C++ Solutions for Mathematical Problems

Arun Ghosh

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Preface

The objective of this book is to present a substantial introduction to the ideas, phenomena and methods of the problems that are frequently observed in mathematics, mathematical physics and engineering technology. The book can be appreciated at a considerable number of levels and is designed for everyone from amateurs to research workmen.

Included throughout are applications with appropriate suggestions and discussions, whenever needed, that form a significant and integral part of the text book.

In a word, the text directs at an all-embracing and practical treatment of differential equations with some methods specifically developed for the purpose controlled by computer programs. The effects of this treatment, the computerised solutions for each problem, represented in compact form, sometimes with graphical figures, have been provided for further study.

The operation has been performed by the programming language C++ based on any MS-DOS computer system (version 6.0) applying the classical methods, such as Euler, Simpson, Runge-Kutta, Finite-Difference, etc.

Chapters 4 and 5 provide introductions to first order and second order initial-value problems discussing the possibilities for finding solutions, analytic and computerised. Non-linear types of differential equations have also been considered.

Emphasis has been placed on section 6.3 in Chapter 6 that deals with the problems concerning Fourier series, because the subject matter of numerical evaluation for Fourier series is a predominant topic.

Special attention is focussed on Chapter 7 in view of the fact that the differential equations with boundary conditions are the kernels of the physical and technical problems. The Difference Method for the solution of a two point second-order boundary value problem has been applied. Chapter 8 contains numerically developed methods for the computer solution of elliptic, hyperbolic and parabolic partial differential equations.

A variety of solved examples in each chapter has been given for the students who cannot get any difficulty to understand the conceptual text. Chapter 11 entitled "A Short Review On C++" provides an opportunity to recapitulate the fundamental points on C++.

I prepared the text on personal computer from Compaq (type *Presario CDTV528*) and on Laptop from Toshiba (type *Satellite 110CS*) operated on MS-DOS 6.0 and Windows 95. MS-Workgroups (version 3.0) have also been used in the preparation.

A software supplement consisting of a set of programs designed in C++ (Turbo) is provided for the reader to work out the related mathematical models referred to the text. A Diskette (3.5 inch/1.44 MB) in standard PC-compatible form containing this supplement will accompany the book.

I think, my anticipation to believe is not indifferent, that the students, instructors and other readers who use this text can enjoy the development just as well the author has taken joy in the preparation.

I wish to express my very great thanks to my beloved Professor P. Sinharay of Calcutta University who has motivated me with interest. I am grateful to Professor G. Bertram of Hannover University who has guided me to complete this book also to the friends of the Computer center (RRZN) in Hannover for their helpful suggestions for improving the edition. In this connection I should not forget to mention the names of the twins Steven and Benjamin who have contributed much to prepare this volume.

Finally, I express my gratitude to Mr S. Gupta of New Age Int. (P) Ltd, publisher in New Delhi.

ARUN GHOSH

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Preliminary Mathematical Viewpoints

1.1 Environment Numerically Developed

Almost all the physical problems and many practical applications of technology are dependent on mathematical formulations, especially in terms of *balanced* equations, one of the important implements of mathematics.

If I approach the problem of differential equation with analytical knowledge, I should be really almost certain as to my ability to determine the solution of the problem, but the question arises always at this stage, how many particulars and how many steps are required for the complete solution.

The operations of arithmetic and the algebraic moves in the intermediate steps, necessary for the solution of the problem, can be sometimes complicated and cumbersome.

A large number of problems that have been met casually in the practical field of mathematics and technical science, would be sometimes difficult to solve analytically. *Numerical analysis* would thus be a necessary and an inevitable fitting for the purpose of obtaining the numerical solutions of mathematical equations which cannot be solved or rather difficult to be solved by some standard methods that I want to discuss in the following chapters.

In view of that some methods or techniques are needed for the solutions of the equations that are numerically developed and these methods are known as *numerical methods*.

The numerical methods provide the most impressive and favourable results when they have been applied to difficult problems. There are many numerical methods, some of them are widely known because of precision and accuracy. With precision and accuracy is meant that they produce precise and accurate results by degrees.

The numerical methods available at present age, are very significant and predominant because of widespread acclamation in practice. In particular, the advent of digital computer has brought a revolution to the system of numerical analysis that involves the methods for the numerical development of the solutions.

In other words, numerical methods have been successfully employed for numerically developed solutions of mathematically formulated problems.

I consider the numerical procedures throughout this textbook analytically and use them in solving the problems of mathematics and the related problems with the aid of Personal Computer.

The approximation of the solution of mathematical equations has been performed by the program instructions designed by the programming language C++ from BORLAND International using the appropriate numerical methods that are well fitted.

1.2 Error Spread

In attempting to determine the numerical solutions of mathematical problems the confrontation with the number system which generally involve errors is conventional. We want to discuss now the error spreading caused by the number system.

Error spread is defined as the means of developing at a certain phase of a calculation further flaw that arises out of the error at a previous point of calculation. Unfavourable propagation of error might be sometimes serious and could influence the computed results. The approach by which indefinite assumptions of initial and boundary conditions propagate in the final results can be manipulated properly applying a good numerical method.

Basically, the errors are *inherent* in the formulation of mathematical or physical problems. Mathematical constants like π (Pi), *e* (exponential function), Ln 2 (logarithmic function) and physical parameters, such as, force of attraction, velocity of light (*c*), etc. cause the formulation of the problem inaccurate when the parameters rounding to some decimal places have been taken.

Most of the arithmetical of algebraic calculations produce numbers that contain unlimited decimal representations. Such long representation cannot be accepted in the approximation process because of limited word-length capacity of the computer. For the sake of convenience, I restrict myself to a limiting decimal representation rounding the number up or down. The error caused by rounding a number is known as *roundoff error*.

The roundoff error plays an important role in the approximation system. In order to determine the accurate representation of a number, the value of the error, that is rounded, must be added to approximate value.

We experience from our common mathematical practice another type of *error* called truncation error that arises generally in the definition of an elementary mathematical function. Applying the series expansion theorem to a trigonometric function it is obtained for

$$\text{Sin}\, x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x7}{7!} + \ldots$$

To approximate the function sin x by a numerical procedure considered in Chapter 6, the series must be bounded by limiting the value of k. The truncation error is the result of truncating an infinite series in order to obtain a finite series.

In view of the above considerations let us now formulate the definition of error.

Error = True value – Approximate value.

Assuming the values Tv = 0.96543425 and Av = 0.96542310 the definition of an error yields.

Error = 0.96543425 – 0.96542310 = 0.00001115 = 1.115×10^{-5}.

Dividing the error by the true value we obtain the relative error that is defined by

$$\text{Relative error} = \frac{\text{error}}{\text{true value}} = 1.154 \times 10^{-5}.$$

The use of relative errors is significant when the true value of a quantity is very small or too large. Sometimes the error is expressed by the symbol |error| that is meant for *absolute error*. The floating-point representation of a number that is rounded or chopped in a computational procedure often causes the development of an error. The error that committed once, can affect the intermediate results of the operations leading to erratic final results.

Let us consider a simple equation. It is nothing but a problem of finding the two roots of a quadratic equation

$$ax^2 + bx + g = 0, \quad \text{where} \quad a, b, g \quad \text{are constants.}$$

Example 1.1 $x^2 + 185.21x + 12.451 = 0$.

Applying the quadratic formula from algebra for determining the roots that state

$$x_{1,2} = \frac{-b \pm \sqrt{(b^2 - 4ag)}}{2a}.$$

Here $b^2 = 34302.744$ (rounding to 3 decimal places)

$b^2 - 4ag = 34252.940$ (rounding to 3 decimal places)

$\sqrt{(b^2 - 4ag)} = 185.075$ (rounding to 3 decimal places)

True roots $x_{1,2} = -0.0672508$ and -185.1427492

Approximate roots $x_{1,2} = -0.06750$ and -185.14250.

It is evident that there are discrepancies in the values between true and approximate roots. The discrepancies are due to roundoff errors that arise in the decimal representation for the quantity b^2 considered at start. The error developes step by step in further rounding a quantity and produce the unwanted results in subsequent operations. In this manner an error propagates through succeeding estimations that yield finally inaccurate results. At this stage I like to make another approach to roundoff error spread considering an Initial Value Problem (IVP) that will be discussed now.

Example 1.2 $Dy/dx = 2y\,(x \text{ Ln } x) + 1/x$ with the initial condition $y(e) = 1.0$ ($e = 2.71828182$).

The exact solution of the IVP is $y(x) = \text{Ln } x\,(2\text{Ln } x - 1)$.

The initial condition $y(e) = 1.0$ satisfies the exact solution by setting $x_0 = e$, initially chosen value of x. The approximation of $y(x)$ developes with a rounding error when I set the representation of e with 5 places after decimal places, that means, accepting the value for $x_0 = e = 2.71828$. All the succeeding approximations starting with y_0 provide inaccurate results because of rounding errors once determined for x_0.

Putting the chosen value of e, rounded, in the exact solution yields a result that contradicts the initial conditions.

In the exact solution I could substitute the accurate incremental values of x in order to get the exact values for $y(x_0)$, $y(x_1)$, etc. But the acceptance of the rounded initial condition for $x_0 = 2.71828$ and its imposition on the formula for the determination of the approximations produce error in the computed solutions. The errors develop gradually in unallowed magnitudes.

1.2.1 Error Reduction

I have so far discussed about the possible types and sources of errors occurred in the approximate solutions. I want to point out now some techniques in order to reduce the errors. The escape from errors in toto is almost out of the question, even if a realistic estimation is achieved.

A numerical method is a complete and unambiguous process for the constructive solution of mathematical problems that require particular numerically developed solutions together with a suitable error analysis. A numerical method is an integral part of numerical analysis, an important subject matter of mathematics, that deals with the various aspects of error development in numerical methods.

The most important fundamental requirement for computable error estimate is the *convergence* of an algorithm. An algorithm, a set of well-defined steps for the solution of a problem, is said to be convergent, if the approximations are more closely to the sequence of true solutions of the problem. The efficiency of the convergence of algorithm is as well an important factor. The size of the error after a finite number of operations must be estimated when the condition of convergence is not satisfied in some cases.

The error estimates are extensively accepted by the numerical methods that are developed step-by-step for the approximation of differential equation presented in the example. In order to make the estimate accurate, the value of width (w) must be considered very small.

Forward error analysis and *backward* error analysis are the two known methods that are also useful in numerical analysis. The procedures have been preferred to successful analysis of errors.

The most simple case of error estimate is to make use of *double precision*. The representation of a number that is double the usual number of bits, is known as double precision representation. The double precision arithmetic can be used to achieve a required level of accuracy.

2

Computing Surface Areas

2.1 Evaluation of the Definite Integral

Let us consider a function $g(x)$ that is supposed to be finite and continuous on a closed interval $[\alpha, \beta]$, assuming $\beta > \alpha$.

The function

$$y = g(x) \qquad \ldots(2.1)$$

is integrable on $[\alpha, \beta]$ based on the aforesaid considerations and with the finite values on the interval represents a curve that is moving continuously in a path curving inwards or outwards.

Now we make some mathematical observations on the function $g(x)$ with some presumed values of x within the range of the interval $[\alpha, \beta]$. The base of the field of observation amounting to $\beta - \alpha$, is divided into m parts each of width w providing the values for x finite in quantity:

$$\alpha, \alpha + w, \ \alpha + 2w, \ldots, \alpha + (m-1)w, \ \beta = \alpha + mw.$$

We are now at liberty to erect perpendiculars on these points in the x-axis which aid us to form rectangular figures connecting the curve represented by the equation (2.1). The aggregate of the rectangles in the limits yields the area of the space enclosed by the curve $y = g(x)$, the x-axis and the two fixed ordinates corresponding to $x = \alpha$ and $x = \beta$.

The spatial area can now be represented by the numerical measure

$$I = \int_{\alpha}^{\beta} g(x)\, dx = G(x) \Big|_{\alpha}^{\beta} = G(\beta) - G(\alpha). \qquad \ldots(2.2)$$

The relation (2.2) can be interpreted as the integral of $g(x)\,dx$ from α to β, where α is the lower bound and β is the upper bound of the integral. The function $g(x)$ to be integrated is called the integrand and the differential dx indicates that x is the *index of integration*. The integrand can be termed also as the subject of integration as suggested by the mathematician G.H. Hardy of Cambridge University.

The measure $G(\beta) - G(\alpha)$ that depends only on the limits α and β and the shape of the function $g(x)$, is named as the integral sum or the *definite integral* or simply, the integral. It is also known as the Riemann integral after the name of a renowned German mathematician. The operational technique by which we get at the integral is called the *numerical integration*. This is a brief study for the evaluation of definite integral from the point of view of the limit of a sum, provided that the function $g(x)$ satisfies certain fundamental necessary and sufficient conditions, such as, existence, continuity, integrability in certain domain.

The geometrical interpretation of definite integral exposes a clear view when it is described graphically as shown in Fig. 2.1.

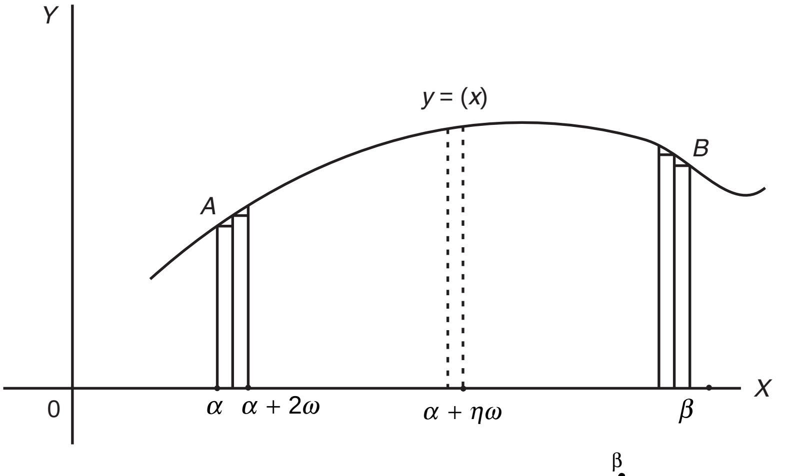

Fig. 2.1 *Graphic representation of the integral* $\int_{\alpha}^{\beta} g(x)\,dx$.

Because of the fact that the indefinite integral

$$\int g(x)\,dx \qquad \ldots (2.3)$$

supplies consistently a definite value provided that the integral (2.3) is defined by finite limits, it is called the definite integral.

I need not be concerned here in detailed discussion about the general properties of definite integrals and also about the proofs of the theorems related to the theory of integration. The availability of standard books concerning the subject matter is so widespread that I make use of the advantage to refer only some of them to this treatise at the end.

The evaluation process for numerical integration can be accomplished with the help of widely known methods, that have well worth mentioning at this point. The methods for the numerical solution of integration due to Gauss, Simpson, Euler, Newton, Maclaurin and Stirling are most usable amongst others.

To determine the general indefinite integral (2.3) two standard methods of integration are generally employed

1. Integration by Substitution.
2. Integration by Parts.

Case 1

Integration by substitution means only a change of the independent variable in an indefinite integral that reduces the integral to another readily integrable form as shown in the example.

Example 2.1 Find the integral of $\int xe^{-x^2}\,dx$.

The substitution $-x^2 = t$ makes a change in the indefinite integral and by means of differentiation we obtain $-2x\, dx = dt$.

Hence $$I = \int xe^{-x^2}\, dx = \int e^{-x^2} \cdot x\, dx = -1/2 \int e^t\, dt = -1/2\, e^t + C.$$

In terms of original variables it becomes

$$I = -1/2\, e^{-x^2} + C,$$

where C is the constant of integration, so-called *guide factor*.

Case 2

Integration by Parts can be applied to the integration of a product of multiple differentiable functions. One has to apply the formula for Integration by Parts that reads

$$uv = \int u\, dv + \int v\, du \qquad \ldots (2.4)$$

u and v being two differentiable functions.

The proper choice of the first function to be integrated is to some extent a matter of practice. In some cases of integration we have to use the formula several times in a process in order to get at the final results.

Example 2.2 Integrate $\int x^2 \text{ Ln } x\, dx$.

Considering Ln x as the first function and $x^2\, dx$ as the second, we put

Ln $x = u$ and $x^2\, dx = dv$.

On differentiating the first function and integrating the second we obtain

$1/x\, dx = du$ and $1/3\, x^3 = v$.

Now if we apply the formula for Integration by Parts given in (2.4) selecting the particulars we find

$$I = \int x^2 \text{ Ln } x\, dx = 1/3\, x^3 \text{ Ln } x - 1/3 \int x^3 \times 1/x\, dx$$

$$= 1/3\, x^3 \text{ Ln } x - 1/3 \int x^2\, dx.$$

As we see that the evaluation of the integral is not complete we can apply the formula (2.4) again by considering the integral $\int x^2\, dx$ as the product of the function given and unity.

Selecting unity as the first function and proceeding as before the final solution of integral becomes

$$I = 1/3\, x^3 \text{ Ln } x - 1/9\, x^3 + C = 1/9\, x^3\, (3 \text{ Ln } x - 1) + C.$$

As a matter of fact the evaluation of the definite integral from the practical viewpoint is regardless of concerning the basic rules and the guide factor C, constant of integration, that is remained in the open.

The evaluation from the practical viewpoint is meant for a computerised solution of a mathematical problem by means of a computer software. The method, I want to describe here, is the analytic method for the numerical solution for the definite integral.

The *Simpson's* extended rule for the approximation of definite integral is represented by the relation

$$I = \frac{w\left[y_0 + 4\left(y_1 + y_3 + \ldots\, y_{m-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{m-2}\right) + y_m\right]}{3} \qquad \ldots (2.5)$$

where m = total number of divisions (even), w = width of each division and $y_0, y_1, y_2, \ldots$, are the values of y i.e., the ordinates at points $x_0, x_1, x_2, \ldots$, on the x-axis.

Many techniques for the approximation of definite integrals are known in the field of numerical analysis. To mention some of them are the Midpoint rule, Trapezoidal rule, Simpson's rule, Gaussian rules and Adaptive quadrature.

Here at this stage, I limit myself to the general presentation of the formula form Simpson without a detailed consideration. If the reader wants to make much of it, and of other methods just mentioned it is advisable to look at the literatures on analysis.

By the relation (2.5) the definite integral of any function or the area is expressed in terms of any number of ordinates within the prescribed range of interval, provided that the function or the area within each of the small confines, can be represented to an adequate degree of approximation by a parabolic function. On that ground Simpsons's rule can be sometimes termed as parabolic rule.

Let us try to elaborate the formulated relation of Simpson by a graphical representation.

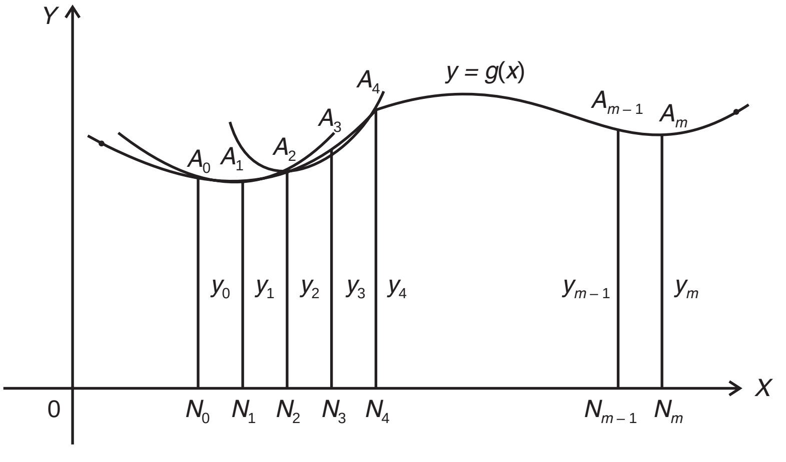

Fig. 2.2

A curve $A_0\,A_m$ defined by $y = g(x)$ is bounded by the interval $x = \alpha$ and $x = \beta$. The interval $N_0\,N_m$ is divided into an even number (= m) of parts, each equal to w by the points $N_1, N_2, \ldots, N_m$, so that $w = (\beta - \alpha)/m$.

Through each successive set of three points (A_0, A_1, A_2); (A_2, A_3, A_4); etc. are drawn arcs of parabolas with vertical axes. Evidently, $y_0, y_1, y_2, \ldots\, y_m$ are the corresponding ordinates.

Now a parabola with a vertical axis is drawn through the first set of points (A_0, A_1, A_2), the equation of which becomes

$$y = ax^2 + 2bx + c \qquad \ldots (2.6)$$

where a, b and c are constants. Now the required area $A_0\,N_0\,A_2\,N_2$ becomes

$$A = \int_{-w}^{w} \left(ax^2 + 2bx + c\right) dx = 2w\left(aw^2/3 + c\right). \qquad \ldots (2.7)$$

The relation (2.6) takes the form

1. $y_0 = N_0\,A_0 = aw^2 - 2bw + c$, when $x = -w$;

2. $y_1 = N_1 A_1 = c$, when $x = c$,
3. $y_2 = N_2 A_1 = aw^2 - 2bw + c$, when $x = -w$;

In view of these relations when combined we obtain

$$1/3w\,(y_0 + 4y_1 + y_2) = 2/3aw^3 + 2cw = 2/3w\,(aw^2 + 3c)$$

that is the area of first parabolic strip $N_0 A_0 A_1 A_2 N_2$ according to the equation (2.7).

In a similar manner we obtain

area of the second strip $N_2 A_2 A_3 A_4 N_4 = 1/3w\,(y_2 + 4y_3 + y_4)$

… … …

area of the last strip $N_{m-2} A_{m-2} A_{m-1} A_m N_m = 1/3w\,(y_{m-2} + 4y_{m-1} + y_m)$.

Summing up all together, the area under the curve denoted by

$$\int_{\alpha}^{\beta} g(x)\,dx$$

is approximately given by the relation

$$I = \frac{w\left[y_0 + 4\left(y_1 + y_3 + \ldots\, y_{m-1}\right) + 2\left(y_2 + y_4 + \ldots + y_{m-2}\right) + y_m\right]}{3}$$

which is the same as represented in equation (2.5).

Thus we have derived the Simpson's formula for the approximation of definite integral and to solve a definite integral I apply the mathematical formula according to the extended Simpson's rule represented by the relation (2.5).

Example 2.3 Evaluate $\int_0^1 x e^x\, dx$ by Simpson's rule considering 10 equal intervals.

Given $a = 0$, $b = 1$, $m = 10$, $w = 1/10 = 0.1$, and the functional relation $y = g(x) = x\,e^x$.

Substituting the values for x = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 in

$$y = x\,e^x$$

we obtain the approximate values

$y_0 = 0$,

$y_1 = 0.11051709$, $y_2 = 0.24428055$, $y_3 = 0.40495764$,

$y_4 = 0.59672987$, $y_5 = 0.82436063$, $y_6 = 1.09327128$,

$y_7 = 1.40962689$, $y_8 = 1.78043274$, $y_9 = 2.21364280$,

$y_{10} = 2.71828182$.

Hence, The estimated area A

$$= \frac{w\left[y_0 + 4\left(y_1 + y_3 + y_5 + y_7 + y_9\right) + 2\left(y_2 + y_4 + y_6 + y_8\right) + y_{10}\right]}{3}$$

$$= 1.00000436.$$

On integrating the original function yields

$$I = \int_0^1 x\,e^x\,dx = x\,e^x \Big|_0^1 - \int_0^1 e^x\,dx = e^x\,(x - 1)\Big|_0^1 = 1.00000000.$$

The estimated value thus obtained for the area can be eventually compared with the exact value determined by integration in order to find the error for approximation.

If we have a look at both the resulting values, we see that the values for estimation and exact are correct to 5 decimal places. A better approximation reduces the percentage of error and by raising the number of divisions of the interval we can get at a better approximation.

The approximate value obtained by using the analytic method from Simpson which is already discussed, can be reasonably acceptable considering a carefully suggested computational width.

The program g2CSINTG written in C++ (Turbo) language is used to approximate the definite integral. Before starting to work out the examples by means of the program, it is recommendable to look at this brief layout referred to section 2.4.

Example 2.4

DI: $\int_0^1 \text{Ln}\,(x^2 + 1)\,dx$

ES: 0.2639435

W: 0.03125, K: 5, M: 32.

The abbreviations, such as DI, ES, W, K and M have been explained in the Program. Therefore, before operating on this problem by means of the Program one should read the instructions mentioned in the Program.

When the process of integration executed by the program is over we obtain the following on the screen:

Estimated solution of the integral:	0.2639435
Exact solution of the integral:	0.2639435
Error for approximation:	–2.06035267e–09

The significant digit of the error is at the 9th decimal place.

The approximate value of the definite integral along with the actual value has been determined. The approximation is correct to 8th place after decimal and the amount of error caused by the approximation begins to propagate at the 9th decimal place.

Let the program run with other examples.

Example 2.5

DI: $\int_1^2 (x - 3)/x^2\,(x + 1)\,dx$

ES: 4 Ln (4/3) – 3/2 [put 1.33333 for 4/3 and 1.5 for 3/2]

W: 0.015625, K: 6, M: 64.

Estimated solution of the integral:	–0.34927172
Exact solution of the integral:	–0.34927172
Error for approximation:	6.02869e–09.

The significant digit of the error is at the 9th decimal place.

Example 2.6

$$\text{DI: } \int_0^1 1/(1+x)\text{Sqrt }(1+2x-x^2)\ dx$$

ES: $\pi/5.656854$

W: 0.015625, K: 6, M: 64.

Estimated solution of the integral:	0.55536039
Exact solution of the integral:	0.55536039
Error for approximation:	8.18047663e–09.

The significant digit of the error is at the 9th decimal place.

Example 2.7

$$\text{DI: } \int_0^{\sqrt{2}} x^2 e^{x^2}\ dx$$

ES: 4.19452800

W: 0.00552427, K: 8, M: 256.

Estimated solution of the integral:	4.19452806
Exact solution of the integral:	4.19452800
Error for approximation:	–6.3578204e–08.

Integration of Trigonometric Functions

Before I enter upon the evaluation process for the approximate solution of definite integrals involving trigonometric functions, I have the intention to consider some standard indefinite integrals containing trigonometric functions or transcendental functions of related functions.

Example 2.8 Find the integral of $\int \sin^2 x \cos^5 x\ dx$.

The integral can be rearranged as

$$I = \int \sin^2 x \cos^5 x\ dx = \int \sin^2 x \cos^4 x\ \cos x\ dx. \quad \ldots (2.8)$$

As it is known from Trigonometry that $\sin^2 x + \cos^2 x = 1$ and putting it in (2.8) for $\cos^4 x$ yields a simplified form

$$I = \int \sin^2 x\ (1-\sin^2 x)^2 \cos x\ dx$$

$$= \int \sin^2 x\,(1 - 2\sin^2 x + \sin^4 x)\cos x\; dx$$

$$= \int \sin^2 x - (2\sin^4 x + \sin^6 x)\cos x\; dx.$$

Setting sin $x = u$ yields of differentiation cos $x\,dx = du$.

By means of the transformation we have

$$I = \int \left(u^2 - 2u^4 + u^6\right) du.$$

On integrating and rearranging we get

$$I = 1/3\, u^3 - 2/5\, u^5 + 1/7\, u^7 + c$$
$$= 1/3 \sin^3 x - 2/5 \sin^5 x + 1/7 \sin^7 x + c$$
$$= \sin^3 x\,(35 - 42\sin^2 x + 15\sin^4 x)/105 + c$$

where c, the guide factor, can be found out when the limits of integration are known.

Example 2.9 Integrate $\int \tan^4\theta\; d\theta$.

From trigonometrical relation we know $\sec^2\theta - \tan^2\theta = 1$.

Arranging the integral properly and using the trigonometric formula we obtain

$$\int \tan^4\theta\; d\theta = \int \tan^2\theta \cdot \tan^2\theta\; d\theta$$
$$= \int \tan^2\theta\,(\sec^2\theta - 1)\; d\theta$$
$$= \int \tan^2\theta \sec^2\theta\; d\theta - \tan^2\theta\; d\theta$$
$$= 1/3 \tan^3\theta - \int (\sec^2\theta - 1)\; d\theta$$
$$= 1/3 \tan^3\theta - \tan\theta + \theta + C$$

is the general solution of the integral.

Let us consider another illustrative example applying the formula for integration by parts stated in equation (2.4) in section 2.1.

Example 2.10 Evaluate $\int e^{2x} \sin 3x\; dx$.

Let

$u = e^{2x}$ and $dv = \sin 3x\, dx$,

$du = 2\, e^{2x}\, dx$ and $v = -1/3 \cos 3x$.

Then applying the formula for integration by parts we expand the integral

$$\int e^{2x} \sin 3x\; dx = -1/3\, e^{2x} \cos 3x + 2/3 \int e^{2x} \cos 3x\; dx \qquad (\alpha)$$

Again let

$u = e^{2x}$ and $dv = \cos 3x\, dx$,

$du = 2\, e^{2x}\, dx$ and $v = -1/3 \sin 3x$.

Then applying the relation (2.4) again for the integral we get

$$\int e^{2x} \cos 3x \; dx = 1/3 \, e^{2x} \sin 3x - 2/3 \int e^{2x} \sin 3x \; dx. \qquad (\beta)$$

Substituting (β) in (α) and rearranging we get finally

$$\int e^{2x} \sin 3x \; dx = e^{2x} / 13 \left(2 \sin 3x - 3 \cos 3x\right) + c.$$

The approximation process for definite integrals involving trigonometric functions can now be performed by applying the mathematical formula from Simpson and using the same program g2CSINTG.

Example 2.11

DI: $\int_0^{\pi} (1 - \cos x)^2 \, dx$

ES: $3\pi/2$

W: 0.39269908, K: 3, M: 8.

Estimated solution of the integral:	4.71238898
Exact solution of the integral:	4.71238898
Error for approximation:	8.97954067e–09.

Example 2.12

DI: $\int_0^{\pi/2} \sin x \; \text{Ln} \, (\sin x) \; dx$

ES: –0.30611106

W: 0.09817477, K: 4, M: 16.

Estimated solution of the integral:	–0.30611106
Exact solution of the integral:	–0.30611106
Error for approximation:	–8.83163792e–09.

Example 2.13

DI: $\int_0^{1} (\cos^{-1} x)^2 \, dx$

ES: $\pi - 2.0$

W: 0.015625, K: 6, M: 64.

Estimated solution of the integral:	1.14159265
Exact solution of the integral:	1.14159265
Error for approximation:	–4.45322357e–09.

The significant digit of the error is at the 9th decimal place.

Example 2.14

DI: $\int_0^{\pi} x \sin x \cos^2 x \, dx$

ES: $\pi/3$

W: 0.01227185, K: 8, M: 256.

Estimated solution of the integral:	1.04719755
Exact solution of the integral:	1.04719755
Error for approximation:	–3.96789335e–09.

The higher-order error term makes the Simpson's rule significantly predominant to the rules in almost all states, provided that the width of the interval is very small.

We mention now some approximated results in tabular form applying Simpson's rule to various integral functions. The reader should verify these approximations by means of the Program mentioned before and try to find the amount of error in each case.

Table 2.1

$g(x)$	$e^x/(1+e^x)$	x Ln$(1+2x)$	$1/[x(1+2x)^2]$	$1/\sin 2x$	$x \tan x/(\sec x + \tan x)$
Interval	[0, Ln2]	[0,1]	[1,2]	$[x/6, \pi/3]$	$[0, \pi]$
Exact	Ln(3/2)	3Ln 3/8	Ln(6/5)–2/15	Ln$\sqrt{3}$	$\pi(\pi-2)/2$
Division (m)	2^6	2^6	2^5	2^6	2^7
Estimated	0.40546511	0.41197961	0.04898824	0.54930615	1.79320952

2.2 Areas of Curves

Consider two fixed points, $x = \alpha$ and $x = \beta$ on an interval $[\alpha, \beta]$, that is closed, in cartesian coordinates system. Assume a plane curve that is moved continuously in a path and defined by the relation

$$y = \phi(x). \quad \ldots (2.9)$$

In section 2.1 we have already discussed about some mathematical observations on the function $\phi(x)$ with the supposition that the function is bounded and continuous on the interval $[\alpha, \beta]$, accepting $\beta > \alpha$. The area of the figure formed by the curve, by the two extreme lines (so-called ordinates) at the points $x = \alpha$ and $x = \beta$ and by a segment of the base (x-axis) is given by the formula

$$\text{Area} = \int_{\alpha}^{\beta} \phi(x)\, dx = \psi(x) \Big|_{\alpha}^{\beta} = \psi(\beta) - \psi(\alpha) \quad \ldots (2.10)$$

Hence, the area, in general, is the outcome of indefinite integral with finite limits and by definition, it is the definite integral that can be measured numerically by the relation (2.10). In fact, it is simply a geometrical representation with the corresponding data. About more consideration of the subject matter regarding areas of plane curves I wish to refer to the standard text mentioned in the reference.

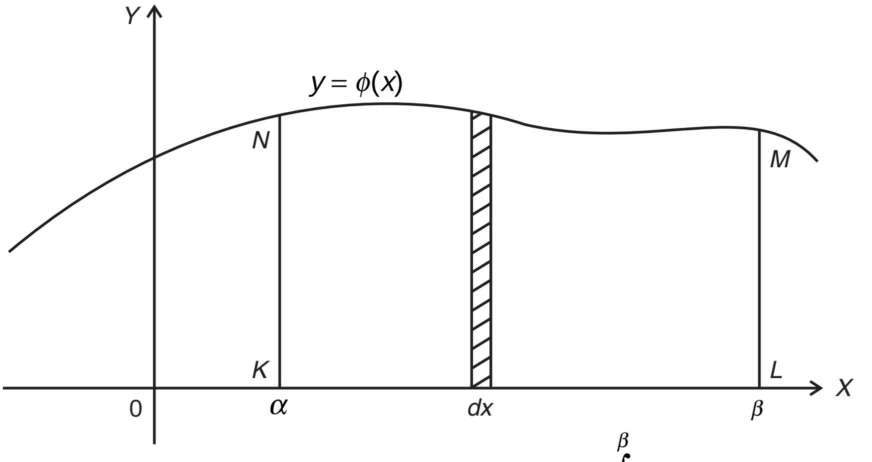

Fig. 2.3 *Graphical Figure for the integral* $\int_{\alpha}^{\beta} \phi(x)\, dx$.

The estimation process for evaluation of areas of curves can be performed by the program `g2CSAREA` and the sequence of events follows its proper course according to the instructions outlined in the program.

The program just mentioned that is used here to approximate the definite integral arisen from the problem of area of plane curve is designed by the programming language C++ applying the method of Simpson's formula already described in section 2.1.

Example 2.15 Compute the area bounded by a single half-wave of the sine curve represented by $y = \sin x$ and the x-axis. The limits of integration are $(0, \pi)$.

DI: $\int_{0}^{\pi} \sin x \; dx$

ES: 2.0

W: 0.00306796, K: 10, M: 1024.

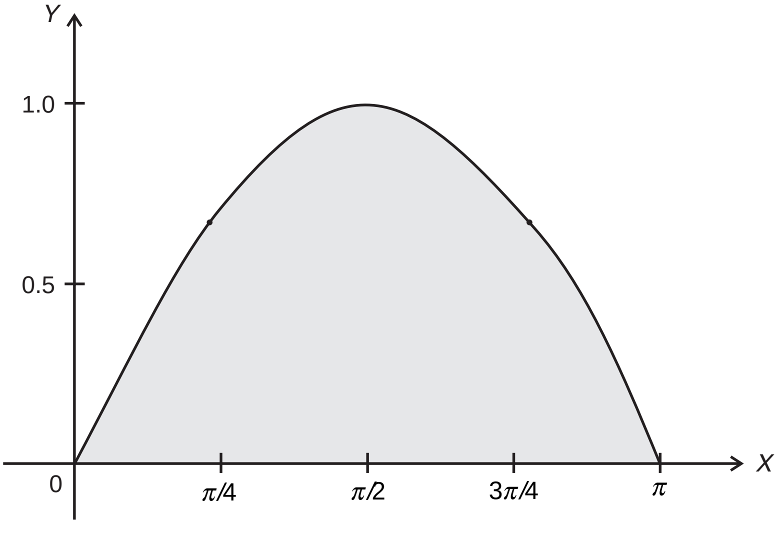

Fig. 2.4

Estimated solution of the integral:	2
Exact solution of the integral:	2
Error for approximation:	–9.82325332e–13.

The significant digit of the error is at the 13th decimal place.

Here, the estimation is highly satisfied in which the error for approximation is very negligible. Thus, we can conclude that this numerical method has a prevailing character in order to obtain the approximate solution.

Example 2.16 Compute the area of a segment cut off by the straight line $y = 3 - 2x$ from the parabola $y = x^2$.

The problem can be arranged in computational form as follows:

DI: $\int_{-3}^{1} (x^2 + 2x - 3)\, dx$

ES: –32/3

W: 0.0625, K: 6, M: 64.

Estimated solution of the integral:	–10.66666666
Exact solution of the integral:	–10.66666666
Error for approximation:	6.66666544e–09.

The significant digit of the error is at the 9th decimal place.

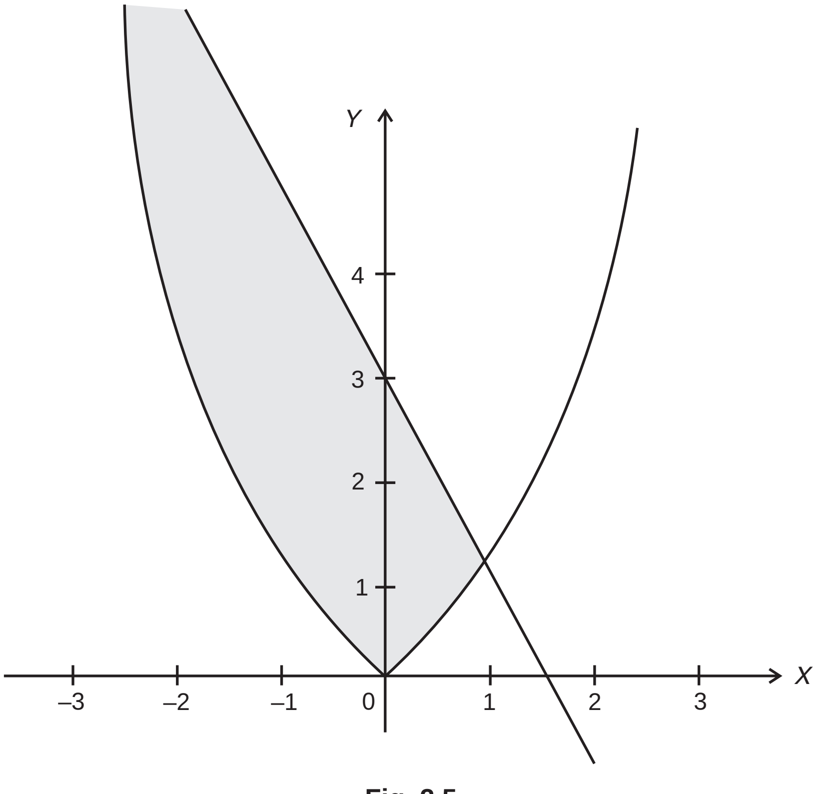

Fig. 2.5

Example 2.17 Find the area held between the curve by $y = \tan x$, the x-axis and the straight line at $x = \pi/3$.

DI: $\int_0^{\pi/3} \tan x \, dx$

ES: Ln 2

W: 0.00409062, K: 8, M: 256.

The computerised solution of the indefinite integral bounded by finite limits $[0, \pi/3]$ yields

Estimated solution of the integral:	0.69314718
Exact solution of the integral:	0.69314718
Error for approximation:	1.95125338e–09.

The significant digit of the error is at the 9th decimal place.

The graphical figure for the area held between the curve produced by $y = \tan x$, the x-axis and the straight line at $x = \pi/3$.

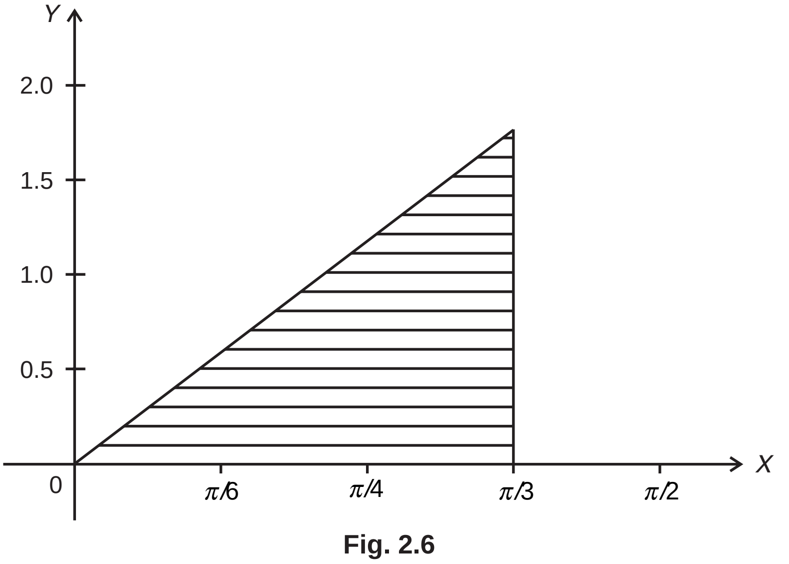

Fig. 2.6

Example 2.18 Compute the area bounded by curved surface $y = \text{Ln } x$, the x-axis and the straight line at the point $x = e$.

DI: $\int_1^{e} \text{Ln } x \, dx$

ES: 1.0

W: 0.02684815, K: 6, M: 64.

Estimated solution of the integral:	0.99999999
Exact solution of the integral:	1
Error for approximation:	–1.3938905e–08.

Here, in this case the integrand is a logarithmic function that is operated within the limiting bounds [1, *e*]. The approximation can be accepted.

The area of the curved surface has been measured within the limits and represented graphically:

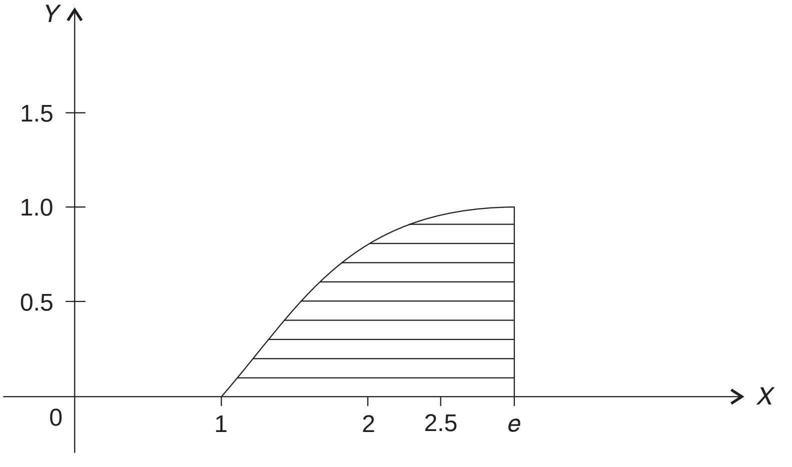

Fig. 2.7

Example 2.19 Find the area included between the graphs of the equations $y^2 = 4x$ and $x^2 = 4y$ which have two points of intersections when $x = 0$ and $x = 4$.

The problem has the following plot for the estimations. Have a try to find out the corresponding data for the approximations forming the equation for the program.

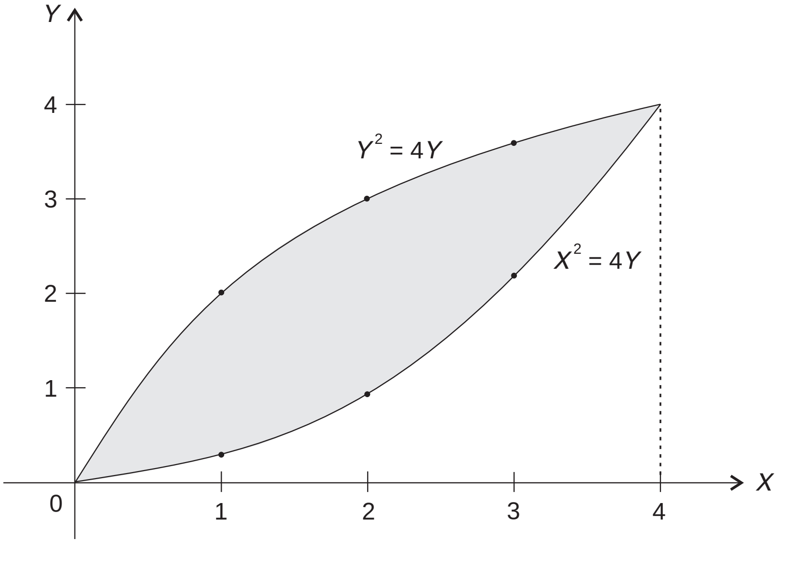

Fig. 2.8

2.3 Surfaces in 2-Dimensional Space

In the previous section we have solved the problems of areas of plane curves by simple integration. The methods discussed in previous section can be applied directly with some adjustments for the determination of numerical approximation of double integrals.

Let us consider a bounded closed region *R* on the *xy*-plane surrounded by the curves $y = \psi_1(x)$ and $y = \psi_2(x)$. The region of integration *R* is bounded by the lines $x = \alpha$ and $x = \beta \; (\beta > \alpha)$.

The considerations can be represented graphically as shown in Fig. 2.9.

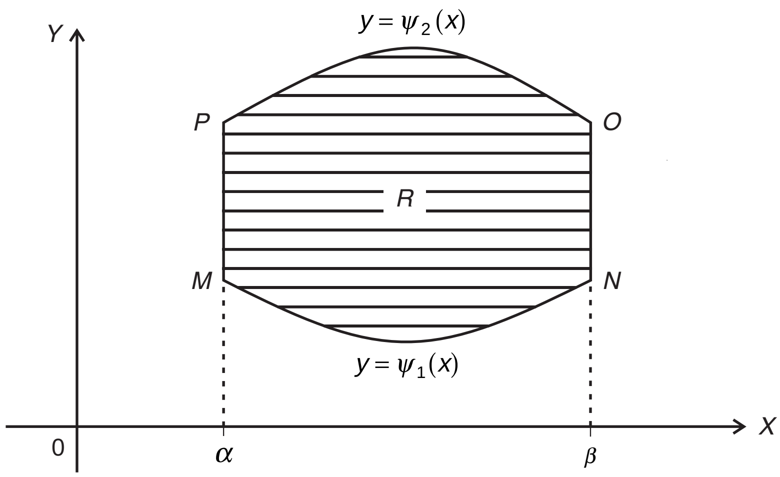

Fig. 2.9 *Graphical representation of the double integral* $\iint_R g(x, y)\, dx\, dy$.

The functions $\psi_1(x)$ and $\psi_2(x)$ are continuous everywhere on the closed interval $[\alpha, \beta]$ and $\psi_2(x) \geq \psi_1(x)$. In this region the variable *x* has a set of functional values that vary from α to β, while the variable *y* varies from $\psi_1(x)$ to $\psi_2(x)$ (for constant *x*).

If the considerations are justifiable, then the above can be represented by expression

$$\iint_R g(x, y)\, dx\, dy = \int_{\psi_1(x)}^{\psi_2(x)} dy \int_{\alpha}^{\beta} g(x, y)\, dx$$

$$\int_{\psi_1(x)}^{\psi_2(x)} \left[\int_{\alpha}^{\beta} g(x, y)\, dx \right] dy \qquad \dots (2.11)$$

and is called the *definite double integral* of the function $g(x,y)$ over the region *R*. The double integral (2.11) is also known as the repeated integral.

The procedure for evaluating a definite double integral such as in (2.11) is to first integrate the inner function with respect to the variable *x* taking *y* as a constant and then to integrate the obtained result with respect to the variable *y*. The order of integration is immaterial provided certain conditions are fulfilled.

Example 2.20 Evaluate the integral $\int_0^2 dx \int_x^{x\sqrt{3}} \dfrac{x}{x^2 + y^2}\, dy$.

Integrating first with respect to the variable y gives

$$\int_0^2 dx \left[\tan^{-1}(y/x)\right]\Big|_x^{x\sqrt{3}} = \left[\tan^{-1}(\sqrt{3}) - \tan^{-1}(1)\right]\int_0^2 dx$$

$$= 0.26179938 \times x\Big|_0^2$$

$$= 0.26179938 \times 2 = 0.5235987.$$

The domain of integration can be represented graphically as shown in Fig. 2.10.

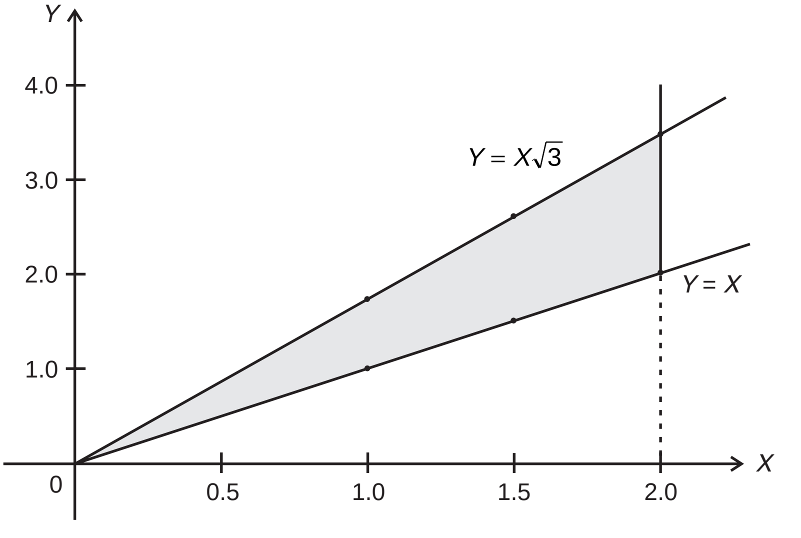

Fig. 2.10

Example 2.21 Evaluate the double integral $\iint_R x^2/y^2 \; dx\, dy,$ if the region of integration R is bounded by the lines $y = x$, $y = 1/x$, $x = 1$ and $x = 2$.

According to the formula (2.11) we can write

$$\psi_1(x) = 1/x, \quad \psi_2(x) = x, \quad \alpha = 1, \quad \beta = 2.$$

Therefore, the definite double integral can now be formulated as

$$I = \iint_R x^2/y^2 \, dx\, dy = \int_1^2 dx \left(\int_{1/x}^{x} x^2/y^2 \, dy\right)$$

$$= \int_1^2 x^2 \left[(-1/y)\right]\Big|_{1/x}^{x} dx = \int_1^2 x^2\left[-1/x + x\right] dx$$

$$= \int_1^2 (x^3 - x)\, dx = \left(x^4/4 - x^2/2\right)\Big|_1^2$$

$$= 2.25.$$

The shaded area in the following graphical figure represents the region of integration *R*.

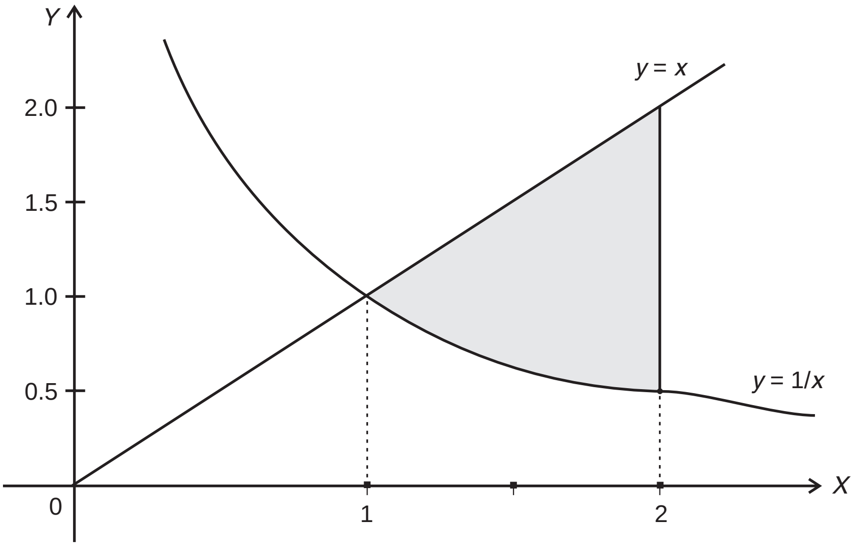

Fig. 2.11

Example 2.22 Computer the double integral $\int_0^{\pi} \sin x \, dx \int_0^{1+\cos x} y^2 \, dy.$

By means of the relation (2.11) we have

$$\psi_1(x) = 0, \ \ \psi_2(x) = 1 + \cos x, \ \ \alpha = 0, \ \ \beta = \pi.$$

Here, the region of integration is shown in Fig. 2.12.

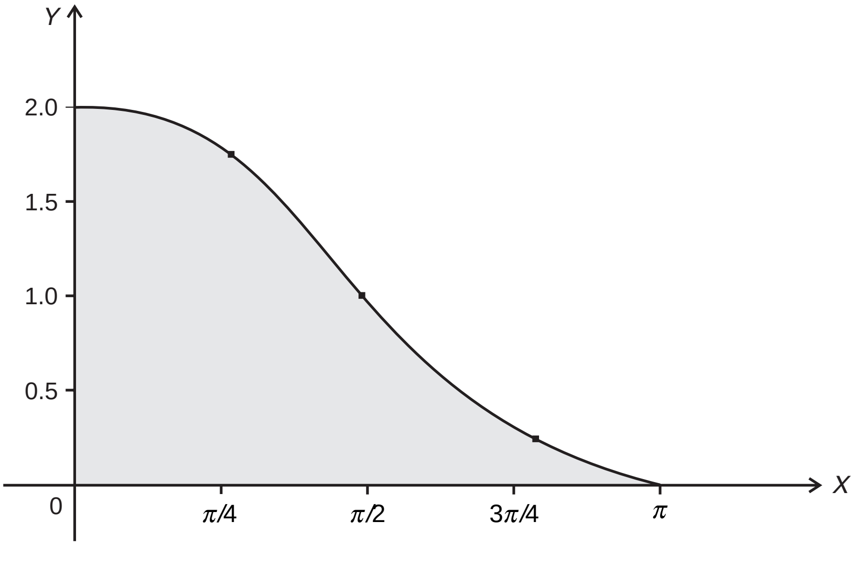

Fig. 2.12

Hence, the definite double integral given

$$I = \iint_R y^2 \sin x \; dx \; dy = \int_0^{\pi} dx \left(\int_0^{1+\cos x} y^2 \sin x \; dy \right)$$

$$= 1/3 \int_0^{\pi} \sin x \, (1 + \cos x)^3 \, dx = -1/12 \, (1 + \cos x)^4 \Big|_0^{\pi}$$

$$= -1/12 \, (-16) = 1.3333.$$

Now we start off with this type of integral by employing the composite Simpson's formula with respect to the variables x and y in the program named g2CS2DIM designed in C++ (Turbo) programming language.

In section 2.4 some suggestions how to use the program with the necessary data have been given. The number of functional evaluations required for the approximation is so large that the processing takes much time to get at the final results. If we select $k = 8$ for superscript, so that m (total number of divisions) becomes $2^8 = 256$. The number needed for functional operations is, in the case, $256 \times 256 = 65536 = 2^{16}$.

Let us follow some examples although the approximations are not so satisfactory as desired.

Example 2.23

DI: $\int_{-0.1}^{0.1} dy \int_{1.3}^{1.5} \sqrt{x y^2} \; dx$

ES: 0.00015775

Wx: 0.00078125, Wy: 0.00078125, K: 8, M: 256.

Estimated solution of the integral:	0.00015775
Exact solution of the integral:	0.00015775
Error for approximation:	2.226094e–09.

Example 2.24

DI: $\int_{-1}^{1} dy \int_{0}^{3} \cos \pi/2 \cos \pi/2 \; dx$

ES: $-8/\pi^2$

Wx: 0.00585937, Wy: 0.00390625, K: 9, M: 512.

Estimated solution of the integral:	–0.8105593
Exact solution of the integral:	–0.81056947
Error for approximation:	–1.01724714e–06.

Example 2.25

$$\text{DI: } \int_0^{0.5} dy \int_0^{0.5} \sin(xy)/(1+xy)\ dx$$

ES: 0.01406353

Wx: 0.00097656, Wy: 0.00097656, K: 9, M: 512.

Estimated solution of the integral:	0.01401245
Exact solution of the integral:	0.01406353
Error for approximation:	5.10811269e–05.

Example 2.26

$$\text{DI: } \int_0^{0.1} dy \int_0^{0.1} e^{y-x}\ dx$$

ES: 0.01000830

Wx: 0.00019531, Wy: 0.00019531, K: 9, M: 512.

Estimated solution of the integral:	0.01000638
Exact solution of the integral:	0.01000830
Error for approximation:	1.9172e–06.

2.4 Related Software for the Solution

I have considered in this chapter the approximating integrals of one-dimensional and two-dimensional functions. Surface areas enclosed by plane curves or by segments of curves have been determined also by considering the integrals within the extent of integration.

The subject matter has been extended to the surfaces in 2-dimensional space (Double integral) discussing the cases with the help of graphical representation of the figures.

In section 2.1 some of the techniques for the approximation have been mentioned. But the mostly acclaimed Simpson's extended method can be preferred as general method provided that the integrand function $g(x)$ is sufficiently smooth.

In other words, it produces accurate approximations if the function $g(x)$ does not oscillate in a subinterval of the domain of integration. After all, the application of the rule stated in equation (2.5) of section 2.1 is very easy.

The Computer programs used to approximate the definite integrals have been designed in programming language: C++ (Turbo).

The program g2CSINTG implements the extended Simpson's rule and is used to determine the approximations for the problems of definite integrals considered in section 2.1.

The program g2CSAREA evaluates the areas of surfaces when the functional relation is of the form

$\int_\alpha^\beta \phi(x)\ dx$ involving a single variable x.

The program g2CS2DIM is made use of determining the approximations for double integrals discussed in section 2.3.

The detailed description of each program can be found when one uses the floppy disc prepared and supplied for the purpose.

3

Systems of Simultaneous Equations

3.1 Preliminary Concepts

Generally the systems of equations are the analytic representations of physical problems. The system of equations of some order is the common occurrence in connection with mechanical, dynamical and astronomical problems. The mathematical model of the mechanical problem dealing with small vibrations or small derivations is of immense importance in the context of simultaneous system of equations.

Simultaneous system is a dependent system of equations. In short, simultaneous means combined. In order to make it free from the state of being dependent, we have to study the system eliminating the variables from the equations that represent quantitative properties. When the equations in the system have possessed all linear properties, it is the simplest to study.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m &= b_2 \\
\cdots \qquad \cdots \qquad \cdots \qquad \cdots & \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mm}x_m &= b_m.
\end{aligned}
$$

It is evident that the number of the variables $x_1, x_2, \ldots, x_m$ being carried by each equation is the same as the number of equations being considered.

In this case it is presumably assured that there exists a unique solution. Our concern here is to find the solution of m simultaneous linear equations in m unknowns.

For determining the solution of linear equations the most elementary method introduced by *Gauss* can be applied. It is based on the principles of eliminating the variables $x_1, x_2, x_3, \ldots, x_m$ from the selected equations with proper manipulation leaving at least one equation in the final stage that contains only one variable.

By substituting the value of the admitted variable in other reduced equation containing two variables we determine the value of the next unknown variable. We continue the procedure until the values for all the variables are known.

Let us illustrate an example.

Example 3.1 Consider a set of 4 equations denoting by

$$x_1 - x_2 - 2x_3 + 4x_4 = 3 \quad \text{(a)}$$
$$2x_1 + 3x_2 + x_3 - 2x_4 = -2 \quad \text{(b)}$$
$$3x_1 + 2x_2 - x_3 + x_4 = 1 \quad \text{(c)}$$
$$5x_1 - 2x_2 + 3x_3 + 2x_4 = 0. \quad \text{(d)}$$

First of all we consider the equation (a) to eliminate the unknown variable x_1 from the equations (b), (c) and (d) accomplishing the operations $(b - 2a) \to b$, $(c - 3a) \to c$ and $(d - 5a) \to d$.

As a result of that we get a new system of equations with (a) remaining unchanged

$$x_1 - x_2 - 2x_3 + 4x_4 = 3 \quad \text{(a)}$$
$$5x_2 + 5x_3 - 10x_4 = -8 \quad \text{(b)}$$
$$5x_2 + 5x_3 - 11x_4 = -8 \quad \text{(c)}$$
$$3x_2 + 13x_3 - 18x_4 = -15. \quad \text{(d)}$$

For the sake of convenience the new system has been marked with the same labels. Now we consider the equation (b) to eliminate the unknown x_2 from (c) and (d) applying the operations $(5d - 3b) \to c$ and $(c - b) \to d$.

Accepting the equation (a) unaltered the resulting system takes the form as represented

$$x_1 - x_2 - 2x_3 + 4x_4 = 3 \quad \text{(a)}$$
$$5x_2 + 5x_3 - 10x_4 = -8 \quad \text{(b)}$$
$$50x_3 - 60x_4 = -51 \quad \text{(c)}$$
$$x_4 = 0. \quad \text{(d)}$$

Finally we get at another new system of equations that has been represented in reduced form. If we use at this stage the process of backward substitution for the solution of the unknown variables we obtain from the equation (c) taking into account from (d)

$$x_4 = 0,\ x_3 = (-51 + 60 \times 0)/50 = -51/50 = -1.02.$$

Accordingly, the equation (b) yields

$$x_2 = [-8 - 5 \times (-51/50)]/5 = -29/50 = -0.58$$

and (a) yields

$$x_1 = 3 - 29/50 - 51/25 = 19/50 = 0.38.$$

Hence, the solution of the system becomes

$$(x_1/x_2/x_3/x_4) = (0.38/-0.58/-1.02/0)$$

In the treatment of a number of simultaneous equations we have not faced any formal difficulties in connection with algebraic processes as the above example shows. But the considerable effort involved at each step in writing down the variables x_1, x_2, x_3 and x_4 throughout and in manipulating the unknowns imply that we need a developed method to deal with them in a simple way.

On this ground matrix representation has been introduced to carry out the complicated calculations in a simple manner. A linear system is often represented by matrix form that contains all the properties of the system.

3.2 Matrix Relationship

A matrix is a rectangular array of numbers, usually real, arranged in rows and columns. A system of p linear equations in q unknowns has the general form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1q}x_q &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2q}x_q &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad\quad \ldots & \\ a_{p1}x_1 + a_{p2}x_2 + a_{p3}x_3 + \ldots + a_{pq}x_q &= b_p. \end{aligned} \qquad \ldots (3.1)$$

The relation (3.1) can be conveniently written as

$$\sum_{1=1}^{q} a_{k1}\, x_1 = b_k, \quad k = 1, 2, \ldots, p. \qquad \ldots (3.2)$$

In matrix representation the equation (3.2) takes the compact form

$$Ax = b \qquad \ldots (3.3)$$

where the coefficients $[a_{k1}]$ form a matrix A defined by as follows:

$$A = [a_{k1}] = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1q} \\ a_{21} & a_{22} & \ldots & a_{2q} \\ \ldots & \ldots & & \ldots \\ a_{p1} & a_{p2} & \ldots & a_{pq} \end{bmatrix}. \qquad \ldots (3.4)$$

The matrix A in (3.4) has p rows and q columns. It is customary to write, A is of order $p \times q$ (read as p by q) matrix. In brief, A is $p \times q$ matrix. The (k, l) refers to an *entry* of a_{k1} that is located at the intersection of the kth row and the lth column of the matrix A. When $p = q$, that means, the number of rows equals to the number of columns, the matrix A is said to be a *square matrix* of order p.

The $1 \times p$ matrix $A = [a_{11}\ a_{12}\ \ldots\ a_{1p}]$ is known as p-dimensional row vector and $p \times 1$ matrix

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{p1} \end{bmatrix}$$

is known as p-dimensional column vector.

Accordingly, the unknowns x_1 $(l = 1, 2, \ldots, q)$ and the constants b_k $(k = 1, 2, \ldots, p)$ are both column vectors as shown

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_q \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_p \end{bmatrix}. \qquad \ldots (3.5)$$

The equation (3.1), so far introduced, can now be written in matrix form arranging in proper order

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_q \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_p \end{bmatrix} \qquad \dots (3.6)$$

that suggests that the $p \times q$ matrix A is combined with the $q \times 1$ column matrix x and the combination is equal to the $p \times 1$ column matrix b as has been shown in compact form (3.3).

3.2.1 Matrix Product

The multiplication of two matrices: A of order $p \times n$ and B of order $n \times q$ is a resulting matrix C of order $p \times q$, the entries of which are given by

$$C_{kl} = \sum_{m=1}^{n} a_{km}\, b_{m1} \qquad \dots (3.7)$$

$$= a_{k1}\, b_{11} + a_{k2}\, b_{21} + \dots + a_{kn}\, b_{nl}$$

for $k = 1, 2, \dots, p$ and $1 = 1, 2, \dots, q$.

The matrix product can be defined only when *the number of columns of A equals the number of rows of B.*

Under this condition it can be declared that the matrices A and B are conformable for the product AB.

Let us consider the product of two arbitrary matrices A and B, in which A $(p \times q)$ is conformable with B $(q \times r)$.

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ a_{31} & a_{32} & \dots & a_{3q} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ b_{31} & b_{32} & \dots & b_{3r} \\ \dots & \dots & \dots & \dots \\ b_{q1} & b_{q2} & \dots & b_{qr} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1r} \\ c_{21} & c_{22} & \dots & c_{2r} \\ c_{31} & c_{32} & \dots & c_{3r} \\ \dots & \dots & \dots & \dots \\ c_{p1} & c_{p2} & \dots & c_{pr} \end{bmatrix}.$$

Now, the coefficient c_{32} can be found as shown

$$c_{32} = a_{31}b_{12} + a_{32}b_{22} + \dots + a_{3q}b_{q2}.$$

Example 3.2

$$A = \begin{bmatrix} 4 & 5 & -1 \\ -5 & 0 & 6 \\ 1 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}.$$

We want to show that the product exists. When we multiply A by B we obtain

$$AB = \begin{bmatrix} 4+10-3 & -8+5+1 & 4+15-2 \\ -5+0+18 & 10+0-6 & -5+0+12 \\ 1+6+15 & -2+3-5 & 1+9+10 \end{bmatrix} = \begin{bmatrix} 11 & -2 & 17 \\ 13 & 4 & 7 \\ 22 & -4 & 20 \end{bmatrix}.$$

It is to be noted here that even the product BA is defined, it need not be equal to the product AB. Hence, if we multiply B by A we get

$$BA = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 \\ -5 & 0 & 6 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 8 & -8 \\ 6 & 19 & 19 \\ 19 & 21 & 1 \end{bmatrix}.$$

The product holds true, but $AB \neq BA$ when we compare the results of multiplication. A square matrix A has the same number of rows as the columns that has already been defined.

For example,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \qquad \ldots (3.8)$$

is a square matrix of order 4.

Now we can define a *diagonal matrix* when the entries or elements of A other than those down the *leading* or *principal* diagonal are zero. In other words, a diagonal matrix D is in this case a square matrix of order 4 with off diagonal elements being zero and is represented by

$$D = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix}.$$

The *identity* matrix of order 4 is a diagonal matrix of order 4, in which all the diagonal entries equal to 1 and all other off-diagonal entries equal to 0.

Hence

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The elements (a_{k1}) of a square matrix A of order p are known as *superdiagonal* when $k < 1$ and the elements with $k > 1$ are known as *subdiagonal*. When $k = 1$, the elements are diagonally placed.

In a word, a square matrix A consists of superdiagonal, diagonal and subdiagonal entries. From the relation (3.8) we see that

the entries	$a_{11}, a_{22}, a_{33}, a_{44}$	are diagonal	when $k = 1$,
the entries	$a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}$	are superdiagonal	when $k < 1$ and
the entries	$a_{21}, a_{31}, a_{32}, a_{41}, a_{42}, a_{43}$	are subdiagonal	when $k > 1$.

An *upper-triangular* matrix U is formed when all the subdiagonal entries of the coefficient matrix A are zero. On the other hand, when all the superdiagonal entries of the matrix A are zero we get a matrix called *lower-triangular*, denoted by L.

3.2.2 Nonsingular Matrix: Inverse Matrix

A square matrix A of order p is *nonsingular* or invertible, if the following property holds

$$AA^{-1} = A^{-1}A = I$$

provided that the *Inverse Matrix* A^{-1} exists. A square matrix whose determinant vanishes is known as singular matrix. We will show the validity of this property just mentioned later in section 3.2.3.

The determinant of a matrix is defined as a single number obtained from the elements of a square matrix of some order.

Example 3.3

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 8 \end{bmatrix}$$

is a matrix of order 3.

Now the determinant $\det A = |A| = 1\begin{vmatrix} 2 & 3 \\ 4 & 8 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

$$= 1\,(2 \times 8 - 4 \times 3) - 1\,(1 \times 8 - 2 \times 3) + 1(1 \times 4 - 2 \times 2)$$
$$= 4 - 2 + 0 = 2.$$

Hence, the determinant of the matrix A of order 3 is found to be a number, other than 0. Because of the property that det A does not vanish, we can conclude that the matrix A is nonsingular.

I need not be concerned here to discussing the elementary properties of determinants for the evaluation. In these circumstances I confine myself to conversing some main points that we need for the purpose regarding determinant.

A *minor* of a particular elements a_{k1} in the coefficient matrix A is defined as the value of the determinant $|A|$ obtained by deleting the kth row and lth column of the matrix A. The *cofactor* of the element a_{k1} is defined by the relation

$$A_{k1} = (-1)^{k+1} m_{k1} \qquad \dots (3.9)$$

where m_{k1} is the minor of the element a_{k1}. Considering the 4th order coefficient matrix presented by the relation (3.8) the cofactors can be found as shown

$$A_{11} = (-1)^{1+1}\, m_{11} = m_{11} = \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$A_{12} = (-1)^{1+2}\, m_{12} = -\, m_{12} \;-\begin{bmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{bmatrix}$$

and so on … .

The cofactors play an important role for solving the system of linear equations by matrix methods.

Example 3.4 Let us consider a matrix of 3rd order

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}.$$

Here the determinant of the matrix A becomes $|A| = -24$ which implies that the matrix is nonsingular. Now, the minors of the elements of the matrix appear to be

$$m_{11} = 23, \quad m_{12} = 4, \quad m_{13} = -13$$
$$m_{21} = 7, \quad m_{22} = -4, \; m_{23} = -5$$
$$m_{31} = -11, \quad m_{32} = -4, \; m_{33} = 1.$$

With these admitted values we can construct a matrix of the minors:

$$\begin{bmatrix} 23 & 4 & -13 \\ 7 & -4 & -5 \\ -11 & -4 & 1 \end{bmatrix}.$$

The cofactors can be determined by means of (3.9) and have been represented by

$$A_{11} = 23, \quad A_{12} = 4, \quad A_{13} = -13$$
$$A_{21} = -7, \quad A_{22} = -4, \; A_{23} = 5$$
$$A_{31} = -11, \; A_{32} = 4, \quad A_{33} = 1.$$

Now the cofactors are arranged in a matrix form that is a matrix of 3rd order

$$\text{Cof (A)} = \begin{bmatrix} 23 & -4 & -13 \\ -7 & -4 & 5 \\ -11 & 4 & 1 \end{bmatrix}.$$

3.2.3 Matrix Transposition

The *transpose* of a matrix A of order $p \times q$ is a matrix A^t of order $q \times p$, that is obtained by simply interchanging row and columns. Recalling the relation in (3.6) for the coefficient matrix A we can write again

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \dots & \dots & \dots & \dots \\ a_{p1} & a_{p2} & \dots & a_{pq} \end{bmatrix}.$$

Now the transpose of A becomes

$$A^t = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{p1} \\ a_{12} & a_{22} & \dots & a_{p2} \\ \dots & \dots & \dots & \dots \\ a_{1q} & a_{2q} & \dots & a_{pq} \end{bmatrix}. \quad \dots (3.10)$$

The *adjoint* of A is defined as the transpose of Cof (A). Considering the values of the cofactors found at the end of the section 3.2.2 a matrix can be formed following the definition which is the adjoint of A.

Hence,

As I have mentioned earlier in section 3.2.2 that the inverse of matrix A can be defined only if the square matrix A is nonsingular. That means, det A ($|A|$) does not vanish. Under these circumstances we can derive an important relation that reads

$$\text{Adj}\,(A) = \begin{bmatrix} 23 & -7 & -11 \\ -4 & -4 & 4 \\ -13 & 5 & 1 \end{bmatrix}.$$

$$A^{-1} = \frac{\text{Adj}(A)}{|A|} = -\frac{1}{24}\begin{bmatrix} 23 & -7 & -11 \\ -4 & -4 & 4 \\ -13 & 5 & 1 \end{bmatrix} \qquad \text{... (3.11)}$$

It is easy to show, now, that the following relation is valid.

Hence,

$$AA^{-1} = A^{-1}A = I \qquad \text{... (3.12)}$$

where I is the identity matrix.

A square matrix A is called *symmetric* if the transpose of A is equal to A, i.e., $A^t = A$. Otherwise, it is known as *skew-symmetric* when the relation $A^t = -A$ holds.

The following relations in connection with the transpose of a matrix are sometimes useful.

(α) $(A^t)^t = A$

(β) $(AB)^t = B^tA^t$ [the reverse order is to be noted]

(γ) $(A^{-1})^t = (A^t)^{-1}$, provided that A is nonsingular and the inverse matrix A^{-1} exists.

Let us consider two matrices A and B. The matrix A is a square matrix of order 3 and the matrix B is order 3 × 2 as represented by the expressions

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 6 \\ 0 & 8 \end{bmatrix}.$$

Relation (α)

Evidently the transpose of the matrix A becomes

$$A^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 2 & 5 \end{bmatrix}.$$

Hence, operating on the transposed matrix again we obtain the original matrix as shown

$$(A^t)^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 2 & 5 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{bmatrix} = A.$$

Relation (β)

$$(AB)^t = \left[\begin{matrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 1 & 5 \end{matrix}\right]\left[\begin{matrix} 3 & -1 \\ 2 & 6 \\ 0 & 8 \end{matrix}\right]^t = \left[\begin{matrix} 7 & 35 \\ 16 & 44 \\ 11 & 43 \end{matrix}\right]^t$$

$$= \left[\begin{matrix} 7 & 16 & 11 \\ 35 & 44 & 43 \end{matrix}\right].$$

Similarly,

$$B^t = \left[\begin{matrix} 3 & 2 & 0 \\ -1 & 6 & 8 \end{matrix}\right].$$

Now,
$$B^t\ A^t = \left[\begin{matrix} 3 & 2 & 0 \\ -1 & 6 & 8 \end{matrix}\right]\left[\begin{matrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 2 & 5 \end{matrix}\right] = \left[\begin{matrix} 7 & 16 & 11 \\ 35 & 44 & 43 \end{matrix}\right].$$

Hence, the relation $(AB)^t = B^tA^t$ holds true.

Relation (γ)

We have already found out the elements of the inverse matrix. Recalling the expression presented by the relation (3.11) we have

$$A^{-1} = -\frac{1}{24}\left[\begin{matrix} 23 & -7 & -11 \\ -4 & -4 & 4 \\ -13 & 5 & 1 \end{matrix}\right]$$

Evidently, the transpose of the inverse matrix

$$(A^{-1})^t = -\frac{1}{24}\left[\begin{matrix} 23 & -4 & -13 \\ -7 & -4 & 5 \\ -11 & 4 & 1 \end{matrix}\right].$$

We want to determine now the inverse of the transposed matrix A, that is,

$$(A^t)^{-1} = C^{-1},$$

where

$$C = \left[\begin{matrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 2 & 5 \end{matrix}\right] = A^t.$$

Now, det $C = |C| = -24$. The matrix of the minors becomes

$$\left[\begin{matrix} 23 & 7 & -11 \\ 4 & -4 & -4 \\ -13 & -5 & 1 \end{matrix}\right]$$

The matrix of the cofactors yields

$$\text{Cof}\,(C) = \begin{bmatrix} 23 & 7 & -11 \\ -4 & -4 & 4 \\ -13 & -5 & 1 \end{bmatrix}$$

Now by definition we have

$$\text{Adj}\,(C) = [\text{COf}\,(C)]^t = \begin{bmatrix} 23 & -4 & -13 \\ -7 & -4 & 5 \\ -11 & 4 & 1 \end{bmatrix}$$

Therefore,

$$C^{-1} = \frac{\text{Adj}(C)}{|C|} = -\frac{1}{24}\begin{bmatrix} 23 & -4 & -13 \\ -7 & -4 & 5 \\ -11 & 4 & 1 \end{bmatrix} = (A^t)^{-1}$$

Thus, the relation γ, that reads, $(A^{-1})^t = (A^t)^{-1}$ holds true.

3.3 Systems of Linear Equations

We suppose a set of p linear equations associated with p unknown variables $x_1, x_2, \ldots, x_p$ that is expressed in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad &\ldots \\ a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pp}x_p &= b_p \end{aligned} \qquad \ldots (3.13)$$

where the coefficient a_{k1}, $1 \le k \le p$ and $1 \le 1 \le p$ and the right-hand constants b_k, $1 \le k \le p$ are selected.

The problem, now, is to determine the values of the unknowns x_k, $1 \le k \le p$, which satisfy the system of p linear equations represented by the relation (3.15). The system of linear equations can be written in a compact and practical form as

$$\sum_{1=1}^{p} a_{k1}\,x_1 = b_k,\ k = 1, 2, \ldots, p.$$

In matrix representation we have a more convenient form

$$Ax = b \qquad \ldots (3.14)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p1} & a_{p2} & \ldots & a_{pp} \end{bmatrix},\ x = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_p \end{bmatrix} \text{ and } b = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_p \end{bmatrix}$$

A is called the $p \times p$ coefficient matrix and x as well as b is the column matrix. Numerous methods to compute the system of linear equations are available. Some of them can be directly applied to solve the system. The problems of simultaneous linear equations have been extensively practised with the application of well-developed matrix methods.

3.3.1 Gauss Elimination

For the systematic determination of the solution to a system of linear equations the method of Gauss can be introduced. It is the most elementary method by which the solution of simultaneous linear equations can be obtained by the successive elimination of the unknown variables from different equations. On that ground it is called *Gaussian elimination*.

We have already generalised this method in section 3.1 considering a specific example. Actually speaking, in Gauss's method a sequence of well-arranged operations has been performed to transform the given system of linear equations to an equivalent system of equations that can be readily solved.

Let us consider the set of linear equation presented by equation (3.15) again

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1p}x_p &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2p}x_p &= b_2 \\ \dots \quad \dots \quad \dots \quad \dots & \\ a_{p1}x_1 + a_{p2}x_2 + \dots + a_{pp}x_p &= b_p. \end{aligned} \qquad \dots (3.15)$$

As we have mentioned before that the method of Gaussian elimination eliminates the unknown variables $x_1, x_2, \dots, x_p$ in succession, one at a time until we get at an equivalent triangular system.

First of all we consider the coefficient a_{11}, called the pivot element of the first equation, the *pivotal equation*, assuming $a_{11} \neq 0$. If $a_{11} = 0$, we try to find an a_{k1} that is not equal to zero ($k > 1$) and select the kth equation as the pivotal equation interchanging the first and kth rows.

Now we admit the *multiplier* $m_{k1} = a_{k1}/a_{11}$ ($k = 2, 3, \dots, p$) which can change the original system to a simplified one. Multiplying the pivotal equation by m_{k1} and subtracting it from the original kth equation leads to eliminate the coefficient of x_1 from the kth equation.

Following the principle and selecting for $k = 2, 3, \dots, p$ yields a system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1p}x_p &= b_1 \\ a_{22}x_2 + \dots + a_{2p}x_p &= b_2 \\ \dots \quad \dots \quad \dots \quad \dots & \\ a_{p2}x_2 + \dots + a_{pp}x_p &= b_p \end{aligned} \qquad \dots (3.16)$$

Now, we subtract $m_{k2} = a_{k2}/a_{22}$ times the second equation from the kth equations in (3.16), $k = 3, 4, \dots, p$ to eliminate the coefficient of x_2 in each of these rows, provided that a_{22} is nonzero. Continuing with this procedure performing the operations we get the derived systems of equations and after $(n - 1)$ steps the new triangular system is formed:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots \quad \dots \quad + a_{1p}x_p &= b_1 \\ a_{22}x_2 + \dots \quad \dots \quad + a_{2p}x_p &= b_2 \\ a_{33}x_3 + \dots \quad + a_{3p}x_p &= b_3 \\ \dots \quad \dots \quad \dots \quad \dots & \\ a_{pp}x_p &= b_p \end{aligned} \qquad \dots (3.17)$$

with the nonzero *diagonal* elements.

For solving this triangular system (3.17) we apply the backward substitution principle. According to this principle we select at first the last equation as it contains one unknown and solve for x_p. Then the $(p – 1)$st equation has been treated for the variable $x_{p–1}$ and so on. In this manner we determine the values for the unknowns.

To demonstrate the Gaussian elimination process let us consider a specific example.

Example 3.5 Solve the linear system

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - x_4 &= 10 \\ 3x_1 + 4x_2 - x_3 + x_4 &= 4 \\ 5x_1 + 2x_2 - 3x_3 + 2x_4 &= 9 \\ 2x_1 - x_2 + 5x_3 - 3x_4 &= 7. \end{aligned}$$

The multiplier is, in this case, 3/1 = 3. As the pivot element $a_{11} \neq 0$, we select the first equation as pivotal equation.

We multiply the pivotal equation by 3 and subtract it from the second to eliminate the coefficient x_1 in the new second equation. The elimination process is continued using the appropriate multipliers that eliminate x_1 from the 3rd and 4th equations.

As a result of that we obtain an equivalent reduced system of equations as follows:

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - x_4 &= 10 \\ -25x_2 + 7x_3 - 4x_4 &= 26 \\ -37x_2 + 13x_3 - 7x_4 &= 41 \\ -13x_2 - x_3 + x_4 &= 13 \end{aligned} \qquad \text{... (3.18)}$$

which corresponds to the system of equations (3.16).

Now we consider the second equation in system (3.18) as pivotal equation and eliminate the unknown x_2 from the third and fourth equations using the proper multiplier to obtain

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - x_4 &= 10 \\ -25x_2 + 7x_3 - 4x_4 &= 26 \\ -2.64x_3 + 1.08x_4 &= -2.52 \\ 4.64x_3 - 3.08x_4 &= 0.52. \end{aligned}$$

Finally we get a linear system of equations in triangular form when we eliminate x_3 from the last equations. –4.64/2.64 is the multiplier in this case.

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - x_4 &= 10 \\ -25x_2 + 7x_3 - 4x_4 &= 26 \\ -2.64x_3 + 1.08x_4 &= -2.52 \\ 1.18x_4 &= 3.91. \end{aligned}$$

To solve this triangular system we apply the principle of backward substitution, so that we can determine the variable $x_4 = 3.31$ from the last equation.

Substituting the value x_4 in the last but one, that means, in the third equation we find another unknown $x_3 = 2.31$.

In this manner we obtain $x_2 = -0.92$, $x_1 = 2.25$.

Hence, the solution of the system is $x_1/x_2/x_3/x_4 = 2.25/–0.92/2.31/3.31$ that satisfy the given system of linear equations.

By pivoting strategy the method used in this section to solve the system of linear equations needs many calculations in each step which we have experienced so far. Now we try to determine the values for the unknown variables by means of our computational technique that can certainly perform the large number of arithmetic computations in well throughout range of time.

The program g3GAUELI implements the computational technique for the solution of the system of linear equations. As usual, the program designed in C++ (Turbo) programming language is applied to determine the unknown variables involved in the linear equations. A short review of this program has been given in section 3.4.

Example 3.6 Solve the system by Pivoting method

$$\begin{aligned} 2x_1 - x_2 + x_3 &= 1 \\ 2x_1 + 2x_2 + 2x_3 &= 2 \\ -2x_1 + 4x_2 - x_3 &= 5 \end{aligned}$$

Pertaining to computational technique we represent the system in a convenient form given in relation (3.14), that means,

$$Ax = b,$$

where $$A = \begin{bmatrix} 2 & -2 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & -1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Let the program run.

Selecting, first of all, the order of the matrix A that is, in this case, 3 we proceed by entering the values for each coefficient with proper mathematical sign rowwise as needed for the input.

For example, for a_{11} putting 2, for a_{12} putting –2, …, for b_1 putting 1, etc. we follow the execution of the program. Finally we get the results accepting only five significant places of accuracy in any computation.

The Linear System of order 3:

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & 2 & 2 \\ -2 & 4 & -1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 4$, $x_2 = 2$ and $x_3 = –5$.

Example 3.7 Solve the 4 × 4 linear system using Gaussian elimination

$$\begin{aligned} 5x_1 + 3x_2 - x_3 \qquad &= 11 \\ 2x_1 \qquad + 4x_3 + x_4 &= 1 \\ -3x_1 + 3x_2 - 3x_3 + 5x_4 &= -2 \\ 6x_2 - 2x_3 + 3x_4 &= 9. \end{aligned}$$

The coefficient matrix A can be deduced from the system

$$A = \begin{bmatrix} 5 & 3 & -1 & 0 \\ 2 & 0 & 4 & 1 \\ -3 & 3 & -3 & 5 \\ 0 & 6 & -2 & 3 \end{bmatrix}$$

and the column matrix b becomes

$$b = \begin{bmatrix} 11 \\ 1 \\ -2 \\ 9 \end{bmatrix}.$$

The performance of the computing technique leads to the final resulting system that reads the Linear System of order 4

$$\begin{bmatrix} 5 & 3 & -1 & 0 \\ 2 & 0 & 4 & 1 \\ -3 & 3 & -3 & 5 \\ 0 & 6 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ -2 \\ 9 \end{bmatrix}$$

yields the values for the Unknowns: $x_1/x_2/x_3/x_4 = 1.0/2.0/0/-1.0$.

Example 3.8 Consider the most simple system with 2 unknowns

$$3x_1 + 3x_2 = 6$$
$$2x_1 + 4x_2 = 8.$$

In matrix notation it can be written as

$$\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.$$

The program used for the purpose of Gaussian elimination will produce the following results:

The Linear System of order 2

$$\begin{bmatrix} 3 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 0$ and $x_2 = 2$.

3.3.2 Cramer's Rule

It is a very simple, but well-known method for computation of the system of linear equations. According to some technicians of mathematics the Cramer's rule is generally only of theoretical interest as it is fundamentally based on the evaluations of determinants.

The process of evaluating determinants is sometimes cumbersome because of the long-winded shape of the system involving many unknowns. *Cramer's rule* can be used for the direct computation of the solution just as in the case of Gauss elimination in which the exact results of the system can be obtained after a finite number of steps.

The basic idea of this rule for computing the compact matrix form presented by (3.14)

$$Ax = b,$$

is to find the unknowns x_k for $1 \le k \le p$ by evaluating the determinants. Each unknown can be expressed as the quotient

$$x_k = \det A_k / \det A \qquad \ldots (3.19)$$

where det A is the determinant of the coefficient matrix A and A_k is the matrix with the kth column replaced by the right side column matrix b.

The familiar relation (3.19) for Cramer's rule cannot be applied when the matrix A is singular, i.e., det $A = 0$. The rule fails to operate also, when the number of unknowns is not equal to the number of equations.

Example 3.9 Solve the system using Cramer's rule

$$\begin{aligned} 5x_1 + 2x_2 + \ \ x_3 &= 0 \\ x_1 + 7x_2 - 3x_3 &= 1 \\ 2x_1 + 2x_2 - 7x_3 &= 2. \end{aligned}$$

The system can be written in matrix representation as $Ax = b$ that means

$$\begin{bmatrix} 5 & 2 & 1 \\ 1 & 7 & -3 \\ 2 & 2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

where the coefficient matrix A is of order 3.

The system should have a unique solution when the condition is fulfilled, that means, the matrix A is nonsingular. In other words, the determinant of the matrix is nonzero, that we want to show, first of all.

Now,
$$\det A = 5\begin{vmatrix} 7 & -3 \\ 2 & -7 \end{vmatrix} - 2\begin{vmatrix} 1 & -3 \\ 2 & -7 \end{vmatrix} - 1\begin{vmatrix} 1 & 7 \\ 2 & 2 \end{vmatrix}$$

$$= -5 \times 43 + 2 \times 1 - 1 \times 12$$

$$= -215 + 2 - 12 = -225.$$

The determinant of the matrix A_1 becomes

$$\det A_1 = \begin{vmatrix} 0 & 2 & 1 \\ 1 & 7 & -3 \\ 2 & 2 & -7 \end{vmatrix}$$

$$= 0\begin{vmatrix} 7 & -3 \\ 2 & -7 \end{vmatrix} - 2\begin{vmatrix} 1 & -3 \\ 2 & -7 \end{vmatrix} + 1\begin{vmatrix} 1 & 7 \\ 2 & 2 \end{vmatrix}$$

$$= 0 + 2 \times 1 - 1 \times 12 = -10.$$

Similarly,

$$\det A_2 = \begin{vmatrix} 5 & 0 & 1 \\ 1 & 1 & -3 \\ 2 & 2 & -7 \end{vmatrix} = -5 \text{ and } \det A_3 = \begin{vmatrix} 5 & 2 & 0 \\ 1 & 7 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 60.$$

Hence, from the relation (3.19) we obtain the unknowns

$$x_1 = \det A_1/\det A = 2/45,$$
$$x_2 = \det A_2/\det A = 1/45,$$
$$x_3 = \det A_3/\det A = -4/15.$$

The Cramer's rule in contrast to the method of elimination by Gauss is less significant because of many more steps to be performed in terms of simple determinants. To determine the solution for the system $Ax = b$ of nth order we need the $(n + 1)$ determinants to evaluate.

The program g3CRALIS is used to solve the system of linear equations by the method of Cramer's rule.

Example 3.10 We consider the system

$$x_1 + 3x_2 + 2x_3 = 17$$
$$x_1 + 2x_2 + 3x_3 = 16$$
$$2x_1 - x_2 + 4x_3 = 13$$

We see that the order of the matrix is 3. Inputting the number 3 we follow the program directions by inserting the coefficients or elements of the matrix for determination of the determinants. The following picture will be obtained after the program execution:

Computer Solution of the System following Cramer's rule

The Linear System of order 3.

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 4.0$, $x_2 = 3.0$ and $x_3 = 2.0$.

Example 3.11 Solve the set of linear equations

$$2x_1 + x_2 + 2x_3 + x_4 = 6$$
$$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$$
$$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$$
$$2x_1 + 2x_2 - x_3 + x_4 = 10.$$

The coefficient matrix A can be obtained from the system

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}$$

and the column matrix b becomes

$$b = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}.$$

Evidently, the order of the coefficient matrix is 4. For the evaluation of the determinants we need all numerical values from the coefficient matrix and the values from the right side with proper mathematical signs. By entering the numerals we obtain the values for the unknowns by means of the program.

The Linear System of order 4:

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 2.0$, $x_2 = 1.0$, $x_3 = -1.0$ and $x_4 = 3.0$ which satisfy the set of linear equations.

3.3.3 Matrix Inversion

Earlier in this chapter we discussed on the necessity of matrix for the solution of linear systems of equations. In the study of matrix relationship in section 3.2 we have introduced all the fundamental properties of matrix that we need for the study of linear systems of equations. One of the important properties is the *matrix inversion* that we want to apply here for the determination of the unknown variables involved in the system. The matrix inversion method is also a direct method.

Let us consider again the linear system of equations presented by the relation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1p}x_p &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2p}x_p &= b_2 \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pp}x_p &= b_p \end{aligned}$$

It is to be supposed here that a system of p linear equations involving p unknowns $x_1, x_2, \ldots, x_p$ has been concerned. The left-hand members of the system, so-called *elements*, can be separated and arranged in a square array, so that we can write these in a convenient form that is called the *coefficient matrix* as shown

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1p} \\ a_{21} & a_{22} & \ldots & a_{2p} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p1} & a_{p2} & \ldots & a_{pp} \end{bmatrix},$$

where A is a square matrix of order p. Similarly, the unknowns can be set up in a $p \times 1$ column matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_p \end{bmatrix}$$

and the members on the right-hand side can be put in a $p \times 1$ column matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_p \end{bmatrix}$$

Thus the convenient matrix form $Ax = b$ has been originated from the relation (3.14) which can be written as

$$x = A^{-1}b \qquad \text{... (3.20)}$$

The relation (3.20) is a matrix representation by which we can find a solution of the system, provided that the matrix A is nonsingular, that means, the determinant of A is a nonzero number and b is a nonzero matrix.

The method to find the inverse of matrix A, i.e., A^{-1} was demonstrated earlier in section 3.2.2. To recapitulate this we have given a brief outline here.

The *minor* of any element a_{k1} in the coefficient matrix A is the value of the determinant that can be obtained by deleting the kth row and lth column of the matrix.

The *cofactor* of the element a_{k1} is simply determined from the relation

$$A_{k1} = (-1)^{k+1} m_{k1},$$

m_{k1} being the minor of the element a_{k1}.

By means of the predefined values of the cofactors we can define the adjoint of A that is actually the transpose of Cof (A), a matrix formed by the cofactors as follows:

$$\text{Cof}\,(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix}$$

and

$$\text{Adj}\,(A) = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{p1} \\ A_{12} & A_{22} & \cdots & A_{p2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1p} & A_{2p} & \cdots & A_{pp} \end{bmatrix} = \left[\text{Cof}\,(A)\right]^t.$$

The inverse of the matrix A is finally defined by the relation

$$A^{-1} = \text{Adj}\,(A)/\det A.$$

Thus, we solve the system of equations (3.15) using the relation (3.20) that will yield the values of the unknowns.

Example 3.12 Using the method of inversion solve the system

$$x_1 + 2x_2 + 3x_3 = 1$$
$$2x_1 – 3x_2 + 4x_3 = –1$$
$$3x_1 + 4x_2 + 6x_3 = 2.$$

Writing the system of equations in matrix notation yields

$$Ax = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = b.$$

The coefficient matrix A is, at this stage, the main operating field that is the first concern here. Evidently, the determinant of the matrix A is det A = 17.

As described the minors of the elements in the matrix have been found

$$\begin{bmatrix} -34 & 0 & 17 \\ 0 & -3 & -2 \\ 17 & -2 & -7 \end{bmatrix}$$

and the cofactors of the matrix becomes

$$\text{Cof }(A) = \begin{bmatrix} -34 & 0 & 17 \\ 0 & -3 & 2 \\ 17 & 2 & -7 \end{bmatrix}.$$

By taking the transpose of Cof (A) we obtain the adjoint of A by means of which the inverse of the matrix A can be found.

$$A^{-1} = 1/17 \begin{bmatrix} -34 & 0 & 17 \\ 0 & -3 & 2 \\ 17 & 2 & -7 \end{bmatrix}$$

Now, considering the relation (3.20) for the solution of the given system of linear equations stating

$$x = A^{-1}b,$$

we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1/17 \begin{bmatrix} -34 & 0 & 17 \\ 0 & -3 & 2 \\ 17 & 2 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

which after evaluation we find the unknowns.

$$x_1 = 0, \quad x_2 = 0.41176 \quad \text{and} \quad x_3 = 0.05882.$$

For computations of the system of linear equations by the method of matrix inversion the program g3MATINV is directly used. The determination of the unknown variables involved in the system will be followed by entering the coefficients of the matrix and the constant values for b.

Example 3.13 The system of equations

$$\begin{aligned} 6x_1 + 9x_2 + 1.56x_3 &= 34.1 \\ 9x_1 + 19.8x_2 + 2.15x_3 &= 46.2 \\ 1.56x_1 + 2.15x_2 + 2.23x_3 &= 9.39. \end{aligned}$$

Here, the order of the coefficient matrix is 3. Therefore, the computation needs a number 3 for the *Order* to be entered, so that the execution of the program succeeds in accomplishing the desired 3rd order inversion of the matrix. Finally we obtain Computer Solution of the linear system of order 3 by Matrix Inversion.

The Linear System of order 3:

$$\begin{bmatrix} 6 & 9 & 1.56 \\ 9 & 19.8 & 2.15 \\ 1.56 & 2.15 & 2.23 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 34.1 \\ 46.2 \\ 9.39 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 6.83$, $x_2 = -0.79$ and $x_3 = 0.189$.

Example 3.14 Solve the system of equations

$$\begin{aligned} 554.11x_1 - 281.91x_2 - 34.24x_3 &= 273.02 \\ -281.91x_1 + 226.81x_2 + 38.1x_3 &= -63.965 \\ -38.24x_1 + 38.1x_2 + 80.221x_3 &= 34.717. \end{aligned}$$

Here, the order of the coefficient matrix is 3. Hence, we need a number 3 for the order, so that the execution of the program succeeds in accomplishing the desired 3rd order inversion of the matrix. As a result of that we obtain for the linear system of order 3

$$\begin{bmatrix} 554.11 & -281.91 & -34.24 \\ -281.91 & 226.81 & 38.1 \\ -38.24 & 38.1 & 80.221 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 273.02 \\ -63.965 \\ 34.717 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 0.92083$, $x_2 = 0.78654$ and $x_3 = 0.45224$.

Example 3.15

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 + x_4 &= 1 \\ x_1 + 2x_2 \qquad\quad + x_4 &= 0 \\ 2x_1 + 3x_2 + x_3 - x_4 &= 2 \\ x_1 - 2x_2 - x_3 - x_4 &= 3. \end{aligned}$$

The coefficient matrix A can be obtained from the system

$$A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & -1 \\ 1 & -2 & -1 & -1 \end{bmatrix}.$$

Evidently, the order of the coefficient matrix is 4. For the evaluation of the inverse matrix we need all the numerical values from the coefficient matrix with proper mathematical signs. By entering the numerals we obtain the values for the unknowns by means of the program.

Computer Solution of the System by Matrix inversion for the linear system of order 4

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & -1 \\ 1 & -2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$$

yields the values for the Unknowns: $x_1/x_2/x_3/x_4 = 1.5/-0.5/0.0/-0.5$ which satisfy the set of linear equations.

3.3.4 LU-Decomposition

In the study of Gaussian elimination in section 3.3.1 we solved the system of linear equations directly by reducing the system to a triangular form. In this section the coefficient matrix A has been factorised into a product of matrices that we can manipulate easily. The product is symbolised by

$$A = L \times U \qquad \text{... (3.21)}$$

where L is the lower triangular matrix and U is the upper triangular matrix that we already defined in section 3.2.1.

The LU-Factorisation or decomposition can be performed, subject to the condition that the matrix A is nonsingular.

Hence, the matrix A can now be factorised into

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ l_{p1} & l_{p2} & l_{p3} & \cdots & l_{pp} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & \cdots & \cdots & u_{1p} \\ 0 & 1 & 0 & \cdots & u_{2p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = LU.$$

By means of the known coefficients of the matrix A we determine the elements of the matrices L and U. Considering the related equation (3.14) which states

$$Ax = b$$

we can write now on the basis of the relation (3.21)

$$LUx = b \qquad \text{... (3.22)}$$

that will be solved in two stages. Substituting $Ux = Z$ in equation (3.22) where Z is a column matrix as shown next

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_p \end{bmatrix}$$

we obtain a new equation $LZ = b$ that yields the value of z_1, z_2, …, z_p by the method of forward substitution.

Finally, we consider equation $Ux = Z$ and by means of the determined values of Z we obtain the unknowns $x_1, x_2, \ldots, x_p$ by the method of backward substitution.

Example 3.16 Solve the system by the factorisation method

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \\ 3x_1 + 3x_2 + 5x_3 &= 4. \end{aligned}$$

The coefficient matrix A can be factorised into the product

$$\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}. \qquad \ldots (3.23)$$

With the help of matrix product we find

$$l_{11} = 2,\ l_{21} = 3,\ l_{22} = 9/2,\ \ l_{31} = 3,\ l_{32} = 9/2,\ l_{33} = -4;$$

and

$$u_{12} = -1/2,\ u_{13} = 1/2,\ u_{23} = 5/3.$$

Hence, the equation (3.23) can be rewritten with the values just found

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 9/2 & 0 \\ 3 & 9/2 & -4 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

Now, selecting an equation $LZ = b$, where Z is a 3×1 column matrix, yields

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 9/2 & 0 \\ 3 & 9/2 & -4 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

that provides the determined values for z_1, z_2 and z_3.

Therefore
$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/3 \\ -1 \end{bmatrix}.$$

Now on account of the relation $Ux = Z$ we obtain for the solution to linear system.

$$\begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 1 & 5/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/3 \\ -1 \end{bmatrix}.$$

Solving the matrix equation yields $x_1/x_2/x_3 = 1.0/2.0/-1.0$.

For computing the system of linear equations by the method of LU-decomposition the program `g3MLUFAC` is used.

Example 3.17 Solve the system

$$\begin{aligned} 3x_1 + 4x_2 - x_3 &= 7 \\ 4x_1 + 12x_2 + 6x_3 &= -4 \\ -x_1 + x_2 + 4x_3 &= 4. \end{aligned}$$

Here the order of the matrix is 3.

According to the program instructions we enter the values for the coefficient matrix and for *b* after having mentioned the order of the matrix. For convenience, we write down

$$\begin{bmatrix} 3 & 4 & -1 \\ 4 & 12 & 6 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix}$$

from which we obtain the values for the unknowns.

The Linear System of order 3

$$\begin{bmatrix} 3 & 4 & -1 \\ 4 & 12 & 6 \\ -1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 23$, $x_2 = -13$ and $x_3 = 10$.

Example 3.18 Compute the system

$$\begin{aligned} 6.1x_1 + 2.2x_2 + 1.2x_3 &= 16.55 \\ 4.4x_1 + 11.0x_2 - 3.0x_3 &= 21.10 \\ 1.2x_1 - 1.5x_2 + 7.2x_3 &= 16.80 \end{aligned}$$

Evidently, the order of the coefficient matrix is 3. Following the related instructions of the program we obtain the computerised values for the unknowns as shown

The Linear System of order 3

$$\begin{bmatrix} 6.1 & 2.2 & 1.2 \\ 4.4 & 11 & -3 \\ 1.2 & -1.5 & 7.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16.55 \\ 21.10 \\ 16.80 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 1.5$, $x_2 = 2.0$ and $x_3 = 2.5$.

Example 3.19 Compute the 2nd order system

$$\begin{aligned} 0.0003x_1 + 1.566x_2 &= 1.569 \\ 0.3454x_1 - 2.436x_2 &= 1.018. \end{aligned}$$

For the system of linear equations we have the following picture as solution.

The Linear System of order 2

$$\begin{bmatrix} 0.0003 & 1.566 \\ 0.3454 & -2.436 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.569 \\ 1.018 \end{bmatrix}$$

yields the values for the Unknowns: $x_1 = 10.0$ and $x_2 = 1.0$.

Example 3.20 Solve the system

$$\begin{aligned} x_1 + x_2 \qquad\quad + 3x_4 &= 4 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4. \end{aligned}$$

The coefficient matrix A can be obtained from the system

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix}.$$

Here we deal with a coefficient matrix of 4th order. By the method of factorisation we get the linear system of order 4

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

that yields the values for the Unknowns: $x_1 = -1.0$, $x_2 = 2.0$, $x_3 = 0$ and $x_4 = 1.0$ which satisfy the set of linear equations.

3.4 Related Software for the Solution

In this chapter we have directed our attention towards direct methods for solving the system of linear equations. A linear system as described in the beginning of this chapter is a combination of m equations in m unknowns expressed in matrix notation as $Ax = b$.

The linear system in matrix representation $Ax = b$ provides a unique solution when the inverse matrix A^{-1} exists. The main condition for the existence of the inverse matrix is that the determinant of the matrix A is not equal to zero, i.e., $A \neq 0$.

Gaussian elimination has been introduced for the determination of the solution to a system of linear equations using pivoting technique. The pivoting strategy enables us to transform the system of equations into a triangular system that can be easily solved. The program g3GAUELI implements the computational technique for the solution of the system.

Cramer's rule, the most fundamental method for direct computation of the solution, has been applied using the techniques of determinants. The program g3CRALIS is thought for solving the system by the method of Cramer's rule discussed in section 3.3.2.

The method of *Matrix Inversion*, that I have described earlier in this section, is used to compute the system of equations when the system is represented in matrix form $Ax = b$. The solution of the linear system becomes $x = A^{-1}b$, if we can find out the inverse matrix, A^{-1}. The software for the matrix operations and the direct solution of the system is based on the program g3MATINV.

The method *LU-factorisation* or LU-decomposition can be applied to the linear system in view of the condition that the coefficient matrix A is nonsingular. Considering that the matrix A can be factorised into the product of matrices, symbolised by $A = L \times U$, where L is the lower triangular matrix and U is the upper triangular matrix with 1s on the diagonal. The program g3MLUFAC implements the factorisation method.

Some mathematical problems have been provided in Chapter 10 for further study. A Floppy disc (3.5 inch/1.44 MB) containing the required programs will be supplied for working out the selected mathematical models.

4

First Order Initial-Value Problems

4.1 Preliminary Concepts

As it is known to us from our commonplace practice that the selection for method of solving a differential equation depends on the order and the degree of the differential equation to be solved. Based on that ground we want to introduce the definitions of order and degree of an equation. The order of a differential equation is the same as the order of the highest derivative the equation contains. The degree of a differential equation is the power to which the highest derivative is raised when the differential equation is brought into conformity with a rational form.

Let us now elaborate the formal statements with some examples.

1. $dy/dx = x(x + y)/(x - y)$
2. $yy'' + (y')^2 - 2yy' = 0$
3. $(y')^2 - rx^2 - sx = t$
4. $e^x(1 + x)\,dx = (xe^x - ye^y)\,dy$
5. $x\,\partial t/\partial x + \partial t/\partial y = 0$
6. $\partial^2 x/\partial p^2 + \partial^2 x/\partial q^2 = 0.$

The equations 1, 4 and 5 are of first order and first degree; equation 3 is of first order, but second degree; the equations 2 and 6 are of second order and first degree.

The differential equation having partial differentials with two or more independent variables is known as *partial differential* equation.

We consider in this chapter only the differential equations that are ordinary and first order. There are many types of ordinary differential equations which can be classified according to their forms. I think, it is worthwhile now to review at this stage some of the standard forms of differential equations of the first order.

A. An equation expressed in the form … (4.1)

$$P(x)\,dx + Q(y)\,dy = 0$$

is a first order differential equation that is easily separable when it is written in the form

$$dy/dx = -P(x)/Q(y).$$

$P(x)$ denotes a function of x only and $Q(y)$ is a function of y only. The equation (4.1) is a representation in differential form.

Example 4.1 $\cos^2 x \, dy + \sin x \cos^2 y \, dx = 0.$

The equation can be written in another form

$$1/\cos^2 y \, dy + \sin x/\cos^2 x \, dx = 0$$

or

$$\sec^2 y \, dy + \sec x \tan x \, dx = 0$$

which on integrating both sides we get $\int \sec x \tan x \, dx + \int \sec^2 y \, dy = c,$ c is the constant of integration. The parameter c is a so-called guide factor.

The general solution is $\sec x + \tan y = c$.

After rearranging we can write $\tan y = c - 1/\cos x$ or $y(x) = \tan^{-1}(c - 1/\cos x)$.

B. The equation of the form $P(x, y)\, dx + Q(x, y)\, dy = 0$ … (4.2)

is a first order differential equation that can be reduced to a simplified form applying the method of substitution.

Here $P(x, y)$ is a combined function of x and y. The choice of introducing a new specification for the dependent variable reduces the function to another form that is solvable.

Example 4.2 $xy \, dy - (x^2 + y^2)\, dx = 0.$

The equation can be written as $dy/dx = (x^2 + y^2)/xy$.

The proper setting with another variable

$$y = ux$$

transforms the above equation to another form $x \, du/dx = 1/u$.

On due arrangement and integration we obtain $\text{Ln}\, x = u^2/2 + c$

that can be written explicitly for y: $y^2 = (\text{Ln}\, x^2 - 2c)\, x^2$ as general solution.

C. A differential equation $P(x, y)\, dx + Q(x, y)\, dy = 0$

can be termed as *exact*, provided that, there exists a function $1(x, y)$ that makes the equation exact. So we can write

$$dl\,(x, y) = P(x, y)\, dx + Q(x, y)\, dy. \quad \dots (4.3)$$

The condition for exactness $\partial P/\partial y = \partial Q/\partial x$ is satisfied only if the first order partial derivatives of the continuous functions $P(x, y)$ and $Q(x, y)$ are continuous on some rectangular system of the (x, y) plane.

Example 4.3 $Dy/dx = (e^x \cos y + 3x^2)/(e^x \sin y).$

In differential form we write $(e^x \cos y + 3x^2)\, dx - e^x \sin y \, dy = 0.$

Applying the conditions for exactness we obtain

$$\partial P/\partial y = -e^x \sin y$$

and

$$\partial Q/\partial x = -e^x \sin y$$

which fulfil the conditions in this case.

Recalling the original equation and rearranging we get

$$3x^2 \, dx + d\,(e^x \cos y) = 0$$

which on integration yields

$$y = \cos^{-1}[(c - x^3)/e^x]$$

as general solution.

D. Let us consider a linear differential equation

$$Dy/dx + M(x)y = N(x). \qquad \dots (4.4)$$

On account of the fact that $d/dx\,(ye^{\int Mdx}) = dy/dx\, e^{\int Mdx} + My\, e^{\int Mdx}$

$$= e^{\int Mdx}\left[dy/dx + M(x)y\right]$$

the expression $e^{\int Mdx}$ is called an *integrating factor* (I.F.) of the equation (4.4).

The integrating factor is very effective in transforming an equation to a simplified form that can be readily solvable.

The linearity is based on the principle that the differential equation is of the first degree in the unknown function y and its derivative y'. In this sense the functions $M(x)$ and $N(x)$ are always integrable.

Now we consider a problem of linear differential equation that is nonhomogeneous. When the right side of the equation (4.4), i.e., the function $N(x)$ is equal 0, the differential equation is homogeneous.

Example 4.4 Solve $dy/dx - y\cos x + \cos x = 0$. … (4.5)

Here, $\text{I.F.} = e^{-\int \cos x\, dx} = e^{-\sin x}$.

Multiplying the equation (4.5) by the factor yields

$$e^{-\sin x}\, dy/dx - y\, e^{-\sin x}\cos x = -\cos x\, e^{-\sin x}.$$

On integrating we obtain $y\, e^{-\sin x} = -\int \cos x\, e^{-\sin x}\, dx + c = e^{-\sin x} + c.$

In other words it can be expressed as $y = 1 + c\, e^{\sin x}$ as general solution.

There is another method of solving the nonhomogeneous linear equation that is called *Variation of Parameters*. The method of Variation of Parameters can be used to solve all the differential equations that are *linear*.

According to the method we determine at first the general solution of the associated homogeneous linear equation that involves the constant factor c. Considering the parameter c at this point as a function of x we try to find a solution of the nonhomogeneous equation (4.5).

Example 4.5 Recalling the linear equation in (4.5) we have

$$y' - y\cos x + \cos x = 0.$$

The homogeneous from yields $y' - y\cos x = 0$.

On integrating we obtain the general solution of the homogeneous equation

$$y = c_1\, e^{\sin x}. \qquad \dots (4.6)$$

Let us now consider c_1 as *a function of* x according to the principle of Variation of Parameters.

Substituting $c_1 = a(x)$ and differentiating we get

$$y' = a'(x)\, e^{\sin x} + a(x)\cos x\, e^{\sin x}.$$

Now, the given nonhomogeneous equation can be written by putting the determined value of y and y'

$$a'(x)\, e^{\sin x} + a(x)\cos x\, e^{\sin x} - a(x)\cos x\, e^{\sin x} + \cos x = 0$$

that gives $a'(x)\, e^{\sin x} = -\cos x$.

By means of integration we get $a(x) = c + e^{-\sin x}$.

Hence, from the relation (4.6) the general solution of the given equation is

$$y = (c + e^{-\sin x})\, e^{\sin x} = 1 + c\, e^{\sin x}.$$

E. There are some differential equations which are not directly belonged to the linear form as they explicitly appear. One classic type in generalised (indirect) linear form as represented by

$$Dy/dx + M(x)y = N(x)y^n \qquad \dots (4.7)$$

is due to *Bernoulli*. The value of the parameter n, in this case, is different from 0 and 1. The equation (4.7) can be transformed to a normal (direct) linear form on appropriate setting of the dependent variable y.

In the generalised (4.7) if we set $n = 0$, we achieve a linear nonhomogeneous equation. The equation reduces to a linear homogeneous form when $n = 1$.

In order to get a reduced linear form we divide the equation by y^n throughout and set $u = y^{1-n}$, where u is a new variable.

Example 4.6 $x\,dy + y\,dx = x^3y^6\,dx$.

The equation may be written as $dy/dx + y/x = x^2y^6$

which is a Bernoulli equation with $n = 6$.

In order to approach the linearity we divide both sides of the equation by y^6.

The equation now becomes

$$y^{-6}\,dy/dx + 1/xy^{-5} = x^2 \qquad \dots (4.8)$$

As proposed, setting the new variable u for y^{-5}, i.e. $u = y^{-5}$,

yields on differentiation

$$du/dx = -5y^{-5}\,dy/dx.$$

On substituting these values in equation (4.8) we obtain a new linear equation

$$du/dx - 5u/x = -5x^2.$$

This is a linear *nonhomogeneous* differential equation in u that can be solved with the help of the integrating factor that reads

$$\text{I.F.} = e^{-5\int 1/x\,dx} = 1/x^5.$$

The solution to the last equation is

$$u = 5x^3/2 + cx^5.$$

Hence, the general solution of the equation is

$$y^{-5} = 5x^3/2 + cx^5$$

or

$$y = 1/(5x^3/2 + cx^5)^{1/5}.$$

F. The differential equation

$$Dy/dx = M(x) + N(x)y + P(x)\,y^2 \qquad \dots (4.9)$$

is known as *Riccati* equation.

The functions M, N and P are functions of x or may be constants.

Fundamentally, the equation was first considered by J. Bernoulli whose concepts had been further developed.

The equation (4.9) can be transformed to a Bernoulli equation with the substitution

$$y = y_1(x) + 1/u \qquad \ldots (4.10)$$

when the function $y_1(x)$ is a particular solution that satisfies the original equation (4.9). Now, without loss of generality we can write

$$(M + Ny_1 + Py_1^2) - 1/u^2\, du/dx = dy_1/dx - 1/u^2\, du/dx$$

Differentiating (4.10) we have

$$dy/dx = dy_1/dx - 1/u^2\, du/dx$$

that is equal to

$$M + N\,(y_1 + 1/u) + P\,(y_1 + 1/u)^2.$$

Hence, equating both sides we obtain a new equation in u

$$du/dx + (N + 2Py_1)u = -P.$$

This is a linear nonhomogeneous form that can be solved easily as described before.

Here, $$\text{I.F.} = e^{\int (N + 2Py_1)dx}$$

Multiplying by the I.F. we get a solution in u

$$u\, e^{\int (N + 2Py_1)dx} = C - \int P\, e^{\int (N + 2Py_1)dx}\, dx.$$

The substitution of the value of u in the setting for y yields the general value of y that satisfies the equation (4.9).

Example 4.7 Solve $dy/dx = 1 - xy - y^2$ … (4.11)

This is a Riccati form where $M(x) = 1$, $N(x) = -x$, $P(x) = -1$.

Now we assume an appropriate setting for y by putting

$$y = 1/x$$

and take as particular solution when it satisfies the equation (4.11).

Therefore, we can substitute

$$y = 1/x + 1/u$$

that converts the original equation (4.11) in u.

This substitution transforms to

$$dy/dx = -1/x^2 - 1/u^2\, du/dx = 1 - x\,(1/x + 1/u) - (1/x + 1/u)^2$$

Rearranging and equating we obtain

$$du/dx = u\,(x + 2/x) + 1$$

or $$u/dx - (x + 2/x)\, u = 1$$

which is a linear nonhomogeneous equation, integrating factor of which becomes

$$\text{I.F.} = e^{-\int (x + 2/x)dx} = 1/x^2\, e^{-x^2/2}.$$

Finally, the solution of the equation is

$$u/x^2\, e^{-x^2/2} = C + \int \left(e^{-x^2/2} \big/ x^2 \right) dx \qquad \ldots (4.12)$$

Expanding $e^{-x^2/2}$ by means of power series yields

$$e^{-x^2/2} = 1 - x^2/2 + x^4/8 - x^6/48 + \ldots \text{ (considering only 4 terms).}$$

Dividing by x^2 throughout and integrating we obtain

$$\int \left(e^{-x^2/2} \big/ x \right) = \int (x^{-2} - 1/2 + x^2/8 - x^4/48) = -x^{-1} - x/2 + x^3/24 - x^5/240.$$

The equation (4.12) can be written then $u/x^2\, e^{-x^2}/2 = C - 1/x - x/2 + x^3/24 - x^5/240.$

Converting in y and rearranging yields

$$x/(xy - 1) = e^{x^2/2}\left[Cx^2 - x - x^3/2 + x^5/24 - x^7/240 \right].$$

4.2 Methods for Solution

Let us assume a functional differential relation $G(x, y, y') = 0$

which can be written in explicit form

$$y' = f(x, y) \qquad \ldots (4.13)$$

A general solution of an ordinary differential equation of the kind (4.13) can be represented by another functional relation

$$y = y(x, C)$$

that we have already experienced earlier in our examples. Here the parameter C is an arbitrary constant, so-called guiding factor, also mentioned earlier. The factor C, involved in the general solution plays an important role. The number of constant factors that is furnished with the general solution, by hypothesis, equal to the order of the differential equation.

By imposing some conditions, called the *Initial Conditions*, on the variables

$$y(x_0) = y_0 \qquad \ldots (4.14)$$

where x_0 and y_0 are the starting values, we can evaluate the factor C in order to achieve the particular solution of the equation. The differential equation (4.13) together with the initial conditions (4.14) is known as the first-order *Initial Value Problem* (IVP).

The present chapter has the concern only with the first-order initial value problems of different types. Many of the numerical methods for the approximation of the solution to initial value problems had been proposed and some of them had been accepted as standard methods because of precision and accuracy. I want to outline some of the standards next. For readers having great interest on the development of the standard methods, I wish to request them to consult the textbooks mentioned in the reference.

Euler's Improved Method

The Euler's formula for the approximation is represented by the equation

$$y_{m+1} = y_m + w\,[\,f(x_m, y_m) + f(x_m + w, y_m + w\, f(x_m, y_m))]/2 \qquad \ldots (4.15)$$

where y_m is the *m-th* approximation at $x = x_m$; $f(x_m, y_m)$ is the functional relation at the *m-th* approximation; $m = 0, 1, 2, \ldots$, and w is the width.

In order to approximate a differential equation let us use the modified formula.

Example 4.8 $dy/dx = -2xy^2$

We accept the initial conditions for $y_0 = 1.0$ when $x_0 = 0$ bounded on [0, 2].

Here we can formulate the problem

$$f(x_m, y_m) = -2x_m\, y_m^2 \text{ where } m = 0, 1, 2, \ldots, 9.$$

We calculate $w = (2 - 0)/10 = 0.2$ and $x_0 = 0$, $y_0 = 1.0$

The first approximation results from the equation (when $m = 0$)

$$y_1 = y_0 + w\,[\,f\,(x_0, y_0) + f\,(x_0 + w, y_0 + w\; f\,(x_0, y_0))]/2$$

Now, $f(x_0, y_0) = -2 \times 0 \times (1.0)^2 = 0$ (putting the values of x and y)

$$f(x_0 + w, y_0 + w\, f(x_0, y_0)) = -2 \times 0.2 \times (1.0 + 0.2 \times 0)^2 = -0.4$$

Hence $y_1 = 1 + 0.2[0 - 0.4]/2 = 0.96$

Similarly, the second approximation yields (when $m = 1$)

$$y_2 = y_1 + w\,[\,f\,(x_1, y_1) + f\,(x_1 + w, y_1 + w\; f\,(x_1, y_1))]/2$$

$$= 0.96 + 0.2\,[-0.36864 - 0.62838]/2 = 0.86029.$$

Proceeding in this manner the other approximations of the differential equation can be found out easily. The exact solution of the differential equation is obtained as

$$y(x) = 1/(1 + x^2).$$

imposing the initial conditions on the general solution.

The difference between the results of the actual values obtained from exact solution and the approximate values obtained from (4.15) yields the ERROR at each step. This is the simplest method and is not so useful in the field of practical numerical analysis.

Picard's Method

It is a process of successive approximations. The successive approximations $y_m(x)$ are determined from the formula

$$y_{m+1}(x) = y_0 + \int_{x_0}^{x} f\,(x, y_m(x))\; dx \qquad \ldots (4.16)$$

where $m = 0, 1, 2, \ldots$.

The second term on the right side is a simple integral that can be determined by the process of integration. For each approximation we need the determined (approximate) value of y every time. The method can be employed to find out the approximations of the differential equation using the same initial conditions as taken before.

As the first approximation we have

$$y_1 = 1 + \int_0^{0.2} \left[-2x \times (1.0)^2\right] dx = 0.96$$

The other approximate values for y can be obtained as usual.

Heun's Method

The approximation process can be performed applying the formula

$$y_{m+1} = y_m + w[f(x_m, y_m) + 3f(x_m + 2w/3, y_m + 2w\ f(x_m, y_m))/3]/4 \qquad \ldots (4.17)$$

where $m = 0, 1, 2, \ldots, N - 1$ and N = number of divisions. The approximation is qualitatively better than that of Euler's.

Milne's Method

The formula according to Milne reads

$$y_{m+1} = y_{m-3} + 4w[2\ f(x_m, y_m) - f(x_{m-1}, y_{m-1}) + 2f(x_{m-2}, y_{m-2})]/3 \qquad \ldots (4.18)$$

where $m = 0, 1, 2, \ldots, N - 1$ and N = number of equal parts.

Simpson's Method

For approximation Simpson has proposed the following equation

$$y_{m+1} = y_{m-1} + w[f(x_{m+1}, y_{m+1}) + 4f(x_m, y_m) + f(x_{m-1}, y_{m-1})]/3 \qquad \ldots (4.19)$$

where $m = 0, 1, 2, \ldots, N - 1$. The formulas of Milne and Simpson are reasonably acceptable.

Adams-Moulton Method

The formula for Adams-Moulton Method reads

$$y_{m+1} = y_m + w[9f(x_{m+1}, y_{m+1}) + 19f(x_m, y_m) - 5f(x_{m-1}, y_{m-1}) + 5f(x_{m-2}, y_{m-2})]/3 \qquad \ldots (4.20)$$

where $m = 2, 3, \ldots, n - 1$.

Because of the fact that $(m + 1)$th approximation for y depends partly on the functional relation $f(x_{m+1}, y_{m+1})$, it is known as an *implicit* method. Simpson's method is similarly an implicit method.

Example 4.9 Returning to the initial value problem considered by Euler's method $dy/dx = 2x(1 + y)$ with the initial conditions for $y_0 = 0$ when $x_0 = 0$, we accept the auxiliary conditions $w = 0.1$ and $m = 2, 3, \ldots, 9$.

The implicit equation in (4.20) from Adams-Moulton takes the reduced form when $m = 2$ $x_n = w \times n$, $n = 0, 1, 2, \ldots$

$$y_3 = y_2 + 0.1[9 \times 2x_3(1 + y_3) + 19 \times 2x_2(1 + y_2) - 5 \times 2x_1(1 + y_1) + 2x_0(1 + y_1)]/24$$

$$= y_2 + 0.1(5.4y_3 + 7.6y_2 - y_1 + 12.0)/24.$$

Rearranging we obtain an equation explicitly for y_3

$$y_3 = 1.0553y_2 - 4.2625y_1 + 0.0511$$

that can be solved by using the method of backward difference formula. As a matter of fact this technique is termed as *Predictor-Corrector* method because of the prediction of an approximation made by the explicit method and the implicit method corrects the prediction.

Method Due to Runge-Kutta

The classical mathematical technique form C. Runge for approximating the solution to initial value problems was later developed by W. Kutta, both being German applied mathematicians. The main mathematical achievement in applying the method involves the function evaluations of the functional *G*.

Each estimation of *G* is the result of certain mathematical operations.

The algebraically developed technique of evaluation for the formula needs many more calculations leading to a simple representation that states

$$y_{m+1} = y_m + (G1 + 2G2 + 2G3 + G4)/6 \qquad \ldots (4.21)$$

where $m = 0, 1, 2, \ldots, n-1,$

and

$$G_1 = wf\,(x_m, y_m)$$
$$G_2 = wf\,(x_m + w/2, y_m + G1/2)$$
$$G_3 = wf\,(x_m + w/2, y_m + G2/2)$$
$$G_4 = wf\,(y_m + w, y_m + G_3).$$

Evident from the equation in (4.21) that the approximation requires in each case four function evaluations. In other words, we consider the functional relation $f(x, y)$ four times for a single integration process. In view of this consideration the method due to Runge-Kutta is called a fourth-order method.

Example 4.10 We recall the initial-value problem $dy/dx = 2x(1 + y)$ with $x_0 = 0$, $y_0 = 0$, $w = 0.1$ and $m = 0, 1, 2, \ldots, 9$.

$$G_1 = wf\,(x_0, y_0) = 0.1 \times 0 = 0$$
$$G_2 = wf\,(x_0 + w/2, y_0 + G1/2) = 0.1 \times 2 \times 0.1/2 \times (1 + 0 + 0) = 0.01$$
$$G_3 = wf\,(x_0 + w/2, x_0 + G2/2) = 0.1 \times 2 \times 0.1/2 \times (1 + 0 + 0.01/2) = 0.01005$$
$$G_4 = wf\,(x_0 + w, y_0 + G_3) = 0.1 \times 2 \times 0.1 \times (1 + 0 + 0.01005) = 0.020201.$$

Hence, by inserting the *G*-values in (4.21) we have the first approximation

when $m = 0$

$$y_1 = 0.01005016$$

When $m = 1$ we obtain the following numerical values for *G*

$$G_1 = 0.1 \times 2 \times 0.1 \times (1 + 0.1) = 0.022$$
$$G_2 = 0.1 \times 2 \times (0.1 + 0.1/2) \times (1 + 0.1 + 0.02/2) = 0.03333$$
$$G_3 = 0.1 \times 2 \times (0.1 + 0.1/2) \times (1 + 0.1 + 0.3333/2) = 0.0334999$$
$$G_4 = 0.1 \times 2 \times (0.1 + 0.1) \times (1 + 0.1 + 0.0334999) = 0.04533999.$$

Hence, for the second approximation we have

$$y_2 = 0.01005016 + 0.03349996 = 0.04355012.$$

The other approximate value for *y* can be determined by calculating the corresponding *G*-values. The error estimate is then ascertained by comparing the numerical values of the actual results obtained from the particular solution

$$y(x) = e^{x^2} - 1$$

imposing the corresponding step size, *w*.

We have considered so far the most standard method far solution to the initial-value problems by mathematical procedures on theoretical basis. Now we try to attack the differential equations considering this an appropriate numerical method that should run on a computer program.

The approximate values obtained by employing the fourth-order Runge-Kutta method to the initial-value problems are reasonably acceptable assuming a carefully suggested computational width.

The Runge-Kutta method is a more efficient method that can be used for approximating the solution of ordinary differential equation. In the next following sections it is shown that the computational achievement of this method is so profound that it is not unreasonable to use the method. It is very friendly to a computer program, too.

Separation of Variables

I have already introduced this particular form of differential equation in section (4.1) defining as standard form *A* and have obtained the theoretical result containing the constant of integration, *c*. The computational achievement in finding the practical solutions with the help of initial conditions needs a real software, a program.

The program g4IVPIDE used to approximate the solution of initial value problems of the type, when the variables are separated. This program has been employed, also for the approximations of other different problems that are considered in other sections of this chapter.

The program instructions have been designed by the programming language C++ (Turbo) using the classical Runge-Kutta method described in previous section and the complete program has been executed on any MS-DOS operating system (version 6.0) furnished with an appropriate Turbo C++ (version 3.0) compiler from BORLAND International.

Example 4.11

DE: dy/dx – y/tan x = cot x

PS: y(x) = sin x – 1

C++: dy [sin(x)] = dx [1 + y) × cos(x)]

IC: = y(0.5235987) = –0.5 Sts: 0.01 ($\pi/6 = 0.5235987$)

Here, DE = Differential Equations, PS = Particular Solution, C++ = Computer Equation, IC = Initial Conditions and Sts = Step size.

The initial value problem yields the following results.

The Computer Solution of the IVP of order 1:

x-value	Approx. value	Exact value	Error
y (0.53359878):	–0.49136489	–0.49136489	–5.31010e–10
y (0.54359878):	–0.48278064	–0.48278064	–5.43717e–10
y (0.55359878):	–0.47424812	–0.47424812	–5.56266e–10
y (0.56359878):	–0.46576817	–0.46576817	–5.68660e–10
y (0.57359878):	–0.45734164	–0.45734164	–5.80907e–10
y (0.58359878):	–0.44896938	–0.44896938	–5.93005e–10
y (0.59359878):	–0.44065222	–0.44065222	–6.04958e–10
y (0.60359878):	–0.43239099	–0.43239099	–6.16768e–10
y (0.61359878):	–0.42418653	–0.42418653	–6.28440e–10
y (0.62359878):	–0.41603964	–0.41603964	–6.39973e–10

Now, if we throw a glance at the results, we get a general conception of the initial value problem in this example that is represented here by the numerical values with 8 digits after decimal. In this case, for the corresponding increments of *x* the exact values and the approximate values are the same.

The table provides the exact values of *y* that have been simultaneously determined by the program in order to compare the results readily. The equation for the particular solution (PS) is used for the actual values. The errors caused by the approximate solutions are also tabulated.

The graphical figure that can be drawn with the aid of numerical data from the exact solution and can be compared with the graphical representation of the approximations. The comparison has no degree of variation in this case. The errors that arise from the approximate values have also been displayed.

The amount of error in the first case in equal to –5.3101e–10. In floating-point decimal conversion it can be expressed as –0.00000000053101 that indicates the first significant digit at the 10th place after decimal. In view of that we can conclude that the approximate solution for the initial value problem is correct to 9 places after decimal.

For the sake of convenience, I have confined myself to exhibiting only the first ten values of approximations. If you wish to have more from the development of approximation process, you are recommended to increase the value of *m* in the program. In the present case, *m* has been given the value 10.

Evaluation by Use of Substitutions

This form of differential equation has already been introduced in section (4.1) characterising as standard form *B*. The same program has been applied to approximate the solution of initial value problem of the form, when the variables are separable by the method of substitution. The program in C++ that you want to retain in a storage device, is provided in Floppy disc.

Example 4.12

DE: Dy/dx + 1 = (x + y)/2
PS: y(x) = $e^{x/2}$ – x
C++: [2] dy = [x + y – 2) dx
IC: = y(0) = 1 Sts: 0.01

The Computer Solution of the IVP of Order 1

***x*-value**	**Approx. value**	**Extact value**	**Error**
y (0.01):	0.99501252	0.99501252	2.60902e–14
y (0.02):	0.99005017	0.99005017	5.24025e–14
y (0.03):	0.98511306	0.98511306	7.90479e–14
y (0.04):	0.98020134	0.98020134	1.05915e–13
y (0.05):	0.97531512	0.97531512	1.33005e–13
y (0.06):	0.97045453	0.97045453	1.60427e–13
y (0.07):	0.96561917	0.96561917	1.88072e–13
y (0.08):	0.96081077	0.96081077	2.16049e–13
y (0.09):	0.95602786	0.95602786	2.44249e–13
y (0.10):	0.95127110	0.95127110	2.72671e–13

The significant digit of the error starts up at 14th decimal position and ends in 13th decimal position. The field of errors confines to the range [2.60902e–14, 2.72671e–13]. The approximations are correct to within this domain.

4.3 Exact Equations

This particular form of differential equation has been defined in section (4.1) and classified as standard form *C*. The condition for exactness has been defined mentioning the criteria for the differential equation being exact.

Let us start with a differential equation

$$P(x,y)\,dx + Q(x, y)\,dy = 0 \qquad \ldots (4.22)$$

If we assume a function $g(x,y)$ such that

$$\partial g/\partial x = P \quad \text{and} \quad y_m^2\,\partial g/\partial y = Q \qquad \ldots (4.23)$$

then we can write the equation (4.22) in the form

$$\partial g/\partial x\,dx + \partial g/\partial y\,dy = 0 \qquad \ldots (4.24)$$

than can be written in compact form $dg = 0$.

So the equation (4.22) can be termed as an *exact differential equation*.

Now turning the case around and assuming that the relation (4.22) is exact we can take a function $g(x, y)$ that satisfies the equations in (4.23). Taking partial derivatives of the function $g(x, y)$ yields

$$\partial^2 g/\partial y\partial x = \partial^2 g/\partial x dy \qquad \ldots (4.25)$$

that gives finally

$$\partial P/\partial y = \partial Q/\partial x, \qquad \ldots (4.26)$$

is a necessary condition for exactness. The condition is satisfied only if the first order partial derivatives of the continuous functions $P(x, y)$ and $Q(x, y)$ are continuous on some rectangular system of the (x,y) plane.

The program g4IVP1DE used to approximate the solution of initial value problems of the type, when the forms are exact. The program has been written in computer language C++ using the Runge-Kutta method described earlier in this Chapter and the execution of this program has been performed on any MS-DOS operating system furnished with an appropriate C++ (version 3.0) compiler.

Example 4.13

DE: x^y Ln x dy/dx + yx^{y-1} = 0
PS: = y(x) = Ln 2/Ln x
C++: [–pow(x,y) × log(x)] dy = [y × pow[x,y – 1]dx
IC: y(2) = 1; Sts: 0.01

The *initial value problem* (IVP) is in this case the differential equation (DE) together with the initial conditions (IC).

Rearranging the differential equation in differential form for the input of Computer equation, that means, the coefficient of *dy* on the left side of the equation and the coefficient of *dx* on the right, yields a form, we call it C++ equation.

The program needs only the respective coefficients of *dy* and *dx* for input as well as the initial conditions for the execution of the program without concerning with the matter of exactness.

After successful completion of the program we obtain the following results in tabular form:

x-value	Approx. value	Exact value	Error
y (2.01):	0.99285590	0.99285590	6.44373e–13
y (2.02):	0.98584787	0.98584787	1.25011e–12
y (2.03):	0.97897195	0.97897195	1.81966e–12
y (2.04):	0.97222437	0.97222437	2.35523e–12
y (2.05):	0.96560150	0.96560150	2.85916e–12
y (2.06):	0.95909982	0.95909982	3.33344e–12
y (2.07):	0.95271597	0.95271597	3.77975e–12
y (2.08):	0.94644671	0.94644671	4.19997e–12
y (2.09):	0.94028889	0.94028889	4.59555e–12
y (2.10):	0.93423951	0.93423951	4.96814e–12

The significant digit of the error starts up at 13th decimal position and ends in 12th decimal position. The field of errors confines to the range [6.44373e–13, 4.96814e–12].

We have admitted only ten values for the approximations of *y* considering the corresponding increment (Sts) of the independent variable *x*. Now, if we throw a glance at the table we have a general conception of the initial value problem in example 4.13 that is represented here by the numeric values.

For the sake of brevity we restrict ourselves to representing the numbers with 8 digits after decimal.

The table provides the exact values of *y*(*x*) that have been simultaneously determined by the program in order to compare the results readily. The equation for the particular solution (PS) is used for the actual values. The errors caused by the approximate solutions are also tabulated.

The amount of error in the first case which reads 6.44373e–13 arises from the approximation due to the value of $x = 2.01$. This value for error indicates that the first significant digit is at the 13th place after decimal. In view of that we can have a conclusion that the approximate solution for the initial value problem is correct to 12 places after decimal. Now, you can agree with the assertion that I make earlier accepting the fourth-order Runge-Kutta method as accurate and reasonable.

Example 4.14

DE: $dy/dx - 2 \sin y(x + 2 \sin y)/(x^2+1 \cos y) = 0$

PS: $y(x) = \sin^{-1}[x^2 + 1)/(8 - 4x)]$

C++: $[x \times x + 1) \times \cos(y)]dy = [2 \times \sin(y) \times (x + 2 \times \sin(y))]dx$

IC: y(1) = 0.5235987 $(\pi/6)$; Sts: 0.001

x-value	Approx. value	Exact value	Error
y (1.01):	0.53533158	0.53533158	6.67642e–10
y (1.02):	0.54744837	0.54744837	7.42313e–10
y (1.03):	0.55996884	0.55996884	8.22592e–10
y (1.04):	0.57291433	0.57291433	9.08715e–10
y (1.05):	0.58630808	0.58630808	1.00085e–09
y (1.06):	0.60017541	0.60017541	1.09903e–09
y (1.07):	0.61454401	0.61454401	1.20313e–09
y (1.08):	0.62944431	0.62944431	1.31273e–09
y (1.09):	0.64490982	0.64490982	1.42697e–09
y (1.10):	0.66097764	0.66097764	1.54436e–09

The amount of error in the first case which reads 6.67642e–10 arises from the approximation due to the value of $x = 1.01$. This value for error indicates that the first significant digit is at the 10th place after decimal. In view of that we are led to a conclusion that the approximate solution for the initial value problem is correct to 9 places after decimal. Now, one can agree with the assertion that I make earlier accepting the fourth-order Runge-Kutta method as accurate and reasonable.

Example 4.15

DE: $e^x \sin y\ dy/dx - e^x \cos y = 3x^2$
PS: $y(x) = \cos^{-1}(-x^3/e^x)$
C++: dy[exp(x) × sin (y)] = dx [3 × **x** × x + exp(x) × cos (y)]
IC: y(0) = 1.5707963 $(\pi/2)$; Sts = 0.01

The approximations can be found out solving this problem by means of the Program with the field of errors that confines to the range [1.77703e–09, 1.62240e–09].

Example 4.16

DE: $Dy/dx - 2x(1 + \sqrt{(x^2 - y)})\Big/\sqrt{(x^2 - y)} = 0$
PS: $y(x) = x^2 - [5 - 3x^2)/2]^{2/3}$
C++: [sqrt(x × x – y)] dy = [2 × x × (1 + sqrt(x × x – y))] dx
IC: y(1) = 0, Sts: 0.01

The amount of error in the first case which reads –2.59435e–10 arises from the approximation due to the value of $x = 1.01$. This value for error indicates that the first significant digit is at the 10th place after decimal. In view of that we conclude that the approximate solution for the initial value problem is correct to 9 places after decimal.

4.4 Linear Equations

We discussed the linear differential equation as standard from D of section 4.1. To recapitulate the matter, by a linear differential equation of the first order we mean an equation of the form

$$a(x)\ dy/dx + b(x)\ y = c(x)$$

where $a(x)$, $b(x)$ and $c(x)$ are functions of x which are continuous over some interval of values of x.

Now we divide the equation by $a(x)$, provided that $a(x) \neq 0$, to obtain a linear standard form

$$dy/dx + M(x)\ y = N(x) \qquad \ldots (4.27)$$

where $M(x) = b(x)/a(x)$ and $N(x) = c(x)/a(x)$ are also continuous.

The general linear differential equation of first order can be written now

$$dy/dx + M(x)\ y = N(x)$$

where $M(x)$ and $N(x)$ are integrable functions.

Multiplying the given differential equation by $I(x, y) = e^{\int M(x)dx}$ throughout gives

$$e^{\int M(x)dx} dy/dx + e^{\int M(x)dx}\ M(x)\, y = e^{\int M(x)dx}\ N(x),$$

and rearranging the left hand side in terms of exact form yields

$$d/dx\,(e^{\int Mdx}\,y) = N\,e^{\int Mdx}. \qquad \ldots (4.28)$$

On integrating the relation (4.28) we obtain

$$e^{\int Mdx}\,y = \int N\,e^{\int Mdx} + c,$$

or

$$y = e^{-\int Mdx}\left(\int N\,e^{\int Mdx} + c\right) \qquad \ldots (4.29)$$

is the required solution for the equation (4.27).

The function $I(x, y)$ is an *integrating factor* and in this case it amounts to $e^{\int Mdx}$.

We need not be worry about the complicated formula of (4.29). What we have to do is to multiply by the integrating factor

$$e^{\int Mdx}$$

and integrate. In this section we consider only the linear differential equations of nonhomogeneous type, that means, when the equation (4.27) contains a nonzero function of x standing alone.

Example 4.17 Solve $(x + 1)\;y' + y\,(2 + x) - 2\sin x = 0$.

After rearrangement the nonhomogeneous equation can be written as

$$y' + y(2 + x)/(x + 1) = 2\sin x/(x + 1)$$

where $M = (2 + x)/(x + 1)$ and $N = 2\sin x/(x + 1)$.

Now, $$\text{I.F.} = e^{\int Mdx} = e^{\int (x+2)/(x+1)\,dx} = (x + 1)e^{x}.$$

Multiplying the given differential equation by the factor throughout yields

$$(x + 1)e^{x}y' + (2 + x)e^{x}y = 2\,e^{x}\sin x.$$

On integrating we obtain

$$e^{x}(x + 1)y = 2\int e^{x}\sin x\;dx + C = e^{x}(\sin x - \cos x) + C$$

or $$y(x + 1) = \sin x - \cos x + Ce^{-x}.$$

In this stage when I apply the initial conditions to this equation, C can be determined. Accepting $y(0) = 2$ as the initial conditions and applying we get the value for C, that amounts to 3. Hence, the particular solution of the linear nonhomogeneous equation is

$$y(x) = (x + 1)^{-1}(\sin x - \cos x + 3\,e^{-x}).$$

In order to determine the approximated solutions for the initial value problems of nonhomogeneous character I employ the same program g41VPIDE as before. The technique of the programs is based on the fourth-order Runge-Kutta method. To some extent I have discussed on the method in section 4.2. The Program file is saved to the Computer Disc that can be used for the purpose.

Example 4.18

DE: x dy/dx – y = $x^3 + 3x^2 - 2x$

PS: y(x) = $(x^3 + 6x^2 - 4x\,\mathrm{Ln}x - 5x)/2$

C++: [x] dy = [pow (x,3) + 3 × pow(x,2) – 2 × x + y] dx
IC: y(1) = 1; Sts: 0.01.

Earlier in section 4.3 the explanation for the abbreviations were made. What we need only are the coefficients of the differentials *dy* and *dx* for the input of the computer program together with the initial conditions. The C++ equation provides the coefficients confined in brackets and the initial conditions we can get from IC.

The successful program execution requires the correct and well-provided data to be inputted. In the long run we obtain the computerised solutions for approximations along with the respective particular solutions for comparison of the results in order to find out the errors for approximations of the initial value problem.

x-value	Approx. value	Exact value	Error
y (1.01):	1.03035083	1.03035083	9.81104e–12
y (1.02):	1.06140664	1.06140664	1.95346e–11
y (1.03):	1.09317237	1.09317237	2.91722e–11
y (1.04):	1.12565292	1.12565292	3.87252e–11
y (1.05):	1.15885316	1.15885316	4.81946e–11
y (1.06):	1.19277791	1.19277791	5.75815e–11
y (1.07):	1.22743199	1.22743199	6.68876e–11
y (1.08):	1.26282015	1.26282015	7.61140e–11
y (1.09):	1.29894712	1.29894712	8.52620e–11
y (1.10):	1.33581760	1.33581760	9.43332e–11

The significant digit of the error starts up at 12th decimal position and ends in 11th position. The field of errors confines to the range [9.43332e–11, 9.81104e–12].

As we see from the table of values that for convenience we take an account of the selection of ten approximations for the corresponding increments of *x*-values. It would be just as well if we restrict ourselves to represent the value with 8 digits after decimal place.

Example 4.19

DE: x dy/dx = (Ln x + 2y)/Ln x
PS: y(x) = Ln x (2 Ln x – 1)
C++: [x × Ln x) dy = (2 × y + log (x)] dx
IC: y(2.71828182) = 1; Sts: 0.01.

The Computer Solution of the IVP of Order 1

x-value	Approx. value	Exact value	Error
y (2.72828182):	1.01104310	1.01104309	–9.40410e–09
y (2.73828182):	1.02209944	1.02209943	–9.47249e–09
y (2.74828182):	1.03316872	1.03316871	–9.54088e–09
y (2.75828182):	1.04425067	1.04425066	–9.60927e–09
y (2.76828182):	1.05534498	1.05534497	–9.67767e–09
y (2.77828182):	1.06645140	1.06645139	–9.74607e–09

(Contd)

x-value	Approx. value	Exact value	Error
y (2.78828182):	1.07756964	1.07756963	–9.81448e–09
y (2.79828182):	1.08869944	1.08869943	–9.88288e–09
y (2.80828182):	1.09984052	1.09984051	–9.95128e–09
y (2.81828182):	1.11099264	1.11099263	–1.00197e–09

Example 4.20

DE: $(x^2 + 1)\,y' - 4xy = (x^2 + 1)^2$

PS: $y(x) = (1 + \tan^{-1}x)\,(x^2 + 1)^2$

C++: [x × x + 1] dy = [4 × x × y + pow(x × x + 1,2)] dx

IC: y(0) = 1; Sts: 0.01.

In this example the approximation in the first case is 1.01020168 for the step size 0.01 arises an error term 1.15792e–11. The approximation can be well acceptable.

Example 4.21

DE: $Dy/dx + y \tan x = \cos^2 x$

PS: $y(x) = \cos x\,(1 + \sin x)$

C++: [1] dy = [cos (x) × cos (x) – y × tan (x)] dx

IC: y(0) = 1; Sts: 0.01.

In this example the first approximation 1.00994933 for an increment of 0.01 arises an error 8.36220e–13.

Example 4.22

DE: $Dy/dx - 4y = -5e^{-x}$

PS: $y(x) = e^{-x}$

C++: [exp (x)] dy = [4 × y × exp (x) – 5] dx

IC: y(0) = 1; Sts: 0.01.

4.5 Bernoulli's Equation

The technique for determining the solutions of the nonlinear equations in Bernoulli form has been discussed in standard form *E* of section 4.1. For convenience, let us recall here the equation (4.7)

$$dy/dx + P(x)\,y = Q(x)\,y^n \qquad \dots (4.30)$$

where $n \neq 0$ and $n \neq 1$, are the conditions for special cases.

This is a first order differential equation, known as Bernoulli's equation, that can be solved by reducing it to linear form by means of proper substitution.

If we put $n = 0$ in the generalised equation (4.30), we obtain a linear *nonhomogeneous* equation. The equation reduces to a linear *homogeneous* form when $n = 1$.

Dividing the equation by y^n through in order to get a reduced linear form and setting $u = y^{1-n}$, where u is a new variable, we get another equation that can be readily solved.

Let us illustrate an example.

Example 4.23 Solve the equation $xy' + y = 2\sqrt{(xy)}$.

This is Bernoulli's equation with $n = 1/2$.

Dividing both sides of the equation by $y^{1/2}$, as usual, we get

$$xy^{-1/2}\, y' + y^{1/2} = 2\, x^{1/2}. \qquad \ldots (4.31)$$

Now setting $u = y^{1/2}$ yields

$$u' = 1/2y^{-1/2}\;\; y'$$

and the equation (4.31) reduces to

$$u' + u/(2x) = x^{-1/2}$$

that is, of course, a linear nonhomogeneous form.

Here, the
$$\text{I.F.} = e^{\int 1/2x\, dx} = x^{1/2}.$$

In accordance with the linear system we multiply the equation by the I.F. and on integrating we obtain

$$u\, x^{1/2} = \int 1 \cdot dx + c = x + c.$$

Hence, the general solution in y becomes

$$x^{1/2}\, y^{1/2} = x + c$$

or
$$xy = (x + c)^2.$$

At this point the initial conditions for the equation can be imposed for finding the value of c, the constant factor. Considering $y(0.5) = 4.5$ as the arbitrary initial conditions and determining the value for c we obtain the exact solution

$$y = (x + 1)^2/x.$$

The approximations for the initial value problems of Bernoulli type can be determined by means the program g41VPIDE

Example 4.24

DE: dy/dx + xy = e^{x2} y^3
PS: y(x) = sqrt(1/(2 × pow(exp(x),2) × (2 – x))
C++: dy = [exp(x × x) × pow(y,3) –x × y] dx
IC: y(0) = 0.5; Sts: 0.01.

The Computer Solution of the IVP of Order 1

***x*-value**	**Approx. value**	**Exact value**	**Error**
y (0.01):	0.50122965	0.50122965	–4.18776e–13
y (0.02):	0.50241841	0.50241841	–8.43436e–13
y (0.03):	0.50356607	0.50356607	–1.27376e–12
y (0.04):	0.50467237	0.50467237	–1.70963e–12
y (0.05):	0.50573712	0.50573712	–2.15084e–12
y (0.06):	0.50676009	0.50676009	–2.59726e–12

(Contd)

x-value	Approx. value	Exact value	Error
y (0.07):	0.50774111	0.50774111	–3.04856e–12
y (0.08):	0.50867998	0.50867998	–3.50475e–12
y (0.09):	0.50957654	0.50957654	–3.96538e–12
y (0.10):	0.51043063	0.51043063	–4.43057e–12

The significant digit of the error starts up at 13th decimal position and ends in 12th decimal position. The field of errors confines to the range [–4.18776e–13, –4.43057e–12]. The approximations are correct to within this domain.

Example 4.25

DE: $y^2\ dy/dx + x^2 \sin^3 x = y^3 \cos x/\sin x$

PS: $y(x) = \sin x\ (2.67835 - x^3)^{1/3}$

C++: [y × y] dy = [pow(y, 3) × cos x/sin x –x × x × pow(sin (x),3)] dx

IC: y(1) = 1; Sts: 0.01.

The Computer Solution of the IVP of Order 1

x-value	Approx. value	Exact value	Error
y (1.01):	1.00027765	1.00027762	–2.57027e–08
y (1.02):	1.00017820	1.00017817	–2.60664e–08
y (1.03):	0.99969066	0.99969063	–2.64404e–08
y (1.04):	0.99880338	0.99880335	–2.68254e–08
y (1.05):	0.99750394	0.99750391	–2.72222e–08
y (1.06):	0.99577910	0.99577908	–2.76316e–08
y (1.07):	0.99361470	0.99361467	–2.80543e–08
y (1.08):	0.99099550	0.99099547	–2.84913e–08
y (1.09):	0.98790514	0.98790511	–2.89433e–08
y (1.10):	0.98432592	0.98432589	–2.94112e–08

The significant digit of the error starts up at 8th decimal position and ends in 8th decimal position. The field of errors confines to the range [–2.57027e–08, –2.94112e–08]. The approximations are correct to within this domain.

Example 4.26

DE: $2x^3\ dy/dx - 2x^2y + y^3 = 0$

PS: $y(x) = x \times (1 + \log(x))^{-1/2}$

C++: [2 × pow(x,3) dy =[y × (2 × x × x – y · y)] dx

IC: y(1) = 1; Sts: 0.01

The Computer Solution of the IVP of Order 1

x-value	Approx. value	Exact value	Error
y (1.01):	1.00501227	1.00501227	9.72555e–13
y (1.02):	1.01004822	1.01004822	1.86806e–12

(Contd)

x-value	Approx. value	Exact value	Error
y (1.03):	1.01510659	1.01510659	2.69385e–12
y (1.04):	1.02018619	1.02018619	3.45635e–12
y (1.05):	1.02528593	1.02528593	4.16134e–12
y (1.06):	1.03040474	1.03040474	4.81393e–12
y (1.07):	1.03554164	1.03554164	5.41855e–12
y (1.08):	1.04069570	1.04069570	5.97966e–12
y (1.09):	1.04586603	1.04586603	6.50080e–12
y (1.10):	1.05105181	1.05105181	6.98508e–12

The significant digit of the error starts up at 13th decimal position and ends in 12th decimal position. The field of errors confines to the range [6.98508e–12, 9.72555e–13].

Example 4.27

DE: x dy/dx = y(x^2 + 3 Lny)

PS: y(x) = e^{x^2} (x – 1)

C++: [x] dy =[y × (x × x × 3 × log (y)] dx

IC: y(1) = 1; Sts: 0.01.

The significant digit of the error starts up at 10th decimal position and ends in 9th decimal position. The field of errors confines to the range [3.23107e–09, 2.23220e–10].

4.6 Riccati Equation

Let us consider a first order linear differential equation as represented by

$$y' = M(x) + N(x)\, y \qquad \dots (4.32)$$

and also the equation represented by the equation (4.9) mentioned in section 4.1

$$y' = M(x) + N(x)\, y + P(x)\, y^2 \qquad \dots (4.33)$$

where $M(x)$, $N(x)$ and $P(x)$ are functions of x or may be constants.

If we make an observation of the two equations, we can deduce that the equation represented by (4.33), known as Riccati Equation, is only a finite extension of the equation (4.32) developed in a natural process.

A direct method cannot be applied to solve the Riccati equation. The general solution of the equation can have the form

$$y(x) = y_1(x) + a(x)$$

where $y_1(x)$ is a particular solution that can be found by inference and $a(x)$ is the general solution of the Bernoulli's equation

$$a' - (N + 2Py_1)\, a = Pa^2.$$

In section 4.1 of this chapter we have already considered the subject matter giving an example which has simply shown how to solve the problem of Riccati type, Now, I apply the same program as mentioned in section 4.5 for the approximations of the initial value problems of Riccati type.

Example 4.28

DE: $Dy/dx = 1 + y/x + (y/x)^2$
CE: $dy = [1 + y/x + (y/x)^2]\, dx$
PS: $y(x) = x \tan(Lnx)$
IC: $y(1) = 0$, Width: 0.01.

The Computer Solution of the IVP of Order 1

x-value	Approx. value	Exact value	Error
y (1.01):	0.01005017	0.01005017	–2.01127e–12
y (1.02):	0.02020132	0.02020132	–3.92257e–12
y (1.03):	0.03045444	0.03045444	–5.73754e–12
y (1.04):	0.04081047	0.04081047	–7.45960e–12
y (1.05):	0.05127036	0.05127036	–9.09198e–12
y (1.06):	0.06183504	0.06183504	–1.06378e–11
y (1.07):	0.07250542	0.07250542	–1.20998e–11
y (1.08):	0.08328242	0.08328242	–1.34807e–11
y (1.09):	0.09416692	0.09416692	–1.47832e–11
y (1.10):	0.10515982	0.10515982	–1.60096e–11

The significant digit of the error starts up at 12th decimal position and ends in 11th decimal position. The field of errors confines to the range [–1.60096e–11, –2.01127e–12]. The approximations are correct to within this domain.

Example 4.29

DE: $x(1 - x^3)\, dy/dx = x^2 + y - 2xy^2$
PS: $y(x) = x^2 + (x - x^4)/(x^2 + 1)$
C++: [x × (1 – pow(x,3))] dy = [x × x + y – 2 × x × y × y] dx
IC: y(–1) = 0; Sts: 0.01.

The significant digit of the error starts up at 12th decimal position and ends in 11th decimal position. The field of errors confines to the range [–1.42286e–11, –1.31512e–12].

Example 4.30

DE: $Dy/dx = 1 + y/x + (y/x)^2$
PS: $y(x) = x \tan(Ln\, x)$
C++: [1] dy = [1 + y/x + y × y/(x × x)] dx
IC: y(1) = 0; Sts: 0.01.

The significant digit of the error starts up at 12th decimal position and ends in 11th decimal position. The field of errors confines to the range [–1.60096e–11, –2.01127e–12].

4.7 System of Simultaneous Differential Equations

So far we have applied the numerical methods for approximating the solution of initial value problems considering only One Single differential equation. But the system of simultaneous differential equations of some order is the common occurrence in connection with mechanical, dynamical and astronomical problems. The physical problem dealing with small vibrations or small deviations are of great importance in relation to simultaneous system.

We make a reference to simultaneous system of differential equations, when several functions are dependent on a single independent variable and upon each other. In short, simultaneous means combined. As for example, a functional relation

$$dy/dx = g(x,y)$$

can be represented as

$$dy/dt = l(x,y) \text{ and } dx/dt = m(x,y)$$

equivalent to a simultaneous system accepting the condition

$$g(x,y) = l(x,y)/m(x,y).$$

In other words, simultaneous system is a dependent system of equations. In order to make it free from the state of being dependent, the condition, just mentioned, can be applied.

An *nth-order* system of the first order initial value problems can be expressed generally in the form

$$\begin{aligned} dv_1/dt &= g_1\ (t, v_1, v_2, \ldots, v_n) \\ dv_2/dt &= g_2\ (t, v_1, v_2, \ldots, v_n) \\ \ldots &\quad \ldots \quad \ldots \quad \ldots \\ dv_n/dt &= g_n\ (t, v_1, v_2, \ldots, v_n). \end{aligned} \qquad \ldots (4.34)$$

on an interval $a \le t \le \beta$ with the initial conditions

$$v_1(\alpha) = \gamma_1,\ v_2(\alpha) = \gamma_2, \ldots,\ v_n(\alpha) = \gamma_n.$$

The solution of the system of equations (4.34) can be defined as the set of n functions

$$\begin{aligned} v_1 &= \psi_1(t), \\ v_2 &= \psi_2(t), \\ \ldots &\quad \ldots \\ v_n &= \psi_n(t) \end{aligned}$$

that satisfies the system (4.34) together with the prescribed initial conditions.

The most simple class of the system is that containing only one independent variable and the other two dependent variables. For the sake of convenience and simplicity, we confine ourselves through the rest of this chapter to considering the system of only two differential equations which may be written in the form

$$\begin{aligned} dx/dt &= f\ (x, y, t) \\ dy/dt &= g\ (x, y, t). \end{aligned} \qquad \ldots (4.35)$$

In this system of equations (4.35) t denotes the independent variable; x and y denoting the dependent variables. The system of functions

$$\begin{aligned} x &= F\ (t) \\ x &= G\ (t) \end{aligned} \qquad \ldots (4.36)$$

is the solution of the original system defined in (4.35). Because of the fact that the equations are linked together, the system can be represented by a particular curve in Cartesian coordinate system, commonly known as *xy*-plane, being described by the path that moves in the plane and determines the phase of the curve at any instant of time.

Let us consider now a nonhomogeneous linear system of the form

$$dx/dt = a_1 x + a_2 y + g_1(t)$$
$$dy/dt = b_1 x + b_2 y + g_2(t). \quad \ldots (4.37)$$

where a_1, a_2, b_1 and b_2 are the given coefficients. The functions $g_1(t)$ and $g_2(t)$ are continuous on a certain closed interval of the t-axis. If these functions are not present in the equations, the system (4.37) in reduced form as expressed below

$$dx/dt = a_1 x + a_2 y$$
$$dy/dt = b_1 x + b_2 y \quad \ldots (4.38)$$

is known as *homogeneous*.

The method I want to describe now for the solution of linear homogeneous system is on the basis of determining linearly independent solutions and can be obtained directly from the given system.

With the help of the exponential function we simply set

$$x = \gamma\, e^{mt}$$
$$y = \delta\, e^{mt} \quad \ldots (4.39)$$

so that,

$$dx/dt = \gamma m\, e^{mt}$$
$$dy/dt = \delta m\, e^{mt} \quad \ldots (4.40)$$

Substituting (4.39) and (4.40) in homogeneous system (4.38) yields

$$\gamma m\, e^{mt} = a_1 \gamma\, e^{mt} + a_2 \delta\, e^{mt}$$
$$\delta m\, e^{mt} = b_1 \gamma\, e^{mt} + b_2 \delta\, e^{mt}$$

Without loss of generality we can eliminate e^{mt} throughout obtaining two new equations in γ and δ

$$(a_1 - m)\gamma + a_2 \delta = 0$$
$$b_1 \gamma + (b_2 - m)\delta = 0 \quad \ldots (4.41)$$

In order to get nontrivial solutions we should find an auxiliary equation from (4.41) accepting the determinant of the coefficients of γ and δ equal to zero.

Now
$$\begin{vmatrix} a_1 - m & a_2 \\ b_1 & b_2 - m \end{vmatrix} = 0$$

or
$$m^2 - (a_1 + b_2)\, m + (a_1 b_2 - a_2 b_1) = 0.$$

Assuming that the equation has two distinct real roots, say m_1 and m_2, we have two series of solutions that are linearly independent

$$x = \gamma_1 e^{m_1 t} \quad \text{and} \quad x = \gamma_2 e^{m_2 t}$$

and
$$y = \delta_1 e^{m_1 t} \quad \text{and} \quad y = \delta_2 e^{m_2 t}.$$

Hence, the general solution of the system (4.38) becomes

$$x = C_1 \gamma_1 e^{m_1 t} + C_2 \gamma_2 e^{m_2 t}$$

and $$y = C_1 \delta_1 e^{m_1 t} + C_2 \delta_2 e^{m_2 t}.$$

Example 4.31 Solve the system

$$dx/dt = 4x - y$$
$$dy/dt = 2x + y.$$

Obviously $a_1 = 4$, $a_2 = -1$, $b_1 = 2$ and $b_2 = 1$.

By adopting the same substitutions as in (4.39) for the system we proceed to solve the given set of equations. The system (4.41) takes the form now

$$(4 - m)\,\gamma - \delta = 0$$
$$2\gamma + (4 - m)\,\delta = 0.$$

The auxiliary equation is $$m^2 - 5m + 6 = 0$$

or $$(m - 3)\,(m - 2) = 0.$$

Case 1

When $m = 3$, we get from the above equation $\gamma = \delta$.

A simple nontrivial solution of this system is $\gamma = 1,\ \delta = 1$.

In this case the given system has the solution $x = e^{3t}$ and $y = e^{3t}$.

Case 2

When $m = 2$, we obtain $\gamma = \delta/2$.

So, when $\gamma = 1,\ \delta = 2$ lead to the results $x = e^{2t}$ and $y = 2e^{2t}$.

Hence, the general solution of the given system in the example is

$$x = C_1 e^{2t} + C_2 e^{3t}$$

and $$y = 2C_1 e^{2t} + C_2 e^{3t}.$$

At this stage when we introduce the initial conditions to the general solution of the system assuming $x(0) = 0$ and $y(0) = 1$, we can find the value of C_1 and C_2.

The exact solution of the system, then, becomes

$$x = e^{2t} - e^{3t}$$

and $$y = 2e^{2t} - e^{3t}.$$

The program g4LINSYS has been constructed to approximate the solutions of initial value problems. For this purpose we have applied the same principles as in the case for the solutions of initial value problems of ordinary differential equations.

Example 4.32

```
 Sys: Dx – x + 2y = 0, Dy + 3x – 2y = 0
  PS: x = exp(–t)[5 – 2 exp(5t)], y = exp(–t)[5 + 3 exp(5t)]
 C++: dx = [x – 2y] dt
      dy = [2y – 3x] dt
  IC: x(0) = 3, y(0) = 8; Sts: 0.01
```

The coefficients of the differential *dt* are considered only for the input of the computer program. In C++, the system of equations provides the coefficients of the differential *dt* confined in brackets. The *values* of *x*, *t* and *y* at the initial stage that can be obtained from IC will be set for the initial conditions.

C++ Solution of the System of DE:

Step-up	*x*-Approxim	*x*-Actual	*y*-Approxim	*y*-Actual
0.01	2.86862762	2.86862762	8.07268149	8.07268149
0.02	2.73441923	2.73441923	8.15085456	8.15085457
0.03	2.59723397	2.59723396	8.23471821	8.23471822
0.04	2.45692546	2.45692545	8.32447980	8.32447981
0.05	2.31334162	2.31334161	8.42035538	8.42035540
0.06	2.16632438	2.16632437	8.52257010	8.52257012
0.07	2.01570949	2.01570947	8.63135851	8.63135854
0.08	1.86132622	1.86132620	8.74696500	8.74696502
0.09	1.70299712	1.70299710	8.86964414	8.86964417
0.10	1.54053772	1.54053769	8.99966115	8.99966118

The graphical figure drawn below represents the values of *x*- and *y*-approximations (Fig. 4.1)

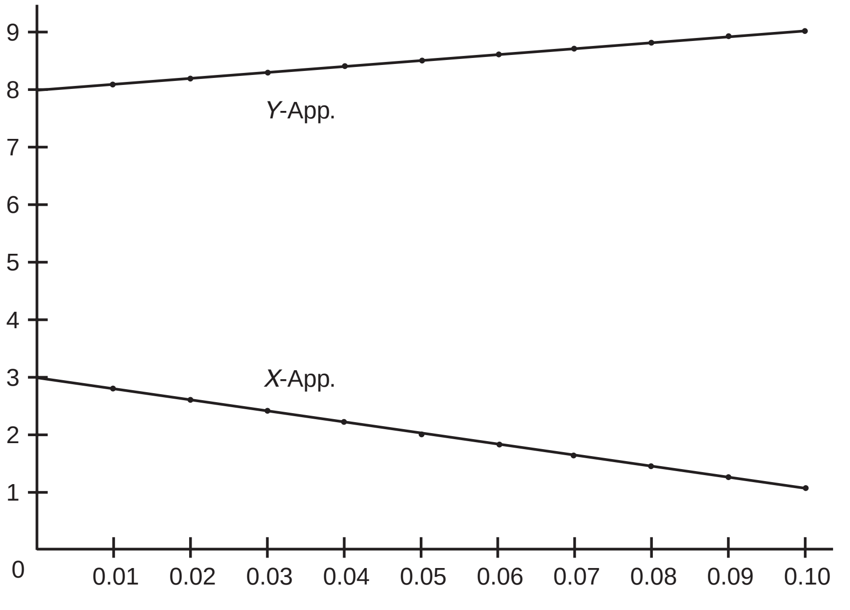

Fig. 4.1

Example 4.33

Sys: Dx = 4x + y, Dy = 3x + 2y
PS: x(t) = exp(t)[exp(4t) – 2], y(t) = exp(t)[exp(4t) + 6]
C++: dx = [4x + y] dt
dy = [3x – 2y] dt
IC: x(0) = –1, y(0) = 7; Sts: 0.01.

C++ Solution of the System of DE:

Step-up	x-Approxim	x-Actual	y-Approxim	y-Actual
0.01	–0.96882924	–0.96882924	7.11157210	7.11157210
0.02	–0.93523177	–0.93523176	7.22637895	7.22637896
0.03	–0.89907483	–0.89907483	7.34456144	7.34456145
0.04	–0.86021880	–0.86021879	7.46626739	7.46626740
0.05	–0.81851679	–0.81851678	7.59165198	7.59165199
0.06	–0.77381431	–0.77381429	7.72087807	7.72087809
0.07	–0.72594884	–0.72594881	7.85411661	7.85411664
0.08	–0.67474947	–0.67474944	7.99154707	7.99154710
0.09	–0.62003642	–0.62003638	8.13335785	8.13335789
0.10	–0.56162061	–0.56162057	8.27974674	8.27974678

The values of *x*- and *y*-Approximations have been plotted in the graphical figure drawn below. The *x* approximations have a range between –1 and 0 and the values of *y* approximations lie between 7 and 8.

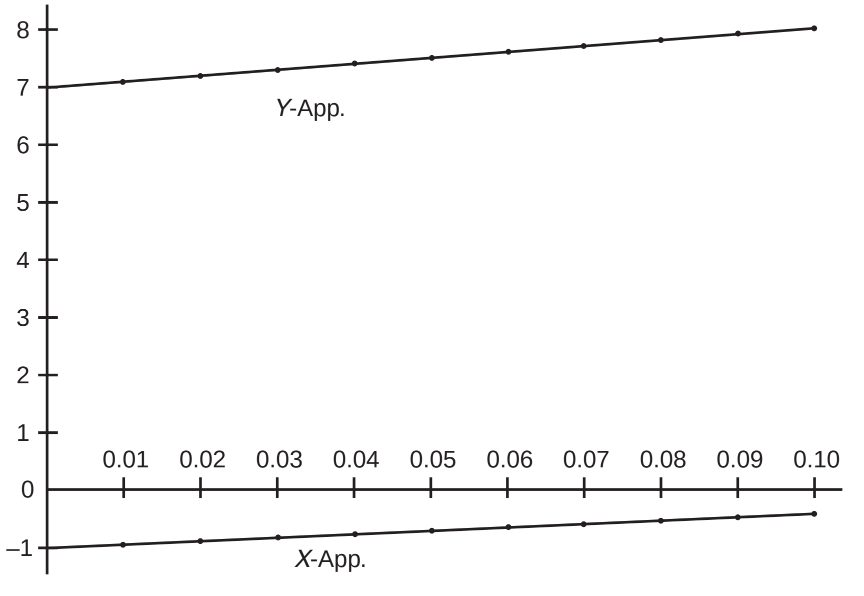

Fig. 4.2

Example 4.34

Sys: dx/dt + 7x – y = 0, dy/dt + 2x + 5y = 0

PS: x(t) = exp (–6 × t) × (cos(t) + sin (t)), y(t) = 2 × exp (–6 × t) × cos(t)

C++: dx = [y – 7 × x] dt

dy = [–(2 × x + 5 × y] dt

IC: x(0) = 1, y(0) = 2; Sts: 0.01.

Step-up	*x*-Approxim	*x*-Actual	*y*-Approxim	*y*-Actual
0.01	0.95113493	0.95113493	1.88343490	1.88343489
0.02	0.90448028	0.90448028	1.77348613	1.77348612
0.03	0.85994872	0.85994872	1.66978876	1.66978874
0.04	0.81745537	0.81745537	1.57199732	1.57199729
0.05	0.77691787	0.77691787	1.47978482	1.47978478
0.06	0.73825635	0.73825635	1.39284181	1.39284177
0.07	0.70139344	0.70139344	1.31087547	1.31087542
0.08	0.66625423	0.66625423	1.23360873	1.23360868
0.09	0.63276628	0.63276628	1.16077949	1.16077943
0.10	0.60085961	0.60085960	1.09213979	1.09213973

We illustrate now the nonhomogeneous linear system expressed in (4.37) by use of some examples. The same method has been applied for determining the approximation of the system of equations

Example 4.35

Sys: dx/dt = 3y + 6 sin(t), dy/dt + 3x = 0
PS: x(t) = [3 cos(t) + cos(3t)]/4, y(t) = –[9 sin (t) + sin (3t)]/4
C++: dx = [3 × y + 6 × sin (t)] dt
dy = [–3 × x] dt
IC: x(0) = 1, y(0) = 0; Sts: 0.01.

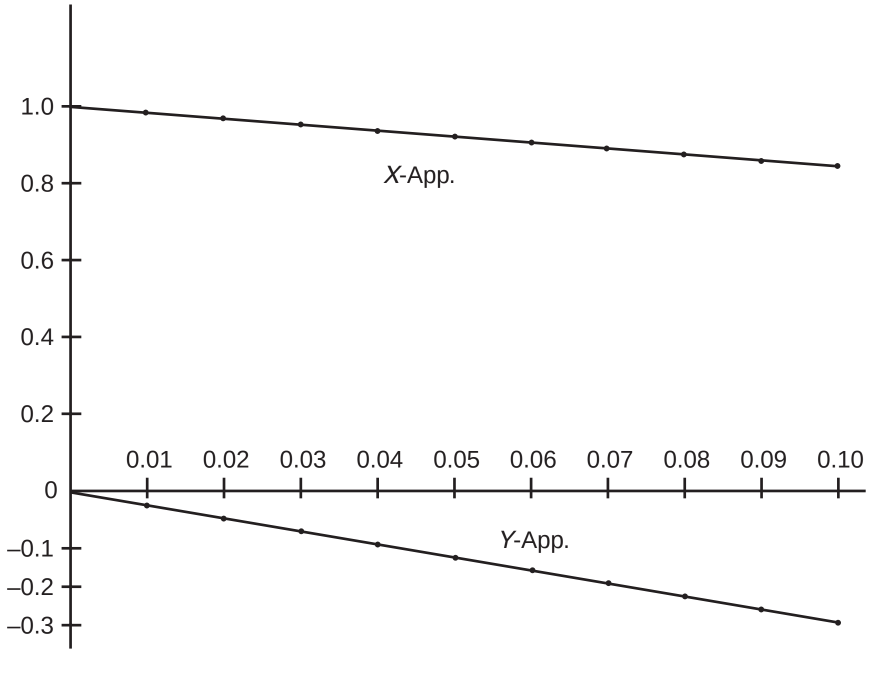

Fig. 4.3

Step-up	x-Approxim	x-Actual	y-Approxim	y-Actual
0.01	0.99985001	0.99985001	–0.02999850	–0.02999850
0.02	0.99940014	0.99940014	–0.05998800	–0.05998800
0.03	0.99865071	0.99865071	–0.08995951	–0.08995951
0.04	0.99760224	0.99760224	–0.11990405	–0.11990405
0.05	0.99625546	0.99625546	–0.14981266	–0.14981266
0.06	0.99461133	0.99461133	–0.17967641	–0.17967641
0.07	0.99267098	0.99267098	–0.20948638	–0.20948638
0.08	0.99043577	0.99043577	–0.23923372	–0.23923372
0.09	0.98790727	0.98790727	–0.26890959	–0.26890959
0.10	0.98508725	0.98508725	–0.29850524	–0.29850524

The values of x and y-approximations have been plotted in Fig. 4.3.

Example 4.36

Sys: dx/dt + 3y + y = 10 exp(–t), dy/dt + 2y + 2x = 4

PS: x = 0.33333333 × (10t X + 14X – Y – 3)

y = 0.33333333 × (–20t X – 8X – Y + 9)

[exp(–t) = X, exp(–4t) = Y]

C++: dx = [10 × exp(–t) –3 × x – y] dt, dy = [4 –2 × x – 2 ×y] dt

IC: x(0) = 3.33333333, y(0) = 0; Sts: 0.01

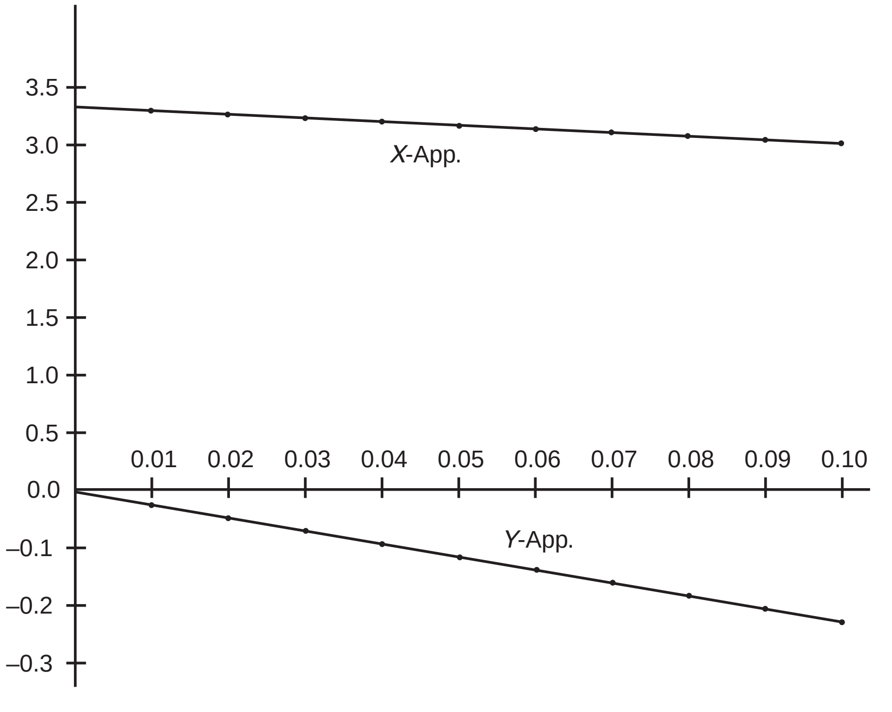

Fig. 4.4

Step-up	*x*-Approxim	*x*-Actual	*y*-Approxim	*y*-Actual
0.01	3.33297107	3.33297104	–0.02639936	–0.02639936
0.02	3.33190160	3.33190157	–0.05226173	–0.05226173
0.03	3.33015023	3.33015020	–0.07758401	–0.07758401
0.04	3.32774137	3.32774134	–0.10236362	–0.10236362
0.05	3.32469863	3.32469860	–0.12659852	–0.12659852
0.06	3.32104477	3.32104474	–0.15028719	–0.15028719
0.07	3.31680180	3.31680177	–0.17342855	–0.17342855
0.08	3.31199096	3.31199093	–0.19602199	–0.19602199
0.09	3.30663277	3.30663275	–0.21806731	–0.21806731
0.10	3.30074707	3.30074704	–0.23956474	–0.23956474

The values of *x*- and *y*-Approximations have been plotted in the graphical figure drawn (Fig. 4.4).

Example 4.37

Sys: dx/dt –x – y = –cos t, dy/dt + 2x + y = sin t + cos t

PS: x(t) = sin t – cos t – t cos t, y(t) = 2cost + t(sin t + cos t)

C++: dx = [y] dt

dy = [exp(2 × t) + x] dt

IC: x(1.57079632) = 1, y(1.57079632) = 1.57079632; Sts: 0.01

C++ Solution of the System of DE:

Step-up	*x*-Approxim	*x*-Actual	*y*-Approxim	*y*-Actual
1.5808	1.02575753	1.02575752	1.54490991	1.54490994
1.5908	1.05161248	1.05161246	1.51866703	1.51866706
1.6008	1.07756222	1.07756220	1.49206833	1.49206835
1.6108	1.10360411	1.10360410	1.46511451	1.46511454
1.6208	1.12973548	1.12973547	1.43780635	1.43780638
1.6308	1.15595363	1.15595361	1.41014467	1.41014470
1.6408	1.18225581	1.18225580	1.38213035	1.38213037
1.6508	1.20863928	1.20863927	1.35376432	1.35376434
1.6608	1.23510125	1.23510123	1.32504757	1.32504760
1.6708	1.26163889	1.26163887	1.29598116	1.29598119

4.8 Related Software for the Solution

In this chapter I have discussed the standard forms of ordinary differential equations of first order. The methods for approximating the solutions to initial value problems have also been considered beginning with the most fundamental techniques like Euler and Simpson. The discussion is carried on through more general methods, rather to say, more powerful methods, such as Adams-Moulton and Runge-Kutta.

The fourth-order Runge-Kutta technique, a set of classical mathematical formulas, is more efficient for this purpose. The efficiency is due to the computational achievement of significant results having a desired accuracy of approximation. In order to achieve the accurate approximation we need a programming technique that should be well-organised.

The computer programs required for the purpose have been written in C++ (Turbo) language. The program used to approximate the initial value problems of order 1 employing an appropriate standard method has been so constructed that it produces the approximation having an error within the extent of reasonable tolerance.

The program g4IVP1DE implements the Runge-Kutta method and determines the approximations for the problems of exact differential equations discussed in section 4.3; the linear differential equations discussed in section 4.4; the problems of Bernoulli and Riccati considered in sections 4.5 and 4.6.

The program g4LINSYS is used for the linear system of equations, homogeneous and non-homogeneous, discussed in section 4.7.

Chapter 10 provides some mathematical models that can be worked out by use of these programs built for the purpose.

A software supplement consisting of a set of programs and subprograms designed in C++ (Turbo) language is prepared by the author. A Diskette (3.5 inch/1.44 MB) in standard PC-compatible form containing this supplement will be provided for the reader to work out the related mathematical models.

5

Second Order Initial-Value Problems

5.1 Preliminary Concepts

In the preceding chapter we considered the standard forms of ordinary differential equations of first order discussing the methods for solution with some examples for elaboration. The methods for approximation of solutions to initial value problems were the generalised methods that helped us to achieve the significant results of the problems.

The present chapter contains an account of fundamentals and applications involving the differential equations of second order. The second order differential equation can be represented in compact form

$$G(x, y, y', y'') = 0$$

or explicitly $$y'' = \phi\,(x, y, y') \qquad \ldots (5.1)$$

The equation (5.1) describes a relationship involving the differentials of the first and second order with the variables, dependent and independent, and the general function $y(x)$.

Let us consider a functional polynomial of dy/dx that can be expressed as

$$d^m y/dx^m + p_1 d^{m-1}\, y/dx^{m-1} + p_2 d^{m-2}\, y/dx^{m-2} + \ldots + p_{m-1}\, dy/dx + p_m y = Q \qquad \ldots (5.2)$$

where $p_1, p_2, \ldots, p_m$ and Q are functions of x or constants.

The equation (5.2) is the general form of a *linear* differential equation of the m-th order. If we put $m = 1$ in the above equation, it reduces to the form

$$dy/dx + P_1 y = Q$$

which is equivalent to the first-order linear form (4.4) mentioned in previous chapter. The relation (5.2) furnishes another equation by putting $m = 2$

$$d^2\, y/dx^2 + P_1 dy/dx + P_2\, y = Q \qquad \ldots (5.3)$$

that is the required linear differential equation of order 2.

Linear differential equations of second order are extremely important in many branches of mathematics and mathematical physics, especially in mechanics and quantum mechanics and in electrical engineering.

There exist various types of second order linear differential equations. But this chapter concerns with only three types of equations. The second order *linear homogeneous* and *linear non-homogeneous* differential equations that include the equations from Euler, Bessel, Lagrange, Abel and the *nonlinear* differential equations which include the equations from Van der Pol have been dealt with next.

5.2 Linear Homogeneous Type

Let us define the functions

$$y_1 = g_1(x),\ y_2 = g_2(x),\ \ldots,\ y_m = g_m(x)$$

on an arbitrary interval $[a, b]$. They are said to be *linearly dependent* on the interval $[a, b]$ if one function is a constant multiple of the other. Otherwise, these functions are linearly *independent*.

The equation (5.2) written more simply in the form

$$y^m + p_1 y^{m-1} + p_2 y^{m-2} + \ldots + p_m y = Q(x) \qquad \ldots (5.4)$$

has the term $Q(x)$ on the right that involves no dependent variable y or any part of its derivatives.

By putting $Q(x) = 0$ yields a linear homogeneous form which has a general solution of the form

$$y(x) = c_1 y_1 + c_2 y_2 + \ldots + c_m y_m,$$

where $y_1, y_2, \ldots, y_m$ are linearly independent solutions of the homogeneous equation and $c_1, c_2, \ldots, c_m$ are constants not all equal to zero.

Let us recall the second order equation (5.3).

If $Q(x)$ is identically zero the equation reduces to a simple linear form

$$d^2 y/dx^2 + P_1 dy/dx + P_2\, y = 0 \qquad \ldots (5.5)$$

in which P_1 and P_2 are constant coefficients.

As the right side is free of function it is called a *linear homogeneous* equation.

Several methods are available by which the equation of type (5.5) can be readily solved. For the simplest possible method we accept a trial solution for (5.5) when we set

$y = e^{\lambda x}$, λ is a constant quantity.

Putting $y = e^{\lambda x}$ along with its derivatives in (5.5) yields

$$e^{\lambda x}(\lambda^2 + P_1\lambda + P_2) = 0,$$

the factor $e^{\lambda x}$ being common to all.

Hence,
$$\lambda^2 + P_1\lambda + P^2 = 0,$$

since $e^{\lambda x}$ can never be zero.

This equation is called the *auxiliary or characteristic* equation of (5.5) and has two roots, namely λ_1 and λ_2.

Obviously, $y = C_1 e^{\lambda_1 x}$ and $y = C_2 e^{\lambda_2 x}$ are the solutions of (5.5) and as the general solution we have expression

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \qquad \ldots (5.6)$$

containing two arbitrary constants according to the order of the equation. The general solution is dependent on the nature of the roots λ_1 and λ_2 which we want to discuss now.

If $\lambda_1 = \lambda_2$ (*equal roots*), the equation (5.6) takes the shape as

$$y = e^{\mu x}(C_1 + C_2) = Ce^{\mu x},$$

where $C = C_1 + C_2$ and $\mu = \lambda_1 = \lambda_2$,

that is not the general solution because it contains only one constant.

The general solution will be, thus, $y = (C_1 + C_2)e^{\mu x}$

accepting $y = e^{\mu x}\, u$ as a trial solution, u being a function of x.

In the case of *real and distinct* roots $(\lambda_1 \neq \lambda_2)$ the solution (5.6) satisfies the original differential equation (5.5).

If $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ (*complex roots*),

then the general solution of (5.5) is

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$

Using the trigonometric relations we get another form as a solution

$$y = C_3 \cos \beta x + C_4 \sin \beta x.$$

Example 5.1 Solve $d^2y/dx^2 - dy/dx - 2y = 0$.

If $y = e^{\lambda x}$ is a solution for some value of λ, then we get

$$d^2y/dx^2 - dy/dx - 2y = e^{\lambda x}(\lambda^2 - \lambda - 2) = 0$$

which is satisfied when $\lambda = -1$ or 2 (*real and distinct roots*)

Hence, the general solution is

$$y = C_1 e^{-x} + C_2 e^{2x}.$$

Example 5.2 Solve $y'' - 2y' + y = 0$.

By setting $y = e^{\lambda x}$ and rearranging we obtain

$$y'' - 2y' + y = e^{\lambda x}(\lambda^2 - 2\lambda + 1) = 0$$

which is satisfied when $\lambda = 1$ twice.

Here, the characteristic equation $\lambda^2 - 2\lambda + 1 = 0$ has *repeated* roots and the general solution is

$$y = e^x(C_1 + C_2 x).$$

Example 5.3 $d^2y/dx^2 + 4dy/dx + 13y = 0$.

As a trial solution let us write $y = e^{\lambda x}$.

Then $$d^2y/dx^2 + 4dy/dx + 13y = e^{\lambda x}(\lambda^2 + 4\lambda + 13) = 0$$

yields $\lambda = -2 \pm 3i$ (*complex roots*).

According to the theoretical discussion

$$y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x)$$

is the general solution.

As in the case of initial-valued first-order differential equations there are many numerical methods for the approximation of the solution to initial value problem of order 2 available. In the following I confine myself to discussing about the *classical Runge-Kutta* method of order 4, which is definitely a more efficient method that can be applied for the approximation of ordinary linear differential equations of the second order.

The main mathematical achievement in using the fourth-order Runge-Kutta method involves the function evaluations of G. The technique of evaluation for the formula presented here requires more operations.

For the equation $y' = g(x, y)$ with the initial conditions $y(x_0) = y_0$, generate approximations y_ν to $y(x_0 + \nu w)$ for fixed w (width) and for $\nu = 0, 1, 2, \ldots$, applying the extended formula for second order equations

$$y_{\nu+1} = y_\nu + wy'_\nu + w(G_1 + G_2 + G_3)/6 \qquad \ldots (5.7)$$

and

$$y'_{\nu+1} = y'_\nu + (G_1 + 2G_2 + 2G_3 + G_4)/6 \qquad \ldots (5.8)$$

where $\nu = 0, 1, 2, \ldots.$

Taking w as width the functions are defined by

$$G_1 = w\, G(x_\nu, y_\nu, y'_\nu)$$
$$G_2 = w\, G(x_\nu + w/2, y_\nu + w\, y'_\nu/2, y'_\nu + G_1/2)$$
$$G_3 = w\, G(x_\nu + w/2, y_\nu + w\, y'_\nu/2 + G_1/4, y'_\nu + G_2/2)$$
$$G_4 = w\, G(x_\nu + w, y_\nu + w\, y'_\nu + w\, G_2/2,\ y'_\nu + G_3).$$

As a matter of fact, the approximations are performed on the basis of fixed w, that is so small in each case. I can carry on the approximations accepting a large value of w to get the accurate results, but the development of the process needs a definite number of repetitions of the technique.

In this connection I want to mention that the formulae (5.7) and (5.8) are widely used in practical field with significant outcome.

Example 5.4 Solve $d^2 y/dx^2 - 6dy/dx + 9y = 0$ applying Runge-Kutta method.

We accept $y_0 = y(x_0) = 0.0$, $y'_0 = y'(x_0) = 2.0$ when $x_0 = 0$ and $w = 0.01$ for the initial conditions.

On writing the equation in the form $y'' = 6y' - 9y$

we determine G-values according to the principle of G-functions

$$G_1 = 0.01(12 - 0) = 0.12$$
$$G_2 = 0.01(12.36 - 0.09) = 0.1227$$
$$G_3 = 0.01(12.36810 - 0.36) = 0.120081$$
$$G_4 = 0.01(83.8630785) = 0.838630785.$$

By means of the formulae (5.7) and (5.8) for the approximate y-values we obtain the estimations of G.

$$y_1 = 0.02060463$$

and

$$y'_1 = 2.24069879.$$

when $v = 0$. These are the first approximations for *y*.

Now to approximate the initial value problem of order 2 by means of C++ program named g5LI2IVP applying the method from Runge-Kutta is used. After processing the program calculates simultaneously the exact values of $y(x)$ and $y'(x)$ with the corresponding increments of the independent variable *x*, so that the errors caused by the approximations can be found out.

Let us consider some problems of homogeneous type that can be approximated by means of the program just mentioned using Runge-Kutta principles.

Example 5.5

DE: $y'' + 4y' + 4y = 0$

CE: $y'' = -4y' - 4y$

PS: $y = (7 + 5x)e^{-2(x+1)}$, $y' = -(9 + 10x)e^{-2(x+1)}$

IC: $y(-1) = 2$, $y'(-1) = 1$, Width: 0.01.

DE: Differential Equation, CE: Computer Equation, PS: Particular Solution, IC: Initial Conditions, Width = Step size.

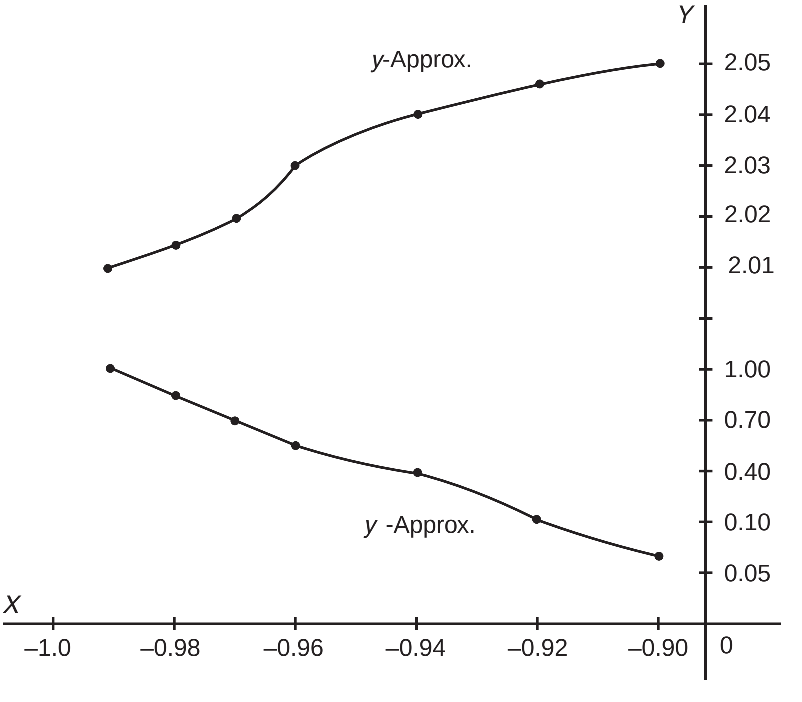

Fig. 5.1

The representation of Computer Solution of the IVP of Order 2 in tabular form:

x-value	*y*-Approx.	*y*-Exact	Error	y′-Approx.	y′-Exact	Error
–0.99	2.00940728	2.00940728	2.788e–10	0.88217881	0.88217881	–6.906e–10
–0.98	2.01765772	2.01765782	1.065e–07	0.76862117	0.76863155	1.038e–05
–0.97	2.02479344	2.02479375	3.116e–07	0.65921510	0.65923517	2.008e–05
–0.96	2.03085535	2.03085596	6.091e–07	0.55384068	0.55386981	2.912e–05
–0.95	2.03588320	2.03588419	9.926e–07	0.45238116	0.45241871	3.754e–05
–0.94	2.03991555	2.03991700	1.456e–06	0.35472281	0.35476817	4.537e–05
–0.93	2.04298986	2.04299185	1.994e–06	0.26075485	0.26080747	5.262e–05
–0.92	2.04514249	2.04514509	2.600e–06	0.17036942	0.17042876	5.933e–05
–0.91	2.04640875	2.04641202	3.269e–06	0.08346150	0.08352702	6.552e–05
–0.90	2.04682289	2.04682688	3.997e–06	0.07121844	0.07271789	9.945e–05

With the determined values for *y*- and y′-approximation a graphical figure has been designed (Fig. 5.1).

Example 5.6

DE: $y'' + y' - 6y = 0$

CE: $y'' = -y' + 6y$

PS: $y = (e^{2x} + 4e^{-3x})/5$, $y' = 2(e^{2x} - 6e^{-3x})/5$

IC: $y(0) = 1$, $y'(0) = -2$, Width: 0.01

The representation of Computer Solution of the IVP of Order 2 in tabular form:

x-value	*y*-Approx.	*y*-Exact	Error	y′-Approx.	y′-Exact	Error
0.01	0.98039670	0.98039669	–1.558e–10	–1.92098875	–1.92098874	4.943e–10
0.02	0.96157384	0.96157378	–5.747e–08	–1.84390488	–1.84391057	–5.689e–06
0.03	0.94351243	0.94351226	–1.699e–07	–1.76868905	–1.76870023	–1.117e–05
0.04	0.92619410	0.92619376	–3.353e–07	–1.69527776	–1.69529422	–1.646e–05
0.05	0.90960112	0.90960056	–5.520e–07	–1.62360922	–1.62363078	–2.165e–05
0.06	0.89371636	0.89371554	–8.180e–07	–1.55362329	–1.55364977	–2.648e–05
0.07	0.87852329	0.87852216	–1.132e–06	–1.48526144	–1.48529267	–3.123e–05
0.08	0.86400595	0.86400446	–1.491e–06	–1.41846669	–1.41850252	–3.583e–05
0.09	0.85014896	0.85014707	–1.896e–06	–1.35318357	–1.35322384	–4.027e–05
0.10	0.83693747	0.83693513	–2.343e–06	–1.28935806	–1.28940263	–4.457e–05

The field of errors in *y*-approximation confines to the range [–1.558e–10, –2.343e–06] and the field of errors in *y*′-approximation is limited to the range [4.943e–10, –4.457e–05].

The estimated approximations have been represented in a diagram as follows (Fig. 5.2):

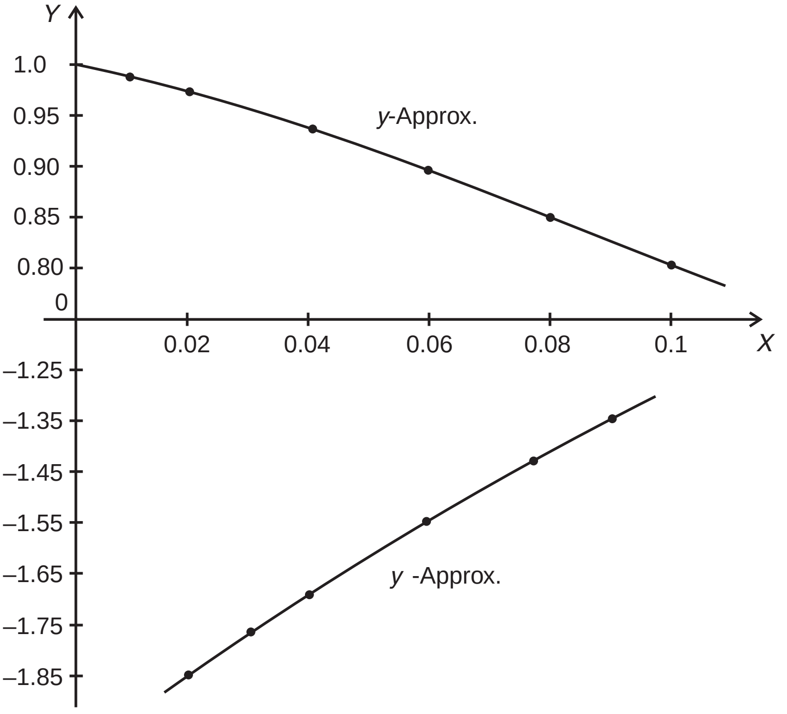

Fig. 5.2

Example 5.7

DE: $(1 + x^2)y'' - 2y = 0$

CE: $y'' = 2y/(1 + x^2)$

PS: $y = \text{Tan}^{-1} x(x^2 + 1)/2 + x^2 + x/2 + 1$, $y' = 1 + 2x + x \text{ Tan}^{-1}x$

IC: $y(0) = 1$, $y'(0) = 1$, Width: 0.01.

The representation of Computer Solution of the IVP of Order 2:

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
0.01	1.01010033	1.01010033	1.667e–12	1.02010000	1.02010000	8.366e–12
0.02	1.02040267	1.02040267	–5.005e–09	1.04040045	1.04039995	–5.009e–07
0.03	1.03090901	1.03090900	–1.510e–08	1.06090074	1.06089973	–1.010e–06
0.04	1.04162136	1.04162133	–3.038e–08	1.08160068	1.08159915	–1.528e–06
0.05	1.05254170	1.05254165	–5.091e–08	1.10249997	1.10249792	–2.054e–06
0.06	1.06367203	1.06367195	–7.678e–08	1.12359828	1.12359569	–2.588e–06
0.07	1.07501433	1.07501422	–1.081e–07	1.14489515	1.14489202	–3.131e–06
0.08	1.08657059	1.08657045	–1.449e–07	1.16639008	1.16638640	–3.681e–06
0.09	1.09834279	1.09834261	–1.873e–07	1.18808248	1.18807824	–4.240e–06
0.10	1.11033290	1.11033267	–2.353e–07	1.20997167	1.20996687	–4.807e–06

Graphical representation for the approximations in example 5.7 (Fig. 5.3):

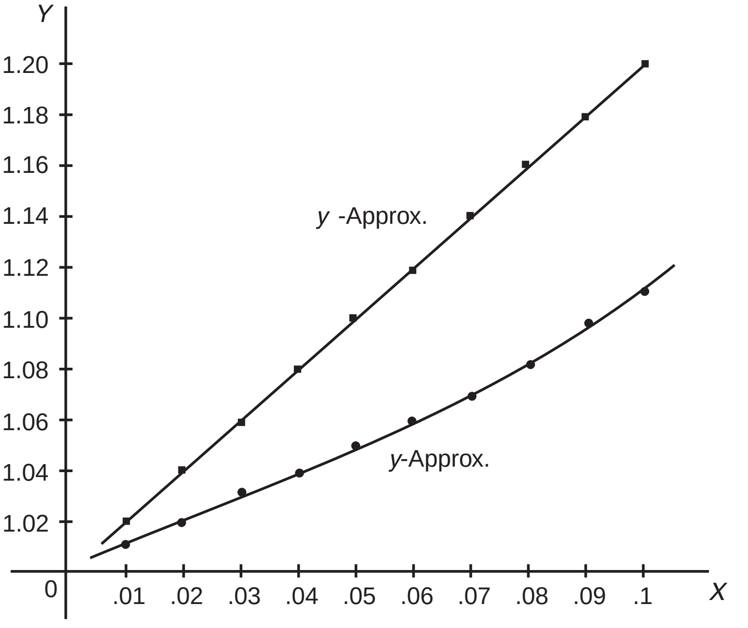

Fig. 5.3

Example 5.8

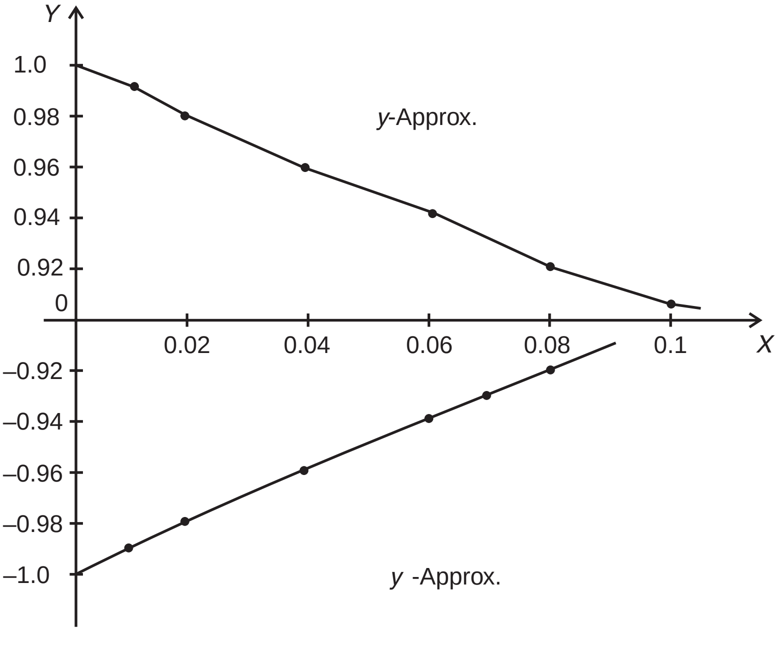

Fig. 5.4

Example 5.8 *(Contd)*

DE: $y'' + 3y' + 2y = 0$

CE: $y'' = -3y' - 2y$

PS: $y(x) = e^{-x}$, $y' = -e^{-x}$

IC: $y(0) = 1.0$, $y'(0) = -1$, Width: 0.01.

Graphical sketch for the estimations of $y(x)$ and $y'(x)$ when example 5.8 is processed Fig. 5.4 (the data are not given):

Example 5.9

DE: $y'' + 2y' + 5y = 0$

CE: $y'' = -2y' - 5y$

PS: $y = e^{-x}\sin(2x)/4$, $y' = e^{-x}[2\cos(2x) - \sin(2x)]/1$

IC: $y(0) = 0$, $y'(0) = 0.5$, Width: 0.01

The field of errors in y-approximation confines to the range [–7.932e–12, –2.031e–07] and the field of errors in y'-approximation is limited to the range [–9.070e–12, –3.805e–06].

5.3 Linear Nonhomogeneous Type

The equation (5.3) in section 5.1 is a linear nonhomogeneous differential equation of order 2. The general solution of a linear nonhomogeneous differential equation

$$d^2y/dx^2 + P_1\,dy/dx + P_2\,y = Q(x)$$

may be expressed as $y = y_c + y_p$ … (5.9)

where y_c is the general solution of the linear *homogeneous* form

$$d^2y/dx^2 + P_1\,dy/dx + P_2\,y = 0$$

that has already been introduced in section 5.2. y_c is also called the *complementary* solution. Here, y_p is a particular solution of the equation (5.3) and the function y_p can be found out by the method of *undetermined coefficients*.

5.3.1 Method of Undetermined Coefficients

The function $Q(x)$ is, now, the main point of study that we want to discuss with some particular cases. It may be present in the right side of the equation as one of the following functions:

(a) The right side contains exponential functions
(b) The right side contains polynomial with exponentials
(c) The right side contains trigonometric functions
(d) The right side contains a combination of terms

Let us discuss on the cases with examples.

Case A Solve $y'' - y' - 2y = e^{3x}$.

Here, $P_1 = -1$, $P_2 = -2$ and $Q(x) = e^{3x}$.

The *auxiliary* equation $\lambda^2 - \lambda - 2 = 0$ of the homogeneous differential equation

$$y'' - y' - 2y = 0$$

has two roots $\lambda_1 = -1$ and $\lambda_1 = 2$. The roots are obviously real and distinct.

According to the case in section (5.2) the complementary solution can be obtained as

$$y_c = C_1e^{-x} + C_2e^{2x}.$$

Let us assume a particular solution

$$y_p = C_0e^{3x}$$

where C_0 is the *undetermined coefficient* that we want to determine, so that the given equation will be satisfied by the values of y_p.

Substituting y_p and the derivative of y_p into the given differential equation yields

$$9C_0e^{3x} - 3C_0e^{3x} - 2C_0e^{3x} = e^{3x}$$

or
$$4C_0e^{3x} = e^{3x}, \quad C_0 = \frac{1}{4}.$$

Therefore, in conformity with the relation in (5.6) the general solution becomes

$$y = C_1e^{-x} + C_2e^{2x} + e^{3x}/4.$$

Case B *Solve* $y'' - y' - 2y = (6x^2 + 8x + 7)e^x$.

The complementary solution of the characteristic equation is, as usual,

$$y_c = C_1e^{-x} + C_2e^{2x}.$$

We try a particular solution $\quad y_p = (C_3x^2 + C_4x + C_5)e^x,$

according as the function on right side that reads $(6x^2 + 8x + 7)e^x$.

On differentiating the expression with respect to x we get

$$y_p' = (C_3x^2 + 2C_3x + C_4x + C_4 + C_5)e^x$$

and
$$y_p'' = (C_3x^2 + 4C_3x + C_4x + 2C_3 + 2C_4 + C_5)e^x.$$

On substitution of particular functions into the differential equation and equating the coefficients of like powers of x, we obtain

$$-2C_3 = 6 \quad \text{or} \quad C_3 = -3$$
$$2C_3 - 2C_4 = 8 \quad \text{or} \quad C_4 = -7$$
$$2C_3 + C_4 - 2C_5 = 7 \quad \text{or} \quad C_5 = -27/2$$

Hence, according to the equation (5.9) the general solution is

$$y = C_1e^{-x} + C_2e^{2x} - (3x^2 + 7x + 27/2)e^x.$$

Case C Solve $dy^2/dx^2 + 4dy/dx + 5y = 2(\cos x - \sin x)$.

The complementary solution of the reduced homogeneous equation is

$$y_c = e^{-2x}(C_1\cos x + C_2\sin x).$$

We accept a form as a particular solution, such as,

$$y_p = A\sin x + B\cos x.$$

for the function $Q(x)$, A and B being new constants.

Taking the derivatives of y_p twice and putting all these values into the given differential equation we get after some manipulation

$$(4A - 4B) \sin x + (4A + 4B) \cos x = -2 \sin x + 2 \cos x$$

Equating the coefficients of like terms yields

$$4A - 4B = -2 \quad \text{and} \quad 4A + 4B = 2$$

A simple calculation gives $A = 0$, $B = 1/2$.

Thus, the general solution of the original euqation is

$$y(x) = e^{-2x} (C_1 \cos x + C_2 \sin x) + 1/2 \cos x.$$

Case D Solve $y'' + 2y' + 4y = 8x^2 + 12e^{-x}$.

Here

$$y_c = e^{-x} (C_1 \cos \sqrt{3x} + C_2 \sin \sqrt{3x}).$$

In order to obtain a particular solution we take

$$y_p = ax^2 + bx + c + de^{-x}$$

corresponding to the polynomial and exponential on the right side of the equation

$$8x^2 + 12e^{-x}$$

which is equivalent to $8x^2 + 0x^1 + 0x^0 + 12e^{-x}$.

Substituting, rearranging and equating we get the values for the constants

$$a = 2, \quad b = -2, \quad c = 0, \quad d = 4$$

that furnish the particular solution in the form

$$y_p = 2x^2 - 2x + 4e^{-x}.$$

The general solution is obtained accordingly as (5.9) that suggests

$$y(x) = y_c + y_p = 2x(x - 1) + e^{-x} (C_1 \cos \sqrt{3x} + C_2 \sin \sqrt{3x} + 4).$$

For the computerised solutions of the second order initial value problems of the type just discussed we apply the same rules and regulations as for the previous section.

5.3.2 Euler's Equation

An equation of the form

$$a_m x^m y^m + a_{m-1} x^{m-1} y^{m-1} + \ldots + a_1 x y' + a_0 y = f(x) \qquad \ldots (5.10)$$

where $a_m, a_{m-1}, \ldots, a_0$ are constants,

is a particular type of differential equation with constant coefficients and is known as *Euler's* equation.

An equation of the form (5.10) can be reduced to a linear equation with constant coefficients by a transformation. The substitution $x = e^{\lambda}$ causes the reduction. By introducing the new independent variable λ we find the derivatives as follows:

$$dy/dx = dy/d\lambda \cdot d\lambda/dx = 1/x \cdot dy/d\lambda = e^{-\lambda} \cdot dy/d\lambda$$

$$d^2y/dx^2 = d/dx\,(dy/dx) = dy/d\lambda\,(-1/x^2) + 1/x \cdot d/dx\,(dy/d\lambda)$$

$$= -1/x^2\; dy/d\lambda + 1/x \cdot d/d\lambda\,(dy/d\lambda) \cdot d\lambda/dx = e^{-2\lambda} (d^2y/d\lambda^2 - dy/d\lambda).$$

Similarly we get,

$$d^3y/dx^3 = e^{-3\lambda} (d^3y/d\lambda^3 - 3d^2y/d\lambda^2 + 2\,dy/d\lambda)$$

and so on.

The calculated derivatives are inserted in the equation (5.10) resulting a linear differential equation with constant coefficients, that can be solved as in the case of nonhomogeneous type.

We want to simplify the matter by giving some examples.

Example 5.10 Let us consider an Euler nonhomogeneous equation

$$x^2 y'' - 3xy + 5y = 3x^2 \qquad \dots (5.11)$$

Setting $x = e^{\lambda}$, i.e., $\lambda = \text{Ln}\, x$ we get

$$dy/dx = e^{-\lambda}\, dy/d\lambda$$

and
$$d^2y/dx^2 = e^{-2\lambda}\,(d^2y/d\lambda^2 - dy/d\lambda)$$

Now, by means of the derivatives found the given differential equation (5.11) becomes

$$e^{-2\lambda} \cdot e^{-2\lambda}\,(d^2y/d\lambda^2 - dy/d\lambda) - 3e^{-\lambda} \cdot e^{-\lambda}\, dy/d\lambda + 5y = 3e^{2\lambda}$$

or
$$d^2y/d\lambda^2 - 4dy/d\lambda + 5y = 3e^{2\lambda}.$$

This is a linear nonhomogeneous differential equation that can be solved by a method described earlier in section 5.3.1.

The general solution y_c, so-called complementary solution of the corresponding homogeneous equation is

$$y_c = e^{2\lambda}\left(A \cos\lambda + B \sin\lambda\right).$$

The particular solution of the nonhomogeneous form can be found when we select

$$y_p = Ce^{2\lambda}$$

in order to obtain

$$dy/d\lambda = 2C\, e^{2\lambda} \text{ and } d^2y/d\lambda^2 = 4C\, e^{2\lambda}.$$

Substituting these determined values into the transformed nonhomogeneous equation yields

$$4C\, e^{2\lambda} - 4 \times 2C\, e^{2\lambda} + 5C\, e^{2\lambda} = 3\, e^{2\lambda}$$

or
$$4C - 8C + 5C = 3, \text{ since } e^{2\lambda} \neq 0$$

or
$$C = 3.$$

Thus, the general solution of the nonhomogeneous differential equation is

$$y = y_c + y_p = e^{2\lambda}\left(A \cos\lambda + B \text{ Sin}\,\lambda + 3\right)$$

Returning to the original independent variable x we obtain

$$y = x^2\,[A \cos(\text{Ln}\, x) + B \sin(\text{Ln}\, x) + 3]$$

as the final solution.

Example 5.11 Solve $(1 + x)^2 y'' - 3(1 + x)y' - 4y = (1 + x)^3$

Let us assume $1 + x = e^{\lambda}$.

Then
$$y' = e^{-\lambda}\, dy/d\lambda \text{ and } y'' = e^{-2\lambda}\,(d^2y/d\lambda^2 - dy/d\lambda)$$

Now the given equation has been transformed into

$$d^2y/d\lambda^2 - 4\, dy/d\lambda + 4y = e^{3\lambda}.$$

For complementary solution we have the expression

$$y_c = \left(A + B\lambda\right) e^{2\lambda}.$$

Selecting $y_p = C\, e^{3\lambda}$

we obtain for particular solution

$$y_p = e^{3\lambda},$$

suggesting $C = 1$.

Hence, the general solution of the original equation is

$$y = y_c + y_p = \left(A + B\lambda\right) e^{2\lambda} + e^{3\lambda}.$$

Returning to the original position we get finally

$$y = [1 + x]^2 \, [A + B \text{ Ln}(1 + x)] + (1 + x)^3.$$

5.3.3 Method of Variation of Parameters

All linear differential equations can be solved by the method of variation of parameters. On the basis of that ground it is termed as a more powerful method than the method of undetermined coefficients described in section 5.3.1, which can be applied only to linear differential equations having constant coefficients and the particular simple form of the function $Q(x)$.

The method of *variation of parameter* can always be employed without concerning the *character* of the coefficients and the function $Q(x)$. The only requirement for determining the particular solution of the nonhomogeneous equation is that the general solution of the associated homogeneous differential equation should be known.

Let us consider a general linear nonhomogeneous differential equation of the form

$$y'' + P_1(x)y' + P_2(x)y = Q(x). \quad \dots (5.12)$$

We assume that the general solution of the corresponding homogeneous equation

$$y'' + P_1(x)y' + P_2(x)y = 0 \quad \dots (5.13)$$

with $Q(x) = 0$ is

$$y_c = c_1\, y_1(x) + c_2\, y_2(x) \quad \dots (5.14)$$

A particular solution to the nonhomogeneous form (5.12) can be defined by the relation

$$y_p = u_1\, y_1 + u_2\, y_2 \quad \dots (5.15)$$

where u_1 and u_2 are the unknown functions of x, that are replaced by the constants c_1 and c_2 in the general solution represented by (5.14). At this stage the parameters c_1 and c_2 have been varied and so is the name.

Differentiating the considered relation (5.15) with regard to x yields

$$y_p = u_1\, y_1' + u_1'\, y_1 + u_2\, y_2' + u_2'\, y_2.$$

This equation can be rearranged as

$$y_p' = (u_1\, y_1' + u_2\, y_2') + (u_1'\, y_1 + u_2'\, y_2).$$

For simplification of further work let us impose the helpful condition

$$\sum_{k=1}^{2} u_k' \, y_k = 0 \qquad \ldots (5.16)$$

that furnishes $y_p' = u_1 y_1' + u_2 y_2'$.

Another differentiation of this expression gives $y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$

On substituting the values of y_p, y_p' and y_p'' into the original equation (5.12) we obtain

$$u_1(y_1'' + Py_1' + Qy_1) + u_2(y_2'' + Py_2' + Qy_2) + u_1' y_1' + u_2' y_2' = Q(x) \qquad \ldots (5.17)$$

As $y_1(x)$ and $y_2(x)$ are the solutions of the equation (5.13) the relation (5.17) can be reduced to the form

$$u_1' y_1' + u_2' y_2' = Q(x) \qquad \ldots (5.18)$$

accepting the other expressions equal to 0.

Now we have two equations (5.16) and (5.18) in the unknowns u_1' and u_2' that can be solved easily.

The equation (5.16) suggests $u_1' y_1 + u_2' y_2 = 0$

from which we obtain

$$u_1' = -(y_2 \times u_2')/y_1. \qquad \ldots (5.19)$$

On substituting the value of u_1' into the equation (5.18) yields

$$u_2' = (y_1 \times Q(x))/(y_1 y_2' - y_1' y_2). \qquad \ldots (5.20)$$

Hence, arranging in other way we get

$$u_1' = -(y_2 \times Q(x))/Y$$

and $$u_2' = (y_1 \times Q(x))/Y$$

where $$Y = y_1 y_2' - y_2' y_2 \neq 0.$$

On integrating both the equations (5.19) and (5.20) to determine the unknown functions $u_1(x)$ and $u_2(x)$ we obtain finally the particular solution from the relation (5.15)

$$y_p = y_1 \int -(y_2 \times Q(x))/Y \, dx + y_2 \int (y_1 \times Q(x))/Y \, dx$$

It seems that this generalised formula with two integrals for the particular solution to be a difficult one as it is assumed.

Let us attempt to clarify the matter with an example.

Example 5.12 Solve $y'' - 5y' + 6y = x^3 e^{2x}$.

Evidently the general solution of the homogeneous equation

$$y'' - 5y' + 6y = 0$$

becomes $$y_c = c_1 e^{3x} + c_2 e^{2x}$$

where c_1 and c_2 are constants, as usual.

Now to determine the particular solution we define a new relation as discussed before

$$y_p = u_1\, e^{3x} + u_2\, e^{2x}. \qquad \ldots (5.21)$$

Considering the unknown function $u_1(x)$ and $u_2(x)$ for c_1 and c_2 respectively.

On differentiation we obtain

$$y'_p = 3u_1 e^{3x} + 2u_2 e^{2x} + (u'_1 e^{3x} + u'_2 e^{2x})\,.$$

The condition represented by (5.16) has been introduced here, so that

$$u'_1 e^{3x} + u'_2 e^{2x} = 0,$$

which reduces the equation involving the first derivative of y_p to

$$y'_p = 3u_1 e^{3x} + 2u_2 e^{2x} \qquad \ldots (5.22)$$

Differentiating the relation (5.22) we observe

$$y''_p = 9u_1 e^{3x} + 4u_2 e^{2x} + 3u'_1 e^{3x} + 2u'_2 e^{2x}. \qquad \ldots (5.23)$$

Taking into account of the derivatives y'_p and y''_p from (5.22) and (5.23) we get the original equation rearranged

$$3u'_1 e^{3x} + 2u'_2 e^{2x} = x^3 e^{2x}.$$

The unknown functions $u_1(x)$ and $u_2(x)$ can be found by considering only the two equations

$$u'_1 e^{3x} + u'_2 e^{2x} = 0$$

and

$$3u'_1 e^{3x} + 2u'_2 e^{2x} = x^3 e^{2x}\,,$$

which supply the values for u'_1 and u'_2 as follows

$$u'_2 = -x^3 \text{ and } u'_1 = x^3 e^{-x}.$$

Introduction of the rule *Integration by Parts* yields the unknowns

$$u_2 = -x^4/4 + c_3$$

and

$$u_1 = e^{-x}(x^3 + 3x^2 + 6x + 6) + c_4$$

where c_3 and c_4 are constants of integration.

The relation (5.21) for particular solution can, now, be treated with the determined values in order to get the final results for y_p

$$y_p = -e^{2x}(x^3 + 3x^2 + 6x + 6) + c_4 e^{3x} - (x^4 e^{2x})/4 + c_3 e^{2x}$$

$$= -(x^4/4 + x^3 + 3x^2 + 6x + 6 + c_3)\, e^{2x} + c_4\, e^{3x}.$$

The general solution of the nonhomogeneous equation becomes

$$y = y_c + y_p = -(x^4/4 + x^3 + 3x^2 + 6x + 6)\, e^{2x} + Ae^{2x} + Be^{3x}$$

A and *B* being two new constants.

For the computerised solutions of second order initial value problems of nonhomogeneous type discussed so long we use the same program g5LI2IVP applying the necessary rules and regulations.

Example 5.13

DE: $x^2y'' - xy' + y = 2x$

CE: $y'' = (2x - y + xy')/x^2$

PS: $y = x\,(\text{Ln } x + (\text{Ln } x)^2)$, $y' = 1 + 3\text{Ln } x + (\text{Ln } x)^2$

IC: $y(1) = 0$, $y'(1) = 1$, Width: 0.01

The abbreviations for DE, CE, PS, IC and Width have already been made comprehensible in section (5.2).

The Computer Solution for Initial Value Problem.

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
1.01	0.01014983	0.01014983	–2.883e–12	1.02995000	1.02995000	–4.459e–13
1.02	0.02059866	0.02059867	4.961e–09	1.05979953	1.05980003	4.991e–07
1.03	0.03134549	0.03134550	1.486e–08	1.08954913	1.08955013	9.947e–07
1.04	0.04238931	0.04238934	2.965e–08	1.11919892	1.11920040	1.487e–06
1.05	0.05372913	0.05372918	4.931e–08	1.14874900	1.14875097	1.976e–06
1.06	0.06536395	0.06536402	7.380e–08	1.17819953	1.17820199	2.461e–06
1.07	0.07729278	0.07729289	1.031e–07	1.20755070	1.20755364	2.942e–06
1.08	0.08951463	0.08951477	1.371e–07	1.23680271	1.23680613	3.420e–06
1.09	0.10202850	0.10202868	1.759e–07	1.26595579	1.26595968	3.895e–06
1.10	0.11483341	0.11483363	2.194e–07	1.29501020	1.29501457	4.366e–06

With the determined values for *y*-approximation and *y′*-approximation a graphical figure has been designed for further study:

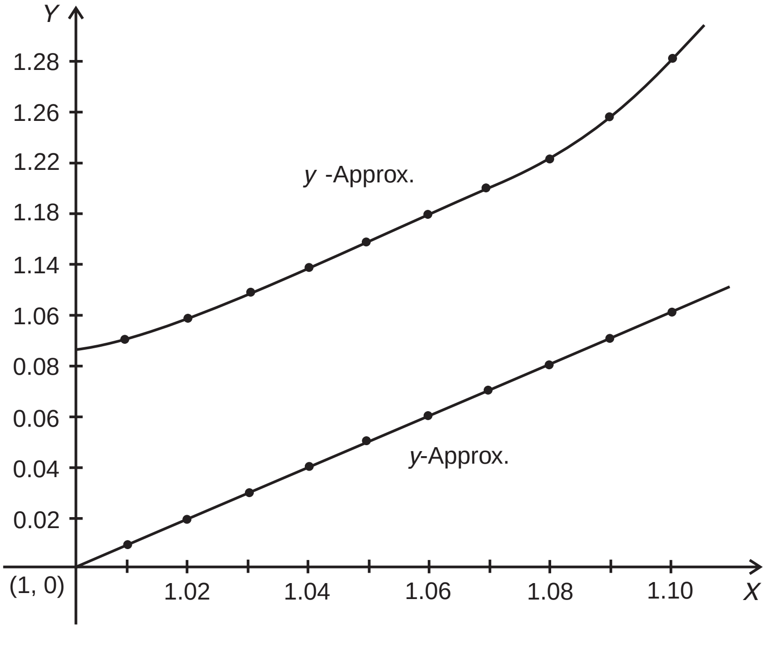

Fig. 5.5

Example 5.14

DE: $y'' + 2y' + 5y = e^{-x} \sin x$

CE: $y'' = e^{-x} \sin x - 2y' - 5y$

PS: $y = e^{-x}(\sin x + \sin 2x)/3$, $y' = e^{-x}(\cos x - \sin x + 2\cos 2x - \sin 2x)/3$

IC: $y(0) = 0$, $y'(0) = 1$, Width: 0.01.

Graphical figure for the estimations of $y(x)$ and $y'(x)$ when example 5.14 is processed.

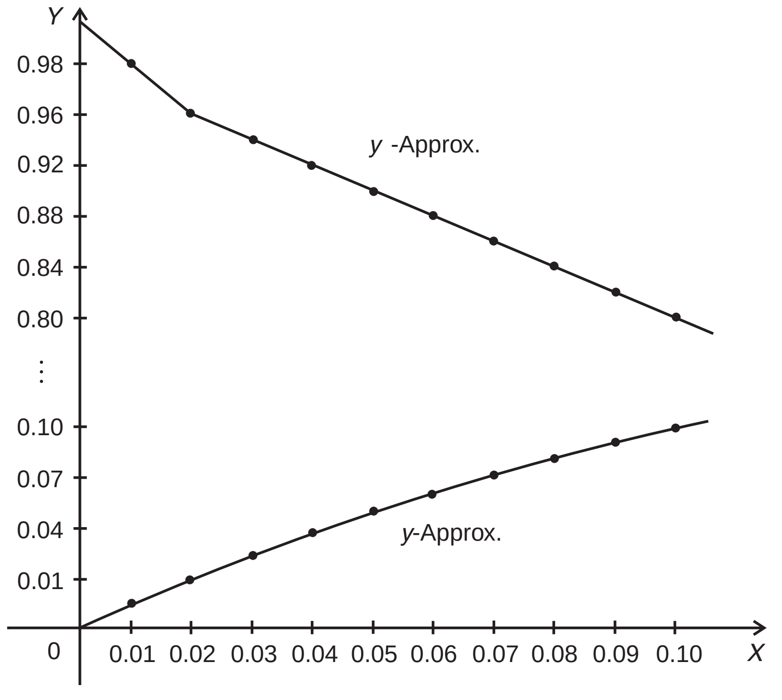

Fig. 5.6

The following is the table of the approximations with the errors. We have used the data for the diagram sketched in Fig. 5.6.

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
0.01	0.00990000	0.00990000	–1.723e–11	0.98000133	0.98000133	–9.892e–10
0.02	0.01960006	0.01960005	–9.196e–09	0.96001148	0.96001057	–9.078e–07
0.03	0.02910029	0.02910027	–2.721e–08	0.94003731	0.94003552	–1.782e–06
0.04	0.03840090	0.03840084	–5.373e–08	0.92008645	0.92008383	–2.624e–06
0.05	0.04750214	0.04750205	–8.844e–08	0.90016642	0.90016299	–3.434e–06
0.06	0.05640436	0.05640423	–1.310e–07	0.88028458	0.88028037	–4.211e–06
0.07	0.06510799	0.06510781	–1.811e–07	0.86044813	0.86044317	–4.957e–06
0.08	0.07361351	0.07361327	–2.385e–07	0.84066415	0.84065848	–5.671e–06
0.09	0.08192148	0.08192117	–3.027e–07	0.82093955	0.82093320	–6.353e–06
0.10	0.09003253	0.09003215	–3.736e–07	0.80128113	0.80127412	–7.004e–06

Example 5.15

DE: $y'' + y = 2x - \pi$

CE: $y'' = 2x - \pi - y$

DE: $y = 2x - \pi(1 - \cos x) + \sin x$, $y' = 2 - \pi \sin x + \cos x$

IC: $y(0) = 0$, $y'(0) = 3$, Width: 0.01.

Pictorial representation for the estimations of $y(x)$ and $y'(x)$ determined by the program:

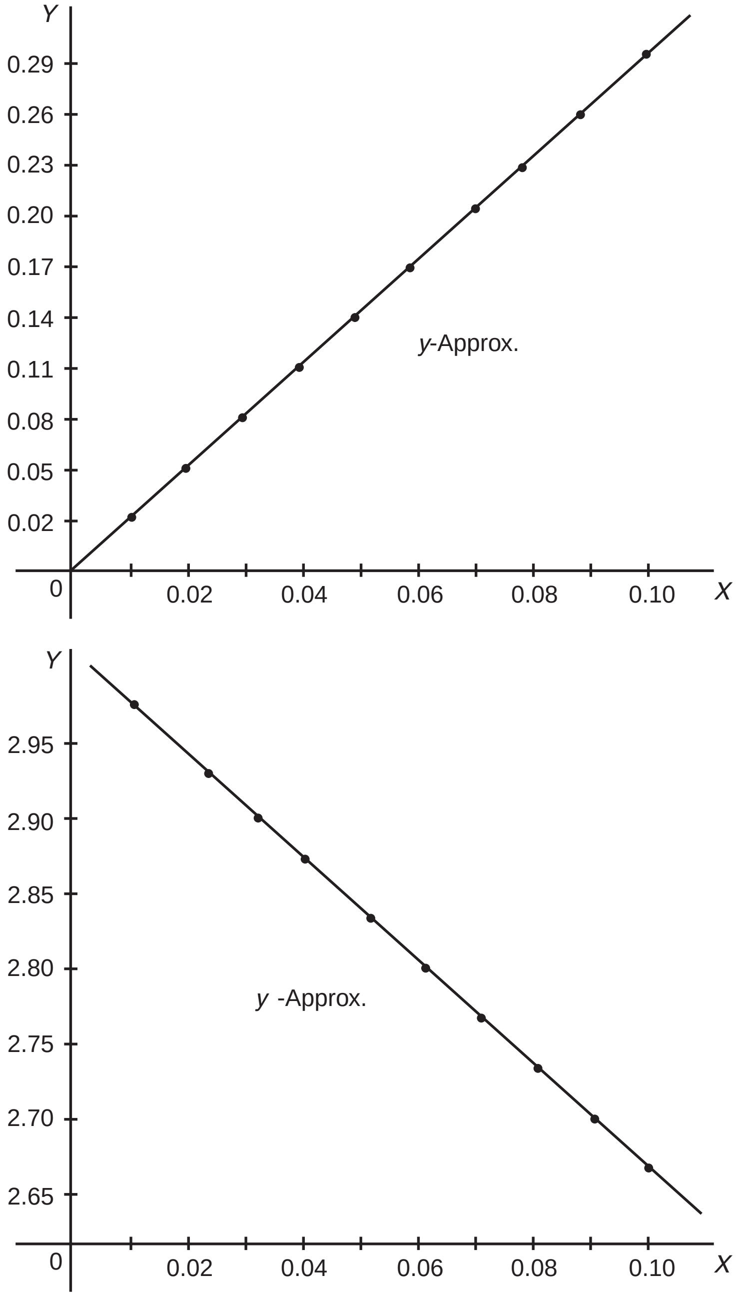

Fig. 5.7

The computer solutions for the estimations in tabular form

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
0.01	0.02984276	0.02984276	8.290e–13	2.96853460	2.96853460	–2.619e–12
0.02	0.05937037	0.05937037	–2.612e–09	2.93697260	2.93697234	–2.614e–07
0.03	0.08858190	0.08858189	–7.847e–09	2.90531691	2.90531639	–5.235e–07
0.04	0.11747641	0.11747640	–1.571e–08	2.87357069	2.87356991	–7.864e–07
0.05	0.14605302	0.14605300	–2.621e–08	2.84173712	2.84173607	–1.050e–06
0.06	0.17431088	0.17431084	–3.936e–08	2.80981937	2.80981805	–1.314e–06
0.07	0.20224914	0.20224909	–5.515e–08	2.77782064	2.77781906	–1.579e–06
0.08	0.22986703	0.22986696	–7.360e–08	2.74574413	2.74574229	–1.844e–06
0.09	0.25716378	0.25716368	–9.470e–08	2.71359305	2.71359094	–2.110e–06
0.10	0.28413866	0.28413854	–1.185e–07	2.68137061	2.68136823	–2.377e–06

The field of errors in *y*-approximation confines to the range [8.290e–13, –1.185e–07] and the field of errors in y'-approximation is limited to the range [–2.619e–12, –2.377e–06].

Example 5.16

DE: $D^2y/dx^2 + 2dy/dx - 3y = 2 \cos x - 4 \sin x$

CE: $D^2y/dx^2 = 2 \cos x - 4 \sin x - 2dy/dx + 3y$

PS: $y(x) = 2e^{-3x} + \sin x$, $y'(x) = \cos x - 6e^{-3x}$

IC: $y(0) = 2$, $y'(0) = -5$, Width: 0.01.

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
0.01	1.95089090	1.95089090	–4.091e–10	–4.82272320	–4.82272320	1.227e–09
0.02	1.90352787	1.90352773	–1.390e–07	–4.65077352	–4.65078719	–1.368e–05
0.03	1.85785828	1.85785787	–4.089e–07	–4.48401038	–4.48403708	–2.669e–05
0.04	1.81383101	1.81383021	–8.038e–07	–4.32228344	–4.32232251	–3.907e–05
0.05	1.77139644	1.77139512	–1.318e–06	–4.16544676	–4.16549760	–5.084e–05
0.06	1.73050637	1.73050443	–1.944e–06	–4.01335871	–4.01342073	–6.202e–05
0.07	1.69111402	1.69111134	–2.678e–06	–3.86588183	–3.86595448	–7.265e–05
0.08	1.65317393	1.65317042	–3.513e–06	–3.72288272	–3.72296546	–8.274e–05
0.09	1.61664198	1.61663754	–4.446e–06	–3.58423191	–3.58432423	–9.233e–05
0.10	1.58147533	1.58146986	–5.470e–06	–3.44980373	–3.44990516	–1.014e–04

The reader would try for plotting a diagram with the values of estimations for $y(x)$ and $y'(x)$.

5.4 Nonlinear Equations

In previous chapter I mentioned the criteria for linearisation of a first order differential equation. The linearity is based on the ground that the equation is of the first power in the unknown function and its derivative. As to the second order differential equation the principle remains unchanged.

Nonlinear differential equations are the equations that describe the events of interaction. Such equations can be often found in the field of mechanics, quantum physics and electrodynamics. The famous gravitation equation of Newton in such a case. Some forms of nonlinear nature have already been known to us which we considered in Chapter 4. The equations of Bernoulli and Riccati are of such type.

Our concern in this section is to demonstrate the equations of nonlinear nature discussing some of the fundamental points and methods of this subject-matter. The nonlinear equations include the equations from van der Pol.

Evidently the equations

$$(\alpha) \quad y'' - 5x\, y'^4 = e^x + 1$$

$$(\beta) \quad y''^2 - 3y\, y' + xy = 0$$

are nonlinear, because the derivatives of the unknown function y is raised to a power other than the first.

Example 5.17 Solve $x^2 yy'' + (xy' - y)^2 - 3y^2 = 0.$

Simplifying and rearranging, the nonlinear equation can be written in exact form

$$d/dx\,(x^2\, y\; y') - 2d/dx\,(x\, y^2) = 0.$$

Integrating yields $x^2\, y\; y' - 2x\, y^2 = C_1$

than can be transformed into a first-order linear form by putting $y^2 = v$.

The transformed linear equation becomes

$$dv/dx - 4v/x = 2C_1 / x^2 .$$

We need the application of the principles of linear form here.

I.F. = x^{-4} and with the help of this we obtain

$$v\, x^{-4} = 2C_1 \int x^{-6}\; dx \; = -2/5\; C_1 / x^{-5} + C_2 .$$

Hence, the general solution is

$$y^2 = C_2\, x^4 - 2/5\; C_1\, x^{-1}.$$

The nonlinear differential equation can also be solved by means of the method of *Power series solution*. A thorough description of the Power series solution will be given later on in Chapter 6 with some examples. First and second order differential equations have been dealt with along with the Program development. On that cause I admit here a brief outline for the power series solution with an example.

Example 5.18 Find a series solution for the equation $y'' = xy^2 - y'$

when the initial conditions are $y(0) = 2,\; y'(0) = 1$.

Let us assume that

$$y = \sum_{m=0}^{\infty} C_m\, x_m \; = C_0 + C_1\, x + C_2\, x^2 + C_3\, x^3 + \ldots \qquad \ldots (5.24)$$

is a solution of the given differential equation.

On differentiation of the assumed expression (5.24) and using the initial conditions yields

$$C_0 = 2,\; C_1 = 1$$

The equation (5.24) now takes the form

$$y = 2 + x + C_2x^2 + C_3x^3 + \ldots \qquad \ldots (5.25)$$

The series (5.25) and its derivatives can be substituted into the given differential equation to obtain

$$2C_2 + 6C_3x + 12C_4x^2 = x\,(2 + x + C_2x^2 + C_3x^3 + \ldots)^2 - (1 + 2C_2x + 3C_3x^2 + \ldots).$$

Rearranging and equating the coefficients of corresponding power of x one can determine the values

$$2C_2 = -1, \quad 6C_3 = 4 - 2C_2, \quad 12C_4 = 4 - 3C_3, \ldots.$$

from which it follows

$$C_2 = -1/2,\ C_3 = 5/6, \text{ and so on.}$$

Hence the power series solution for y is

$$y = 2 + x - 1/2x^2 + 5/6x^3 + \ldots$$

The approximated solution for the second-order initial value problem of nonlinear type can be determined using the program g5NL2IVP. The program is constructed to estimate the initial value problem by means of *Runge-Kutta* formulas.

Example 5.19

DE: $(1 + x^2)y'' + y'^2 + 1 = 0$
CE: $y'' = -(1 + y'^2)/(1 + x^2)$
PS: $y(x) = 2 \text{ Ln } (x + 1) - x + 1,\ y'(x) = 2/(x + 1) - 1$
IC: $y(0) = 1,\ y'(0) = 1$, Width: 0.01.

The approximated values of the IVP of Order 2:

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
0.01	1.00990066	1.00990066	1.471e–11	0.98019802	0.98019802	8.858e–12
0.02	1.01960525	1.01960525	4.979e–09	0.96078382	0.96078431	4.902e–07
0.03	1.02911759	1.02911760	1.451e–08	0.94174663	0.94174757	9.426e–07
0.04	1.03844140	1.03844143	2.824e–08	0.92307556	0.92307692	1.360e–06
0.05	1.04758028	1.04758033	4.585e–08	0.90476016	0.90476190	1.744e–06
0.06	1.05653775	1.05653782	6.702e–08	0.88679036	0.88679245	2.097e–06
0.07	1.06531721	1.06531730	9.146e–08	0.86915646	0.86915888	2.422e–06
0.08	1.07392196	1.07392208	1.189e–07	0.85184913	0.85185185	2.720e–06
0.09	1.08235524	1.08235539	1.491e–07	0.83485939	0.83486239	2.993e–06
0.10	1.09062018	1.09062036	1.818e–07	0.81817858	0.81818182	3.242e–06

Example 5.20

DE: $2yy'' - 3y'^2 = 4y^2$
CE: $y'' = (4y^2 + 3y'^2)/2y$
PS: $y(x) = 1/\cos^2 x,\ y'(x) = 2 \tan x/\cos^2 x$
IC: $y(0) = 1,\ y'(0) = 0$, Width: 0.01.

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
0.01	1.00010001	1.00010001	1.279e–13	0.02000267	0.02000267	1.674e–12
0.02	1.00040011	1.00040011	1.669e–09	0.04002117	0.04002134	1.669e–07
0.03	1.00090054	1.00090054	5.002e–09	0.06007172	0.60007206	3.335e–07
0.04	1.00160170	1.00160171	9.983e–09	0.08017040	0.08017090	4.985e–07
0.05	1.00250416	1.00250417	1.658e–08	0.10033338	0.10033404	6.610e–06
0.06	1.00360863	1.00360866	2.477e–08	0.12057695	0.12057777	8.196e–07
0.07	1.00491602	1.00491605	3.448e–08	0.14091752	0.14091849	9.732e–06
0.08	1.00642736	1.00642741	4.566e–08	0.16137167	0.16137279	1.121e–06
0.09	1.00814388	1.00814394	5.823e–08	0.18195620	0.18195746	1.260e–06
0.10	1.01006697	1.01006705	7.210e–08	0.20268810	0.20268949	1.391e–06

Example 5.21

DE: $(x + 1)y'' + xy'^2 - y' = 0$

CE: $y'' = y'(1 - xy')/(x + 1)$

PS: $y(x) = 2 \text{ Ln } x - 2/x, \; y'(x) = 2/x + 2/x^2$

IC: $y(1) = -2, \; y'(1) = 4$, Width: 0.01.

The *y*-approximation starts with –1.96029736 for an increment of 1.01 when the corresponding y'-approximation has the value 3.94079012. The field of errors in *y* confines to the range [1.689e–10, 1.635e–06] and the field in y' is limited to the range [–2.916e–10, 2.887e–05].

After the execution of the program we obtain the data with which a diagram can be sketched.

Example 5.22

DE: $y'' - e^{-2y} = 0$

CE: $y'' = e^{-2y}$

PS: $y(x) = \text{Ln}(\cos h(x + \pi/2)), \; y'(x) = \tan h(x + \pi/2)$

IC: $y(-\pi/2) = 0, \; y'(-\pi/2) = 0$, Width: 0.01.

The Computer Solution of the IVP of Order 2:

x-value	*y*-Approx.	*y*-Exact	Error	*y′*-Approx.	*y′*-Exact	Error
–1.5608	4.99991667e–05	4.99991667e–05	1.472e–15	0.00999967	0.00999967	8.331e–13
–1.5508	0.00019999	0.00019999	1.667e–09	0.01999717	0.01999733	1.667e–07
–1.5408	0.00044993	0.00044993	5.000e–09	0.02999067	0.02999100	3.333e–07
–1.5308	0.00079978	0.00079979	9.998e–09	0.03997818	0.03997868	4.997e–07
–1.5208	0.00124946	0.00124948	1.666e–08	0.04995771	0.04995837	6.660e–07
–1.5108	0.00179890	0.00179892	2.498e–08	0.05992727	0.05992810	8.321e–07
–1.5008	0.00244797	0.00244800	3.497e–08	0.06988489	0.06988589	9.978e–07
–1.4908	0.00319655	0.00319659	4.660e–08	0.07982861	0.07982977	1.163e–06
–1.4808	0.00404448	0.00404454	5.989e–08	0.08975646	0.08975778	1.328e–06
–1.4708	0.00499161	0.00499169	7.482e–08	0.09966650	0.09966799	1.492e–06

The field of errors in *y*-approximation confines to the range [1.472e–15, 7.482e–08] and the field of error in y'-approximation is limited to the range [8.331e–13, 492e–06].

Example 5.23

DE: $\cos y\, y'' + y'^2 \sin y - y' = 0$

CE: $y'' = y'(1 - y' \sin y)/\cos y$

PS: $y(x) = 2\tan^{-1}(e^{2(x+1)}) - \pi/3$, $y'(x) = 4e^{2(x+1)}/(1 + e^{4(x+1)})$

IC: $y(-1) = 0.52359878$, $y'(-1) = 2$, Width: 0.01.

The Computer Solutions of the Initial Value Problem of second order will be determined by the program. The approximate values can be controlled comparing with the exact values that are parallely found. The y-approximation starts with 0.54359745 for an increment of –0.99 when the corresponding y'-approximation has the value 1.99960007.

5.5 Related Software for the Solution

The initial value problems of second order can be estimated by different numerical methods. One to the powerful as well popular method, namely standard *Runge-Kutta* method of order four, has been properly chosen for the approximation in this case. The software package, particularly designed for each individual section in this chapter, incorporates the Runge-Kutta principle to find the approximation to $y(x)$.

We have studied in this chapter different types of second-order differential equations: linear type, homogeneous and nonhomogeneous, Euler type and nonlinear type applying several methods for the approximation of initial value problems of second order.

The program needed for the purpose of estimating the dependent variable $y(x)$ after solving the computer equations given in the examples have been designed in C++ (turbo) language.

The program g5LI2IVP implements the Runge-Kutta method and determines the approximations of the functions involved in linear, homogeneous and nonhomogeneous, equations and Euler's equations.

The program g5NL2IVP is used for nonlinear system described in Section 5.4.

6

Computing Operational Series

6.1 Analytic Numerical Series

6.1.1 Infinite Series. Convergence and Divergence

The expression $r_1 + r_2 + r_3 + \ldots + r_n + \ldots.$ … (6.1)

Where the terms $r_1, r_2, r_3, \ldots$, etc. formed according to some regular rule, is known as *series*. The succession of terms $r_1, r_2, r_3, \ldots$, etc. is called a *sequence*. The series is the specified sum of the terms of a sequence. When in a series the number of terms is unlimited, it is known as an *infinite* series. Otherwise, it is finite with limited number of terms.

The infinite series in (6.1) can be written now in a convenient form

$$\sum_{n=1}^{\infty} r_n = r_1 + r_2 + r_3 + \ldots. \qquad \ldots (6.2)$$

By means of the series (6.2) we can form *partial* sums as follows:

$$\begin{aligned} s_1 &= r_1, \\ s_2 &= r_1 + r_2, \\ s_3 &= r_1 + r_2 + r_3, \\ &\ldots \ \ldots \ \ldots \ \ldots \\ s_n &= r_1 + r_2 + r_3 + \ldots + r_n, \\ &\ldots \ \ldots \ \ldots \ \ldots \end{aligned} \qquad \ldots (6.3)$$

where $s_1, s_2, s_3, \ldots, s_n$ are the terms for the partial sums.

When the sequence of partial sums (6.3) has a finite limit as n tends to ∞ (infinity), that means when s_n approaches to a limit, say S, specified by a relation

$$\lim_{n \to \infty} s_n = S,$$

the infinite series (6.2) is said to be *convergent* and converges to a value S.

If s_n does not approach to a limiting value, the series is *divergent*. There is another class of series called *oscillating* that is also convergent, but periodically.

Different types of infinite series are known, which belong to various branches of mathematics. Some of them we mention here:

Geometric series: $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \ldots$

Logarithmic series: $\sum_{n=1}^{\infty} (-1)^{n-1}\, r^n/n = r - r^2/2 + r^3/3 - \ldots$

Exponential series: $\sum_{n=0}^{\infty} r^n/n! = 1 + r/1! + r^2/2! + \ldots$

Trigonometric series: $\sum_{n=0}^{\infty} (-1)^n\, r^{2n+1}/(2n+1)! = r/1! - r^3/3! + r^5/5! + \ldots (= \text{Sin } r)$

In the present chapter we deal with the series that are fundamentally infinite. In view of that we confine our discussion to the application of infinite series that are purposely convergent, because of the fact that the convergent series is of great importance in the field of science and technology.

Example 6.1 We consider a *geometric* series as represented by

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots.$$

Evidently, the *n*-th partial sum of the series is

$$s_n = 1 + x + x^2 + \ldots + x^{n-1}. \qquad \ldots (6.4)$$

Multiplying the equation (6.4) by *x* throughout and subtracting this from the original we obtain after rearrangement

$$s_n(1 - x) = (1 + x + x^2 + \ldots + x^{n-1}) - (x + x^2 + x^3 + \ldots + x^{n-1} + x^n) = 1 - x^n.$$

Hence, the *n*-th partial sum of the series becomes $s_n = (1 - x^n)/(1 - x)$ that can also be written in the form

$$s_n = 1/(1 - x) - x^n/(1 - x).$$

Case 1 If *x* is numerically less than 1, the sum converges then to the limit $1/(1 - x)$ making the series convergent.

Case 2 For $x = 1$, the series is obviously divergent.

Case 3 For $x = -1$, the series becomes $1 - 1 + 1 - 1 + 1 - 1 + \ldots$.

The sum of the series oscillates between the values 0 and 1 according as *n* is even or odd. Considering above we can accept the infinite geometric series as convergent under a particular condition, i.e., when $|x| < 1$, and the sum of the series amounts to

$$\sum_{k=0}^{\infty} x^k = 1/(1 - x).$$

Example 6.2 We want to show that the infinite series in compact form

$$\sum_{k=1}^{\infty} 1/k(k+1) \text{ is convergent.}$$

The series can be rewritten in the form

$$1/k\,(k+1) = 1/k - 1/(k+1)$$

yielding the sequence of partial sums

$$s_k = (1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + (1/4 - 1/5) + \ldots + [1/k - 1/(k+1)]$$
$$= 1 - 1/(k+1).$$

Taking the limit of the sum we get $\lim_{k \to \infty} s_k = \lim_{k \to \infty} [1 - 1/(k+1)] = 1$

which shows that the given series converges and its sum is equal to 1.

6.1.2 Comparison Test. Condition for Convergence

Aim of the test is to determine whether a given series is convergent or not, by comparing it with another known series.

Example 6.3 Test the series

$$\sum_{k=1}^{\infty} 1/k^2. \qquad \ldots (6.5)$$

The given series can be rewritten, without any loss of generality, as

$$\sum_{k=1}^{\infty} 1/k^2 = \sum_{k=0}^{\infty} 1/(k+1)^2.$$

In the preceding example we have shown that the series

$$\sum_{k=1}^{\infty} 1/k(k+1) \text{ is convergent.}$$

Seeing that the equation of inequality

$$1/(k+1)^2 < 1/k(k+1)$$

holds for $k = 1, 2, 3, \ldots$,

We can conclude by means of comparison test, that the given series is convergent.

Comparing the terms of the two inequalities step by step, that means, putting the value of k in each step we find that the values of the known series are *not less than* the corresponding values in the given series (6.5).

This is the *condition* of the comparison test *for convergence*. In the case of *divergence* the terms of the series to be tested are never less than the corresponding terms in the known series.

6.1.3 D'Alembert Test (Test-Ratio Test)

Let us consider the infinite series in (6.1) again which reads

$$r_1 + r_2 + r_3 + \ldots + r_n + r_{n+1} + \ldots.$$

If the terms of the series are presented in such a manner that there exists a finite limit, say α i.e.,

$$\lim_{n \to \infty} r_{n+1}/r_n = \alpha,$$

then it is known as the test ratio limit.

When the limiting value α is less than 1, that means, when

$$|\lim_{n \to \infty} (r_{n+1}/r_n)| < 1,$$

then the series is convergent.

When the limiting value α is greater than 1, the series is divergent.

When the limiting value α is equal to 1, the test does not meet the requirement.

Example 6.4 Investigate the series $\sum_{k=1}^{\infty} (k^3/2^k)$.

Obviously the (k + 1)-th term of the series is

$$r_{k+1} = (k+1)^3/2^{k+1}.$$

Accordingly by test-ratio we have

$$\lim_{k \to \infty} (r_{n+1}/r_k) = \lim_{k \to \infty} [(k+1)^3\, 2^k]/[2^{k+1}\, k^3]$$

$$= \lim_{k \to \infty} [k^3(1 + 1/k)^3\, 2^{k-k-1}]/k^3$$

Thus $$\lim_{k \to \infty} (r_{n+1}/r_k) = \lim_{k \to \infty} (1 + 1/k)^3/2 = 1/2$$

which is obviously less than 1.

Hence, according to the test-ratio test the given series is convergent and converges to 1/2.

Next, we consider some infinite series that are convergent and we wish to determine the partial sums of the series assuming that the series is limited. We make the series limited by accepting the specified number of terms in the series.

In order to find the approximated summation of the infinite series the selected program `g6FUNCSR` is used. In Section 6.4 some suggestions about the related software have been given.

Example 6.5

Isr: $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$

Fsr: $\sum_{x=1}^{100} [1/x\,(x+1)\,(x+2)]$

Ssr: 0.25, Ntr: 100.

For the sake of convenience and simplicity we write down the functional infinite series (Isr) in the form of finite series (Fsr) accepting only the first 100 terms of the series for Computation. The Program calculates the partial sums of the 100 terms yielding finally the resulting limiting sum of the series (Ssr). The abbreviation Ntr means the number of terms selected for calculation.

The successful program execution requires naturally the correct and well-provided data for input. The number of terms in the series has been restricted in each example because of the data limiting capacity. The program sends finally a message in this case that reads:

`The series converges to the value 0.25`

which is the resulting sum of the given series.

Example 6.6

$$\text{Isr: } \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots$$

$$\text{Fsr: } \sum_{x=1}^{100} [1/(2x-1)\,(2x+1)]$$

Ssr: 0.50, Ntr: 100.

The message

`The series converges to the value 0.50`

will appear on the screen, which is the result of the series.

Example 6.7

$$\text{Isr: } \tan^{-1}[1/(2.1^2)] + \tan^{-1}[1/(2.2^2)] + \tan^{-1}[1/(2.3^2)] + \ldots$$

$$\text{Fsr: } \sum_{x=1}^{100} \tan^{-1}[1/(2x^2)]$$

Ssr: $\pi/4 = 0.78539$, Ntr: 100.

`The series converges to the value 0.78.`

Example 6.8

$$\text{Isr: } \frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} + \ldots$$

$$\text{Fsr: } \sum_{x=1}^{100} \left[x/(2x-1)^2\,(2x+1)^2\right]$$

Ssr: 0.125, Ntr: 100.

`The series converges to the value 0.125.`

Example 6.9

$$\text{Isr: } \sin(\pi/2) + \sin(\pi/2^2) + \sin(\pi/2^3) + \ldots$$

$$\text{Fsr: } \sum_{x=1}^{100} [\sin(\pi/2^x)]$$

Ssr: 2.481, Ntr: 100

The series converges to the value 2.481.

6.2 Power Series Solution

6.2.1 Introduction

A *power series* solution is nothing but an infinite series consisting of algebraic terms in ascending positive integral power of a variable. The series of the form

$$\sum_{m=0}^{\infty} C_m x^m = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n + ... \qquad \text{… (6.6)}$$

where the coefficients C_0, C_1, C_2, … are constants is known as a *power series* in power of x.

The series

$$\sum_{m=0}^{\infty} C_m (x - x_0)^m = C_0 + C_1 (x - x_0) + C_2 (x - x_0)^2 + ... \qquad \text{… (6.7)}$$

is also a power series, but in power of $(x - x_0)$. Such type of series is quite frequently encountered in numerical analysis. In particular, the series (6.6) is a special case, when the term $(x - x_0)$ of the series (6.7) is replaced by x. In this Section I shall confine the discussion to the power series of the form (6.6), that is the power series in standard form.

The power series possesses some significant properties which will be considered next.

In the preceding section the discussion on the convergence and the divergence of the series with their respective properties have been made.

The series

$$\sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + ... \qquad \text{… (6.8)}$$

converges for $| x | < 1$ and diverges for $| x | > 1$.

If we divide the series (6.8) by $m!$ {$m!$ means factorial m} and expand, we have

$$\sum_{m=0}^{\infty} x^m / m! = 1 + x/1! + x^2/2! + x^3/3! + ... \qquad [0! = 1]$$

that converges for all values of x.

Finally, Multiplication of the series (6.8) by $m!$ and expanding yields

$$\sum_{m=0}^{\infty} m!\, x^m = 1 + 1!\, x + 2!\, x^2 + 3!\, x^3 + ...$$

which diverges for all $x \neq 0$.

In view of the above we can conclude that to each power series there corresponds a positive real point r on the real axis of the coordinate system satisfying the following properties:

1. the series converges when $| x | < r$
2. the series diverges when $| x | > r$.

The real number r is called the *radius of convergence* of the power series if we draw a concentric circle with r as radius. Each power series in x has a radius of convergence with the properties just mentioned.

Let us assume that the radius r is a finite and nonzero quantity. Then there exists an interval for values of x, $-r < x < r$, known as the *interval of convergence*, such that the power series will converge when x lies in the interval and will diverge for values of x outside the interval. The power series may converge at the endpoints of the interval of convergence $[-r, +r]$, but it is not always the case.

As a matter of fact, $-1 < x < 1$ is the interval of convergence of the power series (6.8), the radius of convergence r being 1.

Example 6.10 The series

$$\text{Ln}\,(1 + x) = x - x^2/2 + x^3/3 - \ldots (-1)^{m-1}x^m/m + \ldots \qquad \ldots (6.9)$$

has the limiting value when we apply the ratio-test.

$$\lim_{m \to \infty} |x^{m+1}/(m+1) \times m/x^m| = \lim_{m \to \infty} |x \times m/(m+1)| = x.$$

The series (6.9) converges for $| x | < 1$ and diverges for $| x | > 1$.

Let us consider the endpoints, that means, the points $x = 1$ and $x = -1$.

When $x = 1$, the series becomes $1 - 1/2 + 1/3 - 1/4 + \ldots$

which is conditionally convergent.

When $x = -1$, the series becomes $-1 - 1/2 - 1/3 - 1/4 - \ldots$

and is evidently divergent.

Hence, the series converges on the interval $-1 < x \le 1$, that is the interval of convergence.

6.2.2 Series Expansion

Recalling the series (6.6) and assuming its convergence for $| x | < r$ with $r > 0$ we can write it in a functional relation

$$f(x) = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \qquad \ldots (6.10)$$

where $f(x)$, the sum of the series is naturally a continuous function which has differential coefficient of all orders for $| x | < r$.

Now differentiating the series step by step we have

$$f'(x) = \sum_{m=1}^{\infty} m\, a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

$$f''(x) = \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} = 2a_2 + 2.3a_3 x + \ldots$$

$$f'''(x) = \sum_{m=3}^{\infty} m(m-1)(m-2)\, a_m x^{m-3} = 2.3a_3 + 2.3.4a_4 x + \ldots.$$

If we proceed further we can find out the m-th derivative.

So, on putting $x = 0$ we have the basic relation for a_m

$$f^m(0) = m!\ a_m$$

or

$$a_m = f^m(0)/m!$$

The equation (6.10) can now be rewritten as

$$f(x) = \sum_{m=0}^{\infty} f^m(0)/m!\ x^m = f(0) + f'(0)x + f''(0)/2!\,x^2 + \ldots \qquad \ldots (6.11)$$

where $a_0 = f(0),\ a_1 = f'(0),\ a_2 = f''(0)/2!$, etc.

The series (6.2) is known as *Maclaurin's* series.

In this connection we mention *Taylor's* formula which reads

$$f(x) = \sum_{k=0}^{n} f^k(0)/k!\ x^k + R_n(x)$$

where the function $R_n(x)$, *Lagrange's* remainder form, is given by

$$R_n(x) = f^n(\nu)/n!\ x^n$$

for some point ν between 0 and x.

This formula can be used for the validity of the series expansion for a particular point x. If we can show that $R_n(x) \to 0$ as $n \to \infty$, then the equation (6.11) is proved to be true.

A function $f(x)$ is analytic at a point x_0 if the infinite series

$$\sum_{k=0}^{\infty} f^k(x_0)\,(x - x_0)^k/k!$$

in relation to Taylor series about x_0 converges to the function $f(x)$ in some neighbourhood of x_0.

Example 6.11 Find an expansion for the following series according to Taylor

$$f(x) = \text{Ln}\,(1 + x) \text{ around } x = 0.$$

We have

$$f(x) = \text{Ln}\,(1 + x),\ f(0) = 0$$

$$f'(x) = 1/(1 + x),\ \ f'(0) = 1$$

$$f''(x) = -1/(1 + x)^2,\ \ f''(0) = -1$$

$$f'''(x) = 2/(1 + x)^3,\ \ f'''(0) = 2!$$

$$f^{\text{iv}}(x) = -2.3/(1 + x)^4,\ \ f^{\text{iv}}(0) = -3!$$

Now by means of the equations (6.11) we can write

$$f(x) = x - x^2/2! + 2!\, x^3/3! - 3!\, x^4/4! + 4!\, x^5/5! - 5!\, x^6/6! + \ldots$$
$$= x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - x^6/6 + \ldots$$

that is the power series expansion for Ln $(1 + x)$.

Considering the six terms of Taylor series we find the approximate values for the given logarithmic series. We compute the errors by comparing with the actual values and represent them in tabular form.

Increment	*f (x)*	$S_6(x)$	*Error*
$x = 0.01$	0.00995033	0.00995033	–1.6e–13
$x = 0.05$	0.04879016	0.04879016	1.076e–10
$x = 0.1$	0.09531017	0.09531016	1.0e–08

Example 6.12 Let us consider another series

$f(x) = \cos x$.

The original function $f(x) = \cos x$ that leads to $f(0) = 1$.

Taking the derivative step by step we find

$$f'(x) = -\sin x \quad \text{that leads to} \quad f'(0) = 0$$

$$f''(x) = -\cos x \quad \text{that leads to} \quad f''(0) = -1$$

$$f'''(x) = \sin x \quad \text{that leads to} \quad f'''(0) = 0$$

$$f^{\text{iv}}(x) = \cos x \quad \text{that leads to} \quad f^{\text{iv}}(0) = 1$$

and so on.

By applying the series expansion formula we get the power series solution for

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - \ldots.$$

The following table provides the calculated values of $f(x)$ and the approximations $S_5(x)$ yielding the difference of results

Increment	*f (x)*	$S_5(x)$	*Error*
$x = 0.1$	0.99500416	0.99500416	0
$x = 0.2$	0.98006657	0.98006657	–4.0e–11
$x = 0.5$	0.87758256	0.87758256	–2.58e–10
$x = 1.0$	0.54030230	0.54030257	–2.73464e–07

We recall the remainder function $R_n(x)$ at this stage and want to show that $R_n(x)$ approaches zero as a limiting value as n becomes infinite for any fixed value of x. In view of the above we can write the n-th derivative in the form

$$f^n(x) = \cos(n\pi/2 + x)\,.$$

Now, $R_n(x)$ takes the form as

$$R_n(x) = \cos(n\pi/2 + x_v)\; x^n/n! \qquad (0 < x_v < x)\,.$$

Here the first factor $\cos\,(n\pi/2 + x_v)$ can never be greater than 1. The term $x^n/n!$ is the n-th term of the series

$$x + x^2/2! + x^3/3! + \ldots + x^n/n! + \ldots$$

that is convergent for all values of x. Hence it approaches zero as a limit when n becomes infinite.

6.2.3 First Order Equations

Many essential differential equations of the order have been encountered in mathematics and physical sciences which cannot be solved easily, sometimes not be possible to solve by means of any elementary methods discussed in Chapter 4.

The solutions for the differential equations can be then expressed in the form of a power series. We want to show now how the power series method is used for the solution of first order equation.

A first order differential equation of the form

$$P(x)\,y' + Q(x)y = R(x) \qquad \ldots (6.12)$$

is considered now for power-series solution. $P(x)$, $Q(x)$ and $R(x)$ are the coefficient functions of the differential equation.

When $R(x) = 0$, the differential equation is termed as *homogeneous*, otherwise it is nonhomogeneous.

Let us assume that the equation (6.12) has a power series solution of the form

$$y = \sum_{k=1}^{\infty} a_k x^{k-1} \qquad \ldots (6.13)$$

where a_k are the coefficients.

The series converges for $|\, x \,| < r$ when r, the radius of convergence, is greater than 0. In other words, we can deduce that the equation (6.12) has a solution which is analytic at the origin, an *ordinary* point for the equation.

A power series can be differentiated term by term in the interval of convergence determined by the ratio-test.

Expanding the series (6.13) yields

$$y = a_1 + a_2x + a_3x^2 + \ldots + a_mx^m + \ldots$$

and on differentiation the series (6.13) becomes

$$y' = \sum_{k=1}^{\infty} ka_{k+1}x^{k-1} \qquad \ldots (6.14)$$

that on expanding we obtain

$$y' = a_2 + 2a_3x + 3a_4x^2 + \ldots + (m+1)\,a_{m+1}x^{m+1} + \ldots$$

The assumption that the power-series expansion of this solution is valid on the given interval is only justified, provided that the power series expansions of the coefficient functions $P(x)$, $Q(x)$ and $R(x)$ are valid on the same interval.

Now, the power series expansions of

$$P(x) = \sum_{k=1}^{\infty} p_k\, x^{k-1} = p_1 + p_2x + p_3x^2 + p_4x^3 + \ldots \qquad \ldots (6.15)$$

$$Q(x) = \sum_{k=1}^{\infty} q_k \, x^{k-1} = q_1 + q_2 x + q_3 x^2 + q_4 x^3 + \ldots \qquad \ldots (6.16)$$

$$R(x) = \sum_{k=1}^{\infty} r_k \, x^{k-1} = r_1 + r_2 x + r_3 x^2 + r_4 x^3 + \ldots \qquad \ldots (6.17)$$

where p_k, q_k and r_k are the coefficients.

Multiplying the series (6.15) by (6.14) we obtain

$$P(x)\; y' = \left(\sum_{k=1}^{\infty} p_k \, x^{k-1} \right) \left(\sum_{k=1}^{\infty} k\, a_{k+1} \, x^{k-1} \right)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} 1\, p_{k-1+1} \; a_{1+1} \right) x^{k-1}. \qquad \ldots (6.18)$$

Similarly, it follows from (6.16) and (6.13)

$$Q(x)\; y = \left(\sum_{k=1}^{\infty} q_k \, x^{k-1} \right) \left(\sum_{k=1}^{\infty} a_k \, x^{k-1} \right)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} q_{k-1+1} \; a_1 \right) x^{k-1}. \qquad \ldots (6.19)$$

Substituting (6.17), (6.18) and (6.19) into (6.12) we obtain

$$\sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} 1\, p_{k-1+1} \; a_{1+1} \right) x^{k-1} + \sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} q_{k-1+1} \; a_1 \right) x^{k-1} = \sum_{k=1}^{\infty} r_k \, x^{k-1}$$

for $k = 1, 2, 3, \ldots$.

After rearrangement it can be written as

$$\sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} [1\, p_{k-1+1} \; a_{1+1} + q_{k-1+1} \; a_1] \right) x^{k-1} = \sum_{k=1}^{\infty} r_k \, x^{k-1}.$$

As x^{k-1} cannot be zero, we get at the results finally

$$\sum_{1=1}^{\infty} [1\, p_{k-1+1} \; a_{1+1} + q_{k-1+1} \; a_1 - r_{k-1+1}] = 0, \qquad \ldots (6.20)$$

for $k = 1, 2, 3, \ldots$, which is known as the *recursion formula* for the coefficients of the general power-series solution of (6.12).

Example 6.13 $\quad y' + 2xy = 0,\; y(0) = 1.0.$

This is a first order initial value problem of homogeneous type with the given initial conditions.

The relations (6.15) and (6.16) suggest

$$p_1 = 1,\ p_2 = p_3 = \dots = 0$$
$$q_1 = 0,\ q_2 = 2,\ q_3 = q_4 = \dots = 0$$
$$r_1 = r_2 = r_3 = r_4 = \dots = 0.$$

Substituting the values of $p's$, $q's$ and $r's$ into the formula (6.20) the coefficients a_k can be determined when $k = 1, 2, 3, \dots$.

So the values of the coefficients are

$$a_2 = 0,$$
$$a_3 = -a_1,$$
$$a_4 = -2/3\ a_2 = 0,$$
$$a_5 = -1/2\ a_3 = 1/2\ a_1,$$
$$a_6 = -2/5\ a_4 = 0.$$
$$\dots.$$

Hence, the formula (6.13) provides the general power-series solution of the given equation by means of the determined values of the coefficients

$$y = a_1(1 - x^2 + x^4/2! - x^6/3! + \dots).$$

Now, $y(0) = a_1 = 1.0$, as the initial conditions imply.

Hence, $y(x) = 1 - x^2 + x^4/2! - x^6/3! + \dots$

which is equivalent to e^{-x^2} .

Here, the computer program g6POW1HM is used for the determination of the solution for the first-order initial value problems in terms of power series.

Example 6.14

DE: $(1 + x)y' - 2y = 0$

P(x): $p_1 = 1,\ p_2 = 1,\ p_3 = p_4 = \dots = 0$

Q(x): $q_1 = -2,\ q_2 = q_3 = \dots = 0$

IC: $y(0) = 1$

Pss: $y(x) = 1 + 2x + x^2$.

The differential equation DE has been, at first, expanded in terms of power series. The coefficient function P(x) relates to the function $1 + x$ and the coefficient Q(x) corresponds to the coefficient of y in the equation.

It has already been explained in the discussion of power-series expansion how to determine the values of p_1, p_2, etc. and q_1, q_2, etc. With the help of the initial conditions (IC) we ascertain the initial constant value of a_1 and the determination leads to the power-series solution (Pss) at last.

Example 6.15

DE: $dy/dx - x^2y = 0$

P(x): $p_1 = 1,\ p_2 = p_3 = p_4 = \dots = 0$

Q(x): $q_1 = 0,\ q_2 = 0,\ q_3 = -1,\ q_4 = q_5 = \dots = 0$

IC: $y(0) = 1$

Pss: $y(x) = 1 + x^3/3 + x^6/18 + \dots = e^{x^2}/3$.

Example 6.16

DE: $(1 + x)y' + 3y = 0$

P(x): $p_1 = 1$, $p_2 = 1$, $p_3 = p_4 = \ldots = 0$

Q(x): $q_1 = 3$, $q_2 = q_3 = q_4 = \ldots = 0$

IC: $y(0) = 1$

Pss: $y = (1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \ldots$

Example 6.17

DE: $y' + y = 0$

P(x): $p_1 = 1$, $p_2 = p_3 = p_4 = \ldots = 0$

Q(x): $q_1 = 1$, $q_2 = q_3 = q_4 = \ldots = 0$

IC: $y(0) = 1$

Pss: $y = e^{-x} = 1 - x + x^2/2 - x^3/6 + x^4/24 - x^5/120 + \ldots$

Example 6.18

DE: $y' - y = x$

P(x): $p_1 = 1$, $p_2 = p_3 = p_4 = \ldots = 0$

Q(x): $q_1 = -1$, $q_2 = q_3 = q_4 = \ldots = 0$

R(x): $r_1 = 0$, $r_2 = 1$, $r_3 = r_4 = \ldots = 0$

IC: $y(0) = 1$,

Pss: $y = -1 - x + 2e^x = 1 + x + x^2 + x^3/3 + x^4/12 + x^5/60 + \ldots$

6.2.4 Second Order Equations

We now turn our discussion to the general second order differential equation of the nonhomogeneous form

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = S(x). \qquad \ldots (6.21)$$

Assuming that the coefficient functions $P(x)$, $Q(x)$, $R(x)$ and $S(x)$ are analytic at a point x_0 we can presume that the power-series solution of the equation (6.21) is valid on the interval.

We accept that the equation (6.21) has a power series solution of the form

$$y = \sum_{k=1}^{\infty} a_k\, x^{k-1}$$

$$= a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \ldots \qquad \ldots (6.22)$$

Now differentiating the power series (6.22) term by term we obtain

$$y' = \sum_{k=1}^{\infty} k\, a_{k+1}\, x^{k-1} = a_2 + 2a_3 x + 3a_2 x^2 + 4a_5 x^3 + \ldots \qquad \ldots (6.23)$$

and

$$y'' = \sum_{k=1}^{\infty} k\,(k+1)\, a_{k+2}\, x^{k-1}$$

$$= 2a_3 + 6a_4 x + 12a_5 x^2 + 20a_6 x^3 + \ldots \qquad \ldots (6.24)$$

On basis of the assumption taken above the power series expansions of the coefficient functions for $P(x)$ and $Q(x)$ become

$$P(x) = \sum_{k=1}^{\infty} p_k \, x^{k-1} = p_1 + p_2 x + p_3 x^2 + p_4 x^3 + \ldots \qquad \ldots (6.25)$$

and

$$Q(x) = \sum_{k=1}^{\infty} q_k \, x^{k-1} = q_1 + q_2 x + q_3 x^2 + q_4 x^3 + \ldots \qquad \ldots (6.26)$$

Again the power series expansion of the coefficient functions for $R(x)$ and $S(x)$ become

$$R(x) = \sum_{k=1}^{\infty} r_k \, x^{k-1} = r_1 + r_2 x + r_3 x^2 + r_4 x^3 + \ldots \qquad \ldots (6.27)$$

and

$$S(x) = \sum_{k=1}^{\infty} s_k \, x^{k-1} = s_1 + s_2 x + s_3 x^2 + s_4 x^3 + \ldots \qquad \ldots (6.28)$$

If we multiply the series (6.25) by (6.24) we get

$$Py'' = \sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} 1(1+1)\, p_{k-1+1} \, a_{1+2} \right) x^{k-1}.$$

In the same manner from the relations (6.23) and (6.26) we have

$$Qy' = \sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} 1 q_{k-1+1} \, a_{1+1} \right) x^{k-1}$$

and finally from the series relations (6.22) and (6.27) we obtain

$$Ry = \sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} r_{k-1+1} \, a_1 \right) x^{k-1}.$$

On substitution of these values together with (6.28) into the given equation (6.21) we get a *recursion formula* for the coefficients as follows:

$$\sum_{k=1}^{\infty} \left(\sum_{1=1}^{k} 1(1+1)\, p_{k-1+1} \, a_{1+2} + \sum_{1=1}^{k} 1 \, q_{k-1+1} \, a_{1+1} + \sum_{1=1}^{k} r_{k-1+1} \, a_1 \right) x^{k-1} = \sum_{k=1}^{\infty} s_k \, x^{k-1}$$

for $k = 1, 2, 3, \ldots$.

Hence, the recursion formula for computing the coefficients of the general power series solution of the nonhomogeneous form in (6.21) becomes

$$\sum_{1=1}^{k} \left[1(1+1)\, p_{k-1+1} \, a_{1+2} + 1 \, q_{k-1+1} \, a_{1+1} + r_{k-1+1} \, a_1 - s_{k-1+1} \right] = 0 \qquad \ldots (6.29)$$

for $k = 1, 2, 3, \ldots, 10$, accepting the first ten values for k.

Example 6.19 $(1-x^2)y'' - 2xy' + 6y = 0$

with the initial condition $y(0) = -0.5$, $y'(0) = 0$.

This is a second-order initial value problem of homogeneous type.

The equations (6.25), (6.26) and (6.27) imply

$$p_1 = 1,\quad p_2 = 0,\quad p_3 = -1,\quad p_4 = p_5 = \ldots = 0,$$
$$q_1 = -2,\quad q_2 = -2,\quad q_3 = q_4 = q_5 = \ldots = 0$$

and
$$r_1 = 6,\quad r_2 = r_3 = r_4 = \ldots = 0.$$

Substituting the calculated values in the general formula (6.28) and rearranging we obtain

$$a_3 = -3a_1,\ 3a_4 = -2a_2,\ a_5 = 0,\ a_6 = -a_2/5.$$

The formula (6.22) yields the power-series solution of the given equation

$$y(x) = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 + \ldots$$

Imposing the initial conditions we have $a_1 = -0.5$ and $a_2 = 0$.

Therefore, $y(x) = (3x^2 - 1)/2$.

Let us determine the approximated solutions of the second-order differential equations in terms of power series employing the program g6POW2HM.

Example 6.20

DE: $y'' + xy = 0$
P(x): $p_1 = 1,\ p_2 = p_3 = p_4 = \ldots = 0$
Q(x): $q_1 = q_2 = q_3 = q_4 = \ldots = 0$
R(x): $r_1 = 0,\ r_2 = 1,\ r_3 = r_4 = \ldots = 0$
IC: $y(0) = 1,\ y'(0) = -1$
Pss: $y(x) = 1 - x - x^3/6 + x^4/12 + x^6/180 + \ldots$

Example 6.21

DE: $y'' + y\cos x = 0$
P(x): $p_1 = 1,\ p_2 = p_3 = p_4 = \ldots = 0$
Q(x): $q_1 = q_2 = q_3 = q_4 = \ldots = 0$
R(x): $r_1 = 1,\ r_2 = 0,\ r_3 = -0.5,\ r_4 = 0,\ r_5 = 1/24$
[$\cos x = 1 - x^2/2 + x^4/24 - x^6/720 + \ldots$]
IC: $y(0) = 1,\ y'(0) = 0$
Pss: $y(x) = 1 - x^2/2! + 2x^4/4! - 9x^6/6! + 55x^8/8! - \ldots$

Example 6.22

DE: $y'' + y' - 2y = 0$
P(x): $p_1 = 1,\ p_2 = p_3 = p_4 = \ldots = 0$
Q(x): $q_1 = 1,\ q_2 = q_3 = q_4 = \ldots = 0$
R(x): $r_1 = -2,\ r_2 = r_3 = r_4 = \ldots = 0$
IC: $y(0) = 1,\ y'(0) = 1$
Pss: $y(x) = e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + \ldots$

Example 6.23

DE: $y'' - xy' + 2y = 0$

P(x): $p_1 = 1,\ p_2 = p_3 = p_4 = \ldots = 0$

Q(x): $q_1 = 0,\ q_2 = -1,\ q_3 = q_4 = \ldots = 0$

R(x): $r_1 = 2,\ r_2 = r_3 = r_4 = \ldots = 0$

IC: $y(1) = 2,\ y'(1) = 3$

Pss: $y = 2 + 3(x-1) - (x-1)^2/2 - 2(x-1)^3/3 - (x-1)^4/6 + (x-1)^5/15 - \ldots$

6.3 Fourier Series Solution

6.3.1 Fourier Coefficients

We know from our experience that many physical and technical problems are treated with the specific trigonometric series of the form

$$a_0/2 + (a_1 \cos x + b_1 \sin x) + \ldots + (a_k \cos kx + b_k \sin kx) + \ldots$$

The power series representation of functions in the preceding section is already known to us. For power series expansion we can consider only *continuous* functions having differential coefficients of all orders. The distinguishing feature of the particular trigonometric series just mentioned is that it can represent the functions with *discontinuities*.

As for example we can mention the obvious cases of discontinuous impulse functions that can be observed in technology, especially in electrical engineering.

The specific trigonometric series just mentioned may now be written in the general form

$$g(x) = a_0/2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \qquad \ldots (6.30)$$

defining the function $g(x)$ on the closed interval $-\pi \le x \le \pi$.

Assuming that the series is, according to Dirichlet criteria, uniformly convergent, which suggests that the series can be integrated term by term.

For the proof of Dirichlet's principles and theorem I need not be concerned here. I want to say only that the availability of the standard texts is extremely large, some of them have been referred in the reference. On the subject of conditions for convergence I shall come back to the Dirichlet's principles in a short while.

In view of the assumption we integrate the series (6.30) from $-\pi$ to π

$$\int_{-\pi}^{\pi} g(x)\, dx = \int_{-\pi}^{\pi} \left[a_0/2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \right] dx.$$

Taking account of the following simple valid trigonometric integrals

$$\int_{-\pi}^{\pi} \sin kx\, dx = 0 = \int_{-\pi}^{\pi} \cos kx\, dx$$

for $k = 1, 2, 3, \ldots,$

the term-by-term integration of the series results

$$a_0 = 1/\pi \int_{-\pi}^{\pi} g(x)\, dx. \qquad \ldots (6.31)$$

Similarly, to find the coefficient a_k we multiply the equation (6.30) by cos *kx* throughout yielding another equation

$$g(x) \cos kx = a_0/2 \cos kx + (a_k \cos^2 kx + b_k \sin kx \cos kx) + \ldots. \qquad \ldots (6.32)$$

With the help of the appropriate trigonometrical formulas for sine and cosine converting the products into sums

$$2 \sin kx \cos lx = \sin (k + 1)x + \sin (k - 1)x$$

$$2 \cos kx \cos lx = \cos (k + 1)x + \cos (k - 1)x$$

$$2 \sin kx \sin lx = \cos (k - 1)x - \cos (k + 1)x$$

we can easily calculate the integrals

$$\int_{-\pi}^{\pi} \sin kx \cos lx\, dx = 0$$

and

$$\int_{-\pi}^{\pi} \cos kx \cos lx\, dx = 0, \qquad \text{when } k \# 1$$

$$= \pi, \qquad \text{when } k = 1; \qquad (k, 1 \geq 1.)$$

Now, the integration, term-by-term, of the series (6.31) from $-\pi$ to $+\pi$ leads to the results

$$\int_{-\pi}^{\pi} g(x) \cos kx\, dx = a_k \int_{-\pi}^{\pi} \cos^2 kx = a_k\, \pi \qquad \ldots (6.33)$$

when we take into consideration of the trigonometric realities which are mentioned before. Rearranging the equation (6.33) we obtain an expression for a_k

$$a_k = 1/\pi \int_{-\pi}^{\pi} g(x) \cos kx\, dx. \qquad \ldots (6.34)$$

By putting $k = 0$, the formula (6.34) is seen to be valid in comparison to the formula in (6.31). On that reason we take $a_0/2$ as the *constant term* instead of a_0.

Following the same procedure the coefficient b_k can be found. So, we multiply the equation (6.30) by sin *kx* throughout and integrate term by term applying the trigonometric identities stated above.

As a result of that we obtain

$$\int_{-\pi}^{\pi} g(x) \sin kx\, dx = b_k \int_{-\pi}^{\pi} \sin^2 kx = b_k\, \pi$$

which can be written in the form

$$b_k = 1/\pi \int_{-\pi}^{\pi} g(x) \sin kx\, dx \qquad \ldots (6.35)$$

that is the coefficient b_k of the series.

These determinations conclude that if the series (6.29) is uniformly convergent, then the coefficient a_k and b_k can be obtained from the function $g(x)$ with the help of the formulas mentioned in (6.33) and (6.34).

The coefficients a_k and b_k are called the *Fourier coefficients* of the function $g(x)$ and the series (6.29) is called the *Fourier series* corresponding to the function $g(x)$ in the interval $[-\pi, \pi]$. Thus we can say a Fourier series is a particular case of trigonometric series.

6.3.2 Dirichlet's Principles

Let us assume that

(a) a function $g(x)$ is defined in the closed interval $[-\pi, \pi]$ and single-valued except possibly at a finite number of points in the interval

(b) $g(x)$ is periodic with 2π period

(c) $g(x)$ and its derivative $g'(x)$ are piecewise continuous functions in $[-\pi, \pi]$.

Under the assumption started the Fourier series for $g(x)$, the series on the right side in equation (6.29) with coefficients given by the relations (6.33) and (6.34), converges to the value of $g(x)$ if x is a point of continuity. At each point of discontinuity of $g(x)$ the series converges to the arithmetic means of the values of $g(x)$ from the right and left, that is,

$$[g(x+0) + g(x-0)]/2.$$

The Fourier series represented by a function $g(x)$ to be approximated should not be necessarily a trigonometric function. If an arbitrary function is defined in an interval of length 2π and then periodically extended beyond the interval to the left and right hand limits in order to satisfy the functional relation

$$g(x \pm 2\pi) = g(x),$$

then a function with period 2π will enter.

It's not easy to understand the Dirichlet's Principles on theoretical basis. We should accustom to these conditions on practical viewpoint when we follow some examples.

Let us now expand a function by means of Fourier series satisfying the Dirichlet's constraints.

Example 6.24 Find the Fourier series of the function $g(x) = x^2$,

over the interval $-\pi < x < \pi$.

Here, by the relation (6.30)

$$a_0 = 1/\pi \int_{-\pi}^{\pi} x^2\, dx = 2\pi^2/3.$$

By means of the equation (6.31) we find the coefficient a_k when $k \geq 1$,

$$a_k = 1/\pi \int_{-\pi}^{\pi} x^2 \cos kx\, dx.$$

An integration by parts yields

$$a_k = 1/\pi \left[x^2/k \sin kx + 2x/k^2 \cos kx - 2/k^3 \sin kx\right]_{-\pi}^{\pi}$$

$$= 4/k^2 \cos k\pi = 4/k^2 (-1)^k, \qquad \text{for } k \neq 0.$$

For b_k we make use of the formula (6.34)

$$b_k = 1/\pi \int_{-\pi}^{\pi} x^2 \sin kx \; dx.$$

Applying the rule of integration by parts we get the coefficient b_k in the form as shown

$$b_k = 1/\pi \left[-x^2/k \cos kx + 2x/k^2 \sin kx + 2/k^3 \cos kx\right]_{-\pi}^{\pi} = 0.$$

Now substituting the results, thus obtained for a_0, a_k and b_k in (6.30) we obtain

$$x^2 = \pi^2/3 + \sum_{k=1}^{\infty} (4/k^2 \cos k\pi \cos kx) \qquad \dots (6.36)$$

$$= \pi^2/3 - 4\,(\cos x - \cos 2x/2^2 + \cos 3x/3^2 - \dots)$$

as *Fourier series expansion* for the given function $g(x)$.

The series converges to the function $g(x) = x^2$ at every point in the interval $(-\pi, \pi)$ for which x is continuous and indeed, throughout the interval. The Fourier series is also periodic with the period 2π.

At the end points $\pm k\pi$, for k = 1, 2, 3, …, the series represented by the formula (6.35) must converge to the average value $(\pi^2 + \pi^2)/2 = \pi^2$.

Beyond this interval $-\pi \le x \le \pi$, this series represents the periodic extension of the original function as shown in Fig. 6.1.

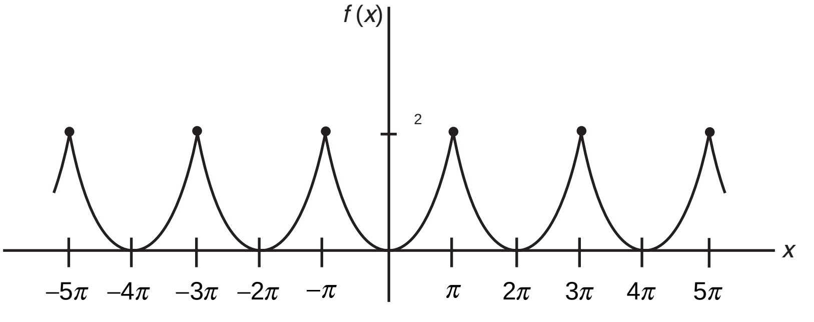

Fig. 6.1

It is interesting to observe the results of the function $g(x) = x^2$ denoted by the relation (6.35). This leads us to further logical conclusions reached by the results of the function.

Let us put $x = \pi$ in the Fourier series (6.36) to obtain another series

$$\pi^2 = \pi^2/3 + 4 \sum_{k=1}^{\infty} (\cos k\pi \times \cos k\pi)^2 / k^2 = \pi^2/3 + 4 \sum_{k=1}^{\infty} 1/k^2$$

$$\pi^2/6 = \sum_{k=1}^{\infty} 1/k^2 = 1/1^2 + 1/2^2 + 1/3^2 + \dots.$$

By setting $x = 0$ in the series (6.36) yields

$$0 = \pi^2/3 + 4 \sum_{k=1}^{\infty} (-1)^k / k^2$$

or

$$\pi^2/12 = \sum_{k=1}^{\infty} (-1)^{k+1}/k^2 = 1/1^2 - 1/2^2 + 1/3^2 - \ldots.$$

6.3.3 Even and Odd Functions

A function $g(x)$ defined on the interval $-\pi \le x \le \pi$ of length 2π is said to be *even* if the functional equality, such as,

$g(-x) = g(x)$ holds.

As a result of that the product $g(x) \times \cos kx$ is even for $\cos kx$ is even and the product $g(x) \times \sin kx$ is odd, because $\sin kx$ is odd.

Sine and Cosine Series

According to the properties just mentioned, the relations (6.33) and (6.34) become

$$a_k = 1/\pi \int_{-\pi}^{\pi} g(x) \cos kx \, dx = 2/\pi \int_{0}^{\pi} g(x) \cos kx \, dx$$

and

$$b_k = 1/\pi \int_{0}^{\pi} g(x) \sin kx \, dx = 0$$

that let us know that the Fourier series corresponding to an even function consists of terms with *cosines* only and the coefficient a_k may be required for expansion. The series in this case has symmetric property.

On the other hand if $g(-x) = -g(x)$, then the function $g(x)$ is said to be *odd*.

Accordingly, the integrand $g(x) \cos kx$ is an odd function and the integrand $g(x) \sin kx$ is even.

Therefore,

$$a_k = 1/\pi \int_{-\pi}^{\pi} g(x) \cos kx \, dx = 0$$

and

$$b_k = 1/\pi \int_{-\pi}^{\pi} g(x) \sin kx \, dx = 2/\pi \int_{0}^{\pi} g(x) \sin kx \, dx,$$

so that we can conclude that the Fourier series corresponding to an odd function, has *only sine* terms and the coefficient b_k may be required for computation. The series representing an odd function has skew-symmetric property.

Example 6.25 Expand $g(x) = \sin x$, when $g(x)$ is even periodic character on the open interval $0 < x < \pi$.

We have already observed that a Fourier series consisting of cosine terms alone is represented only for an even function, that means,

$$g(-x) = g(x).$$

The series converges to $g(x) = \sin x$ everywhere on $(0, \pi)$.

As $\sin x$ is even we can write for the coefficients

$$b_k = 0$$

and
$$a_0 = 2/\pi \int_0^{\pi} \sin kx\ dx = 4/\pi\ .$$

Now for the coefficient a_k, we have

$$a_k = 2/\pi \int_0^{\pi} \sin x \cos kx\ dx.$$

Applying the trigonometric identity for the expression $\sin x \cos kx$ at this point yields

$$a_k = 1/\pi \int_0^{\pi} \left[\sin (x + kx) + \sin (x - kx)\right] dx$$

$$= 1/\pi \left[1/(k - 1)\cos (k - 1)x - 1/(k + 1) \cos (k + 1)x\right]_0^{\pi}$$

$$= -\cos(k\pi + 1) / \pi \left[1/(k - 1) - 1/(k + 1)\right]$$

$$= -2(1 + \cos k\pi)/(\pi(k^2 - 1)), \quad \text{for } k \# 1.$$

For control if we put $k = 0$ it follows the same computation we have already found for a_0.
For $k = 1$ we have

$$a_1 = 2/\pi \int_0^{\pi} \sin x \cos kx\ dx$$

$$= 1/\pi \int_0^{\pi} \sin 2x\ dx - 1/2\pi\left[\cos 2x\right]_0^{\pi} = -1/2\pi[1 - 1] = 0\,.$$

Hence, $$g(x) = 2/\pi - 2/\pi \sum_{k=2}^{\infty}[(1 + \cos k\pi)/(k^2 - 1) \cos kx]$$

$$= 2/\pi - 4/\pi[(\cos 2x)/3 + (\cos 4x)/15 + (\cos 6x)/35 + \ldots]\,.$$

The Graphs of three partial sums for the Fourier series when computed are sketched together with the original function. The successive partial sums of the series

$$S_1 = 2/\pi,$$

$$S_2 = 2/\pi - 4/\pi\ (\cos 2x)/3,$$

$$S_3 = 2/\pi - 4/\pi\ [(\cos 2x)/3 + (\cos 4x)/15],$$

have been plotted here in order to have some ideas of the functions.

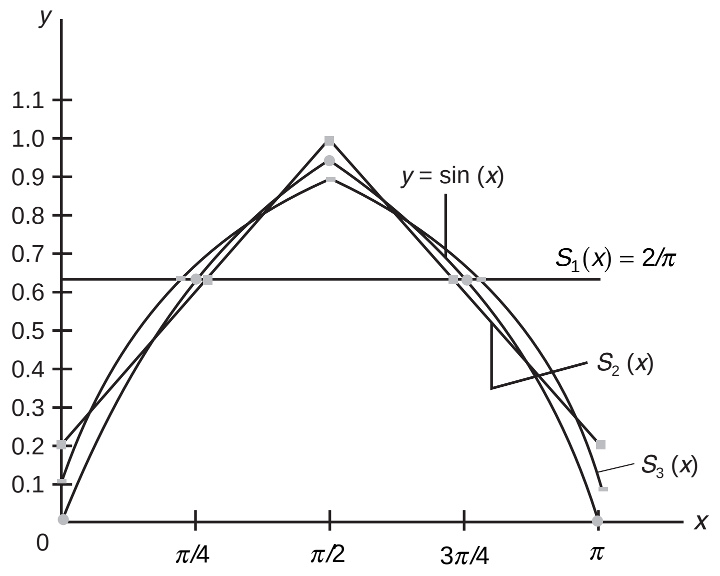

Fig. 6.2

6.3.4 Any Arbitrary Interval [–1, 1]

We have discussed so far the properties of Fourier series in standard form, where the considered function $g(x)$ is defined on the interval $-\pi \le x \le \pi$.

Sometimes it is advantageous to modify the form of a Fourier series to a function $g(x)$, as required in many problems of mathematics and physics. For that purpose we consider the interval [–1, 1], that is arbitrary, 1 being and real number.

Introducing a new variable λ and substituting $\lambda = \pi x/1$ yields the transformed function $G(\lambda)$ as follows:

$$g(x) = g(\lambda 1/\pi) = G(\lambda)\,,$$

so that λ varies in $[-\pi, \pi]$ as x varies in [–1, 1].

The new transformed function $G(\lambda)$ satisfies also the Dirichlet conditions if the assumption for convergence in the case of the standard function $g(x)$ is valid. Hence, we can expand the function $G(\lambda)$ in the form of Fourier series.

$$G(\lambda) = a_0/2 + \sum_{k=1}^{\infty}[a_k \cos(k\lambda) + b_k \sin(k\lambda)]$$

with the corresponding coefficient integrals

$$a_k = 1/\pi \int_{-\pi}^{\pi} G(\lambda)\cos(k\lambda)\, d\lambda$$

and

$$b_k = 1/\pi \int_{-\pi}^{\pi} G(\lambda)\sin(k\lambda)\, d\lambda\,.$$

Now we can define a function $g(x)$ on the interval $-1 \le x \le 1$ as follows

$$g(x) = a_0/2 + \sum_{k=1}^{\infty}[a_k \cos(k\pi x/1) + b_k \sin(k\pi x/1)]$$

with the corresponding coefficient integrals with respect to x,

$$a_k = 1/1 \int_{-1}^{1} g(x) \cos(k\pi x/1)\, dx \quad \ldots (6.37)$$

and

$$b_k = 1/1 \int_{-1}^{1} g(x) \sin(k\pi x/1)\, dx. \quad \ldots (6.38)$$

Example 6.26 Find the Fourier series corresponding to the function

$$g(x) = 3, \quad \text{for} \quad 0 < x < 5$$
$$= -3, \quad \text{for} -5 < x < 0.$$

First of all, we want to determine the coefficient a_0, a_k and b_k of the series. It follows from the equation (6.37)

$$a_k = 1/1 \int_{-1}^{1} g(x) \cos(k\pi x/1)\, dx$$

that can be transformed into another form considering the interval (–5, +5)

$$= 1/5 \int_{-5}^{5} g(x) \cos(k\pi x/5)\, dx$$

$$= 1/5 \left[\int_{-5}^{0} -3\cos(k\pi x/5)\, dx + \int_{0}^{5} 3\cos(k\pi x/5)\, dx\right]$$

$$= 3/5 \left[-\int_{-5}^{0} \cos(k\pi x/5)\, dx + \int_{0}^{5} \cos(k\pi x/5)\, dx\right]$$

$$= 3/(k\pi) \left[-\sin(k\pi x/5) \Big|_{-5}^{0} + \sin(k\pi x/5)\Big|\right]_0^5$$

$$= 0, \quad \text{for } k \neq 0.$$

It is to be noted if $k = 0$, then by (6.31) we get

$$a_0 = 1/1 \int_{-1}^{1} g(x)\, dx$$

$$= 1/5 \int_{-5}^{5} g(x)\, dx = 1/5 \int_{-5}^{0} -3\, dx + 1/5 \int_{-5}^{5} 3\, dx = 0.$$

Similarly, from the equation (6.38) the coefficient b_k gives rise to

$$b_k = 1/1 \int_{-1}^{1} g(x) \sin(k\pi x/1)\, dx = 1/5 \int_{-5}^{5} g(x) \sin(k\pi x/5)\, dx$$

$$= 1/5 \left[\int_{-5}^{0} -3 \sin(k\pi x/5)\, dx + \int_{0}^{5} 3 \sin(k\pi x/5)\, dx \right]$$

$$= 3/(k\pi) \left[\cos(k\pi x/5) \Big|_{-5}^{0} - \cos(k\pi x/5) \Big| \right]_0^5$$

$$= 6/(k\pi)\,[1 - \cos k\pi].$$

Considering the values of the coefficients a_0, a_k and b_k the corresponding Fourier series becomes

$$g(x) = a_0/2 + \sum_{k=1}^{\infty} [a_k \cos(k\pi x/1) + b_k \sin(k\pi x/1)]$$

$$= \sum_{k=1}^{\infty} 6/(k\pi)\,(1 - \cos k\pi) \sin(k\pi x/5)$$

$$= 12/\pi\,(\sin \pi x/5 + 1/3 \sin 3\pi x/5 + 1/5 \sin 5\pi x/5 + \ldots).$$

To compute the approximated solutions of Fourier series let us introduce now the program g6FOUSER that performs the Fourier series with the corresponding expansion of the coefficient function a_k and b_k for a given set of data.

The following examples have been treated by using the program and the numerical results are represented in tabular form. The graphical figures in connection with the approximations are accordingly plotted.

Example 6.27 $Fx = x \times \cos(x)$ on (0, 1)
$m = 5, \quad n = 4, \quad x = 0.1 \times 1$

Example 6.28 $Fx = \cos(2 \times x)$ on (0, 1)
$m = 5, \quad n = 3, \quad x = 0.1 \times 1$

Example 6.29 $Fx = \cos(pi \times x) - 2 \times \sin(pi \times x)$ on (0, 1)
$m = 5, \quad n = 4, \quad x = 0.1 \times 1$

Example 6.30 $Fx = x \times x \times \cos(x)$ on (0, 1)
$m = 5, \quad n = 4, \quad x = 0.1 \times 1.$

6.4 Related Software for the Solution

I have discussed in this chapter the approximation of functions that are simple. Among simple functions there are polynomials, geometric and harmonic progressions and trigonometric polynomials. The particular trigonometric series has been used for the formation of a Fourier series, which is without any doubt an indispensable tool in modern mathematical physics, especially in the branch of quantum mechanics.

The program which have been designed for the purpose have been written in C++ (Turbo) language. The programs used to approximate the series, namely, analytic numerical series, the power series and the Fourier series with appropriate methods have been so constructed that the approximated series have the field of error within the extent of reasonable tolerance.

The program `g6FUNCSR` has been written for the sake of approximated summation of the infinite numerical series in analytic form. It enables us to obtain the calculated sum of the series accepting the arbitrary choice of finite number of terms of the series. In section (6.1) we have considered the subject matter: Analytic numerical series by taking over some examples which explain the tests, such as, comparison, ratio-test, etc.

The discussion on the power series solution has been elaborated on the basis of Taylor's formula and Maclaurin's expansion method. First order and Second order initial value problems have also been solved by means of power series. The programs `g6POW1HM` and `g6POW2HM` have been thought for the purpose.

For Fourier series expansion and Fourier series solution of particular function we have accepted specified trigonometric polynomial, rather to say, a special trigonometric series. The constant or unknown coefficient functions: $a_0, a_1, a_2, \ldots b_1, b_2, \ldots$ of the series have been determined and on substitution of these determinations into the series we obtain the Fourier series solution for the particular function. For the computation of approximate series solution as expressed by Fourier the program `g6FOUSER` is used.

A software supplement consisting of a set of programs and subprograms designed in C++ (Turbo) language is prepared. A Diskette (3.5 inch/1.44 MB) in standard PC-compatible containing this supplement will be provided for the reader to work out the related mathematical models.

7

Boundary-Value Problems for Ordinary Differential Equations

7.1 Introduction

In the previous chapters we have discussed on the solutions of differential equations that are of first order and contain one initial condition to satisfy. Later in Chapter 5 the developed technique for solution would be extended to second-order equations, but using the specified conditions at the endpoints of an interval. These are all the examples of initial value problems.

In this chapter we are concerned only with the differential equations in which the specific conditions are given at different points on the interval. Contrary to this in the case of first-order initial value problem only one condition, i.e., the starting value is chosen and the approximation process has been developed.

The differential equations having the prescribed conditions at the endpoints have often been encountered in many branches of physics and mathematics, engineering and other related practical sciences. These problems cannot be treated as initial value problems because of the specified boundary. Such problems are known as the *boundary value problems*.

In order to solve a boundary value problem the general solution of the given differential equation will be first found and then a system of equations can be obtained from the prescribed boundary conditions to determine the constants involved in the equations. The general solution of the boundary value problem is thus obtained after evaluating the constants.

Let us consider a two-point value problem

$$y'' = g(x, y, y') \qquad \dots (7.1)$$

for $a \le x \le b$, a and b being the end points. The prime denotes differentiation with respect to the independent variable x, as usual. We accept the boundary conditions prescribed for the equation (7.1) by the relations

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

for some constant values of α and β.

Such a problem can be solved provided that the function g and its partial derivatives with respect to y and y' are continuous and the boundary conditions exist.

Thus, assuming that the boundary value problem has a unique solution we try to determine the solution of the problem.

Example 7.1 Find the solution of the equation $y'' + y = 1.$

satisfying the boundary conditions $y(0) = y(\pi/2) = 0.$

The general solution of the above differential equation can be written as

$$y(x) = C_1 \sin x + C_2 \cos x + 1.$$

Applying the mentioned conditions, that means $y(0) = 0$ and $y(\pi/2) = 0.$

we obtain $C_1 = C_2 = -1.$

Hence, the function $y(x) = 1 - \sin x - \cos x$ satisfies the boundary conditions.

The numerical methods for the approximation of the solution to boundary value problems of order 2 can be classified fundamentally in the following two types: The Shooting method and the Finite Difference method.

The Shooting method is not so stable because of round-off error instability often occurred in the procedure. The method I want to present here for solving the second-order boundary value problems is the method having much stable character. It needs more functional operations in order to get proper accuracy. The method containing finite differences replace each of the derivatives in the differential equation with the appropriate difference-quotient approximation.

7.2 Linear Boundary Value Problems

Assuming that we have a linear boundary value equation of second order with conditions specified at the endpoints on a closed interval $[a, b]$ which we divide into $m + 1$ subintervals, equally placed, of width $u = (b - a)/(m + 1)$.

Setting the endpoints $x_0 = a$ and $x_m = b$

we define

$$x_k = x_0 + ku$$
$$k = 0, 1, 2, \ldots, m + 1.$$

Now the dependent variable y has the corresponding values at the points

$$y_k = y(x_0 + ku),$$

for $k = 0, 1, \ldots, m.$

To solve a boundary value problem by Finite Difference method the derivatives in the differential equation should, in principle, be replaced by the appropriate difference-quotient approximations, for example,

$$y'(x) \approx [y(x_0 + u) - y(x_0)]/u$$

eliminating the errors.

A linear two-point boundary value problem for the unknown function y can be presented by

$$y'' = p(x)y' + q(x)y + r(x). \qquad \ldots (7.2)$$

for $a \le x \le b,$

with the boundary conditions $y(a) = \alpha,\ y(b) = \beta$.

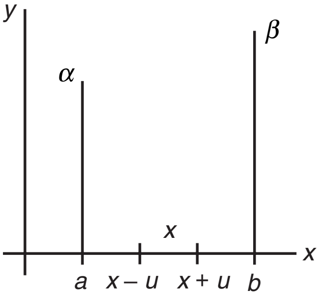

Fig. 7.1

If $x - u$, x and $x + u$ are the points in the close interval $[a, b]$ as shown in the representation, then we can write for the first order and second order derivatives

$$y'(x) = \frac{y(x+u) - y(x-u)}{2u} = \frac{y(x_{k+1}) - y(x_{k-1})}{2u} \qquad \ldots (7.3)$$

and

$$y''(x) = \frac{y(x+u) - 2y(x) + y(x-u)}{u^2} = \frac{y(x_{k+1}) - 2y(x_k) + y(x_{k-1})}{u^2} \qquad \ldots (7.4)$$

which are called *finite difference* approximations to the derivatives.

Choosing an integer m we subdivide the interval $[a, b]$ into $m + 1$ equal parts accounting the endpoints as the mesh points

$$x_k = a + ku$$

for $k = 0, 1, \ldots, m + 1$ and $u = (b - a)/(m + 1)$.

At the interior mesh point x_k, for $k = 1, 2, \ldots, m$, the differential equation (7.2) to be approximated becomes

$$y''(x_k) = p(x_k)y'(x_k) + q(x_k)y(x_k) + r(x_k). \qquad \ldots (7.5)$$

Using a third-degree Taylor polynomial

$$y(x_{k+1}) = y(x_k + u)$$

$$= y(x_k) + uy'(x_k) + u^2/2!\, y''(x_k) + u^3/3!\, y'''(x_k) + u^{(4)}/4!\, y^{(4)}(\xi_k^+)$$

for some ξ_k^+ in (x_k, x_{k+1})

and

$$y(x_{k-1}) = y(x_k - u) = y(x_k) - u\,y'(x_k) + u^2/2!\, y''(x_k) - u^3/3!\, y'''(x_k) + u^{(4)}/4!\, y^{(4)}(\xi_k^-)$$

for some ξ_k^- in (x_{k-1}, x_k)

Adding the equations determined for $y(x_k + u)$ and $y(x_k - u)$ and eliminating the terms involving $y'(x_k)$ and $y''(x_k)$ yields

$$y''(x_k) = [y(x_{k+1}) - 2y(x_k) + y(x_{k-1})]/u^2 - u^2/4!\,[y^{(4)}(\xi_k^+ + \xi_k^-)]$$

$$= [y(x_{k+1}) - 2y(x_k) + y(x_{k-1})]/u^2 - u^2/12\,[y^{(4)}(\xi_k)] \qquad \dots (7.6)$$

for some ξ_k in (x_{k-1}, x_{k+1})

Similarly, the *centered difference* formula for $y'(x_k)$ can be obtained

$$y'(x_k) = [y(x_{k+1}) - y(x_{k-1})]/(2u) - u^2/3!\,[y'''(\eta_k)] \qquad \dots (7.7)$$

for some η_k in (x_{k-1}, x_{k+1}).

Putting the formulae (7.6) and (7.7) in equation (7.5) results

$$[y(x_{k+1}) - 2y(x_k) + y(x_{k-1})/u^2 = p(x_k)\,[y(x_{k+1}) - y(x_{k-1})]/(2u) + q(x_k)\,y(x_k)$$
$$+ r(x_k) - u^2\,[2p(x_k)\,y^{(3)}(\eta_k) - y^{(4)}(\xi_k)]/12\,.$$

This equation can be applied together with the boundary conditions

$$y(a) = \alpha,\ y(b) = \beta,$$

to define $s_0 = \alpha$, $s_{m+1} = \beta$,

and $$(2s_k - s_{k+1} - s_{k-1})/u^2 + p(x_k)\,[s_{k+1} - s_{k-1}]/(2u) + q(x_k)\,s_k = -r(x_k)$$
for $k = 1, 2, \dots, m$. The error terms are omitted.

Rearranging the equation we obtain finally

$$-[1 + u\,p(x_k)/2]\,s_{k-1} + [2 + u^2\,q(x_k)]\,s_k - [1 - u\,p(x_k)/2]\,s_{k+1} = -u^2\,r(x_k). \qquad \dots (7.8)$$

This is the so long required equation for the approximation which can be simplified with an example.

The equation (7.8) can be rewritten in a system of equations and the resulting system can be expressed in matrix form

$$As = b,$$

where A is a coefficient matrix and b is a column matrix for the right side. By means of the tridiagonal matrix the nonlinear system should have a unique solution on condition that p, q and r are continuous functions on $[\alpha, \beta]$.

Now the coefficient matrix A with its elements can be expressed as

$$A = \begin{bmatrix} 2 + u^2\,q(x_1) & -1 + u\ p(x_1)/2 & 0 & \dots & 0 \\ -1 + u\ p(x_2)/2 & 2 + u^2\,q(x_2) & -1 + u\ p(x_2)/2 & \dots & 0 \\ 0 & 0 & 0 & \dots & -1 + u\ p(x_{m-1})/2 \\ 0 & 0 & -1 + u\ p(x_m)/2 & \dots & 2 + u^2\,q(x_m) \end{bmatrix}$$

and the column matrices s and b take the forms

$$s = \begin{bmatrix} s_1 \\ s_2 \\ \dots \\ s_{m-1} \\ s_m \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} u^2\,r(x_1) + (1 + u\ p(x_1)/2)\ s_0 \\ -u^2\,r(x_2) \\ \dots \\ -u^2\,r(x_{m-1}) \\ -u^2\,r(x_m) + (1 - [u\ p(x_m)]/2)s_{m+1} \end{bmatrix}.$$

The system has a unique solution on condition that p, q and r are continuous functions on $[\alpha, \beta]$.

Example 7.2 Solve $y'' + x^2 y + 2 = 0$

with the boundary conditions $y(-1) = y(1) = 0$ on the interval $[-1, 1]$.

To solve the boundary value problem we use the method of finite differences.

Now dividing the interval into 4 equal parts we have

$$n = 4,\ u = (b - a)/(n + 1) = 0.4, \text{ the endpoints } y_0 = 0 \text{ and } y_4 = 0.$$

Setting up the system of equations (7.3) and (7.4) we can write

$$y'(x) = \frac{y(x+u) - y(x-u)}{2u} = \frac{y(x_{k+1}) - y(x_{k-1})}{2u} \quad \ldots (7.9)$$

$$y''(x) = \frac{(x+u) - 2y(x) + y(x-u)}{u^2} = \frac{y(x_{k+1}) - 2y(x_k) + y(x_{k-1})}{u^2}. \quad \ldots (7.10)$$

Putting $k = 1, 2, 3$ in order and manipulating the equations we obtain a system of another equations.

When $k = 1$, $\quad (y_2 - 2y_1 + y_0)/w^2 + x_1^2\, y_1 + 2 = 0$

or $\quad (y_2 - 2y_1 + y_0) + 0.0256\, y_1 + 0.32 = 0$

or $\quad y_2 - 1.9744\, y_1 + y_0 = -0.32. \quad \ldots (7.11)$

When $k = 2$, $\quad (y_3 - 2y_2 + y_1)/w^2 + x_2^2\, y_2 + 2 = 0$

or $\quad y_3 - 2y_2 + y_1 + 0.1024\, y_2 + 0.32 = 0$

or $\quad y_3 - 1.8976\, y_2 + y_1 = -0.32 \quad \ldots (7.12)$

When $k = 3$, $\quad (y_4 - 2y_3 + y_2)/w^2 + x_3^2\, y_3 + 2 = 0$

or $\quad y_4 - 2y_3 + y_2 + 0.2304\, y_3 + 0.32 = 0$

or $\quad y_4 - 1.7696\, y_3 + y_2 = -0.32. \quad \ldots (7.13)$

Solving the equations (7.11), (7.12) and (7.13) yields

$$y_1 = 0.5685, \quad y_2 = 0.8026, \quad y_3 = 0.6344$$

which are the approximations along with $y_0 = y_4 = 0$.

Let us now consider the approximations of linear boundary value problems by means of C++ program applying the method of Finite-Difference using Taylor series.

Now to approximate the boundary value problem of order 2, the program `g7BVP2LI` is used applying the method of Finite Differences. For the approximation of a boundary value problem processed by the computer program the following points should be noted:

(a) The initial boundary conditions of the mathematical problem, xB/xF/yB/yF are to be put in the DEFINITION Block of BC of the program.

(b) The Particular Solution for $y(x)$ given in the example is inserted in the MEMBER Function setACT.

(c) The MEMBER Function setDE is furnished with the values of P, Q and R mentioned in the example.

After the execution the program calculates simultaneously the exact values of $y(x)$ with the corresponding increments of the independent variable x, so that the errors arisen from the approximations can be found out.

Example 7.3

BVP: $D^2y/dx^2 - y(1 + \tan^2 x) = 0$

$P = 0$, $Q = 1 + \tan^2 x$, $R = 0.0$

PS: $y(x) = 1/\cos x$

I C: $y(0)$: 1, $y(1)$: $1/\cos(1)$, $0 \le x \le 1$

W: 0.01

The abbreviations BVP/PS/IC/W used in the example are self-explanatory. They are meant respectively Boundary Value Problem/Particular Solution/Initial Condition/Width.

The graphical figure can be drawn now with the aid of numerical data represented underneath. The approximate values as well as the exact solutions have been displayed separately and the development of the errors arisen from the approximations has also been plotted (Fig. 7.2).

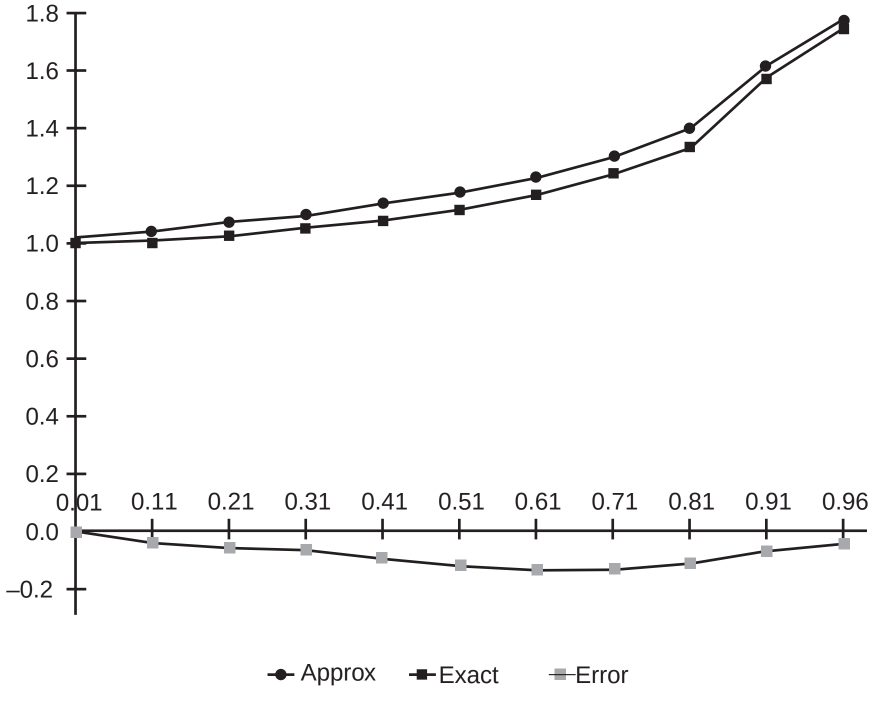

Fig. 7.2

The following table lists the approximations of the BVP of Order 2 when the given problem is processed by the computer program accepting the reasonable increments of x:

***x*-value**	**Approx. value**	**Exact value**	**Error**
$y(0.01)$:	1.00130211	1.00005000	–0.00125
$y(0.06)$:	1.00931861	1.00180270	–0.00752
$y(0.11)$:	1.01986906	1.00608065	–0.01379
$y(0.16)$:	1.03300194	1.01293797	–0.02006
$y(0.21)$:	1.04878636	1.02246257	–0.02632

(Contd)

x-value	Approx. value	Exact value	Error
y(0.26):	1.06731384	1.03477894	–0.03253
y(0.31):	1.08870066	1.05005224	–0.03865
y(0.36):	1.11309106	1.06849385	–0.04460
y(0.41):	1.14066144	1.09036888	–0.05029
y(0.46):	1.17162569	1.11600604	–0.05562
y(0.51):	1.20624227	1.14581071	–0.06043
y(0.56):	1.24482317	1.18028205	–0.06454
y(0.61):	1.28774579	1.22003589	–0.06771
y(0.66):	1.33546845	1.26583532	–0.06963
y(0.71):	1.38855104	1.31863169	–0.06992
y(0.76):	1.44768287	1.37962240	–0.06806
y(0.81):	1.51372077	1.45032962	–0.06339
y(0.86):	1.58774210	1.53271394	–0.05503
y(0.91):	1.67112019	1.62933919	–0.04178
y(0.96):	1.76563393	1.74361840	–0.02202

The error field confines to the range [–0.00125, –0.02202]

Example 7.4

BVP: $D^2y/dx^2 - \pi^2 y = -2\pi^2 \sin(\pi x)$

$P = 0,\ Q = \pi^2,\ R = -2\pi^2 \sin(\pi x)$

PS: $y(x) = \sin(\pi x)$

IC: y(x): 0, y(1): 0, $0 \le x \le 1$

W: 0.01.

The error field confines to the range [–1.29172e–06, –5.15100e-06].

Example 7.5

BVP: $x^2 y'' = \sin(\mathrm{Ln}\,x) - 2xy' + 2y$

$P = -2/x,\ Q = 2/x^2,\ R = \sin(\mathrm{Ln}\,x)/x^2$

PS: $y = 1.139207x - 0.039207/x^2 - [3\sin(\mathrm{Ln}\,x) - \cos(\mathrm{Ln}\,x)]/10$

IC: y(1): 1, y(2): 2, $1 \le x \le 2$

W: 0.01.

x-value	Approx. value	Exact value	Error
y(1.01):	1.00918446	1.00918450	3.87004e–08
y(1.06):	1.05536407	1.05536426	1.96107e–07
y(1.11):	1.10199104	1.10199134	3.04627e–07
y(1.16):	1.14907925	1.14907963	3.76290e–07
y(1.21):	1.19663204	1.19663246	4.19989e–07
y(1.26):	1.24464543	1.24464587	4.42363e–07

(Contd)

x-value	Approx. value	Exact value	Error
y(1.31):	1.29311051	1.29311096	4.48419e–07
y(1.36):	1.34201505	1.34201550	4.41966e–07
y(1.41):	1.39134471	1.39134514	4.25919e–07
y(1.46):	1.44108385	1.44108425	4.02526e–07
y(1.51):	1.49121614	1.49121651	3.73530e–07
y(1.56):	1.54172501	1.54172535	3.40287e–07
y(1.61):	1.59259394	1.59259424	3.03859e–07
y(1.66):	1.64380666	1.64380693	2.65078e–07
y(1.71):	1.69534734	1.69534756	2.24599e–07
y(1.76):	1.74720062	1.74720080	1.82936e–07
y(1.81):	1.79935173	1.79935187	1.40495e–07
y(1.86):	1.85178650	1.85178659	9.75947e–08
y(1.91):	1.90449138	1.90449144	5.44871e–08
y(1.96):	1.95745346	1.95745347	1.13683e–08

The error field confines to the range [3.87004e–08, 1.13683e–08].

Example 7.6

BVP: $D^2y/dx^2 - y + x + e^x + e^{-x} = 0$

$P = 0,\ Q = 1.0,\ R = -x - e^x + e^{-x}$

PS: $y(x) = x + (1 - x)\,(e^x - e^{-x})/2$

IC: y(0): 0, y(1): 1, $0 \le x \le 1$

W: 0.01.

The Computer solution of the BVP of Order 2

x-value	Approx. value	Exact value	Error
y(0.01):	0.01990001	0.01990017	1.58480e–07
y(0.06):	0.11643294	0.11643385	9.01689e–07
y(0.11):	0.20809599	0.20809755	1.56480e–06
y(0.16):	0.29497202	0.29497417	2.14999e–06
y(0.21):	0.37711940	0.37712206	2.65893e–06
y(0.26):	0.45457195	0.45457505	3.09277e–06
y(0.31):	0.52733901	0.52734246	3.45220e–06
y(0.36):	0.59540525	0.59540899	3.73737e–06
y(0.41):	0.65873047	0.65873442	3.94796e–06
y(0.46):	0.71724931	0.71725339	4.08313e–06
y(0.51):	0.77087078	0.77087493	4.14151e–06
y(0.56):	0.81947783	0.81948196	4.12122e–06
y(0.61):	0.86292668	0.86293070	4.01986e–06
y(0.66):	0.90104614	0.90104997	3.83445e–06
y(0.71):	0.93363676	0.93364032	3.56144e–06
y(0.76):	0.96046998	0.96047318	3.19672e–06

(Contd)

x-value	Approx. value	Exact value	Error
y(0.81):	0.98128701	0.98128974	2.73553e–07
y(0.86):	0.99579773	0.99579990	2.17249e–06
y(0.91):	1.00367942	1.00368092	1.50156e–06
y(0.96):	1.00457536	1.00457607	7.15995e–07

7.3 Nonlinear Equations

Earlier in section 5.4 the subject matter on the nonlinear differential equations was studied and the suitable solutions were found. These are the equations that describe the events of interaction. Such equations have been encountered casually in the real world.

The nonlinear differential equations of order 2 have been occurred in higher branches of modern physics, for example, in quantum mechanics, electrodynamics etc.

The solution to a nonlinear boundary-value problem cannot be expressed as a linear combination of the solutions which we have experienced already in the case of initial-value nonlinear problems of order 2.

The difference method for the general nonlinear boundary value problem

$$y'' = g(x, y, y')$$

for $a \le x \le b$, a and b being the endpoints,

accepting the boundary conditions

$$y(a) = \alpha \text{ and } y(b) = \beta$$

is similar to the method applied to the linear problems in section 7.2.

Here, the system of nonlinear equations of second order is dealt with. We need, in this case, the solutions to a series of initial-value problems and on that ground the requirement of an *iterative procedure* for the solution is inevitable.

As in the case of linear system the interval [a, b] is divided into n + 1 subdivisions, equally placed, having endpoints at $x_k = a + ku$, for $k = 0, 1, \ldots, m + 1$.

On the assumption that we have a fourth-order derivative for the exact solution, we can replace $y''(x_k)$ and $y'(x_k)$ in each of the equations by the appropriate finite difference approximations in order to obtain

$$[y(x_{k+1}) - 2y(x_k) + y(x_{k-1})]/u^2 = g[x_k, y(x_k), \{y(x_{k+1}) - y(x_{k-1})\}/2u - (u^2\, y^{(3)}(\gamma_k)/6)] + u^2\, y^{(4)}(\delta_k)]/12 \quad \ldots (7.14)$$

for each $k = 1, 2, \ldots m$, and for some γ_k and δ_k in (x_{k-1}, x_{k+1}).

The centered difference regulation yields the results deleting the errors and adding the given boundary conditions as in the case of linear system. Finally we get at the $m \times m$ system that is nonlinear

$$\begin{aligned} &2s_1 - s_2 + u^2\, g[x_1, s_1, (s_2 - \alpha)/(2u)] - \alpha = 0, \\ &-s_1 + 2s_2 - s_3 + u^2\, g[x_2, s_2, (s_3 - s_1)/(2u)] = 0, \qquad \ldots (7.15) \\ &\quad \ldots \\ &-s_{m-1} + 2s_m + u^2\, g[x_m, s_m, (\beta - s_{m-1})/(2u)] - \beta = 0 \end{aligned}$$

The solution to this nonlinear system of equations can be obtained by tridiagonal matrix method with the help of L-U factorisation.

The general implementation of Finite Difference method for the system of nonlinear equations can be performed using the program g7BVP2NL. The approximation to the boundary value problem of second order has been accordingly carried out with the application of the program designed in C++.

The results for approximation of a BVP would be appropriate if we follow the points mentioned below:

(a) The values of *P*, *Q* and *R* will be created in the Function DEFINITION Block placed at the end of the program.

(b) The boundary conditions of the problem, in this case the values for *x*0, *x1*, *yB*, and *yF* are to be put in the DEFINITION Block of BC of the program when called for.

Example 7.7

BVP: $Dy^2/dx^2 + y'^2/2 + 1 = 0$

$P = -y'/2, Q = 0, R = -y'^2/2 - 1$

PS: $y(x) = 2 \text{ Ln}[\cos(2x-1)(\sqrt{2}/4)/\cos(\sqrt{2}/4)]$

IC: y(0): 0, y(1): 0, $0 \le x \le 1$

W: 0.01.

The Computer Solution of the BVP of Order 2

***x*-value**	**Approx. value**	**Exact value**	**Error**
y(0.01):	0.00495	0.00516259	0.00021
y(0.06):	0.02820	0.02929116	0.00109
y(0.11):	0.04895	0.05066046	0.00171
y(0.16):	0.06720	0.06932886	0.00213
y(0.21):	0.08295	0.08534636	0.00240
y(0.26):	0.09620	0.09875509	0.00256
y(0.31):	0.10695	0.10958984	0.00264
y(0.36):	0.11520	0.11787836	0.00268
y(0.41):	0.12095	0.12364167	0.00269
y(0.46):	0.12420	0.12689430	0.00269
y(0.51):	0.12495	0.12764440	0.00269
y(0.56):	0.12320	0.12589386	0.00269
y(0.61):	0.11895	0.12163829	0.00269
y(0.66):	0.11220	0.11486700	0.00267
y(0.71):	0.10295	0.10556289	0.00261
y(0.76):	0.09120	0.09370226	0.00250
y(0.81):	0.07695	0.07925460	0.00230
y(0.86):	0.06020	0.06218223	0.00198
y(0.91):	1.04095	1.04243992	0.00149
y(0.96):	1.01920	1.01997441	0.00077

The computational diagram for the approximate values together with actual values has been plotted here for a clear review (Fig. 7.3):

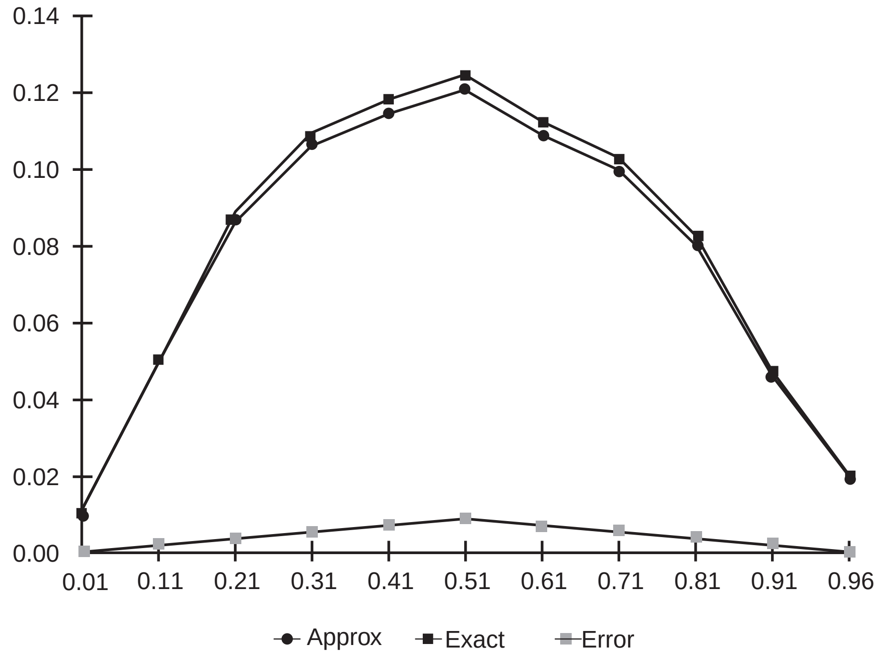

Fig. 7.3

Example 7.8

BVP: $2y'' = (1 + x + y)^3$

$P = 0$, $Q = [y^2 + 3y(1 + x) + 3(1 + x)^2]/2$, $R = (1 + x + y)^3/2$

PS: $y(x) = 2/(2 - x) - x - 1$

IC: $y(0)$: 0, $y(1)$: 0, $0 \le x \le 1$

W: 0.01.

The following is the table that lists the results when the problem is processed by the Computer program accepting only the first twenty increments of *x* beginning with *W* = 0.01. The calculated errors caused by the approximate are also tabulated.

***x*-value**	**Approx. value**	**Exact value**	**Error**
y(0.01):	–0.00486542	–0.00497487	–0.00011
y(0.06):	–0.02841523	–0.02907216	–0.00066
y(0.11):	–0.05059544	–0.05179894	–0.00120
y(0.16):	–0.07129918	–0.07304348	–0.00174
y(0.21):	–0.09041068	–0.09268156	–0.00227
y(0.26):	–0.10780301	–0.11057471	–0.00277
y(0.31):	–0.12333531	–0.12656805	–0.00323
y(0.36):	–0.13684953	–0.14048780	–0.00364
y(0.41):	–0.14816657	–0.15213836	–0.00397
y(0.46):	–0.15708176	–0.16129870	–0.00422

(Contd)

x-value	Approx. value	Exact value	Error
y(0.51):	–0.16335944	–0.16771812	–0.00436
y(0.56):	–0.16672665	–0.17111111	–0.00438
y(0.61):	–0.16686560	–0.17115108	–0.00429
y(0.66):	–0.16340484	–0.16746269	–0.00406
y(0.71):	–0.15590862	–0.15961240	–0.00370
y(0.76):	–0.14386443	–0.14709677	–0.00323
y(0.81):	–0.12666789	–0.12932773	–0.00266
y(0.86):	–0.10360489	–0.10561404	–0.00201
y(0.91):	–0.07382989	–0.07513761	–0.00131
y(0.96):	–0.03634005	–0.03692308	–0.00058

The error field confines to the range [–0.00011, –0.00058].

Example 7.9

BVP: $y'' - 2y' + y'^2 - e^y = -2 - e^x(\cos x + \sin x)$

$P = 2 - y'$, $Q = 0.0$, $R = 2y' - y'^2 - 2 + e^y - e^x(\cos x + \sin x)$

PS: $y(x) = \text{Ln}\,[e^x(\cos x + \sin x)]$

IC: y(0): 0, y(π/2): π/2, $0 \le x \le \pi/2$

W: 0.01570796.

The computer solution of the second order BVP in tabular form considering only the first ten values of the approximation:

x-value	Approx. value	Exact value	Error
y(0.01570796):	0.02931292	0.03117173	0.00186
y(0.17278759):	0.31959357	0.31865125	0.00094
y(0.32986722):	0.60084781	0.56888631	0.03196
y(0.48694685):	0.86652117	0.78830654	0.07821
y(0.64402648):	1.10873878	0.98057363	0.12817
y(0.80110611):	1.31848876	0.14755633	0.17093
y(0.95818574):	1.48600655	0.28975668	0.19625
y(1.11526537):	1.60142913	0.40641649	0.19501
y(1.27234500):	1.65580144	0.49535523	0.16045
y(1.42942463):	1.64253191	0.55246044	0.09007

Example 7.10

BVP: $D^2y/dx^2 - 2y^3 = 0$

$P = 0$, $Q = 0$, $R = 2y^3$

PS: $y(x) = 1/x$

IC: y(1): 1, y(2): 0.5, $1 \le x \le 2$

W: 0.01.

The computational diagram for the approximate values together with actual values has been plotted here for a clear view (Figure 7.4):

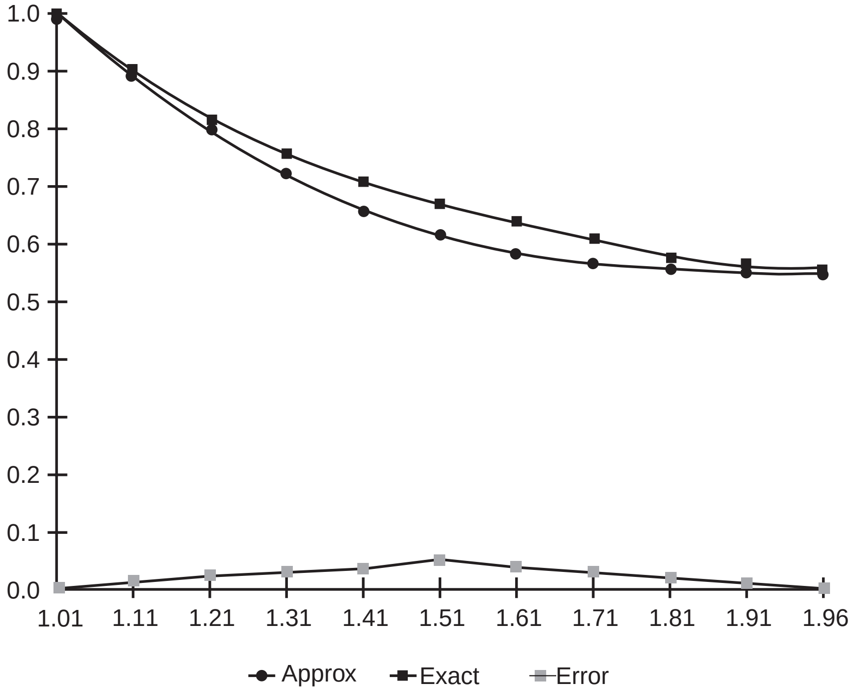

Fig. 7.4

The computer solution of the Boundary Value Problem represented in example 7.10:

***x*-value**	**Approx. value**	**Exact value**	**Error**
y(1.01):	0.98897460	0.99009901	0.00112
y(1.06):	0.93674419	0.94339623	0.00665
y(1.11):	0.88907860	0.90090090	0.01182
y(1.16):	0.84563396	0.86206897	0.01644
y(1.21):	0.80608415	0.82644628	0.02036
y(1.26):	0.77012027	0.79365079	0.02353
y(1.31):	0.73745020	0.76335878	0.02591
y(1.36):	0.70779816	0.73529412	0.02750
y(1.41):	0.68090419	0.70921986	0.02832
y(1.46):	0.65652372	0.68493151	0.02841
y(1.51):	0.63442706	0.66225166	0.02782
y(1.56):	0.61439898	0.64102564	0.02663
y(1.61):	0.59623823	0.62111801	0.02488
y(1.66):	0.57975703	0.60240964	0.02265
y(1.71):	0.56478065	0.58479532	0.02001
y(1.76):	0.55114692	0.56818182	0.01703
y(1.81):	0.53870576	0.55248619	0.01378
y(1.86):	0.52731872	0.52763441	0.01032
y(1.91):	0.51685850	0.52356021	0.00670
y(1.96):	0.50720848	0.51020408	0.00300

Example 7.11

BVP: $D^2y/dx^2 + y'^2 + y = \text{Ln } x$

$P = -y', Q = -1, R = \text{Ln } x - y'^2 - y$

PS: $y(x) = \text{Ln } x$

IC: y(1): 0, y(2): 0.693147, $1 \le x \le 2$

W: 0.01.

The error field confines to the range [–0.00073, 0.00038].

Example 7.12

BVP: $8\, d^2y/dx^2 + y\, dy/dx = 32 + 2x^3$

$P = -y/8, Q = -y'/8, R = (32 + 2x^3 - yy')/8$

PS: $y(x) = x^2 + 16/x$

IC: y(1): 17, y(3): 14.333333, $1 \le x \le 3$

W: 0.02.

The Computer Solution of the Boundary Value Problem represented in example 7.12:

***x*-value**	**Approx. value**	**Exact value**	**Error**
y(1.02):	16.72082667	16.72667451	0.00585
y(1.22):	14.57111726	14.60315410	0.03204
y(1.42):	13.29048715	13.28400563	–0.00648
y(1.62):	12.57999089	12.50094321	–0.07905
y(1.82):	12.25378237	12.10360879	–0.15017
y(2.02):	12.19748301	12.00119208	–0.19629
y(2.22):	12.34239952	12.13560721	–0.20679
y(2.42):	12.64948524	12.46797025	–0.18151
y(2.62):	13.09932687	12.97127023	–0.12806
y(2.82):	13.68588061	13.62615887	–0.05972

7.4 Related Software for the Solution

We have discussed in this chapter the method of Finite Difference for approximating the solutions to the boundary value problems. In the Finite Difference method the differentials,

$$d^2y/dx^2 \quad \text{and} \quad dy/dx$$

are replaced by difference approximations for solving the linear boundary value problem

$$d^2y/dx^2 = p(x)\, dy/dx + q(x)y + r(x),$$

for $a \le x \le b$, and with the boundary conditions $y(a) = \alpha,\ y(b) = \beta.$

Another numerical method, the Shooting method, for the approximation of the solution to the boundary value problems of order 2 is also available. But the method is not so stable because of round-off error instability often occurred.

For the nonlinear differential equations of second order we apply the same principles of the Finite Difference method with replacement.

The program g7BVP2LI written in C++ language is used to solve the linear boundary value problems and the program g7BVP2NL written in the same language is applied for solving the nonlinear boundary value problems.

8

Partial Differential Equations

8.1 Introduction

The partial differential equation is the basic tool for representing physical phenomena. Many physical problems appeared in Mathematical Physics, have more than one variable and have often been expressed by the equations involving partial differential coefficients. The boundary value problems involving partial differential equations are such type.

The aim of this chapter is to give a brief introduction to some of the basic concepts to approximate the partial differential equations having two variables with the application to particular problems in the field of physics.

A linear second-order partial differential equation is an equation of the form

$$A\, \partial^2 u/\partial x^2 + 2B\, \partial^2 u/\partial x \partial y + C\, \partial^2 u/\partial y^2 + D\, \partial u/\partial x + E\, \partial u/\partial y + Fu = G \qquad \ldots (8.1)$$

where A, B, C, D, E, F and G are functions of (x, y).

The equation becomes *homogeneous* if the quantity G on the right side is equal to zero. Denoting the partial derivatives of the equation (8.1) by subscripts we have

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0. \qquad \ldots (8.2)$$

Some frequently occurred relations that are very useful as well important to Physics are

1. $u_{xx} + u_{yy} = 0$ (Laplace equation)
2. $u_{tt} - c^2 u_{xx} = 0$ (Wave equation)
3. $u_t - C u_{xx} = 0$ (Heat equation)
4. $u_{xx} + u_{yy} = G$ (Poisson equation).

In (2) and (3) we have introduced the time function t in place of y, because of the original interpretation in Physics.

The partial differential equation (8.2) can be classified into three types, depending on the coefficient functions A, B and C:

(α) if $AC - B^2 > 0$, the equation is *elliptic*,

(β) if $AC - B^2 < 0$, the equation is *hyperbolic*,

(γ) if $AC - B^2 = 0$, the equation is *parabolic*

Considering the above relations we can conclude that the equations of Laplace and Poisson are elliptic; the wave equation is hyperbolic; and the heat equation is parabolic. The detailed study on this classification as well as general analytical theory concerning partial differential equation is beyond the scope of this book.

Among the various methods available to approximate the partial differential equations the *finite difference* method is the most common one. The principles of this method are very simple. The derivatives occurred in the differential equation and in the boundary conditions have been replaced by the appropriate difference equations.

Let us consider a rectangular *region* R in the *xy*-plane and select the positive integers m and n in order to define the step sizes $h = (b - a)/m$ and $k = (d - c)/n$. The interval $[a, b]$ on the x-axis is divided into m equal parts of width h and the interval $[c, d]$ on the y-axis is divided into n equal parts of width k.

On the region R we construct a network by the vertical and horizontal lines through the points (x_i, y_j). where $x_i = a + ih$ and $y_j = c + jk$, for $i = 0, 1, ..., m$ and $j = 0, 1, ..., n$ as shown in Fig. 8.1.

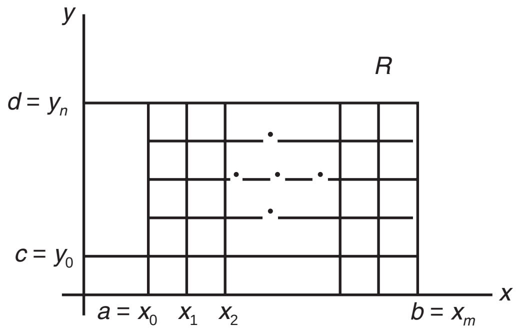

Fig. 8.1

The straight lines are known as grid lines and the intersections are the mesh points of the grid. At each interior mesh point of the grid we get the centered difference formula using Taylor polynomial

$$\partial u/\partial x = u_x = [u(x + h, y) - u(x - h, y)]/2h + E(h^2)$$

$$= [u(x_{i+1}, y_j) - u(x_{i-1}, y_j)]/2h + E(h^2)$$

$$\partial u/\partial y = u_y = [u(x, y + k) - u(x, y - k)]/2k + E(k^2)$$

$$= [u(x_i, y_{i+1}) - u(x_i, y_{i-1})]/2h + E(k^2)$$

$$\partial^2 u/\partial x^2 = u_{xx} = [u(x - h, y) - 2u(x, y) + u(x + h, y)]/h^2 + E(h^2)$$

$$\partial^2 u/\partial y^2 = u_{yy} = [u(x, y - k) - 2u(x, y) + u(x, y + k)]/\text{k}^2 + E(k^2)$$

where $E(h^2)$ and $E(k^2)$ are error terms.

Using the difference quotients in the above relations and eliminating the error terms we can write

$$u_x = (w_{i+1, j} - w_{i-1, j})/2h \qquad \dots (8.3)$$

$$u_y = (w_{i, j+1} - w_{i, j-1})/2k \qquad \dots (8.4)$$

$$u_{xx} = (w_{i-1, j} - 2w_{i, j} + w_{i+1, j})/h^2 \qquad \dots (8.5)$$

$$u_{yy} = (w_{i,\, j-1} - 2w_{i,j} + w_{j,j+1})\big/k^2 \qquad \ldots (8.6)$$

where $w_{i,j}$ approximates $u(x_i, y_j)$.

8.2 Wave Equation

The well-known Wave equation is expressed by the partial differential equation

$$\partial^2 u(x,t)/\partial t^2 - c^2\, \partial^2 u(x,t)/\partial x^2 = 0 \qquad \ldots (8.7)$$

for $0 < x < 1$ and $t > 0$,
subject to the initial conditions

$$u(x,0) = f(x) \text{ and } \partial u/\partial d\ t(x,0) = g(x),\ 0 \le x \le 1 \text{ on } [a, b],$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0, \text{ for } t > 0,$$

where c is a constant parameter.

The expression in (8.7) is an example of hyperbolic partial differential equations.

At any interior mesh point (x_i, t_j) the Wave equation becomes

$$\partial^2 u(x_i, t_j)/\partial t^2 - c^2\, \partial^2 u((x_i, t_j))/\partial x^2 = 0 \qquad \ldots (8.8)$$

for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$.

Considering the equations (8.5) and (8.6) we can write

$$u_{xx} = (w_{i-1,j} - 2w_{i,j} + w_{i+1,j})/h^2 \qquad \text{along the } x\text{-axis}$$

and

$$u_{tt} = (w_{i,j-1} - 2w_{i,j} + w_{j,j+1})/k^2 \qquad \text{along the } t\text{-axis (time).}$$

Substituting these values of u_{xx} and u_{tt} into the equation (8.8) we obtain an equation with finite differences after some manipulation

$$w_{i,j-1} - 2w_{i,j} + w_{j,j+1} = c^2k^2/h^2\,(w_{i-1,j} - 2w_{i,j} + w_{i+1,j}). \qquad \ldots (8.9)$$

Putting $\alpha = c^2k^2/h^2$ the equation (8.9) can be rewritten which directs

$$w_{j,\, j+1} = 2(1-\alpha)w_{i,\, j} + \alpha(w_{i-1,\, j} + w_{i+1,\, j}) - w_{i,\, j-1}. \qquad \ldots (8.10)$$

This equation is valid for $i = 1, 2, \ldots, m-1$ and $j = 1, 2, \ldots$, yielding the approximated boundary conditions $w_{0,j} = w_{m,j} = 0$, for $j = 1, 2, \ldots$
and the initial conditions $w_{i,0} = f(x_i)$, $i = 1, 2, \ldots, m-1$.

The equation (8.10) can be put in matrix form

$$\begin{bmatrix} w_{j,\, j+1} \\ w_{2,\, j+1} \\ \cdots \\ w_{m-1,\, j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\alpha) & \alpha & 0 & \ldots & 0 \\ \alpha & 2(1-\alpha) & 0 & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \alpha \\ 0 & \ldots & 0 & \alpha & 2(1-\alpha) \end{bmatrix} \begin{bmatrix} w_{1,\, j} \\ w_{2,\, j} \\ \cdots \\ w_{m-1,\, j} \end{bmatrix} - \begin{bmatrix} w_{1,\, j-1} \\ w_{2,\, j-1} \\ \cdots \\ w_{m-1,\, j-1} \end{bmatrix}.$$

Now replacing the initial condition

$$\partial u/\partial t\ (x,0) = g(x),\ \ 0 \le x \le 1,$$

by the difference equation we obtain

$$\partial u/\partial t = (w_{j,\,j+1} - w_{i,\,j})/k = g(x).$$

Rearranging yields $w_{j,\,j+1} = w_{i,\,j} + k\,g(x)$ at $t = 0$.

or $$w_{j,\,1} = w_{i,\,0} + k\,g(x) \quad \text{for} \quad j = 0. \qquad \ldots (8.11)$$

Again from the initial condition we have

$$w_{i,\,0} = f(x). \qquad \ldots (8.12)$$

Now, putting the value of $w_{i,\,0}$ into the equation (8.11) gives a new equation

$$w_{j,\,1} = f(x) + k\,g(x). \qquad \ldots (8.13)$$

The approximations for $w_{j,j+1}$ gives values at the $(j + 1)$th level. The values for the mesh points at the jth and $(j - 1)$th levels can be found from the relations (8.12) and (8.13), (vide Fig. 8.2).

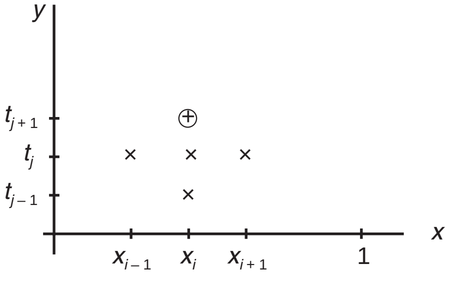

Fig. 8.2

For the computational solution of the approximations we can apply the program: g8PARWAV at this stage.

8.3 Heat Equation

We have mentioned in section 8.1 that one-dimensional heat conduction or diffusion equation is an example of the *parabolic* partial differential equations. The propagation of heat through a material and the case of unsteady flow problems concerning heat or gas direct to the parabolic equations.

To approximate the solution to the problem of heat equation we use finite differences similar to those mentioned in Section 8.1.

The one-dimensional heat equation is expressed by the partial differential relation

$$\partial u(x,t)/\partial t = C\,\partial^2 u(x,t)/\partial x^2 \qquad \ldots (8.14)$$

subject to the conditions $u(0, t) = u(1, t) = 0$, for $t > 0$,

and $u(x, 0) = f(x)$ for $0 \le x \le 1$ on $[a, b]$,

where c is a *constant* quantity, known as thermal conductivity of the material through which the heat propagates.

First of all, we select two fixed parameters h and k in the xy-plane and determine the integer $m = 1/h$.

Now, the mesh points, in this case are (x_i, t_j),
where $x_i = ih$, for $i = 0, 1, \ldots, m$ and $t_j = jk$, for $j = 0, 1, \ldots$.

The parabolic differential equation (8.14) becomes at the interior mesh point (x_i, t_j)

$$\partial u\,(x_i, t_j)/\partial t = C\,\partial^2 u((x_i, t_j))/\partial x^2 \qquad \ldots (8.15)$$

for $i = 0, 1, \ldots, m$ and $j = 0, 1, \ldots$

Replacing the partial derivatives of the equation (8.15) by finite differences yields

$$(w_{i,j-1} - w_{i,j})/k - C\,(w_{i-1,j} - 2w_{i,j} + w_{i+1,j})/h^2 = 0 \qquad \ldots (8.16)$$

where $w_{i,j}$ approximates $u(x_i, t_j)$.

Solving for the difference equation for $w_{i,j-1}$ yields

$$w_{i,j-1} = (1 - 2Ck/h^2)\,w_{i,j} + Ck/h^2\,(w_{i-1,j} + w_{i+1,j}) \qquad \ldots (8.17)$$

for $i = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots$.

The condition $u(x, 0) = f(x)$ suggests

$$w_{i,0} = f(x_i) \qquad \ldots (8.18)$$

for $i = 0, 1, \ldots, m$.

The difference equation can find the values for $w_{i,1}$, for $i = 1, 2, \ldots, m - 1$, by the use of the equation (8.18).

The boundary conditions $u(0, t) = 0$ and $u(1, t) = 0$ suggest that

$$w_{0,1} = w_{m,1} = 0.$$

In this way all the values of the form $w_{i,1}$ can be found and as a result of that the values of $w_{i,2}$, $w_{i,3}$, ... $w_{i,m-1}$ can be determined.

The equation (8.17) can be rearranged in the form

$$w_{i,j-1} = (1 - 2\lambda)\,w_{i,j} + \lambda\,(w_{i-1,j} + w_{i+1,j}) \qquad \ldots (8.19)$$

selecting a constant $\lambda = Ck/h^2$.

The resulting equation (8.19) can be solved by $(m - 1) \times (m - 1)$ matrix form

$$A = \begin{bmatrix} 1-2\lambda & \lambda & 0 & \ldots & 0 \\ \lambda & 1-2\lambda & \lambda & \ldots & 0 \\ 0 & \ldots & 0 & \ldots & 0 \\ 0 & \ldots & 0 & \ldots & \lambda \\ 0 & \ldots & \lambda & \ldots & 1-2\lambda \end{bmatrix}$$

and the approximate solution can be obtained by the relation presented now

$$A\,W_{i,j-1} = w_{i,j} \qquad \ldots (8.20)$$

Now to approximate the partial differential equation of order 2 of this type we use the program g8PARDIF written in the programming language C++ applying the method of Finite Difference.

8.4 Laplace Equation

The partial differential equation which is considered now

$$\nabla^2 u(x, y) = \partial^2 u(x, y)/\partial x^2 + \partial^2 u(x, y)/\partial y^2 = 0 \qquad \ldots (8.21)$$

is an example of the elliptic equations, known as Laplace equation. In general, the steady state flow of heat and the potential functions are expressed by the Laplace equation. Equations of this type arise mainly in the study of different time-independent physical problems.

We choose a closed interval $[a, b]$ that is partitioned into m equal parts of width $h = (b - a)/m$. Similarly, another interval $[c, d]$ in y-direction is partitioned into n equal parts of width $k = (d - c)/n$, m and n being positive integers.

At this stage we can construct a grid or network inside an assumed region R with boundary S by drawing the straight lines, vertically and horizontally through the points x_i, y_j. The intersections of the lines are the *mesh points* of the grid. In figure 8.3 we have shown the corresponding neighbouring points of an interior point.

The region R has been constrained by some conditions on the boundary S satisfying the equation (8.21). The conditions, known as *Dirichlet* conditions are represented by

$$u(x, y) = g(x, y)$$

for $$(x_i, y_j) \in S.$$

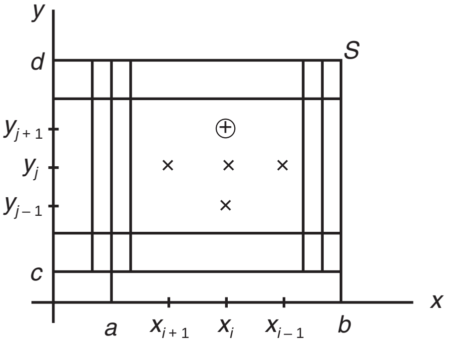

Fig. 8.3

At any interior mesh point (x_i, y_j) the Laplace equation (8.21) becomes

$$\partial^2 u(x_i, y_j)/\partial x^2 + \partial^2 u((x_i, y_j))/\partial y^2 = 0 \quad \dots (8.22)$$

which generates the centered difference formula

$$(w_{i+1,j} - 2w_{i,j} + w_{i-1,j})/h^2 + (w_{i,j-1} - 2w_{i,j} + w_{i,j+1})/k^2 = 0 \quad \dots (8.23)$$

where $i = 1, 2, \dots, m - 1$ and $j = 1, 2, \dots, n - 1$, and error terms $E(h^2 + k^2)$ are neglected.

Here, $w_{i,j}$ approximates $u(x_i, y_j)$.

The equations (8.23) can be taken for computation after rearrangement

$$2[h^2/k^2 + 1]w_{i,j} - (w_{i+1,j} - w_{i-1,j}) - h^2/k^2(w_{i,j+1} + w_{i,j-1}) = 0 \quad \dots (8.24)$$

where $i = 1, 2, \dots, m - 1$ and $j = 1, 2, \dots, n - 1$,

with the conditions

$$w_{0,j} = g(x_0, y_j), \quad w_{m,j} = g(x_m, y_j)$$
$$w_{i,0} = g(x_i, y_0), \quad w_{i,n} = g(x_i, y_n)$$

for $i = 1, 2, \dots, m - 1$ and $j = 0, 1 \dots, n$, and the function g is continuous.

Now, we are left with a linear system $(m - 1)(n - 1) \times (m - 1)(n - 1)$ presented by the relation (8.24) involving the unknown functions for matrix computation. For that purpose the interior mesh points are arranged in order labeling the points from left to right and from top to bottom by

$$T_r = (x_i, y_j) \text{ and } w_r = w_{i,j} \qquad \dots (8.25)$$

where $r = i + (m - 1 - j)$, for each $i = 1, 2, \dots, m - 1$ and $j = 1, 2, \dots, n - 1$.

For the sake of convenience accepting the equal step sizes, i.e., $k = h$ yields a reduced system

$$4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = 0. \qquad \dots (8.26)$$

We use here the reduced equation (8.26) for the solution of the partial differential equations.

Example 8.1 Solve the partial differential equation $u_{xx} + u_{yy} = 0$
for (x, y) on $R = [0 < x < 0.5,\ 0 < y < 0.5]$,
satisfying the boundary conditions given by

$$u(0, y) = 0, \qquad u(x, 0) = 0$$

and

$$u(x, 0.5) = 200\,x, \quad u(0.5, y) = 200y.$$

The differential equation (8.26) is of the form when we accept $h = k$, that means, equal step sizes

$$4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = 0$$

for each $i = 1, 2, 3$ and $j = 1, 2, 3$.

The grid has been set up with the interior points by means of the relation (8.25). The labeling is drawn as in figure 8.4, where the points T_1 through T_9 are the interior mesh points which correspond to the approximate values of w_1 through w_9.

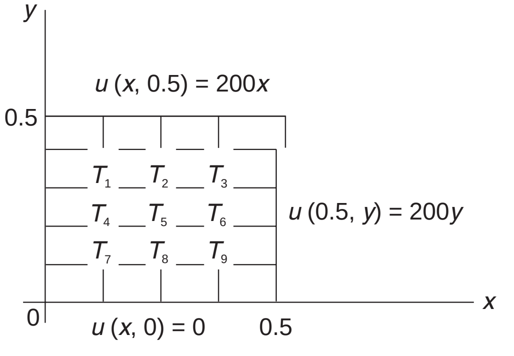

Fig. 8.4

Now, we obtain a set of simultaneous equations derived from the relation (8.26) for each interior point T_r defined in (8.25) represented by

$$T_1: 4w_1 - w_2 - w_4 - w_{0,3} - w_{1,4} = 0 \qquad (i = 1,\ j = 3)$$
$$T_2: 4w_2 - w_3 - w_1 - w_5 - w_{2,4} = 0 \qquad (i = 2,\ j = 3)$$
$$T_3: 4w_3 - w_2 - w_6 - w_{4,3} - w_{3,4} = 0 \qquad (i = 3,\ j = 3)$$
$$T_4: 4w_4 - w_5 - w_1 - w_7 - w_{0,2} = 0 \qquad (i = 1,\ j = 2)$$
$$T_5: 4w_5 - w_6 - w_4 - w_2 - w_8 = 0 \qquad (i = 2,\ j = 2)$$
$$T_6: 4w_6 - w_5 - w_3 - w_9 - w_{4,2} = 0 \qquad (i = 3,\ j = 2)$$
$$T_7: 4w_7 - w_8 - w_4 - w_{0,1} - w_{1,0} = 0 \qquad (i = 1,\ j = 1)$$
$$T_8: 4w_8 - w_9 - w_7 - w_5 - w_{2,0} = 0 \qquad (i = 2,\ j = 1)$$
$$T_9: 4w_9 - w_8 - w_6 - w_{3,0} - w_{4,1} = 0 \qquad (i = 3,\ j = 1)$$

The boundary conditions suggest that

$$w_{1,0} = w_{2,0} = w_{3,0} = w_{0,1} = w_{0,2} = w_{0,3} = 0$$
$$w_{1,4} = w_{4,1} = 25,\ w_{2,4} = w_{4,2} = 50,\ w_{3,4} = w_{4,3} = 75.$$

Substituting the values and rearranging the equations yields a 9 × 9 linear system

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{bmatrix} = \begin{bmatrix} 25 \\ 50 \\ 150 \\ 0 \\ 0 \\ 50 \\ 0 \\ 0 \\ 25 \end{bmatrix}$$

that can be solved by various methods. We accept here the method of LU-factorisation which had already been elaborately described with examples in Chapter 3. The general principle of this is the decomposition of the coefficient matrix A into L and U, where L is the *lower* triangular matrix and U is the *upper* triangular matrix.

The matrix method gives the values for w_1 through w_9 as shown

$$w_1 = 18.75,\quad w_2 = 37.50,\quad w_3 = 56.25,\quad w_4 = 12.50,\ w_5 = 25.00,$$
$$w_6 = 37.50,\quad w_7 = 6.25,\quad w_8 = 12.50,\quad w_9 = 18.75,$$

that can be well verified by comparing with the true values arisen in exact solution

$$u(x, y) = 400xy.$$

Now to approximate the partial differential equation of order 2 of Laplace type we use the program g8PARLAP designed in the programming language C++ (Turbo) applying the method of Finite Difference.

Example 8.2 Solve the Laplace equation $u_{xx} + u_{yy} = 0$
for $0 < x < 1$ and $0 < y < 1$;

$$u(0, y) = 0,\quad u(1, y) = y,\quad 0 \le y \le 1$$
$$u(x, 0) = 0,\quad u(x, 1) = x,\quad 0 \le x \le 1.$$

The exact solution is $u(x,y) = xy$.

The assumed boundary conditions applied are

$$m = 4 = n \quad \text{and} \quad h = k = 1/4 = 0.25.$$

For the execution of the program we provide for the difference equations and boundaries with the required data in order to get a clear picture of the problem as shown in figure 8.5.

The boundary values imply that

$$w_{1,0} = w_{2,0} = w_{3,0} = w_{0,1} = w_{0,2} = w_{0,3} = 0$$
$$w_{1,4} = w_{4,1} = 0.25,\ w_{2,4} = w_{4,2} = 0.50\ w_{3,4} = w_{4,3} = 0.75.$$

The figure represented here shows the interior mesh points w_1 through w_9 together with the values of the boundary conditions just mentioned.

The matrix solution yields the values for w_1 through w_9 as shown

$$w_1 = 0.1875,\ w_2 = 0.375,\ w_3 = 0.5625,\ w_4 = 0.125,\ w_5 = 0.25,$$
$$w_6 = 0.375,\ w_7 = 0.625,\ w_8 = 0.125,\ w_9 = 0.1875,$$

that can be well verified by comparing with the true values from exact solution

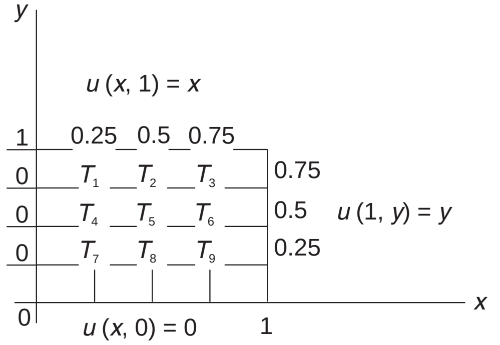

Fig. 8.5

8.5 Poisson Equation

The partial differential equation represented by

$$\nabla^2 u(x,y) = \partial^2 u(x,y)/\partial x^2 + \partial^2 u(x,y)/\partial y^2 = f(x,y) \qquad \ldots (8.27)$$

is an example of the elliptic equations, known as Poisson equation. Poisson equation occurs in the field of electricity and magnetism and also in fluid mechanics.

As in the case of Laplace equation in the previous section we set up a network on the region R surrounded by the finite intervals $[a, b]$ and $[c, d]$.

Here, on R we have $\{a < x < b\}$ and $\{c < y < d\}$ with the continuous functions

$$u(x, y) = g(x, y),$$

for $(x, y) \in S$, where S is the boundary of the region R.

As a general principle, at any interior mesh point (x_i, y_j) the Poisson equation (8.27) changes to

$$\partial^2 u(x_i, y_j)/\partial x^2 + \partial^2 u((x_i, y_j))/\partial y^2 = f(x_i, y_j) \qquad \ldots (8.28)$$

which can be approximated to $w_{i,j}$ yielding

$$(w_{i+1,j} - 2w_{1,j} + w_{i-1,j})/h^2 + (w_{i,j-1} - 2w_{i,j} + w_{i,j+1})/k^2 = f(x_i, y_j) \qquad \ldots (8.29)$$

for $i = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, n - 1$,

without considering the error terms.

Finally, we get at a useful relation rearranging the equation (8.29)

$$2[h^2/k^2 + 1]w_{i,j} - (w_{i+1,j} + w_{i-1,j}) - h^2/k^2\,(w_{i,j-1} + w_{i,j+1}) = -\,h^2 f(x_i, y_j) \qquad \ldots (8.30)$$

with the conditions

$$w_{0,j} = g(x_0, y_j),\ w_{m,j} = g(x_m, y_j)$$
$$w_{i,0} = g(x_i, y_0),\ w_{i,n} = g(x_i, y_n)$$

for $i = 1, 2, \ldots, m - 1$ and $j = 1, 2, \ldots, n$.

We have now a linear system $(m - 1)(n - 1) \times (m - 1)(n - 1)$ presented by the relation (8.30) involving the unknown functions for matrix computation. For that purpose the interior mesh points are arranged in order labeling the points from left to right and from top to bottom by

$$T_r = (x_i, y_j) \quad \text{and} \quad w_r = w_{i,j} \qquad \dots (8.31)$$

where $r = i + (m - 1 - j)$, for each $i = 1, 2, \dots, m - 1$ and $j = 1, 2, \dots, n - 1$.

For the sake of convenience accepting the equal step sizes, i.e., $k = h$ yields a reduced system

$$4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = -h^2 f(x_i, y_j) \qquad \dots (8.32)$$

For the solution of the partial differential equations of Poisson type the reduced equation (8.32) can be applied

Now to approximate the partial differential equation of order 2 of Poisson type we apply the program g8PARPOI designed in C++ by the use of the method of Finite Difference.

Example 8.3 Solve the partial differential equation

$$u_{xx} + u_{yy} = x/y + y/x$$

for $(x, y)\ R = [1 < x < 2,\ 1 < y < 2]$,

satisfying the boundary conditions given by

$$u(1, y) = y \text{ Ln } y,\ u(2, y) = 2y \text{ Ln } (2y) \ (1 \le y \le 2)$$

and $$u(x, 1) = x \text{ Ln } x,\ u(x, 2) = x \text{ Ln } (4x^2) \ (1 \le x \le 2) .$$

The exact solution $u(x, y) = xy \text{ Ln } (xy)$.

The assumed boundary conditions applied are

$$m = 4 = n \quad \text{and} \quad h = (2 - 1)/4 = 1/4 = 0.25 = k.$$

Now, we obtain a set of simultaneous equations derived from the relation (8.32) for each interior point T_r defined in (8.31) represented by

$$T_1: 4w_1 - w_2 - w_4 - w_{0,3} - w_{1,4} = -0.0625(x_i, y_j) \quad (i = 1,\ j = 3)$$
$$T_2: 4w_2 - w_3 - w_1 - w_5 - w_{2,4} = -0.0625(x_i, y_j) \quad (i = 2,\ j = 3)$$
$$T_3: 4w_3 - w_2 - w_6 - w_{4,3} - w_{3,4} = -0.0625(x_i, y_j) \quad (i = 3,\ j = 3)$$
$$T_4: 4w_4 - w_5 - w_1 - w_7 - w_{0,2} = -0.0625(x_i, y_j) \quad (i = 1,\ j = 2)$$
$$T_5: 4w_5 - w_6 - w_4 - w_2 - w_8 = -0.0625(x_i, y_j) \quad (i = 2,\ j = 2)$$
$$T_6: 4w_6 - w_5 - w_3 - w_9 - w_{4,2} = -0.0625(x_i, y_j) \quad (i = 3,\ j = 2)$$
$$T_7: 4w_7 - w_8 - w_4 - w_{0,1} - w_{1,0} = -0.0625(x_i, y_j) \quad (i = 1,\ j = 1)$$
$$T_8: 4w_8 - w_9 - w_7 - w_5 - w_{2,0} = -0.0625(x_i, y_j) \quad (i = 2,\ j = 1)$$
$$T_9: 4w_9 - w_8 - w_6 - w_{3,0} - w_{4,1} = -0.0625(x_i, y_j) \quad (i = 3,\ j = 1)$$

The boundary values suggest

$$u(0,1) = 0.2789,\ u(0,2) = 0.6082,\ u(0,3) = 0.9793,$$
$$u(1,0) = 0.2789,\ u(2,0) = 0.6082,\ u(3,0) = 0.9793,$$
$$u(4,1) = 2.2907,\ u(4,2) = 3.2958,\ u(4,3) = 4.3847,$$
$$u(1,4) = 2.2907,\ u(2,4) = 3.2958,\ u(3,4) = 4.3847.$$

The approximated solutions of the problem in the example are the following:

Table 8.1

	Approximation	*Exact solution*	*Error*
$u(1.3)$:	1.71275525	1.71228605	–0.00046920
$u(2.3)$:	2.53385541	2.53333735	–0.00051805
$u(3.3)$:	3.42801290	3.42764670	–0.00036620
$u(1.2)$:	1.17925400	1.17864214	–0.00012760
$u(2.2)$:	1.82530470	1.82459299	–0.00071172
$u(3.2)$:	2.53385541	2.53333735	–0.00051805
$u(1.1)$:	0.69784172	0.69732360	–0.00051812
$u(2.1)$:	1.17925400	1.17864214	–0.00012760
$u(3.1)$:	1.71275525	1.71228605	–0.00046920

8.6 Related Software for the Solution

In this chapter in considering the approximated solutions to partial differential equations we restrict ourselves to particular types of equations. We have chosen the Wave equation as an example of hyperbolic partial differential equation, the diffusion equation as an example of parabolic partial differential equation and the Laplace of Poisson equation as examples of elliptic partial differential equations.

Among many other methods for the solution we have discussed the method of Finite Differences thoroughly for all these types.

The program g8PARWAV has been written for the sake of approximated solution of Wave equation.

The program g8PARDIF is used for the computation of Heat or Diffusion equation.

The program g8PARLAP has been thought for the purpose of approximating the Laplace equation.

The program g8PARPOI implements the Poisson equation.

9

Laplace Transforms

9.1 Introduction

In Chapter 2 in discussing the subject matter pertaining to definite integrals we have considered the interval of integration as finite. That means, the range of the interval is bounded by finite limits provided that the function of the integrand being continuous.

If the integrand has an infinite discontinuity in the interval or the integrand approaches to infinity for some isolated values of the variable lying within the range of interval, the integral is known as Improper integral or Infinite integral.

The concept of infinite integral, such as

$$\int_0^\infty g(x)\, dx$$

subject to the condition that $g(x)$ is integrable,

has been applied to the study of Laplace transforms in solving particular types of differential and integral equations.

It is known to us that each and every transformation operates on functions to originate other functions. For example,

$$D\{g(x)\} = g'(x) \quad \ldots (9.1)$$

and

$$I\{g(x)\} = \int_0^x g(t)\, dt. \quad \ldots (9.2)$$

In (9.1) a case of differentiation that transforms the function $g(x)$ into another function $g'(x)$, the result of differentiation and in (9.2) it is an example for integration transform, naturally, on condition that the two relations satisfy the principles of transformation.

The conception of transformation, *Laplace transform* besides others, has in recent time become an essential part of discussion as the need for transformed functions has a great demand in technology and science. Laplace transforms should have an important position in mathematical environment on some reasonable grounds that I want to describe now. In Chapter 6 we have discussed on power series of the form:

$$\sum_{m=0}^{\infty} g(m)\, x^m$$

that is analogous to the improper integral in the form

$$\int_0^{\infty} g(t)\, x^t\, dt.$$

Making a simple change by replacing x^t with e^{-st} to obtain

$$\int_0^{\infty} e^{-st}\, g(t)\, dt.$$

which is, of course, the Laplace transforms of the function $g(t)$. Hence, we can conclude, by analogy principle, that the Laplace transforms are important as the power series in numerical analysis.

9.2 Laplace Transforms of Functions

The *integral transformation* can be, in general, expressed by

$$I\{g(x)\} = \int_a^b C(t,x)\, g(x)\, dx \qquad \ldots (9.3)$$

where $C(t, x)$ is the *core* function of the transformation and $g(x)$ is defined on the closed interval $[a, b]$.

Now, let us change the status of the relation (9.3) assuming a = 0, $b = \infty$ and $C(t, x) = e^{-tx}$ in order to obtain the particular form for the relation (9.3), the Laplace transformation L as defined

$$L\{g(x)\} = \int_0^{\infty} e^{-tx}\, g(x)\, dx = G(t) \qquad \ldots (9.4)$$

This equation relates that the transformation L operates on the function $g(x)$ to originate the function $G(t)$, a function with the variable t.

The improper integral in (9.4) can without any loss of generality be written as

$$\int_0^{\infty} e^{-tx}\, g(x)\, dx = \lim_{r \to \infty} \int_0^{r} e^{-tx}\, g(x)\, dx \qquad \ldots (9.5)$$

which exists only when the limiting integral on the right side exists and is said to converge to a value.

Let us consider some elementary functions for Laplace transforms.

1. $g(x) = 1$.

   Acting on the function $g(x)$ by Laplace transformation we have

$$L\{1\} = \int_0^{\infty} e^{-tx}\, (1)\, dx.$$

   When $t = 0$,

$$\int_0^{\infty} e^{-tx}\, dx = \lim_{r \to \infty} \int_0^{r} dx = \lim_{r \to \infty} |x|_0^r = \infty,$$

which is a case of divergence.

When $t \neq 0$,

$$\int_0^\infty e^{-tx}\,dx = \lim_{r\to\infty}\int_0^r e^{-tx}\,dx = \lim_{r\to\infty} -1/t\,|e^{-tx}|_0^r.$$

Case 1 If $t < 0$, i.e., $-tr > 0$, then the limit is ∞ and the integral diverges.

Case 2 If $t > 0$, i.e., $-tr < 0$, then the limit is $1/t$ and the integral converges.

Hence, $L\{1\} = 1/t$, for $t > 0$.

2. $g(x) = x$.

$$L\{x\}\int_0^\infty e^{-tx}\,x\,dx = \lim_{r\to\infty}\int_0^r e^{-tx}\,x\,dx.$$

With the help of integration by parts we have

$$L\{x\} = \lim_{r\to\infty} |-xe^{-tx}/t - e^{-tx}/t^2|_0^r = 1/t^2, \quad \text{for} \quad t > 0.$$

3. $g(x) = e^{mx}$, where m is a constant.

Now,

$$L\{e^{mx}\} = \int_0^\infty e^{-tx}\,e^{mx}\,dx = \lim_{r\to\infty}\int_0^r e^{(m-t)x}\,dx$$

$$= \lim_{r\to\infty} |e^{(m-t)x}/(m-t)|_0^r = 1/(t-m)\ (x\to\infty \text{ for } e^{-(t-m)x}\to 0).$$

The integral converges for $t > m$.

4. $g(x) = x^m$, where m is a constant.

$$L\{x^m\} = \int_0^\infty e^{-tx}\,x^m\,dx = \lim_{r\to\infty}\int_0^r e^{-tx}\,x^m\,dx$$

$$= \lim_{r\to\infty} |-x^m e^{-tx}/t|_0^r + \lim_{r\to\infty} m/t\int_0^r e^{-tx}\,x^{m-1}\,dx \quad \text{(integration by parts)}$$

$= m/t\,L\{x^{m-1}\} = m/t\,(m-1)/t\,L\{x^{m-2}\}$.

In this manner we can proceed and obtain finally

$$L\{x^m\} = m!/t^m\,L\{1\} = m!/t^{m+1}$$

substituting the value of $L\{1\}$.

The results of (1) and (2) can be verified when we take $m = 0$ and $m = 1$ in the result of (4) respectively.

5. $g(x) = \sin mx$.

Here, $$L\{\sin mx\} = \int_0^\infty e^{-tx}\sin mx\,dx = \lim_{r\to\infty}\int_0^r e^{-tx}\sin mx\,dx.$$

With the aid of integration by parts two times and useful integral relation from Calculus

$$\int e^{mx} \sin nx \, dx = e^{mx} (m \sin nx - n \cos nx)/(m^2 + n^2)$$

we can write the Laplace transform

$$L\{\sin mx\} = \lim_{r \to \infty} \left| -e^{-tx} (t \sin mx + m \cos mx)/(t^2 + m^2) \right|_0^r$$

$$= m/(t^2 + m^2), \quad t > 0.$$

6. $g(x) = \cos mx$.

By means of the principle of integration by parts and integral relation form Calculus

$$\int e^{mx} \cos nx \, dx = e^{mx} (m \cos nx + n \sin nx)/(m^2 + n^2)$$

we obtain

$$L\{\cos mx\} = \int_0^\infty e^{-tx} \cos mx \, dx = \lim_{r \to \infty} \int_0^r e^{-tx} \cos mx \, dx$$

$$= \lim_{r \to \infty} \left| -e^{-tx} (t \cos mx - m \sin mx)/(t^2 + m^2) \right|_0^r$$

$$= t/(t^2 + m^2), \quad t > 0.$$

7. $g(x) = \sinh mx$ (Hyperbolic sine function).

$$L\{\sinh mx\} = \int_0^\infty e^{-tx} \sinh mx \, dx = \lim_{r \to \infty} \int_0^r e^{-tx} (e^{mx} - e^{-mx})/2dx$$

$$=1/2\ [1/(t - m) - 1/(t + m)]$$

$$= m/(t^2 - m^2), \quad t > |m|.$$

8. $g(x) = \cosh mx$ (Hyperbolic cosine function).

$$L\{\cosh mx\} = \int_0^\infty e^{-tx} \cosh mx \, dx = \lim_{r \to \infty} \int_0^r e^{-tx} (e^{mx} + e^{-mx})/2 \, dx$$

$$= 1/2\ [1/(t - m) + 1/(t + m)]$$

$$= t/(t^2 - m^2), \quad t > |m|.$$

9. $g(x) = e^{mx} h(x)$.

$$L\{e^{mx} h(x)\} = \int_0^\infty e^{-tx} e^{mx} h(x) \, dx = \lim_{r \to \infty} \int_0^r e^{-(t-m)x} h(x) \, dx$$

$$= G(t - m).$$

This is known as the *shifting formula* that yields some useful results when applied to other functions.

(a) $L\{e^{mx} x^n\} = n!/(t - m)^{n+1}$.

(b) $L\{e^{mx} \sin nx\} = n/[(t - m)^2 + n^2]$.

(c) $L\{e^{mx} \cos nx\} = (t - m)/[(t - m)^2 + n^2]$.

9.3 Evaluation of Derivatives and Integrals

Let us consider the generalised Laplace transform

$$L\{g(x)\} = \int_0^\infty e^{-tx}\, g(x)\, dx = G(t) \qquad \ldots (9.6)$$

where G is a function with the parameter t.

The differentiation of the equation (9.6) with respect to t under the integral sign applying the Leibnitz's rule yields

$$dG/dt = G'(t) = \int_0^\infty e^{-tx}\, (-x)\, g(x)\, dx = -\int_0^\infty e^{-tx}\, x\, g(x)\, dx \qquad \ldots (9.7)$$

or
$$-L\{x\, g(x)\} = G'(t)$$

or
$$L\{x\, g(x)\} = -G'(t). \qquad \ldots (9.8)$$

Differentiating (9.7) we get after rearrangement

$$L\{x^2\, g(x)\} = G''(t).$$

Finally, on repeated differentiation a generalised formula can be found

$$L\{x^m\, g(x)\} = (-1)^m\, G(m)\, (t) \qquad \ldots (9.9)$$

for any positive m.

The formula (9.9) can well be applied in finding the Laplace transforms of functions of type $x^m\, g(x)$ when $G(t)$ is known.

Example 9.1 Find $L\{x^3\, e^{4x}\}$.

From (3) we can deduce $L\{e^{4x}\} = 1/(t - 4)$.

Applying (9.9) we have

$$L\{x^3\, e^{4x}\} = (-1)^3\, d^3/dt^3\, (1/(t - 4) = 6/(t - 4)^4.$$

Example 9.2 Find $L(x^2 \cos mx\}$.

Since $L\{\cos mx\} = t/(t^2 + m^2)$, we have

$$L\{x^2 \cos mx\} = (-1)^3\, d^2\, /dt^2\, (t/(t^2 + m^2)) = 2t/(t^2 - 3m^2)/(t^2 + m^2)^3.$$

Example 9.3 Find $L\{x \sin mx\}$.

Since $L\{\sin mx\} = m/(t^2 + m^2)$, we have

$$L\{x \sin mx\} = -d/dt\, (m/(t^2 + m^2)) = 2mt/(t^2 + m^2)^2.$$

Example 9.4 Find $L\{x^{-1/2}\}$.

Here,
$$L\{x^{-1/2}\} = \int_0^\infty e^{-tx}\, x^{-1/2}\, dx.$$

Now we put $tx = s$ that yields

$$L\{x^{-1/2}\} = t^{-1/2} \int_0^\infty e^{-s}\, s^{-1/2}\, ds.$$

Another substitution $s = r^2$ leads to

$$L\{x^{-1/2}\} = 2t^{-1/2} \int_0^\infty e^{-r^2}\, dr. \qquad \ldots (9.10)$$

From the treatise of improper integrals we have

$$\int_0^\infty e^{-x^2}\, dx = \sqrt{\pi}/2$$

and inserting this in relation (9.10) we obtain

$$L\{x^{-1/2}\} = 2t^{-1/2} \times \sqrt{\pi}/2 = \sqrt{(\pi/t)},$$

which is a very useful result.

Example 9.5 Find $L\{x^{1/2}\}$.

Without loss of generality writing $x \times x^{-1/2}$ for $x^{1/2}$ we can find the Laplace transform applying the relation (9.9)

$$L\{x^{1/2}\} = L\{x \times x^{-1/2}\} = -\,d/dt\,(\sqrt{(\pi/t)} = \sqrt{\pi}\ t^{-3/2}/2.$$

Example 9.6 Find $L\{x^{7/2}\}$.

Considering $g(x) = x^{1/2}$ we can arrange $x^{7/2} = x^3\, x^{1/2} = x^3\, g(x)$.

Now, $L\{x^{7/2}\} = L\{x^3 \times g(x)\} = (-1)^3\ d^3/dt^3\,(\sqrt{\pi}\ t^{-3/2}/2) = 105\sqrt{\pi}\ t^{-9/2}/16.$

Example 9.7 Find $L\{x \cosh 3x\}$.

We know that $L\{\cosh 3x\} = t/(t^2 - 9)$.

Hence, $L\{x \cosh 3x\} = -d/dt\ t/(t^2 - 9) = (t^2 + 9)/(t^2 - 9)^2$.

Now, we turn our attention to evaluate the *integrals* by Laplace transforms.

Let us consider a relation

$$L\{g(x)/x\} = \int_t^\infty G(t)\, dt. \qquad \ldots (9.11)$$

Substituting $L\{g(x)/x\} = F(t)$ and applying the relation (9.9) yields

$$F'(t) = -L\{x\ g(x)/x\} = -\,L\{g(x)\} = -\,G(t).$$

On integration we obtain

$$F(t) = -\int_c^t G(t)\, dt$$

for some c. Putting $c = \infty$ we have

$$F(t) = \int_t^\infty G(t)\, dt$$

which shows that the relation considered in (9.11) is justified. It can now be written as

$$\int_0^\infty e^{-tx}\, g(x)/x\, dx = \int_0^\infty G(t)\, dt \qquad \ldots (9.12)$$

taking simply $t \to 0$ and is valid when the integral on the left exists.

This equation is sometimes useful to evaluate the integrals by Laplace transforms.

Example 9.8

Evaluate $\int_0^\infty \sin x/x \, dx$ by Laplace transforms.

Hence, $L\{\sin x\} = 1/(t^2 + 1)$,

the relation (9.12) yields

$$\int_0^\infty e^{-tx} \sin x/x \, dx = \int_0^\infty 1/(t^2+1) \, dt = |tan^{-1} t|_0^\infty = \pi/2.$$

Example 9.9

Evaluate $\int_0^\infty x \sin x \, dx$ by Laplace transforms.

Here, $L\{\sin x\} = 1/(t^2 + 1)$,

Now, $$\int_0^\infty e^{-tx} x \sin x \, dx = L\{x \sin x\} = -d/dt \, (1/(t^2+1)$$

$$= 2t/(t^2 + 1)^2$$

$$= 1/2,$$

for $t = 1$.

9.4 Inverse Laplace Transforms

We have so far discussed on a few functions, that are involved in Laplace transforms and thereby found in each case a resulting form which can be used as a standard formula. Every formula produced in a Laplace transform can be taken as a formula for an inverse Laplace transform.

For the sake of convenience we formulate some of the inverse Laplace transforms in tabular forms. The output is nothing but the original function $g(x)$ in each case as given.

Table 9.1

Record	$G(t)$	$g(x) = L^{-1}\{G(t)\}$
1	$1/t,\ t > 0$	1
2	$1/t^2,\ t > 0$	x
3	$1/(t-m),\ t > m$	e^{mx}
4	$m!/t^{m+1},\ t > 0$	x^m
5	$m/(t^2 + m^2),\ t > 0$	$\sin mx$
6	$t/(t^2 + m^2),\ t > 0$	$\cos mx$
7	$m/(t^2 - m^2),\ t > \|m\|$	$\sinh mx$

(Contd)

Record	*G(t)*	$g(x) = L^{-1}\{G(t)\}$
8	$t/(t^2 - m^2)$, $t > \|m\|$	$\cosh mx$
9	$n!/(t - m)^{n+1}$, $t > m$	$e^{mx} x^n$
10	$n/[(t - m)^2 + n^2]$, $t > m$	$e^{mx} \sin nx$
11	$(t - m)/[(t - m)^2 + n^2]$, $t > m$	$e^{mx} \cos nx$
12	$2mt/(t^2 + m^2)^2$, $t > 0$	$x \sin mx$
13	$(t^2 - m^2)/(t^2 + m^2)^2$, $t > 0$	$x \cos mx$
14	$2m^2/[t(t^2 + m^2)^2]$ $t > 0$	$\sin^2 mx$
15	$1/(1 + mt)(1 + nt)$	$(e^{-x/m} - e^{-x/n})/(m - n)$
16	$1/(1 + mt)^2$	$(m - x) e^{-x/m}/m^3$
17	$m(t^2 + 2m^2)/(t^4 + 4m^4)$	$\sin mx \cosh mx$
18	$t^3/(t^4 + m^4)$	$\cos(mx/\sqrt{2}) \cosh(mx/\sqrt{2})$
19	$mt/(t^2 - m^2)^2$	$x \sinh mx/2$
20	$t/(1 + mt)(1 + nt)$	$(me^{-x/n} - ne^{-x/m})/[mn(m - n)$

Example 9.10 Find $L^{-1}\{2t/(t^2 + 1)^2\}$ as a function of x.

Record 12 of the table shows $G(t) = 2mt/(t^2 + m^2)^2$, $t > 0$.

When $m = 1$ we have $L^{-1}\{G(t)\} = L^{-1}\{2t/(t^2 + 1)^2\}$

$$= x \sin x.$$

Example 9.11 Find $L^{-1}\{1/(t + 3)\}$ as a function of x.

According to record 3 of the table we have $G(t) = 1/(t - m)$, $t > m$.

Putting $m = -3$ yields inverse Laplace transform

$$L^{-1}\{G(t)\} = L^{-1}\{1/(t + 3)\} = e^{-3x}.$$

Example 9.12 Find $L^{-1}\{6/(t - 9)^4\}$as a function of x.

We should make use of the record 9 to obtain

$$L^{-1}\{G(t)\} = L^{-1}\{n!/(t - m)^{n+1}\}, \quad t > m.$$

Putting $m = 9$ and $n = 3$ yields

$$L^{-1}\{G(t)\} = L^{-1}\{3!/(t - 9)^4\} = x^3 e^{9x}.$$

Example 9.13 Find $L^{-1}\{(2t - 18)/(t^2 + 9)\}$ as a function of x.

Resolving the quantity $(2t - 18)/(t^2 + 9)$ into partial fractions yields

$$(2t - 18)/(t^2 + 9) = 2t/(t^2 + 9) - 6 \times 3/(t^2 + 9).$$

With the help of the tabular records 5 and 6 we obtain

$$2L^{-1}\{t/(t^2 + 9)\} - 6L^{-1}\{3/(t^2 + 9)\} = 2\cos 3x - 6\sin 3x$$

accepting $m = 3$ in both cases.

Example 9.14 Find $L^{-1}\{(t - 3)/(t^2 - 6t + 25)\}$ as a function of x.

The expression $t^2 - 6t + 25$ is resolved into $(t - 3)^2 + 16$ in order to formulate

$$L^{-1}\{(t - 3)/[t - 3)^2 + 16]\} = e^{3x} \cos 4x$$

by means of record 11 with $m = 3$ and $n = 4$.

Example 9.15 Find $L^{-1}\{3/(t^2 + 4t + 6)\}$ as a function of x.

Now, $$t^2 + 4t + 6 = (t + 2)^2 + 2.$$

Using record 10 with $m = -2$ and $n = \sqrt{2}$ we have

$$L^{-1}\{3/(t^2 + 4t + 6)\} = 3/\sqrt{2}\ L^{-1}\{\sqrt{2}/[((t + 2)^2 + 2)]\} = 3/\sqrt{2}e^{-2x} \sin \sqrt{2}x.$$

Example 9.16 Find $L^{-1}\{(4t + 12)/(t^2 + 8t + 16)\}$ as a function of x.

Here,
$$\begin{aligned} & L^{-1}\{(4t + 12)/(t^2 + 8t + 16)\} \\ &= L^{-1}\{[4(t + 4) - 4)/(t + 4)^2\} \\ &= 4L^{-1}\{1/(t + 4)\} - 4L^{-1}\{(1/(t + 4)^2\} \\ &= 4e^{-4x} - 4x\, e^{-4x} = 4\, e^{-4x}\, (1 - x). \end{aligned}$$

Example 9.17 Find $L^{-1}\{(t + 1)/(t^2 + t + 1)\}$ as a function of x.

Here, $$(t + 1)/(t^2 + t + 1) = (t + 1/2 + 1/2)/[(t + 1/2)^2 + 3/4].$$

We can express the inverse Laplace transform now

$$L^{-1}\{(t + 1/2)/[(t + 1/2)^2 + 3/4]\} + 1/\sqrt{3}\ L^{-1}\{\sqrt{3}/2/[(t + 1/2)^2 + 3/4]\}.$$

$$= e^{-x/2} \cos \sqrt{3}\,x/2 + 1/\sqrt{3}\ e^{-x/2} \sin \sqrt{3}\,x/2$$

$$= e^{-x/2}/\sqrt{3}\ (\sqrt{3} \cos \sqrt{3}\,x/2 + \sin \sqrt{3}\,x/2).$$

9.5 Convolutions

Let us consider a function defined by

$$\theta(u) = \int_0^u g(v)\ h(u - v)\ dv \qquad \ldots (9.13)$$

The Laplace transform of the relation (9.13) yields

$$L\{\theta(u)\} = \int_0^\infty e^{-tu} \left[\int_0^u g(v)\ h(u - v)\ dv\right] du$$

$$= \int_0^\infty \int_0^u e^{-tu}\ g(v)\ h(u - v)\ dv\ du. \qquad \ldots (9.14)$$

The relation (9.14) relates that the range of integration is the extent lying between the lines $v = 0$ and $v = u$ as shown in Fig. 9.1.

Reversing the order of integration as suggested in the Figure we can write the equation (9.14) rearranged

$$L\{\theta(u)\} = \int_0^\infty \int_0^\infty e^{-tu}\ g(v)\ h(u - v)\ du\ dv$$

$$= \int_0^\infty e^{-tv}\ g(v) \left[\int_0^\infty e^{-t(u-v)}\ h(u - v)\ du\right] dv.$$

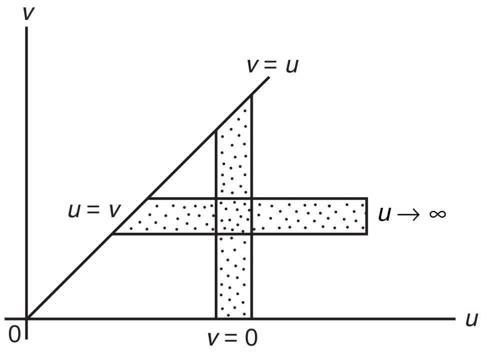

Fig. 9.1

Introducing a new variable x in the linear integral by putting $u - v = x$, so that $du = dx$ (taking v as constant) we have

$$L\{\theta(u)\} = \int_0^\infty e^{-tv}\, g(v) \left[\int_0^\infty e^{-tx}\, h(x)\, dx\right] dv$$

$$= \int_0^\infty e^{-tv}\, g(v)[L\{h(x)\}]\, dv$$

$$= L\{g(v)\} \times L\{h(x)\} = G(t)\, H(t).$$

Now, without loss of generality we can write

$$L\{\theta(u)\} = L\left\{\int_0^u g(v)\, h(u-v)\, dv\right\} = G(t)\, H(t) \qquad \ldots (9.15)$$

The integral is known as the *Convolution* of the two functions. In other words, the integral is the outcome of the generalised *product* of these functions. The convolution theorem, stating that the product of Laplace transforms of two functions is the transform of their convolution, can be well applied to find inverse transforms.

Example 9.18 Find $L^{-1}\{1/(t-1)^2\}$ as a function of x by convolution theorem.

Here, $L^{-1}\{G(t)\, H(t)\} = L^{-1}\{1/(t-1)]\, [1/(t-1)] = g(x) \times h(x)$.

The record 3 from the table gives $g(x) = e^x$ and $h(x) = e^x$.

Now, $$\int_0^x g(u)\, h(x-u)\, du = \int_0^x e^u\, e^{x-u}\, du = e^x \int_0^x du = x\, e^x.$$

Example 9.19 Find $L^{-1}\{t/(t^2+m^2)^2\}$ as a function of x by convolution.

Defining $G(t) = t/(t^2+m^2)$ and $H(t) = 1/(t^2+m^2)$ we have

$$L^{-1}\{t/(t^2+m^2)\} = g(x) = \cos mx \text{ and } L^{-1}\{1/(t^2+m^2)\} = h(x) = \sin mx/m.$$

Hence, by means of convolution theorem we obtain

$$L^{-1}\{t/(t^2+m^2)^2\} = g(x) \times h(x) = \int_0^x g(u)\, h(x-u)\, du$$

$$= \int_0^x \cos mu \sin m(x - u)/m \, du$$

$$= 1/m \int_0^x \cos mu(\sin mx \cos mu - \cos mx \sin mu) \, du$$

$$= 1/m \sin mx \int_0^x \cos^2 mu \, du - 1/m \cos mx \int_0^x \cos mu \sin mu \, du$$

$$= 1/m \sin mx[x/2 + \sin(2mx)/(4m)] - 1/m \cos mx(1 - \cos 2mx)/(4m)$$

$$= x \sin mx/(2m).$$

Example 9.20 Evaluate $L^{-1}\{1/t^2(t^2 + 3)\}$ as a function of x by convolution.

Defining $G(t) = 1/t^2$ and $H(t) = 1/(t^2 + 3)$ we have

$$L^{-1}\{1/t^2\} = g(x) = x \quad \text{and} \quad L^{-1}\{1/(t^2 + 3)\} = h(x) = (\sin \sqrt{3}\, x)/\sqrt{3}.$$

Hence, by means of convolution theorem we obtain

$$L^{-1}\{1/t^2 (t^2 + 3\} = \int_0^x (x - u)\,(\sin \sqrt{3}\, u)/\sqrt{3}\, u \, du$$

$$= -1/3\,|(x - u) \cos \sqrt{3}\, u \Big|_{u=0}^{x} - 1/3 \int_0^x \cos \sqrt{3}\, u \, du$$

$$= x/3 - 1/3\sqrt{3}\,|\sin \sqrt{3}\, u \Big|_{u=0}^{x}$$

$$= x/3 - \sin \sqrt{3}\, x/(3\sqrt{3}).$$

9.6 Application to Differential Equations

Let us consider a nonhomogeneous differential equation

$$y'' + \alpha y' + \beta y = f(x) \qquad \dots (9.16)$$

with the initial condition $y(0) = y_0$ and $y'(0) = y'_0$.

Applying the Laplace transformation to both sides of (9.16) gives

$$L\{y'' + \alpha y' + \beta y\} = L\{f(x)\}.$$

It can be written, by principle of linearity, as

$$L\{y''\} + \alpha L\{y'\} + \beta L\{y\} = L\{f(x)\}.$$

The Laplace transform of y' can be expressed using the integration by parts

$$L\{y'\} = \int_0^\infty e^{-tx} y' dx = |y\, e^{-tx} \Big|_0^\infty + t \int_0^\infty e^{-tx} y \, dx \; (\lim_{x \to \infty} y\, e^{-tx} = 0)$$

$$= -y(0) + t\, L\{y\}$$

or $$L\{y'\} = t\,L\{y\} - y(0)\,. \qquad \ldots (9.17)$$

In a similar manner we have for the second-order

$$L\{y''\} = L\{(y')'\} = t\ L\{y'\} - y'(0)$$

or $$L\{y''\} = t^2\ L\{y\} - t\ y(0) - y'(0). \qquad \ldots (9.18)$$

The expressions for $L\{y''\}$ and $L\{y'\}$ can now be inserted in the equation (9.16) together with the initial conditions in order to obtain

$$t^2 L\{y\} - t\,y_0 - y_0' + \alpha\,t\,L\{y\} - \alpha\,y_0 + \beta\,L\{y\} = L\{f(x)\}$$

that gives

$$L\{y\} = [L\{f(x)\} + (t+\alpha)\ y_0 + y_0']/(t^2 + \alpha\ t + \beta) \qquad \ldots (9.19)$$

where α, β, y_0 and y_0' are known constants and the function $f(x)$ is also known and $L\{y\}$ can be found as a function of t only.

Example 9.21 Find the solution of $y' - 5y = e^{5x}$

that satisfies the initial conditions $y(0) = 2$.

Taking the Laplace transforms of both sides of this differential equation yields

$$L\{y'\} - 5\ L\{y\} = L\{e^{5x}\}.$$

Now, the relation (9.17) that states

$$L\{y'\} = t\ L\{y\} - y(0).$$

substituting this and the initial conditions in the Laplace transforms of the equation and after rearrangement we obtain

$$t\ L\{y\} - y(0) - 5\ L(y) = L\ \{e^{5x}\} = 1/(t-5)$$

or
$$\begin{aligned} L\{y\} &= 1/(t-5)^2 + 2/(t-5) \\ &= L\{xe^{5x}\} + L\{2e^{5x}\} \\ &= L\{xe^{5x} + 2e^{5x}\}. \end{aligned}$$

Taking the inverse Laplace transforms yields

$$y = x\ e^{5x} + 2\ e^{5x} = e^{5x}\ (x+2),$$

which is the required solution of the differential equation.

We can easily verify the result accepting the differential equation as linear differential equation and applying a method described in Chapter 4. The general solution of the equation becomes

$$y\ e^{-5x} = x + c,$$

that on imposing the initial condition yields $c = 2$.

The particular solution will be $y = e^{5x}\ (x + 2)$.

From now on the reader of this book should check the results of the examples solved by the Laplace transforms.

Example 9.22 Solve the differential equation $y' + y = \sin x$

with $y(0) = 0$.

Taking the Laplace transforms of both sides of this differential equation yields

$$L\{y'\} + L\{y\} = L\{\sin x\}.$$

On application of (9.17) we obtain

$$L\{y\}\ (t + 1) = L\{\sin x\} = 1/(t^2 + 1)$$

or $$L\{y\} = 1/(t + 1)\ (t^2 + 1).$$

Taking the inverse Laplace transforms yields

$$y = (e^{-x} + \sin x + \cos x).$$

Example 9.23 Solve $y' + py = 100p$

with $y(0) = 50$ and p is a constant.

Taking the Laplace transforms of both sides of the differential equation yields

$$L\{y'\} + p\,L\{y\} = 100p\,L\{1\}.$$

On application of (9.17) we obtain

$$L\{y\}\ (t + p) = 100p/t + 50$$

or $$L\{y\} = (100p + 50t)/t(t + p)$$

Taking the inverse Laplace transforms gives

$$\begin{aligned} y &= L^{-1}\{50/(t + p) + 100p/t(t + p)\} \\ &= L^{-1}\{100/t - 50/(t + p)\} \\ &= 100 - 50\,e^{-px} = 50(2 - e^{-px}). \end{aligned}$$

Example 9.24 Solve $y'' - 3y' + 4y = 0$

that satisfies the initial conditions $y(0) = 1$ and $y'(0) = 5$.

Taking the Laplace transforms of both sides of the differential equation yields

$$L\{y''\} - 3L\{y'\} + 4L\{y\} = L\{0\},$$

so that we get from the relation (9.19) along with the related initial conditions and $\alpha = -3$, $\beta = 4$ and $f(x) = 0$

$$\begin{aligned} L\{y\} &= [L\{0\} + (t - 3) \times 1 + 5]/(t^2 - 3t + 4) \\ &= (t + 2)/(t^2 - 3t + 4) \\ &= [(t - 3/2) + 7/2]/(t - 3/2)^2 + (\sqrt{7}/2)^2]. \end{aligned}$$

The inverse Laplace transforms gives

$$\begin{aligned} y &= L^{-1}\{(t - 3/2)/[(t - 3/2)^2 + (\sqrt{7}/2)^2]\} + L^{-1}\{(7/2)/[(t - 3/2)^2 + (\sqrt{7}/2)^2]\}. \\ &= e^{3x/2}(\cos\sqrt{3}x/2 + \sqrt{7}\sin\sqrt{7}\,x/2). \end{aligned}$$

Example 9.25 Solve $y'' + 2y' + 5y = e^{-x}\sin x$ that satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

Taking the Laplace transforms of both sides of the differential equation yields

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-x}\sin x\},$$

so that $$t^2\,L\{y\} - t\,y_0 - y_0' + 2t\,L\{y\} - 2y_0 + 5L\{y\} = 1/(t^2 + 2t + 2)$$

or $L\{y\} = (t^2 + 2t + 5) - 1 = 1/(t^2 + 2t + 2)$

or $L\{y\} = (t^2 + 2t + 3)/[(t^2 + 2t + 2)\ (t^2 + 2t + 5)]$.

Hence,

$$\begin{aligned} y(x) &= L^{-1}\{(1/3)/(t^2 + 2t + 2) + L^{-1}\{(2/3)/(t^2 + 2t + 5) \\ &= 1/3\ L^{-1}\{1/[(t + 1)^2 + 1]\} + 2/3\ L^{-1}\{1/[(t + 1)^2 + 4]\} \\ &= e^{-x}\ (\sin x + \sin 2x)/3. \end{aligned}$$

Example 9.26 Solve $y'' + y = x$ with $y(0) = 1$ and $y'(0) = -2$.

Taking the Laplace transforms of both sides of the given differential equation yields

$$L\{y''\} + L\{y\} = L\{x\},$$

from which we get

$$t^2\ L\{y\} - t\ y(0) - y'(0) + L\{y\} = 1/t^2$$

or $L\{y\}\ (t^2 + 1) = 1/t^2 + t - 2$

or

$$\begin{aligned} L\{y\} &= 1/t^2(t^2 + 1) + (t - 2)/(t^2 + 1) \\ &= 1/t^2 + t/(t^2 + 1) - 3/(t^2 + 1). \end{aligned}$$

Hence,

$$\begin{aligned} y\{x\} &= L^{-1}\{1/t^2 + t/(t^2 + 1) - 3/(t^2 + 1)\} \\ &= L^{-1}\ \{1/t^2\} + L^{-1}\ \{t/(t^2 + 1)\} - L^{-1}\{3/(t^2 + 1)\} \\ &= x + \cos x - 3\sin x, \end{aligned}$$

is the required solution.

10

Problems to be Worked Out

Find the approximations of the Definite Integrals in Chapter 2 using the program g2CSINTG.CPP.

1. DI: $\int_0^1 e^{-x^2}\,dx$, ES: 0.7468241, K: 5

2. DI: $\int_0^1 (1-x)/(1+x)\,dx$, ES: 2 Ln2 – 1, K: 6

3. DI: $\int_{-10}^{-6} 1/(x+2)\,dx$, ES: Ln2, K: 5

4. DI: $\int_{1.0}^{1.6487212} x\,\mathrm{Ln}\,x\,dx$, ES: 1/4, K: 3

5. DI: $\int_0^1 x/\sqrt{(1+x^2)}\,dx$, ES: $\sqrt{2}-1$, K: 5

6. DI: $\int_1^e \mathrm{Ln}\,x\,dx$, ES: $x(\mathrm{Ln}\,x - 1)$, K: 6

7. DI: $\int_0^1 1/\sqrt{(x^4+1)}\,dx$, ES: 0.9270374, K: 5

8. DI: $\int_0^1 e^{-x}/(x+1)\,dx$, ES: 0.4634221, K: 5

9. DI: $\int_{-2}^{3} e^{-x/2}\,dx$, ES: $2(e - e^{-3/2})$, K: 5

10. DI: $\int_{0}^{1.0} \text{Ln}(x^2 + 1)\, dx$, ES: 0.2639435, K: 5

11. DI: $\int_{3}^{5} 1/\sqrt{(x^2 + 16)}\, dx$, ES: Ln 0.5, K: 4

12. DI: $\int_{-1}^{2} 1/(x^2 - 9)\, dx$, ES: Ln (0.1)/6, K: 6

13. DI: $\int_{1}^{3} \text{Ln}\,(x + \sqrt{(x^2 - 1)})\, dx$, ES: 2.458019, K: 5

14. DI: $\int_{0}^{\pi/4} \text{Ln}\,(1 + \text{Tan}\theta)\, d\theta$, ES: $\pi/8$ Ln2 , K: 5

15. DI: $\int_{0}^{1} 1/(x^2 + 1)^2\, dx$, ES: $(\pi + 2)/8$, K: 5

16. DI: $\int_{0}^{1} 1/(1 + x)\,\sqrt{(1 + 2x - x^2)}\, dx$, ES: $\pi/5.656854$, K: 6

17. DI: $\int_{0}^{1} x^6\,\sqrt{(1 + x^2)}\, dx$, ES: $5\pi/256$, K: 10

18. DI: $\int_{0}^{1} 1/(4x^2 - 9)\, dx$, ES: –1/12 Ln 5, K: 6

19. DI: $\int_{0}^{1} 1/e^{3x}\, dx$, ES: 0.3167377, K: 6

20. DI: $\int_{0}^{1} x\, e^{-x^2}\, dx$, ES: 0.3160603, K: 6

21. DI: $\int_{2}^{4} (x^3 - 2)/x^2 (x - 1)\, dx$, ES: 2.7876821, K: 6

22. DI: $\int_{1}^{3} (2 - x^2)/(x^3 + 3x^2 + 2x)\, dx$, ES: Ln 9/10, K: 6

23. DI: $\int_{2}^{3} (3 - \theta)/(\theta^3 + 4\theta^2 + 3\theta)\, d\theta$, ES: Ln 81/80, K: 6

24. DI: $\int_0^1 (3x^2 + 7x)/(x+1)(x+2)(x+3)\,dx$, ES: Ln 4/3, K: 6

25. DI: $\int_0^4 9x^2/(2x+1)(x+2)^2\,dx$, ES: 5 Ln 3 – 4, K: 8

26. DI: $\int_1^4 (5x^2+4)/x(x^2+4)\,dx$, ES: 3 Ln 4, K: 6

27. DI: $\int_{\pi/4}^{\pi/2} \text{Sin}\,t/t\,dt$, ES: $x - x^3/3 \times 3! + x^5/5 \times 5! - x^7/7 \times 7! + \ldots$, K = 5

28. DI: $\int_0^{\pi/3} 1/(1-\text{Sin}x)\,dx$, ES: 2.7320480, K: 6

29. DI: $\int_{-1}^{1} x^2\text{Cos}\,x\,dx$, ES: 0.4782672, K: 6

30. DI: $\int_0^{\pi/2} 1/(3+\text{Cos}\,2x)\,dx$, ES: 0.5553602, K: 6

31. DI: $\int_0^{\pi/3} x^2\,\text{Sin}3x\,dx$, ES: 0.2173927, K: 6

32. DI: $\int_0^{\pi/2} \text{Sin}^6\theta\,\text{Cos}^3\theta\,d\theta$, ES: 2/63, K: 6

33. DI: $\int_0^1 \text{Sin}^{-1}x\,dx$, ES: $\pi/2 - 1$, K: 10

34. DI: $\int_{\pi/4}^{\pi/2} 1/\text{Sin}^2 x\,dx$, ES: 1.0, K: 6

35. DI: $\int_0^{\pi/2} (1-\text{Sin}x)/\text{Cos}x\,dx$, ES: Ln 2, K: 5

36. DI: $\int_0^{\pi/2} e^{\sin x}\,dx$, ES: 3.1041172, K: 4

37. DI: $\int_0^{\pi/4} 1/\text{Cos}^2 x\,dx$, ES: 1.0, K: 4

38. DI: $\int_0^{\pi} \text{Sin}^5\theta\,\text{Cos}^4\theta\,d\theta$, ES: 0.0507938, K: 5

39. DI: $\int_0^{\pi} (1 - \cos x)^2\, dx$, ES: $3\pi/2$, K: 3

40. DI: $\int_0^{\pi/2} \mathrm{Sin}^2\theta\ \mathrm{Cos}^3\theta\ d\theta$, ES: 2/15, K: 6

41. DI: $\int_0^{\pi/4} 1/\mathrm{Cos}^4 x\ dx$, ES: 4/3, K: 6

42. DI: $\int_0^{\pi/2} (\mathrm{Cos}\,x - \mathrm{Sin}\,x)/(1 + \mathrm{Sin}\,x\ \mathrm{Cos}\,x)\ dx$, ES: 0, K: 6

43. DI: $\int_0^{\pi} 1/(2\mathrm{Sin}\,x + \mathrm{Cos}\,x + 3)\ dx$, ES: $\pi/4$, K: 8

44. DI: $\int_0^{\pi/2} \mathrm{Sin}\,x/(\mathrm{Sin}\,x + \mathrm{Cos}\,x)\ dx$, ES: $\pi/4$, K: 8

45. DI: $\int_0^{2\pi} \mathrm{Sin}^4(\theta/2)\ \mathrm{Cos}^5(\theta/2)\ d\theta$, ES: 0, K: 5

46. DI: $\int_0^{\pi/2} \mathrm{Cos}^6 x\ dx$, ES: $5\pi/32$, K: 5

47. DI: $\int_0^{\pi} x\ \mathrm{Sin}^2 x\ dx$, ES: $\pi^2/4$, K: 5

48. DI: $\int_0^{\pi} x\ \mathrm{Sin}x\ \mathrm{Cos}^2 x\ dx$, ES: $\pi/3$, K: 7

49. DI: $\int_0^{\pi/2} \mathrm{Sin}^3 x\ \mathrm{Cos}^3 x\ dx$, ES: 0.08333333, K: 6

50. DI: $\int_0^{\pi/4} 1/\mathrm{Cos}^4 x\ dx$, ES: 1.33333333, K: 6

Find the area of the region from Chapter 2 using the program: `g2CSAREA.CPP`.

1. Find the area of the region enclosed by the curves
   $y^2 = 4bx + 4b^2$ and $x + y = 2b$ (when $b = 1$).
   ES: 64/3.

2. Find the area of the region enclosed by the curves
   $xy = 4$ and $x + y = 5$.
   ES: (15 – 16 Ln2)/2.

3. Compute the area of the region bounded by the surfaces $y = 8/(x^2 + 4)$, $x = 2y$ and $x = 0$.
   ES: $\pi - 1$.
4. Find the area contained between the parabolas $y = x^2/3$ and $y + 2x^2/3 = 4$.
   ES: 32/3.
5. Compute the area between the catenary $y = 2 \cosh x/2$, the y-axis and the straight line $y = e + 1/e$.
   ES: $8/e$.
6. Find the arc length of the semicubical parabola $y^2 = x^3$ from the origin to the point $x = 4$.
   ES: 9.07341528.
7. Compute the arc length of the curve $y = \sin^{-1}(e^{-x})$ from $x = 0$ to $x = 1$.
   ES: $\text{Ln}\left[e + \sqrt{(e^2 - 1)}\right]$.
8. Find the arc length of the curve $x = y^2/4 - \text{Ln}\, y/2$ from $y = 1$ to $y = e$.
   ES: $(e^2 + 1)/4$.
9. Solve the equation of the parabolas represented by $y^2 = 10x + 25$ and $y^2 + 6x - 9 = 0$ and find the area.
   ES: $16\sqrt{(5/3)}$.
10. Find the area enclosed by the ellipse $(x - 2y + 3)^2 + (3x + 4y - 1)^2 = 100$.
    ES: 10π.
11. Find the area of the domain enclosed by the ellipse $x^2/2 + y^2/2 = 1$.
    ES: 4π.
12. Find the area of the domain bounded by the curve $(x^2/4 + y^2/9)^2 = (x^2 + y^2)/25$.
    ES: $39\pi/25$.
13. Compute the equations to find the area enclosed by $y = x^3 - 2x$ and $y = 6x - x^3$.
    ES: 16.
14. Find the area in the first quadrant bounded by the x-axis and the curves $x^2 + y^2 = 10$ and $y^2 = 9x$.
    ES: 6.75.

15. Calculate the finite area bounded by the pair of curves $x^{2/3} + y^{2/3} = 1$ and $x + y = 1$.
    ES: $(16 - 3\pi)/32$.
16. Compute the area contained between the two parabolas $y = x^2$ and $y = x^2/2$ and the straight line $y = 2x$.
    ES: 4.0.
17. Find the area of the curve $y = 1/x$ bounded by the lines $y = x$ and $x = 2$.
    ES: 2.25.
18. Calculate the area between the part of the circle $x^2 + y^2 = 1$ and the straight line $x + y = 1$.
    ES: $\pi/2 - 2$.

Find the approximations of the Double integrals in Chapter 2 using the program g2CS2DIM.CPP.

1. DI: $\int_0^2 dy \int_0^1 xy\, dx$

   ES: 1
   Wx: 0.001953, Wy: 0.003906, K: 9, M: 512

2. DI: $\int_0^{\pi} dy \int_0^{\pi/2} x\, \mathrm{Sin}(x + y)\, dx$

   ES: $\pi - 2$
   Wx: 0.003068, Wy: 0.006136, K: 9, M: 512

3. DI: $\int_1^3 dy \int_2^5 1/(x + 2y)^2\, dx$

   ES: Ln (14/11)/2
   Wx: 0.005859, Wy: 0.003906, K: 9, M: 512

4. DI: $\int_0^1 dy \int_0^1 1/(x + 1 + y)^2\, dx$

   ES: Ln (4/3)
   Wx: 0.001953, Wy: 0.001953, K: 9, M: 512

5. DI: $\int_0^2 dy \int_0^1 (x+2)\, dx$

   ES: 5.0
   Wx: 0.001953, Wy: 0.003906, K: 9, M: 512

6. DI: $\int_0^1 dy \int_0^1 e^y \, dx$

ES: $e - 1$

Wx: 0.001953, Wy: 0.001953, K: 9, M: 512

7. DI: $\int_1^5 dy \int_1^5 1/\sqrt{(x^2 + y^2)} \, dx$

ES: 3.95183553

Wx: 0.007813, Wy: 0.007813, K: 9, M: 512

8. DI: $\int_1^2 dy \int_0^1 2xy/(1 + x^2)(1 + y^2) \, dx$

ES: 0.31756667

Wx: 0.001953, Wy: 0.001953, K: 9, M: 512

9. DI: $\int_0^2 dy \int_0^1 (x^2 + y) \, dx$

ES: 8/3

Wx: 0.001953, Wy: 0.003906, K: 9, M: 512

10. DI: $\int_0^1 dy \int_0^2 (x^2 + 2y) \, dx$

ES: 4.66666666

Wx: 0.003906, Wy: 0.001953, K: 9, M: 512

11. DI: $\int_1^{1.5} dy \int_{1.4}^2 \mathrm{Ln}(x + 2y) \, dx$

ES: 0.4295545

Wx: 0.001172, Wy: 0.000977, K: 9, M: 512

12. DI: $\int_0^1 dy \int_0^1 x^2/(1 + y^2) \, dx$

ES: $\pi/12$

Wx: 0.001953, Wy: 0.001953, K: 9, M: 512

13. DI: $\int_0^{\pi/2} dy \int_0^2 x^2 \cos y \, dx$

    ES: 8/3

    Wx: 0.003906, Wy: 0.003068, K: 9, M: 512

14. DI: $\int_0^2 dy \int_2^4 (x^2 + y^2) \, dx$

    ES: 70/3

    Wx: 0.003906, Wy: 0.001953, K: 9, M: 512.

The Programs: g3GAUELI.CPP/g3CRALIS.CPP/g3MATINV.CPP/g3MLUFAC.CPP used to approximate the simultaneous system of linear algebraic equations in Chapter 3.

1. $x_1 - x_2 - 2x_3 + 4x_4 = 3$
   $2x_1 + 3x_2 + x_3 - 2x_4 = -2$
   $3x_1 + 2x_2 - x_3 + x_4 = 1$
   $5x_1 - 2x_2 + 3x_3 + 2x_4 = 0$

   $x_1/x_2/x_3/x_4 = 0.38/-0.58/-1.02/0.$

2. $x_1 - 7x_2 + 2x_3 - x_4 = 10$
   $3x_1 + 4x_2 - x_3 + x_4 = 4$
   $5x_1 + 2x_2 - 3x_3 + 2x_4 = 9$
   $2x_1 - x_2 + 5x_3 - 3x_4 = 7$

   $x_1/x_2/x_3/x_4 = 2.23077/-0.92308/2.30769/3.30769.$

3. $2x_1 - x_2 + x_3 = 1$
   $2x_1 + 2x_2 + 2x_3 = 2$
   $-2x_1 + 4x_2 - x_3 = 5$

   $x_1/x_2/x_3 = 4/2/-5.$

4. $5x_1 + 3x_2 - x_3 = 11$
   $2x_1 + 4x_3 + x_4 = 1$
   $-3x_1 + 3x_2 - 3x_3 + 5x_4 = -2$
   $6x_2 - 2x_3 + 3x_4 = 9$

   $x_1/x_2/x_3/x_4 = 1/2/0/-1.$

5. $3x_1 + 3x_2 = 6$
   $2x_1 + 4x_2 = 8$

   $x_1/x_2 = 0/2.$

6. $5x_1 + 2x_2 + x_3 = 0$
   $x_1 + 7x_2 - 3x_3 = 1$

$2x_1 + 2x_2 - 7x_3 = 2$

$x_1/x_2/x_3 = 0.04444/0.2222/-0.26667.$

7. $x_1 + 3x_2 + 2x_3 = 17$
$x_1 + 2x_2 + 3x_3 = 16$
$2x_1 - x_2 + 4x_3 = 13$

$x_1/x_2/x_3 = 4/3/2.$

8. $2x_1 + x_2 + 2x_3 + x_4 = 6$
$6x_1 - 6x_2 + 6x_3 + 12x_4 = 36$
$4x_1 + 3x_2 + 3x_3 - 3x_4 = -1$
$2x_1 + 2x_2 - x_3 + x_4 = 10$

$x_1/x_2/x_3/x_4 = 2/1/-1/3.$

9. $x_1 + 2x_2 + 3x_3 = 1$
$2x_1 - 3x_2 + 4x_3 = -1$
$3x_1 + 4x_2 + 6x_3 = 2$

$x_1/x_2/x_3 = 0/0.41176/0.05882.$

10. $6x_1 + 9x_2 + 1.56x_3 = 34.1$
$9x_1 + 19.8x_2 + 2.15x_3 = 46.2$
$1.56x_1 + 2.15x_2 + 2.23x_3 = 9.39$

$x_1/x_2/x_3 = 6.79901/-0.7795/0.20604.$

11. $554.11x_1 - 281.91x_2 - 34.24x_3 = 273.02$
$-281.91x_1 + 226.81x_2 + 38.1x_3 = -63.965$
$-38.24x_1 + 38.1x_2 + 80.221x_3 = 34.717$

$x_1/x_2/x_3 = 0.9124/0.76705/0.50589.$

12. $2x_1 + 3x_2 + 4x_3 + x_4 = 1$
$x_1 + 2x_2 + x_4 = 0$
$2x_1 + 3x_2 + x_3 - x_4 = 2$
$x_1 - 2x_2 - x_3 - x_4 = 3$

$x_1/x_2/x_3/x_4 = 1.5/-0.5/0/-0.5.$

13. $2x_1 - x_2 + x_3 = -1$
$3x_1 + 3x_2 + 9x_3 = 0$
$3x_1 + 3x_2 + 5x_3 = 4$

$x_1/x_2/x_3 = 1/2/-1.$

14. $3x_1 + 4x_2 - x_3 = 7$
$4x_1 + 12x_2 + 6x_3 = -4$
$-x_1 + x_2 + 4x_3 = 4$

$x_1/x_2/x_3 = 23/-13/10.$

15. $6.1x_1 + 2.2x_2 + 1.2x_3 = 16.55$
$4.4x_1 + 11.0x_2 - 3.0x_3 = 21.10$
$1.2x_1 - 1.5x_2 + 7.2x_3 = 16.80$

$x_1/x_2/x_3 = 1.5/2/2.5.$

16. $0.0003x_1 + 1.566x_2 = 1.569$
$0.3454x_1 - 2.436x_2 = 1.018$

$x_1/x_2 = 10/1.$

17. $$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 2 \\ -1 & 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

$x_1/x_2/x_3/x_4 = -1.0/2/0/1.$

18. $$\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 51 \end{bmatrix}$$

$x_1/x_2 = -2.0/11.$

19. $$\begin{bmatrix} 10 & 2 & 1 \\ 1 & 8 & 2 \\ 2 & -1 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 69 \\ -3 \\ 76 \end{bmatrix}$$

$x/y/z = 7/-2/3.$

20. $x + 2y + 3z = 14$
$2x + 5y + 2z = 18$
$3x + y + 5z = 20$

$x_1/x_2/x_3 = 1/2/3.$

21. $$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$x/y/z = -0.28571/-0.92857/-0.21429.$

22. $$\begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 10 \\ 12 \end{bmatrix}$$

$x_1/x_2/x_3 = 3/2/1.$

23. $$\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 2\\ 3 & 5 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}8\\ 6\\ 14\end{bmatrix}$$

$x_1/x_2/x_3 = 5/1.66667/1.33333$.

24. $$\begin{bmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}9\\ 52\\ 9\end{bmatrix}$$

$x_1/x_2/x_3 = -2.33333/11.3333/0$.

25. $$\begin{bmatrix}3 & 2 & -1 & 1\\ 1 & -1 & -2 & 4\\ 2 & 3 & 1 & -2\\ 5 & -2 & 3 & 2\end{bmatrix}\begin{bmatrix}u\\ v\\ w\\ x\end{bmatrix}=\begin{bmatrix}1\\ 3\\ -2\\ 0\end{bmatrix}$$

$u/v/w/x = 0.38/-0.58/-1.02/0$.

26. $$\begin{bmatrix}2.63 & 5.21 & -1.694 & 0.938\\ 2.16 & -2.95 & 0.813 & -4.21\\ 5.36 & 1.88 & -2.15 & -4.95\\ 1.34 & 2.98 & -0.432 & -1.768\end{bmatrix}\begin{bmatrix}x\\ y\\ z\\ w\end{bmatrix}=\begin{bmatrix}4.23\\ -0.716\\ 1.28\\ 0.419\end{bmatrix}$$

$x/y/z/w = 1.51513/0.207651/1.05025/1.00474$.

27. $$\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 2\\ 3 & 1 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}14\\ 18\\ 20\end{bmatrix}$$

$x_1/x_2/x_3 = 1/2/3$.

28. $$\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & -1\\ 4 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}5\\ -3\\ 7\end{bmatrix}$$

$x/y/z = 1/-1/2$.

29. $$\begin{bmatrix}1.23 & 4.56 & 9.87\\ -9.61 & 6.02 & 11.1\\ 7.31 & 2.89 & 5.04\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4.12\\ 5.34\\ -3.56\end{bmatrix}$$

$x/y/z = -0.44904/-4.74342/2.66488$.

Determine the approximations of all Initial Value Problems in Chapter 4 using the program g4IVP1DE.CPP.

1. DE: $Dy/dx = 2xy$
   PS: $y(x) = e^{x^2}$
   IC: $y(0) = 1.0$, Width: 0.01
2. DE: $Dy/dx - 2x(1 + y) = 0$
   PS: $y(x) = 3e^{x^2-1} - 1$
   IC: $y(1) = 2$, Width: 0.01
3. DE: $dy/dx + xy^2 = 0$
   PS: $y(x) = 2/(x^2 + 1)$
   IC: $y(0) = 2$, Width: 0.01
4. DE: $dy/dx + 2xy = 0$
   PS: $y(x) = e^{-x2}$
   IC: $y(0) = 1.0$, Width: 0.01
5. DE: $y' + y = 0$
   PS: $y(x) = e^{1-x}$
   IC: $y(1) = 1.0$, Width: 0.01
6. DE: $Dy/dx = xy/2$
   PS: $y(x) = e^{x^2/4}$
   IC: $y(0) = 1.0$, Width: 0.01
7. DE: $Dy = (2y + 5)\ dx$
   PS: $y(x) = (11\ e^{2x} - 5)/2$
   IC: $y(0) = 3.0$, Width: 0.01
8. DE: $Dy/dx - x^2y = 0$
   PS: $y(x) = e^{x^3/3}$
   IC: $y(0) = 1$, Width: 0.01
9. DE: $(1 + x)\ dy/dx + 3y = 0$
   PS: $y(x) = (1 + x)^{-3}$
   IC: $y(0) = 1.0$, Width: 0.01
10. DE: $(x + y)\ dy/dx + (x - y) = 0$
    PS: $\text{Ln}\ (x^2 + y^2) + 2\tan^{-1} y/x = 0$
    IC: $y(1) = 0$, Width: 0.01
11. DE: $e^{3x}\ dx - 2dy = 0$
    PS: $y(x) = (e^{3x} - 1)/6$
    IC: $y(0) = 0$, Width: 0.01
12. DE: $xy' - y^2 = 1$
    PS: $y(x) = \tan\ (\text{Ln}\ x)$
    IC: $y(1) = 0$, Width: 0.01

13. DE: $x\, dy/dx + y^2 = 1$
PS: $y(x) = (x^2 - 1)/(x^2 + 1)$
IC: $y(1) = 0$, Width: 0.01

14. DE: $y'(x - 2y) = 2x + y$
PS: $\text{Ln}\,(x^2 + y^2) \tan^{-1} y/x$
IC: $y(1) = 0$, Width: 0.01

15. DE: $8y\, y' - (3x + 2)x = 0$
PS: $y(x) = x\sqrt{(x+1)}/2$
IC: $y(3) = 3$, Width: 0.01

16. DE: $dy/dx + xy = 0$
PS: $y(x) = e^{-x^2/2}$
IC: $y(0) = 1.0$, Width: 0.01

17. DE: $x(1 + x^2y^4)\, dy = y(1 - x^2y^4)\, dx$
PS: $x^3y^4 + 2y = 3x$
IC: $y(1) = 1$, Width: 0.01

18. DE: $Dy/dx + 2xy^2 = 0$
PS: $y(x) = 1/(1 + x^2)$
IC: $y(0) = 1$, Width: 0.01

19. DE: $Dy/dx = y^2$
PS: $y(x) = 1/(1 - x)$
IC: $y(0) = 1$, Width: 0.01

20. DE: $Dy/dx = y^3$
PS: $y(x) = 1/\sqrt{(1-2x)}$
IC: $y(0) = 1.0$, Width: 0.01

21. DE: $Dy/dx = e^y$
PS: $y(x) = \text{Ln}[1/(1 - x)]$
IC: $y(0) = 0$, Width: 0.01

22. DE: $Dy = [\sqrt{xy} + 1]\, dx$
PS: $y(x) = x + x^2/2 + x^3/12 + \ldots$
IC: $y(0) = 0$, Width: 0.01

23. DE: $(xe^{-y} + e^x)\, dy = (e^{-y} - ye^x)\, dx$
PS: $ye^x - xe^{-y} = 0$
IC: $y(0) = 0$, Width: 0.01

24. DE: $[x + 2\,\text{Ln}\, y - (\text{Ln}\, y)^2]dy = y\, dx$
PS: $x = y + (\text{Ln}\, y)^2$
IC: $y(1) = 1$, Width: 0.01

25. DE: $x[\text{Ln } y - \text{Ln } x]\, dy = y\, dx$
PS: $y[\text{Ln } y - \text{Ln } x - 1] = -0.6137053$
IC: $y(1) = 2$, Width: 0.01

26. DE: $\left[t\sqrt{t^2 + y^2} - y\right] dt = t - y\sqrt{(t^2 + y^2)}\; dy$
PS: $\sqrt{(t^2 + y^2)^3} - 3ty = 89$
IC: $y(4) = 3$, Width: 0.01

27. DE: $[3(x-1)^2\, y + 2]\, dy = -2\, [(x-1)\, y^2]\, dx$
PS: $y^3(x-1)^2 + y^2 = 9$
IC: $y(1) = 3$, Width: 0.01

28. DE: $2xy\, dx = (y^2 - x^2)\, dy$
PS: $3x^2y - y^3 = 1$
IC: $y(0) = -1$, Width: 0.01

29. DE: $x(2 - 9xy^2)\, dx + y(4y^2 - 6x^3)\, dy$
PS: $x^2 - 3x^3y^2 + y^4 = 1$
IC: $y(0) = 1$, Width: 0.01

30. DE: $y/x\, dx + (y^2 + \text{Ln } x)\, dy = 0$
PS: $4y\, \text{Ln } x + y^4 = 1$
IC: $y(1) = 1.0$, Width: 0.01

31. DE: $(3x^2/y^2 + 1)\, dx = (2x^3/y^3 + 5/y^2)\, dy$
PS: $x + x^3/y^2 + 5/y = 5$
IC: $y(0) = 1$, Width: 0.01

32. DE: $3x^2(1 + \text{Ln } y)\, dx = (2y - x^3/y)\, dy$
PS: $x^3(1 + \text{Ln } y) - y^2 = -1$
IC: $y(0) = 1$, Width: 0.01

33. DE: $\cos x \sin y\, dy/dx = \cos y \sin x$
PS: $y(x) = \cos^{-1}[\cos x/\sqrt{2}]$
IC: $y(0) = \pi/4$, Width: 0.01

34. DE: $(x^2 + 1)\; y' = 1 + y^2$
PS: $y(x) = (1 + x)/(1 - x)$
IC: $y(0) = 1$, Width: 0.01

35. DE: $y(x^2 - 1)\, dy/dx + x(y^2 + 1) = 0$
PS: $y(x) = \sqrt{[(1 + x^2)/(1 - x^2)]}$
IC: $y(0) = 1$, Width: 0.01

36. DE: $xy' \cos(y/x) = y \cos(y/x) - x$
    PS: $y(x) = x \sin^{-1}[\text{Ln}\,(1/x)]$
    IC: $y(1) = 0$, Width: 0.01.

Find the approximations of the differential equation in Chapter 4 using the program g4IVP1DE.CPP.

1. DE: $Dy/dx + 3xy/(1 + x^2) = 5x$
   PS: $y(x) = 1 + x^2$
   IC: $y(1) = 2.0$, Width = 0.01
2. DE: $(1 + x^2)dy/dx = xy + (1 + x^2)^{3/2}$
   PS: $y(x) = (1 + x)\,(1 + x^2)^{1/2}$
   IC: $y(0) = 1$, Width = 0.01
3. DE: $x\,dy/dx = x^2 - y$
   PS: $y(x) = (x^3 - 1)/3x$
   IC: $y(1) = 0$, Width = 0.01
4. DE: $Dy/dx = x + y$
   PS: $y(x) = 2e^x - x - 1$
   IC: $y(0) = 1.0$, Width = 0.01
5. DE: $2y + (x^2 y + 1)xy' = 0$
   PS: $y(x) = \left[2x - \sqrt{(3x^2 - 2)}\right]\Big/x$
   IC: $y(1.0) = 1.0$, Width = 0.01
6. DE: $y\,dx + x(2xy + 1)\,dy = 0$
   PS: $y^2 e^{-1/xy} = 0.3678795$
   IC: $y(1.0) = 1.0$, Width = 0.01
7. DE: $(x + 2y)\,dx = x\,dy$
   PS: $y(x) = x(x - 1)$
   IC: $y(1.0) = 0.0$, Width = 0.01
8. DE: $Dy + (3y - 8)\,dx = 0$
   PS: $y(x) = 2(4 - e^{-3x})/3$
   IC: $y(0) = 2.0$, Width = 0.01
9. DE: $y' = 8 - 3y$
   PS: $y(x) = 4(2 + e^{-3x})/3$
   IC: $y(0) = 4.0$, Width = 0.01
10. DE: $x\,dy + (1 - y)y\,dx = 0$
    PS: $y(x) = 1/(x + 1)$
    IC: $y(1.0) = 0.5$, Width = 0.01

11. DE: $x\ dy = \left(y + \sqrt{(xy)}\right) dx$
    PS: $y(x) = x(\text{Ln } x + 2)^2/4$
    IC: $y(1.0) = 1.0$, Width = 0.01
12. DE: $Dy/(2 - y) = \tan x\ dx$
    PS: $y(x) = 2 - 3 \cos x$
    IC: $y(0) = 1.0$, Width = 0.01
13. DE: $2xy\ dy = (y^2 - x^2)\ dx$
    PS: $y(x) = \sqrt{(2x - x^2)}$
    IC: $y(1.0) = 1.0$, Width = 0.01
14. DE: $Dy/dx + y/x = x$
    PS: $y(x) = (x^3 - 1)/3x$
    IC: $y(1.0) = 0$, Width = 0.01
15. DE: $x\ dy = (x - y)\ dx$
    PS: $y(x) = x/2 + 1/x$
    IC: $y(1.0) = 1.5$, Width = 0.01
16. DE: $Dy/dx - y \cos x + 1 = 0$
    PS: $y(x) = 1 - x^3/6 - x^4/24 + x^6/72 + \ldots$
    IC: $y(0) = 1.0$, Width = 0.01
17. DE: $(x + y)\ dy/dx = y$
    PS: $y(x) = x/\text{Ln } (y/2)$
    IC: $y(0) = 2.0$, Width = 0.01
18. DE: $(x + y)\ dy/dx = y$
    PS: $y(x) = x/(\text{Ln } y + 1)$
    IC: $y(1.0) = 1.0$, Width = 0.01
19. DE: $Dy/y = (y^2 e^{x^2} - x)\ dx$
    PS: $y^2(x) = e^{-x^2}/(4 - 2x)$
    IC: $y(0) = 0.5$, Width = 0.01
20. DE: $2x^3 y\ dy/dx + 3x^2 y^2 = -7.0$
    PS: $x^3 y^2 + 7x + 8 = 0$
    IC: $y(-1.0) = 1.0$, Width = 0.01
21. DE: $(1 - 2xy)\ dy/dx = y/x$
    PS: $y(xy - 1) = 2x$
    IC: $y(-1.0) = 1.0$, Width = 0.01
22. DE: $Dx/dy - x \cos y = \sin 2y$
    PS: $x(y) = 3e^{\sin y} - 2(1 + \sin y)$
    IC: $y(1.0) = 0$, Width = 0.01

23. DE: $Dy = (x/y\ e^{2x} + y)\ dx$
    PS: $y(x) = \sqrt{[(x^2 + 1)\ e^{2x}]}$
    IC: $y(0) = 1.0$, Width = 0.01

24. DE: $(x^2 + y^2)\ dy/dx = y^3/x$
    PS: $y(x) = 2x/(x + 1)$
    IC: $y(1.0) = 1.0$, Width = 0.01

25. DE: $y'\ x/y + 2xy\ \text{Ln}\,x = -1$
    PS: $y(x) = 1/x[(\text{Ln}\ x)^2 + 1]$
    IC: $y(1.0) = 1.0$, Width = 0.01

26. DE: $xy\ \text{Ln}x\ dy = (y^2 - \text{Ln}\ x)\ dx$
    PS: $y^2 = \text{Ln}\ x(2 - 0.804021 \times \text{Ln}\ x)$
    IC: $y(2.0) = 1.0$, Width = 0.01

27. DE: $x\ dy/dx = y\ (y\ \text{Ln}\ x - 1)$
    PS: $y(x) = 1/(1 + \text{Ln}\ x - 0.5x)$
    IC: $y(1.0) = 2.0$, Width = 0.01

28. DE: $x\ dy/dx - y\ (1 - y) = 0$
    PS: $y = x/(x + 0.5)$
    IC: $y(-1.0) = 2.0$, Width = 0.01

29. DE: $y\ dy/dt + (y^2 + t^2 + t) = 0$
    PS: $y(t) = -\sqrt{(5e^{-2(t+2)} - t^2)}$
    IC: $y(-2.0) = -1.0$, Width = 0.01

30. DE: $t\ dy/dt + 2y = 3t$
    PS: $yt^2 = t^3 + 200$
    IC: $y(-5.0) = 3.0$, Width = 0.01

31. DE: $t\ dx/dt - x - \text{Ln}\ t = 0$
    PS: $x(t) = 101t - 1 - \text{Ln}\ t$
    IC: $x(1.0) = 100$, Width = 0.01

32. DE: $2xt\ dx/dt = 3t^2 + x^2$
    PS: $x(t) = \sqrt{(3t^2 + 2t)}$
    IC: $x(2.0) = -4.0$, Width = 0.01

33. DE: $2yy' + x^2(1 + y^2) = 0$
    PS: $y(x) = \sqrt{(2e^{-x^3/3} - 1)}$
    IC: $y(0) = 1.0$, Width = 0.0174

34. DE: $y' + y = e^{-x}$
 PS: $y(x) = e^{-x}(1 + x + 3e^{-1})$
 IC: $y(-1.0) = 3.0$, Width = 0.01

35. DE: $xy^2\, dy\, (x^3 + y^3)\, dx$
 PS: $y^3 = 3x^3 \text{ Ln } (Cx)$, $C = 1.39561242$
 IC: $y(1.0) = 1.0$, Width = 0.01.

Solve the following system of equations and find the approximations of x and y using the program: g4LINSYS.CPP.

Example 10.1

SYS: $dx/dt = 2ty/(t^2 - x^2 - y^2)$, $dy/dt = 2ty/(t^2 - x^2 - y^2)$

PS: $t^2 + x^2 + y^2 = 2x$, $y(t) = x$

IC: $x(0) = 1$, $y(0) = 1$, w: 0.01

Example 10.2

SYS: $dx/dt = x + 2y$, $dy/dt = 4y - x$

PS: $x(t) = \text{Exp}(2t)\,[2 - \exp(t)]$, $y(t) = \text{Exp}(2t)\,[1 - \exp(t)]$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.3

SYS: $dx/dt = x - 2y$, $dy/dt = 2y - 3x$

PS: $x(t) = [2\, \text{Exp}(4t) + 3\, \text{Exp}(-t)]/5$, $y(t) = 3[\text{Exp}(-t) - \text{Exp}(4t)]/5$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

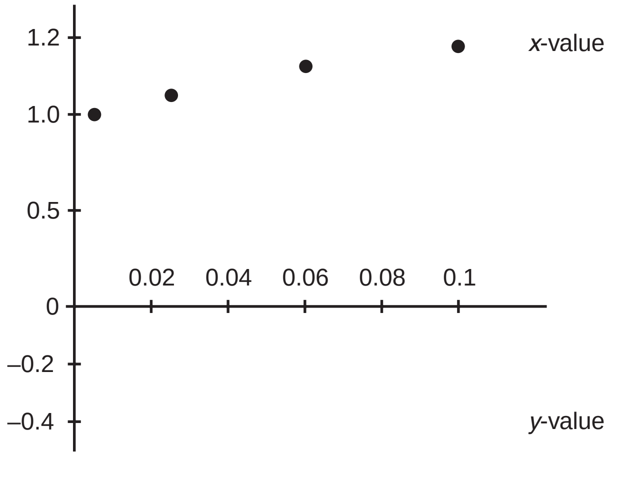

Fig. 10.1

Example 10.4

SYS: $dx/dt = t - y - x$, $dy/dt = t + y + x$

PS: $x(t) = t^{**}2/2 - t^{**}3/3$, $y(t) = t^{**}2/2 + t^{**}3/3$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

Example 10.5

SYS: $Dx = -3y - 3\ \text{Exp}(t)$, $Dy = 4y + 6\ \text{Exp}(t)$

PS: $x = -0.5 + 3\ \exp(t)[1 - 0.5\ \exp(3t)]$, $y = 2\ \exp(\text{t})[\text{Exp}(3t) - 1]$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.6

SYS: $Dx = x + y + t$, $Dy = 2t - 3y - 4x$

PS: $x = -9(1 - \exp(t)) + t(5 + 4\ \exp(-t)$, $y = 14(1 - \exp(-t)) - 2t(3 + 4\ \exp(-t))$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

Example 10.7

SYS: $Dx = x + 3y$, $Dy = 5y - x$

PS: $x(t) = x\ \text{Exp}(2t)$, $y(t) = \text{Exp}(2t)$

IC: $x(0) = 3$, $y(0) = 1$, w: 0.01

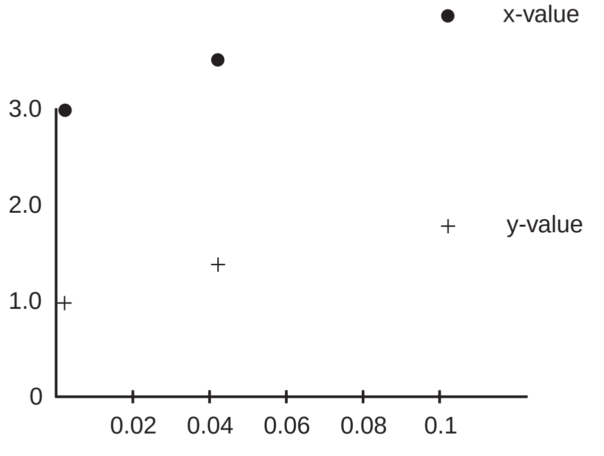

Fig. 10.2

Example 10.8

SYS: $Dx/dt = 2x - 3y$, $dy/dt = x - 2y + 2\sin t$

PS: $x(t) = 3(2\sin t - e^{t} + e^{-t})/2$, $y(t) = (4\sin t - 2\cos t - e^{t} + 3e^{-t})/2$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

Example 10.9

SYS: $Dx = 4x + y$, $Dy = 3x + 2y$

PS: $x(t) = \text{Exp}(t)\ [\text{Exp}(4t) - 2]$, $y(t) = \text{Exp}(t)\ [\text{Exp}(4t) + 6]$

IC: $x(0) = -1$, $y(0) = 7$, w: 0.01

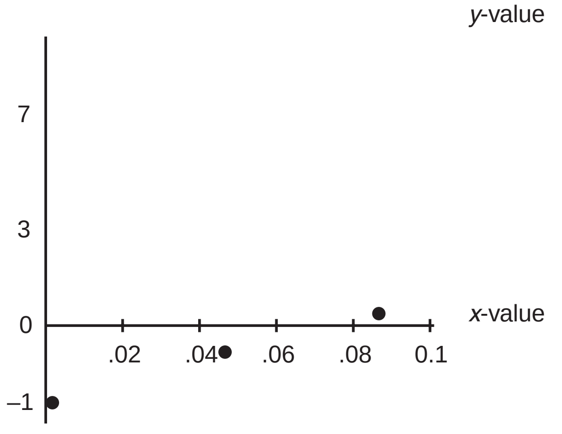

Fig. 10.3

Example 10.10

SYS: $Dx = 5x - 2y$, $Dy = 2x + y$

PS: $x(t) = \text{Exp}(3t)\ (1 + 2t)$, $y(t) = 2t\ \text{Exp}(3t)$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.11

SYS: $Dx = 2x + y$, $Dy = 3x + 4y$

PS: $x(t) = 0.5\ [\text{Exp}(t) + \text{Exp}(5t)]$, $y(t) = 0.5\ [3\ \text{Exp}(5t) - \text{Exp}(t)]$

IC: $x(0) = 1$, $y(0) = 1$, w: 0.01

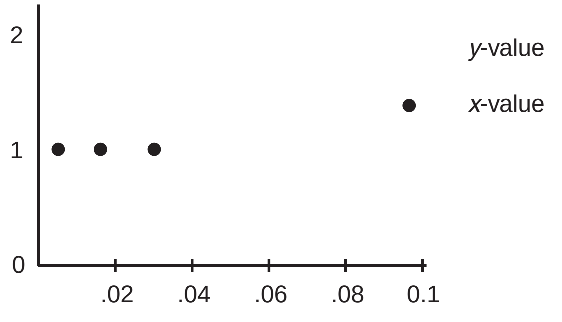

Fig. 10.4

Example 10.12

SYS: $Dx = 3y + 6\sin(t)$, $Dy = -3x$

PS: $x(t) = [3\cos(t) + \cos(3t)]/4$, $y(t) = -[9\sin(t) + \sin(3t)]/4$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.13

SYS: $Dx = x - y$, $Dy = y - 4x$

PS: $x = [3\ \text{Exp}(-t) + \text{Exp}(3t)]/4$, $y = [3\ \text{Exp}(-t) - \text{Exp}(3t)]/2$

IC: $x(0) = 1$, $y(0) = 1$, w: 0.01

Example 10.14

SYS: $4dx/dt + 5x - 3y = 4\,\text{Exp}(t)$, $4dy/dt - 5y + 3x = 4\,\text{Exp}(t)$

PS: $x = 3X/8 + t\,\text{Exp}(t)/4$, $y = X/8 + 3t\,\text{Exp}(t)/4$ $[\text{Exp}(t) - \text{Exp}(-t) = X]$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

Example 10.15

SYS: $dx/dt + 3x + y = 10\,\text{Exp}(-t)$, $dy/dt + 2y + 2x = 4$

PS: $x = (10tX + 14X - Y - 3)/3$, $y = (-20tX - 8X - Y + 9)/3$ $[\text{Exp}(-t) = X, \text{Exp}(-4t) = Y]$

IC: $x(0) = 3.333$, $y(0) = 0$, w: 0.01

Example 10.16

SYS: $Dx + x + 5y = 0$, $Dy - y - x = 0$

PS: $x(t) = \cos(2t) - 3\sin(2t)$, $y(t) = \cos(2t) + \sin(2t)$

IC: $x(0) = 1$, $y(0) = 1$, w: 0.01

Example 10.17

SYS: $Dx - 5x - 3y = 0$, $Dy + y + 3x = 0$

PS: $x(t) = \text{Exp}(2t)\,(1 + 6t)$, $y(t) = \text{Exp}(2t)\,(1 - 6t)$

IC: $x(0) = 1$, $y(0) = 1$, w: 0.01

Example 10.18

SYS: $dx/dt - x + y = 0$, $dy/dt - tx = 0$

PS: $x(t) = t{**}2/2 + t + 1$, $y(t) = t{**}2/2$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.19

SYS: $dx/dt - y = 0$, $dy/dt - x = \text{Exp}(2t)$

PS: $x = [2\,\text{Exp}(2t) - 3\,\text{Exp}(t) + \text{Exp}(-t)]/6$, $y = [4\,\text{Exp}(2t) - 3\,\text{Exp}(t) - \text{Exp}(-t)]/6$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

Example 10.20

SYS: $dx/dt - 4x + y = 0$, $dy/dt - 2x - y = 0$

PS: $x(t) = \text{Exp}(2t)[1 - \text{Exp}(t)]$, $y(t) = \text{Exp}(2t)[2 - \text{Exp}(t)]$

IC: $x(0) = 0$, $y(0) = 1$, w: 0.01

Example 10.21

SYS: $Dx = \text{Exp}(t)/3 - x - 2y/3$, $Dy = 4x/3 + y - t$

PS: $x = 11X/2 - 9Y/2 - 6t$, $y = -11X + 9Y/2 + \text{Exp}(t)/2 + 9 + 9$ $[\text{Exp}(t/3) = X, \text{Exp}(-t/3) = Y]$

IC: $x(0) = 1$, $y(0) = 3$, w: 0.01

Example 10.22

SYS: $Dx - 4x + 2y = 0$, $Dy - 3x + y = 0$

PS: $x = \text{Exp}(t)\,[3\,\text{Exp}(t) - 2]$, $y = 3\,\text{Exp}(t)\,[\text{Exp}(t) - 1]$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.23

SYS: $Dx - 4x + y = \text{Exp}(t)$, $Dy - 2x - y = 0$

PS: $x = \text{Exp}(2t)\ [3\ \text{Exp}(t) - 2]$, $y = \text{Exp}(t)\ [3\ \text{Exp}(2t) - 4\ \text{Exp}(t) + 1]$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.24

SYS: $Dx - x + 2y = 0$, $Dy + 3x - 2y = 0$

PS: $x = \text{Exp}(-t)\ [5 - 2\ \text{Exp}(5t)]$, $y = \text{Exp}(-t)\ [5 + 3\ \text{Exp}(5t)]$

IC: $x(0) = 3$, $y(0) = 8$, w: 0.01

Example 10.25

SYS: $dx/dt = 4x + y - 36t$, $dy/dt = -2x + y - 2\ \text{Exp}(t)$

PS: $x = 10X - Y - Z + 6t - 1$, $y = -20X + Y + 3Z + 12t + 10$ [$\text{Exp}(2t) = X$, $8\text{Exp}(3t) = Y$, $\text{Exp}(t) = Z$]

IC: $x(0) = 0$, $y(0) = 1$, w: 0.01

Example 10.26

SYS: $dx/dt = 3x - 2y$, $dy/dt = 4x + 7y$

PS: $x = \text{Exp}(5t)\ [\cos(2t) - \sin(2t)]$, $y = 2\ \text{Exp}(5t)\ \sin(2t)$

IC: $x(0) = 1$, $y(0) = 0$, w: 0.01

Example 10.27

SYS: $dx/dt = 1 + 2x/t$, $dy/dt = x + y - 1 + 2x/t$

PS: $x = t/3$, $y = -t/3$

IC: $x(1) = 0.333333$, $y(1) = -0.333333$, w: 0.01

Example 10.28

SYS: $dx/dt - x$**2$/y = 0$, $dy/dt + x = 0$

PS: $x = -1/(2\ \text{Sqrt}(t))$, $y = \text{Sqrt}(t)$

IC: $x(1) = -0.5$, $y(1) = 1$, w: 0.01

Example 10.29

SYS: $dx/dt = x(t + y)/(t$^2 $- xy)$, $dy/dt = y(y - x)/(t$^2 $- xy)$

PS: $x(t) = t - y$, $y(y - 2t)$^3 $= (t - x)$^2

IC: $x(0) = -1$, $y(0) = 1$, w: 0.01

Example 10.30

SYS: $dx/dt = (x - t) / (y - x)$, $dy/dt = (t - y) / (y - x)$

PS: $x + y + t = 0$, x^2 + y^2 + t^2 = 6

IC: $x(1) = -2$, $y(1) = 1$, w: 0.01

Example 10.31

SYS: $dx/dt = t/y$^2, $dy/dt = t/xy$

PS: $x - y = 0$, y^3 $- 3t$^2/2 = 1

IC: $x(0) = 1$, $y(0) = 1$, w: 0.01

Example 10.32

SYS: $dx/dt = 1.0 / (y - t)$, $dy/dt = (x - 1.0)/x$

PS: $x(t) = \text{Exp}(-t)$, $y(t) = t - \text{Exp}(t)$

IC: $x(0) = 1$, $y(0) = -1$, w: 0.01

Example 10.33

SYS: $dx/dt = xy + t + 2$, $dy/dt = tx - 1.0$

PS: $x = t(2 + t/2 - 2/3t\text{^}2 - 1/8t\text{^}3 + \ldots)$, $y = t(-1 + 2/3t\text{^}2 + 1/8t\text{^}3 - 1/6t\text{^}4 + \ldots)$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

Example 10.34

SYS: $Dx - xy - 2t = 1.0$, $Dy - x\text{^}2 - t = 2.0$

PS: $x = t(1 + t + 2/3t\text{^}2 + 5/8t\text{^}3 + \ldots)$, $y = t(2 + t/2 + 1/3t\text{^}2 + 1/2t\text{^}3 + \ldots)$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01

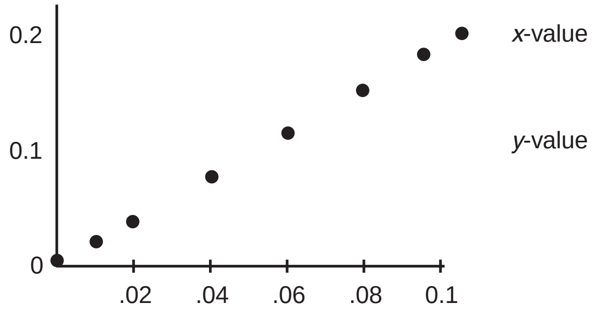

Fig. 10.5

Example 10.35

SYS: $Dx - xy = 1.0$, $Dy - x - t = 0$

PS: $x = 1 + 3X + 4X\text{^}2 + 16/3X\text{^}3 + \ldots$, $y = 2 + 2X + 2X\text{^}2 + 4/3X\text{^}3 + \ldots$ [$t - 1 = X$]

IC: $x(1) = 1$, $y(1) = 2$, w: 0.01

Example 10.36

SYS: $Dx - y - t = 2.0$, $Dy - y - x\text{^}2 = 1.0$

PS: $x = 2 + 4X + 3.5X\text{^}2 + 11/3X\text{^}3 + \ldots$, $y = 1 + 6X + 11X\text{^}2 + 41/3X\text{^}3 + \ldots$ [$t - 1 = X$]

IC: $x(1) = 2$, $y(1) = 1$, w: 0.01

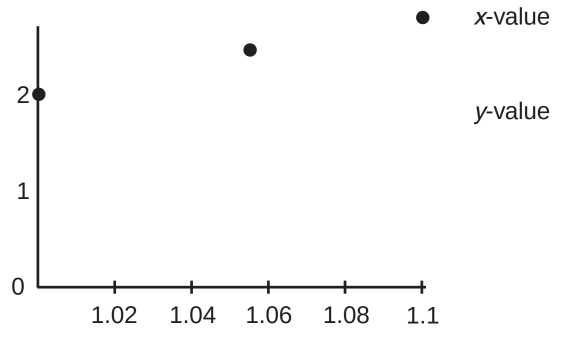

Fig. 10.6

Example 10.37

SYS: $Dx - y\hat{}2 - 2t = 1.0$, $Dy - x - t = 2.0$

PS: $x = t(1 + t + 4/3t\hat{}2 + t\hat{}3 + \ldots)$, $y = t(2 + t + 1/3t\hat{}2 + 1/3t\hat{}3 + \ldots)$

IC: $x(0) = 0$, $y(0) = 0$, w: 0.01.

Find the approximations of second order Initial Value Problems in Chapter 5 by the programs: g5LI2IVP.CPP and g5NL2IVP.CPP, selecting the appropriate one.

1. DE: $y'' - y = 0$

   PS: $y(x) = e^{-x}$, $y'(x) = -e^{-x}$

   IC: $y(0) = 1$, $y'(0) = -1$

   W: 0.01

2. DE: $y'' + 2y' = 0$

   PS: $y(x) = 1.0$, $y'(x) = 0$

   IC: $y(0) = 1$, $y'(0) = 0$

   W: 0.01

3. DE: $y'' + xy = 0$

   PS: $y = 1 - x - x^3/3! + 2x^4/4! + \ldots$, $y' = -1 - x^2/2 + x^3/3 + \ldots$

   IC: $y(0) = 1$, $y'(0) = -1$

   W: 0.01

4. DE: $y'' - y^2 = x^2$

   PS: $y = 1 + x^2/2! + x^4/3! + \ldots$, $y' = x + 2x^3/3 + \ldots$

   IC: $y(0) = 1$, $y'(0) = 0$

   W: 0.01

5. DE: $y'' - 2 = 0$

   PS: $y(x) = x^2 + 2x + 1$, $y'(x) = 2(x + 1)$

   IC: $y(0) = 1$, $y'(0) = 2$

   W: 0.01

6. DE: $y'' - y'e^y = 0$

   PS: $y(x) = -1.0$, $y'(x) = 0$

   IC: $y(0) = -1$, $y'(0) = 0$

   W: 0.01

7. DE: $yy'' + y'^2 - yy' = 0$

   PS: $y(x) = (e^x + 2)$, $y'(x) = 0.5e^x/\sqrt{(e^x + 2)}$

   IC: $y(\text{Ln } 2) = 2$, $y'(\text{Ln } 2) = 0.5$

   W: 0.01

8. DE: $y'' - y' - 2y = 0$
   PS: $y(x) = 2e^{-x} + e^{2x}$, $y'(x) = 2(e^{2x} - e^{-x})$
   IC: $y(0) = 3$, $y'(0) = 0$
   W: 0.01

9. DE: $(D^2 - 4)y = 0$
   PS: $y(x) = 5e^{2x}/8 + 3e^{-2x}/8$, $y'(x) = (5e^{2x} - 3e^{-2x})/4$
   IC: $y(0) = 1$, $y'(0) = 0.5$
   W: 0.01

10. DE: $D^2y - 4\,Dy + 4y = 0$
    PS: $y(x) = e^{2x}(3 - 2x)$, $y'(x) = 4e^{2x}(1 - x)$
    IC: $y(0) = 3$, $y'(0) = 4$
    W: 0.01

11. DE: $y'' - 5y' + 6y = 0$
    PS: $y(x) = e^{3x-1}$, $y'(x) = 3e^{3x-1}$
    IC: $y(1) = e^2$, $y'(1) = 3e^2$
    W: 0.01

12. DE: $y'' + 4y' + 2y = 0$
    PS: $y = e^{-0.5857x} - 2e^{-3.4142x}$, $y' = -0.5857\,e^{-0.5857x} + 6.8284\,e^{-3.4142x}$
    IC: $y(0) = -1$, $y'(0) = 6.2427$
    W: 0.01

13. DE: $y'' + 4y' + 13y = 0$
    PS: $y(x) = e^{-2x}(4\cos 3x + 3\sin 3x)$, $y'(x) = e^{-2x}(\cos 3x - 18\sin 3x)$
    IC: $y(0) = 4$, $y'(0) = 1$
    W: 0.01

14. DE: $y'' + 4y' + 4y = 0$
    PS: $y = 2e^{-2x}(1 + x)$, $y' = -2e^{-2x}(1 + 2x)$
    IC: $y(0) = 2$, $y'(0) = -2$
    W: 0.01

15. DE: $y'' - y^2 - 1 = 0$
    PS: $y = x^2/2 + x^6/120 + \ldots$, $y' = x + x^5/20 + \ldots$
    IC: $y(0) = 0$, $y'(0) = 0$
    W: 0.01

16. DE: $y'' + y - 4e^x = 0$
    PS: $y(x) = 2\cos x - 5\sin x + 2e^x$, $y'(x) = -2\sin x - 5\cos x + 2e^x$
    IC: $y(0) = 4$, $y'(0) = -3$
    W: 0.01

17. DE: $y'' - 2y' - 2e^x = 0$
    PS: $y = e^{2x-1} - 2e^x + e - 1$, $y' = 2(e^{2x-1} - e^x)$
    IC: $y(1) = -1$, $y'(1) = 0$
    W: 0.01

18. DE: $y'' + 2yy' = 0$
    PS: $y(x) = \tanh x$, $y'(x) = \text{sech}^2\, x$
    IC: $y(0) = 0$, $y'(0) = 1$
    W: 0.01

19. DE: $(1 - x^2)\, y'' - 2xy' + 6y = 0$
    PS: $y(x) = (3x^2 - 1)/2$, $y'(x) = 3x$
    IC: $y(0) = -0.5$, $y'(0) = 0$
    W: 0.01

20. DE: $(1 - x^2)\, y'' - 2xy' + 20y = 0$
    PS: $y = (35x^4 - 30x^2 + 3)/8$, $y' = 5x(7x^2 - 3)/2$
    IC: $y(0) = 0.375$, $y'(0) = 0$
    W: 0.01

21. DE: $yy'' + y'^2 - y^4 = 0$
    PS: $y(x) = 1/\cos x$, $y'(x) = \sec x \tan x$
    IC: $y(0) = 1$, $y'(0) = 0$
    W: 0.01

22. DE: $2yy'' - y'^2 = 1$
    PS: $y(x) = (x^2 + 1)/2$, $y'(x) = x$
    IC: $y(1) = 1$, $y'(1) = 1$
    W: 0.01

23. DE: $xy'' - y' = 0$
    PS: $y(x) = (x^2 - 1)/2$, $y'(x) = x$
    IC: $y(1) = 0$, $y'(1) = 1$
    W: 0.01

24. DE: $y^2 y'' - 1 = 0$
    PS: $2y^2 - 4x^2 = 1$, $y'(x) = 2x/y$
    IC: $y(0.5) = 1$, $y'(0.5) = 1$
    W: 0.01

25. DE: $4\, d^2y/dx^2 + 4\, dy/dx + y = \cos(x/2)$
    PS: $y(x) = \sin(x/2) - xe^{-x/2}/4$,
    $dy/dx = \cos(x/2)/4 - e^{-x/2}(1 - x/2)/4$
    IC: $y(0) = 0$, $y'(0) = 0$
    W: 0.01

26. DE: $D^2y/dx^2 - 3dy/dx + 2y = e^{2x}(3x^2 + 2\cos x)$
PS: $y(x) = e^x + e^{2x}(x^3 - 3x^2 + 6x + 1 + \sin x - \cos x)$,
$dy/dx = e^x + e^{2x}(2x^3 - 3x^2 + 6x + 8 + 3\sin x - \cos x)$
IC: $y(0) = 1$, $y'(0) = 8$
W: 0.01

27. DE: $D^2y/dx^2 - 5dy/dx + 6y = 3x^2 - 4$
PS: $y(x) = 9e^{2x}/4 - 10e^{3x}/9 - 5/36 + 5x/6 + x^2/2$,
$dy/dx = 9e^{2x}/2 - 10e^{3x}/3 + 5/6 + x$
IC: $y(0) = 1$, $y'(0) = 2$
W: 0.01

28. DE: $yy'' + y^2y' + y'^2 = 0$
PS: $y = 3e^{3x}/(2 + e^{3x})$, $y' = 18e^{3x}/(2 + e^{3x})^2$
IC: $y(0) = 1$, $y'(0) = 2$
W: 0.01

29. DE: $2yy'' - y'^2 + y^2 = 0$
PS: $y(x) = \sin x + 1$, $y'(x) = \cos x$
IC: $y(0) = 1$, $y'(0) = 1$
W: 0.01

30. DE: $y'' + y' - 2y = 0$
PS: $y(x) = 2e^x + e^{-2x}$, $y'(x) = 2(e^x - e^{-2x})$
IC: $y(0) = 3$, $y'(0) = 0$
W: 0.01

31. DE: $y'' - 4y = 0$
PS: $y(x) = (e^{-2x} + 3e^{2x})/4$, $y'(x) = (3e^{2x} - e^{-2x})/2$
IC: $y(0) = 1$, $y'(0) = 1$
W: 0.01

32. DE: $Q''(t) + 4Q'(t) + 20Q(t) = 16e^{-2t}$
PS: $Q(t) = e^{-2t}(1 + \sin 4t + \cos 4t)$,
$Q'(t) = e^{-2t}(2\cos 4t - 6\sin 4t - 2)$
IC: $y(0) = 2$, $y'(0) = 0$
W: 0.01

33. DE: $y''(x) = 3x - 2$
PS: $y'(x) = (3x^2 - 4x - 5)/2$, $y(x) = x^3/2 - x^2 - 5x/2 + 2$
IC: $y(0) = 2$, $y'(0) = -2.5$
W: 0.01

34. DE: $y'' - y = x$
PS: $y(x) = -x + e^x/2 - 3e^{-x}/2$, $y'(x) = -1 + e^x/2 + 3e^{-x}/2$

IC: $y(0) = -1,\ y'(0) = 1$
W: 0.01

35. DE: $y'' - 3y' + 2y = 4e^{2x}$
PS: $y = -7e^x + 4e^{2x} + 4xe^{2x},\ y' = -7e^x + 12e^{2x} + 8xe^{2x}$
IC: $y(0) = -3,\ y'(0) = 5$
W: 0.01

36. DE: $D^2q/dt^2 + 8\, dq/dt + 25q = 150$
PS: $q(t) = 6 - e^{-4t}(6 \cos 3t + 8 \sin 3t),\ q'(t) = 50e^{-4t} \sin 3t$
IC: $q(0) = 0,\ q'(0) = 0$
W: 0.01

37. DE: $D^2y/dx^2 - xdy/dx + y^2 = 0$
PS: $y(x) = 1 + 2x - x^2/2 - x^3/3 - x^4/3 + \ldots,$
$y'(x) = 2 - x - x^2 - 4x^3/3 + \ldots$
IC: $y(0) = 1,\ y'(0) = 2$
W: 0.01

38. DE: $D^2y/dx^2 - (dy/dx)^2 - xy = 0$
PS: $y(x) = 4 - 2x + 2x^2 - 2x^3 + 19x^4/6 + \ldots,$
$y'(x) = -2 + 4x - 6x^2 + 38x^3/3 + \ldots$
IC: $y(0) = 4,\ y'(0) = -2$
W: 0.01

39. DE: $D^2y/dx^2 - 2x(dy/dx)^2/y = 0$
PS: $y = [8(x + 2)/(3 - x)]^{1/5},\ y' = [8/(3 - x)^2]\,[8(x + 2)/(3 - x)]^{-4/5}$
IC: $y(2) = 2,\ y'(2) = 0.5$
W: 0.01

40. DE: $y'' - 5y' + 6y = 2e^x$
PS: $y(x) = e^x,\ y'(x) = e^x$
IC: $y(0) = 1,\ y'(0) = 1$
W: 0.01

41. DE: $y'' + y = \sin 2x$
PS: $y(x) = 2 \sin x/3 - \sin 2x/3,\ y'(x) = 2(\cos x - \cos 2x)/3$
IC: $y(0) = 0,\ y'(0) = 0$
W: 0.01

42. DE: $y'' + y' - 2y = 0$
PS: $y(x) = e^x,\ y'(x) = e^x$
IC: $y(0) = 1,\ y'(0) = 1$
W: 0.01

43. DE: $D^2y/dx^2 + y = x$
    PS: $y(x) = x + \cos x - 3 \sin x,\ y'(x) = 1 - \sin x - 3 \cos x$
    IC: $y(0) = 1,\ y'(0) = -2$
    W: 0.01
44. DE: $D^2y/dx^2 + 4y = 0$
    PS: $y(x) = 2 \cos 2x + \sin 2x,\ y'(x) = 2(\cos 2x - 2 \sin 2x)$
    IC: $y(0) = 2,\ y'(0) = 2$
    W: 0.01
45. DE: $D^2y/dx^2 - Dy - 2y = 4x^2$
    PS: $y(x) = 2e^{2x} + 2e^{-x} - 2x^2 + 2x - 3,\ y'(x) = 4e^{2x} - 2e^{-x} - 4x + 2$
    IC: $y(0) = 1,\ y'(0) = 4$
    W: 0.01.

Use the program: g6FUNCSR.CPP for Analytic Numerical Series in Chapter 6:

Example 10.38

$$\text{Fsr: } \sum_{n=1}^{100} [(2 \times n - 1)/\text{pow}(2,n)],\ \text{Ssr: } 3.0.$$

Example 10.39

$$\text{Fsr: } \sum_{n=1}^{100} [\text{pow}(-1,n-1)/(2 \times n - 1) \times \text{pow}(3,n-1)],\ \text{Ssr: pi} \times \text{sqrt}(3)/6.$$

Example 10.40

$$\text{Fsr: } \sum_{n=1}^{100} [1/\text{n} \times (n+1) \times (n+2) \times (n+3)],\ \text{Ssr: } 0.5555.$$

Example 10.41

$$\text{Fsr: } \sum_{x=1}^{100} [\text{pow}(-1,x) \times \log(x)/x],\ \text{Ssr: } 0.183.$$

Example 10.42

$$\text{Fsr: } \sum_{x=1}^{100} [\text{pow}(-1,x-1) \times \tan(1/x \times \text{sqrt}(x))],\ \text{Ssr: } 1.308.$$

Example 10.43

$$\text{Fsr: } \sum_{x=1}^{100} [\log((x \times x + 1)/x \times x)],\ \text{Ssr: } 1.292.$$

Example 10.44

$$\text{Fsr: } \sum_{n=1}^{100} [(\sin(pi \times n/4))/n],\ \text{Ssr: } 1.19.$$

Example 10.45

$$\text{Fsr: } \sum_{n=1}^{100} [\sin(n)/\text{pow}(\log(3), n)],\ \text{Ssr: } 0.906.$$

Example 10.46

$$\text{Fsr: } \sum_{n=1}^{100} [\text{pow}(-1, n+1)/\text{sqrt}(n)],\ \text{Ssr: } 0.555.$$

Example 10.47

$$\text{Fsr: } \sum_{n=1}^{100} [\text{pow}(-1, n+1) \times (n \times n)/\text{pow}(2, n)],\ \text{Ssr: } 0.074.$$

Example 10.48

$$\text{Fsr: } \sum_{n=1}^{100} [n \times \text{pow}(a\sin(1/n), n],\ \text{Ssr: } 2.255.$$

Use the programs: g6POW1HM.CPP and g6POW2HM.CPP for Power series in Chapter 6:

1. DE: $y' - y = x$
   P(x): $p_1 = 1$, $p_2 = p_3 = p_4 = \ldots = 0$
   Q(x): $q_1 = -1$, $q_2 = q_3 = q_4 = \ldots = 0$
   R(x): $r_1 = 0$, $r_2 = 1$ $r_3 = r_4 = \ldots = 0$
   IC: $y(0) = 1$,
   Pss: $y = -1 - x + 2e^x = 1 + x + x^2 + x^3/3 + x^4/12 + x^5/60 + \ldots$

2. DE: $(x^2 - 1)y'' + xy' - y = 0$
   P(x): $p_1 = -1$, $p_2 = 0$, $p_3 = 1$, $p_4 = p_5 = \ldots = 0$
   Q(x): $q_1 = 0$, $q_2 = 1$, $q_3 = q_4 = \ldots = 0$
   R(x): $r_1 = -1$, $r_2 = 1$, $r_3 = r_4 = \ldots = 0$
   IC: $y(0) = 1$, $y'(0) = 2$,
   Pss: $y(x) = 1 + 2x - 0.5x^2 - 0.125x^4 - 0.0625x^6 - \ldots$

3. DE: $y'' - 2x^2y' + 4xy = x^2 + 2x + 2$
   P(x): $p_1 = 1$, $p_2 = p_3 = p_4 = p_5 = 0$
   Q(x): $q_1 = 0$, $q_2 = 0$, $q_3 = -2$, $q_4 = q_5 = 0$
   R(x): $r_1 = 0$, $r_2 = 4$, $r_3 =$, $r_4 = \ldots = 0$
   S(x) $s_1 = 2$, $s_2 = 2$, $s_3 = 1$, $s_4 = s_5 = 0$
   IC: $y(0) = 1 = y'(0)$
   Pss: $y(x) = 1 + x + x^2 - 0.33333x^3 - 0.0833x^4 - 0.0222x^6 - \ldots$

4. DE: $y'' - xy' + y = x$

   P(x): $p_1 = 1$, $p_2 = p_3 = p_4 = p_5 = 0$

   Q(x): $q_1 = 0$, $q_2 = -1$, $q_3 = q_4 = q_5 = 0$

   R(x): $r_1 = 1$, $r_2 = r_3 =$ $r_4 = \ldots = 0$

   S(x): $s_1 = 0$, $s_2 = 1$, $s_3 = s_4 = s_5 = 0$

   IC: $y(0) = 0 = y'(0)$

   Pss: $y(x) = x^3/3! + 2x^5/5! + 2!\,2^2\,x^7/7! + \ldots$

Use the program: g5FOUSER.CPP for Fourier Series in Chapter 6:

1. $Fx = x \times \cos(x)$ on (0, 1)

   $m = 5$, $n = 4$, $x = 0.1 \times 1$

2. $Fx = \cos(2 \times x)$ on (0, 1)

   $m = 5$, $n = 3$, $x = 0.1 \times 1$

3. $Fx = \cos(\text{pi} \times x) - 2 \times \sin(\text{pi} \times x)$ on (0, 1)

   $m = 5$, $n = 4$, $x = 0.1 \times 1$

4. $Fx = x \times x \times \cos(x)$ on (0, 1)

   $m = 5$, $n = 4$, $x = 0.1 \times 1$.

Solve the following Boundary value problems using the programs: g7BVP2LI.CPP and g7BVP2NL.CPP.

1. BVP: $D^2y/dx^2 - y\cos^2 x + \sin x\, e^{\sin x} = 0$

   $P = 0$, $Q = \cos^2 x$, $R = -\sin x\, e^{\sin x}$

   PS: $y(x) = e^{\sin x}$

   IC: $y(0)$: 1, $y(\pi)$: 1, $0 \le x \le \pi$

   W: 0.03141593

2. BVP: $D^2y/dx^2 + 2xy' - y = 2(1 + x^2)\cos x$

   $P = -2x$, $Q = 1$, $R = 2(1 + x^2)\cos x$

   PS: $y(x) = x \sin x$

   IC: $y(0)$: 0, $y(\pi/2)$: $\pi/2$, $0 \le x \le \pi/2$

   W: 0.01570796

3. BVP: $D^2y/dx^2 + 9y = \sin x$

   $P = 0$, $Q = -9$, $R = \sin x$

   PS: $y(x) = (\sin x + \sin 3x)/8 + \cos 3x$

   IC: $y(0)$: 1, $y(\pi/2)$: 0, $0 \le x \le \pi/2$

   W: 0.01570796

4. BVP: $y'' - (2x - 1)y' - 2y = 1 - 4x$

   $P = 2x - 1$, $Q = 2$, $R = 1 - 4x$

   PS: $y(x) = e^{x(x-1)}$

   IC: $y(0)$: 1, $y(1)$: 2, $0 \le x \le 1$

   W: 0.01

5. BVP: $y'' + 4y = 2$
   $P = 0,\ Q = -4,\ R = 2$
   PS: $y(x) = (1 - \cos 2x/\cos 2)/2$
   IC: $y(-1)$: 0, $y(1)$: 0, $-1 \le x \le 1$
   W: 0.02

6. BVP: $y'' + y + 1 = 0$
   $P = 0,\ Q = -1,\ R = -1$
   PS: $y(x) = \cos x/\cos(1) - 1$
   IC: $y(-1)$: 0, $y(1)$: 0, $-1 \le x \le 1$
   W: 0.02

7. BVP: $y'' + y - 1 = 0$
   $P = 0,\ Q = -1,\ R = 1$
   PS: $y(x) = \sin x/\sin(1) + 1$
   IC: $y(0)$: 1, $y(1)$: 2, $0 \le x \le 1$
   W: 0.01

8. BVP: $y'' + \pi^2 y = 0$
   $P = 0,\ Q = -\pi^2,\ R = 0$
   PS: $y(x) = 2\cos \pi x$
   IC: $y(0)$: 2, $y(1)$: –2, $0 \le x \le 1$
   W: 0.01

9. BVP: $y'' - 10y' + 25y = 0$
   $P = 10,\ Q = -25,\ R = 0$
   PS: $y(x) = e^{5x}(1 - x)$
   IC: $y(0)$: 1, $y(1)$: 0, $0 \le x \le 1$
   W: 0.01

10. BVP: $y'' + y = 0$
    $P = 0,\ Q = -1,\ R = 0$
    PS: $y(x) = \cos(x) - \sin x/\tan(1)$
    IC: $y(0)$: 1, $y(1)$: 0, $0 \le x \le 1$
    W: 0.01

Solve the partial differential equations in Chapter 8 using the programs: g8PARWAV.CPP, g8PARDIF.CPP, g8PARLAP.CPP and g8PARPOI.CPP.

1. DE: $u_{tt} - 4u_{xx} = 0,\ (0 < x < 1,\ 0 < t)$
   BC: $u(0, t) = u(1, t) = 0,\ 0 < t,$
   IC: $u(x, 0) = \sin(\pi x)$ and $u_t(x, 0) = 0,\ 0 \le x \le 1,$
   ES: $u(x, t) = \sin(\pi x)\cos(2\pi t),\ m = 4,\ n = 4.$

2. DE: $u_{tt} - 1/(16\pi^2)\, u_{xx} = 0$, $(0 < x < 0.5, 0 < t)$
   BC: $u(0, t) = u(0.5, t) = 0$, $0 < t$,
   IC: $u(x, 0) = 0 < x < 0.5$,
   $u_t(x, 0) = \sin\,(4\pi x)$, $0 \le x \le 0.5$,
   ES: $u(x, t) = \sin t \sin\,(4\pi x)$, $m = 4$, $n = 4$.

3. DE: $u_t - u_{xx} = 2$, $(0 < x < 1, 0 < t)$
   BC: $u(0, t) = u(1, t) = 0$, $0 < t$,
   IC: $u(x, 0) = \sin\,\pi x + x(1 - x)$, $0 \le x \le 1$,
   ES: $u(x, t) = e^{-\pi^2 t} \sin\,\pi x + x(1 - x)$, $h = k$.

4. DE: $u_t - 1/16\, u_{xx} = 0$, $(0 < x < 1, 0 < t)$
   BC: $u(0, t) = u(1, t) = 0$, $0 < t$,
   IC: $u(x, 0) = 2 \sin\, 2\pi x$, $0 \le x \le 1$,
   ES: $u(x, t) = 2e^{-\pi^2/4} \sin\, 2\pi x$, $h = k$.

5. DE: $u_{xx} + u_{yy} = 0$, $1 < x < 2$, $0 < y < 1$;
   BC: $u(1, y) = \log\,(y^2 + 1)$, $u(2, y) = \log\,(y^2 + 4)$, $0 \le y \le 1$,
   $u(x, 0) = 2 \log x$, $u(x, 1) = \log\,(x^2 + 1)$, $1 \le x \le 2$,
   ES: $u(x, y) = \log\,(x^2 + y^2)$, $m = 4$, $n = 4$.

6. DE: $u_{xx} + u_{yy} = 0$, for $0 < x < 1$, $0 < y < 1$;
   BC: $u(0, y) = 0$ $u(1, y) = y$, $0 \le y \le 1$,
   $u(x, 0) = 0$, $u(x, 1) = x$, $0 \le x \le 1$,
   ES: $u(x, y) = xy$, $m = 4$, $n = 4$.

7. DE: $u_{xx} + u_{yy} = -\pi^2 \sin \pi x \sin \pi y$, for $1 < x < 2$, $0 < y < 1$;
   BC: $u(0, y) = 1$, $u(1, y) = y$, $0 \le y \le 1$,
   $u(x, 0) = 1 - x$, $u(x, 1) = 1$, $0 \le x \le 1$,
   ES: $u(x, y) = 1 - x + xy\ (\sin \pi x \sin \pi y)/2$, $m = 4$, $n = 4$.

8. DE: $u_{xx} + u_{yy} = -2$, for $0 < x < 1$, $0 < y < 1$;
   BC: $u(0, y) = 0$, $u(1, y) = \sinh \pi \sin \pi y$, $0 \le y \le 1$,
   $u(x, 0) = x(1 - x) = u(x, 1) = 1$, $0 \le x \le 1$,
   ES: $u(x, y) = \sinh \pi x \sin \pi y + x(1 - x)$, $m = 4$, $n = 4$.

9. DE: $u_{xx} + u_{yy} = (x^2 + y^2) \cos xy - \cos \pi x$, for $0 < x < 1$, $0 < y < 1$;
   BC: $u(0, y) = 1/\pi^2 - 1$, $u(1, y) = -1/\pi^2 - \cos y$, $0 \le y \le 1$,
   $u(x, 0) = \cos \pi x/\pi^2 - 1$, $u(x, 1) = \cos \pi x/\pi^2 - \cos x$, $0 \le x \le 1$,
   ES: $u(x, y) = \cos \pi x/\pi^2 - \cos xy$, $m = 4$, $n = 4$.

11

A Short Review on C++

C++ is an Object-Oriented Programming language (OOP) that allows us to take advantage of modern methodology for labour-saving features. Turbo C++ (version 3.0) uses the industry-standard DPMI (DOS Protected Mode Interface). It requires a 286-processor or higher, DOS version 3.11 or higher and 1 MB extended memory.

The language C++, Initially named “C With classes”, was developed by Bjarne Stroustrup in 1980, while engaged at AT & T Bell Laboratories in Murray Hill, New Jersey, USA. In 1983 it had been changed to a standard name C++.

Specifically, C++ is built on the foundation of C language. All C++ compilers can also be used to compile C programs. In fact, C++ is a superset of C. That is to say, it is an enhanced version of C. The language C was first implemented by Dennis Ritchie in the 1970s on the computer type PDP-11 from Digital Equipment Corporation.

Turbo C++ is a powerful compiler. It is very fast and efficient compiler for creating practically any application program. Turbo C++ provides an incorporate atmosphere for further development of the program. It includes C++ and ANSI C. ANSI is the abbreviation for American National Standards Institute, founded in 1918.

Basically, the object is the main attraction of object-oriented programming. An *object* is a data type that has structure and state. Every object defines operations which can have access on that state.

The Integrated Development Environment (IDE), the Programmer’s platform, has everything one needs to write, edit, compile, link and debug the programs. Most of what we see and do in the IDE happens in a Window, that is nothing but a screen area we can open, close, move, resize, zoom, tile and overlap. Only one window can be active at any time. There are three visible components to the IDE: the Menu bar at the top, the window area in the middle and status line at the bottom of the screen. The IDE menu

File Edit search Run Compiler Debug Windows Project Options, etc.

11.1 Basic Structure of a C++ Program

A typical program written in C++ language contains basically five sections as represented by the constructional shown in Fig. 11.1.

Preprocessor Directives: INCLUDE files
Common definition block defined by the symbol #
CLASS declaration
Class: definition of member functions and data type items
Main function with Main (): entry of the program

Fig. 11.1

Every function and every file must be signified by a name followed by a parenthesized parameter list. The list consisting of the names and types of variables will show what type of information it contains. If the parenthesis is empty (void), the function works without any parameter. The body of the function is enclosed "{ }". The function terminates and returns to the next calling function after the closing brace is reached. A return statement has the same function as the closing brace.

11.2 The Keywords from C++

Keywords in C++ are the reserved words that are used for some special purposes and must not be employed as identifier names in C++ program. *Identifier* names are the user-defined names. In the discussion about identifiers afterwards you can find the distinction between the keywords and identifiers. I make use of the following very common keywords in the programs designed for the approximation of the Differential Equations. Some of them are apparently simple to understand and the others have been explained in this chapter in course of time.

auto	bool	break	case
catch	char	class	const
const_cast	continue	default	delete
do	double	else	endl
enum	extern	explicit	false
float	for	friend	goto
if	inline	int	interrupt
long	mutable	namespace	new
operator	private	protected	public
register	return	short	signed
sizeof	static	struck	switch
template	throw	true	try
typedef	union	unsigned	using
virtual	void	volatile	while

The case of the keywords is very significant. All the keywords must be typed in lowercase, that is one of the requirements of C++ program.

11.3 Identifiers in C++

In C++ program an identifier is defined as a name that is used for a function, a variable or any other defined item. An Identifier is allowed to begin only with any letter of the alphabets or the underscore character. The rest may be letters (a through z) or digits (0 through 9). Here I mention some names as acceptable identifiers.

Temp_file ; myProgram ; first_name ; _file01 ; macro_type ; Name10; address3 ; etc.

Identifiers are case sensitive. This means that *amount* and *Amount* are completely separate variables. 1989_taxpayment and stop! are not legal identifiers, because of beginning with a digit and containing an exclamation point.

You can use letters or underscores to separate words in a long identifier. PricePer100 or price_per_100 are much easier to read than priceper100.

11.4 The Features of C++

The characteristics of C++ can be described with the help of the examples that we want to represent now.

Example 11.1

```
# include <iostream.h>
main ()
{
   inc rice ;
   float rupees,rate ;
   cout <<"How many rupees did you get Rs.  ";
   cin >> rupees ;
   cout <<"For how may kilos rice? ";
   cin >> rice ;
   rate = rupees/rice ;
   cout.precision(2) ;
   cout <<"You have received Rs. "<<rate <<"for one kilo rice /n" ;
   count <<"Type (1) to end the program :";
   cin >> rice;
   return 0 ;
}
```

The first line represented in a comment. A *comment* is identified by the symbol // (double slash). Basically, a comment followed by double slash is a single line comment. A multiline comment would be surrounded by the symbols, starting with /* (a slash followed by an asterisk) and ending with */ (asterisk followed by a slash). The compiler disregards the text that is enclosed by these symbols.

The second line, # include <iostream.h>, tells the compiler to add the C++ *header file* iostream.h to Example 11.1 before compiling. This file contains the declarations and functions for the input and output library. A *library* is a collection of ready to use functions which the program can call to take care of programming chores. The file allows us to apply the *cout* and *cin* operators.

The next line, main (), defines a function. A *function* is a group of related program instructions. Every C++ program must have a main function, which is where program execution begins. Functions are the building blocks of C++ program. The open "{"indicates the beginning of a group of program instructions or statements. In this case the statements which define what will happen when the *function* *main* is executed. Each group of statements ends with a closing brace”}”. Notice that each statement ends in a semicolon (;).

The operators << and >> are the insertion (“put to”) and extraction (“get form”) operators. The symbols define the *cout* statement and the *cin* statement respectively. The cin statements get the values for rupees and rice and store them in the variables named. Cout is the standard output stream and cin is the standard input stream. The code “/n”, the most commonly used escape sequence, is the new line character that places the cursor at the beginning of the next line.

The statement “return 0” terminates the main program and causes it to return the value 0 to the calling process that is typically the operating system. For most operating system, a return value of 0 signifies that the program terminated normally.

11.5 Loop Statements

Example 11.2

/* How a long series of *if and else if* statements is tedious to write, confusing and prone to error the next program will show you. */

```
#include <conio.h>
#include <iostream.h>
#include <ctype.h>

int main()
{   char cmd;
    cout <<“Menu desired : Rice Mutton Chicken Hilsafisch Prawn”;
    cout<<“\n Press first letter of the Menu you want: ”;
    cmd = toupper(getch());
    cout <<‘\n’;
    if (cmd == ‘R’)
    cout <<“Cooking Biriyani with rice\n”;
    else if (cmd == ‘M’)
    cout <<“Do you need Mutton for Biriyani? \n”;
  else if (cmd ==’C)
    cout <<“Doing chap with Chicken\n”;
  else if (cmd ==’H’)
    cout <<“Can you make boiled Hilsa ?\n”;
  else if (cmd == ‘P’)
    cout <<“Doing Prawn malaicurry\n”;
  else cout <<“Invalid choice.\n”;
  cin >> cmd;
  return 0;
}
```

The first header file conio.h signifies that we use getch() or getche(). The program displays a menu line, then gets a value for cmd through the function getch. Along the way the value is passed to the toupper function. This ensures that you only need to deal with *uppercase* characters. The other header file ctype.h supports character handling. The series of if and else if branches then test for each valid value and execute the corresponding function. The last else serves as the *default* case, handling invalid values.

Example 11.22 can be rewritten with a *switch* statement that makes these multipath branches easier to code.

Example 11.3

```
#include <conio.h>
#include <iostream.h>
#include <ctype.h>

int main()
{
    char cmd;
    cout << “Menu desired: Rice Mutton Chicken Hilsafisch Prawn”;
    cout << “\n Press first letter of the Menu you want: ”;
    cmd = toupper(getch());
    cout <<‘\n’;
    switch (cmd)
     {
       case ‘R’: cout <<“Cooking Biriyani with rice. \n”; break ;
      case ‘M’: cout <<“Do you need Mutton for Biriyani? \n”; break ;
       case ‘C’: cout <<“Doing chap with Chicken. \n”; break ;
       case ‘H’: cout <<“Can you make boiled Hilsa ?\n’: break ;
       case ‘P’: cout <<“Doing Prawn malaicurry. \n”; break ;
         default: cout <<“Invalid choice. Try again \n”;
     }
      cin >> cmd;
      return 0;
}
```

The value is tested against the value for the first *case*. This is executed until the end of the *switch* statement or until the special statement break is reached. Then all the different values for cases are tested. If the desired value is not found the *default* statement is executed. The *default* is optional. When no conditional choice is met, then it switches to *default*.

The *break* statement at the end of each *case* is very important. It causes execution to jump past the end of the *switch* statement. When we remove the break at the end of case ‘H’ and select ‘H,’ we will see

```
Can you make boiled Hilsa?
Doing Prawn malaicurry.
```

In other words, leaving out the break statements execution continues until it finds a break or the end of the *switch* statement.

The most significant characteristic of the if, else if and switch statements in that they perform their test only once. But many computer tasks involve repetition. The same process on each item in the file needs repeated executions, a kind of looping. Loops cause a statement or a series of statements to be executed repeatedly, monitoring a specified condition in order to determine when to stop. C++ provides three kinds of loops: while, do and for.

The while loop executes one or more statements as long as a specified condition is true.

11.6 The For Loop

The *for* loop is one of C++'s three loop statements. It allows one or more statements to be repeated. The *for* loop is considered by many C++ programmers to be its most flexible loop.

It allows a large number of variations, which makes it applicable to a much wider array of programming tasks than you might otherwise think. In its most common form, the *for loop* is used to repeat a statement or block of statements a specified number of times.

Example 11.4

```
/* Introduction for nested loop. */

#include <iostream.h>
#include <string.h>
#include <conio.h>

int main()
{
   int i;
   char text[80];
   cout <<Type \"end\" to quit\n";
   for(cin.getline(text,80);strcmp(text,"end") !=0; cin.getline(text,80))
  {
     for(i = 1; i <= strlen(text); i++)
     cout <<"–"
     cout << '\n';
  }
     return 0;
}
```

A new header file *string.h* has been included for the string characters. char text[80] which reserves 80 characters in the variable named *text* type char.

include is a *preprocessor* directive. It is an instruction to the compiler for inclusion into another source file. The expression <string.h> is termed as *header file* that contains information to support C++ I/O system. C++ language has been provided with several header files that are very useful for the development of the program.

Each preprocessor directive begins with a symbol #. The C++ preprocessor contains the following directives:

# if	# ifdef	# else
# else if	# endif	# include
# define	# line	# error

The first *for* loop gets the input string with the function cin.getline (text, 80), initialising the *for* loop and storing the string in the character array text. Then it compares this string to the string "end" with the strcmp function. If the comparison yields a 0, the strings are identical and the loop exists. If the value is other than 0, the nested for loop is executed.

The nested *for* loop then prints a number of hyphens equal to the length of the string that was originally entered using the *strlen* function to find out how long it was. The cout statement that ends the body of the initial for loop, positions the cursor for entry of the next line.

11.7 Pointers

Every variable has a unique memory address that indicates the beginning of the memory are occupied by its value. The amount of area used depends on the type of data involved.

In the case of an int, this area is 2 bytes long, while a float uses 4 bytes. For an array the area occupied is equal to the number of elements times the size needed for one value of the declared data type. For a structure the area used is equal to the sum of the areas needed for the structure's members, plus some padding needed.

Because in all cases data is stored in an orderly, predictable way, it is possible to access data by using a variable that contains the relevant address. Such a variable is known as *Pointer*.

The pointers are very useful. They allow us to access and manipulate structured data easily without having to move the data itself around in memory. Pointers can also be used to allow a function to receive and change the value of a variable. This can avoid the need for declaring global variables.

A pointer declaration takes the form

```
type *name
```

where type is any data type. Some examples for pointer declarations are as follows:

```
int *intptr;          // Points to an integer
float *fltptr;        // Points to a floating-point value
char *string;         // Points to a character value
```

*intptr yields the value stored at the address in the pointer intptr.

Let us see how to handle strings declaring a pointer to character and using it to manipulate the string.

Example 11.5

```
#include <iostream.h>
int main()
{
   char name[40];
   char *str_ptr = name;
   int pos, num_char;
   cout << "Enter a string(name) for the character array : ";
   cin.get(name,40,'\n');
   cout <<"How many characters do you want to extract? ";
   cin >> num_char;
   for (pos = 0; pos < num_char; pos++)
   cout  <<*str_ptr++;
   cout <<'\n';
   return 0;
}
```

If we enter the name "kolkatawala" the variable name[0] is *k* and name[7] is a and so on. *str_ptr++ means that the pointer is incremented each time so that it points to the next value.

Example 11.6

/* The program uses a two-dimensional array of pointers to create a string table that links apple varieties with their colours. Enter the name of apple and the program will tell you its colour.

```
#include <iostream.h>
#include <string.h>
#include <stdio.h>

char *p[][2] =
{
  "Red delicious", "red",
   "Golden delicious", "yellow",
   "Winesap",  "red",
   "Gala", "reddish orange",
   "Lodi", "green",
   "Mutsu", "yellow",
    "Cortland", "red",
   "Jonathan", "red",
    "", ""                      // terminate the table with null strings
};
main()
{   int i;
   char apple[80];
   cout <<"Enter the name of apple:";
   gets(apple);
   for(i=0, *p[i][0]; i++)
   {
     if(!strcmp(apple, p[i][0]))
     cout <<apple<<"is"<<p[i][1] <<"\n";
   }
   cout <<"Type (1) to exit:"; cin >>i;
   return 0;
}
```

Look carefully at the condition controlling the for loop. The expression *p[i] [0] gets the value of the first byte of the *i*-th string. Since the list is terminated by null strings, this value will be zero (false) when the end of the table is reached. In all other cases it will be non-zero and the loop will repeat.

At this stage we represent the layout of a complete program that is written in C++ Turbo, is provided to solve the *area of plane curves* involved in definite integral by means of the SIMPSON's Formula and is processed by the operating system MS-DOS.

Example 11.7

/* *Program for solving area of plane curves involved in definite integral*.

Programname: G2SURINT.CPP

Lb = lower limit, Ub = upper limit, K = superscript of m, total number of equal divisions ($2^{**}K = m$) and w = (Ub – Lb)/m = width of each division. We suppose the interval [Lb, Ub] to be divided into m equal parts each of length w.

Run-time ERROR messages arise after the program has successfully compiled and when it is set for running. I have dealt with the error handling in the program to intercept the mathematical functions, such as sqrt(–1), log(–1), division by 0, sqrt (cos pi), etc. The program does not complete its proper course when an error in the mathematical problem has been detected by it. An error message stops the program running.

The estimation of the Definite Integral has been performed by setting the lower and upper limits, Lb & Ub, and the integrand Fx in the program.

```
    ++++++++++++++++++++++++++++++++++++++++++++++
                        Important to note !!!
    ++++++++++++++++++++++++++++++++++++++++++++++
    1) Update the initial Conditions Lb & Ub & K    (symbolised by $)
    2) Change the Functions Fx & ES                 (symbolised by $)
    ++++++++++++++++++++++++++++++++++++++++++++++
*/
//######################################  HEADER Files
#include <iostream.h>
#include <conio.h>
#include <math.h>
#include <errno.h>
#include <stdio.h>
#include <stdlib.h>
#include <dos.h>
#include <float.h>
//#####################################   Define the Symbols
#define Prgr "Program :"
#define Atitle "ESTIMATION OF AREA OF PLANE CURVE"
#define pi 3.14159265
#define eps 2.71828182
######################################### MAIN function
main()
{   _clear87();
   _control87(MCW_EM,MCW_EM);
   int i,j,m,Ck;
   char ans,rep;
   double ES,Esol,Fx,Farea,Aerr,limc,w;
   double F0=0.0,sumE=0.0,sumO=0.0,sumF=0.0,x=0.0;
   double Lb,Ub,K;
   double Ftemp[4096];
   char Pgnam[]="g2SURINT";
//########################################   FUNCTIONS control
   clrscr();
```

```
    gotoxy(60,1);  cout  <<Prgr;
    textcolor(0);textbackground(15);
    cprintf(Pgnam);
    textcolor(15);textbackground(0);
  SetFunc:
    gotoxy(3,8);
    cout <<"\aHave you set up the Functions";
    gotoxy(3,9);
    cout <<"Fx & Es with the ICs (";
    textcolor(0);textbackground(15);
    cprintf("y/n");
    textcolor(15);textbackground(0);
    cout <<")?";
    ans=getche();
    switch(ans)
    { case  'y':
          break;
      case 'n';
          cout << '\a';
        sleep(1);
        return 0;
      default:
          gotoxy(3,11);
          cout <<"\aWRONG INPUT ... Try again"; sleep(1);
          gotoxy(3,11);
          cout <<"           ";
          goto SetFunc;
}
clrscr();
//$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$          Initial Setting
    Lb = 1.0;
    Ub = eps;
    K = 6;
//#####################################          Escalator Block
    Ck = 13;
    limc = Ub–Lb;
    m = pow(2,K);
    w = limc/m;
    For (i=1; i<=m+1; i++)
    {   x = Lb + (i-1)*w ;
//$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$    UPDATING Center for FX & ES
      Fx = log(x)   ;
      ES = 1.0   ;
```

```
//########################################    Confine  Test
   if ( limc <= 0 )
   {   gotoxy(3,8);
       cout <<""
       gotoxy(25,8);
       textcolor(0);textbackground(15);
       cprintf("\a\aWrong Boundary CONDITIONS ");
       sleep(2);
       gotoxy(25,10);
       cprintf("\a\aCheck");
       cout<<"    Lb & Ub";
       sleep(3);
       return 0;
   }
   if (K >=Ck)
   { gotoxy(3,8);
     cout <<"     ";
     gotoxy(25,8);
     textcolor(0);textbackground(15);
     cprintf("\a\aWrong SUPERSCRIPT");
     sleep(2);
     gotoxy(25,10);
     cprintf)"\a\aCheck");
     cout<<" K (K <=12)";
     sleep(3);
     return 0;
   }
//########################################        Function Calculation
     Ftemp[i-1]= Fx;
     if (i==1)
     F0=Ftemp[i-1];
   }
   sumF=F0 + Ftemp[m];
   sumO=Ftemp[1];
   for(j=2; j<=m-1; j=j+2)
   {
   sumE=sumE + Ftemp[j];
   sumO=sumO + Ftemp[j+1];
   }
   Farea= (sumF + 4*sumO + 2*sumE)*w/3.0;
   Esol= ES;
   Aerr=Esol-Farea;
//########################################        Print out the Heading
   clrscr();
```

```
   gotoxy(60,1);
   cout << Prgr;
   textcolor(0);textbackground(15);
   cprintf(Pgnam);
   gotoxy(5,3);
   cprintf(Atitle);
   textcolor(15);textbackground(0);
//######################################          Print out the RESULTS
   cout.precision (5);
   gotoxy(5,5);
   cout <<" Lower Limit :" << Lb;
   gotoxy(25,5);
   cout <<', Upper Limit : :<< Ub;
   gotoxy(5,6);
   cout <<"Width : "<< w ;
   gotoxy(25,6);
   cout <<", Division     : " << m;
   gotoxy(5,7);
   cout <<"==================================================";
   cout.precision(8);
   gotoxy(5,9);
   cout <<"Estimated Value of the Integral : " <<Farea ;
   gotoxy(5,11);
   cout <<"Exact Value of the Integral : " <<Esol;
   cout.precision(5);
   gotoxy(5,13);
   cout <<"Error for approximation : " << Aerr;
   gotoxy(5,14);
   cout <<"==================================================";
//############################################        WAY to go OUT
TryOnce:
   gotoxy(12,16);
   cout << "\aType ( ";
   textcolor(0);textbackground(15);
   cprintf("e");
   textcolor(15);textbackground(0);
   cout <<" ) to exit : " ;
   rep=getche();
   switch(rep)
   {
      case 'e':
         gotoxy(9,20);
         cout <<"\aThat's rigth ..."; sleep(1);
         gotoxy(9,20);
```

```
          cout <<"          ";
          break ;
       default :
          gotoxy(10,20);
          cout <<"\aWRONG CHOICE ...";sleep(1);
          gotoxy(15,19);
          cout <<"          ";
          gotoxy(10,20);
          cout <<"          ";
          goto TryOnce;
   }
Exit:
   return 0;
}
```

A multiline comment has been used here in this program surrounded by symbols, starting with /* and ending with */. The compiler has no influence on this text that describes the programname, the important points to note, the possible difficulties to handle the program errors, etc.

#include is a preprocessor directive. It is an instruction to the compiler for inclusion into another source file. The header file <errno.h> contains the information to support the system reporting error messages.

C++ language has been provided with several header files that are very useful for the development of the program. In the end we mention some of them with their properties.

A *library function* has been conducted by the corresponding *header file*. The header files provide among other matters the prototypes for the library functions. Hence, whenever we use of library function in the program, the corresponding header must be included.

#define is also a directive that is employed to define an identifier for a character sequence or a definite value in the source file. This is called macro identifier. #define Prgr "Program" is a command that yields the instructions to the preprocessor to substitute the string: Program when the macro name is encountered. The substitution process is known as macro substitution. Each directive begins with the symbol #.

```
//############################################   MAIN function
```

Main is the gate of the entry to the program. Each C++ program must have a single external function named *main* that is fundamentally a group of related program instructions. If a program consists of many function including *classes* and other program elements the execution of the program begins with the functions called main(). This is the real entrance of the complete program and without the part the linker does not work at all. About classes we discuss then.

The variables required for the execution of the program are set, defined and initialised. They are always accessible for the required function.

```
double Ftemp[4096] ;
```

is a one-dimensional array with data type double. With this instruction C++ stores 4096 elements under the name Ftemp in a contiguous memory location. The arrays have been placed in order, such as, Ftemp[0], Ftemp[1], ..., Ftemp[4095].

In *Confine Test* section two controls have been undertaken. If the value of *limc* is less than or equal zero or the value of *K* is greater than fixed number, then the error messages such as *wrong boundary conditions* or *wrong superscript* will appear on the screen.

Print out the Results section prints the final results of the problem with the necessary variables. Cout.precision (8) means a number with 8 decimal places. Finally the program ends when it finds the statement: return 0.

In addition to C++'s built-in data types we may also define our own types of data. There are several ways to do this. The most common are the *structure*, *union* and the *CLASS*. How does C++ change the way we work with code and data? One important way is *encapsulation*: the welding of code and data together into a single class-type object.

In C++ a single-class identity (defined with *struct*, *union* or *class*) combines functions (known as member function) and data (known as data members). We usually given a class a useful name that is user-defined. With C++ we can control access to class members (code and data) by declaring individual members as *public*, *private* or *protected*.

A class defined with struct is simply a class in which all the members are public by default. But this arrangement can be varied. A class defined with union has all its members public. This access level cannot be changed. In a class defined with class the members are private by default.

The following example shows how the *Class* with public class members works.

Example 11.8

/* The program is constructed to approximate an *Initial Value problem* applying the Runge-Kutta Formulas. The Initial Value Problems pertaining to several differential equations have been treated by this program that is written in C++ and is executed on the system MS-DOS.

Estimation of the differential equation has been performed by setting the function PSy in the program and the selected function for Ivdx and Ivdy in MEMBER Function CE/PS. The initial values *x*0 and *y*0 that are the respective initial conditions for *x* and *y* have been put when called for during the execution of the program. The width *w* is the step size or increment at each interval and *m* is number of divisions.

Sometimes in the final results the two values: approximate and exact do not tally within certain measure. It is due to the wrong input of the functions, set up by the variables: PSy/Ivdx/Ivdy. Each of the functions has been checked before getting a move on.

```
-----------------------------------------------------------------------------------------------------
Set the Function PSy and provide the terms Ivdx and Ivdy in MEMBER Functions
set DE & setACT (symbolised by $)
-----------------------------------------------------------------------------------------------------
*/
//####################################### The Header Files
#include <iostream.h>
#include <fstream.h>
#include <iomanip.h>
#include <conio.h>
#include <math.h>
#include <errno.h>
#include <stdio.h>
#include <stdlib.h>
#include <dos.h>
```

```
#include <float.h>
//########################################          Defines Titles / m / w
#define Prgr "Program : "
#define Pgnam "g4IVP1DE"
#define Xval "y(x-value)"
#define Appval "Approximation"
#define Extval "Exact value"
#define Aerr "Error for Appx."
#define m 10
#define w 0.01
#define pi 3.14159265
#define eps 2.71828182
extern int errno;
//########################################          Define CLASS
class Ivpeqn
{   int k;
    double  Fxy,Ivdx,Ivdy,Rgxy,PSy,Dx,Dy,x0y0;
    double  ACT[11],RK[11],Vx[11],Ecterr[11];
    double  x,y,X1Y1,Ftemp,F0,F1,F2,F3,F4;

public:
// $$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$    MEMBER Functions setDE/setACT
    void  setDE()
    {                                           // Coefficient functions of dx & dy
        IVdx  =  exp(x*x)*pow(y,3)-x*y  ;
        IVdy  =  1.0  ;
    }
    void  setACT()
    {                                           // Equation for exact solution
    PSy  =  exp(-x*(-x))/(4-2*x)  ;
    }
//############################################          Temporary setting
    void  setIC()
    { gotoxy(15,12);   cout <<"x0 =         ";
      cin >> x0 ;
      if (x0 == 99) { exit(-1); }
      gotoxy(15,13);       cout <<"y0 =    ";
      cin >> y0 ;
      if (y0 == 99) { exit(-1); }
    }
    void  setx0y0()
    { x=x0;  y=y0;           }
    void setTemp()
    {   RGxy  =  IVdx/IVdy;
```

```
      Fxy = RGxy*w;
   }
//####################################          Calculation Block
   void setVAL0()
   { k=0;  }
   void setVAL00()
   { Dx=x; Dy=y;  }
   void setVAL1()
   { X1=x; Y1=y;  }
   void setVAL2()
   { Ftemp=Fxy;
     F1=X1+w/2.0;
     F2=Y1+fxy/2.0;
     x=F1; y=F2;
   }
   void setVAL3()
   { Ftemp=Ftemp+2.0*Fxy;
     F3=Y1+Fxy/2.0;
     x=F1; y=F3;
   }
   void setVAL4()
   { Ftemp=Ftemp+2.0*Fxy;
     F0=X1+w;
     F4=Y1+Fxy;
     x=F0; y=F4;
   }
   void setVAL5()
   { Ftemp=Ftemp+Fxy;
     y=Y1+Ftemp/6.0;
     x=X+w;
   }
   void setVAL6()
   { k=k+1;
     ACT[k]=PSy;
     RK[k]=y;
     Vx[k]=x;
   }
//####################################          Test Block
   void set_yTEST()
   { if ( y == 0 )
     {    clrscr();
          gotoxy(60,1);   cout << Prgr;
          textcolor(0);textbackground(15);
          cprintf(Pgnam);
```

```
        gotoxy(25,8);   cprintf("\a\aUnexpected APPROXIMATE value"); sleep(3);
        gotoxy(25,10); cprintf("\a\aCheck :") cout<<"\a   IC / "; sleep(1);
        cout <<"\aIVdx / "; sleep(1);
        cout <<"\aIVdy / "; sleep(2);
        exit (-1};
    }
  }
  void Ectrlsec()
  { if (_status87() == 4|| errno != 0 )
    { clrscr();
        gotoxy(60,1);   cout << Prgr;
        textcolor(0);textbackground(15);
        cprintf(Pgnam);
        gotoxy(25,8);   cprintf("\a\aMath DOMAIN error"); sleep(2);
        gotoxy(25,10); cprintf("\a\aCheck");
        cout<<": IC / ";sleep(1);
        cout <<"\aIVdx / "; sleep(1);
        cout <<"\aIVdy"; sleep(2);
        exit (-1};
    }
  }
//#####################################           Heading Block
  void Sevtitle()
  { gotoxy(10,4);
    cout <<"[y( " <<setprecision(8) <<Dx <<" ) = " <<Dy <<",            ";
    cout <<"Step size : " << w << " ]" << endl;
  }
  void Extitle()
  { gotoxy(9,k+6);
    cout << Xval;
    gotoxy (29,k+6);
    cout<< Appval;
    gotoxy(47,k+6);   cout << Extval <<"    " << Aerr;
    gotoxy(9,k+7);        cout <<"____________";
    gotoxy(29,k+7);       cout<<"_____________";
    gotoxy(47,k+7);      cout <<"___________      ___________" << endl;
  }
  void Subdata()
  { k=k+1;
  Ecterr[k]=ACT[k]-RK[k];
  gotoxy(9,k+7);        cout <<"y( " <<setprecision(8) << Vx[k];
  gotoxy(23,k+7);    cout <<") : " << RK[k];
  gotoxy(47,k+7);    cout << ACT[k];
  gotoxy(62,k+7);    cout << setprecision(5) << Ecterr[k] <<endl;
```

```
   }
};
//####################################        CENTER PLACE of the Program
main()
   { lvpeqn deqn:
     _clear87();
     control87(MCW_EM,MCW_EM);
     int i,j,k;
     double F0,F1,F2,F3,F4,RGxy,ACTy;
     double x,y,x0,y0,X1,Y1;
     double ACT[11],RK[11],Vx[1];
     double Ecterr[11];
     deqn.setVAL0()
//####################################        Data Modification Block
   clrscr();
   gotoxy(60,1); cout << Prgr;
   textcolor(0);textbackground(15);
   cprintf(Pgnam);
   textcolor(15);textbackground(0);
Allset:
   Gotoxy(5,8)  cout << "\aHave you setup C++ & Ps Eqns. ( ";
   textcolor(0);textbackground(15);
   cprintf("y/n");
   textcolor(15);textbackground(0);
   cout << " ) ? ";
   ans=getche();
   switch (ans)
   { case 'y':
         break;
     case 'n':
         gotoxy (8,10);  cout <<"\aSorry ! will be back again"; sleep(2);
         gotoxy (8,10);   cout <<"          ";
         return 0;
   default :
         gotoxy (8,10);   cout <<"\aINVALID CHOICE ... Try again";sleep(1);
         gotoxy (8,10);   cout <<"          ";
         goto Allset;
   }
//##########################################        INITIAL Setting
Funcset:
   clrscr();
   gotoxy(60,1);          cout << Prgr;
   textcolor(0);textbackground(15);
   cprintf(Pgnam);
```

```
    gotoxy(5,8);  cprintf("Update the initial conditions ( IC )");
    textcolor(15);textbackground(0);
    gotoxy(5,9); cout <<"\a(Press ENTER after setting each value/ EXIT= 99)";
    deqn.setIC();
    PutIC:
    gotoxy(5,16);
    cout <<"\aAny CORRECTION( ";
    textcolor(0);textbackground(15);
    cprintf("y/n");
    textcolor(15);textbackground(0);
    cout <<" ) ?  ";
    ans=getche();
    switch(ans)
    { case 'y':
            got Funcset;
      case 'n':
            break;
      default :
            gotoxy (10,17);   cout <\aTry again ..."; sleep(1);
            gotoxy(10,17);    cout <<"         ";
            goto PutIC;
    }
    deqn. setx0y0();
    deqn.setVAL00();
//#######################################          The function F1
    for (i=1; i<=m;i++)
    { deqn.setVAL1();
      deqn.setDE();
      deqn.setTemp();
      deqn.Extr1sec();
//#######################################          The function F2
      deqn.setVAL2();
      deqn.setDE();
      deqn.setTemp();
//#######################################          The function F3
      deqn.setVAL3();
      deqn.setDE();
      deqn.setTemp();
//#######################################          The function F4
      deqn.setVAL4();
      deqn.setDE();
      deqn.setTemp();
//#######################################          PS-calculation Block
      deqn.setVAL5();
```

```
      deqn.set_yTEST();
      deqn.setACT();
      deqn.setVAL6();
    }
//#######################################          PRINT OUT THE RESULTS
    clrscr();
    deqn.setVAL0();
    gotoxy(60,1);
    cout <<Prgr;
    textcolor(0);textbackground(15);
    cprintf(Pgnam);
    gotoxy(9,3);
    cprintf ("C++ SOLUTION OF THE INITIAL VALUE PROBLEM OF ORDER 1");
    textcolor(15);textbackground(0);
    deqn.Sevtitle();
    deqn.Extitle();
    for(j=1;j<m;j++)
    {
       deqn.Subdata();
    }
//#######################################          Way to EXIT
Tryagain:
    gotoxy (9,18);
    cout <<"_______________________________________________________________";
    gotoxy(9,19);          cout <<"\aType ( ";
    textcolor(0);textbackground(15);
    cprintf("e");
    textcolor(15);textbackground(0);
    cout << " ) to exit : ";
    cmd=getche();
    switch(cmd)
    {
          case 'e':
          gotoxy(9,20);   cout <<"\aThat's right ..."; sleep(1);
          gotoxy(9,20);   cout <<"                    ";
          break ;
      default :
          gotoxy(10,20); cout <<"\aWRONG CHOICE ...";sleep(1);
          gotoxy(15,19);   cout <<"                      ";
      gotoxy(10,20);      cout <<"                    ";
      goto Tryagain;
    }
Exit:
```

```
    return 0;
}
```

The Class named IVPeqn with members functions *setDE*, *setACT*, *setIC*, etc. that are public in this example have been defined. Each member can be accessed by another function defined in the main (). Here the expression deqn is used in the main to have an access on any member function of the class.

For the print out of the results the members *Sevtitle*, *Extitle* and *Subdata* are used defining the variables Xval, Appval, Extval, Aerr of the member functions in an arranged way.

The functions F1 through F4 and PS-calculation Block in the main() have been set up for the development of the program with functions and data accessing the member functions of class with the aid of the expression *deqn*.

In the section *INITIAL Setting* there is a provision for updating the initial conditions of a new problem. A member *setIC* has been recalled for putting the values of the variables necessary for the purpose with an opportunity for correction of the values.

#include is a preprocessor directive. As has been told it is an instruction to the compiler for inclusion into another source file. C++ language has been provided with several headers files. That are very useful for the development of the program.

A *library function* has been conducted by the corresponding *header file*. The header files provide among other matters the prototypes for the library functions. Hence, whenever we use a library function in the program, the corresponding header must be included.

<assert.h>:	Supports debugging; test a conditions and possibly aborts; includes the files <stdio.h> and <stdlib.h>
<conio.h>:	Declares various functions used in calling DOS I/O operations
<ctype.h>:	Supports character handling; contains information for character classification and character conversion
<dir.h>:	Contains macros, structures and functions for working with directories and path names
<dos.h>:	Defines various constants and declares specific call necessary for DOS and 8086-processoror
<errno.h>:	Reports error message
<float.h>:	Defines related floating-point values
<fstream.h>:	Declares the C++ streams that support file I/O; includes <iostream.h>
<graphics.h>:	Declares prototypes for the graphics functions, such as, getcolor, setcolor, getx, gety, etc.
<io.h>:	Contains information about the structures and declarations for low-level I/O routines like access, eof, open write, etc; includes the file <stdarg.h>
<iomanip.h>:	Declares the C++ streams I/O manipulators and contains macros
<iostream.h>:	The basic C++ (earlier version) streams I/O routines are defined
<math.h>:	Provides all possible mathematical functions
<memory.h>:	Memory manipulation functions have been declared
<stdio.h>:	Defines types and macros required for standard I/O (gets, puts, etc.)
<stdlib.h>:	Supports commonly used routines: conversion routines, search/sort routines and others
<string.h>:	Deals with string handling (strcpy, strlen, movedata, etc.)
<time.h>:	Provides time and data functions

Appendix I

Mathematical Relations

Greek Alphabets

α	alpha	ν	nu
β	beta	ξ	xi
γ	gamma	o	omicron
δ	delta	π	pi
ε	epsilon	ρ	rho
ζ	zeta	σ	sigma
η	eta	τ	tau
θ	theta	υ	upsilon
ι	iota	ϕ	phi
κ	kappa	χ	chi
λ	lambda	ψ	psi
μ	mu	ω	omega

Fundamental Constants

e (exponential function) having the value 2.71828182.

e is the number whose logarithm is 1, i.e. Ln *e*= 1 (also called Euler's number).

Ln means natural logarithm (logarithms naturalis). Natural logarithms are those which have the number *e* for base.

π (pi) = 3.14159265.

Some of the important formulas and relations of mathematics that are often required for the formulation of an equation in the problem. They are well mentionable.

Relations from Trigonometry

- $\sin^2\theta + \cos^2\theta = 1$

  $\sec^2\theta - \tan^2\theta = 1$

  $\text{cosec}^2\theta - \cot^2\theta = 1$

- $\sin(\alpha + \beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta$

  $\sin(\alpha - \beta) = \sin\alpha\,\cos\beta - \cos\alpha\,\sin\beta$

  $\cos(\alpha + \beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$

  $\cos(\alpha - \beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$

  $\tan(\alpha + \beta) = (\tan\alpha + \tan\beta)/(1 - \tan\alpha\,\tan\beta)$

  $\tan(\alpha - \beta) = (\tan\alpha - \tan\beta)/(1 + \tan\alpha\,\tan\beta)$

- $\cos 2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1 = (1 - \tan^2\alpha)/(1 + \tan^2\alpha)$

  $\sin 2\alpha = 2\sin\alpha\,\cos\alpha = 2\tan\alpha/(1 + \tan^2\alpha)$

  $\tan 2\alpha = 2\tan\alpha/(1 - \tan^2\alpha)$

  $\tan^2\alpha = (1 - \cos 2\alpha)/(1 + \cos 2\alpha)$

- $\sin\alpha\,\sin\beta = [\cos(\alpha - \beta) - (\alpha + \beta)]/2$

  $\cos\alpha\,\cos\beta = [\cos(\alpha + \beta) + \cos(\alpha - \beta)]/2$

  $\sin\alpha\,\cos\beta = [\sin(\alpha + \beta) + \sin(\alpha - \beta)]/2$

  $\cos\alpha\,\sin\beta = [\sin(\alpha + \beta) - \sin(\alpha - \beta)]/2$

- $\sin\alpha = (e^{i\alpha} - e^{-i\alpha})/2i,$

  $\cos\alpha = (e^{i\alpha} - e^{-i\alpha})/2$

  $e^{a+1\beta} = e^{\alpha}(\cos\beta + i\sin\beta), \quad i = \sqrt{(-1)},\ i^2 = -1$

- $\sin^{-1}x + \cos^{-1}x = \pi/2$

  $2\sin^{-1}x = \sin^{-1}\left[2x\sqrt{(1 - x^2)}\right] = -2i\ \text{Ln}\left[ix + \sqrt{(1 - x^2)}\right]$

  $2\cos^{-1}x = \cos^{-1}(2x^2 - 1) = -2i\ \text{Ln}\left[x + i\sqrt{(1 - x^2)}\right]$

  $2\tan^{-1}x = \tan^{-1}\left[2x/(1 - x^2)\right] = -i\ \text{Ln}\left[(1 + ix)/(1 - ix)\right]$

- $\cosh^2 x - \sinh^2 x = 1$

  $\text{sech}^2 x + \tanh^2 x = 1$

  $\coth^2 x - \text{cosech}^2 x = 1$

- $\sinh x = (e^x - e^{-x})/2$
  $\cosh x = (e^x + e^{-x})/2$
- $\sinh^{-1} x = \text{Ln}\left[x + \sqrt{(x^2+1)}\right]$

  $\cosh^{-1} x = \text{Ln}\left[x + \sqrt{(x^2-1)}\right]$

  $\tanh^{-1} x = \text{Ln}\left[(1+x)/(1-x)\right]/2$
- $\sin(ix) = i \sinh x$
  $\cos(ix) = \cosh x$
  $\tan(ix) = i \tanh x$

Algebraic Relations

- $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k}\, b_k$ (Binomial theorem)

  where m is a positive integer and

  $$\binom{m}{k} = \frac{m(m-1)(m-2)\ldots(m-k+1)}{1.2.3.\ldots k}$$
- $\text{Ln}\,(ab) = \text{Ln}\, a + \text{Ln}\, b$
  $\text{Ln}\,(a/b) = \text{Ln}\, a - \text{Ln}\, b$
- $\text{Ln}\, a^n = n\, \text{Ln}\, a$
  $\text{Ln}\, 1 = 0$
  $0\,! = 1\,! = 1$ (! Means factorial)
  $a^x = e^x\, \text{Ln}\, a$

Series Expansion

- $e^{-x} = 1 - x/1! + x^2/2! - x^3/3! + \ldots = \sum_{k=0}^{\infty} (-x)^k/k!$ $[|x| < \infty]$

  $\text{Ln}(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots = \sum_{k=0}^{\infty} (-1)^{k+1}\, x^k/k!$ $[-1 < x \le 1]$

  $(1+x)^{-n} = 1 - nx + n(n+1)\, x^2/2! - n(n+1)(n+2)\, x^3/3! + \ldots$

  $= 1 + \sum_{k=1}^{\infty} \left[n(n+1)\ldots(n-k+1)\right] (-x)^k/k!$ $[n > 0]$

  $\sin x = x/1! - x^3/3! + x^5/5! - x^7/7! + \ldots = \sum_{k=0}^{\infty} (-1)^k\, x^{2k+1}/(2k+1)!$

$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k}/2k!$

- $\sin^{-1} x = x + x^3/6 + 3x^5/40 + 15x^7/366 + \ldots$

  $= x + \sum_{k=1}^{\infty} [1.3.5 \ldots (2k-1)]\, x^{2k+1}/[2.4.6 \ldots 2k(2k+1)] \quad |x| < 1$

  $\cos^{-1} x = \pi/2 - \sin^{-1} x$

  $\tan^{-1} x = x - x^3/3 + x^5/5 - x^7/7 + \ldots = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}/(2k+1)$

- $\sinh x = x + x^3/3! + x^5/5! + x^7/7! + \ldots = \sum_{k=0}^{\infty} x^{2k+1}/(2k+1)!$

  $\cosh x = 1 + x^2/2! + x^4/4! + x^6/6! + \ldots = \sum_{k=0}^{\infty} x^{2k}/(2k)!$

  $\tanh x = x - x^3/3 + 2x^5/15 - 17x^7/315 + \ldots$

- $\sinh^{-1} x = x - x^3/6 + 3x^5/40 - 15x^7/336 + \ldots \quad [|x| < 1]$

  $= \text{Ln}\ 2x + x^{-2}/4 - 3x^{-4}/32 + 15x^{-6}/288 - \ldots \qquad [|x| > 1]$

  $\cosh^{-1} x = \text{Ln}\ 2x - x^{-2}/4 - 3x^{-4}/32 - 15x^{-6}/288 - \ldots$

  $\tan^{-1} x = x + x^3/3 + x^5/5 + x^7/7 + \ldots = \sum_{k=0}^{\infty} x^{2k+1}/(2k+1) \qquad [|x| < 1]$

- $f(x + w) = f(x) + wDf(x)/1! + w^2D^2f(x)/2! + \ldots + w^{m-1} D^{m-1}f(x)/(m-1) + R_m$

  R_m denotes remainder (*Taylor series*),

  $w^m D^m f(x + \lambda w)/m!$ (*Lagrange's form*),

  $w^m (1 - \lambda)^{m-1} D^m f(x + \lambda w)/(m-1)!$ (*Cauchy's form*),

  λ being a number in the interval $0 < \lambda < 1$.

Differential Calculus (Function with One Variable)

- $D(c) = d/dx\ (c) = 0,$ c being a constant.

  $D(f/g) = (g\ Df - fDg)/g^2$ (Quotient rule)

  $Dx^m = m\ x^{m-1}$ $[m > 0]$

  $De^x = ex;\ \ D(\text{Ln}\ x) = 1/x$

  $Dy = dy/du \cdot du/dx$ (chain rule)

- D sin x = cos x
  D cos x = –sin x
  D tan x = sec^2 x
- $\mathrm{D}\sin^{-1} x = 1\big/\sqrt{(1-x^2)}$

  $\mathrm{D}\cos^{-1} x = -1\big/\sqrt{(1-x^2)}$

  $\mathrm{D}\tan^{-1} x = 1\big/\sqrt{(1+x^2)}$
- D sin$h$ x = cos$h$ x
  D cos$h$ x = sin$h$ x
  D tan$h$ x = sech^2 x
- $\mathrm{D}\sin h^{-1} x = 1\big/\sqrt{(1+x^2)}$

  $\mathrm{D}\cos h^{-1} x = 1\big/\sqrt{(x^2-1)}$

  $\mathrm{D}\tan h^{-1} x = 1\big/(1+x^2)$

Relations from Integral Calculus

- $$\int_{\alpha}^{\beta} f(x)\,dx = G(x)\Big|_{0}^{\beta} = G(\beta) - G(\alpha)$$

  $$\int_{\alpha}^{\beta} f(x)\,dx + \int_{\beta}^{\alpha} f(x)\,dx = 0$$

  $$\int_{\alpha}^{\beta} f(x)\,dx = \int_{\alpha}^{\gamma} f(x)\,dx + \int_{\gamma}^{\beta} f(x)\,dx \qquad [\alpha < \gamma < \beta]$$

- $$\int_{\alpha}^{\beta}\int_{\gamma}^{\delta} g(x,y)\,dx\;dy = \int_{\alpha}^{\beta} dy\left[\int_{\gamma}^{\delta} g(x,y)\,dx\right] = \int_{\alpha}^{\beta}\left[\int_{\gamma}^{\delta} g(x,y)\,dx\right]dy$$

  $$\int_{-\alpha}^{\alpha} f(x) = \int_{0}^{\alpha}\left[f(x) + f(-x)\right]dx$$

  $$= 2\int_{0}^{\alpha} f(x)\,dx, \text{ when } f(x) \text{ is even } = 0, \text{ when } f(x) \text{ is odd}$$

  $$\int_{0}^{m\alpha} f(x)\,dx = m\int_{0}^{\alpha} f(x)\,dx, \text{ when } f(x) = f(\alpha + x).$$

- $D\int g'(x)\,dx = g'(x)\,dx$

  $\int C\,dx = Cx,$ C being a Constant.

  $\int x^n\,dx = x^{n+1}/(n+1)$ [n # –1]

  $\int x^{-1}\,dx = \text{Ln } x$

  $\int e^{mx}\,dx = e^{mx}/m$

  $\int Dg(x)/g(x) = \text{Ln } g(x)$

  $\int \text{Ln } x\,dx = x\,(\text{Ln } x - 1).$

- $\int u\,dv = uv - \int vdu,$

  u and v are differentiable functions of x (Integration by Parts)

- $\int \sin x\,dx = -\cos x$

  $\int \cos x\,dx = \sin x$

  $\int \tan x\,dx = \text{Ln} \sec x$

  $\int \cot x\,dx = \text{Ln} \sin x$

  $\int \sec^2 x\,dx = \tan x$

  $\int \sec x \tan x\,dx = \sec x$

  $\int \sec x\,dx = \text{Ln}\,(\sec x + \tan x) = \text{Ln}\,\tan(\pi/4 + x/2)$

- $\int \sinh x\,dx = \cosh x$

  $\int \cosh x\,dx = \sinh x$

  $\int \tanh x\,dx = \text{Ln}\,(\cosh x)$

  $\int \text{sech}\,x\,dx = 2\tan^{-1}(e^x)$

- $\int 1/(x^2 + \alpha^2)\,dx = -1/\alpha\,\tan^{-1}(x/\alpha)$ [α # 0]

  $\int 1/(x^2 - \alpha^2)\,dx = 1/2\alpha\,\text{Ln}\left[(x-\alpha)/(x+\alpha)\right]$ [$|x| > \alpha$]

$$\int 1/(\alpha^2 - x^2)\; dx = 1/2\alpha \text{ Ln}\left[(\alpha - x)/(\alpha - x)\right] \qquad [|x| > \alpha]$$

$$\int 1/\sqrt{(x^2 + \alpha^2)}\; dx = \text{Ln}\left[x + \sqrt{(x^2 + \alpha^2)}\right]$$

$$\int 1/\sqrt{(x^2 - \alpha^2)}\; dx = \text{Ln}\left[x + \sqrt{(x^2 - \alpha^2)}\right]$$

$$\int 1/\sqrt{(\alpha^2 - x^2)}\; dx = \sin^{-1} x/\alpha$$

- $$\int \sqrt{(x^2 + \alpha^2)}\; dx = x\sqrt{(x^2 + \alpha^2)}\Big/2 + \alpha^2 \text{ Ln}\left[x + \sqrt{(x^2 + \alpha^2)}\right]\Big/2$$

$$\int \sqrt{(x^2 - \alpha^2)}\; dx = x\sqrt{(x^2 - \alpha^2)}\Big/2 - \alpha^2 \text{ Ln}\left[x + \sqrt{(x^2 - \alpha^2)}\right]\Big/2$$

$$\int \sqrt{(\alpha^2 + x^2)}\; dx = x\sqrt{(\alpha^2 - x^2)}\Big/2 + \alpha^2 \sin^{-1}(x/\alpha)/2$$

- $$\int_0^{\pi/2} \text{Ln} \sin x\; dx = \int_0^{\pi/2} \text{Ln} \cos x\; dx = \pi/2 \text{ Ln }(0.5)$$

$$\int_0^{\pi/2} \text{Ln} \tan x\; dx = 0$$

- $$\int_0^{\pi/2} \sin^n x\; dx = \int_0^{\pi/2} \cos^n x\; dx = (n-1)/n.(n-3)/(n-2).(n-5)/(n-4).3/4.1/2.\pi/2,$$ when n is even

$$= (n-1)/n.(n-3)/(n-2).(n-5)/(n-4).4/5.2/3.1, \text{ when } n \text{ is odd}$$

$$\int_0^{\pi/2} \sin^m x \cos^n x\; dx = \int_0^{\pi/2} \cos^m x \sin^n x\; dx = \frac{1.3.5.\ldots(m-1).1.3.5\ldots(n-1)}{2.4.6\ldots(m+n)}, \; (m \text{ and } n \text{ even})$$

$$= \frac{2.4.6.\ldots(m-1)}{(m+1)(n+3)\ldots(n+m)}, \text{ when } m \text{ is odd}$$

Relations from Differential Equations

- First-order differential eqaution

Separation of variables: $g_1(x)f_1(y)\; dx + g_2(x)f_2(y)\; dy = 0$

Method of substitution: $yG(xy)\; dx + xF(xy)\; dy = 0.$

Homogeneous equation: $y' = g(y/x).$

Exact equation: $M(x,y)\; dx + N(x,y)\; dy = 0,$

where $\partial M/\partial y = \partial N/\partial x$.

Linear homogeneous: $y' + P(x)\,y = 0$.

Linear nonhomogeneous: $y' + P(x)\,y = Q(x)$.

Bernoulli form: $y' + P(x)\,y = Q(x)\,y^n$.

Riccati form: $y' + P_1\,y + P_2\,y^2 = Q$.

Lagrange form: $y = x\,f\,(y') + \phi\,(y')$.

Clairaut equation: $y = p(x) + g(p),\ p = dy/dx$.

D'Alembert equation: $y = x\,g(p) + f(p)$.

- Second-order different equation

Linear homogeneous: $y'' + \alpha\,y' + \beta\,y = 0$, α and β are constants.

Linear nonhomogeneous: $y'' + \alpha\,y' + \beta\,y = R(x)$.

Euler or Cauchy equation: $x^2y'' + \alpha xy' + \beta\,y = R(x)$.

Legendre's equations: $(1 - x^2)y'' - 2xy' + n(n + 1)\,y = 0$.

Bessel's equation: $x^2y'' + x\,y' + (\lambda^2\,x^2 - n^2)\,y = 0$.

- *The general form of a linear of n-th order*:
  $d^ny/dx^n + P_1\,d^{n-1}y/dx^{n-1} + P_2\,d^{n-2}y/dx^{n-2} + \ldots + P_{n-1}dy/dx + P_ny = Q$, where $P_1, P_2, \ldots, P_{n-1}, Pn$ and Q are constants or functions of x.

Partial Differential Equations

- $u_{xx} + u_{yy} = 0$ (Laplace equation)
  $u_{tt} - c^2\,u_{xx} = 0$ (Wave equation)
  $u_t - C\,u_{xx} = 0$ (Heat equation)
  $u_{xx} + u_{yy} = g$ (Poisson equation)

Appendix II

Using the Program Disc (CPPSOLMP)

The Program Disc

The disc (3.5 inch/1.44 Mb) in the package contains a set of programs designed in C++ (C plus plus) for each of the mathematical problems presented in the textbook: *C++ Solution For Mathematical Problems* by Arun Ghosh. Each program is provided with the conditions and data of the problem making it ready to be run. At the initial stage you have an idea of the form of the input and output formats. For other problem to solve some changes in the program are required that are also described and designed in each program.

The programs are designed in such a way that they can run on a MS-DOS Computer with appropriate configuration. The corresponding Compiler system for C++ version 3.0 from Borland Turbo C++ Compiler must be available. Using some other version of C++ the equivalent compiler manual should be consulted.

The Turbo C++ Program

The C++ program is so designed that it runs properly under C++ Compiler (version 3.0) from BORLAND International.

1. Boot the system
2. Type at the command prompt (C:\>) tree tc

   A list of directories and subdirectories will be seen as follows:

```
C: TC
 ├──── Bin
 ├──── Include
 │  └──── Sys
 ├──── Lib
 ├──── Bgi
 ├──── Classlib
 │  ├──── Source
 │  ├──── Objs
 │  └──── Lib
 │
 ├──── Examples
 │  └──── Tcalc
 └──── Doc
```

3. Place the Disc named CPPSOLMP in drive A.
   Type dir a:
   A list of programs will appear on the screen
4. Type the command
   copy a:*.cpp c:\tc\examples*.cpp
   With this command all the C++ programs in Floppy disc have been transferred to drive C, i.e. to hard disc.
5. Replace the disc from drive A.
6. Type the command tc (to invoke Turbo C++).
   The screen will display the following MENU selection.

   -File Edit Search Run Compile Debug Project Options Window Help

7. Press Alt+F to view the file-management command
   Select the highlighted submenu "Open".
   Press Enter to locate the file.
   Enter the name of directory path the file-name and then press Enter.
   (You can also search for the required file with the aid of TAB key).
8. The required file (the complete program you need for the solution) will appear on the screen. Follow the instructions for updating or creation of data of the problem mentioned in the formal review section "IMPORTANT TO NOTE" of the Program.
9. To run the Program:
   Press Alt+R. Press Enter.
10. To exti TC:
   Press Alt+F to view the file-menu. Highlight Quit. Press Enter

Program for Chapter 2

The program g2CSINTG.CPP implements the extended Simpson's formula and is used to determine the approximations for the problems of definite integrals. The specimen problem use the function

$$f(x) = \ln (x^2 + 1).$$

Input:

$Fx = \log (x^2 + 1);$
$ES = 0.2639425;$
$Lb = 0.0; Ub = 1.0;$
$K = 5.$

The program g2CSAREA.CPP uses the extended Simpson's rule to evaluate the areas of surfaces. The specimen problem uses the function

$$f(x) = \sin x$$

Input:

$Fx = \sin x;$
$ES = 2.0;$

Lb = 0.0; Ub = pi;

K = 10.

The program g2CS2DIM.CPP implements the extended Simpson's formula and is used to determine the approximations for the problems arising in 2-dimensional surfaces. The specimen problem uses the

$$\text{DI: } \int_{-0.1}^{0.1} dy \int_{1.3}^{1.5} \sqrt{x\ y^2}\, dx.$$

Input:

$$Fx = \sqrt{x \cdot y^2};$$

ES = 0.00015775;

Llx = 1.3; Ulx = 1.5; $L1y$ = –0.1; Uly = 0.1;

K = 8.

Program for Chapter 3

The program g3GAUELI.CPP implements the computational technique of Gauss method for the solution of the system of linear equations. The specimen problem uses the linear system

$$\begin{aligned} 5x_1 + 3x_2 - x_3 \quad &= 11 \\ 2x_1 + 4x_3 + x_4 &= 1 \\ -3x_1 + 3x_2 - 3x_3 + 5x_4 &= -2 \\ 6x_2 - 2x_3 + 3x_4 &= 9. \end{aligned}$$

Input:

$$\text{Coefficient matrix } A = \begin{bmatrix} 5 & 3 & -1 & 0 \\ 2 & 0 & 4 & 1 \\ -3 & 3 & -3 & 5 \\ 0 & 6 & -2 & 3 \end{bmatrix} \text{ and matrix } b = \begin{bmatrix} 11 \\ 1 \\ -2 \\ 9 \end{bmatrix}.$$

The program g3CRALIS.CPP is used to solve the system of linear equations by the method of Cramer's rule. The specimen problem uses 3 × 3 linear system

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 17 \\ x_1 + 2x_2 + 3x_3 &= 16 \\ 2x_1 - x_2 + 4x_3 &= 13. \end{aligned}$$

Input:

$$\text{Coefficient matrix } A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix} \text{ and matrix } b = \begin{bmatrix} 17 \\ 16 \\ 13 \end{bmatrix}.$$

The program g3MATINV.CPP is used for computations of linear equations by the method of Matrix Inversion. The specimen problem uses the 3 × 3 linear system

$$x_1 + 2x_2 + 3x_3 = 1$$
$$2x_1 - 3x_2 + 4x_3 = -1$$
$$3x_1 + 4x_2 + 6x_3 = 2.$$

Input:

Coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ and matrix $b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

The program g3MLUFAC.CPP uses the method of LU-decomposition or LU-facorisation for computing the system of linear equations. The specimen problem uses the 3 × 3 linear system

$$3x_1 + 4x_2 - x_3 = 7$$
$$4x_1 + 12x_2 + 6x_3 = -4$$
$$-x_1 + x_2 + 4x_3 = 4.$$

Input:

Coefficient matrix $A = \begin{bmatrix} 3 & 4 & -1 \\ 4 & 12 & 6 \\ -1 & 1 & 4 \end{bmatrix}$ and matrix $b = \begin{bmatrix} 7 \\ -4 \\ 4 \end{bmatrix}$.

Program for Chapter 4

The program g4IVP1DE.CPP implements the Runge-Kutta method and determines the approximations of all types of Initial Value problems. We have discussed mainly the *exact*, *linear*, *Bernoulli* and *Riccati* equation. The specimen problem uses the following differential equation

$$dy/dx + 1 = (x + y)/2.$$

Input:

$IVdx = x + y + 2$; $IVdy = 2.0$;

$Psy = e^{x/2} - x$;

$y(0) = 1$.

The program g4LINSYS.CPP is used for the solution of linear system of differential equations. The specimen problem uses the following linear system

$$dx - x + 2y = 0,\ dy + 3x - 2y - 0.$$

Input:

$Sys\ Dx = x - 2y$; $Sys\ Dy = 2y - 3x$;

$Extx = e^{-t}(5 - 2e^{5t})$; $Exty = e^{-t}(5 + 3e^{5t})$;

$x(0) = 3$; $y(0) = 8$.

Programs for Chapter 5

The program g5LI2IVP.CPP implements the Runge-Kutta technique and determines the approximations of the functions involved in second-order linear, homogenous and nonhomogeneous differential equations. *Euler's* equations have also been considered. The specimen problem used the differential equation

$$y'' + 4y' + 4y = 0.$$

Input:

$LNHxy = -4' - 4y;$

$PSy = (7 + 5x)e^{-2(x+1)}$; $PSy1 = -(9 + 10\mathrm{x})e^{-2(x+1)}$;

$y(-1) = 2$; $y'(-1) = 1.$

The program g5NL2IVP.CPP is used for the approximated solution of the second-order initial value problem of nonlinear type. The same technique has been applied. The specimen problem uses the nonlinear differential equation.

$$(1 + x^2)\, y'' + y'^2 + 1 = 0.$$

Input:

$NLxy = -(1 + y'^2)/(1 + x^2);$

$PSy = 2\log(x + 1) - x + 1$; $PSy1 = 2/(x + 1) - 1$;

$y(0) = 1$; $y'(0.0) = 1.$

Program for Chapter 6

The program g6FUNCSR.CPP enables to obtain the approximated summation of the infinite numerical series in analytic form accepting the arbitrary choice of finite number of terms of the series. The specimen problem uses the following infinite series

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$$

Input:

$FnX = 1/x(x + 1)(x + 2)$;

$m = 100$;

$Ssr = 0.25$.

The program g6POW1HM.CPP has been designed for the purpose of solving the first-order initial value problems by means of Power series. The evaluation of the power series is based on Taylor's formula and Maclaurin's expansion method. The specimen problem uses the following first-order differential equation

$$dy/dx - x^2y = 0.$$

Input:

$p_1 = 1, p_2 = p_3 = p_4 = 0$;

$q_1 = q_2 = 0, q_3 = -1\; q_4 = q_5 = 0$;

$r_1 = 0, r_2 = 1, r_3 = r_4 = 0$;

$y(0) = 1$.

The program g6POW2HM.CPP has been designed for the purpose of solving the second-order initial value problems by means of Power series. The evaluation of the power series is based on Taylor's formula and Maclaurin's expansion method. The specimen problem uses the differential equation

$$y'' + xy = 0.$$

Input:

$p_1 = 1, p_2 = p_3 = p_4 = 0;$

$q_1 = q_2 = q_3 = q_4 = 0;$

$r_1 = 0, r_2 = 1, r_3 = r_4 = 0;$

$y(0) = 1;\ y'(0) = -1.$

The program g6FOUSER.CPP computes the approximated solutions of Fourier series performing the expansion of the coefficient functions a_k and b_k. The specimen problem uses the following function

$g(x) = x \cos x$ on the interval (0, 1).

Input:

$Fx = x \cos x;$

$m = 5; n = 4$

Program for Chapter 7

The program g7BVP2LI.CPP is used to approximate the linear boundary value problem order 2 applying the method of Finite Difference with Taylor series expansion. The specimen problem uses the boundary value problem

$$y'' - y\,(1 + \tan^2 x) = 0.$$

Input:

$P = 0;\ Q = 1 + \tan^2 x;\ R = 0;$

$Ps = 1/\cos x;$

$y(0) = 1;\ y(1) = 1/\cos(1).$

The program g7BVP2NL.CPP implements the Finite Difference method for the approximation to the nonlinear boundary value problem of second order. The specimen problem uses the boundary value problem

$$y'' + y'^2 + 1 = 0.$$

Input:

$P = -y'/2;\ Q = 0;\ R = -y'^2/2 - 1;$

$PS = \log\left[\sqrt{2/4}\ \cos(2x - 1)/\cos\left(\sqrt{2/4}\right)\right];$

$y(0) = 0;\ y(1) = 0.$

Program for Chapter 8

The program g8PARWAV.CPP is applied for the sake of approximated solution of Wave equation, an example of hyperbolic partial differential equation. The specimen problem uses the equation.

$$u_{tt} - 4u_{xx} = 0.$$

Input:

BC: $u(0,t) = u(1,t) = 0$;

IC: $u(x,0) = \sin \pi x \;\; u_t(x,0) = 0$;

ES: $\sin \pi x \; \cos(2\pi t)$;

$m = n = 4$.

The program g8PARDIF.CPP is applied for the computation of one-dimensional heat conduction of diffusion equation, an example of parabolic partial differential equation. The specimen problem uses the partial differential equation

$$u_t - u_{xx} = 2.$$

Input:

BC: $u(0,1) = u(1,t) = 0$;

IC: $u(x,0) = \sin \pi x + x(i - x)$

ES: $e^{-\pi^2 t} \sin \pi x + x(1 - x)$;

$h = k$.

The program g8PARLAP.CPP uses the Finite Difference method to approximate the partial differential equation of second order of Laplace type. The specimen problem uses the equation

$$\nabla^2 u = u_{xx} + u_{yy} = 0.$$

Input:

BC: $u(0,y) = u(x,0) = 0$;

$u(x,0.5) = 200x$; $u(0.5,y) = 200y$;

ES $400xy$;

$m = n = 4$.

The program g8PARPOI.CPP implements the Poisson equation, an example of elliptic partial differential equation. The specimen problem used the following equation

$$\nabla^2 u = u_{xx} + u_{yy} = x/y + y/x.$$

Input:

BC: $u(1,y) = y \log y$; $u(2,y) = 2y \log(2,y)$;

$u(x,1) = x \log x$; $u(x,2) = x \log(4x^2)$;

ES $xy \log(xy)$;

$m = 4 = n$.

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