Lecture 5: Polarimetry 1

Outline

1. Polarized Light from the Sun
2. Fundamentals of Polarized Light
3. Descriptions of Polarized Light

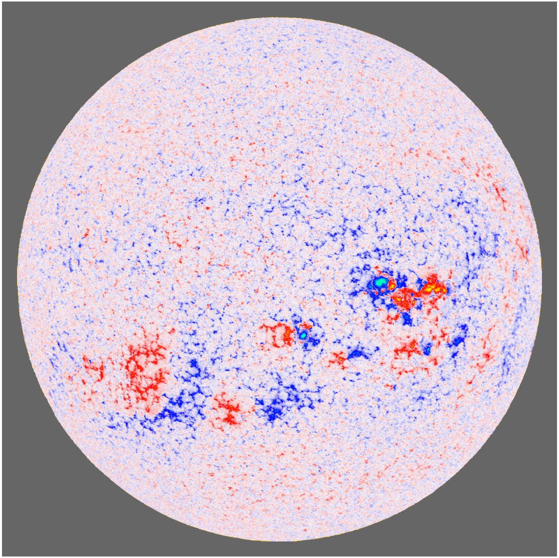
Polarized Light in the Universe

Polarization indicates *anisotropy* $\Rightarrow$ not all directions are equal

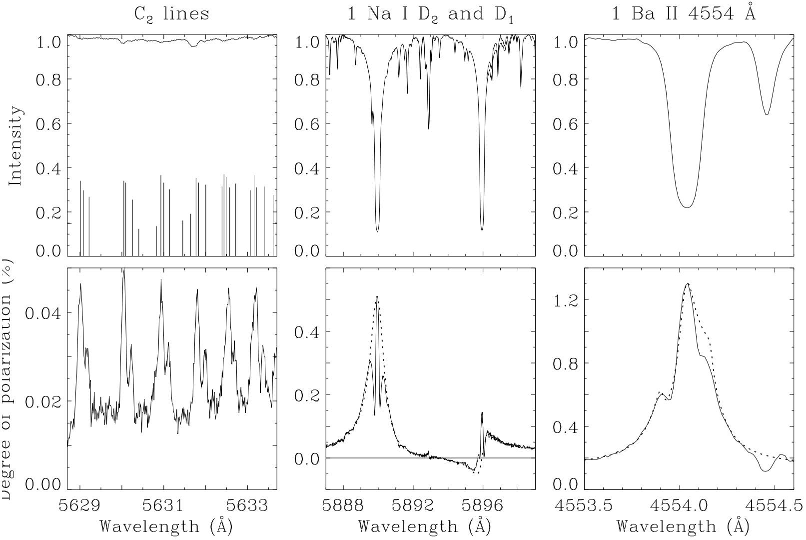
Typical anisotropies introduced by

- geometry (not everything is spherically symmetric)
- temperature gradients
- magnetic fields
- electrical fields

Solar Magnetic Field Maps from Longitudinal Zeeman Effect



Second Solar Spectrum from Scattering Polarization



Fundamentals of Polarized Light

Electromagnetic Waves in Matter

- *Maxwell's equations* $\Rightarrow$ electromagnetic waves
- optics: interaction of electromagnetic waves with matter as described by *material equations*
- polarization of electromagnetic waves are integral part of optics

Maxwell's Equations in Matter

$$\nabla \cdot \vec{D} = 4\pi\rho$$

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = \frac{4\pi}{c}\vec{j}$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$$

$$\nabla \cdot \vec{B} = 0$$

Symbols

$\vec{D}$	*electric displacement*
ρ	*electric charge density*
$\vec{H}$	*magnetic field*
c	*speed of light in vacuum*
$\vec{j}$	*electric current density*
$\vec{E}$	*electric field*
$\vec{B}$	*magnetic induction*
t	time

Linear Material Equations

$$\vec{D} = \epsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$
$$\vec{j} = \sigma \vec{E}$$

Symbols

- ϵ *dielectric constant*
- μ *magnetic permeability*
- σ *electrical conductivity*

Isotropic Media

- isotropic media: ϵ and μ are scalars
- for most materials: $\mu = 1$

Wave Equation in Matter

- static, homogeneous medium with no net charges: $\rho = 0$
- combine Maxwell, material equations $\Rightarrow$ differential equations for damped (vector) wave

$$\nabla^2 \vec{E} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{E}}{\partial t} = 0$$

$$\nabla^2 \vec{H} - \frac{\mu\epsilon}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial \vec{H}}{\partial t} = 0$$

- damping controlled by conductivity σ
- $\vec{E}$ and $\vec{H}$ are equivalent $\Rightarrow$ sufficient to consider $\vec{E}$
- interaction with matter almost always through $\vec{E}$
- but: at interfaces, boundary conditions for $\vec{H}$ are crucial

Plane-Wave Solutions

- Plane Vector Wave ansatz

$$\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$\vec{k}$	spatially and temporally constant *wave vector*
$\vec{k}$	normal to surfaces of constant phase
$\|\vec{k}\|$	*wave number*
$\vec{x}$	spatial location
ω	*angular frequency* ($2\pi\times$ frequency)
t	time
$\vec{E}_0$	(generally complex) vector independent of time and space

- could also use $\vec{E} = \vec{E}_0 e^{-i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$
- damping if $\vec{k}$ is complex
- real electric field vector given by real part of $\vec{E}$

Complex Index of Refraction

- temporal derivatives $\Rightarrow$ Helmholtz equation

$$\nabla^2 \vec{E} + \frac{\omega^2 \mu}{c^2} \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right) \vec{E} = 0$$

- *dispersion relation* between $\vec{k}$ and ω

$$\vec{k} \cdot \vec{k} = \frac{\omega^2 \mu}{c^2} \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right)$$

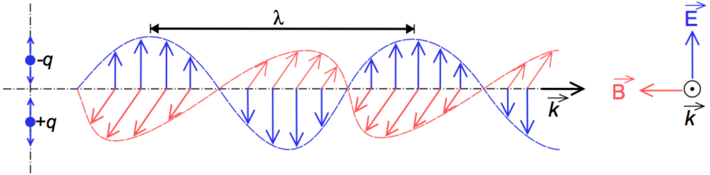
- *complex index of refraction*

$$\tilde{n}^2 = \mu \left(\epsilon + i \frac{4\pi\sigma}{\omega} \right), \quad \vec{k} \cdot \vec{k} = \frac{\omega^2}{c^2} \tilde{n}^2$$

- split into real (n: *index of refraction*) and imaginary parts (k: *extinction coefficient*)

$$\tilde{n} = n + ik$$

Transverse Waves



- plane-wave solution must also fulfill Maxwell's equations

$$\vec{E}_0 \cdot \vec{k} = 0, \quad \vec{H}_0 \cdot \vec{k} = 0, \quad \vec{H}_0 = \frac{\tilde{n}}{\mu} \frac{\vec{k}}{|\vec{k}|} \times \vec{E}_0$$

- isotropic media: electric, magnetic field vectors normal to wave vector $\Rightarrow$ transverse waves
- $\vec{E}_0$, $\vec{H}_0$, and $\vec{k}$ orthogonal to each other, right-handed vector-triple
- conductive medium $\Rightarrow$ complex $\tilde{n}$, $\vec{E}_0$ and $\vec{H}_0$ out of phase
- $\vec{E}_0$ and $\vec{H}_0$ have constant relationship $\Rightarrow$ consider only $\vec{E}$

Energy Propagation in Isotropic Media

- *Poynting vector*

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E} \times \vec{H}\right)$$

- $|\vec{S}|$: energy through unit area perpendicular to $\vec{S}$ per unit time
- direction of $\vec{S}$ is direction of energy flow
- time-averaged Poynting vector given by

$$\left\langle \vec{S} \right\rangle = \frac{c}{8\pi} \text{Re}\left(\vec{E}_0 \times \vec{H}_0^*\right)$$

Re	real part of complex expression
$*$	complex conjugate
$\langle . \rangle$	time average

- energy flow parallel to wave vector (in isotropic media)

$$\left\langle \vec{S} \right\rangle = \frac{c}{8\pi}\frac{|\tilde{n}|}{\mu}\left|E_0\right|^2 \frac{\vec{k}}{|\vec{k}|}$$

Polarization

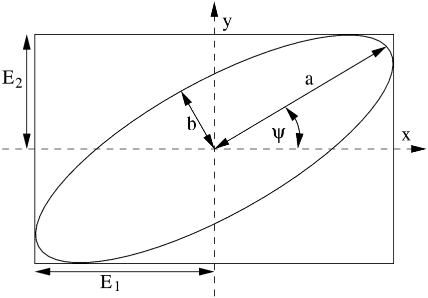
- Plane Vector Wave ansatz $\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$
- spatially, temporally constant vector $\vec{E}_0$ lays in plane perpendicular to propagation direction $\vec{k}$
- represent $\vec{E}_0$ in 2-D basis, unit vectors $\vec{e}_1$ and $\vec{e}_2$, both perpendicular to $\vec{k}$

$$\vec{E}_0 = E_1 \vec{e}_1 + E_2 \vec{e}_2 .$$

  E_1, E_2: arbitrary complex scalars
- damped plane-wave solution with given ω, $\vec{k}$ has 4 degrees of freedom (two complex scalars)
- additional property is called *polarization*
- many ways to represent these four quantities
- if E_1 and E_2 have identical phases, $\vec{E}$ oscillates in fixed plane

Polarization Ellipse

Polarization Ellipse



Polarization

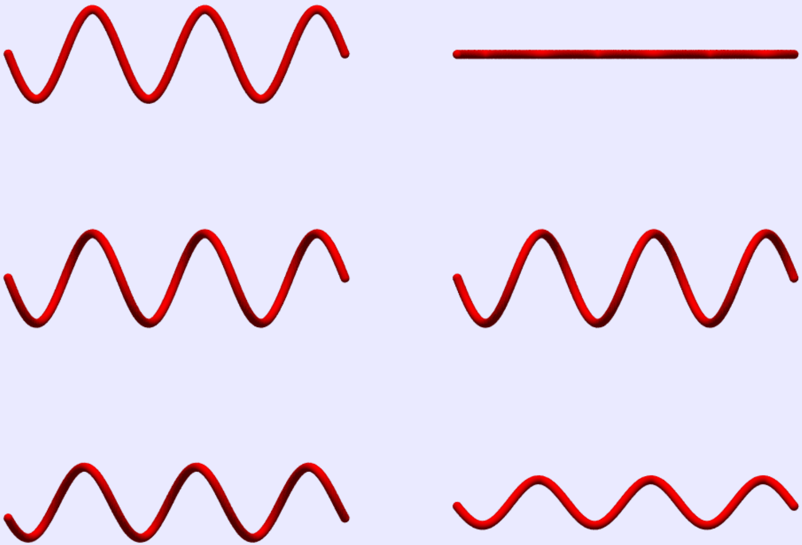
$$\vec{E}(t) = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

$$\vec{E}_0 = E_1 e^{i\delta_1}\vec{e}_x + E_2 e^{i\delta_2}\vec{e}_y$$

- wave vector in z-direction
- $\vec{e}_x$, $\vec{e}_y$: unit vectors in x, y
- E_1, E_2: (real) amplitudes
- $\delta_{1,2}$: (real) phases

Polarization Description

- 2 complex scalars not the most useful description
- at given $\vec{x}$, time evolution of $\vec{E}$ described by *polarization ellipse*
- ellipse described by axes a, b, orientation ψ



Jones Formalism

Jones Vectors

$$\vec{E}_0 = E_x \vec{e}_x + E_y \vec{e}_y$$

- beam in z-direction
- $\vec{e}_x$, $\vec{e}_y$ unit vectors in x, y-direction
- complex scalars $E_{x,y}$
- Jones vector

$$\vec{e} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

- phase difference between E_x, E_y multiple of π, electric field vector oscillates in a fixed plane $\Rightarrow$ *linear polarization*
- phase difference $\pm\frac{\pi}{2}$ $\Rightarrow$ *circular polarization*

Summing and Measuring Jones Vectors

$$\vec{E}_0 = E_x \vec{e}_x + E_y \vec{e}_y$$

$$\vec{e} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

- Maxwell's equations linear $\Rightarrow$ sum of two solutions again a solution
- Jones vector of sum of two waves $=$ sum of Jones vectors of individual waves if wave vectors $\vec{k}$ the same
- addition of Jones vectors: *coherent* superposition of waves
- elements of Jones vectors are not observed directly
- observables always depend on products of elements of Jones vectors, i.e. intensity

$$I = \vec{e} \cdot \vec{e}^* = e_x e_x^* + e_y e_y^*$$

Jones matrices

- influence of medium on polarization described by 2×2 complex *Jones matrix* J

$$\vec{e}' = \mathsf{J}\vec{e} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix} \vec{e}$$

- assumes that medium not affected by polarization state
- different media 1 to N in order of wave direction $\Rightarrow$ combined influence described by

$$\mathsf{J} = \mathsf{J}_N \mathsf{J}_{N-1} \cdots \mathsf{J}_2 \mathsf{J}_1$$

- order of matrices in product is crucial
- Jones calculus describes coherent superposition of polarized light

Linear Polarization

- horizontal: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
- vertical: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- 45°: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Circular Polarization

- left: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$
- right: $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Notes on Jones Formalism

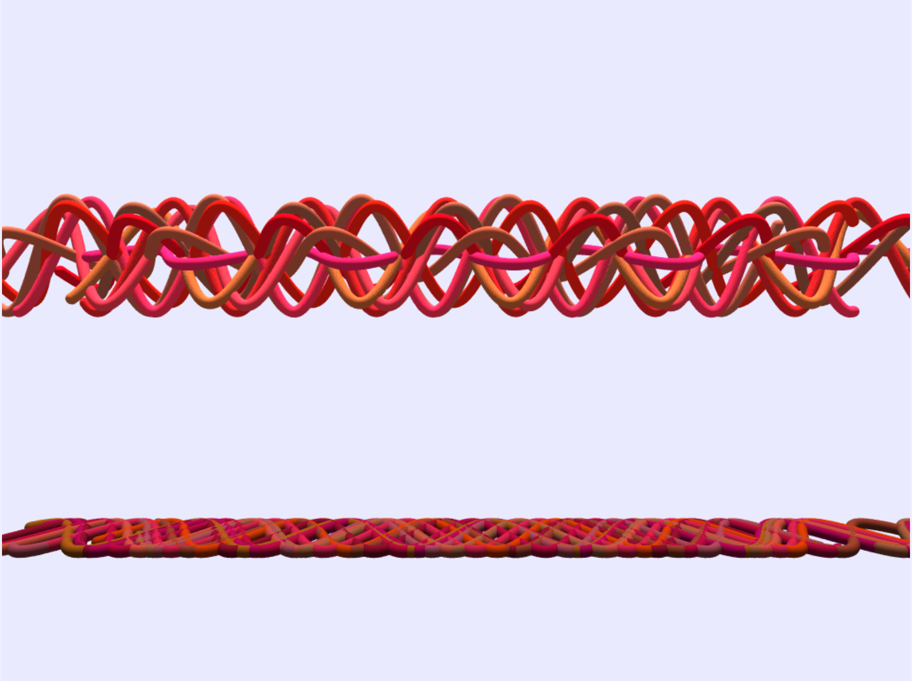
- Jones formalism operates on amplitudes, not intensities
- coherent superposition important for coherent light (lasers, interference effects)
- Jones formalism describes 100% polarized light

Quasi-Monochromatic Light

- monochromatic light: purely theoretical concept
- monochromatic light wave always fully polarized
- real life: light includes range of wavelengths $\Rightarrow$ *quasi-monochromatic light*
- quasi-monochromatic: superposition of mutually incoherent monochromatic light beams whose wavelengths vary in narrow range $\delta\lambda$ around central wavelength λ_0

$$\frac{\delta\lambda}{\lambda} \ll 1$$

- measurement of quasi-monochromatic light: integral over measurement time t_m
- amplitude, phase (slow) functions of time for given spatial location
- *slow*: variations occur on time scales much longer than the mean period of the wave



Polarization of Quasi-Monochromatic Light

- electric field vector for quasi-monochromatic plane wave is sum of electric field vectors of all monochromatic beams

$$\vec{E}(t) = \vec{E}_0(t)\, e^{i\left(\vec{k}\cdot\vec{x}-\omega t\right)}$$

- can write this way because $\delta\lambda \ll \lambda_0$
- measured intensity of quasi-monochromatic beam

$$\left\langle \vec{E}_x \vec{E}_x^* \right\rangle + \left\langle \vec{E}_y \vec{E}_y^* \right\rangle = \lim_{t_m -> \infty} \frac{1}{t_m} \int_{-t_m/2}^{t_m/2} \vec{E}_x(t)\vec{E}_x^*(t) + \vec{E}_y(t)\vec{E}_y^*(t) dt$$

  $\langle \cdots \rangle$: averaging over measurement time t_m
- measured intensity independent of time
- quasi-monochromatic: frequency-dependent material properties (e.g. index of refraction) are constant within $\Delta\lambda$

Polychromatic Light or White Light

- wavelength range comparable to wavelength ($\frac{\delta\lambda}{\lambda} \sim 1$)
- incoherent sum of quasi-monochromatic beams that have large variations in wavelength
- cannot write electric field vector in a plane-wave form
- must take into account frequency-dependent material characteristics
- intensity of polychromatic light is given by sum of intensities of constituting quasi-monochromatic beams

Stokes and Mueller Formalisms

Stokes Vector

- formalism to describe polarization of quasi-monochromatic light
- directly related to measurable intensities
- Stokes vector fulfills these requirements

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} E_x E_x^* + E_y E_y^* \\ E_x E_x^* - E_y E_y^* \\ E_x E_y^* + E_y E_x^* \\ i\left(E_x E_y^* - E_y E_x^*\right) \end{pmatrix} = \begin{pmatrix} E_1^2 + E_2^2 \\ E_1^2 - E_2^2 \\ 2E_1 E_2 \cos\delta \\ 2E_1 E_2 \sin\delta \end{pmatrix}$$

  Jones vector elements $E_{x,y}$, real amplitudes $E_{1,2}$, phase difference $\delta = \delta_2 - \delta_1$

- $$I^2 \geq Q^2 + U^2 + V^2$$

Stokes Vector Interpretation

$$\vec{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} \text{intensity} \\ \text{linear } 0^\circ - \text{linear } 90^\circ \\ \text{linear } 45^\circ - \text{linear } 135^\circ \\ \text{circular left} - \text{right} \end{pmatrix}$$

- *degree of polarization*

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}$$

  1 for fully polarized light, 0 for unpolarized light
- summing of Stokes vectors $=$ *incoherent* adding of quasi-monochromatic light waves

Linear Polarization

- horizontal: $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$
- vertical: $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$
- 45°: $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

Circular Polarization

- left: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$
- right: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$

Mueller Matrices

- 4×4 real Mueller matrices describe (linear) transformation between Stokes vectors when passing through or reflecting from media

$$\vec{I}' = \mathrm{M}\vec{I}\,,$$

$$\mathrm{M} = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

- N optical elements, combined Mueller matrix is

$$\mathrm{M}' = \mathrm{M}_N \mathrm{M}_{N-1} \cdots \mathrm{M}_2 \mathrm{M}_1$$

Vertical Linear Polarizer

$$M_{\rm pol}(\theta) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

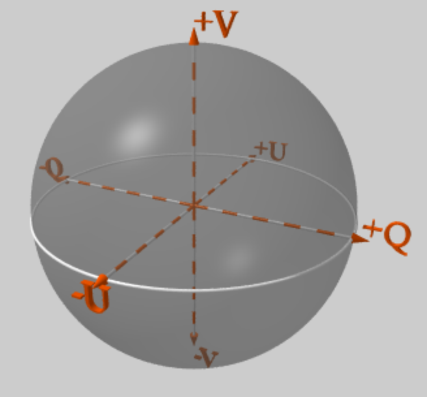
Horizontal Linear Polarizer

$$M_{\rm pol}(\theta) = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Mueller Matrix for Ideal Linear Polarizer at Angle θ

$$M_{\rm pol}(\theta) = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \sin 2\theta \cos 2\theta & 0 \\ \sin 2\theta & \sin 2\theta \cos 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

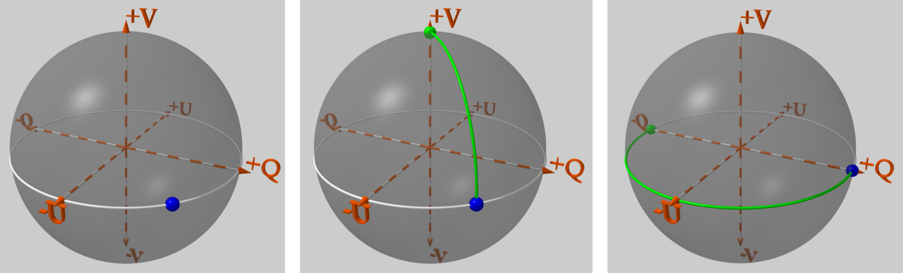
Poincaré Sphere



Relation to Stokes Vector

- fully polarized light: $I^2 = Q^2 + U^2 + V^2$
- for $I^2 = 1$: sphere in Q, U, V coordinate system
- point on Poincaré sphere represents particular state of polarization
- graphical representation of fully polarized light

Poincaré Sphere Interpretation



- polarizer is a point on the Poincaré sphere
- transmitted intensity: $\cos^2(l/2)$, l is arch length of great circle between incoming polarization and polarizer on Poincaré sphere