NEW DIRECTIONS IN MATHEMATICS AND SCIENCE EDUCATION

Theorems in School

From History, Epistemology and Cognition to Classroom Practice

Paolo Boero

Edited Volume

SensePublishers

Theorems in School

NEW DIRECTIONS IN MATHEMATICS AND SCIENCE EDUCATION
Volume 1

Scope

Mathematics and science education are in a state of change. Received models of teaching, curriculum, and researching in the two fields are adopting and developing new ways of thinking about how people of all ages know, learn, and develop. The recent literature in both fields includes contributions focusing on issues and using theoretical frames that were unthinkable a decade ago. For example, we see an increase in the use of conceptual and methodological tools from anthropology and semiotics to understand how different forms of knowledge are interconnected, how students learn, how textbooks are written, etcetera. Science and mathematics educators also have turned to issues such as identity and emotion as salient to the way in which people of all ages display and develop knowledge and skills. And they use dialectical or phenomenological approaches to answer ever arising questions about learning and development in science and mathematics.

The purpose of this series is to encourage the publication of books that are close to the cutting edge of both fields. The series aims at becoming a leader in providing refreshing and bold new work—rather than out-of-date reproductions of past states of the art—shaping both fields more than reproducing them, thereby closing the traditional gap that exists between journal articles and books in terms of their salience about what is new. The series is intended not only to foster books concerned with knowing, learning, and teaching in school but also with doing and learning mathematics and science across the whole lifespan (e.g., science in kindergarten; mathematics at work); and it is to be a vehicle for publishing books that fall between the two domains—such as when scientists learn about graphs and graphing as part of their work.

Theorems in School
From History, Epistemology and Cognition to Classroom Practice

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SENSE PUBLISHERS
ROTTERDAM / TAIPEI

A C.I.P. record for this book is available from the Library of Congress.

Paperback ISBN 978-90-77874-21-9
Hardback ISBN 978-90-77874-22-6

Published by: Sense Publishers,
P.O. Box 21858, 3001 AW Rotterdam, The Netherlands

Printed on acid-free paper

Cover image: Partial of Raphael's fresco "The school of Athens", courtesy of Vatican Museums, Vatican City.

CONTENTS

Preface

The ongoing value of proof
Gila Hanna 3

Introduction

Theorems in school: An introduction
Paolo Boero 19

Part I: The historical and epistemological dimension

1 Origin of mathematical proof: History and epistemology
Gilbert Arsac 27

2 The proof in the 20th century: From Hilbert to automatic theorem proving
Ferdinando Arzarello 43

3 Students' proof schemes revisited
Guershon Harel 65

Part II: Curricular choices, historical traditions and learning of proof: Two national case studies

4 Curriculum change and geometrical reasoning
Celia Hoyles and Lulu Healy 81

5 The tradition and role of proof in mathematics education in Hungary
Julianna Szendrei-Radnai and Judit Török 117

Part III: Argumentation and proof

6 Cognitive functioning and the understanding of mathematical processes of proof
Raymond Duval 137

7 Some remarks about argumentation and proof
Nadia Douek 163

Part IV: Didactical aspects

8	Making possible the discussion of "impossible in mathematics" *Greisy Winicki-Landman*	185
9	The development of proof making by students *Carolyn A. Maher, Ethel M. Muter and Regina D. Kiczek*	197
10	Approaching and developing the culture of geometry theorems in school: A theoretical framework *Marolina Bartolini Bussi, Paolo Boero, Franca Ferri, Rossella Garuti and Maria Alessandra Mariotti*	211
11	Construction problems in primary school: A case from the geometry of circle *Maria G. Bartolini Bussi, Mara Boni and Franca Ferri*	219
12	Approaching theorems in grade VIII: Some mental processes underlying producing and proving conjectures, and conditions suitable to enhance them *Paolo Boero, Rossella Garuti and Enrica Lemut*	249
13	From dynamic exploration to "theory" and "theorems" (from 6th to 8th grades) *Laura Parenti, Maria Teresa Barberis, Massima Pastorino and Paola Viglienzone*	265
14	Geometrical proof: The mediation of a microworld *Maria Alessandra Mariotti*	285
15	The transition to formal proof in geometry *Ferdinando Arzarello, Federica Olivero, Domingo Paola and Ornella Robutti*	305

PREFACE

GILA HANNA

THE ONGOING VALUE OF PROOF

Over the past thirty years or so proof has been relegated to a less prominent role in the secondary mathematics curriculum in North America. This has come about in part because many mathematics educators have been influenced by certain developments in mathematics and in mathematics education to believe that proof is no longer central to mathematical theory and practice, and that in any case its use in the classroom will not promote learning. As a result many educators appear to have sought relief from the effort of teaching proof by avoiding it altogether.

In mathematics itself the use of computer-assisted proofs, the growing recognition accorded mathematical experimentation, and the invention of new types of proof that do not fit the standard mould have led some to argue that mathematicians will come to accept such forms of mathematical validation in place of deductive proof. The influence of these developments in mathematics has been strongly reinforced by the claims of some mathematics educators, inspired in part by the work of Lakatos, that deductive proof is not central to mathematical discovery, that mathematics is "fallible" in any case, and that proof is an authoritarian affront to modern social values.

This state of affairs has caused great concern among other mathematics educators. One of them was Greeno (1994), who laid the blame squarely on misconceptions as to the nature of proof:

> Regarding educational practice, I am alarmed by what appears to be a trend toward making proofs disappear from precollege mathematics education, and I believe that this could be remedied by a more adequate theoretical account of the epistemological significance of proof in mathematics. (pp. 270–271)

This chapter holds that none of the developments mentioned really undermines the value of proof, and that many of the assertions made in their wake are either simply wrong or based upon misunderstandings (primarily on the part of mathematics educators). It maintains that proof deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding.

THE INFLUENCE OF DEVELOPMENTS IN MATHEMATICS

A number of recent developments in mathematical practice, most of them reflecting in some way the growing use of computers, have caused some mathematicians

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 3–16.

and others to call into question the continuing importance of proof or indeed to announce its imminent death. John Horgan (1993), a staff writer of *Scientific American*, makes this prediction in his article “The death of proof” that appeared in its October 1993 issue.

Computer Proofs and a Potential Semi-Rigorous Culture

One of the developments that prompted Horgan’s announcement is the use of computers to create or validate enormously long proofs, such as the recently published proofs of the four-colour theorem (Appel and Haken) or of the solution to the party problem (Radziszowski and McKay). These proofs require computations so long they could not possibly be performed or even verified by a human being. Because computers and computer programs are fallible, then, mathematicians will have to accept that assertions proved in this way can never be more than provisionally true.

This is a limitation in principle, but computing also has practical limitations, for all its ever-increasing power. There will always be tasks that take too long or are thought too expensive. Computer proofs are no exception, and so mathematicians have explored the implications that these limitations might have for mathematical practice. One prediction is that mathematicians, in the face of impractical times or prohibitive cost, will come to settle for “semi-rigour.”

In an article published in 1993 in the *Notices of the American Mathematical Society* entitled “Theorems for a price: Tomorrow’s semi-rigorous mathematical culture,” the mathematician Doron Zeilberger predicts that with the advent of computer proofs a “new testament is going to be written.” As “absolute proof becomes more and more expensive,” he maintains, mathematicians will use proofs which are less complete, but cheaper. He points to the example of algorithmic proof theory for hypergeometric identities, where there is no lack of well-known algorithms. The problem is that some cases require computations which even on tomorrow’s computers would take so long that they would exhaust the budget, if not the lifetime, of the researcher. He concludes that mathematicians will choose to limit the amount of computation allocated even to theorems which, in principle, are easily provable, opting for a less costly “almost certainty.” Furthermore, he predicts that mathematicians as a whole will come to accept such “semi-rigour” as a legitimate form of mathematical validation.

A mathematical conjecture has always been considered no more than a conjecture until proven, so it is not surprising that Zeilberger’s comments were quickly challenged by another mathematician. In an article published in the *Mathematical Intelligencer* (1994) with the dismissive title: “The death of proof? Semi-rigorous mathematics? You’ve got to be kidding!” George Andrews maintains that Zeilberger’s evidence is simply not convincing. That certain algorithms may prove too expensive to execute, he says, does not mean that mathematicians will now give up the idea of absolute proof with its “concomitantly great insight and, dare I say it, beauty” (p. 17).

And others have already pointed out that cheaper, non-rigorous proofs may prove costly in the long run. Saunders MacLane (1996) reported that in Italy during the years 1880–1920 several results in algebraic geometry were published without careful proving. The situation became so bad that "unverified rumour seems to have it that a real triumph for an Italian algebraic geometer consisted in proving a new theorem and simultaneously proposing a counter-example to the theorem" (p. 2). Italian results in algebraic geometry were discredited until several mathematicians, including Emmy Noether, cleared up the difficult points by applying much more rigorous standards of proof.

New Types of Proof

Doubts about proof as a whole have also been raised by new types of proof that have little in common with traditional forms. A particularly fascinating development is the recently introduced concept of zero-knowledge proof (Blum, 1986), originally defined by Goldwasser, Micali and Rackoff (1985). This is an interactive protocol involving two parties, a prover and a verifier. It enables the prover to provide to the verifier convincing evidence that a proof exists, without disclosing any information about the proof itself. As a result of such an interaction the verifier is convinced that the theorem in question is true and that the prover knows a proof, but the verifier has zero knowledge of the proof itself and thus is not in a position to convince others.

In principle a zero-knowledge proof may be carried out with or without a computer. In terms of our topic, however, the most significant feature of the zero-knowledge method is that it is entirely at odds with the traditional view of proof as a demonstration open to inspection. This clearly thwarts the exchange of opinion among mathematicians by which a proof has traditionally come to be accepted.

Another interesting innovation is that of holographic proof (Babai, 1994; Cipra, 1993). Like zero-knowledge proof, this concept was introduced by computer scientists in collaboration with mathematicians. It consists of transforming a proof into a so-called transparent form that is verified by spot checks, rather than by checking every line. The authors of this concept have shown that it is possible to rewrite a proof (in great detail, using a formal language) in such a way that if there is an error at any point in the original proof it will be spread more or less evenly throughout the rewritten proof (the transparent form). Thus to determine whether the proof is free of error one need only check randomly selected lines in the transparent form.

By using a computer to increase the number of spot checks, the probability that an erroneous proof will be accepted as correct can be made as small as desired (though of course not infinitely small). Thus a holographic proof can yield near-certainty, and the degree of near-certainty can be precisely quantified. Nevertheless, a holographic proof, like a zero-knowledge proof, is entirely at odds with the traditional view of mathematical proof, because it does not meet the requirement that every single line of the proof be open to verification.

Experimental Mathematics

Zero-knowledge proofs, holographic proofs and the creation and verification of extremely long proofs such as that of the four-colour theorem are feasible only because of computers. Yet even these innovative types of proof are traditional, in the sense that they remain analytic proofs. More and more mathematicians appear to be doing all their work outside the bounds of deductive proof, however, confirming mathematical properties experimentally. A case in point is the Geometry Center at the University of Minnesota, where mathematicians use computer graphics to examine the properties of four-dimensional hypercubes and other figures, or to study transformations such as the twisting and smashing of spheres.

Even today one does not usually associate mathematics with empirical investigations, yet mathematicians have long carried out experiments to formulate and test conjectures (knowing full well that such testing did not constitute proof). Earlier mathematicians, limited to testing a small number of cases, would undoubtedly have done even more extensive experimentation if they had had the means. Thus today's experimental mathematics would not seem to differ in principle from what has been done all along.

What does seem to be new is that more and more mathematicians spend their time almost exclusively on experimentation, and so naturally wish to assert a claim to its importance in its own right. Horgan quotes several mathematicians who assert that experimental methods have acquired a new respectability. These methods have certainly received increased attention and funding following the growth of graphics-oriented fields such as chaos theory and non-linear dynamics.

Certainly many more mathematicians have come to appreciate the power of computers in communicating mathematical concepts. Some of them are going well beyond communication, however. In a clear departure from previous practice, some now see it as legitimate to engage in experimental mathematics as a form of mathematical justification. Horgan maintains that:

> … some mathematicians are challenging the notion that formal proofs should be the supreme standard of proof. Although no one advocates doing away with proofs altogether, some practitioners think the validity of certain propositions may be better established by comparing them with experiments run on computers or with real-world phenomena. (p. 94)

The implication of such a view is that experimentation is not only a prestigious mathematical activity, but also an alternative to proof, an equally valid form of mathematical confirmation. This would seem to redefine "experimental mathematics" as a new discipline, one which is no longer subject to the criteria by which mathematical truth has traditionally been judged.

The founding of *Experimental Mathematics* in 1991 might be seen as a portent of such a new and independent discipline. This new quarterly does differ markedly from traditional journals, in that it publishes the results of computer explorations rather than theorems and proofs. But does this mean that its editors think proof is dead? This would not seem to be the case. In their paper "Experimentation and proof in mathematics" the editors of *Experimental Mathematics*, Epstein and Levy, first point out the enhanced potential of experimentation in the age of the computer: "the use of computers gives mathematicians another view of reality and another tool for investigating the correctness of a piece of mathematics through investigating examples" (1995, p. 674). They then go on, however, to make very clear how they believe experimentation fits into the mathematical scheme of things:

> Note that we do value proofs: experimentally inspired results that can be proved are more desirable than conjectural ones.… The objective of *Experimental Mathematics* is to play a role in the discovery of formal proofs, not to displace them (p. 671).… We believe that, far from undermining rigor, the use of computers in mathematics research will enhance it in several ways. (p. 674)

A New Division of Labour within Mathematics?

Many mathematicians are nevertheless very concerned that the recognition of experimentation as a valid full-time mathematical activity may obscure the fact that its results cannot be considered to have been proven. They do not agree on what, if anything, should be done about this. Some propose separation: that heuristic results be isolated as a clearly separate category.

Jaffe and Quinn (1993), for example, in their paper "Theoretical mathematics: Toward a cultural synthesis of mathematics and theoretical physics," stress how important it is to distinguish unequivocally between results based on rigorous proof and those based on heuristic arguments. They even suggest labels for the two activities, proposing the former be called "rigorous mathematics" and the latter "theoretical mathematics," by which they mean heuristic or speculative.

Jaffe and Quinn are motivated by a concern for standards of rigour, which they propose to preserve by isolating rigorous from non-rigorous mathematics through a new division of labour. They suggest that non-rigorous mathematics ("theoretical mathematics") be considered a valid branch of mathematics in its own right, and that mathematicians be evaluated by the standards of the branch to which they choose to belong.

The suggestion that mathematicians be divided into two camps brought swift and varied reactions, sixteen of them in the *Bulletin of the American Mathematical Society* (1994). William Thurston, for example, responded in an eighteen-page essay entitled "On proof and progress in mathematics," in which he opposes the division suggested by Jaffe and Quinn. In his view the important question is not "how do mathematicians prove theorems?" or "how do mathematicians make progress in mathematics?" but how they "advance human understanding of mathematics," and accordingly he believes it wrong to split mathematics on the basis of standards of rigour. Though he does not question the role of proof in validation, he sees its main value in its ability to communicate ideas and generate understanding. Accordingly he proposes to mathematicians, who have traditionally gained recognition among their peers primarily by proving theorems, that they all undertake to recognize and value the entire range of activities that advance understanding in their common discipline.

Fifteen other prominent mathematicians gave shorter responses. Most rejected the proposal put forward by Jaffe and Quinn to recognize two separate branches of mathematical activity (Atiyah et al., 1994). James Glimm wrote that if mathematics is to cope with the "serious expansion in the amount of speculation" it will need to adhere to the "absolute standard of logically correct reasoning [which] was developed and tested in the crucible of history" (p. 184).

Though driven, as were Jaffe and Quinn, by the growth of experimental mathematics and by a concern for rigour, it is clear that Glimm has come to precisely the opposite conclusion. While Jaffe and Quinn seem to believe that identifying and welcoming heuristic mathematics as a separate (though perhaps lesser) discipline would prevent it from establishing itself as a method of mathematical confirmation equal in value to rigorous proof, Glimm appears to fear that such isolation would have the opposite effect of allowing heuristics to stake this parallel claim.

But the responses also revealed differing views on the role of rigorous proof. Saunders MacLane stated that "mathematics does not need to copy the style of experimental physics. Mathematics rests on proof—and proof is eternal" (p. 193), while Atiyah conceded that "Perhaps we now have high standards of proof to aim at but, in the early stages of new developments, we must be prepared to act in more buccaneering style" (p. 178). And, not surprisingly, Mandelbrot asserted that rigour is "besides [sic] the point and usually distracting, even when possible."

Mandelbrot also takes exception in his response to the customary practice of awarding credit only to those who prove conjectures, slighting those who came up with them in the first place. Indeed, one cannot ignore that the recent controversies over the place of experimentation and other heuristic approaches may be motivated as much by a concern for professional recognition as by disagreement over the nature of mathematical truth.

Certainly in these controversies the issue of the importance and prestige of heuristics has become intertwined, often confusingly, with the issue of the role of proof as the arbiter of mathematical truth. In the recent discussion triggered by Jaffe and Quinn, however, there is a perhaps surprising degree of agreement. All the participants would seem to agree with Albert Schwartz that heuristic mathematics is an important and legitimate part of their discipline. But none suggested that mathematicians carry out their work without a view to the ultimate test of proof. Those who agreed, as most did, that mathematicians should accord more recognition to those who come up with interesting and productive heuristic results, were nevertheless of the opinion that such results remain conjectures until validated by proof.

THE INFLUENCE OF LAKATOS

Mathematics educators in North America have been propelled in the direction of a diminished role for proof in the curriculum, however, not only by the recent developments in mathematical practice discussed above, but also by interpretations given to the work of Imre Lakatos. His thinking, published first as a dissertation in 1961 and finally as *Proofs and refutations* in 1976, provoked much discussion among philosophers, and in particular among philosophers of mathematics (Agassi, 1981; Feyerabend, 1975; Hacking, 1979; Lehman, 1980; Steiner, 1983). Whatever their assessment of his claims as a whole, they tended to accept Lakatos' principal insight that the critique of mathematical results by others has been the motive force in the growth of mathematical knowledge.

Practising mathematicians were impressed by his work as well, in particular by his detailed study of how the proof of Euler's theorem had evolved over time. This study shed light upon many previously unappreciated aspects of mathematical activity, and for many mathematicians Lakatos' account of the dynamics of mathematical discovery rang true.

Lakatos' ideas were brought to the attention of North American mathematics educators primarily by Davis and Hersh (1981) in their book *The Mathematical Experience*. Their enthusiastic exposition of Lakatos' approach gained for it broad acceptance among these educators, who assumed this approach to be more widely applicable in mathematics itself than in fact it is.

It is not surprising that such a fascinating new way of looking at mathematical discovery diverted attention from its weaknesses. The method of proof analysis is admittedly engaging, but the case for it as a general method rests upon two examples, one of which is the study of polyhedra—an area in which it is relatively easy to suggest the counterexamples required. This method does not even begin to explain some important cases of mathematical discovery, however. It has nothing to say about set-theory research and the acceptance of the Zermelo-Fraenkel axioms, or about the emergence of non-standard analysis, or in fact about the many mathematical discoveries that did not start with a primitive conjecture.

It is not difficult, in fact, to cite cases in which a proof was found or a mathematical discovery made in a way radically different from the process of heuristic refutation described in *Proofs and refutations*. Even in the proof of Euler's theorem cited by Lakatos, for example, refutation is redundant; as soon as adequate definitions have been formulated the theorem can be proved for all possible cases without further discussion. Indeed, whenever mathematicians work with adequate definitions (or an adequate "conceptual setting," to use Bourbaki's term), the process of proof is not one of heuristic refutation. In "A renaissance of empiricism in the recent philosophy of mathematics" (1978, p. 36), Lakatos himself says:

> Not all formal mathematical theories are in equal danger of heuristic refutations. For instance, *elementary group theory* is scarcely in any danger; in this case the original informal theory has been so radically replaced by the axiomatic that heuristic refutations seem to be inconceivable.

In *Proofs and Refutations* Lakatos defines proof as a "thought experiment…a decomposition of the original conjecture into subconjectures or lemmas" (p. 9). For example, in his interpretation of the history of Euler's theorem for a polyhedron ($V–E+F=2$, where V is the number of vertices, E the number of edges, and F the number of faces), Lakatos describes a thought experiment in which one imagines stretching a rubber polyhedron and observing the effects of its manipulation. He goes on, however, to describe a broader process which allows proofs and refutations to interact, generates counter-examples and "informal falsifiers," gives rise to happy guesses, and ends with a well-formulated result.

This approach can be viewed as an attempt to examine mathematics from Popper's point of view, to erect a critique of deductivism in mathematics parallel to Popper's critique of inductivism in the physical sciences. Taking "induction" to mean the verification of general laws on the basis of observational data, Popper hoped to show that "empirical science does not really rely upon a principle of induction" (Putnam, 1987). Similarly, Lakatos hoped to show that verification in mathematics does not rely on "Euclidean deductivism." In describing the heuristic process, Lakatos constantly attacks what he calls the "Euclidean programme," which in his opinion aims at making mathematics "certain and infallible."

But the truth is, first of all, that when mathematicians have undertaken the heuristic method which Lakatos describes, or one similar to it, it has almost always been for the purpose of arriving at certainty. In the case of Euclid's theorem, for example, the long heuristic process did lead, in fact, to a proof which satisfies the accepted criteria of mathematical certainty. As Ian Hacking (1979) put it: "Critical discussion can enable a conjecture to evolve into logical truth. In the beginning Euler's theorem was false; in the end it is true. The theorem has been 'analytified'."

Secondly, the concept of fallibilism would seem to be a red herring. Mark Steiner has shown that in the eyes of present-day topologists Euler's theorem is "not about a polyhedron so much as about the underlying space the polyhedron divides" (p. 514). (He also shows that the modern proof is more explanatory than the one from the 19th century which Lakatos studied.) Steiner comes to the conclusion that the history of Euler's theorem in the 20th century not only provides a case in which Lakatos' model does not work, but, more importantly, demonstrates that we "can have progress without fallibilism" (p. 521). He also states that "despite the title of his book, Lakatos' mathematical realism can be profitably disengaged, not only from his fallibilism, but from the method of proofs and refutations itself!" (p. 510).

John Conway has remarked recently that Lakatos' *Proofs and Refutations* "is a very interesting book, but I fear is definitely misleading as regards mathematics in general" (Sept. 1995, request for advice, www.forum.swarthmore.edu). And in words which seem to sum up the present discussion, Conway adds:

> It is misleading to take this example (Euler's) as typical of the development of mathematics. Most mathematical theorems do get proved, and stay proved; the original proof may not be quite satisfactory according to later standards of proof, but that is a fairly trivial matter. In many cases there has been a significant omission or error in the first attempt at a proof, which later had to be corrected; but there have been very few cases like Euler's theorem, in which the discussion continued for several centuries.

Let us now turn to the manner in which Lakatos' ideas have come to influence the curriculum, at least in North America. Lakatos chose, perhaps with good reason, to put some of his ideas rather dramatically. Some mathematics educators would seem to have taken such assertions literally and sought to translate them directly into classroom practice (Dawson, 1969; Lampert, 1990).

Lakatos dismissed certainty and infallibility with the dramatic assertion "we never know, we only guess," for example, and this has led some to consider mathematical knowledge to be provisional. Ernest (1996), for example, stated that "mathematics knowledge is understood to be fallible and eternally open to revision, both in terms of its proofs and its concepts" (p. 808). In addition, the very terms "informal falsifier" and "fallibility" of mathematics seem to have led many mathematics educators to propose downplaying "formal" mathematics in the curriculum (Dossey, 1992; Hersh, 1986).

Echoes of Lakatos' thinking can be heard quite clearly in the curriculum guidance developed in the United States by the National Council of Teachers of Mathematics. In response to wide-spread concern for the quality of the mathematics curriculum, the NCTM has published *Standards for curriculum and evaluation* and *Professional standards for teaching mathematics*, covering the entire range from kindergarten through Grade 12 (NCTM, 1989, 1991). There is no national curriculum in the United States or Canada, where education is the responsibility of each state or province, but the NCTM *Standards*, though not binding, are very influential in both countries.

The authors of these guidelines, in their desire to reflect a modern view of mathematics, incorporated into them a position of relativism. According to John Dossey (1992), president of NCTM at the time the *Standards* were being drafted, "[T]he leaders and professional organisations in mathematics education are promoting a conception of mathematics that reflects a decidedly relativistic view of mathematics" (p. 45).

It may have been these views that led the NCTM *Standards* to give short shrift to proof, avoiding almost all mention of the term. The only explicit reference to proof, in fact, is in the context of preparation for post-secondary education, where the document states that "… college-intending students can … construct proofs for mathematical assertions, including indirect proofs and proofs by mathematical induction." The implication is that students who do not intend to pursue post-secondary studies need not encounter the concept of proof.

Some of its recommendations (such as the development of short sequences of theorems, and the use of deductive arguments expressed orally and in sentence form) do offer a faint glimmer of proof (pp. 126–127). But the NCTM explicitly recommended decreased attention to proof even in the geometry curriculum, suggesting, as topics to be de-emphasized, two-column proofs, proofs of incidence, proofs of betweenness theorems, and Euclidean geometry as an axiomatic system.

This tendency to downplay the role of proof in mathematics is surely misguided. In the first place, formal proof arose as a response to a persistent concern for justification, a concern reaching back to Aristotle and Euclid, through Frege and Leibniz. There has always been a need to justify new results (and often previous results as well), not always in the limited sense of establishing their truth, but rather in the broader sense of providing adequate grounds for their plausibility. Formal mathematical proof has been and remains one quite useful answer to this concern for justification.

Secondly, it is a mistake to think that the curriculum would be more reflective of mathematical practice if it were to limit itself to the use of informal counterexamples. The history of mathematics clearly shows that it is not the case, as Lakatos seems to have implied, that only heuristics and other “informal” mathematics are capable of providing counterexamples. Indeed, formal proofs themselves have often provided counterexamples to previously accepted theories or definitions. For instance, as Mark Steiner (1983, p. 502-521) points out, Peano provided a counterexample to the definition of a curve as the path of a continuously moving point by showing formally that a moving point could fill a two-dimensional area.

Gödel’s famous incompleteness proofs are another example, with an interesting and ironic twist. In this case formal proofs were employed to demonstrate that the axiomatic method itself has inherent limitations. Gödel could not have produced these proofs without using a comprehensive system of notation for the statements of pure arithmetic and a systematic codification of formal logic, both developed in the *Principia* for the purpose of arguing the Frege–Russell thesis that mathematics can be reduced to logic. His proofs could certainly not have been produced in informal mathematics or reduced to direct inspection.

Nor does it seem reasonable to assume that Gödel’s conclusions could have been arrived at through a discovery of counterexamples (“monster-barring”) followed by a denial (“monster-adjusting”), or by finding unexplained exceptions (“exception-barring”) or unstated assumptions (“hidden lemmas”). Curiously enough, however, when some educators make a case that formal proof and rigour should be downplayed in the curriculum they rest their case on Gödel’s most formal proof.

THE INFLUENCE OF SOCIAL VALUES

In the minds of many mathematics educators in North America the status of proof has also been called into question by the claim put forward, primarily by other educators, that it is a key element in an authoritarian view of mathematics (Confrey, 1994; Ernest, 1991; Nickson, 1994). This claim owes something to Lakatos (1976), who not only challenged the “Euclidean programme” for an “authoritative, infallible, irrefutable mathematics,” as noted, but also wrote of the dangers of elitism in mathematics. But it surely owes its prominence and its degree of acceptance primarily to the prevailing wind of “relativism” that seems to dominate the North American “intellectual” climate.

Indeed, supporters of this claim would say that the “Euclidean” view is in conflict with the present values of society, which dictate not only that one need not defer to authority, but also that one should not regard any knowledge as infallible or irrefutable. Some even appear to see proof in general, and rigorous proof in particular, as a mechanism of control wielded by an authoritarian establishment to help impose upon students a body of knowledge that it does regard as infallible and irrefutable.

It must be stressed that such views are not in the first instance a protest against authoritarianism in the classroom, but rather a projection upon the curriculum debate of attitudes that have their origins in the popular culture of the United States. Discussing these attitudes, the philosopher of science Larry Laudan says:

> The displacement of the idea that facts and evidence matter by the idea that everything boils down to subjective interests and perspectives is—second only to American political campaigns—the most prominent and pernicious manifestation of anti-intellectualism in our time. (1990, p. x)

Of course mathematics has sometimes been taught in an authoritarian way, as have other subjects, but one could hardly maintain that there has been a recent consensus among educators that it should be. One can only despair to find that proof has become the target of what would seem to be no more than a misguided desire to impose a sort of “political correctness” on mathematics education.

It is not easy to refute such a view of mathematics. In the first place, it is not easy to understand what it means to say that mathematics or a mathematical proof is “authoritative.” Certainly a proof offered by a very reputable mathematician would initially be given the benefit of the doubt, and in that sense the fact that this mathematician is considered an “authority” by other mathematicians would play some role in the eventual acceptance of the proof. But the claim seems to be that the very use of proof is authoritarian, and this claim is hard to fathom.

In fact the opposite is true. A proof is a transparent argument, in which all the information used and all the rules of reasoning are clearly displayed and open to criticism. It is in the very nature of proof that the validity of the conclusion flows from the proof itself, not from any external authority. Proof conveys to students the message that they can reason for themselves, that they do not need to defer to authority. Thus the use of proof in the classroom is, if anything, actually anti-authoritarian.

In the second place, it is hard to understand how the use of proof strengthens the idea that mathematics is infallible. Looking at the issue first from the point of view of theory, it is clear that any mathematical truth arrived at through a proof or series of proofs is contingent truth, rather than absolute truth, in the sense that its validity hinges upon other assumed mathematical truths and rules of reasoning. Nor would infallibility seem to be an issue from the point of view of mathematical practice. Mathematicians are as prone to making errors as almost anyone else, in proof and elsewhere. The history of mathematics can supply many examples of erroneous results which had to be subsequently corrected. Thus the concept of “infallibility” would seem to be irrelevant to the teaching of mathematics in general and the teaching of proof in particular.

The use of proof in the classroom has also been called into question on the grounds that it would encourage the idea that mathematics is an a priori science. The supporters of this claim see a conflict between this idea and their own view that mathematics is “socially constructed” (Ernest, 1991). Though their use of the term a priori is not entirely clear, it would seem that what they reject is not that mathematics is a priori in the sense of being analytic (non-empirical), but rather that it is a priori in the sense of given, pre-existing, waiting to be discovered. Of course this is a view of mathematics that they might well see as standing in opposition to "socially constructed."

On this point, however, Kitcher (1984) is surely right when he says that the pursuit of proof and rigour in mathematics does not carry with it a commitment to looking at mathematics as a body of a priori knowledge. Nor need it do so in mathematics education. As Kitcher put it: "To demand rigor in mathematics is to ask for a set of reasonings which stands in a particular relation to the set of reasonings which are currently accepted" (p. 213). Whether the set of reasonings currently accepted is regarded as given a priori or as socially constructed has no bearing on the value of proof in the classroom.

Those who challenge the use of proof in general would challenge even more strongly the use of rigorous proof in particular. Yet in mathematical practice the level of rigour is often a pragmatic choice. Kitcher states that it is quite rational to accept unrigorous reasoning when it proves its worth in solving problems, as it has in physics. Mathematicians worry about defects in rigour, he adds, only when they "come to appreciate that their current understanding is so inadequate that it prevents them from tackling the urgent research problems that they face" (p. 217). When is it rational to replace non-rigorous with rigorous reasoning? Kitcher's answer is: "when the benefits it [rigorization] brings in terms of enhancing understanding outweigh the costs involved in sacrificing problem-solving ability." (Mathematics educators, whose goal is surely to enhance understanding, would be well advised to adopt this guideline.)

Rigour is a question of degree in any case. In the classroom one need provide not absolute rigour, but enough rigour to achieve understanding and to convince. An argument presented with sufficient rigour will enlighten and convince more students, who in turn may convince their peers. It is the teacher who must judge when it is worthwhile insisting on more careful proving to promote the elusive but most important classroom goal of understanding.

CODA: PROOF IN THE CLASSROOM

With today's stress on making mathematics "meaningful," teachers are being encouraged to focus on the explanation of mathematical concepts and students are being asked to justify their findings and assertions. This would seem to be precisely the right climate to make use of proof, not only in its role as the ultimate form of mathematical justification, but also as an explanatory tool. But for this to succeed, students must be made familiar with the standards of mathematical argumentation; in other words, they must be taught proof. (The value of proof as an explanatory tool has been explored in some detail by a number of authors, among them Hanna (1990, 1995), who discusses "proofs that explain;" Wittmann and Müller (1990), who talk about the "inhaltlich-anschaulicher Beweis;" and Blum and Kirsch (1991), who emphasize the use of "preformal proofs.")

Teaching students to both recognize and produce valid mathematical arguments is certainly a challenge. We know all too well that many students have difficulty following any sort of logical argument, much less a mathematical proof. But we cannot avoid this challenge. We need to find ways, through research and classroom experience, to help students master the skills and gain the understanding they need. Our failure to do so will deny us a valuable teaching tool and deny our students access to a crucial element of mathematics.

NOTES

Previously published in *Zentralblatt für Mathematik Didaktik*, (97), 2/3, 171–185.

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INTRODUCTION

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THEOREMS IN SCHOOL: AN INTRODUCTION

WHY THIS BOOK?

The idea of this book emerged during the Annual Conference of the International Group for the Psychology of Mathematics Education (PME-XXI) in Lahti, 1997.

The Forum Presentation on “Theorems in school” by some of the authors of this book, and related discussions involving other authors, showed that there were suitable conditions to start preparing a book that meant to support the renewed interest for proof and proving in mathematics education.

In the meantime, reconsideration of the importance of proof in mathematics education was leading to important changes in the orientations for curricula in different countries all over the world. In particular, this movement led, in the NCTM Standards published in 2000, to revalue proof and proving in mathematics curricula, and to recommendations to develop proof-related skills since the beginning of primary school.

The general reasons for these changes are presented in the chapter written by Gila Hanna, the Preface of this book.

But how to approach the development of proving skills (by students) and teach proof in school?

Old teaching models (essentially based on learning and repetition of proofs of relevant theorems as they are written in textbooks) do not fit the current needs of students and teachers. Moreover those models showed their inefficiency in the attempt to understand the role of proof in mathematics and the development of skills related to the production of conjectures and the construction of proofs by students. Such inefficiency was one of the reasons for getting rid of proof or reducing its importance in secondary school curricula in some countries, like the USA at the beginning of the last decade, or Sweden, Italy and other European countries in the last two decades.

Therefore entirely new approaches are needed. And these approaches must take into account the actual complexity of the subject: it is not wise to replace the old, structured teaching of proof with naive alternatives; the unavoidable bad results would bring teachers back to old methods!

This book, addressing mathematics educators, teacher-trainers and teachers, is published as a contribution to the endeavour of renewing the teaching of proof (and theorems) on the basis of historical-epistemological, cognitive and didactical considerations.

What led us to choose such a broad scope, embracing so different disciplines and perspectives? How does this choice affect and shape the plan of this book?

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 19–24.

PART I: THE HISTORICAL AND EPISTEMOLOGICAL DIMENSION

First of all, both teacher training and mathematics teaching and learning need to consider how theorems and proofs (and the ways of conceiving them in mathematics) developed in the history of mathematics.

Both teacher trainers and teachers need to be aware that proof and proving were viewed under different perspectives by mathematicians, and are still viewed from different perspectives in the schools of different countries as well as within mathematics.

This awareness is necessary in order to avoid that one particular epistemological position (that might be related to a peculiar situation—e.g., the diffusion of some textbooks—or depend on peculiar theories in the field of mathematics education) induces teachers to make educational choices that might be in contrast with the needs of students to become familiar with one of the crucial aspects of mathematics. Another crucial reason for the relevant role of epistemology and history in a book like this, is to prepare the ground for an in-depth analysis of the differences between proof and proving.

In this context proof is considered as a product shaped in school by the constraints posed by the community of mathematicians and by the community of mathematics educators; proving is viewed as an individual process that develops under the constraint of yielding a product with given characteristics (according to students' conceptions and their understanding of proof).

For the listed reasons, we believe that the historical and epistemological dimensions need to be dealt with by authors who look at specific research results already produced in the history and epistemology of mathematics with an eye to crucial issues related to educational choices.

In his chapter, G. Arsac deals with different hypotheses elaborated by historians about the origin of proof in Greek mathematics, in particular those accounting for internal necessities related to the development of mathematics, and those referring to external influences related to the development of Greek society and culture (in particular, Greek philosophy). The crucial role of the problem of irrationality of the square root of 2, and of the incommensurability between the diagonal and the side of the square is stressed. It appears as a possible common point of interest for mathematics and philosophy, with relation to the early development of proof in Greece, as it raises simultaneously the problem of the kind of reasoning appropriate to mathematics and that of the status of mathematical objects, not belonging to the sensible world. Some links are made with educational problems (in particular, those related to the approach to proof—in what field of mathematics? With what aims, "convincing" or "explaining"?).

F. Arzarello's chapter deals with mathematical proof in the 20th century. The notion of proof is investigated from an epistemological point of view. Its meaning is scrutinised through a comparison of the contributions of a number of different philosophies about the nature of mathematical knowledge. The main idea underlying this chapter is that only by knowing what a proof is (or can be) can one face the didactical problem of its teaching in the classroom.

Therefore this chapter tries to make some aspects of the debate about the nature of proof, relevant for educational choices, accessible to mathematics educators and mathematics schoolteachers. Technical formalisms are avoided, although some technical aspects must be taken into account in order to understand the real content of the debate.

The goal of the chapter by G. Harel is to describe how Harel and Sowder's psychological framework of "proof schemes," elaborated for examining students' understanding of mathematical proof, was revised with an almost exclusive focus on historical and philosophical considerations. The chapter provides an example about how history and epistemology of mathematics can be exploited to develop tools that are useful to analyse students' performances in the domain of proof.

Other contributions on the historical and epistemological ground are brought by authors of chapters which mainly focus on other subjects. In particular, I suggest considering the links with epistemological issues that are made in the chapters written by R. Duval and N. Douek, and some epistemological assumptions and reflections that are made in the chapters in Part IV.

PART II: CURRICULAR CHOICES, HISTORICAL TRADITIONS AND LEARNING OF PROOF: TWO NATIONAL CASE STUDIES

How does epistemology of proof influence curricular choices? How do learning of proof in school, and in particular students' conceptions about proof, relate to historical traditions and epistemological assumptions underlying curricula? The chapters by C. Hoyles and L. Healy, and by J. Szendrei-Radnai and J. Török deal with the relationships between curriculum choices concerning proof (and the related implicit or explicit epistemological assumptions and historical traditions and values) in a given school system (within UK and Hungary, respectively), and the effective teaching and learning of proof in schools in those countries.

In their systematic study, C. Hoyles and L. Healy deal with the conceptions of proof held by students who had followed the National Curriculum introduced in England and Wales since the late 1980s. The authors aimed to investigate the characteristics of arguments recognized as proofs by high-attaining students, aged 14–15 years, the reasons behind their judgements and the ways they constructed proofs for themselves. The reported data concern proof in geometry (while the major focus of the curriculum was on reasoning and its separation from geometry).

J. Szendrei-Radnai and J. Török provide the reader with a historical glance at the (intended and real) situation of proof in the Hungarian school, with some recent data that provide a partial, yet interesting image concerning students' conceptions about proof when entering university. This chapter also provides a look at the existence, in Hungary, of alternative situations and "agents" (outside the school setting: mathematical contexts, mathematical journals for students) which contribute to providing good opportunities for some students to deal with proof in a consistent way.

PART III: ARGUMENTATION AND PROOF

Cognitive aspects (in a broad sense, embracing both the individual cognition and the social construction or transmission of knowledge in the classroom) are important in order to avoid the didactical choices not fitting the needs and the potentialities of learners.

As concerns cognitive aspects, the choice was to first deal with the features of reasoning related to proof within two chapters (those by R. Duval and N. Douek) mainly concerning the relationships between argumentation and proof.

The natural continuation was then to concentrate on some crucial cognitive aspects of the development of proof from the early approach in primary school, to high school and university in Part IV of this book.

In his chapter, R. Duval analyses the cognitive working of reasoning in the case of proof. He deals with students' difficulties in understanding both the mathematical processes characterising a proof and the double awareness inherent in it (concerning how a proof really works, and how one becomes truly convinced by proofs). The importance of the formal aspects of proof (those related to the logical-deductive enchaining of propositions according to their operational status) is stressed. Some educational implications are derived regarding the variables to be used in order to give rise to the above mentioned double awareness.

In her chapter, N. Douek takes R. Duval's analysis of the functioning of proof as a reference point to stress the need for considering other aspects of the process of proof construction within mathematics. Some examples are provided to show how the concrete process of proof construction develops out of a strict reference to the ideal shape of the formal proof. Some educational implications are derived. As such, Duval's and Douek's chapters present two complementary perspectives: the former depending on the logical constraints that bind the final product (proof), the latter concerning the real process of proof construction.

PART IV: DIDACTICAL ASPECTS

Didactical aspects are clearly related in a very direct way to the title of this book. The choice was to present an example of a stimulating activity related to proof performed at the level of teacher training (in the chapter by G. Winicki-Landman), then examples taken from teaching experiments and projects developed in primary and secondary schools. The aim of these examples is to encourage teachers and teacher trainers to change their view about students' difficulties in accessing theorems in school. We will show how suitable didactical proposals within appropriate educational contexts can match the great (yet, underestimated!) students' potentialities in approaching theorems and proof and developing mathematical theories from primary school onwards.

In her chapter, G. Winicki-Landman presents an exploratory study aimed at collecting, describing and analysing student-teachers' understanding of the notion of "mathematical impossibility". This notion is strongly connected with the idea of proof and the author's approach relies on students' claims as well as on their performances when producing proofs and refutations of mathematical statements involving impossibility. The contribution brought by this chapter can suggest productive activities in teacher training programmes to encourage teachers to reflect on the meaning of “truth” and the role of “proof” in mathematics.

Coming to classroom activities, the chapter by C. A. Maher, E. M. Muter and R. D. Kiczek presents some results (concerning the approach to proof) of a long-term longitudinal study (from primary to high school) regarding the development of mathematical ideas in students. Reported results concern how students come to construct proofs and the identification of some conditions that support the development of proof making.

The subsequent six chapters deal with a co-ordinated set of teaching experiments that were performed in Italy in primary and secondary schools by the research groups of Genoa, Modena, Pisa and Turin. These chapters constitute an expansion of the Forum Presentation on “Theorems in school” at Lahti (during the PME-XXI Conference), which originated the idea of this book.

The short chapter by M. Bartolini Bussi, P. Boero, F. Ferri, R. Garuti and M. A. Mariotti presents the common framework and general guidelines of the teaching experiments reported in the other chapters.

The examples concerning primary and lower secondary schools show the feasibility of an early approach to geometry theorems and proving skills, provided that some conditions are fulfilled regarding the choice of specific didactical situations and the previous development of argumentative skills.

M. G. Bartolini Bussi, M. Boni and F. Ferri present classroom activities dealing with geometric construction problems in grade V. Students had already performed (under the guidance of the teacher) the collective construction of a germ-theory (an embryo of theory that has an expansive potency to develop into a fully-fledged one). In this context students are able to manage construction problems concerning the geometry of a circle.

In their chapter, P. Boero, R. Garuti and E. Lemut deal with an approach to geometry theorems in school in grade VIII. They stress the importance of some conditions inherent in the choice of the context of where to set appropriate tasks (mainly in order to avoid low-level forms of validation, like measurements, and enhance the dynamic exploration of the problem situation), and of other conditions inherent in the a-priori analysis of the solving process. In particular, they point out how, in a situation of continuity between conjecturing and proving, students can be facilitated in their approach to the proving process.

In their chapter, L. Parenti, M. T. Barberis, M. Pastorino and P. Viglienzone report a long-term teaching experiment showing the feasibility of an early approach (in grades VII–VIII) to geometry theorems and proving in the geometric domain. In particular, students move progressively from the observation of regularities in the relationships between three-dimensional objects and their two-dimensional representations (for instance, the transformation of straight segments into straight segments or points), to the formulation of some axioms, the production of conjectures and the construction of related proofs concerning the geometry of central projection and the geometry of parallel projection.

The two subsequent chapters deal with the transition (in a suitable learning environment) to more advanced levels of proof making and awareness about the functioning of proof and mathematical theories in secondary school.

M. Alessandra Mariotti analyses the potential of the Cabri learning environment, on the one hand to develop a dialectic interaction between the figural and the conceptual aspects in dealing with definitions and geometry theorems, on the other hand to construct a theoretical framework within which this interaction is accomplished.

F. Arzarello, C. Micheletti, F. Olivero, D. Paola and O. Robutti present the main ideas and some experimental outcomes concerning proof within a project at the secondary level aimed at approaching theories in the field of geometry. A theoretical model (based on the notion of ascending and descending control) is elaborated in order to analyse the transition to formal proof and compare students' behaviours in different learning environments.

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PART I: THE HISTORICAL AND EPISTEMOLOGICAL DIMENSION

GILBERT ARSAC

1. ORIGIN OF MATHEMATICAL PROOF

History and Epistemology

INTRODUCTION: WHAT IS A MATHEMATICAL PROOF?

Mathematical proof occupies a central place in mathematics as it is the validation method whose systematic use characterizes this discipline among other scientific ones. Consequently, it appears as a privileged object of study for mathematical education all the more so as it is at the origin of difficulties for many pupils. Any research about its teaching raises the problem of its history, as for any other mathematical concept, even if proof is not exactly a concept but rather a technique. This chapter is devoted to investigating the origin of mathematical proof, from a point of view which will be specified later. We shall also tackle the question of its subsequent evolution, that is to say the question of the history of rigour in mathematics. But on this problem, we refer the reader to Lakatos (1976) and its rich bibliography.

First, me must specify what we mean by mathematical proof. The generally suggested definitions gather about two poles:

- a formal pole in which mathematical proof is characterized by its form, as a text which respects some precise rules, as for instance Balacheff (1977) states: a statement is known to be true, or is deduced from the precedents using an inference rule taken in a well defined set of rules;
- a social, or cultural, pole in which mathematical proof is characterized as the proceedings for validation used by mathematicians. So a text is a mathematical proof if it is recognized as valid by mathematicians.

We can remark that the formal definition emphasizes that a mathematical proof must be written, and that the cultural one adds that it must be published, and, as a rule, within the reach of everyone.

The two poles are not independent: the rules that a proof must fulfil arise from an agreement between mathematicians. It is a fact that there exist debates about some mathematical proofs, but in fact, they fall into two categories:

- does this particular proof fulfil the rules usually admitted by mathematicians?
- this proof uses new rules, can they be accepted?

P. Boero (Ed.), Theorems in schools: From history epistemology and cognition to classroom practice, 27–42.

The existence of this second kind of debate emphasizes the fact that the rules that a mathematical proof must fulfil have historically changed.

Of course, each of the two poles can be specified and give rise to several definitions; for instance, the most celebrated formal definition, that of Hilbert, requires writing in an entirely formalized language, as well regarding logical rules as mathematical content, but this is only an extreme position in the formal family in which we can class all definitions or practices which put emphasis on conformity with certain rules. Reading Euclid shows that, in his elements, mathematical proof satisfies a very precise form (Netz, 1999), but this form appears only through practice, not in a treatise on what a proof must be.

On the other hand, historic and contemporary experience of mathematics shows that, as soon as we leave a theoretical point of view, as Hilbert's one, the concrete implementations are quite varied, depending on mathematical context. Agreement on the rules is often a theoretical agreement.

If we follow the cultural definition of mathematical proof, such proofs appear in all the mathematic traditions, in the sense that there are necessarily means of validation on which mathematicians agree and which are relatively constant, as can be seen for instance at the moment in studies on Chinese mathematics. We will restrict our study to the western tradition whose origin takes place in Greece, and which is continuing in Arabic science then in western countries, and this for several reasons: it is the origin of all contemporary mathematics, and it is the first that makes systematic use of proof in a fixed and even stereotyped way. Actually, Greece is the place of a radical transformation of mathematics which simultaneously characterised the objects of this science by defining them axiomatically as idealities, ideal objects, and rules of their handling, particularly mathematical proof which allows true statements to be distinguished. In the western mathematical tradition, the pattern provided by Euclid's elements, the proof "in the way of ancients" was considered up to the eighteenth and even nineteenth century as a standard which could not be exceeded. As a matter of fact, we see Leibniz, the founder of differential calculus, and its promoter the Marquis de L'Hôpital, asserting that they would be able to prove in the style of Euclid, despite the fact that their practice was very different. This tradition contributed to make geometry the privileged place for teaching and learning mathematical proof.

CHARACTERISTICS OF THE EUCLIDEAN PROOF

It is out of the question to study this extensively here; for this see Heath (1908) or Vitrac (1990) or Netz (1999), but we will recall, from the point of view that is of interest for us, some features of the organization and writing of the Euclidean treatise.

There is a list of axioms:

- There are no undefined terms: some words or expressions of the mathematical language are defined (point, line, angle …) and others (greater than, …) appear without comment, their signification is obviously supposed to be known by the reader. This imprecision is curious as Euclid's work is subsequent to Aristotle's, who clearly specifies that it is impossible to define all the words used in a given science.

- The proofs given by Euclid use not only the preceding results and axioms: some properties are identified through the drawing; this is well known as it appears as early as in the first proposition of book I in which it is admitted, for the purpose of constructing an equilateral triangle, that, given two distinct points A and B, the two circles centred in A (resp B) and passing through B (resp A) necessarily meet. Similarly, Pasch's axiom has its origin in the observation that Euclid implicitly uses the result stated in this axiom. So, in his proofs, Euclid refers to what is seen on the drawing.
- The logic employed is merely that underlying usual language (Gardies, 1997, pp. 49–75), with a large use of reductio ad absurdum. There are no reasoning errors, but an analysis using propositional calculus shows that the author appears to be unaware of some logical rules such as the equivalence between a statement and its contraposition (pp. 49-75). There is some evidence that Euclid does not separate in his proofs "what is pure logic and the part which is not" (pp. 49-75). We can also notice that Hilbert's "foundations of geometry" follow the same way, using a logic not made explicit, underlying current language.
- The very writing form of proofs is very stereotyped, as is easy to see with a rapid glance at Euclid's work (cf. Netz, 1999), and we know from the work of Autolycus de Pitane (1979) that this form is in fact prior to Euclid.

CONTEXT OF APPEARANCE OF MATHEMATICAL PROOF IN GREECE

Historians of Thought point out that the emergence of mathematical proof in Greece is quite contemporary to that of democracy and philosophy.

> Certainly, it is not a matter of chance if reason arises in Greece as a consequence of this so original form of political institutions called the city. With the city, and for the first time in history, the human group considers that its common affairs can be settled, decisions of general interest taken, only after a public and contradictory debate, open to everybody, in which argumented discourses conflict with each other. If rational reasoning appeared in Greek cities of Asia Minor such as Miletus, it is because the rules of the political game—public debate, argued and contradictory—had also begun the rules for the intellectual game. (Vernant, 1979, p. 97, translated from French.)

Szabo (1978) specifies this idea by ascribing to the Eleatic school of Parmenides and Zeno the origin of the radical transformation of mathematics which took place in Greece. This thesis appears to be essentially externalist as it searches for this origin not first in internal necessities of mathematical development, but in external influences.

Nevertheless, this explanation which finds in an influence external to mathematics the origin of their transformation and even, in some sense, of their emergence as an autonomous area of thinking, with its proper rules of validation, conflicts with the idea, commonly admitted among mathematicians, that it is problem solving which is at the origin of advances in mathematics, particularly advances in rigour (Lakatos, 1976). More widely, that explanation is contradictory with the "logic economy principle" which says that "one does not use more logic than needed for the practical use" (Bourdieu, 1980). Now, history tells us that the emergence of mathematical proof in Greece is contemporary with the solution of the irrationnality and incommensurability problem, that is to say a double discovery: on one hand, 2 has no rational square root, on the other hand, square's diagonal is incommensurable with its side. And precisely, we shall see that the mathematical proof, but also the whole Greek approach to mathematics and to the status of their objects, is an indispensable tool to overcome these difficulties in the manner of Greek mathematicians.

Now there comes a problem: no historical document tells us the period of emergence of mathematical proof in Greece, nor the reasons for it. It is true that Proclus relates that Thales was the first to "prove" that a diameter divides a circle into two equal parts, but nobody knows what "prove" means here, especially because Proclus reports that Thales went further than the Egyptians "sometimes in abstract generalization, sometimes in empirical investigation". Similarly, Simplicius left us some passages of Eudeme's mathematics history, describing the quadrature of three kinds of lunes by Hippocrates of Chio, and it really seems that in this work there was a deductive process predating Euclidean proof, but we are far from being able to restore the original text of Hippocrates (Heath, 1921). This lack of historical documents is in particular due to the fact that in opposition to the Babylonians, whose tablets can still be studied, Greeks wrote on perishable materials so we have no direct document about origins: we know them only through what is said by the Greek historians and philosophers whose works have been preserved. So we will begin our study with an a priori analysis, that is to say, without direct reference to documents, of the links between irrationality and mathematical proof, using only mathematics and general historical knowledge about the evolution of Greek thought; only later shall we go back to the few historical indications directly concerning our subject.

AN ANALYSIS OF THE QUESTIONS RAISED BY THE INCOMMENSURABILITY PROBLEM

Mathematical Analysis

Let us consider the figure formed by a square and its diagonal. A priori, examining this drawing by no means leads to the discovery of the incommensurability between the side and the diagonal. On the contrary, as Aristotle notes, one will rather conclude that it is actually possible to measure the diagonal using the side as a unit: according to measuring precision, we will find 1.4 or 1.41.

Nevertheless, Pythagoras' theorem shows that, if the diagonal's length, taking the side as unity, is d, then $d^2=2$. A classical argument, that we find in Euclid, shows then that if d could be written p/q, where the fraction is supposed to be irreducible, number q would be even and odd at the same time: indeed, since $p^2=2q^2$, p is even, $p=2p'$, so q must be odd, as the fraction is irreducible, but $q^2=2p'^2$, so q must also be even. So we have proved the two results quoted above: the number 2 has no rational square root and the diagonal is incommensurable with the side. This proof clearly implies a "change of setting" (Douady, 1986), that is to say that one has to go from the geometric frame to the arithmetic frame, and a reductio ad absurdum, not easy to avoid when one has to show that a number is not rational.

As we saw that this incommensurability property cannot be verified by direct reading on the figure, it necessarily concerns ideal segments, represented by those we can see on the figure but which cannot be identified with them. So the mere statement of the incommensurability property supposes a change of status for geometrical objects, a problem to which their axiomatic definition, in the Greek sense, gives a solution. We can note that we find in the *Meno* by Plato a proof dealing with square duplication which takes place entirely in a geometrical setting and rests on evidence read on the drawing: it is easy to see on it (Figure 1) that to obtain a square whose area is twice that of a given square, one must take the square having the diagonal of the given square as its side. The figure also shows why the idea of doubling the side leads to a square whose area is actually four times that of the given square.

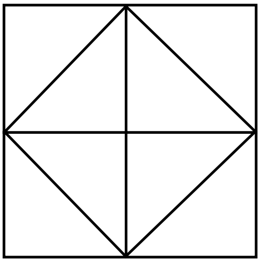

Figure 1.

This type of proof, using the subdivision of areas, that we abundantly find in Euclid from Book I, is also known in prehellenic mathematics (for instance, there are Chinese and Indian proofs of Pythagoras' theorem resting on this principle). The incommensurability proof that we recalled cannot be directly reduced to such a proof by geometric evidence; so the drawing in *Meno* gives an example of a figure on which Euclid will allow himself to read certain geometric evidence related to area ratios, but on which appear squares and their diagonals without possibility of reading on this same figure the incommensurability property.

But this last remark raises a question: how did the idea occur to man that incommensurable segments can exist and even how is it possible to think of a definition of incommensurability, as the hypothesis that two segments are always commensurable is indeed (implicitly) present in the whole of prehellenic mathematics. So the first question is: how does the incommensurability problem occur? To answer this question, we have to make a detour by the history of mathematics looking into what we know about Pythagorean mathematics.

Analysis According to Pythagorean Mathematics

It is possible to summarize Pythagorean very briefly thought from the point of view which is of interest to us, by the following phrase: all things in nature can be expressed by ratios (logos) of integers. Moreover, Pythagoreans base their study of numbers on the dialectics between odd and even. So there is nothing in this thought leading one to imagine the phenomenon of incommensurability. Greek tradition nevertheless says that it is really to Pythagoreans that we owe the discovery of irrationality of $\sqrt{2}$, but also mentions that it was considered a secret. So we have to understand how Pythagoreans had the opportunity to face the question of irrationality.

The answer usually accepted among historians appeals to the anthyphairesis process, that is to say to the geometrical version of Euclid's algorithm which consists in operating by "successive subtractions". This is what we shall now develop.

Given an initial pair of two segments AB and CD, with AB≥CD, we "replace" AB by AB–CD that is to say we consider a new pair (A_1B_1,C_1D_1) where A_1B_1=AB–CD and C_1D_1=CD, if CD≤AB–CD, and A_1B_1=CD, C_1D_1=AB–CD in the opposite case.

If initial segments were commensurable, repeating the same process leads, through a finite number of such operations, to a pair (A_kB_k,C_kD_k) with A_kB_k=C_kD_k, and after that to (A_kB_k,0). Then, A_kB_k is the largest segment contained an integer number of times both in AB and CD. It is then easy to deduce, by reading the operations in an inverse order, what was the value of the ratio of initial segments. If AB and CD are incommensurable, the process is infinite. But in this case also, it is necessary to distinguish carefully between theory and practice: if we consider it as a graphic process, it always converges very quickly. For instance, if we apply it to two segments, one being the diameter of a circle and the other the length of its circumference, we rapidly find that π=22/7. The infinite characteristic of anthyphairesis is not easy to see on the drawing!

For Pythagoreans, as all things can be expressed by way of ratios (logos) of integers, the process is always finite. Actually, there exist in geometry some cases in which one can meet "naturally," that is to say, without presupposing it, an infinite anthyphareisis, and, by an irony of fate, pointed out by K. Von Fritz, this case actually occurs when studying the emblematic figure of Pythagoreans: the star with five branches, that is to say the star-shaped regular pentagon.

Using notations of Figure 2 below, applying anthyphairesis to the pair (AD,AE) leads to replace it by the pair (B′E′,B′A′), using only one intermediary (B′E,EA′), and mathematical knowledge attributed to that age is enough to perform the two stages; in fact we can consider that it is a matter of proof by visual evidence of the same kind as that of *Meno*. So the initial problem is reduced to the same problem for the smaller regular pentagon A′B′C′D′E′, hence the process is infinite, unless assuming the existence of an absolute minimum among lengths, which is dismissed by knowledge of the fact that there exists a middle for each segment.

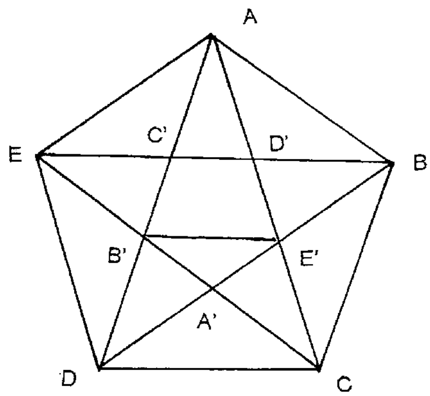

Figure 2.

Nevertheless, recognizing the infinite character of the process supposes one admits that all regular pentagons share the same properties. It is what we do teaching geometry when we speak about the properties of “the figure,” assuming implicitly properties of similar figures, which are particularly evident in the case of regular polygons.

So, thinking about the pentagon case leads one to consider that the two segments AD and AE have no common measure using a thought process which shows, and in the meanwhile partially exploits, the fact that the ratio AD/AE is equal to A′D′/A′E′ and so recurs indefinitely with smaller and smaller terms. Each reflection about properties of similarity aiming to improve more rigorous proofs of the used properties will necessarily be based on the hypothesis of universal commensurability of length ratios and hence leads either to a vicious circle or to the fearful task of defining a new notion of ratios for segments. We shall see further that this was probably tried first in a not quite rigorous way.

Regarding the square, the infinite character of the anthyphareisis applied to its diagonal and side is less evident, but can be put in evidence with the mathematical tools of the epoch with the same problems of circular reasoning (Caveing, 1982). On the other hand, in the case of the square, a change of setting leads, as we saw, to a proof in the frame of arithmetic, which, all things considered, rests uniquely on properties of the couple (even, odd) familar to Pythagoreans and on the use of reductio ad absurdum. So, it allows one to escape geometric setting, a place where contradictions and questions about evidence arise, and to join the arithmetic field, a place where evident facts are more stable. Assuming that the irrationality of $\sqrt{2}$ has been proved in the arithmetic setting, one has, if not an explanation, at least a result which makes less unlikely for Greek mathematicians the difficulties encountered with the diagonal of the square. Nevertheless, in the geometrical setting, contradiction remains until its solution using Eudoxus’ ratios theory explained in Euclid’ Book V. We shall now examine, using the history of Greek thought, possible means of raising the contradiction .

POSSIBLE ANSWERS ACCORDING TO THOUGHT HISTORY

The Sophistic School

The sophistic school was interested in the whole set of knowledge and although sophists were never considered as mathematicians, they worked in mathematics in a sufficiently explicit manner to give us a rather precise idea of the status they gave to geometrical figures.

- For instance, Protagoras maintains “against geometers” that, in accordance with what we can notice on the drawing, a circle and its tangent have more than one common point. We can remark that pupils around twelve have the same point of view: the circle and its tangent touch together on a whole length. For Protagoras, the world is really contradictory, we must accept meeting contradictions without taking refuge in an ideal world. Adopting this position, we can confine ourselves to the real figure, and refuse firmly to consider an ideal figure; for Protagoras, there is nothing behind appearance.
- Other sophists, such as Hippias or Antiphon, probably preferred to confine themselves to the possibility of actually solving concrete problems: Hippias invents a curve, that we can trace in an approached manner since we can construct an infinite number of points of it, which succeed in resolving the problem of squaring a circle; as for Antiphon, aiming to compute the length of a circle, he does not hesitate to confuse it with an inscribed regular polygon when the latter has a great number of sides. Graphically, such approximations are quite acceptable, and such practical solutions are still used now in certain trade associations. Similarly, vedic geometry, whose goal, building altars made of bricks, is both practice and ritual, mixes exact and approached solutions.

We must notice that these sophistic solutions are clearly linked with the philosophical positions of their authors: the sophist’s opposition to handling ideal objects is linked to their disagreements with Plato. Among the sophists there is a refusal of the research of an absolute truth, which for them is necessarily of religious essence, they prefer a relative truth which agrees better with democracy and the search for arguments which lead to the success of a thesis over another but without paying great attention to the problem of truth. There is also, on another level, the refusal, shared between sophists and Pythagoreans, to question the knowledge resulting only from the use of our senses.

The Eleatic School and Mathematical Proof

Founded by Parmenides, the Eleatic school conflicts with the sophists, and foreshadows Plato who, although he differs from this school, is its heir. For the Eleatics, the sensible universe, that of appearances and phenomena, cannot be the object of real knowing, without contradictions. Truth is inaccessible to observation, it is accessible only by pure thinking: for the Eleatics, whenever evidence given by our senses is contradictory with our reasoning necessities, these necessities will prevail (Zafiropulo, 1950). More detailed information about eleatism can be found in Zafiropulo (loc. cit.) or Caveing (1982).

So, according to the Eleatics, the objects of the sensation are continually transforming and changing into their contradictory, they are essentially contradictories. Conversely, the objects of pure thinking are the only ones that are not contradictory, so it is possible to prove a statement about them by showing that its negation implies contradiction (reductio ad absurdum). So, reductio ad absurdum appears as the convenient tool for handling these objects, whereas it would not be the case for sensible objects. On this point, there is an agreement between Eleatics and Plato that Szabo regards to be an heir of Eleatic thought. In "the Parmenides," Plato presents Parmenides, Zeno and Socrates, and attributes to Parmenides the following praise of Socrates:

> I must say that I was delighted with a remark that you did, when you said that you did not want to let the survey become lost in visible things and apply to them, but focus on objects of thought those that we grasp first by thinking and that we can consider as forms.

And Socrates answers:

> Indeed, I consider that it is not at all difficult to prove in this way that visible things are at the same time similar and dissimilar and susceptible of all the opposites. Quite true, says Parmenides; but there is still another thing to do. It is not enough to assume than an object exists and examine the consequences of this assumption, we must as well assume that this object does not exist, if you want to carry on with your gymnastics …

We can remark how the program described in this text, which foreshadows the study of the problem of the « one » and of the « being », applies exactly to the problem of incommensurability: if the segments which come into play in the figure constituted by the square and its diagonal, or in that of the pentagon, belong to the visible world, there is nothing surprising in the fact that they are "susceptible of all contraries," and agreeing on this point would have been sufficient for a sophist. On the other hand, transferring geometrical objects to the world of forms will allow them to be objects of true knowledge, whose existence is ensured by the fact that their properties are not contradictory. Breaking away from reality is the condition which allows geometry to rank among real knowledge, perhaps against what a contemporary mind would maintain!

There remains the question of reductio ad absurdum, which is necessary to solve the irrationality problem. The first primitive example appears in the "poem" by Parmenides where it is used in the theory of being: "as regards the decision on that, it is contained in the following alternative: either it is or it is not". Further, this kind of reasoning is systematically used by Zeno that Aristotle considers as the father of dialectic. What is to be understood by this? Aristotle himself gives a detailed definition of dialectic from which we extract what follows: dialectic trains reasoning, it is a deduction which starts with granted ideas and draws from them contradictory consequences aiming to refute the initial hypothesis. Since Plato, dialectic has also been known as a means to raise thoughts from sensible things to intelligible ones. We must note, and we will go back to this later, that Aristotle claims that Pythagoreans did not know about dialectic.

Well, we can be tempted to conjecture, as Szabo (1978) does, that the solution of the incommensurability problem was based on an "application" of eleatics ideas to mathematics. Nevertheless we have no proof of direct intervention of eleatics in mathematics, unlike the sophists.

TRACES OF TRIALS AIMING TO SOLVE THE IRRATIONALITY PROBLEM

As we said before, it is quite unlikely that only being aware of the problem immediately leads to solving it. Many trials and errors must have preceded, around the fifth century BC, the working-out of the final solution. As far as these trials mean steps on the way to rigour, they are of interest for us. Our intellectual guide here will be Lakatos as the irrationality concept clearly falls into the category of the "proof generated concept" (Lakatos, 1976). Indeed, as we saw, the phenomenon of incommensurability cannot be expected in a geometrical setting if it has not been met, and in some sense proved; it is even only at the end of the whole building of deductive mathematics, building caused by the problem that the concept can take place, and that the initial contradiction is overcome. Meanwhile, in accordance with the description of the play between conjectures and refutations, mathematics will have continued developing, either by leaving aside this awkward case, or by getting round the obstacle, or elaborating partial solutions, or going completely wrong. If we accept Knorr's dating (Knorr, 1975), this phase would have lasted for a century, from –430 (acknowledgement of incommensurability phenomenon) to –330 (axiomatization by Eudoxus).

Despite the difficulty already mentioned of finding historical documents about the beginning of mathematics, the quest in this field is not quite unsuccessful and we have traces of trials which preceded the final solution by Eudoxus to the conceptual problem of incommensurability, rigorously defining the ratio of two magnitudes, which allows one to assert that the ratio of two segments always exists.

A first indication is given by a text from Plato in "the Republic" (book VIII, 546c). This famous text, dealing with the "wedding number," is of interest for us as it uses the length of the square diagonal in the following passage: "one hundred squares of rational diagonals of five, each reduced of one, or one hundred squares of irrational diagonals, reduced of two …" Interpreting this text does not bring differences between translators: the square of side 5 has a diagonal of length $5\sqrt{2}$, irrational, whose approached integer value is 7 since $7^2=49$, and the quoted sentence expresses that 100(72–1)=100(50–2). At the expense of certain anachronisms, we can say briefly that the "irrational diagonal" refers to $\sqrt{50}$, while the "rational diagonal" refers to 7.

But what is interesting and escapes the reader of this translation, is actually the Greek word used by Plato and translated by irrational: it is not the word employed later by Euclid which means "incommensurable regarding to the length". Plato uses a word which etymologically means "impossible to enunciate". Similarly, the word translated in this text by rational means "possible to enunciate" (cf. Szabo, 1977, or Caveing, 1977, p. 1205, who translates into French by “exprimable,” that is to say “expressible”). A very attractive idea is to see here a trace of a stage of mathematics where the impossibility to attribute a rational measure to the diagonal of the square of side 5 was discovered, but without any definition of the incommensurability concept. Moreover, we can remark that the text avoids the problem by reasoning on the square of the diagonal which is rational. This leads us to the second indication.

This second indication is underlined by Caveing (1977, p. 1270 and 1982, p. 183). He remarks that several traces remain from a time in which mathematicians attempted to circumvent the difficulty of incommensurability by reasoning on areas of squares constructed upon segments: the work of Hippocrates of Chio, the author of the famous squaring of lunes, of which fortunately we have an extract, shows evidence of this stage (Caveing, 1977, p. 692). The very possibility of this stage implies a rigour level which is not that of the Euclidean proof, as such mathematicians simultaneously use properties of similar magnitudes while they are not able to define clearly similarity. Certainly, this time is a period in which incommensurability difficulty is recognized, but not cleared up. A trace of this time can probably be found in Euclid (book X, definitions) when, given a unit segment, he partitions, by a definition, other segments into rational and irrational ones, but not in the contemporary way, as he ranks in the first class not only segments commensurable with the unit, but also those for which the area of the square constructed on them is rational. This confirms the existence of the stage which can be illustrated by Hippocrates of Chio, and also suggests the idea that a first hope of surpassing the incommensurability problem may have been the hypothesis that two segments are always commensurable by means of their squares, which partly saved the Pythagorean ideas, and was naturally suggested by the example of the square and its diagonal.

A third indication is related by Aristotle (*Topics*, 158B30) when he quotes the identity of their anthyphareisis as a definition for equality of two ratios. Clearly, this third indication supports the idea that geometric incommensurability was discovered by the way of anthyphareisis and confirms this process was of practical use. Nevertheless, without the Eudoxian definition of ratios of magnitudes, it gives evidence of a stage of mathematics in which dealing with ratios was certainly spoilt by circular reasonings, as we explained on the pentagon example, as a result of the inevitable use of ratios and similarity (for a reconstruction of this stage, cf. Fowler, 1979 and 1980, and Knorr, 1975, VIII and IX).

We shall now conclude this brief survey which, in this particular case, supports the assertion that overcoming a contradiction is not contained in the terms in which it is expressed. This study could gone into more deeply by trying, on the pattern of Lakatos (1976), to refine the analysis of various behaviours adopted to surpass the contradiction; it is not the principal matter of the present work. Nevertheless we must emphazise one point: the classical solution by Eudoxus was not the only conceivable one, and its success did not solve all the problems: that of the ratio between the circumference and the diameter of a circle remained open, and still remained so for a long time.

CONCLUSION: A DIALECTIC BETWEEN MATHEMATICS ANS PHILOSOPHY AT THE ORIGIN OF MATHEMATICAL PROOF?

The set of historical data leaves the door open to divergent conclusions. We'll now specify Szabo's conclusions (Szabo,1978), then we'll set out Caveing's.

Szabo's thesis specifies the idea that the rise of mathematical proof is the result of philosophical debate; it recognizes in the incommensurability problem a question internal to mathematics whose solution would have been supplied by philosophy, and more precisely by the eleatic school. In Szabo's opinion indeed, it is unimaginable that refusal of empiricism and the use of reductio ad absurdum would have spontaneously appeared among Greek mathematicians. As these ways of thinking are characteristic of the eleatic philosophy whose prominent representatives are Parmenides and Zeno, Szabo is naturally induced to refer to them for the origin of the change in mathematics. This leads him to propose the following outline.

- In a first stage, contradictory properties according to the very old theory of even and odd, and concerning the rational number supposed to represent the ratio between the diagonal and the side of a square, would have been noted. The ancient terms whose English translations are "impossible to enunciate" or "possible to enunciate" would go back to that stage, that would indicate an obstacle within mathematics.
- In a second stage, characterized by the rejection of sensory experience and intuition and the call on a way of thinking, inspired by the philosophers of the eleatic school, conceiving mathematical objects as purely ideal ones, dealt by rigorous reasoning, first uniquely the reductio ad absurdum, one could have succeeded in defining and proving incommensurability (both processes coincide) in that case. So the obstacle, internal to mathematics, was overcome with the help of an external contribution, but at the expense of a total overhaul of mathematics.

To support his thesis concerning the introduction of abstraction, the reductio ad absurdum, in mathematics by eleatic thought, Szabo (1978) supplies some further arguments, belonging essentially to the philological field. Nevertheless, it is not so easy to decide if the eleatic influence appeared directly or essentially through the intermediary of Plato; Theaetatus and Eudoxus were his disciples and are precisely the mathematicians who, according to the Greek tradition of the history of mathematics, finally solved the irrationality problem (Knorr, 1975, ch. II, III). So we will simply admit the hypothesis of an eleatic, that is to say external origin, of the reductio ad absurdum, but we shall not follow Szabo in his extreme conclusions when he goes so far as to state that mathematics were merely a part of dialectic and that the Eleatics were mathematicians (Szabo's opinion seems dangerous: cf. Caveing's appreciation, 1979).

Contrary to Szabo's, Caveing's work does not deal with the rise of mathematical proof, but rather with the origin of the status of mathematical objects, of their "ideality," "putting together the four characteristics of objectivity, not belonging to the sensible world, perfection and intelligibility" (Caveing, 1977). Nevertheless, as we noticed in the introduction, this definition of mathematical objects, which at the same time constitutes mathematics as an autonomous science, is necessarily contemporary with the rise of mathematical proof used systematically as a validation tool.

Caveing's point of view on the interaction between mathematics and philosophy is rather opposed to that of Szabo, whom he moreover criticizes (Caveing, 1979). For him, the irrationality problem, to which he equally recognizes a crucial role, led to the creation of mathematical idealities which would have been used later as models by philosophers, essentially Plato and Aristotle. So, the trend is reversed, here mathematics imparts philosophy. Caveing also makes a detailed study of Zeno's thought: after doing so, he concludes that Zeno's argumentation was essentially directed against Pythagoreans and especially against physico-mathematical Pythagorean syncretism: arguments like Achilles and the turtle, or the arrow, which are the best known, apparently aimed to show, using the reductio ad absurdum, that is to say dialectic in the Greek sense, that the Pythagorean ideas did not enable one to conceive motion or a geometric continuum without contradictions. Even restricted to the preceding facts, the Eleatic's influence and particularly Zeno's would remain of great importance, as all in all, Zeno would have, perhaps independently of the incommensurability problem, but certainly simultaneously, challenged Pythagorean principles and especially their ideas about geometrical objects. We can also remark that the dialectical technique, which always begins with clarifying the opponent's presuppositions, as completely as possible, led to express completely, and so to specify, the foundations of Pythagorism, the initial nature of which we already showed to be probably fuzzy and largely implicit.

We will retain Caveing's idea that a change in mathematics is first due to the internal problems of incommensurability and irrationality; first because of the arguments that we set out at the end of the introduction against Szabo's excessively externalist position, but also because the historical reconstitution in the mathematics setting of the appearance and the attempts to solve the incommensurability problem, appears to us to be the more likely.

We do not know any convincing historical argument enabling us to choose between the two previous theses. Nevertheless, it appears that a minimal conclusion can be drawn, compatible with both, taking the form of two negative statements:

- without the irrationality problem, the transformation of mathematics would not have occurred, even in Greek society;
- in another social context, even faced with the same problem, mathematics would not have changed in the same way.

SOME REMARKS ABOUT THE SUBSEQUENT EVOLUTION OF THE MATHEMATICAL PROOF

The mathematical proof, "in the way of the Ancients," as Euclid and Archimedes did it, remained a model for centuries, but the history of mathematics shows that mathematicians are largely unfaithful to it, first in their practices, then by their criticisms. As it is out of the question to give here a detailed account of this complex story, we'll only recall some limits of the Euclidean proof which explain the quick rise (from Diophant's time onwards) of divergent practices. We'll also recall the criticism of the Euclidean thought processes.

The Limits of the Euclidean Way of Proving

One of the first typical features and limits of the Euclidean mathematical proof is its quite linguistic nature, which completely distinguishes it from a contemporaneous proof in algebra or arithmetic and even in geometry. Indeed, it does not include any notation, or any ideogram, which allows calculation. The expansion of algebra, which is essentially a calculation upon symbols, was not possible while respecting the Euclidean tradition.

A second limit is the lack of a method for discovery and the large place given to the reductio ad absurdum. Both things are linked, as reductio ad absurdum is possible only given a conjecture whose truth is to be proved, but does not offer any way of checking this conjecture. This problem is particularly evident when reasoning by exhaustion: it allows one to prove rigorously area equivalences, but it does not give any way of discovering them. Archimedes, a great authority of the subject, confirms: he left us a book, "about the method," in which he describes the "mechanical" reasoning he uses to discover the formulas he is looking for, and he completely distinguishes these reasonings from the proofs by exhaustion, that he gives later in accordance with the Euclidean tradition. The progress in calculating plane areas and tangents to curves which has been made since the seventeenth century, and culminating in the equivalent methods of Leibniz and Newton's also led to replacing Euclidean proof by a calculation, and give a larger place to effectiveness in solving problems rather than to the Euclidean rigour. So, in analysis also, looking for calculation methods and effective formulas will overcome the rules of Euclidean proof.

DIDACTICAL REMARKS

Several kinds of questions arise following this study. We shall restrict ourselves to three of them. They are simple but important ones.

Convincing or Explaining Proofs?

The Euclidean proof clearly aims to convince the reader; later on it will be criticized as it does not shed light on the solution, or give a method to find it. The debate reappears when dealing with the teaching of mathematical proof: what function of proof must first be emphasized, especially for an initiation? Concerning this question, we can remark that the first proofs of irrationality and incommensurability, to which we attributed a historically basic role, are not so easy to classify in these categories. It is possible indeed, to claim that the indirect proof of the irrationality for $\sqrt{2}$, considered separately, convinces without explaining, whereas if we look at it in its historical context, it can appear as an explanation for the phenomenon of infinite anthyphareisis. At least it shows a link between irrationality and the incommensurability problem, or even suggests a first idea of connection between geometry and the properties of an underlying set of numbers. So we can conclude that convincing, explaining, enlightening, are not only mutually exclusive, but also have value only in respect to a certain state of knowledge.

Proof and Evidence

Euclidean mathematics is also characterized by wanting to prove everything, and contemporaries were already arguing against this: what is the point in proving triangle inequality, Epicureans asked, when it is evident, even for a donkey who is going for its food, that the shortest way is the straight line? The debate re-emerges at the level of the teaching of mathematical proof: what shall we choose to prove? What is to be considered evident? Regarding this point, we must recall that, even in Euclid's work, there is a call to evidence, even if it is implicit.

Limits of Geometrical Proof

Finally, is it necessary to follow the tradition according to which geometry is the privileged place for mathematical proof, despite the difficulty in defining the role of figures clearly, to delimit what can be noted on them and what must be proved, when it is also possible to see it on the figure? We met several examples of these very complicated subtleties in using drawings: proofs such as that of *Meno*, which use visual evidence, are numerous in Euclid's work, whereas triangle inequality needs a proof, and irrationality contradicts visual evidence. Moreover, as we remarked, geometrical proofs make little use of calculation, lay stress on the linguistic aspects of mathematical proof, and can even lead pupils and teachers to believe that proof only exists in geometry.

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2. THE PROOF IN THE 20TH CENTURY

From Hilbert to Automatic Theorem Proving Introduction

In this chapter the notion of proof is investigated from an epistemological point of view. Specifically, its meaning will be scrutinised comparing the contributions of some different philosophies about the nature of mathematical knowledge. The main idea is that only knowing what a proof is (or can be), one can face the didactical problem of its (possible) teaching in the class.

The chapter is divided into five sections: in the first the dichotomy *formal–informal* in mathematics is tackled; in the second the notion of *logical consequence* is defined in a precise way as the core of mathematical proofs; in the third, the discussion developed in the first two is used to criticise (quasi-) *empiristic* positions; the fourth and fifth sections sketch two important topics in proof: the role of *computers* and that of *perception.*

PROOFS AND DERIVATIONS

The Notion of Formal in Mathematics

The word *formal* is used in mathematics with different meanings.

One (*form*1) concerns the form of mathematical sentences, as structured syntactical objects independently of their intertextual contexts: for example, Aristotle's theory of syllogism considers only the structure of sentences like (a) "every A is B," (b) "some C are not B," (c) "some A are not C" and explains the reason why (c) is a consequence of (a) and (b) not in terms of their semantics but in virtue of their syntactic form1.

A second (*form*2) concerns the way mathematics is presented as a final product, in a formalised language, generally contrasted with that after which results are found by mathematicians: the arguments in the creative phase may be informal, semi-formal, formalised, drawn, sketched, based on intuition and so on, while the final product is built up in a neat and very formal way.

A third (*form*3) concerns the very notion of logical consequence, which will be discussed in §2, namely what does it mean that a mathematical sentence, e.g., "2+2=4," the theorem of Pythagoras or Riemann hypothesis, is valid within arithmetic, geometry, analysis and so on. This must be contrasted with the analogous problem of sentences in experimental sciences, e.g., what does it mean that the First Principle of Thermodynamics holds?

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 43–63.

All these and other aspects have been scrutinised with different tools and ideas in the course of centuries, sometimes without distinguishing carefully among them; in particular, the nature of formal mathematics has been investigated studying its logical aspects. Roughly speaking, we can classify such researches with respect to the given solutions in three main streams (see Cellucci, 1998; Feferman, 1978 and Lolli, 1987):

A. Some scholars have seen the formal1,2 logic only as a justificative tool, claiming that intuitive and creative aspects of mathematicians' work elude a formal scaffolding and generally leave them to *psychology* (Frege, Feferman). For example, according to Frege, the formal1,2 logic is his Begriffschrift, that is the science which studies the laws of correct inference, whilst the natural logic concerns the ways after which inference concretely is performed and as such pertains more to psychology than to logic and is based upon empirical principles and not upon necessary and universal rules (Frege, 1969, Logik, p. 4 and Grundgesetze der Arithmetik, vol. 1, p. XIV). The relationship between the acknowledgement of the truth, which is a thought, as such not purely formal, and the developing of the proving process, is very complex. Frege, in a letter to Hilbert in 1895 (Frege, 1976, p. 58) uses the metaphor of lignification: a proof to develop must be built by the truth acknowledgement, as a tree to develop in those points where it lives and grows up must be soft and juicy; but the inference must become something mechanical to develop strongly, as a tree to become high must lignify its juices. In this sense Frege develops the ideas of Leibniz about a characteristica universalis and a calculus ratiocinator (as he explicitly says in the Introduction to his Begriffschrift).
B. Some people have argued in favour of a scientific and formal1 logic, which does capture the essence of mathematics, namely its justificative aspects as well its creative features (Aristotle, Leibniz, Couturat, Hilbert,[1] Gentzen, Hintikka). For example, the difference between Frege and Leibniz consists mainly in the fact that for Leibniz formal logic concerns also the discovery of new results: for the latter logic is useful "not only for judging what is proposed but also for discovering what is hidden" (Leibniz, 1965, VII, p. 523, letter to G. Wagner, 1696). The formal method is a mechanical substitute of thought, insofar it "discharges imagination".
C. Many scholars distinguish between a scientific logic, which generally is formal1 (but has also some of the other features), and a natural logic, which goes beyond all aspects of formality (Descartes, Frege, Peirce, Dedekind). Some have also tried to study the natural logic, as a distinct "discipline" from formal1,2 logic; they have investigated both the origin of mathematical ideas (Dedekind) as well as the features of mathematical discovery (Plato, Descartes, Peirce, Poincaré, Polya, Lakatos, Hintikka and Remes). For example, Descartes illustrates the needs of a new logic of discovery (see Barbin, 1988) which cannot be embodied any longer in the formal1 logic of Scholastics: the Aristotelian logicians "cannot skilfully form a syllogism, which entails the truth, if they have not previously had its matter, that is, if they have not already known in advance that very truth

which is deduced in it" (Descartes, 1998, p. 47, Regulae ad directionem ingenii, Regula X). The Aristotelian logic is "useless for investigating the truth of things, but it can only be useful for exposing to the others the reasons which are already known, hence it must be shifted from philosophy to rhetoric" (p. 47, here the form2 aspect is stressed more). To ascend to the top of human knowledge people need a new logic (Regula II). The new logic has roots different from Aristotle and Euclid, in particular is not formal1,2, but based on the intuitive grasping of ideas and elementary facts; in fact it goes back to Pappus and Diophantus (Regula IV), namely to the so-called analytic method.

The same root, namely the analytic method of Pappus, is invoked by many people who pursue anti-formalistic issues. For example, the last chapter of Lakatos' Dissertation at Cambridge (1961) is devoted to the method of analysis-synthesis, as well as an address at a Conference in Finland (1973), in reply to a paper of Hintikka on the subject (all together, they constitute chapter 5 of Lakatos, 1978). Lakatos uses Pappus' and Proclos' definition of analysis to describe the process of discovery in mathematics, in particular that of criticising proofs and improving conjectures (Lakatos, 1976, pp. 9 and 75). The method, that Lakatos calls thought-experiment or quasi-experiment (from Szabo, 1958), consists in decomposing "the original conjecture into sub conjectures or lemmas, thus embedding it in a possible quite distant body of knowledge" (Szabo, 1958.).

Polya (1954, 1990) rephrases Pappus' method (he was called a second Pappus by Hintikka and Remes): for him analysis is not a method upon which one can build up criticism to the formal way mathematical truths are presented in books after Euclid, namely with "finality-certainty requirements [which] survive in mathematics until today as the requirement of necessary and sufficient conditions" (from Lakatos, 1978, p. 75). Instead, it is an auxiliary method, which helps in building up the rigorous formal proof: it helps in generating "a better understanding of the mental operations which typically are useful to solve problems" (p. 75.), and as such it is introduced as a useful pedagogical tool.

Formal and Informal Mathematics

In the historical development of ideas about proof in this century, Hilbert is crucial, insofar as his position, known as (a variety of) formalism, is a landmark in the history of logic and moreover his claims are the target of many people akin to quasi-empirical or anti-formalistic positions (e.g., Lakatos).[2] "Formalism denies that mathematics is knowledge of some reality, and claims that is more akin to a deductive activity. There are many variants of formalism. … Hilbert's formalism was a shrewd position: for him the presentation of theories as formal systems was only a technical move to allow the use of formal tools of mathematical logic to prove their consistency. Notwithstanding the failure of Hilbert's program, by his authority most mathematicians became convinced that his approach (or, loosely, formalism) was the right mathematically acceptable foundations (which anyway is another matter than practice)" (Lolli, 2000, p. 15). This conviction was at the origin of a comedy of errors, according to which formalism would say that mathematical practice is formal (in the sense 1 and 2): and many times the criticism of anti formalists, in particular that of Lakatos, was against this last claim. Hilbert asserted only the possibility of a technical translation of such a practice into suitable formal systems. In fact, Hilbert, as well as Frege, faced the problem of the relationships between proofs, that is conceptual proofs with a semantic content, as usual in mathematical practice, and derivations, that is syntactic objects of some formal system.

Let us give a very well known example, to illustrate the situation; namely, consider the sentence that prime numbers are infinite and contrast:

i. the proof given by Euclid (Book IX, Proposition 20; see Heath, 1956);
ii. the proof given by Euler (see Ribenboim, 1988, chap. 1);
iii. proofs given in books of analytic number theory (for an example see Rademacher, 1964);
iv. its derivation in a formalised version of Peano arithmetic (e.g., as in Schütte, 1977, chap. VII).

If one tries to translate proofs (i), (ii), (iii) into some formal system, one must face increasing difficulties. For example, the notion of finite cannot be fully captured within systems of arithmetic using first order predicate language (because of Lowenheim-Skolem theorems, which imply the non categoricity in power of the models of the theory): so there is a derivation within Peano arithmetic which mimics proof (i), but its semantic counterpart requires some reflection (for example to formalise everything within set theory and to use it as a basic theory to found arithmetic, but this makes the derivation very far from the original Euclid proof). Proof (ii) requires many translations of new concepts within arithmetic (for example the notion of rationals, of infinite sum, of its convergence), so that the derivation becomes involved and very far from the original proof of Euler; proof (iii) poses more problems, as all proofs which are not elementary (Hilbert called them impure): in fact, to formalise analysis one must introduce more powerful machinery, which again makes the derivation in (iii) far from the original proof. On the other hand, the relationships between formal systems, let us say of Peano arithmetic and analysis, may be investigated with logical machinery. The so-called conservativity results allow us to investigate when, given a derivation of an arithmetic statement within analysis (that is by impure methods), there also exists a derivation within elementary arithmetic (even if one does not always know how to produce it). This is a truism for the Euclid proposition, but it is not always so: for example the fundamental theorem on prime numbers (the number of primes up to N is asymptotically of the same order as N/logN) had an impure proof in 1896 by de la Vallée Poussin and only in 1949 an elementary proof was found by Erdös and Selberg (see Ribenboim, 1988).

The example illustrates the differences between proofs and derivations. At this point of the discussion, a precise definition of what is a proof is not possible:[3] it is possible to enumerate its functions[4] or to investigate its epistemic contribution to the validity of mathematical sentences,[5] but only at an informal level. On the contrary, it is possible to give a precise definition of a derivation within a formal system (generally as a finite sequence of sentences of its language, each obtained from the previous ones or from the axioms of the system by means of precise deduction rules).

As Rav points out, “the relation between proofs and derivations is in a limited sense analogous to the relation between the non-technical term of effectively computable function and the technical term of partially recursive function” (Rav, 1999, p. 11). In the latter case, the adequacy between the two notions is assured by the so-called Church Thesis. Is there an analogous bridge between proofs and derivations? (the terminology is from Rav). Hilbert thought in the affirmative; his claim was that, not only every formal derivation can be seen as a proof (soundness of the formal system with respect to mathematical reasoning: the derivation can seem far from human reasoning, but at the end one must recognise it as a proof), but also every proof is represented by a formal derivation within a suitable formal system (completeness): in other words, formal derivations capture the human reasoning, at least indirectly. In fact, as we have seen, the derivation may be very far from the original proof and from its ideas; sometimes it can be so far that it should be impossible to make the reverse way, namely to restore the original proof from the derivation self.

Anti-Formalistic Criticism

It is well known that Hilbert’s programme failed when Gödel proved his incompleteness theorems: in fact, for any formal system which formalises some piece P of mathematics (containing at least some part of arithmetic), there are statements in the language of P, that are mathematically meaningful but are also undecidable (namely neither the statement nor its negation is provable in the system).

Now, the main criticism of anti-formalists, in particular of quasi-empiricists, is to challenge the bridge between proofs and derivations and to deny the interest of the latter and sometimes of the former. To support their argument, they generally invoke Gödel theorems and the failure of Hilbert’s programme; say that the formal aspects (generally form1 and form2) do not concern proofs; what is important consists in the informal aspects of genuine proving processes of mathematicians. Emphasis is on the different epistemic functions and semantic features of proofs which are not incorporated in derivations, which for such a reason become uninteresting.

Here are some typical examples of their arguments:

- Lakatos underlines that the dialectic between conjectures and refutations is essential to the never-terminating genesis of proofs but is missing in the crystallised formal derivations according to the Euclidean style.
- Other people attack derivations (but generally they do not distinguish them from proofs) insofaras they do not convince of the truth of what they prove: they are not feasible, not holistic, nor surveyable and so on (for an extreme position, see Horgan, 1993).

The conclusion is sometimes that (not only derivation but) the same process of proof is not interesting and needs deep revision (Kolata, 1976). This has been stressed by influential scientific journalists as well as by mathematicians: Horgan

(1993) says that Wiles proof of Fermat's last theorem was a "splendid anachronism," while Zeilberger (1993) says that the age of semi-rigorous mathematical culture and of theorems which are "probably true" is coming.[6]

The manifesto of people arguing against the proof can be sketched as follows:

i. derivations and proofs (generally confused together) are useless for grasping mathematical truths;
ii. formalistic aspects are not relevant for genuine mathematical practice; on the contrary, this has an intuitive character which is lost if one shifts to the formal; possibly only reasons of communication are at the root of the final products of mathematicians structured according to semi-formal2 standards;
iii. the nature of mathematics is essentially empiric:[7] "trial and error procedures, conjectures and refutations, searches with computers, quasi-experimental methods are sufficient to establish mathematical results … the main consequence of the empiricists' outlook is that mathematics is mutilated with the elimination of proofs … at best, proofs are seen as part of a falsificationist procedure, a Popperian conjectures-and-refutations strategy, as in Lakatos" (Lolli, 2000, p. 14);
iv. Gödel's theorems show the uselessness of formalisation.

Let us discuss such claims. First of all, the distinction between proofs and derivations is fundamental, as it is important to remember that proof theory is a branch of logic which studies derivations as mathematical structures (as indicated by Hilbert). The study of proofs requires different methodologies, even if the results of the former discipline may be interesting for the latter. If one has not clarified this distinction, it is easy to throw away the baby with the water. "The usual proof generated by a mathematician does not involve the careful application of a specifically formalised rule of inference, but rather involves a somewhat large jump from statement to statement based on formal technique and on intuitions about the subject matter at hand" (Arbib, 1990, p. 55). Hence the problem is how to analyse such large jumps from a chunk of mathematics to another. To do that, it is necessary to discuss briefly the notion of logical consequence.

THE NOTION OF LOGICAL CONSEQUENCE[8]

Lolli (2000, p. 54) with arguments similar to those of Arbib, points out that "to expose, or to find, a proof people certainly argue, in various ways, discursive or pictorial, possibly resorting to rhetorical expedients, with all the resources of conversation, but with a special aim which is foreign to the lawyer, that of letting the interlocutor see a certain pattern, a series of links connecting chunks of knowledge. The chunks may be more or less large, according to the mathematical sophistication of the discussants. The links are logical … A logical link is not a small step corresponding to some rule, but the subsistence of the relation of logical consequence".

This notion was given by mathematicians (Pasch, Poincaré, Hilbert, Enriques, Peano) at the turning of the century: they said more or less that every theorem is a statement B for which there is another statement A (generally a conjunction of statements, which possibly are among the axioms of some theory), such that B is a logical consequence of A (which we can write "A→B," where the arrow has only an iconical function and does not mean formal implication). Another way of saying that is that A→B is "logically true". "Logically true means 'true under any interpretation whatsoever'" (Lolli, 200, p. 54.): that is in whichever way the relationships, functions, etc. of the statement A→B are interpreted (e.g., within set theory or a system of objects or also a translation of a language into another), each time that A is satisfied in that structure then also B is satisfied.[9] This same notion of theoremhood used informally by mathematicians has been taken and investigated by logicians. It is precisely in this sense that mathematics is formal3: in fact our definition of logical consequence gives a reason why the mathematical statements hold. In this sense, it is certainly true that they are timeless truths very different from empirical ones.

A statement B can be a theorem only relative to some theory; it is senseless to say that it is a theorem (or a truth) in itself: even a proposition like "2+2=4" is a theorem in a theory A (e.g., some fragment of arithmetic).

The activity of proving made by mathematicians consists in entering into the relationship A→B and for doing that they argue: "in trying to understand the author's claim, one picks up paper and pencil and tries to fill up the gaps; after some reflection on the background theory, the meaning of the terms and using one's general knowledge of the topic, including eventually some symbol manipulation, one sees a path from A to A1, from A1 to A2, …, and finally from An to B. This analysis can be written schematically as follows:

A→A1, A1→A2,......, An→B.

Explaining the structure of the argument to a student or non-specialist, the other may still fail to see why, for instance A1 ought follow from A. So again we interpolate A→A′, A′→A1. But the process of interpolations for a given claim has no theoretical upper bound. In other words, how far has one to analyse a claim of the form 'from property A, B follows' before assenting it depends on the agent". A similar picture is given in Polya (1954, 1990), where he points out the non linear and multi-directional features of this process.

This is the basis for the following definition of proof (see Lolli, 2000): A proof is an ordered set of statements of the form Ai→Ai+1, which are linked by transitivity. There is only one rule: write Ai→Ai+1, whenever Ai+1 is a logical consequence of Ai. The set may have a linear order, but also more complex types of order, like the observations of Polya stress (see Arzarello et al., 1999).

The real problem consists in the fact that the relationship of logical consequence is undecidable. That is, there does not exist any machine like that dreamed of by Leibniz, which can compute for any statement A→B, if this is logically true or not. Hence the mathematicians look for good reasons why a statement like A→B holds, or not, and this job requires ingenuity. These reasons constitute the proof: "the proof is nothing more than a decomposition of the consequence relation in a chain of instances of the same relation (granted transitivity) which are easier to see, until people agree that they see them". In other words a proof is a discourse, which can refer to every possible chunk of mathematical knowledge that the agent(s) think useful and productive for ascertaining that B is indeed a consequence of A. Proving is a dialogic process of the subject with an interlocutor (possibly virtual): this has important consequences for the didactics of proof (see: Balacheff, 1988; Duval, 1992).

It is clear that a priori a proof has no finitary character: it is not necessarily a finite object. This point must be discussed briefly with respect to two basic results of logic, both due to Gödel, namely the completeness and the incompleteness theorems.[10]

Because of the former, for each proof of A→B there is a derivation of the same statement, that is a finitary object (see the definition of derivation at p. 51 of this chapter) which "proves" formally A→B within a formal system. As we have underlined, this does not mean that the formal proof corresponds verbatim to the informal one (even it may be so in many cases). Moreover, the finiteness of the derivation is such in line of principle; in concrete cases the length of the derivation would be enormous, hence the derivation would not be perspicuous, surveyable, elegant, etc., properties which mathematicians like the best in their informal proofs.

Because of the latter, the mathematical truths can be only approximated step by step by the finitary means of formal systems. As Rav says: "There is no theoretical reason to warrant the belief that one ought to arrive at an atomic claim C⇒D which does not allow or necessitate any further justifying steps between C and D. This is one of the reasons for considering proofs as infinitary objects. Both Brouwer and Zermelo, each for different reasons, stressed the infinitary character of proofs". As Rav observes, it was Kreisel (1970, footnote 22, p. 511) who pointed out that "properly interpreted, Gödel's theorems can be used to support this insight [that proofs are of infinite character], just as they are used to refute Hilbert's assumption that finite formal derivations reflect faithfully the structure of mathematical reasoning".[11]

The first consequence of such an analysis is of a didactical nature. In fact, our discussion shows that the conditionality of statements characterises the same process of proof and that to prove a conditional statement one must enter into an infinitary game of interpretation (see also note 15 below). Hence a major problem in the class becomes the generation of conditionality and the transition to proving processes. For further elaboration about that, see Arzarello et al. (1999), Boero et al. (1999), and their chapter in this book.

The second consequence is a radical criticism of those who claim the uselessness of proof.

A RADICAL CRITICISM TO QUASI-EMPIRICISTS (AND OTHERS)

Let us remember the headlines of the anti-proof manifesto:

i. derivations (proofs) are useless for grasping mathematical truths;
ii. formalistic aspects are not relevant for genuine mathematical practice (possibly except communication);
iii. the nature of mathematics is (quasi-)empiric;
iv. Gödel's theorems show the uselessness of formalisation.

The previous technical discussion has put forward the following points:

i. distinction between proofs and derivations;
ii. proofs concern assertions like "B is a logical consequence of A," where form3 is crucial (form1, form2 are not so pertinent);
iii. proofs as such are infinitary objects, which concern the undecidable relationship of logical consequence: hence it is necessary to look for perspicuous proofs.

Critical consequences:

1. the argument (iv) is false: in fact (c) is the theoretical basis which gives sense to the process of continuous research of proofs made by mathematicians.[12]
2. (ii) is false, because of (b): proofs are formal3; assertion (ii) is based on other notions of formal (namely formal1, formal2), which do not capture the genuine mathematical notion. In fact proofs have a lot of functions (see note 2): it is an ill posed problem to investigate if these belong also to derivations: derivations are another thing from proofs and can be used essentially to study a mathematical counterpart of proofs; the connections between the two worlds are very complex (see: completeness–incompleteness).
3. Claim (i) can be accepted in a weak form, that is only restricted to derivations and with some proviso. It is certainly true that a proof gives meaning to a mathematical statement, putting it into the network of mathematical knowledge. The points (b)+(c) show that there is no previous crystallised form of entering into the interpretive process of producing an assertion like A⇒B and to prove it. All heuristics are good! The point is that a proof has not the aim of convincing but of entering in one (or more) perspicuous way(s) into the formal3 reasons why such a logical consequence is valid. Convincing is not to be one of the functions of the proof (see note 2). This is a crucial point for the didactic of proof, pointed out by Balacheff (1982).[13] Some people seem not to be always aware of that; for example they compare proofs with legal arguments, and seem to forget the essentially different epistemic values of statements in the two fields (see Alchourron and Martino, 1987). On the contrary, the (different levels of) form2 aspects are important in the communication process, according to a complex pattern of interaction among mathematicians, which has been described in a fascinating way in Thurston (1994). Moreover, one must not forget the main point, discussed above: the completeness theorem, which assures that the semantic notion of logical consequence makes sense, is provable because derivations exist. Paradoxically, we do not need derivations because, using them, we have proved that informal proofs are enough.
4. Claim (iii) is contradicted by facts.[14] Our previous discussion and particularly sentences under (b) and (c) show that the essence of mathematical practice consists in conjecturing and proving statements of the form A Æ B, namely in pursuing suitable hypotheses and reasons for the so-called mathematical truths (whose origin may be very variegate, also from empirical facts).

The game of hypothesis and of interpretation which features mathematical research[15] has a completely different dynamic than that described by Lakatos and his supporters (see Boero et al., 1999). The proof whose death has been announced in a big rumour some years ago (Horgan, 1993) is not the proof but some virtual creature, a pale imitation of real proofs. The examples in papers of Rav (1999) and Lolli (1985, 2000) and the increasing numbers of proofs published each year in the scientific magazines give both theoretical and experimental support to this observation.

The main consequence of the above discussion is that if one's concern is the teaching of mathematics, one must teach proofs, as it is also argued widely in the contribution by Hanna. The reason is not because one wants students to mimic what professional mathematicians do, but because under the list of the functions of proof (note 2) one finds many of the headlines of a reasonable agenda for the learning of mathematics (from computations to powerful ideas, through problem solving). Our epistemological analysis gives a solid basis to the proof as a crucial activity within mathematical practice, putting also in the right perspective its connections with the formal systems, which are studied by logicians. In fact, our foundational basis is different from Lakatos' claim that mathematics is purely conjectural.[16] "Mathematical knowledge is cohesively soldered thanks to the methodological and logical components of proofs. Proofs as we know them are the heart of mathematics, the generators, bearers, and guarantors (modulo collective verifications) of mathematical knowledge" (Rav, 1999, p. 31). The didactical analysis made in the contribution by Boero et al. (Boero et al., 1997), shows the essential role of proof within the didactics of mathematics, as a route to theoretical thought in the students.

But if our epistemological discussion and the contributions by Hanna and Boero in this volume show the opportunity of teaching the proof, it leaves open the question of what it means to teach proof in the class, which implies the necessity of considering the problem from a didactical and a cognitive perspective.

The chapter by Boero et al., answers this question within our epistemological framework. However, it is still necessary to discuss here two general points, which are of interest when approaching proof in the class.

First, which is the role of derivations in the didactics of proof? In fact, if it is true that they are very different from proofs, in many elementary cases there is an isomorphism between proofs and derivations. In such cases the use of a suitable formalism could help students in understanding the meaning of what they are proving. In fact, as is well known,[17] the capability of using a suitable symbolism can help and support students in their way to understanding mathematics, provided that the symbolism is acquired through suitable experiences which do not prune its semantic counterpart.[18]

Second, which is the role that perception can play in the game of hypothesis and interpretation, within which pupils build up their proofs?

Such questions are particularly interesting nowadays, when the new technologies allow one to tackle them with fresh and promising ideas. These points will be elaborated in subsequent chapters, where the discussion will also face some cognitive problems.

THE MACHINE AS A METAPHOR AND AS A REAL OBJECT

Approaching proofs, the notion of logical consequence is basic,[19] but it is difficult to grasp and to communicate it, even in the long run (see for example, the discussions in Harel and Sowder, 1998). In fact it has no ostensive[20] counterpart (except possibly in mathematical logic[21]). This may explain its substitutes, which we find both at epistemological and at didactical level, namely the rhetorical form2 as well as the syntactic form1, which both can be based on some language of representation and manipulations (for example that of mathematical logic, or the double columns proofs used in some schools). Didactics of proof based on such "shifted" meanings can be dangerous in the long run, as Duval (1992) and Mariotti (1996) have pointed out. As appears in their experimentations, a possible way to create a didactic situation suitable for proof is to introduce in the class some kind of ostensives to interpret and manipulate,[22] which are meaningful for the students.

In this chapter I shall discuss briefly how the computer can give some help for teaching proofs in the class.

A major question is: how are computers linked with proofs? Here are some possible answers:

i. Each derivation within a formal system is a computation and vice versa;[23] that is, a derivation can be represented by a program (and conversely, each program is a derivation).
ii. Like there is a complex relationship between proofs and derivations, so there is an analogous correspondence between algorithms and programs.[24]
iii. Computers allow students to access and manipulate specific ostensives concerning proofs.[25]
iv. The interactions students–computers in the context of proof (e.g., in software for dynamical geometry[26]) may facilitate the generation of dynamic ways of reasoning, that is of those transformational schemes which are so important for developing mathematical reasoning abilities (see: Simon, 1996; Harel & Sowder, 1998 and Harel's contribution in this volume).
v. The interaction students–computers entails the perceptual level in a massive way, hence opening new routes to theoretical knowledge within a concrete environment, which is meaningful for the student.

One of the reasons why the computer is so intriguing in the didactics of proof is that it can be used as a cultural artefact (see Saxe, 1992, 1994) which mediates in a "natural" way the problem of coding. Hence it puts on the table some relevant questions, which risk remaining hidden, when looking exclusively at proofs. As we have seen discussing the bridge between proofs and derivations, coding makes things mathematically precise but sometimes anti-intuitive: this contradiction is one of the main features when switching back and forth from proofs to derivations or from algorithms to programs. To solve a problem in a feasible way,[27] both the generation of an algorithm and its implementation with a program are required. The dynamic is similar to the proving strategies: finding an algorithm is a particular case of finding a proof (see note 24). In fact, people describe both with the same words. For example, a key point is given by the so-called "strategic moves": see for example the concept of pivot in Leron (1985) for the proofs and that of strategic moves within the generation of algorithms in Arzarello et al. (1993).

A second key point is the interplay between the two types of languages that enter in the game, when thinking algorithmically.[28] Namely, that of the problem where one has (a more direct representation of) the involved objects (for example, numbers, graphs, and so on) and that of pseudo-programs. That is a general language of programs (not yet a specific language), within which one generally writes down algorithms and that will be the starting point for writing a concrete program within a specific language.[29] At a first glance, the language of objects has a more semantic nature, whilst in that of programs the syntactic aspects prevail: the interplay between the two is complex, since they are not isomorphic. But a general point which concerns us is the following: the above interplay does not happen in two steps, namely first there are the strategic moves and then their translation into the general language of programs; the two are both alive from the very beginning and the one deeply influences the other. It is the cognitive and epistemological nature of this interplay (see Arzarello et al., 1993) which makes possible and visible a game of settings, in the sense of Douady (1991), and that can have positive consequences for the learning of mathematics.

Summarising, one can say:

1. for algorithms the general language of programs as well the computer upon which one imagines to make dry runs naturally mediate amongst the problem, students' knowledge and the algorithm to find; that is, they can support the students cognitively, while scaffolding the idea-pivot upon which to develop the algorithm and successively the program. One can assert that such an environment may create suitable fields of experience (in the terminology of Boero et al., 1995) as well as didactic space-time of production and communication (SP in short, see Arzarello et al., 1995).
2. For proofs, no natural mediator seems to exist at first glance: the researches of Boero and his school are concerned exactly with the problem of finding suitable fields of experience for the teaching of proof: the problem is discussed in his contribution to the volume. In what follows, I shall elaborate some complementary idea, focusing on the possibility of using the computer as a possible source of fields of experiences or SP[30] and as a suitable mediator within them. There are practical and theoretical reasons for this. Practically, the process has already begun: symbolic manipulators as well as dynamic software for geometry are forcing all teachers to approach mathematics within these new technological environments. Theoretically, the computer, as a universal interpreter, incorporates the abstract notion of algorithm-derivation and, as a cultural artefact, makes accessible:

   i. ostensives concerning the abstract notion of logical consequence;
   ii. perceptual objects which with their dynamicity are ostensives for the same notion of 'generic' and 'general' object,[31] which is crucial in the transition of students from empirical to theoretical arguments.

I shall sketch only the last points. First some comments about the fact that a computer incorporates algorithms-derivations. The revolution of computers has fascinating technological and mathematical roots, namely transistors, microcircuits and the idea of a universal interpreter. Roughly speaking, the latter means that one does not need different machines, one for each type of task; in fact, there exists a machine that can mimic every other machine. This is possible via the representation of algorithms and machines themselves through programs (in some language): because of this, one can code each program say with a number, and build an algorithm that can de-code such a number, get the real program and execute it (universal interpreter). This conceptual revolution was made by Turing in 1937 (see Hodges, 1983), when he invented the (virtual) machine which has his name. The physical revolution was realised with programmable computers: a PC is nothing more than a concrete universal machine, which accomplishes its task more or less efficiently, according to its technological sophistication.

The idea that a machine incorporates some piece of abstract mathematics is very old. For Greek geometry, (ideal) ruler and compass incorporate the meaning of geometric constructions, of which many propositions of Euclid constitute the abstract and discursive counterpart. Descartes invented a new (ideal) machine (see Figure 1) for drawing curves:[32] it incorporated a wider abstract notion of geometric object (namely, that of algebraic curve), whose discursive counterpart are the manipulations on the algebraic equations, that is the new ostensives of such objects.[33]

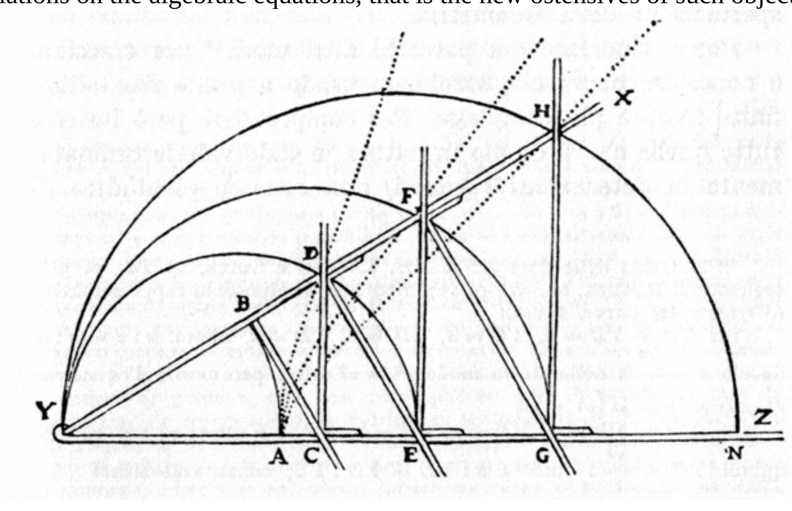

Figure 1

Before Turing revolution, the machines incorporated some mathematical and abstract ideas, but proofs were outside them.[34] With Turing a piece of hardware (namely a specific machine) could be interpreted also as a piece of software (namely a program): via coding, every machine could become the input of another machine (the universal interpreter). Hence proofs could enter the machines as programs, in the same way as they could be translated into derivations. As a consequence, proofs (or at least derivations) are no longer outside the machines. The personal computer that I am using to type this manuscript incorporates also mathematical derivations (hence, indirectly, proofs).[35] But this is not so explicit at a first glance. Some specific software is required that makes this visible, so that the instructions corresponding to the axioms of a theory become explicit. This has been realised at different levels by software like Sketchpad, Cabri, Cabri-Euclid (see Luengo, 1997), GEX (see Gao, 1998). Some examples that are in Chapter 15 show a concrete didactical translation of this idea.

PERCEPTION AND PROOF

Let us now discuss the interplay between perceptual and conceptual features in proofs, particularly in geometry. The problem has been studied by many people in the far and near past and has recently been discussed in Otte (1999): "Mathematical perception depends on representations. This implies that mathematics deals with intensional objects. … For instance, in Cabri-geometry two triangles which seem to be completely the same (or congruent) may behave differently when being pulled around, because they have been constructed in different ways (they are intensionally different). [But] Mathematics is interested in truths about real objects and therefore is fundamentally interested in extensionality. … [It] depends on ostensive demonstration and indexicality and thus, like any empirical science, once more on perception". According to Otte, it has been our humanistic and philosophic culture which has pointed out the prevailing role of the language in mathematics, namely its discursive aspects, underestimating its intuitive and perceptive aspects.[36]

On the contrary, as we have seen, the mathematics develops through "large jumps from statement to statement based on formal technique and on intuitions about the subject matter at hand" (p. 7), namely the game of hypothesis and proving goes on through a dialectic between formal and perceptual or observational facts (or anything else that, within the intended matter, can help in that moment to make the large jump). Among the heuristics used by mathematicians many times the visual and perceptual aspects are crucial, as has been very well known for a long time (a detailed discussion concerning Euclidean geometry is made in Arsac, 1998; see also his contribution to this volume).

But modern technology also points out other aspects where perception is involved: for example the interaction between subjects and the figures at the computer screen through a mouse within an environment for dynamic geometry involves not only vision but also motion and tactile senses. So the interplay between the activity of conceptualisation, proving etc. and perception becomes more complex and intriguing. As Otte (1999) underlines, "the function of the logical development of mathematical concepts … consists above all in transforming a dynamic blend, a chaotic motion of activities and temporal processes, into images or forms which can be examined". The books by Jakobs (1990) and Berthoz (1997) and the project Geometry Cognition (see Longo et al., 1999) give some examples of this type of analysis.

This approach seems very promising both from an epistemological and from a practical point of view. A concrete worked-out example is given in the contribution of my group to this volume. The focus on the relationships between perception (in a wide sense), cognition and mathematical concepts, which is now entering the didactics of mathematics, underlines a new line of research, which still needs theoretical analysis and experimental research.[37]

NOTES

[1] The position of Hilbert is particular (see § 1.2): for him symbolism does not coincide with mathematical thinking; it is only a representation through a code, which in its turn can be studied mathematically (for a wide discussion see Lolli, 1987).

[2] The different epistemological positions concerning mathematics (see: Benacerraf and Putnam, 1964; Schirn, 1998) are usually divided into Platonism (Gödel), Logicism (Frege), Formalism (Hilbert), Constructivism (Brouwer), Empiricism (Lakatos), Naturalism (Quine). In the debate within the didactics of mathematics there is much discussion, particularly about Formalism and Empiricism, above all for the relative success of Lakatos' thesis (see Ernest, 1994).

[3] Rav (1999, p. 13) proposes the following definition, which we accept as a working hypothesis: "Proofs are the mathematician's way to display the mathematical machinery for solving problems and to justify that a proposed solution to a problem is indeed a solution"; for more a technical definition, see p. 8.

[4] Lolli (2000) enumerates the following functions of proof and says that the list is not exhaustive: "avoiding or replacing computations, controlling instruments, anticipating outcomes of experiments, prevision, economy of thought and of memory, reliability through visualisation, suggesting generalisations, generalising, explanation (through axioms or by subsuming under a general case), transportability, correctness of algorithms, problem solving, extracting algorithms, constructing figures, making life easier, having fun, showing impossibility, disproving, finding counterexamples, inventing concepts, conceptual analysis, refining and correcting intuition, validating intuition".

[5] Rav (1999, p. 20), observes: "The whole arsenal of mathematical methodologies, concepts, strategies, and techniques for solving problems, the establishment of connections between theories, the systematisation of results—the entire mathematical know-how is embedded in proofs. … Theorems indicate the subject matter, summarise major points, and as every research mathematician knows, they are usually formulated after a proof-strategy was developed, after innovative ideas were elaborated in the process of 'tossing ideas around'. Proofs are for the mathematician what experimental procedures are for the experimental scientist: in studying them one learns of new ideas, new concepts, new strategies. … Think of proofs as a network of roads in a public transportation system, and regard statements of theorems as bus stops; the site of the stops is just a matter of convenience".

[6] For a discussion of this point and particularly of misunderstandings concerning probabilistic algorithms see Rav (1999); for more discussion see also: Andrews (1994), Kranz (1994), Thurston (1994).

[7] On quasi-empiricism and antiformalistic trend see: Hersh (1979), Davis and Hersh (1981), Feferman (1981), Lolli (1985), Tymoczko (1986).

[8] I am deeply indebted to G. Lolli for the elaboration of this chapter.

[9] As a simple example, consider the statements of groups. To say that the statement B="the inverse of an element is unique" is a theorem means to show that in every system of objects that satisfy the axioms A for groups also the statement B is satisfied (and this is achieved by making some computations, possible within that system of objects, since it satisfies the axioms for groups).

[10] Roughly speaking:

- The former says that whenever $A \rightarrow B$ is logically true, then there is a derivation of it within a logical system for predicate logic.

- The second (at least one of the versions of the so-called first incompleteness theorem, see also p. 5) says that arithmetical truths (namely statements of arithmetic which are satisfied by the system of natural numbers) cannot all be obtained as theorems within any formal theory of numbers. This result applies to a lot of mathematical structures.

[11] Kreisel quotes the following papers as a source of the idea that proofs are infinitary: Brouwer (1927), footnote 8; Zermelo (1935), p. 145.

[12] This continuous elaboration is complex and does not fit with the picture given in Lakatos (1967); in fact it happens at two levels:

- First, within a theoretical context, by seeking new ideas for simpler and more perspicuous proofs of some statement. This may also cause the search for new and deeper definitions of the mathematical objects involved in the statement. At a certain point the crystallisation within the theory is reached and a proof of that statement is acquired in a definitive way. That is, the process of approximation of the infinitary aspects of proofs comes to an end (at least within that theoretical context).
- Second, at the level of theoretical contexts and theories, which can be changed because of more general theoretical reasons. As an example think of the three different proofs of the infinity of primes: each points out some relevant reason why such a statement holds, within different mathematical contexts.

[13] Especially when an approach is pursued, which is based on the so-called “scientific debate” in the class: see the discussion in the last part of the chapter and Alibert and Thomas (1991).

[14] This point is discussed in detail in Arzarello (1992).

[15] I take the expression “game of hypothesis” from Ferrari (1992), who uses it in the context of proof, and the expression “game of interpretation” from Arzarello et al. (1995), who discuss the semiotic process of interpretation within an algebraic context; for an example of semiotic interpretation within the context of proof, see Duval (1992).

[16] Our analysis shows also the weakness of the notion of heuristic falsifier (apart the unclear way in which Lakatos defines it, see Lakatos, 1967): since the picture of a dialectic between proofs and derivations is not genuine, since only the proving process is essential in mathematics, the idea of a heuristic falsifier loses its interest. In fact, heuristic falsifiers live within the concrete process of proof; generally, they mean the necessity of deepening the analysis in a more rigorous way, for example changing the definitions of the mathematical objects involved (like in the well-known examples on polyhedra or uniform continuity, illustrated in Lakatos, 1976) or even changing the theory of reference (the story of the Last Fermat Theorem illustrates this widely: see Aczel, 1998).

[17] G. Hanna writes (Hanna, 1991, p. 61): “Formalism should not be seen as a side issue, but as an important tool for clarification, validation and understanding. When a need for justification is felt, and when this need can be met with an appropriate degree of rigour, learning will be greatly enhanced”.

[18] A typical example is the language of algebra: in fact any algebraic computation is a derivation—possibly a meaningful proof—within a formal system, for example the theory of fields.

[19] It is the mathematical counterpart of what Boero et al. (1997) call the theoretical sense of mathematics; some other people say simply that this means to know why mathematical sentences are true: see Dreyfus (1996).

[20] The notion of ostensive is discussed widely in Chevallard et al. (1997).

[21] Compare the situation for proofs and mathematical logic with that for numbers and algebra: our discussion above shows that mathematical logic is not for proof what algebra is for numbers and other structures; to put it succinctly “mathematicians’ errors are corrected, not by formal symbolic logic, but by other mathematicians” (De Millo et al., 1979). Mathematical logic and derivations are used mainly to make meta-assertions about theorems, axiom systems, and so on. Of course, this position would not be shared by those people who think that logical sytems incorporate the proofs of mathematical practice (see list of names under b, p. 44); but as we have seen, this strong position is not tenable.

[22] This point of view is already present in Descartes (see the first pages of his Geometrie, 1983): the idea of substituting figures with algebraic equations in geometry means also the possibility to have suit-

able ostensives and explicit manipulation rules which could substitute the proofs made in the Euclidean style; it was usual in the books of analytic geometry made in different places to diffuse the new ideas to underline the perfect isomorphism between the new method (algebraic calculations) and the Euclidean proofs (see Freguglia, 1999).

[23] In fact, via coding, the set of theorems of a formal system becomes a recursively enumerable set of numbers (that code the provable sentences), that is the set of images of a total recursive function (things can be arranged so that the first generated numbers are the axioms of the system, provided this is finitely axiomatizable); conversely every total recursive function can be interpreted as a suitable formal system (see Schütte, 1977, Chap. I).

[24] An algorithm corresponds to a strategy for solving a problem (generally for computing a certain function defined in some conceptual way), with the proof that it really does that. As such, an algorithm is made of ideas, strategic moves and so on, exactly like a proof. The program is a translation of the found algorithm into some precise language, which can be "read" by a machine. It corresponds to derivations; hence we can pursue the following analogy: proof: derivation=algorithm: program.

[25] See for examples Luengo (1997), Gao et al. (1998).

[26] For example, Cabri, Sketchpad, GEX.

[27] That means the concrete production of an algorithm which solves the problem, with the justification that it really does that job; feasible can be contrasted with effective: a pure existential proof of such an algorithm shows that there is an effective solution, but possibly the concrete algorithm is not yet known. A further problem, which generally is faced when writing the program, is the complexity of the algorithms, for example how its computation time grows as a function of the length of the inputs.

[28] The presence of a double register is less evident with proofs, where the impression may be of working mainly at a semi-informal level within a single language. For example in synthetic geometry, the discourse seems to remain within one language (but there is also the language of figures, which acts at different levels, see Laborde, 1993); on the contrary, within elementary arithmetic, the double register between the semi-formal languages of arithmetic and of algebra is more visible (see Arzarello, 1996).

[29] For examples, see the book by Maurer and Ralstons (1991).

[30] Also the geometry of coordinates which was introduced by Descartes as a powerful substitute to Euclidean proofs is not seen in the school as such, at least in Italy.

[31] See Harel and Tall (1991) for a discussion on these concepts, and Balacheff (1988), for a detailed analysis of the notion of generic as crucial in the transition from an empirical to an intellectual approach to mathematical truths.

[32] The so-called compass with shifting straight-edges, of which he already speaks in his *Cogitationes Privatae* (1619–20).

[33] The dialectic between mathematics and machines is described widely in Bartolini et al. (1997) and llustrated in Bartolini (1998); for further comments, see her contribution to this volume.

[34] It was the dream of Leibniz to have a theorem prover and it was the ingenuity of Descartes to introduce the algebraic language to represent the geometric objects and the algebraic computations as a new revolutionary method of proof (see Freguglia, 1999).

[35] Of course there are serious physical limitations which make such an assertion true only within a limited range of sentences. Neither does the claim mean that, in principle, all mathematical sentences are decidable using a computer (which is false).

[36] He writes: "after Greek Antiquity, mathematics are a science of the eye and of the form, hence a visual art. However, since the Renaissance ideas about the matter have started to get confused" (*ibid.*).

[37] For an example of a theoretical analysis see Arzarello et al. (1999), where, following Peirce (1960) (Vol. II, Book III, Chap. 5, pp. 372–388), the concept of abduction is considered from a "perceptual" point of view.

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3. STUDENTS' PROOF SCHEMES REVISITED

INTRODUCTION

A psychological framework, informed by historical, philosophical, and cultural analyses, for examining students' conceptions of proof (called *proof schemes*) was offered in Harel and Sowder (1998). The framework was based on extensive observations from a sequence of teaching experiments with college students (mostly mathematics majors) in three mathematical areas: geometry, linear algebra, and elementary number theory. The goal in Harel and Sowder (1998) was to characterize students' *proof schemes* and to provide evidence for the existence of such schemes. It was designed to sketch the landscape and define vocabulary to describe findings. Additional empirical data—mainly from interview analyses and classroom observations—and historical and epistemological considerations have led to a refinement of this *proof schemes* framework. The goal of this chapter is to describe the revised *proof schemes* framework, focusing mostly on historical-philosophical observations that have contributed to the revision of the original framework.

PROVING, PROOF, AND PROOF SCHEME

The triad "*proving, proof, proof scheme*" is an instantiation of a more general triad, "*mental act, way of understanding, way of thinking*." The latter triad is discussed extensively in Harel (in press a, in press b), and so it will only be mentioned briefly in this chapter.

The notion of *mental act* refers to such acts as interpreting, conjecturing, inferring, proving, explaining, generalizing, applying, predicting, classifying, searching, and problem solving. This chapter concerns one single mental act—that of proving. Proving is defined as the process of removing or instilling doubts about an assertion. It is worth distinguishing between two processes of proving: the process of *ascertaining* and the process of *persuading*. *Ascertaining* is a process an individual employs to remove her or his own doubts about the truth of an assertion, and persuading is a process an individual employs to remove others' doubts about the truth of an assertion. Clearly, these two processes are not independent—in ascertaining for oneself, one is likely to consider how to persuade others, and in the process of persuading others one's own certainty is likely to be affected.

Proving, by this definition1, is not unique to mathematics; people prove assertions in many areas of their everyday and professional life. A critical difference

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 65–78.

among professionals in relation to the proving act is in the characteristics of this act. For example, while both a biologist and mathematician prove assertions, the characteristics of their proving are different: a biologist's proving is characteristically empirical while a mathematician's proving is characteristically deductive. A characteristic of a mental act is referred to as a way of thinking, and a *way of thinking* associated with the proving act is called a *proof scheme*. Thus, proving empirically and proving deductively are examples of proof schemes (other proof schemes will be discussed later in this chapter).

Mental acts may be inferred by observing people's statements and actions. A person's statements and actions are products of her or his mental acts; they represent the person's ways of understanding associated with those mental acts. In this respect, a proof—a particular statement one offers to ascertain for oneself or convince others—is a way of understanding: it is the person's *ways of understanding* why an assertion is true or false.

The above definitions are deliberately student-centered for the important reason that instruction must take into account students' current proof schemes in designing and implementing mathematics curricula. Despite the subjective stance taken in these definitions, the goal of instruction must be unambiguous; namely, to gradually refine students' current proof schemes toward the proof scheme shared and practiced by contemporary mathematicians. This claim is based on the premise that such a shared scheme exists and is part of the ground for advances in mathematics.

THE NEW PROOF SCHEME FRAMEWORK

The proof scheme framework presented here is a revision of the framework presented in Harel and Sowder (1998). The two frameworks will be referred to as the *new framework* and *original framework*, respectively. In this section the focus is on the new framework. A discussion of the changes made and the rationale for making them will be presented in the next section.

While both empirical and theoretical considerations have led to the revision of the original proof scheme framework, in this chapter I will address almost exclusively historical–epistemological considerations. Some of the definitions here are repetitions of the definitions given in Harel and Sowder (1998); others are new. It is recommended that in reading this chapter, the reader consult Harel and Sowder (1998).

As the old framework, the new framework consists of three main classes of proof schemes. In the new framework they are labeled the external *conviction proof scheme class*, the *empirical proof scheme class*, and the *deductive proof scheme class*.

The External Conviction Proof Scheme Class

Proving within the *external conviction proof schemes* class depends (a) on an authority such as a teacher or a book, (b) on strictly the appearance of the argument (for example, proofs in geometry must have a two-column format), or (c) on symbol manipulations, with the symbols or the manipulations having no potential coherent system of referents (e.g., quantitative, spatial, etc.) in the eyes of the student (e.g., $(a + b)/(c + b) = (a + \not{b})/(c + \not{b}) = a / c$). Accordingly, we distinguish among three proof schemes within the external conviction proof scheme class: the *authoritative proof scheme,* the *ritual proof scheme*, and the *non-referential symbolic proof scheme*.

The Empirical Proof Scheme Class

Schemes in this class are marked by their reliance on either (a) evidence from examples (sometimes just one example) of direct measurements of quantities, substitutions of specific numbers in algebraic expressions, etc. or (b) perceptions. Accordingly, we distinguish between two proof schemes: the *inductive proof scheme*, and the *perceptual proof scheme*.

The Deductive Proof Scheme Class

The *deductive proof scheme* class consists of two subcategories, each consisting of various proof schemes: The *transformational proof scheme* category and the *axiomatic proof scheme* category.

All the *transformational proof schemes* share three essential characteristics: generality, *operational thought*, and *logical inference*. The generality characteristic has to do with an individual's understanding that the goal is to justify a "for all" argument, not isolated cases and no exception is accepted. Evidence that operational thought is taking place is shown when an individual forms goals and subgoals and attempts to anticipate his/her outcomes during the proving process. Finally, when an individual understands that justifying in mathematics must ultimately be based on the rules of logical inference, the logical inference characteristic is being employed. It should be noted that, although evidence in mathematics rests on logical inferences rules, it is almost never the case that one develops a proof by reasoning with these rules alone; inductive and abductive reasoning, for example, are often integral parts of the proving process. In the course of proving arguments, as well as in generating hypotheses, one applies mental operations to transform images from one state of knowledge into another. These transformations and the entities to which they apply are part of one's mathematical reality.

The *axiomatic proof schemes* share the three characteristics that define the transformational proof schemes, and include others. For now, it is sufficient to define an axiomatic proof scheme as a transformational proof scheme which acknowledges that in principle any proving process must start from accepted principles (axioms). The situation is more complex, however, as we will see shortly.

We will now discuss the different instantiations of the transformational proof scheme category and the *axiomatic proof scheme* category.

TRANSFORMATIONAL

The images in the transformational proof schemes are of human idealized physical reality; they govern the deduction process and usually include at least one of three restrictions: (a) restriction of the context of the argument, (b) restriction of the generality of the argument's justification, or (c) restriction of the mode of the justification. Accordingly, depending on the kind of restriction, we called the proof scheme *contextual*, *generic*, or *causal*. We begin with *causal restriction*.

Causal

"We do not think we understand something until we have grasped the why of it. … To grasp the why of a thing is to grasp its primary cause," asserts Aristotle in *Posterior Analytics*. Some 16–17th-century philosophers argued that mathematics is not a perfect science because "implication" in mathematics is a mere logical consequence rather than a demonstration of the cause of the conclusion. To illustrate the nature of this argument, consider Euclid's Proposition I.32 and its proof:

Theorem: The sum of the three interior angles of a triangle is equal to 180º.

Proof: Construct CE parallel to AB (Figure 1). Then the alternate angles BAC and ACE are congruent and the corresponding angles ABC and ECD are congruent. Hence, m(ABC)+m(BAC)+m(ACB)=m(ECD)+m(ACE)+m(ACB)=180º.

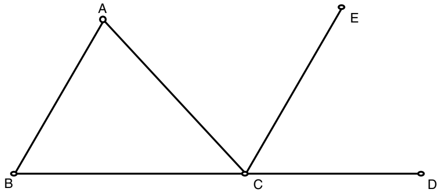

Figure 1. Euclid's construction.

What is the cause of the property that is proved here, asked these philosophers? The proof appeals to two facts about the auxiliary segment CE and the external angle ACD. But these facts, they argued, cannot be the true cause of the property. For the property holds whether or not the segment CE is produced and the angle ACD considered.

Another argument against the scientific nature of mathematics was this: If mathematical proofs were scientific (i.e., causal), then proofs for "A if and only if B" statements entail that A is the cause of B and B is the cause of A. Hence, A is the cause of itself, which is absurdity, because nothing can be its own cause. Since the basis of proof by exhaustion is proof by contradiction, it too was unsatisfactory to many mathematicians of the 16th and 17th centuries. They felt that the ancients, who broadly used proof by exhaustion to avoid explicit use of infinity, failed to convey their methods of discovery.

If we are to draw a parallel between the individual's epistemology of mathematics and that of the community, the following questions are of paramount importance: Was the causality issue of marginal significant concern to the mathematics of the 16th and 17th centuries? To what extent did the practice of mathematics in the 16th and 17th centuries reflect global epistemological positions that can be traced back to Aristotle's specifications for perfect science? Mancosu (1996) argues that the practice of Cavalieri, Guldin, Descartes, Wallis, and other important mathematicians reflects a deep concern with these issues. He shows, for example, how two of the major works of the 1600s—the work by Cavalieri on indivisibles and that by Guldin, his rival, on centers of gravity—aimed at developing mathematics by means of direct proofs. These two mathematicians, argued Mancosu, explicitly avoided proofs by contradiction in order to conform to the Aristotelian position on what constitutes perfect science—a position Aristotle articulated in his *Posterior Analytics*.

Mancosu (1996) also argued convincingly that Descartes, whose work represents the most important event in 17th-century mathematics, was heavily influenced by these developments. Descartes appealed to a priori proofs against proofs by contradiction because they show how the result is obtained and why it holds, and they are causal and ostensive.

It should be noted, however, that not all philosophers of the time held this position. Barozzi, for example, argued that some parts of mathematics are more scientific (causal) than others; but that proof by contradiction is not a causal proof, and therefore it should be eliminated from mathematics. Others, like Barrow for instance, argued that all mathematics proofs are causal including proof by contradiction.

Examples of students' causal proof schemes. The proofs of the Linear Dependence Theorem and the Eigenvalues Eigenvectors Theorem below are examples of proofs to which some students—always the more able students in the class—responded in a manner that has perplexed us. (These examples were also discussed in Harel, 1999.)

Linear Dependence Theorem: Prove that any three vectors in $\mathbb{R}^2$ are linearly dependent.

Proof: (given by the instructor)
Proof : (given by the instructor)

Let $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}$ be three vectors in $\mathbb{R}^2$,

and consider the system $AX = 0$, where $A = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$.

The system $AX=0$ has at least one free variable: therefore, it has a non-zero solution, $X = [x_1 \quad x_2 \quad x_3]$. Since $x_1A^{(1)} + x_2A^{(2)} + x_3A^{(3)} = 0$, and $x_1 x_2 x_3$ are not all zero, one of the columns of A must be a linear combination of the others. Hence the columns of A are linearly dependent.

These students seemed to understand each step in the proof, and yet responded something to the effect: What if the system weren't homogeneous? Your answer is dependent on the fact that the system is homogeneous. If the system weren't homogeneous, you wouldn't be able to prove that vectors are dependent.

We encounter a similar situation when the following theorem and proof were presented:

Eigenvalues Eigenvectors Theorem: Let T be a square matrix, and let $v_1, v_2, \ldots, v_k$ be eigenvectors of T that correspond to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are all distinct, then $v_1, v_2, \ldots, v_k$ are linearly independent.

Proof : Assume $\sum_{i=1}^{k} a_i v_i = 0$. We should show that $a_i = 0$ for each i.

Let $p_j(x) = \prod_{i \neq j}^{k} (x - \lambda_j)$ for $j = 1, 2, \ldots, k$.

By the Spectral Mapping Theorem,

$$0 = \sum_{i=1}^{k} a_i p_j(\lambda_i) v_i = a_j p_j(\lambda_j) v_j$$

Since $p_j(\lambda_j) v_j \neq 0$, $0 = p_i(T)0 = \sum_{j=1}^{k} a_i \delta_{ij} v_i = a_i v_i$,

Hence, $a_i = 0$ for each i.

The response by some of the students to this proof was something to the effect: "What if you chose different polynomials? Just because you chose these polynomials, you could prove that $v_1, v_2, \ldots, v_k$ are independent. Maybe if we choose different polynomials, the vectors wouldn't be independent."

What is the conceptual base for these responses? What really is the question these students are asking? While further research is needed to answer these questions, in what follows we will offer a conjecture. The history of the development of the concept of proof may suggest that our current understanding of proof was born out of an intellectual struggle during the Renaissance about the nature of proof—a struggle in which Aristotelian causality seem to have played a significant role. If the epistemology of the individual mirrors that of the community, we should expect the development of students' conception of proof to include some of the major obstacles encountered by the mathematics community through history. We conjecture that Aristotelian causality is one of these obstacles. Causality is more likely to be observed with able students, who seek to understand phenomena in depth, than with weak students, who usually are satisfied with whatever the teacher presents. It is possible that, for example, students' responses to the Linear Dependence Theorem Problem and the Eigenvalues Eigenvectors Theorem are a manifestation of the causality phenomenon. The students who responded to the proof by saying "What if the system weren't homogeneous?" had interpreted the homogeneous system $AX = 0$ to be the cause for the independency of the vectors

$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix}$, and so they desired to understand the exact causality relationship. Similarly, the students who responded to the other proof by saying "What if you took different polynomials" sought to understand the cause–effect relationship between the Lagrange polynomials,

$$p_j(x) = \prod_{i \neq j}^{k} (x - \lambda_j)$$

, and the theorem's assertion about the independency of the eigenvectors.

I will conclude this section with another observation that can be interpreted in terms of effect of the causality scheme. A group of eight inservice teachers were presented with two proofs of Proposition I.32: Euclid's original and the following proof which was originally offered by a preservice teacher (Amy) taking a course in college geometry (reported in Harel and Sowder, 1998):

> Amy demonstrated to the whole class how she imagines the theorem, "The sum of the measures of the interior angles in a triangle is 180°." Amy said something to the effect that she imagines the two sides *AB* and *AC* of a triangle *ABC* being rotated in opposite directions around the vertices *B* and *C*, respectively, until their angles with the segment *BC* are 90° (Figure 2a, b). This action transforms the triangle *ABC* into the figure *A′ BCA″*, where *A′B* and *A″C* are perpendicular to the segment *BC*. To recreate the original triangle, the segments *A′B* and *A″C* are tilted toward each other until the points *A′* and *A″* merge back into the point *A* (Figure 2c). Amy indicated that in doing so she "lost two pieces" from the 90° angles *B* and *C* (i.e., angles *A′ BA* and *A″ CA*) but at the same time "gained these pieces back" in creating the angle *A*. This can be better seen if we draw *AO* perpendicular to *BC*: angles *A′ BA* and *A″ CA* are congruent to angles *BAO* and *OAC*, respectively (Figure 2d).

All eight teachers preferred Amy's proof to the standard Euclid's proof, saying that it shows why the sum of the angles in a triangle is 180°. They indicated that through Amy's proof they could see how the construction of the triangle "made" the sum of the angles 180°. For these teachers, I suggest, Amy's proof was a causal proof—an enlightening proof that gives not just mere evidence for the truth of the theorem but the cause of the theorem's assertion.

Constructive

In the constructive proof scheme, students' doubts are removed by actual construction of objects—as opposed to mere justification of the existence of objects. This scheme is reminiscent of the constructivist mathematics philosophy founded by Brouwer at the turn of the 20th century, which viewed the natural numbers as the fundamental objects that are irreducible to further basic notions, and so any meaningful mathematical proof must ultimately be based constructively on the natural numbers. A corollary of this premise is that one cannot establish the truth of an argument by showing that its negation leads to a contradiction, for no construction that is based on the natural numbers is involved in such a demonstration.

Indeed proof-by-contradiction was another reason for the denial of the scientific nature of mathematics by 16–17th century philosophers, in the eyes of whom this method of proof did not qualify as causal proofs. When a statement "A implies B" is proved by showing how not-B (and A) leads logically to an absurdity, we do not learn anything about the causality relationship between A and B. Nor, these philosophers continued to argue—do we gain any insight of how the result was obtained.

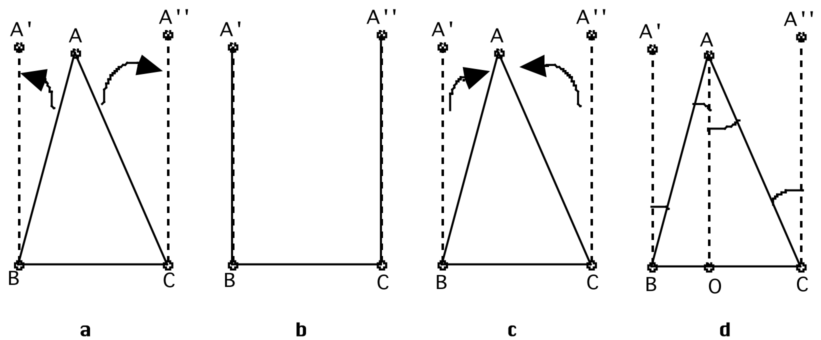

Figure 2. Student's demonstration.

Generic

In a generic proof scheme, conjectures are interpreted in general terms but their proof is expressed in a particular context. This scheme reflects students' inability to express their justification in general terms, as is demonstrated in several episodes in Harel and Sowder (1998).

Contextual

In his book, *Greek mathematical thought and the origin of algebra*, Klein (1968) argues that the revival and assimilation of Greek mathematics during the 16th century resulted in fundamental conceptual changes that ultimately defined modern mathematics. He focuses primarily on the ancient Greeks' conception of number and its crucial transformation during the Renaissance. The process of this conceptual transformation, Klein argues, culminated in Vieta's work of symbolic algebra, where the distinction between modern mathematics and Greek mathematics began to crystallize. In Greek science, concepts are formed in continual dependence on their "natural" foundations, and their scientific meaning is abstracted from "natural," pre-scientific experience. In modern science, on the other hand, what is intended by the concept is not an object of immediate insight. Rather, it is an object whose scientific meaning can be determined only by its connection to other concepts, by the total edifice to which it belongs, and by its function within this edifice.

One of the questions we addressed in our studies was: Do undergraduate mathematics majors possess the axiomatic conception at any level? For example, do students understand that axioms in geometry require no specific interpretation? In particular, can students consider their own intuitive space (i.e., the Euclidean space) as a specific system that may or may not satisfy the structure at hand? Our research has shown that the answers to these questions are negative (Harel and Sowder, 1998). When students are unable to detach from a specific context, whether it is the context of intuitive Euclidean space in geometry or the context of R^n in linear algebra, we call that conception "contextual." And so, with the contextual conception, general statements are interpreted (and proved) in terms of a specific context.

Greek Axiomatic

The axiomatic method—that is, the notion of deductive proof from some accepted principles—was conceived by the Greeks. However, it is important to note that the Greeks had one single type of mental objects in mind, namely, objects that are idealizations of physical reality, such as a line, plane, triangle, etc. Accordingly, with the intuitive axiomatic proof scheme, the student is able to handle only axioms that correspond to her/his intuition. For example, the statement "One and only one line goes through two points" is understood only in the context of personal geometric intuition. Here the objects, which are derived from an idealization of the physical reality, determine the set of axioms.

Arithmetical Symbolic

In the arithmetical symbolic proof scheme, letters used to express the unknowns are isolated signs that obey no independent operation rules, but the rules dictated by the specific meaning of the quantities they represent. Diophantine algebra, for example, obeys the arithmetical proof scheme conditions.

In Diophantus' conception, a number must always mean a number of something. Accordingly, "a fraction" always refers to a number of fractional parts of whatever the unit of calculation happens to be. By way of this conception, it is quite impossible, therefore, to arrive at the concept of "negative numbers" and "irrational numbers." In order to avoid these kinds of numbers as solutions to an equation, Diophantus introduced a restrictive condition by which an equation that leads to such solutions is declared "impossible." In Diophantus' sense, an unknown is defined as an indeterminant multitude of units, but it is indeterminant only for the solvers. In each problem a completely determined number of units exists a priori as its solution. The Diophantine procedure operates with the number sought as with something already given or granted. The construction of an equation is a process by which the conditions of a problem are expressed in a form which enables us to ignore whether the magnitudes occurring in the problem are known or unknown. The equation then is transformed into a canonical form from which the number sought is obtained. If the final computations result in an "impossible number" (e.g., a negative solution), then the problem itself is impossible.

MODERN AXIOMATIC

Generally speaking, the modern axiomatic proof scheme is a scheme by which one understands that in principle any deductive proof must start from accepted principles (axioms). This definition, however, does not capture other critical cognitive and epistemological processes. Like the transformational proof scheme, the modern axiomatic proof scheme is characterized by three conditions: (a) consideration of the generality aspects of the conjecture, (b) application of mental operations that are goal-oriented and anticipatory—an attempt to predict outcomes on the basis of general principles—and most critically (c) a set of (arbitrary) rules that governs the transformations of images in the evidencing process. Note that the transformational proof scheme and the modern axiomatic proof scheme share exactly the same first two conditions, but differ diametrically in the third. In addition, unlike the transformational proof scheme, the modern axiomatic is free from all three restrictions in the transformational proof scheme: contextual, generic, and causal.

The discussions in the previous sections together with the subsequent ones will capture the essential differences between the transformational proof scheme and the modern axiomatic proof scheme.

Structural

The idea that the objects are determined by a set of axioms was a revolutionary way of thinking in the development of mathematics, and has shed light on some of the difficulties we observed with students. A clear manifestation of this revolution is the distinction between Euclid's *Elements* and Hilbert's *Grundlagen*. While the *Elements* is restricted to a single interpretation—namely that its content is a presumed description of human spatial realization—the *Grundlagen* is open to different possible realizations, such as Euclidean space, the surface of a half-sphere, ordered pairs and triples of real numbers, etc.—including the interpretation that the axioms are meaningless formulas. In other words, the *Grundlagen* characterizes a structure that fits different models. This obviously is not unique to geometry. In algebra, a group or a vector space is defined to be any system of objects satisfying certain axioms that specify the structure under consideration. The structural proof scheme, therefore, is the understanding that definitions and theorems represent situations from different realizations that share a common structure determined by a permanent set of axioms.

Structural Symbolic

The structural symbolic proof scheme is better understood by contrasting it with the "algebra" of Diophantus discussed earlier. The latter is merely "algebra of abbreviations" not "symbolic algebra"; the symbol the Greeks used for an unknown is merely a word abbreviation, not a symbolic representation. That is to say, a letter is never a "symbol" in the sense that that which is signified by the symbol is in itself a "general" object. A letter does not symbolize a value that may vary, a variable; rather, it merely names the value that is a priori determined. Nor does a letter lend itself to being an object that can be operated on.

Symbolic algebra was born with the inception of the idea of representing the problem statement symbolically (e.g., by an equation) and, similarly, representing each operation by a special sign. Thus, in the structural symbolic proof scheme the focus is on the symbolic representation, not on the problem statement, in that the symbolic representation alone dictates the possible solutions, which may be meaningless in the context of the problem statement. This proof scheme has two distinctive characteristics that set it apart from the ancient method: (a) the object is identified with, rather than referenced to by, its representation and (b) the representing symbol signifies possible determinacy, rather than real determinacy, of the object.

The modern notion of "number," is a critical case in point. This notion was born when symbols representing no specific objects were conceived as conceptual entities, as a cognitive object for which the mental system has procedures that can take the object as an argument, as an input, independent of any specific reference. Further, the procedures are the operation rules—constituting an axiom system—that define the objects to which they apply. A number in this sense is no longer a number of units that is always determined but an object with unlimited possibilities of ciphering, according to rules of calculation.

The modern notion of "number" was born when symbols representing no specific reality were treated as objects that can be operated upon by certain rules. These objects are defined not by what they represent but by an a priori set of rules. Not all mathematicians of the 17th century shared this new way of thinking; some raised serious doubts about its intelligibility and viability. How is it possible to reason about symbols without a concrete referent and especially without a geometrical referent, as in the case of imaginary numbers and negative numbers? How is it possible, asked Arnauld, a 17th-century mathematician, to subtract a greater quantity from a smaller one, where the mental image of "quantity" is nothing else but a physical amount or a spatial capacity? Moreover, how is it possible to understand such a statement as $(-1)/1=1/(-1)$, where the quantity 1 is larger than the quantity –1, and therefore, the division of 1 by –1 must be smaller than the division of –1 by 1?

Axiomatizing

In the structural conception, the axioms that define the structure are permanent, and one studies the structure itself, not just the axiom system. So, for example, one studies real analysis on the basis of the axioms of a complete ordered field, or one studies the theory of vector spaces on the basis of the vector space axioms, etc. Our data suggest that the structural conception is a cognitive prerequisite to the axiomatizing proof scheme—a conception by which a person is able to investigate the implications of varying a set of axioms, or to understand the idea of axiomatizing a certain field.

SUMMARY AND CONCLUSIONS

The new framework we present here is a revision of the framework presented in Harel and Sowder (1988), which will be referred to as the original framework. Each of the frameworks consists of three classes of proof schemes. The first two classes—the external proof schemes class and the empirical proof scheme class—are identical, except that the symbolic proof scheme was renamed non-referential symbolic proof scheme. The third class was renamed “deductive” because logical deduction is an essential ingredient in each of its proof scheme categories. This change reflects our position that a proof scheme can—and in a certain stage in the student’s intellectual development should—be deductive without necessarily being axiomatic.

Underlying the change in the third class is our observation that certain epistemological obstacles with our students seem to parallel critical obstacles in the development of mathematics. Of particular importance are two such observations: The first observation is that some students’ proof schemes are reminiscent of the 16th–17th century conception of mathematics, where the practice of mathematics reflects global epistemological positions that can be traced back to Aristotle’s specification that explanations in science must be causal. The addition of the causal proof scheme to the transformational proof scheme category is an expression of this observation. The second observation is that students’ proof schemes—viewed in relation to those of their instructors—seem to parallel the Greek conception of mathematics—viewed in relation to that of modern days. The latter observation is expressed (a) by renaming the axiomatic proof scheme modern axiomatic proof scheme, (b) by renaming the intuitive axiomatic proof scheme Greek axiomatic proof scheme, and (c) by relocating the latter as an instantiation of the contextual proof scheme.

Finally, the internalized proof scheme and the interiorized proof scheme in the new framework are viewed as stages of conceptual development rather than as instantiations of the transformational proof scheme. They are part of the new framework in that each of the proof schemes is either internalized—encapsulated into a proof heuristic—or interiorized—has been reflected upon by the person possessing it so that he or she becomes aware of it.

Analyses of data from teaching experiments (Harel and Sowder, 1998; Harel, 1999) and from a longitudinal study (Sowder and Harel, in press), show that the proof scheme framework remains stable; namely, with the exception of one proof scheme—"causality," which is discussed here—no additional categories of proof schemes were discovered and none of the existent categories of proof schemes has been altered. Further, the works of other researchers who have used our framework to interpret their findings of students' conceptions of proof (e.g., Csikos, 1999) provide additional confirmation of the validity and stability of the framework.

The new framework, therefore, continues to serve the old framework's original purpose—a tool for identifying and encoding students' proof schemes. Further, the new framework goes beyond this function in that it suggests a conceptual basis for the epistemological obstacles that students encounter in developing and restructuring their proof schemes.

Historical/philosophical analyses of the concept of mathematical proof point to possible parallelisms between certain obstacles encountered by students and those encountered in the course of the history of mathematics. We say "possible parallelism" because our research has not fully established such a parallelism. Further research, which we hope this chapter will generate, is still needed to fully understand the conceptual and epistemological basis for students' proof schemes in relation to the historical epistemology of proof.

NOTE

[1] Here and elsewhere in this chapter the verb "to prove," in all forms, is in the sense of the definition of "proving" stated above.

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PART II: CURRICULAR CHOICES, HISTORICAL TRADITIONS AND LEARNING OF PROOF: TWO NATIONAL CASE STUDIES

CELIA HOYLES AND LULU HEALY

4. CURRICULUM CHANGE AND GEOMETRICAL REASONING

INTRODUCTION: STUDENTS' CONCEPTIONS OF PROOF

Proof has traditionally appeared in school mathematics in exercises involving the formal confirmation of Euclidean geometry theorems (see for example, Hanna, 1995). As Harel and Sowder (1998, p. 428) noted: "the idea of proof as a reductive process where hypotheses lead to conclusions, has traditionally been stressed in the teaching of geometry but not in the teaching of algebra". Despite numerous attempts to teach students to prove conjectures in the context of geometry, there is overwhelming evidence of persistent confusion and misunderstandings (for reviews see Harel & Sowder, 1998; Hoyles, 1997). To summarise these difficulties: students tend to regard proof as no more than the confirmation of a few examples (see Martin & Harel, 1989), tend not to appreciate the significance and generality of a deductive proof, interpret proof as no more than evidence (Chazan, 1993; Fischbein & Kedem, 1982; Vinner, 1983) and, most fundamentally, tend to see proving not as a process that is central to mathematical activity but rather as a ritual disconnected from the construction of knowledge or understanding (see Schoenfeld, 1989).

Several attempts have been made to describe the interrelated facets that make up students' conceptions of proof. Harel and Sowder (1998) have proposed a typology of students' proof schemes that consists of external proof schemes (including ritual authentication and symbolic argument), empirical proof schemes (including inductive and perceptual argument) and analytical proof schemes (including transformational, generic and axiomatic justifications). An alternative approach starting from the mathematical practices of students is suggested by Balacheff (1988), who distinguished pragmatic from conceptual proofs, where the former are controlled by actions and the latter by properties and theory. Within these two general categories, Balacheff identified several subcategories; in the former, taking an empirical approach and using crucial experiments; in the latter, engaging in thought experiments and making inferences from statements, with the use of generic examples suggested as important in negotiating the passage between pragmatic and conceptual proofs.

Proving in mathematics is undoubtedly a complex process. It not only involves logical and deductive argument coordinated with visual or empirical evidence and mathematical results and facts, but is also influenced by intuition and belief, by perceptions of authority and personal conviction, and by the social norms that regulate what is required to communicate a proof in any particular situation. The

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 81–115.

failure of traditional geometry teaching in schools stemmed at least partly from a lack of recognition of this complexity. In particular, the standard practice was simply to present formal deductive proof (often in a ritualised two-column format) without regard to its function or how it might connect with students' intuitions of what might be a convincing argument: "deductivity was not taught as reinvention, as Socrates did, but [that it] was imposed on the learner" (Freudenthal, 1973, p. 402).

In the light of the problems students experienced, new approaches to introducing proof have been proposed. These new approaches aim to guard against the danger of proof becoming an empty ritual. One popular line of attack has been to prioritise, for teaching purposes, the role of proof in explaining and illuminating mathematical ideas and to place rather less emphasis on its role in verification (see for example deVilliers, 1990; Hanna & Jahnke, 1993). One method, proposed by Whittman (1998), is systematically to exploit non-symbolic representations of phenomena that can serve an explanatory function within what he calls operative proofs. Another method seeks to avoid the fragmentation of problem solving from proving through the use of extended and collaborative projects in geometry: for example by introducing a theorem as a cognitive unity of three elements, statement, proof and theory (see Mariotti et al., 1997). By contrast, research in the U.S into geometrical reasoning, influenced by the van Hiele model (see Clements & Battista, 1992, for a summary) proposes that, for instruction to be effective, it must take account of the hierarchical levels of student understanding in geometry and build in sequence from the lowest level of perceptual recognition up to analysis, deduction and axiomatic proof.

From this brief review it is apparent that a huge amount is known about students' conceptions of proving and proof in geometry. It is also clear that students can be led to engage in geometrical reasoning and proof within particular teaching situations, as evidenced in chapters in this book. What more needs to be investigated that might contribute to either theoretical or didactical knowledge in this area? The contribution of the study reported in this chapter is to move beyond analyses that focus only on the individual student or classroom and to begin to identify curricula and school influences on geometrical reasoning. The research set out to analyse students' geometrical reasoning after following a curriculum approach that was rather different from the traditional Euclidean methods mentioned at the beginning of this chapter; that is where the verification role of proof in geometry was introduced only after several years during which students had been encouraged to explain patterns in data arising from non-geometric contexts.

A New Approach to Proof in the Curriculum

In England and Wales, the main response in the 70s and 80s to the evidence of students' poor grasp of mathematical proof was to develop an approach to proving in which students would have opportunities to test and refine their own conjectures and gain personal conviction of why they were true alongside experiences of presenting generalisations and evidence of their validity. In the language of Goldenberg, Cuoco and Mark (1998), the curriculum aimed to promote the development of mathematical habits of mind where students would tinker, conjecture, test informally and explain, but in numerical or algebraic contexts rather than geometric ones. The aim of the research reported in this chapter was to analyse how this approach to proving impacted upon students' proof conceptions in geometry. Thus we not only analysed individual student conceptions but also looked systematically at school and curriculum factors that might have played a role in shaping them. In order to make sense of our analysis, it is important for the reader to understand the position of proof and of geometry in the curriculum in operation at the time of the study.

At the time of the study that forms the main focus of this paper (1995–1999), the National Curriculum in Mathematics for England and Wales (Department of Education and Science, 1995) had been in operation for almost 10 years. This curriculum is statutory, and "delivered" by all state schools across the country. The curriculum in 1996 was organized into four attainment targets, with geometry located in the target, Space, Shape and Measures (AT3). In order to encourage teachers to pay specific attention to the process objectives of conjecturing, testing and explaining, rather than simply concentrating on the delivery of procedures and content, mathematical reasoning appeared under a separate attainment target, Using and Applying Mathematics (AT1). Associated with each attainment target were eight level descriptors, against which students would be tested, with descriptors added to classify students of "exceptional performance".[1] This imposition of levels on AT1 inevitably meant that proof would be introduced in a hierarchical way with an initial focus on empirical investigation with formulating, testing and refining conjectures coming later, to be followed finally for older children by explanation and proof.

In the National Curriculum documentation, there were strong exhortations that teachers should help students to make connections across levels and across targets in the curriculum, and in particular to link the reasoning processes developed in AT1 with the applications of proofs (in AT3). But what were the consequences of the emphasis in the curriculum on reasoning but its separation from geometry? Did the habits of mind that AT1 aimed to cultivate actually spill over into the geometry context and if they did what were the outcomes in terms of student conceptions? Were students who had followed AT1 able to exploit the expertise in argumentation and explanation that they had gained in this attainment target as glue to link conjecture and deduction in geometry? Did students no longer display the major confusions identified in the past between empirical verification and general truth? What also were the consequences for student attitudes to proof and their judgements as to the nature of mathematical proof?

Since the curriculum set in place in England and Wales was to some extent unique, not fitting neatly into any of the categories described by Goldenberg, Cuoco and Mark (1998),[2] there was rather little evidence to draw on as a basis from which to address these questions. The aims of the study we designed were therefore broad and overarching: to identify shifts in the landscape of student conceptions of geometrical reasoning following systemic curriculum change where students were engaged in proving activities largely outside geometry; and, given this analysis, to point to implications for future curriculum planning.

In 2000, the National Curriculum was revised. Although the structure of the curriculum was not radically altered and it continued to be organised in terms of the same four attainment targets with their corresponding level descriptors, the programmes of study relating to the first attainment target, Using and Applying Mathematics, became integrated into the programmes of study for the other three attainment targets. This chapter focuses mainly on data collected before the curriculum change. However, given its potential influence on students' conceptions in geometry (as reasoning was now intended to be developed in the attainment target, Space, Shape and Measures), we have included in a postscript at the end of this chapter some data from a subsequent research project that included some of the questions described here, in order to throw some light on the impact of this curriculum change.

RESEARCHING STUDENTS' GEOMETRICAL REASONING

In the research project, Justifying and Proving in School Mathematics,[3] we set out to analyse the conceptions of proof held by students who had followed the National Curriculum, described above. Specifically, we aimed to investigate the characteristics of arguments recognised as proofs by high-attaining students, aged 14–15 years, the reasons behind their judgements and the ways they constructed proofs for themselves. We focused on high-attainers (around the top 20% of the age cohort), as it was this group of students who would have covered most of the process objectives and levels of reasoning specified in the curriculum. The study investigated proof in two domains, arithmetic/algebra and geometry. This chapter reports the findings from the latter study only (for a discussion of the results from the algebra study, see Healy & Hoyles, 2000). We used a combination of quantitative and qualitative methods: a nationwide survey to obtain a picture of student conceptions set against teacher and school data and student and teacher interviews to seek interpretations of the patterns found in the quantitative data.

The Research Instruments

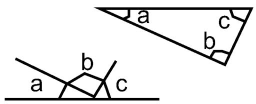
To collect the quantitative data, we designed a student proof questionnaire to provide, first, an overview of students' views of what comprised a proof, its role, and its generality and, second, an indication of students' competencies in constructing proofs.

G1. Amanda, Barry, Cynthia, Dylan, Ewan and Yorath were trying to prove whether the following statement is true or false:

When you add the interior angles of a triangle the sum is always 180°.

Amanda's answer

I tore the angles up and put them together.

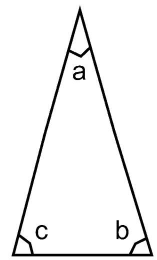


It came to a straight line which is 180°. I tried for an equilateral and an isosceles as well and the same thing happened.

So Amanda says it's true.

Barry's answer

I drew an isosceles triangle with *c* equal to 65°.

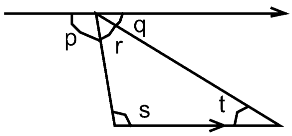


Statements	Reasons
$a = 180° - 2c$	Base angles in isosceles triangle equal
$a = 50°$	$180° - 130°$
$b = 65°$	$180° - (a + c)$
$c = b$	Base angles in isosceles triangle equal
$a + b + c = 180°$	

So Barry says it's true.

Cynthia's answer

I drew a line parallel to the base of the triangle



Statement	Reasons
$p = s$	Alternate angles between two parallel lines are equal
$q = t$.	Alternate angles between two parallel lines are equal
$p + q + r = 180°$	Angles on a straight line
$s + t + r = 180°$	

So Cynthia says it's true.

Dylan's answer

I measured the angles of all sorts of triangles accurately and made a table.

a	*b*	*c*	*total*
	34	36	180
	43	42	180
	72	73	180
	27	143	180

They all added up to 180°.

So Dylan says it's true.

Figure 1: continued next page

Figure 1 continued

Ewan's answer
If you walk all the around the edge of the triangle, you end up facing the way you began. You must have turned a total of 360˚.

You can see that each exterior angle when added to the interior angle must give 180˚ because they make a straight line. This makes a total of 540˚. 540˚ – 360˚ = 180˚.

So Ewan says it's true.

Yorath's answer
I drew a tessellation of triangles and marked all the equal angles. I know that the angles round a point add up to 360˚.

So Yorath says it's true

Figure 1: Amanda, Barry, Cynthia, Dylan, Ewan and Yorath's answerst

Students' Views of Proof

The proof survey included several items designed to probe student views of proof from a variety of perspectives. First, students were asked to describe in an open format what they thought about proof and its purposes. Their responses were coded according to a simplified version of deVillier's classification of the functions of proof, using the categories, truth (verification), explanation and discovery together with a fourth category "none/other" if students wrote nothing or if their contributions appeared to be irrelevant. Second, students were presented with mathematical conjectures and a range of arguments in support of them in a multiple-choice format. They were asked to make two selections from these arguments: the argument that would be nearest to their own approach and the argument they believed would receive the best mark from their teacher. Third, students were asked to judge all the arguments according to their validity and how far they were convincing and explained the conjecture. Two conjectures were included in geometry, one familiar and the other unfamiliar. In the interests of simplicity, we present in this chapter only the analysis of student responses to the familiar conjecture, G1; that the sum of the interior angles of a triangle is 180° (see Figure 1).[4]

The theoretical framework that governed the choice of arguments presented as proofs to this conjecture drew on the analyses of van Dormolen (1977) and Balacheff (1988) and the operationalisation of Balacheff's framework by Coe and Ruthven (1994). There were pragmatic arguments characterised as specific, empirical, or requiring an action or concrete demonstration (Amanda's and Dylan's arguments); an argument that relied on common properties or a generic case (Yorath's argument); and deductive proofs that presented a logical argument with links explicitly made between premises and conclusions. Since we were interested in the extent to which students distinguished the logical structure of a proof from the form in which it was presented, we included two arguments in this final category, one that was valid (Cynthia's) and one that was not (Barry's). Barry's argument displays a common problem; namely that the statement to be proved is used as part of the argument. We also included a narrative argument (Ewan's) by means of exterior angles that is commonly used in English textbooks, follows rather naturally from the curriculum definition of angle as "turn," and is easily generalised to proofs about the sum of the interior angles of other polygons.

We developed these arguments over three phases prior to the pilot study. First, we studied the National Curriculum specifications and looked through the textbooks in widespread use. Second, we asked 68 high-attaining 14–15-year-old students to prove the conjecture given in G1 in order to obtain a bank of appropriate arguments. And, third, if there were no arguments in our bank that fitted a category in our framework, we filled the gap by modifying a student production or writing an argument from scratch.

To obtain more evidence of students' views of the functions of proof, we asked them to assess the correctness and generality of each of the arguments presented in G1. Did they think it contained a mistake? Did they believe that it was always true or only held for a specific case or cases? These questions were inspired by our wish to test how far students exposed the confusions found in previous research and summarised earlier. The format used, as applied to Amanda's argument, is shown in statements 1 to 3, in Figure 2. The correctness of students' evaluations was scored by what was called a student's validity rating (VR): an entirely correct profile of responses for any given argument scored 2, a profile in which the student correctly noted if the argument was general, specific or wrong but was unsure of other factors obtained a rating of 1, and all other profiles scored 0.

Because of the importance accorded to the function of explanation in proving, both in the curriculum and in the research literature, we asked students to assess how far each argument in G1 explained the conjecture and convinced them of its truth. An example, again relating to the assessment of Amanda's argument, is shown in statements 4 and 5 in Figure 2. These assessments were combined to give a score, called the argument's explanatory power (EP). If students agreed with both statements, their EP for that argument was 2, if they agreed with one or other of the statements that the EP was 1, otherwise the EP was 0.

Finally, we were interested in following up Chazan's finding (Chazan, 1993) that many students in the context of geometry believed proof merely to be evidence. We therefore sought to assess students' feelings for the generality of a proven statement by asking them whether or not a valid proof for the conjecture G1 automatically held for a given subset of cases. The actual question, G2, is presented in Figure 3.

	agree	don't know	disagree
Amanda's answer			
Has a mistake in it	1	2	3
Shows that the statement is always true	1	2	3
Only shows that the statement is true for some triangles	1	2	3
Shows you why the statement is true	1	2	3
Is an easy way to explain to someone in your class who is unsure	1	2	3

Figure 2. Assessing the validity and explanatory power of Amanda's answer.

G2. Suppose it has now been proved that, when you add the interior angles of any triangle, your answer is always 180°.

Zoe asks what needs to be done to prove whether, when you add the interior angles of any right-angled triangle, your answer is always 180°.

Tick either A or B.
(A) Zoe doesn't need to do anything, the first statement has already proved this.
(B) Zoe needs to construct a new proof.

Figure 3. Assessing the generality of a proven statement in geometry.

Student Proof Constructions

The proof survey included questions with an open format where students were asked to construct their own proofs, again one for a familiar conjecture, G4, and another for an unfamiliar one, G7. These two conjectures are presented in Figure 4. The order of the questions in the survey was such that it was possible to use the conjecture presented in the multiple-choice question or to modify one of the arguments presented in this question as part of a later proof construction. For example, arguments used in G1 for proving that the sum of the angles of a triangle is 180° could be modified to build a proof about the sum of the angles of a quadrilateral.

The familiar statement to be proved was

G4. Prove that if you add the interior angles of any quadrilateral, your answer is always 360°.

The unfamiliar statement to be proved was:

G7. A is the centre of a circle and AB is a radius. C is a point on the circumference where the perpendicular bisector of AB crosses the circle. Prove whether it is true or false that triangle ABC is always equilateral. Write your answer in a way that would get you the best mark.

All students' constructed proofs were scored for correctness using a modification of Coe and Ruthven's (1994) classification: 0 for no basis for proof, 1 for relevant information (such as confirming examples) but no deductions, 2 for a partial proof with some attempt to present reasons or an explanation, and 3 for a complete proof, in which sufficient relevant facts, results and deductions were mentioned but the presentation did not necessarily follow any particular format. The main form of argument used in the student response was also recorded as empirical, formal, narrative, or visual.

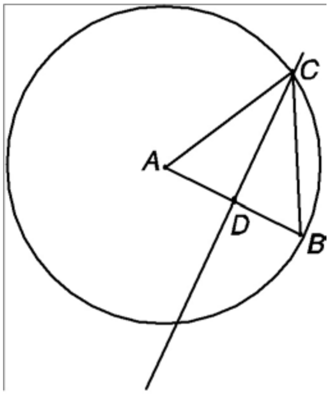


Figure 4. The familiar and unfamiliar statements to be proven.

The School and Teacher Surveys

In order to tease out any causes of different types of student conceptions that went beyond either the individual or the curriculum, we needed to set student responses against information about their school and about their teachers. The proof survey was to be administered to whole classes of students so it was reasonable to hypothesise that there would be a class or teacher effect. Thus, simultaneously with the development of the student proof survey, we designed school and teacher questionnaires. The first obtained data about a school—the type of school, its selection and setting procedures, the hours spent on mathematics per week, the textbooks adopted and the examinations entered. The teacher of the class who had answered the proof survey completed the other questionnaire. It sought data on teacher gender and qualifications, the percentage of the class who were to be entered for the GCSE higher tier,[5] the teacher's reactions to the place of proof in the National Curriculum, and the approach adopted to introducing proving and proof in the classroom. The teachers were also asked to consider each of the conjectures presented with multiple proofs and asked to select the proof nearest to the one they personally would use, and the proof that they predicted their students would believe would receive the best mark from them. Finally, the schools provided the Key Stage 3 test scores[6] (KS3) of all the students who completed the proof survey in order to provide a baseline assessment of mathematics attainment. This national test of attainment in mathematics predominantly tests procedures and calculations and includes no items on proving and proof.

METHODS

After piloting the survey with 182 students in eight schools and making appropriate modifications, it was administered to 2459 students from 94 classes in 90 schools, with the 94 class mathematics teachers completing the teacher and school questionnaires. The schools were spread across England and Wales, 29 in urban, 25 in rural and 36 in suburban settings. The sample of 2459 students was made up of 1305 girls and 1154 boys, with a mean Key Stage 3 score of 6.56.[7]

The purposes of the analyses of the data from the survey were fivefold: to describe students' views of the role and generality of proof, to categorise their choices and assessments of different arguments, to score their constructions and evaluations of proofs, to establish factors associated with these responses, and finally, to examine how these factors varied between schools. To achieve these goals, descriptive statistics based on frequency tables, simple correlations and tests of significance were produced, followed by multilevel statistical analysis with a two-level structure; student variables at Level 1 and class, teacher, school and curriculum variables at Level 2 (see Goldstein, 1995). The latter analysis was used to identify first statistical correlates and second deviations from the general picture of responses at school and student level (for example, schools in which performance was significantly better than predicted, or students who gave rather different responses from those characteristic of his/her class). Finally, we sought to contextualise these trends in the quantitative data through interviews with a sample of teachers and students.

In the following sections we present an overview of the findings and then a selection of interview data along with the multilevel analysis to suggest underlying causes.

Choosing a Proof of a Familiar Conjecture

In Table 1 we present the distribution of students' and teachers' responses to the multiple-choice question, G1 (see Figure 1).

Clearly there was marked variation between the choices students made when selecting an argument they felt most closely resembled their own approach and when choosing the one they believed would receive the best mark from their teacher. The arguments that were the most popular for the students' own approach turned out to be the least popular when it came to choosing for best mark, and vice versa. In G1, nearly half the students chose pragmatic proofs (Amanda or Dylan) for their own approach, while only 9% made these choices for best mark. These differences were statistically significant. Conversely, students were less likely to choose as the way they would approach the question, a proof that was presented as a chain of deductions (either Cynthia's correct argument or Barry's incorrect one), although these arguments dominated choices for best mark. Again the differences were significant in both cases.

Table 1. Distribution of Students' and Teachers' Choices of Proofs of Familiar Conjecture

	Argument chosen											
	Amanda		*Dylan*		*Cynthia*		*Barry*		*Yorath*		*Ewan*	
Criterion for choice	No.	%	No.	%	No.	%	No.	%	No.	%	No.	%
Students' own approach	615	25	516	21	516	21	369	15	246	10	197	8
Students' best mark	123	5	98	4	1180	48	369	15	369	15	320	13
Teachers'own approach[a]	24	26	5	5	47	50	0	0	*	*	16	17
Teachers' best mark prediction	21	22	3	3	59	63	1	1	*	*	10	11

*Notes: *Teachers were not given Yorath's option. [a]Missing data for 2 teachers' own approach.*

These descriptive statistics show that the distribution of choices of argument was significantly influenced by the criterion of choice—own approach or best mark. However, the two distributions were not completely independent, as evidenced from the construction of cross-tables of student choices where significant correlations were found. Thus there was a reciprocal influence between the argument believed to receive the best mark and a student's choice for his/her approach, with the exact nature of this influence varying between the arguments in question.

We also investigated whether either the validity rating, VR, or the explanatory power, EP, accorded to any argument were associated with a student's choice for his/her own approach. In both cases, the pattern was consistent: students who chose a valid argument for their own approach were more likely to appreciate its generality than those who had not made this choice. Additionally, irrespective of its validity, an argument that was felt to convince or explain was more likely to be selected as a student's own approach than one that was not.

These simple statistics indicate that students seemed simultaneously to hold two views of proof in geometry; they preferred a pragmatic approach for themselves while recognising that a more formal deductive presentation would be required to receive a good mark. In fact nearly two-thirds of the students had "picked up" that a presentation that included statements and reasons was regarded highly by their teachers, a finding that was somewhat surprising given that rather little emphasis on this way of presenting geometric proofs was evident in the curriculum the students had followed. However, it seems that presentation was not the only factor influencing choice, as many more students chose the correct rather than the logically confusing formal approach. This finding contrasts with the situation we found in algebra, where students chose any proof for best mark that appeared to include algebra (or rather included letters), regardless of the logic of the argument (see Healy & Hoyles, 2000).

These descriptive statistics suggest that the curriculum orientation to geometric proof had to some extent been modified in practice and values instilled that may not have been specified in the curriculum documents. As further evidence, we predicted that Ewan's proof that sets out a reasoned argument in a way that was familiar and encouraged in the curriculum would have proved popular for both own approach and for best mark. In neither situation was this the case. Given the preference among students' choices of argument nearest to their own approach for pragmatic approaches, we might conjecture that rather little of the explanatory narratives encouraged in AT1 had spilled over into a geometry context. We seek further evidence for this conjecture in the next section.

OPENING WINDOWS ON STUDENTS' GEOMETRICAL REASONING

To shed light on the trends identified in the data and to provide a more multifaceted view on student conceptions, we first turn to student responses to other questions in the survey along with some interview responses and then report the data on the mathematics teachers in the survey.

Reasons for Student Choices

The question G2 (see Figure 3) was included in the survey to obtain data about how students assessed the generality of a valid proof. The majority of students (84%) agreed that having proved that the sum of the interior angles of a triangle was 180°, no further work was necessary to prove that this held for right-angled triangles. This finding may at first sight appear to be of rather little interest. Yet, in a comparable study in Taiwan where students were asked the same question (see Lin & Chen, 1999) the number of students who judged that they had to produce a new proof was considerably increased (from 16% to 24%). How far these differences in comparative statistics are significant is difficult to assess given inevitable differences in sample (although the age of the students was the same) Nonetheless they do at least suggest that students who have been taught about geometry proof in the traditional Euclidean way (as in Taiwan) may have a different meaning for specialising the results of a given proof than students who had followed the more process-oriented approach.

Second, to obtain further insight into how students may have perceived the two most popular choices for their own approach to G1, Dylan and Amanda's arguments, and their most popular choice for best mark, Cynthia's argument, we turn to the students' validity scores for these three questions. Were students able to assess whether each of these arguments showed that the statement was true for all triangles or just for some triangles? In fact, nearly three quarters of the students believed that Dylan's and Amanda's argument showed the statement was true for all triangles (73% and 72% respectively) indicating some confusion about the limitations of pragmatic demonstrations. Additionally, nearly one quarter of the students (21%) thought Cynthia's proof was only true for specific triangles, a finding that is food for thought given that so many students had chosen this as the argument that would receive the best mark! These data suggest that it may not have been the generality of the formal proof that had appealed to the students. Was it therefore its logical character? This seems doubtful given that the analysis of validity scores showed that only 1% of the students believed that Barry's argument was incorrect.

Third, following the survey we interviewed several students who had made these common choices in G1, that is selected Amanda's or Dylan's argument for own approach and Cynthia's for best mark. Our aim in the interviews was to probe why they had made these decisions. The interviews showed still more clearly the complexity of the proving process and how the reasons underlying choices not only concerned an appreciation of proof but also were related to beliefs about mathematics and teaching mathematics. Take T, whose choices were based neither on the logic or the generality of the argument selected. He chose Cynthia's argument for best mark because he did not understand it and Amanda's argument for own approach, because it was easy, as illustrated in the extract below:

> Q. Let's go on. Geometry, why would you do Amanda? You chose Amanda, you said the best mark would go to Cynthia.
> T. Probably because I wouldn't be as clever as Cynthia—I don't know!
> Q. Let's have a look.
> T. Probably because that's the easiest to do. Amanda's.
> …
> Q. So you thought Cynthia's was probably the best one, but you didn't think you'd be able to do it?
> T. No, I just think I couldn't be bothered to do that, I just don't quite follow it … Amanda's would be the easiest.

Later, when asked which of the arguments presented were indeed proofs, T seemed to understand the importance of the generality of an argument but exhibited some confusion about whether Cynthia's argument was general or not, apparently triggered by the presence of the diagram. The letters in Cynthia's response also proved to have implications that we did not predict. It was these letters that suggested to T that the argument would have high status.

Q. Let's go again, which ones do you think are proofs? (pointing to the arguments presented in G1)
T. That's not a proof (Amanda's), what I said.
Q. Because why?
T. It only shows for some. This (Barry's) only shows it for an isosceles. This … Cynthia's. It doesn't really, no, Cynthia's doesn't show it for every triangle. But she's saying why p, why q there's proof in there somehow and that is important … but it doesn't really prove that for all of the triangles—she's just done that one (pointing to diagram) … but it does kind of …

A similar concern for the generality of Cynthia's argument was expressed by another student, again because of the presence of a diagram that was interpreted as showing just one case. In contrast to T, C's was a more sophisticated response, that did not accord high status to Cynthia's proof because of the letters in the diagram but rather because the argument mentioned properties she had come across in mathematics lessons.

C. This one (Cynthia's) … I think this one's better, but I'm not sure. I think it's a proof in a way, but because they only use … it's only showing that one, that one triangle. Actually it does show for all … I'm not sure really. … I think that one's … It is a proof because just by using these reasons to show that that … these are like parallel lines … alternate angles …

At this point C seemed to gain conviction about the nature of a proof as she shifted her attention during the interview to the properties used in the argument and away from the particular characteristics of the diagram:

Q. OK, so now you said before that you thought it was only true for some.
C. No, I don't agree with that now.
Q. You don't agree with that now, you think it's true for all? And why do you think it's true for all?
C. Because it's using, like, because it's got, using things like the parallel lines to show that the angles are equal, and alternate angles as well, it doesn't actually matter what the angles are. I think I got confused because you see they put only one triangle in the picture there, only one …

These extracts throw light on some of the reasons why students might have chosen Cynthia's argument for best mark, not only as we might have predicted because of the generality of a proof that follows the deductive process, but rather because of the algebra used or the explicit mention of geometrical properties. A mixture of these points was alluded to by another girl, A:

Q. But why did you think Cynthia would … [receive the best mark]?

A. Because she was like using algebra, and talking about alternate angles and how their lines …

A had also chosen Amanda's argument as nearest to her own approach, because she "knew that way" and was convinced by it. She also thought that Amanda's argument was general, partly because Amanda had tried several cases, but also because she was beginning (as shown in her last comment) to "see" how it would always work:

A. Because I already knew that [referring to Amanda's argument], and it was more straightforward, I thought.
Q. You knew that? How do you mean, you knew it, you'd done it?
A. Yes, I'd done that before, I'd come across it.
Q. What, you've torn up angles, and so you liked it?
A. Yes.
Q. What we're interested in, you chose Amanda, and you said … that Amanda's argument shows that the statement is always true. You're happy that this shows it's always true? Is that right? You stick by that? It does show it's always true?
A. Yes, I think so, because she's tried it with an equilateral triangle and isosceles, and I think this angle …
Q. She's tried it for equilateral and isosceles.
A. Yes. And even if this angle got bigger that one would get smaller as well, so it would always get to 180.

So A's choice was guided by her familiarity with the argument but also by the fact that she could "see" its generality. This was also true for her assessment of Dylan's argument which seemed so perfectly accurate, although in this case she was less sure:

Q. What about Dylan? You also say this shows that it's always true. Dylan did measuring, drew lots of angles, drew lots of triangles, measured them and made a table, and you thought that was always true, showed that it's always true. Do you still think it's always true, from Dylan?
A. It kind of shows, I mean, it's probably not like, you couldn't say that it's always going to be true, that you're not going to have one case, or something, but it looks quite, looks like it does show … because, I don't know, they're all …
Q. They're all what?
A They're all perfectly 180.

K, from the same school, chose Amanda's argument for her own approach, again for reasons of simplicity, arguing that she could follow it, she understood it and found it convincing—even though she was sure that it did not prove the statement for every triangle. The lack of generality was not a priority for her.

K. (long pause) I think that the one I would choose [Amanda] … it is pretty simplistic but it seems to show me, but it can't show for every single one.
Q. Which one's that?

K. I think Amanda's answer would help me to understand that … it would help understand it, but it doesn't prove it for every single one.

K's thinking about Cynthia's argument was interesting. She was impressed by the letters but this threw up another difficulty; she was unsure whether the argument was true, at least partly because she could not follow the logic in the algebra:

K. Because even though she's put a,b,c, they are set … Well, with the z it will be always true, and so that … I don't know how she can tell what they are though. Oh because they'll make … adding them all up will equal 180.
Q. You don't see why that follows?
K. Yes. "alternate angles between 2 parallel lines …"
Q. You're happy with p=s aren't you?
K. Angles on a straight line.
Q. Do you know that?
K. p, q and r … yes, but I don't see how it would make s, t and r.
Q. You're happy that p+q+r=180? You don't see that it therefore follows that s+t+r=180?
K. Exactly, yes. I mean, I know that, but …
Q. You know that because you know the sum of the angles …
K. But I don't see how it can just follow on like that.

We were surprised that K could not see that if she substituted a variable in an expression with an equal variable then she could deduce that the expression remained unchanged. Clearly this way of thinking and making "small" deductions using known results in geometry was not familiar to K and indeed many other students.

In summary, student choices of argument were influenced by criteria for choice (own approach or best mark) and by a range of factors, some central to proof such as generality and logic, but others more concerned with presentation, such as the existence of a diagram, references made to properties taught in mathematics lessons or letters used as labels for unknown angles. We now turn to how teachers evaluated the arguments presented in the multiple-choice questions as another possible influence on student choices.

The Influence of the Teacher

Referring to the teachers' choices of argument in G1, presented in Table 1, the first point that can be made is that in contrast to the student data, for the teachers the differences between the two distributions of their own approach and prediction of their students' choice of best mark were not significant: the most popular choice for the teacher's own approach, Cynthia's formal correct argument, was also the most frequently selected as the argument which was believed to be the students' choice for highest mark. This latter prediction certainly matched the students' responses. It is heartening that no teacher failed to see the incorrect logic in Barry's argument. However, it is of interest that about one quarter of the teachers chose Amanda's 'folding' approach as what they would do. This result may reflect a lack of familiarity with geometrical reasoning amongst teachers, many of whom would have learnt rather little geometry themselves because of the decline in geometry in the school curriculum since the 70s. It might also stem from the importance teachers give to arguments that are felt to explain, with teachers believing Amanda's actions to be a particularly good way of thinking about the property being considered and valuing this over a deductive argument. Also of interest is that 22% of the teachers chose Amanda's argument as the prediction for students' choice of best mark, a prediction that is strange given the students were nearly 15 years old and high attainers. It is even stranger when one notes (in Table 1) that only 5% of students actually made this choice for best mark themselves. Teachers in this case were apparently "not in touch" with the aspirations of their students. Some idea as to why this might have been the case can be gleaned from the interviews with a sample of teachers who chose Amanda's argument. We present some illustrative extracts:

Q. Here, you've gone for Amanda [choice for her view of students' choice for best mark]. Why was that?
A. Because that's the way I teach it.
Q. So that's actually within the curriculum … tearing up…
A. Because they can always remember a whole turn and a half turn therefore the straight line, it's a nice, simple … and they can actually physically do it.
Q. Yes. Do you think they would choose Amanda mainly because they've actually done it, so it's recognition …
A. Yes, probably because they recognise it.
Q. What about the issue of the fact that there's, I mean, it is one triangle?
A. Ah well, I normally get the whole class to draw a different triangle.
Q. Right, so there's lots of them …
A. Cut it out, yes, so I make a point that I didn't know what triangles you were going to draw, and you've all done different and …

This teacher herself seems to accept the generality of "trying many cases" although in her final remark she hinted at the issue of making a random choice of triangle. Nonetheless, it does appear that the main influence on this teacher's choice was the fact that Amanda's proof had been taught and (presumably) Cynthia's had not.

This same argument was echoed in another teacher interview:

Q. Another thing that I was quite interested in is that, when you filled in yours, again we picked up that you were quite a formal person, but here you thought that a lot of them (your students) would go for Amanda [for best mark], which is the ripping up one.
A. Mmm.
Q. In fact, they don't seem to like that at all.
A. Interesting that. In Year 7, we started teaching—I don't normally teach Year 7—but when they come to that, they normally do it that way with them.
Q. Right. So that was the reason that you thought they'd go for that.
A. Yes.
Q. In fact they didn't, they were much more likely to go for the formal.

A. That's interesting (muttering: We're wasting our time!)

So a large minority of teachers not only chose a pragmatic argument for themselves but also chose it for best mark, apparently because that was how they had taught the students so they could not (should not?) expect anything else. It was a matter of coverage. Many students though had somehow picked up another message, as evidenced in their choices of best mark, and it would be interesting to investigate if students were aware of the fact that more often than not they had not actually been taught how to make the deductive presentations that they believed would receive the best marks.

PROOF CONSTRUCTIONS IN GEOMETRY

We now turn to our analysis of students' own proof constructions in order to assess how well they were done from a mathematical perspective, how far they satisfied the criteria implicit in the students' own choices and how they were presented. The distributions and means of the students' scores for both the familiar and unfamiliar conjectures, G4 and G7 respectively, are presented in Table 2.

Not surprisingly, students constructed better proofs for the familiar conjecture than for the unfamiliar one. However, even in the former case, only 19% presented what we assessed as a complete proof and a remarkably small additional number (5%) included at least some deductive reasoning in their argument. Where the content was unfamiliar, 62% of students were unable to provide even the basis for a correct proof; they produced nothing at all or wrote nothing of any relevance. From a comparison of the total number of students who selected an argument representing what we deemed to be a correct proof with the total number of students who constructed either a partial or complete proof, we can deduce that students were significantly better able at choosing correct proofs than constructing them (χ^2=961.29, df=1, p<0.0001). To construct a proof, students need to mobilise mathematical knowledge as well as organise an argument. What were the major obstacles in proving these geometrical conjectures? Did students have no intuitions about the relevant premises in the conjecture, or were they simply not able to come up with any sort of chain of argument?

To try to tease out some explanations, we looked in more detail at what students actually produced. As well as assigning a score to the student proofs, we had also classified them according to the argument used: whether it only included relevant examples, whether it was presented in a narrative style (that is comprised an argumentation written in everyday language that did not make the logical process explicit) or whether it was formal (that is attempts were made to present the argument in deductive steps). In the familiar question, the most common approach was empirical, adopted by 35% (861) students. Given our interest in the effects of the National Curriculum structure on geometrical reasoning, we noted an influence of AT1 on these empirical student responses—not in terms of their reasoning, as we would have hoped, but rather in terms of their presentation. In AT1, data have to be collected, tabulated, a pattern spotted and if possible explained and proved. We found that this "ritual" was transposed by many students to the geometry context:

artificial data that satisfied the statement were tabulated and then used to show the statement to be true! A typical example is given in Figure 5.

Table 2. Distribution of Students' Scores for Proofs to Familiar and Unfamiliar Conjectures

	Constructed proof score	*Familiar conjecture (G4)*		*Unfamiliar conjecture (G7)*	
		No.	*%*	*No.*	*%*
0	No basis for the construction of a correct proof	586	24	1531	62
1	No deductions but relevant information presented	1289	52	690	28
2	Partial proof, including all information needed but some reasoning omitted	118	5	121	5
3	Complete proof	466	19	117	5
	Total	2459	100	2459	100
	Mean score	1.188	0.522		
	Standard deviation	1.005	0.796		

For the proof of the unfamiliar conjecture, the distribution of responses was rather different. As mentioned earlier, 62% made no relevant responses but, in contrast to answers to G4, few students (5%) tried to justify the conjecture through empirical examples, by for instance measuring the sides and showing that they all were the same. We can only speculate as to why this was the case. It may have been that in this unfamiliar situation the students were unable to identify the starting points or "givens" as these were properties embedded in the figure. Or maybe they were unsure of the properties necessary for an equilateral triangle.

In contrast to the differences in numbers of students adopting empirical approaches in the two questions, we did find some similarities in the distribution of scores for narrative and formal approaches in the two cases. For both questions, by far the most popular mode of response was narrative (28% and 23% in G4 and G7 respectively) with only a small number of students attempting any formal presentation (6% and 10% respectively). Thus most of the students who did more than produce data tried to present a reasoned argument in a narrative style—maybe a subtle influence of the process approach to proving in the curriculum.

Given the relatively large number of narrative responses to both questions, we now consider the percentage distribution of scores, (1, 2, 3), when proofs were presented in this narrative mode, as shown in Table 3.

If we first look at narrative responses to G4, we find that it was among these arguments that the majority of correct proofs were located: 60% of the narrative arguments were complete proofs. In fact by far the majority of the narrative proofs

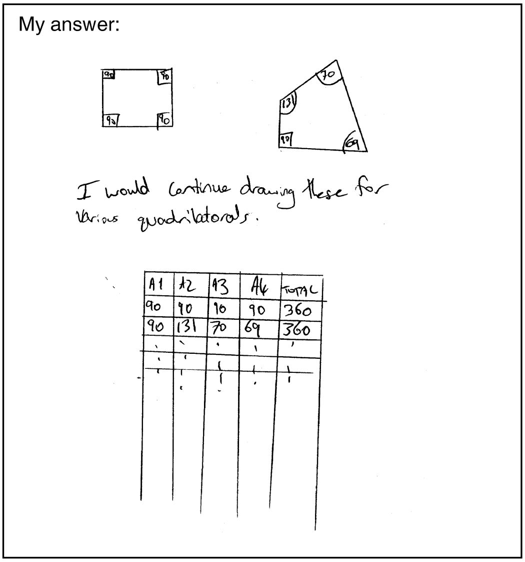


Figure 5. An empirical argument for the proof of the familiar conjecture, G4.

Table 3. Distribution of Scores for Proofs in Narrative Mode

	Familiar conjecture (G4)		*Unfamiliar conjecture (G7)*	
	No.	*%*	*No.*	*%*
1	193 (8)	28	453 (18)	80
2	76 (3)	11	68 (3)	12
3	416 (17)	60	45 (2)	8
Total	689 (28)		566 (23)	

Note. The numbers in parentheses represent the percentage of the total student population in this category.

for G4 (400 of the 463 correct proofs) used the familiar method by which the quadrilateral is divided into two triangles, as illustrated in Figure 6. The remaining 63 students modified Ewan's argument in G1 and used exterior angles as illustrated in Figure 7. We interpret that it was in these narrative responses to G4 that students showed how they were striving to achieve the general proof that they knew was required. They used the battery of argumentation at their disposal but in an informal way, giving little attention to premises and deductive steps. Yet, we would argue that the lines of argument and the geometric facts used in the examples presented are clear to the reader.

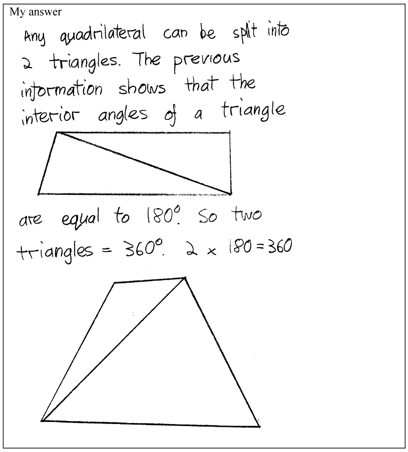


Figure 6. A narrative proof of G4.

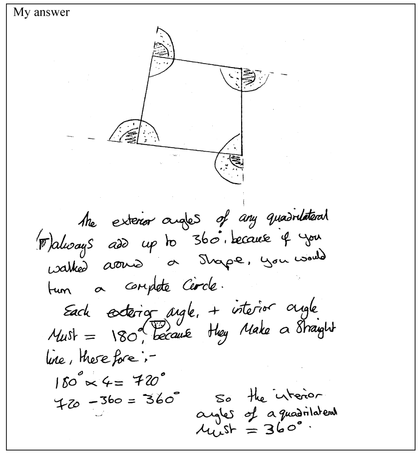


Figure 7. Adapting Ewan's proof in G1 to prove G4.

There was in fact considerable ingenuity exhibited in many answers to G4 that is hard to capture in bare statistics. This creativity has been illustrated in the figures already presented but we show a few more examples to provide the reader with a taste for the wealth of different approaches taken. Many students attempted to generalise from a specific case, as illustrated in Figure 8.

Others tried to adapt the arguments given previously in G1. Figure 9 shows an adaptation of Amanda's enactive method and Figure 7 (shown previously) is an adaptation of Ewan's proof.

In trying to prove the second unfamiliar geometry conjecture, again the most common construction was narrative in style (28% of the sample). But, in contrast to G4, only 8% of these narrative arguments could be assessed as complete proofs. One example of such a proof is given in Figure 10, in which again we would argue that the line of argument is clear. The response indicates that the student appreciates the steps in the proof—although there are certainly flaws in the way the reasoning is presented.

Eighty percent of the responses described some (but not all) of the relevant properties of the figure (see Table 3), for example, simply specifying the equality of the radii. Again what the numbers are unable to show is the range and richness of the student proofs—or in this case the limited range and absence of richness. In contrast to responses to G4, there was rather little creativity and insight displayed in the student responses to G7; few students attempted to adapt the arguments presented in the previous multiple-choice question (which would have been possible), and few attempted idiosyncratic but relevant solutions. In this question, possibly due to its unfamiliarity, the students seemed only able to mobilise a small number of arguments or intuitions to help them grapple with the problem, in contrast to the wealth of methods used in the familiar question, G4.

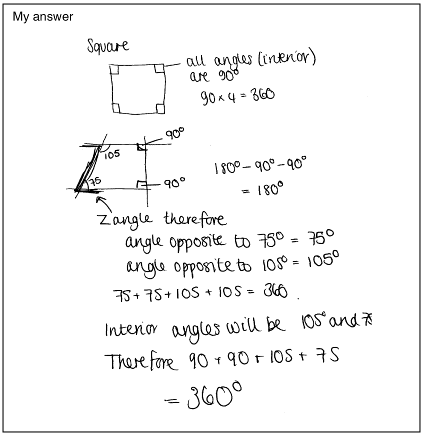


Figure 8. Generalising from a specific case.

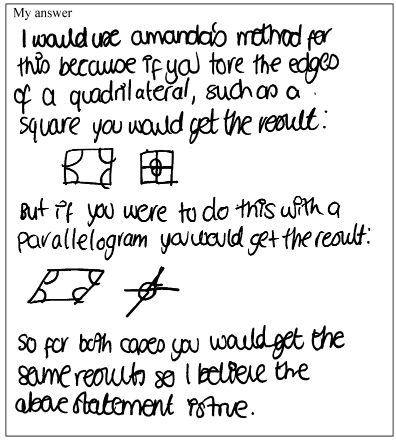


Figure 9. Adapting Amanda's proof in G1 to prove G4.

Given the importance of formal presentation for best mark in the eyes of the students as shown by our analysis of the multiple-choice question, we next consider the distribution of scores (1, 2, 3) for G4 and G7, when the proofs were presented formally (see Table 4).

These data indicate that, although more students attempted a formal presentation as a proof for the unfamiliar conjecture than for the familiar one (10% as compared to 8%), the percentages scoring 1, 2 and 3 were surprisingly consistent across the two questions, in particular a similar percentage (around one third) of those attempting a formal proof for the two conjectures, achieved a maximum score.

We now turn to the multilevel analysis to try to draw out factors that might explain some of the trends in these responses by reference to teacher and school data collected in the survey.

> Side AC and AB are both radiuses of the circle so AC = AB.
>
> Triangle ACD and CDB are congruent because:
>
> they have AD and DB in common because they are halved by the perpendicular bisector.
>
> AD = DB because they have CD in common because it's the same line and because the p. bisector cuts it in half.
>
> Angles ABC and CDB both = 90°. So SAS proves them to be

Figure 10. A narrative proof of G7 (typed from original student work).

Table 4. Distribution of Scores for Proofs in Formal Mode

	Familiar conjecture (G4)		*Unfamiliar conjecture (G7)*	
	No.	*%*	*No.*	*%*
1	5 (2)	40	113	46
2	36 (1)	24	59(2)	24
3	52 (2)	35	74 (3)	30
Total	148 (8)		246 (10)	

Note. The numbers in parentheses represent the percentage of the total student population in this category.

Explaining Students' Responses

From the analysis of the descriptive statistics given earlier, we found that student choices for their own approaches correlated with their choices for best mark, and their views about the generality and explanatory role of the arguments. However, this analysis may be misleading as interactions between input variables are not identified and any clustering of responses associated with shared classroom experiences was not taken into account. We did not wish only to seek explanations for student responses in terms of student "ability" or even attainment in the curriculum, but rather to try to take account of individual differences, such as student gender or attitude to proof. We also wanted to look beyond the individual, to attempt to assess whether variables concerned with teachers, curriculum and school were related to student performance, to look at whether student performance was uniform across the sample or whether there were variations according to school. As mentioned earlier this involved the use of multilevel modelling techniques to take into account the two-level structure of the data set: with school (including school, curriculum, teaching and teacher factors) at Level 2[8] and students at Level 1.

First we describe the models for the following output measures: students' choices on the multiple-choice question for their own approach, and students' scores on the two constructed proofs. A total of 34 input variables were tested for association with each output measure and those that were significantly associated with at least one output and which suggested a relationship that we can interpret from a broader or theoretical perspective, are shown in Table 5.

The analysis presented in Table 5 shows that only one variable, Key Stage 3 test score, was significantly associated with all the output measures and its estimated effects on proof scores were much larger than the other variables. It might only be expected that being able to prove would be strongly associated with a general measure of mathematics attainment, or with knowledge of mathematical facts and procedures. Key Stage 3 test score also clearly influenced the choice of argument for a student's own approach; as this score increased so did a student's preference for an argument that was not empirical.

Table 5. Variables Significantly Associated with Output Measures

Variables	*Output measures*		
	Choices for own approach[a]	*Constructed proof scores*	
	Familiar conjecture, G1	*Familiar conjecture, G4*	*Unfamiliar conjecture, G7*
Level 1:			
Student characteristics			
Gender	*		
Key Stage 3 test score	*	*	*
Responses to questionnaire			
Best mark	*		
Proof as general (geometry)		*	*
Validity ratings (VR)	*		
Explanatory power (EP)	*		
Level 2:			
Curriculum factors			
% GCSE higher tier			*
Approaches to teaching proof			
Write geometry proofs		*	

[a] *The construction of a multinomial model of a categorical outcome involves selecting one category as a fixed base or comparison category and comparing responses to this with responses to the other categories. In fact, estimates of the logarithms of the ratio of the number of students choosing any category to the number of students choosing the comparison category are obtained. In all cases we chose the empirical category as the basis for comparison.*

However, it is of interest that other factors apart from general mathematics competence were also exerting significant influence on proof responses. For example, gender was a factor, with girls making significantly different choices than boys with similar Key Stage test scores in G1. Many of the trends indicated in the descriptive statistics mentioned earlier were also supported by the models: students were more likely to choose an argument for their own approach if they believed it would receive the best mark, if it could be correctly evaluated and if it was felt to convince or explain.

Turning to Level 2 variables, we found no variation in student response according to teacher qualifications, gender and teaching experience. Additionally, no significant associations were found when the teachers' choices were added to the multinomial models, suggesting students' choices were not the same as their teachers', and student choice patterns seemed not to be influenced by differential preferences amongst their teachers.

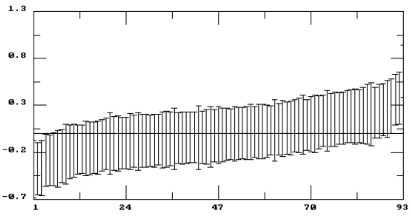
We also modelled students' choices for best mark in question G1. It is not necessary to present all the findings here but it is worth noting that we found a similar set of explanatory variables, with the exception that Key Stage 3 test score did not play a significant role. Somehow, for this familiar conjecture, there was substantial agreement among all students regardless of attainment about the answers that would gain the best marks from their teachers. At Level 2, a range of factors proved to be significant, but again teachers' predictions of their students' choices for best mark were not significant, against a plausible expectation that there would be some agreement between teachers and students as to the most highly valued proof. The previous discussion over teachers' responses to Amanda's argument throws some light on this paradox. Clearly some teachers' preferences for the pragmatic proof both as a predicted choice for best mark and for their own approach were out of line with their students' partiality for formal presentation.

Turning to the constructed proof scores, it is noteworthy that the variable, proof as general, was significantly associated with both constructed proofs. This refers to the responses to question G2 discussed earlier. Maybe it is this appreciation of generality that provided some basis for success in the narrative style proofs in G4 and the formal presentations in G7. It is also of interest that students in classes with a larger proportion taking the higher- rather than the middle-tier GCSE paper[9] were better at constructing proofs for the unfamiliar conjecture and the experience of writing proofs improved scores for the proofs of the familiar conjecture. Both factors point to the influence of teaching. The latter points to the importance of familiarity in setting down an argument in geometry. The former suggests that teachers may have a rather different approach to teaching about proving if there is a majority of students in the class who are highly motivated to take a challenging test at the end of the year. The point is that this influence can be detected among all students in the class regardless of prior mathematics attainment and regardless of whether (or not) the student was actually due to sit the challenging test. Maybe it is only in these classes that students have gained enough familiarity with simple geometrical content and language to be able to begin to engage in argumentation? Maybe it is only in these classes that students are given some instruction in organising an argument in Euclidean geometry?

School Differences

To test if there were schools in which students obtained higher (or lower) scores than would be expected after adjustment according to the significant variables in our models, we analysed the variation in response between students from different schools.

We found that for both the constructed proof scores, there was substantially more variation in the performances of students within schools than between schools. This is illustrated in Figure 11, where 95% uncertainty intervals around the residual estimates for each school have been plotted, showing clearly that, after adjusting for the significant variables, there was considerable overlap between schools.



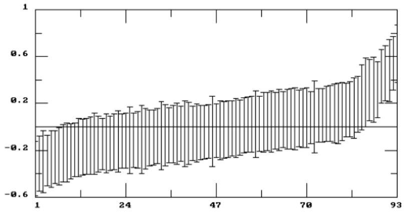


Figure 11. School deviations from predicted means: 95% uncertainty intervals around estimates for each school.

In Table 6 we show the results of random effects of the models of scores in the two questions G4 and G7. It shows that the highest intra-school correlation of schools was obtained for the scores on the unfamiliar geometry question, G7, indicating that there was a larger proportion of the total unexplained between-schools variance for this question than for G4. In fact, the plot of school residuals associated with student scores on G7 (given in Figure 11), shows a rather sharp increase in the gradient at the upper extremes, suggesting that, in a handful of schools, students were performing especially well. It is interesting to note that this pattern mirrored that identified by the multinomial modelling of student choices. In this model it was again found that unexplained school variation was largest among the choices of argument for the unfamiliar geometry conjecture, and that the school attended could enhance students' preferences for a formal argument (whether or not it was correct).[10] These findings together suggest that, for unfamiliar geometry questions, schools were influencing students' responses in ways not captured by any variables in the models. Thus, although there was little between-school variation in students scores for constructed proofs, there were some schools whose students scored better (or worse) than predicted. It is these "outlier" schools which would be interesting to study in depth to find out more about their policies and practices.[11]

Table 6. Random Effects in Models of Scores for Proofs to the Familiar and Unfamiliar Conjectures

Random effects	*Score for proof of familiar conjecture (G4)*	*Score for proof of unfamiliar conjecture (G7)*
Level 1 variation	0.85 (0.026)	0.52 (0.016)
Level 2 variation	0.05 (0.013)	0.05 (0.011)
Intra-school correlations	5.6%	8.7%

Note. Standard errors shown in parentheses.

CONCLUSIONS

So what light does our research throw on the questions raised at the beginning of this chapter? Are students still exhibiting the same difficulties with proof as documented in studies of their responses after following a more traditional course where proof is introduced in the context of geometry and taught with little emphasis on explanation? How far do the proof processes explicitly addressed in a statutory National Curriculum spill over and shape responses to geometrical content in the separate attainment target?

With respect to the first question, it is clear that students were still experiencing many of the same difficulties in constructing proofs that have been identified in previous research. Even among our high-attaining sample, the fact that so few could use their experience in explanation to come up with any deductive argument in geometry can only be disappointing. Most students seemed unable to describe and distinguish mathematical properties relevant to proving a given geometry statement and fewer still could construct logical arguments to connect them. The majority appeared content to rely on inductive inference rather than any logical argument to determine the truth of mathematical statements. When this was not possible, that is, when they could not come up with examples for themselves, the vast majority were unable to begin the process of proving.

Unsurprisingly, the students revealed many of the same problems around appreciation of the generality of proofs as reported in previous research and reviewed earlier—with diagrams often being a source of confusion when a formal geometry proof was merely presented. Students also rarely produced watertight deductive arguments or noted flaws in a given deductive argument. Yet other factors were revealed by our interviews as possibly conferring status on a proof, quite apart from its generality or logical nature, such as the presence of named geometrical facts or relationships and the inclusion of algebra. Ironically, these same facets seemed to have rendered the proofs harder to follow, and paradoxically resulted in them gaining high status.

What may well be different from responses in other countries is that students who had followed our curriculum preferred to construct a proof to a familiar conjecture in a narrative style, and in the process frequently showed considerable individuality and creativity. They rarely if ever acted out meaningless formal rituals when producing proofs, as reported in studies when students had followed a more traditional curriculum. Rather, their proofs were the products of struggle to mould their informal, even private, explanations into a more public communication of the lines of their arguments. The students displayed less creativity in the face of unfamiliar conjectures in geometry: they wanted their own proofs to satisfy the criterion of convincing and explaining but, unlike in the familiar case, could not find ways to express these needs.

It seems that the majority of our sample were not entirely satisfied with pragmatic verification of conjectures. At least, very few of them felt that their teachers would value such arguments, although surprisingly, nearly a quarter of their teachers would indeed have done so. Most students felt that formally expressed arguments would receive the best marks, regardless of their own attainment and the fact that this approach was likely not to have been taught. However students rarely chose or use such an approach themselves—they would find it too hard but also it would not satisfy their need to stay engaged, or to be convinced.

This study does point to the effects of the National Curriculum—not surprising given its statutory nature and the fact that many textbooks at that time were divided into sections according to the National Curriculum attainment targets. Before the curriculum changes in 2000, the curriculum specifications had meant that geometry had been largely *separated from* mathematical reasoning. Reasoning on the other hand had become strangely transformed and became dominated by data-driven activity and pattern-spotting, where the majority of students did not seem be motivated to think about the structures underpinning the patterns—many of which ironically are geometrical. The influence of this curriculum change was mainly evident in the survey in many of the students' empirical proofs to a familiar conjecture.

On the positive side, those students who did not opt for an empirical approach or a familiar pragmatic proof, did display a myriad of creative attempts to prove a familiar conjecture, which, while sometimes falling short of a complete proof, showed their willingness to engage with the problem in experimental and thoughtful ways. Our statistical models also indicated that success in proving was enhanced if students had gained a feeling for the generality of a proven statement, and if there were high expectations for the classes' performances (that is, they were in a group with a larger proportion entering higher-tier examinations). Scores in proving were also related to general mathematics attainment. This competence measure would not only be more likely to guarantee a basic knowledge of geometry facts and language (not widespread in the student population), but also might well be associated with confidence and risk taking.

The findings of this research raise questions for curriculum planning. They suggest that if proving in whatever form or function is to have a place in geometry, (as opposed to or as well as in other areas of the mathematics curriculum), then the curriculum must explicitly be designed to achieve this goal. Building skills in argumentation might have considerable benefits for students in encouraging them to seek to understand and explain their mathematical ideas, but in order to exploit these skills most effectively considerable work has to be done in the context of geometry. How this might be done is a matter of drawing on research that has documented a variety of effective approaches and building systematic plans around these approaches that are long term and progressive. Our teacher data also pointed to the need for teachers to develop their own expertise and confidence in geometry—an issue that might face other countries where, as in England, geometry has long been neglected in the curriculum.

Overall the research has confirmed the complexity of the process of proving in geometry; there are no short cuts or easy solutions. For any teaching to be successful in the complex area of proof, it must build connections between informal intuitive argumentation and more formal proof practices, between analytic and synthetic approaches. Teaching has to enable students to control their work by theoretical considerations while not losing sight of their intuitive problem solving. But what our study has shown is that consideration of the conceptual and cognitive issues alone might be necessary but not sufficient for success and other factors must be taken into account, not least the content and structural organisation of the curriculum and the expectations it engenders about basic geometry, the process of proof and what is “good mathematics”.

Proving for our students was *not* a ritual as was so often the case with traditional geometry teaching, and may still be the case in other countries. Proving was about explaining in general terms; it was part of seeking understanding. Clearly this emphasis along with confidence in investigative practices shaped students’ choices and attempts at proofs. But our evidence suggests that students also recognise a high status geometry proof and value a deductive approach, paradoxically even despite what they might have been taught or what their teachers think they would value. The benefits accruing from a process approach to proving with a focus on explanation can be found in situations concerning generalisation and algebra, as documented in Healy and Hoyles (2000). In this domain, a way forward can clearly be envisaged where students might learn to use algebra as their language to explain phenomena. Geometry presents a separate and more complex issue. But unless activities and curriculum are designed with clear mathematical and pedagogical goals in this context, the general introduction of a process approach could actually turn out to be to the detriment of developing geometrical intuition and explanation.

POSTSCRIPT

As mentioned earlier, the National Curriculum for Mathematics was revised in the year 2000, with changes introduced that particularly affected the attainment target related to proof and reasoning. As a result of this revision, geometrical reasoning was given a more explicit place in the statutory document, since the programmes of study associated with the attainment target *Using and Applying Mathematics* were integrated in sections on *Number and Algebra, Shape, Space and Measures* and *Handling Data*. In 2002, as part of a second research project, the Longitudinal Proof Project,[12] the development of student proof conceptions over a three year period was investigated through annual proof surveys (for an overview of this project, see Hoyles et al., 2005) . A proof test, including some of the items used in 1996 reported here, was administered to 1512 students, aged 14–15 years in the final year of this later study. Although the curriculum revisions were still relatively recent, we present some comparisons of the distribution of responses of teachers and students in the two studies to throw light on what might be the effects of the changes. Clearly such comparisons must be treated with caution given different samples, but nonetheless we believe they are worthy of attention. Figure 12 shows the distributions of student choices on the familiar multiple-choice question (Figure 1) for own approach in 1996 and in 2002. Generally speaking the two distributions are remarkably similar. Amongst both student groups, the two pragmatic proofs were the most popular choices, although there seems to have been a slight shift away from Amanda’s enactive argument in 2002 in favour of Dylan’s set of examples. The arguments chosen least frequently in both 1996 and 2002 were Ewan's reasoned argument and Yorath's visually presented one, with the biggest change between the two samples being the drop in the percentage of students opting for the visual argument (from 10% in 1996 to 1% in 2002). The number of students indicating they were likely to attempt to construct arguments presented as a chain of deductions did in fact increase from 1996 to 2002, but only marginally from 36% to 39%.

Figure 12 also suggests no major changes in the distributions of best mark choices in the two data sets: the two formally presented arguments, and particularly Cynthia's correct deductive proof, continued to be the most popular and the pragmatic proofs were chosen only rarely by the students in both samples. Perhaps surprisingly, it was students in the 1996 sample who seemed to be slightly better at determining which of the two formally presented proofs was correct.

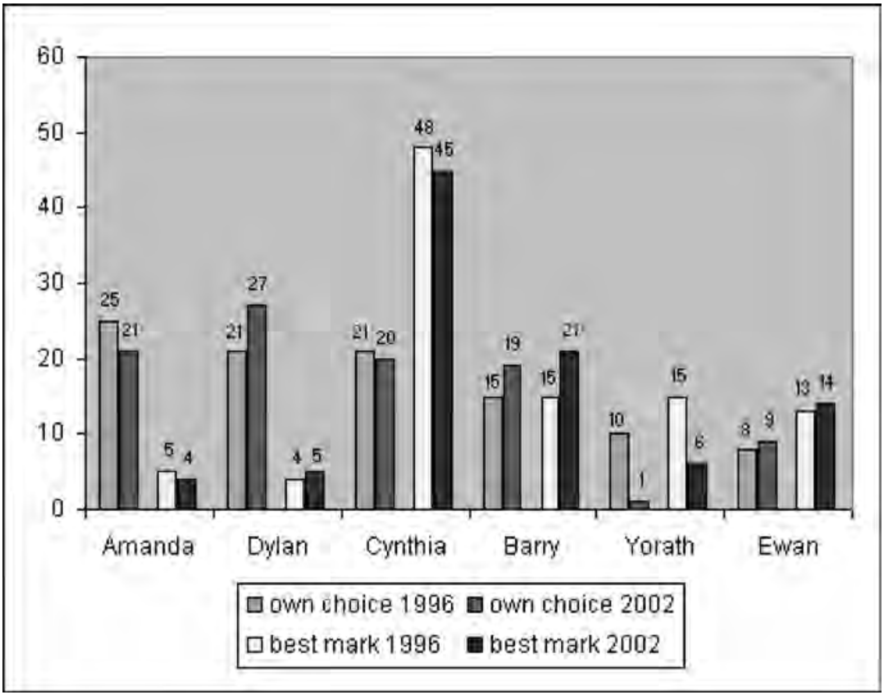


Figure 12. Percentage distributions of students' choices for the familiar conjecture in 1996 and 2002.

While the distributions in students' choices were similar in 1996 and 2002, the same was not true for their teachers, as shown in Figure 13. Amongst the 1996 sample, as discussed earlier, a substantial minority (26%) opted for Amanda's pragmatic proof, which involved tearing up a triangle and organising the angles to form a straight line, as the argument closest to the own approach. In the 2002 sample this percentage dropped substantially to only 10%. Similarly, in 2002 only 6% of the teachers felt their students would choose this argument as best mark in comparison to 22% in 1996. To compensate for the drop in the percentage selecting a pragmatic proof, either for own approach or best mark in 1996, there was a corresponding increase in teachers opting for Cynthia's deductive argument. It could be that, whereas the curriculum changes had not yet motivated substantial changes in the mathematics classroom, the more explicit emphasis on deductive reasoning in the programmes of study for geometry had some impact on how teachers judged geometrical proofs. It should be noted, however, that for the choice of best mark, the 2002 group of teachers substantially overestimated the preference, as choice for best mark, for the correct option of the two arguments presented as deductive steps, with 76% believing that their students would choose Cynthia and only 1% indicated Barry as the probable choice. Figure 12 shows as well that there was a slight increase in the proportion of students who chose Barry in the second sample (21% as compared to 15% in the 1996 sample). This points to the risk that an increased attention to geometrical can lead students to value form at the expense of content.

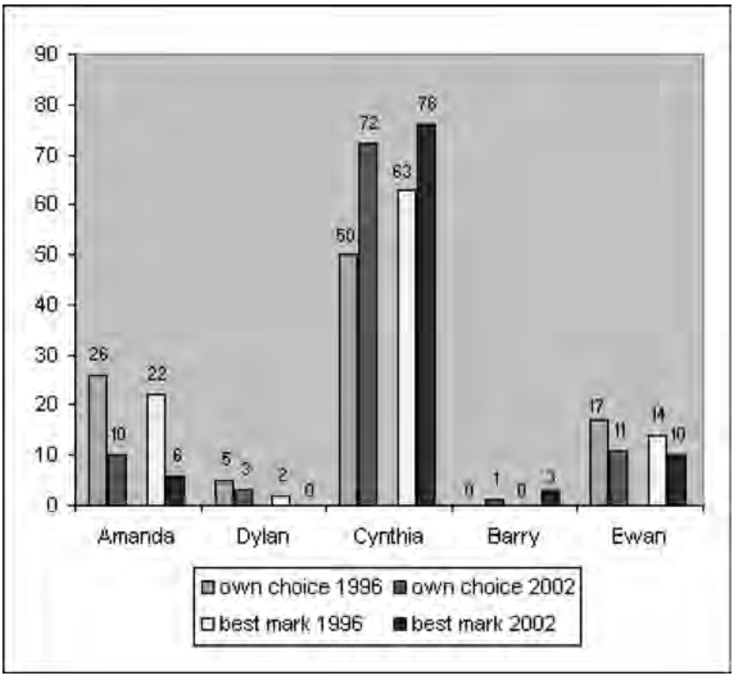


Figure 13. Percentage distributions of teachers' choices for the familiar conjecture in 1996 and 2002. (Yorath's argument has been excluded from the graph as it was not given to teachers in 1996. In 2002, it was chosen by 1% of teachers for both own approach and best mark.)*

Turning to student proof constructions, Figure 14 shows the distribution of student scores for the familiar and unfamiliar conjectures (G4 and G7, as presented in Figure 4) in both 1996 and 2002 (for details of the scoring system, see Table 2).

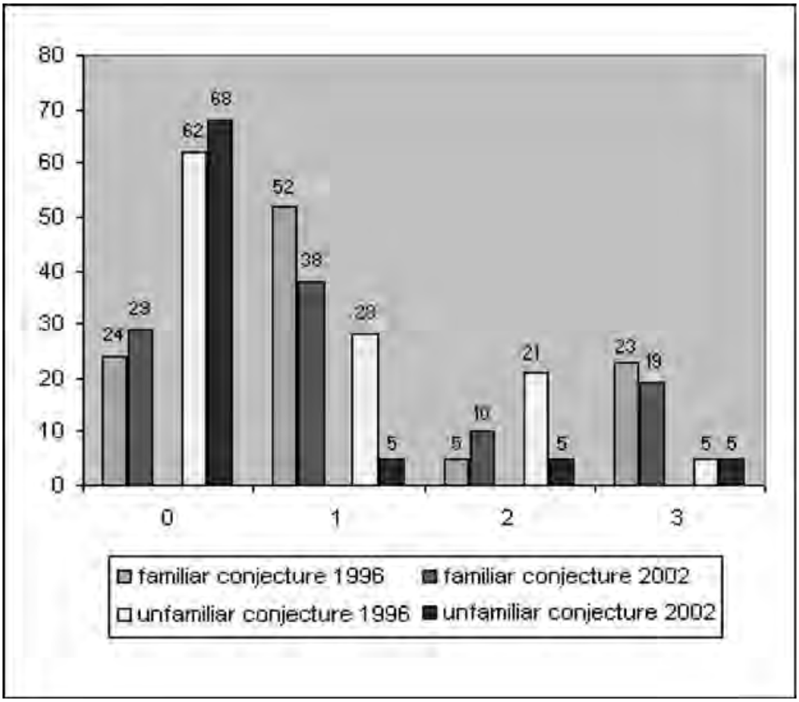


Figure 14. Percentage distributions of scores for students' proof constructions for the familiar and unfamiliar conjectures in 1996 and 2002

In both samples the percentage of students who managed to construct complet- proofs (scored 3) was low. In relation to the familiar conjecture, there was a slight improvement with 23% managing a correct proof in 2002 as compared to 19% in 1996. For the unfamiliar proof there was no difference in the percentage of students producing complete proofs (5% in both groups). There were however some further differences worth noting given the particular curricula revision made and its effects on the contexts in which reasoning would first be met by students; that is a change away from proving being introduced in the context of investigations where empirical data are collected. Among the students who completed the proof test in 1996, 52% scored 1 for the familiar conjecture, and this group included a large number of students (42% of the overall sample) who constructed an argument similar to Amanda's or Dylan's pragmatic proofs (7% and 35% respectively). In 2002 the percentage obtaining a score of 1 fell to 38%. This fall also corresponded to a drop in arguments similar to the two pragmatic options (5% like Amanda's and 26% like Dylan's). The number of students obtaining a score of 1 in relation to the unfamiliar conjecture also fell, from 28% in 1996 to only 5% in 2002. One more positive trend was the increase in students who included at least some deductive reasoning in their attempts to prove the unfamiliar conjecture, that is obtained a score of 2, from 5% in 1996 to 21% in 2002. The percentage of students who scored 2 for their attempts to prove the familiar conjecture also increased from 1996 to 2002, although this increase (from 5% to 10%) was not as marked. On the other hand, the percentage of students who scored 0 also increased from 1996 to 2002, up from 24% to 29% for the familiar conjecture and from 62% to 68% for the unfamiliar.

Despite the limitations in this comparative analysis because of obvious problems in equivalence of samples of students and teachers, it does appear that the process approach to proof that characterised the pre-2000 version of the curriculum encouraged the production of empirical examples, which, though limited as not necessarily giving any focus to analytical or deductive argument, did have the advantage of affording to students some entry point into examining conjectures in geometry. The comparison also points to some risks in simply giving increased curriculum emphasis to geometrical reasoning, and leaves us with the challenge of finding a curriculum approach that allows teachers to support students in negotiating the passage from evidence that illuminates geometrical conjectures to reasoning which justifies them.

ACKNOWLEDGEMENT

We gratefully acknowledge the support of the Economics and Social Research Council (ESRC), Project R000236178.

NOTES

[1] It is worth noting that this division of the targets into eight levels was not based on any analysis of stages of progression in a subject area. Rather it was imposed on all subjects in the National Curriculum, in order that the levels could serve as a mechanism to measure and compare the achievement of students, teachers and schools.

[2] In an analysis of different geometry courses, Goldenberg, Cuoco and Mark (1998) distinguished three types: “faithful replicas of Euclid,” “Euclid without proof with emphasis on applications,” and “inductive geometry where conclusions are drawn from experiments” (p. 41).

[3] Research funded by the Economic and Social Sciences Research Council (ESRC), grant number R000236178.

[4] The analysis of the responses to the unfamiliar conjecture in fact adds rather little to that presented here. It can be found in Healy and Hoyles (1998).

[5] GCSE (General Certificate of Secondary Education) is the public examination taken by students in England and Wales at the end of their compulsory schooling (age 16 years). Teachers decide which students are to be entered to one of three levels in the examination—foundation, middle or higher. Although there is overlap in the grades obtainable from taking the different tiers, there are ceiling grades for the lower tiers.

[6] Key Stage 3 tests are national tests administered in the summer term to all Year 9 students (age 13/14 years). The scores are organised into Levels 1 to 8. At Key Stage 3, about 20% of students achieve each of Levels 5 and 6, 10% Level 7, and 2% Level 8.

[7] The distribution of scores according to National Curriculum Levels was: 1 Level 4, 133 Level 5, 920 Level 6, 1109 Level 7, 162 Level 8 and 133 unknown.

[8] Since we obtained responses from 2 classes in only 4 schools, it is impossible to distinguish between school and class effects.

[9] The GCSE is the public examination taken by students in England and Wales at the end of their compulsory schooling (age 16 years). Students are entered to one of three levels in the examination, the foundation, middle or higher tier. Although there is overlap in the grades obtainable from taking the different tiers, there are ceiling grades for the lower tiers.

[10] This model is not shown here in the interests of simplicity.

[11] The identification and study of outlier schools over a period of 4 years was one of the focuses of the Longitudinal Proof Project (1999-2003), funded by the Economic and Social Sciences Research Council (ESRC), project number R000237777.

[12] This project was directed by one of the authors, Celia Hoyles, together with Dietmar Küchemann.

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5. THE TRADITION AND ROLE OF PROOF IN MATHEMATICS EDUCATION IN HUNGARY

When we were offered the opportunity to write this paper for a book on “Theorems in school,” we considered three main reasons for it: the relevance of mathematics in Hungarian culture, the good reputation for teaching of mathematics in Hungary, and the important contributions brought by Hungarian mathematicians to the development of mathematics in the last two centuries. Then we considered the importance of teaching proof in our schools. In 1983–84 a program “Budapest Semesters in Mathematics,” started with the intention to offer undergraduate courses conveying the tradition of Hungarian mathematics for foreign students. A one-semester course had the title “Conjecture and Proof ”; it referred to the late Hungarian mathematician Paul Erdös’ slogan, or imperative: “conjecture and prove!” (Lackovich 1998).

By analyzing the syllabuses, curricula and more than one hundred textbooks of the last century and considering also the implicit content of them as well as the unwritten shared knowledge of professionals, we could get a picture about the official and informal features of proof in Hungarian schools. A questionnaire and our personal experiences about proof in school will help us to better appreciate the gap between official aims and students’ knowledge of proof.

This chapter focuses on teaching and learning proof in Hungary. Aims of school teaching of proof are compared with results and then reasons for unsatisfactory results and students’ related difficulties are discussed.

Alternative routes to proof (through journals and books for interested students) are considered. The personal experience of one of the authors is revisited, as a synthesis for the arguments presented in this chapter.

PROOF: THE OFFICIAL SIDE AND THE REALITY

During the last century in Hungary the style and the content of mathematics education has changed a lot. Quite naturally all these changes enhanced the role and importance of proofs and reasoning. However at present the two major areas where proofs are concerned seem to be teaching the proofs of classical theorems as a cultural knowledge, and using proper arguments during problem solving tasks as development of individual proving skills.

According to most teachers and documents, in Hungary the teaching proofs in school mathematics is important for many reasons:

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 117–134.

- it is a major characteristic and inherent part in mathematics as a discipline;
- classical theorems and their proofs are part of human culture;
- proofs provide tools and methods for reasoning;
- proofs can clarify the meaning and domain of validity of statements for both the author and the "audience" of the proof;
- a proof can work as a generic example of a logical structure;
- crucial aspects of mathematics can be "taught by proofs" where the meaning of concepts and their relations are revealed by the context of the proof;
- techniques or even art of proving provide the possibility of mental, social etc. development of individuals;
- proof as a tool can help communication between human minds;
- the teaching of proving raises the level of clarity and helps to avoid ambiguity;
- proofs are examples where decisions are based neither on authority nor on democracy.

But what is Present Reality about Proof in School?

Using the items of a parallel inquiry performed by Paolo Boero in Italy, we had the possibility of getting some information (concerning a small sample of students) about this question: does the teaching of proof satisfy some of the above expectations? Moreover there was the opportunity of a rough comparison between the answers of Italian and Hungarian first-year university students. Populations were rather different, but some relevant analogies and differences emerged. The administration of questionnaire in Italy—University of Eastern Piedmont in Alessandria, first-year chemistry students—happened at the beginning of the first year, while in Hungary it happened in the middle of the first semester. In Hungary three groups of first-year students were involved (65 students altogether): two groups of prospective primary teachers (for grades I–IV) and one group of prospective mathematics teachers (for grades V–VIII).

The Questionnaire

Please decide for each statement if it is true or not. Give a short explanation of your choices.

A. A fifteen years old student produces a new proof of the Pythagorean Theorem, needing only some elementary geometrical constructions. In order to establish if the proof is valid,
A1. is the competence of a 17-year-old student, good in mathematics, sufficient?
A2. is the competence of a high school mathematics teacher sufficient?
A3. is it necessary to ask the opinion of a mathematician?
B. In order to establish in a rigorous way that a conjecture is not true,
B1. is it sufficient to find a counter-example?
B2. is it necessary to prove that the conjecture is false in general?

C. In order to establish that a conjecture is true,
C1. is it sufficient to find no counter-example?
C2. is it sufficient to verify it in a large number of cases?
C3. is it necessary to prove it in general?

D. "To prove by reductio ad absurdum" means:
D1. to prove that the contrary of the hypothesis is false?
D2. to prove that the contrary of the thesis is false?
D3. to prove that if the thesis is assumed as false, also the hypothesis is false?

In Hungarian

Kérem, egyenként válassza ki, hogy igazak-e az alábbi állítások! Adjon rövid indoklást is!

A Egy tizenöt éves tanuló a Pitagorasz tétel új bizonyítását készítette el úgy, hogy a bizonyítás során csak elemi geometriai eszközöket használt fel. Annak eldöntésére, hogy a bizonyítás helyes-e
A1. elegendõ egy 17 éves, a matematikában kiváló tanuló kompetenciája.
A2. elegendõ egy középiskolai matematikatanár kompetenciája.
A3. Az szükséges, hogy megkérdezzük egy matematikus véleményét.

B. Ahhoz, hogy tudományosan bebizonyítsuk azt, hogy egy sejtés nem igaz
B1. elegendõ egy ellenpéldát találnunk.
B2. általános bebizonyítást kell adnunk arra, hogy a sejtés nem igaz.

C. Ahhoz, hogy egy sejtésrõl bebizonyítsuk azt, hogy igaz,
C1. elegendõ az, hogy nem találunk ellenpéldát.
C2. elegendõ elég sok esetre megmutatni, hogy a sejtés igaz.
C3. általános bizonyítást kell rá adni.

D. Az indirekt bizonyítás azt jelenti, hogy
D1. azt bizonyítjuk be, hogy nem teljesül a tételben szerepl Õ kiindulási feltétel.
D2. azt bizonyítjuk be, hogy nem teljesül a tételben szereplõ következmény.
D3. azt bizonyítjuk be, hogy amennyiben feltesszük, hogy a következmény nem teljesül, akkor a feltételek sem teljesülhetnek.

Students' Answers

Answers for question A in Hungary showed that many students think that the validity of a proof depends on the authority (of a teacher or a mathematician), not on a logic check accessible to a sufficiently competent student.

Comments gave an insight about students' expectations toward the role of a mathematics teacher.

> A high school teacher has to be able to decide if the proof is correct or not. (A2)

> We have to ask one who has the largest knowledge. (A3)

> The truth of a new proof must be validated at the highest level. (A3)

> Who has to decide? I do not know. (This student did not choose any of the answers.)

> The pupil obviously did not prepare the proof for his/her own purposes. If this is true then a scientifically appreciated person is needed in order to make the proof known for others. (A3)

> The proofs of theorems are made by mathematicians. (A3)

> One cannot ask the opinion of anybody while preparing a new proof. (A3)

> I have chosen this because I think that a 17-year old (however clever he or she is) does not have as much mathematical knowledge as a teacher. On the other hand this is not serious enough to turn to a mathematician. (A2)

Answers for question B showed significant difference between the groups of students; 100% of the group of prospective mathematics teachers chose B1 while 69% of the prospective primary teachers and 30% of the Italian chemistry students think that it is necessary to prove in general that a conjecture is not true. The reason for 100% for the first group can be the result of their university lessons where the “one counter-example is sufficient” slogan is frequent. The 69% seems to be surprisingly high and shows strong misconceptions. On one hand there is a gap between theory and practice in their minds. They probably think that one counter-example can be enough in practice, in ordinary problem solving, but from a theoretical point of view mathematical truth must be proven in general. On the other hand this phenomenon shows the overestimation of the role of general proofs in mathematics. One of the underlying reasons can be that during their school experiences they had encountered only statements which required general proofs.

> It is not sufficient to find a counter-example because a scientific verification was asked for.

> General proof is necessary because we have to consider all the possibilities. (Naturally this might come from another slogan: one example is not evidence.)

> Counter-example is only evidence but not a scientific proof.

> One counter-example is not sufficient because it is possible that we have found only one exception.

> I have chosen B1 because we had done something similar during the course.

> If we are not talking about a theorem but only a conjecture then one single counter-example is enough to refuse the conjecture. For a theorem we should give a general proof.
>
> For scientific proof this (general proof) is necessary, for a non-professional one counter-example is obviously sufficient.

It is also possible that for some students the meaning of “the conjecture is not true” is that the opposite of the conjecture is true.

Answers for question C showed some differences between the Hungarian and Italian students:

C1 was chosen by 36% of Italian and 9% of Hungarian students.
C2 was chosen by 15% of Italian and 3% of Hungarian students.
C3 was chosen by 49% of Italian and 80% of Hungarian students.
8% of Hungarian students chose two answers.

Some students remarked that if they have checked all the possibilities without finding a counter-example then the statement is true, and we have to give a general proof only if it is impossible to check all the possibilities.

Choosing C2 can be quite reasonable if someone thinks an existence-type statement.

Answers for question D showed that most Hungarians did not understand the meaning of the given sentences. Many of them, instead of choosing one of the given possibilities, gave his or her own definition. Some others made their choices but their comments were not coherent with their choices. The situation was completely different in Italy: most students engaged in the answer and showed themselves familiar with the technical expressions (even if several answers revealed lack of reflection about “proof by reductio ad absurdum”).

Summary: We can summarize the information obtained through the questionnaire by saying that in Hungary the situation of the mastery of proof at the end of secondary school seems to be rather far from the claims of official documents and the expectations of teachers and mathematics educators. The situation seems better in Hungary than in Italy for some aspects (see results about C) and worse for others.

The most relevant difference (in favor of Italian students) concerns the answers to question D. The reasons for Hungarian students’ difficulties mostly concern the language.

LANGUAGE ASPECTS

From the answers to the questionnaire and from other experiences we think that (with some exceptions; see later) Hungarian students are not very conscious about the logical structure of (theorems and) proofs. Even those who are quite skilful at constructing proofs would be embarrassed if one asked them to explain what they did and why. Expressions like “condition,” “hypothesis,” “thesis” are missing from the vocabulary of secondary school teaching. Some of them have no Hungarian equivalent. In 60–80-year old books we can find Latin words with Hungarian spelling (at that time Latin was compulsory in secondary schools), but nowadays these words are absolutely out of fashion: even most of the mathematics teachers would not understand them. Some other expressions have Hungarian translations but the meaning of them is different from the mathematical meaning or ambiguous. For example the Hungarian word "hypothesis" means "conjecture" rather than "hypothesis" of a theorem, etc. The lack of proper expressions makes the teaching of several subjects and mathematical discussions difficult. For example it is difficult to deal with negation of statements, opposite of statements, a theorem and its converse theorem, etc. Probably this is the reason why George Polya dedicates a chapter on "How to solve it?" to these logic-linguistic issues:

> The principal parts of a "problem to find" are the unknown, the data, and the condition". Condition links the unknown of a "problem to solve" to the data. In this meaning, it is a clear, useful and unavoidable term. It is often necessary to decompose the condition into several parts. Now each part of the condition is usually called a condition. This ambiguity which is sometimes embarrassing could be easily avoided by introducing some technical term to denote the parts of the whole condition; for instance, such a part could be called a "clause" (p. 155).

> Hypothesis denotes an essential part of a mathematical theorem of the more usual kind … The term, in this meaning, is perfectly clear and satisfactory. The difficulty is that each part of the hypothesis is also called a hypothesis so that hypothesis may consist of several hypotheses. The remedy would be to call each part of the whole hypothesis a "clause" or something else. (1.201).

With the lack of proper and common expression of the mother tongue, the language of teaching of proving greatly enlarges the role of the teacher in the learning process of students.

When students are requested to reproduce proofs of classical theorems one can experience a great amount of anxiety. Pupils cannot discuss the learned proof in their own words partly because of the language and partly because they are uncertain even of the proper order of the words. Instead of giving them a useful access to mathematics, this practice means only a piece of "memoriter" (=text to learn by heart) to learn by heart for many of them. The expression "mathematical proof" frequently becomes the symbol of school anxiety. We read in a pamphlet of the famous Hungarian writer, Frigyes Karinthy,

> Write it, then, [says the teacher]

> The bad learner turns to the blackboard.

> "minus b plus minus square root b squared minus four times a times c divided by two times a"

> And the bad learner obediently starts to write (…), He knows what it is. The theorem reminds him of something that he read that night when he fell asleep

> over the book, and when he had no idea what it was all about. Yes, he vaguely suspects, some quadratic equation—but what will come out from all this?

The word "proof" sounds frightening even for university students. Before each written examination they ask whether there will be proof tasks or only problem solving. This question shows that the meaning of proof for them is restricted to the proofs of known theorems or similar statements. Justifications of the steps of normal problem solving tasks are easier for them and do not belong to their conception of proof. One element of the underlying reasons can be the usage of language. Careful examination of mathematics books written in the Hungarian language shows two different types of phenomena.

In the case of rules, theorems, definitions, or explanations the language of the book (and consequently the teacher) uses exclamation mode quite often. Commands like prove (Bizonyítsuk be! Mutassuk meg! Lássuk be! Igazoljuk! etc.) are more common concerning reproduction of traditional pieces of knowledge than concerning problem solving situations. Statements show certainty. At the same time questions can be detected only in the preparatory process or applications of traditional statements. On the contrary in problems many questions appear. One can observe a major difference between the level of grammatical complexity of sentences and the "polish" of them.

The language of textbooks concerning proof is rather an archaic Hungarian (it does not reflect today's language).

SOME INTERESTING EXAMPLES OF REASONING, ARGUING AND PROVING IN SCHOOL PRACTICE IN THE LAST DECADE

Primary School (Grades I–VIII)

A typical feature of the first appearance of logical reasoning concerning arithmetical calculations is the following.

During the period devoted to exercises concerning arithmetical calculations (in order to reach automatism) in a usual class situation the teacher asks the pupils to explore the method of obtaining the result of e.g. 7+8 (naturally several pupils are not able to construct or reconstruct their argument).

On a video taken in a first-grade class (age 6–7) at the end of the school year the pupils gave the following answers:

Pupil A: *7+8 makes 15 because 8 is 3+5, I know that 7+3 makes 10, so 7+8 makes 15.*

Pupil B: *7+8 makes 15 because I know that 7+7 makes 14 and 8 is one bigger than 7, so the result is 15.*

We can see that both examples refer to some earlier stable knowledge that students are sure of. They thought of a number in the form of a sum or a difference and they tried to link this with the original problem. Besides developing strategies of calculations the consequent demand of arguments from the part of the teacher creates an atmosphere which is similar to the one of deductions. The continuous appearance of verbalization of reasoning helps the pupils to learn ideas from each other, and makes them more aware of their own possible thinking processes. In a communication situation, being able to express one's own way of thinking produces "a chain of thoughts," and provides the possibility of doing the same in the group as will be done later in the form of a conscious individual "inner speech". This is a necessary step for the interiorization of the need of the "chain of thoughts" in learning mathematics and perhaps in everyday life (see Vygotsky, 1978).

In the upper primary classes (grades VI–VIII) students often encounter problems which connect different topics of mathematics. The following example, which is quite popular amongst Hungarian math teachers, can be considered as a problem in geometry (concerning area and perimeter) as well as an exercise for Diophantine equations.

> The lengths of the sides of a rectangle are a cm and b cm, where a and b are integers. The area is the same number in square cm as the perimeter in cm. What are the lengths of the sides?

The usual solution is to solve the Diophantine equation $ab=2(a+b)$. This can be done in several ways. For example, we discover that the equation can be separated in the form $(a–2)(b-2)=4$. From the factorizations of 4 we get two positive solutions for a and b, namely, $a=4$, $b=4$ and $a=6$, $b=3$ (or vice versa).

There are other types of solution, however. The following reasoning has also occurred.

The measure of the area of the rectangle is equal to the number of all the unit squares within the borderline. The measure of the length of the perimeter of the rectangle is equal to the number of the unit squares that lie along the borderline, with the condition that the unit squares at the corners should be counted twice. We have 4 corners; consequently, the measure of the area is equal to the measure of the length of the perimeter if and only if there are exactly 4 squares in the inside of the rectangle. The 4 squares must not have any side which lies on the borderline of the rectangle. This can be done in two ways: either there is a 2×2 square in the inside, and the whole rectangle is a 4×4 square; or there is a 1×4 rectangle in the middle, and the whole rectangle is a 3×6 rectangle.

Another problem situation that can stimulate a proof reasoning is the following:

> Two taxicab companies are working in our town. At the first one, the bill of fare starts from 70 Hungarian Forints (HUF for short), and a further 160 HUF for each kilometer. At the other company, the data are 2000 HUF and 140 HUF, respectively. I have three favorite spots where I usually go by taxi, one is 5 km from home, the other 10 km, and the third 15 km. Shall I stick to one company or not?

There is a wide variety of reasoning on the part of the students. Some of them determine the value of the bill for each route. Others solve the relevant inequality algebraically or graphically to prove that up to 6 km the first company is the good choice; from 6 km up, the second.

PROOFS IN TEXTBOOKS

By analyzing the textbooks of the 20th century one can see the following. Until 1950 the high school mathematical textbooks were rather similar to the books for mathematician both in the level of complexity of language and in typographical solutions. In 1950 a new series of textbooks started in mathematics by Rózsa Peter and Tibor Gallay. The major change affected the topic “proof”. This was the first, and until the early 80s, also the only textbook which dedicated a special chapter to proving methods for 14-year old pupils. The language of this book was much more informal; by using the language of everyday life it started a dialog with the reader instead of declaring definitions, theorems, proofs and problems for application of the theorems. The typographical solutions were designed to underline the meaning and facilitate the understanding of the topics. Unfortunately the teachers of that period were not ready to teach according to the spirit of this book and even if the textbook was present in the schools the teaching methods followed the style of previous books. The following series of textbooks again became more traditional. In the early 80s the series of textbooks (for grades V–VIII) prepared for the new curriculum under the guidance of the late Tamás Varga and one series of secondary books (for grades IX–XII) prepared under the guidance of the late Rózsa Peter tried to deeply involve the learner to understand the idea and practice of proving, but at the end of the 20th century this kind of textbook was rarely used in compulsory mathematics education.

PROOFS IN JOURNALS FOR STUDENTS

For more than 100 years we have had the monthly journal *Középiskolai Matematikai Lapok* (*Mathematical Journal for Secondary Schools*) for the talented student in mathematics and physics. From the very beginning this journal, the second of its kind all over the world, published high-level mathematical problems and articles, with detailed solutions of the problems which were published in the previous issue. In addition those pupils who were successful in solving problems have their name published every month. Year by year the journal runs a competition in problem solving (problems and exercises). Besides these there are both easier and more difficult mathematics problems. The editors publish various solutions regarding the same problem and report the names of students who produced those solutions.

The fact that pupils can buy this journal and read the solutions to problems offers the opportunity to increase their problem solving abilities and learn the style and level of proofs that is required by the journal. Many great Hungarian mathematicians remember the journal as a part of their training. For example we can read in an interview with Paul Erdös:

> Do you feel that your mathematical development was affected by the high school mathematics newspaper (*Középiskolai Matematikai Lapok*)?
>
> Yes, of course. You actually learn to solve problems there. And many of the good mathematicians realize very early that they have ability.

(Peter J. O'Halloran, *Journal of the World Federation of National Competitions*, Vol. 5, No. 1, June 1992)

BOOKS FOR SPECIAL COURSES AND INTERESTED STUDENTS

Besides journals, in Hungary the special mathematics courses were and still are popular in secondary schools. A great number of books were published especially for the purposes of these courses. These books are written in a style which can be understood by 10–18-year old readers without the help of the teacher. Pupils who start reading these books have a better opportunity not to see mathematical proofs as magic things. The position of weaker (or less interested) students who use only the textbooks is much more difficult! Usually the language of these books is easier than the language of textbooks even if the mathematical content is rather deep. Not textbooks, but this kind of alternative books, continued the style and spirit of the Peter–Gallay textbook!

COMPETITIONS, CONTEXTS

Each year there are several type of contests in mathematics. The problems of the Eötvös Competition (it started in 1894 and ended in 1947) were published with their detailed solutions for the first time in 1929, under the title *Matematikai versenytételek* (*Problems for Mathematical Contests*), by József Kürschák. He used the best solutions of the students and added his remarks. Problems and solutions for the period 1929–1947 were collected in further volumes.

At the beginning of the 50s the National Olympiads were announced for the upper two grades of the secondary schools. At the beginning of the 60s the János Bolyai Mathematical Society organized a nationwide contest under the name Dániel Arany Contest for the lower two grades and at the beginning of the 90s the Tamás Varga Contest for elementary schools (up to year 14). All these contests allowed pupils to use any kind of book during the solution of problems. When a student uses a not-well-known theorem he is obliged to refer to the book in which he has found the proof of that particular theorem. This rule creates a climate in which the ability of reproducing the proofs of known theorems is considered marginal.

A PROOF-CV

Finally, one of the authors of this paper, Julianna, would like to provide the reader with an excursus about the role of proof in her mathematical thinking; "a proof-CV," if you prefer. We hope that such an account gives you a clearer picture of the atmosphere of mathematics teaching in Hungary during recent decades (in particular as concerns proof). In particular, we will illustrate again (in a personal case) the gap between the opportunities offered in normal school courses and the need for understanding proof and experiencing challenging proof situations.

Kind of School

My high school was a "humanistic secondary school" (in contrast with "science secondary school"), with more stress on arts and literature, and less on mathematics. We studied psychology and logic in the last grade, when we were 18-year old.

Kind of Learner

I was a prototype of "conformist" learner at that time. I was able to solve a complex logarithmic equation, although I was aware that logarithms meant nothing for me but an abstract word, a magic thing without any reference to real-world applications in physics or chemistry or other disciplines. It remained for me an enigmatic thing, a strange word even in my undergraduate days. I did not cease to formally understand it; and I did not feel the need for a deeper understanding. That was just enough to please my teachers, and to gain good marks at my exams.

Proof 1 (first year at secondary school, age: 14) At this age I already had the feeling that proving was necessary.

This feeling was aroused by my math teacher who urged me, a good grader, to participate in special math courses. The teacher during his course did his best to demonstrate that proving was important. "*You must not believe your eyes,*" he said, and showed us the following two figures which apparently "prove" that 64=65:

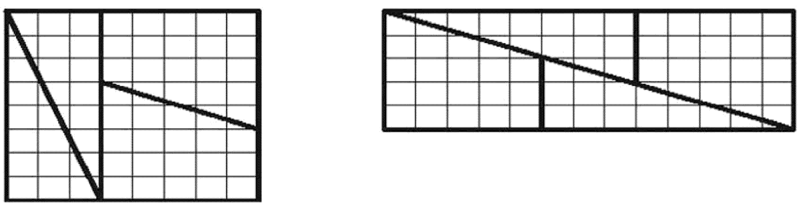

Figure 1. The square paradox.

I was convinced of not believing my eyes any more. He introduced the process of proving to me at a mathematical ceremony. We were shown elegant and understandable proofs, mainly in the field of Combinatorics. This was the first time for me to encounter the expression "Q. E. D." which seemed to me to be a magic key to the world of adults. The other person who introduced the proving ceremony to me was my teacher in Hungarian literature. She knew a lot about Mathematics Olympiad problems. Her personality was a message for me to view proofs in problem solving situations as part of human culture as a whole.

Proof 2 (at the age of about 18–19 years, last year in secondary school, first year at the university) Proving ceremony was a question of good manners for me, like eating with fork and knife. For the oral part of the maturity examination I learned by core all the proofs that were required at that time. After the maturity examination I went to study maths and physics at the Roland Eötvös University at Budapest to become a secondary school math and physics teacher. Proofs played a central role in our studies.

Naturally, our professors demonstrated for us the aims and goals of proofs; but I was totally bewildered by questions which I asked but found nobody to answer them: “Who were the people who could create the statements to be proved?” “Who were the people whose truths we must prove?” “How did these questions become important for them?” But as a conformist learner I quickly overcame these doubts, I learned the proofs as I was expected to learn, and my professors were quite satisfied. I had only one teacher who took great pains during her algebra lectures to create the need for proofs. She showed us that a series of preliminary theorems or lemmas were indispensable before we could carry out the proof for the main theorem.

Proof 3 (approximately second year at the university) At this stage, proof became a marketing tool, something that was necessary to sell your theory to another person, in mathematics and in physics. The task was to convince another person, not myself. This was something that reminded me of my studies on models in physics. We were not interested if a model was proper, we were careful only about being not improper.

Proof 4 (approximately in the third university year) The proving ceremony had grown to be something like the following: I was delighted with myself because I was able to use what I had learned in logic at 18. The subject of logic initially appeared to me as learning about some trivial facts which were obvious to good sense. Later on, logic became a tool to manage with in geometry and other subjects; to create counter-examples about certain properties of a figure without any help from the teacher; and to slowly and gradually convince myself that I was going in the right direction. Proving became a kind of compass for me. But I was not really satisfied with my progress. I only felt safe if I was expected to reproduce an accomplished proof. I focused on my presentation regarding the proof; I wanted to convince my professors that I deeply understood the central ideas and was able to use the right mathematical terms in the right place.

Proof 5 One of my compulsory courses at the fourth university year was The Foundations of Geometry. At that time the major part of our studies consisted of compulsory courses. After the first two lectures I realized that I knew nothing of the proving ceremony as related to the course. For the first time in my life I had to break away from the conformist learner attitude, and create a philosophy for my personal use, as an aid for a conformist learner in an emergency. The philosophy was: Concepts do not exist as such; concepts are only sentences and conventions. In order to see, to draw the concept or to have a pragmatic use of it, one does not need to really understand it. Concepts are created by people; they are but parts of a nice intellectual game. This approach helped me have some ideas about Bolyai-Lobachevskian geometry.

Proof 6 Then I began to study probability. We started with the concept of the relative frequency of an event, followed (after some lectures) by the Kolmogorov probability theory. Mathematics totally collapsed with me at that time. I had no idea about what "mathematics" could mean. I found no way of filling the gap between "tossing the coin" and the Kolmogorov theory.

Proof 7 I tried to hide my failure to handle my philosophical problems, while I was teaching mathematics at secondary level, and writing math papers on various topics.

The problems were solved, and the gap filled, with the help of Professor Rózsa Péter. I met her when I and a colleague of mine were invited to give a four-day-long summer course on probability for primary teachers. The course seemed to remain "on the side of coin tossing" of the gap, so I felt quite secure about my problems. At the beginning of the first session, Professor Rózsa Péter entered the room. She was a striking personality, sometimes very kind, sometimes terrifying, but always ready to overtly express her opinions. I knew very well that she hated probability in schools. She always said: "*Probability teaching is not mathematics teaching, it is like physics.*" In the fifth minute of my lecture, her presence inspired me to say to my audience that "*all our experiments would only serve for creating a mathematical model of a real coin and other real events, by our experiments we will help ourselves to create the mathematical coin in our thinking*." I added: "*Actually, we know nothing about the probability of tossing a real coin. Math has nothing to say about that. We will predict the behaviour of a real coin by replacing it with the mathematical one in our thinking. But math is not to be blamed, because probability theory is not about a real coin*". When I finished the course I definitely felt better than before.

Proof 8 During the years teaching at different level of schools and teaching courses for undergraduate students involved in initial teacher training, I realized that the "enjoyment of mathematical proofs during the understanding process of Bolyai-Lobachevskian geometry" are quite independent from the learning process of mathematics. Such kinds of proofs still please me, but like a nice piece of music. I had to realize that the polished proofs of genius minds gives a misinterpretation about mathematics itself in the mind of pupils/students. And I started to love less the books with nice problems and their elegant proofs. I started to search for the works of teachers, writers who put a strong stress on the meaning of the concepts itself, and on the context from which the problems may arise. Even in the cases of traditional problems of mathematics history I tried to describe or reinvent situations for my students which could clarify why these problems could be relevant and interesting for those who first had created them. In those years I had a lot of doubts, started to believe that I am not a real mathematician, maybe I do not really like mathematics, etc. The appearance of the Hungarian translation of the book by Imre Lakatos, "*Proofs and Refutations*," let me "recover my breath" again.

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APPENDIX

The obligatory theorems of the maturity examinations for proof are the following:

1. Prove the following identities (a and b are real numbers, n and k are positive integers):

a) $(ab)^n = a^n b^n$

b) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, $(b \neq 0)$

c) $(a^n)^k = a^{nk}$

2. Prove that $\sqrt{2}$ is irrational.

3. Prove the following identities:

a) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$;

b) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$;

c) $\left(\sqrt[n]{a}\right)^n = \sqrt[k]{a^n}$

What are the conditions for a, b, n and k?

4. Prove the following identities:

a) $\log_a xy = \log_a x + \log_a y$;

b) $\log_a \frac{x}{y} = \log_a x - \log_a y$;

c) $\log_a x^k = k\log_a x$

What are the conditions for x, y, a and k?

5. Prove the formula for the roots of the general quadratic equation.

6. Prove the connections between the roots and the coefficients of the general quadratic equation.

7. Prove that in a triangle opposite to a bigger side there is a bigger angle.

8. Prove that the perpendicular bisectors of the sides of a triangle are concurrent.

9. Prove that the bisectors of the inner angles of a triangle are concurrent.

10. Prove that the altitudes of a triangle are concurrent.

11. Prove Thales theorem and its reverse.

12. Prove that a quadrilateral in the plain is a tangent quadrilateral if and only if the sums of its opposite sides are equal.

13. Prove that a quadrilateral is a chord quadrilateral if and only if the sums of its opposite angles are 180°.

14. Prove that the angle at the circumference of a circle belonging to an arc is half of the angle at the center belonging to the same arc.

15. Prove that the sum of interior angles of an n-sided convex polygon is $(n-2)180^{o}$ and the number of its diagonals is $n(n-3)/2$.

16. Prove the theorem for the arithmetic and geometric means of two positive numbers.

17. Prove that the medians of a triangle are concurrent.

18. Prove Pythagoras' theorem and its reverse.

19. Prove that a bisector of the angle in a triangle divides the opposite side with the ratio of the neighboring sides.

20. Let's consider two similar polygons and two similar pyramids where the ratio is k in both cases. Prove that the ratio of the areas of the polygons is k^2 and the ratio of the volumes of the pyramids is k^3.

21. What is the connection between the base area of a pyramid and the area of a section of it parallel to the base? Prove it.

22. Prove that in a right triangle the length of a side is the geometric mean of the length of the hypotenuse and the length of the segment which is the perpendicular projection of the side to the hypotenuse.

23. In a right triangle the altitude belonging to the hypotenuse divides the hypotenuse into two segments. Prove that the length of the altitude is the geometric mean of the lengths of the two segments.

24. Draw a tangent line and an intersecting line to a circle from the same outer point. Prove that the length of the tangent segment is the geometric mean of the lengths of the two segments of the intersecting line.

25. Prove the following identity:

$\sin^2\alpha+\cos^2\alpha=1$ for any real α.

26. The angle of planes S_1 and S_2 is α. The area of a triangle in S_1 is t_1, the area of its perpendicular projection to S_2 is t_2. Prove that $t_2=t_1\cos\alpha$.

27. Prove that if r is the length of the radius of a circle, a is the length of a chord and α is the angle at the circumference belonging to the chord, then $a=2r \sin \alpha$.

28. Prove the sine rule.

29. Prove the cosine rule.

30. Prove the following identities:

$\sin(\alpha+\beta)=\sin \alpha \cos \beta+\cos \alpha \sin \beta$ and

$\cos(\alpha+\beta)=\cos \alpha \cos \beta-\sin \alpha \sin \beta$.

31. Prove that any vectors a, b, c $(a+b)c=ac+bc$.

32. Write a formula for the distance between points A(a_1; a_2) and B(b_1; b_2) and prove its validity.

33. Express the coordinates of the half point and the third point of a segment with the coordinates of the endpoints of the segment and prove the validity of your formulae.

34. The coordinates of the vertices of a triangle are given. Prove that the coordinates of the centroid of the triangle are the arithmetic means of the corresponding coordinates of the vertices.

35. Prove that the equation of a line going through the point $P_o(x_o; y_o)$ with the direction vector $\boldsymbol{v}(v_1; v_2)$ is

$$v_2x-v_1y=v_2x_o-v_1y_o.$$

36. Prove that the equation of a line going through the point $P_o(x_o; y_o)$ with the normal vector to it $\mathbf{n}(n_1; n_2)$ is

$$n_1(x-x_o)+n_2(y-y_o)=0.$$

37. Prove that the equation of a line going through the point $P_o(x_o; y_o)$ with the direction tangent m is $y-y_o=m(x-x_o)$.

38. Prove that the equation of a circle with the center $C(u; v)$ and radius r is

$$(x-u)^2+(y-v)^2=r^2.$$

39. Prove that if the focus of a parabola is $F(0; p/2)$, its vertex is the origin and its axes is the y-axes, then its equation is $x^2=2py$.

40. The length of the major axes of an ellipse is $2a$, the length of the minor axes is $2b$ $(a>b)$, the major axes lies on the x-axes and the minor axes lies on the y-axes. Prove that the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

41. Prove that the sum of the first n square numbers is

$$\frac{n(n+1)(2n+1)}{6}$$

42. The first element of an arithmetic sequence is a_1, the common difference is d. Prove that $a_n = a_1 + (n-1)d$ and $S_n = n\frac{a_1+a_n}{2}$.

43. The first element of a geometric sequence is a_1, the quotient is q. Prove that

$a_n = a_1q^{n-1}$ and $S_n = a_1\frac{q^n-1}{q-1}$.

44. Prove that if $|q|<1$ then the geometric series $1 + q + q^2 + \dots + q^n + \dots$ is convergent. Determine the S sum to infinity.

45. Prove that $\lim_{x \to 0}\frac{\sin x}{x} = 1$.

46. Prove that the derived function of $\cos x$ is $\sin x$ for all real numbers.

47. Prove that the derived function of x^n is nx^{n-1}.

48. Prove that the derived function of $\sin x$ is $\cos x$ for all real numbers.

49. Prove the Newton–Leibniz rule for integration.

50. Prove that if the base area of a prism is T and its height is m, then its volume is $V = Tm$.

51. Prove that the volume of a cylinder with base of radius r and height m is

$V = r^2\pi m$.

52. Prove that the volume of a pyramid with base area T and height m is $V = Tm/3$.

53. Prove that the volume of a frustum of a pyramid with bases T and t and height m is

$$V = \frac{m}{3}\left(T + \sqrt{Tt} + t\right).$$

PART III: ARGUMENTATION AND PROOF

RAYMOND DUVAL

6. COGNITIVE FUNCTIONING AND THE UNDERSTANDING OF MATHEMATICAL PROCESSES OF PROOF

Proof constitutes a crucial threshold in the learning of mathematics. Why do so many students not succeed in truly crossing it? Though proving cannot be reduced to reasoning, this major didactical problem concerns the variety of approaches for what is designated commonly as "reasoning," mainly when reasoning is required within the framework of a scientific or a mathematical activity. Three great trends have gradually emerged in research on the development of student's reasoning.

- The psychological strand in which the models of reasoning are logical forms of **valid reasoning** such as Aristotelian syllogisms or material implication with use of connectors for truth-functions (Piaget & Inhelder 1955; Johnson-Laird 1983, Rips 1988).
- The didactical trend in which the models are those of explanation-researching, mainly in geometrical situations requiring an interaction between a visual exploration of figures and an application of few theorems and definitions which have to be used as "tools" of "justification". The goal is to determine **the truth of a statement** which is put forward as a conjecture at first and **to convince** other people. In this strand considerable attention is given to the successive attempts and explanations of students and therefore to their discursive productions (Lakatos 1976; Balacheff 1987).
- The Artifical Intelligence strand in which the models of reasoning are **rules of conditions-action** working as "inference engines". This trend must be further subdivided into a cognitive model of proof for tutor conceptions (Anderson 1987) and the construction of micro-worlds for dialectic interactions with students (Luengo 1997).

What is common to these different approaches is that they start from some external characteristics of reasoning—either logical or mathematical—and take them as references in order to model the reasoning activity. Consequently, the real understanding of how this activity works, how it can be different from spontaneous reasoning in everyday life or in areas other than mathematics, is completely forgotten. The cognitive working of reasoning is not the image or the reproduction of logical or mathematical patterns. Thus Schoenfeld, after an experiment in the classroom made in 1984 over one semester, correctly pointed out: "perhaps what is needed, and what has been lacking, is an understanding of how proof really works" (Schoenfeld 1986, p. 253). Such an understanding is based on the perception of the

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 137–161.

meaning of propositions in any proof: the difficulty of this perception is due to the multidimensionality of the meaning of propositions. Students must become aware of the different components of the meaning of propositions. And this multidimensionality is closely connected to the different ways of organizing propositions in one discourse which can be either an ordinary argumentation or a proof or a formal reasoning, though the wording is sometimes similar. It is why something like a double awareness is required. It is what makes students able to understand "how proof really works" and to become truly convinced by proofs. This chapter will set out the main characteristics of the cognitive functioning of reasoning. Then it will examine the consequences for the didactical problem of learning to prove. Finally, it will examine the variables to use in order to give rise to the double awareness.

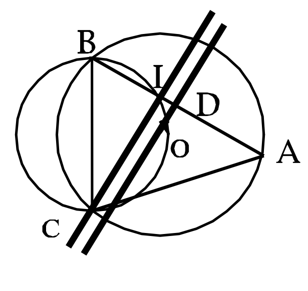
OVERVIEW OF THE COGNITIVE COMPLEXITY OF REASONING WORKING

In order to analyze the cognitive complexity of mathematical reasoning activity, some prior distinctions are required. Some are well known such as the distinction of operative status of proposition in deduction (hypotheses, theorem, etc.), but the main ones, which can seem needless, such as the distinction between truth value and epistemic value, have come to the surface through observation of students during proof teaching experiments. The students' explanations, the sudden change of their text productions, showed that the gap they had to bridge was to become aware of the implicit complexity of proposition meaning within the different possible organizations of propositions which underlie the various types of reasoning (Duval 1991, pp. 247–253).

Characteristics of Reasoning: A Meaning Space for the Discursive Organization of Propositions

Any reasoning, implicitly or explicitly, works with propositions, that is with statements which have a value for themselves and a status in relationship to one another. Value and status are specific components of any proposition's meaning.

The internal components of the meaning of a proposition First of all, the meaning of any proposition is more complex than the meaning of any word. The meaning of a proposition is determined with respect to several dimensions: a semantical dimension through its content, a knowledge dimension through its epistemic value (obvious, likely, absurd, unreal, possible, necessary, etc.) and a logical dimension through its truth values (true, false, undecidable, etc.). The epistemic value is closely connected to the way somebody understands the content of a proposition: it depends on the subject's knowledge basis. For example, this way of understanding can be "theoretical," that is with a background of definitions, theorems and deductive practice, if the subject is an expert mathematician, or it can be only "semantical," that is reflecting ordinary language understanding, if the subject is a young learner. For example, any proposition whose content focuses on mathematical properties which can be immediately seen on a figure (parallelism, perpendicularity, etc.) can have quite different epistemic values: visually obvious for the student but only possible or, maybe, impossible from a mathematical point of view. A mathematical property involves the necessity of its statement. How to become aware of its necessity and how to make students aware of it?



On a circle of center O, mark three points A, B, C. The circle of diameter [BC] intersects (AB) at I. Let D be the midpoint of [AB].
Prove that the straight lines (OD) and (CI) are parallel.

Figure 1. Problem given to 13–14-year-old students.

The content of the proposition "the straight lines (OD) and (CI) are parallel" can be checked on the figure. Understanding what this proposition means, it is just a matter of seeing something obvious. What else can be wanted in order to know that it is true?

The distinction between epistemic value and truth value is important. It enables us to explain what reasoning achieves. The outcome of any reasoning is not only to produce new information but also, and above all, **to change the epistemic value of a proposition** whose truth we want to prove or attempt to convince somebody else of. If the truth of a proposition seems possible, reasoning shows it to be necessary or, on the contrary, impossible; if a proposition is thought to be absurd, reasoning makes its claim likely or necessary, and so on. The key point for proving and for understanding how a proof works in mathematics **is the connection between the various epistemic values and the logical value "true"**. Here we must take into account two specific features.

The first one is epistemological. While in other fields such as botany, in history, etc. the logical value "true" may be connected to different epistemic values in relationship with data from perception or from some technical devices or from testimonies, in mathematics the only agreed connection is this one between the logical value "true" and the epistemological value "necessary"! The second one is cognitive. While usually epistemic values are directly connected to the understanding of the proposition content, it is quite different in mathematics: epistemic values depend on the status of propositions and not first of all on their content. That means we cannot change a spontaneous epistemic value of a proposition into the value "necessary" by reasoning if there is no comprehension of status as one of the components of the proposition meaning.

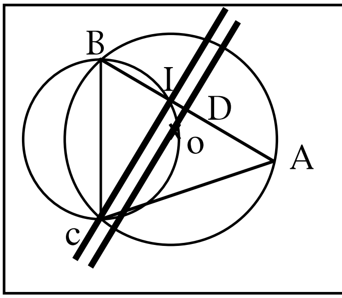
Status and functional differences between propositions within a discursive development (reasoning, argumentation, proof, etc.) Thus, reasoning can be described as being like steps from propositions to other ones, or like a "logical linking" of propositions, like putting forward propositions in order to justify a claim, etc. In order to understand reasoning, we need to perceive the functional differences between each proposition it mobilizes. There is no reasoning without a discursive organization ruled by functional differences between its constituent propositions. We shall call "status" the specific function, the particular role of each proposition within the set of the other propositions which are required or stated to get a proof or to produce an argumentation. For instance, "hypothesis," "premise," "conclusion," "claim" "argument," etc. refer to the possible status of proposition in a reasoning. The status is the third meaning component of a proposition with regard to discursive proposition organization (Figure 4 below). Therefore, we must distinguish status which is intrinsic to any reasoning organization like premise, hypothesis, conclusion, etc. from that which is intrinsic to a theoretical framework like axiom, definition, theorem, conjecture, principle, rule, etc. We call the first one "operative" status and the second one "theoretical" status. Operative status refers to the level of a local proof, that is to an organization of propositions. Theoretical status refers to a higher level of organization, such as an axiomatic linking of local proofs, like the one in the first book of Euclid or in Hilbert's Grundlagen der Geometrie. Of course there are interactions between these two levels. But anyone who does not understand how a local proof works cannot understand why a proof proves, just like someone who cannot understand any page or episode of a book cannot understand the whole book (Duval 2001).

In the classroom, students were trained to write proofs in which they made the status of each proposition explicit, by using three terms: hypotheses, property, conclusion.

The real problem for teaching is that such a lack of discrimination between the different operative status of propositions remains even when there is no longer any superficial confusion or circularity in expressions. Many students, without making any apparent mistakes, do not grasp exactly how functional differences between propositions inside a discourse or an "explanation" work in a mathematical proof. They do not see why and how the operative statuses, and not only the theoretical statuses (definitions, theorems), are tools to develop reasoning in a quite different way from argumentation in natural language. The same incomprehension has been observed with 15–16-year-old students.

The Specific Cognitive Functioning of a Mathematical Proof

There are different ways to step from one or several propositions to another. But they do not all allow a legitimate proof to be constructed. A proof requires a valid reasoning. That means **the conclusion of each step must be necessary and between two steps no gap can be found**. It is this kind of reasoning, often called "deductive," which is used in any geometry proof. So from the previous distinctions we must ask two questions: what kind of proposition organization does such a valid reasoning require and what meaning component of propositions does it bring to the fore?



Hypothesis: (OD) is perpendicular to [AB]
Property: If a straight line is perpendicular to a segment in its midpoint then it is the median of this segment
Conclusion: (OD) is perpendicular to [AB]

Figure 2. Script of a 13-year-old student.

Any deductive reasoning involves two quite different levels of discursive organization: the level of organization of several propositions in one deduction step and the level of the organization of several steps into a proof.

At the **level of a deduction step**, each proposition gets one of the following three operative status categories: premise, conclusion, third statement. Very often, teachers used to say "property" instead of "third statement". But this is a misleading term, because what is called "property" is in fact a theorem, that is a statement which has the *bipartite organization of any IF–THEN rule*: one or several conditions to check and whenever they are fulfilled, an action must be performed or a proposition must be brought out. In this way, contrary to most psychological models (Johnson-Laird 1983, Rips 1988) a deduction step operates quite differently from the classical syllogisms or from the explanations in ordinary speech with the background of semantical networks. And this frequent assimilation corresponds to one of the blind spots for many students. For them using a theorem means only referring to a simple argument, it is not using a bipartitioned statement in order to check the required premises and to assert the conclusion (Bourreau et al., 1998, p. 13, 25). The lack of discrimination between a theorem and its converse is a symptom of this blind spot (Duval 1991, pp. 237–239).

This operative way of using theorems, definitions, or axioms, involves a crucial semantical consequence. The ***links between the propositions*** inside any step ***depend only on their operative status***, which means that connectors (if, then, therefore, etc.) between premise and conclusion are not required. When connectors are used, they are only linguistic cues of the operative status. And this operative status is previously determined by the theoretical status.

At the ***level of step organization, steps are linked by proposition overlapping***: some conclusions of the latest steps are taken up as premises for the next step. It is because of this that there is no gap between two steps. At this level the use of connectors is not relevant. This specific way of linking produces a discursive treelike expansion and not a linear sequential organization.

We can visualize the articulation of these two quite different levels of deductive organization, with their specific kind of links, by the following propositional graph.

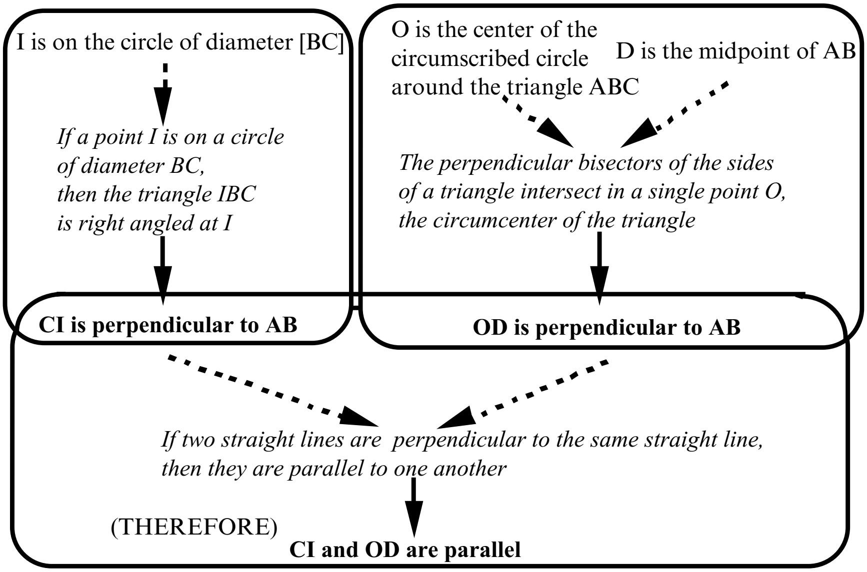


Figure 3. The two levels of deductive organization in any valid reasoning.

A real understanding of mathematical proof requires both comprehension of the operative way of using theorems within each step (roman, italic and bold type in the figure above corresponds to the three operative status categories) and comprehension of steps linking by overlapping each other. Proof construction involves a continual bottom-up and top-down focusing between these two levels, which obviously presupposes a previous awareness of their particular organization.

However, from an epistemological point of view, one very often looks at proofs from a third level: ***the theoretical level***. This level involves a scale change: from the local deduction of a proposition to the global deduction within a set of propositions. Thus, for example, we jump from the understanding of any proposition proof in Euclid’s Elements to the evaluation of the deductive derivation, without gaps or external support, of all the propositions of the first book of the Elements. At this scale we can find some gap in the string of local proofs. But this third level is beyond the capabilities of learners, because it requires, at the same time, that they have already understood how a deductive organization of propositions runs and that they can take into account a set of local proofs! This epistemological requirement comes up against the didactical problem classically known as “hermeneutical circle”.

Shifting the Focus within the Meaning Space

Comprehension of this specific functioning of a mathematical proof requires a change of focus on the predominant component of a proposition’s meaning. According to the kind of discourse (ordinary talk or debate, description, explanation, argument, or valid deduction, etc.) they are not the same features of the different components of a proposition's meaning which are taken into account. For example, epistemic values do not matter within description or explanation, but they are in the foreground with every kind of reasoning. And between argument and valid deduction the difference lies in the role assigned to status. So, in any debate, we can get convincing argument without proving, that is to say without making it necessary to assert a proposition. For such a necessity to exist, status must be prevailing, because functional differences then become the operating process, as we have just pointed out. Generally whenever we change the kind of discourse, we change both the meaning dimension predominant in each proposition and the way the propositions are organized into a purposeful thought process.

In ordinary speech and social interactions the only meaning features activated for any uttered proposition are its content (report) and its pragmatic value of communication. On the other hand, epistemic values become the predominant meaning features in reasoning, because reasoning plays with differences of epistemic values between propositions. And then the question arises of where the different epistemic values come from. From content? From logical values? From status? The cognitive ways of functioning for reasoning are as many as the components and features in the space of meanings of a proposition. So we can easily describe the gap between reasoning as argument and reasoning as valid deduction.

In deductive organization, it is the status of propositions which is the predominant meaning component, rather than their content. The operative status of each proposition is fixed by its theoretical status and therefore its epistemological value becomes dependent on its theoretical value and not on its content. It is almost the opposite of what occurs in an argumentation or an explanation in ordinary speech.

Unlike ordinary speech, reasoning mobilizes the three possible components of each proposition's meaning, but they are combined in quite specific ways for argument and for mathematical proof. Argument does not run like a valid deduction, because content overrides the other components as it does in ordinary speech. In argument or in ordinary speech, the "warrant" does not operate like any other mathematical third statement. No specific operation is required to verify first whether the different clauses of the IF-part of a theorem are met and next to detach the THEN-part as conclusion. A semantical inclusion or a verbal association is sufficient to "draw conclusion or make claim" (Toulmin 1958, p. 98). Moreover, "conditions of exception or rebuttal" are possible too and are integrated in the step to draw the conclusion (p. 101). In fact, within the Toulmin model of the use of arguments, there is no distinction between two levels of deduction organization. For mathematical proof, on the left side of Figure 5, we can recognize all the meaning features which we have circled in Figure 4 above. Thus we can see why the cognitive process underlying the comprehension of valid reasoning required in mathematical proof is not primarily a matter of logic or of formal language as it was assumed in some mental models or in some cognitive researches.

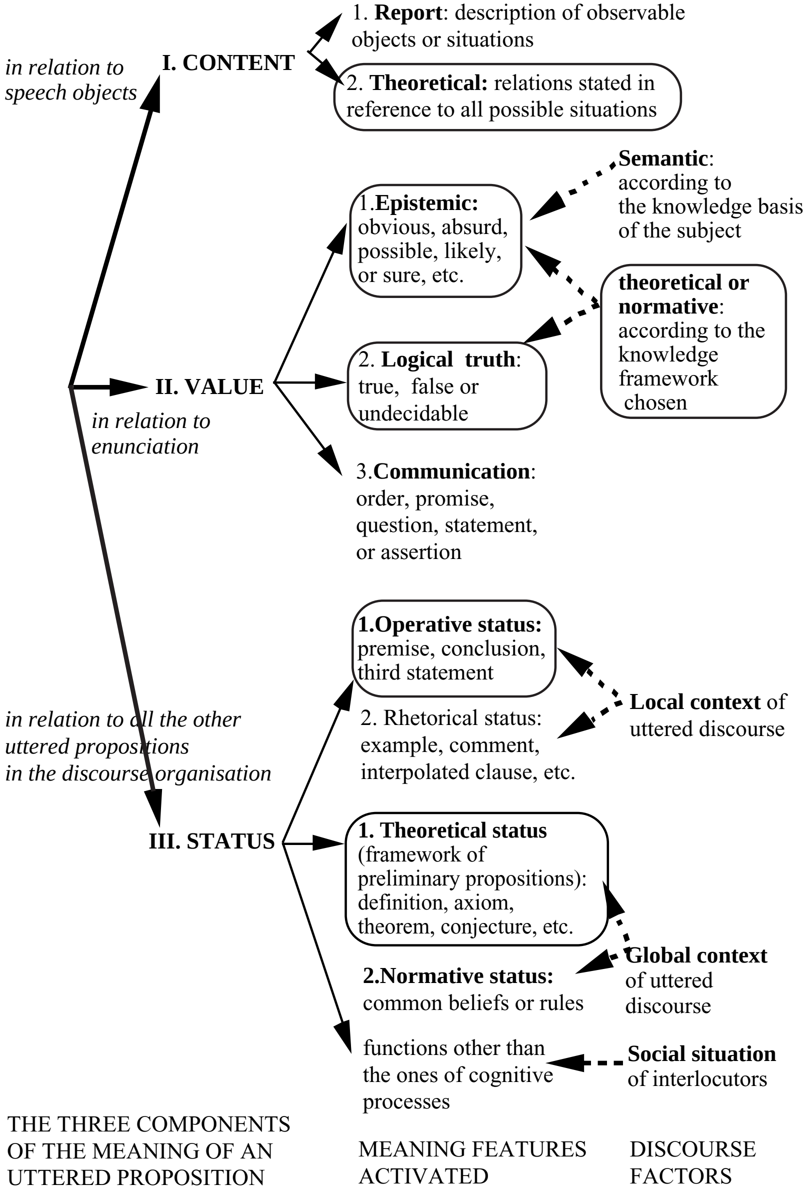


Figure 4. Possible variations within a proposition's meaning space.

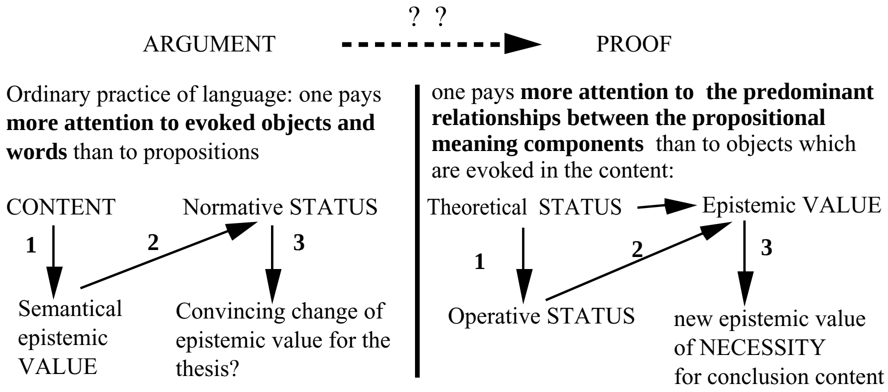


Figure 5. Required shifting of focus on meaning components for understanding deductive reasoning.

This raises immediately the crucial problem from a cognitive point of view and therefore for teaching: does the passage from one way of operating to the other happen naturally? Is it easy for students to notice it or does it require a specific approach to learning (Duval 1993)? Very often teachers believe that this passage is easy because they feel the main difficulty is to discover what "properties" to use, or to get the right ideas (the suggestive images) in order to provide a proof. And they believe that next it is only necessary to explain or to write in as few words as possible. But in fact things happen differently.

Reasoning and Language: Two Kinds of Variation

This is the most controversial topic in mathematics education. We have two opposite claims: "mathematics is independent of any language," "mathematics intrinsically needs some symbolic or representational device for object-processing (computation, visualization, reasoning, etc.) and not only for communication". To support the first claim reference is made to mathematical introspection or to conceptual Piagetian theory, and, for the second claim, the difficulties are highlighted that most students encounter systematically with the variety of the representational and symbolic devices used in mathematics. This debate underlies the choices made in the teaching of mathematical proof. For example, if one believes that mathematics is independent of any language, one can consider that learning to prove lies in solving problems. Then, the basic activity is heuristic: when an answer is found, nothing significant remains to be done, only to communicate this answer. But if we take into account that mathematical thinking involves, even in its mental representation, some semiotic activity, we must consider that learning to prove also requires specific work in order to discover the changes in proposition meaning and discursive organization needed to move from standard argumentation to proof (Figures 3 and 5 above). Either way, there is no valid reasoning without language, because only propositions can be true and because there is no proposition without statements. Therefore the problem of the relationship between reasoning and explicit wording cannot be ignored, especially in mathematics education. What are the interactions between reasoning as specific proposition organization and wording, or expression, as explicit utterance in a particular language?

Two kinds of variation characterize these interactions:

1. At the level of deduction step, the expression in native language can involve many degrees of freedom towards some proposition organization. First, it is not necessary that each proposition of a step should be explicitly uttered in one sentence, or even in one clause. Very often, for economic reasons, some proposition, either a premise or the third statement, is left implicit. Or all the propositions of a step can be uttered in one sentence (Duval and Egret 1993, pp. 127–129). We must utter quickly in order to grasp the whole step organization and to not get lost in the details of each step. There are also many ways to express the status of each proposition. We can use logical connectors, propositional attitudes or even the only succession order of propositions. These linguistic cues are also used in ordinary argumentation and explanation. Sometimes the connective "if … then …" is used in order to point out the premises and the conclusion. But this connective also expresses the entailment relation, that is an organization which is intrinsic to a proposition and not to the step organization. Thus these free variations can generate very different texts for the same proposition organization and can be a source of misunderstanding for learners. And all the more so as great variations can be observed from one teacher to another. Hence this understanding problem: how can a learner distinguish between valid and not valid reasoning, if the expression of both has a similar surface structure?
2. Other kinds of representation registers are also used for expression in mathematics: formal or symbolic language, networks, configurations, etc. Technically formal languages and algebraic symbols are more powerful and rigorous than native language, and in some areas are essential. In these registers proof is performed through computation and proof methods can be described. That is why proving in these more technical registers can appear less complicated than deductive reasoning in a native language. But it has often been noticed that students lose the meaning of the processes and operations they are performing within these registers, because, for them, there is often no coordination between these different representation registers (Duval 1995a, 1996).

It is through native language that learners can become aware of what is required for and what produces a mathematical proof. For a simple reason. Reasoning in a native language requires that one takes explicitly into account the status and the epistemic value of every proposition at once! It is only in this way that reasoning can work as a real reasoning for a subject, that is to say as a convincing reasoning. In contrast to this, in computation the important thing is to focus on the rules of use and substitution for every symbol (variables, quantifiers, operators, relations, etc.). Status and the epistemic value of symbolic expressions do not matter. From a cognitive point of view it is the deep difference: computation is easier than reasoning. And it is the reason why the claim that reasoning is "nothing more than computational processes" (Johnson-Laird 1983, p. 12) is false from a cognitive point of view and useless from an educational point of view.

HOW CAN THE PROBLEM OF LEARNING TO PROVE BE STATED?

The cognitive processes which make a student able to understand how a mathematical proof works, and able to prove, depend on a double awareness: one concerns the discrimination between different causes for the feeling of necessity which can be experienced and the other concerns the discrimination between different organization processes in a discursive development. But this double awareness goes against two familiar and common practices.

Becoming Aware of the Discrepancy between a Valid Reasoning and a Non-Valid Reasoning: A Shift in the Common Epistemological Practice

Everybody knows that the feeling of necessity has been the main line of Piaget's research into the child's cognitive development. It is a change in the sensitivity to new kinds of causes in the child's experience of necessity which indicates the stages of concrete and formal operations. But one knows also that reaching the formal operations stage is induced from spontaneous experimental activity (dissociation or combination of parameters in application of the principle of the control of variables: to make a variable vary while keeping the others constant (Piaget & Inhelder 1955), and that is far from being sufficient in order to understand mathematical proof. And since the goal of proof is not only to find out further information but also to change the epistemic value of the information stated in a proposition, we cannot avoid the question: what makes the necessity of asserting some proposition?

That question is not a matter of logic, but it is a matter of the subject's cognitive structures: what are the prerequisites for sensitivity to what makes a mathematical proof? In order to find these prerequisites we must start from these two requirements:

1. only a valid reasoning can produce the necessity of the utterance of a proposition;
2. in mathematics, truth can be only connected to an intrinsic discursive derivation of this epistemic value.

But, and here lies the rub, we have different possible causes which drive a person to recognize the necessity of an utterance, and therefore different kinds of necessity meaning.

Three quite different experiences leading to awareness of necessity The first experience leading to the awareness of the necessity of some proposition is that its content corresponds to sensory data, perceived with or without instruments. There we can check what is said by looking at what can be perceived. That is the common epistemological practice. In this way, the best common proof is direct observation: "regardez, ça se voit sur le figure!," "look, it is obvious from the figure!". Nothing else is needed. For everyone this practice is the most natural and it is difficult to understand why it cannot be used in geometry as, for example, in botany, geology, etc. Here, the cognitive roots of necessity are extrinsic and not intrinsic to reasoning. Here, the roots of necessity and conviction are in experience, and reasoning must work as an accurate description of observed relations. It is what Leibniz called the "physical necessity" as opposed to "logical or geometrical necessity" (Leibniz 1969, p. 51; Piaget 1967, pp. 60, 188).

The second experience leading to the assertion of the necessity of some proposition lies in the fact that others agree to its truth. This cause can be strong enough to change the individual judgement in a group as experiments have shown: each subject changes his or her estimation of the autocinectic illusion when he or she is informed of the others' estimation. Here the roots of the feeling of necessity are in the normative regulation of interactions between members of any group: each one must reduce divergences and conflicts in order to keep the group's cohesion or their own individual integration. What is thus acknowledged becomes a consensual necessity.

The third experience occurs whenever one sees that uttering some proposition is the only possible conclusion from what has previously been asserted, even though it goes against perceptual evidence or general agreement. But here teaching can lead students astray. When teachers emphasize problem-solving, what is highlighted is the search for appropriate theorems to use in the proof. In this case, the meaning of necessity can be tied to the use of such or such a theorem to solve a given problem: in this problem, one "must" use these theorems as tools. Here it is only a "methodological necessity," because we can find other mathematical ways to solve a problem: such a necessity concerns only what is relevant in order to get the solution when we take up some theoretical framework. It does not concern the way a theorem leads to necessarily uttering one proposition as conclusion. Here we get an intrinsic discursive necessity: whatever the used theorem, from what has already been said and agreed there is no choice but to utter this proposition. Instead of "logical necessity" we prefer to call it "discursive operating necessity". It is on such a necessity immanent in thinking that any theoretical explanation can be developed or derived. But this kind of necessity can remain hidden to students, even when they quote the relevant theorems, if they have never had the opportunity of engaging in a specific activity to realize it.

A functional classification of proofs In the first systematic investigation of mathematical proof from an educational point of view, Balacheff distinguished between four kinds of proofs (1987, pp. 163–166; 1988, p. 55): naïve empiricism, crucial experiment, generic experiment and mental experiment. The two first ones are pragmatic proofs because they focus on the observation: "it works". The two last ones are intellectual because they aim at the "necessary character" of assertion. Between the pragmatic and the intellectual proofs, there is a "break" (1988, p. 55). A distinction between two kinds of explanation must be added to the distinction between four kinds of proofs (1987, pp. 147–148; 1988, pp. 28–30) according to the fact that intellectual proofs require both “language tools” and the control of possible contradictions. Thus we get the distinction between “proof” as “an explanation acknowledged by a community … in relation to a validation system common to interlocutors” and “demonstration” (apodeixis) as “a sequence of statements organized according to specific rules”.

This classification corresponds to the different kinds of necessity which one can get experience of. If we exclude naïve empiricism from this classification, because it is confined to the obviousness of any immediate perception, we can notice that “crucial experiment” rests on “physical necessity,” “proof as an acknowledged explanation” rests on “consensual necessity,” and “demonstration” rests on “discursive operating necessity”. Such a classification is mainly functional and it leaves the means and the process for proving aside. However, these means and processes not only depend on the kind of proof but they change according to the area of knowledge too.

In the framework of such a functional classification, the issue becomes one about the cognitive passage from one kind of proof to the other. In a Piagetian way, the hypothesis of a cognitive “hierarchy and direct line” between the kinds of proof is put forward (1988, pp. 565–566). But the expected passage from “mental experiment” to “demonstration” (mathematical proof), or from a social interaction to demonstration, raises difficulties (1987, p. 166; 1988, pp. 451, 461). Why?

Becoming Aware of what is Specific in Valid and Creative Organization of Propositions: A Shift in the Discursive Practice of Speech

In any debate, in any discussion, and, more generally, **whenever social interactions are oral interactions**, we never argue in the way which is required by a mathematical proof. Oral social interactions run according to a quite different organization of propositions than that of a sequence of statements according to rules of valid reasoning (Duval, 1993, 2001).

We have highlighted above that deductive reasoning joins two levels of discursive organization and in particular that the way of functioning is different for each level. At the first level there is a reversal in the usual predominance between content and status of propositions (Duval, 1993, pp. 44–45). But this reversal is difficult to realize, because it involves an implicit substitution: the theoretical epistemic value must repress the pregnant semantical value! For a learner, the right and significant use of any theorem depends at first on the awareness of this reversal. Otherwise, steps are understood as binary organizations in which one does not need to check the premises in order to apply theorems. Hence, among other significant mistakes, there is the risk of making circular arguments without noticing them or of confusing a theorem and its converse (Figure 6). At the second level, the linking between two steps is based on the explicit or implicit re-using of propositions already uttered as conclusions or as given hypotheses, but with a change of status from one step to the next, what we have called a “recycling”. Therefore reasoning can move forward from one conclusion to another conclusion without a gap. Hence another quite different failure: proof is not really a proof because there is a gap which has not been noticed.

All that which concerns the cognitive conditions of comprehension is in bold type (right hand column in Figure 6). All these kinds of misunderstanding are connected with the organization specific to a mathematical proof and its way of running.

	The functioning of valid reasoning	Kinds of misunderstanding
I. Organizing propositions into a deductive step by taking into account three kinds of status	(1) A shift of focusing about what is taken as the first component of the meaning of proposition: status instead of content. (2) Making the theorem work: detachment of its "then" part. A theorem is not an argument.	Dysfunctional: - **Mixing up** hypothesis (given) with conclusion, - **Confusing** a statement with its converse, or inverse -**Non-checking** of application conditions of theorems
II. Organizing deductive steps into a proof of …	Making deductive steps overlap. Two conditions: (1) The conclusion of one step must be the premise of another one. Hypothesis and premise do not refer to the same level. (2) Use all the mathematical properties relevant to the problem.	**-No discrimination of the mechanism of substitution** **Gap in the progress** of proof: - No perception of all the constraints of the problem to be solved.

Figure 6. Indicators of incomprehension of the "demonstration" organization.

At the surface level of natural language valid deduction cannot be differentiated from spontaneous argument The deductive organization of a proof and its specific way of functioning are not visible through explanation in natural language. For example, the two levels of deductive organization are evidently mixed up in the linear surface expressions. But the spontaneous way of wording obscures this. Two features characterize the spontaneous way of wording. First, the speaker describes what he or she has seen or has made, by making explicit only what he or she was planning over its action or what he or she noticed. And the speaker is led by associations, which are often activated and guided by semantic networks. In this spontaneous way the speaker focuses only on the outcomes of his or her operations whereas mathematical proof requires focusing on discursive operations which the speaker is performing. These discursive operations cannot be confused with wording, although this is very often done in the field of mathematics education through oppositions between concepts, or mental representations, and language. Discursive operations are operations which are not directed towards the objects but towards the different possibilities of naming objects, to assert propositions about objects and, mainly, about the multidimensional space of meaning opened up by propositions. In this spontaneous way of wording, taking into account the status of propositions can only be of no significance.

We can see, then, why wording a mathematical explanation does not depend on the same cognitive processes and does not correspond to similar short cuts in a mathematician and in a young learner. When the wording is used by a teacher or by a mathematician there is the concern to make the top-down and bottom-up focusing as fluent as possible, while also taking the theoretical level into account. In these circumstances there is little chance for any young student to discover what an intrinsic valid discursive production is. Even though one asks students to state explicitly the status of propositions by naming them (hypotheses, property, etc.) or by using connectors, it could be only a screen. The specific organization of a deduction step can only be realized through the articulation of the two different proposition organizations.

Finally, outside mathematics, the only really conceived organization of propositions for a step is a binary one and not a ternary one. We have either a statement and its justification as the presentation of an exhibit or the word of a property and its natural derivation as a semantic inference. But there is no differentiation between different kinds of organization of propositions, that is between different kinds of structures for the process of reasoning and proving. So it is amazing to see that mental models of thinking refer to the classical syllogisms taken as patterns of deduction (Johnson-Laird 1983) whereas these syllogisms have a binary organization and work as semantic inferences, without any theorem or another third statement (Duval, 1995a, pp. 238–241, 251–255)!

The key point: a complete reversal of the predominant component meaning of propositions We have highlighted above (1.3) that the kind of discourse organization depends on the predominant component and features of the propositions' meanings uttered within the discourse. Being able to discriminate a mathematical proof from an argument under similar wording, or under the same verbal marks (grammatical and logical connectors), involves a reversal of focusing about what is taken as the first component of the propositions' meanings: their status instead of their content. Becoming aware of this reversal is the condition for understanding how a mathematical proof runs and what changes it brings about in knowledge.

Now we can remind ourselves that the different kinds of proof can be classified according to the various experiences of necessity they are based on. We must add that such experiences require specific means and processes and cannot be dissociated from them. *In other words the jump from an experience of physical necessity, and/or from experience of consensual necessity, to an experience of "logical or geometrical necessity," is a change in the kind of proof.* This change involves a structural break in the way of reasoning because "logical or geometrical necessity," which is in fact a discursive operating necessity, can only be experienced in the understanding of a valid deduction. In order to realize that a valid reasoning produces the necessity of the utterance of propositions as conclusion, a shift in the focus of attention is required, but this shift goes against the common epistemological practice: the necessity of asserting some proposition cannot come from experience as is usual but from an intrinsic valid discursive production, which is not the case in the other fields of knowledge.

This whole analysis raises a crucial educational issue: what is the implication of this structural reversal for the introduction of proof within mathematics teaching and curriculum? We have quite opposite alternatives. N. Balacheff emphasized the importance of teachers themselves taking charge of all that concerns the status of propositions uttered during a debate (1988, pp. 450, 462, 531). In the same vein J. R. Anderson put forward a geometrical tutor in which students did not need to take into account the status of propositions, and where the construction of a proof graph focused mainly on "subgoals" (1987, pp. 113–117). In this way students cannot be faced with the possibility of dysfunctional misunderstandings (Figure 6 above). In the opposite alternative, making students aware of the decisive role of status becomes a decisive objective for teaching. Thus V. Luengo integrated this into the interplay between students and the Cabri-geometry tutor (1997).

The distinction between different kinds of proof raises *the educational issue of the cognitive passage from one kind of proof to another, and primarily from argument within a social interaction to a mathematical proof*. It is a deep change in the kind of proof because mathematical proof calls for the experience of a quite different kind of necessity. Such an experience cannot take place or be discovered within oral interactions. And being asked to write about what has been explained in any debate, in order to make explicit the status of the uttered propositions, is of no use (Figure 2 above). We can now state the problem of proof learning: what factors must be brought into play to make the students experience a shift both in their discursive practice of speech and in their common epistemological practice, and thus make them achieve the double awareness? That is, this double awareness which is the intrinsic source of conviction and the real heuristic guide. The factors must depend on the subject's cognitive architecture and must match with the basic conditions for mathematics learning.

HOW TO LEAD STUDENTS INTO THE COGNITIVE FUNCTIONING OF DEDUCTIVE REASONING?

This way of stating the requirements for learning to prove leads us away from the classical conceptions on this topic. Thus in order to learn to prove many teachers believe it is necessary and sufficient to learn various proof methods (reductio ad absurdum, division of possible cases, etc.) or various mathematical ways to prove a proposition (geometrical, vectorial, analytic, etc.). And if it is not quite sufficient, they believe it is what matters most. We do not contest that. But it requires the acquisition of several capabilities. For example, producing different mathematical proofs of a proposition requires different mathematical frameworks and to change the framework very often involves a representation register change. But here we come up against a well-known difficulty: the different representation registers remain for most students isolated one from another. Moreover that does not resolve the initial and basic problem of discriminating a valid deductive reasoning from a non-valid one, mainly in native language. Another classical conception emphasizes the research activity about stimulating problems. Here, it can be difficult to discriminate between the ways to make a conjecture and those to prove or to refute a conjecture. These classical conceptions, which refuse to stand back a bit from the mathematical processes, lie beyond the real problems of learning to prove.

The Need of a Detour in Order to Respect the Two Basic Conditions for Mathematics Learning: The Differentiation and Coordination between Semiotic Representation Registers

When one examines closely the mathematics learning of 10–16-year-old students, a fact is always compelling: many students do not think to perform, or do not understand how to perform, the different actions which are required to solve a problem, or to apply some already-acquired knowledge even though the asked tasks can seem simple, obvious, natural to teachers and to mathematicians! What seems simple or natural for the fulfillment of any mathematical activity involves in fact an implicit complex differentiation and coordination of semiotic representation registers in a way which is not generally required in other fields of mental activity (Duval, 1996).

Learning mathematics occurs through the construction of a subject's cognitive architecture, that is never, or too rarely, achieved as the outcome of learning such or such content (concepts, algorithms, or even ways of representation as graphics, numerical systems, etc.). In other words, understanding does not follow the order of mathematical construction of knowledge, but supposes the development of some specific skills, which are also fruitful for other fields. One cannot teach mathematics, at the lower level, without taking into account the basic requirements to develop the subject's cognitive architecture. And that is particularly true for mathematical proof.

According to the mathematical field and according to its elementary or complex character, a proof can be constructed in native language or can require specific notations of formal language, for example the use of quantifiers. That is one of the two great kinds of variations in the interactions between reasoning and language, as discussed above. First, if we confine ourselves to native language, we have just seen that two quite different discursive expansion processes generate different deep proposition organizations. And that is not always visible through their surface expression. One cannot imagine a reliable teaching of proof which avoids having *these two uses of native language differentiated by students*. However, in some cases, reasoning depends on using a symbol system in order to make explicit the extensional aspect of sentences: connectors of negation, of material implication for propositions, universal and existensial for variables and predicates (Carnap, 1958).

Here we change semiotic representation registers for the discursive processes: something else becomes necessary besides the understanding of how to use if–then rules for an operation of detachment. But if a mathematical proof can play with different semiotic registers, it remains the case that the specific way discursive expansion works must be discovered. And for symbol systems or formal language, the understanding of how a proof functions *requires too a coordination with the native language* (Duval, 1995a, pp. 151–155).

In order to lead students into understanding how a proof works, mathematical activities must be organized or split into three stages: a first stage of free exploration, a second stage of specific investigation into the deductive organization of propositions in a non-discursive register, and a last stage of verbal description or of verbal explanation of the deductive organization which has been discovered. This amounts to split first on what is ordinarily considered as heuristic activity, or a matter of intuition, and then on what is considered either as a logical activity or a communication activity. Why is there such a double splitting and what awareness does it make possible for students and also for teachers?

The Variable Triggering this Double Splitting: To Change the Representation Register of Working

Gathering together the relevant properties or theorems for proof construction The first stage of free exploration is standard. Very often this exploration takes place in small groups. That can help the many students who do not succeed in discriminating the relevant "properties" and theorems to use, or even realize why some theorem is relevant and another is not! But it is not always sufficient and a general confrontation is necessary in order to make the key ideas emerge from the various productions of each small group. Then, all seems nearly over, since there is nothing left, except for producing a written record of the proof. That can be true from mathematical perspective. But it is deceptive from the learner's perspective. Knowing all the theorems which are to be used in a proof does not help the learner to understand why they prove and thus gain insight into why a proposition is true and thus become really convinced. The real usefulness of this stage is to make the students enter into the problem and to provide them with the relevant "properties" as data for a specific research into the deductive organization of propositions. We are at the starting point.

Research into the deductive organization and its functioning If we confine this to the field of elementary geometry, the mathematical activity in this first stage is carried out in the mixing of two registers: the geometrical figure register in order to "see" and the natural language register in order to "explain" (most often in an oral way!). We must remember that theorems and definitions, which are uttered or formulated in the natural language, do not work as statements in ordinary practice. This mixing is very often inextricable for most students. So a third register appears necessary in order to make visible the discursive operations involved in organizing propositions into a deduction.

The most natural one might seem to be the graph representation register. At least, it has often been put forward in psychological or didactical research, under various names. But this register has no more value than the others: it is a blind alley. What really matters is what this register is used for and what the student has to do with it. Briefly, **propositional graph construction** can be **undertaken from a heuristic perspective**: the propositional graph is used in order to trigger forward and backward processes (Anderson et al., 1987; Rips 1988), and what the student has to do is only to find the “path” between the hypotheses and the conclusion by choosing the relevant theorems. In this case the frame of the propositional graph is already fixed, since hypotheses and conclusion are already placed at the top and at the bottom of the screen: the task is focused on the choice of the relevant theorems to find out the links. That means that the task of taking account of the status of propositions becomes a dormant activity which disappears from view: the student has only to choose the relevant theorems, that is to go no further than the content of propositions.

But a propositional graph construction can also be **asked in a deductive organization perspective**: graph representation construction is used in order to discriminate the status from the content of propositions and also to differentiate between the use of theorems from natural argumentative justifications or from physical explanations, etc. In this case no frame is fixed beforehand! Then what the student has to do is to choose the propositions according to their status in order to construct the whole graph, since the relevant theorems are known from the first stage. The student only has to cope with three rules of construction which deal only with the discrimination of status. These construction rules focus exclusively on the representation of a proposition’s status:

1. From an **hypothesis**, an arrow starts but ***an arrow can never arrive***.
2. One or several arrows arrive at a **theorem** but ***only one can start from it***.
3. One or several arrows arrive at the **target conclusion** (what is to be proved) and no arrow can start from it.

To construct a graph which represents how the use of theorems solves the problem, students have only to use arrows ***in order to link two statements according their status***. Through this task, which rests on a change of representation register, the awareness process is started.

In every teaching experiment, one finds the same evolution in the behavior of students and the same deep transformation in their production over several didactical sequences. First of all, students can be disconcerted by this kind of task, but above all they seldom succeed. *All the latent misunderstandings about mathematical deduction, often hidden by linguistic formulations which are neither false nor precise nor explicit, appear*: confusion between hypotheses and conclusion, between a theorem and its converse or, more subtle and deeper, confusion between class inclusion (natural part–whole relation) and propositional implication, non-taking into account the conditions that apply to a theorem, reduction of reasoning to only the linear linking of sentences through connectors, unmindfulness of the possibility of gaps, etc. For example, among the first productions of students we have found graphs of this kind, which corresponds to a wrong understanding about the mathematical organization of proof:

Hypothesis. 1 → Hyp.2 → Hyp. N → Theorem A → Theorem B → Conclusion

The construction rules provide students with the means to construct the propositional graph and also to check for themselves the validity of their construction, but, above all, they help students to become aware of where and why they were mistaken (Egret & Duval, 1989). All these misunderstandings appear through the constructed graphs and become obvious just as easily to students as to teachers. And it is at this moment that students become aware of the specificity of deductive reasoning and can begin a true investigation of how a propositional deductive organization works through new graph constructions. And for the choice of proofs, the teacher can make use, through graph constructions, of variations in the complexity of the organization: the proof is more or less arborescent, the givens are, or are not, needed only for the initial steps, etc. (Bourreau-Billerait et al., 1998).

A new representational situation for wording The third stage can begin only when the students can organize a whole propositional graph, that is to check themselves the validity of links, according to each proposition status, and to realize what is a gap in a proof reasoning. Then the teacher can ask a second change register: to describe or to explain the propositional graph they have constructed. It is a quite different cognitive situation for wording. There is a shifting in the data reference for wording: it is no longer the geometrical figure as in the first stage but the discursive organization that is represented by a diagram. Thus a step back from the visual obviousness of the geometrical figure is created by the way of this transitional representation. Now wording makes students become aware of the epistemic value of propositions and, above all, of the epistemic value transformation which is carried out through the deductive reasoning: what was only visually obvious, or what seemed not possible, is becoming theoretically necessary. This is not the place to speak about the processes of this new awareness (Duval, 1995a, pp. 223–231). What matters is perhaps the following question: why resort to natural language for that? For two well-known reasons. First, it is only in natural language that epistemic values can be uttered. Second, it is only in natural language that a subject can become aware of what is involved in his or her activity (here, the propositional graph construction) as Piaget (1967) explained. Understanding, in mathematics learning, cannot be truly reached through the exclusion of natural language.

For this third stage one also finds an evolution. When confronted by the daunting sight of the deductive organization they have found, students may write more than necessary and thus their proof texts seem wordy. But gradually their expression becomes more concise and they do not feel the need for the propositional graph construction. Then students have attained ***the ability to master a valid deductive reasoning*** which is more complex than any simple syllogism (besides, most syllogisms are not valid reasoning, as Aristotle (1964) has explained at length!).

What is Aimed at through this Double Splitting: A Register Coordination

Operations which appear simple from a mathematical standpoint are very often like submerged summits. Below we have the synergy between several heterogeneous cognitive systems, some of them requiring a transitional specific practice. In other words, what is simple does not lie at the start of learning sequences but at the end. Hence what we have called the needful cognitive detour.

In the field of elementary geometry, proof requires the coordination between two representation registers: the geometrical figure register in order to "see" and the natural language register in order to "explain" (most often in an oral way!). The introduction of a third register, for a temporary detour, seems to highlight the discursive presentation of proof to the detriment of its exploration and construction which are often reduced to the first stage. But nothing of the sort happens; rather it is the opposite. In this way students get to truly discriminate the discursive apprehension of a geometrical figure (through given hypotheses, definitions, etc.) from its merely perceptive apprehension and realize the priority of the discursive apprehension over the perceptive one. In other words, they gain a framework to guide their investigations in the field of geometry: there is no real figural intuition without some deductive basis. With that perspective the introduction of a third register supports the development of a coordination between the geometrical figure register and natural language. When such an explicit coordination begins, students feel released from mental mutism, which can bring about an irreversible lack of interest in mathematical problem-solving or even a complete withdrawal in many young students.

But if we want also to develop an intuition skill, mainly for the first stage, when the aim is to find the relevant properties or theorems needed for constructing a proof, or even for problem solving, specific training is likewise required in the geometrical figure register. For example how does one find the relevant theorems for solving the problem given above (Figure 1)? Perhaps we cope here with some "hermeneutical circle" about the role of the figure in finding the relevant theorems.

Each of these three subfigures corresponds to one of the three theorems used to construct a proof (Figure 3). If these subfigures are needed to make properties come to mind, how can students discriminate and recognize them in the starting figure? If, on the other hand, the theorems are needed to see the subfigures, then what are the subfigures used for? This raises the more global issue about the cognitive interactions between visualization, construction and reasoning which are involved in any geometrical activity (Duval, 1998, 2005). And from this more global viewpoint proving in geometry requires the ability to activate fluently either statements or their figural representations. But most students cannot untangle this complex and hidden interplay, which is completely unconscious or automatic for mathematicians!

Whatever geometrical task (analyzing figures, proving, construction, etc.) is commonly given to students, there overlaps a very broad spectrum of heterogeneous processes. Learning in geometry requires tasks that are designed to make students discriminate and practice each of theses heterogeneous processes. Thus it is the same for geometrical figures as for natural language: the mathematical way of looking at a figure, or of describing its construction, differs from the perceptual way of looking at and interpreting it. In a figure belonging to a given task, there are different factors which trigger or inhibit the visibility of the pertinent subfigures which show the key ideas for solving the problem. Students must become aware of the play of these factors in any geometrical visualization too (Duval, 1995b).

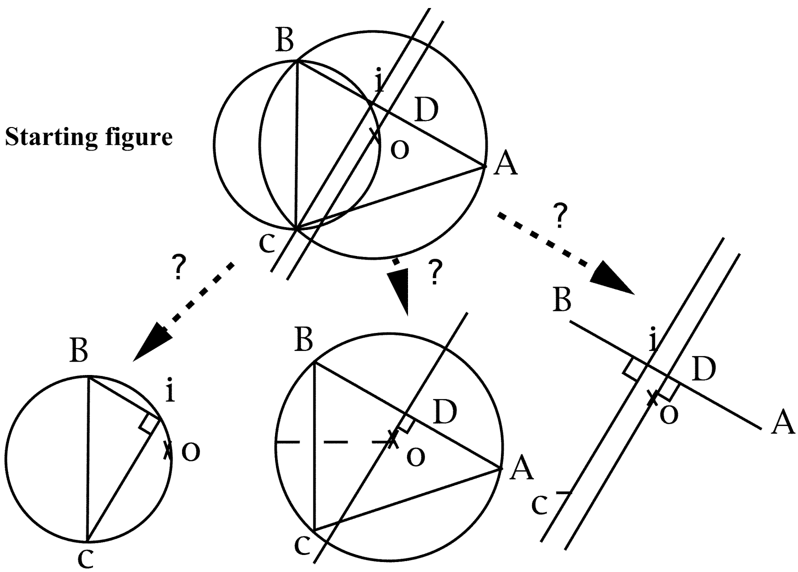


Figure 7. What are the cognitive conditions for a heuristic role of the figure?

Of course, the question is whether there is an order of acquisition or some hierarchy in the “skills”. Unlike others models which assume this, we can see that there is no order of acquisition between visualization and reasoning, since geometrical activity is based on a synergy between various cognitive systems which must run in parallel. And it would be an illusion and cul de sac in mathematics education to promote one register as being easier than all the others. Comprehension of geometrical activity mobilizes, implicitly or explicitly, several representation registers and emerges in the same time as their synergy.

CONCLUSION

In mathematics education, understanding as well as learning must be examined not only from a mathematical point of view but also from a cognitive point of view, because there may be a discrepancy in the conditions of understanding between one point of view and the other: what can appear simple from one point of view can hide a true complexity that is visible from the other. In this chapter we have highlighted the underlying complexity of the cognitive processes for the steps involved in learning to prove, even for mathematical situations which seem easy because they seem close to natural perceptual situations and do not require technical tools or a specific representation register (logical symbolic notations, algebraic writing, etc.). It can appear like a complicated detour. But the nature of students' recurrent mistakes and failures highlights the necessity of such a detour. We have distinguished two kinds of failures:

1. Dysfunctions in valid reasoning, such as status confusion, non-distinction between a statement and its converse, etc. They can be explicit or remain implicit, hidden by omissions or by clumsily expressed explanations or even by clumsy formal presentation of proofs.
2. Gaps or deficiencies in the progress of a proof: some can be obvious and easy to detect while others may require close scrutiny.

We must add to these failures, this well-known and widespread behavior:

3. Mental block and mental mutism in response to being asked to construct a proof which can lead students to withdraw from any proof activity or to develop a greater or lesser aversion.

Mathematicians and teachers focus mainly on the second kinds of failure (2) because these reflect the complexity of mathematical properties and objects. From this point of view the difficulties can change with each mathematical situation: also, it seems possible to find situations or problems where proofs are within the reach of everyone. And from this point of view, one tries to overcome mental blocks (3) by suggesting key ideas. In contrast to this, we have focused on the first kinds of failure (1) because they are persistent whatever the mathematical problem given to students! As long as students remain unaware of the specific way deductive reasoning runs, they cannot go beyond latent dysfunction and therein lies the true deep reason for mental blocks. Moreover, a proof cannot work as proof as long as there is no comprehension of the specific deductive organization of discourse which determines even the mathematical way of defining.

The first advantage of the double splitting and of the change register is to make them apparent before the eyes of students and teachers. The second advantage is to provide a tool to reveal what lies beneath seemingly natural ways of wording and visualizing.

The issue here is not to oppose mathematical and cognitive points of view in mathematics education, but to articulate them. One can learn to prove only in mathematical situations. But one cannot learn to prove if the learning situations are not organized according to the cognitive variables. Each time these variables have been taken into account, students have experienced an overstepping in their practice of reasoning and research.

The significance of elementary geometry for discovering what is a mathematical proof is due to the fact that it *mobilizes two multifunctional registers*: the one of natural language and the one of gestalt configurations. In this way what is at stake first of all in proof learning is to discover that reasoning in mathematics does not function in the same way as reasoning within discussion that aims to convince other people, outside mathematics. Moreover, becoming aware of the functioning of valid reasoning is absolutely essential **whenever deduction has to compensate for the limitations of vision and visualization**. This is the case, for example, for reasoning ad absurdum (and for three-dimensional geometry, where the support of figures shows itself to be more complex and more limited than in plane geometry). It is within their practice of speech organization that students can truly experience original change and strength of mathematical proving.

The use of quantifiers is distinctive of discursive registers and cannot be considered separately from the **use of negation**. Omnipresent in natural language, but often implicitly, they become explicit in formal language. But difficulties of reasoning with quantifiers in relation to negation (no language without negation) arise within the monofunctional register where the treatments are those of predicate calculus. Here we are facing a specific problem of learning in order to make students both connect and disconnect the ways of referring to objects and to quantify within both natural and formal languages. And that is especially needed since statements in calculus (for definitions, theorems) employ a mixed use of natural and formal language. But such a learning of quantification can be meaningless for students who have not yet realized what valid reasoning is and how it functions.

It is obvious that proofs in most fields of mathematics are not founded and developed as in elementary geometry, because one does not work with the same representation registers, i.e., with geometrical figures and natural language. So what can the contribution of this learning be to the general mathematical education of students? Two experiences seem basic for further learning. First, the discovery of what valid reasoning is, which is as important as accuracy is in computation. Second, the awareness of different ways of working with natural language and with configurations of gestalts. Natural language and gestalt representation are not technical representation registers in mathematics. But no technical representation register can be introduced in mathematics education without a coordination with one or other of these two primitive registers, in order to highlight similarities and differences, congruence and non-congruence, in the ways of referring to objects and of processing information.

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7. SOME REMARKS ABOUT ARGUMENTATION AND PROOF

INTRODUCTION

The general motivation for this study comes from the need to call into question the idea, widely shared among teachers and mathematics educators, that there are profound differences between mathematical thinking and thinking in other domains, and that these differences produce many difficulties in learning mathematics. In particular, some mathematics educators think that one of the main difficulties students face in approaching mathematical proof (one of the most characteristic and important mathematics subjects) lies in their inability to grasp the differences between ordinary argumentation and mathematical proof. This position has been clearly presented by Duval (1991):

> Deductive thinking does not work like argumentation. However these two kinds of reasoning use very similar linguistic forms and propositional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proof.

I will refer to that article, since Duval's analysis offers a precise cognitive perspective for "formal proof" (i.e., proof reduced to a logical calculation). And this is very helpful to widen the analysis to the various activities involved by mathematical proof. It suggests the following questions:

- What are the relationships between formal proof and proofs really performed in mathematics, in school mathematics as well as in the history of mathematics and the mathematics of modern-day mathematicians?
- What are the relationships between mathematical proof (as a written communication product), and the working mathematician's process of proving?
- In spite of the superficial analogies and profound differences between argumentation and formal proof, aren't there some deep connections between argumentation and mathematical proof (as products and as processes)?
- If those connections do exist, how can we take them into account, in order to manage the approach of students to mathematical proof?

In this chapter I will try to explore only some aspects of these questions and show their relevance for the "culture of mathematical proof," which should be developed in teacher training and also for some direct educational implications. In particular I will try to show how proving and arguing, as processes, have many common as-

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 163–181.

pects from the cognitive and epistemological points of view, though significant differences exist between their outcomes as socially situated products.

Regarding the process of construction of proof in school context, we meet the problem of recourse to "meaning". This problem is also part of mathematicians' concerns since the beginning of the last century. Meaning was negated as a possibility or even a necessity by some mathematicians, see Whitehead (1925) for instance. According to Whitehead, mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about.

But other mathematicians did not deny its importance, like Hardy, who says that a formal proof is a kind of X-ray picture of an actual or possible piece of reasoning, revealing the bones [the form] but making the flesh [the content, the meaning] invisible (Hardy, 1929).

I will develop the analysis of the need for meaning amongst mathematicians by referring to more recent positions (specially by Thurston and Lakatos). I will also show the need for semantically rooted arguments in students' production, and argue that this need is part of the mathematical activity. This position results from my interest in Duval's hypothesis that argumentation relies on need for meaning. I came to compare proof and argumentation from this perspective. Particularly, I compared the cognitive activity involved in the process of searching for a proof and elaborating it, on one hand, with the process of elaborating an argumentation, on the other.

My analysis will be mainly inspired by Thurston (1994) as concerns modern-day mathematical proofs and the way mathematicians debate them. I will also refer to Lakatos (1985) as concerns definitions and proofs in the history of mathematics; Balacheff (1988), Mariotti et al. (1997), Arzarello et al. (1998), Arzarello (2000), Bartolini et al. (1999), Simon (1996), Boero et al. (1996) and Harel and Sowder (1998) as concerns some epistemological, cognitive and educational aspects of proving; Lakoff and Nunez (1997) as concerns the idea of everyday experience as "grounding metaphor" for mathematics concepts; and Granger (1992) as concerns the relationships between formal proof and verification in mathematics.

The theoretical construct of Theorem, by Mariotti, seems to be appropriate to frame theorems in this chapter. According to her (see Mariotti et al., 1997) a "theorem" is a statement, its proof and the reference theory—distinguishing between axioms, definitions and theorems of the specific theory in play, on the one hand, and general meta-knowledge about proving and theorems, on the other. In the same perspective I will consider "Cognitive unity of theorems": this theoretical construct of Garuti's (Garuti et al., 1996, 1998) concerns the links that can exist between the activity of conjecturing (especially as concerns the production of arguments for the plausibility of the conjecture) and the activity of proving.

I will consider how argumentation and mathematical proof "live" in different settings, today and in the past.

I shall start by a case study concerning the mathematical activity of conjecturing and proving. I will exploit a corpus of texts written by Italian undergraduate mathematics students; they wrote their reasoning while trying to generalise a property concerning the system of natural numbers and then prove the generalised property. In particular I will try to seek for the ways students exploited and represented their mathematical knowledge. This case study will offer some hints about the links between mathematical argumentation, conjecturing and proving and related educational implications. Then I will move to more general considerations about argumentation and proof by considering both mental processes and their outcomes. Some educational implications will be discussed in the final section.

A CASE STUDY

The Educational Context

I will study written production of conjectures and their proofs in a task related to elementary number theory. The output in question was produced by 43 university students over four consecutive years (from 1996 to 1999) while completing their undergraduate studies in mathematics at Genoa University. At this level the students master the mathematical knowledge and the rules of algebraic calculation they must deal with. They are following a mathematics education course and work under a contract (explicitly established with their teacher) that requires them to write down every idea that comes to them during their work, even if they change their mind about its validity or its usefulness. This contract is intended to obtain productions regularly for use by the whole group for didactical and cognitive analyses of problem solving activities.

The Task

The students were to generalise a proposition (*The sum of two consecutive odd numbers is divisible by four*), then prove the generalised proposition. The fact that they had to build up their own conjectures makes their work very different from ordinary school proving, where students have to gather arguments to support a proposition they might never have thought of before. In our case we may suppose that the act of forming a conjecture fixes the conjecture very firmly in their minds, and the proof can be strongly influenced by the steps that led to the insight of the conjecture (see Garuti et al., 1998: "cognitive unity of theorems").

Modes and Criteria of Analysis of Students' Performances

I considered 14 texts (by the 1997/98 students) in particular detail, and then checked analogies and possible differences with the whole set of 43 texts. Reference will only be made to the 14 texts analysed in detail, but the aspects described are recurrent in the other texts as well. Some excerpts from two texts (by Students [1] and [2]), chosen as representatives of opposite behaviours, are reported (see Appendix).

Bearing in mind the aim of this study and the theoretical framework, each text has been analysed according to the following modes and criteria:

- Overall account of student's conjecturing and proving (global effectiveness of their performance, etc.).
- Implicit and explicit reference knowledge backing students' argumentation. I distinguished (see Mariotti et al., 1997: "theorems") between: content reference knowledge; meta-knowledge about the operations that the task called for (generalising, etc.). I also analysed the external representation of explicit reference knowledge. Concerning this issue, our attention focused particularly on personal (verbal, schematic, etc.) expressions that would be unusual in a normally acceptable written mathematical production. This kind of analysis was meant to explore in depth how these undergraduate mathematics students used their knowledge, as it might reveal their personal implication in the search of meaning or of tools for interpretation.
- Occurrence of algebraic-syntactic or semantically based steps of reasoning and the relationships between them. This analysis was needed in order to understand better how the two kinds of reasoning are functionally linked and connected to the resolution of the problem.
- Relationships between the proving process and the proof as a product (and the consequences of matching the former with the latter).

Students' Behaviour

Overall account of students' work Within the 14 texts, only four (Students [1], [2], [11], [13]) tried to prove something distinctly: two (Students [1] and [11]) prove their conjectures; and Student [13] a partial result of a confused conjecture. Student [2] (see Doc. 2) tries to prove a result that is stronger than the conjecture expressed in words; his proof lacks a fundamental step (justification of the formula used, which derived by generalisation from numerical examples). Let us call these four students the "proof group". But as we can hardly distinguish the processes of construction of conjectures from construction of proofs in the work of the students, we may as well study more texts from the perspective of proof construction. Another important argument to support this shift in the study from proof to conjecture construction is that five students do not achieve their proofs (even though they were on the right track) probably because of a lack of active mathematical practice combined with the unusual situation of having to build their own conjectures. So we can consider the constructive work of nine students (we may call "conjecture group," which includes the "proof group") and take, as comparative examples, elements of the work of the other five ("failure group").

Reference knowledge and its representation The task called for elementary content reference knowledge: elementary arithmetic, algebraic language and its rules of calculation. Some students tried to use other reference knowledge such as functions and series. Concerning algebra, we may remark that the process of formalization (i.e., the passage from content to formula) was not easy for many students, especially when they wanted to write the sum of K odds: for instance, some of them wrote (2n+1)+(2n+3)+…+(2n+?) and then stopped; few were able to express? as 2K-1: see (**E**) in Doc. 1. Writing the result of the sum was not easy either: it demanded a semantically rooted conversion of a known formula (the formula for the sum of the first n natural numbers—cf. Szeredi & Torok, 1998), or the reconstruction of an ad-hoc formula: see (**F**) in Appendix, Doc. 1.

As concerns the external representation of content reference knowledge, I have found various organisations of data and schemas with visual effects that reveal regularities and help to express some arithmetic relations in an algebraic way; for instance, symmetries in the disposition of data and formulas provided hints for the calculus (see figures in Appendix, Doc. 1 for two examples).

We may remark that re-organising data reflects a type of knowledge that is important for solving problems, but not always recognised or valued, though it is itself constructed knowledge (cf. Briand, 1993, for similar remarks concerning counting strategies). We may also remark that in other fields of mathematics (such as numerical analysis or category theory) schemas and organisational schemes are crucial tools.

Meta-mathematical knowledge was made explicit especially when it was almost algorithmic (see Student [2]) or referred to the task ("*What does it mean 'to generalize'* "), but appeared only implicitly when it was complex (actually richer) and nearer to the mathematicians' behaviour (see Thurston, 1994). Summing up the analyses dealing with meta-mathematical knowledge, I may say that the range of shared explicit knowledge was much narrower than the actual knowledge used globally by the group. I found that eight students referred explicitly to methods for solving problems of this kind, but, to take an example, "organization of data" was never mentioned even when methods were partially made explicit, though it was a key strategy for four students and useful for three of them. Only one of the fourteen students (Student [12]) seemed to have no idea of possible strategies for solving problems of this kind: she seemed lost, mixed up different steps undertaken and produced several unfinished propositions. For Students [1] and [13] ("proof group"), I detected very rich implicit meta-mathematical knowledge about how to solve the problem.

The implicit problem-solving methods I could detect globally were: change of representation; interpreting calculations in words and vice versa; visually organizing data and calculations, up to a geometrical regularity. I could also detect changes of mathematical frames: arithmetic, algebra, series, etc, which is common in the process of proving for mathematicians.

Algebraic-syntactic or semantically based steps of reasoning I have listed numerous breaks during calculations. They were needed to re-interpret the mathematical content of calculus in words. This can be seen as a sign of the primacy of semantical content over algebraic calculation during the process of conjecture and proof construction. As an example, we can consider the need of Student [1] to express algebraic propositions in words when seeking to recognise possible conjectures. This attitude displays the search for a semantically consistent grasp of the algebraic signs. We can interpret it by saying that constructive-reflective work in mathematics cannot evolve only within formal expression.

On the other hand, if we observe the students who did not express the results of their calculation in words richly (five of the fourteen students), three of them (Students [3], [4], [12]) are in the "failure group" of five students and two ([2] and [13]) in the "conjecture group" of nine students. So the majority of the "conjecture group" (seven out of nine) needed semantic interpretations to pursue their work. I recall that Student [2] did not recognise the strong result obtained, and that [13] was confused in expressing his conjecture—it was not clear to this student what was proved by the calculation.

Proof as product and proof as process Let us compare two examples that are representative of some others in the whole group of 43: in the first (student [2]), proof as a product is close to proof as a process, while in the other (student [1]) the distance is great.

Student [2] is considered skilful (good notes, etc.), but sticks very closely to her explicit method and her presentation is very close to that of a final presentation. This approximation to a formally correct mathematical text (cf. Hanna, 1989) seems to bear negative consequences on the productivity of the student's work: her research is linear and no change of strategy is found at any level. There are long repetitive arithmetic calculations, quite astonishing for the only student in the group who usually managed algebraic tools very well; more remarkably, the student arrives algebraically at a strong conjecture and interprets it in words as much weaker. And finally she does not produce a complete proof.

Analysing the text of *Student [1]*, we can observe frequent changes of strategy, organisation of data and calculations, as well as a frequent effort to interpret in words. This variety, this need for change might help technically, but these were also "interpretation" efforts. They helped understanding and often stimulated the development of new ideas. These moves in students' ideas can be recognized as "transformational reasoning" (see Harel & Sowder, 1998; Simon, 1996). Some of these very useful forms disappeared in the final draft of the proof (P), where the logical link between the propositions became a priority. In addition, justification of the research method disappeared from the product (while examples of the interwoven presence of meta-mathematical arguments in mathematical reasoning were frequent in the construction stage). Her conjecture is strong and her proof is almost complete.

Some Hints from the Case Study

We have seen that important reference knowledge remained implicit in students' proving processes and that some of the references concerned the content, while others related to the meta-knowledge about the activity to be performed. These references are not all part of an institutionalised (or "institutionalisable") corpus of mathematical references, or axioms.

We have also seen how non-standardised, appropriate representation of explicit reference knowledge had an important role in students' performances. In particular we may infer that they were means of interpretation. In this same perspective, we have seen that when elaborating a productive process many students found syntactic arguments insufficient, and so semantically rooted arguments became critical, thus revealing search of meaning of the various steps.

Finally, we have collected some experimental evidence about the negative consequences of subordinating the proving process to the requirements of proof as a final product.

ABOUT ARGUMENTATION AND PROOF

This section is intended to provide the reader with reference definitions and basic ideas for the following sections. The general aim is to relate the results of students' behaviour in our case study to mathematicians' behaviour and try to develop a general view about the need for meaning in mathematical proof.

What Argumentation are we Talking About?

We cannot accept any discourse as an argumentation. In this chapter the word "argumentation" will indicate both the process which produces a logically connected (but not necessarily deductive) discourse about a given subject: (from *Webster Dictionary:* "1. The act of *forming reasons, making inductions, drawing conclusions, and applying them to the case under discussion*") and the text produced by that process (*Webster Dictionary:* "3. Writing or speaking that argues")—the linguistic context will allow the reader to select the appropriate meaning.

The word "argument" will be used as "A reason or reasons offered for or against a proposition, opinion or measure" (*Webster*), and may include verbal arguments, numerical data, drawings, etc. In brief, an "argumentation" consists of one or more logically connected "arguments," and the discoursive nature of argumentation does not exclude the reference to non-discoursive (for instance, visual or gestural) arguments.

Formal Proof

This chapter will consider "formal proof" as a proof that approaches a logical calculation. Concerning its cognitive characteristics, it will refer to Duval's description (1991) by quoting his "*cognitive analysis of deductive organisation versus argumentative organisation of reasoning*" as concerns "inference steps": in argumentative reasoning, "semantic content of propositions is crucial," while in deductive reasoning "propositions do not intervene directly by their content, but by their operational status" (defined as "their role in the functioning of inference"); as concerns "enchaining steps": argumentative reasoning works "by reinforcement or opposition of arguments" (p. 233). "Propositions assumed as conclusions of preceding phases or as shared propositions are continuously reinterpreted". "The transition from an argument to another is performed by extrinsic connection". On the contrary, in deductive reasoning "the conclusion of a given step is the condition of application of the inference rule of the following step". The proposition obtained as the conclusion of a given step is "recycled" as the entrance proposition of the following step. Enchaining makes deductive reasoning similar to a chain of calculations; as concerns the "epistemic value" (defined as the "degree of certainty or conviction attributed to a proposition"): in argumentative reasoning "true propositions have not the same epistemic value," while in mathematics "true propositions have only one, specific epistemic value […] that is, certainty deriving from necessary conclusion"; and "proof modifies the epistemic value of the proved proposition: it becomes true and necessary". This modification constitutes the "productivity of proof".

Mathematical Proof

We could start by saying that mathematical proof is what in the past and today is recognised as such by people working in the mathematical field. This approach covers Euclid's proof as well as the proofs published in high-school mathematics textbooks, and current modern-day mathematicians' proofs, as communicated in specialized workshops or published in mathematical journals (for the differences between these two forms of communication, see Thurston, 1994). Furthermore, we can recognize some common features between argumentation and proof, in particular: a common function, i.e., the validation of a statement; the reference to an established knowledge (see the definition of "theorem" as "*statement, proof and reference theory*" in Mariotti et al., 1997); and some common requirements, like the enchaining of propositions.

The distinction between the process of proof construction (i.e., "proving") and the result (as a socially acceptable mathematical text, as proposed in the introduction, and preceding considerations point out the fact that mathematical proof and the proving process can be considered as particular cases of argumentation (according to the preceding definition). However, in this chapter "argumentation" will exclude "proof" when comparing them.

Concerning the relationships between formal proof and proofs currently produced by mathematicians, I quote Thurston:

> We should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and that they are quite different from formal proof. For the present, formal proofs are out of reach and mostly irrelevant: we have good human processes for checking mathematical validity. (p. 121)

We may also consider some examples of theorems in mathematical analysis (e.g., Rolle's Theorem, Bolzano-Weierstrass' Theorem, etc.) whose usual proofs in current university textbooks are formally incomplete: completion would bring students far from understanding; for this reason semantic (and visual) arguments are frequently exploited in order to fill the gaps existing at the formal level.

Argumentation in Mathematics

Argumentation can be performed in pure and applied mathematical situations, as in any other area. Argumentations are usually held informally between mathematicians to develop, discuss or communicate mathematical problems and results, but are not recognised socially in a research paper presenting new results: in that case proofs and "rigorous" constructions (or counter-examples) are needed. As concerns "communication," Thurston (1994) writes:

> Mathematical knowledge can be transmitted amazingly fast within a subfield of mathematics. When a significant theorem is proved, it often (but not always) happens that the solution can be communicated in a matter of minutes from one person to another within the subfield. The same proof would be communicated and generally understood in an hour's talk to members of the subfield. It would be the subject of a 15- or 20-page paper, which could be read and understood in a few hours or perhaps days by the members of the subfield. Why is there such a big expansion from the informal discussion to the talk, to the paper? One-to-one, people use wide channels of communication that go far beyond formal mathematical language. They use gestures, they draw pictures and diagrams, they make sound effects and use body language. (p. 166)

As concerns "rigour," it is considered here because it appears as a requirement of mathematical texts although it needs to be defined, or rather to be questioned and put into a historical perspective—see Lakatos (1985). The problem of rigour will be reconsidered later with the question of the epistemic value of statements.

Reference Corpus

The expression "reference corpus" will include not only reference statements but also visual and, more generally, experimental evidence, physical constraints, etc. assumed to be unquestionable (i.e., "reference arguments," or, briefly, "references," in general). In the Subsection "*About the Reference Corpus*", I will discuss the social determination of the fact that a "reference" is not an object of doubt, as well as the necessary existence of references that are not made explicit.

The knowledge the students referred to or used implicitly, in our case study, forms examples of the existence of such reference corpus.

Tools of Analysis and Comparison of Argumentation and Proof

Aren't there some criteria (even implicit ones) that enable us to accept or refuse an argumentation, as it happens for a proof? And are they not finally related to logical constraints and to the validity of the references, even if entangled with complex implicit knowledge? If we follow Duval's analysis, for argumentation it seems as if there is no recognised reference corpus for argumentation, whereas for proof it exists systematically. I do not think that this distinction is correct. The following criteria of comparison, inspired by Duval's analysis, will help me to discuss this point, in the next section: the existence of a "reference corpus" for developing reasoning; the means by which doubts about the "epistemic value" of a given statement can be dispelled; and the form of reasoning.

ANALYSIS AND COMPARISON OF ARGUMENTATION AND MATHEMATICAL PROOF AS PRODUCTS

About the Reference Corpus

No argumentation (individual or between two or more protagonists) would be possible in everyday life if there were no reference corpus to support the steps of reasoning. The reference corpus for everyday argumentation is socially and historically determined, and is largely implicit. "Reference corpus" for mathematical proof may seem to be completely explicit and not socially determined, but we will see that this is not so: it obeys specific social and historical influence.

Social and historical determination of the "reference corpus" for proof In this subsection I will try to support the idea that the "reference corpus" for mathematical proof is socially and historically determined. In order to do so, I will exploit arguments of different nature (historical and epistemological considerations as well as reflections on ordinary school practices) that are not easy to separate.

The reference corpus used in mathematics depends strongly on the users and their listeners/readers. For example, in secondary school some detailed references can be expected to support a proof, but in communication between higher-level mathematicians those may be considered evident and as such disregarded. As Yackel (1998) pointed out, the existence of jumps related to "obvious" arguments in the presentation of a proof can be considered as a sign of familiarity with knowledge involved in that proof. On the other hand, some statements accepted as references in secondary school are questioned and problematised at higher levels; questions of "decidability" may surface. We may remark that today problems of "decidability" are dealt with by few mathematicians and seldom encountered in mathematics teaching (although in my opinion simple examples concerning Euclidean geometry vs non-Euclidean geometries could be of great pedagogical value). To quote Thurston (1994):

> On the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do. Most mathematicians adhere to foundational principles that are known to be polite fictions. (p.170)

Thus for almost all the users of mathematics in a given social context (high school, university, etc.) the problem of epistemic value does not exist (with the exception of the case: "true" after proving, or "not true" after counter-example) although it was and it still is an important question for mathematics as for any other field of knowledge. Mathematics concepts are the most stable, giving an experience of "truth" which should not be necessarily taken for truth.

Let us now consider other aspects of the social determination of the reference corpus which concern the nature of references. If we consider the “references” that can back an argumentation for validating a statement in primary school, we see that at this level of approach to mathematical work references can include experimental facts. And we cannot deny their “grounding” function for mathematics (see Lakoff & Nunez, 1997), both for the long-term construction of mathematical concepts and for establishing some requirements of validation which prepare proving (e.g., making reference to acknowledged facts, deriving consequences from them, etc.). For instance, in primary school geometry we may consider the superposition of figures for validating the equality of segments or angles, and superposition by bending for validating the existence of an axial symmetry. Later on in secondary school, these references no longer have value in proving; they are replaced by definitions or theorems (see Balacheff, 1988). In general, at a higher level it is a hypothesis or a partial result of the problem to solve that informs us of equalities and not “experimental” validation (see Balacheff, 1988). At such a level the meaning of equality is not questioned as might (and should) happen at “lower” levels. We may note that, in the history of mathematics, visual evidence supports many steps of reasoning in Euclid’s *Elements*. This evidence was replaced by theoretical constructions (axioms, definitions and theorems) in later geometrical theories.

Implicit and explicit references The reference corpus is generally larger than the set of explicit references. In mathematics, as in other areas, the knowledge used as reference is not always recognised explicitly (and thus appears in no statement): some references can be used and might be discovered, constructed, or reconstructed, and stated afterwards. The example of Euler’s theorem discussed by Lakatos (1985) provides evidence about this phenomenon in the history of mathematics. The same also occurs for argumentation within areas other than mathematics. Let us consider the interpretations made by a psychoanalyst: we cannot fathom his ability unless we believe that he bases his work on chains of reasoning that refer to a great deal of shared knowledge about mankind and society, this knowledge being obviously impossible to reduce to explicit knowledge. And, in general, we could hold no exchange of ideas, whatever area we are interested in, without exploiting implicit shared knowledge. Implicit knowledge is a source of important “limit problems” (especially in non-mathematical fields, but also in mathematics) when we come to bring them (or their effects) to consciousness: in the “fuzzy” border of implicit knowledge we can meet the challenge of formulating more and more precise statements and evaluating their epistemic value. Lakatos (1985) provides us with interesting historical examples about this issue.

Note that in our case study we point out some implicit, as well as some explicit, references.

How to dispel doubts about a statement and the form of reasoning

Thurston writes:

> Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas. (p. 170)

And considering the example of Wiles's proof of Fermat's Last Theorem:

> The experts quickly came to believe that his proof was basically correct on the basis of high-level ideas, long before details could be checked.

These quotations raise some interesting questions concerning the ways by which doubts about mathematics statements are dispelled. Formal proof "produces" (according to Duval's analysis) the reliability of a statement (attributing to it the epistemic value of "truth"). But what Thurston argues is that "*reliability does not primarily come from mathematicians formally checking formal arguments*". In Thurston's view, the requirements of formal proof represent only guidelines for writing a proof—once its validity has been checked according to "substantial" and not "formal" arguments. The preceding considerations directly concern the form of reasoning: the model of formal proof as described by Duval and based on the "operational status" of propositions rather than on their "semantic content" does not seem to fit the description of the activities performed by many working mathematicians when they check the validity of a statement or a proof. Only in some cases (for instance, proofs based on chains of transformations of algebraic expressions) does Duval's model neatly fit proof as a product.

Despite the distance between the ways of dispelling doubts (and the forms of reasoning) in mathematics and in other fields, the preceding analysis shows many points of contact—even between mathematical proof process and argumentation in non-mathematical fields. Granger (1992) suggests the existence of deep analogies that might frame (from an epistemological point of view) these points of contact. Naturally, as concerns the form of reasoning visible in the final product, argumentation presents a wider range of possibilities than mathematical proof: not only deduction, but also analogy, metaphor, etc. Another significant difference lies in the fact that an argumentation can exploit arguments taken from different reference corpuses that may belong to different theories with no explicit, common frame ensuring coherence. For instance, the argumentation developed in this chapter derives its arguments from different disciplines (history of mathematics, epistemology, cognitive psychology); at present there is no mean to tackle the problem of coherence between reference theories belonging to these domains. On the contrary, mathematical proof refers to one or more reference theories explicitly related to a coherent system of axioms. But I would prefer to stress the importance of the points of contact (especially from an educational point of view: see the next Section).

THE PROCESSES OF ARGUMENTATION AND CONSTRUCTION OF PROOF

In the Introduction I proposed distinguishing between the process of construction of proof ("proving") and the product ("proof"). Of what does the "proving" process consist? I will refer to some theoretical elaboration that can help us answer this question, and, in parallel, return to the case study for confirmation.

Experimental evidence has been provided about the hypothesis that "proving" a conjecture entails establishing a functional link with the argumentative activity needed to understand (or produce) the statement and recognising its plausibility (see Mariotti et al., 1997). This was reflected in the case study by the important recourse to semantic hints within the mathematical explorations and also by the use of a variety of external representation. These were part of the efforts of interpretation and developing meaning.

Proving needs an intensive argumentative activity, based on "transformations" of the situation represented by the statement, as it was through the use of a variety of external representation in our example. Experimental evidence about the importance of "transformational reasoning" in proving has been provided by various, recent studies (see Arzarello et al., 1998; Arzarello; 2000, Boero et al., 1996; Simon, 1996; Harel & Sowder, 1998). Simon defines "transformational reasoning" as follows:

> the physical or mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated.

It is interesting to compare Simon's definition with Thurston (1994):

> People have amazing facilities for sensing something without knowing where it comes from (intuition), for sensing that some phenomenon or situation or object is like something else (association); and for building and testing connections and comparisons, holding two things in mind at the same time (metaphor). These facilities are quite important for mathematics. Personally, I put a lot of effort into "listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quietening and focusing of my mind. Words, logic and detailed pictures rattling around can inhibit intuitions and associations. […] We have a facility for thinking about processes or sequences of actions that can often be used to good effect in mathematical reasoning. (p. 165)

These quotations suggest some interesting reflections about the role of *metaphors* and their links with transformational reasoning in the proving process.

Metaphors can be considered as closely related to transformational reasoning. For a metaphor we may consider two poles (a known object, an object to be known) and a link between them. In this case the "creativity" of transformational reasoning consists in the "choice" (which may not be totally conscious) of the known object and the link—which allows us to know some aspects of the unknown object as suggested by the knowledge of the known object ("abduction") (cf. Arzarello et al., 1998).

Concerning the possible metaphors: mathematics is apparently only concerned by mathematical objects. Metaphors where the known pole is not mathematical are not acknowledged. But in many cases the process of proving needs these metaphors, sometimes with material, physical or even bodily referents (their traces can be detected when a mathematician produces an informal description of the ideas his proof is based on: see Subsection "*Modes and Criteria of Analysis of Students' Performances*", quotation from Thurston). In general, Lakoff and Nunez (1997) suggest that such metaphors have a crucial role in the historical and personal development of mathematical knowledge ("*grounding metaphors*"). The example of continuity is illuminating. Simon (1996) discusses the importance of a physical enactment in order to check the results of a transformation in transformational reasoning. In some situations the mathematical object is the known object and the other pole concerns a non-mathematical situation: the aim is to validate some statements concerning this situation, exploiting the properties of the mathematical model. Let us also note that; in other cases it happens to relate two non-mathematical objects (one known, the other not) by a mathematical, metaphoric link which sheds light on the unknown object and/or on its relationships with the known object. By these means, argumentative activities concerning non-mathematical situations rely upon mathematical creations (metaphors), an observation that should be taken into account in mathematics education.

When our students re-organize data in a way they can better handle, we can recognize a transformational reasoning from a pure algebraic reading of the signs to a space reading of the relation to be built between the signs, thus leading to a possibility of metaphor of the calculus procedure. As if the elements were put on a table and moved to be associated together.

The example of metaphors shows the possible "semantic" complexity of the process of proving—and suggests the existence of a variety of links with various mathematical activities and also with non-specific mathematical activities. It also shows the importance of transformational reasoning as a free activity (in particular, free from usual boundaries of knowledge).

Induction in general is also relevant—and the need to produce a deductive chain guides the search for arguments to "enchain" when coming to the writing process (see Boero et al., 1996).

SOME EDUCATIONAL IMPLICATIONS

Let us return to the processes of argumentation and proof construction as opposed to the final static results. An important part of the difficulties of proof in school mathematics comes from the confusion of proof as a process and proof as a product (see Garuti et al., 1996). Students are expected to learn proof through the learning of text-proof writing (and organising); the research process needed to produce the ideas of proof were totally left to the responsibility of the students. For instance, we made the remark that organisation of data, which was important in students' performances in our case study, is a piece of knowledge that is generally not recognised in school context. This piece of knowledge is seldom (if ever) studied in a meta-cognitive reflective perspective. The difficulties are often overcome by an authoritarian approach. Frequently, mathematics teaching is based on the presentation (by the teacher, and then by the student when asked to repeat definitions and theorems) of mathematical knowledge as a more or less formalised theory based on rigorous proofs. In this case, authority is exercised through the form of the presentation (see Hanna, 1989); in this way school imposes the form of the presentation and values it over the thought, leading to the identification between them, and demands a thinking process modelled by the form of the presentation (eliminating every "dynamism"). This analysis may explain the strength of the model of proof, which promotes an idea of "linearity" of mathematical thought as a necessity and a characteristic aspect of mathematics.

If a student (or a teacher) assumes such "linearity" as the model of mathematicians' thinking without taking the complexity of conjecturing and proving processes into account, it is natural to see "proof" and "argumentation" as extremely different. This may have consequences in other fields: it can reinforce a style of "thinking" for which no "sacred" assumption is challenged, only "deductions" are allowed (obviously, also school practice of argumentation may suffer from authoritarian models!). On the contrary, giving importance to "transformational" reasoning (and, in general, to the non-deductive aspects of argumentation needed in constructive mathematical activities—including proving) can develop different potentials of thinking. On the possibility of educating manners of thinking other than deduction, Simon considers "transformational reasoning" and hypothesizes:

> […] transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas. The inclination […] must be nurtured and developed.[…]school mathematics has failed to encourage or develop transformational reasoning, causing the inclination to reason transformationally to be expressed less universally.

I am convinced that Simon's assumption is a valid working hypothesis, needing further investigation not only regarding "*the role of transformational reasoning in classroom discourse aimed at validation of mathematical ideas*" but also its functioning and its connections with other "creative" behaviours (in mathematics and in other fields).

As concerns possible educational developments, the analyses performed in this chapter suggest some immediate consequences:

- classroom work should include (before any "institutionalisation") systematic activities of argumentation about work to be done (for instance, producing and backing a conjecture), as well as work that has been done (in this case, explaining, justifying and validating);

- in particular, validation, in mathematical work and in other fields, should be demanded whenever it can be meaningful;
- the fact that validation has not been done or was unsatisfactory or impossible should be openly recognised;
- and, finally, references as such should be explicitly recognized, be they statements, experiments or axioms (this does not mean that references are fixed as true once and for all, but rather that for at least a certain time we have to consider them as "references" for our reasoning). This demands meta-cognitive reflection to become part of the usual discussion activity.

The passage from argumentation to proof about the validity of a mathematical statement should openly be constructed on the basis of limitation of the reference corpus (see Subsection "*How to dispel doubts about a statement and the form of reasoning*", last paragraph). It could be supported by exploiting different texts, such as historical scientific and mathematical texts, and different modern mathematical proofs (see Boero et al., 1997, for a possible methodology of exploitation).

NOTES

[1] He seems to refer to Godel's proof that ordinary arithmetic contains propositions that cannot be proved within the system and cannot be disproved either, making them *formally undecidable*.

[2] Following the Pythagorean tradition, in his *Mysterium Cosmographicum* (1621), Kepler discussed the existence of precisely six planets by connecting their orbits to the existence of exactly five platonic solids. He believed that between each pair of spheres containing the orbits of adjacent planets, one of the platonic solids was inscribed.

[3] The problem she mentioned is discussed also by Grenier (2000).

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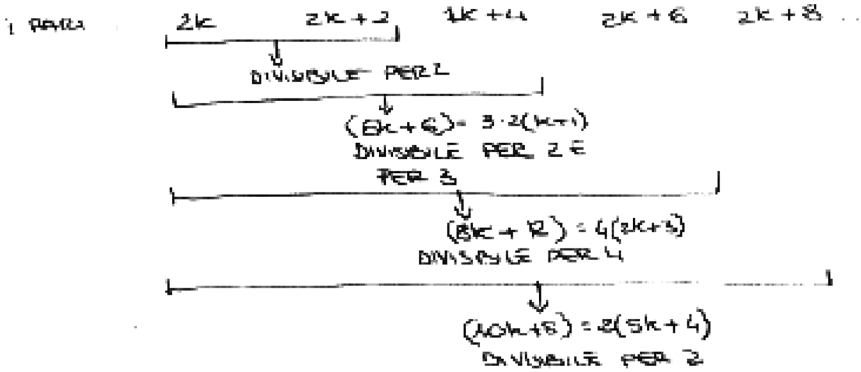
APPENDIX

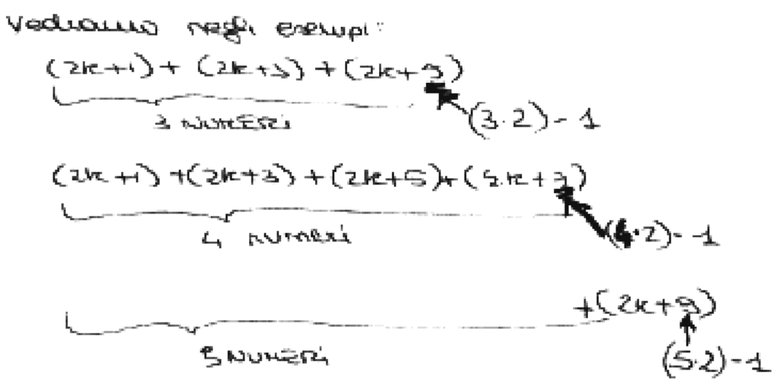
DOC. 1: Excerpts from the text of Student [1]; it contains seven large, spatially organized pieces, like the two reported below, and many arrows, connecting lines and encirclings.

"*I have some difficulties in understanding in what direction I must generalize. It might be: 'by adding two odd or even consecutive numbers I get a number divisible by 4'* [she performs some numerical trials]. *This does not work. I shall try to generalize in another way*:



I was looking for something that could help me but I got nothing.
[other trials, with a rich spatial organisation: two consecutive even numbers, two consecutive odd numbers—here she gets divisibility by 4; then three, four, five, six, seven consecutive odd numbers. By performing calculations, she gets the following formulas: 3(2K+3); 8(K+2) 10K+25=5(2K+5); 12K+36=12(K+3); 14K+49=7(2K+7)]. Is the result of the addition of n consecutive odd numbers (n odd) divisible by n? (2K+1)+(2K+3)+…(2K+ What must I put here?

(E)



[she performs an unsuccessful trial by induction; then she considers n numbers in general]
n numbers: (2K+1)+(2K+3)+…(2K+(2n-1))=2nK+1+3+5+…+(2n–1)=2nK+ (I am thinking of the anecdote of "young Gauss":
(**F**) it makes 2n.n/2=n2) = 2nK+n2= n(2K+n) OK!!
[Trials performed by applying the preceding formula 2nK+n2 in the cases n=2, n=4, n=6, n=8: she gets: 4K+4 divisible by 4; 8K+16 divisible by 8; 12K+36=12(K+3); 16K+64=16(K+4) divisible by 16]. Then if I add n consecutive odd numbers (n even), I get divisibility by 2n. Let us try a proof: (P) (2K+1)+(2K+3)+ …(2K+2n-1)=2nK+(1+3+2n–1)=2nK+(2n.n)/2=2nK+n2=2n(K+ n/2); n even implies that n/2 is an integer number: so I get divisibility by 2n. […]

DOC. 2: Excerpts from the text of Student [2]; spatial organization is almost linear, like that in the following transcript.

Student [2] starts her work by checking (on numerical examples: 3+5; 5+7; 101+103) the validity of the given property, then proves it. Then she writes: "*When I must tackle a problem, I try to see how it works in particular cases and then I generalize, as I have done in this case—although I knew the solution. I reason in this way because the particular case allows me to understand better how I can reach the solution of the problem in general (and this method works even when I do not know the solution). Thinking in arithmetic terms and then in algebraic terms helps me to solve the problem. For the original property the generalization comes fairly automatically, because* [she explains why in detail].

What does it mean "to generalize"? It means considering a property in which there are some closed variables (two odd numbers, or divisibility by 4) and getting a property in which variables are open. I change the number of odd consecutive numbers to add. For instance, I consider 3 [crossed out] 4 consecutive odd numbers 2n+1, 2n+3, 2n+5, 2n+7 and make the addition:
2n+1+2n+3+2n+5+2n+7=8n+16=8(n+2)=4(2n+4). Then I find a number that is divisible by 8, so it is divisible by 4. I perform the addition of 6 consecutive odd numbers [similar calculations]=12n+36=6(2n+6). Then I find a number that is divisible by 12, so it is divisible by 6. I try with 8: [similar calculations]=8.2n+64=8(2n+8) Then I find a number that is divisible by 8, so it is divisible by 4. Following my reasoning, for an even number K of odd consecutive numbers I get: 2n+1+2n+3+.…+2n+15+.…=K(2n+K)=2K(n+K/2); but K is an even number, so it is divisible by 2 and (n+K/2) is an integer number. Then 2K is divisible by 4 (because K is odd). So I have found that the given property is still valid if I add up an even number of odd consecutive numbers.

PART IV: DIDACTICAL ASPECTS

GREISY WINICKI-LANDMAN

8. MAKING POSSIBLE THE DISCUSSION OF "IMPOSSIBLE IN MATHEMATICS"

Alice laughed: "There's no use trying," she said; "one can't believe impossible things."

"I dare say you haven't had much practice," said the Queen. "When I was younger, I always did it for half an hour a day. Why, sometimes I've believed as many as six impossible things before breakfast."

Alice in Wonderland, Lewis Carroll

This exploratory study is a first attempt to collect, describe and analyze students' understanding of the notion of "mathematical impossibility". This notion is strongly connected to the idea of proof and the approach chosen to the study relies on students' declarations as well as on their performances of proofs and refutations of mathematical statements involving impossibility.

INTRODUCTION

Instead of asking students "Show that there is no" I generally ask them "Find a" without giving any additional information. The reader may consider I lied to them … I don't. With the task formulated affirmatively, the solver will try to accomplish the task and only after some attempts (sometimes many attempts or even never …) he will suspect that the task is "very hard," or almost "impossible". Then, he is in front of a meta-mathematical decision: to keep on trying to do what he was asked to do or to be "insolent" and change direction trying to prove that he had been asked to do something that is not possibly done. From an educational point of view, I think it is very important to have mathematics students discuss the role of impossible things in mathematics.

First of all, it is necessary to acknowledge that there exist impossible "things" in mathematics. Some of the questions that lead to such "things" were asked from antiquity (i.e., Trisecting an angle, Doubling a cube, Squaring a circle—all of the above with straightedge and compass alone) and the attempts to give them an answer led to the development of very important branches of mathematical knowledge. As Hilbert wrote in 1900: "Sometimes it happens that we seek the solution under unsatisfied hypotheses or in an inappropriate sense and are therefore unable to reach our goal. Then the task arises of proving the impossibility of solving the problem under the given hypotheses and in the sense required. Such impossibility proofs were already given by the ancients, in showing, e.g., that the hypotenuse of

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 185–195.

an isosceles right triangle has an irrational ratio to its leg. In modern mathematics the question of the impossibility of certain solutions has played a key role, so that we have acquired the knowledge that such old and difficult problems as to prove the parallel axiom, to square the circle, or to solve equations of the fifth degree in radicals have no solution in the originally intended sense, but nevertheless have been solved in a precise and completely satisfactory way." (quoted in Davis, 1973) (my emphasis).

Second, the fact that there are impossible things in mathematics may be quite surprising. As exposed by Delahaye, "disconcerting are the mathematical results demonstrated since 1930 concerning the impossibility of certain demonstrations.[1] because they can assign a sort of limit to thought itself" (Delahaye as quoted in Young 1992). The analysis of results about impossibility may lead students to a more real perspective of the subject. There are even those who consider impossibility theorems as "the most remarkable theorems in mathematics. For they have a mystifying quality" (Richards, 1975, p. 250).

Third, the discussion of results of the form "It is impossible to …" may constitute a good opportunity to clarify the distinction between unsolved problems and unsolvable problems. As Davis said, "There seems to be a time element at work in such [impossibility] statements. Actuality is here, actuality is now, it is complete; an impossibility seems to bargain with an uncommitted future" (Davis, 1986, p. 67).

In the frame of a course for pre-service secondary school mathematics teachers, the classic proof that there exist at most five regular solids was presented. Later on, during the same lesson, Kepler's model of the Solar System[2] was exposed and I discussed Kepler's connections between the Platonic Solids and the orbits of the planets. Then the following dialog occurred:

> S [student]: At Kepler's time only some of the planets were known. But nowadays, we know that there are nine planets.
> T [teacher]: Yes, …
> S: If so, isn't it possible that in the future a new regular solid may be discovered?
> T [to the class] What do you think?

Of course, the answer to the student's question is one and only one: if the definition of regular solid is the one used in the presented proof, there are no more than five regular solids, so it is impossible to find another one. For the students involved in the lesson the word "impossible" sounded "too strong" and "very dramatic". It turned on a red light on my head: How do they understand impossible statements? How do they prove such statements? From these questions arises another reason to study the students' conceptions about impossibility in mathematics: its potential to highlight the students' conceptions of proofs and to expose their approaches while proving mathematical impossibilities.

THE STUDY

To learn more about these questions, an open questionnaire was designed and delivered to the students some time later. They were asked to work individually, to reflect about the following questions and to write down their opinions.

1. What is the meaning of "impossible" in mathematics?
2. Enunciate three examples of mathematical correct statements involving the idea of impossibility.
3. How do you explain to your math students that something is impossible?
4. In your opinion, do we—as mathematics educators—have to expose our students to mathematical statements involving impossibility? If so, to what purposes? If not, why?
5. Let us define a new concept: A *centrified* triangle is a right triangle whose circumcenter is also its baricenter. Enunciate some properties of a *centrified* triangle.
6. Prove or disprove the following statements:

   i. It is impossible for a kite to have exactly one right angle;
   ii. It is impossible for a non-special parallelogram to be a cyclic quadrilateral;
   iii. It is impossible to find three collinear points in the same circle;
   iv. It is impossible for a square to have a diagonal whose length is a rational number;
   v. It is impossible for a straight line which is not tangent to a parabola to have only one common point with the parabola;
   vi. It is impossible for a function to be odd and even at the same time.

The questionnaire was built taking into account the following considerations:

a) An open questionnaire allowed the students to express themselves freely, using their own words to expose their conceptions.
b) The questions were diverse in order to expose different aspects of the students' notion of impossibility: Question (i) explicitly asked for the students' "definition" of the notion, Question (ii) asked for examples, enabling the exposition of their free associations in their own language; Questions (iii) and (iv) dealt with the teaching of the notion; and Questions (v) and (vi) exposed the students' approaches while proving (or disproving) mathematical statements involving impossibility.
c) Six statements to be proved or disproved were included in order to give the students the opportunity to think about concrete impossibility statements, to *decide* about their correctness, and to prove or disprove them. These statements allowed collecting data about the processes in which the students get engaged when they prove that an impossibility statement is true or false.
d) Following Hadas and Hershkowitz (1999), the statements were chosen in order to have the students feel the *uncertainty* concerning their truth. Some of the statements presented are true while the others are not. The proof of (ii) and (iii)

may involve *proof by contradiction* while the refutation of the other statements may evoke the use of *counterexamples*.

e) In mathematics, nonexistence usually is a matter of impossibility; with this idea in mind, the statements were chosen also considering the possibility to construct relatively elementary examples/counterexamples.

f) The mathematical contents involved in the statements to be proved or disproved are known by the students and are directly connected to the contents they will teach at the secondary school (e.g., Elementary Algebra and Euclidean Geometry).

RESULTS AND ANALYSIS

This study tries to examine students' understanding of the notion of "impossible" and the ways in which they prove or disprove impossibility statements. A profile of four students—Abi, Bernie Carmen and Dalia—will be presented, according to their responses to the first five questions. The analysis of the students' answers to the last question will be exposed in a future work (Winicki-Landman, in preparation).

Abi

Abi wrote that for him impossible is "when there is no object that fulfills the requirements". He developed his ideas saying that he identified impossible with the empty set: "If each one of the requirements is translated into the set of objects that fulfill it, then the objects that belong to the intersection set fulfill all the requirements. If this intersection set is empty, then I call the situation impossible." In his search for properties of the *centrified* triangles (Question 5), Abi explicitly wrote that he identified the set of triangles for which their baricenter is also their circumcenter as the set of equilateral triangles. He wrote:

> If the baricenter of a triangle is also its circumcenter, it follows that its medians are congruent. We proved some time ago that if two medians in a triangle are congruent, it must be isosceles. So, our triangle is "isosceles twice," that means it's equilateral.

Then he continued by saying:

> Since there is no triangle that is both equilateral and right-angle, I conclude that the set of centrified triangles is empty. So, it is impossible for a right triangle to be centrified.

Abi approached the task in a deductive way: he constructed a general triangle to allow his reasoning flow and he reached his conclusion in a very concise and elegant way. He is a very good student, in general his proofs are original and he is very exigent with his own mathematical language.

The examples of impossibility statements Abi produced to answer Question 2 were:

> Since the solution set of the equation sin x=2 is empty, it is impossible to find a real number x for which sin x=2.

Then he added:

> In fact, every equation without solutions is an example of an impossibility statement.

When asked to explain the idea of impossibility to his students (Question 3), he chose two examples

> Solve the equation x=x+1

and

> It is impossible to divide 5 by 0, since there is no number x that fulfills the condition 0x=5.

In all his examples, the central idea is the idea of “nonexistence,” that is, the empty set.

When asked whether he took into account the possibility of proposing to students mathematical statements involving impossibility, he declared he was not sure, but he believed

> … the discussion of “possible” and “impossible” in mathematics may help students understand better the meaning of the theorems they prove and those they use in class. But I’m not sure that I always understand them myself. It seems to me that impossibility statements may be proved, in general, by using the indirect method of proof. This method is very potent but at the same time is very non-intuitive. But, for this kind of statements, it may constitute the sole way of proving, making it an essential tool in mathematics.

Abi believes that proving by using the reductio ad absurdum method is not always clear for the students. Although he himself might have experienced the lack of clearness embedded in such proofs, he is able to appreciate it as a “potent” method of proof. He raised an interesting question that needs to be verified: Impossibility statements are proved only by this method? I believe the answer is not, but this belief needs logical support.

Another important point that arises from his words is the controversy about indirect proof, in which you prove “*A must be true*” by proving “*not A is impossible*”. Indirect proofs accepted by classical mathematicians are rejected by intuitionists and constructivists (Hersh, 1997, p. 85) and I believe this controversy has to be studied in order to learn more about the logical and the psychological aspects of mathematical proof.

Bernie

Bernie wrote that impossible is

> … when you cannot obtain an answer to the question, when you cannot find a way to solve a problem, or when you cannot prove that something indeed exists.

It seems that, for Bernie, impossibility is a subjective property, meaning that some mathematical task may be impossible for him but possible for a friend. One of the examples of impossibility statements he mentioned was Fermat's Last Theorem. He explained his choice:

> It was impossible for almost four centuries, but now it is possible.

It may be important to point out that the impossibility Bernie wrote about is not the same as the impossibility that Fermat himself wrote about. Fermat wrote in Latin:

> On the other hand, it is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as a sum of two fourth powers or, in general for any number which is a power greater than the second to be written as a sum of two like powers. For this I have discovered a truly wonderful proof, but the margin [in his copy of Diaphantus' *Arithmetic*] is too small to contain it. (quoted by Young, 1992, p. 42).

Fermat wrote about the non-existence of three integers that fulfill certain requirements and Bernie was thinking about the fact that the problem was open. It seems that this student confused the terms "unsolved" and "unsolvable".

When Bernie was asked how to prove that a result is impossible he wrote:

> I don't know if you can do that at all … You only prove positive statements. I think you cannot prove that something is impossible. If you prove something, then it may be, it may exist. So, I think it is impossible to prove that something is impossible.

When asked to provide examples of impossibility statements he wrote

> 5–2=7 is an impossibility statement

and

> a>a is an impossibility statement for every real number a.

It seems that Bernie confused the terms "false" and "impossible," and this is consistent with his reluctance to discuss the idea of impossible with his students because for him this discussion may confuse them more and they will lose their confidence in themselves and in the subject as a scientific endeavor.

His consideration of students' anxiety is very laudable, but his comment exposed another misunderstanding, this time concerning the scientific endeavor.

In my opinion, the fact that there are mathematical statements concerning the impossibility of certain constructions—for example—must be seen as natural as the statements of the form "For all x in X, P(x)" or "Exists an x in X for which P(x)". It seems that, for Bernie, the former kind of statements is not so natural, even considering the fact that he met them before. For example, it's sensible to assume that he proved at least once that *the sum of two even numbers is an even number*. In other words, he may have proved that it is impossible to find a pair of even numbers whose sum is an odd number. But he seems not to identify the equivalence between the two statements. In some sense, Bernie's answers may reflect the lack of his former teachers' emphasis on developing their students' understanding of the *meaning* of what had been proved. The gap between the logical perspective and the *pedagogical* perspective to proofs needs to be considered seriously by teachers, and the discussion of the meaning of "impossible" in mathematics and the identification of impossibilities may constitute a springboard to skip this gap.

Bernie's considerations may lead teachers' educators to question the mathematical culture of future teachers. The impossibility of each one of the three classical constructions with just straightedge and compass are important historical facts, but in order to understand each one of these proofs the learner needs a background that is not usually available until university studies. On the other hand, there are impossibility statements that may be explicitly discussed in high school, for example the statement "It is impossible to find three collinear points in the same circle". It is an intuitive result, it may be formulated in different ways, and it may be proved with elementary synthetic geometry tools. Moreover, some impossibility statements may be refuted quite easily by finding a counterexample. Following the spirit of Zaslavsky and Ron (1998) and Peled and Zaslavsky (1997), this kind of tasks should be more frequent in secondary-school mathematics lessons since they may create appropriate opportunities to discuss different approaches to the production of counterexamples to false statements.

Carmen

Carmen wrote that, for her, impossible is

> … something that contradicts mathematics laws, principles and definitions, something that if you do it, it leads you to an absurd, something you cannot do in a specific framework of definitions, axioms and theorems.

In Carmen's words, very important aspects of the impossibility in mathematics may be identified: the contradiction to the mathematical structure built and the relativity of the notion possibility–impossibility to the system of axioms and definitions chosen. One of the examples of impossibility she mentioned was

> It is impossible to take the square root of a negative number if you are talking about Real numbers. If you think about the Complex Numbers, it is a different story. The same idea is true if you think about other operations defined in more simple sets. For example, you cannot subtract 10 from 7 if you are thinking of natural numbers. This operation is impossible in N but possible in Z.

Carmen's examples involve the idea that in some cases the non-existence of one thing is equivalent to existence of another thing.

> Euclid proves there's no largest prime number—no prime number greater than all other primes. Non-existence! Today the usual statement of this fact is: There exist infinitely many primes. Infinite existence! (Hersh, 1997, p. 84).

In Carmen's case, the non-existence of the square root of a negative real number, enables the creation of a new set of numbers, in which this operation is indeed possible. Yet the case of division by zero is another story.

When she studied the *centrified* triangle, she wrote:

> The centrified triangle is a right triangle, so it has all the properties of a right triangle: the Pythagorean Theorem, …

Then she continued reasoning:

> Since the triangle is right-angle, the median of the hypotenuse is a radius of the circumscribed circle. The baricenter of a general triangle belongs to every one of its medians and it divides each one of them into two segments of ratio 2:1. So, the baricenter is an inner point of each one of the medians of every triangle, especially in the case of the right triangle. But the circumscribed circle of a right triangle is one of the extremes of the median of the hypotenuse. Contradiction! It means that these two points cannot coincide in a right triangle. But if the latter condition is cancelled, these triangles may exist. I think that in other triangles it may be possible. But in the right-angle triangle, no! This case is impossible.

When asked if students should be exposed to the discussion of impossibility statements Carmen answered:

> I'm not sure we have this kind of tasks in our textbooks. I believe they contain just statements of the form 'Prove that …' But I find the idea of facing an uncertain statement much more charming. Some of the statements may be true, while others may not. I remember when you [the teacher] gave us a geoboard and you asked us to construct there a regular pentagon.[3] The fact that I found it difficult to construct it led me to suspect that such a pentagon may not exist. I thought that I might have chosen a wrong approach to the construction. But after a lot of thinking, I started to feel that this construction is impossible. Then, I had to decide. It may have been the first time I decided whether I was going to prove that there is no such a pentagon or by the contrary, I was going to keep on looking for such a pentagon. I could not prove it, but I was almost sure that this construction is impossible. Then, we discussed it in class, and you [the teacher] showed us how to prove it. In that case, I remember that I understood from the proof why such pentagons do not exist. I believed it before the proof, but after it, I knew why it was true.

Carmen's words remind us of one of George Polya's rules of inductive reasoning:

> You are moved to give up a task that withstands your repeated efforts. You desist only after many and great efforts if you are stubborn or deeply concerned. You desist after a few cursory trials if you are easygoing or not seriously concerned. Yet in any case there is a sort of inductive conclusion. The conjecture under consideration is:

> A. It is impossible to do this task.
> You observe:
> B: Even I cannot do this task.
> This, in itself, is very unlikely indeed. Yet certainly
> A implies B
> and so your observation of B renders A more credible, by the fundamental inductive pattern. (Polya, 1968, p. 17).

Carmen felt comfortable with this kind of thinking, she managed it quite well, and she was aware that her difficulties in finding an example of a regular pentagon with integral coordinates do not prove that such an example does not exist. She was convinced that such an example does not exist and she explicitly declares that for her, the proof does not provide *conviction* but *explanation* (De Villiers, 1990). Her description of the situation constitutes another testimony that conviction is not the unique role of proof. In that sense, a proof of an impossibility statement may convince someone that the statement is really impossible, but it may also explain why it is so. Such a conviction may come as a result of several unsuccessful attempts to find an example, but a proof of this impossibility may enlighten the reasons for this non-existence.

Dalia

Dalia wrote that

> It is impossible to prove an axiom or to define a fundamental concept.

Her example does not belong to the same category of the other mentioned examples. She wrote about an impossibility statement of the language in which mathematics is written, while the other examples are theorems of mathematics (Davis, 1986, p. 68). One of the examples she presented was:

> It is impossible for two parallel lines to meet.

In this case, it seems she used the definition of "parallel lines" and built a statement of the form "It is not the case …". This algorithm to build an impossibility statement in mathematics was frequent among other students too.

She wrote:

> In some sense, almost every theorem can be seen as a case of impossibility: it is impossible that its negation is true.

Her understanding of the logic involved in proof making is very strong and she explicitly wrote:

> If we remember that P is equivalent to *Not (Not P)*, we may change any statement P for It's impossible that Not P. For example, since we know that the sum of the interior angles of any triangle is equal to 180°, it is impossible to find a triangle with angles 30º, 30º, 100º. Well, it is a trivial example … But … it is surprising, I haven't thought about it before: If I take any three numbers (a,b,c) I cannot always construct a triangle, I can do that only if the

biggest is smaller than the sum of the other two. It is also an impossibility statement concerning the triangle.

Surprising is the fact that when asked if she would expose her students to this kind of statements she answered that she saw them just as a curiosity, as a logical game but without any serious value in itself. I prefer to have them learn how to prove "normal" statements.

Dalia mentioned an interesting point that needs further discussion: the logical equivalence between any statement P and the statement Not (Not P). This logical equivalence does not imply that the statements are alike from the psychological point of view. For example, one may be clearer or more convincing than the other, and while trying to construct a proof, one of the formulations may be more fruitful than the other. This point may be studied further, including also statements concerning impossibilities.

CONCLUDING REMARKS

From the profiles presented many questions arise concerning proofs and the processes involved in proving. One of the problematic issues may be the connections between examples and proof. That is, a) the roles played by examples in the formulation of a conjecture, b) their role performing existence proofs, c) their power to disprove a statement, d) the processes involved in the production of counterexamples to a statement. Another topic for further study may be the different roles played by proofs specially while treating an impossibility statement. Following De Villiers' framework (De Villiers, 1990), I suggest investigating: a) proof as a means for *verification* of the impossibility of the statement; b) a proof as a means for *explanation*, enlightening the reasons why the statement is impossible; c) a proof as a means of *discovery*, since the analysis of the proof of a certain result may lead to the a priori discovery—without conjecturing it before, without experimenting or without the trial an error involved in the inductive process—of an impossibility concerning the mathematical objects studied; d) and the *systematization* role that a proof may play when studying the robustness of group of theorems.

Yet another area for further investigation, concerning proofs and proving, may be the one that was opened by Abi's reflections concerning the different ways of proving an impossibility statement. He suggested that the indirect method may be the sole way to prove such statements. But in cases in which the problem studied allows an approach of a case by case exhaustive verification, the indirect method is not the only method available. Although in many cases there seems to be no choice, impossibility statements are not proved by using reductio ad absurdum only.

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9. THE DEVELOPMENT OF PROOF MAKING BY STUDENTS

INTRODUCTION

How do students come to make proofs? What are some of the conditions that support the development of proof making? As part of a long-term longitudinal study of the development of mathematical ideas in students, we investigated these and other questions about student learning of mathematics. Our data came from a large collection of videotapes of students working together in a community within which sense making was a cultural norm. We were particularly interested in observing the development of representations and explanations provided by the students, as well as the reasoning offered to support their ideas. Some arguments, justifications, and proof-making contributions of five of the students are reported in this chapter. The data span a seven-year period, elementary through high school.[1] Individual cognition with respect to the movement of student ideas within this group was also traced. In this report, we present examples of: (1) the representations produced by the students; (2) the reasoning exhibited and justifications made for their solutions; and (3) the connections they made between their earlier and later ideas.

THEORETICAL FRAMEWORK

Our work is based on the view that children are capable of building powerful and lasting images of mathematical ideas when challenged to investigate problematic situations which are interesting to them. Facilitators play an important role in the process by implementing carefully crafted mathematical investigations in their classrooms and by being attentive to the developing ideas of the students. As students engage in thoughtful activities and are later presented with similar tasks and/or extensions of them, earlier built images can be retrieved, modified, and extended (Davis, 1984, 1992a, 1992b; Davis & Maher, 1990, 1997; Maher, 2002, 2005; Maher & Davis, 1990, 1995). Teachers' awareness of their students' thinking makes possible the posing of tasks that can facilitate the building of connections between and among representations. Further, a timely seizing of opportunities for students to revisit earlier ideas helps them to refine their thinking and, as appropriate, to generalize their ideas. In soliciting explanations from their students, teachers can challenge them to provide appropriate justifications of their solutions. In this way, students' thinking can be nurtured and extended as they continue to do

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 197–209.

mathematics. Hence, fostering opportunities for students to think mathematically is not the objective of a single lesson, but a long-range goal.

Under these conditions, we study modifications and changes to earlier representations of student ideas. By tracing the development of student thinking over time, it is possible to understand how original ideas are built, refined, and connected as students attempt to solve similar, as well as new, problems. Pirie and Kieren (1992) also suggest that students revisit ideas and refine their conjectures. When faced with a new situation, students fold back to an inner level of understanding and reflect upon and reorganize earlier ideas in light of new information (Maher, 2005; Pirie & Kieren, 1992).

BACKGROUND

In 1984 a long-term partnership was initiated between Rutgers University and a school in a working-class New Jersey community. This led to several classroom studies of the development of mathematical ideas in students. The long-term study has its origin in 1988 with a class of first-grade students (Martino, 1992). Aspects of the study continue into its eighteenth year as new data are collected to trace the impact of the study on its participants. When Martino's students entered grade four, they became the focus group of two National Science Foundation research projects through high-school.[2]

An essential condition for the study was to provide children with an opportunity to construct mathematical ideas as they engaged in problem-solving activities (Maher & Martino, 1996). The students, since grade one, were encouraged to make conjectures, develop theories, and justify their solutions for a variety of mathematical problems. In explaining their ideas to each other, opportunities arose for them to invent strategies and build justifications. As their ideas were challenged, individual positions were sometimes reconsidered and modified, or reinforced and consolidated (Maher, 1991; Maher & Martino, 1997; Martino, 1992). In the course of these exchanges, some students built convincing arguments that took the form of proofs to validate their results.

METHODOLOGY

Classroom Investigations

Students routinely were organized to work on their tasks with a partner. Pairs of students sometimes joined to form small groups. Activities varied in time from fifty-minute regular class sessions to two-and-a-half to three-hour extended class periods. Usually, the activity took place over three school days.

The classroom explorations provided by the researcher took the following form: (1) presentation of the task; (2) questioning of student pairs and small groups about their ideas; (3) posing task extensions on the basis of assessment of student thinking; (4) facilitating the sharing of ideas between pairs and/or among groups; and (5) encouraging whole class sharing and discussion. The expectation for argument and challenge among the students was the norm. The decision to allow the students to have additional time to think about the problems was deliberate. Sometimes, original solutions were re-examined within a few days or weeks. At other times, the ideas were revisited over longer periods of time.

After-School Mathematics

When leaving their elementary school, some of the original cohort of students volunteered to continue doing mathematics together after school on Fridays. The students rearranged part-time work hours and accommodated busy sports schedules over several seasons (football, basketball, baseball and softball) to participate. The format for the investigations was: (1) presentation of the task by the researcher; (2) organization by the students into pairs and/or small groups; (3) sharing of ideas; and (4) posing of task extensions by the student(s) and/or researcher. Each session lasted for two-and-a-half to three hours.

Data Source

This report focuses on the development of mathematical thinking of a group of sixteen-year-old tenth-grade students: Ankur, Brian, Jeff, Michael and Romina.

Videotapes from grades four, five, and ten provide data that capture the children engaged in investigations in the area of combinatorics. The students explored combinatorial ideas by investigating particular problems as well as variations and extensions of them over the seven-year period. The investigations in grades four and five were whole class activities. For these sessions, three cameras were used to videotape student activity. The after-school grade-ten sessions were videotaped with two cameras, one directed on the actions of the group of students and the other focusing on their written work. For all sessions, a videographer and sound technician operated each camera and graduate students took field notes. The videotapes, researcher notes, and student work provided the data for the study.

The Tasks

A strand of investigations dealing with ideas in combinatorics was chosen because these problems: (1) were not a part of the regular mathematics curriculum that the children studied in school at that grade level; (2) afforded students opportunities to invent individual strategies, notations, rules and justifications; (3) provided for the building of isomorphisms; and (4) were later revisited at deeper levels of abstraction.

For this report, two tasks and variations of them are central in the investigations by the students. They involve building towers of varying heights when selecting from cubes of two colors and finding all pizza combinations when selecting from various topping choices. Variations of tower and pizza problems have been given extensively to many students (Maher, 2002, 2005; Maher & Martino, 1996; Maher & Speiser, 1997; Martino, 1992; Muter, 1999).

TWO EXAMPLES

Example 1: Michael's Use of Binary Code

The data for this example will be reported in three episodes that come from videotapes of problem-solving sessions in grades five and ten. The Pizza Problem (see Appendix 1), a task originally explored in the fifth grade, was revisited as the first of a series of related combinatorics investigations in grade ten. The students were asked to determine the number of different pizzas that could be made when there is the option of selecting from among four toppings, and then to find a way to convince each other that they had accounted for all possible choices. In the fifth grade, the students were able find a solution and justify their results. In the tenth grade, a coding scheme invented by one student, Michael, prompted the students to move beyond the solution for a particular case to a proof for the general case of n toppings.

Episode 1: Ankur, Brian, Jeff, Michael and Romina's Solution, Grade five—4/2/93
Romina, Jeff, Brian and Ankur worked as a group on the Pizza Problem as fifth graders, although they used a variety of strategies and representations to produce the sixteen combinations. These included a partial tree diagram, lists, and an organization that systematically controlled for variables. Michael developed his own solution, drawing circles to represent the various pizzas and labeling each "pizza" with its toppings. All of the students created codes using letters or abbreviations to represent the four toppings (for example, pepperoni = pe; m = mushrooms) and all decided to code for a pizza with no toppings (plain = pl or c = cheese).

Ankur explained the method that was used by the students to the researcher.

> *Ankur:* Okay. You start with the first one, that's P, and you mix it with the second one. That's P slash S [P/S]. And then you start with the first one again, skip the second one and go to the next one. That's M. P slash M [P/M]. Then you start with P again and mix it with the fourth one, PE. And then you start with the S since that's the—'cause you can't use plain. We start with S and mix it with M.
> *T/R:* Where's that?
> *Ankur:* S M [Ankur pointed to S/M on his paper.]
> *T/R:* That's this one. Okay.
> *Ankur:* Then we start with S and PE, right here. And we start with M, and PE. S and P is right here. The first one. [He pointed to P/S.]
> *T/R:* Okay. So why is it you can't go M with P?
> *Ankur:* Because you already have it: P M [He pointed to P/M.].

The pizzas were categorized as "whole" (plain and one-topping pizzas) and "mixed" (two or more toppings). The students found all sixteen pizzas, justifying their solution by the way they organized their results.

Episode 2: Ankur, Brian, Jeff and Romina's Solution, Grade ten—12/12/97 In the tenth-grade session, the students were asked if they recalled solving the Pizza Problem in elementary school. After some discussion, they were able to reconstruct the problem but chose to add a fifth topping. In the course of the work that ensued, they considered the cases of both four and five toppings.

Ankur, Brian, Jeff and Romina each started to write their own solutions, but soon began to collaborate. They talked aloud about combinations of toppings and of the patterns they were observing. They initially used a code of letters to represent the toppings. As they began to compare their lists of combinations, they switched their notation, using the numerals one through five. They decided that if five toppings were available, thirty different pizzas could be made with at least one topping, plus one plain cheese pizza, for a total of thirty-one.

Episode 3: Michael's Solution, Grade ten—12/12/97 While the other four students worked together, Michael spent at least fifteen minutes quietly developing his own solution. He found there were thirty-two pizzas when choosing from five toppings, disagreeing with his classmates. He invented a symbolic representation based on a binary coding scheme which enabled him to prove his results.

> *Michael:* I think it's thirty-two—with that cheese. And without the cheese, it would be thirty-one. I'll tell you why.
> *Ankur:* Mike, tell us the one we're missing then.

Michael responded by explaining what the zeros and ones meant in his representation and how they are used to write base ten numbers in base two.

> *Michael:* Okay, here's what I think. You know like a binary system we learned a while ago? Like with the ones and zeros—binary, right? The ones would mean a topping; zero means no topping. So if you had a four-topping pizza, you have four different places—in the binary system, you'd have—the first one would be just one. The second one would be that [he wrote 10]; that's the second number up. You remember what that was? This was like two, and this was three [he wrote 11].

Jeff recalled that they had seen this before.

> *Jeff:* know exactly what you're talking about. It's the thing we looked at in Mr. Poe's class [Mr. Poe was the students' teacher in grade eight.]; it was with computers.

Michael continued to relate his coding scheme to the pizza problem.

> *Michael:* Well, you get, I think—I have a thing in my head. It works out in my head … You've got four toppings. This is like four places of the binary system. It all equals up to fifteen. That's the answer for four toppings.

Romina sought clarification about the meaning for the zeros and ones. Jeff responded that one indicated a single topping choice and zero meant no toppings.

Romina: So is the one—is that your topping?
Jeff: Yeah. Each one is a topping. The zeros are no toppings. The ones are toppings.

Michael then summarized his conclusions.

Michael: So you go from this number [He wrote 0001], which is in the binary system; it's one, to this number [He wrote 1111], which is fifteen, and that's how many toppings you have. There's fifteen different combinations of ones and zeros if you have four different places.
Brian: Wow! [He indicated his enthusiasm for Michael's solution.]
Michael: I don't know how to explain it, but it works out. That's in my head—these weird things going on in my head. And if you have an extra topping, you just add an extra place and that would be sixteen, that would be thirty-one.

Michael's representation using binary numbers did not include the representation for a plain cheese pizza [0 0 0 0]. However, he corrected for this by adding one to the fifteen combinations (for making pizzas when selecting from four toppings) and to the thirty-one combinations (for making pizzas with five toppings available), thus accounting for all possibilities.

Jeff: And then you add the cheese?
Michael: Plus the cheese would be thirty-two.

With the assistance of the other students, Michael presented his binary coding scheme to the researcher, saying, "This is the way I interpret it into the pizza problem." When the researcher asked questions about Michael's solution, other students responded.

Researcher: What's the difference between one, zero, zero, zero [1 0 0 0] and zero, one, zero, zero [0 1 0 0]?
Jeff: Well, that would be the difference between an onion pizza and a pepperoni pizza.

Jeff suggested that the others label each column with the name of a topping. Michael agreed, noting that the entry of one in a column indicated that pizza had that particular topping. As an extension, the students were asked to consider the case where ten toppings were available. While investigating this extension, the group recognized that a string containing all zeros represented a plain cheese pizza. Finally, they were asked to generalize to the case of n toppings. After working on the problem for a few minutes, they determined that there would be 2n different pizzas when there are n topping choices.

Example 2: Michael and Ankur's Connection

Two episodes, one from grade four and the other from grade ten, describe the reasoning the students used to justify their answer of ten for the number of towers,[3] five-tall, that could be made using exactly two red cubes, when selecting from red and yellow cubes. Data from both grade four and grade ten indicate that students were able to build a convincing argument for their solution. However, as tenth graders, the students utilized a binary code to represent the tower cases in justifying their solution. Their use of the new representation enabled them to generalize their solution to towers n-tall.

Episode 1: Ankur, Michael, and classmates, Grade four—2/6/92 After working on the task for approximately one hour, the students were asked to share their ideas. The researcher led the discussion of the question: “How can you be sure that all towers, five-tall with exactly two red have been built?” Following a short period of time during which the students worked together to find an answer, the researcher again questioned the class as a whole.

Researcher: Now somebody at this table told me that when I looked at all the towers with exactly two red, there would be how many of them?
Ankur: Ten.
Researcher: How many got ten? Towers with exactly two red floors? How many? Okay. Now [what] I want you to think [about for] tomorrow is [about] how you can convince me that you found the ten, that there can’t be eleven, or twelve, or eight, or nine, or six.

The students responded immediately to the challenge and developed a proof by cases. They first determined that there were exactly four where the two red blocks were “stuck together,” moving down one position from each tower to the next. When asked if “stuck together” was the only way that a tower five-high with exactly two red blocks could be built, they responded by finding the three towers which can be built with the two red blocks separated by one yellow block. The following explanation was offered by Michael for Ankur’s assertion that there are only two towers five-tall with two red blocks separated by two yellow blocks.

Researcher: I’m asking you to find me exactly two reds separated by two.
Michael: Here’s a third one, here’s a third one. [He showed another tower.]
Ankur: There’s only two.
Researcher: I got that.
Ankur: There’s only two.
Researcher: Yes? [indicating Ankur].
Ankur: There’s only two.
Researcher: Why are there only two? I see a lot of hands here. [indicating student enthusiasm].
Michael: Because if you needed one more, you would need more than five, because you need another one. Because
Researcher: Wonderful! You’d need another block. So let’s put this here. Is there another way to have two reds?

The discussion concluded with students determining that there could only be one tower with the two red blocks separated by the three yellow blocks and that no further towers with this particular set of requirements could be built. They determined that exactly ten towers could be built with the condition of including exactly two red cubes and they made a convincing argument using a proof by cases: all towers with two red "stuck together," all towers with the two red cubes separated by one yellow cube, all towers with the two red cubes separated by two yellow cubes, and all towers with the two red cubes separated by three yellow cubes. While the full argument was not voiced individually by either Michael or Ankur in this particular episode, it was convincingly put forth by several members of the classroom community in which the two boys were active participants.

Episode 2: Michael and Ankur, Grade ten—12/19/97 During an after-school session, the problem posed to them as fourth graders was again presented. The group quickly answered "ten" to the question of how many five-tall towers could be built with exactly two red cubes when selecting from red and yellow cubes. They were then asked to justify their answer.

Michael and Ankur used the coding scheme developed by Michael to solve the pizza problem the previous week (see Example 1, Episode 3). Using a zero to represent a yellow block and a one to represent a red block, they constructed an array (see Figure 1) that parallels the organization used in their fourth grade activity (see Example 2, Episode 1).

1	0	0	0	1	1	1	0	0	0
1	1	0	0	0	0	0	1	0	1
0	1	1	0	1	0	0	0	1	0
0	0	1	1	0	1	0	1	0	0
0	0	0	1	0	0	1	0	1	1

Figure 1. Ankur's and Michael's first array.

Ankur, while explaining their solution, referred to a different arrangement of zeros and ones (see Figure 2), which indicated a red cube fixed in the top position and a second red block moved into successively lower positions until it reached the bottom position. The fixed red was then moved into the second position and the process was repeated until they accounted for all possible towers.

What is interesting to observe is that the same students, both as fourth and tenth graders, provided a justification for their solution that took the form of a proof by cases (Muter & Maher, 1998; Kiczek & Maher, 1998; Muter, 1999). However, the representations across years differed. It seems that the notation used in grade ten facilitated the building of a general solution.

1	1	1	1	0	0	0	0	0	0
1	0	0	0	1	1	1	0	0	0
0	1	0	0	1	0	0	1	1	0
0	0	1	0	0	1	0	1	0	1
0	0	0	1	0	0	1	0	1	1

Figure 2. Ankur's and Michael's second array.

CONCLUSIONS

Videotape data enable us to view the same students working on problems in grades four and five, and again in grade ten. We have observed fourth-grade students build an argument for a proof by cases of a combinatorics tower-problem they had been investigating earlier.[4] When those same students revisited the problem in grade ten, we observed that earlier ideas were retrieved, modified and extended. The tenth-graders utilized arguments parallel to those seen in the fourth-grade classroom. However, their representations were expressed symbolically, rather than with concrete objects.

In the fifth-grade pizza problem-solving session, the original representations displayed by the students made use of a notation that enabled them to keep track of their ideas and to account for all possibilities to reach a solution. This system enabled them to justify their solution for finding all possible pizzas that could be made when selecting from four different topping choices. In the tenth grade session, representations displayed by Romina, Jeff, Brian and Ankur were similar to those used earlier. Michael's representation, however, was different, drawing from an image he retrieved from his eighth-grade mathematics class - a binary number notation using zeros and ones. The students readily adopted Michael's notation and mapped their representation using numerals for topping choices into Michael's representation to develop a justification for their solution, and then, to generalize to n-topping choices.

In a subsequent session, Michael's binary notation was applied by the students to solve the isomorphic combinatorics tower problem. They were quickly able to generalize their solution to towers n-tall selecting from two colors. Later, they extended their reasoning to account for all towers n-tall when selecting from r colors (Muter, 1999).

These students worked together to build convincing arguments for their solutions to problems. The findings support the importance of introducing rich investigations to young children, challenging them to support their ideas, and providing opportunities to revisit tasks as they grow older and have more tools available to build upon their earlier ideas.

The findings suggest the following hypotheses for further study: (1) students propose thoughtful and strong arguments at a young age and build upon those ideas in later years; (2) student representations become more symbolic and abstract over time; (3) the structure of student arguments remains durable; and (4) student representational systems become more elegant and powerful over time.

NOTES

[1] We report here on the mathematical behavior of the children during the interval of grade 4 (as nine-year olds) through grade 10 (as sixteen-year olds).

[2] The research on learning and reasoning has been funded, in part, by National Science Foundation awards MDR-9053597 and REC-9814846. Any opinions, findings, conclusions and recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[3] A tower is an ordered sequence of Unifix cubes, snapped together. Each cube is called a *block*. The height of the tower is the number of its blocks.

[4] The students worked on tower problems in grades three and four (Maher & Martino, 1996).

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APPENDIX 1: PIZZA PROBLEMS

The Pizza Problem. Pizza Hut® has asked us to help design a form to keep track of certain pizza choices. They offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. How many choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possibilities.

Variations include:

The Capri Pizza Problem. Capri Pizza® in Kenilworth has asked us to help them design a form to keep track of certain pizza sales. Their standard "plain" pizza contains cheese. On this cheese pizza, one or two toppings could be added to either half of the plain pie or the whole pie. How many choices do customers have if they could choose from two different toppings (sausage and pepperoni) that could be placed on either the whole pizza or half of a cheese pizza? List all possibilities. Show your plan for determining these choices. Convince us that you have accounted for all possibilities and that there could be no more.

Another Pizza Problem. Pizza Hut® was so pleased with your help on the first problem that they have asked us to continue our work. Remember they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. Pizza Hut® now wants to offer a choice of crusts: regular (thin) or Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

A Final Pizza Problem. At customer request, Pizza Hut® has agreed to fill orders with different choices for each half of a pizza. Remember they offer a cheese pizza with tomato sauce. A customer can then select from the following toppings: peppers, sausage, mushrooms and pepperoni. There is a choice of crusts: regular (thin) or Sicilian (thick). How many different choices for pizza does a customer have? List all the possible choices. Find a way to convince each other that you have accounted for all possible choices.

APPENDIX 2: TOWER PROBLEMS

Building Towers Five-Tall. Your group has two colors of Unifix Cubes. Work together and make as many different towers five cubes high as is possible when selecting from two colors. See if you and your partner can plan a good way to find all the towers five cubes high.

Guess My Tower. You have been invited to participate in a TV Quiz Show and the opportunity to win a vacation to DisneyWorld. The game is played by choosing one of four possibilities for winning and then picking a tower out of a covered box. If the tower you pick matches your choice, you win. You are told that the box contains all possible towers that are three tall that can be built when you select from cubes of two colors, red and yellow.

You are given the following possibilities for a winning tower:

7. All cubes are exactly the same color.
8. There is only one red cube.
9. Exactly two cubes are red.
10. At least two cubes are yellow.

1. Which choice would you make and why would this choice be better than any of the others?

   Assuming you won, you can play again for the Grand Prize which means you can take a friend to DisneyWorld. But now your box has all possible towers that are four tall (built by selecting from the two colors yellow and red). You are to select from the same four possibilities for a winning tower.
2. Which choice would you make this time and why would this choice be better than any of the others?

MARIOLINA BARTOLINI BUSSI, PAOLO BOERO, FRANCA FERRI, ROSSELLA GARUTI AND MARIA ALESSANDRA MARIOTTI

10. APPROACHING AND DEVELOPING THE CULTURE OF GEOMETRY THEOREMS IN SCHOOL

A Theoretical Framework

INTRODUCTION

This chapter is based on research work regarding the approach to geometry theorems and theories in schools carried out over the last ten years by teams in Genoa, Modena and Pisa. These studies have involved students of different age groups (from grade 5 to grade 12) and different thematic contexts. Although the specific goals of these projects differed to some extent, they did share some common features such as general goals, research methodology, epistemological analysis and cultural, cognitive and educational hypotheses. A common framework has emerged as a result of a longstanding dialectic discussion dating back to the design of our teaching experiments: this framework has brought to light some of the deep yet implicit common motives and theoretical perspectives of our independent research designs. This chapter provides a unified framework of the research studies which have been reported in other papers (Bartolini Bussi, 1996; Bartolini Bussi et al., 1999; Boero et al., 1996, 1999; Garuti et al., 1996, 1998; Mariotti, 1995; Mariotti et al., 1997). The same studies will be partially reported in the next chapters. We will focus on the following general points: the function of the different contexts in approaching geometry theorems; the role of the teacher in classroom interaction; and the idea of theorem as an unity of statement, proof and theory.

Our studies take into account the following issues in current research into the school approach to theorems: the present-day value of proof in mathematics and mathematics education (Hanna, 1989 and this book), even for very young pupils (Maher, 1995 and this book); the social dimensions of the approach to proof (Balacheff, 1991) and the distinction between argumentative reasoning and deductive reasoning (Balacheff, 1988; Duval, 1991); the classification of student proof schemes (Harel & Sowder, 1998 and this book) and the relevance of 'transformational reasoning' in the production of statements and the construction of proofs (Simon, 1996); the study of the potentialities of geometrical software (Goldenberg & Cuoco, 1995; Laborde, 1993). Inspired by the seminal work of Balacheff (1988) and other studies (e.g. De Villiers, 1991; Hanna & Jahnke, 1993) on the pragmatic of proof, we focused on the link between epistemological, cognitive and didactic analysis.

P. Boero (Ed.), Theorems in schools: From history epistemology and cognition to classroom practice, 211–217.

THEORETICAL FRAMEWORK

The general theoretical framework of our research studies is based on the construct of '*field of experience*' and the construct of '*mathematical discussion*'.

As reported in Boero et al. (1995), a field of experience can be metaphorically defined as a system of three evolutive components (external context, the student's internal context and the teacher's internal context) referred to a sector of human culture, which the teacher and students can recognize and consider as unitary and homogeneous. Classroom activities within any field of experience can have different goals: in this chapter we shall limit ourselves to those related to approaching geometry theorems. In this perspective, the features of a field of experience that are meaningful for us can be described as follows:

- the presence of 'concrete' and semantically pregnant referents (external context) for performing concrete actions that allow the internalization of the visual field where dynamic mental experiments are carried out; this feature is consistent with Vygotski's general theory on mental processes and with specific findings on the function of dynamic processes both in the production of conjectures and in the construction of proofs (see Polya's idea of variational strategies, as well as the recent consideration of 'transformational reasoning' by Simon, 1996);
- the presence of semiotic mediation tools (including excerpts from historical sources, documents, meaningful linguistic expressions), chosen by the teacher from the cultural heritage with the aim of introducing the mathematical idea of theorem;
- the construction of an evolving student internal context, rooted in the dynamic exploration, where different processes such as conjecturing, arguing, proving and systematizing proofs as formal deduction are given sense and value.

These points are consistent with general ideas about the production of geometry statements and the construction of proofs relying on the one hand on 'reified' (Sfard, 1991) pieces of knowledge produced by the historical evolution of mathematics and, on the other, on figural (Fischbein, 1993) referents, which may be either static or dynamic.

As concerns mathematical discussion, we refer to the metaphorical definition given by Bartolini Bussi (1996): mathematical discussion is a polyphony of articulated voices on a mathematical object, which is one of the motives of the teaching–learning activity. In this case the motive of the discussion is a specific theorem together with the idea of theorem itself (see below). Therefore the complex of conjecturing, arguing, proving and systematizing proofs related to a specific problem situation is taken into account by the teacher by means of mathematical discussion. The continuity between argumentation and proof is naturally emphasized by argumentative behaviours, but at the same time the distance between argumentation and proof (Balacheff, 1988; Duval, 1991; Moore, 1994) is taken into account by the teacher's careful management of discussion with the specific aim of the social construction of the sense and value of a theorem. Concerning this issue, we believe that two crucial points emerge from current literature: on the one hand, the problem of the motivation to proof; and on the other, the distinction between argumentation and mathematical proof. These two aspects are linked to each other in a complex way. Motivation to proof can be expressed at different levels. At the first level the truth of the fact is central: Is a fact true? At the second level, truth may no longer be in question, but a foundation of truth is needed: Why is a fact true? Hence the sense and the need for this grounding process (validation) is detached from the truth of the fact. In the first question, the truth of the fact is uncertain whilst in the second the truth of the fact may be certain. In our opinion, the uncertainty status of the truth of a statement is crucial for the initial construction of the meaning of theorems and calls for the careful selection of problem-solving situations, where the production of a conjecture is required. A third level, which is not considered in the research studies reviewed in this chapter, concerns the release of theorems from the issue of truth search. In other words, we do not deal with formal proofs and their release from semantics.

Within this general framework, we introduce two specific theoretical constructs, the ‘cognitive unity’ and the ‘mathematical theorem’, which may help the management of class work on geometry theorems, the functioning of problem-solving situations and the interpretations of student behaviour. These constructs may also represent instruments for analysing some difficulties students meet when following the traditional school approach to geometry theorems.

Cognitive Unity of Theorems

Analysis of work done by past and present geometers highlights the continuity that can exist between the process of statement production and the construction of its proof, as well as providing meaningful examples. This continuity is not evident at all in the theoretical systematization of ancient classical geometers such as Euclid and Apollonius, but is emphasized as from the 17th century, in documents that reveal the process by which a result has been obtained (Barbin, 1988). What is in play is the relationship between conjecturing and looking for a proof, in particular specifying the objects of the conjecture and determining stricter hypotheses or stating a new weaker conjecture (Alibert & Thomas, 1991; Lakatos, 1976; Thurston, 1994). More generally, the development of the relationship between conjecturing and proving witnesses the longstanding process of elaboration of the idea of rigour.

Does a cognitive counterpart of this analysis exist? A metaphorical definition may be useful in analysing student processes. The continuity that can exist between the processes of conjecture production and proof construction, recognizable in the close correspondence between the nature and the objects of the mental activities involved, expresses a cognitive phenomenon, which will henceforth be referred to as ‘cognitive unity’. Some hints about ‘cognitive unity’ are given in Harel and Sowder’s investigation into student behaviour (Harel & Sowder, 1998). Some of the following chapters provide experimental evidence about ‘cognitive unity’ and the fact that cognitive unity can play an important role in facilitating students’ approach to construction of proofs (see Chapters 12 and 13 in this book).

Mathematical Theorem

However, in mathematicians' mathematics the aforementioned continuity between statement and proof is always considered in a theoretical context, even if the context can change over time; the existence of a reference theory as a system of shared principles and deduction rules is needed if we are to speak of proof in a mathematical sense. Principles and deduction rules are intimately interrelated so that what characterizes a mathematical theorem is the system of statement, proof and theory. Historical-epistemological analysis highlights important aspects of this complex link and shows how it has evolved over the centuries.

Subsequent papers will show how the theoretical construct of "Mathematical theorem" can play an important role in planning long term teaching experiments (see Parenti et al., this book)

TOWARDS TEACHING EXPERIMENTS

According to the theoretical framework presented in the previous section, two crucial elements characterize the approach to geometry theorems in our teaching experiments: the function of a particular field of experience, and the role of the teacher as a cultural and cognitive mediator.

Every field of experience has to be analysed in terms of limits and potentialities in fostering cognitive unity and a systemic approach to geometry theorems. Historical and epistemological analysis has allowed us to identify the following criteria which, in the presence of a culturally relevant piece of mathematical knowledge, make it possible to choose a field of experience and particular activities within it: the need for concrete and semantically pregnant referents that promote dynamic processes; and the availability of tasks, meaningful to the field of experience, that foster cognitive unity. Dynamic exploration of the problem situation can determine the production of conditional statements and the construction of proofs, with strong functional relationships between these processes (Boero et al., 1996; Bartolini Bussi et al., 1999; Garuti et al., 1996, 1998). The conditional form of most geometry statements, from Euclid to the present day, represents the functional connection between statement and proof: actually, a proof develops, in the form of a deductive chain, the link (which is implicit in the statement) between facts that are assumed as starting points in the frame theory and the 'thesis' of the theorem, under some conditions that are given as 'hypotheses'.

As far as the role of the teacher is concerned, we assume that the process of construction of the meaning of theorems, although rooted in the field of experience, requires cultural and cognitive mediation. Actually, the teacher is responsible for introducing pupils to a theoretical perspective, which, although not spontaneous, is needed for a systemic view of mathematical theorems. In our teaching experiments, the construction of a theory is pursued in the form of accepted principles: invariants in perspective representation; the evident properties of shadows produced by vertical nails; and the underlying logic of Cabri.

The research methodology is typical of long-term teaching experiments: classrooms are observed for several months (or even years), by collecting individual texts and transcripts of collective discussions, together with teachers' reports. The length of the process determines evolution in the general assumptions, until specific hypotheses are reached. The specific aim of these studies is on the one hand to single out the conditions under which students can approach geometry theorems, and on the other to study the mental processes involved in such an approach. For these reasons, the direct and productive involvement of teachers in all the phases of research is called for in each of the three experiments. In spite of the common features they share, the studies deal with different didactic problems, and actually concern different school levels (5th, 6th, 7th, 8th, 10th, 12th grades). This requires completely different approaches to geometry theorems for two reasons: the different levels of cognitive development and geometrical knowledge that pupils have reached (geometry is taught in Italian schools from the 1st grade); and the general attitude towards mathematics and its methods derived from their past experiences. At the outset of work on geometry theorems, younger students do not yet have a sufficient grasp of geometry notions. For them, the approach to geometry theorems is a fundamental step in the process by which geometry gradually becomes a 'field of experience' (Boero et al., 1995) and a corpus of mathematical knowledge as well. At high-school level, where students have a grounding in geometry, the problem is how to manage the delicate relationship between their geometrical background and a new approach to this knowledge from a deductive point of view. (See Chapter 14 and 15 in this book for further details about these issues.)

An Historical Digression: the Birth of a Theory

The history of geometry gives meaningful examples of the development of fully-fledged theories from a long-standing tradition of spatial practices. In this section we shall explore a paradigmatic example: the birth of projective geometry from the long-standing process of assuming properties of space and vision as axioms and modelling definitions, and of proving practical rules of painting as theorems.

Natural perspective was developed from the classical age (Euclid's Optics) onward with the aim of representing objects with illusionistic effects. Practical rules for painting were transmitted in artists' workshops and collected in treatises of practical geometry. In the 15th century natural perspective gradually gave way to artificial perspective. This was based first of all on the idea of the (central) vanishing point or point of flight: if we consider the picture plane as a vertical window the spectator stands in front of, the central vanishing point is the point of the picture plane where a line from the spectator's eye, orthogonal to the picture plane, cuts it. This definition is taken from a more recent treatise (by Brook Taylor) where the genesis from practice was already somewhat hidden. The genesis is more evident if we consider that in early treatises, which contain also a theory of vision in space (e.g. Piero della Francesca) the central vanishing point was named 'eye'. The history of the theoretical development of artificial perspective up to projective geometry is actually the history of its progressive independence from painting practices, from Desargues' first introduction of invariants (Field & Gray, 1987) to the 18th century treatises of linear perspectives of Brook Taylor and Lambert (Bessot

& Le Goff, 1992): the incidence axioms listed by Brook Taylor gave birth to a projective approach to problems, and Lambert's use of perspective to prove properties of plane configurations stated definite autonomy from painting. Within the theory of projective geometry, based on incidence axioms, practical rules of painting assumed the status of theorems.

A similar analysis could be made for the genesis of other theories in the history of geometry (see the analysis of sunshadows in Serres, 1993 and the analysis of geometrical construction in Lebesgue, 1950). Actually, the above perspective has guided our experimental research studies into the school approach to geometry theorems.

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11. CONSTRUCTION PROBLEMS IN PRIMARY SCHOOL

A Case From the Geometry of Circle

Proofs arrange propositions. They give them connection

(Wittgenstein, Bemerkungen über die Grundlagen der Mathematik, Appendix 6, 1)

INTRODUCTION

The approach to theoretical thinking in primary school has been focused by our research team from the early nineties within the project *Mathematical Discussion*. We have studied, in the recent past, two cases: the kinematics of gears (Bartolini Bussi et al., 1999) and the representation of visible world by means of perspective drawing without (Bartolini Bussi, 1996) and with instruments (Bartolini Bussi et al., 2005). The case studies refer to two complementary ways of modelling spatial experience, which produced two different geometries (mutually exclusive for centuries): the geometry of metric-mechanical activity and the geometry of light and sight. The former, related from the classical age to the development of technology, was represented in the 17th century by Descartes, who worked in Euclid's tradition, whilst the latter was represented by Desargues, who laid the foundations of modern projective geometry, introducing into academic geometry hints from extramathematical fields, such as architecture and painting. The language developed is partially overlapped: for instance the word 'line' is used in both cases, but the quality of experiences is quite different, to the extent that we might well say that the concept of 'line' in the geometry of metric-mechanical activity is quite different from the corresponding one in the geometry of light and sight (see the different roles played by measure and infinity). This difference is mirrored in the different axiomatic theories that were developed independently and reconciled only at the end of the 19th century (Otte, 1997).

In the classroom activity, designed and implemented by our research team for all the experiments, the detachment from conceiving the empirical verification as the only tool suitable to solve conflictual situations was carefully managed by the teacher by means of the collective construction of *germ-theories* (a germ-theory is an embryo of theory that has an expansive potency and a tendency to develop into a fully-fledged one). The basic elements of germ-theories in both cases were

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 219–247.

chosen from historical sources (respectively, Euclid and Heron of Alexandria for the former and Piero della Francesca and Alberti for the latter), some excerpts of which were introduced in classes too, with the purpose of guided reading and interpretation.

The approach to theorems in primary school is conceived by our team as a paradigmatic case of approach to theoretical thinking: on the one hand, it requires the progressive refinement of language to avoid ambiguity, that is meant as an introduction to the problem of definition; on the other hand, it requires the shift from empirical to argumentative validation, that is meant as an introduction to the problem of proof. In this process, new statements together with more and more refined argumentative explanations are produced and functionally interlaced with the collective construction of the reference theory, that allows one to change their status into the status of *theorems* (i.e. triples (S, P, T)), given by a Statement and a Proof in a Reference Theory (see Mariotti, in this volume).

No research study on a didactical approach to geometry theorems could avoid the issue of construction problems. The distinction between problems and theorems (Heath 1925–1956) dates back to Euclid's Elements and is maintained meticulously by the authors of the principal treatises of classical geometry. However, a strict link between theorems and problems is natural: on the one side, each problem contains at least an implicit theorem, because it contains the proof that, within the given theory and under the specified conditions, the figure obtained by the construction has the required properties; on the other side, most geometry theorems imply the solution of some construction problem, at least whenever an auxiliary construction is required.

In this chapter we shall deal with a paradigmatic construction problem concerning the geometry of circle. By analysing the class processes raised by the assignment of this task, we shall touch some relevant issues, such as the management of the delicate relationship between concrete practices and theoretical thinking, and we shall analyse how the shift from each other was intentionally provoked during class interaction.

Circles were introduced in the project *Gears* from the 3rd grade in a dynamic way in the process of modelling toothed wheels in gears, up to the appropriation of the definition of circle in both Euclid's and Heron's forms (see the following section). As toothed wheels were modelled as circles ('toothless' wheels), the cogs of two wheels in gear with each other were naturally modelled as 'the' point of contact of two tangent circles.

The theory was constructed collectively, by means of some introductory problems, up to the statement of some 'postulates' that described properties shared among all the pupils (Ferri et al., 1998). In this situation, a construction problem, that pla yed a crucial role in the sequence, was given to 5th graders: on a sheet where two external circles were already drawn, the pupils were asked to draw another circle with a given radius touching the given ones and to explain carefully and justify the method used.

As we shall argue in the following, this problem could be solved by an adult as a standard application of the classical method of analysis and synthesis (see

Arzarello, in this volume). For expert problem solvers this process appears so natural, that it is easy to identify it as for the universal mental process that underlies the production of the procedure and its validation (actually this misunderstanding is supposed to permeate Heath's presentation of the method, that is focused only on the logical relationship between the statements involved as hypotheses, theses or intermediate steps).

However fascinating this idea might be, it is not true, at least for novices. This chapter will defend the following theses; the former focuses the classroom processes, whilst the latter focuses the individual process.

The classical method of analysis–synthesis is not, for beginners, a model of the mental processes underlying the individual solution of a problem but rather a model of the overall long term process that is realised in the class, under teacher guidance; the eventual transformation of this interpsychological process into an intrapsychological one might be an example of the internalization, as it is meant by Vygotskij (1978).

This process of analysis–synthesis is accomplished by means of dynamic exploration of the referents of the problem, provided that we admit both concrete and abstract referents and both physical and ideal explorations with a continuous shift from each other, even if, at the very end of the process, the product might well be a method of construction that has apparently eliminated this dynamic component; in this process, new pieces of theory might be produced. From the analysis of protocols, the relationship between a material tool (the usual compass) and a 'mental' tool (the 'geometric' compass, that orient the definition of circle towards the solution of construction problems) will be emphasized.

The data have been collected until now in some 5th and 8th grade classes. This chapter will consider only the data from two 5th grade classes, which represent two different situations of high and low level pupils. We intend to illustrate also the differences in teachers' management of the 'same' class activities in such different situations.

THE REFERENCE CULTURE: THE GEOMETRY OF GEARS.

The field of experience (Boero et al., 1995) of gears refers to physical objects that can be handled (gears and mechanisms), to their representations (figures, arrows) and to the explicit mathematical theories, by which modelling can be realized. Even if we neglect dynamics problems (like speed problems, out of the reach of young pupils), we have at least two different kinds of elementary problems: (1) construction of either a gear or a mechanism with given parts; (2) motion or functioning of a given gear or mechanism. Examples of the former are also the descriptions by words, drawing, signs of mechanisms and gears as far as shapes are concerned (a kind of deconstruction problem): we can have actual construction (or deconstruction) with physical objects and graphical construction (or deconstruction) with drawing representing objects (see later in this chapter). Examples of the latter are the formulations of either previsional or interpretive hypotheses about the motion of a mechanism or of some part of it. A combination of a construction problem and of a motion problem is represented by (3) the design of a mechanism with given parts that are to fit in a given space to realize some given operation (the inverse problem concerns the study of a real mechanisms with reference to shapes and to motion as well). A didactical approach to the theory of motion and functioning in primary school is described in Bartolini Bussi et al. (1999).

Modelling two wheels in gear requires circles tangent either externally or internally: an example of the former is the tape cleaner; an example of the latter is the kitchen centrifuge (vegetables-drier) with a large toothed wheel with teeth inside and a small toothed wheel with teeth outside.

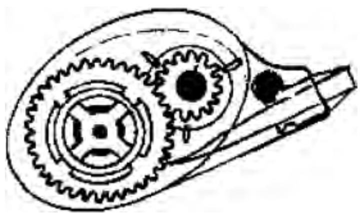

Figure 1. A tape cleaner.

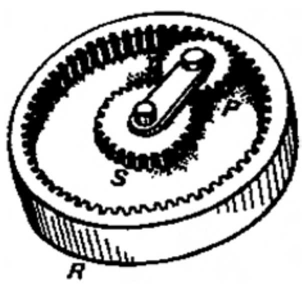

Figure 2. The gear of a vegetables-drier.

The crucial situation of inserting any third wheel between two given separate coplanar wheels is the mechanical counterpart of the geometric problem of drawing a circle tangent to two given circles. The germ-theory within which to search for a solution is constructed around one property, that can be expressed in one of the following ways:

1. If two wheels are in gear with one another, the straight line joining their centres will pass through the point of gear (referents: wheels).
2. If two circles touch one another, the straight line joining their centres will pass through the point of contact (referents: circles).

In Euclid's Elements (III, 12), the latter is the statement of a theorem about circles, but in our teaching experiment it is assumed as a *postulate* of the germ-theory.

Euclidean geometry is an answer to the need of rationalizing the shapes of gears and mechanisms. Sometimes, plane geometry is enough, yet in other cases space geometry is needed. For instance, corkscrews have a pair of lateral toothed wheels and a central rack that are coplanar, to transform the symmetrical circular motions of wheels downwards into the linear motion of the cork upwards; bicycles have coaxial cog-wheels in parallel planes to change gear; mills have cog-wheels in orthogonal planes, to transform the circular motion of mill water-wheel into the circular motion of the millstone. Geometrical constructions with ruler and compasses are an answer to the need of producing pictures of gears with the purpose of representing or designing, at least for planar objects. As far as spatial objects are concerned the need of forms of technical drawing (such as orthographic projections, cutaway and exploded views) might arise.

In the physical experience with mechanisms (and gears evocative of real mechanisms) the problems of construction and of motion cannot be easily separated: actually while handling mechanisms there is both the perception of shapes and the awareness of the final end determined by the chain of motions. This strict link is actually evident also in ancient descriptions of machines where shapes, materials, practices and embryonic kinematic or dynamic theories were always intertwined (see Bartolini Bussi, 1993, for a discussion of this point). Actually, the practical experience in the fields of gears and mechanisms constitutes the *pragmatic basis* (Hanna & Jahnke, 1996) of at least two complementary theories, namely Euclid's geometry and Heron's kinematics. The above theories, embodied by two scientists of the far past, constitute different historical voices of an ideal dialogue about the modelling of objects in space.

What seems interesting from a didactical perspective is that this dialogue is going to have a new life in the class. The reference to two complementary theories has interesting effects concerning the relationships with each other. Actually kinematics is not conceived in our project as a byproduct of geometry as it could be in a top-down systematization of the field: rather the relationship between geometry and kinematics is dialectical. So, on the one hand, geometry offers a language that can be used also in kinematics; on the other hand, the kinematic approach has effects also on the elements of geometry. The case of circle is exemplary. The static Euclid's definition of the circle, a line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another, the point being called the centre of the circle (Euclid, Definitions 15 and 16), is complemented by Heron's dynamic definition, reconsidered later by Newton, Spinoza and others:

> the circle is the figure described when a straight line, always remaining in one plane, moves about one extremity as a fixed point until it returns to its first position (Heron, as quoted by Heath, 1956).

From a didactical perspective, this dialogue is intended to be appropriated by pupils. By means of suitable tasks, and under teacher guidance, the pupils become able to shift from wheels to circles and vice-versa and to import freely exploration strategies from one context to the other (see Ferri et al., 1997). In the segment of the teaching experiment analysed in this chapter, the appropriation of this dialectial relationship was already realized and will be observed at work in the solution of problems.

THE TEACHING EXPERIMENT: THE CHRONICLE

Introduction

This experiment was carried out in two different 5th grade classes, taught from the 1st grade by two teacher–researchers of our team. The two classes will be referred to in the following as class C1 (teacher T1: Franca Ferri) and class C2 (teacher T2: Mara Boni). Both teachers were implementing the project on *Mathematical Discussion*, with great emphasis on the collective construction of knowledge.

A lot of care had been spent by the teachers in improving the mastery of natural language, verbal and written as well, since the early grades of primary school. The pupils had always been asked (and guided) to explain carefully on the sheet not only the solution but also the process for producing the solution itself: their protocols had always been given great value, becoming the basis of the very important balance discussions (Bartolini Bussi, 1996), i.e. discussions aiming at exposing and evaluating different strategies and solutions, in order to come to one (or more) shared solution.

A byproduct of the teacher's management was in both cases an emphasis on the search for *general methods* whose application might not be limited to an individual situation.

This was the common background of the two classes. Yet some differences were present, concerning the socio-cultural extraction of the pupils: class C1 consisted of pupils with rich, yet disperse, stimuli from their families, whilst class C2 consisted of pupils with learning disabilities and little help, if any, from their families. This difference must be recalled, because it might help to analyze the different management strategies of the teacher for similar tasks.

The sequence of tasks was similar. Actually class C1 implemented the project first, whilst class C2 followed the same project some months later, at a slower pace and with slight intentional differences (these will be partly described in the following), influenced by the analysis of the data collected in class C1. The whole sequence was developed over three school years (3rd, 4th and 5th grade). It consisted of five main phases that are outlined in Table 1.

A dynamic approach to circle (3rd grade) In parallel with the study of the functioning of gears (Bartolini Bussi et al., 1999), a dynamic approach to circle had been introduced in both classes. The circle was introduced by means of rotation problems, concerning wheels and other objects as well. The usual compass was introduced and recognized as a tool that embodies the characteristic feature of the circle, in either static form or genetic form (see the following section. In particular, in class C2 the pupils invented the plane compass, made by any object (e.g. a stick) with a hole for the pencil, rotating around another of its points. They were really struck when they saw that a similar one-stick compass was drawn in an ancient plate together with specimens of more usual compasses The knowledge of the plane compass played a basic role in the further development of the experiment in class C2, as we shall see below.

Table 1. The five main phases

		C1 (Franca Ferri)			C2 (Mara Boni)	
A.	A dynamic approach to circle	3rd	↓			3rd
B.	The germ-theory for construction problems	4th	P0 ↓ T0 ↙	 ↘		5th
C.	The crucial construction problem	5th	P1 ↓ M1 ↓		P2 ↓ M2 ↓	5th
D.	Generalization	5th	Q1 ↓ T1		Q2 ↓ T2	5th
E.	Images from outside the class	5th	↘ D	↙		5th

The Main Phases

The germ-theory for construction problems (4th grade C1/5th grade C2) A construction problem was given. According to the teacher's aim, the pupils should have produced statements about externally tangent circles that model wheels in gear. The problem was intended to pave the way to two main statements, that would later have been expressed with the teacher's help and become the postulate of the germ—theory (see the following section).

Problem P0. In a class the pupils have been given the following problem: 'Draw a wheel S with a radius of 3 cm in gear with the wheel that is drawn in Figure 3 (radius 5 cm).' A pupil, after some attempts, has produced this drawing, but has noticed that there is something wrong. What is wrong? Try to solve the problem yourself and explain to him how to adjust his drawing.

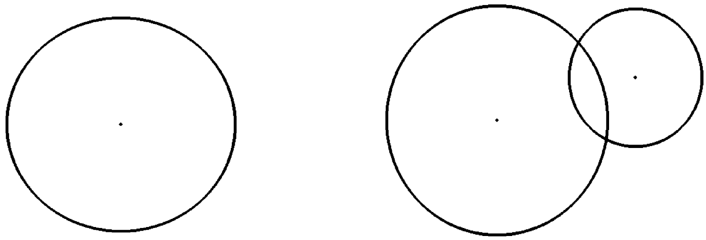

Figure 3. The given (left) and the produced (right) drawing.

The text of the problem was intentionally ambiguous, because it referred verbally to wheels, whilst circles were drawn; moreover also verbal expressions about wheels and circles were mixed. Hence two different modelling theories were recalled, even if actually only one (i.e. circles) would have been enough in the graphical solution of this particular problem.

In both case the individual solution was followed by a balanced discussion, where all the solutions were compared and evaluated. Different dynamic strategies were observed to modify the wrong drawing in order to produce the correct one. After a couple of additional exercises, the following ‘theory’ was summed up (together with the translation from the case ‘circles’ to the case ‘wheels’) and written in pupils’ notebooks with explanatory drawings too.

Theory T0

If I have two tangent circles:

1. the point of tangency is aligned with the two centres;
2. the line segment joining the two centres crosses the point of tangency;
3. the distance of the two centres is equal to the sum of radii.

Vice-versa, if I have three aligned point A, T, B, the circles with centre A and radius AT, with centre B and radius BT are tangent at T.

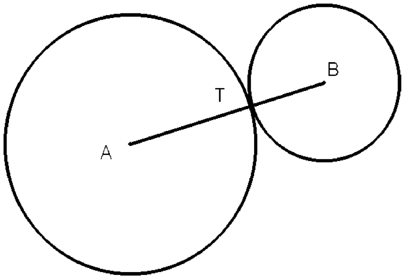

Figure 4. The theory T0.

Obviously, statements 1) and 3) above are equivalent for a cultured adult, but their sense is different, as related to different uses of the ruler (without numbers—alignment—or with numbers—distance) . Hence we considered it important to emphasize both forms, postponing to a forthcoming phase (if any) any reflection on this equivalence. A couple of additional problems were given, concerning also circles/wheels externally tangent (see Ferri et al., 1998).

The crucial construction problem (5th grade) Two different versions of this crucial problem P1–P2 were designed with the same drawing. The differences between the forms of the two texts were intentional and depended on the choice of each teacher, who expressed, in this way, the intention to emphasize one set of referents (wheels vs. circles), that, in her opinion, had risked being overshadowed in the previous weeks. However, as we shall see below, this textual difference alone did not provoke relevant differences in the process of solution. Besides, in class C2, the teacher wanted to shift attention from the simple production of the method to its justification. She said that, because of the more limited linguistic abilities of her pupils (if compared with the pupils of the other class), an explicit question to be answered was to be posed.

Problem P1
(referent: wheels, class C1)
Draw a wheel with a radius of 4 cm in gear with the given wheels.
Explain CAREFULLY the method so that others can use it.

Problem P2
(referent: circles, class C2)
Draw a circle with a radius of 4 cm tangent to the given circles. Explain clearly the method so that others can use it.
Explain carefully why the method works.

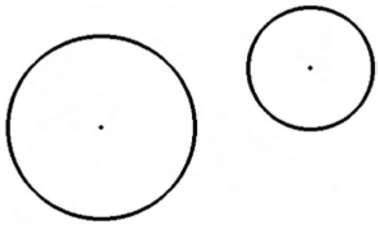

Figure 5. The drawing of problems P1 and P2; Q1 and Q2.

In Figure 5, the radii are 3 cm and 2 cm; the distance between the two centres is 7 cm. Each pupil gave at least one solution (many gave two, symmetrical with respect to the line joining the centres), drawing with care the wheel/circle by means of the compass.

The quality of solution (and the underlying processes) and the quality of arguments will be described in the next section.

A balanced discussion was designed and realized in each class, to produce and justify a shared method for solution. The method was institutionalized. In class C1 Alice's and Enzo's method [M1] was assumed. This method is the standard method of finding, by means of the compass, the third vertex of a triangle with given sides (the protocols are in the Appendix). In class C2 the following text was produced collectively.

> *Method M2*
> We know that the distance between the centres of two tangent circles is equal to the sum of radii. If we rotate (with the centre of the drawn wheels) with the compass the sum-segments, the crossing points of the circumferences will give the centre(s) of the wheel(s). That point will be where the two sum-segments touch each other.

Then the pupils in both classes were asked to apply the method to a problem with different data, and they all succeeded.

The generalization (5th grade) In both classes the problem of generalization was introduced. However the teachers followed different ways. With the same drawing of the Figure 6, two texts were given.

In the class C1, after the individual solutions, in the balanced discussion, the existence of a minimum radius was focused upon, and the conjecture made by most pupils about the alignmennt of all the centres was discussed and contrasted by offering counterexamples.

In class C2, in the balanced discussion, the existence of a minimum radius and the non-existence of a maximum radius was discussed up to the production of a written text, summarising the results. The text T2, produced in the class C2, follows.

Problem Q1
With different radii, draw wheels in gear with the given ones.
In your opinion, can we take any radius? Argue carefully in your answer. After having carefully drawn several cases, make a conjecture about the centres of the wheels: in your opinion, do they have some interesting property?

Problem Q2
1) Draw a circle with radius 3 cm tangent to the given ones.
2) Were the radius smaller than 3 cm, would you succeed all the same? Always? Why?
3) Were the radius larger than 3 cm, would you succeed all the same? Always? Why?

Test some cases and write down your observations.

> *Text T2*
> After having compared our solutions we have understood that:
>
> Given two wheels with the same or different radius, not tangent, at a distance from one another it is not always possible to put (draw) a third wheel with given radius in gear with (tangent to) them.
>
> It is possible to draw tangent wheels if summing the two sums of radii we obtain a number equal to or larger than the distance between the centres.

Among all the tangent wheels, there is one which has the smallest radius; its diameter is equal to the distance between the two circles: all the wheels which have a radius smaller than this one cannot be tangent to both the given ones. The centre of this (smallest) wheel is aligned with both the centres of the given wheels.

If the given wheels have the same radius, the distance between the two centres is equal to twice the sum of radii (of the given and the smallest wheel).

Starting from the smallest wheel, we can always draw another with a larger radius: the number of possible tangent wheels is infinite.

If the given wheels are equal, the centres of the tangent wheels are aligned, i.e. they lie on a straight line.

Images from outside the class (5th grade) At the very end, in both classes a copy of a drawing made by means of the software Cabri II was analyzed and coloured by all the pupils. In the drawing several pairs of circles tangent to two given ones were drawn together with the locus of the centres (i.e. an arch of hyperbola).

Phase Three: Process Analysis

Problem P1/P2 We shall sketch an adult solution of the crucial problem, according to the method of analysis and synthesis. We shall consider a more general problem, where the two given circles are external to each other and the radii of the circles are not fixed. We shall present the method of construction by repeating (and filling the lacking steps) the method used by Euclid in his first Problem (I, 1), that concerns a similar task.

Given two circles mutually external, to construct a circle tangent to both

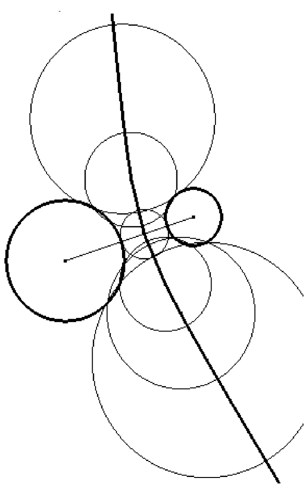

Figure 6. The Cabri drawing.

ANALYSIS

Suppose the problem solved and the circle (C,c) drawn, with the given radius c and the centre C, tangent to the circle (A,a) at H and to the circle (B,b) at K.

TRANSFORMATION. The points ABC are vertices of a triangle, with sides given: AB, AC=a+c, BC=b+c.

RESOLUTION. But the triangle ABC is given, because its sides are given. C lies on two circles, the first with centre A and radius (a+c) and the second with centre B and radius (b+c).

SYNTHESIS

CONSTRUCTION. Suppose the circles (A,a) and (B,b) and the radius c given. With centre A and radius a+c let the circle (A,a+c) be described. Again with centre B and radius b+c let the circle (B,b+c) be described. Let C be one point in which the circles cut one another. With centre C and radius c let the circle (C,c) be described. I say that (C,c) is tangent to the circle (A,a) at H and to the circle (B,b) at K.

DEMONSTRATION. Since the points AHC are on the same line and again the points BKC are on the same line, and since AC=AH+HC+a+c and again BK+KC=b+c, the circles (A,a) and (C,c) are tangent and again the circles (B,b) and (C,c) are tangent.

CONDITIONS OF POSSIBILITY. In the analysis, I see that the solution is only possible on certain conditions, given by the existence of a triangle with sides of lengths AB, a+c, b+c (the so called triangular inequality: any triangle has two sides (together) greater than the remaining side).

This means that the circle (C,c) cannot be drawn when when c is too small, i.e. when 2c<EF=AB–(a+b). When 2c=EF we can draw only one circle, with the centre on the line AB. When 2c>EF we can draw two circles (C,c) and (C′,c) where C and C' are symmetrical with respect to the line AB. In other words, there is a minimum radius (i.e. c0=½ (EF) to have at least one solution C0. When c>c0, there are always two solutions C and C′.

ABOUT THE NUMBER OF SOLUTIONS

It is easy to prove that, in general, the locus of centres is an arch of hyperbola, with foci A and B. In fact |AC–BC|=|(a+c)–(b+c)|=|a–b|, that is constant.

If a=b, the locus of centres is a line (the axis of the segment AB).

If we accepted also the case of a circle (D,d) that contained the given ones (so that (A,a) and (B,b) were internally tangent to (D,d)), the other arch of hyperbola would be obtained.

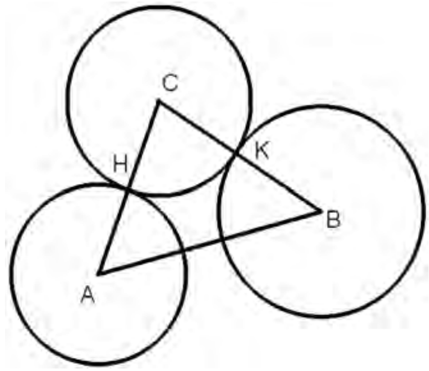

Figure 7. From Euclid.

In the TRANSFORMATION step, abduction (see Arzarello, page 310, footnote 66 in this book) plays a fundamental role: by abduction, the solver recognises the figure as a case of triangle and chooses, amongst the available knowledge, the part that is expected to be useful in the solution, i.e. the method of constructing a triangle with given sides, together with the conditions of application. As we shall see, at this point we find one of the differences between the script of this proof and the process constructed in the classroom by 5th graders.

Class processes: the attack In both classes, all the pupils attacked the problem by trying to draw a wheel/circle with the due radius, by means of the compass. Nobody tried to draw free-hand, because of the habit of using the compass to draw circles precisely.

The process was clearly by trial and error, as the number of small holes in the sheet clearly showed, and as the pupils themselves admitted in the discussion. However, trials were not completely random: they were rather oriented towards a small zone of the sheet, where the centre of the wheel-circle was expected to be. The zones were nearly the same for all the pupils (i.e. one of the two symmetrical regions over and under the line joining the two centres A and B), whilst the processes to determine it were different.

The processes could be roughly classified according to the pupils' representations of the problem. Because of the activity in the field of experience of gears, both wheels and circles were evoked: no pupil (not even in class C1) tried to solve the problem as a wheel problem only, because, after all, a drawing was required; however, independently of the particular version of the problem (wheels as in C1 or circles as in C2), in each class there were pupils who from the very beginning preferred a circle interpretation, and others who maintained a dialectical interpretation wheels/circles with a continuous shift from one to the other in the search phase, evident from the choice of metaphors used. Some protocols are enclosed in the Appendix: Alessandro (C1) and Maddalena (C2) waver between wheels (i.e. solids, which cannot interpenetrate each other) and circles (i.e. drawings, which can be overlapped), whilst Alice (C1), Enzo (C1) and Veronica (C2) choose the circle interpretation.

Class processes: germ of analysis without synthesis In class C1, after having drawn a solution by trial and error and having justified it by means of the known theory, the pupils' efforts were directed towards stating a general procedure to find the centre. Nearly all failed (only two pupils succeeded, see below), but nevertheless they stated the goal explicitly.

In class C2, the theory T0 played a twofold role: first, it was explicitly used to attack the problem, calling attention to the sum of radii; second to justify the drawing obtained. However, the process was stopped with the description of the way in which a particular centre had been actually found, by adjusting in practice the sum of radii in order to have a suitable point, without any attempt to state a general procedure.

In both classes, after having drawn a solution by trial and error, some pupils drew the line segment joining the vertices of the three wheels–circles, obtaining a triangle. This transformation of the problem had different effects, such as shifting again the focus on the segments obtained by summing radii or suggesting the existence of a symmetrical solution.

Class processes: synthesis without analysis In class C1, two pupils succeeded in stating a correct procedure, by means of the compass (see Alice's and Enzo's protocols in the Appendix). Both pupils claimed (in the protocol and in the further discussion) to have started by trial and error. However, in one case (Enzo), this process was completely hidden in the verbal part of the protocol and also the justification was elliptical. Not even in Alice's protocol did we have any information on the process by which she had succeeded in finding the method, even if she explicitly recalled theory T0 of the sum of radii.

Class processes: the reconstruction of analysis and synthesis In both classes the situation was very favourable to the implementation of a collective discussion, where the individual solutions could have been compared and oriented towards the collective production of a method and the collective construction of its proof. This is a typical situation of collective construction of knowledge. Several complementary ways of attacking the problems had been produced and documented; no pupil alone had yet produced a complete text, even if embryos of promising reasonings were available in the protocols. Hence the didactical problem concerned the orchestration of the discussion, as a polyphony of voices.

Even if, in both cases, the aim was the reconstruction of the whole process of analysis and synthesis, the fine grain design had to be different in the two classes.

In class C1 the teacher had to exploit the embryos of analysis and to laboriously, at the same time, inhibit Alice and Enzo from offering their ready-made synthesis; actually if the method had been proposed immediately, it could have been automatized (as usually happens in the technical drawing exercises that are proposed in 6th or 7th grade), but not linked by the individual pupils to their own ways of attacking the problem (see the collective protocols on the *general method* and on *gestures and other arguments* in the Appendix).

On the contrary, in class C2, the teacher had first to nurture the shared awareness that the statement of a general method was needed and to orient the embryos of analysis produced towards it. She knew in advance that she could eventually have used, to enrich the discussion, Enzo's or Alice's protocols, which were available, telling the pupils that the solutions had been produced by 5th graders of another class; yet this 'emergency way out' typical of our projects was not used in this case. Rather, the class succeeded, under the teacher's guidance, in producing their own method, where the relationships between the analysis and the synthesis phases became very evident (see the protocol from *Veronica's analysis to collective synthesis* in the Appendix).

DISCUSSION

The Pupil's Internal Context: the 'Mental' Compass

As the excerpts of the discussions show, in both classes, with different communicative strategies, the teachers intentionally attained the same goal, i.e. the shift from a practice-oriented to a theory-oriented use of the compass. In the former, typical of all the previous activities of the sequence, the compass was used as a precision tool to draw objects (either circles or wheels) with round shapes. In the latter, the compass was used as a geometry tool to select the points of the plane that are at a given distance from a given point. This use emphasizes also the relationships between Euclid's and Heron's definitions, and orients the definition of circle towards the solution of construction problems.

Especially, in the excerpt of the discussion of class C2, we are observing in real time the emergence of the latter use. The method of using the compass (i.e. the gesture of handling and of tracing) is the same for both precision and geometry tool, but the senses given by the pupils to the process (gesture) and to the product (drawing) are very different. When the compass is used to produce a round shape its main goal is communication; when the compass is used to find the points which satisfy a given relationship it becomes a tool of semiotic mediation (Vygotsky, 1978), that can control—from the outside—the pupil's process of solution of a problem, by producing a strategy that:

- can be used in any situation,
- can produce and justify the conditions of possibility in the general case,
- can be defended by argumentations referring to the accepted theory.

The geometric compass, embodied by the metal tool stored in every school-case, is no longer a material object: it has become a mental object, whose use may be substituted or evoked by a body gesture (rotating hands or arms). This fact is especially evident in class C2, where the shared implicit reference to the plane compass makes it possible to shortcut the process, by evoking the compass together with experiencing the gesture.

The collective construction of the 'mental' compass is very important in this approach to the theoretical dimension of geometry with young learners: even if the link with the body experience is not cut (it is rather emphasized), the loss of materiality allows distance oneself to one from the empirical facts, transforming the empirical evidence of the drawing that represents a solution (whatever was the initial way of producing it) into the external representation of a mental process.

The realization of this learning process (guided by the teacher) is consistent with the epistemological analysis carried out by Longo (1997), on the basis of neurological findings, regarding 'geometrical abstraction': the (geometrical) circle is not a generalization of the perception of round shapes, but the reconstruction, from memory, of a variety of acts of spatial experiences (a 'library' of trajectories and gestures).

The Classroom Process: Analysis and Synthesis

For this particular problem, the categories of analysis and synthesis were used to classify pupils' solutions of the problem P1/P2 and to design the following balanced discussion up to the statement of a general method. However, it is necessary to emphasize that the shared method was established by a process that only roughly matches the adult solution of Section 4.1.

First the task given to the pupils was a particularization (for given radii of the three circles) of the problem discussed in Section 4.1, hence no condition of possibility needed to be discussed overtly, at least at the very beginning.

Second, and most important, even when some pupils drew the triangle with vertices in the three centres of the given wheels and of the sought one, no result regarding the method of construction of a triangle with given sides was yet available. Hence the problem had simply been transformed into another problem whose solution was not yet ready made. The pupils needed to produce some additional pieces of theory, still lacking from their store, concerning the interpretation of the well-known compass in a new way, i.e. as a geometry tool aiming to select the points of the plane that are at a given distance from a given point.

Whilst the adult method by analysis and synthesis applied to this particular (and very elementary) problem could be schematized according to the directional interpretation, i.e. to the backward reduction, the pupils method better fitted the configurational interpretation, where the hypothesis of the synthesis is augmented, during the process, by an additional piece of theory (see Arzarello, in this volume).

This conclusion is reinforced by the a posteriori analysis of the balanced discussions which effectively, in both classes, allowed the collective production of the method and the collective construction of the proof. The teachers did not aim at creating the logical chain between the statements, as we have done in the adult solution (reducing backward transformation and resolution into construction and demonstration), but rather aimed at fostering the pupils to express clearly their own strategies for attacking the problem and of giving time to the pupils, who have failed, to relive the dynamic explorations that might have produced the method. As a byproduct, the (cognitive) unity between the phases of analysis (the search for a method) and the phases of synthesis (the justification of the method) was guaranteed by the recourse to the same gestures and gazes.

We could say that the method of analysis and synthesis for early construction problems proved to be within the zone of proximal development of most 5th graders. Surely attacking a problem by means of analysis seems to be consistent with attacking by trial and error typical of young pupils, who aim at producing the figure by means of all the instruments they have—both practical and theoretical. The further task is to state explicitly the method, and to offer justifications for its validity.

FURTHER DEVELOPMENTS AND OPEN PROBLEMS

In the classes we have not designed any additional task to observe whether the method of analysis and synthesis could have been applied to other problems. Actually it was not a goal of this research study to explore whether the method itself could be learned and applied elsewhere.

This pilot experiment, realized in two 5th grade classrooms, was later repeated in another 5th grade classroom (teacher: Rita Canalini) and in an 8th grade classroom (teacher: Rossella Garuti). In these classrooms some corrections were made, because of the age and previous experience of the pupils.

For instance, in the 8th grade classroom, first, the activity to establish theory T0 was enriched to explore also the relationships between the different statements, which mirrored statements produced by the pupils themselves: in this way the dependence relationships between one statement and another were made explicit. Second, the approach to the method of analysis and synthesis was emphasized by introducing an explicit reference to the different steps, by means of the reading of historical sources too. The intention was to foster the interiorization (in the Vygotskian sense, 1978) of the method as a powerful tool to produce procedures of construction and to construct proofs of their validity with reference to a shared theory: in this way the method of analysis and synthesis was expected to become a tool of semiotic mediation to control students' behaviour from the outside, in the process of solving construction problems. The data have not yet been analysed, but the anecdotal evidence seems to confirm our theory.

The germ-theory (T0) proved to be a good training ground for approaching theoretical thinking: however, there is no doubt that it is really limited. A realistic expansion could be the introduction into primary classrooms of an explicit reference to the geometrical use of the ruler (without numbers), accompanied by the reading of Euclid's first postulates. Making pupils aware of the differences and the relationships between geometry with measure and without measure (like the one experienced in perspective drawing, see Bartolini Bussi, 1996) is surely an ambitious, yet realistic, aim, that is considered now in the further development of the project *Mathematical Discussion*.

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APPENDIX

Individual protocols

Pupil's texts are in italic. Numbering in square brackets has been added in the translation to show the sequence in the pupil's text; comments and field notes are in square brackets too.

C1: Alessandro

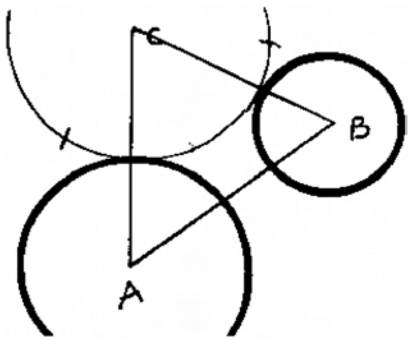

Figure 8. Alessandro's drawing.

[THE ATTACK: WHEEL–CIRCLE]

[1] I try desperately to make this wheel touch the other two.

[2] I try again and again, planting the compass.

[3] At last a fine circle comes out with the radius of 4 cm.

[4] As I have said, I paid much attention to make my circle touch the others.

[5] The most difficult part is the one I have marked [i.e. close to the points of tangency] because I must make the circle touch the others, but I must go on rotating [the compass].

[6] In practice I must make the compass rotate and in its rotation it must touch the other wheels.

[THE METHOD]

Here is the method: I have opened the compass 4 cm and I have made the circle. But now I have discovered the TRUE METHOD (I think and hope so!). As all my fellows draw the triangle in the circles I try and understand what information it can give. Now I have understood.

I have joined the two centres in the circles drawn by Franca. From the centre of my circle I make two lines leave to join the other two centres and a triangle comes out. Now we have two radii for each wheel; yet we must be aware that my wheel is not yet done. We must consider one of the two radii of my not-done wheel.

I take the compass opened 4 cm and I plant the point in the centre. I put the lead close to circumference of one of the circles and then I make my circle.

The radius of A is 3 cm.

The radius of B is 2 cm.

The radius of C must be 4 cm.

3+4=7

2+4=6

… and I find the point.

If I consider the radius of the A wheel I must go on 4 cm. The same for the wheel B. Beyond each radius I must go on 4 cm. To sum up, I say that I must draw a triangle with the vertices in the centres of the wheel. The radii of C must be 4 cm.

I take the compass, I open it 4 cm and I plant it in the centre C where I draw the wheel.

C1: Alice

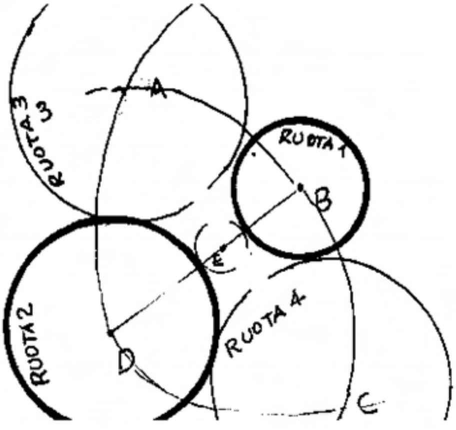

Figure 9. Alice's drawing.

[CIRCLE: THE ATTACK AND THE METHOD]

From the very beginning I have understood two ways of solving the problem, one over and one under.

I have opened the compass 4 cm.

FIRST METHOD

I have planted the compass close to the wheels 1 and 2 and the lead on the circle 2 and then according to the way of touching I have moved the point so that it could touch both 1 and 2 and in this way I draw the circle 3.

SECOND METHOD

I measure the radius of the wheel 2, it is 3 cm and then I add 4 cm, because I must draw a circle with a radius of 4 cm+3 cm=7 cm.

I open the compass 7 cm, I plant it in D and I draw the circle around the 2. The same procedure for the wheel 1, with a different radius, that is 6 cm.

That done, I look at the points where the two circles cross each other. I plant the compass open 4 cm in the crossing point and draw the circle 4 and you will see that it touches perfectly both 1 and 2.

This way is useful also to see whether the circle above is precise, because if the centre A is the crossing point it is right.

In this way I have proved that we have two solutions.

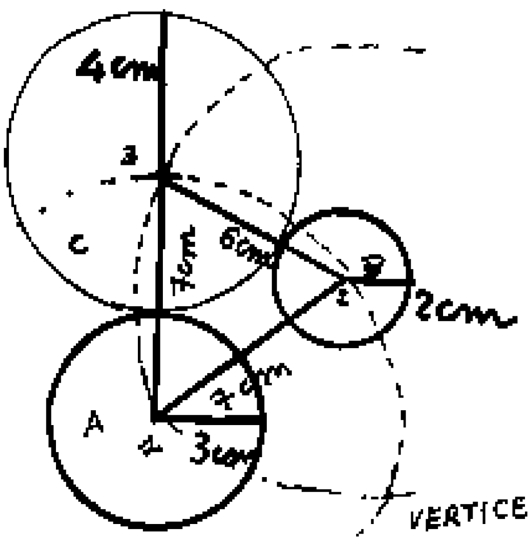

Figure 10. Enzo's drawing.

[Then she repeats the method in formulas]

Were the circle with a radius smaller than 4, i.e. 1, it is possible to make all the same: to open the compass 1 cm and to draw the circle between the circles 1 and 2.

Were the wheels 1 and 2 precisely horizontal, the points A E C would be on the same straight line one under each other.

C1: Enzo

[CIRCLE—THE METHOD]

I explain carefully the method.

- first I open the compass 4 cm
- then I name the wheels: A, B, C

I name also the centres: 1, 2, 3.

I name also the points of tangency: O, E.

- Then I have measured the radii: 6 cm, 7 cm, 7 cm.
- Then I have measured the radii: 4 cm, 2 cm, 3 cm.
- Then I have found the vertex, i.e. the point of crossing of the two not-tangent circles. The vertex is the very centre of the wheel I have drawn. That must have a radius of 4 cm.

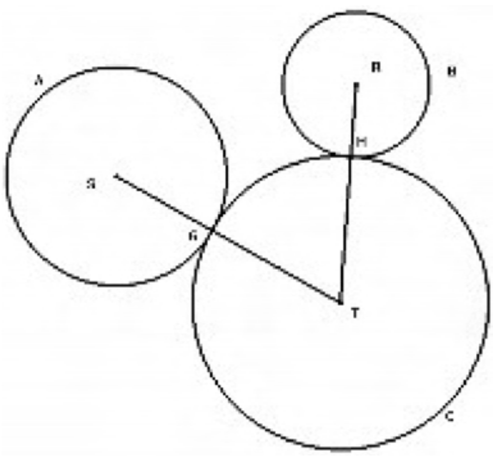

Figure 11. Veronica's drawing.

To sum up:

- To draw wheels in gear they must be perfectly tangent and with the points aligned.
- On the contrary to find the vertex we must draw not-tangent circles, in the crossing points there are the centres of the circles.

Observation

I think that there are two solutions because if you draw a circle under the circles A and B; it is a different solution and I have discovered it by looking for the vertices, because with the crossing points you find the centres and you have two solutions

C2: Veronica

[THE ATTACK: WHEEL–CIRCLE]

The first thing I have done was to find the centre of the wheel C.

I have made by trial and error, in fact I have immediately found the distance between the wheel B and C. Then I have found the distance between A and C and I have given the right 'inclination' to the two segments, so that the radius of C measured 4 cm in all the cases. Then I have traced the circle.

JUSTIFICATION

I am sure that my method works because it agrees with the three theories we have found:

I) The points of tangency H and G are aligned with St and TR;
II) The segments St and TR meet the points of tangency H and G;
III) The segments ST and TR are equal to the sum of the radii SG and GT, TH and HR;
IV) The wheels ABC are in gear.

C2: Maddalena

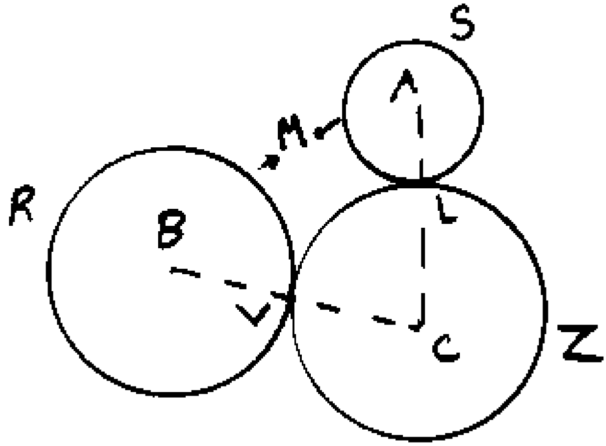

Figure 12 Maddalena's drawing.

[THE ATTACK: WHEEL–CIRCLE]

Before starting, I have thought, looking at the drawing, that the wheels R and S had been put there to make the wheel Z pass in the space M, but, reading again the text, I have seen that it did not stay in the space M.

So, I have looked/tried below and, after '3.000' trials I have succeeded in finding the centre of the circle; then I have traced the segment from the centre C to the centre A and from the centre C to the centre B and I have constructed the circle.

JUSTIFICATION

My method works because, first, I have controlled again; then I know that to be tangent the three points ALC are aligned like the points BVC.

Discussion Protocols

C1: Discussion (excerpt): The general method. January the 20th, 1998.

Present pupils: 17 (Matteo and Marco are absent).

Teacher: Franca Ferri—External observers: Maria Bartolini Bussi, Guershon Harel

[Some pupils have presented some ways of attacking the problem by trial and error]

54 Dario: I think that there is a method, even if I have not found it, but surely it exists, a method to see where the circle goes. I believe it exists.

[…]

61 Dario: To proceed by trial and error, it's possible and one can even find the solution well, but this is not a method. We must try and find a precise method.

62 Chiara: First you draw it, then you maybe understand the method later.

63 Alessandro: It's not a method and you do not understand well, not even later. I have made the wheel, then I have realized that I had to explain. I cannot say: OK, I have opened the compass 4 cm and I have drawn the wheel. Why is the centre there?

64 Chiara: Yes, in this way you do not understand anything.

65 Alessandro: You might even avoid the exact method, but you must know and understand it.

66 Enzo: That is from the method you have to start. To go by trial and error is nothing.

67 Chiara: When you know the method, the solution is easier.

68 Emanuele: . . . and faster.

69 Dario: I remember a problem you [the teacher] gave us about several solutions and a method to find them all. We could go by trial and error, but we had to discover how to be sure of the procedure. This is the reason why we explain the reasoning when you give us a problem. You ask us to explain for this very reason, don't you?

70 Teacher: I do.

71 Enzo: [Otherwise] it would be too easy.

72 Isabella: It would be not useful, because if you are lucky you might succeed all the same, but you do not understand. You must try again and again and look for a method, a method.

73 Chiara: By a stroke of luck you do not learn if you do not understand.

74 Enzo: You could even copy, but if you do not know a method you do not understand what you have copied. Like that joke you told us, if you do not understand you laugh because you are copying, but you have not understood and you need an explanation, a reasoning.

C1: Discussion (excerpts): Gestures and other arguments. January the 20th, 1998.

Present pupils: 17 (Matteo and Marco are absent).

Teacher: Franca Ferri—External observers: Maria Bartolini Bussi, Guershon Harel.

[The pupils have presented and commented ways of attacking the problem. They have talked a lot about the second (symmetrical) solution, but they are still discussing how to find the first centre. Alice has already presented her solution—interventions 116–120—by carefully repeating her written protocol. Some pupils have correctly repeated the procedure but there is still somebody not convinced.]

250 Chiara [she is at the blackboard]: We could make a big circle [with the sum of radii] … and then control. No [puzzled]. It does not work. I have written that I do not know how to make it.

[…]

252 Chiara: This is 7, this is 6 [she seems to speak to herself pointing at two segments obtained as sums of radii]. I do not know: where have I to plant the compass.

253 Alessandro [from his seat, planting his elbows on the desk and rotating the forearms until the hands meet] You must find the meeting point.

[There is a lot of noise, everybody wants to come and speak].

254 Stefano: I am trying, I am not sure. First step: you measure the radii of the two circles and add 4 cm. And from the radii you go on until they meet here [he gestures pointing at the blackboard where he has reproduced his drawing].

255 Enzo [laughing]: It's not right. May I come?

260 Teacher: Come, Enzo, it's your turn eventually.

[…]

263 Enzo [very happy]: First. Here you have a radius of 4 cm but you must have it also here [He points at the two zones of tangency]. You cannot lengthen them to make them meet. [He draws on the blackboard two big circles with 6 cm and 7 cm radii around the two given ones and points out the two intersections.]

[The pupils are struck but still incredulous.]

274 Stefano: Yet it's not clear yet for me how one finds the two big circles. Does Enzo find them by chance?

275 Enzo: No, I don't. [to the teacher], Franca, please, show him my sheet that is better than the drawing at the blackboard.

276 Chiara: We would need to enlarge the drawing at the blackboard and to go through the drawing process by the rallentee.

[Some pupils repeat the method but the implicit question is always: why does it work?]

[…]

283 Emanuele: We have a circle with, say, 5 cm radius and then you draw another, say, 6 cm. The circle is the whole of possible radii of 5 or 6 cm. . . . There are all the possible radii of 5 and 6 inside the two circles and in the overlapping zone. There is the contact, when you put the circles at that distance.

[…]

336 Alessandro: Look at my fingers. The radii look for the inclination for meeting each other [he now rotates only the hands with straight forefingers].

[…]

345 Alessandro: Look at me [he indicates with the eyes his two forefingers]. These two segments do not meet each other, but we put them nearer and nearer until—PAM!—they meet and you find the centre.

346 Teacher: Well, until now there are three justifications.

[And she sums up the three arguments: Emanuele claims that the two big circles contain all the possible radii of 6 and 7; Alice insists that the sum of radii is maintained; Alessandro has explained by gestures; the discussion goes on by linking these three arguments.]

C2: Discussion (excerpt): From Veronica’s analysis to collective synthesis: the shift from practice oriented to theory oriented compass. May the 8th, 1998.

Present pupils: 16 (Samuele is absent).

Discussion

a) 10 minutes, individual reading of Veronica’s protocol (copied for each pupil),
b) Andrea, Vincenzo and Jessica say that they have not understood,
c) excerpt of the discussion.

Transcripts and [field notes]	Comments
1) Teacher: I wish that those who have understood ask Veronica some questions to see whether they have understood well or to clarify some small doubts.	The teacher asks the pupils to ask questions. However she decides to involve first the pupils who have understood. This is not a typical strategy. In this case it is used because the teacher thinks that the pupils who have not understood are not yet able to express questions clearly and exactly
2) Elisa: I am not sure to have understood well the piece where you say: ‘I have made by trial and error, in fact I have immediately found the distance between the wheel B and C’.	This first part of the discussion (2–6) concerns segments, and namely the ‘sum-of-radii’ segments.
3) Veronica: I have said that I have made by trial and error, because it was not sure that the segments were perpendicular to each other and you would have already had the solution, you had rather to find the right inclination.	However, they are drawn on the sheet but they can be given different inclinations. These pupils have no familiarity with dynamic softwares like Cabri, where the very drawing can be dynamically transformed.
4) Teacher: Veronica has tried to give the right inclination. Which segments is she speaking of? Many of you open the compass 4 cm. Does Veronica use the segment of 4 cm? What does she say she is using? [Veronica’s text is read again.]	Hence we interpret this excerpt as if the pupils have materialized the segment (as a stick) and try to give the stick the right inclination.
5) Jessica: She uses the two segments …	
6) Maddalena: . . . given by the sum of	

radii	
7) Teacher: How did she make?	
8) Giuseppe: She has rotated a segment.	
9) Veronica: Had I used one segment, I could have used the compass.	Veronica recalls the use of the plane compass, made by only one bar, that had been produced by the classroom while approaching the circle.
[Some pupils point at the segments on Veronica's drawing and try to 'move' them]	The pupils pick up an ideal segment as if it were a stick and try to move it.
10) Francesca B: From the wheel B have you thought or drawn the sum?	Francesca is clearly posing the question about which referents Veronica has used: an ideal (thought) referent or a physical (drawn) referent. Veronica claims to have drawn.
11) Veronica: I have drawn it.	
12) Giuseppe: Where?	
13) Veronica: I have planned to make RT perpendicular [she points to the base of the sheet] and then I have moved ST and RT until they touched each other and the radius of C was 4 cm.	But to have allowed herself to move the static drawing.
14) Alessio: I had planned to take two compasses, to open them 7 and 6 and to look whether they found the centre. But I could not use two compasses.	Alessio states the link between the rotation of the segments (either thought or drawn) and the compasses that are nothing but materialized segments. But he had only one.
15) Stefania P: Like me; I too had two compasses in the mind.	Surely the previous experience of the plane compass offers a good bridge between physical and ideal experience.
16) Veronica: I remember now: I too have worked with the two segments in this way, but I could not on the sheet.	And Veronica too refers her strategy to the plane compass.
[All the pupils 'pick up' the segments on Veronica's drawing with thumb–index of the two hands and start to rotate them.]	The shared experience is strong enough to capture all the pupils.
17) Elisabetta [excited]: She has taken the two segments of 6 and 7, has kept the centre still and has rotated: ah I have understood!	Elisabetta and Stefania together by words and gestures repeat the procedure.
18) Stefania P: … to find the centre of the wheel . . .	

19) Elisabetta: … after having found the two segments …	
20) Stefania P: … she has moved the two segments.	
21) Teacher: Moved? Is moved a right word?	The teacher encourages the correction of an ambiguous word.
22) Voices: Rotated … as if she had the compass.	
23) Alessio: Had she translated them, she had moved the centre.	
24) Andrea: I have understood, teacher, I have understood really, look at me … [The pupils continue to rotate the segments picked up with hands.]	And Andrea too has understood and shows it by gesturing.
25) Voices: Yes, the centre comes out there, it's true.	
26) Alessio: It's true but you cannot use two compasses.	Alessio still has his problem (no. 14): only one physical compass whilst the two rotations are contemporaneous
27) Veronica: You can use first on one side and then on the other.	But Veronica breaks the time of contemporaneity using the same compass twice.
28) Voices: Yes, good, let's try.	
29) Teacher: Good boys! Try and draw the two circles! Today we stop here. [All the pupils take the compass and find the solutions by intersecting the circles of radii 7 cm and 6 cm.]	

PAOLO BOERO, ROSSELLA GARUTI AND ENRICA LEMUT

12. APPROACHING THEOREMS IN GRADE VIII

Some Mental Processes Underlying Producing and Proving Conjectures, and Conditions Suitable to Enhance Them

INTRODUCTION

This chapter presents the experimental study which served as a source for some ideas and theoretical constructs indicated in the chapter by Bartolini, Boero, Ferri, Garuti and Mariotti (especially the role of the dynamic exploration of the problem situation in the production of conjectures and the construction of their proofs, and the "cognitive unity" of theorems).

The research project reported in this chapter went on analysing mental processes underlying the production and proof of conjectures in mathematics. We believed that such analysis could give us some hints on suitable problem situations and the best class-work management modality for an extensive involvement of students in the construction of conjectures and proofs.

SOME HYPOTHESES CONCERNING PRODUCING AND PROVING CONJECTURES IN THE EARLY APPROACH TO THEOREMS

First we took into consideration the conditionality of the statements (i.e. the implicit or explicit formulation of most statements in terms of "if … then …"), to which the logical structure of the proving process is connected. We have tried to formulate some hypotheses concerning the production of conditional statements and related proving developments. In order to do this, reference has been made to preceding studies, which suggested: the importance of the exploratory activity during the production of conjectures (cf. Polya's "variational strategies"; see also Schoenfeld, 1985); the relevance of mental images (as "a pictorial anticipation of an action not yet performed," Piaget & Inhelder, 1967—see Harel & Sowder, 1998) in the anticipatory processes in geometry; the possibility of deriving the hypothetical structure "if … then …" from the dynamic exploration of a problem situation (cf. Boero et al., 1999).

We therefore came to the following hypotheses referred to a didactic situation where students are requested to solve an open problem through the production and proof of a conjecture. The hypotheses concern the crucial role that can be taken on by the dynamic exploration of the problem situation both at the stage of conjecture production and during the construction of proof. The hypotheses were organized as follows (see Boero et al., 1996):

P. Boero (Ed.), Theorems in schools: From history epistemology and cognition to classroom practice, 249–264.

– as to the **conjecture** production,
  A) the conditionality of the statement can be the product of a dynamic exploration of the problem situation during which the identification of a special regularity leads to a temporal section of the exploration process, which will be subsequently detached from it and then "crystal" from a logic point of view ("if … then …");
– as to the **proof** construction,
  B) for a statement expressing a sufficient condition ("if … then …"), proof can be the product of the dynamic exploration of the particular situation identified by the hypothesis;
  C) for a statement expressing a sufficient and necessary condition ("… if and only if …"), proving that the condition is necessary can be achieved by resuming the dynamic exploration of the problem situation beyond the conditions fixed by the hypothesis.

We can consider these hypotheses as a partial answer to a claim by M. A. Simon, published in the same year (see Simon, 1996) and concerning the need for studying the potentialities of "transformational reasoning" in conjecturing and proving.

Another hypothesis stems from our previous research on the feasibility of a constructive approach to theorems by students. In particular, during a teaching experiment concerning arithmetic theorems, students were engaged in the production and proof of conjectures. It was observed that students kept a keen coherence between the text of the statement produced by them and the proof constructed to justify it (see Garuti et al., 1995). This textual coherence brought forward the problem of a possible cognitive continuity between the statement production process and the proving process. A similar behaviour in a problem solving situation implying the necessity of formulating and justifying conjectures was observed by C. Maher in very young students (grade IV) (see Maher, 1995).

As concerns the links between conjecturing and proving, we have elaborated the following (see Garuti):

> Hypothesis D ("cognitive unity of theorems" as a facilitator of the student approach to theorems): the majority of grade VIII students can produce theorems (statements and proofs) if they are placed in a condition so as to implement a process with the following characteristics:
>
> - during the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingling with the justification of the plausibility of his/her choices;
> - during the subsequent statement proving stage, the student links up with this process in a coherent way, organizing some of the justifications ("arguments") produced during the construction of the statement according to a logical chain.

This hypothesis, if validated and thoroughly investigated by other studies (cf. Garuti et al., 1998), might have important didactic consequences as to the school approach to theorems, radically calling into question the teaching traditions (see the last section).

Hypothesis D has an interesting counterpart in the history of mathematics. Indeed history of mathematics shows remarkable similarities between the holistic way of producing theorems by the student, described in our hypothesis, and the way of producing theorems by mathematicians: despite important differences (as to reasoning, cultural experience, institutional bonds, etc.—see Hanna & Jahnke, 1993), we can detect the existence of common features, in particular as to the intermingling between the progressive focusing of the statement and the argumentative activity aimed at justifying its plausibility. At times, in the case of the history of mathematics, this is a long process, that involves many people for many years (cf. Lakatos, 1976); at times it is a personal process, traces of which are found in the notes or memoirs of one mathematician (cf. Alibert & Thomas, 1991).

Despite the undeniable differences between "deductive organization of thinking" and "argumentative organization of thinking" (Duval, 1991), by hypothesis D we want to stress some aspects of continuity, concerning the production, during the conjecturing phase, of the elements ("arguments") that will be used later during the construction of the proof. Most of the studies on mathematical proof within mathematics education research, on the contrary, above all point out the elements of difference between argumentative reasoning and deductive reasoning (see Balacheff, 1988; Duval, 1991). It seems to us that the existence of differences, epistemological obstacles, etc. is not incompatible with the fact that students can construct the proof using elements arising during the argumentation that accompanied the conjecture construction process. But every element of continuity implies the risk for students to identify processes of a different nature (cf. Duval, 1991). These reflections were helpful to us for the planning of our teaching experiment and for the analysis of students' behaviours; in particular:

- at the stage of construction of the teaching experiment we tried to create favourable conditions for the appearance of the cognitive unity assumed by us, but also for separation by students of the conjecture production stage from the proving stage, insisting in particular on the reasons for the necessity of proof as "proof of the statement truth";
- in the analysis of protocols we tried to catch the signs of attained change in students between the perspective of the argumentation to construct the conjecture and the persuasion of its plausibility, and the perspective of its proof.

The teaching experiment is described in the next section. The analysis and conclusions of the teaching experiment are shown in the two subsequent sections. The discussion (final section) contains some reflections on our findings and indicates some of the developments suggested by our research.

THE TEACHING EXPERIMENT

The main difficulty which we had to face was that of finding experimental confirmation for our hypotheses. It was necessary, in particular, to create an experimentation and observation context suitable to “reveal” the nature of processes of statements and proofs production and verify the potentiality conjectured by us.

The teaching experiment was carried out in two grade VIII classes of 20 and 16 students, at the beginning of the third school year with the same teacher. Students had already interiorized the habit of producing argued hypotheses in different domains (mathematical and non-mathematical), writing down their reasoning. Students had already experienced situations of statements production in arithmetic and geometry; they had approached proof production in the arithmetic field (see Boero & Garuti, 1994; Garuti et al., 1995).

The task concerning the production and proof of a conjecture was contextualized in the “field of experience” (Boero et al., 1995) of sun shadows. Students had already performed about 80 hours of classroom work in this field of experience. They had observed and carefully recorded the sun shadows phenomenon over the year (in different days) and over the morning of some days. They had approached geometrical modelling of sun shadows and solved problems concerning the height of inaccessible objects through their sun shadows.

The field of experience of sun shadows was chosen because it offers the possibility of producing, in open problem solving situations, conjectures which are meaningful from a space geometry point of view, not easy to be proved and without the possibility of substituting proof with the realization of drawings. The field of experience of sun shadows is a context in which students can naturally explore problem situations in different dynamical ways. In order to study the relationships between sun, shadow and the object which produces the shadow, one can imagine (and, if necessary, perform a concrete simulation of) the movement of the sun, of the observer and of the objects which produce the shadows. In particular, students had already realized some activities which needed the imagination of different positions of the sun and of the observer in order to produce hypotheses concerning the shape and the length of the shadows.

In the two classes the activities were organized according to the following stages (whole amount of time for classroom work, about 10 hours):

a) *Setting the problem:* individual work or work in pairs, as chosen by the students.

   In recent years we observed that the shadows of two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement.

   Some thin, long sticks and three polystyrene platforms were provided in order to support the dynamic exploration process of the problem situation.

b) *Producing conjectures:* many students started to work with the thin sticks or with pencils. They started to move the sticks or to move themselves to see what happened. Other students closed their eyes. The absence of sunlight or spotlight in the classroom hindered the experimental verification of conjectures they were formulating: it was the mind’s eyes that were “looking”. Students individually wrote down their conjectures.

c) *Discussing conjectures*: the conjectures were discussed, with the help of the teacher, until statements of correct conjectures were collectively obtained which reflected the different approaches to the problem by the students.
d) *Arranging statements*: through different discussions, under the guidance of the teacher, the following statements, made more precise from a linguistic point of view than those produced by students at the beginning, were collectively attained:
   - If sun rays belong to the vertical plane of the oblique stick, shadows are parallel.
   - If the oblique stick moves along a vertical plane containing sun rays, then shadows are parallel.
   - The shadows of the two sticks will be parallel only if the vertical plane of the oblique stick contains sun rays.

The first two statements stand for two different ways of approaching the problem on the part of the students: the movement of the sun and the movement of the sticks; the third statement makes explicit the uniqueness of the situation in which shadows are parallel.

After further discussion the collective construction of the two statements below was attained:

- If sun rays belong to the vertical plane of the oblique stick, shadows are parallel. Shadows are parallel only if sun rays belong to the vertical plane of the oblique stick.
- If the oblique stick is on a vertical plane containing sun rays, shadows are parallel. Shadows are parallel only if the oblique stick is on a vertical plane containing sun rays.

In order to help the students in the proving stage we preferred not to express the statement in its standard, compact mathematical form “if and only if …” (its meaning in common Italian cannot be distinguished from the meaning of “only if …”).

e) *Preparing proof*: the following activities were performed:
   - individual search for analogies and differences between one’s own initial conjecture and the three “cleaned” statements considered during stage d);
   - individual task: “What do you think about the possibility of testing our conjectures by experiment?”
   - discussion concerning students’ answers to the preceding question. During the discussion, gradually students realize that an experimental testing is “very difficult,” because one should check what happens “in all the infinite positions of the sun and in all the infinite positions of the sticks”.

This long stage of activity (about 3 hours) was planned in order to enhance students’ critical detachment from statements, to motivate them to prove, and to state that since then classroom work would have concerned the validity of the statement “in general”.

f) *Proving that the condition is sufficient* (activity in pairs, followed by the individual wording of the proof text).
g) *Proving that the condition is necessary* (short discussion guided by the teacher, followed by the individual wording of the proof text).
h) *Final discussion,* followed by an individual report about the whole activity (at home).

The following materials were collected: videotapes of the initial stages (a and b); tape-records of discussions and teacher–students interactions; all the students' individual, written texts. The data which we are about to consider mainly concern stages b), f) and g).

For each type of students' behaviour one example of written texts individually produced by students during stages b), f) and g) will be reported entirely. At this stage of the research we deem it important to dwell on typical behaviour that can justify the plausibility of our hypotheses and to examine it more deeply (in view of its subsequent and more extensive confirmation).

SOME FINDINGS

The teaching experiment analysis seems to confirm the validity of our hypotheses, as proved by the behaviour of the great majority of the students of the two classes. All students actively took part in the production of the initial conjecture; 29 students (over 36) were able to follow the activities (from c to h) in a productive way.

The elements found which confirm our hypotheses can be summarized as follows:

Relevance of the dynamic exploration on the problem situation during the conjecture production stage (Hypothesis A)

The analysis of the videotape shows that at least one half of students (in reality, probably more) performs the dynamic exploration of the problem situation in different ways: indicating with their hands the imagined movement of the sun, or moving themselves, or moving the oblique stick, or moving the platform supporting the sticks, etc.

On the other hand, in 14 individual texts (out of 36) there is explicit evidence of the passage from the imagined (and/or simulated) dynamic exploration of the problem situation to focusing on a temporal section, with successive transition to the formulation of a statement "crystallized" from a logic point of view:

> EX. 1 (Simone): If we took into consideration two sticks, of which one is vertical, the shadows will be parallel when the two sticks are seen parallel by the sun. If we suppose that the person is in the position of the sun and looks at the sticks, by going round the sticks we can observe that the sticks are parallel in a certain position and the shadows are also parallel since the difference in position of the two sticks cannot be seen from that position. Thinking about the shadow space we can say that the non-vertical stick seems to be within the shadow space. Let's imagine an imaginary vertical stick representing the oblique one, in line with the sun rays and the same stick, the oblique

> one cannot be seen so it seems to be vertical, forming parallel shadows. The shadows can be parallel if the sun is situated along the direction of the oblique stick [with a gesture he indicates the vertical plane of the oblique stick].

During the subsequent discussion, Simone explains how he produced this conjecture: he moves the polystyrene plane supporting the sticks "at random" (notice should also be paid to the generality of his reasoning) after identifying himself with the sun. Then, he places a new stick (which he calls "imaginary stick") in the same position he described in the written text, making the polystyrene plane rotate until the non-vertical stick is completely hidden by the "imaginary" vertical one. At this point he says "well, now in this position the shadows are parallel because …".

Finally, it is interesting to analyse the way in which certain initially wrong conjectures are overcome: at the beginning of stage b) some students hypothesize that shadows are always parallel, on the basis of a kind of "principle":

> EX. 2 (Lucia): I think shadows are parallel because the oblique stick functions like a normal object perpendicular to the ground, so if the rays are equal for all the objects, the shadows will be parallel.
>
> (I've changed my mind)
>
> By making a small model [they had fixed sticks to the desks with adhesive tape] we found that the parallelism of shadows depends on the position of the sun, that is, if we put the sun behind (or in front of) the sticks, the shadows are parallel but if the sun is placed on the side of the sticks then the shadows form an angle, spread apart and are no longer parallel.

This conjecture is overcome by imagining and/or simulating the movements of the sun. In other cases it is overcome by moving the sticks. Those movements allow students to explore new alternatives.

Relevance of the dynamic exploration of the situation determined by the hypothesis during the construction of the proof that the condition is sufficient (Hypothesis B). The following texts represent well the individual texts produced by most students:

> EX. 3 (Giovanni): The sun "moves". At a given moment it "sees" the two parallel sticks and relative shadows. As the sun is far away it "sees" the two shadows parallel, so it imagines the oblique stick to be vertical (imaginary stick) [introduced by Simone during the discussion phase]. But if the imaginary stick were real its shadow would cover that of the oblique stick, that is they are on the same line. Well, now we know that the shadows of the two vertical sticks are parallel and at this moment it is as if we saw two parallel shadows because that of the oblique stick is "under" that of the imaginary one. Now, if we removed the imaginary stick, the shadow of the oblique stick would appear again since it was "under" the parallel shadow of the imaginary stick, so the shadow of the oblique stick is also parallel to that of the vertical one.

Giovanni imagines being the sun and puts himself in the position identified by the hypothesis; he exploits the vertical stick as a "subsidiary construction," then he imagines removing it. The imagined movement allows him to establish a link between the hypothesis and the property to be validated (contemporaneity of two different positions of the sticks).

> EX. 4 (Fabio): If we take two vertical sticks we know that their shadows are, of course, parallel. If we moved, that is inclined one of the two sticks along the vertical plane of the rays, the situation will not vary since the oblique stick along this plane seems to be another vertical stick, lower than the first. Consequently, their shadows are parallel.

In this case the student moves the inclined stick along the vertical plane, then he identifies two consecutive positions of the same stick (the vertical position, that corresponds to the imaginary stick in the previous example, and the inclined one). In this case too the movement allows him to establish a link between the hypothesis and the property to be validated (the same stick takes two different positions in two different times).

In the "Conclusions" section we will consider the different functions of the dynamic exploration (realised through the movements of the sticks) in the conjecturing phase and in the proving phase.

Resuming the dynamical exploration of the problem situation during the construction of the proof that the condition is necessary (Hypothesis C)
We observe that:

1. In some cases the sun or its rays are moved:

> EX. 5 (Stefania): If the sun rays no longer belong to the vertical plane of the oblique stick, the sun would "see" three sticks: one vertical, one oblique and an imaginary vertical one that casts shadow. Taking for granted that the shadows of the two vertical sticks are always parallel independently from the position of the sun or its rays, then the sun would cast three shadows, of which two parallel and one oblique with respect to the other two. And if this shadow of the oblique stick were not aligned with that of the imaginary stick, it won't be parallel with the shadow of the vertical stick, so the shadows would not be parallel and the hypothesis would not be true"

2. In other cases students moved the stick (beyond the vertical plane identified by the hypothesis):

> EX. 6 (Sandra): In order to prove the second part of the statement [the shadows are parallel only if the stick moves along a vertical plane containing sun rays] we can move and place the oblique stick in another vertical plane so as to obtain two vertical planes, that of the oblique stick and that of the imaginary vertical stick. With this operation the two shadows are no longer situated in the same line so the shadow of the oblique stick and that of the vertical stick are no longer parallel. In this way, I've denied the previous statement so

the shadows will be parallel only if the oblique stick is placed again along the vertical plane of the sun rays.

Links between conjecturing and proving ("cognitive unity of theorems"— Hypothesis D)
We were interested in analysing possible links between conjecturing end proving, concerning linguistic aspects, kinds of movements, etc.; we considered different groups of students.

Correct conjecture with justification (21 students)
Underlining indicates traces of connections between conjecture production and proof construction.

Formulation of the conjecture with shifting of the stick

EX. 7 (Beatrice): I tried to put one stick straight and the other in many positions (right, left, back, front) and with a ruler I tried to create the parallel rays. I sketched the shadows on a sheet of paper and I saw that: if the stick moves right or left shadows are not parallel; **if the stick is moved forward and back shadows are parallel**. Shifting the stick along the vertical plane, forward and back, the two sticks are always on the same direction, that is to say **they meet the rays in the same way**, therefore shadows are parallel. Whereas shifting the stick right and left the two sticks are not on the same direction anymore and therefore **do not meet the sun rays in the same way** and shadows in this case are not parallel. Shadows are parallel if the oblique stick is moved forward and back in the direction of sunrays.

Proof: Shadows are parallel because, as we already said, sun rays belong to the vertical plane of the oblique stick.

But all this does not explain to us why this is true. First of all, though the sticks stand one in an oblique and the other in a vertical position, they are **aligned in the same way and if the oblique stick is moved along its vertical plane** and is left in the point in which it becomes vertical itself we see that they are parallel and, as a consequence, their shadows must naturally be also parallel, and also parallel with the shadow of the oblique stick, which has the same direction as that produced by the imaginary, vertical stick.

In this case the justification produced at the beginning ("meet the sun rays in the same way") is the one that in the following proof makes Beatrice imagine the oblique stick moving along the vertical plane containing sun rays.

Formulation of the conjecture with the movement of the sun

EX. 8 (Sara): They could be parallel **if I imagine to be the sun that sees and I must place myself in the position so as to see two parallel sticks**. In this way the sun sends its parallel rays to light the sticks. But if the sun changes its position it will not see the parallel sticks and, therefore, their shadows will not be parallel either. Shadows can be parallel if the oblique stick is on the same vertical plane as the sun rays.

> Proof: **If the sun sees the straight stick and the oblique stick parallel** it is as if there were another vertical stick at the base of the oblique stick. If this stick is in front of the oblique stick its shadow covers the shadow of the oblique stick. These shadows are on the same line, therefore, the oblique and vertical sticks shadows are parallel.

In this case the initial idea “I imagine to be the sun” seems to suggest the main argument of the proof (the shadow of the imaginary, vertical stick covers the shadow of the oblique stick).

Concerning production of the statement, Beatrice’s and Sara’s texts give evidence of complex mental processes corresponding to our hypothesis.

Concerning proof, both texts show interesting traces of detachment from the problem situation (e.g.: “I imagine to be …” becomes “If the sun sees”) and the original statement. Students seem to be aware that it is necessary to validate the statement by a reasoning process (“But all this does not explain to us why this is true.”). Many other texts show similar aspects.

Wrong conjecture (9 students)
Nine students, some of high level and some of low level, produce wrong conjectures probably suggested by the principle “sun rays are parallel, then …” or by drawings that, owing to their bidimensional nature, may be misleading, and are also static and so they may stick at particular situations.

> EX. 9 (Vincenzo) Conjecture: In my opinion shadows cannot be parallel if the two sticks are one vertical and the other not vertical. I took the two sticks, I put them in a vertical position and shadows were parallel, then slowly I moved the right-hand side stick and noticed that its shadow moved. In my opinion they do not remain parallel, because if I have two vertical sticks, their shadows are parallel because rays are parallel, that is to say they come across the obstacle and form the shadow. But if I move slowly, rays that were hindered before now pass by, though they are hindered from another point, that is to say the shadow moves and, therefore, it is not parallel anymore.

At the proving stage, after classroom discussions, 6 of these students “make up for” the lost grounds and it can be noticed how their proof is full of constructions and argumentations, as if these students had to reconstruct the conjecture to be proved:

> (Vincenzo) Proof: The statement is true because: let us imagine to have an oblique stick and a vertical stick. Let us imagine to draw an imaginary line, perpendicular to the horizontal plane, starting from the point of the oblique stick. Let us do the same thing with the vertical stick but the other way round, meaning that I draw an imaginary oblique line parallel to the oblique stick.

It happens that I get two vertical lines with parallel shadows and two oblique lines with parallel shadows. The imaginary stick casts a shadow into the direction of the oblique stick, as a consequence the shadows between the oblique stick and the vertical stick are parallel.

Correct conjecture without justification (6 students)
Six students out of 36, some of high level and some of low level, formulated the conjecture correctly, but during the formulation did not manage to produce arguments backing up their hypothesis. This fact seems somehow to affect the subsequent proof that turns out to be lacking in “arguments” and rather confused.

> EX. 10 (Elisabetta) Conjecture: In some cases, although the oblique stick is in a position different from that of the vertical stick, the parallelism is kept, whereas in other cases the parallelism in shadows is not kept. Therefore, shadows can be parallel only if the oblique stick [meaning with a gesture the vertical plane] is parallel to the direction of the straight stick shadow, that is to the sun rays.
>
> Proof: Our statement is true because if the vertical plane of the oblique stick receives the sun rays as the vertical plane of the vertical stick, then the two shadows will be projected on the same line.

CONCLUSIONS

It appears to us that the data illustrated above are consistent and make our hypotheses plausible.

We notice that, in the cases of Beatrice and Sara, just as for the majority of students, the dynamic process that led to the production of the statement (movement of the sun or movement of the stick) is found again in the proving process. Yet the dynamic exploration implemented during the construction of the proof, though it shows remarkable similarities with the one implemented during the production of the conjecture as to the type of movement, differs deeply as to the function assumed in the thinking process: from a support to the selection and the specification of the conjecture, to a support for the implementation of a logical connection between the property assumed as true (“vertical sticks produce parallel shadows”) and the property to be validated. The movement of the stick keeps the direction of its shadow (since it happens in the vertical plane containing sun rays) and, therefore, opens the possibility to reason in a transitive way (e.g.: the real, vertical stick produces a shadow parallel to the one of the imaginary, vertical stick; the oblique stick produces a shadow aligned with that of the imaginary, vertical stick; therefore the oblique stick produces a shadow parallel to that of the real, vertical stick). It also seems interesting to underline the fact that the hypothesis fixes the vertical plane on which the movement takes place that allows one to relate logically the property to be proved with the property assumed as known.

The teaching experiment suggests some interesting hints about the links between argumentative reasoning in the phase of the production of the conjecture and proof construction. Actually, as concerns the production of the statement, argumentative reasoning fulfils a crucial function: it allows students to consciously explore different alternatives, to progressively specify the statement and to justify the plausibility of the produced conjecture (see Simone and Lucia). On the other hand, students who produced wrong conjectures later show the need for reconstructing the valid conjecture in order to produce the proof (see EX. 9). The fact that poor argumentation during the production of the statement always corresponds to lack of arguments during the construction of the proof seems to confirm the close connection that exists between production of the conjecture and construction of the proof (see EX. 10). Moreover, the consistency among personal arguments provided during the production of statements and the ways of reasoning developed during the proof seems to be confirmed:

- by the fact that the type of argumentative reasoning made during the production of the statement by one student is resumed by him/her (often also with similar linguistic expressions) in the justification of the statement subject to proof;
- by the fact that the kind of dynamic process (movement of the sun or the stick) recorded at the conjecture stage is almost always the same as the one used at the proof stage.

DISCUSSION AND FURTHER DEVELOPMENTS

In our teaching experiment, the "dynamic" learning environment of sun shadows was chosen in order to enhance the dynamic exploration of the problem situation on the part of students (taking into account their background related to the same field of experience). The great majority of the students (29 out of 36) has productively taken part in the statement construction and subsequent proof. This fact raises the problem of searching for learning environments similar or even more effective than that of the sun shadows, as well as the problem of the transfer to "static" mathematics situations.

As regards the problem of finding suitable learning environments to develop the conjectures processes (dynamic exploration of problem situations), there are many learning environments which can be usefully compared with that of sun shadows (in particular, in the perspective of the "dynamic geometry" indicated by Goldenberg & Cuoco, 1995): Cabri or Geometric Supposer or Geometer's Sketchpad, even the "mathematical machines," "gears" (Bartolini Bussi et al., 1999) and the "representation of the visible space" (Bartolini Bussi, 1996). Comparisons like these could propose different potentials and limits for the different learning environments.

With regard to the problem of the transfer from strongly contextualized theorems in a dynamic environment such as that of the sun shadows geometry to the theorems of "context-free" mathematics, a number of confirmations derive from the observations that followed the teaching experiment in the two classes during activities with traditional geometry theorems, as well as in other classes which moved from activities of conjecture production and proof construction in other dynamic environments to "context-free" geometry (see Parenti et al., in this book).

A delicate matter concerns the variety of possible approaches to the conditionality of statements (and related connections with proving process). In fact, in Boero and Garuti (1994), a report dealing with the "Thales Theorem" and concerning the same learning environment of "Sunshadows," the following type of reasoning was identified in 3 students out of 34: "The length of the shadows is proportional to the height of the sticks due to the parallelism of the sun shadows. … If the lines are parallel, the lengths of the segments cut on another two lines shall be proportional". The process appears to be very different from that considered in our hypothesis, since in this case the student passes from a recognition of causal dependency between parallelism and proportionality in the physical phenomenon, to the conditional statement that takes into account the possibility that lines cannot be parallel. This process requires therefore a detachment from the physical phenomenon (that on the contrary can be deferred in the case of the approach to conditionality studied in this report). It is for this reason that we have formulated our hypothesis A) by emphasizing the possibility ("can") that the conditionality of statements were originated in the dynamic exploration of the problem situation without excluding other possibilities. Further research has supplied interesting indications in this field: in Boero et al. (1999) four different processes of generation of conditionality were described.

As mentioned in the second section, hypothesis D) seems to have important didactic implications, since it calls into question the traditional school approach to theorems. In fact, usually in Italy and in other countries the teacher asks the students to understand and repeat proofs of statements supplied by him, which appears to be one of the most difficult and selective tasks for grade IX–X students. The teacher may ask students to prove statements, generally not produced by students but suggested by the teacher: this is a possible last stage, often reserved to the top level students or students choosing an advanced mathematical curriculum. Even more seldom students are asked to produce conjectures themselves. If our hypothesis is valid, during this traditional path students' difficulties can at least partly depend on the fact that they should reconstruct the cognitive complexity of a process in which mental acts of different natures functionally intermingle, starting from tasks that by their nature bring them to partial activities that are difficult to reassemble in a single whole. Our teaching experiment suggests an alternative didactic path.

Just for the importance of such didactic implications we deem it opportune to critically analyse some possible limits of the study made so far and to sketch further developments of it.

First we must consider in what sense students have performed a mathematical activity concerning theorems.

The object of the experiment is a hypothesis concerning the physical phenomenon of sun shadows; it has as a geometric counterpart, at the level of model, a statement of parallel projection geometry. Students produce their conjecture as a hypothesis concerning the phenomenon of sun shadows; when they verify their conjecture most of them seem to be aware of the fact that they must get the truth of the statement by reasoning, starting from true facts. Most of them produce a validation realized through deductive reasoning. Actually their reasoning starts from properties considered as true ("two vertical sticks produce parallel shadows") and comes to the truth of the statement in the "scenario" determined by the hypothesis.

In this way, students produce neither a statement of geometry "strictu sensu," nor a formal proof: objects are not yet geometric entities, deduction is not yet formal derivation. But their deductive reasoning shares some crucial aspects with the construction of a mathematical proof. Moreover, the whole activity performed by students shares many aspects with mathematicians' work when they produce conjectures and proofs in some mathematics fields (e.g.: differential geometry): mental images of concrete models are frequently used during those activities. As to proof, mathematicians frequently come near to realizing the ideal of the formal proof only during the final stage of proof writing. During the stage of proof construction, the search for "arguments" to be "set in chain" in a deductive way is frequently performed through heuristics, the reference to analogical models and taking into account the semantics of considered propositions (cf. Alibert & Thomas, 1991; Hanna, 1989; Thurston, 1994).

For these reasons we think that the activity performed during our teaching experiment may represent an approach to mathematics theorems which is correct and meaningful from the cultural point of view.

In our opinion, the continuity aspects highlighted by us represent a huge potentiality for the development of the students' ability to prove conjectures; nevertheless, this potentiality needs an adequate educational context in order to surface successfully. In planning our teaching experiment we singled out some conditions that are probably necessary to this end; they concern:

- *the didactic contract set up in the classroom* (the production of a conjecture to solve an open problem, the value of an hypothesis as an "argued choice");
- *the didactic path in which the task is inserted* (particularly, in our case, the choice of the field of experience of sunshadows as a long term learning environment);
- *the management of classroom work after the task* (individual activities alternating with activities in pairs and discussions; activities to prepare the proof stage—see item e).

We are not yet able to establish whether all the conditions that we singled out are actually necessary and sufficient for the extensive implementation of the process that we recorded in our teaching experiment. It is necessary to ascertain what the actual weight of the didactic contract is, through comparisons with classes having a different history behind them. It is necessary to find out how much, and how, the cognitive unity of theorems appears also in mathematical fields other than geometry (and, in particular, that of "shadows geometry"). It also appears important to ascertain the consequences of experiences linked to the cognitive unity of theorems on the activity of standard theorems proving, proposed through their statements (cf. Arzarello, 2000).

Garuti et al. (1998) addressed another problem: Can the "cognitive unity" construct be a tool allowing teachers and researchers to predict and interpret students' difficulties when they have to prove a given statement? Can it be a tool that allows the teacher to select appropriate tasks that increase in difficulty, in relation to the increasing difficulty in establishing continuity between the statement and the proving process?

They produce some partial answers to these questions. In particular, they formulate the following hypothesis:

> the greater is the gap between the exploration needed to appropriate the statement and the proving process, the greater is the difficulty of the proving process (p. 347).

They present some examples that illustrate and support the validity of this perspective. These were chosen in a non-geometrical field (the elementary theory of numbers) in order to avoid the perspective of "cognitive unity of theorems," which was elaborated in the geometrical domain, being regarded as context specific (and in this way the second problem is also addressed).

In recent years, B. Pedemonte has performed a thorough experimental investigation of the relationships between the conjecturing process and the proving process. She has introduced the theoretical construct of "structural unity of theorems" in order to study the possible continuity between the structure of argumentation in the conjecturing and early proving phases, and the final structure of proof. Discontinuity can be an obstacle for students when they try to construct the proof (see Pedemonte, 2001).

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13. FROM DYNAMIC EXPLORATION TO "THEORY" AND "THEOREMS" (FROM 6TH TO 8TH GRADES)

INTRODUCTION

Our long-term teaching experiment, which started in 1995, was designed to enquire how far and in what circumstances an entire lower secondary school class could be involved in production of conjectures and in an early approach to theories, theorems and their proofs.

Our present research what peculiar conditions enabled the classes to reach good levels in taking part in theoretical discourses; it is also studies to study some mental processes which are involved in these activities.

Our analysis confirms the important role the teacher plays in the approach to theoretical aspects of mathematics (see Bartolini Bussi et al., in this book): in the classroom as a cultural mediator, who raises and coordinates discussions; in the research group, as a member who takes part in planning activities and analysing the students' mental processes.

This work involved (from the 6th to the 8th grades) about 80 students in four classes engaged in the Genoa Project for Low Secondary School (the number changed slightly, due to some small changes in the composition of the classes). Students were engaged in the experimental activities for about 15 hours in the 6th grade, 25 hours in the 7th grade and 16 hours in the 8th grade. As usual in Italy, they worked with the same teacher for three years. About one third of the students were school-integrated, interested, but with scarce results in learning (in mathematics as well as in the other subjects).

We worked on problems concerning geometry subjects which arise from the field of experience of "plane representation of space situations" (in the 6th and 7th levels) and in the field of experience of "sun shadows" (in the 8th level). All chosen problems offered the possibility to be tackled with *dynamic explorations* (see theoretical framework below, and Bartolini et al., in this volume). From the geometric point of view, these problems can be framed, respectively, within the fields of experience of "geometry of the space representation" (elements of geometry of central projection) and "geometry of sun shadows" (elements of geometry of parallel projection), that is in two fields of experience that within

P. Boero (Ed.), Theorems in schools: From history epistemology and cognition to classroom practice, 265–284.

mathematics help to build the field of experience of "geometry" (meant as "geometric theories").

We planned a sequence of problem situations, which cannot be solved merely on an empirical basis (that is just through physical experiments, measurements, etc.) and therefore require a theoretic proof as the only source of validation. Since every theoretical validation must take place within a theory of reference (see theoretical framework: *theorem* as *statement*, *proof* and *theory of reference*), we will analyse in some detail how the students worked together (under the guidance of the teacher) in order to build the elements of theory of central projection, that they had to master. In this chapter we will show also how students' awareness of their own mental dynamics and their practice in managing them can have a good influence on the production of conjectures and proofs. We will also stress how the mental processes which are involved in proving statements (meant as an activity related to the theory of reference and aimed at producing a deductive chain of arguments) can be influenced by the dynamic approach and take place in the awareness of the dynamic character of the proving process and the static character of its results.

REFERENCE TO SOME ELEMENTS OF THE THEORETICAL FRAMEWORK

The theoretical framework of our research is based upon the ideas of *theorem, field of experience* (Boero et al., 1995), *mathematical discussion* (Bartolini Bussi, 1996) and cognitive unity (as the continuity between the conjecture production process and the proof construction process, which can exist in the most favourable situations (Garuti et al., 1996, 1998), in accordance with Boero et al., in this book.

A few words about some issues related to the theoretical framework.

The *external context* of the *fields of experience* is related to meaningful problem solving situations, which require the production of conjectures and allow mental experiments based on active observations. Particularly the external context includes specific linguistic expressions the teacher is supposed to introduce in order to build a geometric language, which is useful to express statements properly, and texts beside the ones the class normally uses, in order to compare the notions of statement, theorem and proof which are gradually drawn up by the class, under the teacher's direction, with "official" formulations.

The *students' internal context* develops especially as concerns the subject knowledge, the awareness to approach a culturally demanding content suitable to be developed further at high school level, the organization of thought and language which are necessary to manage this content.

With the teacher's help, who leads the *discussion* in class, students gradually come to formulate general and abstract statements, by expressing them in a conditional form and in a geometric language. They can also reach the awareness that mathematical proofs and theory are necessary to validate some statements (in accordance with Mariotti et al., 1997).

Most students succeed in producing rough draft of proofs and show (in successful cases) the *cognitive unit* between their statement production and proof construction processes (Garuti et al., 1996).

THE TEACHING EXPERIMENT AND STUDENTS' BEHAVIOURS

Our teaching experiment tries to make the students achieve the following main aims:

- *Acquiring awareness of the existence of mental dynamics and learning how to manage them*. Through dynamic explorations of problem situations we have accustomed the students to observe reality carefully and apply graphic and/or mental representations to find the solution of problems concerning the representation of the visible world. The critical analysis of remarkable mistakes in their production has enabled the students to stress in their own mental processes the existence of mental representations/dynamics which have an influence on the productions themselves. Further discussions and activities (comparisons between drawings, between photos and drawings, etc.) have allowed students to learn how to manage one's own mental representations/dynamics, in order to improve one's production too (for further details see Parenti & Tizzani, 1996).
- *Being able to produce statements expressed in a conditional form*. According to our hypotheses, *students* can formulate their first forecasting or interpretative conjectures, still expressed in a personal language linked to perception, when they manage to "isolate" within dynamic explorations of problem situations a peculiar "status" of one of the variables that are taken into account. We have therefore privileged one of the possible processes of developing conditionality, the one Boero et al., describe (for a survey of this and other processes of generation of conditionality of statements, see Boero et al., 1999). For this purpose we proposed activities of exploration of reality (representations and shadows), followed by the request to determine "invariants" related to the analysed changes. The discussions in the class have improved students' language and progressively allowed them to build a common language used by the whole class, through which conjectures could be reformulated in a relational form, with the peculiar characteristics of most geometry statements (generality, abstraction and conditionality).
- *Approaching a "theory building" process*. The elements of theory of reference which are necessary to validate statements that cannot be validated empirically (elements of geometry of central projection in the 7th level and of parallel projection in the 8th level) are built in the class by students under the guidance of the teacher. Through collective considerations and discussions, a few statements are chosen among the produced ones that are unanimously considered "true". These statements are reformulated in a relational form, by using the geometric language that is partly known, partly suggested by the teacher: these will be the "postulates" of the "theory of reference" the students will have to master.

- *Early approach to theorems and their template for proof.* The students learn how to use the statements of the "theory of reference" within a logic deductive process, in order to validate other statements. The proof schema (or "template") is introduced by the teacher as a common synthesis of individual trials of proofs, some of which have been successful. The template has been an occasion for everybody for an arrangement of a mathematical speech within a mathematical theory.
- *Succeeding in making brief remarks concerning the comparison between the two introduced theories.* The students are requested to think about "relativity" of mathematical truths, by discussing the fact that a statement may be either true or false depending on the theory it is referred to.

From the Field of Experience of the "Plane Representation of Space Situations" to the Field of Experience of "Geometry of Space Representation"

Our teaching experiment starts in the 6th level in the field of experience of the "plane representation of space situations". Students were guided in investigating "How does the eye see reality?" by suggesting experiences of observation of objects, drawings and/or photos, by varying the observer's point of view and requiring a plane representation of observed objects. Through reading/drawing activities, the students learn to recognize the most frequent mistakes in their drawings and to stress remarkable differences between what they see and what they sometimes draw. For instance many of them draw the "roof" of the cabinet in the classroom, even though they cannot see it from their point of view.

Through a critical analysis of significant drawings, the students are gradually involved in a *discovery/awareness about how drawings can be affected by mental representations/dynamics*; they learn how to discuss them, by formulating conjectures and by looking for means to manage them in order to improve their own representations as well. According to our hypothesis (Parenti & Tizzani, 1996), the mental dynamics which play a role in the field of experience of the "plane representation of space situations" can be read as a dynamic relationship either between "eye" and "moving object," or between "moving eye" and "object," or "moving eye" and "moving object". They can be divided into:

- *space dynamics*: movements a mind makes in order to represent to itself the subject moving in relation to the object, or vice versa in order to represent to itself some rotations or displacements of the object and/or movements of mental sizing of the drawing in order to make it fit the sizes of the sheet of paper;
- *time dynamics*: these are connected with acquaintances one gained in the past ("when I had observed a cabinet like that …," etc.) and acquaintances one is gaining presently through perception, and the relationships between them (Guala & Boero, 1999);
- *producer/reader dynamics*: these show up when there are improvements during the various stages of the production of drawings (while they are usually lacking

when the final production shows great differences between what I "see" and what it actually "is").

In this phase of classroom work, a crucial task for students consists in observing a parallelepiped satchel standing on the teacher's desk and to draw it as he/she can see it from his/her seat. Secondly, the student must discover the positions of the pupils who did the drawings chosen by the teacher.[1]

The conjectures on the drawer's position are at first formulated in a *personal language which is still anchored to perception*, such as:

Sara: The drawer was in the row near the door, because we can see the thickness of the satchel in the right side in his/her drawing.
Mara: In my opinion the drawer was in the second row on the door side, I understood it by seeing the satchel drawn only in its central front.

All the problems connected with seeing have arisen: what I can see from the right side, from the front; no one should draw the roof, because it cannot be seen while sitting, but what must these lines be like? (when "upwards," "downwards," etc.); the nearest side is longer, the farthest one is shorter, etc.

At this moment, teachers organize a comparison between the photo of the satchel from one's seat and the drawing one produced. The photo has shown "in an objective way" the course of the lines of the satchel and has enabled the students to re-discuss their own conjectures, possibly to a mend their drawings, by stressing the differences between what I "see" and what it actually "is".

The following discussion concerning the conjectures that have been produced about the relationships between the observer's position and the content of the observations is aimed to share the students' explanatory hypothesis and to build a *shared language in the class*. At this stage the produced statements are of the kind: "If I am opposite to the satchel, I can see only a side," "If in reality figures stand opposite, they maintain their own shape in perspective," or "If a rectangle lies parallel to one's sight level, it is still a rectangle". These formulations have shown the awareness of the existence of dynamic relationships between the observer's "eye" and the "observed" object and the attempt to fix a state of these variables in order to produce statements. The language is influenced by perceptive aspects ("opposite to," "I can see," "sight level," etc.), but it is more general. Indeed it uses words such as rectangle, level, parallel, which belong to geometry and show the attempt to generalize to rectangles some argumentations, which fit the case that has been examined. The reasoning is richer and the conjectures are more general.

The work goes on in the 7th grade, by analysing two perspective representations (of a street in a town) and asking the students to recognize which of the observed figures were in reality rectangles, what kind of geometric figures they had been transformed into through perspective and "when" a "rectangle" in reality is still a rectangle in perspective.[2]

The students recognize conditions under which parallelism/perpendicularity among straight lines is preserved; everybody concludes that a rectangle is still such even in perspective "if it lies on a plane that is opposite". The teacher explicates the connection between the meaning of "opposite to" and the concept of "visual field": he/she builds a shared meaning in the class, by evaluating approximately its wideness, by stressing relationships between the visual sphere and the plane which approximates it, etc. The teacher promotes a classroom discussion about the conditions produced by the students. The teacher helps reformulating the hypotheses arising in the shared language of the class, classifying them in categories and drawing a "table of invariants" (Bartolini Bussi, 1996).

Later the teacher proposes a problem situation that is undecidable by an empirical approach and enhances the discussion: "If a rectangle is looked at non-frontally, is it still a rectangle in perspective?" .

Then an open discussion follows; students share the idea that if t*he answer is Yes, it is sufficient to find an example* and try to find examples which may help find a positive answer.

The teacher urges further: "If we cannot find any examples, are we merely unlucky, or does that mean the answer is No?". Many students maintain they will succeed in finding an example with a large number of trials; but how many trials will be necessary? A few students recollect previous activities on properties of numbers, where it became clear that it was impossible making infinite trials. Thus the teacher helps understanding that, as we cannot make infinite trials, it is necessary to find a few *objective* "*rules*" that enable us to come to a conclusion. He points out that perception may help, but it cannot give certain results, as it is subjective. In this regard he reminds the students of previous activities about proportional drawings, where objectivity came out from the rules for scale reduction and from measuring and activities on photographs, where, on the contrary, the objectivity of the photograph on the situation which had been taken into account had been invalidated by the subjective reading of the photograph itself.

As in the field of experience of the "plane representation of space situations" measuring is of no help; it is necessary to find new *objective* "*rules,*" which are formulated in terms that do not depend on measuring or perceiving. They allow one to validate conjectures by convincing arguments which are independent of experience, in those cases that cannot be decided empirically. We need a proof: a logical deductive process, which is valid within a reference theory and which, therefore, uses arguments that are independent of experience.

The personal meaning everyone assigns to the words *perspective*, *theorem* and *proof* is the objective of a preliminary consideration. The students are required to hang on a board their own definitions of these three words with the help of an Italian dictionary in order to express the meanings in a better way and to communicate their thoughts properly. Later on every definition was discussed collectively. As far as the word *perspective* is concerned, its definition was also applied to direct and accurate observation of tridimensional objects or to following outlines of houses on the window panes with a finger. In this way the formulations and the words that were too vague, imprecise or wrong arose and were replaced or integrated. The produced formulations were later compared with the "official" ones, which can be found in several high school books, in order to work out final definitions, a synthesis of their thought and what is in the books.

Under the guidance of the teacher, the class can arrive at producing the following definitions:

- perspective representation is a picture drawn according to perspective technique, which transfers figures into a vertical plane (plane of projection), as they are seen by a human eye;
- theorem is a statement that is deduced by reasoning from true statements;
- to prove is to show that a statement is true by reasoning and starting from true statements.

As far as the words *theorem* and *proof* are concerned, the teacher's explanatory interventions were more numerous; it was not possible to explain clearly the "deductive" character the proof has.[3]

After explaining why a proof is necessary, the students under the teacher's guidance started building a *"theory" of reference* (elements of projective geometry). They take into account the statements which have been produced and choose among them those which are equivalent to one another, trying to show up connections not immediately evident. The teacher helps students to reformulate these statements in a general, abstract and conditional way, by using the perspective *language, which is independent of both experience and perceiving*.

We report below the list of the "primitive" statements, called *known properties*, as they are recognized to be true by everybody.

P1: if a rectangle lies on a plane that is parallel to the plane of projection in reality, then it is still a rectangle in perspective.
P2: straight lines that are parallel both to one another and to the plane of projection are still parallel or are coincident in perspective.
P3: straight lines that are parallel to one another but not to the plane of projection converge or are coincident in perspective.
P4: if a rectangle is not parallel to the plane of projection in reality, then at least two of its sides are not parallel to the plane of projection.

Thus students reach an elementary "*modelling*" of perspective. The discussions that have led to building the statements have displayed both the "ideal" character of the geometric model (the "model" of geometry of representation is an approximation of the phenomenon of vision—and later we will show how the "model" of geometry of sun shadows will be an approximation of the phenomenon of sun shadows) and its "usefulness". Indeed it allows one to interpret a phenomenon in a better way, to transfer reasoning from the analysed situation to a simplified abstract one (Dapueto & Parenti, 1999), to reason by deduction, by using statements that are valid within the "*theory*" *of reference*, thus independently of experience, as "*sure*" facts.[4]

We have also verified that the knowledge of theoretical elements has had a good influence on the later drawing production and on recognition of mistakes in the representations themselves.

The following work is thus carried out in the theoretical field of experience of "geometry of representation" in order to answer the question that has been put off: "Only if a rectangle lies on a parallel plane to the plane of projection is it still a rectangle in perspective?," which is reopened by the teacher.

> The first to answer is Riccardo, middle level: In my opinion a rectangle *which is not opposite in reality*, cannot be still a rectangle in perspective, because the rectangle would not be parallel to the plane of projection, otherwise in perspective straight lines that are not parallel to the plane of projection would be coincident or convergent, so the rectangle could not keep as such.
>
> Luca comes next (high level): *If a rectangle is not parallel* to the plane of projection, two of its sides are not parallel to the plane of projection and so they converge in perspective and this two sides are no longer parallel.

The teacher resumes and engages in making these formulations shared by students. He also shows how some students' reasoning presents characteristics which are similar to those of the *reasoning by reductio ad absurdum*.

He shows a shared template for proof as a synthesis of individual attempts to produce proofs, a few of which have proved to be successful too. It was the moment for arrangement of a mathematical reasoning within a mathematical theory. At last the teacher helps reformulating the statement in its complete form:

> Theorem 1: A rectangle ABCD is still a rectangle in perspective if and only if ABCD lies on a parallel plane to the plane of projection.

He analyses it together with the class and shows how every conjecture, once validated, increases the number of statements which can be used to validate others.

New empirically undecidable problem The work goes on and students discuss the following new empirically undecidable problem: "Some students had recognized a few parallelograms which are not rectangles among the rectangles that had been transformed by perspective. Can a point in the space be found, from which a rectangle is seen in perspective as a parallelogram non-rectangle?". After a few minutes of reflection the teacher declares the statement is false and asks the students for a personal proof that makes use of the statements. Significant answers*:*

- Alessandro, high level: A rectangle cannot be a parallelogram in perspective. If it were parallel to the plane of projection, we could see *only* a rectangle according to theorem 1; if it were not parallel to the plane of projection, it would not be possible, because the lines joining at a point are <u>not</u> parallel to one another; and it is not true also because it would be in conflict with P3.
- Lara, middle level: If a rectangle ABCD lies on a plane that is parallel to the plane of projection, then it will be still a rectangle in perspective, according to P1. If we suppose a rectangle not parallel to the plane of projection, at least two of its sides are not parallel to the plane of projection (according to P4) thus (according to P3) its sides either converge or are coincident in perspective, but in this case it is not a parallelogram.

- Isabella, middle/low grade: No, because according to P4 if a rectangle is not parallel to the plane of projection in reality, then at least two of its sides are not parallel to the plane of projection and P3 says that straight lines parallel to one another in reality that are not parallel to the plane of projection either converge or are coincident in perspective, so the statement is false.

Each production is characterised by a great effort to look for logic relationships and to justify statements theoretically, by using "true" shared statements and a geometric language.

Many students, about one half, reveal an ability to build a deductive reasoning and to use geometric language correctly; very few do not write anything. During the formulation of the proof, many students moved pencils or a card in the air, others declared they had imagined moving a card in the space and had “found out” the relationships that they had later expressed in the proof.

These facts confirm that the *dynamic exploration* of the problem situation and *mental dynamics* are constitutive aspects for the construction of proof.

From the “Sun Shadows” Field of Experience to the “Sun Shadow Geometry” Fields of Experience

In the 8th grade students were involved in activities aimed at a “transfer” of the method they had followed in the preceding two years (see previous subsection). The aims of the research were similar and were raised by the wish to verify after a while whether the students had acquired a method in activities related to “theory” building and proving. Our didactic purpose was also to conclude the three years’ work by setting up a reflection on possible comparisons between “theories”.

In the “sun shadows” field of experience we set the problem: “*how can the sun draw its shadows?*”; and we asked to find out the rules according to which the sun “*drew its shadows*”

The students worked on geometric problems which were introduced as dynamic problem situations to be explored.

At first we asked them to watch how the shadow of a card varied according to its position in the space and to make out what geometric figures the shadow might take on in the analysed situation, “when” the shadow of the card was still a rectangle and “what” geometric properties of the rectangle the shadows of the card still maintained in every case or in some cases.

The aim of this task is to identify the Euclidean geometric properties that are in conflict with the sun shadows geometry, or that are in accordance with it, in order to rationalize the observed transformations by formalizing a few rules which “govern them”.

The starting conjectures were of this sort: “When the rays are perpendicular to the card, its shadow is a rectangle,” “When the card is opposite to the sun, …,” “if the sun is lateral, the shadow becomes a parallelogram,” “when the card follows the same direction as the sun rays, its shadow is a segment,” “when the card is perpendicular to the sun, …"; many of these conjectures were supported by drawings that display the students' opinions.

The dynamic exploration led the students to make it clear that every result depends on the position of the object (or of the geometric figures) in relation to the inclination of the sun rays and on the plane where the shadow is thrown. In the work they implicitly defined a vertical plane (wall) or a horizontal one (desk or floor) as planes where shadows are thrown and their works are affected by these choices, which will later be made explicit and clear. The students singled out the border-line position by themselves; they used the expressions "sun rays direction" or "plane containing the sun rays" or "parallel plane to the sun rays," according to circumstances. We decided to fix the horizontal plane as the plane where the shadow is thrown.

We immediately noticed that such productions, which were expressed in a language strongly connected with reality, made use of both concept-words such as "frontal," "lateral," whose meaning was common to the class, and words belonging to the geometric language they had learned the previous year. Some make contradictory statements, others formulate conjectures equivalent to one another. In every class the discussion converges on parallelism of sun rays and the reflection on the produced conjectures defines the border-line case clearly and causes the class's language to develop quickly in order to sort out conjectures everybody agrees on (only three hours' work).

When they think of the card sides, the students say "straight lines parallel two by two," when they think of the card changing into a segment, they say "when the straight line is in the same direction as the sun rays …". The physics support, which was useful to explore the situation, is soon overcome as such and the reformulation of the statement is no longer based on the phenomenon of shadows, but it uses the geometric language (straight line, direction, etc.).

The teacher engages in making the statements that have arisen shared by students; then in cooperation with the class he sorts out the ones everybody recognizes as "true," he helps to classify and reformulate these statements in a general, abstract and conditional way, starting to build those elements of "theory" of sun geometry, through which the students will be required to validate a few theorems:

P1: The shadow of a straight line is still a straight line (or a point when the straight line follows the same direction as the rays).
P2: Parallel straight lines project either distinct parallel shadows or coincident ones.

During the discussion an unexpected incident happens in the class (and is suggested in the other classes by the teachers): Riccardo (middle level, but already extremely interested in the activity last year) says "I see: the shadow of a rectangle is always either a pallelogram or a segment."

The teacher asks the class to prove "*Riccardo's theorem*" by using the *known properties*. So the problem of "how to prove" is made concrete through a simple example. We could verify that almost all students remembered how to build a simple proof.

Andrea, one of the best student: *Except for the border-line case*, when the shadows of opposite sides are coincident, the shadows a rectangular figure throws are always quadrilaterals. I suppose by absurdum that the shadow is not a parallelogram: in this case the shadow must be a trapezium, or a trapezoid. But according to P2 I know that parallel straight lines always project parallel shadows, therefore it is not possible for parallel sides to project non-parallel sides, or to draw the shadow of a quadrilateral with non-parallel sides.

In other words [see Figure 1 below]: it is not possible for sides AB and CD to project non-parallel segments as well as it is absurd that projected lines BC and DA are not parallel.

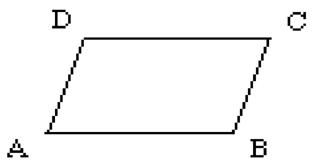

Figure 1. Picture by Andrea.

It is useful to make use of P2 again in order to demonstrate the second part of the statement.

When the rays lie on the same plane as the sides AB and DC or AD and BC, the shadows they project are segments (P2), therefore the shadow of the whole rectangle will be a segment, when it lies on the same plane as the sun rays.

The teacher discusses the students' work and enhances the discussion on the theoretical ground: "How can we be sure that a quadrilateral casts a quadrilateral as a shadow? If we say parallel lines throw parallel shadows, that does not necessarily mean that, for instance, some sides may not become parallel to one another in their projection."

The students state that a closed figure must cast a closed shadow, consecutive segments throw consecutive (or superposed) shadows, etc.

The teacher suggests the idea they need to increase the number of basic statements, in order to demonstrate Riccardo's statement in a correct and complete way. They must find more statements that everybody recognizes as "true," referred to the simplest geometric elements.

A discussion follows and the students suggest statements for the necessary completion. The following stage concerning the choice and the reorganization of the simplest elements leads to defining all the "primitive" statements of the "sun geometry".

P0: The shadow of a point is a point.

P1: The shadow of a straight line that does not follow the same direction as the sun rays is a straight line; the shadow of a straight line that follows the same direction as the sun rays is a point.

> P2: If a point lies on a straight line, the shadow of the point lies on the shadow of the straight line.
> P3: Two straight lines locating a parallel plane to the rays cast superimposed shadows (except in the case they are both in the same direction as the sun rays).
> P4: Two distinct straight lines locating a plane non-parallel to the rays cast distinct shadows.
> P5: Two parallel straight lines cast parallel (either superimposed or distinct) shadows.
> P6: Two intersecting straight lines locating a plane non-parallel to the rays cast intersecting shadows.
> P7: The shadow of a quadrilateral lying on a plane non-parallel to the rays is a quadrilateral.[5]

After assuming the previous statements to be true, teachers asked the pupils to prove the following *theorems* individually:

> T1: The shadow of a parallelogram lying on a plane non-parallel to the direction of the rays is a parallelogram.
> T2: If the shadow of a four-sided figure is a parallelogram, then that quadrilateral is a parallelogram individually.

The association of (T1) and (T2) is an example of necessary and sufficient condition, which requires reasoning by reductio ad absurdum in order to prove. We decided to divide our work into two phases (working on P1 before and later on P2) and to have a discussion, checking and communicating both the results and the ways to reach them between the former stage and the latter one. Eventually the teacher suggested a common proof frame.

A new theorem within "sun shadow geometry" One of the aims of this work was to set a significant problem about the "sun shadows," whose solution could be validated within "sun shadow geometry". We chose the following problem from Boero et al., 1996 (see also Boero et al., 2006), also in order to compare the students' behaviours in two experimental situations, which were very different as to developing explicit elements of "theory":

> *New problem*: In the previous years we learnt that the shadows of two vertical sticks on the horizontal ground are always parallel. What about parallelism of shadows in the case of a vertical stick and a not-vertical one? Can their shadows be parallel? Sometimes? When? Always? Never? Make some conjectures and express your conclusions in statement forms.

Possibly by moving two pens on a desk (no sun ray in the classroom!), the students easily sort out situations when oblique non-parallel lines have parallel shadows, they describe them also in drawings, many produce a verbal description as well, but only a few manage to describe the identified situation by formulating a correct and complete statement. For example Stella (high level): "If we have two non-parallel lines, each of which lying on one of two planes parallel to each other and to the sun rays, the shadows of the two lines will be lines parallel to each other. Therefore parallel shadows can be originated by non-parallel lines."[6]

The statements that have been produced are discussed in class until students come to a *common formulation* like the following one:

> If two straight lines lie on two parallel planes, both parallel to the direction of the sun rays, then their shadows are parallel.

Then teachers *ask for a proof of this statement within the "sun shadow geometry"*.

We report a few significant examples of proofs, adding that the teacher had suggested, after a few minutes of free reflection in the class, the students should fix one of the two lines in a vertical position.

> Andrea (high level): First of all I fix one of the two lines so that it is upright, in order to make my reasoning simpler. Thus only the other one can be moved. As the rotation of the latter one takes place on a parallel plane to the rays, it belongs to an infinite sequence of lines which, according to P3 (two straight lines locating a parallel plane to the rays throw superimposed shadows, except in the case they are both in the same direction as the sun rays) throw superimposed shadows. But in this infinite sequence of straight lines there is a vertical line like the first one, which is fixed. As they are both vertical, they must be parallel to each other, but according to P5 (two parallel straight lines have either superimposed or distinct parallel shadows) they throw parallel shadows. But all the shadows the latter line throws rotating are superimposed on the shadow of the latter line when it is vertical, thus they are parallel to the shadow of the former line as well, which is vertical too. At this point I rotate the former line too, which is therefore no longer vertical and according to P3 belongs to an infinite sequence of lines, which have parallel lines. But all these shadows, as they individualize a single straight line, will be parallel to the superimposed shadows which are thrown by the latter line one after the other.

> Maria (middle level): If two straight lines lie on two vertical planes both parallel to the direction of the sun rays, then their shadows are parallel. First of all the straight lines are still such according to P1 (the shadow of a straight line that does not follow the same direction as the sun rays is a straight line, the shadow of a straight line that follows the same direction as the sun rays is a point). According to P3, the two straight lines lying on two planes parallel to the rays throw superimposed shadows. Therefore, if I rotate only one line, fixing the other one, there will be a case in which it will be parallel to the fixed line and so, according to P5, the shadows are parallel; if I do not fix the other one, the lines rotating will be parallel in one case and so the reasoning is the same.

Most low level students follow correct reasoning, supported by drawings as well and showing they have understood the problem, but they do not succeed in shifting the reasoning to the theoretical level of proving. For instance:

> Omar (low level): The straight lines in the picture 1 (see Figure 2) look like intersecting, but they are oblique and they are parallel to the sun rays as the picture 2 shows. From the picture 2 (see Figure 3) we can also see that the shadows are parallel.
> Statement: If two oblique straight lines are parallel to the sun rays, their shadows are parallel. Proof: it is true, because even though they are oblique, the shadows are parallel to one another.

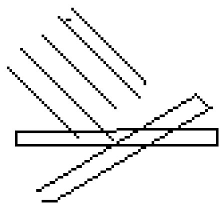

Figure 2. Picture 1 by Omar.

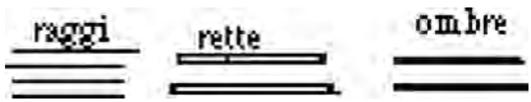

Figure 3. Picture 2 by Omar.

Students' productions show how *dynamic exploration* represents the fulcrum of the students' thinking process during the proving phase, in accordance with Boero et al., 2006.

They also show how their awareness of a reference theory allows many students to formulate several quite exhaustive proofs.[7]

Comparison of "Theories"

At the end of our work we wanted to stress that *mathematical truths are "relative"*: a statement of a theorem can be either true or false according to the reference theory that has been chosen. For this purpose we asked the following question (as an individual task):

> Last year we proved that a rectangle never changes into a parallelogram non-rectangle in perspective. This year we have proved that the shadow of a rectangle is always a parallelogram (except in the border-line case). These results may seem contradictory, but they are both true. How can you explain that?

In general, students write: "They are two completely different things" and most of them explain that an eye sees parallel straight lines to converge to a point in perspective, while shadows depend on the sun rays that are parallel to one another. In particular:

> Michela (high level): Geometry is not a unitary subject at all, it includes several disciplines which have their own rules; we have space geometry, the sun shadow geometry and many more which have different properties, even though they are connected with one another. They cannot be compared with one another, because they originate from different observations and starting data. Therefore I can say it is logical that results are different, because as to geometry we studied last year the main element was perspective: in the drawing the parallel lines of an object had to converge towards a point and thus the sides of a rectangle, which were parallel in reality, became converging and they could not draw a parallelogram. The sun shadow geometry is based upon different rules and so the ones we found out last year could not fit, as the latter is based upon parallelism of the sun rays that keeps parallelism of the sides of an object in its shadow.

> Alessandro (high level) after drawing two proofs related to perspective lines converging and parallelism concludes: in different geometries two different statements can be both true, even if they may sound in contradiction, because they follow different rules and laws.

> Simone (middle/low level): They are two completely different things, referring to two different theories. Starting points for reasoning are different.

CONCLUSION

At the end of the reported long-term teaching experiment we tried to understand what circumstances had allowed teachers to involve the whole classes actively in mathematical argumentation/proof activities and to reach quite satisfactory results. Our analysis was performed in the perspective of performing wider experiments. We took the following elements into account:

General conditions not depending on the planning of the teaching experiment

- *School environment*. Class team's cooperative work: surely it had a positive influence on the students' classroom work (in particular as concerns the strong engagement in improving verbal performances, under the joint pressure of mathematics and language teachers); composition of classes: classes without any disturbing elements and with at least one particularly interested student in every class. They used to echo the teacher's voice, provide an appropriate mediation for difficult issues through their texts and sometimes let the teacher change some aspects of the planned activities, in order to favour students' understanding.

- *Extra school environment*. Teachers worked in small town schools, where demanding school education is highly appreciated. The families, which were rather homogeneous as to social and cultural status, showed confidence in the choices the class teams made.

Specific and intentional conditions
In our opinion, the main factors that let this long-term teaching experiment work out, and the reasons for it, are the following ones:

- *Choice of suitable experience fields*. The experience fields of "plane representation of visible space" and "sun shadows" turned out to be particularly stimulating in every stage of our research, because they enabled teachers to carry out numerous activities implying observation, rationalization of reality and an early meaningful approach to tridimensional geometry (not restricted to studying traditional solids: cube, cylinder, etc.); they helped students to build (under the direction of the teachers) the statements of the reference theory, as significant situations are easy to be tested; they allowed us to choose problem situations which are suitable for dynamic exploration and, on the other hand, do not allow an empirical solution (through measurement, physical testing, etc.); they allowed as to display easily the "ideal" character of geometric models.
- *Choice to face significant geometric problems which require dynamic explorations of problem situations*. Dynamic exploration is one of the terms that most contributed to pursue our aims, also because it allows us to carry out complex mental experiments. We verified how analysing geometric figures on varying their position in the space (or on varying the observer's point of view) helped the students to discover/to become aware of mental dynamics influencing their production, but above all it led them to recognize naturally the "rules" regulating the examined geometric transformations, considering the border-line situations as well. The following formulation of the invariant table (see Bartolini Bussi, 1996) and of the terms that will make up the reference "theory," was positively influenced by that. For instance, *"The shadow of a straight line (not lying in the same direction as the sun rays) is a straight line; the shadow of a straight line lying in the same direction as the sun rays is a point"* is a formulation suggested by the students. Within the chosen fields of experience, the dynamic exploration helped to formulate conjectures that are impossible to be proved only through physical experiences and thus to justify mathematical proof as the only possible mean to validate them. The reasoning that supported the students' proofs was positively affected by the dynamical explorative feature as well. Moreover, we verified that all the students, when describing a specific "state" of the examined situation, often properly used notions such as parallelism, perpendicularity etc. (which had been built in the same "fields of experience" through previous activities). The habit of analysing geometric figures by varying their position in the space and their specific shape, contributed to overcome conceptual ambiguities/problems (which are often met in the traditional approach to geometry) between a drawn figure and the concept it represents.

- *Choice of the problem situations and their management in the classroom. The experience*. We suggested different problems which, being significant as to their mathematical content and interesting as to the research of their solution, enabled teachers to keep the students’ interest alive for a long time. The problems were formulated as questions, whose answers were to be sought within diversified activities: in the 6th level observations, drawings, photos and arguments; in the other classes we followed the stages: to observe → to produce conjectures and throwing doubts upon the formulated conjectures and suggesting knowledge should be organized in a different way in order to prove them (due to the fact that the possibility of an empirical test failed).
- *Choice of the didactic contract*. The students accepted the intellectual challenge and the special didactic contract that was explicitly proposed. In effect the teacher declared they would carry out an “experimental” work on high school issues, thus of high standard. The students knew they would be evaluated, only favourably, on the grounds of their efforts and application, not of their outcome. Hence, every student could feel a protagonist: the low level students were not afraid of unfavourable evaluation and took an active part; the naughtiest students endeavoured to follow the class work, because they were aware they could be as prominent as the best ones. Everybody immediately realized the best intuitions did not always come from the best students, so kept their initial interest in the long run. For instance in a class it was important that Giada and Sabrina, two high level girls, found it difficult to represent mentally the situations that were to be analysed and, therefore, by making a wrong use of a statement, they drew unrealistic conclusions without realizing it.

Intentional conditions relating to the general choices of the project

- *Paying attention to the language*. The students were already accustomed to formulate motivated hypotheses, to discuss them and to compare individual texts.

  This ability relating to language, which had been built during previous activities of the project (for instance, in the 6th level within activities concerning properties of numbers, shadows or observations of objects from different points of view) enabled the class language to develop rapidly from a unrefined one, based on experience, to the geometric language. We paid great attention to logical refinement of language as to its vocabulary and its correct use (of course taking account the students’ age). In particular the use of words and expressions like “every,” “all,” “some,” “it may be that,” “it must me,” “if … then,” etc. was put into discussion. Every refinement of hypothesis coincided with a refinement of the language that was used to reformulate the hypothesis. So the students got accustomed, gradually but constantly, to revising and to improving the formulations they already knew, gradually using a more and more precise language.

  The attempt to build “theory” together with the students was very helpful to refine their language. In these work stages the whole class’s attention shifted consciously to the words that were to be used, to their meanings and to the possible ambiguity of formulation. The low attainers also succeeded in mastering many elements of a correct and proper language they managed to use consciously, so kept their starting interest in the long run. For instance in a class this was important.

- *The teacher's role*. In our work this is very important, not only as a cultural reference for families and students, but as a determinative factor in every "condition" we have taken into account. The teacher was able to establish relationships based on mutual trust with the students; he was able to act as a cultural mediator who coordinates and enhances the mathematical discussion (Bartolini Bussi, 1996) and leads it from the empirical ground to the theoretical one. Namely he was able to be the "voice" of the reference culture during the stages of gradual and slow construction of language and of the meanings of the words "statement," "theorem," "mathematical proof".

  In the examined fields of experience we verified the teacher can coordinate the students' attempts to issue statements, to take an active part in building a reference "theory" and in producing proof frames without pushing or forcing the phases of their learning process.

  In our research group the teacher's role is very important in order to plan activities and analyse the students' mental processes (see Malara & Zan, 2002).

In conclusion we think the activities that were carried out during our "long-term teaching" experiment may be a correct and culturally significant approach to "theory" and mathematical "theorems". Our outcomes seem to confirm students can be introduced to theorems early and unselectively, provided that activities are carried out in suitable contexts (fields of experience) and with a correct mediation.

The theoretical reference frame within which we worked (fields of experience, mathematical discussion and cognitive unit), together with the process we found, seem to have important didactic implications (seem to be didactically helpful) in order to help the high school 1st and 2nd grade students to overcome the difficulties they may have to face in the traditional school approach to theorems.

OPEN PROBLEMS

One of the problems our group mostly debated was the level of generalization, abstraction and awareness the low attainers of the classes (about one third of the whole number) reached in the activities we had planned.

These students took part in the activities constantly and with great interest, and acquired quite satisfactory results as concerns mastery of geometric language.

They actively took part in the open discussions related to the activities of "theory" construction, sticking constantly with the semantics of the statements we discussed. On the other hand we noticed that their processes of abstraction/ generalization met big obstacles during proving activities: the low grade's participation arose only on an empirical level, thanks to continuous concrete references.

We think it could be interesting to check whether an appropriate management of this empirical escape may be useful to allow low attainers to move towards the theoretical level; whether the planned activities may have better results if carried on for longer periods (thus attributing blockages to lack of time for approaching the more demanding tasks and levels of performance); whether the lowest attainers in grades 7–8 of compulsory school need specifically designed activities or whether they are precluded from the intended levels of performance.

NOTES

[1] The former task enhances the students' ability to communicate meanings (position), by stressing the producer's role; while the latter one puts the importance of the continuous shift of the producer/reader roles into evidence.

[2] This task is aimed at identifying the Euclidean geometry properties that are in conflict with the geometry of representation, or that are in accordance with it, in order to rationalize the observed transformations by formulating a few rules which "govern them".

[3] These activities have contributed to building the first elements of a shared idea of theorem and of a strict mathematical language, which is useful to find/to get rid of possible ambiguities in the formulation of the statements which have been previously produced.

[4] As far as these aspects are concerned, our teaching experiment can also be considered as a prototypical situation where students can understand and take part in the development process of abstraction/formalization which a mathematician and, more generally, a scientist must undertake, when from the observation of particular *facts* he formulates conjectures about general behaviours that he must justify later within a theory of reference.

[5] These statements are superabundant and are not independent (in particular P6 and P7 can be proved by using P1 and P2).

As teachers had little time at their disposal, they had not been considering that for long, also because the class had learnt since the previous year that whenever a statement is demonstrated as true, it can be used in order to validate some more statements.

Because of lack of time, teachers could not complete building the geometric model and introduce the projective geometry elements that are necessary to compare the model with the geometry of representation (elementary statements, correspondence between a point on a plane and its shadow, etc.).

[6] We notice that, in the experiment described in Boero et al., 1996, most students had come to correct and complete statements only thanks to the discussion led by the teacher.

[7] As regards this aspect we can notice remarkable progress in comparison with the preceding teaching experiment reported in the above-mentioned paper.

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14. GEOMETRICAL PROOF

The Mediation of a Microworld

INTRODUCTION

Intuition and Theory

It is common opinion that geometry is a science, deeply rooted in common experience and based on empirical data: much of the feeling of certitude and the guarantee of consistency that geometry inspires has always come from these origins. Recent history witnesses a deep change in the attitude of mathematicians towards geometry, and mathematics in general: a complete autonomy of the axioms from their interpretation in reality was stated and a clean separation was accomplished between the logical and the psychological level.

Certainly, geometry has lost its centrality as a corpus of truths about the physical world that it had maintained through the centuries; after the theoretical reorganization of mathematical knowledge geometry is now a piece of theory among others.

As soon as the theoretical aspect of mathematics, and in particular of geometry, prevailed a break occurred separating the formal and the intuitive level. The clarification of the relationship between the two levels was stated in the formal distinction between truth and validation and as far as mathematics is concerned, the problem seemed to be solved.

Nevertheless, as the following passage shows, the complex link between the intuitive and the theoretical level cannot be neglected and mainly at the educational level.

> […] the choice of the basic elements of geometry is not determined "a priori": the simplest elements are selected with respect to *psychological intuition*. That is, they are those elements, the notion of which is already formed in one's mind as the content of the concept of space, for instance, the point, the line and the plane. Geometrical properties which are the basis of the axioms gather around a number of notions, unquestionable, but intuitively understandable by themselves.
> (Enriques, 1920, p. 3).

Although it is reasonable to think that for a mathematician the notion of primitive elements is already formed, this is not the case for our pupils. Managing the com-

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 285–304.

plex relationship between intuitive and theoretical dimensions constitutes one of the main difficulties of geometry teaching/learning.

Although the concepts of Euclidean geometry taught at school derive from the experience of realistic physical space and, in principle, do not contradict ordinary beliefs, complete congruence between the two systems is not always assured.

As far as geometrical reasoning is concerned a particular kind of mental object is involved; these are commonly referred to as *geometrical figures*. A geometrical figure—as it is used in geometrical reasoning—is neither a pure image nor a pure concept. In this sense and for this reason, Fischbein (1993) introduced the notion of ***figural concepts***, referring to geometrical figures as mental entities which possess, simultaneously, both conceptual and figural properties. According to the theory of figural concepts, any simple geometrical reasoning deals with:

> a mixture of two independent, defined entities that is abstract ideas (concepts), on the one hand, and sensory representations reflecting some concrete operations, on the other.
> (Fischbein, 1993, p. 140)

Geometry maintains its own specificity: what is particular about geometry is that it conserves *in the reasoning process an objective, pictorially representable property of reality which is* ***space***. Actually, as previous studies show, harmonizing the two components of figural concepts is neither spontaneous nor simple (Mariotti, 1991, 1995, 1996a): difficulties in geometrical reasoning can be interpreted in terms of a rupture in the fusion between figural and conceptual aspects.

Besides the possible discrepancies between spontaneous conceptualization of space and geometrical concepts (Mariotti, 1996a), one of the main characteristics distinguishing geometry from intuitive cognition is the way in which they are made acceptable. A geometrical statement becomes acceptable only because it is systematized within a theory, with a complete autonomy from any verification or argumentation at an empirical level; in other word it becomes a theorem (Mariotti et al., 1997).

Although it is impossible to reduce mathematics (and mathematical activities) to formal deduction, its theoretical dimension constitutes a fundamental aspect: mathematical knowledge is characterized by its organization according to axioms, definitions and theorems.

> What does mathematics *really* consist of? Axioms (such as the parallel postulate)? Theorems (such as the fundamental theorem of algebra)? Proofs (such as Gödel proof of undecidability)? Definitions (such as the Menger definition of dimension) […] Mathematics could surely not exist without these ingredients; they are essential. It is nevertheless a tenable point that none of them is the heart of the subject, that a mathematician's main reason for existence is to solve problems and that, therefore, what mathematics *really* consists of is problems and solutions.
> (Halmos, 1980, p. 519)

Thus, the complexity of the educational problem is linked to the difficulty of harmonizing the intuitive and the theoretical level and can be summarized as follows: the main objective is twofold: on the one hand to develop a dialectic interaction between the figural and the conceptual aspects, on the other hand to construct a theoretical framework within which this interaction is accomplished.

School Geometry: Intuitive versus Deductive

The complexity of the relationship between the intuitive and theoretical dimension of geometrical knowledge leads to a long-standing controversy between two opposite perspectives, reflecting the basic dichotomy empirical/deductive. (Schoenfeld, 1985). The opponents of the deductive approach hold that deductive geometry is meaningless without a deep intuitive understanding and this should be rooted in the empirical world. Thus often the accent on intuition determines a radical shift to an empirical approach. That means that a generic and ambiguous criterion of intuitiveness becomes the inspiring principle: observing and discovery become the basic geometrical activities, not much attention is paid either to developing a theoretical attitude, or to elaborating a coherent system of geometrical properties. The gap between spontaneous conceptualization of space and geometry is underestimated; generally speaking geometry is considered as being a natural development from physical experiences.

As a consequence, the great difficulties that students show are interpreted as caused by the "formal classroom exposure to geometry in its deductive–axiomatic form" (p. 249); the remedy is simple: banish any theoretical element and simply refer to intuition.

Actually a deductive approach to geometry has become very rare; it disappeared from the curricula and even in those countries, e.g. Italy, where the curricula did not change, it almost disappeared from school practice.

In recent years the didactic problem of "proof," and more generally the educational problem of a theoretical approach to mathematics, has exploded, as the flourishing of research projects, reports and discussions testify. A shared opinion emerged:

> [proof] deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding.
> (Hanna, 1995, pp. 21–22)

A Didactic Problem

In the Italian school, geometry is one of the basic topics at any school level.

Referring to the previous discussion, it is interesting to read what official Italian programs state. An implicit assumption about Geometry education is recognizable: according to a natural development from intuition to theory, the first level of instruction—corresponding to 1st to 8th grades —is based on an intuitive approach for which reference to reality and observation is constant and is not questioned, the second level—corresponding to 9th to 12th grades—is devoted to a rational systematization of intuitive knowledge according to a deductive approach.

For instance, at the primary school level the official programs suggest the following: “The introduction to the study of geometry must be *naturally* related to different stimuli coming from the perception of reality. […] A rich and various geometrical activity is to be promoted, starting from manipulation of concrete objects and from *observing and describing* their transformations and mutual positions (emphasis is mine).”

As for the secondary level, the programs for junior schools state as the first topic of geometry: “From the objects to the geometrical concepts: study of 2-D and 3-D figures *starting from concrete models* (emphasis is mine)”.

At the upper secondary level, the new programs, proposed and not yet approved, remind us: “The basic aim of geometry is that of progressively guiding the student from intuition and discovery of geometrical properties to rationally describing them […].”

In the tradition of the Italian school, the beginning of the 9th grade (at the beginning of high school) coincides with the introduction to deductive geometry. In the previous grades geometry is usually thought at an “intuitive” level, that is a collection of geometrical facts is presented to pupils, together with a number of definitions.

At the beginning of high school, a very difficult didactic problem arises:

> *how to manage such a delicate relationship between the geometrical background that pupils have and the new theoretical approach to this knowledge.*

This relationship is usually very difficult to be managed, as everybody (and teachers better than others) knows very well. Often, the pupils fail to grasp what is new in respect to the old, thus it becomes impossible for them to make sense of the new way of doing geometry.

A crucial point is that of changing the status of justification.

Justifying

One of the main points in the comparison between the intuitive and the deductive approach to geometry is the role played by “justification”; that is by explaining, arguing, corroborating, verifying a particular statement.

A deductive approach is deeply rooted in a practice of justification; in fact, a deductive approach to geometry means the construction of a system of geometrical properties, coherently related through appropriate argumentation. In particular, it means constructing a system, which is based on a number of primitive assumptions, often related to an intuitive interpretation of evidence (=axioms), and can be enlarged by introducing new statements, related to the previous ones through a proof (=theorems).

Where proving is concerned, it is commonly accepted that arguing and proving do not have the same nature; arguing has the aim of convincing, but not always does the necessity of convincing somebody coincide with the need of stating the theoretical truth of a sentence.

> […] une très grande distance cognitive entre le fonctionnement d'un raisonnement qui est centré sur les valeur épistémiques lieés au statut théoriques des propositions. "…" Passer de l'argumentation à un raisonnement valide implique une décentration spécifique qui n'est pas favorisée par la discussion ou par l'interiorisation d'une discussion. […] Le developpement de l'argumentation même dans ses formes les plus élaborées n'ouvre pas une voie vers la démostration.
> (Duval, 1992–93, p. 60)

The gap between these two modalities may be very deep and sometimes arguing can become even an obstacle to a correct evolution of the very idea of proof (Balacheff, 1987; Duval, 1992–93, see also Mariotti, in press)

According to a so-called "intuitive approach" to geometry, based on observation and measure, pupils intuitively "discover" certain facts, most of them with a high degree of evidence. Often the teacher introduces geometrical facts through a "justification," but still such supporting arguments have the specific aim of convincing pupils of the evidence of those facts, and in this perspective they are far from providing pupils with a basis for approaching the deductive method. Moreover, pupils are never asked to justify their knowledge, they must simply know the "fact," the truth of which is considered immediate and self-evident, i.e. intuitive (Fischbein, 1987). Thus, according to pupils' experience, justifying pertains to the teacher, and has the aim of convincing one of the "evidence" of a certain fact; as a consequence, when a certain knowledge is attained, its justification is no longer necessary and is quickly forgotten.

Sometimes this phenomenon is particularly evident. Consider the case of the Pythagorean theorem. The common justification used by teachers to introduce it is based on one or two of the traditional "visual proofs" and has the aim of convincing pupils of its obviousness. However, a few months later, everybody knows the Pythagorean theorem and is able to apply it; nevertheless once the statement is learned, very few of the pupils can remember any "proof"! Actually there is no need to remember any justification: as soon as a statement reaches the status of evidence, any argument becomes useless and ready to be forgotten. According to its nature, intuition contrasts the very idea of justification. Stressing intuitiveness may become an obstacle to developing a need for justifying if the correct perspective from which the arguments make sense is not achieved. In other words, when intuitive properties are concerned, it is senseless to ask for supporting arguments, and the only way to give sense to a "proof" becomes to consider such arguments from a theoretical perspective. In order to express this complex of relationships a notion "Mathematical Theorem" was introduced (Mariotti et al., 1997). In the following for mathematical theorem we mean the triplet consisting in a statement, its proof and the theory within which the proof make sense.

In summary, from a didactic point of view, when a deductive approach is concerned, there are two interwoven aspects to be developed: the need of *justification* and at the same time the idea of a *theoretical system* within which that justification may become a *proof*. Proof makes sense in respect to a theory and vice versa; thus, the introduction of a deductive approach presents two problems of sense, which are interrelated: *the sense of proof and the sense of theory*.

In other words, the first difficulty to be overcome is related to developing the need of a justification, and this contrasts with the intuitive approach to which pupils are used, the second difficulty is related to the possible cognitive rupture between argumentation, i.e. a set of arguments supporting the acceptance of a statement, and mathematical proof, validating a statement within a theory.

The following discussion aims to face these crucial educational points and presents the choice of a specific "field of experience" (Boero et al., 1995): geometrical constructions within a particular Dynamic geometry Environment (Cabri-géomètre).

THE FIELD OF EXPERIENCE OF GEOMETRICAL CONSTRUCTIONS IN THE CABRI ENVIRONMENT

Theorems and Constructions in a DGE

A geometrical construction consists of a procedure that, through the use of specific tools and according to specific rules, produces a drawing. A construction is considered correct if the tools have been used according to the stated rules.

Despite the fact that there is a concrete counterpart of a geometrical construction that can be accomplished on a sheet of paper, geometrical constructions have a theoretical meaning, which overcomes the apparently practical objective. It is possible to state a correspondence between specific tools and their use and a set of axioms characterizing a piece of theory. Within this theory the validity of a construction will correspond to a theorem. Construction problems belong to the classic tradition of geometry: the *impossible* problems, so important in the history of mathematics, clearly illustrate the theoretical aspect of constructions (Heath, 1956, pp. 124–31).

Actually, the theoretical meaning of geometrical construction problems is very complex, and certainly not immediate for students to be grasped (Schoenfeld, 1985); it seems that the very nature of the construction problem makes it difficult to take a theoretical perspective, as shown in a completely different school context (Mariotti, 1996b).

Let us consider the drawing activity in a geometrical environment such as that provided by a Dynamic geometry Environment, such as "Cabri-géomètre" (Baulac et al., 1988).[1]

The internal logic of a Cabri-figure is not directly evident, but appears at once when one of the elements of the figure is moved. The particular "dragging" function permits one to move one of the elements, whilst maintaining all the geometrical relationships defined by the menu commands used in its construction. The movement becomes an essential component of the meaning of a Cabri-figure (Laborde, 1993).

Cabri-figures possess an intrinsic logic, which is the logic of their construction; the elements of a figure are related in a hierarchy of relationships, corresponding to the procedure of construction.

Figures produced by "Cabri-géomètre" are seen, but they must be conceptualized in order to be managed. On the one hand, the sense of a Cabri-figure consists of *conceiving a figure in terms of its own (characterizing) geometrical properties and accepting the dragging function as an intrinsic defining element of the environment*.

But there is something more. The dynamic system of Cabri-figures embodies a system of relationships consistent with the broad system of a geometry theory. Thus, solving construction problems in Cabri means not only accepting all the potentialities of the software, but also accepting a logic system within which the correctness of a construction can be validated.

In conclusion, two main aspects characterize the Cabri environment: one concerns the correspondence between the primitives of the software and the basic geometrical properties, the other concerns the dynamic of manipulating Cabri-figures which corresponds to a specific criterion of validation within a coherent system of those properties. According to these two basic aspects, which link the Cabri environment and a geometry theory, it is possible to build a correspondence between a construction and a theorem, so that *justifying* the correctness of a construction corresponds to *proving* a particular statement in a specific theory.

The following discussion aims to show how the specificity of the Cabri environment is determinant in order to make the sense of justification arise and evolve towards a theoretical proof.

In particular I'm going to discuss the construction task as it is presented within the Cabri environment; the analysis will refer to a long-term experimental project carried out over few years (Mariotti, 2000; 2001).

Semiotic mediation in the Cabri environment The term "field of experience" is used after Boero et al. (1995) to mean "the system of three evolutive components (external context; student internal context; teacher internal context), referred to a sector of human culture which teacher and students can recognise and consider as unitary and homogeneous" (p. 153).

As far as our project is concerned, the external context is determined by the concrete objects of the activity (paper and pencil; the computer with the commands of the Cabri software; signs—e.g. gestures, figures, texts, dialogues).

The following example will focus on the functioning of particular objects of the external context, offered by the Cabri environment, in relation to the evolution of the internal context:

> The Cabri-commands (primitives and macros), realising the geometrical relationships which characterise geometrical figures; the dragging function which provides a perceptual control of the correctness of the construction, corresponding to the theoretical control consistent with a geometry theory.

In the concrete realization of classroom activity, both elements may be used by the teacher as instruments of semiotic mediation (Mariotti, 2002; Vygotskij, 1978) in order to make the internal context evolve towards a theoretical attitude to geometrical knowledge.

The evolution of the field of experience is realized over time through the social practices of the classroom. In our experiment, classroom verbal interaction is realized by means of "*mathematical discussion, i.e. a polyphony of articulated voices on a mathematical object, that is one of the objects—motives of the teaching—learning activity* (Bartolini Bussi, 1996, 1998). The polyphony of voices in this case concerns the dialogue between the voice of practice and the voice of theory about graphical construction. On the one side, the concrete production of a drawing on a sheet of paper is a practical activity, whose correctness is definitely controlled by empirical verification, on the other side, geometrical constructions have a theoretical meaning that overcomes the apparent practical objective. The main motive of classroom activities proposed to the students is the development of the idea of geometrical construction at a theoretical level. Through the dialogue between the voice of the practice and the voice of the theory, the ability in creating and reading configurations that is continuously practised in the production of drawings in both environments has to be enriched with the theoretical control.

According to the previous discussion the crucial point is the status of justification, both in what concerns its need and its adequacy in terms of a fixed theory. Let us consider a construction task, as it is conceived within the Cabri environment. Certainly there are figures to be drawn, but that must be done using the available commands on the menu, moreover the figure is controlled by the dragging function, i.e. a construction task is solved if and only if the image drawn on the screen passes the dragging test. As soon as the control by dragging is accepted, the necessity of a justification for the solution comes from the need for explaining why a certain construction works (that is, it passes the dragging test), whilst other constructions do not.

Thus, a justification comes from the need of validating one's own construction, in order to explain why it works and/or foresee that it will function.

The key point is that what must be validated is the correctness of the construction; that is, *it is not the product of a procedure, but the procedure itself that must be validated*.

In other words, the problem is shifted from validating by dragging, to explaining the "validation by dragging" itself.

According to our experience (Mariotti, 1996b) this change of focus is hard to be achieved spontaneously. The results of our teaching experiment show that the nature of the particular environment may facilitate this shift, providing a context within which a request about the procedure becomes meaningful; nevertheless the context itself is not sufficient and the intervention of the teacher becomes crucial. The following analysis of a collective discussion aims at showing the complexity of the process as well as pointing out the main elements contributing to its development. In particular, in the following section an example will be given, describing the functioning of the history command.

THE MEDIATION OF THE HISTORY COMMAND

The episode that we are going to discuss involved one of the experimental classes of the project (9th grade in a scientific high school (Liceo Scientifico)). The long-term teaching experiment was based on a sequence of activities centred on the use of Cabri, the following example concerns the first activity that was divided into two parts.

The Scenario

The first part took place in the computer laboratory, where pupils sat in pairs at the same machine. Pupils had a general expertise of the computer, but they had never used Cabri; after a short acquaintance with the Cabri environment—they were allowed to freely explore the software for about half an hour—the following task was presented.

> *Construct a segment on the screen. Construct a square which has the segment as one of its sides*.

As the teacher explained, pupils were asked to realize a figure on the screen and write down a description of both the procedure and their reasoning. At that point, the interpretation of the term *construction* remained ambiguous, but we did it on purpose: different interpretations of the task were expected. Actually, the protocols collected after the activity in the computer lab contained solutions obtained differently, some of them referring to geometrical properties, others referring to perceptual control, so that using dragging function those solutions would have been differently transformed.

The second part of the activity took place in the classroom. The teacher opened the discussion suggesting analysis the solution given by Group 1 (Giovanni and Fabio). The solution was obtained drawing four consecutive segments, carefully arranged in a square, using perceptual control.

When the teacher asked pupils to judge this solution, everybody agreed that the control must be exerted on the particular drawing. According to the well-known definition of a square, pupils suggested measuring sides and angles. The main elements, arising from the discussion, were the use of measure and the precision related to it. All the interventions showed that the shared objective concerned the control on the drawn square. At this point the discussion was interrupted and resumed the day after. Let us analyse what happened in this second part of the discussion.

A Semiotic Game

After a brief summary of the previous discussion, the teacher presents the drawing proposed by Group 1 (Giovanni and Fabio) and drags it. The realization of the drawing was based on the perceptive adaptation of four segments; thus it is upset: everybody agrees that it is not a square any more.

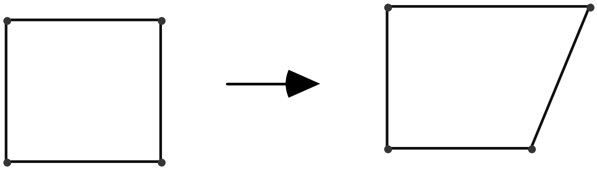

Figure 1. The construction does not stand the dragging test.

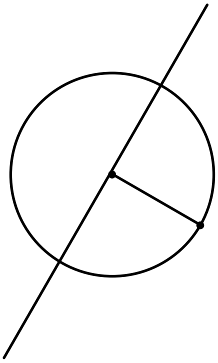

Figure 2. Group 3. The first steps of the construction.

Another solution (Group 3, Dario and Mario) is proposed by the teacher.

21 I: Well, I'd like to know your opinion about the construction of Dario and Mario

22 Marco: They did a circle then two perpendicular lines …

23 I: Do you know from what did they start?

24 Michele: We can use the command "history".

25 I: Let's do it . They took a segment, then they …

[… *the construction step by step follows*]

They drew a line perpendicular to the segment, then the circle … in your opinion, what is it for? What the use of it?

SILENCE Is there a logic in doing so, or did they do it just because they fe-like drawing a perpendicular line … a circle … Alex, tell us … 26 Alex: the measure of the segment is equal to the measure reported by the circle on the perpendicular line.

27 I: You mean that the circle is used to assure two equal consecutive segments, the first one and that on the perpendicular line … and the perpendicular …
28 Chorus: is used … to obtain … an angle of 90°
29 I: I know that the square has an angle of 90° and four equal sides or three equal angles … then let's see if it is true … let's go on. Intersection between line and circle. They (Dario and Mario) determined the intersection point between the line and the circle—why did they need that point?
30 Chiara: the intersection point between the line and the segment …
31 I: and what should you draw from there?
32 Chiara: a segment, perpendicular to the line
33 I: what else??
34 Chorus: parallel to the segment …
35 I: let's see what did they do …

After a first attempt to describe the procedure, Michele suggests using the command "history". The teacher immediately catches the suggestion and puts in execution the command showing the screen to the whole class. The first steps of the construction are successively repeated and the corresponding elements are redrawn on the screen; in the meantime, the teacher describes what has been done. At a certain point, she interrupts the description and asks the pupils to reflect and try to detect the "motivations" for those actions. This intervention (we call it the "**interpretation game"** (23)) aims to provoke the first *shift from the procedure to a justification of the procedure itself.*

Despite the difficulty and the artificiality of the move leading from action to explicating the motivation of this action, the teacher presses the pupils to detect a "logic" in the procedure described (25). In the following interchange the functioning of the discussion is clearly shown in respect to its main "motive": shifting the control from the description of the procedure to the motivation of the procedure.

Alex (26) expresses the relationship between two of the segments according to the series of commands previously executed and the teacher (27) reformulates his statement in terms of motivations: "You mean that the circle **is used for** assuring two equal consecutive segments …". The Chorus appropriates the terms used by the teacher and continues in terms of motivation.

The discussion continues: the pupils are now asked to foresee the next step, motivate it and then compare it with the step recorded in the history. That is what we have called the **prediction game**.

In the following different solutions are compared, repeating the script of the interpretation and prediction game; this makes it possible to negotiate the acceptance of a Cabri figure as the correct solution of a construction task. What is most interesting is the fact that, together with the acceptance of a solution in terms of the dragging test, a new relationship to drawing is achieved: it is possible to explain the correctness of a construction controlling the "logic" of the procedure. In other terms, a new meaning of the term construction emerges, related to the use of Cabri, but also consistent with its geometrical meaning.

The Role of the Teacher

According to our basic hypothesis pupils' relation to drawing is modified by the mediation of the Cabri environment, but as clearly shown in the previous section the role played by the teacher is fundamental. Because of the fact that the discussion is developed in the special context of Cabri constructions, in addition to the standard strategies (see Bartolini Bussi, 1996, 1998), specific strategies are there available to the teacher to manage discussions.

Besides the role of the dragging function which mediates the generality of a figure, the previous analysis reveals the key role of the "history" command; through its mediation the teacher may put in practice the **interpretation** and the **prediction games**.

1. the interpretation game, led by questioning which could have been the intention or the goal of the author in making such a construction; for instance the teacher can ask: why did the authors choose this command? what is its use for?
2. the prediction game, led by questioning which could have been the following step in this construction; for instance the teacher can ask how would you go on from this point?

The presence of the history command, that is the presence in the computer of a decontextualized and detemporalized copy of the construction, allows the realization of the two games avoiding explicit comments or implicit information (gestures, and so on) towards the expected answer.

Although not simple and spontaneous, the shift from "drawing as a product" to "drawing as a procedure" occurred. But, although necessary, this first shift is not sufficient to gain a theoretical perspective and the development of the sense of justification into the meaning of proof is still far from being achieved.

The Construction of the Theory: The Mediation of a Microworld

As discussed above, the world of geometrical construction has seen a new revival in the use of dynamic softwares such as Cabri-géomètre. As a microworld (Hoyles, 1993), Cabri embodies Euclidean geometry; in particular, it refers to the classic world of "ruler and compass" constructions.

However, the novelty of a dynamic environment such as Cabri consists in the possibility of direct manipulation of its figures through the dragging tool and in the fact that the functioning of the dragging tool is coherent with the system of Euclidean geometry. The dynamics of the Cabri-figures, realized by the dragging function, preserves its intrinsic logic, i.e. the logic of its construction; the elements of a figure are related in a hierarchy of properties, and this hierarchy corresponds to a relationship of logic conditionality.

Because of the intrinsic relation to Euclidean geometry, the control "by dragging" can be interpreted as the theoretical control—"by proof and definition"—within the system of Euclidean geometry. In other terms, it is possible to state a correspondence between the world of Cabri constructions and the theoretical world of geometry (for a discussion see also Mariotti, 2000, 2001).

In our project, the evolution of the Field of Experience is based on the potential correspondence between Cabri construction and geometric theorems. Once a construction problem is solved, i.e. if the Cabri-figure passes the dragging test, a theorem can be proved within the geometry theory. Thus, solving construction problems in the Cabri environment means not only accepting all the graphic facilities of the software, but also accepting a logic system in which its observable phenomena will make sense. The explicit introduction of this interpretation and the continuous reference to the parallel between Cabri enviroment and geometry theory constitutes the basis of our teaching project. Our basic hypothesis is consistent with the idea of microworld (Hoyles, 1993) as an environment for solving problems where pupils can experience the constraints of the underlying mathematical system and in so doing construct their own mathematical system.

As far as Cabri is concerned, the complexity of the underlying mathematical system is very high: in fact, the complete set of commands corresponds to the whole of Euclidean geometry.

As a consequence, the logic control on the underlining system may become too difficult to be achieved; in particular, because of the richness of the “geometrical tools” available, it is difficult to state what is given (axioms) and what must be proved (theorems). The richness of the environment might emphasize the ambiguity about intuitive facts and theorems and constitute an obstacle to the choice of correct hypotheses.

In order to overcome this difficulty, the basic idea of working inside a microworld was adapted to our specific objective. Instead of giving pupils an already-made Cabri menu, we made pupils participate (Leont’ev, 1976/1964) in the construction of an axiomatization through the construction of the corresponding menu, step by step. At the beginning an empty menu is presented and the choice of the first commands discussed, stating the correspondence with the specific statements selected as axioms. Then the other elements of the microworld are added, according to new constructions and in parallel with corresponding theorems.

In this way the system is slowly built up, and step by step the complexity increases: the aim is that of providing a complexity which can be managed by pupils; if the whole system is present since the beginning, there is the risk that pupils are not able to control the relationship between what is given and what is deduced. On the other hand, if the menu commands are changed too frequently it is impossible to grasp any systematic order.

According to our hypothesis, geometrical construction constitutes the key of accessing to a theoretical perspective. The analysis of pupils’ protocols shows the slow evolution of the meaning of construction. At first, a construction is conceived as a concrete process to reach a drawing, which has its own justification in the acceptability of the product; then, a construction is conceived as a theoretical procedure which has its own justification in a theorem.

On the one hand, the descriptions of the procedure change, improving in clarity through an increasing mastery of correct terms; on the other hand, the arguments supporting the correctness of the procedure approach the status of theorems; that is the justifications provided by the pupils assume the form of a statement and a proof.

An Example: The Construction of the Perpendicular Line

According to our basic aim of introducing pupils to "deductive" geometry, we decided to build a dialectic relationship between Cabri constructions and geometrical theorems. A detailed analysis of the protocols is not possible here (more details can be found in Mariotti, 2000, 2001), none the less, the following examples aim to give an idea of the evolution of the meaning of justification as it is mediated by the Cabri environment.

In analogy with the Euclidean axioms, besides the primitives of the creation menu, in the construction menu the commands are reduced to include only "Intersection of objects," "Compass" (i.e. "Report of length") and the "Report of angle".[2] From the theoretical point of view, this situation corresponds to the three criteria of congruence for the triangles and actually, that is what pupils had in their first germ-theory (see Chapter 11 in this volume).

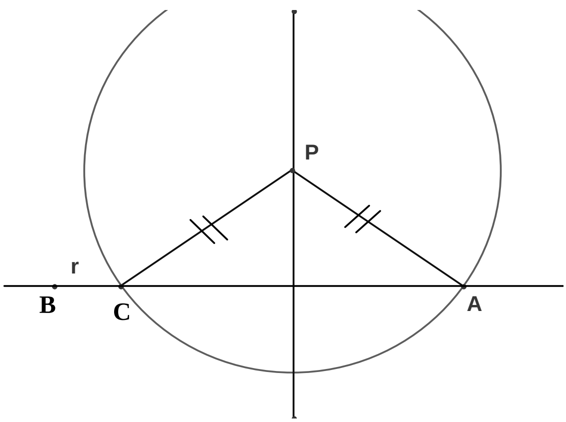

Figure 3. Group 1. The construction of the perpendicular line.

The first construction was that of the "angle bisector"; after its validation and the enlargement of the theory by the corresponding theorem, the new command "angle bisector" was introduced in the Cabri "menu". Afterwards, a perpendicular line was defined as the angle bisector of a straight angle. A long and articulated activity introduced the theorem of the isosceles triangle, which states that in a triangle the angle bisector of an angle is perpendicular to the opposite side if and only if the two sides of the angle are equal. All that represented the "theory" to which one can refer in his/her justifications. The following task was presented to the pupils.

Given a line r and a point P, construct a line passing through P and perpendicular to r. Describe the construction and justify your solution geometrically.

The pupils are grouped in pairs at the computer and asked to provide a common text for the solution. Let us consider the following protocol.

Group A (9th grade)

- Point P
- Construction of a line through two points A and B
- Circle centre P and point A
- Intersection of two objects: circle radius PA and line AB, it comes out at point C
- Construction of a triangle PAC
- Angle bisector (angle) APC

[The line] is perpendicular because:

PC=AP because radii of the same circle thus one knows that the angle bisector of CPA is the a. bisector of an isosceles triangle that divides the opposite side perpendicularly (in Italian, perpendicolarmente).

This is a good exemplar of solution. The theoretical meaning of a construction is achieved: the pupils show themselves aware of the necessity of justifying the Cabri construction. The parallel between the Cabri commands used in the construction and the hypotheses of the argument is clearly shown and that is accomplished by referring to pertinent pieces of theory (i.e. radius of a circle); similarly the reference to the previous theorem of the isosceles triangle is correctly linked to the use of the command "angle bisector".

Although some details are missing, the solution is basically correct and witnesses a good level of appropriation of the deductive approach.

The complete description of a construction, even of a simple one, reveals its difficulty: in fact, not all the pupils produce a schematic list of the commands used, followed by a justification of the correctness of the procedure; on the contrary, often the description of the single steps is interwoven with their motivations, with the result of a confusing text, difficult to interpret. That means that the elaboration of the solution is not yet separated from its systematization in a deductive frame.

It is in the collective discussion that such distinction can be achieved: discussing the constructions produced by others, pupils become aware of the necessity of a clear description of the procedure as a prerequisite to the control of the correctness. At the same time, during the collective discussion modalities of communication can be negotiated. As in every mathematical discussion (Bartolini Bussi, 1998), the role played by the teacher is fundamental: pupils must be introduced to the mathematical world with its specific way of communication, its specific discourse (Mariotti, 2001). A new goal emerges, strictly linked to the general objective of introducing pupils to a theoretical approach; the analysis of this new point overcomes the aim of this exposition.

I'd now like to come back to the basic didactic problem from which we started: the delicate relationship between pupils' intuitive knowledge and a deductive approach.

The Control of Intuition

As previously discussed, pupils have a geometrical background, that is impossible and unreasonable to cancel; on the contrary, one of the basic aims of the project is that of constructing pupils' awareness of a theoretical control of geometrical knowledge as it can be achieved by a deductive approach.

The following protocol is an example of the complexity of this process; it shows how pupils achieved a good theoretical control of the situation, but at the same time reveals the persistence of the intuitive background.

Group B (9th grade Liceo Scientifico)

What we constructed on the screen of the computer.

We created the line R and put the point P on it, and we put the point P on it. We created a circle of centre P. We intersected this circle with two more circles, which are equal and opposite to one another. The points of intersection between the line and the first circle are the centres of these last circles, and the radiis are equal to the distance of point P to the "end" (in Italian, *estremo*) of the first circle, that is it is equal to its radius.

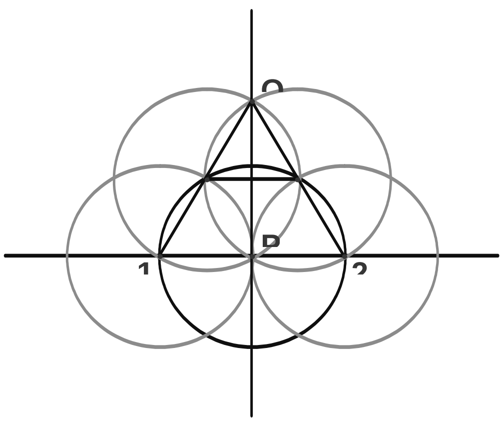

Figure 4. Group B. The construction of perpendicular line.

With these two circles we found two more points, which are the centres of two circles, which have the radius equal to the radius of the first one. These last circles intersect in a point (Q). If we join this with the point P, we find the perpendicular to the given line.

> We can say that the line that we found is perpendicular to R because if we join point Q with the ends of the first circle (1,2), we find two triangles ΔQP2 and ΔQP1, *opposite in respect to the perpendicular*.
>
> Given that the definition of perpendicular is: *the perpendicular through P to a given line is the angle bisector of the straight angle determined by the line with the vertex P,* we must see whether the two triangles are equal.
>
> Let say that minor *catheti* (in Italian, *cateti minori*) are equal because they are radii of the same circle. The major *cathetus* is in common, and consequently is equal and the *hypotenuses* are equal because they are two diameters of two circles, which are distinct but have the same radii; we can state that the third criterion of congruence of triangles is valid, because given three segments there is only one triangle which has those segments as its sides. Thus, because of the fact that the two triangles have equal sides, they are equal and the *perpendicular line* divides the straight angle into two equal parts.

The first description of the construction refers to the symmetry of the figure (*opposite in respect to the "perpendicular"*). Symmetry is not available as a theoretical element, it imposes itself because of its perceptual evidence, nevertheless the description of the procedure is formulated correctly in terms of circles and radii.

There is a general difficulty in verbalization, that is in finding the appropriate geometrical terms to describe the construction, thus words like "cathetus" and "hypotenuse" may appear helpful to overcome this difficulty. Although the pupils seem to maintain the control on what must be proven and do not use the fact that the triangles are right-angled, they cannot ignore the fact that they are right-angled. Similarly, since the beginning the line that is going to be proven to be the perpendicular, is called "the perpendicular". Nevertheless this property is not used in the argumentation, and despite the use of inappropriate terms this protocol shows a good level of theoretical control.

That shows how evident is the truth of the fact to be proven and how difficult it may be to accept the "fiction" of proof. Proving requires the suspension of truth's evaluation, that is reasoning and behaving as if the truth of the statement to be proven were not so evident. Such suspension of judgement requires a control and a detachment from reality that takes time to be completely achieved.

CONCLUSIONS

The examples discussed above show that the evolution of a justification in a proof is possible, but they also show that this evolution is not expected to be simple and spontaneous.

The basic modification we were interested in concerned the change of status of justification in geometrical problems. This modification is strictly related to the passage from an "intuitive" geometry, as a collection of facts submitted to empirical verification, to a 'theoretical' geometry, as a system of relations among statements, validated by proof. According to our basic hypothesis the relation to geometrical knowledge is modified by the mediation of the computer. Our results (Mariotti, 2000, 2001) confirm that the specificity of the Cabri environment is determinant in order to make the sense of justification evolve from an empirical verification towards a theoretical proof.

Nevertheless the process of semiotic mediation, exploited by the teacher through the use of particular Cabri tools, remains fundamental. For instance, in previous example, the “history” command provides the basis for discussion, but it is not sufficient to accomplish the shift from actions to intentions: the software only shows the sequence of the steps, whilst the interpretation game introduces the point of view of motivation, which is reinforced by the prediction game. Both games are based on the facilities offered by the software; the history command allows the reconstruction “step by step” of a figure, and consequently gives access to an objectification of the construction procedure (decontextualised and depersonalized), through which the actions with their motivations can be reconstructed. The software provides the teacher with a tool of semiotic mediation through which to introduce the pupils into the games of motivations and to the new meaning of geometrical construction: keeping control, she leaves it to the pupils to make explicit the required operations and their motivations

It is no surprise that the interpretation and the prediction games are so effective. Similar games are used by the teacher in the “voices and echoes game” described by Boero et al., 1997, where students are asked to interpret an historical source and to predict which solution could have been produced by the same author in a given situation. In order to start the process what is needed is a “text” (the historic source, in the case of the “voices and echoes game” the objectified sequence of operations reproduced by the history command, in the case of Cabri constructions) that can be analysed by the pupils in a detached way, in order to play the author’s role and to guess the author’s intentions. It is the very presence of the software that transforms a personal construction into a depersonalized logical sequence of instructions that can be looked at by the author himself/herself in a detached way.

Coming back to the main educational problems previously pointed out, it is possible to state the following conclusions. Geometrical constructions within Cabri-géomètre provide a rich field of experience where the harmony between the figural and the conceptual aspects can be achieved together with the development of a sense of theory.

The construction task may develop a theoretical meaning in relation to the logic of the software environment and provide a powerful means for introducing pupils to geometry.

NOTES

[1] I will discuss the example of Cabri-géométre, but other software may be used as well, providing screen images controlled by geometrical logic.

[2] These are macros to copy respectively a segment and an angle; they were available in Cabri version 1 in use at that time.

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15. THE TRANSITION TO FORMAL PROOF IN GEOMETRY

INTRODUCTION

The document of the Catania ICME Congress on the teaching of geometry (edited by Mammana and Villani, 1998) underlines that the teaching of geometry today must be re-thought considering its connections with modern technology. Modern technology, as well as old mechanical machines (see the contribution of Bartolini Bussi), make powerful conceptual tools accessible to students at a friendly and comfortable level. Moreover such tools incorporate at an incredibly deep level pieces of cultural revolutions which have happened in the far and near past and that are very difficult to explain without the mediation of such tools (see Arzarello's contribution in the epistemological section). For example, the analysis of proof in the domain of (elementary) geometry today shows very deep connections between the synthetic and the analytic approach. Technology links the old dispute among Carnot, Poncelet, Staudt, Pasch, etc. with modern research in algebraic geometry as well as with symbolic computation tools, since they represent geometrical figures and polynomials with the same data structure, namely, acyclic oriented graphs: connections among ideas are based on more concrete connections realised in hardware.

Our project for teaching geometry at the secondary level (from 8th to 13th grade) is aimed at approaching geometry in order to embed pupils into this stream of thought, giving them the consciousness of what a proof means within mathematics as well as in the history of our civilisation. In such a sense we are pursuing, in collaboration with other teams, the deep connections among mechanical machines (see Bartolini Bussi), computers (see Mariotti) and theorem proving.

Our approach takes into account epistemological, cognitive and didactical features.

In particular, we will illustrate from these points of view the following aspects.

A. The delicate phase of transition to the formal side, exploiting its connections with the informal one (see Arzarello's contribution in the epistemological section and the didactical experiments of Boero and Mariotti). Our research group has been studying the above problem for two years in the area of elementary geometry, making experiments in different environments with high-school and college students, as well as with their teachers. We have elaborated a theoretical model to investigate the transition to the formal side. It is used as a key to inter-

P. Boero (Ed.), Theorems in schools: From history, epistemology and cognition to classroom practice, 305–323.

pret processes of thought in pupils of different levels who solve geometrical problems in different environments and with different modalities. The model underlines an essential continuity of thought which rules the transition from the conjecturing phase to the proving one, through exploration and suitable heuristics. The essential points are the different types of control of the subject with respect to the situation, namely ascending vs. descending and the switching from one to another. Its main didactic consequence consists in the change that the control provokes on the relationships among geometrical objects (see Section "*Towards a Model for Analysing the Transition to Formal Proof in Geometry*").

B. The way a software like Cabri incorporates at a feasible level delicate conceptual points which are essential in the transition to formal proofs. The epistemological and cognitive analysis of what concretely happens with students using Cabri shows that the transition is deeply marked by those analytic-algebraic aspects which are under the apparent synthetic-Euclidean structure of the software. In such a sense computer acts here as a cognitive tool which modifies the learning of mathematics because of its specificity and situativeness as a learning environment. Dragging, which has a complex feedback with the visual perception and the movements of the mouse, is a crucial instrument of mediation between the figural and conceptual level. While dragging, pupils who make constructions or explore geometric situations often switch back and forth from figures to concepts and an evolution of their attitudes from the empirical to the theoretical level can possibly be generated in the long run. This switching (and the generated evolution) can also be observed in pencil and paper environments, particularly in experts' and clever students' performances; it is crucial in all environments insofar as it makes it possible for pupils to manage the big gap between the status of knowledge based on drawings and the one which refers to geometrical concepts, sustaining them in the solution process and avoiding stumbling-blocks. We find that the different modalities of dragging are crucial for determining a productive shift to a more "formal" approach. By exploiting our model (see A), we classify such modalities and use them to describe processes of solution in the Cabri setting, comparing it with the pencil and paper ones. In fact, dragging behaviours change according to the specific epistemological and cognitive modalities after which pupils develop their control (and consequently make their actions); hence, looking at dragging modalities can give an insight into other inner and more theoretical variables. Moreover, in some pupils, particularly in those who produce good conjectures while exploring open situations, the modality of dragging involves different specific features, which show a genetic structure, which underlines the evolution from conjecturing to proving in a very detailed way (see the Subsection "*Explorations and Constructions in Cabri*").

C. Dragging modalities are also the key tool after which complex geometrical relationships are grasped by pupils in an easy way (see Subsection "*Dragging by Lieu Muet as a Reorganiser*"). Typically, the new relationship consists in the fact that:

i. a certain locus L is empirically built up thanks to a feedback given by the preservation of some ‘regularity’ in drawn figures and the movement of the mouse dragging a (draggable) point P in a suitable way;
ii. when the point P runs on L some corresponding figures F(P) satisfy some regularity, invariance or rule.

To use a mathematical language, the (usually algebraic) variety L (usually of dimension 1, that is a curve) parameterises (all or some of) the figures of the situations in a way which is perspicuous for the problem to solve.

This parameterisation of course is given only through dragging and not by equations: if made explicit they would express the relationships found empirically by dragging in the language of algebra; that is to say, dragging makes relationships of logical inclusion between algebraic varieties accessible to pupils at a perceptual level. Different modalities of dragging support and help students to make conjectures: they are a perceptual counterpart of the above algebraic relationship. This dragging dialectic makes accessible a jeu de cadre, in the sense of R. Douady between Euclidean geometry and algebraic varieties. Clarifying the algebraic counterpart, in a suitable way, also makes accessible to students the idea of proof as a computation, that is the most formal aspect of proof (see Arzarello’s contribution in the epistemological section); but this happens in a concrete way, because of the mediation of machines (also mechanical machines are crucial, insofar as they incorporate in a very suitable way the notion of parameters and variables and that of degree of freedom, which is essential in this algebraic approach to theorems and proofs).

In all our experience, the role of the teacher is crucial, who helps students to achieve the new knowledge, supporting them within the environment where they are embedded and helping them to listen to the voices that come to them from history and technology (see Boero and Bartolini Bussi).

The chapter will also frame research in the existing literature and will expose the main points more through the analysis of paradigmatic cases than developing a theoretical discourse, which would be too heavy.

TOWARDS A MODEL FOR ANALYSING THE TRANSITION TO FORMAL PROOF IN GEOMETRY

In the current literature on mathematics education, the concept of proof is examined in a wide sense, which goes beyond the narrow formal one; in fact, explorations, conjectures, argumentations produced by novices and experts while solving problems, as well as *semi-rigorous, zero-knowledge, holographic proofs* (see Hanna, 1996), are also taken into account, because of their interest in the pragmatic of proof (Hanna & Jahnke, 1993). However, in real school life, even if proof is generally considered central in the whole mathematics, it does not enter all the curriculum, but it is restricted almost exclusively to geometry (Hanna, 1996). The processes through which pupils and experts approach proofs are analysed from different points of view and by means of different tools. First, at least two components are considered crucial for focusing the meaning of proof, namely a *cognitive* and a *historic-epistemological* one (Balacheff, 1988, 1991; Barbin, 1988; Harel, 1996; Mariotti et al., 1997; Simon, 1996). Of course the two components can be separated only for reasons of theoretical analysis; on the contrary, they are deeply intertwined in reality (Hanna, 1996) and both must be considered in order to tackle suitably the didactic of proof (even if different authors underline one of them more than the other.) Second, the production processes of proofs are analysed pointing out *continuity* and *discontinuity* features both from the epistemological and the cognitive point of view; the question is particularly intriguing when one considers the relationships between the argumentative, informal side and the discursive, formal one of a proof (in the wide sense of the word). For example, the *transformational reasoning* of Simon, 1996, the *cognitive unity* of Mariotti et al., 1997, all underline a substantial continuity from a cognitive point of view. The issue of continuity from an epistemological point of view has been faced in Polya, 1957; Barbin, 1988 and Thurston, 1994. Moreover, some authors, e.g., Duval, 1991, mark the dramatic epistemological and cognitive gap between argumentation and proof: Duval tackles it from a didactic point of view using suitable semiotic mediators, namely graphs for representing the formal deductions. In an intermediate position we find Harel, 1996 with his *students' proof schemes* and Balacheff, 1988, who, following the analysis of Lakatos, stresses the big epistemological discontinuities, which can be overcome by pupils, insofar as they become able to pass from the *naive-empiricist* way of looking at mathematical sentences towards a more *formal* approach, through the discovering of the so-called *generic example*. The explicit or implicit attitude of the teacher towards the question continuities vs. discontinuities, both from a cognitive and an epistemological point of view, proves to be crucial for planning the didactic of proof in the class (for examples of concrete approaches see Balacheff, 1988; Duval, 1991; Mariotti et al., 1997). The problem becomes even more intriguing when new technologies are taken into account and such softwares as Geometer's Sketchpad, Cabri-Géomètre, Derive, Excel or others are used in the class as tools for exploring situations, making conjectures and validating the same process of proving theorems.

Within this research issue, a crucial point consists in analysing the delicate phase of transition to the formal side, exploiting its connections with the informal one. Important variables for such an analysis are: the mathematical area, for example geometry, algebra, analysis, etc.; the modalities after which the problem is given, namely exploring an open situation vs. proving a given statement; the environment, namely paper and pencil vs. computer (for example a Cabri setting). In this report we expose a theoretical model we elaborated to investigate the transition to the formal side. The main sources for this model are the papers quoted above, which analyse the relationships between conjecturing and proving under the issue of continuity. In particular, we are indebted to Gallo, 1994, for the notion of *ascending/descending control* and to Mariotti et al., 1997, for that of *dynamic exploration*, which supports the *selection/specification of conjectures* in the form of conditionals and rules the passage to proof construction, by implementing the logical connections of sentences. We illustrate the model by means of a paradigmatic example, which is exposed in Subsection "*The Theoretical Model*" and commented upon in Subsection "*Some Partial Solution and Related Problems*".

A Paradigmatic Example

We expose the protocol of solution given by a teacher to the following problem:

Problem. *Given a quadrilateral ABCD and a point P0, construct the point P1, symmetric of P0 with respect to A, P2 symmetric of P1 with respect to B, P3 symmetric of P2 with respect to C, P4 symmetric of P3 with respect to D. Determine which conditions the quadrilateral ABCD must satisfy so that P0 and P4 coincide* [see Figure 1].

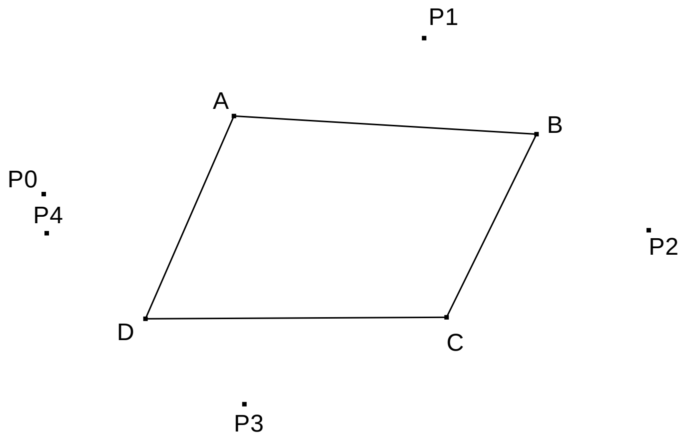

Figure 1.

The subject solving the problem used pencil, paper and (sometimes) ruler; he was invited to use only elementary mathematics and to think aloud: an observer took notes of his comments (which are written in italics, while the observer's comments are in square brackets). The solution process has been divided into 15 episodes, which lasted about six minutes in total; a minor episode (n.6) has been skipped, because it is a detour not relevant to our analysis; references are to figures at the end of the protocol. The comments on the protocol are given in the next Subsection (S=subject).

1. [S draws very rapidly and sketchily, without using the ruler].
2. "*I'll check for a simpler case, with only three points*" [S sketches a figure with triangles instead of quadrilaterals, i.e. D and P4 disappear] "*I do not see anything*".
3. "*I consider a particular case, which is easier: the rectangle*"

[S sketches Figure 2].
"*Perhaps it closes in the rectangle's case*" [in the figure it is dubious if P0 and P4 coincide (P0, P1, P2, P3, P4 close) or not, because the figure has been drawn by hand].

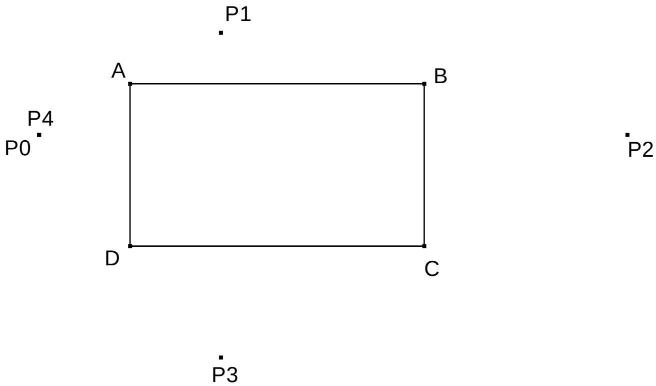

Figure 2.

4. "*I can't see that in this way. I redraw a very different case, always with rectangle: P0 far away from A*" [S draws Figure 3 without ruler but with more attention, with a smaller rectangle but with P0 far from A: P0 and P4 seem to coincide].
5. "*I see the Varignon's case in the opposite way*" [Varignon's theorem is a classic Cabri problem, well known to S; it says that, given a quadrilateral, if K, L, M, N in order are the middle points of its sides, then KLMN is a parallelogram; successively asked, S says that he meant that he saw the Varignon configuration, with K, L, M, N as the rectangle of Figure 2] .
[S looks carefully at the figure] "*However I realise it's not so*".
6. [In this short episode S tries to follow another idea, but he soon abandons it.]
7. "*Let me draw it better*" [S draws with the ruler and with great care].
"I see Varignon's case applied to crossed quadrilaterals, 'cause I've drawn all segments completely" [S drew full segments between P0, P1, P2, P3, P4] .
8. "*Now I am going to use the analytic method. I imagine the problem has already been solved. In my mind I anticipate that it's Varignon*" [By analytic method S means the method of Analysis due to Pappus (see Panza, 1996); S redraws a figure like Figure 3, using the ruler; but now he first draws points P0, P1, P2, P3, then A, B, C, D as midpoints of the sides P0P1, P1P2, P2P3, P3P0; in all previous drawings S drew A, B, C, D first and then P0, P1, P2, P3, P4].
9. "*I see it's a rectangle again*" [In fact, in Figure 3, even if S started from 'generic' P0, P1, P2, P3, P4, the quadrilateral ABCD looks like a rectangle.]
"*I conjecture that if it is a rectangle it will close*".
10. "*I'll prove it. I'm guided by Varignon's proof. It results that AB // CD // P0P2 and BC // AD // P1P3. Now I look at the figure again to prove it's a rectangle*." [He looks at the figure … he draws AC, BD …].

11. "*… hem … I reconsider AB // P0P2 … and I observe that [the angle] ABC is equal to [the angle formed by the lines] P0P2, P1P3. I come to believe that in general it isn't a rectangle: I look for a counterexample. I start from P0, P1, P2, P3 and draw ABCD carefully*".
12. "*It's a parallelogram. The proof is done! I write it down*".

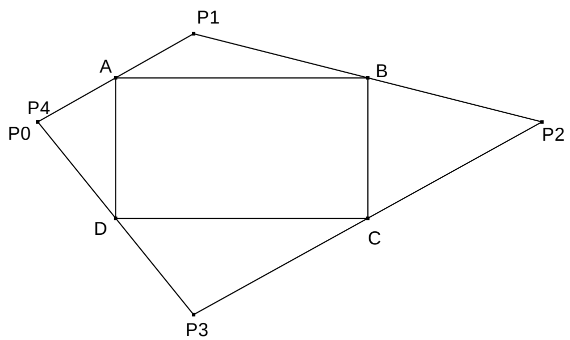

Figure 3.

13. "*I know that given a quadrilateral ABCD (even crossed), the quadrilateral constructed on the midpoints is a parallelogram 'cause of Varignon. Now let us consider a parallelogram ABCD. If P0, P1, P2, P3, P4 are built as symmetric then the thesis is that the points P0 and P4 coincide. I go back to the figure to prove it*" [S writes the key words of his theorem arranged as hypothesis and thesis on another sheet of paper; then he comes back to the sheet with his drawing …].
14. "*It's a synthesis!*" [… S takes a new sheet of paper and draws a figure that is deliberately 'wrong', by hand without a ruler].
"*I consider P0, P1, P2, P3 with the resulting quadrilateral: I construct the first three midpoints which are A, B, C. [While speaking, he draws] Then I construct D', midpoint of P0P3. ABCD' is a parallelogram because of Varignon*".
15. "*ABCD is a parallelogram too, by hypothesis. If D (D' then CD (CD', but they are both parallel to AB. It's absurd! Then ABCD and ABCD' are congruent. Therefore P0 and P4 coincide*".

The Theoretical Model

It is now time to explain our model of transition from conjecturing to proving (see the protocol as a paradigmatic example).

As a first working hypothesis, which we shall modify during the exposition, we use that of Mariotti et al., 1997 (but the responsibility of the interpretation is only due to this chapter's authors). High and middle level subjects, who explore geometrical problems in different environments in order to conjecture and to prove theorems (within their own theoretic framework) show successively two main modalities of acting, namely: *exploring/selecting a conjecture and concatenating sen-*

tences logically. In fact, any process of exploration–conjecturing-proving is featured by a complex switching from one modality to the other and back, which requires a high flexibility in tuning to the right one. Our aim is to analyse carefully how the transition from one modality to the other does happen, using the protocol above: its dynamic has been divided into four main phases, each corresponding to a different modality or transition. At the end, the picture of the transition will appear and we shall rephrase the working hypothesis in a suitable way (Subsection "*Some Partial Solution and Related Problems*").

PHASE 1. Episodes 1–3 show a typical exploring modality, with the use of some heuristics to guess what happens working on a particular example (ep.3), hence selecting a conjecture. The conjecture in reality is a working hypothesis to be checked: its form is far from a conditional statement and to confirm it new explorations are made by using a new heuristic principle (namely: *choose very different data, to check the validity of the conjecture*, ep.4). The phase culminates in ep.5: some of its general features are described in Mariotti et al., 1997, specifically the *internalisation of the visual field* (the subject 'sees'), and the *detachment* from the exploration process (which is seen from the outside); the situation is described by the subject in a language which has a logic flavour (ep.5), but it is not phrased in a conditional form (if … then), nor it is crystallised in a logical form: in fact, the subject expresses his hypothesis (which is more stable and sure than that of ep.3) not yet as a deductive sentence, but as an *abduction*, namely a sort of *reverse deduction*, albeit very different from an induction (Peirce, 1960).[1] In fact the subject sees (with his mind's eyes, because of the internalisation of his visual field) *what rule it is the case of*, to use Peirce language. Namely, he selects the piece of his knowledge he believes to be right; the conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form: the direction after which the subject sees the things explored in the previous episodes is still in the stream of the exploration: the control of the meaning is *ascending* (we use this term as in Saada-Robert, 1989 and Gallo, 1994). It is in the stream of the preceding exploration that the negative validation at the end of ep.5 happens. Ep.7 is still in the same stream of thought; now the heuristic is: *draw better to see better*; indeed it is the last drawing (Figure 3) which allows the second abduction (ep.7): it is interesting to observe that the hypothesis changes (now the quadrilateral is crossing) but it is still in the reversed abductive form.

PHASE 2. Ep.8 marks the switching from the abductive modality to the deductive one: the meta-comments in the protocol show this clearly; but this change is shown also by the way in which the figure is drawn: see the observer's comments. Now the control is *descending* and we have an exploration of the situation, where things are looked at in the opposite way. Ep.9 shows this: exploration now produces as output the figure which in previous explorations was taken as input. The reversed way of looking at figures leads the subject to formulate the conjecture in the conditional form. Now the modality is typically that of a logical concatenation.

PHASE 3. Now in the new modality suitable heuristics can be used, namely look for *similar proofs* (ep.10): this task seems straightforward for the subject and so does not generate any further exploration, at least as far as parallelism of sides is concerned. Some exploration (with descending control) starts at the end of ep.10, for proving that it is a rectangle, but it does not work, so a new exploration, after a new selection (concerning angles) starts with ep.11. Here the descending control is crucial: it allows the *detached* subject to interpret in the 'right' way what is happening: it is not an abduction (what rule it is—possibly—the case of) but a counterexample (what rule it is not—surely—the case of); the switched modality has started a new exploration process, which culminates in the final conjecture of ep.12.

PHASE 4. This is the real implementation of logical connections in a more global and articulated way than the local concatenation of statements, which featured the previous conjecturing phase. Here detachment means to be a true *rational agent* (Balacheff, 1982), who controls the products of the whole exploring and conjecturing process from a higher level, selects from this point of view those statements which are meaningful for the very process of proving and rules possible new explorations. In this last phase, conjectures are possibly reformulated in order to combine better *logical concatenations* (ep.13) and new explorations are made to test the latter: looking at what happens word by word, this exploration is not very far from those made under ascending control. It is the sense attached to them by the rational agent to change deeply the meaning of what happens. A typical example is ep.14, where a 'wrong' figure is drawn in order to explore the situation, anticipating that it is an impossible case: during the episode a second figure is drawn, where the 'logical impossibility' has changed the relationships among the points: in fact the old point D has been substituted by a new point D′, which incorporates in a positive way the logical impossibility. The control is typically descending and global; in fact a proof by contradiction is tackled: the sense of the logical relationships among the drawn objects produces a 'new' situation, which is explored. In ep.15, the 'old' and the 'new' situations are put together by the rational agent, who can draw the conclusion by contradiction.

Some Partial Conclusions and Related Problems

Our model is somehow different from the starting working hypothesis: in fact the exploration and selection modality is a constant in the whole conjecturing and proving processes; what changes is the different attitude of the subject towards her/his explorations and the consequent type of control with respect to what is happening in the given setting. *It is the different control to change the relationships among the geometrical objects, both in the way they are 'drawn' and in the way they are 'seen'*. This seems essential for producing meaningful arguments and proofs. Also detachment changes with respect to control: there are two types of detachment. The first one is very local and marks the switching from ascending to descending control through the production of conjectures formulated as conditional statements (that is *local logical concatenations*) because of some abduction, like in ep. 5 and 7. The second one is more global and we used the metaphor of the rational agent to describe it: in fact it is embedded in a fully descending control, produces new (local) explorations and possibly proofs (that is *global logical combinations*), like in ep. 10, 11, 12, 14. *The transition from the ascending to the descending control is promoted by abduction*, which puts on the table all the ingredients of the conditional statements: it is the detachment of the subject to reverse the stream of thought from the abductive to the deductive (i.e. conditional) form, but this can happen because an abduction has been produced. *The consequences of this transition are a deductive modality and the new relationships among the geometrical objects of the figures*, as pointed out above (ep.8). The inverse transition from descending to ascending control is more 'natural': in fact as soon as a new exploration starts again (ep.14), control may change and again become ascending, even if at a more local level (with the rational agent who still controls the global situation in a descending way). In short, the model points out an essential continuity of thought which rules the successful transition from the conjecturing phase to the proving one, through exploration and suitable heuristics, ruled by the ascending/descending control stream. The most delicate cognitive point is the process of abduction, crucial for switching the modality of control; the most relevant didactic aspect is the change in the mutual relationships among geometrical objects, which are the essential product of such a switching. Many scholars, with a different language, exploited carefully various aspects of the way in which the switching can be realised by pupils in the class. In other ongoing research we use our model to study how the Cabri environment can support pupils in getting the above switching and changing of the relationships among the geometric objects.

DRAGGING IN CABRI AND MODALITIES OF TRANSITION FROM CONJECTURES TO PROOFS IN GEOMETRY

The literature on computers as cognitive tools (Dörfler, 1993) which modify the learning of mathematics because of their specificity and 'situativeness' as learning environments (Hoyles & Noss, 1992) is especially rich for Cabri-géomètre (Laborde, 1993; Balacheff, 1993; Hölzl, 1995, 1996). In particular, several studies which analyse specific components of Cabri's *epistemological domain of validity* (Balacheff & Sutherland, 1994) point out that for learning geometry in Cabri environments the dialectic figures vs. concepts (Mariotti, 1995; Laborde, 1993) and perceptual activity vs. mathematical knowledge (Laborde & Strässer, 1990) is essential. Typically, a geometrical problem cannot be solved only remaining at the perceptual level of figures on the screen, even if their graphical space is provided with movement as a further component (Laborde, 1993): a conceptual control is needed, and it requires some pieces of explicit knowledge. Dragging, which has a complex feedback with the visual perception and the movements of the mouse, is a crucial instrument of mediation between the two levels (Hölzl, 1995): its function consists in validating procedures and constructions built up using the menu commands (Laborde & Strässer, 1990, p. 174; Mariotti et al., 1997). While dragging, pupils who make constructions or explore geometric situations often switch back and forth from figures to concepts and an evolution of their attitudes from the *empirical* to the *theoretical* level can possibly be generated in the long run (Balacheff, 1988; Mariotti et al., 1997; Laborde, 1997). In this section we use our model (see previous Section) for describing the switching modalities in pupils who use Cabri[2] to solve geometric problems and for contrasting them with the modalities of pencil and paper environments. Such a description will isolate in a transparent way some components of Cabri's *epistemological domain of validity*, which become important didactic variables for our project of teaching geometry at high-school level with a multi-medial approach (pencil and paper, Cabri, geometrical machines, etc.).

Explorations and Constructions in Cabri

Before discussing some concrete examples, we sketch very briefly the main points of our model. We consider tasks of exploring open geometric problems (Arsac et al., 1988) in order to select/formulate conjectures and possibly to prove them. The model points out an essential continuity of thought, which features the successful transition from the conjecturing phase to the proving one, through exploration and suitable heuristics, ruled by what we call an ascending/descending control stream (see Saada-Robert, 1989 and Gallo, 1994). The process of switching from one control modality to the other is a delicate cognitive point, which also has a relevant didactic aspect: in fact it is deeply intermingled with the change in the mutual relationships after which the geometrical objects of the situation are seen. It is precisely in these two aspects that one can observe different dynamics between 'pencil and paper' and 'Cabri' environments. In both, the transition is ruled by abduction, which will be explained below; but while in the former the abductions are produced because of the ingenuity of the subjects, in Cabri the dragging process can mediate them: our model allows us to describe how Cabri can support pupils in getting the above transition.

We distinguish between 'constructions' and 'open problems' explorations', which correspond to two different modalities of using Cabri. The former consists in drawing figures through the available commands of the menu, because of a construction task, which is considered solved if the figure on the screen passes the *dragging test*: the Cabri figure will not be messed up by dragging it (this has been studied by Mariotti et al., 1997). For the latter, let us illustrate it with an example. Consider the following problem to be solved in Cabri:

> Let ABCD be a quadrangle. Consider the bisectors of its internal angles and their intersection points H, K, L, M of pairwise consecutive bisectors. Drag ABCD, considering all its different configurations: what happens to the quadrangle HKLM? What kind of figure does it become? [Figure 4].

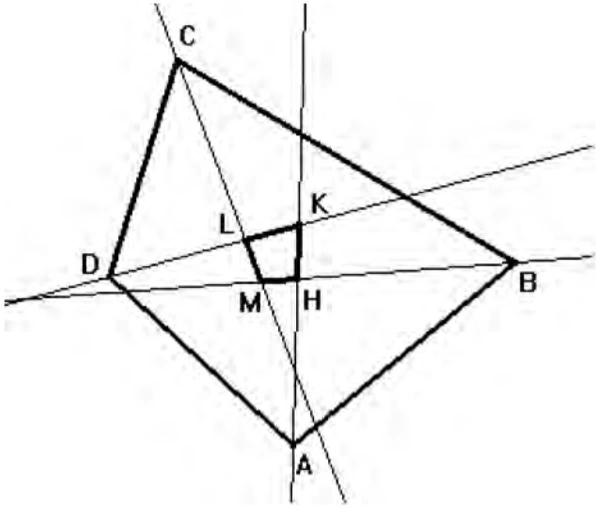

Figure 4.

This is a typical example of the open problems we use in our experiment, which is being carried out in the second year of a 'Liceo Scientifico' (pupils aged 15), aimed at teaching geometry with a multi-medial approach (in this specific case, with Cabri). Our example will illustrate other modalities of dragging, namely: (i) *wandering dragging*, that is dragging (more or less) randomly to find some regularity or interesting configurations; (ii) *lieu muet* dragging, that means a certain locus C is built up empirically by dragging a (draggable) point P, in a way which preserves some regularity of certain figures.

We analyse the data collected from a class of 27 students, who have already learned some Euclidean geometry the year before; the exposed activity takes place in a two-hour lesson. One hour is devoted to the work with Cabri (two students for each computer): having created a paper and pencil drawing of the geometrical situation, the pupils go on working in Cabri and making conjectures. The second hour is devoted to a mathematical discussion about the groups' discoveries: groups show their discoveries to the class, using a data-show, and the teacher orchestrates the discussion (according to the methodology illustrated in Bartolini Bussi, 1996) so that students can move towards more general statements. The analysis of the collected material shows three different ways of using Cabri in order to solve the problem, corresponding to the three dragging modalities mentioned above: *lieu muet*, *dragging test*, *wandering dragging*. A case in point of the first two is the protocol of Group A (high-level students):

1. The pupils start to shape ABCD into standard figures, apparently following an implicit order: when ABCD is a parallelogram, HKLM is also a parallelogram; then they draw ABCD as a trapezium and then as a square.

2. As soon as they see that HKLM becomes a point when ABCD is a square, they consider it an interesting fact, therefore they drag ABCD (from a square) so that H, K, L, M keep on being coincident (*lieu muet* exploration).
3. They realise that this kind of configuration can also be seen with quadrilaterals that apparently have not got anything special; so they look for some common properties to all those figures which make HKLM one point. Paying attention to the measures of the sides of the figure ABCD (which appear automatically next to the sides and change in real time, while dragging along the *lieu muet*), they see that the sum of two opposite sides equals the sum of the other two; they remember that this property characterises the quadrilaterals that can be circumscribed to a circle.
4. Using the Cabri menu, they construct the perpendicular lines from the point of intersection of the angle bisectors to the sides of ABCD: they 'see' that this point has the same distance from each side of ABCD, then they draw the circle which has this length as radius: it is the circle inscribed in ABCD. They formulate the following 'conjecture': *If the external quadrilateral can be circumscribed to a circle, then its internal angle bisectors will all meet in one point, so the distances from this point are equal and the sum of the opposite sides is equal too*.
5. They wonder whether this works even if they begin their construction with the circle: they construct a circle, a quadrilateral circumscribed to this circle, its angle bisectors and they observe that all of them meet in the same point; afterwards, they write down this 'conjecture': *If the internal angle bisectors of a quadrilateral all meet in the same point then the quadrilateral can be circumscribed to a circle.*

Let us examine carefully ep. 2 and 3. First, pupils look for an object more *generic* than a square (which is thought as 'trivial' and 'too easy a figure'), such that points H, K, L, M still coincide: they do that by *lieu muet* dragging. Then, their attitude changes: they look at the figure in Cabri without moving anything, try to discover some rule or invariant property under the *lieu muet* dragging, select 'which rule it is the case of' in their geometrical knowledge; this phase is marked by a continuous switching from figures to theory and back. Some general features of this new attitude are typical and described also in Mariotti et al., 1997: specifically we see in these pupils the *internalisation of the visual field* (the subjects 'see'), and the detachment from the exploration process (which is seen from the outside). Moreover, it is typical that again the subjects express their hypothesis not yet as a deductive sentence, but as an *abduction* (Peirce, 1960). In fact the subjects 'see' what 'rule' this is the case of, to use Peirce language. Namely, their visual field has been internalised in order to find a property which can help them to classify the figures into something they know; they select the part of their geometrical knowledge they judge as the right one. The conditional form is virtually present: its ingredients are all alive, but their relationships are still reversed, with respect to the conditional form; the direction after which the subjects 'see' things is still in the stream of the exploration through dragging, the control of the meaning is ascending, namely they are looking at what they have explored in the previous episodes in an abductive way. Control direction changes in ep.4: here students use the construction modality (and the consequent *dragging test*) to check the hypothesis of abduction and at the end they write down a sentence in which the way of looking at figures has been reversed. By *lieu muet* dragging, they have seen that when the intersection points are kept to coincide the quadrilateral is always circumscribed to a circle; now they formulate the 'conjecture' in a logical way, which reverses the stream of thought: 'if the quadrilateral is circumscribed then the intersection points coincide'. It is not a mistake! This episode marks the switching from the abductive to the deductive modality: now the control is *descending* and things are looked at in the opposite way. In ep.5 the descending control continues; exploration now produces as output the figure which in previous dragging was taken as input: the pupils now construct a figure with the underlined property in order to validate the conjecture itself and check whether the figure on the screen passes the *dragging test*. In this way they formulate a conjecture expressing a sufficient and necessary condition "if … and only if …" in a conditional form, even if they are not able to summarise it into one statement only. Hence, at the end of their resolution process they have all the elements they need to prove the statement.

We can find some interesting elements also in the discussion which immediately followed the activity in Cabri (St 8, 9=students of Group A):

[…] St 9: "*Well, we can find many other figures in which all the bisectors meet in the same point, in some quadrilaterals that apparently haven't got anything special*. (1) [he moves the figure by *lieu muet* in order to have a generic quadrilateral in which H, K, L, M are coincident] *But, if we draw a circle … no, first of all let us draw a perpendicular line through one of these points* [H, K, L, M] *to one of the sides of* ABCD (2) [he draws the perpendicular from L to DC] *and consider the intersection point* … [he draws], *we notice that this quadrilateral is circumscribed to a circle, then since it is a circle all the radii are equal and all the distances from the sides of ABCD are equal too …*"
St 8: "… all these centres are coincident …" […]
St 9: "If a quadrilateral can be circumscribed to a circle, all its angle bisectors meet in the same point."
St 8: "We proved the same thing but starting from a circle too (3); we drew the tangent lines and we came to the same conclusion." […]

These students recollect what they have just found out reversing the exploration process: the descending control is ruling their thinking in the discussion phase. It is important to underline which concerning Cabri elements are still present in their words, which now are spoken from a detached point of view (numbers refer to the sentences in the discussion): (1) The *lieu muet* dragging, which allows them to move from a square to a more generic object that keeps H, K, L, M coincident [they are probably moving along a diagonal of the square]. (2) The construction activity (perpendicular line), with the *dragging test*, which supports their reasoning towards proof. (3) The 'only if' form of their conjecture. Here we have a second form of detachment, fully embedded in the descending control stream, which we call the rational agent (Balacheff, 1982): they control the products of the whole exploring and conjecturing process from a higher level, selecting from this point of view those statements which are meaningful for the very process of proving and rule possible new explorations. They are again reversing the way of looking at the relationships among the objects: however this is not an abduction, but a logical concatenation of the 'only if' part (see Mariotti et al., 1997 as regards the 'only if' reasoning).

We also found another modality of dragging (*wandering dragging*), which we will illustrate sketching Group B strategies. These students (of middle level) have a dynamic approach to the problem as well: they begin by dragging the vertices of ABCD at random and observing what happens to HKLM. As soon as they see something interesting about HKLM, such as a known or a 'strange' shape (for example a crossing quadrilateral), they stop moving. Then they go on by (*lieu muet*) dragging ABCD so that HKLM keeps the same shape and they look at ABCD, trying to find out what kind of quadrilateral it is. We notice an evolution in their way of using the drag mode in Cabri: at first they seem to move the drawing just because Cabri allows them to do so, they haven't got any plan in their mind and move points at random; then they change their behaviour and move points in such a way as to keep a certain property of the figure, e.g., along a fixed direction. They continue switching from the first mode to the second one, every time they find an 'interesting' shape of HKLM. Hence the *lieu muet* dragging can be seen as a *wandering dragging* which has found its path, as some possible regularity has been discovered, at least at a perceptual level: both the dragging modalities are in the same stream of thought, namely in the ascending control one; at the opposite side we find dragging test, which is typical of descending control (albeit it can be used at different levels of sophistication).

Dragging by Lieu Muet as a Reorganiser

The protocols above are very important, because they clearly show how the dialectic between the different modalities of dragging can deeply change the relationships among the geometrical objects of the situation; so through the analysis of the dragging modalities used by pupils we can observe how such a shift takes place. In particular we shall concentrate on *lieu muet* modalities. A *lieu muet* can act both as a *logical reorganiser* (Pea, 1987; Dörfler, 1993) and as a producer of new powerful heuristics (Hölzl, 1996). The former shows a new, intriguing way after which dragging can act as a mediator between figures and concepts (Hölzl, 1996), namely at a deeper and unexpected level of conceptual knowledge; the latter makes accessible some aspects of such a reorganised knowledge at a perceptual level and in a strongly 'situated' way, so it seems to support a 'situated abstraction' in the sense of Hoyles & Noss, 1992 (compare Group A protocol with the example in Hölzl, 1996).

Let us sketch the kind of logical reorganisation that the *lieu muet* encompasses: it shows a new and wide component of Cabri's *epistemological domain of validity*. The example of exploration showed in our protocols illustrates this in a paradigmatic way; a lot of explorations described in the literature seem to be coherent with our analysis: e.g. the cases discussed in Hölzl, 1996, where he observes a shifting of perspective in students 'from the constructions of certain points to the interpretation of certain loci' (p. 181).

The *lieu muet* 'shows' a new logical relationship between points and figures, which adds to the usual functional dependence of the kind variables–parameters, where some constructed objects depend in their construction on others which are considered as 'given'.

The new relationship consists in the fact that: (i) a certain locus C is empirically built up (see Group A protocol, as well as example at p. 176 in Hölzl, 1996) thanks to a feedback given by the preservation of some 'regularity' in drawn figures and the movement of the mouse dragging a (draggable) point P in a suitable way (which means precisely that P describes C as a *lieu muet*); (ii) when the point P runs on C some corresponding figures F(P) satisfy some regularity, invariance or rule (in the example, for each P belonging to the empirical curve C, the bisectors of the corresponding quadrilateral Q(P) meet in the same point). To use a mathematical language, the (usually algebraic) variety C (usually of dimension 1, that is a curve) parameterises (all or some of) the figures of the situations in a way which is perspicuous for the problem to solve. This parameterisation of course is given only through dragging and not by equations: if made explicit they would express the relationships found empirically by dragging in the language of algebra; that is to say, dragging makes relationships of logical inclusion between algebraic varieties accessible to pupils at a perceptual level. The role of *lieu muet* in the dynamic of ascending/descending control supports and helps students to produce abductions and provokes the switching between ascending and descending control modalities. A *lieu muet*, as a perceptual counterpart of the above algebraic relationship, *expresses an abduction in a figural and perceptive way*: C is indeed the 'rule' which the figures F(P) are the case of, given the functional dependence among the constructed objects. In fact, the successive dynamics of pupils' actions have the same structure as those in pencil and paper environments: namely, first the pupils formulate a conjecture in a conditional way (which is a regularity produced by the *lieu muet* dragging), then they make explorations and constructions to validate the hypothesis, as we have seen in the protocols above. The latter are ruled by a descending control; the function of dragging changes: it is now used as a test for validating the hypothesis. This dragging dialectic makes accessible a 'jeu de cadre', in the sense of R. Douady between Euclidean geometry and algebraic varieties. The former becomes explicit for pupils through constructions and *dragging test* ruled by descending control; the latter remains implicit, at the perceptual level of *lieu muet* dragging, but the dragging test makes accessible abductions (and possibly conjectures and proofs), which concern more difficult problems than those that they can tackle in paper and pencil environments.

NOTES

[1] The example given by Peirce is illuminating (Peirce, 1960, p. 372). Consider a situation where we have a bag of beans and some beans on the floor. Consider the following sentences : A) These beans (on the floor) are white; B) The beans of that bag are white; C) These beans (on the floor) are from that bag. A *deduction* is a concatenation of the form: B and C, hence A. An *abduction* is: A and B, hence C (Peirce called *hypothes*is the abduction). An *induction* would be: A and C, hence B.

[2] All our experiences refer to Cabri I, MS-DOS version.

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