2w^3(6-w^2)$. Numerical evaluation gives $\partial(\partial fx/\partial a)/\partial w > 0$ for $a : w < 0.605$ and $\partial(\partial fx/\partial a)/\partial w > 0$ for $a : w > 0.605$. These values of a are permissible since $w^2 = y \approx 0.367 < y\,\text{max} \approx 0.67$. $\partial(\partial fx/\partial a)/\partial w = \text{max}$ at $w \approx 0.34$. Hence $\partial fx/\partial a$ is increasing at an increasing rate for $a : w < 0.34$.

(c) Total wealth is given by $v = gx + fx$. So $v = (1 - f)x + fx = x = 2\ (y^2/a\sqrt{y} - y)$ in equilibrium. Hence $v = 2(w^3/a - w^2)$. This gives $\partial v/\partial a =$ $[(6w^2 \partial w/\partial a)a - 2w^3]a - 4w\partial w/\partial a$. Substituting for a and evaluating gives $\partial v/\partial a < 0$ for all a and $\partial^2 v/\partial a^2 < 0$.

5
Problems of Reputation and Markets for Protection – Non-Co-operative Games

5.1 Introduction

It would be necessary for the state to enforce rules of property only if protection for property could not be provided as a market good or in some other manner. This chapter investigates some aspects of this possibility. Nozick argues, for example, that, in a state of nature without Hobbesian problems, something like a market for protection could emerge and produce a solution to the general problem of security for property.[1] In this case, if the state provided physical security, the market could provide security for private property through a process of selling and buying protection. The state would thus remain an ultra-minimal institution that only protects life. The market could then be seen as a quasi-autonomous mechanism that could operate without external support once physical security is provided. That is, it could be seen as a self-sustaining mechanism that could produce its own inputs as outputs. Nozick avoids the security problem by assuming that individuals have rights in the state of nature and that these rights would be respected by others. It is difficult to see how this assumption might be justified. This problem will be disregarded for the sake of the argument. If a market for protection cannot plausibly arise in the absence of security problems, then it would not be plausible under more rigorous conditions.

This chapter investigates the question of whether repeated games can lead to self-supporting markets. It also considers the characteristics of a market for security. As might be expected, the main characteristic of markets for protection is that, with sufficient information, the supplier can capture most of whatever is to be protected.

5.2 Promises and reputation

The first question to consider is whether any form of market could arise without rules of property. Since many exchanges cannot be made simultaneously, why would traders keep their promises and deliver the good? If the game is played only once, the problem is that individuals will only keep promises when this is an optimal move. As there is no authority to enforce contracts, and the good costs something to provide, it is immediately obvious that the trade has the structure of a game with defect as the single dominant strategy. In this case individuals might hire another agency to enforce contract keeping, but this duplicates the problem at a higher level.

From the analysis of prisoner's dilemma games it is clear that the best possibility of trade is for the interaction to be repeated without a known finite termination time.

Two sorts of games will be considered. In the first players are of the same type. In the second the characteristics of the opponents vary.

5.2.1 Sub-game perfect equilibrium in a game of promises with similar players

The problem of whether players can be expected to keep their promises over things such as trades can be considered by looking at repeated games in which players are able to punish for defection. Consider, for example, a number of players trading goods. The traders can co-operate by providing the good or the money stipulated, or they can defect. It was argued in Chapter 2 that trigger strategy mechanisms would be unreliable in a repeated prisoner's dilemma because the costs of co-operating while the other player defected were large and because permanent defection is not a sub-game perfect equilibrium. In a trading game the costs of co-operating against a defector may not be so large. Is there any condition under which punishment strategies could produce a sub-game perfect equilibrium?

The outcome of punishment strategies depends on what assumptions are made about what the players are trying to optimize. A particularly strong assumption which allows for a refinement of the folk theorem is given in the following example. Assume that there is a repeated game of infinite length and that players apply a time averaging criterion to their pay-offs rather than attempt to maximize any individual series of pay-offs. That is, they seek to maximize the expected value of the sum of their average pay-offs. This gives

$$E[s_i] = [\sum v_i^k(s_i^k, s - i^k)/n]$$

where the summation is for k between 1 and $n \to \infty$. It will be observed that the pay-off is meaningful since the series converges rather than expands without limit. This is because

$$E[s_i]n\alpha/n = \alpha$$

where $\alpha = \max v_i^k$.

In this case, players are not concerned about the timing of the pay-offs and they are not concerned about any finite sequence of pay-offs. This is because a finite sequence will have zero impact on a pay-off of infinite duration.

Let s_i^c be co-operate and s_i^d be defect. Assume the prisoner's dilemma pay-off relations $v_i(s_i^d, s_{-i}^c > v_i(s_i^c, s_{-i}^c)$ and $v_i(s_i^d, s_{-i}^d) > v_i(s_i^c, s_{-i}^d)$.

Theorem 5.1: (Aumann and Shapley, 1976). The infinitely repeated game with a time averaging criterion has a sub-game perfect equilibrium $\varphi = s^*$ with $v_i^* \geq v_i$, where v is an attainable pay-off.[2]

Proof: The strategy s^* is defined by a co-operation phase and a punishment phase. In the co-operation phase, player i plays s_i^c on round t if j has played s_j^c on round $t-1$. If player j plays s_j^d it is punished by playing the sequence s_i^d. This punishment continues until the gains for j from the original move s_j^d are eliminated. Defections during punishment are ignored. Player i then returns to the co-operative phase. ■

Sub-game perfection is proven as follows. For the player who defects, the punishment is such that

$$v_j(s_i^*, s_j^d) \leq v_j(s_i^*, s_j^c).$$

For the player who punishes the cost of punishment is zero from the time averaging criterion.

Since the punishment strategy is sub-game perfect, neither player has an incentive to defect. This means that

$$\varphi = (s_i^*, s_{-i}^*)$$

with outcome (s_i^c, s_{-i}^c). It will also be observed that the argument easily extends to include punishment for those players who fail to carry out the task of punishing defectors.

The idea of time averaging over an infinite series can also be applied to hawk–dove games to allow players to do better than the dove, dove pay-off. Consider the pay-offs for the hawk–dove game in Figure 3.1. If both players use s^d the pay-off (1, 1) is attainable. If players are able to communicate then they could make the Pareto superior move into the shaded region to get $v_i 1$. For example, player i could play $s_i^* = (s_i^h)$ on every odd move and (s_i^d) on every even move of the game with the punishment strategies outlined above. If player j matches this with the sequence $s_j^* = (s_i^d), (s_i^h)$ then each player can get a pay-off $(3+0)/2 = 1.5$. This possibility is illustrated in Figure 5.1.

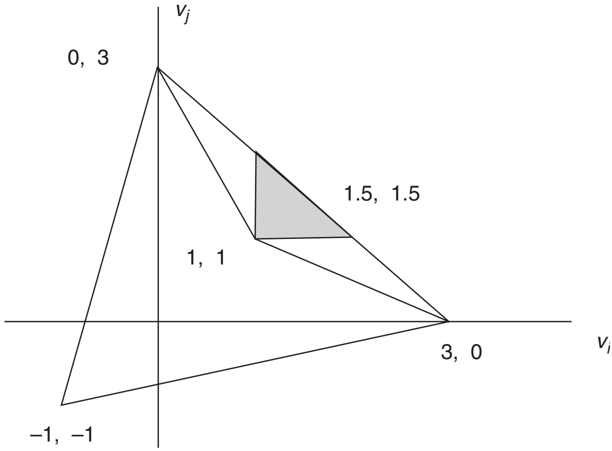


Figure 5.1 Pay-offs for the hawk–dove game with punishment for non-co-operation

The difficulty with the sub-game perfect Nash equilibrium in these examples is that the strong assumptions about the game having infinite length and time averaging makes the conclusion weak. It says that players will co-operate because there is nothing to be lost from carrying out any finite sequence of punishments. If life is short and players do care about a finite series of sub-optimal pay-offs, then the sub-game perfectness of this strategy is lost.

An alternative to relying on punishment for not keeping promises is to allow players to build reputation on past performance.

5.2.2 Reputation

The optimum strategy for a player trying to build a reputation on past performance will depend on the best response strategies to a player sending signals, or producing a performance, of that type. The analysis of credible signals is the analogue of that of credible threats. Hence there is a difficulty that past performance may not necessarily be taken as a guide to future action if there is no means of committing to continuing to behave in the same way. I will ignore this problem and assume that, without guarantees, performance tends to cause other players to impute a reputation.

Since we are now dealing with a situation in which a player may invest in establishing a reputation because it provides future pay-offs it might be the case that a player will take a loss at time t in order to establish a reputation that gives a greater pay-off at time $t + k$. In order to evaluate strategies for a pay-off stream the idea of sequential rationality is used.

Definition: A strategy s_i is sequentially rational for player *i* in a game where *i* can send signals about its type if s_i maximises the sum of pay-offs for that game. ■

Let k^j be the reputation of player *i* for all moves including the move on round *j*. The optimal response function for *i* at round *j* is to set r_i^j to maximize the sum of pay-offs for the game.

$$s_i^j = r_i^j : v_i(r_i^j) = \text{argmax} \sum_{j-1}^{n} v_i[k^j,\ s_i^j, r_{-i}(k^{j-1})]^j \tag{5.i}$$

where $k^j = k^j(k^j - 1, s_i^{j-1})$.

A strategy of only trading with players that have a good reputation might be adopted if players interact frequently. If the numbers are small however, then it may be optimal to trade with a player with a poor reputation if this player is the only source of a desirable good. If the numbers are large, then it may be difficult to get reliable information. In some cases it will pay players to give out information about the reputation of a rival in a strategic manner.

One interesting example of a reputation game that avoids these problems is where there are some large, or long-run players, and a number of smaller players. The games have been studied by Kreps and Wilson and Miligrom and Roberts (Fudenberg and Tirole, 1992, p. 369).

5.2.3 A reputation game with a long-run trader

The players in this game are a long-run trader 1, and a large number of customers or short-run players. Let a short run player be in where $m \in N - 1$. The long-run player is concerned with pay-offs in a repeated game without a termination date and wishes to establish a reputation that facilitates trade. One argument for the existence of a firm, for example, is that it acts as a long-run trader with a reputation and facilitates transactions. Similar arguments apply to brand name products.

The short-run player moves first. It makes a decision whether to enter into a transaction with the long-run player, such as buying a good, on the basis of this reputation. If the long-run trader were to move first, the optimum strategy of the short-run player would be to defect. Suppose the long-run player hands over the good. Since the short-run player has nothing to gain from reputation, its optimum strategy would be to not pay.

The short-run player has a best response that maps the reputation of the long-run player into an optimum response strategy

$$r_m : r(k_m, 1) \in s_m.$$

$k_{m,1}$, is the reputation that player $m \in N-1$ assigns to player 1. The reputation operator can be seen as a mapping of past performance into a probability that the long-run player will use some strategy s_1^o on the next round. Hence

$$k^j_{m,1} : (k^{j-1}_{m,1}, s_1^{j-1}) \to p^{qj}_{m,1}$$

where $p^{qj}_{m,1}$ is the probability assigned by m that player 1 uses strategy q on round $j, j = 1, \ldots$.

Assume that the long-run trader can be honest or cheat and that p^j_m is some function of the number of times the trader has been honest in the past. For the short-run player let s^c_m be trade and s^d_m not trade. For the trader s^c_1 is be honest and s^d_1 is cheat. The customers are uniform and each gets utility, a, from trading. For the purpose of the exercise assume that the trader must sell at a purchase price of c. Both players get zero from not trading. The trader can produce the good at a cost less than c and wishes to maximize its profits.

Suppose the trader is long lived and plays a number of rounds without a known termination date and the trades are sequential. From equation (5.i) the pay-off is maximized by choosing s_1 to satisfy

$$v(s_1) = \text{argmax}\, v_1((s^c_1, s^c_m), (s^d_1, s^c_m))^1 + ((s^c_1, s^c_m(p^2_m)),\ (s^d_1, s^c_m\,(p^2_m))^2 + \ldots$$

where the superscripted numbers are the rounds of play and m is a player taken at random from $N-1$.

The game is considerably simplified, and there is no loss in its qualitative properties, if we take the case where the probability is uniform. The long-run player will maximize its profit if the customer pays c and no good is provided. The constraint is that it must choose s_1 such that all players co-operate. If not the pay-off is zero. This gives

$$\max_s v_1 \text{ subject to } v_m(s^c_m) \geq v_m\,(s^d_m)$$

The lower bound for $p = p^*$ is where $v_m(s^c_m) = v_m(s^d_m)$. This requires the purchaser with utility a and purchase price c sets

$$p^*(a-c) - (1-p^*)c = 0.$$

This gives

$$p^* = c/a$$

For trade to be strictly worthwhile $a > c$ hence $p^* < 1$.

Assume that p is a one-to-one function and that $p \leq 1$ if player 1 has always co-operated. Let $\sum s_1^c$ be the number of co-operative moves in n games. It follows that for $a - c$ sufficiently high there is a unique equilibrium

$$\varphi = (s_1^*, s_m^c)$$

where

$$s_1^* = \sum s_1^c : p = p^* \text{ and } \sum s_1^c \leq n.$$

As $a - c$ increases the value of p^* decreases. This means that the optimum amount of cheating by the trader increases as the value of the good to the purchaser increases. In other words, the more sought after or vital the good, the less reliable the supply.

Consider a life saving drug, for example. Essentially the trader is capturing some of the difference between the price of the good and its value by setting an actual cost that depends on the probability of supply. The trader would be indifferent between doing this and increasing c until $p^* = c/a$ for $p = p$ max.

The pay-offs for this game are illustrated in Figure 5.2. v_1 is the total pay-offs to some sequence of n games. $\sum s_1^c$ is the numbers of games for which the trader has cooperated and is represented as points on the horizontal axis and pay-offs as points on the line $v_1 = \sum v_1^j$.

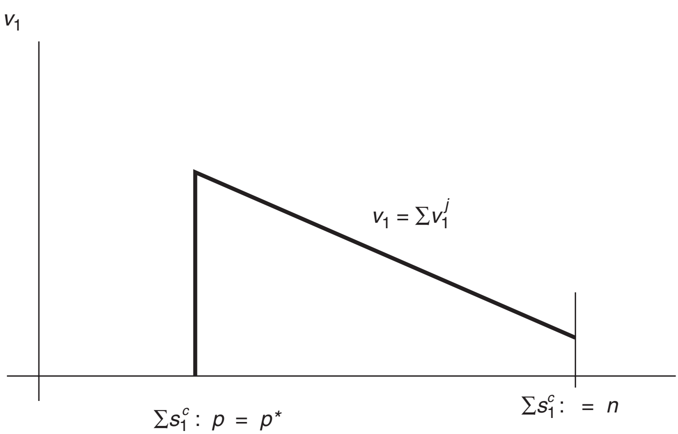


Figure 5.2 Pay-offs for the long run trader

For a specific example, assume that the probability a customer assigns to the trader being honest is the proportion of honest transactions in the past. Then

$$p = k(s_1^c) = \sum s_1^c / n.$$

Pay-offs are given in Figure 5.3.

In this case the customer will trade if $p > \frac{2}{3}$. Hence the trader uses the mixed strategy

$$s_1^* = \left(\frac{2}{3}, \frac{1}{3}\right)$$

at each round of the game.

If we make the assumption that there is always some uncertainty in the game then $p < 1$ for all $\sum s_1^c$. It will be noted that,

$$p \to 1 \text{ as } (a - c) \to 0.$$

This means that many trades that are Pareto superior will not be made. In other words, even with an honest long-run trader, uncertainty may restrict trade to exchanges in which gains relative to potential utility losses are large.

Similarly, governments may not be able to make welfare improving moves unless gains relative to costs are large. It follows that one of the strategies of a welfare maximizing government may be to create institutions that reduce uncertainty and increase the potential for welfare improvement.

In what follows the problem of reputation will be ignored and it will be assumed that protection agencies keep their promises. This still leaves the interesting question of whether individuals with property rights would buy these services and how the market will behave.

	s_1^c	s_1^d
s_m^c	1, 1	–2, 2
s_m^d	0, 0	0, 0

Figure 5.3 Pay-offs in the one-shot trading game

5.3 A market for protection with benefits for non-contributors

Assume that there is a security company, individuals with holdings of material goods, and pirates who steal. The security company punishes offenders and pirates know that some individuals have security, but the pirates do not have perfect information. If security is purchased by some individual *i* the probability of being caught increases for the pirates and this will deter attempts to steal. Hence individual *j* benefits from the purchase of security by individual *i*. It follows that the optimum response for *j* is to purchase less security than it would if it did not benefit from *i*'s action. Protection agencies might try to avoid this by identifying those that it protects. One response to this would be a secondary market in whatever identity stickers are used by the agency. An example is a 'beware of the dog' sign on houses without a dog. It follows that lack of information will mean that some members of the population will benefit from security without paying the costs.

The case where the purchase of security benefits some non-purchasers is a straightforward problem of strategic interaction producing positive pay-offs for a response of do not contribute. This is often referred to as a free-rider problem.

Consider the case where the protection agency protects the rights of all its members. If this protection is extended to everyone in the society then the structure is the basic prisoner's dilemma and the discussion in section 2.7 applies.

Here is a more interesting possibility. Assume that the protection agency only protects those who pay. As the number of paying customers increases existing members are made better off because the agency gets economies of scale, disputes are lessened because it is easier to arbitrate between members and so on. Non-members do not benefit when the membership of the agency is small. As the membership increases non-members begin to benefit because pirates are not certain who is protected.

The pay-off for each member is

$$v_i = f(mc) - c > f(0)$$

since an individual benefits from joining even if no-one else joins. m is the total number of members. The pay-off to a non-member is $h[(m-1)c]$.

$$f(mc) - c > h[(m-1)c]$$

for m small, since the probability that any individual is a member is small and there is little deterrence for pirates.

$$h[(m-1)c] > f(mc) - c$$

for m large.

This gives a variant of the problem in section 2.5. Let join be s_i^c, and not join be s_i^d. The pay-offs are set out in Figure 5.4. The functions are illustrated in Figure 5.5.

Note that the pay-offs will depend on m. Hence the mixed strategy analysis used for the hawk – dove game with constant pay-offs used in Chapter 3 will not work because we do not have constant values to calculate the probabilities. We already know that players cannot have a pure strategy in this game. Since all players are equal they must all have the same mixed strategy. The proof uses the technique for calculating equilibrium that was used for the war of attrition in Chapter 3. Let n be the number of players.

Proposition: The solution for each player is $\varphi = p^* = (m^*/n, 1 - m^*/n)$ where p^* is a probability vector.

Proof: Suppose that players use a probability density function p'. The pay-off for any pure strategy against the mixed strategy p must be equal. Therefore

$$E[0] = \int_0^1 v(0)p(x)'dx = E[1] = \int_0^1 v(1)p(x)'dx$$

where $v(0)$ and $v(1)$ are the pay-offs from contributing with a probability 0 and 1 respectively. Letting $a = v(0)p(x)'dx$ and $b = v(1)p(x)'dx$ we can write the above as

$$\int_0^u a + \int_{u'}^1 a + \int_u^{u'} a = \int_0^u b + \int_u^1 b + \int_u^1 b + \int_u^{u'} b$$

where $u = y: \; yn = m^* - \varepsilon, u' = y: \; yn = m^* + \varepsilon$.

	s_{-i}^c	s_{-i}^d
s_i^c	$f(mc) - c$	$f(1) - c$
s_i^d	$h[(m-1)c]$	0

Figure 5.4 Pay-offs in the security market

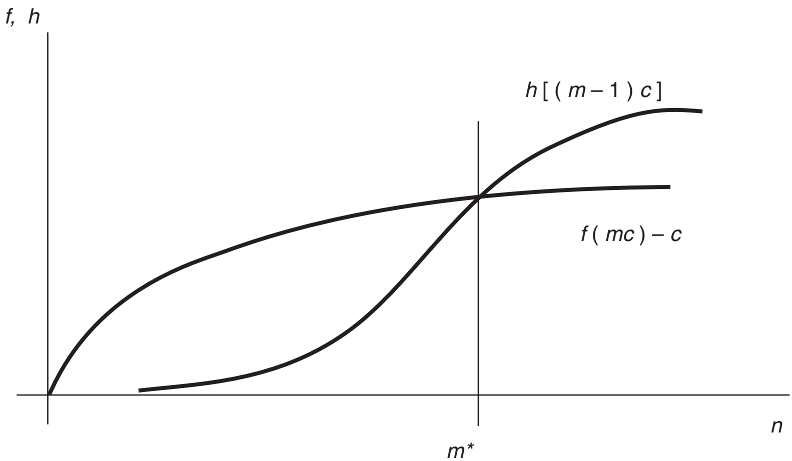


Figure 5.5 Pay-offs for the security game

$v(0) = h[(yn - 1)c]$ and $v(1) = f(yn) - c$. This means that

$$\int_u^{u'} a \rightarrow \int_u^{u'} b \text{ as } \varepsilon \rightarrow 0 \text{ since } v(0) \rightarrow v(1)$$

around $yn = m^*$. Therefore

$$\int_0^u a + \int_{u'}^1 a = \int_0^u b + \int_{u'}^1 b$$

Since $h[(yn - 1)c]$ and $f(yn) - c$ are in general not equal for all $n > 0$, this condition can only be satisfied in general if the probability density function $p' = 0$ in the stated intervals. This means that probability

$$\{0 \leq y \leq y : \ yn = m^* - \varepsilon\}$$

and the probability

$$\{y : \ yn = m^* + \varepsilon \leq y \leq 1\}$$

must go to zero for $\varepsilon \to 0$. Therefore

$$y = y: \; yn = m^* \text{ or } y = m^*/n.$$

To check that this is a Nash equilibrium consider the optimum response of i to all other players using $s_{-i} = p^* = (m^*/n, 1 - m^*/n)$. We have $v_i(p^*, p^*) \geq v_i(0), \; v_i(1)$ by definition. $v_i(p^*, p^*) \geq v_i(p, p^*)$ since $s_i = (p, p^*)$ gives a pay-off in the straight line $\alpha v_i(0) + (1 - \alpha)v_i(1)$. ■

This tells us is that all players contribute to the scheme with some positive probability and that the average level of protection is a proportion $y = m^*/n$ which is less than full protection. Note that y decreases as m^* decreases and as n increases, It will also be noted that the outcome is not on the Pareto frontier since everyone could be made better off with an average level of protection greater than m^*/n.

Remark: The essential difference between this game and the conditional co-operation games analysed in Chapter 2 is the nature of the pay-off structure around $m = 0$. One implication for the free-rider problem is that, if it is worthwhile for each player to co-operate if no-one else co-operates, then an equilibrium at some level will result.

5.4 The basic enforcement problem in a Nozickean state of nature

Nozick assumes that protection agencies would provide security in a manner similar to suppliers of other consumer goods and that the normal efficiency results for markets could be expected. Is this plausible? One of the characteristics of security is that there is an externally imposed penalty for not consuming the good. Nozick's assumption will be investigated by examining the strategies of a price setting agency. It is asked whether it is possible for the agencies to manipulate the price, and how consumer demand reacts to changes in wealth. I also briefly consider a price taking agency

The market for protection would plausibly give rise to a single agency with a natural monopoly because protection has increasing economies of scale. Nozick says that this would be the case, for example, because a larger protection agency will always have an advantage over a smaller agency and each individual will wish to join the most effective service (Nozick, 1974, pp. 15–17). Assume this is the case.

It is easy to show that a monopoly profit setting agency that is free to set either prices or the probability of enforcement can extract much of the available material wealth from the population. Consider the price setting agency.

5.4.1 The price setting agency

The protection agency is free to set any price for enforcement and there are n individuals in the system. Individual i has wealth w_i. The subscripts $1, \dots, n$ index the individuals in an ordered sequence of increasing wealth.

$$w_i + 1 - w_i = \varepsilon.$$

In other words wealth increases uniformly by some increment ε. It is assumed that the agency knows the wealth of each individual. The individuals have two strategies. Buying protection is strategy s_i^p and not buying is s_i^{np}.

Assume that the agency always enforces the claims of its clients. Since there are no rules there is no basis for Nozick's assumption that the agency will adjudicate between just and unjust claims (Nozick, 1974, p. 13). An agency that enforces all claims will be more attractive to clients than one that only enforces just claims.[3]

A player with protection beats one without with a probability p. A conflict between individuals with protection results in no loss to each. For simplicity the probability of enforcement and the proportion of wealth lost is set at one.

If player i gets into a dispute with another player at random, then the pay-off is the average wealth

$$a = \sum w_i/n.$$

Let c_i be the price charged to individual i for enforcement. The pay-offs for i are given in Figure 5.5.

Suppose the protection agency has information on the wealth of every individual

Proposition 5.a: The agency t maximizes its profit by setting the following schedule of prices.

$$c_i = \begin{vmatrix} w_i - \eta \ \textit{if} \ w_i \leq a \\ w_i - x\varepsilon - \eta \ \textit{if} \ w_i > a \end{vmatrix}$$

where $x = (i - k)/2$ and $\eta, \epsilon, > 0$ and k is the index on the individual such that $w_k = a$.

Proof: At price $w_i - \eta$, all individuals with wealth $w_i \leq a$ have a dominant strategy s_i^p. The proportion of the population that buy at this price is $\frac{1}{2}$. An individual with wealth greater than a is now faced with the following pay-offs.

$$\begin{aligned} v_i(s_i^p) &= (w_i - c_i)/2 + (w_i + a - c_i)/2 \\ v_i(s_i^{np}) &= (w_i)/2 \end{aligned}$$

It pays to buy protection provided $v_i(s_i^p) > v_i(s_i^{np})$. This is true if

$$a/2 - c_i + (w_i)/2 > 0$$

Substituting $w_i = a + (i - k)\varepsilon$, and $c_i = w_i - x\varepsilon$, gives

$$x\varepsilon - (i - k)\varepsilon/2 = 0$$

At this point all individuals with $w_i > a$ are indifferent between buying and not buying protection. Hence for $c_i = w_i - x\varepsilon - \eta$ everyone buys the service.

This gives the agency the pay-off to

$$v = \sum_1^n w_i - n\eta - (n - k)x\varepsilon$$

which tends to $\sum w_i$ as $\varepsilon, \eta \to 0$. It follows immediately that c_i is the profit maximizing price schedule.

Hence the agency gets almost the total wealth of the population. ■

This analysis could easily be generalized by dropping the assumption that wealth increases by some fixed increment ε. Assume, for example, that $w : i \to \boldsymbol{R}$ is approximated by a continuous function which increases with i and there is a small interval Δi between individuals. Let the proportion of the total population who buy at the initial price be α. In this case the problem is to set x such that $a - c_i - \alpha(w_i - a) = 0$ for $w_i = w_{k+i-1} \cdot d(w_{k+i-1})\Delta i$.

5.4.2 Price taking and other agencies

It is fairly simple to analyse other forms of agency such as price taking agencies that allow buyers to bid for services. These games will generally exhibit the characteristic, mentioned in the introduction, that the agency can capture the surplus. For example, it might be considered that the agency controls the quality of enforcement. What is also characteristic of these cases is that the surplus tends towards the entire wealth. This is because the outcome without protection is the loss of most material wealth to those with protection.

6
Evolutionary Stable Strategies

6.1 Introduction

The problem which is considered in this chapter is that of how the equilibria that have been calculated for games of security and struggles over property might emerge in a system of interaction where individuals follow some local optimization strategy, such as trial and error. This is important for two reasons. The first is that the models set out in the previous chapters depend on the assumption that agents can optimize globally and calculate their optimum strategies. Although these models are complete as descriptions of the equilibria that would be attained by optimizing agents, they do not address all the issues that are of concern in discussions of order and security. Where the games are straightforward, and the pay-offs sufficiently high, it might be expected that players would calculate the equilibria directly. Where things are more complicated it would take the players considerable effort and technical skill to calculate the optimum strategies. The second is that the assumption of global optimization may not be a completely adequate model of decision makers in the context of much of the discussion of spontaneous order. The implicit model that underlies much of this discussion seems to be of some sort of unplanned outcome. In this case it is of interest to investigate the consequences of treating individuals as following a local optimization rule. It might be the case, for example, that individuals use a trial and error approach.

The outcomes of trial and error type approaches are best dealt with in game theory under the heading of evolutionary stability. This idea is an extension of the notion of stable strategies previously discussed.

This chapter considers the idea of evolutionary stable strategies in general. It is less oriented to specific problems than previous chapters. This general treatment is warranted because of the importance of evolutionary stability to the idea that co-operation might emerge from interactions in large groups. The notion of evolution may also include the evolution of cultural traits that may be conducive to co-operative outcomes. An example would be the emergence of a co-operative or a retaliatory strategy in a mixed population of players.[1]

Evolutionary stability is also important for the problem of choosing between candidates when several possible equilibria exist. It has so far been suggested that an equilibrium should only be considered acceptable if it exhibits some sort of stability properties.[2] So far stability has been explored using an updating process that depended on assuming that all other players' previous moves remained fixed. The concept of evolutionary stability provides a more justifiable approach.

I will concentrate particularly on the case of games which tend to exhibit cycles or spirals even though they may be stable under some definitions.[3] This illustrates some of the consequences of allowing interactions to be dynamic. It also demonstrates that the existence of a stable equilibrium is not sufficient to ensure stable behaviour in the sense of behaviour that converges rapidly to a fixed set of strategies. Such behaviour might oscillate over a wide range.

6.2 Evolutionary stable strategies

The main idea behind evolutionary stability is that a Nash equilibrium in a static game would be stable if it could be explained as the outcome of the actions of players following some process that starts at random and then only changes in a direction that improves the outcome. Hence evolution is analogous to some sort of trial and error process under incomplete information.

The parallels between evolutionary stability and Darwinian theory are not accidental. The major early application of evolutionary stable strategies is Smith's *Evolution and the Theory of Games*, introduced in Chapter 3, where the war of attrition was analysed. Animals do not calculate in the sense of consciously solving optimization problems. At the same time they compete, or play a game, over resources. The blind process of evolution should lead to some sort of local optimal strategy for each species in this game. Roughly, it is argued that this results from the more successful strategies giving greater pay-offs. This increases the ability of the carriers of these strategies to breed and hence increases the proportion of these strategies in the next generation.

The parallel with strategies in games played by calculating agents is that there will be a tendency for more successful strategies to be repeated or copied and the less successful to be dropped. This process provides an analogue to breeding or reproduction.

The analysis of struggles over material goods as a war of attrition in Chapter 3 serves as an illustration. The strategies of the other players may not be known or information may be partial in such a game. When the game is played many times there is some pressure to eliminate strategies that are sub-optimal, or to eliminate players that adopt a sub-optimal strategy. Since the outcome for the war of attrition is an evolutionary stable strategy, there is an argument that players would converge on this strategy through repeated plays of the game.

One of the characteristics of real evolution is that the strategies of all players are evolving simultaneously and it may be the case that the environment is evolving as well. Thus, in the evolution of grass eating animals the grass eating animals themselves change as do their predators and the characteristics of the grass. In struggles between settlers and pirates the strategies of each group change as do the technologies of predation and defence, the technologies of production and some environmental factors, such as forest cover.

An analysis of evolutionary stability can be set up in a number of ways. For simplicity I shall concentrate on a standard approach in the literature known as replicator dynamics.[4]

6.2.1 Replicator dynamics (Smith 1982, Weibull 1996, Samuelson, 1997)

The analysis is restricted to a single class of players in a static game. By a static game is meant something like the prisoner's dilemma or the hawk–dove game. This game is played repeatedly but under the same conditions. It might be thought of as a case where the game is played in a large population and the players do not repeat their interactions sufficiently to use strategies based on retaliation. Repeated interaction games could also be subjected to an analysis in terms of evolutionary stability.

Consider the situation where the players are only allowed to play their pure strategies. In a population of players different pure strategies might be played by different players. This would correspond to a mixed strategy.

Strategies are updated or reproduced continuously to give a smooth process of change. Strategies are reproduced or changed according to some function of their success or fitness. This updating or reproduction process is treated as an analogue of the birth–death process. Since birth and death takes place, the total population will change over time. This might be considered as an analogue to the total population playing the game and revising its strategies, although this analogue may be questionable. To avoid this problem I shall follow the literature and write in terms of the birth and death process.

A pure strategy is written s^k and the number of individuals using s^k at time t is $m^k(t) \geq 0$. It is important to remember that everything that is not a constant changes through time. Notation such as $m(t)$ will usually be abbreviated to m once the point is established. Let K be the set of pure strategies. Hence for $k \in K$ the number of players using pure strategies at time t is

$$\sum_{k \in k} m^k = m(t)$$

The proportion of the population using strategy s^k at time t is

$$m^k/m = x^k$$

This gives a population state which is the proportions using the available pure strategies. Let

$$x = (x^1, x^2, \dots x^r) \text{ where } \sum_{k=1}^{r} x^k = 1$$

Hence x can be treated in the same way as a probability distribution over mixed strategies.

The pay-off to i from a pure strategy s^k, if the population state is x, is $v_i(s^k, x)$. The average pay-off is for an individual who plays s^k with probability x^k against a population with distribution x. So

$$v_i\,(x, x) = \sum x^k v_i(s^k, x)$$

Since the process of replication takes place continuously the number of individuals programmed to use pure strategy s^k will change continuously. What we need is a mapping from the success of s^k to the proportion of s^k users in the population at each instant in time.

Assume that the rate of change, or the birth and death process, at time t for players programmed to use strategy s^k is given by the function

$$a + v_i(s^k, x) - b \tag{6.i}$$

where a and b are constants and b is the death rate. We do not have to worry about the values for the constants a and b since they disappear in the calculation.

The change in the number of players in the population using strategy s^k is dm^k/dt. A time derivative is usually written $\dot{f}$.

Since expression (6.i) is the birth and death process the change in the total number of players using s^k at any instant of time is simply the rate multiplied by that proportion. Hence

$$\dot{m}^k = [a + v_i(s^k, x) - b]m^k$$

6.2.3 Analysis

The previous definitions can be put together to calculate the dynamics of the population shares. What is needed is the change in x^k. Since $m^k = x^k m$

$$\dot{m}^k = \dot{x}^k m + x^k \dot{m}$$

From the fact that the growth rate of m depends on the average fitness of the population and is given by $[a + v_i(x, x) - b]$

$$\dot{x}^k m = [a + v_i(s_i^k, x) - b]m^k - [a + v_i(x, x) - b]x^k m$$

Since $m^k = x^k m$ subtraction gives

$$\dot{x}^k = [v_i(s_i^k, x) - v_i(x, x)]x^k \tag{6.ii}$$

This says that the rate of increase or decrease in the proportion of the population playing strategy s^k depends on the difference between the pay-off for s^k and the pay-off for the average strategy. If s^k gives a better than average pay-off, the proportion of s^k players would increase. If it is worse than average then the proportion of s^k players will decrease.

6.3 Applications of replicator dynamics to hawk–dove games

The two static games that are of most interest are the prisoner's dilemma and the chicken game. It is almost immediate that the prisoner's dilemma will have an evolutionary stable strategy in replicator dynamics

$$\varphi = (s_i^d, s_{-i}^d)$$

since $v_i(s_i^c, x) < v_i(x, x)$ in equation (6.ii) and the proportion of players using s^c must continuously decline. The only exception is if the starting point is $x: \; x^c = 1$ and $s = s^c$ for all players. In this case $dx^c/dt = 0$ and there is no change in the replicator. Hence the equation remains constant.

The fact that $dx^c/dt = 0$ gives a trivial equilibrium in the replicator dynamics does not mean that $\varphi = (s_i^c, s_{-i}^c)$ is an evolutionary stable strategy in a more general sense. One of the characteristics of the replicator equation is that strategies that are not being played cannot be introduced. If a more general concept of evolutionary stability that allowed for perturbations were used we would get $dx^c/dt < 0$ and $x^c \to 0$.

Consider the hawk–dove game given in Figure 3.1, reproduced here for convenience.

What gives the game (a) the hawk–dove structure is $c_1 > a_1$ and $b_1 > d_1$. As previously discussed, provided this structure is preserved, the characteristics of the game remain unchanged. The matrix in (a) can be replaced by (b) without changing the nature of the game provided that $z_1, z_1 > 0$.

This gives the rate of change in the proportion of players using s^h as

$$\dot{x}^h = [v_i(s_i^h, x) - v_i(x, x)]x^h$$

The values are,

$$v_i(s_i^h, x) = x^d z_2 \text{ and } v_i(x, x) = x^d x^h z_1 + x^d x^h z_2$$

Since $x^h = 1 - x^d$ we have

$$\dot{x}^h = [x^d z_2 - x^h z_1]x^d x^h$$

x^h will increase if $[x^d z_2 - x^h z_1] > 0$ and decrease if $[x^d z_2 - x^h z_1] < 0$. For $[x^d z_2 - x^h z_1] = 0$. Hence $(1 - x^h)z_2 - x^h z_1 = 0$ and

$$x^h = z_2/(z_1 + z_2)$$

Hence x^h increases if $x^h < z_2/(z_1 + z_2)$ and decreases if $x^h > z_2/(z_1 + z_2)$. This is set out in Figure 6.2.

What the analysis says is that the proportion of the population playing the hawk strategy and the proportion playing the dove strategy tends towards the Nash equilibrium in the long run. Hence the mixed strategy is evolutionary stable.

These proportions never reach the Nash equilibrium for the static game, however, as they can only tend towards it as t goes to infinity. This is because as

$$[x^d z_2 - x^h z_1] \to 0$$

the rate of change becomes slower and slower. This makes sense. It says that as the differences between switching strategies become less and less the incentive to switch also declines.

Where such numbers are meaningful, this analysis can be repeated using numerical values in Figure 6.1(a) to get values for the proportions of hawks and doves in the population.

6.4 Cyclic evolutionary dynamics in two move asymmetric games

Smith also suggests that some games might not converge to an equilibrium but may have cyclical dynamics. What this would require is either that no Nash equilibrium

	s_2^d	s_2^h
s_1^d	a_1	b_1
s_1^h	c_1	d_1

(a)

	s_2^d	s_2^h
s_1^d	0	z_1
s_1^h	z_2	0

(b)

Figure 6.1 The hawk–dove game

exists or that the Nash equilibrium is not evolutionary stable. Smith gives an example of a game with two strategies and unequal pay-offs (1982, p. 201). So far many of the games that have been analysed have been symmetric. There is no reason that players should get equal pay-offs in general, however. Smith's analysis uses the same set of equations developed in equation (6.ii) above and gives either a cyclical dynamic or spiral path that converges to an equilibrium.

The pay-off matrix for this game is set out in Figure 6.3. The pay-offs have been normalized to 0, 1 in the same manner as the previous game. The strategies are labelled s^a and s^b.

This game has the same pay-offs as a hawk–dove for player 1. For player 2 the best strategy is to mimic player 1.

Equation (6.ii) gives

$$v_i(s_i^h, x) = y^b$$

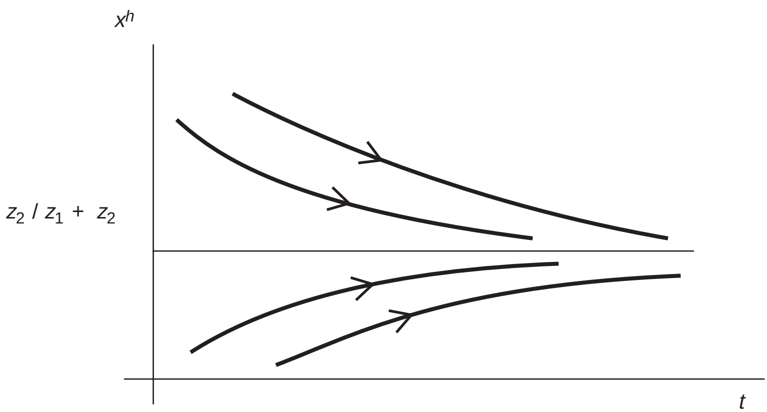


Figure 6.2 Evolutionary stable strategies in the hawk–dove game

	s^a	s^b
s^a	0, 1	1, 0
s^b	1, 0	0, 1

Figure 6.3 Pay-offs for the asymmetric two player game

and

$$v_i(x, x) = x^a y^b + x^b y^a$$

Using the similar calculation for player 2 this gives the dynamics as follows

$$\dot{x}^a = [1 - x^a + 2x^a y^a - 2y^a]x^a$$
$$\dot{y}^a = [2x^a - 2x^a y^a + y^a - 1]y^a$$

The problem is to find the signs for dx/dt and dy/dt. The stationary points for this system of equations are given where $x^a = y^a = 0.5$ and $x^a = y^a = 0$. It is obvious that the second stationary point is unstable. Hence it is not interesting for this exercise.

A standard way of analysing the equilibria around a stable point is to look at the linear system around this point. Let $x + 0.5 = x^a$ and $y + 0.5 = y^a$. Then the system becomes

$$\dot{x} = -y/3 - 4x^2y/3$$
$$\dot{y} = x/3 - 4y^2x/3$$

Consider the non-linear terms

$$h = 4x^2y/3 \text{ and } g = 4y^2x/3$$

If $h/x \to 0$ as $x \to 0$ and $g/y \to 0$ as $x \to 0$ then the stability properties of the system are given by the behaviour of the linear terms $-y/3$ and $x/3$. A solution to the linear system is

$$x = \cos(t/3) \text{ and y} = \sin(t/3)$$

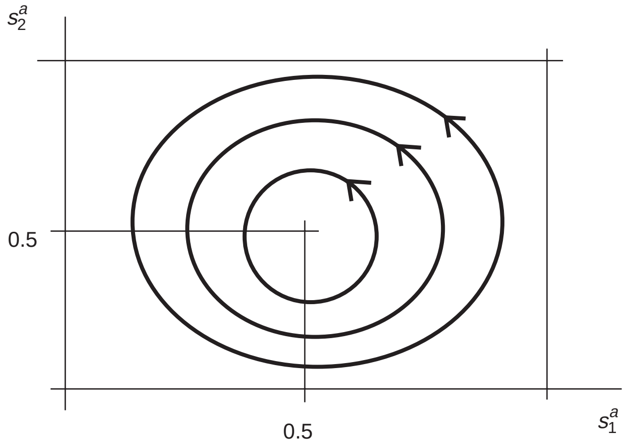


Figure 6.4 Evolutionary dynamics of the asymmetric game

omitting arbitrary constants. This gives a circle, or for the complete system, a series of closed loops in the two dimensional space x^a, y^a. This is illustrated in Figure 6.4.

This dynamic is particulalrly interesting because such loops require the special property that all the real parts of the roots of the charactereistic equation for this system be zero. It is usually considered that such dynamics are unstable and cannot be maintained and that the only stable dynamic properties are for systems to converge or diverge. It can be shown that this is also the case for the system analysed here. For slight variations these closed loops form a spiral towards the equilibrium point at (0.5, 0.5) (Smith, 1982, p. 201).[5]

6.5 Cycles in three move symmetric games

The case where there are three strategies might also be expected to produce cyclic dynamics even when the pay-offs are the same for both players. Consider a situation where each of the strategies dominates one of the others. Such three sided contests often arise in the design of constitutions with the division of power between an executive, legislative and judiciary, for example. It is also possible to imagine a struggle in which the players have three strategies such as become a land power, become a defensive power, become a sea power. In this case, sea power defeats defensive power defeats land power defeats sea power. Alternatively, the choice may be between different technologies.

	s^r	s^s	s^p
s^r	1	2 + *a*	0
s^s	0	1	2 + *a*
s^p	2 + *a*	0	1

Figure 6.5 Pay-offs for the rock, scissors-paper-game

Three player games with this characteristic are generically know as rock-scissors-paper games. Smith (1982, pp. 19–20) suggests that these games may cycle indefinitely or have a stable equilibrium depending on the nature of the pay-offs.

Consider a game with the pay-offs in Figure 6.5 where s^r, s^s, s^p have the obvious meanings. This set up and solution follows Weibull (1996, p. 77). *a* is a constant term introduced to allow the characteristics of the game to be analysed with different payouts.[6]

Equation (6.ii) gives the following dynamics.

$$\begin{aligned} \dot{x}^r &= [x^r + (2+a)x^s - v_i(x,\ x)]x^r \\ \dot{x}^s &= [x^s + (2+a)x^p - v_i(x,\ x)]x^s \\ \dot{x}^p &= [x^p + (2+a)x^r - v_i(x,\ x)]x^p \end{aligned} \tag{6.iii}$$

To find out when dx/dt is greater or less than zero use the fact that dx/dt has the same sign as $dlnx/dt$. Let $f = ln(x^r x^S x^p)$. Then

$$\dot{f} = \dot{x}^r x^s x^p / x^r + x^r \dot{x}^s x^p / x^s + x^r x^s \dot{x}^p / x^p$$

Substituting from equation (6.iii)

$$\dot{f} = 3 + a - 3v_i(x, x)$$

Consider $v_i(x, x)$. This is the expected pay-off from playing $s = (x^r, x^s, x^p)$ against itself. This is given by calculating

$$v_i(x, x) = E[x^r] + E[x^s] + E[x^p]$$

From the matrix in Figure 6.5 we have

$$E[x^r] = (x^r)^2 + (2 + a)x^r x^s$$

and so on. Note that $x^r + x^5 + x^p = 1$ and

$$(x^r + x^s + x^p)^2 = \| x \|^2 + 2(x^r x^s + x^s x^p + x^p x^r)$$

Summing gives

$$v_i(x, x) = 1 + a(x^r x^s + x^s x^p + x^p x^r) = 1 + a(1- \| x \|^2)/2$$

Hence

$$\dot{f} = a(3 \| x \|^2 - 1)/2$$

Note that $\| x \|^2$ will be at a maximum if all the population plays any one strategy. If more than one strategy is used $\| x \|^2 \leq 1$. For $x^r = x^s = x^p$

$$\| x \|^2 = (x^r)^2 + (x^s)^2 + (x^p)^2 = \frac{1}{3}$$

and is at a minimum.

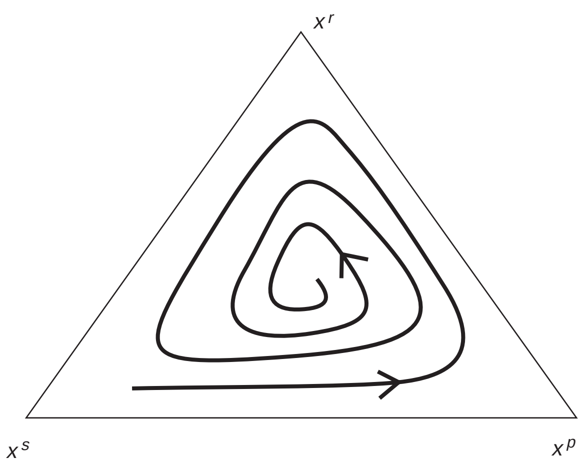


Figure 6.6 Inward spiral in rock-scissors-paper

This is rather neat. It gives three trajectories.

[1] $a = 0$. Then $df/dt = 0$ and $x^r x^s x^p$ is constant. Hence the system cycles.

[2] $a > 0$. Then $df/dt > 0$ and $x^r x^s x^p$ increase. Since $x^r x^s x^p$ is the minimum for $x^i = 0, i = r, s, p$, the system spirals inward.

[3] $a < 0$. Then $df/dt < 0$ and $x^r x^s x^p$ decreases. Hence the system spirals outward.

A trajectory for the game when $a > 0$ is illustrated in Figure 6.6. These paths are unique. In other words, for any two starting points not on the same path, their trajectories never cross. In this case the system has a stable equilibrium although the equilibrium point $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ is never reached.

7
Collective Decisions and the Core – Co-operative Games

7.1 Introduction

The problems of collective decision-making are now considered. It is assumed that individuals have solved the problems of non-co-operation and individual choice by setting up some mechanism which allows them to make collective choices about rules and outcomes, and to enforce those choices. It will be recalled from the analysis of collective optimality in section 1.8 that this means, in principle, that individuals could agree to move to an outcome on the Pareto frontier. This does not mean that individual choice mechanisms are precluded. The collective choice over production and distribution, for example, might be to have a central planner direct the process, or it might be to set up rules of property and exchange and accept whatever outcome results from individual choices. This analysis is concerned with the criteria that might be used to make these decisions.

As part of this analysis some of the cases in which individuals might wish to co-operate in the collective provision of goods are analysed. Such co-operation might be worthwhile where rules that protect life and property and enforce contracts are not sufficient to eliminate strategic considerations. In these cases, a move that increases the total pay-off may not be the optimum strategy for every individual player. Examples were encountered previously in the discussion of security and the market for protection. Other goods with similar characteristics are pollution, public health and basic research.

A characteristic of the analysis of collective choice is that it tends to be abstract. This is because the focus is on the properties of choice mechanisms in general, rather than the analysis of mechanisms under specific institutional constraints. Since it is the broader applications of the theory that are most interesting, I shall concentrate on these.

In order to consider problems of collective choice it is necessary to extend the analysis to co-operative game theory. This is done primarily through a study of the theory of the core. This covers some of the material from the earlier chapters in a different manner.

7.2 The core and efficient outcomes: transferable utilities

The process of co-operative decision making with a number of players can be thought of in terms of a number of offers or outcomes supported by different coalitions. The problem for any one player is to choose the coalitions it wants to join in order to get the specified outcome. The problem faced in an analysis of this is that there is a very large number of potential coalitions that might be formed, and each of these will affect the pay-off to all players. For n players the number of possibilities is $2^n - 1$. It follows that it may not be practical to look at the strategies for bargaining for every individual in, and between, every possible coalition. One way out of this difficulty is to get a theory that makes the pay-off for every coalition depend on its power in a bargaining game. This is done using the theory of the core.

The theory of the core is set out by starting with the simplest case. This is where utilities are transferable. This means that utilities are measurable in some good, say money, which can be transferred among the players. In this case it is only the total value of each coalition which is of concern. This case is the most easy to analyse because the only problem to be solved is that of maximizing the pay-off for each coalition. Since utilities are transferable, coalitions are then free to make whatever internal transfers they wish.

An advantage of transferable utilities is that it allows collective decisions in cases where a move that increases the total pay-off may not benefit every individual player. It was assumed, when discussing the Hobbesian problem, that a move from a state of nature to a structure of authority could make all players better off. This is a special case, however. Consider a choice between two points on the Pareto frontier such as a' and a'' in Figure 1.4. A move from a' to a'' makes at least one individual worse off. Such a move may be acceptable to all players, however, if transfer payments are possible.

The condition for an agreement to be in the core is that it makes each player at least as well off as it would be in any alternative agreement. For example, the argument that all players would be better off with an institution that gets them out of the Hobbesian problem means that the arrangement is a core solution. It will be observed that the core of a game has the appealing property that it is stable. Since the core makes all players better off than they could be in any alternative arrangement, no one would have an incentive to defect.

This condition can be set out formally by letting N be the all player coalition and B be a sub-coalition of N. $v(B)$ is the value of B and $v(N)$ is the value of the all player coalition. Then a pay-off a_i is in the core of a game if and only if

$$\sum_{i=1}^{n} a_i \leq v(N)$$

$$\sum a_i \geq v(B) \text{ for all } i \in B \text{ and all } B \subset N$$

The first condition is a feasibility condition. It says that the sum of the pay-offs to all players cannot exceed the total pay-offs available. For $\sum a_i \leq v(N)$, say $\sum a_i = v(N) - \varepsilon$, the optimal strategy would be for players to agree to distribute ε to make some, or all better off. This would give $\sum a_i = v(N)$ and the core would be on the Pareto frontier.

What the second condition says is that the pay-off in the core must give a player at least as much as it is possible to get in any possible sub-coalition $B \subset N$.

This immediately raises the question of how the alternative arrangements that give $v(B)$ are to be assessed. To see the problem consider the example of individuals trying to get out of the Hobbesian state of nature in section 2.6. Let the strategies for i be to join, or not join, the grand coalition $N - 1$. If all other players agree to co-operate and i does not, then i is in a state of war with the grand coalition. This gives a pay-off

$$v(B) = v(i) = v_i(s_i^d, s_{N-i}^d) \leq v_i(s_i^d, s_{-i}^d)$$

in the previous state of nature since i will always lose against the coalition $N - i$. If the alternative is that z does not enforce and no players co-operate, then

$$v(i) = v_i(s_i^d, s_{-i}^d)$$

It follows that the values of the pay-offs to any sub-coalition may depend on the rule used for assessing the strategies of other coalitions.

The usual way to solve this problem is the method proposed by von Neumann and Morgenstern. $v(B)$ is taken to be the best that the members of B can do acting together against the strategy of the coalition of all other players that minimizes $v(B)$. In other words $v(B)$ is no higher than the strategy that maximizes the minimum return against a malevolent opponent.

One concern with this solution is whether malevolence is a credible assumption. This problem was previously discussed in terms of sub-game perfect strategies in Chapters 1 and 2. In the case above $v(i) \leq v_i(s_i^d, s_{-i}^d)$ is plausible. Now imagine that $n = 261$ million and 260 million individuals form a coalition $N - i$ to provide defence against external threat or to reduce air pollution and i refuses. Would it be credible for $N - i$ to threaten to remain undefended or to continue suffering lung damage? Alternatively, maybe $N - i$ can ensure that i does not get any more than the pay-off $v_i(s_i^d, s_{-i}^d)$ by expelling i. Would the rules of the game allow this?[1]

It should also be noted that the necessary condition for a core to exist is that the coalition of players can get more than the players could get acting in any sub-coalition. If not, there would be no point in combining to make a collective choice.

Such a game is superadditive. This means that the pay-offs for any two coalitions add to an amount less than or equal to the pay-off for the combined coalition.

$$v(B_i) + v(B_j) \leq v(B_i \cup B_j)$$

for $B_i \cap B_j = \phi$.

An example of a superadditive game might be a boss with capital and some workers with labour power where the boss and the workers can produce more by co-operating than each can produce alone. Another example is where there are economies of scale in production.

The superadditivity condition $\sum v(B) \leq v(N)$ is specified for $B_1, \ldots, B_n$ a partition of N. It is important to note that superadditivity is necessary but not sufficient for $\sum a_i \geq v(B)$ for all $i \in B$ where $B \subset N$.

To see why superadditivity is not sufficient for an outcome to be in the core, imagine three players (i, j, k). Let

$$v(B_1) + v(B_2) = v(i, j) + v(k) \leq v(N)$$

and

$$v(B_3) + v(B_4) = v(i) + v(j, k) \leq v(N)$$

It is possible that the coalition (i, j) may give i amount b_i and the coalition (j, k) may give j amount b_j and k amount b_k with

$$b_i + b_j + b_k \geq v(N)$$

This seems odd. The puzzle is resolved if it is noted that the coalitions $B_1 = (i, j)$ and $B_3 = (j, k)$ do not both enter into the inequality $\sum v(B) \leq v(N)$ where $B_1, \ldots, B_n$ is a partition of N.

The core requires that players must be able to get more than they can get in any coalition. This is not the same as the maximum each player can get in a coalition that partitions the players.

Given the assumption of transferable utilities the problem of the existence of the core and the pay-offs that are in the core can be solved as a maximization problem. Consider the following example.

7.2.1 Example: core of agreements over property rules

The problems of struggles over possessions in Chapter 4 raises the question of whether it would be in the interests of all players to reduce guarding costs by agreeing to enforce some set of property rights. One way to approach this might be to consider that property rights take goods out of the no-ownership state of nature where there is no collective barrier to everyone taking whatever they can. These rights would be acceptable to all if they make everyone better off. Clearly it will be the case that, under some circumstance, players that are efficient at stealing will be worse off with such rules. If so, how much should they be compensated for foregoing the right to take goods?

Suppose that there are three individuals *i*, *j* and *k* and the production stealing and guarding equations in Chapter 4 give the following values:

$$v(i) = 4,\ v(j) = 3,\ v(k) = 2,\ v(i, j) = 9,\ v(i, k) = 8,\ v(k, j) = 7$$

and

$$v(i, j, k) = v(N) = 13$$

These values meet the condition that the game is superadditive.

The problem that has to be solved is to choose a_1, a_2, a_3 such that $\sum a_i \leq v(N)$ and

$$\begin{aligned} &a_i \geq 4, a_j \geq 3, a_k \geq 2 \\ &a_i + a_j \geq 9 \\ &a_i + a_k \geq 8 \\ &a_j + a_k \geq 7 \end{aligned}$$

This is solved for

$$a_1 \geq 5, a_2 \geq 4, a_3 \geq 3$$

with $\sum a_i \leq 13$. The solution is not unique since there is a surplus of one unit to be distributed amongst the players.

One way to think of the core is to imagine that the previous inequalities take slices from some pay-off space by ruling certain outcomes as ineligible. This is illustrated in Figure 7.1. The line $v(i \cup j) = 9$, for example, gives a triangle that allocates 9 units between *i* and *j*. Anything to the left of this line is not acceptable.

Observe that if $v(N) = 11$ the core does not exist. There is no system of side payments that would make all agree to form a coalition and stop stealing.[2]

7.2.2 Example: divide the dollar voting game

The divide the dollar game is a simple example of where the core does not exist under a majoritarian voting rule. Despite its simple structure it has interesting and much broader applications to distribution problems.

Imagine that three players have to make a collective decision about dividing some amount of money. Take the case where there is one dollar to divide and it is

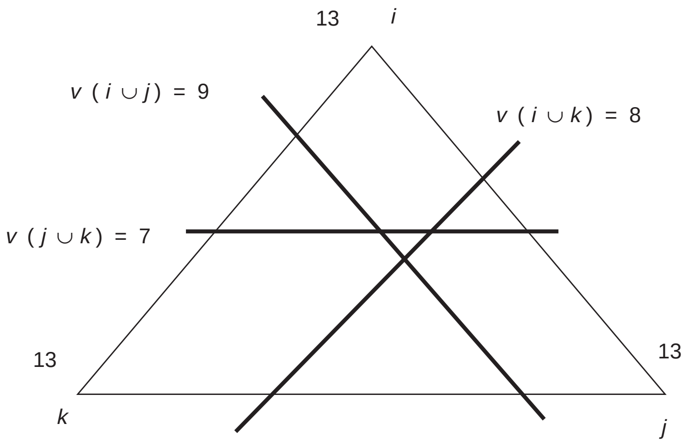


Figure 7.1 The core

allocated by majority vote. It is obvious that the condition of superadditivity is met.

It is easy to show that, for this game, there is no distribution in the core. Consider the following pay-offs, $v(i \cup j) = 1$ with division (0.5, 0.5, 0), $v(i \cup k) = 1$ with division (0.75, 0, 0.25), $v(k \cup j) = (0, 0.5, 0.5) \ldots$. Observe that each coalition defeats its predecessor. In this case the players must move to some other choice rule to get a stable allocation.

7.3 Non-transferable utility games

The non-transferable utility case involves, as it says, pay-offs that cannot be transferred between players. The Hobbesian problem gives a non-transferable utility core with the pay-offs measured in security. It is obvious that the non-transferable utility core is a subset of the transferable utility core. If a game has a core without any transfers between players, then it will certainly have a core if transfers are possible. On the other hand, a game may have a solution with transferable utilities but not with non-transferable utilities.

The conditions for the existence of a non-transferable utility core are written by letting b_i be the pay-off for any individual i in any coalition B

$$\sum_{i=1}^{n} a_i \leq v(N)$$

$$a_i \geq b_i \text{ for each } i \in B \text{ and all } B \subset N$$

It follows that to determine the existence of this core it is necessary to be able to specify b_i.

The non-existence of a core can again be illustrated with the stealing production and guarding example from Chapter 4. The situation where all stop stealing is obviously on the Pareto frontier. Now take a player with production function $m(x) = 0$ for all x and $b_i = v(i) = g_i(f_i) > 0$. In this case

$$a_i = 0 < b_i$$

Hence there will be no core without transfer payments.

More generally, consider the question in Chapter 1 of whether it would be possible to get an allocation in the core as the result of individual choice without a collective choice mechanism. The trivial example in Chapter 1 was where players were strategically independent and $\partial v_{-i}(s^*)/\partial s_i = 0$. An example of this independence is a pure market with a large number of traders. This will be considered first, since it gives some insight into the characteristics of the core and into those cases where the core does not exist without a collective decision mechanism.

7.3.1 Markets and the core

The core for a market with enforceable contracts and in which the number of traders tends to infinity is the same as the set of outcomes in the Walvasian equilibrium studied in neo-classical economics. This means that there are no further trades that can satisfy all parties simultaneously. These trades satisfy the Pareto optimality criterion and the economy is on the Pareto frontier. Such results can be made intuitive by noting that if a trade that would improve the position of two parties simultaneously could be made, then it would take place.

To illustrate the nature of the core solution in a trading game, consider the following example.

There are two traders with bundles of goods $(m, 0)$ and $(0, c)$. The utility functions are $v_1(m_1, c_1)$ and $v_2(m_2, c_2)$. These functions are continuous, monotonic and concave in each good. Trades are Pareto efficient and will allocate

$$(m - w, c - y)$$

and

$$(w, y)$$

to traders 1 and 2 respectively.

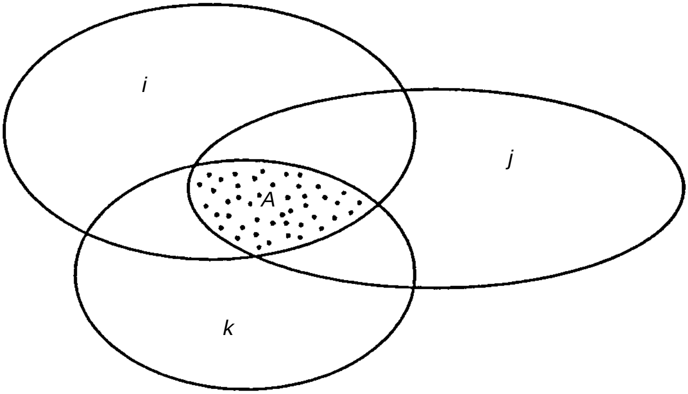


Figure 7.2 Intersection of core solutions

The coalition is $N = (1 \cup 2)$ and $a = (a_1, a_2)$ is an allocation with free trade. Since

$$a_1 = v_1(m - w, c - y) > v_1(m, 0)$$

and

$$a_2 = v_2(w, y) > v_2(0, c)$$

we have $a_i > b_i$. Hence the condition for the existence of the core is not violated.

It is plausible that as the number of traders increases the set of trades that can satisfy $a_i \geq b_i$ for $i \in B$ may decrease. By decrease is it meant that the same elements that were previously in the set are lost. More formally there is the following.

Theorem 7.1: The set of distributions $A : a_i \geq b_i$ cannot increase as the number of traders increases.

Proof: Immediate.

■

Think of this in terms of an intersection of sets. As we keep adding sets for the trades that will satisfy each trader the intersection of these sets, A, cannot increase. See Figure 7.2.

Since there were only two traders any coalition which gives trader 1 and 2

$$(m - w/k, c - y/g)$$

and

$$(w/k, y/g)$$

for $k, q > 1$ is in the core since it meets the condition $a_i > b_i$. Consider the addition of a third trader with initial endowment $(m, 0)$ and the same indifference curves as trader one. This trader will benefit from increasing its holdings of c up to the level $c - y$. Hence the new core is

$$\begin{aligned} a_1 &= v_1(m - w/k, c/2 - y/q) \\ a_2 &= v_2(2w/k, 2y/q) \\ a_3 &= v_3(m - w/k, c/2 - y/q) \end{aligned}$$

with $1 < k < 2$ and $q > 2$ and

$$a_i \geq b_i, \sum a_i = v(N)$$

as required.

The amount of good m in the system has increased, the gains from trade decrease as m is exchanged for c until a new equilibrium is reached. A coalition $B_o = (1 \cup 2)$ that offers player 1 and player 2 the old a_1 and a_2 values can be defeated by a coalition $B_i = (1 \cup 2 \cup 3)$ that now offers to exchange m for c. This can be thought of in terms of prices. Player 3 will pay more in m prices for the good c than player 1 if 3 is given an amount of c less than $c/2 - y/q$. Hence it pays player 2 to join coalition B_i.

This outcome is in the original core, since $a_1 > v_1(m, 0)$ and $a_2 > v_2(0, c)$. Note that it excludes the previous core solution and all other solutions that contain amounts of $c > c/2 - y/q$ for traders 1 and 2. This is because, for an amount of $c > c/2 - y/g$ it pays to trade c for m. Hence the core has shrunk.

Remark: Suppose that the rules of trade are given. A core containing any outcome on the Pareto frontier can be generated under these rules by an appropriate initial distribution of endowments.

This says that, given a set of rules and a core solution, it is possible to select an initial endowment that will contain that solution. It follows that the choice of the rules for a given initial endowment, or an initial endowment for given rules, is also a choice of the set of final distributions if traders all follow their optimum strategies. Hence decisions over rules and starting points are also decisions over outcome sets in a deterministic system.

This observation can be illustrated by letting the initial distributions be altered so that traders one and two start with (rm, qc) and $[(1 - r)m, (1 - q)c]$, $r, q \in [0, 1]$. The feasible Pareto frontier for this starting point is not the same as that which improves on $(m, 0)$, $(c, 0)$. Hence each different set of initial distributions may give a core in a different region.

Since the core shrinks to a small set for a sufficiently large number of players, there is a number of players and a set of initial endowments that map into an arbitrarily small region on the Pareto frontier.[3] In the extreme this region can be reduced to a point by setting the initial endowments at the desired point on the frontier.

7.3.2 Games and the non-transferable utility core

Strategic interaction will also give a solution in the core with voluntary compliance when the games have a structure of type [Ai] or [Aii] in Chapter 1. An obvious example here would be a game in which each individual is decisive in co-operating to procure some good such as safety, or the benefits of public health. Let s_i^c stand for co-operate. Since none of the good will be produced unless all players co-operate $\varphi = (s_i^c, s_{-i}^c)$ and no player can do better in any other coalition of non-co-operators. Alternatively, it might be the case that deviation from the strategy that leads to an optimal outcome gives a lesser pay-off for the individual that deviates. Examples would be agreements among a number of players to standardize equipment to reduce input costs, or among airline pilots to obey the directions of the traffic control.

The core will exist in many games with non-transferable utilities in the trivial sense that the coalition of all players gives each individual the same pay-off as the single person coalition. That is

$$a_i = b_i = v\{i\}$$

The core will not exist when the addition of more players reduces $v(N)$ by, say, increasing conflict. In the case superadditivity is violated. For an illustration of a game without a core consider the following example from Shapley and Shubik (Shubik, 1984, p. 541).

7.3.3 Example: garbage game. Shapley and Shubik

Imagine there are n players who each own some land and one bag of garbage. There is no free land and the pay-off from having a bag of garbage on a player's land is – 1. Let the number of players in a coalition B be m. Then the pay-off for any coalition is given by the number of bags it gets dumped on its land. This gives.

$$v(B) = \left| \begin{array}{l} -(n-m) \text{ if } m < n \\ -n \text{ if } m = n \end{array} \right.$$

Observe that, if all the players gang up and decide to dump their garbage on the remaining player, i, then i can retaliate by threatening to dump garbage on any individual in the $n-1$ player coalition B. In this case the threatened player may do no worse than picking another victim. This gives reason to suspect that the game might not have a core.

From the conditions given for the core, for B an $n-1$ player coalition $v(B) = -1$. Hence the core requires

$$\sum_{1}^{n-1} a_i \geq v(B) = -1$$

Note that there will be n coalitions of size $n-1$. Hence there must be n pay-offs of $\sum a_i$. Adding these gives

$$\sum_{k=1}^{n} [\sum a_i]_k \geq -n$$

i will be in $n-1$ coalitions. So $\sum [\sum a_i]_k = (n-1)\sum a_i = -n$. Feasibility requires that $\sum a_i = v(N)$. Therefore the core requires that

$$(n-1)(-n) \geq -n$$

Which means that $2n \geq n^2$, or $n \leq 2$. Hence the core does not exist in this game if it has more than two players.

7.4 Collective goods as specific examples of strategic interaction problems

The case where a player benefits from the contribution of some other player to the provision of a good is now considered. The optimum strategy for each individual i in this case is to set its contribution, x_i, such that x_i is less than the Pareto optimal level. This situation has already been discussed in 2.7 as a subset of strategic interaction.

To deal with the problem of interdependence let the wealth of i be w_i and s_i^j a strategy of spending some amount x_i^j on the good. A strategy of spending nothing is

$$s_i^0 = x_i^0 = 0$$

A public good is a good that all players can consume. The quantity of the collective good provided is y, where $y = f(x)$.

Definition: Pure collective good. y is a pure collective good if the value for player i is

$$v_i = v_i(w_i - x_i, y)$$

Where v_i is a utility function. That is y is a pure collective good if each individual consumes the entire amount produced. ■

The case where the contribution by player i always costs more than the gains in the amount of the good provided for all levels of the good gives

$$v(x_i^j) \leq v(x_i^o)$$

In this case the game will not have a core. The more interesting case is where the strategies of each player have some influence on the value of y such that

$$v(s_i) > v(s_i^o)$$

for some $x_i > 0$. This continues the analysis in sections 2.5 and 5.3.

7.4.1 Comparison of central government and individual provision of the good

The first case to be considered is where individuals get some benefit from contributing to the good. How does this compare with the allocation by a central government? The following theorem establishes that a central government can give pay-offs that are at least as great as any alternative.

Theorem 7.2: (Sharkey, 1979) The central government can provide a Pareto optimal solution where there is a single collective good, the production function is continuous and increasing and utilities in the collective good are continuous and strictly increasing.

Proof: The amount of collective good provided by a coalition B is $f(\sum x_i)$, $i \in B$. Then $v_i = v_i[w_i - x_i, f(\sum x_i)]$. For $i \in \{B + k\}$

$$v_i{}' = v_i[w_i - x_i, f \sum x_i + x_k)]$$

From the assumption of increasing utilities in the collective good, $v_i{}' > v_i$. $a_i = v_i(N)$. Therefore $a_i \geq v_i(B)$ for all $B \subset N$. ■

This is the same as saying that the non transferable utility game has a core.

Where the central government controls the allocation of every player it has to solve the following optimization problem

$$\max\ v_i(w_i - x_i, y) \tag{7.i}$$

$$\text{subject to} \sum x_i = c(y)$$

where $c(y)$ is the cost of the good. It is assumed that the utility function is continuously differentiable. Indifference curves have the usual characteristics. Form the Lagrangian

$$L_i = v_i + \lambda(x_i - \alpha_i c)$$

where α_i is the share of the cost to player i, $\sum \alpha_i = 1$.

Solving this problem piecewise gives

$$\partial v_i/\partial y = \lambda \alpha_i c'$$

and

$$\partial v_i/\partial x_i = -\lambda$$

Therefore

$$\partial v_i/\partial y / \mid \partial v_i/\partial x_i \mid = \alpha_i c'$$

Summing

$$\sum [\partial v_i/\partial y / \mid \partial v_i/\partial x_i \mid] = c' \tag{7.ii}$$

This result gives the Pareto optimal allocation in the previous theorem.

The Nash equilibrium strategy for the collective good game is for each player to solve the equation holding the contribution of all other players constant. This gives:

$$\max\ v_i(w_i - x_i, y_{-i}, y_i)$$
$$\text{subject to } x_i = c(y_i)$$

Since y_{-i} is given $dL/dy = dL/dy_i$. This gives

$$\partial v_i/\partial y = \lambda \partial c/\partial y$$

$$\partial v_i/\partial x_i = -\lambda$$

Therefore

$$\partial v_i/\partial y/ \mid \partial v_i/\partial x_i \mid = c' \qquad (7.iii)$$

Hence $\partial v_i/\partial y/ \mid \partial v_i/\partial x_i \mid$ is greater in the Nash game than $\partial v_i/\partial y/ \mid \partial v_i/\partial x_i \mid$ in the central government game for any value of y.

Since $\partial v_i/\partial y$ is the same in equations (7.ii) and (7.iii) $\mid \partial v_i/\partial x_i \mid$ is less in the Nash game. Since v_i is concave x_i must be less in the Nash game in equilibrium. Therefore each player contributes less towards the common good than they would under the Pareto efficient allocation.[4]

7.4.2 The core with a continuous production function and no side payments

The existence of a continuous production function and a positive utility for the good does not mean that everyone will contribute. Consider the case of a good with a value function $v(y) = f(m)$ similar to that in Figure 2.5, reproduced here as Figure 7.3. m is the number of contributors. Each individual must either contributes x such that $v(x) = 1$ or contribute nothing. n is sufficiently great to allow f to be approximated by a smooth continuous curve. $df/dm = 1$ at m^* and m^{**}.

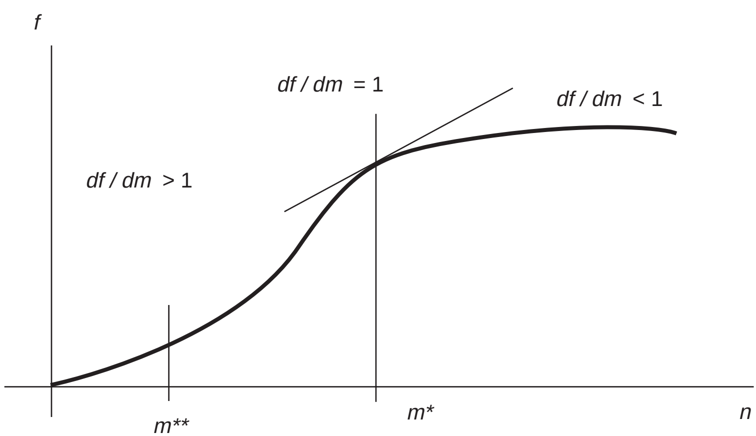


Figure 7.3 Production function for a public good

At a point with $df/dm > 1$ we have

$$f(m) \approx f(m-1) + (df/dm)1$$

for $df/dm = 1 + \varepsilon$. Hence

$$f(m) - 1 \approx f(m-1) + \varepsilon$$

Therefore

$$f(m) - 1 > f(m-1)$$

for ε sufficiently small and it pays player $m - 1$ to contribute.

Assume that the population is greater than m^{**} but less than m^*. As with the previous analysis the game has a core because the optimum strategy for each individual is to contribute.

Now assume that $n > m^*$. $df/dm < 1$ is evaluated at $m > m^*$ and it does not pay player $m - 1$ to contribute if m^* others have contributed. This gives a core in which the set M with m^* members all of whom contribute one unit and M^c with $n - m^*$ members who contribute nothing. For $i \in M$ we have $a_i \geq b_i$ since

$$f(m) - 1 > f(m-1)$$

For $j \in M^c$

$$a_j = f \geq b_i$$

This coalition structure meets the conditions for the existence of a core, but its stability is questionable. Note that it is not consistent with the Nash equilibrium strategies in the one shot game analysed in section 2.5. Each player would do strictly better in M^c provided that some other player would join M. One reason for these different solutions is that the game in which players are allowed to keep promises and the one shot non-co-operative game have different structures.

Part of the problem here is that there is not enough structure in the theory of the core to tell us about the process of coalition forming. This would have to be explicitly modelled. The coalition formation game might be similar to the hawk–dove game, for example, where hawk does not contribute. If this were the case the outcome would depend on the assumptions made about pre-play commitment. Alternatively, players may choose a mixed strategy.

7.5 Free rider problems

The free rider problem is a generalization of the case where it is not possible to block consumption. The cost imposed by free riding is that less than the optimum quantity of the good will be provided. The seriousness of this depends on the nature of the goods that are susceptible to free riding and the extent of the activity. The obvious loss is through non-contribution. It may also be the case that the optimum strategy for the remaining contributors is to reduce their contribution. Consider the following case.

Assume that the benefit from the good for an individual is given by

$$v_i = v_i(y, c_i(y))$$

where v is concave and $c_i(y)$ is the cost to individual i. The good has increasing economies of scale in the relevant range, so c is also concave. As an example of this, imagine a public transport system. As expenditure is increased the greater the return for each additional unit of expenditure because equipment can be used more efficiently, trams and buses run more frequently and become more attractive. Restoring the environment after damage may exhibit the same characteristics.

The Nash equilibrium strategy for an individual, where there is an internal solution, is to contribute up to the point where

$$\partial v_i/\partial y = | \partial v_i/\partial c_i | (\partial c_i/\partial y)$$

As the number of contributors decreases, the cost of each unit increases and $\partial c_i/\partial y$ increases for each unit of y. To maintain the identity $\partial v_i/\partial y$ must increase. Hence less y must be provided by the assumption of concavity. Hence the equality occurs at a lower level of contribution for each of the remaining contributors than if all paid.

7.5.1 Example of free riding and the core

The case where some players get a psychological benefit from their own contribution is now considered. Perhaps individuals wish to act as good citizens or attach a value to participation. Let $f(x)$ be the production function for the good. Psychological benefit can be dealt with by introducing a function $q_i(f)$ which gives the value that player i gets for any level of the good.

Let $\sum x_i = x$. Let x_i be the contribution of player i and f is a continuous and differentiable function of x. A player gets some benefit from contribution for some level of x. Assume that v is linear in x so that the pay-off function looks like

$$v_i(x_i) = q_i[f(x)] - x_i$$

where

$$q_i[f(x)] - x_i > q_i[f(x - x_i)]$$

The necessary condition for player *i* to make an additional contribution is

$$dv_i/dx_i = (\partial q_i/\partial f)(\partial f/\partial x_i) - 1 > 0$$

which means

$$\partial q_i/\partial f > 1/(\partial f/\partial x_i)$$

This says that the rate of increase in the psychological benefits that player *i* gets from its contribution to the collective good must exceed the inverse of the rate at which the value of the collective good increases. This means that if player *i*'s next contribution doubles the value of the good, the rate of psychological benefit that player one receives must increase by one half.

Assume that q is concave. The optimum strategy for players with lower values for $q(f)$ is to stop contributing at low values of x. Hence the burden will be carried by players with high values of q.

Take, for example, a two player game. Player 1 has $q_1 = \sqrt{f}$. Player 2 has $q_2 = 2\sqrt{f}$ Let $f = 2\sqrt{x}$ This gives

$$df/dx_i = 1/\sqrt{x},$$

$$dq_1/df_1 = \sqrt{2}/4x^{\frac{1}{4}},$$

$$dq_2/df_2 = \sqrt{2}/2x^{\frac{1}{4}},$$

Hence player 1 stops contributing at $\sqrt{2}/4^{\frac{1}{4}} < \sqrt{x}$ which gives

$$x_1 \approx 0.27$$

Player 2 stops for

$$x \approx 0.64$$

Thus the Nash equilibrium strategies are

$$\varphi = (0, 0.64)$$

Consider the problem of whether there is a core for this game. Proceed as follows. $B_1 = \{1\}$ and $B_2 = \{2\}$. The maximum values for B_1 and B_2 are calculated by using the Nash equilibrium values for $x = x_1 + x_2 = 0.64$. Since both players consume the good

$$b_1 + b_2 = v(1) + v(2) = [2\surd(0.64)]^{\frac{1}{2}} - \ 0.64 = 3.15$$

$$b_1 = \surd f = 1.26$$

For a coalition of all players, $v(N) = \max \ v(1 \cup 2)$. To get this calculate the x that maximizes $v = v_1 + v_2$. $dq/dx = 0$ gives the first order condition

$$3(\surd 2/4x^{\frac{3}{4}}) = 1$$

Thus $x \approx 0.957$. This gives

$$v(N) = 3[2(0.957)^{\frac{1}{2}}]^{\frac{1}{2}} - 0.957 = 3.24$$

It will be seen that if costs are shared between players and utilities are not transferable there is no core. This is because the pay-off in the core is

$$a_1 = \surd f - 0.957/2 = 0.918 < b_1$$

For the transferable utility game there is a core. This follows from the fact that

$$\sum a_i > v(B) = b_1 + b_2$$

and $\sum a_i \leq v(N) = 3.24$ as required.

7.5.2 Club goods and assurance games

The term 'club goods' is meant to apply to the case where collective goods are provided by some sub-set of individuals that form an association (Cornes and Sandler, 1996). These associations are interesting examples of groups smaller than the whole society that operate as sub-state collective decision making units. Examples might be a tennis club or the Mafia. A general result for the analysis of clubs where all individuals benefit equally from the good is given in 7.4. I now consider some more specific cases with differential benefits.

M is a sub-set of N with m players, where $m < n$. Each member of M benefits from the provision of the collective good more than the remaining $n - m$ players. The m

	s^c_{-i}	s^d_{-i}
s^c_i	3,3	–2,1
s^d_i	1, –2	0,0

Figure 7.4 Assurance game

players might, for example, have more intense problems caused by pollution or unstable markets. Each of the members of M only benefits if all others contribute. If only some contribute by, say, producing less acid, costs outweigh benefits. The core of the game has m producers and $n - m$ free riders. For the m players in M, such games have the structure of an assurance game. They are of type [Aii] in Chapter 1.

Consider the example in Figure 7.4. s^c_i is to provide the good and s^d_i is not provide. i is an individual who may wish join M and $-i$ is an individual from the $M - 1$ player coalition.

It will be observed that there are two Nash equilibria

$$\varphi = (s^c_i, s^c_{-i}) \text{ and } \varphi = (s^d_i, s^d_{-i})$$

and no dominant strategy. The outcome of this game depends on expectations. To make it worthwhile playing s^c_i player one must assign a probability $p \geq 1/2$ that $-i$ will play s^c_i. For $p < 1/2$, i's best response is to play s^d_i.

It is obvious that if pre-game communication is allowed players in the set $B - 1$ can promise to play s^c_{-i}, if player i plays s^c_i. Hence the game has a core in which the pay-off for players $i \in M$ is

$$a_i = v_i(s^c_i\ s^c_{-i}) > b_i = v_i(s^d_i, s^c_i) > v_i(s^d_i, s^d_i) > v_i(s^c_i, s^d_i)$$

Unlike the chicken game, this game will have a Nash equilibrium in which the m players contribute. The promise is credible because $v_i(s^c_i\ s^c_{-i}) > v_i(s^d_i, s^c_i)$ for all $i \in M$.

7.5.3 Example of a club good

Assume that the pay-off function for each of the m players who wish to form the club is a function of the total expenditure on the good less the cost to that player. x_i is expenditure for player i. For simplicity, each player contributes the same amount. Hence $x = mx_i$.

$$v_i(x_i) = \begin{vmatrix} f(x) - x_i & \text{for a member of the club} \\ 0 \text{ otherwise} \end{vmatrix}$$

For a player to make a contribution

$$\partial v/\partial x_i = m\partial f/\partial x_i - 1 > 0$$

which means

$$\partial f/\partial x_i > 1/m$$

This inequality says that for a player to contribute to the club two conditions must be met. The first is that $f - x_i > 0$. The second is that the rate at which the benefit increases is greater than the inverse of the number of members in the club.

There may not be any point which satisfies these two conditions simultaneously. Consider, for example, a good with the production function f concave and

$$df/dx \to 0 \text{ as } x \to qx_i$$

Then for df/dx evaluated at

$$m : x \geq qx_i$$

the required condition will not be met.

This parallels the analysis of the game in section 7.4.2. Strictly speaking the core conditions are met for individuals with

$$m^* : \ \partial f/\partial x_i \ = \ 1/m^*$$

but there is no model of the process whereby the m players are selected from the n candidates.

It is also obvious that, for any concave production function with

$$df(0)/dx > 1$$

there is some minimal number of potential members m for which the conditions required for contribution will be met.

8
Welfare and the Strategies of Preference Revelation

8.1 Introduction

The purpose of this chapter is to consider the case where the state acts as a benevolent dictator and attempts to maximize the welfare of the population. The problem of finding the maximizing set of transfers would be easy to solve if the utilities of each individual and the device for aggregating these utilities, were known. All that would then be required is to specify the welfare operator and to pick the outcomes that maximized the specified measure of aggregate utilities. This may all seem straightforward. Why not simply ask each individual how much each outcome is worth? The values they report could then be aggregated and the outcome selected accordingly. It should be obvious from the previous analysis, however, that the strategies of the individuals reporting their values are unlikely to bring about an outcome on the Pareto frontier. This is because a strategy of lying cannot be prohibited and the optimum strategy for each player will be to report whatever gives the best chance of a favourable outcome. An outcome based on these reports is not necessarily optimal. An example is a game where the prize is given to the individual who reports the highest utility for some desirable outcome. Since it is optimal for everyone to give the maximum number allowable the information content of the reports is zero.

This chapter will investigate the problem of designing mechanisms that induce individuals to report the true value of collective goods and will set out the most important of these mechanisms. This problem can be seen as that of determining an optimal set of rules in a collective decision game.

I concentrate on strategy-proof mechanisms for reporting preferences over the outcomes of collective decisions. I have given these mechanisms a fairly extensive explanation because the underlying principle of mechanism design is important for thinking about collective choice and co-operative action. This is partly to offset the negative approach to collective decision problems generated by impossibility theorems. It is also because the principle is somewhat opaque.

I only deal with one of a large class of mechanisms that come under the general heading of mechanism design and principal agent problems. Other forms of revelation mechanism include auction schemes that attempt to get buyers to reveal the true value they place on the good and contracts that get principals to reveal their true characteristics.[1]

8.2 The general problem and the revelation principle

The task of a benevolent dictator would be to maximize the welfare of the population where the maximum welfare depends on some aggregating device called a social welfare functional. The social welfare functional can be thought of as a mapping from the set of values that individuals attach to states of the world to some aggregate measure of welfare.

Let A be the set of states of the world with $a^i \in A$. The value individual i attaches to some state of the world is v_i. Let $v_i(a^i)$ be the utility of individual i for outcome (a^i). Then

$$v(a^i) = (v_1(a^i), \ldots, v_n(a^i))$$

The welfare functional ω aggregates the values the individuals assign to each possible outcome in some fashion to give a real number for each outcome.

$$\omega: \; v \rightarrow \boldsymbol{R}$$

That is, for outcome a^i, ω aggregates all the $v_i(a^i)$ values into a single value, say 3. It does the same for the $v(a^j)$ vector, and so on. The welfare mapping can be represented as in Figure 8.1.

The problem is to choose an outcome a^* that gives the highest value for ω.

$$\varphi = a^* : \; a^* \text{ maximizes } \omega[v(a)]$$

The functional ω could be specified in a number of ways. This specification is exogenous to the mapping itself. In other words it is normative. Among the norms that might be considered desirable are that:

[1] ω must be sufficiently discriminating to allow a choice between different states of the world.

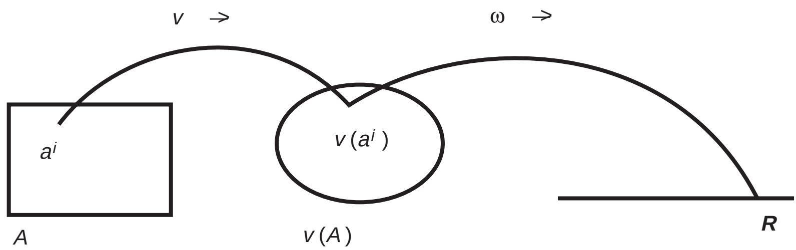


Figure 8.1 A social welfare mapping

[2] ω increases if the value of an outcome to any one individual increases, all other values held constant. If ω is a continuous and differentiable function of v and v is infinitely divisible, this condition requires $\partial\omega/\partial v_i > 0$.

Condition [2] prevents aggregate welfare increasing as some individual is made more miserable. For example the value for player *i* could not be entered into ω such that $\omega = \sum v_j - v_i$. It also means that the outcome is on the Pareto frontier. This follows immediately from the properties of φ.

These conditions can be derived from more basic assumptions about desirable properties of the choice mechanism, but they are simply stated here.

8.2.1 Examples of social welfare functions

[1] The utilitarian welfare functional is written

$$\omega = \sum_{i=1}^{n} v_i$$

[2] The weighted utilitarian functional assigns non-negative weights to the pay-offs for each player. For $\alpha_i \geq 0$, this is written

$$\omega = \sum_{i=1}^{n} \alpha_i v_i$$

An example of a weighted functional of interest to political philosophers is the Rawlsian welfare functional. John Rawls argues in *A Theory of Justice* that inequalities are just only if they improve the position of the 'least advantaged members of society' (1973, p. 15). This could be taken into account by letting M be the set of least advantaged members. Then, for comparisons of two states of the world, the relevant functional is $\omega = \sum \alpha_i v_i$ where $\alpha_i > 0$ if $i \in M$ and $\alpha_i = 0$ otherwise.

	a^1	a^2	a^3	a^4
v_1	13	2	11	7
v_2	6	17	15	9
v_3	1	14	8	21

Figure 8.2 Example of values for different outcomes

[3] It is also possible to aggregate the utilities of individuals in other ways, such as by multiplication. In addition increasing or declining marginal utilities might be taken into account. For example

$$\omega = (1n(v_1 + 1))(1n(v_2 + 1)) \ldots (1n(v_n + 1))$$

If ω is well specified it is straightforward to calculate a^* for a given v. This raises the problem of getting individuals to report their utilities truthfully. To investigate this it will be assumed that the dictator tries to maximize the utilitarian welfare functional.

Since the conditions imposed on ω mean that the probability of an outcome being chosen will increase as the value that is reported increases the optimum strategy is not necessarily to report the true value. If so an outcome on the Pareto frontier would not be expected. This is illustrated in Figure 8.2.

Figure 8.2 gives three individuals and four outcomes. The outcomes are along the top and the utilities are written down the side. v_i is the true value of an outcome for an individual i. s_i^k is a reported value. The social welfare function is $\omega = \sum s_i$. In the case of a tie between m outcomes $a^p, p = 1, \ldots, m$ the choice procedure is

$$\varphi = a^j \text{ with probability } p_j = 1/m$$

Assume that the players know the choice procedure. They are asked to report a value $v_i(a^k) \geq 0$ with $\sum v_i^k \leq q$. They are not allowed to report values such that the total is greater than q.

There are two different cases which depend on the information available to the players:

[1] No information on the utilities of others and no communication. The optimum strategies are

$$s_1 = (q, 0, 0, 0), s_2 = (0, q, 0, 0), s_3 = (0, 0, 0, q)$$

The information the dictator gets is

$$\omega(a^1) = q = \omega(a^2) = \omega(a^4) \text{ and } \omega(a^3) = 0$$
$$\varphi = (a^i) \text{ with possibility } p_i = \tfrac{1}{3}$$

where $i = 1, 2, 4$.

[2] The players can communicate. In this case the game does not have an equilibrium and there is no core. The strategy above gives payoffs

$$v_1 = 22/3, v_2 = 32/3 \text{ and } v_3 = 12$$

This is dominated by

$$s_2 = s_3 = (0, q, 0, 0)$$

with $v_2 = 17$ and $v_3 = 14$. This is dominated for player 1 and player 3 by

$$s_1 = s_3 = (0, 0, 0, q)$$

Similarly, this is dominated for players 1 and 2.

8.2.2 Strategy-proof mechanisms

The problem of getting players to report truthfully when they are not allowed to form coalitions is the following. What sort of mechanism will ensure that the Nash equilibrium strategies of the players is to give a truthful report of their utilities over all states of the world? That is, for s_i^* a strategy that gives a truthful report we need $v_i(s_i^*, s_{-i}^*) \geq v_i(s_i, s_{-i}^*)$.

The answer to this question involves two considerations.

[1] If players are to report truthfully, the pay-offs in Figure 8.2 have to be altered. This can be done by adding some rewards or penalties to the values reported.

[2] It is only necessary to alter the pay-offs to individual *i* if *i*'s report changes the outcome. If the report of an agent makes no difference to the outcome, it is not of interest.

Regarding the design of the mechanism, it is possible to imagine any number of reporting procedures. For example, move one might be to allocate a number of points. Move two might be to buy out unfavourable outcomes by using whatever points were not allocated on move one and so on. It turns out that such complicated mechanisms are not required. This is because any mechanism in which agents directly report their preferences can achieve the same outcome as an indirect mechanism. This is known as the revelation principle.

Let a general mechanism be any game in which the players participate. A direct mechanism is one where players participate in the game and directly reveal their preferences. Then

Theorem 8.1: (Revelation principle) Any strategy-proof Nash equilibrium that can be obtained in a general mechanism can be obtained in a direct mechanism.

Proof: Suppose that the general mechanism has a Nash equilibrium set of strategies

$$\varphi_i = s_i^* = (s_i^{k1}, \ldots, s_i^{*km})$$

and s_i^{*km} is tell the truth on move m. Assume that the Nash equilibrium in a one move game is to report $s_i^k \neq s_i^*$. The one move game can be treated as the last move in the general mechanism. This means that there must be a path

$$s_i^k = (s_i^{k1}, \ldots, s_i^{k(m-1)})$$

such that

$$v_i(s_i^{km}) \geq v_i(s_i^*).$$

This establishes the contradiction.[2] ■

Alternatively, suppose n players are engaged in a general game mechanism. If this mechanism is strategy-proof the Nash equilibrium must be that they truthfully report their preferences on the last move. This can be duplicated by a strategic form game in which each path leading to this outcome is represented by one move. To see this, consider the way in which extensive form games were collapsed into strategic form in Chapter 1. The list of all responses to every possible response by another player that leads to a particular outcome is represented by a single strategy. Since the mechanism is strategy-proof this must be to follow the path that leads to a truthful report. Any one move game is a direct mechanism.[3]

This proof can be generalized to show that any outcome a that is implementable by an indirect mechanism is implementable by a direct mechanism.[4]

Although simple, the revelation principle is quite useful in that it allows us to concentrate on a direct mechanism. Moreover, if no strategy-proof direct mechanism exists, it follows that there is no strategy-proof mechanism.

A strategy-proof mechanism that altered the pay-offs to players by adding a penalty or a tax to any player who changed the outcome by its presence was proposed by Clark and Groves in the early 1970s.

8.3 The pivot mechanism: Clarke-Groves

The idea behind a pivot mechanism is to impose a tax on an agent whose choice changes the decision from what it would have been in the absence of that agent. Consider the case where utilities are additively separable. To simplify the exposition ignore the possibilty of a draw between outcomes. An agent that changes a decision is pivotal. The true value for agent i is v_i^* and a report of v_i^* for agent i is s_i^*. Let the tax for agent i in outcome a^o be

$$t_i^o = \begin{vmatrix} t_i(a^o) \text{ where agent i is pivotal} \\ 0 \text{ otherwise} \end{vmatrix}$$

Since the preferences for the good and for the tax are additively separable the total welfare of i at outcome a^o will be

$$\upsilon = v_i(a^o) + t_i(a^o)$$

where $\upsilon_i = v_i(a^o)$ if agent i is not pivotal. This gives the following definition of a strategy-proof mechanism.

Definition: A mechanism will be strategy-proof if

$$v_i(s_i^*) + t_i \geq v_i(s_i) + t_i \tag{8.i}$$

where $v_i(s_i^*) = v_i(s_i^*, s_j^*)$ and $v_i(s_i) = v_i(s_i, s_j^*)$. ■

This condition is met if every pivotal agent is taxed by an amount that equals the total loss in welfare of all the other agents from the change in outcome from the outcome they would most prefer without i's report. This requires a little demonstration. The idea of taxing an agent by an amount equal to the loss inflicted on others has an intuitive appeal, and seems fair. It is not obvious, however, that it would always pay the pivotal agent to tell the truth. Moreover, since this agent does not know the welfare of others, it might be feared that truth telling could lead to a change in the outcome and a large loss resulting from a high tax.

8.3.1 Details of the strategy-proof pivot mechanism

In order to study the strategy-proof pivot mechanism, a more precise statement is needed. Let the sum of the maximum total welfare for agents $j \in N_{-i}, j \neq i$, be written

$$\max \sum_{j=1}^{n-1} v_j(a) = \max w_j$$

The sum of the total welfare for $j \in N_{-i}$ in the outcome with agent i reporting s_i^k is

$$\sum_{j=1}^{n-1} v_j(s_i^k) = w_j(s_i^k)$$

Note this sum excludes agent i. When everyone tells the truth $w^* = w(s^*)$.

Definition: The strategy-proof pivotal (Spp) mechanism is such that

$$t_i^k = w_j(s_i^k) - \max w_j \tag{8.ii}$$

When agent i does not change the outcome $\max w_j(s_i^k) = \max w_j$ so the tax is zero, as required. Notice also that $t_i^k \leq 0$. This follows from the fact that $w_j(s_i^k) \leq \max w_j$. ■

Theorem 8.2: (Green and Laffont, 1979) The Spp mechanism is Pareto efficient and meets the requirement of strategy-proofness.

Proof: The mechanism must be Pareto efficient at the truth telling equilibrium since it selects the a^* that maximizes $\omega[v(a)]$. The mechanism is strategy-proof if

$$v_i(s_i^*) + w_j(s_i^*) - \max w_j \geq v_i(s_i^k) + w_j(s_i^k) - \max w_j$$

Cancelling the max w_j, it must be shown that

$$v_i(s_i^*) + w_j(s_i^*) \geq v_i(s_i^k) + w_j(s_i^k).$$

This is immediate since $v_i(s_i^*) + w_j(s_i^*) = \max \omega$. ■

The way in which the tax schedule works is illustrated in Figure 8.3. The tax is the difference between the utilities everyone else would get with and without i's report. This report is given value $s_i^o = x$. Observe that the tax does not change either side of the point where x is pivotal. Hence all that has to be shown is that it pays i to make an honest report if i is pivotal. Note that whether i is pivotal depends on x, but the level of the tax is independent of x.

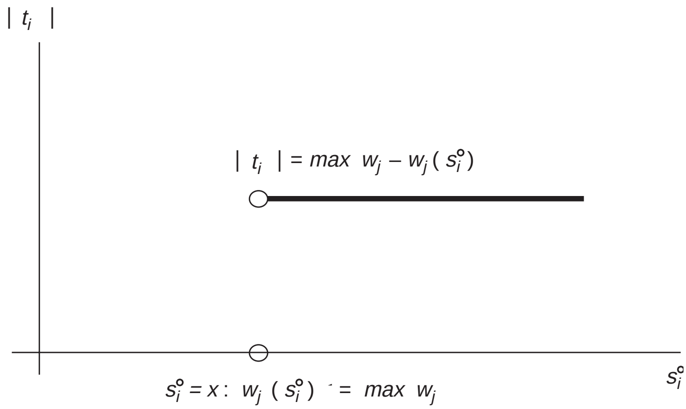


Figure 8.3 taxation for a pivotal player

Assume that max $w_j = w_j(a^k)$. Write $w_j(a^k)$ as w_j^k for compactness. For *i* to report an *x* that changes the outcome to a^o with pay-offs to $N - i$ of w_j^o requires

$$s_i^o = x: \ x + w_j^o > v_i(a^k) + w_j^k.$$

Hence

$$s_i^o > v_i(a^k) + w_j^k - w_j^o.$$

For the mechanism to work when *i* tells the truth

$$v_i(a^0) > v_i(a^k) + (w_j^k - w_j^o).$$

Consider the two possibilities.

(a) *i* prefers a^o but it has value $v_i^o < v_i(a^k) + (w_j^k - w_j^o)$. A dishonest report of $v_i^o = x$ to get outcome a^o gives $v_i^o - (w_j^k - w_j^o) < v_i(a^k)$. Hence $\upsilon_i(x, s_j^*) < \upsilon_i(s_i^*, s_j^*)$.
(b) $v_i^o > v_i(a^k) + (w_j^k - w_j^o)$ but *i* reports $x < v_i(a^k) + (w_j^k - w_j^o)$. Then *i* gets $v_i^k < v_i(a^o) + (w_j^k - w_j^o)$. Hence $\upsilon_i(x, s_j^*) < \upsilon_i(s_i^*, s_j^*)$.

As an example of this, consider the strategies for agent 1 in Figure 8.2.

(a) *Truthful report.* Max w_j, $j = 2, 3$ occurs for outcome a^2, $\max w_j = 31$. The outcome if 1 reports truthfully is $a^* = a^4$ to give $w_j^* = 30$. Hence $t_1 = w_j^* - \max w_j = -1$. This gives $\upsilon_1(s_i^*) = \upsilon_1(a^4) = 7 - 1 = 6$.

(b) *Dishonest report.* Say player 1 wishes to get outcome a^1. To get this requires $\omega[v(a^1)] > \omega[v(a)]$. This is met if player 1 reports $s_1^k =$ $(25, 0, 0, 0)$. $w_j(a^1) = 7$. $\max w_j = 31$ as before $t_1 = w_j^1 - \max w_j = 7 -$ $31 = -24$. Hence $\upsilon_1 = 13 - 24 = -9 < 6$.

8.3.2 Example: public good with costs

The pivot mechanism can easily be generalized to the case of a public good with costs. Suppose that there is some public good with a cost c. The only decision is between producing a units and producing nothing. This gives two outcomes

$$a^* = \begin{vmatrix} a \text{ if } \omega_i \geq 0 \\ 0 \text{ if } \omega_i < 0 \end{vmatrix}$$

The cost of the good is shared equally so that each individual pays c/n. In this case the total welfare for each individual will be

$$\upsilon_i = \begin{vmatrix} v_i(a) + t_i(a) - c/n \text{ if the good is produced} \\ t_i(0) \text{ otherwise.} \end{vmatrix}$$

Since the value of the good is now the utility from consumption less the cost of production it is convenient to introduce a new measure for value. Define this as

$$u_i = \begin{vmatrix} v_i - c/n \\ 0 \text{ if the good is not produced} \end{vmatrix}$$

This means that $\sum u_i \geq 0$ since for $v - c < 0$ the dictator sets $a = 0$.

By direct substitution of u_i for v_i in the definition of the Spp mechanism we get the following.

The Spp mechanism for a public good with costs is given by

$$t_i = \sum u_j(a^i) - \max \sum u_j(a)$$

where a^i is the decision with agent i's contribution and a is the decision otherwise.

This can be expanded to give the following set of tax burdens for agent i. The cases where agent i is not pivotal give

$$t_i = \begin{vmatrix} 0 \text{ if } \max \sum u_j(a) \geq 0, \sum u_j(a^i) \geq 0 \\ 0 \text{ if} \sum u_j(a) = 0, \sum u_j(a^i) = 0 \end{vmatrix}$$

since $a = a^i$ in these cases. Otherwise

$$t_i = \begin{vmatrix} \sum u_j(a^i) \text{ if } \max \sum u_j(a) = 0, \sum u_j(a^i) \geq 0 \\ -\max \sum u_j(a) \text{ if } \max \sum u_j(a) \geq 0, \sum u_j(a^i) = 0 \end{vmatrix}$$

Note that the tax is negative in both cases. In the first case

$$t_i = \sum u_j(a^i) - \max \sum u_j(a).$$

Since $\max \sum u_j(a) = 0$ the value of $\sum u_j(a)$ at a worse position $\sum u_j(a^i)$ must be less than $\max \sum u_j(a)$.

This means that for $\max \sum u_j(a) = 0$ and $\max \sum u_j(a^i) \geq 0$, i is given a tax

$$t_i = \sum u_j(a^i) = \sum v_j(a^i) - (n-1)c/n.$$

For $\max \sum u_j(a) \geq 0, \sum u_j(a^i) = 0$ i is given tax

$$t_i = -\max[\sum v_j(a^i) - (n-1)c/n].$$

8.4 The general class of strategy-proof mechanisms and minimal utility

The results given so far can be extended to give the general form for the class of strategy-proof mechanism for any case where the individuals have quasi-linear utilities in the good and the tax. It can be shown that every such mechanism must be of the form discussed above. The importance of this finding is that the investigation of strategy-proof revelation can concentrate on the properties of this class of mechanisms.

Theorem 8.3: (Green and Laffont)[5] Any strategy-proof direct revelation decision-making mechanism chooses a^* and is of the form

$$\begin{aligned} &a^* \text{ maximizes } \omega[v(a)] \\ &t_i^o = w_j(s^o) - h_i[v_j(a)] \end{aligned} \tag{8.iii}$$

where $j \neq i$ and $h_i[v_j(a)]$ is an arbitrary numerical function that does not depend on s_i.

This generalizes equation (8.ii). Note that h_i in equation (8.iii) is in the same place as max w_j in equation (8.ii) and w_j does not depend on s_i.

To prove this, all that is necessary is to show that, for the Spp mechanism, h_i is independent of the reported preferences of i as claimed. This means that, if i is pivotal then the tax depends only on the cost imposed on others of switching the outcome from a^k to a^o. That is

$$t_i^k - t_i^o = w_j^k - w_j^o$$

since

$$h_i[(s_i^k)] = h_i[(s_i^o)]$$

by the independence of h_i from the reported preferences of i.

Proof: Suppose the mechanism is strategy proof and assume that h_i depends on the strategy of i. Let $s_i^k \neq s_i^o$ and $h_i[(s_i^o)] - h_i[(s_i^k)] = \varepsilon > 0$.

$$t_i^k - t_i^o = w_j^k - w_j^o + \varepsilon \qquad (8.\text{iv})$$

Construct a preference set for agent i with values

$$\begin{aligned} \sigma_i(a^k) &= -w_j(a^k) \\ \sigma_i(a^o) &= -w_j(a^o) + \varepsilon/2 \\ \sigma_i(a^m) &= -c \text{ for } m \neq k, 0 \text{ and } c > \max\ w_j(a) \end{aligned}$$

This is possible because the preferences of agent i are unrestricted. The problem $\max\ \sigma_i(a) + w_j(a)$ is solved at a^o from the preference set.

Since i is pivotal, from equation (8.iv)

$$t_i(a^k) - t_i(a^o) = w_j(a^k) - w_j(a^o) + \varepsilon = \sigma_i(a^o) - \sigma_i(a^k) + \varepsilon/2$$

Therefore

$$\sigma_i(a^k) + t_i(a^k) \geq \sigma_i(a^o) + t_i(a^o)$$

It follows that, whenever the true value for player i is $v_i^o = \sigma_i(a^o)$ it will pay to announce a preference of $s_i^k : \varphi(a) = a^k$. This contradicts the fact that the mechanism is strategy-proof.[6]

When i is not pivotal, the report s_i does not change the outcome. Hence $w_j^k = w_j^o$ and, from equation (8.iv)

$$0 = 0 + \varepsilon = h_i[(s_i^k)] - h_i[(s_i^o)]$$

Contradiction. ■

Remark: The Spp, or pivotal, mechanism set out in equation (8.ii) is a special case of the more general revelation mechanism above. A question which arises is that of whether the pivotal mechanism has any special qualities. One interesting feature of the pivotal mechanism is that it guarantees the best minimum security level of all strategy-proof mechanisms. The worst an agent can do under this mechanism is better than the worst they can do under any other mechanism. This has some attractiveness as an inducement to agents to enter into such a procedure. This is set out as follows.

Theorem 8.4: (Moulin, 1991, p. 214) Let m_i be the final utility of an agent in some strategy-proof mechanism of the type set out in theorem 8.3. Then if

$$m_i \geq \min\ v_i \text{ for profile } v \text{ and agent } i$$

the mechanism is the Spp mechanism.

Proof: Suppose $m_i \geq \min\ v_i$. Since m_i is the final utility of some strategy-proof mechanism, $m_i = v_i + t_i$. Theorem 8.3 gives

$$t_i = w_j(s) - h_i[v_j(a)]$$

Adding v_i to both sides gives

$$m_i = \max\ \omega[v(a)] - h_i[v_j(a)] \tag{8.iii}$$

where $j \neq i$.

Since $m_i \geq \min\ v_i, h_i \leq \max\ \omega - \min\ v_i$. Let $v_i = -w_j, j \neq i$ for v_j fixed. Then $-\min(-w_j) = \max\ w_j$. So $h_i \leq \max\ w_j$, and

$$m_i \geq \max\ \omega - \max\ w_j \tag{8.iv}$$

$\max\ \omega - \max\ w_j$ is a solution to the pivotal mechanism for player i. This is because

$$\upsilon_i = v_i^* + t_i^* = \max(v_i + w_j) - \max\ w_j = \max\ \omega - \max\ w_j$$

What is required is to show $m_i = \upsilon_i$. To do this fix v_j and let a^o be a decision where w_j is maximal. Now construct v_i such that w_{N-i} is maximal at a^o for all i. That is, the summation over all $n-1$ agents is a maximum at a^o no matter which agent is missing. This can be done by setting $v_i(a^o)$ large enough and $v_i(a^k) = 0$ for $a^k \neq a^o$. This means no agent is pivotal. So at a^o

$$\max \omega - h_i\,[v_j(a)] = \max \omega - \max w_j$$

Hence

$$h_i[v_j(a)] = \max w_j$$

This holds for all cases from theorem 7.3 since the special construction was on v_i and not on v_j. Therefore

$$m_i = \max \omega - h_i[v_j(a)] = \max \omega - \max w_j = \upsilon_i$$

as required. ■

Another interesting question is that of individual rationality. A requirement imposed on all collective decision processes by the core is that it must be individually rational, or Pareto superior, for an individual to participate. An individual must be better off entering into the process than abstaining. Otherwise the core constraint that $v\{i\} < \upsilon_i$ is violated. Do the revelation mechanisms so far discussed have this property? It might be wondered whether the tax imposed might make some individuals worse off than they would be if the abstained and accepted the status quo without the decision.

It is easy to show that it is always rational for an individual to participate in any mechanism that gives a transfer greater than or equal to the pivot mechanism.

Theorem 8.5: The requirement of individual rationality in participation is met for any $m_i \geq \upsilon_i$, where υ_i is the final utility of an agent i in the Spp mechanism.

Proof: The Spp mechanism gives $\upsilon_i = v_i^* + t_i^*$ with outcome a^* where $\upsilon_i \geq \max v_i$ with outcome a^k where a^k maximizes w_j without i's report. Hence participation is rational for i for any $m_i \geq \upsilon_i$. ■

It can be seen that an Spp mechanism can be used to construct a choice procedure that overcomes the problems associated with understating preferences for public goods, provided that the value of the good and the taxes are additively separable. These problems include those of free riding when this takes the form of understating the value of the good.

Does this mean that the dictator can maximize social welfare, or meet the weaker criterion that the pay-off is on the Pareto frontier? Unfortunately the answer is 'no' to both questions. To see this it is necessary to consider the surplus that is generated by the Groves mechanism.

8.5 The budget surplus problem

It can be seen that, where at least one agent is pivotal, an Spp mechanism will generate a tax. This means that, even though an efficient level of the public good is provided, the outcome is not socially optimal because the private consumption of some agents is reduced. It would be useful if this sub-optimality could be avoided by using some other revelation mechanism that did not result in a surplus being generated. If this were the case the decision would be both efficient and socially optimal. It turns out that this is not possible, however. This is stated and proven in the following theorem. It is then asked whether this negative result is a cause for concern.

Theorem 8.6: (Green and Laffont, 1979) There is no strategy-proof direct revelation mechanism such that the sum of taxes collected is zero for all preference profiles.

Proof: The proof has to show that there exists a set of preference profiles where $\sum t_i \neq 0$. This is done by induction. Suppose $\sum t_i = 0$. Begin by considering the case for two players. Without loss of generality, assume that every report is honest. Consider the case where there are two preference profiles. This means that players have preferences of type one and then preferences of type two. Write these v_i^j where $i = 1, 2$ and $j = 1, 2$. $v^j = (v_1^j(a), v_2^j(a))$. Let the value of outcome a for player 2 be the same in both preference profiles. That is $v_2^1 = v_2^2$. Let

$$\begin{aligned} v_1^1 + v_2^1 &> 0 \\ v_1^2 + v_2^2 &< 0 \end{aligned} \tag{8.vi}$$

Since the mechanism produces none of the good for the second case, for $\sum t_i = 0$ we must have from equation (8.iii) that $v_i + t_i = \max \omega - h_i$. Adding gives

$$t_1^1 + t_2^1 = v_1^1 + v_2^1 - h_1(v_2^1) - h_2(v_1^1) = 0, \text{ when the good is produced}$$
$$h_1(v_2^2) + h_2(v_1^2) = 0, \text{ where no good is produced}$$

These two equations are added. Recalling that $v_2^1 = v_2^2$ gives

$$v_1^1 - h_2(v_1^1) + h_2(v_1^2) = -v_2^1$$

Observe that the left hand side depends on v_1 and not on v_2. But it is possible to vary v_2^1 within the range required by equation (8.vi). Therefore the equality does not hold. This contradicts the assumption that $\sum t_i = 0$. Thus the proof is established for two players.

For the proof by induction, add another player to equation (8.vi) so that $v_3^1 = v_3^2$ and

$$\sum v_i^1 > 0$$
$$\sum v_i^2 < 0$$

Performing the same addition as before

$$v_1^1 - \sum h_i(v_1^1) + \sum h_i(v_1^2) = -\sum v_i^1(a) \sum \text{ for } i = 2, 3$$

since all the terms $h_i(v_2)$ and $h_i(v_3)$ cancel out as before. Remember that $v_2^1 = v_2^2$ and $v_3^1 = v_3^2$. Once more the left-hand side and the right-hand side are independent.

To complete the proof, assume that $\sum t_i = 0$ for k players with preferences defined such that $v_k^1 = v_k^2$ for all $k - 1$, and the preferences of player 1 as before. Now define the preferences of the $k + 1$ player such that $v_{k+1}^1 = v_{k+1}^2$, and $\sum v_i^2 < 0$. It is obvious that addition will give

$$v_1^1 - \sum h_i(v_1^1) + \sum h_i(v_1^2) = -\sum v_i 1(a) \text{ for } i = 2, 3, \ldots, k + 1$$

as in the previous calculation. This establishes the contradiction for any $k + 1$, as required. ■

Is the fact that any direct revelation strategy-proof decision mechanism will generate a surplus a cause for concern? What would be the expected deviation from the social optimum?

Green and Laffont have argued that the outcome produced by such a mechanism will tend to converge on the social optimum for a very large number of agents (pp. 165–200). Their argument is roughly as follows. As the number of agents increases, the surplus collected is small. Suppose that this surplus is distributed among the population in order to eliminate the social inefficiency. In this case the amount distributed to any particular agent will also be small.

Green and Laffont give the following example (p. 168). Consider a project with a mean value of zero for which 95% of the population has a willingness to pay between – \$200 and + \$200. Willingness to pay has the normal distribution around the mean. For a population of 10,000, the expected per capita rebate would be 20 cents. Where the mean is greater than zero the expected surplus goes to zero as n increases.

It follows that this amount should not cause agents to deviate too far from their honest report. If so, the mechanism produces a result near to the efficient provision of the public good.

8.6 Note on strategy-proof mechanisms and preferences

The assumption that preferences between the public good and private goods such as the tax are separable has been maintained throughout the discussion of the Spp mechanism. That is, the problem has been to choose a strategy that maximizes

$$\upsilon = v(a) + t.$$

The bad news is that, if the assumption of separability is dropped, it is not possible to ensure that a revelation mechanism gives a satisfactory result. This problem is not eliminated when the number of decision makers becomes large, as with the budget surplus problem.

It is shown by Green and Laffont (p. 201) that, when the project has a positive income effect, the optimum strategy of agents will be to understate their net willingness to pay. This suggests that the decision maker should be willing to adopt public projects when the stated value is less than zero.

9 Voting Rules and Strategic Manipulation

9.1 Introduction

In democratic systems some form of voting or scoring method is used to get a collective choice by mapping the preferences of individuals into a final outcome. These methods differ from those used to maximize a social welfare function in Chapter 8. They are simpler and less refined. The attempt to maximize welfare and to make a direct link between utilities and outcomes is abandoned. Instead, the welfare requirements of the choice procedure are much cruder and less demanding and the choice is made on the basis of less information. The players are merely required to order their preferences for outcomes, or for candidates offering a menu of outcomes. Some choice procedure is designed to map this minimal information into an outcome that meets a set of normative conditions. Among these conditions might be that if more individuals prefer an outcome the probability of selection increases, or that the process should not discriminate between individuals on the basis of height or race. An example of such a procedure would be that the outcome with the most first votes wins. Another scheme would be a series of binary competitions between outcomes. The winners from one competition are again paired until a single outcome remains as the social choice. Another would be a scheme whereby each voter vetoes the most disliked candidates at each round.

The question which this gives rise to is, are such schemes strategy-proof? That is, is it in the interests of each voter to report their preferences honestly? This problem has generated a large literature. Most of this is negative and focuses on impossibility results. The most important of these for strategic voting is the Gibbard-Satterthwaite theorem. This says that when three or more candidates are to be compared, and there are no restrictions on preferences, the only strategy-proof mechanism is a dictatorial mechanism. Since a dictator might be considered undesirable, it is worthwhile investigating this result. It is also worthwhile asking whether strategic manipulation might be of less concern than the possibility of a dictator or restrictions on choice.

Another question that arises is whether voting procedures will be stable in the sense that a given set of preferences always gives the same outcome. This is the same as asking whether the voting procedure always has a unique core.

The question of stability has received a great deal of attention from political scientists and seems to have been seen as desirable in the sense that equilibrium results are desirable in a theory of markets. It is not clear, however, whether stability is a desirable property. For example, a core in which $n/2 + 1$ voters always get their preference might be considered a form of tyranny over the remaining $n/2 - 1$. This could conceivably lead to political instability.

It can be seen that different voting rules have different properties and may give different final choices with the same distribution of preferences. Rather than focus on impossibility theorems, the more interesting question is, what type, or mixture, of voting rules is going to have the most attractive characteristics with respect to any particular problem?[1] Strategic manipulation may be an acceptable property if it allows other values, such as no restriction of preferences and no dictatorship, to be maintained.

This chapter deals with some aspects of the social choice theory attack on such problems.[2] It uses both non-co-operative and co-operative game theory. I consider the problems at a fairly general level. By this I mean that little is done by way of specifying particular institutions.

9.2 The voting problem and strategic manipulation

The game theory analysis of voting procedures is concerned with those cases where it pays voters to act strategically, rather than with the larger class of problems about preference aggregation mechanisms. A voting mechanism is an ordering of preferences or an allocation of points for different outcomes. It is not, in general, a statement of the utilities of an outcome. These outcomes might also be candidates in an election. The terms outcome and candidate will be treated as equivalents.

As with the utility reporting system dealt with in Chapter 8, the problem of strategic interaction is that voters may not truthfully report their preferences. The fact that a system is manipulable may mean that the Nash equilibrium strategy for all voters is to give a dishonest report. If this happens, then voting will not give accurate information about voters' preferences. This can be expressed more formally as follows:

Definition: A voting mechanism or rule is a procedure for mapping a set of orderings over possible states of the world, or outcomes, into some final ordering. Let A be the set of m outcomes with $a^k \in A$. A^{mn} is the $m \times n$ matrix of orderings for n voters over these outcomes. This gives $A^{mn} = (A_1, A_2, \ldots A_n)$. An ordering for voter i

is $A_i = \sigma A$ where σ is a permutation of the m elements in the choice set. $A_i = a^r, a^z, \ldots$. with the kth member being at least as preferred as the $k+1$th for all k.

The voting rule φ is a mapping from the ordering of individual preferences A^{mn} to an outcome. This may be a ranking of states of the world or a selection of one or more successful candidates.

$$\varphi : A^{mn} \rightarrow a$$

where a is the selected outcome or ranking.

9.2.1 The voting problem

The voting problem is to pick some rule that satisfies a set of desirable criteria. As with the previous discussion of desirable criteria or axioms these are normative. Among them might be

[1] *Decisiveness*. The rule should be able to make a selection between outcomes. This is usually not considered to be sufficient. A decisive selection could be achieved by using a roulette wheel, for example.
[2] *Monotonicity*. This means that if the support for a candidate increases, then the chances of this candidate being elected cannot decrease.
[3] *Neutrality*. This says that candidates are treated equally.
[4] *Anonymity*. Anonymity says that voters are treated equally.

9.2.2 Voting methods

The two most frequently used methods of voting are scoring methods and Condorcet voting. These will be used for the examples that follow. For all voting mechanisms it will be assumed that a tie is broken by some fair random mechanism such as tossing a coin.

9.2.2.1 Scoring method

This allocates a number of points to each candidate within some predetermined range. The top candidate may get ten, the second nine and so on. The candidate that gets the highest score is elected. This is often known as the Borda method. Note that first past the post or simple plurality voting is a particular case of the scoring method. In this case one point is allocated to the most favoured candidate and zero to all others.

9.2.2.2 Condorcet method

This involves matching the candidates in a pairwise comparison and determining the loser from each by a majority vote. A version of this, where candidates that lose are eliminated, is familiar to those who watch tennis championships. The Condorcet winner is the candidate that defeats every other candidate in a sequence of majority comparison.

The characteristics of these voting methods will often depend on the number of candidates.

9.2.3 Two candidates or less

The problem of strategy-proof voting mechanisms only arises with three or more possible outcomes. To see this consider the case where the number of candidates is two. In this case the scoring method and the Condorcet method meet monotonicity, neutrality and anonymity and are strategy-proof. This is stated in a little theorem as follows. Let p be the probability of a candidate being selected.

Theorem 9.1: Suppose there are no more than two outcomes for an election. Any monotonic choice mechanism is strategy-proof.

Proof: Let s_i^* be the strategy of reporting truthfully and s_i^o the strategy of reporting any other preference. Suppose $v_i(a^i) > v_i(a^k)$ and i supports a^k. This gives

$$p\varphi:\ s \to a^k > p\varphi:\ s \to a^i$$

from monotonicity and

$$E[v_i(s_i^o, s_{-i})] < E[v_i(s_i^*, s_{-i}]$$

Contradiction. ■

This result meets the definition of a strategy-proof mechanism from Chapter 8. It does not hold where the number of candidates exceeds two. Consider the following examples.

9.2.4 More than two candidates

9.2.4a Scoring method

Suppose there is a plurality voting system and three candidates a, b, c. Let P_i mean preferred by i and I_i mean indifferent. Fixed points are allocated to the first second and third preferences. There are n voters. The rankings for voter i and all other voters written $-i$ are

$$\begin{array}{rl} i & aP_ibP_ic \\ -i & cP_{-i}bP_{-i}a \end{array}$$

Assume that the value of the points are such that the distance between the points allocated to first and last for i is greater than the distance between the points allocated for first and second for $-i$. For $i, a = 10,\ b = 5, c = 1$ and for $-i, c = 55, b = 50$ and $a = 20$.

s_i^* is to report true preferences. It is immediate that

$$v_i(s_i^o, s_{-i}^*) \geq v_i(s_i^*, s_{-i}^*)$$

This is because

$$\varphi(s_i^*, s_{-i}^*) = c$$

with 56 points, whereas for

$$s_i^o = [b = 10, a = 5, c = 1]$$

we have

$$\varphi(s_i^o, s_{-i}^*) = b$$

with 60 points. Since $v_i(b) > v_i(c)$, scoring is not strategy-proof.

9.2.4b Condorcet method

Assume a Condorcet voting method with the preferences for three voters i, j, k in Figure 9.1. Preferences run from top to bottom with the higher outcome preferred to the lower.

It will be observed that this ordering can be manipulated by voter i. Let the pairwise comparison be between the winner of a vote between a and b and a vote for c. This gives

$$\varphi = [\varphi(a, b), c]$$

i	*j*	*k*
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

Figure 9.1 Cyclical preferences

Then $\varphi(a, b) = a$ and $\varphi(a, c) = c$. Since i has bP_ic it pays to report bP_ia in the first round. Now

$$\varphi[\varphi(a, b), c] = b$$

with $v_i(b) > v_i(a)$.

This method is not strategyproof since i's stated preferences for b over a depends on i's preference for b over c.

These examples naturally lead to the question, are any methods of voting strategy-proof? Since the number of possible voting methods is very large, this question cannot be decided by examining each possibility.

One way to make a start on this problem is to note that strategic interaction can only take place if the pay-offs for i are influenced by the actions of $-i$. This leads to the following speculation.

Theorem 9.1 shows that, if there are only two outcomes, the optimum strategy is to report truthfully. Moreover if two outcomes were independent, player i's vote for outcome a would not affect the result for any other outcome, say b. Hence it might be thought that if any two outcomes were independent of a third, there would be no point in not reporting the preferences truthfully. It turns out that this is correct. It is given in the following theorem. Independence from irrelevant alternatives means that a choice between any two outcomes, say $\{a, b\}$, is not influenced by preferences for any other outcome.

Theorem 9.2: Suppose a voting mechanism is monotonic and allows a choice from all subsets. If this mechanism satisfies independence from irrelevant alternatives it is strategy-proof.

Proof: Immediate from the definition and theorem 9.1. ■

A corollary of these theorems is the following.

Corollary: If a unique Condorcet winner exists the voting mechanism is strategy-proof.

Proof: Suppose a unique winner exists such that $\varphi: (s^*) \rightarrow a$. Let B be some coalition of players that prefers b. $(s^*) = (s^*_{N-B}, s^*_B)$. From theorems 9.1 and 9.2 $v_B(s^*_{N-B}, s^*_B) \geq v_B(s^*_{N-B}, s^o_B)$. Hence it does not pay B to lie. ■

Theorems 9.1 and 9.2 are very simple, but they have some interesting implications. To see the first of these it is necessary to consider Arrow's famous impossibility theorem.

Arrow's theorem says that, if there are at least three alternatives, and the social choice rules satisfies a number of conditions, including independence from irrelevant alternatives, there is a dictator. That is, there must be a single individual whose preferences are decisive. The reason that Arrow's theorem is not of direct interest from a game theory perspective is that independence of alternatives means that the voting mechanism cannot be manipulated. Hence players do not have interesting strategies.

This leads to the following question. If independence from irrelevant alternatives is required, is it possible to escape the condition that the choice mechanism has a dictator?

9.3 The Gibbard-Satterthwaite theorem

The Gibbard-Satterthwaite theorem shows that, where there are more than two outcomes, any choice mechanism, φ, that is strategy-proof single valued and onto must also be dictatorial. This means that there is a trade-off between truthful reporting and the existence of a dictator. Single valued means that φ picks one and only one outcome $a \in A$. Onto means that there exists a set of preferences such that every member of A can be selected.

Theorem 9.3: (Gibbard (1973), Satterthwaite (1975)) Suppose A contains at least three outcomes and the domain of the preferences is unrestricted. The voting rule φ is strategy-proof if and only if there exists some agent i that is a dictator for all profiles. This means that $\varphi(A^{mn}) = a$ the most preferred outcome for i.

Proof: The argument that if there is a dictator the mechanism is strategy-proof is easy. Since the outcome is a it does not pay i to lie. It does not pay anyone else to lie since their preferences do not affect the outcome. The argument that if the mechanism is strategy-proof there is a dictator is more difficult. A demonstration is to be found in Craven (1992, p. 78) and a proof in Moulin (1983, p. 65). A proof that the dictator is the same in all preference profiles is simple. Let i be the dictator under some preference profile u. Assume $u_i(a) > u_i(b)$. Since i is the dictator, aPb even though $bP_{-i}a$ for all other voters. Now assume some other preference profile $v = (u_i, v_{-i})$. No more individuals can prefer b to a in v. Hence, the preferences for b have remained the same or decreased. Hence, by monotonicity if aPb in u it must be the case that aPb in v. Since aPb, then bPc means aPc even if $cP_{-i}a$. This establishes i is the dictator in v. ■

The intuition underlying the Gibbard-Satterthwaite theorem can be demonstrated by turning to the illustrations in section 9.2.4 that scoring methods and the Condorcet method are not strategy-proof.

Since the domain of the preferences is unrestricted it is legitimate to use the preferences in Figure 9.1. A Condorcet method is not decisive with these preferences because the outcome cycles between a, b and c. To make it decisive, some order must be imposed. Say it is $aPbPc$. Then i is the dictator. If some ordering is imposed that is not in Figure 9.1 we simply take the new example that contains this ordering. This is justified from unrestricted domain.

More generally, it follows from the proof of the first part that one way to make these methods decisive and strategy-proof is to suppose that some voter, i, is a dictator. The theorem shows that there is no other way to ensure strategy-proofness for any possible single valued voting rule with an unrestricted domain.

The reason that there is a single individual who is the dictator for all preference profiles is easy to understand. If the dictator changed with a change in the preference profile, then it is easy to think of an example where it would pay some voter, or coalition of voters, to change their stated preferences in order to alter the identity of the dictator.

It might be thought that the Gibbard-Satterthwaite theorem means that the case where a unique Condorcet winner exists also means that some player must be a dictator. This is not correct. Consider the corollary to theorem 9.2. A voting rule with a unique Condorcet winner is strategy-proof and need not have a dictator. Consider a preference profile like

$$aP_1bP_1c,\ bP_2aP_2c,\ cP_3bP_3a$$

In this case the winner is b. There is no dictator since player one can change to cP_1bP_1a to give the winner c. What went wrong?

The Gibbard-Satterthwaite theorem says that there is a dictator only if preference sets are unrestricted. The stipulation that a unique Condorcet winner must exist places restrictions on the preference set. This violates the conditions of the theorem.

It must be remembered that theorems such as the Gibbard-Satterthwaite and Arrow theorems are impossibility results. They say that no selection procedure of type φ can satisfy some general set of criteria C. All that has to be done to prove them is to find a set of preferences where the procedure fails for some $c \in C$. For example, consider the statement that the function $f = 1/x$ does not have a solution for all points in an unrestricted domain of the real numbers. This is proven by considering $x = 0$.

It is a characteristic of impossibility theorems that they invoke unrestricted domain savagely in the proofs. It is simple to show that this is an important topological property. This is because unrestricted domain means that the space of preferences cannot be continuously retracted or mapped into an outcome represented by a single point. Impossibility problems might be avoided in many specific choice procedures because preferences are restricted. For example, much of the theory of markets assumes that utilities are positive. Preferences are also restricted by institutional rules in many voting arrangements.

9.4 Restricted domain solutions and strategy-proof rules

The most important and well known restriction on the preference set is that the preferences are single peaked. This condition is defined as follows.

Definition: a^k is the most preferred outcome in an array $a^1, \ldots, a^m$, with i, j integers $i, j > 0$ and $i \neq j$ and $v(a^i) \neq v(a^j)$. If the sequence is $\ldots a^i, \ldots a^j \ldots$ then $j > i$. Preferences are single peaked if it is not possible that for $r > k$ we have $v(a^{r+1}) > v(a^r)$ or for $r < k$ we have $v(a^r) < v(a^{r-1})$. ■

Single peaked preferences are illustrated in Figure 9.2(a). The preference in Figure 9.2(b) is double peaked. Flat intervals are prevented by the stipulation that $v(a^i) \neq v(a^j)$.

The property of single peakedness means that the preferences can be set out in a single dimension and the pay-off for an outcome declines monotonically with its distance from the most preferred position. Alternatively, this can be taken as a definition of single peakedness.

An important example of single peakedness is preferences over political parties or programmes when these can be placed on a left–right scale. Another is preferences over the location of a facility on a straight road where players wish to minimize the distance travelled.

Single peaked preference schedules have some simple but important properties. To investigate these put the preferences in an ordered sequence $(a^1, \ldots, a^m)$. Let the number of voters at a^i be $a^{i\#}$. Without loss of generality, assume that the number of peaks is odd to simplify the discussion. Let the median peak be a^m. This is defined as

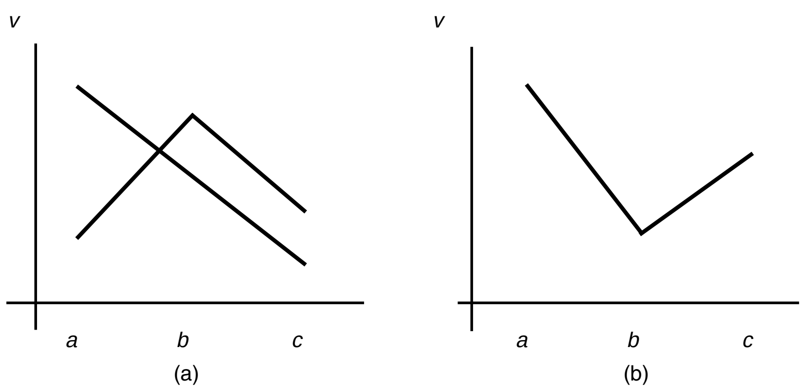


Figure 9.2 Single and double peaked preferences

$$m = x : \sum_{r=1}^{x} a^{r\#} \geq (n+1)/2 \text{ and } \sum_{r=x}^{n} a^{r\#} \geq (n+1)/2$$

It is obvious that the voters at the median peak and the voters on one side or the other can form a majority. Note that majorities will not form by jumping peaks. This gives a lower pay-off than a continuous coalition because it moves the outcome further away from the peak that could be attained with the continuous coalition. This means that the peak a^m is in the intersection of both sets of peaks required for a majority.

Note that the voter, or voters, at the median peak is not a dictator. This voter's position simply coincides with the median outcome.

An example is given in Figure 9.3 for five outcomes and seven voters. ▲ is a voter with a peak at this preference. In this case $a^m = a^4$.

Theorem 9.4: There is always a unique Condorcet winner when preferences are single peaked and there is an odd number of voters. The winner is at the median peak.

Proof (1): The proof follows immediately from the fact that $\varphi(a^m, a^j) = a^m$ from the definition of the median peak. ■

A similar theorem gives the median voter for continuous preferences. Where preferences are discrete and the number of voters is not odd it is possible that ties will occur. The uniqueness result can be preserved by adding a tie breaking procedure.

The proof of theorem 9.4 could also have been constructed using the theory of the core. Suppose that the value of winning is 1 and losing is 0. Then a coalition with the voters at a^m is in the core and a coalition without these voters has no value.

Proof (2): Let B be a coalition of voters from the left or the right of the linear ordering. Write a coalition B without k as $B \setminus k$. Then

$$v(B \setminus a^m) = 0 \text{ and } v(B \cup a^m) > 0$$

■

Any coalition that gives voters with a peak at m less than the value of the outcome a^m is defeated by the alternative coalition made up by the voters with a peak at a^m defecting.

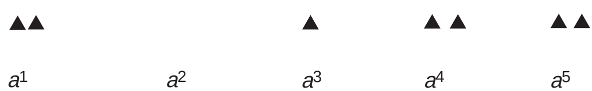


Figure 9.3 Median voter

An interesting corollary of this concerns distribution games and political conflicts where coalitions must form from the left or the right. Examples might be class conflict or ethnic or religious politics. In these cases, the voters in the median position can claim the entire value of the prize for winning.

This raises questions about the relation between the existence of strategy-proof games and the core.

9.5 Core stability

The fact that many voting rules might be open to strategic manipulation and may produce cycles or unstable outcomes is well known in the literature on spatial voting. Some see this finding as the major theme of the public choice literature (Mueller, 1991, p. 88).[3] What instability means is that for any coalition of voters B^k that can form under voting rule φ and express a preference for an outcome a^k such that

$$\varphi(B^k, N\backslash B^k) = a^k$$

it is optimal for some member of B^k to defect to join some other coalition so that $B^k \setminus i$, is losing and $B^o = N \setminus (B^k \setminus i)$ is winning. This gives[4]

$$\varphi(B^o, N\backslash B^o) = a^o$$

where $a^o \neq a^k$. In other words, if a^k is the outcome available from some winning coalition B^k, there is always some i that can join another winning coalition with pay-off a^o such that

$$v_i(a^o) > v_i(a^k)$$

The way in which this violates the condition required for the existence of the non-transferable utility core from Chapter 7 is demonstrated as follows. For the core let the pay-off $v(N) = v(a^*)$. This is the value of the outcome chosen by the all voter coalition and is the same as the value of the outcome chosen by some winning coalition B^k. Then for i

$$v_i(B^k) = v_i(N) = v_i(a^k)$$

But $v_i(a^o) > v_i(a^k)$ in some alternative coalition. Hence the core condition is violated.

An example of cyclic instability for three voters *i, j, k* with a majority voting system is given in Figure 9.4(a). For simplicity it is assumed that preferences are on

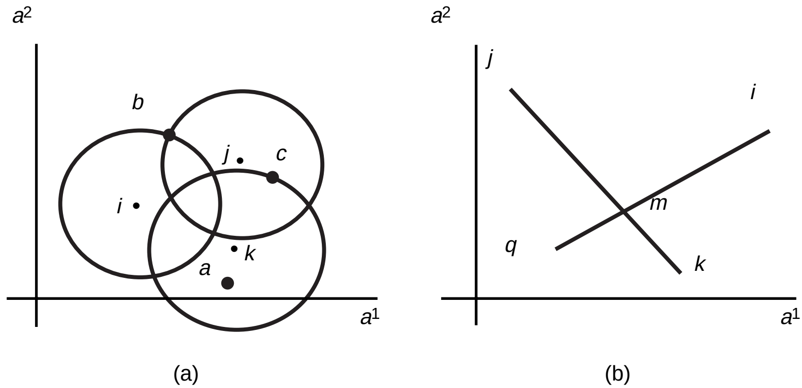


Figure 9.4 Cycling and the Plott condition

a continuous interval. They might be thought of as representing the amount to be spent on some outcome a^k such as health or defence. It is also assumed that utility declines from the ideal point and that the rate of decline is the same in all directions. Hence, preferences can be represented as circles in Euclidean space. Let the outcomes be a^1 and a^2.

It is easy to see that any winning coalition $B^o = (k, i)$ with $\varphi = a$ can be defeated by an alternative winning coalition $B^k = (i, j)$ with $\varphi = b$ if i defects in order to get a point nearer the preferred outcome. Similarly for (j, k) and $\varphi = c$ and so on.

A condition that will make this system stable is that there is a median in all directions. This is usually referred to as the Plott condition. Intuitively, the condition is that any line through the median point m must divide the voters into two sets with equal numbers. In this case any move away from the point m must reduce the pay-offs for $n/2 + 1$ voters. This is the same result as theorem 9.4 for more than one dimension.

An example of a median in all directions is given in Figure 9.4(b) for five voters (i, j, k, q, m) in two dimensions. Since preferences are circular, the contract lines are straight. The same general story would hold for elliptical preferences or preferences described on a different topology. In these cases the contract lines would have different properties.

Is it possible to test for the conditions under which any voting rule will have a stable core? This can be done using the Nakamura number.

9.5.1 Nakamura number

Definition: The Nakamura number of a simple game given by a voting rule φ is the minimum number of winning coalitions with empty intersections. Write this number as η. ■

A simple game is a game in which coalitions are either winning, losing or tied. Coalitions B^k and B^j are said to have empty intersection when there does not exist some individual x such that x is a member of B^k and B^j. This gives $B^k \cap B^j = \phi$.

9.5.1a Example of the Nakamura number

The Condorcet voting rule says that a winning coalition must have a majority. Consider three voters (i, j, k) and a Condorcet voting rule. In this case the minimum winning coalitions are $(i, j), (i, k), (j, k)$. Call these coalitions B^1, B^2 and B^3. Note that

$$B^1 \cap B^2 \cap B^3 = \phi$$

and

$$B^1 \cap B^2 = i$$

but i is not a member of B^3, for example. Therefore $\eta = 3$.

Theorem 9.5: The minimum number of winning coalitions with an empty intersection in a Condorcet majority voting system is three for n odd $n \geq 3$, and three for n even, $n \geq 6$.

Proof: The case where $n = 3$ was proven above. Let $n = 5$.

$$B^1 = (1, 2, 3), B^2 = (3, 4, 5), B^1 \cap B^2 \neq \phi$$

For an empty intersection we need $B^3 = (1, 4, 5)$. Now,

$$B^1 \cap B^2 \cap B^3 = \phi$$

For $n = 6$ we have

$$B^1 = (1, 2, 3, 4), B^2 = (3, 4, 5, 6), B^3 = (1, 2, 5, 6)$$

So the theorem is true for $n = 5$ and $n = 6$. Use induction to prove both the even and the odd cases together. For an odd number a winning coalition has $1 + (n - 1)/2$ and for an even number $1 + n/2$ votes.

Suppose the theorem is true for n even. Then it is true for three coalitions of size $1 + \frac{n}{2}$. Adding an additional voter gives $n + 1$ voters. This means it must be true for three coalitions $1 + (n + 1 - 1)/2$. This is true by definition. So it is true for n odd.

The theorem is true for n odd. Observe that for n even $B^1 \cap B^2$ must contain two voters. Add another voter i. Let the intersection for n even be $B^1 \cap B^2 = \{i, j\}$. For $B^1 \cap B^2 \cap B^3 = \phi$ it must be the case that $B^3 \backslash i, j\}$ is winning. This means that $(n+1) - 2 \geq 1 + (n+1)/2$. This is true for $n \geq 5$ as required. ■

Theorem 9.5 says that if n is even, $n \geq 6$. Consider $n = 4$.

$$B^1 = (1, 2, 3), B^2 = (2, 3, 4), B^3 = (3, 4, 1).$$

$B^1 \cap B^2 \cap B^3 \neq \phi$. This means we need $B^4 = (4, 1, 2)$. Hence $\eta = 4$.

This is used to produce the following elegant theorem. Write the number of outcomes in the set A as $a^{\#}$. Assume that all voters have consistent preferences.

9.5.2 Nakamura's theorem

Theorem 9.6: (Nakamura, 1979) A voting rule φ has a core if and only if η is strictly greater than $a^{\#}$.

Proof: Suppose $\eta > a^{\#} = k$ and there is no core. For some profile of preference orderings there is a cycle

$$a_1 P a_2 P \ldots P a_k P a_1$$

Since $k < \eta$ it follows from the empty intersection property that, for some agent

$$i, a_1 P_i a_2 P_i \ldots P_i a_k P_i a_1$$

because i is a member of every winning coalition. This contradicts the assumption that preferences are consistent.

The second part of the proof is constructed by supposing the core exists and $\eta \leq a^{\#} = k$. Let $\eta = a^{\#} = k$. Since preference profiles are not restricted, it is possible to have a preference profile

$$a_1 P_{B1} a_2 P_{B2} \ldots P_{Bk-1} a_k P_{Bk} a_1$$

for the k winning coalitions $B^1, B^2, \ldots B^k$. From the assumption $\eta = a^{\#} = k$ we have

$$\cap_k B = \phi$$

and there is no i that is a member of every winning coalition. This contradicts the assumption that there is a core for all acceptable preference profiles.[5] ■

Here is an example. Suppose that $\eta = a^{\#} = 3$. The voting rule holds for any distribution of preferences. Hence winning coalition B^1 can have aP_1b and winning coalitions B^2 and B^3 can have bP_2c and cP_3a. This assumption is legitimate because the coalitions have no members in common. Assuming the complete ordering of outcomes is the same as Figure 9.1 it is easy to see that there is at least one individual who will do strictly better in an alternative coalition. This violates the condition required for the existence of a core.

9.5.2a Applications of Nakamura's theorem

The way in which Nakamura's theorem can be used to explore the conditions under which voting is cyclic is illustrated as follows. Assume a Condorcet voting system with unrestricted domain of preferences. Theorem 9.5 shows that, for a Condorcet system with the number of voters $n \geq 5, \eta = 3$. Therefore, for $a^{\#} \geq 3$, there is no core with an unrestricted domain. Hence voting cycles can occur for any number of outcomes greater than three.

To prevent cycling for any $a^{\#} = k$ it is necessary to increase η to give $\eta > k$. This might be done by making the voting rule more strict. For example, the rule might be that a winning coalition has $(n + x)/2$ members where $x > 1$. Alternatively, it might be possible to restrict the way in which coalitions can form. This might be done by restricting preferences. Examples previously given were single peakedness and the Plott condition.

For single peaked preferences and simple majority voting there is no minimal number of winning coalitions with an empty intersection. For any winning coalitions $\cap B = m$. Expressed as a Nakamura number, this means that for single peaked preferences

$$\eta \to \infty$$

Hence cycling cannot occur.

One way to ensure single peakedness is to only vote for one issue at a time. To see how this operates, consider the two dimensional space in figure 9.4(a). A single issue method would be to hold a^i constant and pick the level for a^j. It will be noted that if, say, a^2 is held constant at a^{2*}, then the preferences along the single dimension a^1 are single peaked, as required. This is obvious by drawing the horizontal line at a^{2*}.

Cycling can also be prevented by blocking an alternative winning coalition reconsidering a candidate that has been eliminated by some previous winning coalition. This gives $a^{\#} = 2$ at any round. This ensures the voting rule is stable for any $\eta \geq 3$. This condition is met for any Condorcet compatible method, such as a tennis championship.

Since a candidate is eliminated at each round, the method just discussed is an example of voting by veto. This method probably deserves more attention than it usually gets in the design of voting schemes.

9.6 Voting by veto

The most attractive feature of voting by veto is that it endows a coalition of voters that is not winning, but is over some minimum size, the power to eliminate the most disliked candidates.[6] This may contribute to social stability where the game is being played for high stakes. There may be different ethnic and racial groups, for example, and fear of persecution or violence. Consider the following example of a voting by veto process where candidates can be eliminated in proportion to the number of votes. Write this rule φ^v. There are three voters, *i, j, k* with the following preference orderings. (*i, j*) forms a majority coalition. Writing preferences from the most to the least preferred from left to right

$$\begin{array}{rl} (i,j) & a,b,c,d,e,f,g \\ \{k\} & d,c,e,f,a,g,b \end{array}$$

In a Condorcet system

$$\varphi(A^{7\times 2}) = a$$

In a voting by veto system with elimination according to the portion of votes, (i, j) eliminates two outcomes for each one eliminated by $\{k\}$. Continuing this process of elimination with coalitions eliminating their least liked candidate gives

$$\varphi^v = c$$

It might be thought that this system is strategy-proof because it would never pay a coalition to eliminate an outcome it prefers more than one it prefers less. This is not correct, even though it seems perverse. The example above is not strategy-proof for example. In general a veto rule cannot be strategy-proof whenever it selects a single outcome. This follows immediately from the Gibbard-Satterthwaite theorem, and the fact that a veto process cannot have a dictator.

This raises the problem of the optimum design for voting by veto rule. Let the veto power of a coalition be a function of the number of voters in that coalition. Write this $\upsilon(k^i)$, where k^i is the number of voters in coalition B^i. Clearly, if voters have too much power to veto there may be no outcome at all and the game will not have a core. For example, assume that there are $a^\#$ candidates and that each coalition can veto a number which is equal to the same proportion of the candidates as its proportion of the vote, taken up to the nearest integer. This is the smallest integer

$$z > a^\# k^i/n$$

where $n = \sum k^i$. In this case there is no outcome since all the candidates will be vetoed.

The existence of the core requires that

$$\sum_{k_i \epsilon B} \upsilon(k^i) \leq \upsilon(n) \text{ by superadditivity}$$

$$\upsilon(n) < a^{\#}$$

These conditions are sufficient because a voting by veto rule prevents cycling.

If the veto power of coalitions is too small, the core may be large since insufficient candidates have been eliminated. In this case the rule will also be unsatisfactory.

Moulin (1983, p. 122) suggests a proportional veto solution to the problem of designing a suitable veto rule. It goes as follows.

Define an anonymous veto function as a function from the coalitions of voters to $a^{\#} - 1$ outcomes.

$$\varphi^{v*} : B \rightarrow a^{\#} - r$$

for some integer, r > 0, φ^{v} will be stable if there is at least one outcome that no coalition of voters can block while guaranteeing an increased pay-off to all its members. That is, the core is not empty.

Definition: The proportional veto voting rule is a rule φ^{v*} with a veto function υ defined as

$$\upsilon(k^i) = [x] - 1$$

where $[x]$ is the smallest integer bounded below by $a^{\#} k^i/n$. This gives $[x] - 1$ as the greatest integer strictly less than $a^{\#} k^i/n$. For example if $a^{\#} k^i/n = 3.375$, $[x] = 4$ and $[x] - 1 = 3$. ■

Theorem 9.7: (Moulin, 1983) The proportional veto function has a non-empty core and makes the set of stable outcomes as small as practicable.

Proof: The proof of this is given by Moulin (1983, pp. 126–35). It is intuitively plausible and is omitted due to its length. ■

The idea behind this theorem is easy to grasp. It says that coalitions of voters should be allowed to veto a number of candidates slightly less than the proportion of their votes. This obviously meets the two conditions $\sum \upsilon(k^i) \leq \upsilon(n)$ and $\upsilon(n) < a^{\#}$ with $\upsilon(n)$ as close to the lower integer bound of $a^{\#} - 1$ as practicable.

Notes

Chapter 1

1. See Friedman (1978) for a discussion of this possibility.
2. The papers collected in Barry and Hardin (1982) are a good introduction to this question.
3. See Rousseau, 'The Social Contract' (1979) and Hegel (1976). Goodin (1989) gives an interesting argument for the moral responsibilities of the state from a game theory perspective.
4. Using the term state to cover collective decisions and actions of all kinds Nozick makes the point as follows. The question that 'precedes questions about how the state should be organized, is whether there should be any state at all' (Nozick, 1974, p. 4).
5. It is worthwhile quoting Usher more fully. 'It is a little noticed but nonetheless remarkable fact about economic analysis that there is no role for violence at all. ... This caricature can be maintained through the implicit assumption on traditional welfare economics that property is secure ... each person's entitlements are protected by government. With the abandonment of that assumption comes a recognition of the role of actual or threatened violence in the maintenance of the social order ...' (1992, pp. xvi–xvii)
6. See, for example, de Jasay's claim that 'it is intuitively plausible' that the n-person prisoner's dilemma can be solved when people 'do not instantly club each other to death' (pp. 43–4). This is fun but it is not analysis and intuitions are misleading. Once upon a time it may have been 'intuitively plausible' that prisoner's dilemmas did not exist.
7. See, for example, Auster and Silver (1979).
8. For recent advances in the theory of the firm see Tirole (1989).
9. Rationality should more usefully be reserved for maximizing the material pay-offs to the specific player, if it is to have any substantive content. Nothing hinges on this point here.
10. Weibull (1996) gives a recent introduction to evolutionary game theory.
11. See Myerson (1991) for a discussion of equilibrium concepts.
12. For an introduction to probability measures and probability spaces see Grimmett and Stirzakev (1992, pp. 1–5).

Chapter 2

1. See Sylvan 'Anarchism' in Goodin and Pettit (1993) and Taylor (1982) for recent discussions of the anarchist position on authority.
2. See Kavka (1986) and Hampton (1986) for recent discussions of the game theoretical structure of Hobbes's state of nature.

3. Taylor (1987, p. 13), for example.
4. Rapoport (1968), for example, argues that the prisoner's dilemma reflects a wide variety of human interactions. Hardin (1971) argues that the logic of collective action problems generally is that of the prisoner's dilemma.
5. See, for example, Johansson (1991, p. 68).
6. One problem that is often overlooked is that such normative enforcement is not cost free. Communities often buy co-operation at a price of xenophobia and moral rigidity. Since the concern here is larger groups, these issues are not of interest.
7. See Taylor (1987, ch. 4) for elaborations on decisiveness. In his examples the decisive groups is a sub set of the total number of players, but the general point is not altered.
8. This sometimes known as the chain store paradox (Selten, 1978).
9. Dense in **R** means roughly that if each player is a point on the straight line the gaps between them are infinitely small.
10. Essentially the same solution is given by Hampton (1986, 147–88, 220–47), Kavka (1986) and Okada et al. (1991).

Chapter 3

1. The classical work is Smith (1982).
2. This problem can also be solved by calculating the expected value of the game for player 1. $E(v_1) = xy(1) + x(1-y)(0) + (1-x)y(3) + (1-x)(1-y)(-1)$. Optimizing, $\partial v_1/\partial x = 1 - 3y - 0$. Hence, if two is optimizing against one $y = 1/3$.
3. For a discussion of this see Myerson (1991, pp. 117–22), Axelrod (1984) and Harsanyi (1986).
4. To calculate this note that from the second round $(1-\delta)\sum \delta^{k-1} w_i^k = (1-\delta)\delta w_i^k / (1-\delta)$.
5. For $q(1-q)$ small the approximation used breaks down and the strategy s^q is closer to either s^h or s^d.
6. See Binmore (1992, pp. 429–33) and Smith (1986, pp. 188–91) for an analysis of this type of game.
7. The term $-pq$ takes account of those players who are hawks anyway. If this were not taken out there would be double counting.
8. Restricting strategies to a choice on the real number line gives it some nice mathematical properties. In particular, the strategy set is a compact metric space with the Euclidean metric. See Myerson (1991, p. 140) for a discussion.
9. This is usually called the probability distribution function.
10. This follows Smith (1986).
11. Bishop and Canning (1978).

Chapter 4

1. Friedman (1978) speculates on an outcome, but does attempt to answer the question in a systematic manner.
2. The term stealing is used throughout, even though it is inaccurate. Without rules of property there can be no stealing properly, so-called.

3. Takayama (1985, p. 127) for a discussion of this technique.
4. See Chaing (1984, pp. 387–96) for a discussion of applications of the Arrow Enthoven conditions to quasi-concavity.
5. Schelling (1960).

Chapter 5

1. Nozick (1974, pp. 12–17).
2. Mimeo. Reported in Fudenberg and Tirole (1992, p. 156) and Sorin (1992).
3. It is not even clear what justice means in the no rules state of nature.

Chapter 6

1. See Smith (1982, p. 172) for a discussion of the application of this idea to the evolution of culture and norms.
2. The old way to sort out these equilibria was to refine the Nash equilibrium concept in various ways. This is not very satisfactory. See Binmore (1992, p. 13).
3. The most common definition of stability is Liapunov stability. This roughly says that if a system starts within a space of radius ε around an equilibrium it stays within a space of radius δ around the equilibrium. δ may or may not be greater than ε. The system may converge to the equilibrium in various ways. Alternatively it may orbit within the space δ.
4. The approach loosely follows the discussion in Weibull (1996). Smith (1982) provides a good introduction and should be read. Binmore (1992, pp. 422–34) is very accessible. Samuelson (1997) covers much of the same material.
5. For a discussion of asymptotic stability in evolutionary games see Samuelson (1997, pp. 68–75).
6. For a slightly different generalization of this game see Smith (1982, p. 19).

Chapter 7

1. This problem is discussed further in Myerson (1991, pp. 422–7).
2. The mathematical requirement is that the system must be consistent for a solution to exist. There may, of course, be infinitely many solutions.
3. A more rigorous analysis would show how small the core would become.
4. This is the usual result for public goods. See Mueller (1991, pp. 18–19) for further discussion of this point.

Chapter 8

1. Fudenbberg and Tirole (1992, pp. 243–318) give some of the abstract principles. For a less abstract introduction see Binmore (1992, pp. 501–69). For an introduction to

the general problem of asymetrical information and principal agent problems see Rasmusen (1994).

2. Note that, by the principle of optimality, the path s^* is determined by starting with s_i^{*km} and working backwards.
3. If this is unclear, see Binmore (1992, p. 531) or Kreps (1990, pp. 691–5).
4. Binmore (1992, p. 531).
5. See Green and Laffont (1979, 4.3).
6. See Green and Laffont (1979, p. 61). See Moulin (1991, p. 211) for a different version of this proof.

Chapter 9

1. It may be more reasonable, for example, to allow voters to reject candidate *b* in order to get *a*, rather than the least preferred candidate *c*, even thought this violates independence of irrelevant alternatives.
2. See Mueller (1991) for a discussion of voting and Ordeshook (1986) for an introduction to the application of game theory to voting problems. Moulin (1983) is a mathematically more advanced analysis.
3. See Ordeshook (1986 pp. 144–202) for an introduction to instability.
4. B^c is all the members of N such that $i \in N$ means $i \notin B$. This is N or the complement of B. In a simple game, if B is winning then its complement, that is the rest of the voters, are losing.
5. This is set out in full in Moulin (1993, pp. 185–6). See also Moulin (1991, pp. 296–7)
6. The idea of voting by veto was first discussed by Mueller. See Mueller (1991, pp. 139–44) for a discussion.

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