Abella: A System for Reasoning about Relational Specifications

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The Abella interactive theorem prover is based on an intuitionistic logic that allows for inductive and co-inductive reasoning over relations. Abella supports the λ-tree approach to treating syntax containing binders: it allows simply typed λ-terms to be used to represent such syntax and it provides higher-order (pattern) unification, the ∇ quantifier, and nominal constants for reasoning about these representations. As such, it is a suitable vehicle for formalizing the meta-theory of formal systems such as logics and programming languages. This tutorial exposes Abella incrementally, starting with its capabilities at a first-order logic level and gradually presenting more sophisticated features, ending with the support it offers to the *two-level logic approach* to meta-theoretic reasoning. Along the way, we show how Abella can be used prove theorems involving natural numbers, lists, and automata, as well as involving typed and untyped λ-calculi and the π-calculus.

Contents

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1. INTRODUCTION

Abella is an interactive proof assistant designed to reason about relational inductive and co-inductive specifications, with a particular emphasis on reasoning about relations between terms containing bindings. Abella is available for download from `http://abella-prover.org`.

1.1 A bit of history

The first version of the Abella theorem prover was developed by Andrew Gacek as part of his doctoral work carried out at the University of Minnesota [19]. Kaustuv Chaudhuri and Yuting Wang have subsequently designed and implemented extensions to the system, resulting in an updated release. The various authors of this tutorial and several of their colleagues have been involved in developing the theoretical underpinnings of this system, providing input on its structure and fleshing out examples of its applications.

The Abella prover was originally designed to illustrate the possibility of reasoning directly over the relational judgments that can be constructed in λProlog [39] and in LF [29]. Even back in the 1990's, it was clear that such relational specifications could be extremely useful in a wide range of formalization efforts in the areas of proof systems [3, 15, 49], type systems [2, 14], and programming language semantics [26, 27, 28]. Of course, relational specifications have often been used in a number of other domains such as relational databases and model checking as well.

Reasoning directly with judgments about higher-order objects, examples of which we shall first encounter in Section 6, presents a number of challenges to conventional theorem provers. One of those challenges is providing an abstract and flexible treatment of bindings within syntax. The logic $\mathcal{G}$ is the result of a decade long effort [5, 22, 34, 35, 43, 63, 64, 69] to design an increasingly more flexible and powerful logic that would provide such a treatment of binding. The Abella prover has allowed this work on proof theory to be validated and exploited within an interactive theorem proving setting.

1.2 Abstractions of syntax

Computation and reasoning on linguistic structures goes back at least to Gödel and Church. For them, syntax was encoded as a string of symbols: we usually refer to that approach to syntax as *concrete syntax.* While concrete syntax has the advantage of being readable and writable by humans, it has many disadvantages. Concrete syntax contains too much information that is not important for many manipulations, such as white space, infix/prefix notation, and keywords; and important computational information is not represented explicitly, such as recursive structure, function–argument relationship, and the term–subterm relationship. The field of parsing was developed in part to translate concrete syntax into more meaningful tree structures, often called *parse trees.* In such tree structures, much semantically meaningless information is discarded and the recursive structure is made central.

For those attempting to reason about meta-theoretic properties of programs, type checkers, compilers, theorem provers, etc., it has become clear that parse trees are not abstract enough. In particular, the way that parse trees encode binding structures (e.g., quantifiers, formal parameters, and lexical scopes) is too concrete. Treating binding using named variables (despite the fact that α-conversion makes the choice of actual names that are used unimportant) or as De Bruijn-style offset counters is complex and adds many details to programs and specifications that are, in the end, orthogonal to real semantic content. As evidence of that problem, the POPLMark challenge [4] attempted to raise interest in the theorem proving community to make better tools for dealing with bindings in syntax so that formalized meta-theory could be made manageable.

There have been various attempts to treat bindings in syntax in a more abstract fashion. For example, one of the applications of nominal logic [53] has been to provide a new approach to representing binders [18]. In this tutorial we will focus on using *λ-tree syntax* [41], a third approach to syntactic representation. This approach, which is closely related to the *higher-order abstract syntax* approach [48], is directly supported in λProlog. As we shall see in Section 6, one of the characteristics of Abella is that it make it possible to reason on encodings of syntax at this level of abstraction.

1.3 The organization of this tutorial

Abella is based on the logic $\mathcal{G}$ [21, 22] that contains several features that may appear novel to the uninitiated reader. This tutorial is organized to introduce these features in an incremental fashion.

—At a propositional level, $\mathcal{G}$ makes use of the usual intuitionistic logical constants: true, false, conjunction, disjunction, and implication. In addition $\mathcal{G}$ possesses the typed quantifiers $\forall_\tau$ and $\exists_\tau$ for all simple types τ that do not contain the type of formulas; these quantifiers are drawn from Church's Simple Theory of Types [12]. In Section 2, we present the basic structure of Abella that is built around these connectives and quantifiers.

—Equality at all simple types τ is treated as a logical connective in $\mathcal{G}$. While the proof theory underlying this treatment is natural, it is not well-known in the theorem proving community. We describe reasoning based on this connective in Section 3.

—Relational specifications are introduced into $\mathcal{G}$ via fixed point definitions. Section 4 considers the treatment of such definitions in Abella.

—Inductive and co-inductive definitions of relations are supported in $\mathcal{G}$ by giving definitions a least or greatest fixed point reading. The treatment of induction and co-induction in Abella is presented in Section 5.

—Support for λ-tree syntax in $\mathcal{G}$ is dependent significantly on the ∇ (nabla) quantifier [6, 21, 42, 43] and the closely associated notion of nominal constants [22]. Section 6 introduces these notions and demonstrates how they can be used to treat binding in syntax.

Once the logical features underlying Abella have been presented, we turn to showing how they can be used in practical reasoning tasks. In Section 7 we illustrate how some of the meta-theory of logics and formalisms such as the π-calculus and the (untyped) λ-calculus can be captured in Abella. In Section 8, we finally consider how Abella can be used to reason about relational specifications written in λProlog. We describe here the *two level logic approach* that allows such reasoning to be carried out through an encoding of the derivability relation of the logic underlying λProlog in an Abella definition.

At the outset of this tutorial, we shall assume that the reader is familiar with the natural deduction or sequent calculus style of proof presentation of intuitionistic logic. We shall also assume familiarity with the treatment of the standard logical connectives and quantifiers in such a proof-theoretic setting. Prior exposure to concepts such as λ-tree syntax and the ∇ quantifier can be useful but is not assumed. A reader familiar with the Coq system [9] will also find many points of similarity in the development of proofs in Abella, although we caution that there are significant differences as well.

Before commencing on the tutorial, we comment on the use of the term *higher-order*; see also [39, Section I.3] for a related discussion. This term may or may not apply to the logic $\mathcal{G}$, depending on what exactly is meant by it. A type τ is considered to be higher-order if it has an arrow type to the left of another arrow type. A minimum requirement for a logic to be higher-order is that it permit quantification over higher-order types. The logic $\mathcal{G}$ allows such quantification and may therefore be considered higher-order. However, many people would require a further property for the use of this adjective: quantification must be permitted over types that contain the type of formulas. Such quantification is not permitted in $\mathcal{G}$ and hence it does not pass this test. Given the ambiguity in the term higher-order when applied to $\mathcal{G}$, we eschew its use in this tutorial.

2. THE TOP-LEVEL STRUCTURE OF ABELLA

This section gives an overview of the concrete syntax and command-level interaction with Abella. Keeping to the spirit of a tutorial, we are not concerned here with presenting all details necessary for successfully using Abella: for that, the interested reader can find a user manual at `http://abella-prover.org`

2.1 Types, terms, and formulas

The concrete syntax of Abella is presented in this document using a monospaced font: in addition, keywords are depicted in blue. The types, terms, and formulas used by Abella are described briefly below as well as in the table in Figure 1.

Types in Abella are the *simple types*; such types are either primitive types or built from two types using the arrow type constructor $\to$. The type constructor $\to$ associates to the right, so every type in Abella can be written in the form $\tau_1 \to \cdots \to \tau_n \to b$ (for $n \geq 0$) where b is an atomic type that is called its *target type*, and each τ_i itself has this structure. User-defined atomic types are introduced by means of `Kind` declarations that we shall describe presently.

Terms in Abella are simply typed λ-terms built from variables, constants, λ-abstractions, and applications. Constants are introduced into Abella using the `Type` declarations explained below. A λ-abstraction is written using an infix backslash (\). The scope of a λ-abstraction extends as far to the right as possible while remaining in the syntactic class of terms; thus, `x\ f x` stands for $\lambda x.\,(f\ x)$. Note that the left operand of \ must be a bound variable, possibly with a type ascription, so, even though \ has lower precedence than application, `x\ f y\ y x` is unambiguously parsed as $\lambda x.\,(f\ (\lambda y.\,(y\ x)))$. Types for λ-bound variables may be omitted if they can be inferred; type-inference must be be able to fill in all such missing types for the term to be deemed well-formed.

	Abstract	Concrete	Precedence/ Associativity
Types (τ)			
Atomic types		`prop`, `nat`, `list`, ...	
Arrow types	$\tau_1 \to \tau_2$	`T1 -> T2`	right
Terms (m, n)			
Variables	$x, y, \ldots$	`x`, `y`, ...	
Constants	$c, d, \ldots$	`c`, `d`, ...	
Nominal constants	$\mathtt{n}_1, \mathtt{n}_2, \ldots$	`n1`, `n2`, ...	
		(`n` followed by at least one digit)	
Abstractions	$\lambda x.\, m$	`x\ M`	0, right
	$\lambda x{:}\tau.\, m$	`x:T\ M`	0, right
Applications	$m\ n$	`M N`	5, left
Formulas (A, B)			
Logical constants	$\top, \bot$	`true`, `false`	
	$m = n$	`M = N`	4, none
Atomic formulas	$p\ m_1\ \cdots\ m_n$	`p M1 ... Mn`	5, left
Connectives	$A \wedge B$	`A /\ B`	3, left
	$A \vee B$	`A \/ B`	2, left
	$A \supset B$	`A -> B`	1, right
Quantifiers	$\forall x.\, \forall y{:}\tau.\, \ldots A$	`forall x (y:T) ..., A`	0
	$\exists x.\, \exists y{:}\tau.\, \ldots A$	`exists x (y:T) ..., A`	0
	$\nabla x.\, \nabla y{:}\tau.\, \ldots A$	`nabla x (y:T) ..., A`	0
		(parentheses required for type-annotations)	

The Abella user manual [71] gives details on the lexical structure of identifiers. Parentheses may be used to explicitly indicate groupings, which is otherwise inferred using the precedence and associativity rules in the fourth column.

Fig. 1: Concrete Syntax for $\mathcal{G}$ in Abella.

Formulas in Abella are written using the standard syntax as shown in Figure 1. Formulas are actually terms of type `prop`. Atomic formulas are formulas whose top-level constructor is not a logical symbol (i.e., a logical constant, connective, or quantifier). In other words, the top-level constructor of an atomic formula must be a user defined constant with the target type `prop`. Such a constant is called a *predicate* or *relational* symbol. Predicate symbols can be introduced through `Type` declarations or through the `Define` or `CoDefine` commands explained in Sections 4.1 and 5. The scoping rules for the quantifiers of $\mathcal{G}$ are similar to those for λ-abstractions, and the types of such quantified variables may also be omitted when they can be inferred uniquely. Note that Abella does not allow quantification over variables whose types contain the type `prop`. In particular, quantification is not permitted over predicate variables, i.e., variables that have `prop` as their target type.

In the next several sections, we will not make explicit use of λ-abstractions in terms: as a result, these sections will view Abella as being based on a fairly standard first-order logic. Starting in Section 6, the treatment of λ-bindings will play an important role in our examples. It is also from that section on that we will allow formulas to contain the ∇ quantifier and terms and formulas to contain nominal constants.

2.2 Some top-level commands

Interacting with Abella involves issuing a sequence of *commands*: such commands declare kinds and types, introduce definitions, state theorems and develop their proofs. All such commands end with a *full-stop*. We can also intersperse commands with comments. Comments take the form of *line-comments* that begin with the character `%` that causes the rest of the line to be ignored, or with arbitrary text delimited by a balanced pair of `/*` and `*/` (supporting nesting).

Declarations come in two varieties: declarations of new atomic types, and declarations of new constants. To declare a new atomic type, one uses a `Kind` declaration that has the general form:

```
Kind    b1, ..., bn     type.
```

where the `b1`, ..., `bn` are pairwise distinct valid identifiers that are not already atomic types declared earlier. Each of `b1`, ..., `bn` is then added to the collection of atomic types that may be used to form other types. The final keyword `type` is used to indicate that these atomic types have *kind* `type`; this is somewhat redundant for the simple type system of Abella, but allows for future extensions to the type system with higher kinded atomic types.

To declare a collection of constants `c1`, ..., `cn` of a particular type `T`, one uses a `Type` declaration that has the general form:

```
Type   c1, ..., cn      T.
```

The type `T` must be formed out of atomic types that have been declared earlier by means of `Kind` declarations. Further, the constants `c1`, ..., `cn` must be pairwise distinct valid identifiers that are not already declared using `Type` or defined using the `Define` or `CoDefine` commands that we discuss later. Every Abella development begins with a standard collection of type and constant declarations preloaded, shown in Figure 9 in the appendix.

As an example, here is how we would define a new atomic type `nat` of natural numbers, constants `z` and `s` to construct such numbers, and a type `list` of natural numbers with constants `empty` and `cons` to construct them.

```
Kind nat   type.
Type z     nat.
Type s     nat -> nat.

Kind list  type.
Type empty list.
Type cons  nat -> list -> list.
```

It is important to note that the collection of constants of any type may be extended at any time with new declarations. Thus, there is no guarantee that all terms of type `nat`, for instance, are constructed using just `z` and `s`.

Another command in Abella is the one that proposes a formula to be considered for proof. This command has the following general form:

```
Theorem thm_name : A.
```

where `thm_name` is any valid identifier that is distinct from other constants declared or defined earlier, and `A` is a formula. Abella uses the same keyword `Theorem` for all flavors of provable formulas – lemmas, propositions, corollaries, etc. Every theorem *must* be followed by its proof, which is a series of tactics: we start introducing tactics in Section 2.3. Interactions with Abella are often called *developments*. Such developments can be entered either interactively or via batch files: in the latter case, the files are given names with the suffix `.thm`. Such files can be piped directly to the `abella` program or can be given as the first command line argument to the program. Abella processes the file and outputs its *interaction history* to standard output until it encounters an error (which causes an immediate abort of the program except in the interactive top-level) or it finishes processing all the commands in the file. In the interactive top-level, Abella uses the prompt

```
Abella <
```

to indicate that it is ready for top-level commands, and prompts of the form

```
thm_name <
```

to indicate that it is expecting tactics to prove the theorem named `thm_name`.

2.3 Interactive tactical theorem proving

So far, we have seen a few *top-level commands* of Abella—namely, `Kind`, `Type`, and `Theorem`. In order to prove a theorem in Abella, we use *tactic* commands instead of top-level commands, which are depicted in brown text. The set of tactics in Abella is small and fairly close to the inference rules of $\mathcal{G}$. These tactics will be introduced gradually in this document, and each theorem name will be linked to a website where the effects of the tactics can be browsed without needing to run Abella separately. Nevertheless, the reader is encouraged to follow along by copying the proofs that are presented in this document directly into the Abella top-level, or using one of the provided source files for the accompanying on-line materials with this tutorial.

When proving a theorem, Abella maintains a stack of *subgoals*, with the topmost subgoal of the stack selected as the *current goal*. Initially, there is only a single subgoal which is the theorem itself, but certain tactics may create additional subgoals that are pushed on to the subgoal stack. The current subgoal is presented in the interaction history of Abella in the following general form:

```
Variables: x1 ... xm
H1 : A1
...
Hn : An
============================
 C
```

where `x1`, ..., `xm` are universally quantified variables, `H1`, ..., `Hn` are *hypothesis labels* that are each associated with a unique *hypothesis formula* drawn from `A1`, ..., `An` and `C` is a formula called the *conclusion* of the goal. The collection of variables and hypotheses is called the *context* of the goal. If there are no universally quantified variables, the `Variables` line is omitted entirely. Note also that each of the variables that are shown has a type associated with it, even though this type is not explicitly presented. At each intermediate stage in the development of a proof, only the current goal is shown in full; only the conclusion is shown for the remaining subgoals.

Whenever a tactic is processed by Abella, it transforms the current goal as relevant and then displays the new proof state. As an illustration, suppose that we provide the following declarations

```
Kind i     type.
Type p     prop.
Type q     i -> prop.
```

and then try to prove the following theorem.

```
Theorem extr : forall y, (p -> forall x, q x) -> p -> q y.
```

Abella displays the initial proof state as follows:

```
============================
 forall y, (p -> (forall x, q x)) -> p -> q y

extr <
```

We can use the intros tactic to introduce all the antecedents, which yields:

```
extr < intros.

Variables: y
H1 : p -> (forall x, q x)
H2 : p
============================
 q y
```

Now we can use the apply tactic that takes two hypotheses and uses implication-elimination (modus ponens) on them to get a new hypothesis.

```
extr < apply H1 to H2.

Variables: y
H1 : p -> (forall x, q x)
H2 : p
H3 : forall x, q x
============================
 q y
```

Finally, we can use the backchain tactic to match the conclusion of a goal against the *head* of a hypothesis: if that match works, then the *body* of that hypothesis becomes the conclusion of the new goal.

```
extr < backchain H3.
Proof completed.

Abella <
```

Abella re-enters the top-level command processing state when a proof is complete.

For another example of proving a theorem, consider the following.

```
Kind nat type.
Type z   nat.
Type s   nat -> nat.

Type p   nat -> prop.

Theorem four : (forall x, p x -> p (s x)) ->
                          p z -> p (s (s (s (s z)))).
intros.
```

At this point, there is one current goal:

```
  H1 : forall x, p x -> p (s x)
  H2 : p z
  ============================
   p (s (s (s (s z))))
```

We could use `backchain H1` four times to complete this proof. An alternative is to use the `assert` tactic to introduce a local lemma. The tactic invocation `assert A` causes the current goal to be split into two subgoals: one where the conclusion is changed to `A` and the other that is identical to the current goal except that `A` is added to the hypotheses. For the above subgoal, this would look as follows:

```
four < assert (forall x, p x -> p (s (s x))).
Subgoal 1:

  H1 : forall x, p x -> p (s x)
  H2 : p z
  ============================
   forall x, p x -> p (s (s x))

Subgoal is:
 p (s (s (s (s z))))

four <
```

Observe that there are now two subgoals to prove. The first, which is displayed in full detail, can be proved by two applications of the `backchain H1` tactic.

```
four < intros.
Subgoal 1:

  Variables: x
  H1 : forall x, p x -> p (s x)
  H2 : p z
  H3 : p x
  ============================
   p (s (s x))

Subgoal is:
 p (s (s (s (s z))))

four < backchain H1.
```

```
Subgoal 1:

  Variables: x
  H1 : forall x, p x -> p (s x)
  H2 : p z
  H3 : p x
  ============================
   p (s x)

Subgoal is:
 p (s (s (s (s z))))

four < backchain H1.

  H1 : forall x, p x -> p (s x)
  H2 : p z
  H3 : forall x, p x -> p (s (s x))
  ============================
   p (s (s (s (s z))))

four <
```

Now, having successfully proved that formula, it is available as a hypothesis and we can return to proving our original goal. This proof can be completed by two uses of the `backchain` tactic, but this time using the hypothesis `H3`.

We shall continue to introduce tactics one-by-one as we have occasion to use them in example proofs. The Abella tactics used in this tutorial are briefly described in Figure 7 of the appendix; the full list, including precise documentation of their semantics, can be found in the Abella manual [71].

3. EQUALITY

The most basic relation in Abella, as it is in many theorem provers, is term equality. A distinguishing characteristic of $\mathcal{G}$, the logic underlying Abella, is that equality is treated as a *logical connective* with its own introduction rules and a role in cut elimination [22, 25, 34, 60]. In order to understand the treatment of equality, one must understand the nature of terms: Abella employs the free term algebra, which means that terms are finitely constructed from term constructors that are injective and distinct. Thus, the only way to prove an equality conclusion $t = s$ is for t and s to be syntactically identical.[1] Conversely, an equality hypothesis $t = s$ can be analyzed by considering all the ways in which t and s can be made identical. Unification is useful for this purpose. If t and s fail to unify, then the equality hypothesis is absurd and the goal is immediately proved. Otherwise, we can transform the entire goal into a set of other goals where, first, the hypothesis $t = s$ is removed and, second, a unifier of t and s is applied to all the remaining hypotheses and conclusion formula. The unifiers that are considered in such a transformation must cover all possible unifiers of t and s. In this section, where we restrict our attention to only first-order terms, we use the most general unifiers

[1] The logic assumes the λ-conversion rules so in fact we mean identical with respect to these rules.

that are known to exist for this purpose. Later, when we work with arbitrary simply typed λ-terms, we use the generalization of this notion to *complete sets of unifiers* [31]. In practice, Abella restricts attention to unification problems to the higher-order pattern fragment [37, 47] where, once again, most general unifiers are guaranteed to exist.

3.1 Basic examples

To illustrate the basic rules of equality, consider the following script where we declare a type `i`, equip it with a few term constructors, and set out to prove a few results about terms at type `i`.

```
Kind i         type.
Type a, b, c   i.
Type g         i -> i.
Type f         i -> i -> i.

Theorem ex1 : exists x, x = a.
witness a. search.
```

In the proof of `ex1`, we choose `a` as a witness for the existentially quantified variable `x`. This yields a subgoal with a trivial equality as conclusion, and we discharge it using the `search` tactic.

As another example, consider the following script.

```
Theorem ex2 : forall x y z,
  f x (g y) = f (g y) z -> x = z.
intros. case H1. search.
```

In this example, after introducing the three eigenvariables and the equality hypothesis under the name `H1`, we eliminate the hypothesis using `case`. As a result, Abella computes the most general unifier for the two terms in hypothesis `H1`, namely the substitution $\theta = [x \mapsto (g\ y), z \mapsto (g\ y)]$, and then applies that substitution to all formulas in the full goal. This then gives rise to the trivial conclusion formula `g y = g y` and the `search` tactic is able to complete the proof.

As we have observed above, unification may not always succeed. This is the case in the following example.

```
Theorem ex3 : a = b -> false.
intros. case H1.
```

Here, the application of the `case` tactic to `H1` results in an attempt to unify `a` and `b`. This attempt fails and thereby the proof is trivially completed.

As a slightly more elaborate example of equality reasoning, consider proving that the set $\{a, c\}$ is included in $\{a, b, c\}$. Although Abella does not have a built-in notion of sets[2], we can represent them by their membership predicates, that is $\lambda x.\,(x = a \vee x = c)$ and $\lambda x.\,(x = a \vee x = b \vee x = c)$. (We will discover a nicer way to encode such sets in the next section.) Using this idea, we can express our set inclusion, and prove it as follows.

[2] The curly brackets {, } will be used in Section 8 but not to denote sets.

```
Theorem ex4 : forall x,
  x = a \/ x = c -> x = a \/ x = b \/ x = c.
intros. case H1. search. search.
```

In this instance, the `case` tactic deals with the disjunction and its two equality disjuncts. This tactic generates two subgoals, in which x has been unified with a and b respectively, resulting in the conclusion formulas $a = a \lor a = b \lor a = c$ and $c = a \lor c = b \lor c = c$. Each subgoal is immediately proved using `search`, since its conclusion formula contains a trivial equality as a disjunct.

3.2 Proving that equality is a congruence

Although equality is not directly defined as a congruence, it is easy to show that it is one. First, we show that it is an equivalence.

```
Theorem eq-ref : forall (x : i), x = x.
intros. search.

Theorem eq-sym : forall (x : i) y, x = y -> y = x.
intros. case H1. search.

Theorem eq-trans : forall (x : i) y z,
                     x = y -> y = z -> x = z.
intros. case H1. case H2. search.
```

These three theorems are simple consequences of the way equality is treated in Abella. Now, one way to establish the congruence property for equality is to do it one constructor at a time. For example, the fact that the constructor `f` yields equal terms when applied to equal arguments can be stated and proved as follows.

```
Theorem f-eq-cong :
  forall x y u v, x = y -> u = v -> f x u = f y v.
intros. case H1. case H2. search.
```

Having to organize the proof of congruence this way can be tedious: there can be many constructors to consider. Fortunately, the ability to quantify at higher-order types allows us to formulate a general theorem that is surprisingly easy to prove.

```
Theorem eq-cong :
  forall (F : i -> i) x y, x = y -> (F x) = (F y).
intros. case H1. search.
```

Finally, we can show for any given predicate that it behaves the same on equal terms.[3]

```
Type p i -> prop.
Theorem p-eq-cong : forall x y, x = y -> p x -> p y.
intros. case H1. search.
```

[3] We can actually prove any instance of the substitutivity principle.

3.3 Peano axioms and the open-world assumption

Given the nature of constructors and their treatment by equality, it is possible to prove two of Peano's axioms for describing the natural numbers. In the following development, the type of natural numbers is introduced as are the constructors for zero and successor. The injectivity of successor and the difference between zero and successor are simple to state and prove.

```
Kind nat type.
Type z   nat.
Type s   nat -> nat.

Theorem succ-inj : forall x y, (s x) = (s y) -> x = y.
intros. case H1.  search.

Theorem zero-succ : forall x, (s x) = z -> false.
intros. case H1.
```

In addition, we have a simple proof of the fact that no number is its own successor.

```
Theorem finiteness : forall x, x = (s x) -> false.
intros.  case H1.
```

This theorem holds because x is quantified over finite terms, a fact reflected in the proof through the failure of unification. It is important to note that we did not and could not use an induction on x here. In fact, Abella does not allow case analysis directly on terms—as a result, it is not possible to prove `forall (x : nat), x = z \/ exists y, x = s y`. This goes with another key principle of Abella: we do not assume the signature to be closed, and so all reasoning should remain valid if the signature is extended later. Of course, closedness assumptions are often necessary; they shall be expressed by means of user-defined relations, a topic we discuss in the next section.

4. RELATIONAL SPECIFICATIONS

There are two basic approaches to defining operations such as addition on the terms built from the constructors `z` and `s` declared in the previous section; these approaches can be seen as answers, respectively, to the questions of "how" and "what." The first is the *functional* approach—a response to the "how" question—where a computation `sum` explicitly builds the sum of a pair of `nat`s by analyzing the structure of its inputs. The inputs and outputs of the function are fixed; one cannot use the same function to compute the difference of two numbers. Moreover, the computations interact with the notion of equality on terms; for instance, it must be the case that:

```
sum (s z) (s (s z)) = s (s (s z))
```

It would be incorrect here to observe that the topmost "function" symbol of the two terms being compared are different and therefore that the terms must be different. To compare terms with such embedded computations for equality, we must, therefore, extend the equational theory of terms to account for such computations.

The alternative is the *relational* approach—a response to the "what" question—where we define a relation `plus` between *three* `nat`s that asserts that the third `nat` is the sum of the first two. In other words, `plus` is not a way to construct new `nat`s from old, but is an assertion about `nat`s that are constructed with just `z` and `s`, with no change to their equational theory. Indeed, the `plus` predicate can be defined in terms of a simple inference system with the following inference rules.

$$\frac{}{\texttt{plus z}\ N\ N} \qquad\qquad \frac{\texttt{plus}\ M\ N\ K}{\texttt{plus (s}\ M\texttt{)}\ N\ \texttt{(s}\ K\texttt{)}}$$

To establish that m and n sum to k, it suffices to find a derivation of `plus` $m\ n\ k$ using the inference system above. The `plus` inference system can, moreover, be used just as easily for subtraction: the same derivation establishes that n is the difference of m and k. Finally, if k is *not* the sum of m and n, then there is no derivation of `plus` $m\ n\ k$, so the inference system is a *complete characterization* of addition.

4.1 Defining relations as fixed points

Abella directly realizes the relational approach by means of *relational fixed point definitions* often just called *fixed points.* To illustrate, the inference system for `plus` above is represented in Abella using the following definition.

```
Define plus : nat -> nat -> nat -> prop by
  plus z     N N ;
  plus (s M) N (s K) := plus M N K.
```

The *target type* of the `plus` predicate is the type `prop` of formulas of $\mathcal{G}$; every such definition must have the target type `prop`. Each inference rule above is captured as a *definitional clause* for the `plus` predicate, with all the clauses separated by semi-colons. The *head* of each clause is the atomic formula that occurs to the left of `:=`, while the formula to the right of `:=` is the *body.* (If there is no `:=` in a clause, the body is implicitly set to `true`.) In each clause, the capitalized identifiers—the `M`, `N`, `K`, etc.—are assumed to be universally quantified, so the clause stands for every instance of these identifiers.

This fixed point definition has the same properties as the `plus` inference system. It can establish true facts such as:

```
Theorem plus_two_two :
  plus (s (s z)) (s (s z)) (s (s (s (s z)))).
unfold 2. unfold 2. unfold 1. search.
```

We could have proved this with just `search`, but we have used explicit `unfold`s to show which definitional clause of `plus` was used to unfold the conclusion; see Figure 7 for the semantics of `unfold`. More importantly, the `plus` definition can also be used to show the negation of incorrect summations.

```
Theorem plus_bad : plus (s (s z)) (s z) (s (s z)) -> false.
intros. case H1. case H2. case H3.
```

Note that in the uses of `unfold` or `case` above, the goal or the hypothesis is of the form `plus M N K` which is *matched* against one of the heads of the definitional clauses of `plus`, and the corresponding body is used to replace the goal or the hypothesis. It is possible to view each definition as implicitly constructing a disjunction of all its definitional clauses, with the variables explicitly quantified. Thus, an explicit version of the `plus` relation may be written as follows.

```
Define plus_ex : nat -> nat -> nat -> prop by
  plus_ex M N K :=
     (M = z /\ K = N)
  \/ (exists M' K', M = s M' /\ K = s K' /\ plus M' N K').
```

The two relations `plus` and `plus_ex` are equivalent, but `plus` is easier to use because the disjunctions are left implicit both in its definition and in proofs involving `plus`.

These implicit disjunctions can also be used to enumerate the elements of a finite relation explicitly. For example, the sets $\{a, b\}$ and $\{a, b, c\}$, defined using disjunction and equality in Section 3, can equivalently be defined as relations:

```
Define ab : i -> prop by
  ab a ; ab b.

Define abc : i -> prop by
  abc a ; abc b ; abc c.

Theorem ex4_defs : forall x, ab x -> abc x.
intros. case H1. search. search.
```

The `ex4_defs` theorem, which is the version of `ex4` from Section 3 using definitions, has an identical proof. We will revisit such explicit enumerations when discussing inductive invariants in the next section.

To give a logical meaning to relational fixed point definitions, we have to ensure that the fixed point being declared to exist actually does exist. To see the danger, consider a putative definition such as:

```
Define p : prop by
  p := p -> false.
```

We can prove both `p` and `p -> false` by simply `unfold`ing and `case`-analysis of `p`:[4]

```
Theorem p_true : p.
unfold. intros. case H1 (keep). apply H2 to H1.

Theorem notp_true : p -> false.
intros. case H1 (keep). apply H2 to H1.
```

Abella complains about such a definition by emitting a warning that the definition fails the *stratification* condition, which is a sufficient condition to ensure that the fixed point exists [34, 63]. A definition for a predicate `p` is stratified if in its definitional clauses there are no occurrences of `p` in subformulas to the left of an implication and only those predicates that have previously been defined are used. Note that this condition is *per definition*; any definition that follows the definition of `p` can use `p` without restrictions, just as the definition of `p` can freely use any predicate that has been defined earlier.

[4] The default behavior for the `case` tactic is to remove the hypothesis that has just been analyzed by it. Sometimes, however, it convenient or necessary to keep that hypothesis since it will be used another time. In such cases, we use the expression `(keep)` with the tactic as we have done here. Interestingly, it has been shown in [25, 60] that producing contradictions such as the one under discussion using fixed points requires using a hypothesis more than once.

The stratification condition used by Abella is conservative. For example, the following two definitions fail this condition.

```
Define q : prop by
  q := (q -> false) -> false.

Define r : nat -> prop by
  r z ;
  r (s N) := r N -> false.
```

Recent results [8, 69] suggest that definitions of the first kind may be permitted without loss of consistency. Similarly, a weakened version of stratification has been identified that renders the second definition acceptable [66]. An alternative development permits definitions of the second kind in a form that does not allow case analysis to be done on them, treating them instead as recursive definitions in the style of Coq [8]. A further discussion of this matter is beyond the scope of this tutorial. Suffice it to say that any warnings of the violation of the stratification condition emitted by Abella must be given serious consideration for fear of permitting definitions that introduce inconsistencies.

4.2 Some recursive definitions involving lists

Consider the example of lists of natural numbers below:

```
Kind list   type.

Type empty  list.
Type cons   nat -> list -> list.

Define memb : nat -> list -> prop by
  memb N (cons N L) ;
  memb N (cons M L) := memb N L.
```

We can then directly define the standard set inclusion and equality.

```
Define set_incl : list -> list -> prop by
  set_incl S T :=
    forall E, memb E S -> memb E T.

Define set_eq : list -> list -> prop by
  set_eq S T := set_incl S T
             /\ set_incl T S.
```

Once we have defined the `memb` relation in this way, we can use it in the form `exists E, memb E S /\ pred E`, where it is serves to check membership in a set (encoded as the list S), and in the form `forall E, memb E S -> pred E`, where it is used to define the range of such a set. Observe that if there are repetitions of elements in the list representing a set, then the `memb` relation may enumerate that element more than once.

The sets `ab` and `abc` that we constructed above as definitions can instead be built explicitly as lists and the inclusion can be shown using `set_incl`.

```
Type u,v,w nat.
```

```
Theorem ex4_lists :
  set_incl (cons u (cons v empty))
           (cons u (cons v (cons w empty))).
unfold. intros.
  case H1. /* case of u */ search.
  case H2. /* case of v */ search.
  case H3. /* no more cases */
```

4.3 Finite success and finite failure

Definitions can be used directly to reason about *finite* computations using the case and unfold tactics. This is achieved by specifying the computations as a definition where each clause is, in essence, in the Horn fragment of logic programs [39, chapter 2]. As an example, consider the following definitions of append and reverse of lists.

```
Define append : list -> list -> list -> prop by
  append empty L L ;
  append (cons N L1) L2 (cons N L3) :=
    append L1 L2 L3.

Define reverse : list -> list -> prop by
  reverse empty empty ;
  reverse (cons N L1) L2 :=
    exists L3, reverse L1 L3
            /\ append L3 (cons N empty) L2.
```

Here are two simple proofs for finite lists, where we have used only unfold for the computations and limited search to subgoals with the conclusion true.

```
Theorem append_1 :
   append (cons u empty) (cons v (cons w empty))
          (cons u (cons v (cons w empty))).
unfold. unfold. search.

Theorem reverse_1 :
   reverse (cons u (cons w (cons v empty)))
           (cons v (cons w (cons u empty))).
unfold. witness cons v (cons w empty). split.
  unfold. witness cons v empty. split.
    unfold. witness empty. split. unfold. search.
                                  unfold. search.
    unfold. unfold. search.
  unfold. unfold. search.
```

Both of these proofs could have been reduced to just search.

As a dual to the examples above, the negation of a relation A can often be proved by showing exhaustively that there is no proof of A; this is a technique that is also referred to as *negation by finite failure*. In particular, if the case tactic cannot find a way to match a hypothesis that is an instance of a defined relation with the head of any defitional clause, then assuming that the hypothesis is provable is absurd. As a result, the entire goal can succeed. An example that brings out this style of reasoning is the following.

```
Theorem append_finite_failure :
  append (cons u empty) (cons v (cons w empty))
         (cons u (cons w (cons v empty))) -> false.
intros. case H1. case H2.
```

In this example, the proof state that the tactic `case H2` is applied to is the following.

```
  H2 : append empty (cons v (cons w empty))
              (cons w (cons v empty))
  ============================
   false
```

The tactic application finishes the proof because `H2` does not match with the head of any of the clauses for `append`.

It is important to note that changing `u`, `v`, and `w` from signature constants to universally quantified variables changes the formula from a theorem to a non-theorem.

```
Theorem append_not_failure : forall u v w,
  append (cons u empty) (cons v (cons w empty))
         (cons u (cons w (cons v empty))) -> false.
intros.
```

The goal we have at this point is the following.

```
  Variables: u, v, w
  H1 : append (cons u empty) (cons v (cons w empty))
              (cons u (cons w (cons v empty)))
  ============================
   false
```

Let us now use the `case` tactic twice as before, but with the `(keep)` option that instructs Abella not to delete the hypothesis after case-analysis, so that we can observe its effect:

```
append_not_failure < case H1 (keep).

  Variables: u, v, w
  H1 : append (cons u empty) (cons v (cons w empty))
              (cons u (cons w (cons v empty)))
  H2 : append empty (cons v (cons w empty))
              (cons w (cons v empty))
  ============================
   false

append_not_failure < case H2 (keep).

  Variables: u, w
  H1 : append (cons u empty) (cons w (cons w empty))
              (cons u (cons w (cons w empty)))
  H2 : append empty (cons w (cons w empty))
              (cons w (cons w empty))
  ============================
   false
```

Here, the head of the first clause of `append` *does* match `H2` in the case that `v` and `w` are equal. The subgoal therefore explores this possibility by *instantiating* `v` to `w`, which leaves us in a state from which we cannot complete the proof.

5. INDUCTION AND CO-INDUCTION

Simply unfolding definitions (whether in hypotheses or conclusions of goals) has its limitations. For instance, consider the following definitions of `even` and `odd` natural numbers.

```
Define even : nat -> prop by
  even z ;
  even (s (s N)) := even N.

Define odd : nat -> prop by
  odd (s z) ;
  odd (s (s N)) := odd N.
```

It is clearly the case that `even x -> odd (s x)` holds, but this cannot be shown just by using the `case` and `unfold` tactics:

```
Theorem even_odd : forall x, even x -> odd (s x).
intros. case H1.
  search.
  unfold.
```

The result of the second `unfold` is this subgoal:

```
  Variables: N
  H2 : even N
  ============================
   odd (s N)
```

which is the same as the `even/odd` theorem itself, and therefore cannot be established by finite unfolding.

The above example indicates that to prove interesting properties of types such as natural numbers it is necessary to *reason by induction*. Elementary presentations of natural number induction typically take the form of a base case, where a property is shown for 0, and an inductive step, where the property is assumed for n and then shown for $n + 1$. If the base case and inductive step are proven, then the property is true for all natural numbers. This reasoning is justified by the well-foundedness of the natural numbers: for any particular natural number n, the property could be shown to hold without induction simply by performing the base case and then applying the inductive step n times. In this way, induction allows for guarded cyclic reasoning and avoids unsound circular reasoning.

To introduce induction in the context of Abella, we refine the notion of an arbitrary fixed point definition to that of a *least fixed point definition*. Any predicate introduced via the `Define` command is treated as a least fixed point and is called an *inductively defined predicate* or simply an *inductive predicate*. Roughly, the least fixed point semantics of inductive predicates means that the only instances of the predicate that hold are those which can be obtained by iterating the definition principle a (trans)finite number of times. Thus, we can induct over the number of iterations of the definition for such predicates. Note that induction is applied in Abella to inductively defined *predicates*: there is no direct support for inductively defined types. Of course, since types can generally be encoded as predicates, one can get the effect of an induction on the type by first defining a predicate that identifies the members of the type.

The actual process of inductive reasoning in Abella relies on using the `induction` tactic. This tactic applies when the conclusion has the form of an implication possibly nested under universal quantifiers. The target of the `induction` tactic, called the inductive argument, is one of the hypotheses of the implication which must be an inductively defined predicate applied to some arguments. The tactic takes the conclusion as an inductive hypothesis while adding a restriction that the inductive argument must be reduced, measured in terms of unfoldings of the definition, before the inductive hypothesis may be applied.

We illustrate these ideas by considering how they may be used to realize induction over natural numbers. Towards this end, we first provide an inductive definition of the natural numbers.

```
Define is_nat : nat -> prop by
  is_nat z ;
  is_nat (s N) := is_nat N.
```

Let `p` be a term of type `nat -> prop` representing a property of interest over the natural numbers. To show that `p` holds for all natural numbers, we would have to prove the following.

```
Theorem p_universal : forall n, is_nat n -> p n.
```

At this stage, we can invoke the tactic `induction on 1`, which specifies that we should try induction with the first hypothesis in the conclusion as the inductive argument. This transforms the proof state to

```
IH : forall n, is_nat n * -> p n
============================
 forall n, is_nat n @ -> p n
```

in which the inductive hypothesis is marked as `IH`. Two *size restriction annotations*, `*` and `@`, are used to track the relative sizes of the inductive argument. The restriction `*` is fixed and requires that the inductive hypothesis only applies to arguments which are smaller than the term annotated with `@`. In order to obtain a smaller argument, we first apply `intros` to reach:

```
Variables: n
IH : forall n, is_nat n * -> p n
H1 : is_nat n @
============================
 p n
```

We then proceed with `case H1` which results in two subgoals:

```
Variables: n
IH : forall n, is_nat n * -> p n
============================
 p z
```

and

```
Variables: n, N
IH : forall n, is_nat n * -> p n
H2 : is_nat N *
============================
 p (s N)
```

The first goal corresponds to the base case in traditional natural number induction. In the second goal, we have a hypothesis `is_nat N` which is a candidate for the inductive hypothesis. Using the inductive hypothesis via the tactic `apply IH to H2` produces a state corresponding to the inductive step of traditional natural number induction:

```
Variables: n, N
IH : forall n, is_nat n * -> p n
H2 : is_nat N *
H3 : p N
============================
 p (s N)
```

Thus the notion of induction in Abella subsumes natural number induction.

As a concrete example, let us prove the `even_odd` theorem that we considered in Section 4.3.

```
Theorem even_odd : forall n, even n -> odd (s n).
induction on 1. intros. case H1.
  search.
  apply IH to H2. search.
```

Using `induction on 1` here yields the proof state

```
IH : forall n, even n * -> odd (s n)
============================
 forall n, even n @ -> odd (s n)
```

As before, we must turn the size restricted formula `even n @` into a formula involving `*` before we can apply the inductive hypothesis. This is done by bringing `even n @` into the context using the `intros` tactic. Following this with `case H1`, leaves us with the two subgoals

```
IH : forall n, even n * -> odd (s n)
============================
 odd (s z)
```

and

```
Variables: N
IH : forall n, even n * -> odd (s n)
H2 : even N *
============================
 odd (s (s (s N)))
```

These goals arise from the two ways that `even n` can be proved: the first clause of `even` can be used if the variable `n` is set equal to `z`, while the second clause can be used if n is set to (s (s N)) for some new variable N. In this latter case, we can appeal to the IH because the annotations match. In particular, if we apply IH to H2 on this goal, it will add odd (s N) to the hypothesis set. The goal follows from this by a simple unfolding that can, in fact, be performed automatically by using the search tactic.

The reader is encouraged to try to prove the following theorems at this point, to get a feel for the induction tactic.

```
Theorem even_nat : forall n, even n -> is_nat n.
Theorem odd_nat : forall n, odd n -> is_nat n.
Theorem nat_part : forall n, is_nat n -> even n \/ odd n.
Theorem even_odd_split : forall n, even n -> odd n -> false.
```

5.1 Inductive hypotheses *vs.* inductive invariants

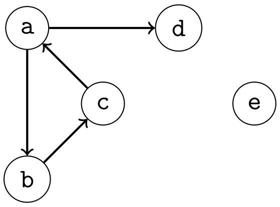
Using inductive hypotheses with size restriction annotations is a more general mechanism than reasoning with explicit invariants. On the other hand, the soundness argument for induction based on annotated inductive hypotheses is harder to justify in terms of invariants, although this can be done formally as well [19].

Despite Abella's commitment to guarded cyclic proofs via the induction tactic, the invariants can be explicitly constructed if needed. As a prototypical example, consider the problem of reachability in finite directed graphs, represented here with a type node of nodes and a finitely enumerated edge relation between the nodes. Let us use the following illustrative example.

```
Kind node        type.
Type a,b,c,d,e   node.

Define edge : node -> node -> prop by
  edge a b ; edge b c ; edge c a ; edge a d.
```

This relation corresponds to the following graph.



For any such graph, the reachability relation reach, which is the reflexive-transitive closure of edge, is easily defined:

```
Define reach : node -> node -> prop by
  reach N N ;
  reach N1 N3 := exists N2, reach N1 N2 /\ edge N2 N3.
```

Let us try to show that the node e is *not* reachable from a. After traversing three edges from a, we would be back to a, so the graph cannot be finitely explored by simply unfolding the edge and reach definitions. However, we can finitely enumerate all nodes reachable from a.

```
Define a_can_reach : node -> prop by
  a_can_reach a ; a_can_reach b ;
  a_can_reach c ; a_can_reach d.
```

Assuming we show that whenever `reach a x` also `a_can_reach x` (which requires induction, of course), it is then a simple matter to observe that `e` is not in this enumeration and is therefore not reachable.

```
Theorem inv : forall x, reach a x -> a_can_reach x.
induction on 1. intros. case H1.
  search.
  apply IH to H2. case H3. search. search. search. search.

Theorem a_cannot_reach_e : reach a e -> false.
intros. apply inv to H1. case H2.
```

Note that the proof `inv` is entirely straightforward and can be efficiently computed by a model checker [7].

5.2 Kinds of induction

5.2.1 *Simple induction.* As should be expected, many properties of inductively defined relations such as `append` and `reverse` from Section 4.3 are proved using the `induction` tactic. Here, for instance, are two examples showing that `append` is deterministic and associative.

```
Theorem append_det : forall I1 I2 O1 O2,
  append I1 I2 O1 -> append I1 I2 O2 -> O1 = O2.
induction on 1. intros. case H1.
  case H2. search.
  case H2. apply IH to H3 H4. search.

Theorem append_assoc : forall A B C AB ABC,
  append A B AB -> append AB C ABC ->
  exists BC, append B C BC /\ append A BC ABC.
induction on 1. intros. case H1.
  search.
  case H2. apply IH to H3 H4. search.
```

We can also show that `append` is total, i.e., for any ground lists in the first two arguments, `append` establishes their concatenation in its third argument. To prove this theorem, we need to induct on the structure of the input lists. In Abella this is achieved by means of a separate definition `is_list` that specifies the recursive structure of the list and can be used to drive the induction.

```
Define is_list : list -> prop by
  is_list empty ;
  is_list (cons N L) := is_list L.

Theorem append_total : forall A B,
  is_list A -> exists C, append A B C.
induction on 1. intros. case H1.
  search.
  apply IH to H2 with B = B. search.
```

Note, however, that the following theorem is *not* provable in Abella

```
forall (A:list) (B:list), exists (C:list), append A B C.
```

because the typing assertion `A:list` does not have an associated induction principle. This is an intentional omission, because Abella allows extensions of the type signature. For example, a new kind of list constructor can be defined:

```
Type cons2   nat -> nat -> list -> list.
```

which the `append` relation does not know how to process. Still, `append_total` would continue to hold, because the `is_list` relation filters out such lists. All inductive theorems in Abella are proved by induction on an explicit inductive definition.

While the induction tactic can prove some theorems directly, some theorems cannot use that tactic until after other lemmas have been proved. For instance, to show that `plus` on natural numbers is commutative requires two lemmas that are proved separately.

```
Theorem plus_zero : forall N, is_nat N -> plus N z N.
induction on 1. intros. case H1.
  search.
  apply IH to H2. search.

Theorem plus_succ : forall M N K,
  plus M N K -> plus M (s N) (s K).
induction on 1. intros. case H1.
  search.
  apply IH to H2. search.

Theorem plus_comm : forall M N K,
  is_nat K -> plus M N K -> plus N M K.
induction on 2. intros. case H2.
  apply plus_zero to H1. search.
  case H1. apply IH to H4 H3.
   apply plus_succ to H5. search.
```

5.2.2 *Mutual induction.* It is common to define a related family of inductive definitions that are mutually recursive in the form of a *definition block.* For example, the `odd` and `even` relations from Section 4.3 can be defined more perspicuously by mutual induction as follows.

```
Define even : nat -> prop,
       odd  : nat -> prop   by
  even z ;
  even (s N) := odd N ;
  odd (s N)  := even N.
```

With such a definition, it becomes natural to write the proofs of the `even_nat` and `odd_nat` theorems earlier by mutual induction. In Abella, this is achieved by first writing the statements of both theorems as a conjunction.

```
Theorem even_odd_nat :
     (forall N, even N -> is_nat N)
  /\ (forall N, odd N  -> is_nat N).
```

To prove this we need *two* inductive hypotheses, one for each conjunct, that may be appealed to whenever we have either a strictly smaller even N or a strictly smaller odd N. The induction tactic in its general form takes a (non-empty) list of antecedent numbers as arguments, one number per conjunct of the theorem.

```
even_odd_nat < induction on 1 1.

  IH : forall N, even N * -> is_nat N
  IH1 : forall N, odd N * -> is_nat N
  ============================
   (forall N, even N @ -> is_nat N) /\
   (forall N, odd N @ -> is_nat N)
```

We can now split the conjunctive goal into two subgoals and use either IH or IH1 (or both) in each subgoal as appropriate. The full proof looks as follows.

```
Theorem even_odd_nat :
     (forall N, even N -> is_nat N)
  /\ (forall N, odd N  -> is_nat N).
induction on 1 1. split.
intros. case H1. search. apply IH1 to H2. search.
intros. case H1. apply IH to H2. search.
```

Given such a theorem, the top-level command Split extracts each conjunct as a separately named theorem to be used in the rest of the development.

```
Split even_odd_nat as even_nat, odd_nat.
```

5.2.3 *Nested and lexicographic induction.* For more complicated inductive definitions, it is sometimes necessary to reason by induction on more than one relation at the same time, with a lexicographic order between the decreasing measures. In Abella, this is achieved by means of *nested induction*, where the nesting expresses the lexicographic ordering.

As an illustration, take the following definition ack that computes the Ackermann relation on natural numbers, i.e., ack M N K if and only if $K = A(M, N)$.

```
Define ack : nat -> nat -> nat -> prop by
  ack z     N     (s N) ;
  ack (s M) z     K     := ack M (s z) K ;
  ack (s M) (s N) K     :=
    exists J, ack (s M) N J /\ ack M J K.
```

Consider the theorem that the relation is total: for any natural numbers as input (the first two arguments to ack), there is a natural number as output (the third argument of ack).

```
Theorem ack_total : forall M N,
  is_nat M -> is_nat N ->
  exists K, is_nat K /\ ack M N K.
```

The inductive argument proceeds as follows: either M decreases in a recursive call to ack (in which case N can grow), or M stays the same and N decreases strictly. We write this using the following nesting.

```
ack_total < induction on 1. induction on 2.

IH : forall M N, is_nat M * -> is_nat N ->
        (exists K, is_nat K /\ ack M N K)
IH1 : forall M N, is_nat M @ -> is_nat N ** ->
        (exists K, is_nat K /\ ack M N K)
============================
 forall M N, is_nat M @ -> is_nat N @@ ->
    (exists K, is_nat K /\ ack M N K)
```

The first inductive hypothesis, IH, handles the case where M decreases strictly—that is, deriving is_nat M requires strictly fewer unfoldings—indicated with *. The second hypothesis, IH1, handles the case where M stays exactly the same (indicated with @) while N shrinks strictly. The doubled annotation (** or @@) on is_nat N indicates that this annotation is *with respect to* an enclosing inductive restriction, in this case for is_nat M. The proof of ack_total is then straightforward.

```
Theorem ack_total : forall M N,
  is_nat M -> is_nat N ->
  exists K, is_nat K /\ ack M N K.
induction on 1. induction on 2.
intros. case H1 (keep).
  search.
  case H2.
    apply IH to H3 _ with N = s z. search.
    apply IH1 to H1 H4. apply IH to H3 H5. search.
```

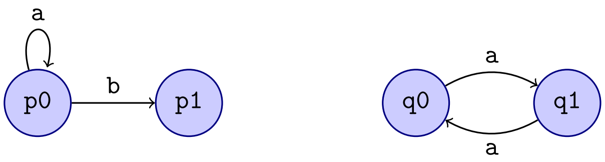
There are standard approaches to realizing *strong induction* or *well founded induction* given the induction principles we have illustrated here. Examples illustrating such induction principles can be found on the Abella website.

5.3 Co-induction

We have thus far seen only recursive definitions and cyclic proofs that were given least fixed point and inductive semantics, respectively. Abella also supports their dual notions: greatest fixed point semantics and co-inductive proofs. Many behavioral properties, most importantly (bi)simulation, are co-inductively defined. Unlike induction, which deals with the *structure* of relations, co-induction reasons about their *behavior*, i.e., the trace of observations that can be made about a co-inductively defined relation as it is iteratively unfolded.

In this subsection we will introduce co-inductive definitions and the coinduction tactic and apply them in the domain of finite automata. Co-induction will be used more heavily in later sections, particularly in Section 7.3, to reason about more sophisticated computational systems such as the π-calculus. In this section, we present a simple example of reasoning co-inductively on a finite automata simulation.

Consider finite automata with labeled transitions, represented with a type st of *states* and lab of *labels*, each of which will have only a finite number of constant constructors. On this basis we define a finitely enumerated step relation that will stand for the labeled transitions between states. For illustration purposes, we use the following two automata that have a disjoint set of states but a common set of labels.



Here is their encoding in Abella:

```
Kind st,lab        type.
Type p0,p1,q0,q1   st.
Type a,b           lab.

Define step : st -> lab -> st -> prop by
  step p0 a p0 ; step p0 b p1 ;
  step q0 a q1 ; step q1 a q0.
```

Two states are said to be *in a simulation relation* if every labeled transition from the first state is matched by an identically labeled transition from the other state and the end points of both transitions are in the simulation as well. For instance, in the above pair of automata, the states `q0` and `p0` are in a simulation, since every trace starting from `q0` is a repetition of the label `a`, which can be matched by the self-transition labeled `a` from `p0`. The *similarity relation* is the greatest relation on pairs of states that is a simulation; equivalently, it is the union of all simulation relations. It coincides with the greatest fixed point of the recursive definition of a simulation, and is written using a `CoDefine` declaration.

```
CoDefine sim : st -> st -> prop by
  sim P Q :=
    forall L Pn, step P L Pn ->
      exists Qn, step Q L Qn /\ sim Pn Qn.
```

Take the problem of showing that `q0` and `p0` are in a simulation, i.e., that `sim q0 p0` is derivable. The proof that would result with just the unfolding tactics `unfold` and `case` is as follows.

```
Theorem q0_sim_p0 : sim q0 p0.
unfold. intros.
case H1. witness p0. split. search.
unfold. intros.
case H2. witness p0. split. search.
/* back to the first case */
abort.
```

Once again, we have a circularity in the proof, but, unlike in the case of induction, there is no measure on the proof state that has strictly decreased. However, there is a measure that has *increased*, viz., the number of times the co-inductive definition `sim` has been `unfolded`. The `coinduction` tactic of Abella, like for induction, adds a *co-inductive hypothesis* with size restriction annotations that enforce this increasing measure. In particular, the proof state changes as follows.

```
q0_sim_p0 < coinduction.


  CH : sim q0 p0 +
  ============================
   sim q0 p0 #
```

The + annotation in the co-inductive hypothesis CH asserts that sim has been unfolded at least once. It therefore cannot be used to match the # annotation in the conclusion, that asserts that sim has not been unfolded. To turn the # to a +, therefore, we have to use the unfold tactic.

```
q0_sim_p0 < unfold.

  CH : sim q0 p0 +
  ============================
   forall L Pn, step q0 L Pn ->
     exists Qn, step p0 L Qn /\ sim Pn Qn +
```

Observe that the annotation on sim Pn Qn in the conclusion has changed. We can now introduce the antecedents and reason by cases on step q0 L Pn. The full proof then looks nearly the same as the attempted proof above:

```
Theorem q0_sim_p0 : sim q0 p0.
coinduction. unfold. intros.
case H1. witness p0. split. search.
unfold. intros.
case H2. witness p0. split. search.
search.
```

The sole difference is that before the final search, the proof state is:

```
  CH : sim q0 p0 +
  ============================
   sim q0 p0 +
```

The annotations match, so the proof can finish by appealing to the CH.

Just as with induction in Section 5.1, it is possible to build the co-inductive invariant set explicitly. In this case, the invariant will be a simulation, i.e., a set of all pairs of states that contains at least the pair $(\mathtt{q0}, \mathtt{p0})$ and progresses[5] to itself. For this simple example, we can construct such a simulation for $(\mathtt{q0}, \mathtt{p0})$ with a finite number of elements:

```
Define sim_ex : st -> st -> prop by
  sim_ex q0 p0 ; sim_ex q1 p0.
```

Note that sim_ex is *inductively* defined. Nevertheless, we can show that it is included in the similarity relation.

```
Theorem sim_ex_incl : forall P Q,
   sim_ex P Q -> sim P Q.
coinduction. intros. unfold. intros. case H1.
case H2. witness p0. split. search.
  assert sim_ex q1 p0. backchain CH.
case H2. witness p0. split. search.
  assert sim_ex q0 p0. backchain CH.
```

[5] A relation R *progresses to* R' if for every $(x, y) \in R$, if $x \xrightarrow{a} x'$ then $y \xrightarrow{a} y'$ and $(x', y') \in R'$.

Thus, to show `sim P Q`, it suffices to show `sim_ex P Q`, which can be done more simply than showing `sim P Q` directly. Yet, except for trivial examples such as this one, such co-inductive invariants are infinite sets with an intricate recursive structure that makes them hard to define, reason about, and maintain.

6. REASONING ABOUT OBJECTS WITH BOUND VARIABLES

Many formal systems whose meta-theory we want to mechanize treat objects whose structures incorporate binding notions. Thus, logical systems often concern formulas with quantifiers, expressions in the λ-calculus include ones that contain abstractions, functional structure is important to capture in programming language syntax, and names that have a scope (and additional relevant properties) are central to the π-calculus; the list goes on. Although the various binding constructs each have their own specific semantics, they also possess a common core of properties, such as equality under renaming and a notion of substitution that avoids inadvertent capture of variables free in a local structure. A proper treatment of such notions is recognized to be both non-trivial and also critical to the correct formalization of the different systems of interest. Considerable research effort has therefore been devoted to developing systematic ways to encode binding constructs within proof assistants and automated theorem provers.

Two common approaches in the computer-based treatment of binding are to use the nameless dummies framework of De Bruijn [13] and the nominal logic framework of Pitts [53]. Abella is based on a third approach that uses the abstraction operator in a typed λ-calculus to encode the binding effect of constructs such as quantifiers in logical formulas and formal parameters in programs. This style of encoding was introduced by Church already in 1940 in his paper formalizing higher-order logic [12], but the potential for Church's idea in a computational setting was not realized until about four decades later [32, 38]. Eventually, this style was elaborated into what is now called *λ-tree syntax* [39].[6]

The realization of λ-tree syntax in Abella is based on two key aspects. First, the λ-calculus that is used to represent syntax is deliberately chosen to be weak, so that it encompasses only the basic notions related to bound variables and substitution. Under this choice, equality and the associated unification computation provide meaningful tools for analyzing syntactic structure. Second, the logic underlying Abella complements the term-level representation of binding with binding notions at the formula and proof-level. Here, the famous slogan of Alan Perlis that "there is no such thing as a free variable" is followed. Rather than converting bound variables in the analysis of syntax into arbitrary structures that correspond to free variables, Abella treats them via a *mobility* of binders: a term-level binder is transformed into a formula-level binder which in turn becomes a proof-level binder.

In this section, we expose the reader to the way the Abella style of treating binding constructs is used in practice. A special role is played in such formalizations by a *generic* quantifier that is written as ∇ and pronounced *nabla* and by *nominal constants* that are a proof-level correlate of such quantifiers. We introduce these notions in the course of our discussion.

[6] Gacek [20] has shown that aspects of nominal logic and of λ-tree syntax can be made to coincide in a significantly weakened subset of $\mathcal{G}$.

6.1 Representing constructs with bound variables

The λ-tree syntax approach identifies types with syntactic categories. Primitive types usually denote basic syntactic classes such as those for terms, formulas, proofs, contexts, etc. Although it is possible to use more sophisticated types that express dependencies such as "the type of all proofs of formula B" [29], Abella limits itself to simple types. In λ-tree syntax, an arrow type denotes just another syntactic class. For example, when encoding first-order logic, one would introduce the primitive types, say, `tm` and `fm`, to denote the syntactic class of (first-order) terms and (first-order) formulas. The arrow type `tm -> fm` would then denote the syntactic category of formulas in which one term variable is abstracted.

Syntactic categories denoted by arrow types play an important role in representing binding constructs in Abella. Consider, for example, the task of encoding the terms of the pure, untyped λ-calculus. We can make use of the type and the constants identified by the following declarations for this purpose:

```
Kind tm   type.
Type abs  (tm -> tm) -> tm.
Type app  tm -> tm -> tm.
```

Note that the constant `abs` that is intended for encoding abstractions takes an argument of arrow type. The intention here is to capture the binding effect of this construct in the language being represented—the untyped λ-calculus—by translating it into an abstraction in the term language of Abella. This is one of the hallmarks of the λ-tree syntax approach. We list below several examples of untyped λ-terms and their representation using the above constants that demonstrate this structure.

$\lambda x.\, x$	`(abs x\ x)`
$\lambda x.\, x\ x$	`(abs x\ app x x)`
$\lambda x.\, \lambda y.\, x$	`(abs x\ abs y\ x)`
$\lambda x.\, \lambda y.\, y$	`(abs x\ abs y\ y)`
$\lambda x.\, \lambda y.\, y\ x$	`(abs x\ abs y\ app y x)`
$\lambda x.\, \lambda y.\, \lambda z.\, x\ z\ (y\ z)$	`(abs x\ abs y\ abs z\ app (app x z) (app y z))`
$(\lambda x.\, x\ x)\ (\lambda x.\, x\ x)$	`(app (abs x\ app x x) (abs x\ app x x))`

Equality between terms in Abella is given by α, β, and η conversion. Noting this fact, it is easy to show that closed terms of type `tm` over the chosen signature correspond directly to the closed terms of the untyped λ-calculus. More precisely, the $\alpha\beta\eta$-equivalence classes of closed terms of type `tm` are in a bijection with α-equivalence classes of closed untyped λ-terms. Later, when we introduce *nominal constants*, this bijection will extend (under a permutation of names) to open untyped λ-terms.

The preferred representation approach in Abella, then, is to encode binding-related aspects of constructs in an object language by using abstraction in the simply typed λ-terms of Abella. The primary virtue of this approach is that it obviates explicit reasoning about binding in the object language: this is accounted for, once and for all, in the logical foundations of Abella. To understand this concretely, let us return to the representation of untyped λ-terms. The equivalence of two such terms under renaming of bound variables can be important to reasoning about them. This equivalence is actually built into the fact that α-conversion holds for Abella terms. Similarly, consider the contraction of a β-redex of the form $((\lambda x.\, P)\ Q)$. To formalize this notion, we need a substitution operation that is capable of replacing free occurrences of x in P with Q while ensuring that no free variables occurring in Q are inadvertently captured by intervening abstractions. Under our representation, the term in question is encoded by an expression of the form `(app (abs P) Q)` and its contraction is given simply by the expression `(P Q)`; the proper realization of substitution is immediate from the understanding of β-convertibility in Abella. Finally, the embedding of simply typed λ-terms within the logic of Abella makes it possible to analyze binding structure in meaningful ways. As a simple example, consider the following definition:

```
Define vac_abs : tm -> prop by
   vac_abs (abs (x\P)).
```

The predicate `vac_abs` that is so defined is capable of recognizing encodings of vacuous abstractions: because P is bound outside of x, the argument to `vac_abs` can only be made equal to an abstraction whose bound variable does not occur in its body. Using it, we can construct proofs such as the following in Abella:

```
Theorem is_vac :
  vac_abs (abs x\ (app (abs y\ y) (abs z\ z))).
search.

Theorem is_not_vac :
  vac_abs (abs x\ (app (abs y\ y) x)) -> false.
intros. case H1.
```

The use of λ-tree syntax that we have illustrated above depends critically on the ability to think of arrow types as corresponding to syntactic categories. It is important to emphasize that this is possible only because of the special properties of the logic underlying Abella. In particular, the term language has limited expressive power so that equality between terms is decidable and unification can be used meaningfully to analyze syntax. Further, although unification in this setting is undecidable in general, the analysis of λ-terms used in syntax representations usually requires a restricted form of unification called (higher-order) *pattern unification* [37]. This unification computation is decidable and constitutes a minimal extension to first-order unification that also treats expressions with bound variables modulo the equality theory of $\alpha\beta\eta$-conversions. Many of these properties are lost if the logic is extended to encompass a richer, more mathematical, notion of equality between functions such as that used, for example, in Church's original formulation of higher-order logic [12].

6.2 The ∇ quantifier and nominal constants

The definition of relations over objects that contain binding often involves a recursion that results in a need to examine *open* terms. For example, consider the relation between (untyped) λ-terms and types. This relation is usually defined by rules such as the following:

$$\frac{\Gamma \vdash t_1 : \alpha \to \beta \qquad \Gamma \vdash t_2 : \alpha}{\Gamma \vdash (t_1\ t_2) : \beta} \qquad\qquad \frac{\Gamma, x{:}\alpha \vdash t : \beta}{\Gamma \vdash (\lambda x.\, t) : \alpha \to \beta}\, x \notin \mathrm{dom}(\Gamma)$$

Observe here that in assigning a type to an expression of the form $(\lambda x.\, t)$ it is necessary to analyze the term t in which x may appear free. To deal with this requirement, the typing relation is generalized to include a *context* in addition to a term and a type: the context constitutes the place to look up the types of the free variables in the term. Notice also that the rule for typing abstractions implicitly assumes a renaming to ensure that the names chosen for the free variables are kept distinct.

Abella provides a logic-based means for treating such relations. A key component of this approach is the ∇ quantifier first introduced by Miller and Tiu [42, 43]. More precisely, the logic underlying Abella includes formulas of the form $\nabla x.\, F$, where x is a variable of some type that appears in F and F is itself a well-typed formula. The logical meaning of such a formula can be understood intuitively as follows: to construct a proof of $\nabla x.\, F$, we need to pick a new constant that does not appear in F and then prove the formula that results from instantiating x in F with this constant. The constants that are to be used in this way are called *nominal constants* and they are depicted in Abella proofs by tokens that consist of the letter `n` followed by digits, i.e., by tokens such as `n1`, `n2`, `n3`, etc. These constants have special properties that we shall discuss presently. One property that is of immediate importance is that each of them has a distinct identity within a formula, and the logic cements their inequality to any other nominal constant appearing in that formula. Such nominal constants are, therefore, different from eigenvariables since they are maintained as distinct by the logic. In particular, both $\mathtt{n1} \neq \mathtt{n2}$ and $\nabla x \nabla y.\ x \neq y$ are provable in Abella while $\forall x \forall y.\ x \neq y$ is not a theorem in Abella.

Using the ∇ quantifier, it is possible to translate the typing relation shown above into an Abella definition. For concreteness, let us assume that the permitted types are those formed by the arrow type constructor from a single base type a. The following declarations then provide an encoding of these types and culminate in a definition of the predicate `of` that formalizes the typing judgment in Abella:

```
Kind ty   type.
Type a    ty.
Type arr ty -> ty -> ty.

Kind vty type.
Type vty tm -> ty -> vty.

Kind assignment  type.
Type vtynil      assignment.
Type vtycons     vty -> assignment -> assignment.

Define varof : tm -> ty -> assignment -> prop by
  varof X Ty (vtycons (vty X Ty) Ass);
  varof X Ty (vtycons VT Ass) := varof X Ty Ass.

Define of : assignment -> tm -> ty -> prop by
  of Gamma X Ty := varof X Ty Gamma;
  of Gamma (app T1 T2) Ty :=
    exists Ty', of Gamma T1 (arr Ty' Ty) /\ of Gamma T2 Ty';
  of Gamma (abs T) (arr Ty1 Ty2) :=
      nabla x, of (vtycons (vty x Ty1) Gamma) (T x) Ty2.
```

The clause to focus on in the definition of `of` is the one for typing abstractions. The body of this clause uses a ∇ quantified formula: an expression of the form $(\nabla x.\, E)$ is written as `(nabla x, E)` in Abella. Viewed procedurally, this clause states that to type an abstraction it is necessary to introduce a new nominal constant that is by design distinct from all others appearing in the formula at that point, to assign it the argument type and, finally, to type the (open) body of the abstraction using the new constant as the name for the abstracted variable. This example vividly illustrates the paradigm of mobility of binders that is used in Abella in formalizing properties of binding constructs in an object language: to type an abstraction in the encoded λ-calculus, we move the λ-binder corresponding to it first into a ∇ quantifier and then into a proof level nominal constant.

It is useful to keep in mind some properties of nominal constants in understanding proofs in Abella that involve such constants. One property that we have already mentioned is that distinct nominal constants in a formula are considered unequal. This property is embodied in the fact that the formula $(\nabla x.\, \nabla y.\, x \neq y)$ is a theorem in the $\mathcal{G}$ logic underlying Abella. Another property is that the scope of these constants is limited to particular formulas. More precisely, two formulas are considered equal if they can be made identical by applying a permutation of nominal constants to one of them. This property is a consequence of three facts about the ∇ quantifier in Abella. First, it admits a strengthening principle: $(\nabla x.\, P)$ is equivalent to P if x does not appear free in P. Second, it admits an interchange principle: $(\nabla x.\, \nabla y.\, P)$ is equivalent to $(\nabla y.\, \nabla x.\, P)$. Finally, the ∇ quantifier distributes over all the propositional connectives. These facts actually underlie the depiction of goals in Abella in the form

`Variables:` $x_1 \ldots\ x_m$

`H1 :` B_1

$\vdots$

`Hn :` B_n

`============================`

B_0

Nominal constants may appear in the formulas $B_1, \ldots, B_n$ and their scopes are then local to each of them but, unlike for the eigenvariables $x_1, \ldots, x_n$ that are scoped over the entire goal, no explicit binder is shown for these constants. Note also that the narrower scope means that if a particular nominal constant, say `n1`, appears in more than one hypothesis and/or the conclusion, then it may be selectively renamed in any one of these formulas in the course of constructing a proof.

The properties we have described for the ∇ quantifier lead to some theorems concerning it that may seem surprising at first. For example, consider proving the following in Abella:

```
  Theorem nabla-drop : (nabla x y, (p x -> q y)) ->
                       (nabla x, p x) -> (nabla y, q y).
```

After the application of the tactics

```
  intros. case H1. case H2. apply H3 to H4.
```

the prover will be in a state given by the following:

```
nabla-drop <

H3 : p n1 -> q n2
H4 : p n1
H5 : q n2
============================
 q n1

nabla-drop <
```

The search command will now complete the proof since the conclusion of the goal is equal, modulo the renaming of nominal constants, to hypothesis H5. The proof we have presented also makes it clear why the theorem in question holds: the distributivity property allows $\nabla x.\, \nabla y.\, ((p\ x) \supset (p\ y))$ to be transformed into $(\nabla x.\, \nabla y.\, (p\ x)) \supset (\nabla x.\, \nabla y.\, (p\ y))$ which, using the strengthening property, can be simplified to $(\nabla x.\, (p\ x)) \supset (\nabla y.\, (p\ y))$.

The strengthening property for ∇ leads naturally to the conclusion that there is a denumerably infinite collection of nominal constants at any type. A consequence of this is that it is possible to prove that any given type is inhabited and that there are a particular number of distinct inhabitants at that type. For example, assuming i to be a primitive type, the following are theorems: the proofs indicate how the existence of an infinite number of nominal constants plays into the derivability of these assertions.

```
Theorem non-empty : exists (x : i), true.
  witness n1. search.

Theorem at-least-3 : exists (x : i) y z,
   ((x = y) -> false) /\ ((x = z) -> false) /\
   ((y = z) -> false).
witness n1. witness n2. witness n3. split.
intros. case H1. intros. case H1. intros. case H1.
```

At this point, we can make some (rather advanced) considerations regarding extensionality. We claim that $\forall x.\, (\lambda w.\, x) \neq (\lambda w.\, w)$ is a theorem in $\mathcal{G}$, for x of any type that makes the formula well-formed. This might be surprising, because that statement is invalid when considered over the singleton domain, where there is only one function. However, we have just seen that any type in $\mathcal{G}$ is inhabited by infinitely many nominal constants. In effect, this allows the logic to treat λ-abstraction just as a syntactic constructor, and use unification on abstraction terms just like on any other term. As a result, we easily prove the theorem in Abella by introducing x and the equality assumption $(\lambda w.\, x) = (\lambda w.\, w)$ and eliminating that assumption: we obtain a contradiction because no unifier can instantiate the universally quantified variable x in such a way that the λ-term $\lambda w.\, x$ is turned into the identity function $\lambda w.\, w$, variable capture not being allowed.

```
Theorem ex1 : forall (x : i), w\x = w\w -> false.
intros. case H1.
```

Although the previous example illustrates an intensional treatment of λ-terms, we observe next that extensionality is actually provable in $\mathcal{G}$. More precisely, we show in Abella that, when M and N have an arrow type, $M = N$ is equivalent to $\forall x. (M\ x = N\ x)$.

```
Theorem ex2 : forall M (N : i -> i),
                (M = N -> forall x, (M x) = (N x)) /\
                ((forall x, (M x) = (N x)) -> M = N).
intros. split. intros. case H1. search.
               intros. apply H1 with x = n1. search.
```

The key step in the above proof is when the universally quantified variable x is instantiated by a fresh name `n1`. In fact, the above proof can be seen as reducing the extensionality principle to its variant expressed with ∇ instead of universal quantification, which is also provable in Abella.

```
Theorem ex3 : forall M (N : i -> i),
                 (M = N -> nabla x, (M x) = (N x)) /\
                 ((nabla x, (M x) = (N x)) -> M = N).
intros. split. intros. case H1. search.
               intros. case H1. search.
```

As another property of Abella, observe that $\forall x.\nabla w. x \neq w$ is provable as also is $\nabla x.\nabla w. x \neq w$, as we have already observed.

```
Theorem ex4 : forall (x : i), nabla w, x = w -> false.
intros. case H1.

Theorem ex5 : nabla (x : i), nabla w, x = w -> false.
intros. case H1.
```

Notice also that the formula $(\nabla x. \nabla y. p\ x\ y) \supset (\nabla z. p\ z\ z)$ is not provable in general. If it were provable, then replacing the predicate p with the expression $\lambda x. \lambda y. x \neq y$ would lead to a contradiction. However, if the three occurrences of ∇ in this formula are replaced by $\forall$, then the resulting formula is a theorem of Abella.

6.3 Induction in the presence of the ∇ quantifier

The style of inductive proofs that is based on using annotations can be used also when formulas contain the ∇ quantifier. As an illustration of this, let us consider proving the following formula, assuming the definition of the `varof` predicate seen in the previous subsection.

```
Theorem varof_inst : forall Gamma T Ty N, nabla (x:tm),
varof (Gamma x) (T x) (Ty x) -> varof (Gamma N) (T N) (Ty N).
```

Note that the ordering of the quantifiers means that, in generating instances of the formula, `Gamma`, `T` and `Ty` must be substituted for by terms that *do not* contain the nominal constant that is substituted for `x`. Thus, the property that the formula states can be understood as the following: if a `varof` formula containing the ∇-quantified variable `x` is provable, then the formula that results from it by replacing `x` uniformly by any term is also provable. This kind of instantiation lemma for ∇ does not hold in general. However, we can prove it in this case using an argument that is inductive on the definition of `varof`.

To spell the proof out in detail, it begins with the application of the following tactic commands.

```
induction on 1. intros.
```

The proof state that this results in is the following.

```
Variables: Gamma, T, Ty, N
IH : forall Gamma T Ty N, nabla x,
       varof (Gamma x) (T x) (Ty x) * ->
         varof (Gamma N) (T N) (Ty N)
H1 : varof (Gamma n1) (T n1) (Ty n1) @
============================
 varof (Gamma N) (T N) (Ty N)
```

If we perform a case analysis on H1 at this stage, we will get two subgoals. The first of these subgoals, which results from using the first clause in the definition of varof, has an obvious proof. The second subgoal is the following.

```
Variables: Gamma, T, N, Ass, VT
IH : forall Gamma T Ty N, nabla x,
       varof (Gamma x) (T x) (Ty x) * ->
         varof (Gamma N) (T N) (Ty N)
H2 : varof (Gamma n1) (T n1) (Ass n1) *
============================
 varof (Gamma N) (T N) (vtycons (VT N) (Ass N))
```

We can apply the induction hypothesis IH to H2 if we can instantiate the quantifiers over Gamma, T, Ty, and N in it with the eigenvariables with the same names in the proof state and we instantiate the ∇ quantifier over x with the nominal constant n1. The quantifier ordering in IH requires that Gamma, T and Ty not depend on n1, but the structure of H2 already imposes this requirement. Thus, the instantiation is acceptable. Once IH has been applied to H2, the proof can be completed using the search command. The full proof script for proving the formula is the following.

```
  induction on 1. intros. case H1.
  search. apply IH to H2 with N = N. search.
```

In the example above, we considered using induction to prove a formula that contained the ∇ quantifier in it. An inductive argument can also be used when ∇ quantifiers appear in the bodies of definitions. For an example of such a proof, consider the following instantiation property for the typing judgment of seen in the previous subsection.

```
Theorem of_inst : forall Gamma T Ty N, nabla (x:tm),
  of (Gamma x) (T x) (Ty x) -> of (Gamma N) (T N) (Ty N).
induction on 1. intros. case H1.
  apply varof_inst to H2 with N = N. search.
  apply IH to H2 with N = N. apply IH to H3 with N = N.
      search.
  apply IH to H2 with x = n1, N = N. search.
```

The proof is by induction on the first typing judgment. Performing a case analysis on this judgment after using an intros tactic command gives rise to three subgoals.

The first subgoal, which pertains to typing a variable, is dealt with by using the varof_inst theorem. The second subgoal, which pertains to typing an application, is also easily dealt with. The interesting case is that of typing an abstraction. The proof state at the start of this case has the following form:

```
Variables: Gamma, N, Ty2, Ty1, T1
IH : forall Gamma T Ty N, nabla x,
       of (Gamma x) (T x) (Ty x) * ->
         of (Gamma N) (T N) (Ty N)
H2 : of (vtycons (vty n2 (Ty1 n1)) (Gamma n1))
        (T1 n1 n2) (Ty2 n1) *
============================
 of (Gamma N) (abs (Ty1 N) (T1 N))
              (arr (Ty1 N) (Ty2 N))
```

The hypothesis H2 has a new nominal constant n2 in it; this nominal constant arises from the ∇ quantifier in the body of the clause for of for the case of abstractions. In order to apply IH to H2, we would need to instantiate the quantifiers in it so that its antecedent matches H2. The substitutions needed for this are the following.

```
Gamma = x\ vtycons (vty n2 (Ty1 x)) (Gamma x)
T     = x\ T1 x n2
Ty    = x\ Ty2 x
x     = n1
```

Note that the substitutions for Gamma and T are terms that contain a nominal constant, namely n2. However, this is not a problem. The only requirement arising from the order of the quantifiers in IH is that these substitution terms must not contain n1, the nominal constant instantiating the nabla quantifier in the formula. Since this requirement is met, the substitutions can be used and the proof can be completed as shown.

We have shown here how the form of induction that uses annotations works in an enriched setting where we have ∇ quantifiers and nominal constants. The other forms of induction, i.e., the strong induction, mutual induction and nested induction that were discussed in Section 5, follow a similar pattern.

6.4 An enhancement to definitions

Our encoding of the typing relation for λ-terms is parameterized by a context. Each such context has the property that it assigns types only to nominal constants and also that it makes at most one assignment to each such constant. These properties may be needed in certain reasoning tasks. For example, suppose that we change our task to formalizing the typing relation for simply typed λ-terms. This relation can be encoded with a few small changes to the declarations shown earlier. In particular, all we need to do is modify the representation of an abstraction so that it takes a type as an additional argument and adapt the definition of the predicate of to this new form:

```
Type abs ty -> (tm -> tm) -> tm.

Define of : assignment -> tm -> ty -> prop by
  of Gamma X Ty := varof X Ty Gamma;
  of Gamma (app T1 T2) Ty :=
    exists Ty', of Gamma T1 (arr Ty' Ty) /\ of Gamma T2 Ty';
  of Gamma (abs Ty1 T) (arr Ty1 Ty2) :=
    nabla x, of (vtycons (vty x Ty1) Gamma) (T x) Ty2.
```

The typing relation that is so defined has the property that it associates a unique type with each (well-formed) λ-term. However, to prove a theorem to this effect in Abella, it is necessary to make explicit the properties of contexts that we have just described.

It is possible to capture these properties in Abella but to do so we need to deploy an enhanced form of definitions that is based on using the ∇ quantifier. In the form for definitions we have seen up to this point, the head of a clause may contain variables that are interpreted as being universally quantified over the entire clause. Thus, the clause defines a relation that holds for any instantiation of those variables. In the enhanced form, it is also possible to ∇-quantify variables over the head of the clause. There are two components to the interpretation of these quantifiers. First, they can be instantiated only by distinct nominal constants. Second, the instantiations of the outer universal quantifiers must satisfy scope restrictions: they must not contain the nominal constants used in instantiating the enclosed ∇ quantifiers.

Perhaps the simplest examples of definitions that use ∇ quantifiers in the head of a clause are the following:

```
Define name : tm -> prop by
  nabla x, name x.

Define fresh : tm -> tm -> prop by
   nabla x, fresh x T.
```

The intention here is that `name` holds exactly of the nominal constant of type `tm` and that `fresh` holds of a nominal constant of type `tm` and a term also of type `tm` exactly when the nominal constant does not appear in the term, i.e., it is *fresh to* the term. The following theorems and their associated proofs show that these requirements are in fact encoded by the shown definitions:

```
Theorem nameex1 : name n4.
unfold. search.

Theorem nameex2 : nabla x, name (app x x) -> false.
intros. case H1.

Theorem freshex1 : nabla x y, fresh x (app y y).
intros. unfold. search.

Theorem freshex2 : nabla y,
     fresh y (app (abs x\ app x x) y) -> false.
intros. case H1.
```

Returning to the problem of establishing the uniqueness of type assignments, we can characterize contexts using the extended form of definitions as follows:

```
Define ctx : assignment -> prop by
```

```
    ctx vtynil;
    nabla x, ctx (vtycons (vty x Ty) L) := ctx L.
```

In fact, we can prove formulas in Abella that make the desired properties explicit for any L of which ctx holds. For example, the following theorem establishes the fact that L assigns types only to nominal constants:

```
Theorem ctx_assigns_to_var :
  forall L T Ty, ctx L -> varof T Ty L -> name T.
induction on 2. intros. case H2.
  case H1. search.
  case H1. apply IH to H4 H3. search.
```

Towards showing that such assignments are unique, we first prove that a context assign types to only the variables that appear in it:

```
Theorem must_appear_for_assignment :
  forall L Ty, nabla x,
   varof x (Ty x) L -> false.
induction on 1. intros. case H1. apply IH to H2.
```

Notice how the non-appearance of a variable in a context is captured in this theorem by the order in which the quantifiers over L and x appear: scoping rules prohibit the nominal constant that instantiates x to appear in any term that instantiates L. Using this theorem, it is easy to show that the type that a context assigns to a variable is unique:

```
Theorem assignment_is_unique :
  forall L T Ty1 Ty2, ctx L -> varof T Ty1 L ->
                varof T Ty2 L -> Ty1 = Ty2.
induction on 2. intros. case H2.
  case H3. search.
    case H1. apply must_appear_for_assignment to H4.
  case H3.
    case H1. apply must_appear_for_assignment to H4.
    case H1. apply IH to H6 H4 H5. search.
```

Having established the described properties of typing contexts, we can proceed to prove the uniqueness of type assignment for arbitrary terms. A natural way to organize this proof is to use induction on the definition of the typing relation which effectively evolves into an induction on the structure of terms. The base case in this proof relies on the uniqueness property that we have just established for the assignments made by a context. When treating the case of an abstraction, it is necessary to consider the extension that is made to a context. In particular, we must show that the addition of a type assignment for the bound variable produces a structure that continues to satisfy the properties defined by ctx. However, this turns out to be easy to do: the use of a ∇ quantifier in the clause that defines of in this case blends into the definition of ctx in just the expected way. The actual proof is shown below.

```
Theorem type_uniqueness :
  forall L T Ty1 Ty2, ctx L ->
    of L T Ty1 -> of L T Ty2 -> Ty1 = Ty2.
induction on 2. intros. case H2.
  apply ctx_assigns_to_var to H1 H4. case H5.
   case H3. apply assignment_is_unique to H1 H4 H6. search.
  case H3.
    apply ctx_assigns_to_var to H1 H6. case H7.
    apply IH to H1 H4 H6. search.
  case H3.
    apply ctx_assigns_to_var to H1 H5. case H6.
    assert (ctx (vtycons (vty n1 Ty3) L)). apply IH to H6 H4
        H5.
     search.
```

This theorem can be specialized to a closed term by picking the (initial) context to be the empty assignment.

All the definitions in the examples we have considered here have had exactly one ∇ quantifier in the head of the clause. It is, of course, possible for there to be several such quantifiers in the head. In this case there may be more than one way to match the variables bound by these quantifiers with the nominal constants that appear in a given atomic formula and we must consider all of them in constructing a proof. More formally, we must use *equivariant unification* [11] to decide all the possible premises to generate when case-analyzing a hypothesis, and equivariant matching to decide all the possible unfoldings of its conclusion. Such unification/matching can be costly. It is therefore important to think carefully about the structure of definitions so that these quantifiers are kept to a minimum. As a practical matter, we have not found it necessary to use more than one or two such quantifiers in the head in all the (meaningful) proofs we have considered.

7. EXTENDED EXAMPLES

We have at this point described all the features of the logic underlying Abella. We now consider a few examples to bring out the use of this logic and its realization in Abella in practical reasoning tasks. One of the strengths of Abella is that it can be used to transparently encode formal systems and to provide succinct proofs of their meta-theoretic properties. We illustrate this aspect here by considering the formalization of three different systems that are of wider interest: the untyped λ-calculus, a sequent calculus and the π-calculus.

7.1 The untyped λ-calculus

The encoding of the untyped λ-calculus introduced in Section 6.1 uses two constructors, `app` and `abs`, to build λ-terms. One often needs to reason by induction on the syntactic structure of λ-terms. The type system of Abella has no built-in induction principles, so such inductive arguments must be mediated by an inductive definition `is_tm` of the structure of λ-terms, just as we did with natural numbers (`is_nat`) and lists (`is_list`) earlier.

One difficulty with describing the inductive structure of λ-terms, that was not present in the context of natural numbers and lists, is that we have to contend with syntax that involves a binding construct. Given the λ-tree representation that we are using, the particular requirement that we need to capture is the following: an expression of the form `(abs R)` must be recognized as a well-formed term provided

R has the structure x\ R' where R' itself is a well-formed term under the relaxed assumption that x may appear at places within it. The nabla quantifier provides a critical part of the machinery needed for expressing such analysis. In particular, it allows for the replacement of the meta-level bound variable with a nominal constant. Such a replacement then yields a simpler syntactic structure over which the needed recursive analysis can be done.

There are two ways to realize the idea described above and to thereby define an is_tm predicate. In the first way, we generalize the relation we want into a binary relation, is_tm' : tmlist -> tm -> prop, where the first argument tracks the nominal constants allowed to appear in the second argument. Each time a λ-abstraction is encountered, a new nominal constant is chosen, it is used to replace the bound variable in the body of the abstraction and also added to the list before we descend into analyzing the body. If a nominal constant is encountered, it is considered a legitimate term only if it also appears in the list. Finally, a term is well-formed and closed only if it can be recognized to be well-formed relative to an empty list.

```
Kind tmlist type.
Type tmnil  tmlist.
Type tmcons tm -> tmlist -> tmlist.

Define tmmemb : tm -> tmlist -> prop by
  tmmemb X (tmcons X G) ;
  tmmemb X (tmcons Y G) := tmmemb X G.

Define is_tm' : tmlist -> tm -> prop by
  is_tm' G X := tmmemb X G ;
  is_tm' G (app M N) := is_tm' G M /\ is_tm' G N ;
  is_tm' G (abs R) := nabla x, is_tm' (tmcons x G) (R x).

Define is_tm : tm -> prop by
  is_tm T := is_tm' tmnil T.
```

This kind of definition is similar to the typing relation of from Section 6.4. However, it tends to be rather heavyweight for the simple case of *untyped* λ-terms, where nothing needs to be assumed about the bound variables besides that they exist. Informal ("pen-and-paper") descriptions of the simply typed λ-calculus therefore dispense with such variable lists entirely and use open λ-terms.

Fortunately, since Abella already has an infinite set of nominal constants at any type, we can use them to encode open λ-terms as well. To be precise, we do not add a new constructor for λ-terms for every nominal constant of type tm; we merely extend the is_tm definition with an additional clause that declares all nominal constants of type tm to be λ-terms.

```
Define is_otm : tm -> prop by
  nabla x, is_otm x ;
  is_otm (app M N) := is_otm M /\ is_otm N ;
  is_otm (abs R) := nabla x, is_otm (R x).
```

This kind of definition avoids the problem of proliferating variable lists while still being stratified. As an illustration, to show is_otm (abs x\ x), we unfold the third clause to change the goal to is_otm n1, which then matches the head of the first clause. Note that because the first clause requires the argument to the predicate to be a nominal constant, we rule out possibilities such as showing forall (f:nat -> tm), is_otm (f z).

It is important to note that the definition of is_otm is compatible with the representation of substitution by application in λ-tree syntax. The following theorem establishes the substitution lemma.

```
Theorem is_otm_subst : forall M N, nabla x,
  is_otm (M x) -> is_otm N -> is_otm (M N).
induction on 1. intros. case H1.
  search. search.
  apply IH to H3 H2. apply IH to H4 H2. search.
  apply IH to H3 H2. search.
```

A more interesting illustration of structural induction on this encoding of λ-terms is the proof that the space of λ-terms can be partitioned into *normal* and *non-normal* terms. Non-normal terms are easily defined to be those terms that contain a β-redex.

```
Define non_normal : tm -> prop by
  non_normal (app (abs R) M) ;
  non_normal (app M N) := non_normal M \/ non_normal N ;
  non_normal (abs R) := nabla x, non_normal (R x).
```

Normal terms, on the other hand, have the form $\lambda x_1. \dots \lambda x_n. x\ t_1\ \cdots\ t_m$ where x is either a bound or a free variable and each of the t_j are themselves normal. Such terms can be characterized by means of a pair of mutually inductive relations, normal and neutral, where the latter is used to define the structure of the body of the term after an initial λ-prefix.

```
Define  normal : tm -> prop,
       neutral : tm -> prop  by
  normal M := neutral M ;
  normal (abs R) := nabla x, normal (R x) ;

  nabla x, neutral x ;
  neutral (app M N) := neutral M /\ normal N.
```

We can then show by induction on the structure of open terms that every such term is either normal or non-normal, and that no term can be both normal and non-normal.

```
Theorem tm_split : forall M,
  is_otm M -> normal M \/ non_normal M.
Theorem tm_disjoint : forall M,
  normal M -> non_normal M -> false.
```

The easy proofs are left as an exercise.

7.2 Meta-theory of minimal intuitionistic logic

As we shall show in this section, the reasoning logic $\mathcal{G}$ can be used to capture provability of a first-order (object) logic as an inductive definition and then to prove properties of provability using induction over such definitions. To illustrate how this

Goal reduction rules

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \Rightarrow B} \qquad \frac{x \notin \mathrm{fv}(\Gamma) \qquad \Gamma \vdash A}{\Gamma \vdash \Pi x.\, A} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B} \qquad \frac{}{\Gamma \vdash \mathtt{tt}} \qquad \frac{A \in \Gamma \qquad \Gamma, [A] \vdash c}{\Gamma \vdash c}\,\mathsf{bch}$$

Backchaining rules

$$\frac{}{\Gamma, [a] \vdash a} \qquad \frac{\Gamma \vdash A \qquad \Gamma, [B] \vdash c}{\Gamma, [A \Rightarrow B] \vdash c} \qquad \frac{\Gamma, [[t/x]A] \vdash c}{\Gamma, [\Pi x.\, A] \vdash c} \qquad \frac{\Gamma, [A_i] \vdash c}{\Gamma, [A_1 \,\&\, A_2] \vdash c} \qquad \text{no rule for } [\mathtt{tt}]$$

Fig. 2: The Inference System $\mathcal{M}$

is possible, we will encode a first-order (minimal) intuitionistic logic consisting of formulas (written $A, B, C, \ldots$) constructed from atomic formulas (written $a, b, \ldots$) using the connectives Π, $\Rightarrow$, &, and `tt`. To specify object-level formulas, we use the two basic types, `tm` and `fm`, to represent terms and arbitrary formulas respectively. The `tm` type is kept abstract and the proof system is complete with respect to any signature extension with new constants of target type `tm`: object-level predicates and function systems will both be seen as being constructors of this one type. The `fm` type, on the other hand, has a finite number of constructors. The definition `is_fm` will enable proofs by induction on the structure of formulas.

```
Kind tm, fm type.

Type atm  tm -> fm.
Type imp  fm -> fm -> fm.
Type fall (tm -> fm) -> fm.
Type and  fm -> fm -> fm.
Type top  fm.

Define is_fm : fm -> prop by
  is_fm (atm P) ;
  is_fm (imp A B) := is_fm A /\ is_fm B ;
  is_fm (fall A)  := forall x, is_fm (A x) ;
  is_fm (and A B) := is_fm A /\ is_fm B ;
  is_fm top.
```

The proof system we use for this logic is derived from *uniform proofs* [40], which corresponds to a focused sequent calculus [1] where all atoms and all connectives have a negative *polarity* [10, 33]. (Note that the formulas we are encoding comprise that subset of hereditary Harrop formulas that can also be used as goal formulas [40].) This proof system, which we call $\mathcal{M}$, uses two kinds of *sequents*:

$$\begin{array}{ll} \Gamma \vdash C & \text{(goal reduction sequents)} \\ \Gamma, [A] \vdash c & \text{(backchaining sequents)} \end{array}$$

where, in each case, Γ is a multiset of formulas, and A and C are arbitrary formulas, and c is an atomic formula. Observe that the conclusions of backchaining sequents are always atomic. The inference rules of the system are depicted in Figure 2. This system is sound and complete with respect to the ordinary *LJ* sequent calculus [33].

The first step we take to encode the inference rules of $\mathcal{M}$ as an inductive definition in Abella involves encoding the contexts that are used on the left-hand side of sequents. Such contexts Γ are encoded using the following type `ctx` and a membership relation `mem`:

```
Kind ctx  type.
Type emp  ctx.
Type add  fm -> ctx -> ctx.

Define mem : fm -> ctx -> prop by
  mem A (add A G) ;
  mem A (add B G) := mem A G.
```

Sequents are represented with the inductively defined relations `red` and `bch`, for goal reduction and backchaining sequents, respectively. To anticipate the later meta-theoretic development, we use an additional `nat` argument to these relations to denote their *height*, i.e., the number of iterated unfoldings of `red` or `bch` used to derive that sequent.[7]

```
Define red : nat -> ctx -> fm -> prop,
       bch : nat -> ctx -> fm -> tm -> prop
by
  red (s N) G (imp A B) := red N (add A G) B ;
  red (s N) G (fall A)  := nabla x, red N G (A x) ;
  red (s N) G (and A B) := red N G A /\ red N G B ;
  red N     G top := is_nat N ;
  red (s N) G (atm P) := exists A, mem A G /\ bch N G A P ;

  bch N     G (atm P) P := is_nat N ;
  bch (s N) G (imp A B) P := red N G A /\ bch N G B P ;
  bch (s N) G (fall A) P := exists t, bch N G (A t) P ;
  bch (s N) G (and A B) P := bch N G A P \/ bch N G B P.
```

It is fairly straightforward to see that the sequent $\Gamma \vdash B$ is provable if and only if there is a natural number n such that `red` n Γ B is true. Similarly, the sequent $\Gamma, [A] \vdash B$ is provable if and only if there is a `nat` n such that `bch` n Γ A B is true.

An important point to note regarding the definition of goal-reduction is that the clause for the universal quantifier `fall` (the second clause of `red`) uses the ∇ quantifier and not the universal quantifier of $\mathcal{G}$. Using ∇ in this way corresponds directly to the use of *eigenvariables* in proof systems originating with Gentzen [24] and it makes it possible to prove that certain sequents are not provable. For example, the following theorem states that the formula $\Pi x.\, \Pi y.\, \mathtt{p}\ x \Rightarrow \mathtt{p}\ y$ is not provable.

```
Type p   tm -> tm.

Theorem not_provable : forall N,
  red N emp (fall x\ fall y\
            imp (atm (p x)) (atm (p y))) -> false.
intros. case H1. case H2. case H3. case H4.
case H5. case H6. case H7.
```

Indeed, there is no proof of the formula $\Pi x.\, \Pi y.\, \mathtt{p}\ x \Rightarrow \mathtt{p}\ y$ in the sequent calculus. If the ∇ quantifier in the definition of `red` were to be replaced by the universal

[7] Instead of decrementing this count on every clause, we can also decrement it only on the final clause for `red`, in which case it would denote the *backchaining height*; it turns out that this coarser count would have been sufficient as well for all the meta-theory of $\mathcal{M}$, but we intentionally choose the finer measure here.

quantifier, then this theorem would only be provable if more information about the type `tm` were to be known. For instance, if `tm` had only zero or one inhabitant, then the theorem would be true, but it would not be true otherwise. The proof using ∇ is much closer to the spirit of how sequent calculus proofs are specified.

A consequence of this complete specification of *provability* rather than *truth* is that we we can analyze the derivations to derive consequences. As an example, suppose the `tm` type had the following constructors.

```
Type a,b   tm.
Type f     tm -> tm -> tm.
Type q     tm -> tm -> tm -> tm.
```

Consider the following *clauses* for `q`:

$$\underbrace{\Pi x.\,\Pi y.\,\mathtt{q}\ x\ x\ y}_{P_1},\quad \underbrace{\Pi x.\,\Pi y.\,\mathtt{q}\ x\ y\ x}_{P_2},\quad \text{and}\quad \underbrace{\Pi x.\,\Pi y.\,\mathtt{q}\ y\ x\ x}_{P_3},$$

and the derivability of the following formula from these three clauses:

$$\underbrace{\Pi u.\,\Pi w.\,\mathtt{q}\ (\mathtt{f}\ t_1\ u)\ (\mathtt{f}\ t_2\ w)\ (\mathtt{f}\ t_3\ w)}_{G}$$

for some arbitrary terms t_1, t_2, and t_3. If $P_1, P_2, P_3 \vdash G$ is provable in the sequent calculus of Figure 2, then it must be because P_3 and not P_1 or P_2 was used to derive G, which must mean that $t_2 = t_3$. This is readily shown in Abella.

```
Theorem two_out_of_three : forall P1 P2 P3 Gam T1 T2 T3 G,
   (P1 = fall x\ fall y\ atm (q x x y)) ->
   (P2 = fall x\ fall y\ atm (q x y x)) ->
   (P3 = fall x\ fall y\ atm (q y x x)) ->
   (Gam = add P1 (add P2 (add P3 emp))) ->
   (G = fall u\ fall w\
          atm (q (f T1 u) (f T2 w) (f T3 w))) ->
   (exists N, red N Gam G) -> T2 = T3.
```

The proof is left as an exercise; besides `intros`, it only requires `case` and `search`.

More generally, we can establish several key meta-theoretic properties of the $\mathcal{M}$ proof system by induction on the `red` and `bch` definitions. The first of these are the theorems that together establish that weakening, contraction, exchange, and instantiation of eigenvariables are admissible for $\mathcal{M}$ sequents and that they do not change the heights of proofs.

```
Theorem monotone :
   (forall N G D C, (forall A, mem A G -> mem A D) ->
      red N G C -> red N D C)
/\ (forall N G D F P, (forall A, mem A G -> mem A D) ->
      bch N G F P -> bch N D F P).

Theorem inst :
   (forall N G C, nabla (x:tm), red N (G x) (C x) ->
      forall t, red N (G t) (C t))
/\ (forall N G A P, nabla (x:tm), bch N (G x) (A x) (P x) ->
      forall t, bch N (G t) (A t) (P t)).
```

We can also establish cut-admissibility for this inference system. To state the theorem, we need to define a context that consists of an extension of another context with a single formula. We write this using the `extend` relation defined as follows:

```
Define extend : ctx -> ctx -> fm -> prop by
  extend (add A G) G A ;
  extend (add A G) (add A D) B := extend G D B.
```

Thus, if `extend G D A`, then `G` stands for `D` extended with `A`. The statement of cut-admissibility uses `extend` for the cut formulas.

```
Theorem cut :
   (forall G D A C N1 N2, is_fm A -> extend G D A ->
      red N1 D A -> red N2 G C ->
      exists N3, red N3 D C)
/\ (forall G D A F P N1 N2, is_fm A -> extend G D A ->
      red N1 D A -> bch N2 G F P ->
      exists N3, bch N3 D F P)
/\ (forall G A P N1 N2, is_fm A ->
      red N1 G A -> bch N2 G A P ->
      exists N3, red N3 G (atm P)).
```

The proof of this theorem is lengthy, requiring *both* mutual and nested induction, but surprisingly straightforward. The outer induction is on `is_fm A` in each case, while the inner induction is on the hypothesis with the cut-formula as an assumption.

The theorems `inst` and `cut` provide powerful ways to reason about object-level provability. In Section 8, we will describe a feature of Abella—its support for the *two-level logic approach*—which directly implements a proof system for the logic and inference rules in this section. As we shall see there, the `inst` and `cut` theorems will give rise to two important and powerful tactics of the same names.

7.3 The π-calculus

We now consider an encoding of the π-calculus [45, 46], covering its operational semantics and some of its meta-theory. We assume that the reader is familiar with the syntax of the π-calculus and its operational semantics, so we shall not describe these in detail. An important feature of the π-calculus is that it allows communication channels to be created and communicated among processes. This enables one to model the notion of *link mobility* [45]. More specifically, communication channels are modelled as *names* and link mobility is modelled via the scoping of names and *scope extrusion*. The role of names is more apparent when considering the behavioral equivalence of processes, formulated in terms of *bisimulation* relations. Unlike the case with CCS [44], there is not a canonical notion of bisimulation in the π-calculus; rather, we have different notions arising from the different treatment of the quantification of names and their relative scoping. We show here how these different notions of bisimulations can be captured with ease in Abella. We illustrate this by encoding two different notions of bisimulation: the late [46] and the open bisimulation [57]. Milner et.al. also considered another bisimulation called early bisimulation [46]. We do not explicitly discuss early bisimulation here, but the interested reader will find both its definition and a development of some of its meta-theory in the Abella repository. The proofs of some of the theorems here are rather lengthy so, in most cases, we shall present only the theorems and then discuss some intermediate steps of the proof in order to illustrate the correspondences with informal proof steps found in the literature. The complete proof scripts are available in the Abella repository. The correctness proof of the encoding of bisimulation presented here can be found in [65].

Encoding channel names. A first step to encoding the π-calculus is to decide on how to encode names. Any satisfactory encoding would need to capture the notion of freshness: given any finite set of names, it is always possible to choose a fresh name that is not in that set. This suggests that names should be encoded as nominal constants. Another important aspect of names, as they are used in the definitions of bisimulation, is that of the identity of names. In late bisimulation, free names in processes are treated as constants: their identity and their relationships to other names are fixed throughout the bisimulation game. This entails, among other things, that given any two free names x and y occurring in a late bisimulation, one can always decide whether they are equal or distinct names. In open bisimulation, the role of names is somewhat more ambiguous, as they can vary between variables and constants, depending on the context of their use. The latter is formalized as a notion of *distinction* of names, i.e., an irreflexive finite binary relation on names. Essentially, a pair (x, y) in a distinction specifies that x and y must stay distinct throughout the bisimulation game. Thus given two names x and y occurring in an open bisimulation, we may not always be able to conclude that they are distinct or not, since that would depend on the context in which those names occur, i.e., whether the two names are specifically declared as distinct. As we shall see in an example that follows, the ability to decide the equality (or dually, the distinction) between two names separates open and late bisimulation.

To differentiate the uses of names in late and open bisimulations, we explicitly introduce a unary predicate `name` that essentially enforces that its argument is a nominal constant. We have seen this definition before in Section 6.4, which we repeat here for a different type.

```
Kind nm type.
Define name : nm -> prop by
  nabla x, name x.
```

It is easy to prove that equality is decidable for elements of this predicate:

```
Theorem eq_nm: forall x y, name x -> name y ->
    x = y \/ (x = y -> false).
intros. case H1. case H2.
  right. intros. case H3.
  left. search.
```

One can think of the predicate `name` as specifying a closed data type for names, and the above theorem gives a complete case analysis on this data type. As we shall see later, names occurring in late bisimulation will be explicitly typed, i.e., their occurrences will be guarded by the predicate `name`.

Syntax of processes. We consider only finite processes in this setting: such processes contain neither recursion nor the `!` operator. Processes are built from names and some basic constructors, such as, action prefixes, parallel composition and the non-deterministic choice operator. The action prefix in the π-calculus allows one to specify a process that sends a name or receives a name on a communication channel. The prefixed processes in the π-calculus can be a free-action prefixed process, e.g., $\bar{a}b.P$ (output of name b on channel a), or an input-prefixed process, e.g., $a(x).P$ (input a name on channel x and binds it to x). For convenience, we also include a τ-action prefix, to denote internal transition. This is strictly speaking not necessary, but it simplifies the discussions on some examples below. An input-prefixed process can be seen as a λ-abstraction and is thus encoded using the same technique as we have seen in the λ-calculus example, i.e., by using the λ-tree syntax. The signature of the process constructors is given below.

```
Kind proc   type.
Type null         proc.
Type taup         proc -> proc.
Type plus, par    proc -> proc -> proc.
Type match, out   nm -> nm -> proc -> proc.
Type in           nm -> (nm -> proc) -> proc.
Type nu           (nm -> proc) -> proc.
```

The constructor `null` represents the ‘zero’ (or deadlocked) process, which does nothing. Action prefixes are encoded using three constructors: `taup` for the τ-prefix, `out` for the (free) output prefix, and `in` for the input prefix. The constructors `match`, `plus`, `par` and `nu` represent, respectively, the match prefix, the non-deterministic choice operator, the parallel composition operator and the ν-operator of the π-calculus.

Operational semantics. There are a couple of ways to specify the operational semantics of the π-calculus: the *early* way and the *late* way [58]. The main difference is in the formulation of the input transition relation. In the early version, an input-prefixed process such as $a(x).P$ can evolve into $P[a/x]$ for every name a, so the transition relation is infinite-branching (one branch for every value of x). In the late version, the parameter x is not instantiated in the transition relation; rather, it allows the transition relation to relate a process and an abstraction. Thus in the late version, the continuation of $a(x).P$ is essentially an abstraction $\lambda x.\, P$. The instantiations of x are instead done when defining the (bi)simulation relations between processes. We consider here only the late version of the operational semantics as it leads to simpler formulations of late/open bisimulations. Our encoding of this late semantics in Abella follows directly from the encoding proposed in [36]. The adequacy of this encoding is discussed in detail in [36, 68].

The encoding of the late operational semantics of the π-calculus is given in Figure 3. The predicate `one` encodes the free transition relation and `oneb` encodes the bound transition. Note that in addition to bound input transition, the π-calculus also features a bound output transition. However, unlike the bound input transition, in the bound output transition $P \xrightarrow{\bar{a}(x)} Q$, the parameter x can be replaced only with fresh names that are not free in P. As we shall see, the distinction between the bound input and bound output in the definition of bisimulation is captured by the use of different quantifiers: in the input case, the parameter x is replaced by a universally quantified variable, where as in the output case, it is replaced by a ∇-quantified variable.

```
Kind action        type.
Type tau           action.
Type up, dn        nm -> nm -> action.

Define
  one  : proc -> action  -> proc  -> prop,
  oneb : proc -> (nm -> action) -> (nm -> proc) -> prop
by
  oneb (in X M) (dn X) M ;
  one (out X Y P) (up X Y) P ;
  one  (taup P) tau P ;

  one  (match X X P) A Q := one  P A Q ;
  oneb (match X X P) A M := oneb P A M ;

  one  (plus P Q) A R := one  P A R ;
  one  (plus P Q) A R := one  Q A R ;
  oneb (plus P Q) A M := oneb P A M ;
  oneb (plus P Q) A M := oneb Q A M  ;

  one  (par P Q) A (par P1 Q) := one P A P1 ;
  one  (par P Q) A (par P Q1) := one Q A Q1 ;
  oneb (par P Q) A (x\par (M x) Q) := oneb P A M ;
  oneb (par P Q) A (x\par P (N x)) := oneb Q A N ;

  one  (nu x\P x) A (nu x\Q x) :=
    nabla x, one  (P x) A (Q x) ;
  oneb (nu x\P x) A (y\ nu x\Q x y) :=
    nabla x, oneb (P x) A (y\ Q x y) ;
  oneb (nu x\M x) (up X) N :=
    nabla y, one (M y) (up X y) (N y) ;

  one (par P Q) tau (nu y\ par (M y) (N y)) :=
    exists X, oneb P (dn X) M /\ oneb Q (up X) N ;
  one (par P Q) tau (nu y\ par (M y) (N y)) :=
    exists X, oneb P (up X) M /\ oneb Q (dn X) N ;
  one (par P Q) tau (par (M Y) T) :=
    exists X, oneb P (dn X) M /\ one Q (up X Y) T ;
  one (par P Q) tau (par R (M Y)) :=
    exists X, oneb Q (dn X) M /\ one P (up X Y) R.
```

Fig. 3: The specification of actions and the one-step predicates.

Late bisimulation. Milner et.al. [46] defined a bisimulation as a symmetric relation which is also a simulation relation. Encoding this definition directly in Abella would require an encoding of the notion of a set or a relation together with its associated theory. Here we follow an alternative approach that avoids specifying symmetry directly, but instead builds it into the definition of bisimulation indirectly [58]. This in turn avoids the problem of having to formalize set theory explicitly, at least for the purpose of presenting bisimulation and a few of its properties that we discuss here. The definition of late bisimulation is given in Figure 4.

Note that in the clauses for bound input transitions, the abstraction `M` and `N` are applied to a universally quantified name `W`. Note also that the name `W` is constrained

```
CoDefine lbisim : proc -> proc -> prop by
  lbisim P Q :=
    (forall A P1, one P A P1 ->
       exists Q1, one Q A Q1 /\ lbisim P1 Q1) /\
    (forall X M, oneb P (dn X) M ->
       exists N, oneb Q (dn X) N /\
         forall W, name W -> lbisim (M W) (N W)) /\
    (forall X M, oneb P (up X) M ->
       exists N, oneb Q (up X) N /\
         nabla w, lbisim (M w) (N w)) /\
    (forall A Q1, one Q A Q1 ->
       exists P1, one P A P1 /\ lbisim P1 Q1) /\
    (forall X N, oneb Q (dn X) N ->
       exists M, oneb P (dn X) M /\
         forall W, name W -> lbisim (M W) (N W)) /\
    (forall X N, oneb Q (up X) N ->
       exists M, oneb P (up X) M /\
         nabla w, lbisim (M w) (N w)).
```

Fig. 4: The specification of late bisimulation.

to `name W`, so it ranges over nominal constants.

Open bisimulation. Open bisimulation is encoded via a co-inductively defined predicate `obisim`, which has the same type as `lbisim`. The body of the definition of `obisim` is just that of `lbisim` but with all occurrences of `name W` removed. Open bisimulation, as defined in the literature [57], is actually a family of relations, indexed by a distinction relation on names, which is an irreflexive finite binary relation on names. An important notion in the definition of open bisimulation is that of a *respectful substitution.* A substitution θ *respects* a distinction D if and only if for every $(x, y) \in D$, $\theta(x) \neq \theta(y)$. One of the conditions in the definition of open bisimulation is that the family of relations indexed by distinctions are closed under arbitrary respectful substitutions.

To encode open bisimulation, one obviously needs to encode the notion of distinctions and respectful substitutions. A straightforward approach would be to represent them directly. This would then require one to define the application of substitutions to processes, in order to satisfy the closure condition under respectful substitutions. In essence, this amounts to a deep embedding of distinction and respectful substitutions. While this is certainly doable, there is a simpler alternative approach via a shallow embedding of distinction and respectful substitutions. Such an approach was proposed in [67, 68] and relies on the use of quantifier alternation to encode distinctions. To see how this is done, consider the following formulas, which are both provable in Abella:

```
forall a, nabla b, forall c, (a = b -> false)

forall a, nabla b c, (b = c -> false)
```

In the first formula, the relative scoping of the quantifiers forbids the identification of `a` and `b`, even though `a` is universally quantified. In the second formula, both `b` and `c` are ∇-quantified and thus they cannot be identified as they denote distinct nominal constants. In general, a quantifier prefix

$$Q_1\ x_1. Q_2\ x_2. \dots Q_n\ x_n.$$

where Q_i is either $\forall$ or ∇, can be seen as encoding a distinction D as follows: $(x_i, x_j) \in D$ if and only if $i \neq j$ and

—$Q_i = Q_j = \nabla$, or
—$Q_i = \forall$ and $Q_j = \nabla$ and $i < j$.

Quantifier alternation obviously does not capture all distinction relations, but as was shown in [68], it is enough to consider distinctions generated by quantifier alternation in proving open bisimilarity. Under this implicit encoding, respectful substitutions are represented indirectly via instantiations of eigenvariables in proof search. Details of the correspondence between the explicit representation and the implicit representation of distinctions can be found in [68].

Some meta theory. Using co-induction, we can prove some simple properties of bisimulation, such as the fact that open bisimulation implies late bisimulation, and that open (late) bisimulation is an equivalence relation. The proofs, except for the transitivity of bisimulation, consist of simple unfolding of definitions, and applications of the co-induction hypothesis.

```
Theorem obisim-implies-lbisim:
   forall P Q, obisim P Q -> lbisim P Q.

Theorem obisim-refl: forall P, obisim P P.

Theorem obisim-sym: forall P Q, obisim P Q -> obisim Q P.

Theorem obisim-trans:
   forall P Q R, obisim P Q -> obisim Q R -> obisim P R.
```

The proof that bisimilarity is a congruence relation—i.e., it is closed under arbitrary process contexts—is more involved. To show that bisimilarity is a congruence, it is enough to show that it is preserved by simple contexts. We discuss below the two main cases that need to be shown—namely, preservation by parallel composition and by the restriction operator.

To prove that bisimilarity is preserved by parallel composition, we prove a slightly more general statement:

```
Theorem obisim-cong-par: forall P Q R S,
   obisim P Q -> obisim R S -> obisim (par P R) (par Q S).
```

The preservation under parallel composition can then be obtained from the above theorem by letting $R = S$ and observing the fact that bisimilarity is reflexive. Proving this theorem requires accounting for all possible interactions between P and R (and respectively, between Q and S) and how their continuations remain related by bisimilarity. To prove this theorem, we first find a co-inductive invariant, or a post fixed point of the definition of (open) bisimulation, that contains these continuations (among others). Following a textbook proof (see e.g., [58]), we define an invariant that includes closure conditions for both parallel composition and the restriction operator (see [58] for details of why we need to consider both operators simultaneously in the invariant).

```
Define inv : proc -> proc -> prop by
  inv P Q := obisim P Q ;
  inv (par P1 P2) (par Q1 Q2) := inv P1 Q1 /\ inv P2 Q2 ;
  inv (nu P) (nu Q) := nabla x, inv (P x) (Q x).

Define obisimInv : proc -> proc -> prop by
  obisimInv P Q :=
    (forall A P1, one P A P1 ->
       exists Q1, one Q A Q1 /\ inv P1 Q1) /\
    (forall X M, oneb P (dn X) M ->
       exists N, oneb Q (dn X) N /\
          forall W, inv (M W) (N W)) /\
    (forall X M, oneb P (up X) M ->
       exists N, oneb Q (up X) N /\
         nabla w, inv (M w) (N w)) /\
    (forall A Q1, one Q A Q1 ->
       exists P1, one P A P1 /\ inv P1 Q1) /\
    (forall X N, oneb Q (dn X) N ->
       exists M, oneb P (dn X) M /\
         forall W, inv (M W) (N W)) /\
    (forall X N, oneb Q (up X) N ->
       exists M, oneb P (up X) M /\
         nabla w, inv (M w) (N w)).
```

The definition `inv` defines the invariant set. The definition `obisimInv` is similar to the definition of `obisim`, except that the occurrences of `obisim` in the body is replaced by `inv`. If one considers the definition of (open) bisimulation as given by a fixed point operator $\mathcal{F}$, then the definition `obisimInv` encodes the application of $\mathcal{F}$ to `inv`. Thus, to show that `inv` is a post-fixed point of open bisimulation ($\mathcal{F}$), it is sufficient to prove the following statement.

```
Theorem inv_obisimInv: forall P Q, inv P Q -> obisimInv P Q.
```

Once the post-fixed point condition for `inv` is established, we can complete the proof of the previously mentioned theorem.

```
Theorem obisim-cong-par: forall P Q R S,
   obisim P Q -> obisim R S -> obisim (par P R) (par Q S).
intros.
assert inv (par P R) (par Q S).
apply inv_obisim to H3.
search.
```

Since the invariant `inv` is closed under the restriction operator, we also get the following property for free:

```
Theorem obisim-cong-nu:
  forall P Q, nabla n, obisim (P n) (Q n) ->
                       obisim (nu P) (nu Q).
intros.
assert inv (nu P) (nu Q).
```

```
apply inv_obisim to H2.
search.
```

A separating example. To illustrate the difference between late and open bisimulation, consider the following example from [57]. Let $R = a(x).(\tau.\tau + \tau)$ and let $T = a(x).(\tau.\tau + \tau + \tau.[x = b]\tau)$. Then R and T are late bisimilar, but not open bisimilar. We briefly discuss a proof of late bisimilarity of R and T below. The proof relies on a case analysis on a name provided to the input prefix of the processes. A crucial point in the proof is that when one process makes an input transition, one may choose the matching transition of the other process based on the input name, i.e., on whether it is a name already occurring in the processes, or a fresh name. In open bisimulation this is not possible; the matching transition has to be chosen without reference to the identity of the input name.

The statement of the late bisimilarity of R and T is given in the theorem below.

```
Theorem lbisim-R-T:
nabla a b,
lbisim (in a (x\ plus (taup (taup null)) (taup null)))
       (in a (x\ plus (taup (taup null))
                      (plus (taup null)
                            (taup (match x b (taup
                               null)))))).
```

This theorem is proved indirectly by using the following lemma, where the input prefix has been replaced by a universal quantifier in the meta level:

```
Theorem lbisim-all-name:
nabla b, forall x, name x ->
  lbisim (plus (taup (taup null)) (taup null))
         (plus (taup (taup null))
               (plus (taup null)
                     (taup (match x b (taup null))))).
```

In a more readable (informal) notation, the above lemma can be written as $\nabla b. \forall x. \texttt{name}\ x \supset \texttt{lbisim}\ R'\ T'$ where $R' = \tau.\tau + \tau$ and $T' = \tau.\tau + \tau + \tau.[x = b]\tau$. The proof is basically just an exhaustive search: for each transition from R', find a corresponding transition for T', and vice versa, so that their continuations are late bisimilar. We show a case below where the transition from T' has to be matched by a transition from R' so that their continuations remain late bisimilar. Suppose the transition from T' leads to the continuation $[x = b]\tau$ (i.e., R' chooses the third subprocess $\tau.[x = b]\tau$ in its non-deterministic choice). The corresponding subgoal when running the Abella proof script is the following:

```
Variables: x, A, Q1
H1 : name (x n1)
============================
 exists P1,
   one (plus (taup (taup null)) (taup null)) tau P1 /\
   lbisim P1 (match (x n1) n1 (taup null))
```

Here the name `n1` represents the ∇-quantified variable b, and the variable `x` represents the eigenvariable for the universally quantified variable x, but raised to account for its relative scoping with respect to `n1`. The variable `A`, which has been instantiated to `tau`, represents the action taken by process R' and the variable `Q1`, instantiated to `(match (x n1) n1 (taup null))`, represents the continuation of T'.

The matching transition from R' in this case depends on the value of x. If $x = b$, then $[x = b]\tau$ is equivalent to τ, so in this case the transition from R' needs to be derived from the first subprocess in its non-deterministic choice, i.e., the process $\tau.\tau$. If $x \neq b$, then $[x = b]\tau$ is equivalent to the null process, so in this case we choose the second subprocess of R' to drive the transition. The continuations of R' for these cases are τ and 0, respectively.

```
lbisim-all-name < case H1.
Subgoal 4.2.2.1:

Variables: x, A, Q1
============================
 exists P1,
   one (plus (taup (taup null)) (taup null)) tau P1 /\
   lbisim P1 (match n2 n1 (taup null))

Subgoal 4.2.2.2 is:
 exists P1,
   one (plus (taup (taup null)) (taup null)) tau P1 /\
   lbisim P1 (match n1 n1 (taup null))
```

In the first subgoal, the variable `(x n1)` is instantiated with a new nominal constant `n2`. This subgoal can be simplified by instantiating `P1` with `null`, followed by the tactics `split` and `search`, resulting in a new subgoal:

```
Variables: x, A, Q1
============================
 lbisim null (match n2 n1 (taup null))
```

This can then be proved by a simple exhaustive search.

The second subgoal (4.2.2.2) can be simplified by instantiating `P1` with `(taup null)`, followed by the tactics `split` and `search`. These reduce the subgoal to

```
Variables: x, A, Q1
============================
 lbisim (taup null) (match n1 n1 (taup null))
```

which can be proved by an exhaustive search.

Let us now look at why open bisimilarity does not hold between R and T. As with late bisimilarity, proving `obisim R T` in this case reduces to proving a lemma similar to `lbisim-all-name`, but without assuming `name x`:

```
Theorem obisim-for-all:
nabla b, forall x,
  obisim (plus (taup (taup null)) (taup null))
         (plus (taup (taup null))
               (plus (taup null)
                     (taup (match x b (taup null))))).
```

To prove this lemma, we would have a similar subgoal as `Subgoal 4.4.2` above, but without assuming `name (x n1)`:

```
Variables: x
============================
 exists P1,
   one (plus (taup (taup null)) (taup null)) tau P1 /\
   obisim P1 (match (x n1) n1 (taup null))
```

The only way the proof search could proceed is to instantiate `P1` with a process. There are two choices for the instantiations of `P1` that would satisfy the first conjunct in the goal: `(taup null)` or `null`. If we take the first choice, we end up having to prove:

```
Variables: x
============================
 obisim (taup null) (match (x n1) n1 (taup null))
```

Since `x` is an eigenvariable, if this subgoal were provable, any instance of it would also be provable. In particular, the instance

```
obisim (taup null) (match n2 n1 (taup null))
```

where `x` is instantiated to `x\n2`, for some fresh nominal constant `n2`, would also be provable. Since `match n2 n1 (taup null)` is (open/late) bisimilar to `null`, this subgoal would not be provable. So we are left with the only other choice for `P1`, i.e., the process `null`. But in this case, we would have to prove the subgoal:

```
Variables: x
============================
 obisim null (match (x n1) n1 (taup null))
```

If this were provable, then `obisim null (match n1 n1 (taup null))` would also be provable, by letting `x` to be `y\y`. But `match n1 n1 (taup null)` is bisimilar to `(taup null)`, which is not bisimilar to `null`, so this subgoal would not be provable either. Having exhausted all the choices for the instantiation of `P1` we conclude that the original goal `obisim R T` is not provable.

The reader may wonder whether the above argument of non-provability of `obisim R T` could be internalized as a proof within Abella of the negation of `obisim R T`. This is unfortunately not the case. The reason has to do with the shallow embedding approach we have taken in formalizing open bisimulation. To prove the negation of `obisim R T`, we would have to reason about non-provability of continuations of `obisim R T` under some respectful substitutions, and one would need to reason explicitly about such substitutions and their effect on bisimilarity. If we were to use a deep embedding, e.g., by formalizing explicitly the notion of distinctions and respectful substitutions, such a proof may be possible.

8. THE TWO-LEVEL LOGIC APPROACH

As Sections 4, 5, 6, and 7 have illustrated, Abella has sophisticated support for reasoning about relations. This makes it particularly well suited for specifications in the *structural operational semantics* style [54, 55] that use inductively defined relations to specify computational behavior. However, Abella's definitions must be *stratified*: the defined predicates cannot be used negatively in their own definitions. This limitation means that when specifying relations that induct on the structure of higher-order λ-terms used in λ-tree syntax, we cannot rely on *hypothetical reasoning*, as is done in systems such as λProlog [39] or Twelf [50]. Instead, we use an explicit context `Gamma` of typing assumptions, as we have illustrated in Sections 6.2 and 7.2. Such definitions, where an explicit context argument has been added, are accompanied by an obvious problem: the meta-theoretic properties of such lists of assumptions have to be established explicitly by the user. For instance, for the version of the `of` predicate from Section 6.2, we had to manually prove the `of_inst` theorem by induction. Moreover, we often need to know that the proof of this kind of theorem is height-preserving. This height is not part of the obvious encoding of the predicate in $\mathcal{G}$. To make it available in constructing inductive proofs, we would need to extend the definition of the `of` predicate with an extra height parameter, together with its associated assumptions using `is_nat` (cf. Section 2.2). In addition to the extra complexity in the definitions and theorems, we would also need to maintain these proofs throughout the evolution of the system being specified. And, finally, after we are done with one definition, we would need to do everything again for a different definition with different kinds of hypotheses.

The *two level logic approach* [23, 35] is a way to alleviate this tedium of meta-theoretic reasoning by defining a canonical hypothetical inductive definition that can capture a wide spectrum of other definitions using hypothetical reasoning. Specifically, it uses the logic of hereditary Harrop formulas [39, Chapter 3] as a *specification logic* where hypothetical definitions are specified using higher-order clauses that may be freely extended without needing to revisit any meta-theorems.[8] This modularity is achieved by encoding derivability of specification logic sequents as an inductive definition in $\mathcal{G}$, for which the meta-theory is proved *once and for all*, as we have done in Section 7.2.

The following are some of the main benefits of the two-level logic approach.

(1) The specification logic operates under an *open world assumption*, where new clauses can always be added without affecting derivability *in* the specification logic. The *reasoning logic* ($\mathcal{G}$), on the other hand, retains its *closed world* reading where the collection of clauses of an inductive relation are fixed and stable with respect to unfolding. In particular, the reasoning logic can reason about both derivability and *non-derivability* of specification logic formulas and sequents, whereas the specification logic itself has no negation connective.[9]

(2) Because all specifications are factored through a common specification logic, the meta-theoretic properties of the specifications are obtained via the meta-theory of the specification logic itself. Indeed, since this meta-theory is fixed and predictable, the tactics of the reasoning logic can often be optimized to utilize the meta-theorems on the fly and without user direction. This is commonly summarized by the slogan "substitution lemmas *for free*."

(3) Although not a property of the two-level logic approach as such, the specification logic of hereditary Harrop formulas underlies the λProlog programming language.

[8] To be more accurate, the two-level logic approach can be used with any specification logic that has certain meta-theoretic properties. For example, the approach is easily adapted to *linear logic* [35] or intuitionistic dependent type theory [16, 17, 61, 62] as the specification logic.

[9] Note that, in a situation where new clauses can always be added, there is no sensible way to prove $\neg A$, as any such proof will not survive extensions with clauses that prove A.

Thus, specifications can be directly executed using λProlog implementations such as Teyjus [56]. We can thus prove properties of specifications that, under a different view, actually constitute code.

We will introduce the two-level logic approach in stages, starting with the simple case of Horn clause specifications and moving on to more complex higher-order specifications.

8.1 Horn clause specifications

Let us begin with a simple Horn clause specification of appending lists. To avoid namespace collisions with the definitions in Section 4.2 we will use the base types `i` (for individuals) and `ilist` for lists of individuals. The appending operation on `ilist`s will then be given as a specification-level relation `iappend`; more concretely, the specification logic has a type `o` of formulas, and `iappend` will have type `ilist -> ilist -> ilist -> o`. Following λProlog conventions, these specification types are defined in a *signature* file (that must have extension `.sig`), here `ilist.sig`:

```
sig ilist.
kind i       type.
type a,b,c   i. % some individuals
kind ilist   type.
type inil    ilist.
type icons   i -> ilist -> ilist.
type iappend ilist -> ilist -> ilist -> o.
```

The clauses defining `iappend` are given in a corresponding *module* file (with extension `.mod`), here `ilist.mod`:

```
module ilist.
iappend inil L L.
iappend (icons X L) K (icons X M) :-
  iappend L K M.
```

These files follow standard λProlog syntactic conventions of capitalizing the universally quantified variables in a clause and using `:-` to separate the *head* of a clause from its body.

In Abella, this specification can be *imported* by means of the `Specification` top-level command.

```
Abella < Specification "ilist".
Reading specification "ilist"
```

If this command is used, it must be the first one in the run of Abella, because every run of Abella is parameterized on a given specification (which may be empty). The effect of this command is to transport the specification signature, without change or interpretation, into the type system of Abella's reasoning logic. In other words, it is as if the following top-level commands were submitted to Abella.

```
Kind i         type.
Type a, b, c   i.
Kind ilist     type.
/* ... and so on */
```

The clauses from the corresponding .mod are also loaded into memory. (The precise nature of the inclusion of the clauses will be made clear in the next subsection.)

To gain access to derivations in the specification logic, Abella uses the notation {F}, where F is a specification logic formula (i.e., a term of type o), to assert that F is derivable in the specification logic from the loaded clauses. It is possible to use Abella as a logic programming interpreter by using the top-level Query command to search for derivations. Here is an example:

```
Abella < Query {iappend L K (icons a (icons b inil))}.
Found solution:
L = inil
K = icons a (icons b inil)

Found solution:
L = icons a inil
K = icons b inil

Found solution:
L = icons a (icons b inil)
K = inil

No more solutions.
```

The logic programming interpreter executes the query by interpreting the capitalized identifiers as logic variables, and outputs their substitutions for each successful solution. It does not terminate if there are infinitely many solutions. This interpreter is provided primarily as a debugging aid; it is not competitive with sophisticated λProlog execution frameworks.

The more interesting way to use such specifications is to prove properties *about* them. As a first example, let us prove that iappend is deterministic: given two lists L and K, there is always at most one M for which iappend L K M may be derived in the specification logic. We can write this theorem statement fairly directly in the reasoning logic:

```
Theorem iappend_det3 : forall L K M1 M2,
  {iappend L K M1} -> {iappend L K M2} -> M1 = M2.
```

Note that the computational behavior of iappend is stated here in terms of what can be derived in the specification logic from the clauses defining this predicate. The proof of this theorem will accordingly be based on an analysis of such derivations. Informally, we reason by induction on the height of one of the antecedent derivations, such as the first one:

```
iappend_det3 < induction on 1. intros.

Variables: L K M1 M2
IH : forall L K M1 M2, {iappend L K M1}* ->
       {iappend L K M2} -> M1 = M2
H1 : {iappend L K M1}@
H2 : {iappend L K M2}
============================
 M1 = M2
```

Observe that the the inductive restrictions `@` and `*` (cf. Section 5) are attached to the `{}` predicates. The hypotheses `H1` and `H2` are now available for case-analysis; whenever we use a clause for the `iappend` predicate in this analysis, the accompanying `@` annotation (if any) is reduced to a `*`. For didactic purposes, let us turn on the `instantiations` flag so that we can see how the variables get instantiated during case-analysis.

```
Set instantiations on.
```

Case-analysis of `H1` yields two subgoals, one for each clause of `iappend`. The first subgoal results from matching the first clause:

```
iappend_det3 < case H1.

Variables: M1 M2
L <-- inil
K <-- M1
IH : forall L K M1 M2, {iappend L K M1}* ->
       {iappend L K M2} -> M1 = M2
H2 : {iappend inil M1 M2}
============================
 M1 = M2
```

We can then case-analyze `H2` to also instantiate `M2` with `M1` which reduces the subgoal to a trivial equality (that can be finished with `search`).

```
iappend_det3 < case H2.

Variables: M2
L <-- inil
K <-- M2
M1 <-- M2
IH : forall L K M1 M2, {iappend L K M1}* ->
       {iappend L K M2} -> M1 = M2
============================
 M2 = M2
```

The second subgoal results from matching the second clause of `iappend`:

```
Variables: K M2 M L1 X
L <-- icons X L1
M1 <-- icons X M
IH : forall L K M1 M2, {iappend L K M1}* ->
       {iappend L K M2} -> M1 = M2
H2 : {iappend (icons X L1) K M2}
H3 : {iappend L1 K M}*
============================
 icons X M = M2
```

Note here that the inductive restriction on `H3` has reduced from that of `H1`. The proof proceeds by case-analysis of `H2` which also instantiates `M2` to `icons X M3` (for a new variable `M3`) to yield:

```
iappend_det3 < case H2.
```

```
Variables: K M L1 X M3
L <-- icons X L1
M1 <-- icons X M
M2 <-- icons X M3
IH : forall L K M1 M2, {iappend L K M1}* ->
       {iappend L K M2} -> M1 = M2
H3 : {iappend L1 K M}*
H4 : {iappend L1 K M3}
============================
 icons X M = icons X M3
```

Since the inductive restrictions now match, we can `apply IH to H3 H4` to get `M = M3`, from which the conclusion follows trivially.

In effect, the theorem proceeds as if Abella had the following inductive definition at the reasoning level (although the syntax is invalid).

```
Define {iappend} : ilist -> ilist -> ilist -> prop by
  {iappend inil L L} ;
  {iappend (icons X L) K (icons X M)} := {iappend L K M}.
```

The correspondence is strong; if we turn the above invalid syntax into an actual definition `iappend_m` (`m` for *meta*):

```
Define iappend_m : ilist -> ilist -> ilist -> prop by
  iappend_m inil L L ;
  iappend_m (icons X L) K (icons X M) := iappend_m L K M.
```

then we can prove both the following theorems by a simple induction (indeed, the proofs are identical).

```
Theorem spec_meta : forall L K M,
  {iappend L K M} -> iappend_m L K M.
Theorem meta_spec : forall L K M,
  iappend_m L K M -> {iappend L K M}.
```

More generally, for Horn clause specifications it is a matter of taste whether to use the specification or the reasoning logic from the perspective of proving meta-theorems. Of course, if we use the specification logic for such specifications, we have the benefit of being able to execute them using an λProlog implementation; thus our meta-theorems state properties of actual code.

Before proceeding further, the reader is encouraged to attempt proving the following meta-theorems to get a feel for reasoning about specification logic derivations.

```
Theorem iappend_det2 : forall L K1 K2 M,
  {iappend L K1 M} -> {iappend L K2 M} -> K1 = K2.
Theorem iappend_assoc : forall L K M LK KM N1 N2,
  {iappend L K LK} -> {iappend K M KM} ->
  {iappend LK M N1} -> {iappend L KM N2} -> N1 = N2.
```

8.2 Hereditary Harrop specifications

The truly interesting applications of the two-level logic approach arise when specifications go beyond Horn clauses by employing universal quantification and implications

	Abstract	Concrete	Precedence/ Associativity
Formulas $(F, G, \ldots)$			
Atomic formulas	$\mathtt{p}\ m_1\ \cdots\ m_n$	`p M1 ... Mn`	5, left
Implication	$F \Rightarrow G$	`F => G`	4, right
		`G <= F`	3, left
Universal quantification	$\Pi x.\, F$	`pi x\ F`	1
	$\Pi x{:}\tau.\, F$	`pi (x:T)\ F`	1
		(parentheses optional)	
Conjunction	$F \,\&\, G$	`F & G`	2, left
Truth	$\mathtt{tt}$	(no concrete syntax)	
Clauses			

$$\underbrace{\mathtt{p}\ m_1\ \cdots\ m_n}_{\text{head}}\ . \qquad \text{stands for} \qquad \Pi\vec{X}.\ (\mathtt{p}\ m_1\ \cdots\ m_n)$$

$$\underbrace{\mathtt{p}\ m_1\ \cdots\ m_n}_{\text{head}}\ \mathtt{:-}\ \underbrace{F_1\mathtt{,}\ldots\mathtt{,}F_k}_{\text{body}}\ . \qquad \text{stands for} \qquad \Pi\vec{X}.\ (F_k \Rightarrow \cdots \Rightarrow F_1 \Rightarrow \mathtt{p}\ m_1\ \cdots\ m_n)$$

(where $\vec{X}$ are the free capitalized identifiers in the clause)

Fig. 5: Concrete Syntax for hereditary Harrop formulas and specification logic clauses.

in the bodies of clauses. Such specifications are particularly apt for λ-tree syntax, where descending under a λ-abstraction is mirrored by introducing a universally quantified variable to stand for the bound variable in the body of the clause, and where associated properties of that variable are encoded as extra antecedents in an implication. This style specification is best illustrated with an example, for which we revisit the simply typed λ-calculus from Section 6.4, but place the clauses in the specification logic. The signature is:

```
sig stlc.
kind ty  type.
type a   ty.
type arr ty -> ty -> ty.
kind tm  type.
type app tm -> tm -> tm.
type abs ty -> (tm -> tm) -> tm.
```

We will encode the typing relation using the specification logic predicate `of` whose type is given by the following declaration.

```
type of  tm -> ty -> o.
```

Recall that `o` is the type of specification logic formulas. These formulas are composed of atomic formulas built out of predicates such as `iappend` and `of`, and the connectives of the logic of hereditary Harrop formulas defined in Section 7.2. Table 5 summarizes the concrete syntax of specification formulas and clauses that is supported by Abella. Note that this concrete syntax is only a fragment of the full λProlog language; in particular, Abella's specification logic is simply typed, lacks disjunction and existential quantification, and only supports atomic heads in clauses.

Using the syntax for formulas described above, the definition of the `of` predicate is written as follows.

```
module stlc.
of (app M N) B :- of M (arr A B), of N A.
of (abs A R) (arr A B) :- pi x\ of x A => of (R x) B.
```

Each clause of the specification is interpreted as a specification logic formula according to the convention described in Table 5. For `of`, these are:

$$\Pi m, n, a, b.\ (\mathtt{of}\ m\ (\mathtt{arr}\ a\ b) \Rightarrow \mathtt{of}\ n\ a \Rightarrow \mathtt{of}\ (\mathtt{app}\ m\ n)\ b)\,,$$
$$\Pi r, a, b.\ ((\Pi x.\, \mathtt{of}\ x\ a \Rightarrow \mathtt{of}\ (r\ x)\ b) \Rightarrow \mathtt{of}\ (\mathtt{abs}\ r)\ (\mathtt{arr}\ a\ b))\,.$$

Once the specification is imported with

```
Specification "stlc".
```

these clauses are collected and stored in a special list, $\mathcal{P}$, called the *program*. We will therefore use the term *program clause* to mean a member of $\mathcal{P}$.

Since the second clause of the `of` predicate contains an embedded implication, using it to backchain `{of (abs R) (arr A B)}` would yield a specification *sequent* with a localized assumption for a new variable `x`, written `{of x A |- of (R x) B}`. The general form of such a specification sequent is:

$$\{\ F_1, \ldots, F_n\ \texttt{|-}\ G\ \}$$

where the case of $n = 0$ is the familiar form $\{\ G\ \}$ from Horn clause specifications. The multiset of formulas to the left of `|-` is called the *context* of the sequent.

We can now use the `Query` command to animate this specification. Here are a few examples:

```
Abella < Query {of (abs A x\ x) B}.
Found solution:
A = ?5
B = arr ?5 ?5

No more solutions.

Abella < Query {of (abs A x\ app x x) B}.
No more solutions.

Abella < Query nabla x,
  {of x (arr A B) |- of (abs A y\ app x y) D}.
Found solution:
A = ?17
B = ?19
D = arr ?17 ?19

No more solutions.
```

In the first query, the solutions are given in terms of a free logic variable `?5`, meaning that any instance of this variable would produce a corresponding instance of this solution. These logic variables are merely used to display the solutions; they are not part of the syntax of terms and are therefore not available for manipulation by users. The second query asks about the term $\lambda x.\,(x\ x)$ that is ill-typed, and Abella responds with no solutions. The third query contains a nabla-quantified variable x. The logic variables A, B, etc. are existentially quantified at an outermost scope, so the solutions for them cannot depend on x.

It is instructive at this juncture to trace the derivation of the solution of the third query step-by-step.

```
Theorem explore : forall A B, nabla x,
   {of x (arr A B) |- of (abs A y\ app x y) (arr A B)}.
intros.
```

This brings us to:

```
Variables: A B
============================
 {of n1 (arr A B) |- of (abs A (y\ app n1 y)) (arr A B)}
```

We can now match the goal of (abs A (y\ app n1 y)) (arr A B) against the head of the clause for abs in stlc.mod, which instantiates the variable R in the head of the clause with y\ app n1 y. The resulting goal[10] should therefore be:

```
Variables: A B
============================
 {of n1 (arr A B) |- pi z\ of z A => of ((y\ app n1 y) z) B}
```

However, this goal is not in a *reduced* form in the sense of $\mathcal{M}$ from Section 7.2. Abella performs this goal reduction automatically, promoting pi-bound variables to nabla-bound variables, and transferring antecedents of => to the context. Abella then also normalizes β-redexes and introduces the new nabla-quantified variables. The result is:

```
Variables: A B
============================
 {of n1 (arr A B), of n2 A |- of (app n1 n2) B}
```

At this point we can continue with the clause for app to obtain:

```
Variables: A B
============================
 exists A1,
   {of n1 (arr A B), of n2 A |- of n1 (arr (A1 n2 n1) B)}
/\ {of n1 (arr A B), of n2 A |- of n2 (A1 n2 n1)}
```

Observe that we now have two goals because the body of the of_app clause has two predicates. Moreover, this clause involves a variable in the body that does not occur in the head, so the result of the unfold is to existentially quantify over that variable. The Query command was able to find a substitution for the variable, but in our case we can provide it explicitly.

```
explore < witness u\ w\ A.
```

[10] To achieve this in the proof, one would use the unfold tactic using the *named clauses* feature. We leave a discussion of named clauses out of this tutorial as this is a new and experimental feature, and is only rarely useful.

```
Variables: A B
============================
   {of n1 (arr A B), of n2 A |- of n1 (arr A B)}
/\ {of n1 (arr A B), of n2 A |- of n2 A}
```

The remainder of the proof is straightforward: in each conjunct, the goal is found among the assumptions, so the subgoal can be closed with `search`.

What we achieved with `unfold` on the conclusion we can also achieve with `case` on a hypothesis of a `{}` predicate; however, instead of selecting a clause, the `case` tactic exhaustively considers all possible derivations, producing a subgoal for each possibility. As an example, suppose we wish to show the negation of the second (unprovable) query above.

```
Theorem explore2 : forall A B,
  {of (abs A x\ app x x) B} -> false.
```

Since there is only one possible clause to backchain (`of_abs`), the `case H1` command produces:

```
explore2 < intros. case H1.

Variables: A B1
H2 : {of n1 A |- of (app n1 n1) B1}
============================
 false
```

At this point, `case H2` produces *two* subgoals, the first of which we will `skip` for now. The second subgoal is:

```
explore2 < case H2. skip.

Variables: A B1 F
H3 : {of n1 A, [F n1] |- of (app n1 n1) B1}
H4 : member (F n1) (of n1 A :: nil)
============================
 false
```

Here we see the *backchaining sequent* form that has the general shape:

$$\{\ F_1, \ldots, F_n, [F]\ \texttt{|-}\ A\ \}$$

where A is restricted to atomic specification-level formulas.[11] This sequent form is the direct representation of backchaining sequents from Section 7.2. The two hypotheses `H3` and `H4` are precisely the two conjuncts for the clause for the goal reduction relation `red` that appeals to the backchaining relation `bch`, which corresponds to the rule `bch` of Figure 2. They also have a natural reading: `F n1` must be a member of the context (`H4`) that can be used to backchain (`H3`).

At this point, if we try to case-analyze `H3`, then the `case` tactic will abort because the exact form of `F n1` is not yet known. The tactic needs to make a decision here: if the head of the backchained clause matches the goal, then the tactic should produce fresh assumptions for the body of the clause; however, if the head does not match,

[11] See [70] for a full explanation of how proof in the specification logic is organized within Abella.

then the tactic should close the subgoal for being vacuous. Neither choice can be made yet, since we don't yet know the head of `F n1`. (Note that the head of a clause must be an atomic formula.)

In order to obtain this information about the structure of `F n1`, we must consider `H4`. Note that in `H4` we have left the specification logic sequent form `{}` and are performing meta-reasoning directly on the form of the context. This is the first point where we need to know how specification contexts are represented in the reasoning logic. Abella represents such contexts as lists of specification formulas (i.e., terms of type `o`) using the type `olist` with two constructors, `nil` and `::`, with the latter written infix. This list type comes equipped with a `member` predicate with this definition:

```
Define member : o -> olist -> prop by
  member X (X :: L) ;
  member X (Y :: L) := member X L.
```

Case-analysis of `H4` will therefore lead to two subgoals for the two definitional clauses of `member`, one where `F n1 = of n1 A` and the other with a fresh hypothesis `member (F n1) nil`. This latter case is trivial, so let us consider the former subgoal:

```
explore2 < case H4.

Variables: A B1
H3 : {of n1 A, [of n1 A] |- of (app n1 n1) B1}
============================
 false
```

At this point, the `case` tactic can finally take a decision on `H3`, since the backchained clause has an atomic head. Indeed, the head actually does not match the goal, since `n1` does not unify with `app n1 n1`, so the subgoal is finished with `case H3`.

This leaves just the subgoal that we skipped earlier, which is:

```
Variables: A B1 A1
H3 : {of n1 A |- of n1 (arr (A1 n1) B1)}
H4 : {of n1 A |- of n1 (A1 n1)}
============================
 false
```

Here, the proof can be finished with a similar argument as before, starting with a case-analysis of `H3` or `H4`. We leave it to the reader as an exercise.

8.3 Dynamic context management

Let us now try to show that the `of` predicate defined in the previous subsection is deterministic in its second argument, i.e., that each term has at most one type. Following the pattern of Section 8.1, we might write the theorem as:

```
Theorem of_det2 : forall M A1 A2,
  {of M A1} -> {of M A2} -> A1 = A2.
```

Like with lists earlier, this theorem is proved by induction on one of the antecedents, such as the first one. If we attempt a direct induction, we will succeed for backchaining the program clause for applications. However, for abstractions, we will end up with the following subgoal:

```
Variables: B R A B1
IH : forall M A1 A2, {of M A1}* -> {of M A2} -> A1 = A2
H3 : {of n1 A |- of (R n1) B}*
H4 : {of n1 A |- of (R n1) B1}
============================
 arr A B = arr A B1
```

The IH cannot be applied to H3 and H4, even though the inductive restrictions match, because the IH only applies to sequents with empty contexts.

The primary issue here is that the contexts of specification sequents *grow* when backchaining clauses, so the of_det2 theorem is not general enough. To generalize it, we need to *quantify over* the possible specification sequent contexts

```
Theorem of_det2' : forall L M A1 A2,
  {L |- of M A1} -> {L |- of M A2} -> A1 = A2.
```

Abella therefore allows specification sequents to have the following structure:

$$\{\ L, F_1, \ldots, F_n\ \texttt{|-}\ G\ \}$$

or

$$\{\ L, F_1, \ldots, F_n, [F]\ \texttt{|-}\ A\ \}$$

where L is a variable of type olist, $F_1, \ldots, F_n, F, G$ are formulas, and A is an atomic formula. We sometimes call L the *context variable*; if it exists, it must be unique and must be the first listed element of the context. Abella does not support multiple context variables within the same {}.

Using this kind of quantification over contexts, the unprovable subgoal earlier now becomes:

```
Variables: L B R A B1
IH : forall L M A1 A2, {L |- of M A1}* ->
       {L |- of M A2} -> A1 = A2
H3 : {L, of n1 A |- of (R n1) B}*
H4 : {L, of n1 A |- of (R n1) B1}
============================
 arr A B = arr A B1
```

We can now apply IH to H3 H4 by instantiating the L in the IH with of n1 A :: L. However, the proof is not yet complete, as we now have a new possibility for proving {L |- of M A1}: backchaining a member of L:

```
Variables: L M A1 A2 F
IH : forall L M A1 A2, {L |- of M A1}* ->
       {L |- of M A2} -> A1 = A2
H2 : {L |- of M A2}
H3 : {L, [F] |- of M A1}*
H4 : member F L
============================
 A1 = A2
```

The hypothesis H3 is not usable by the IH as it involves a backchaining sequent. The only remaining option is to proceed by case-analysis. However, as we saw earlier in the trace through a query, the case tactic cannot yet decide on H3 as it does not know the head of the formula F. Unlike the trace earlier, though, there is no other way to discover the form of F, since all we know is that it is a member of some olist, without any further information on how that olist is built. Hence, this proof attempt for of_det2’ is doomed without further constraining L.

What do we know about L? Let us think back to why we needed L in the first place: we needed to reason about {of (abs A R) (arr A B)}, which in turn required us to reason about {of n1 A |- of (R n1) B}. Thus, the kinds of members that we need in L must include elements such as of n1 A. More generally, to derive of M A, we would need to extend the context by assumptions about bound variables, which are in turn represented by nominal constants. This means that the possible forms of L must include at least the following subset (for any $k \geq 0$):

$$\texttt{of n}_1\ A_1,\ \texttt{of n}_2\ A_2,\ \ldots,\ \texttt{of n}_k\ A_k.$$

Let us therefore attempt to prove the theorem parsimoniously for only these kinds of contexts. We achieve this by writing an inductive definition ctx for recognizing such contexts.

```
Define ctx : olist -> prop by
  ctx nil ;
  nabla x, ctx (of x A :: L) := ctx L.
```

The first clause accounts for the possibility of the context being empty, as is the case for a closed term, while the second clause extends an existing context by a single assumption about of in the form indicated above. Note the use here of nabla at the head (cf. Section 6.4). It guarantees that the ∇-quantified x is fresh for A and L, which are universally quantified over the entire definitional clause.

We can now state an inductively provable form of the of_det2 theorem.

```
Theorem of_det2_lem : forall L M A1 A2, ctx L ->
  {L |- of M A1} -> {L |- of M A2} -> A1 = A2.
induction on 2. intros. case H2.
```

Let us walk through some of the cases of this proof. The first subgoal is for backchaining a program clause for app, which yields the subgoal:[12]

```
Variables: L A1 A2 A N M1
H1 : ctx L
H3 : {L |- of (app M1 N) A2}
H4 : {L |- of M1 (arr A A1)}*
H5 : {L |- of N A}*
============================
 A1 = A2
```

At this point, we can proceed by case-analysis on H3. Unlike the previous proof of of_det2, the case tactic here yields *two* subgoals for the two possibilities for backchaining: the program clause for app or a member of L. The former subgoal is straightforward and identical to what we have already encountered, but the latter subgoal is novel:

[12] In these subgoals, the IH is suppressed unless needed to prevent repetition.

```
Variables: L A1 A2 A N M1 F
H1 : ctx L
H4 : {L |- of M1 (arr A A1)}*
H5 : {L |- of N A}*
H6 : {L, [F] |- of (app M1 N) A2}
H7 : member F L
============================
 A1 = A2
```

We are in a similar situation to what we have seen before, but now we have an additional hypothesis H1 that constrains L. Indeed, we can prove the following lemma about all members of L.

```
Define fresh : tm -> ty -> prop by
  nabla x, fresh x A.

Theorem ctx_mem : forall L F,
  ctx L -> member F L ->
  exists X A, (F = of X A) /\ fresh X A.
```

(The proof of ctx_mem is left as an easy exercise.) Back in the proof of of_det2_lem, we can now apply ctx_mem to H1 H7 to get:

```
Variables: L A1 A2 A N M1 X A3
H1 : ctx L
H4 : {L |- of M1 (arr A A1)}*
H5 : {L |- of N A}*
H6 : {L, [of X A3] |- of (app M1 N) A2}
H7 : member (of X A3) L
H8 : fresh X A3
============================
 A1 = A2
```

At this point, the head of the backchained clause in H6 is atomic, so we can use case H6 to unify X with app M1 N and A3 with A2. This changes H8 to fresh (app M1 N) A2, which is not derivable as app M1 N is not a nominal constant. Thus, case H8 closes this subgoal.

The next subgoal is the case for backchaining a program clause for abs:

```
Variables: L A2 B R A
H1 : ctx L
H3 : {L |- of (abs A R) A2}
H4 : {L, of n1 A |- of (R n1) B}*
============================
 arr A B = A2
```

Like with app earlier, using case H3 will leave us with two subgoals, the first of which is:

```
IH : forall L M A1 A2, ctx L -> {L |- of M A1}* ->
       {L |- of M A2} -> A1 = A2
H1 : ctx L
H4 : {L, of n1 A |- of (R n1) B}*
H5 : {L, of n1 A |- of (R n1) B1}
```

```
============================
 arr A B = arr A B1
```

Now, in order to use the IH, we would need to prove that ctx (of n1 A :: L). Since this follows trivially from H1 by search, Abella allows a special form of the apply tactic with some arguments left unspecified; they are inferred and automatically proved by Abella.

```
of_det2_lem < apply IH to _ H4 H5.

Variables: L R A B1
H1 : ctx L
H4 : {L, of n1 A |- of (R n1) B1}*
H5 : {L, of n1 A |- of (R n1) B1}
============================
 arr A B1 = arr A B1
```

The other subgoal amounts to backchaining a member of L to derive H3:

```
Variables: L A2 B R A F
H1 : ctx L
H4 : {L, of n1 A |- of (R n1) B}*
H5 : {L, [F] |- of (abs A R) A2}
H6 : member F L
============================
 arr A B = A2
```

Here, as before with app, we use the ctx_mem lemma to derive that F must be of the form of n1 A1, and therefore case H5 can decide that H5 is not derivable and hence the goal must be closed.

This leaves us with the final subgoal that defeated our earlier attempt for of_det2':

```
Variables: L M A1 A2 F
H1 : ctx L
H3 : {L |- of M A2}
H4 : {L, [F] |- of M A1}*
H5 : member F L
============================
 A1 = A2
```

Here, we can use ctx_mem with H1 and H5 to derive the structure of F, which lets us refine the goal further.

```
of_det2_lem < apply ctx_mem to H1 H5. case H6. case H4.

Variables: L A2 A3
H1 : ctx (L n1)
H3 : {L n1 |- of n1 (A2 n1)}
H5 : member (of n1 A3) (L n1)
============================
 A3 = A2 n1
```

Now, if we proceed by case-analysis on `H3`, there are no program clauses to backchain as none of these clauses define typing for nominal constants. So, the sole possibility is backchaining a member of `L n1`.

```
of_det2_lem < case H3.

Variables: L A2 A3 F1
H1 : ctx (L n1)
H5 : member (of n1 A3) (L n1)
H7 : {L n1, [F1 n1] |- of n1 (A2 n1)}
H8 : member (F1 n1) (L n1)
============================
 A3 = A2 n1
```

Once again, we can appeal to `ctx_mem` to refine the subgoal further.

```
of_det2_lem < apply ctx_mem to H1 H8. case H7. case H9.

Variables: L A3 A5
H1 : ctx (L n1)
H5 : member (of n1 A3) (L n1)
H8 : member (of n1 A5) (L n1)
============================
 A3 = A5
```

We have now reduced the problem of showing uniqueness of typing to showing uniqueness of typing assumptions for variables. This is easy to show, since our `ctx` definition always extends `L` with typing assumptions with a fresh nominal constant, and hence every typing assumption for a nominal constant occurs at most once. We can prove this as a lemma by induction.

```
Theorem ctx_uniq : forall L X A1 A2, ctx L ->
  member (of X A1) L -> member (of X A2) L -> A1 = A2.
```

The proof is left as an exercise (hint: use the following lemma).

```
Theorem member_prune : forall L E, nabla (x:tm),
  member (E x) L -> exists F, E = x\ F.
```

We are now finally able to close the subgoal above with `backchain ctx_uniq with X = n1, L = L n1`. The reader is encouraged to now re-organize this proof to see that it requires a definition `ctx` of certain kinds of contexts, a lemma `ctx_mem` characterizing members of such contexts, and a lemma `ctx_uniq` that such contexts assign types functionally. Once these lemmas are in place, proving the determinancy of typing (`of_det2_lem`) is straightforward and our original theorem `of_det2` is a trivial corollary.

8.4 Exploiting the meta-theory of the specification logic

As mentioned in the preamble of Section 8, one of the main benefits of using a specification logic is that its meta-theory is proved once and for all (cf. Section 7.2). The following three meta-theorems are the most commonly encountered forms.

Monotonicity. The statement of the monotonicity theorem is as follows.

```
forall L K G, (forall E, member E L -> member E K) ->
  {L |- G} -> {K |- G}.
```

As explained in Section 7.2, this theorem subsumes weakening, contraction, and exchange. One interesting aspect of this theorem is that it is height-preserving: if `{L |- G}` has an inductive restriction, then applying this theorem should preserve the restriction on `{K |- G}`. This is justified in Section 7.2 by the use of the height parameter to `red` and `bch` that is shown to be preserved by the monotonicity meta-theorem. As the `apply` tactic of Abella does not preserve inductive restrictions on arguments, a special `monotone` tactic is used instead. For hypotheses of the form:

```
H1 : {L |- G}      or      H1 : {L |- G} *
```

the invocation `monotone H1 with K` produces a corresponding new hypothesis

```
H2 : {K |- G}      or      H2 : {K |- G} *
```

as appropriate, and an additional subgoal to prove `forall E, member E L -> member E K`. Like with all tactics, the `search` tactic is automatically attempted on this subgoal.

Instantiation. Nominal constants that appear in a specification logic sequent may be replaced by any arbitrary term using the following instantiation theorem.

```
forall L G t, (nabla x, {L x |- G x}) -> {L t |- G t}.
```

Note that this is not perfectly valid Abella syntax because the statement is parametric on the type of `x`. In other words, it is a *polymorphic* theorem. Moreover, just as with monotonicity, this theorem can preserve inductive restrictions, as proved in Section 7.2. Abella uses a specialized `inst` tactic for this theorem; for hypotheses of the form:

```
H1 : {L |- G}      or      H1 : {L |- G} *
```

the invocation `inst H1 with n1 = t` produces a corresponding new hypothesis

```
H2 : [t/n1] {L |- G}      or      H2 : [t/n1] {L |- G} *
```

where `[t/n1]` stands for the capture-avoiding substitution of `t` for `n1` in the specification sequent. (Note: this substitution operation is not legal Abella syntax, just a notation used here to describe the tactic.)

Cut. The final meta theorem corresponds to the use of cut-admissibility of the specification logic:

```
forall L F G, {L |- F} -> {L, F |- G} -> {L |- G}.
```

For hypotheses of the form:

```
H1 : {L |- F}
H2 : {L, F |- G}
```

the invocation `cut H2 with H1` produces a new hypothesis:

```
H3 : {L |- G}
```

If the two premises H1 and H2 do not have a common context L, then both sequents are first put in the least context that contains both premise contexts (by internally appealing to the monotone tactic). However, this operation comes with a limitation: since Abella allows at most a single context variable in specification sequents, the least upper bound (in the inclusion ordering) of two specification contexts may not exist if their context variables differ. In this case, the user must first appeal to the other meta-theorems or lemmas to ensure both premise sequents have a common context variable.

All three meta-theoretic tactics can also be applied to backchaining sequents when the syntax permits it.

Example. As an illustration of the use of the meta-theoretic tactics, consider subject-reduction in the simply typed λ-calculus. We first define the (big step) evaluation relation eval that has this type signature:

```
type eval tm -> tm -> o.
```

and these program clauses:

```
eval (abs A R) (abs A R).
eval (app M N) V :- eval M (abs A R), eval (R N) V.
```

The theorem we wish to prove is:

```
Theorem subjred : forall M V A,
  {of M A} -> {eval M V} -> {of V A}.
```

The proof proceeds by induction on the second antecedent. The case of abstractions is straightforward. For applications, we are in the following subgoal:

```
Variables: V A N R A1 M1 A2
IH : forall M V A, {of M A} -> {eval M V}* -> {of V A}
H3 : {eval M1 (abs A1 R)}*
H4 : {eval (R N) V}*
H5 : {of M1 (arr A2 A)}
H6 : {of N A2}
============================
 {of V A}
```

Here, we apply IH to H5 H3 followed by case-analysis of the resulting hypothesis about abs A1 R to get:

```
Variables: V A N R M1 A2
IH : forall M V A, {of M A} -> {eval M V}* -> {of V A}
H3 : {eval M1 (abs A2 R)}*
H4 : {eval (R N) V}*
H5 : {of M1 (arr A2 A)}
H6 : {of N A2}
H8 : {of n1 A2 |- of (R n1) A}
============================
 {of V A}
```

In order to use the IH with H4, we need to show {of (R N) A}. To get there from H8, we first instantiate the n1 with N, and then cut with H6.

```
subjred < inst H8 with n1 = N. cut H9 with H6.

Variables: V A N R M1 A2
IH : forall M V A, {of M A} -> {eval M V}* -> {of V A}
H3 : {eval M1 (abs A2 R)}*
H4 : {eval (R N) V}*
H5 : {of M1 (arr A2 A)}
H6 : {of N A2}
H8 : {of n1 A2 |- of (R n1) A}
H9 : {of N A2 |- of (R N) A}
H10 : {of (R N) A}
============================
 {of V A}
```

Now we can `apply IH to H10 H4` and finish the proof.

8.5 General context relations

The `ctx` definition in Section 8.3 was tailored to the `of` predicate. For a different predicate that would extend the context in a different way, we would need a different variant of the `ctx` relation. This leads to an obvious question: what if we need to prove a property about two *different* relations? One option might be to try to prove this property in a context that contains both kinds of extensions, but this tends to make the proofs needlessly complicated, with extraneous subgoals in nearly every case to rule out impossible situations. A more scalable option offered by Abella's rather general definition mechanism is to use separate tailored contexts for the various predicates with an explicit link between the separate contexts.

Once again, this technique is best described in terms of an example. Suppose we wish to define a binary relation `copy` on terms such that `copy M N` recursively examines the structure of `M` and recreates it in `N`. It has this type signature:

```
type copy tm -> tm -> o.
```

and these clauses:

```
copy (app M1 M2) (app N1 N2) :-
  copy M1 N1, copy M2 N2.
copy (abs A R1) (abs A R2) :-
  pi x\ copy x x => copy (R1 x) (R2 x).
```

The theorem we are interested in proving is the following:

```
Theorem copy_exists : forall M A, {of M A} -> {copy M M}.
```

(In fact, the theorem is weaker than it can be because `copy` works as well for terms that are ill-typed according to `of`, but we ignore this issue as the example is mainly used to illustrate a technique.)

Now, Section 8.3 should already have made it clear that we cannot hope to prove the `copy_exists` theorem directly by induction. Instead, we will need to prove a generalized lemma that allows for contexts in the two clauses. How should these contexts be characterized? We can examine the program clauses of the `of` and `copy` predicates to observe that the context extensions used to derive `of M A` are indeed related to the those for `copy M M`. When the former has a context of the form:

$$\mathtt{of}\ \mathtt{n}_1\ A_1, \mathtt{of}\ \mathtt{n}_2\ A_2, \ldots, \mathtt{of}\ \mathtt{n}_k\ A_k,$$

then the latter must have a context of the form:

$$\mathtt{copy}\ \mathtt{n}_1\ \mathtt{n}_1, \mathtt{copy}\ \mathtt{n}_2\ \mathtt{n}_2, \ldots, \mathtt{copy}\ \mathtt{n}_k\ \mathtt{n}_k.$$

We can therefore define, inductively, a binary *context relation* `ctx2` that characterizes both contexts at the same time.

```
Define ctx2 : olist -> olist -> prop by
  ctx2 nil nil ;
  nabla x, ctx2 (of x A :: L) (copy x x :: K) := ctx2 L K.
```

The second definitional clause guarantees that `x` is fresh for `A`, `L`, and `K`. We can then prove a generalized lemma using `ctx2` that will have `copy_exists` as a corollary.

```
Theorem copy_exists_lem : forall L K M A, ctx2 L K ->
  {L |- of M A} -> {K |- copy M M}.
```

We proceed by a straightforward induction on the second antecedent; here are some of the interesting cases. For backchaining the program clause for `abs`, we have the subgoal:

```
Variables: L K B R A1
IH : forall L K M A, ctx2 L K ->
       {L |- of M A}* -> {K |- copy M M}
H1 : ctx2 L K
H3 : {L, of n1 A1 |- of (R n1) B}*
============================
 {K |- copy (abs A1 R) (abs A1 R)}
```

Here, we can finish the goal by `apply IH to _ H3`; Abella constructs and proves the omitted goal that `ctx2 (of n1 A1 :: L) (copy n1 n1 :: K)`. The other interesting subgoal arises from backchaining a member of `L`.

```
Variables: L K M A F
IH : forall L K M A, ctx2 L K ->
       {L |- of M A}* -> {K |- copy M M}
H1 : ctx2 L K
H3 : {L, [F] |- of M A}*
H4 : member F L
============================
 {K |- copy M M}
```

Like in Section 8.3, to make progress we need to know something about the members of `L`. We can prove a variant of `ctx_mem` directly with an identical proof.

```
Theorem ctx2_mem1 : forall L K F,
  ctx2 L K -> member F L ->
  exists X A, (F = of X A) /\ fresh X A.
```

Alternatively, we can *project* the binary relation on its first argument and reuse the `ctx_mem` theorem.

```
Theorem ctx2_proj1 : forall L K, ctx2 L K -> ctx L.
```

Whichever option is taken, we will be left with the subgoal:

```
Variables: L K A2
H1 : ctx2 (L n1) (K n1)
H4 : member (of n1 A2) (L n1)
============================
 {K n1 |- copy n1 n1}
```

The proof can finish if we only knew that `member (copy n1 n1) (K n1)`, as there is no program clause for `copy` for nominal constants. This requires a separate lemma showing the relationship between the members of contexts related by `ctx2`. Its easy proof is left as an exercise.

```
Theorem ctx2_sync : forall K L X A, ctx2 L K ->
  member (of X A) L -> member (copy X X) K.
```

8.6 Extended example: Transitivity of subtyping in system F_{sub}

Our examples until now have used only a single nested implication in the bodies of program clauses, which does not fully exploit the significant expressive power of higher-order specifications. As an extended example of the possibilities offered by such specifications, we will give here a version of the proof of transitivity of subtyping for system F_{sub}, which is an extension of system F with subtyping and bounded polymorphism. System F_{sub} is described in more detail in [52, Chapter 26], and transitivity of subtyping in F_{sub} is problem 1a of the POPLMark challenge [4].

The types (written using $T, S, \ldots$) of system F_{sub} are built from type variables (written $\alpha, \beta, \ldots$) using the following grammar.

$$T, S, \ldots ::= \alpha \mid \top \mid T \to S \mid \forall \alpha {\leq} T.\, S$$

The type $\top$ is the greatest element of the subtype relation $\leq$. For the bounded polymorphic type $\forall \alpha {\leq} T.\, S$, valid instances can only be created for type arguments that are $\leq T$. The bound variable can be freely α-varied.

The subtyping relation can be inductively specified using *subtyping contexts* (Γ), which are multisets of elements of the form $\alpha {\leq} T$. The subtyping judgment has the form $\Gamma \vdash S \leq T$ with the following inference rules.

$$\frac{}{\Gamma, \alpha {\leq} T \vdash \alpha \leq T} \qquad \frac{}{\Gamma \vdash \alpha \leq \alpha} \qquad \frac{\Gamma \vdash \alpha \leq S \qquad \Gamma \vdash S \leq T}{\Gamma \vdash \alpha \leq T} \qquad \frac{}{\Gamma \vdash T \leq \top}$$

$$\frac{\Gamma \vdash T_1 \leq S_1 \qquad \Gamma \vdash S_2 \leq T_2}{\Gamma \vdash (S_1 \to S_2) \leq (T_1 \to T_2)} \qquad \frac{\Gamma \vdash T_1 \leq S_1 \qquad \Gamma, \alpha \leq T_1 \vdash S_2 \leq T_2}{\Gamma \vdash (\forall \alpha {\leq} S_1.\, S_2) \leq (\forall \alpha {\leq} T_1.\, T_2)}$$

In the final rule, by standard convention, we require that α is not free in Γ.

In order to encode this as a specification in Abella, we have two possibilities. The first is to have a separate syntactic class of type variables for which the first three subtyping rules apply, but this forces us to reason about α-equivalence. The other choice is to use λ-tree syntax to get the α-equivalence (among other properties) for free, but then the first three rules are difficult to encode as λ-tree syntax has no built in mechanism to detect when something is a variable. Either option seems to present difficulties for the encoding.

Fortunately, there is a third possibility, suggested in [51], of retaining an encoding in λ-tree syntax while using higher-order clauses to construct specialized instances of the first three variable rules whenever a new variable is introduced into the subtyping context. The unscoped form of these three rules are then removed from the system. We therefore use the following type signature for the grammar of types (tp) and for the subtyping relation sub.

```
sig fsub.

kind tp   type.
type top  tp.
type arr  tp -> tp -> tp.
type all  tp -> (tp -> tp) -> tp.

type sub  tp -> tp -> o.
```

As is standard in λ-tree syntax, we use λ-terms to depict binding, so the type $\forall\alpha{\leq}S.\,T$ would be encoded as all S (a\ T).

The clauses for the sub relation are as follows.

```
module fsub.

sub T top.
sub (arr S1 S2) (arr T1 T2) :- sub T1 S1, sub S2 T2.
sub (all S1 S2) (all T1 T2) :-
    sub T1 S1,
    pi a\
      (pi u\ pi v\ sub a u => sub u v => sub a v) =>
      sub a T1 =>
      sub a a =>
        sub (S2 a) (T2 a).
```

The first two clauses are straightforward translations of the corresponding inference rules. The body of the third clause has embedded clauses defining transitivity and reflexivity of subtyping for the bound type variable, represented here with the bound variable a, and the fact that it is $\leq$ T1. Note that the assumption

```
pi u\ pi v\ sub a u => sub u v => sub a v
```

is a Horn clause program which is specialized to the variable a but which can be used with arbitrary types u and v to derive the subtyping relationships for a by transitivity. In other words, it computes the transitive closure of subtyping chains beginning with a. This third program clause of sub is therefore a *third-order* clause, as there are second-order formulas in its body.

To prove that the specified relation sub enjoys transitivity of subtyping amounts to proving the following theorem in Abella:

```
Theorem transitivity' : forall S T U,
  {sub S T} -> {sub T U} -> {sub S U}.
```

However, there is a problem: the proof of the theorem is by induction on the structure of the intermediate type T, which in Abella requires a separate definition, is_tp, as we have seen before. Since the syntax of types is itself untyped,[13] we

[13] Types do not have any classifiers in system F_{sub}.

reuse the technique from Section 7.1 of defining the structure of types in an open fashion, using nominal constants to stand for the type variables.

```
Define is_tp : tp -> prop by
  nabla x, is_tp x ;
  is_tp top ;
  is_tp (arr S T) := is_tp S /\ is_tp T ;
  is_tp (all S T) := is_tp S /\ nabla x, is_tp (T x).
```

Then, the actual transitivity theorem we show is:

```
Theorem transitivity : forall S T U, is_tp T ->
  {sub S T} -> {sub T U} -> {sub S U}.
```

Of course, to prove this theorem we need to generalize the statement to reason about the context extensions resulting from backchaining the program clauses of sub. Like we have already seen in earlier subsections, this requires a context definition ctx characterizing the extensions to the context.

```
Define ctx : olist -> prop by
  ctx nil ;
  nabla a, ctx ((sub a a)    ::
                (sub a T)    ::
                (pi u\ pi v\ sub a u => sub u v => sub a v)
                            :: L) :=
     ctx L.
```

The second clause of ctx adds *three* new members to the context, corresponding to the three antecedents of the main implication in the body of the second program clause of sub.

Although this definition is somewhat more complex than similar definitions in earlier subsections, theorems such as ctx_mem are still easily defined and proved.

```
Define name : tp -> prop by
  nabla x, name x.

Define fresh : tp -> tp -> prop by
  nabla x, fresh x T.

Theorem ctx_mem : forall L F,
  ctx L -> member F L ->
  exists A,
       (F = sub A A /\ name A)
    \/ (exists T, F = sub A T /\ fresh A T)
    \/ (F = (pi u\ pi v\ sub A u => sub u v => sub A v)
         /\ name A).
```

The proof is marginally more complex than proofs of similar theorems in earlier subsections but is nevertheless straightforward and left as an exercise.

We can now state and prove the strengthened transitivity theorem.

```
Theorem transitivity : forall L S T U,
    ctx L -> is_tp T ->
    {L |- sub S T} -> {L |- sub T U} -> {L |- sub S U}.
induction on 2. intros.
```

At this juncture it would be tempting to continue with case H2 but this would have the effect of enumerating all types, regardless of whether those types are related to S or not. Therefore, we instead proceed with case H3, which produces four subgoals: three for the possible ways to backchain a program clause for sub and one for backchaining an element of L.

The first subgoal, produced by backchaining the first program clause for sub, is:[14]

```
Variables: L S U
H1 : ctx L
H2 : is_tp top @
H4 : {L |- sub top U}
============================
 {L |- sub S U}
```

Here, our only option is to continue with case H4, which produces two subgoals: one for the first program clause for sub (which is trivial), and the other for backchaining a member of L, which is:

```
Variables: L S U F
H1 : ctx L
H2 : is_tp top @
H4 : {L |- sub top U}
H5 : {L, [F] |- sub top U}
H6 : member F L
============================
 {L |- sub S U}
```

Now, we can apply ctx_mem to H1 H6; each resulting subgoal would be vacuous because top cannot unify with a nominal constant.

The next overall subgoal is for the program clause of sub involves setting T to arr T1 T2: the resulting argument is straightforward. After that, the most interesting subgoal involves binding T to all T1 T2, which yields:

```
Variables: L U T2 S2 T1 S1
H1 : ctx L
H2 : is_tp (all T1 T2) @
H4 : {L |- sub (all T1 T2) U}
H5 : {L |- sub T1 S1}
H6 : {L, (pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
       sub n1 T1, sub n1 n1 |- sub (S2 n1) (T2 n1)}
============================
 {L |- sub (all S1 S2) U}
```

Once again we start with case H4, which produces three subgoals, two of which are similar to the previous cases, and the third (after a case H2) is:

```
Variables: L T2 S2 T1 S1 T3 T4
H1 : ctx L
H5 : {L |- sub T1 S1}
H6 : {L, (pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
       sub n1 T1, sub n1 n1 |- sub (S2 n1) (T2 n1)}
```

[14] Once again, the IH is suppressed unless relevant.

```
H7 : {L |- sub T4 T1}
H8 : {L, (pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
      sub n1 T4, sub n1 n1 |- sub (T2 n1) (T3 n1)}
H9 : is_tp T1 *
H10 : is_tp (T2 n1) *
============================
 {L |- sub (all S1 S2) (all T4 T3)}
```

From `apply IH to H1 H9 H7 H5` we immediately have `{L |- sub T4 S1}`. Therefore, all that remains is to link `S2` and `T3`, which, by the program clause for `sub`, requires showing:

```
{L, (pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
 sub n1 T4, sub n1 n1 |- sub (S2 n1) (T3 n1)}
```

We can nearly get there by the `IH` applied to `H6` and `H8`, except that their contexts do not match: one has `sub n1 T1` while the other has `sub n1 T4`. But, by `H7`, we know that `sub T4 T1`, which by the embedded clause in the context of `H8` should let us derive `sub n1 T1`. We assert this fact, which Abella can easily prove.

```
transitivity_lem <
  assert {(pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
          sub T4 T1, sub n1 T4 |- sub n1 T1}.

Variables: L T2 S2 T1 S1 T3 T4
H1 : ctx L
H5 : {L |- sub T1 S1}
H6 : {L, (pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
      sub n1 T1, sub n1 n1 |- sub (S2 n1) (T2 n1)}
H7 : {L |- sub T4 T1}
H8 : {L, (pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
      sub n1 T4, sub n1 n1 |- sub (T2 n1) (T3 n1)}
H9 : is_tp T1 *
H10 : is_tp (T2 n1) *
H11 : {L |- sub T4 S1}
H12 : {(pi u\ pi v\ sub n1 u => sub u v => sub n1 v),
       sub T4 T1, sub n1 T4 |- sub n1 T1}
============================
 {L |- sub (all S1 S2) (all T4 T3)}
```

We can then `cut` the assumption `sub T4 T1` using `H7`, and use the result to `cut` the assumption `sub n1 T1` out of `H6`, yielding the required sequent to use with the `IH`.

The final overall subgoal corresponds to backchaining a member of `L` to derive `{L |- sub S T}`:

```
Variables: L S T U F
H1 : ctx L
H2 : is_tp T @
H4 : {L |- sub T U}
H5 : {L, [F] |- sub S T}
H6 : member F L
============================
 {L |- sub S U}
```

We can use `ctx_mem` now to examine all the possibilities for `F`. Most cases are immediate; the sole exception is:

```
Variables: L U T2
H1 : ctx (L n1)
H2 : is_tp T2 @
H4 : {L n1 |- sub T2 (U n1)}
H6 : member (sub n1 T2) (L n1)
============================
 {L n1 |- sub n1 (U n1)}
```

We know that if `sub n1 T2` is in `L n1`, then it must have been added together with a Horn clause for transitivity for `n1` at the same time. In other words, we need the following theorem.

```
Theorem ctx_sync : forall L A T,
  ctx L -> member (sub A T) L ->
    member (pi u\ pi v\ sub A u => sub u v => sub A v) L.
```

The proof is by induction on the first antecedent and left as an exercise as well. Appealing to it for the above subgoal gives us enough information about the contents of `L n1` to derive the goal.

The complete proof is shown in Figure 6.

Fig. 6: full proof of the transitivity of subtyping for system $\mathrm{F}_{\mathrm{sub}}$

```
Theorem transitivity : forall L S T U,
    ctx L -> is_tp T ->
    {L |- sub S T} -> {L |- sub T U} -> {L |- sub S U}.
induction on 2. intros. case H3.

  % T = top
  case H4.
    search.
    apply ctx_mem to H1 H6. case H7.
      case H5. case H8.
      case H5. case H8.
      case H5. case H8.

  % T = arr T1 T2
  case H4.
    search.
    case H2.
     apply IH to H1 H9 H7 H5.
     apply IH to H1 H10 H6 H8.
     search.
    apply ctx_mem to H1 H8. case H9.
      case H7. case H10.
      case H7. case H10.
      case H7. case H10.
```

```
% T = all T1 T2
case H4.
  search.
  case H2.
   apply IH to H1 H9 H7 H5.
   assert { (pi u\ pi v\ sub n1 u => sub u v => sub n1
       v),
            sub T4 T1, sub n1 T4 |- sub n1 T1 }.
   cut H12 with H7. cut H6 with H13.
   apply IH to _ H10 H14 H8.
   search.

  apply ctx_mem to H1 H8. case H9.
    case H7. case H10.
    case H7. case H10.
    case H7. case H10.

% backchain on a member of L
apply ctx_mem to H1 H6. case H7.

  % sub a a
  case H5. search.

  % sub a T1
  case H5. case H8. apply ctx_sync to H1 H6. search.

  % pi u\ pi v\ sub a u => sub u v => sub a v
  case H5. search.
```

9. FUTURE DIRECTIONS

Currently in Abella, only predicates can be inductively defined although it is often convenient to view types as being inductively (or co-inductively) defined. In order to treat induction on types, one must write a predicate that repeats the description of the type and then reason inductively on that predicate. We are considering extending Abella so that all types can be immediately treated as if they were given by predicate definitions.

There have been a number of places in Section 8 where contexts were defined and a number of technical but shallow theorems about contexts needed to be proved. Given that contexts are generally rather simple structures, a variant of Abella has been developed [59] in which contexts can be declared differently than as lists: such specialized declarations allow for many of the technical lemmas to be proved by automatically produced proof scripts.

The core logic of Abella does not allow either predicate quantification or polymorphic typing. Its notion of modules is also rather primitive. By reconsidering modular construction of theorem (.thm) files, it might be possible to view predicate quantification and polymorphic typing as features of a module language instead of (more controversially) the target logic.

The specification logic used in the two-level logic approach described in Section 8 is the intuitionistic logic of hereditary Harrop formulas. This choice is appealing since it is an expressive and popular logic that appears within a number of systems, including λProlog, Isabelle, Minlog, and Twelf. The exact subset of logic and its concrete syntax were chosen so that it corresponds to an executable subset of λProlog and the Teyjus system [56] can be used to automatically search for proofs in that logic. There are, however, other natural choices of specification logics. For example, an early paper on the two-level logic approach [35] illustrated some advantages of using, instead, a linear logic programming language [30]. More recently, a variant of Abella has been designed [62] in which an additional specification logic using the dependently typed λ-calculus LF co-exists with the current specification language [29]. A natural question for the future development of Abella is how far to take this idea: should Abella be made generic over arbitrary specification languages (in which case we would need to consider the question of how specification languages are formally defined) and should it support multiple specifications, possibly in different languages, at the same time?

The Bedwyr [7, 72] model checking system is based on a subset of the logic underlying Abella. Providing articulation between these two systems could prove valuable for several reasons. For example, Bedwyr's automated search can prove theorems that simply require complete state explorations: writing a complete proof for such exploration in Abella is possible in principle but tedious and impractical. Also, Bedwyr could make use of lemmas proved in Abella in order to make its search more effective: for example, searching for a winning strategy in some board game could benefit significantly if it was known that symmetric versions of a winning position are also winning. This latter fact could, in principle, be proved in Abella and imported into Bedwyr.

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A. SUMMARY OF ABELLA SYNTAX AND COMMANDS

The most commonly used basic proof tactics of Abella are summarized in Figure 7. These tactics tend to follow the inference rules of $\mathcal{G}$ closely, and are included in the

Tactic	Effect
`split`	When the conclusion is of the form $C_1 \wedge \cdots \wedge C_n$, replace the current goal by n new subgoals, one for each of the C_i.
`left`, `right`	When the conclusion is of the form $C_l \vee C_r$, replace the current goal by a new subgoal for C_l (for `left`) or C_r (for `right`).
`intros`	From the conclusion, move the outermost `forall`s to new eigenvariables, replace the outermost `nabla`s by nominal constants, and move implication antecedents to new hypotheses.
`witness t`	For a conclusion of the form `exists x, A`, remove the `exists` quantifier and replace `t` for `x` in `A` (avoiding capture).
`case H`	Apply a suitable left-introduction rule for the hypothesis `H`. For conjunctions, this creates new hypotheses for the conjuncts; for disjunctions, it splits the goal per disjunct; for `exists`, it promotes the quantified variable to a new eigenvariable; and for defined atoms (see Section 4.1) it unfolds the definition.
`unfold N`	When the conclusion is a defined atom, unfold the `N`th clause of its definition (`N` must be a number ≥ 1). If `N` is omitted, unfold the first clause whose head matches the conclusion.
`assert A`	Generate two subgoals, one to prove `A` from the current context and another where `A` is added as a new hypothesis for the original conclusion. This corresponds to the cut rule of $\mathcal{G}$. The subgoal for proving `A` is automatically attempted with `search` (see below).

`apply H to H1 ... Hn`
`with x1 = t1, ....`
	When `H` is of the form $\forall \vec{x}.\, \nabla \vec{u}.\, A_1 \supset \cdots \supset A_n \supset C$, and each $\mathtt{H}_i$ is of the form $[\sigma]A_i$ for some substitution σ for the $\vec{x}, \vec{u}$, create a new hypothesis for $[\sigma]C$. The optional `with` clause can give explicit substitutions for some of the $\vec{x}, \vec{u}$. Any of the $\mathtt{H}_i$ can be an underscore (`_`), in which case a suitable candidate is automatically constructed and either proved automatically by `search` or generated as a fresh subgoal.

`backchain H`
`with x1 = t1, ....`
	When `H` is of the form $\forall \vec{x}.\, \nabla \vec{u}.\, A_1 \supset \cdots \supset A_n \supset C$, this matches C with the conclusion and generates new subgoals for each of the A_i; `search` is then attempted on each generated subgoal. The optional `with` clause can give explicit instantiations, as with `apply`.

`search N`	Automatically search for a proof of the conclusion by repeatedly unfolding the conclusion and looking up the result in the context. The optional argument `N` (a number ≥ 1) specifies a maximum number of iterations of the unfold-lookup loop.

Fig. 7: The basic proof tactics

trusted kernel of Abella. In addition to the main tactics, there are a few *meta*-tactics that can be used at any point to manipulate the state of the proof in non-logical ways. Figure 8 lists the most important of these meta-tactics.

Every Abella invocation starts with the declarations in Figure 9 already loaded. These declarations are used to support the *two-level logic approach* described in Section 8. Moreover, the constants `=>` and `::` are treated specially and written as infix. The symbol `pi` is treated polymorphically for any type `T` not containing `o`, or, equivalently, `pi` has an infinite number of types, one for each value of `T`, and once the concrete syntax has been parsed and checked, that type must be established

Tactic	Effect
undo	Undo the effect of the previous ordinary tactic. Can be applied repeatedly up to the initial state of the proof.
skip	Admit the current goal without proof. This is obviously not sound, but is useful during the process of writing a proof. It is also the only mechanism for introducing axioms.
abort	Abort the proof of the current theorem and resets Abella to top-level command processing mode.

Fig. 8: The most important meta-tactics.

```
Kind o          type.
Type =>         o -> o -> o.           % infix, right
Type pi         (T -> o) -> o.         % o not in T

Kind olist      type.
Type nil        olist.
Type ::         o -> olist -> olist.   % infix, right

Define member : o -> olist -> prop by
  member A (A :: L) ;
  member A (B :: L) := member A L.
```

Fig. 9: The declarations and definitions that are implicitly included in every Abella development. Everything in this color is treated as a keyword as it violates the lexical or typing conventions of ordinary declarations and definitions.

unambiguously.

The online user manual for Abella, found at `http://abella-prover.org/`, contains a complete list of commands and options.

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