VSRT MEMO #034

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March 14, 2008

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To: VSRT Group
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Subject: The single baseline LNBF interferometer

A single baseline interferometer is often explained using the sum of plane sinusoid waves of a single frequency received at each element as illustrated in Figure 1A. In this case[1]

$$s(t) = \sin(wt) + \sin(w(t-\tau)) = 2\sin(wt - w\tau/2)\cos(w\tau/2) \quad (1)$$

where w = the frequency in radians/sec

t = time in seconds

τ = time difference of arrival of the plane waves

For the geometry in Figure 1A

$$\tau = (D/c)\sin\theta \quad (2)$$

where D = interferometer baseline m

c = velocity of propagation m/s

θ = angle

λ = wavelength

The term

$\left|\cos(w\tau/2)\right|$ is a maximum when (3)

$$w\tau/2 = n\pi \quad (or\ D\sin\theta = n\lambda) \quad (4)$$

that is when the path difference is an integral number of wavelengths and zero when

$$w\tau/2 = n\pi = n\pi + \pi/2 \quad (or\ D\sin\theta = (n+1/2)\lambda) \quad (5)$$

or when the path difference is an odd number of half wavelengths. In this interferometer a single source can be moved to produce maxima and minima in the detected output. The VSRT interferometer illustrated in Figure 1B does <u>not</u> add the signals directly. Before the signals are added they are shifted down or "downconverted" in frequency to an

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[1]Using trig. Identity $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

intermediate frequency. This is accomplished by multiplying the signal with a local oscillator in each LNB.

The local oscillators are free running and consequently have slightly different frequencies so that equation 1 becomes

$$\begin{aligned} s(t) &= \sin(wt - w_a t) + \sin(w(t-\tau) - w_b t) \\ &= 2\sin((2wt - w_a t - w_b t - w\tau)/2)\cos(w\tau/2 - \delta t/2) \end{aligned} \tag{6}$$

where $\delta = w_b - w_a$ = difference of local oscillators

This shows that the "fringes" of equation 1 are now present for any value of τ and a single source will produce an output independent of the angle θ. Our interferometer uses a detector diode to take the square of $s(t)$ so that[2]

$$s^2(t) = (1 - \cos(2wt - w_a t - w_b t - w\tau))(1 - \cos(w\tau - \delta t)) \tag{7}$$

If we now average over time (or low pass filter) we will be left with only

$$1 + \cos(w\tau - \delta t) \tag{8}$$

that is a constant D.C. term plus a sinusoidal term with a frequency equal to the difference or "beat frequency" between the local oscillators.

If we use 2 equal sources with different differential delays we can demonstrate the interference pattern of maxima and minima. In this case the sinusoidal term becomes

$$\begin{aligned} &\cos(w\tau_x - \delta t) + \cos(w\tau_y - \delta t) \\ &= 2\cos(w(\tau_x + \tau_y)/2 - \delta t)\cos(w(\tau_x - \tau_y)/2) \end{aligned} \tag{9}$$

where τ_x and τ_y are the time difference of arrival at the LNBF from the "*x*" and "*y*" CFL respectively.

Thus we have the same relationship as for the simple adding interferometer except τ becomes the difference between the delays for each source. i.e.

$D(\sin\theta_x - \sin\theta_y) = n\lambda$ for maxima and $D(\sin\theta_x - \sin\theta_y) = (n + \tfrac{1}{2})\lambda$ for minima

[2] Using trig identities $\sin^2 A/2 = (1 - \cos A)/2 \; and \; \cos^2 A/2 = (1 + \cos A)/2$

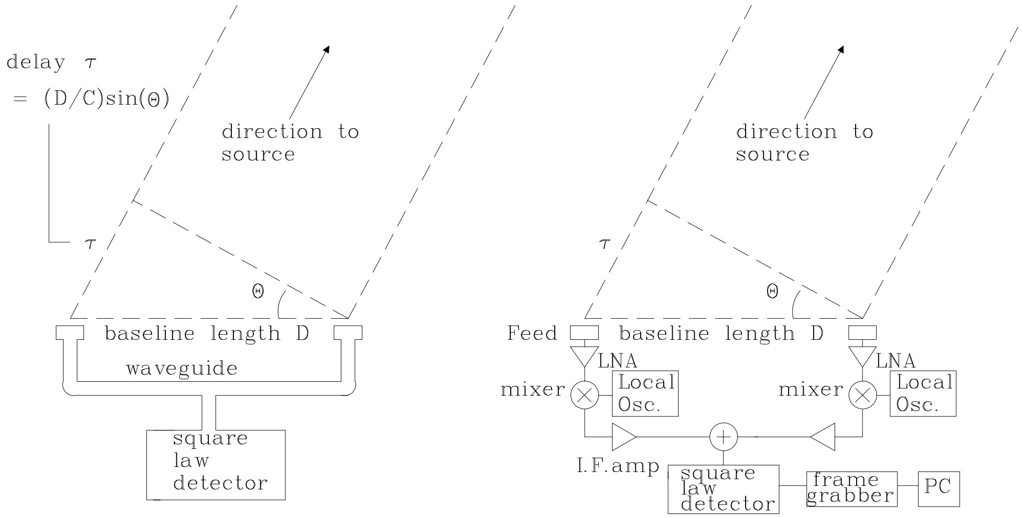

Figure 1A – Simple adding interferometer Figure 1B – LBNF adding interferometer

An alternate analysis of the VSRT interferometer based on the concept of correlation follows:

If we represent the signals from the LNBFs as *x* and *y* the combiner and square law detector produce an output of

$$(x+y)^2 = x^2 + y^2 + 2xy$$

which when low pass filtered equals the sum of the power from each LNBF in the $\left(x^2+y^2\right)$ term added to the correlation given by the term *2xy*. The power can be measured with a voltmeter at the output of the detector while the correlation, which is modulated by the beat frequency, is detected in the spectrum of the output obtained from the Fourier transform of the video signal conveyed to the PC via the USB frame grabber.

If we represent the signal from the CFL as

$$c(t)\sin\left(wt+\phi(t)\right)$$

where $c(t)$ and $\phi(t)$ are Gaussian random variables which make the lamp an "incoherent" source as compared with the near perfect sine wave from a signal generator. After the mixing operation in the LNB this signal becomes

$$c(t)\sin\left((w-w_a)t+\phi(t)\right)$$

and the correlation, $\rho(\tau)$,

$$\rho(\tau) = c^2(t)\sin\left((w-w_a)t+\phi(t)\right)\sin\left(w(t-\tau)-w_b t+\phi(t)\right) \qquad (10)$$

where τ is the path delay into the second LNB. Expanding the equation (10).

$$\rho(\tau) = +c^2\left(\cos\left(w\tau + \left(w_b - w_a\right)t\right)\right)/2$$
$$-c^2(t)\cos\left(2wt - \left(w_a + w_b\right)t - w\tau + 2\phi(t)\right)/2 \qquad (11)$$

The effect of τ on $c(t)$ and $\phi(t)$ is ignored because these have a bandwidth set by the intermediate frequency amplifier (i.e. about 1 GHz) so that values of τ of a few wavelengths at 12 GHz will have little effect.

The second term in the equation averages to zero due to the random phase of the incoherent CFL source and we are are left with

$\rho(\tau) = +c^2(\tau)\left(\cos\left(w\tau - \delta\tau\right)\right)/2$

For 2 equal CFLs with time difference of arrival values of τ_x and τ_y

$$\rho = \cos\left(w\tau_x - \delta t\right) + \cos\left(w\tau_y - \delta t\right)$$
$$= 2\cos\left(w\left(\tau_x + \tau_y\right)/2 - \delta t\right)\cos\left(w\left(\tau_x - \tau_y\right)/2\right) \qquad (12)$$

which is the same as equation (9).

[A more complete theory of two-element interferometers using Fourier transforms is given in Methods of Experimental Physics, **12**, part C, Chapter 5.1 by A.E.E. Rogers.]