Béla Valek

Modern Physics Handbook I.

General Relativity

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bvalek2@yahoo.com

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Introduction

This series presents modern physics through examples and derivations. The organization of the topics does not follow the traditional historical approach, it was determined by practical considerations instead. We build up the mathematical framework of the models first, and then completely derive the most important consequences and compare them to recent experimental results. The volumes can be used in the specialized fields as reference materials, and are suitable for self-study in each topic. They may also serve the needs of university lectures as well.

The first volume deals with the general theory of relativity. This description has only historical relevance by now, since during the last century, much experimental evidence was found for the consequences of Einstein's theory. It is a classical field, where centuries-old scientific and philosophical ideas got mathematical formulation, and the experimental confirmation. It is important to point out, that the traditional mechanical worldview, that is often considered to be easier to grasp, is in fact an incomplete intellectual achievement. The basic assumptions of general relativity draw from everyday experiences, and the recognition of the curved nature of spacetime follows naturally. It is comparable to the understanding of Earth's spherical shape, and if we are familiar with the mathematical foundations, it does not even require too much imagination.

The Reader is assumed to have some basic knowledge in higher mathematics and among the more traditional subjects in physics, but there is only as much mathematical depth in this book, as absolutely necessary. We use the traditional symbols of differential calculus and index notation. The SI system of measurement is used in all physical derivations.

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Observations

Maybe our senses mislead us, but we do not notice that while we sit calmly in a room, Earth is actually speeding around the Sun at nearly 30 km/s. In fact we cannot even tell if we are standing still in the dock or moving on open water, while we are inside an ocean liner. The fact is, that not only us, but (in the ideal case) not even our instruments are able to tell the difference. They may be inaccurate, but it is also possible that there is a principal reason keeping us from telling our absolute velocity. This is the principle of relativity by Galileo Galilei.

Our journey in time in one direction from the past to the future, and the ordering between cause and effect, is such a self-evident experience, that it is surprising that we have to explicitly state it as a condition. Serious logical flaws and paradoxes would arise, if it were not so. But from a practical point of view, we cannot tell anything except that we have not yet observed the contrary. This idea gained great importance, when Rudolf Clausius recognized entropy, the change of which shows the direction of time. We must keep in mind however, that nothing has been said on how fast time passes, or if it passes in a constant manner.

Light is travelling imperceptibly fast to our notions. However already a thousand years ago the Arabic scientist Ibn al-Haytham suggested, that since light is a propagating phenomenon, it may have a finite propagating velocity. We recognize it only at astronomical distances, or with the help of our instruments, having much better reaction time than our naked eyes. It is an important fact that its speed in vacuum is always the same and constant, for all observers, regardless of their motion. The theoretical foundation for this comes from James Clerk Maxwell's equations. We trust this observation so much, that this forms the basis of the definition of meter in the SI metric system. If it were not so, a very fast observer could outrun light, and measure a different value for its velocity, thus measure his/her absolute speed, something impossible, as we believe.

Astronaut candidates in an aeroplane on a paraboloidal path (the “vomit comet”) can experience weightlessness for a short period of time. Fun park simulators tilt back the seats of their visitors, although they feel only their own weights, but they are led to believe that they accelerate. If the simulator would actually move away from its position and not only tilt, the person sitting inside would not be able to tell the difference. There are two indistinguishable phenomena again, let us declare that they are the same, this is Albert Einstein's principle of equivalence.

Bodies accelerating under electromagnetic influence behave as if gravitation would act on them, a similar empirical law describes their motion. However this force depends on whether they have a net charge, moreover it is not only attracting, but it can also be repelling. Nevertheless, the trajectory of a particle moving in a general electromagnetic and gravitational field may be described by purely geometric means, as shown by Theodor Kaluza. Our statement will essentially mean, that a charged instrument does not measure any difference between acceleration under gravitational or electric influence, or the state of weightlessness.

The general theory of relativity is based on these observations, and describes the behavior of spacetime, and its interaction with the matter it contains. Thus it creates a framework where all the other physical models can be described.

Notation and constants

We use index notation in the entire book. Indices are always single letters, and the following table summarizes how and where they are used:

spaces	*free indices*	*summation indices*
3D or general space (1…N)	i, j, k, l, m, n	a, b, c, d, e, f
4D spacetime (0…3)	η, κ, μ, ν, ξ, σ	α, β, γ, δ, ε, ζ
5D spacetime (0…4)	P, Q, R, S, T, U	A, B, C, D, E, F

Both sides of the equations must have the same number of free indices, since in fact we have as many equations as the number of dimensions, multiplied with the number of free indices:

$$v^i = a \cdot u^i + b^i \qquad \rightarrow \qquad v^1 = a \cdot u^1 + b^1 \ \ , \ \ v^2 = a \cdot u^2 + b^2 \ \ , \ \ v^3 = a \cdot u^3 + b^3$$

The terms that contain summation indices are summed, as many times as the number of dimensions:

$$s = v^a \cdot u_a = \sum_{a=1}^{N} v^a \cdot u_a = v^1 \cdot u_1 + v^2 \cdot u_2 + v^3 \cdot u_3$$

The Kronecker delta: $\delta^i_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$

The coordinate systems where the quantities are written down are indicated with a lower-left index. Quantities in different points are always denoted with a different letter, wherever possible.

The determinant of a matrix with two indices can be calculated with the following recursive formula:

$$|M_{ij}| = (-1)^{a+1} \cdot M_{1\,a} \cdot |M_{i \neq 1\ j \neq a}|$$

Calculating the components of the twice contravariant metric tensor from the twice covariant metric tensor:

$$g^{kl} = \frac{(-1)^{k+l} \cdot |g_{i \neq k\ j \neq l}|}{|g_{ij}|}$$

The derivative and integral of the Lambert function:

$$\frac{dW}{dx} = \frac{1}{e^{W(x)} \cdot (W(x)+1)} \qquad \int W(x) \cdot dx = x \cdot \left(W(x) + \frac{1}{W(x)} - 1 \right) + C$$

Natural constants:

Notation and constants

speed of light:	$c=2.99792458\cdot 10^{8}\,\frac{m}{s}$
gravitational constant:	$\gamma=6.67428\cdot 10^{-11}\,\frac{m^{3}}{kg\cdot s^{2}}$

Units of time and distance:

Julian year:	$1\,a=365.25\,days=3.15576\cdot 10^{7}\,s$
astronomical unit:	$1\,AU=1.49597870691\cdot 10^{11}\,m$
lightyear:	$1\,ly=9.4607304725808\cdot 10^{15}\,m$
parsec:	$1\,Pc=3.08567758128\cdot 10^{16}\,m$ $=2.06264806245\cdot 10^{5}\,AU$ $=3.26156377695\,ly$

1. Foundations

In this chapter we introduce the mathematical language used by the general theory of relativity. Our goal is to find measurable quantities that we can use to describe multi-dimensional surfaces. This problem is dealt with by differential geometry, and the methods used are similar to those found in geodesy. The word “geometry” itself means “measuring Earth” by which our topic is related to an ancient science that has been cultivated in the antique world with great expertise. While the geodeses survey the Earth's curved surface, we will explore the curved spacetime, but our goals are exactly the same: to orientate, to measure distances, or to find the shortest path between two points, and so on.

We are going to see that starting with naïve, down to earth assumptions about space, we can build up a geometry with various properties, where we can identify all the necessary geometric quantities. From this we can draw the lesson, that when we think about “simple” flat space, we apply several unspoken assumptions and limitations, that do not follow automatically from our starting conditions.

1.1 Coordinate systems

Let us imagine an arbitrary space, where we identify the points with individual sequences of numbers called vectors, in other words we set up a coordinate system. By definition, we denote each coordinate with an index in the upper-right corner. The *i.*-th coordinate of point *x*:

$$x^i \tag{1.1.1}$$

These sequences of numbers are not exclusive of course, the points in space can be arbitrarily renumbered. If we distribute the numbers according to a different logic, we are setting up another coordinate system, denoted by 2 in the lower left corner, where the coordinates are functions of the coordinates in the first coordinate system:

$${}_2x^i = {}_2x^i({}_1x^j) \tag{1.1.2}$$

Let us now assign a number, or otherwise known as a scalar, to every point in space. A scalar field is defined as a function of the points in space, and the value of this so called scalar function is of course independent of the choice of coordinates:

$$\phi(x^i) \qquad\qquad \phi({}_2x^i)=\phi({}_1x^i) \tag{1.1.3}$$

Let us take a look at two different points in a coordinate system:

$$x^i \qquad\qquad y^i \tag{1.1.4}$$

The difference between the values of the scalar field in those points is independent from our choice of the coordinate system:

$$\phi({}_1x^i)-\phi({}_1y^i)=\phi({}_2x^i)-\phi({}_2y^i) \tag{1.1.5}$$

This is also true if we reduce the distance between the points infinitesimally, which is the total derivative:

$$\frac{\partial\phi}{{}_1\partial x^a}\cdot{}_1dx^a=\frac{\partial\phi}{{}_2\partial x^a}\cdot{}_2dx^a \tag{1.1.6}$$

However the change in the scalar field along a coordinate depends on the choice of the coordinate system. In fact, we are examining how much the change along a coordinate looks along a coordinate of a different coordinate system. To make the equations less crowded, we omit the 1-index most of the time. Let us postulate two transformation rules:

Transforming the partial derivative of the scalar field: $$\frac{\partial\phi}{{}_2\partial x^i}=\frac{\partial\phi}{\partial x^a}\cdot\frac{\partial x^a}{{}_2\partial x^i} \tag{1.1.7}$$

Transforming the coordinate differential: $${}_2dx^i=\frac{{}_2\partial x^i}{\partial x^a}\cdot dx^a \tag{1.1.8}$$

Their scalar product is coordinate system independent. This demonstrates that our postulated transformation formulas are correct, since they behave as expected:

$$\frac{\partial\phi}{{}_2\partial x^a}\cdot{}_2dx^a=\frac{\partial\phi}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^a}\cdot\frac{{}_2\partial x^a}{\partial x^c}\cdot dx^c=\frac{\partial\phi}{\partial x^b}\cdot dx^b \tag{1.1.9}$$

In the case of infinitesimal displacement, coordinates change in both coordinate systems. The ratios of these changes between the coordinate systems form a square matrix, called the transformation matrix:

$$\Lambda^i{}_j=\frac{{}_2\partial x^i}{{}_1\partial x^j}\ \neq\ \Lambda_j{}^i=\frac{{}_1\partial x^i}{{}_2\partial x^j} \tag{1.1.10}$$

The quantities that are transformed like the partial derivatives of the scalar field are called covariant vectors, which are also numbering the points in space, just by a different logic. By definition, the index is placed into the lower right corner:

$$\frac{\partial\phi}{{}_2\partial x^i}=\frac{\partial\phi}{\partial x^a}\cdot\Lambda_i{}^a \qquad v_i=\frac{\partial\phi}{\partial x^i}$$

Covariant vector: a v quantity that transforms like the partial derivatives of the scalar field between coordinate systems:

$${}_2v_i=v_a\cdot\frac{\partial x^a}{{}_2\partial x^i}=v_a\cdot\Lambda_i{}^a \tag{1.1.11}$$

The quantities that are transformed like coordinate differentials are called contravariant vectors. The position of the index remains unchanged:

$$_2dx^i = \frac{_2\partial x^i}{\partial x^a} \cdot dx^a \qquad v^i = dx^i$$

Contravariant vector: a v quantity that transforms like the coordinate differentials between coordinate systems:

$$_2v^i = \frac{_2\partial x^i}{\partial x^a} \cdot v^a = \Lambda^i{}_a \cdot v^a \tag{1.1.12}$$

If we swap the coordinate systems, the reverse transformation formulas are:

$$v^i = \frac{\partial x^i}{_2\partial x^a} \cdot {_2v^a} = \Lambda_a{}^i \cdot {_2v^a} \qquad v_i = {_2v_a} \cdot \frac{_2\partial x^a}{\partial x^i} = {_2v_a} \cdot \Lambda^a{}_i \tag{1.1.13}$$

The reciprocal of the differential transforms like a covariant vector:

$$\frac{1}{_2dx^i} = \frac{1}{\Lambda^i{}_a} \cdot \frac{1}{dx^a} = \frac{\partial x^a}{_2\partial x^i} \cdot \frac{1}{dx^a} = \Lambda_i{}^a \cdot \frac{1}{dx^a} \tag{1.1.14}$$

It follows from above, that the scalar product of the covariant and contravariant vectors is also coordinate system independent, and the result is a scalar:

$$_2v_a \cdot {_2u^a} = v_b \cdot \Lambda_a{}^b \cdot \Lambda^a{}_c \cdot u^c = v_b \cdot u^b \tag{1.1.15}$$

We perform a double transformation, we write down a vector in a different coordinate system, and then we return to the original:

$$v^i = \frac{\partial x^i}{_2\partial x^a} \cdot {_2v^a} = \Lambda_a{}^i \cdot {_2v^a} \qquad {_2v^i} = v^a \cdot \frac{_2\partial x^i}{\partial x^a} = v^a \cdot \Lambda^i{}_a$$

$$v^i = \Lambda_b{}^i \cdot v^a \cdot \Lambda^b{}_a = \delta^i_a \cdot v^a$$

The scalar product of the transformation matrices is the Kronecker delta:

$$\Lambda_a{}^i \cdot \Lambda^a{}_j = \delta^i_j = \frac{\partial x^i}{_2\partial x^a} \cdot \frac{_2\partial x^a}{\partial x^j} \tag{1.1.16}$$

1.2 Tensors

The results of vector products belong to the family of tensors. These are special kinds of tensors, because they depend only on $n \cdot m$ numbers, where m is the number of vectors and n is the number of coordinates:

$$v^i \cdot u^j \cdot w^k \cdot \ldots \cdot p_l \cdot q_m \cdot \ldots = T^{ijk\ldots}{}_{lm\ldots} \tag{1.2.1}$$

On the other hand, a matrix representing such a tensor might have n^m independent components, and this is also true about the general tensor. This suggests the following transformation rule:

Tensor: a T quantity that transforms in the following way between coordinate systems:

$$_2T^{ijk\ldots}{}_{lm\ldots} = \Lambda^i{}_a \cdot \Lambda^j{}_b \cdot \Lambda^k{}_c \cdot \ldots \cdot T^{abc\ldots}{}_{de\ldots} \cdot \Lambda_l{}^d \cdot \Lambda_m{}^e \cdot \ldots \tag{1.2.2}$$

This formula clearly shows, that if all components of the tensor are zero, then it stays zero under any coordinate transformation. The rank of a tensor is the number of indices, a scalar is a tensor of zero rank, the vector is a first rank tensor, and so on. The product of arbitrary tensors is also a tensor, for example:

$$A^{ijk} \cdot B_{lm} = T^{ijk}{}_{lm} \tag{1.2.3}$$

The sum of tensors is interpreted only if they have the same rank, for example:

$$_2A^{ij}{}_k + {}_2B^{ij}{}_k = \Lambda^i{}_a \cdot \Lambda^j{}_b \cdot (A^{ab}_c + B^{ab}_c) \cdot \Lambda^c_k \tag{1.2.4}$$

Let us multiply an arbitrary tensor with vectors in such a way, that we perform a summation for every index. According to the definition above, the result has to be an invariant scalar:

$$T^{abc\ldots}{}_{de\ldots} \cdot v_a \cdot u_b \cdot w_c \ldots p^d \cdot q^e \ldots = s \tag{1.2.5}$$

If we do not know the nature of a quantity having multiple indices, the formula above can determine if it is a tensor. Because if we write it down in a different coordinate system, then the transformation matrices will cancel out only, if the quantity with multiple indices deploys the same number of transformation matrices, as there are for transforming the vectors.

The value of the Kronecker delta is one in the case of corresponding indices, and zero if the indices are different. According to the formula above its a special kind of tensor, whose components are separately invariant:

$$\delta^a_b \cdot v_a \cdot u^b = v_b \cdot u^b = s \tag{1.2.6}$$

Covariant vectors can also be used to number the points in space, these sequences of numbers are made of covariant coordinates:

$$x_i \tag{1.2.7}$$

In fact, this is just another coordinate system, one of many possibilities, therefore the contravariant coordinates of a point certainly can be transformed into covariant ones. Therefore let the other coordinate system be the covariant one, thus the mutual ratios of the coordinate changes form a symmetric g_{ij} quantity:

$$\frac{{}_2\partial x^i}{{}_1\partial x^j} \quad \rightarrow \quad g_{ij} = \frac{\partial x_i}{\partial x^j} = \frac{\partial x_j}{\partial x^i} = g_{ji} \tag{1.2.8}$$

$${}_2v^i = v^a \cdot \frac{{}_2\partial x^i}{\partial x^a} \quad \rightarrow \quad v_i = v^a \cdot \frac{\partial x_i}{\partial x^a} = v^a \cdot g_{ia} \tag{1.2.9}$$

In the reverse direction:

$$\frac{{}_1\partial x^i}{{}_2\partial x^j} \quad \rightarrow \quad g^{ij} = \frac{\partial x^i}{\partial x_j} = \frac{\partial x^j}{\partial x_i} = g^{ji} \tag{1.2.10}$$

$${}_2v_i = v_a \cdot \frac{\partial x^a}{{}_2\partial x^i} \quad \rightarrow \quad v^i = v_a \cdot \frac{\partial x^a}{\partial x_i} = v_a \cdot g^{ai} \tag{1.2.11}$$

The scalar product of the contravariant and covariant representation of the vector is a scalar – the length of the vector – therefore our new quantity is a tensor:

$$v_a \cdot v^a = v_a \cdot v_b \cdot g^{ab} = v^b \cdot v^a \cdot g_{ba} \tag{1.2.12}$$

It is called the metric tensor, and it can raise and lower indices:

$$v_a \cdot u_b \cdot w_c \cdot \ldots \cdot p^d \cdot q^e \cdot \ldots \cdot g^{ai} \cdot g^{bj} \cdot g^{ck} \cdot \ldots \cdot g_{dl} \cdot g_{em} \cdot \ldots = v^i \cdot u^j \cdot w^k \cdot \ldots \cdot p_l \cdot q_m \cdot \ldots$$

$$T_{abc\ldots}{}^{de\ldots} \cdot g^{ai} \cdot g^{bj} \cdot g^{ck} \cdot \ldots \cdot g_{dl} \cdot g_{em} \cdot \ldots = T^{ijk\ldots}{}_{lm\ldots} \tag{1.2.13}$$

We perform a double transformation, we write down a contravariant vector in covariant form, then we transform it back into contravariant form:

$$v^i = v^a \cdot g_{ab} \cdot g^{bi} = v^a \cdot \delta^i_a$$

The scalar product of the metric tensors is the Kronecker delta:

$$g_{ia} \cdot g^{aj} = \delta^i_j = \frac{\partial x_i}{\partial x^a} \cdot \frac{\partial x^a}{\partial x_j} \tag{1.2.14}$$

If we perform a summation on all indices, the result is the number of dimensions:

$$g_{ab} \cdot g^{ab} = \delta^b_b = N \tag{1.2.15}$$

An arbitrary tensor can always be split into the sum of a symmetric and an antisymmetric tensor. In the case of two indices, it is possible to create a symmetric tensor with the averaging of the opposing tensor components:

$$ {}_sT_{ij}=\frac{1}{2}\cdot(T_{ij}+T_{ji})=\frac{1}{2}\cdot(T_{ji}+T_{ij})={}_sT_{ji} \qquad (1.2.16)$$

Subtracting it from the general tensor, the result is an antisymmetric tensor:

$$ {}_aT_{ij}=T_{ij}-{}_sT_{ij}=T_{ij}-\frac{1}{2}\cdot(T_{ij}+T_{ji})=\frac{1}{2}\cdot(T_{ij}-T_{ji})=-\frac{1}{2}\cdot(T_{ji}-T_{ij})=-{}_aT_{ji} \qquad (1.2.17)$$

with having zeroes for diagonal elements: $\quad {}_aT_{ii}=\frac{1}{2}\cdot(T_{ii}-T_{ii})=0 \qquad (1.2.18)$

Their sum recreates the original tensor:

$$ {}_sT_{ij}+{}_aT_{ij}=\frac{1}{2}\cdot(T_{ij}+T_{ji})+\frac{1}{2}\cdot(T_{ij}-T_{ji})=T_{ij} \qquad (1.2.19)$$

Reversing the indices of a general tensor:

$$ T_{ij}=T_{ji}+2\cdot{}_{an}T_{ij} \qquad (1.2.20)$$

1.3 Straight lines

In an arbitrary space, a curve with a constant tangent vector is the closest thing to a straight line. The change of coordinates with respect to an invariant quantity is the following:

$$ u^i=\frac{\partial x^i}{\partial\lambda} \qquad (1.3.1)$$

Let it be the tangent vector of the straight line. We write it down in another coordinate system:

$$ \frac{{}_2\partial x^i}{\partial\lambda}=\frac{{}_2\partial x^i}{\partial x^a}\cdot\frac{\partial x^a}{\partial\lambda} \quad /\frac{\partial}{\partial\lambda} \qquad (1.3.2)$$

The formula which describes that it is not changing, that the derivative is zero, is the equation of the straight line:

$$ \frac{{}_2\partial^2 x^i}{\partial\lambda^2}=\frac{{}_2\partial^2 x^i}{\partial x^a\cdot\partial x^b}\cdot\frac{\partial x^a}{\partial\lambda}\cdot\frac{\partial x^b}{\partial\lambda}+\frac{{}_2\partial x^i}{\partial x^a}\cdot\frac{\partial^2 x^a}{\partial\lambda^2}=0 \quad /\cdot\frac{\partial x^j}{{}_2\partial x^i}$$

$i\rightarrow c:$

$$\frac{\partial x^j}{{}_2\partial x^c}\cdot\frac{{}_2\partial^2 x^c}{\partial x^a\cdot\partial x^b}\cdot\frac{\partial x^a}{\partial\lambda}\cdot\frac{\partial x^b}{\partial\lambda}+\frac{\partial x^j}{{}_2\partial x^c}\cdot\frac{{}_2\partial x^c}{\partial x^a}\cdot\frac{\partial^2 x^a}{\partial\lambda^2}=0$$

We introduce the connection in the first term: $$\Gamma^j{}_{ab}=\frac{\partial x^j}{{}_2\partial x^c}\cdot\frac{{}_2\partial^2 x^c}{\partial x^a\cdot\partial x^b} \qquad (1.3.3)$$

The index changes in the second term: $$\frac{\partial x^j}{{}_2\partial x^c}\cdot\frac{{}_2\partial x^c}{\partial x^a}\cdot\frac{\partial^2 x^a}{\partial\lambda^2}=\delta^j_a\cdot\frac{\partial^2 x^a}{\partial\lambda^2}=\frac{\partial^2 x^j}{\partial\lambda^2}$$

The geodesic equation:

$$\frac{\partial^2 x^j}{\partial\lambda^2}+\Gamma^j{}_{ab}\cdot\frac{\partial x^a}{\partial\lambda}\cdot\frac{\partial x^b}{\partial\lambda}=0 \qquad (1.3.4)$$

We write down the geodesic equation in two different coordinate systems:

$$\frac{\partial^2 x^i}{\partial\lambda^2}+\Gamma^i{}_{ab}\cdot\frac{\partial x^a}{\partial\lambda}\cdot\frac{\partial x^b}{\partial\lambda}=0 \qquad \frac{{}_2\partial^2 x^i}{\partial\lambda^2}+{}_2\Gamma^i{}_{ab}\cdot\frac{{}_2\partial x^a}{\partial\lambda}\cdot\frac{{}_2\partial x^b}{\partial\lambda}=0 \qquad (1.3.5)$$

The transformation of the first and second derivative with respect to the invariant:

$$\frac{{}_2\partial x^i}{\partial\lambda}=\frac{{}_2\partial x^i}{\partial x^a}\cdot\frac{\partial x^a}{\partial\lambda}$$

$$\frac{{}_2\partial^2 x^i}{\partial\lambda^2}=\frac{{}_2\partial^2 x^i}{\partial x^a\cdot\partial x^b}\cdot\frac{\partial x^a}{\partial\lambda}\cdot\frac{\partial x^b}{\partial\lambda}+\frac{{}_2\partial x^i}{\partial x^a}\cdot\frac{\partial^2 x^a}{\partial\lambda^2} \qquad (1.3.6)$$

Insert them into the geodesic equation:

$$a\rightarrow c \qquad b\rightarrow d$$

$$\left(\frac{{}_2\partial^2 x^i}{\partial x^c\cdot\partial x^d}\cdot\frac{\partial x^c}{\partial\lambda}\cdot\frac{\partial x^d}{\partial\lambda}+\frac{{}_2\partial x^i}{\partial x^c}\cdot\frac{\partial^2 x^c}{\partial\lambda^2}\right)+{}_2\Gamma^i{}_{ab}\cdot\left(\frac{{}_2\partial x^a}{\partial x^c}\cdot\frac{\partial x^c}{\partial\lambda}\right)\cdot\left(\frac{{}_2\partial x^b}{\partial x^d}\cdot\frac{\partial x^d}{\partial\lambda}\right)=0 \qquad (1.3.7)$$

Rearrange the formula and multiply with a new factor:

$$\frac{{}_2\partial x^i}{\partial x^c}\cdot\frac{\partial^2 x^c}{\partial\lambda^2}+\left(\frac{{}_2\partial^2 x^i}{\partial x^c\cdot\partial x^d}+{}_2\Gamma^i{}_{ab}\cdot\frac{{}_2\partial x^a}{\partial x^c}\cdot\frac{{}_2\partial x^b}{\partial x^d}\right)\cdot\frac{\partial x^c}{\partial\lambda}\cdot\frac{\partial x^d}{\partial\lambda}=0 \quad /\cdot\frac{\partial x^j}{{}_2\partial x^i}$$

$$i\rightarrow e$$

Write down the change in the first term in detail:

$$\frac{{}_2\partial x^e}{\partial x^c}\cdot\frac{\partial x^j}{{}_2\partial x^e}\cdot\frac{\partial^2 x^c}{\partial\lambda^2}=\delta^j_c\cdot\frac{\partial^2 x^c}{\partial\lambda^2}=\frac{\partial^2 x^j}{\partial\lambda^2}$$

Let us reinsert it:

$$\frac{\partial^2 x^j}{\partial\lambda^2}+\left(\frac{{}_2\partial^2 x^e}{\partial x^c\cdot\partial x^d}\cdot\frac{\partial x^j}{{}_2\partial x^e}+{}_2\Gamma^e{}_{ab}\cdot\frac{{}_2\partial x^a}{\partial x^c}\cdot\frac{{}_2\partial x^b}{\partial x^d}\cdot\frac{\partial x^j}{{}_2\partial x^e}\right)\cdot\frac{\partial x^c}{\partial\lambda}\cdot\frac{\partial x^d}{\partial\lambda}=0 \tag{1.3.8}$$

Our result is in the form of an equation of a straight line. We can recognize the connection, from which we identify the transformation rule:

$$c\rightarrow i \qquad\qquad d\rightarrow k$$

$$\Gamma^j{}_{ik}=\frac{{}_2\partial^2 x^e}{\partial x^i\cdot\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e}+{}_2\Gamma^e{}_{ab}\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e} \tag{1.3.9}$$

The connection is therefore not a tensor-like quantity. The symmetric part of the general connection:

$$C^k{}_{ij}=\frac{1}{2}\cdot(\Gamma^k{}_{ij}+\Gamma^k{}_{ji}) \tag{1.3.10}$$

The antisymmetric part of the general connection is the torsion:

$$S^k{}_{ij}=\frac{1}{2}\cdot(\Gamma^k{}_{ij}-\Gamma^k{}_{ji}) \qquad\qquad \Gamma^k{}_{ij}=\Gamma^k{}_{ji}+2\cdot S^k{}_{ij} \tag{1.3.11}$$

We insert the transformation law of the connection:

$$S^k{}_{ij}=\frac{1}{2}\cdot\left(\left(\frac{{}_2\partial^2 x^e}{\partial x^i\cdot\partial x^j}\cdot\frac{\partial x^k}{{}_2\partial x^e}+{}_2\Gamma^e{}_{ab}\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^j}\cdot\frac{\partial x^k}{{}_2\partial x^e}\right)-\left(\frac{{}_2\partial^2 x^e}{\partial x^j\cdot\partial x^i}\cdot\frac{\partial x^k}{{}_2\partial x^e}+{}_2\Gamma^e{}_{ba}\cdot\frac{{}_2\partial x^b}{\partial x^j}\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{\partial x^k}{{}_2\partial x^e}\right)\right)$$

Simplify, the transformation of the torsion:

$$S^k{}_{ij}=\frac{1}{2}\cdot({}_2\Gamma^e{}_{ab}-{}_2\Gamma^e{}_{ba})\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^j}\cdot\frac{\partial x^k}{{}_2\partial x^e} \tag{1.3.12}$$

Judging from the transformation law of the torsion, it is a tensor with three indices:

$$S^k{}_{ij}={}_2S^k{}_{ij}\cdot\Lambda^a{}_i\cdot\Lambda^b{}_j\cdot\Lambda_e{}^k \tag{1.3.13}$$

The variations of the connection transform like tensors:

first coordinate system: $\delta\Gamma^{j}{}_{ik}=\Gamma^{j}{}_{ik}(x)-\Gamma^{j}{}_{ik}(x+\delta x)$

second coordinate system: ${}_2\delta\Gamma^{j}{}_{ik}={}_2\Gamma^{j}{}_{ik}({}_2x)-{}_2\Gamma^{j}{}_{ik}({}_2x+{}_2\delta x)$ (1.3.14)

Insert into the first equation the transformation of the connection:

$$\delta\Gamma^{j}{}_{ik}=$$
$$\frac{{}_2\partial^2 x^e}{\partial x^i\cdot\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e}+{}_2\Gamma^{e}{}_{ab}(x)\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e}-\frac{{}_2\partial^2 x^e}{\partial x^i\cdot\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e}-{}_2\Gamma^{e}{}_{ab}(x+\delta x)\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e}$$

$$\delta\Gamma^{j}{}_{ik}={}_2\Gamma^{e}{}_{ab}(x)\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e}-{}_2\Gamma^{e}{}_{ab}(x+\delta x)\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e} \tag{1.3.15}$$

The variation of the connection is a tensor-like quantity:

$$\delta\Gamma^{j}{}_{ik}={}_2\delta\Gamma^{j}{}_{ik}\cdot\frac{{}_2\partial x^a}{\partial x^i}\cdot\frac{{}_2\partial x^b}{\partial x^k}\cdot\frac{\partial x^j}{{}_2\partial x^e} \tag{1.3.16}$$

1.4 Parallel displacement

Let us take a look at two infinitesimally close points in space, where we define vectors in each of them, and describe it with two different coordinate systems at the same time. In order to avoid confusion, we summarize the notation here:

	First coordinate system	*Second coordinate system*
First point	x^i	${}_2x^i$
Second point	y^i	${}_2y^i$
Vectors in the first point	v^i, w_i	${}_2v^i$
Vectors in the second point	u^i, q_i	${}_2u^i$

We parallel transfer a vector from the first point to the second, and we define the first coordinate system in such a way, that every components of the new vector are identical to the old one:

$$u^i=v^i \tag{1.4.1}$$

The change in the second coordinate system is more general:

$${}_2u^i={}_2v^i+{}_2dv^i \tag{1.4.2}$$

We transform the coordinates of the vectors in the following way:

$$v^i = \frac{\partial x^i}{{}_2\partial x^a} \cdot {}_2v^a \qquad\qquad u^i = \frac{\partial y^i}{{}_2\partial y^a} \cdot {}_2u^a \tag{1.4.3}$$

Let us insert it into the formula written down in the first coordinate system:

$$\frac{\partial x^i}{{}_2\partial x^a} \cdot {}_2v^a = \frac{\partial y^i}{{}_2\partial y^a} \cdot {}_2u^a \tag{1.4.4}$$

The difference of the two points is the total derivative, where we now differentiate with respect to the coordinates of the second coordinate system:

$$y^i = x^i + \frac{\partial x^i}{{}_2\partial x^a} \cdot {}_2dx^a \quad / \frac{\partial}{{}_2\partial x^j}$$

$$\frac{\partial y^i}{{}_2\partial y^j} = \frac{\partial x^i}{{}_2\partial x^j} + \frac{\partial^2 x^i}{{}_2\partial x^j \cdot {}_2\partial x^a} \cdot {}_2dx^a \tag{1.4.5}$$

Substitute it, and then expand the parentheses:

$$\frac{\partial x^i}{{}_2\partial x^a} \cdot {}_2v^a = \left(\frac{\partial x^i}{{}_2\partial x^b} + \frac{\partial^2 x^i}{{}_2\partial x^b \cdot {}_2\partial x^a} \cdot {}_2dx^a \right) \cdot \left({}_2v^b + {}_2dv^b \right) \tag{1.4.6}$$

$$\frac{\partial x^i}{{}_2\partial x^a} \cdot {}_2v^a = \frac{\partial x^i}{{}_2\partial x^b} \cdot {}_2v^b + \frac{\partial x^i}{{}_2\partial x^b} \cdot {}_2dv^b + \frac{\partial^2 x^i}{{}_2\partial x^b \cdot {}_2\partial x^a} \cdot {}_2dx^a \cdot {}_2v^b + \frac{\partial^2 x^i}{{}_2\partial x^b \cdot {}_2\partial x^a} \cdot {}_2dx^a \cdot {}_2dv^b$$

We simplify and then ignore the last term, where the infinitesimally small quantities are on a higher power:

$$0 = \frac{\partial x^i}{{}_2\partial x^b} \cdot {}_2dv^b + \frac{\partial^2 x^i}{{}_2\partial x^b \cdot {}_2\partial x^a} \cdot {}_2dx^a \cdot {}_2v^b \quad / \cdot \frac{{}_2\partial x^j}{\partial x^i}$$

$$i \to c$$

$$\frac{{}_2\partial x^j}{\partial x^c} \cdot \frac{\partial x^c}{{}_2\partial x^b} \cdot {}_2dv^b = -\frac{{}_2\partial x^j}{\partial x^c} \cdot \frac{\partial^2 x^c}{{}_2\partial x^b \cdot {}_2\partial x^a} \cdot {}_2dx^a \cdot {}_2v^b \tag{1.4.7}$$

On the left side of the equation, we substitute and apply the Kronecker delta, on the right side we substitute the connection:

$${}_2\Gamma^j{}_{ba} = \frac{{}_2\partial x^j}{\partial x^c} \cdot \frac{\partial^2 x^c}{{}_2\partial x^b \cdot {}_2\partial x^a} \tag{1.4.8}$$

$${}_2dv^j = -{}_2\Gamma^j{}_{ba} \cdot {}_2dx^a \cdot {}_2v^b$$

We reinsert this into the parallel transfer formula of the contravariant vector, without the identification numbers of the coordinate systems. It is possible to swap the lower indices, we do so to have the same index convention that is used in a later definition of the connection:

$$u^i = v^i - \Gamma^i{}_{ba} \cdot v^a \cdot dx^b \tag{1.4.9}$$

We take the scalar product of two vectors in the first point. If we parallel transfer them into the second point, the resulting scalar does not change:

$$v^a \cdot w_a = u^a \cdot q_a \tag{1.4.10}$$

The parallel displacement of the contravariant vector:

$$u^i = v^i + dv^i = v^i - \Gamma^i{}_{ba} \cdot v^a \cdot dx^b \tag{1.4.11}$$

The parallel transfer of the covariant vector:

$$q_i = w_i + dw_i \tag{1.4.12}$$

Substitute them into the scalar product:

$$v^a \cdot w_a = \left(v^a - \Gamma^a{}_{dc} \cdot v^c \cdot dx^d\right) \cdot \left(w_a + dw_a\right) \tag{1.4.13}$$

$$v^a \cdot w_a = v^a \cdot w_a + v^a \cdot dw_a - \Gamma^a{}_{dc} \cdot v^c \cdot dx^d \cdot w_a - \Gamma^a{}_{dc} \cdot v^c \cdot dx^d \cdot dw_a$$

Simplify and omit the last term, where the infinitesimal quantities are on a higher power:

$$dw_i = \Gamma^a{}_{bi} \cdot dx^b \cdot w_a$$

We reinsert this into the parallel displacement of the covariant vector:

$$q_i = w_i + \Gamma^a{}_{bi} \cdot w_a \cdot dx^b \tag{1.4.14}$$

Based on these, we can determine the parallel transfer formula of any tensor. We are already not following the notation introduced in the table earlier:

$$v^i \cdot u^j \cdot w^k \cdot \ldots \cdot p_l \cdot q_m \cdot \ldots = T^{ijk\ldots}{}_{lm\ldots}$$

We substitute the parallel transfer formulas, and neglect the terms containing higher powers of infinitesimal quantities. Therefore only the products of vectors remain, and those terms, that contain the connection only once:

$$\left(v^i - \Gamma^i{}_{ba} \cdot v^a \cdot dx^b\right) \cdot \left(u^j - \Gamma^j{}_{ba} \cdot u^a \cdot dx^b\right) \cdot \left(w^k - \Gamma^k{}_{ba} \cdot w^a \cdot dx^b\right) \cdot \ldots \cdot \left(p_l + \Gamma^a{}_{bl} \cdot p_a \cdot dx^b\right) \cdot \left(q_m + \Gamma^a{}_{bm} \cdot q_a \cdot dx^b\right) \cdot \ldots =$$
$$T^{ijk\ldots}{}_{lm\ldots} + \left(-\Gamma^i{}_{ba} \cdot T^{ajk\ldots}{}_{lm\ldots} - \Gamma^j{}_{ba} \cdot T^{iak\ldots}{}_{lm\ldots} - \Gamma^k{}_{ba} \cdot T^{ija\ldots}{}_{lm\ldots} - \ldots + \Gamma^a{}_{bl} \cdot T^{ijk\ldots}{}_{am\ldots} + \Gamma^a{}_{bm} \cdot T^{ijk\ldots}{}_{la\ldots} + \ldots\right) \cdot dx^b \tag{1.4.15}$$

1.5 Connection and metric tensor

Like in the previous chapter, we take the scalar product of two vectors in the first point. If we parallel transfer them into the second point, the resulting scalar does not change:

$$g_{ab}(x)\cdot v^a\cdot w^b = g_{ab}(y)\cdot u^a\cdot q^b \tag{1.5.1}$$

We wrote on the left side of the equation the scalar product in the first point, and on the right side the scalar product in the second point. We approach a metric tensor from another with a Taylor series:

$$g_{ij}(y) = g_{ij}(x) + \frac{\partial g_{ij}(x)}{\partial x^a}\cdot dx^a + \frac{1}{2}\cdot\frac{\partial^2 g_{ij}(x)}{\partial x^a\cdot\partial x^b}\cdot dx^a\cdot dx^b + \ldots \tag{1.5.2}$$

We substitute the parallel displacement formula for vectors and the Taylor series of the metric tensor, neglecting the terms containing products of infinitesimal quantities:

$$g_{ab}(x)\cdot v^a\cdot w^b = \left(g_{ab}(x) + \frac{\partial g_{ab}(x)}{\partial x^c}\cdot dx^c\right)\cdot\left(v^a - \Gamma^a{}_{dc}\cdot v^c\cdot dx^d\right)\cdot\left(w^b - \Gamma^b{}_{dc}\cdot w^c\cdot dx^d\right) \tag{1.5.3}$$

Let us simplify and neglect the higher order terms of infinitesimal quantities again, and then step by step rewrite the summation indices into free indices. We must take care about what indices belongs to what factors:

$$0 = \frac{\partial g_{ab}}{\partial x^c}\cdot dx^c\cdot v^a\cdot w^b - g_{ab}\cdot v^a\cdot\Gamma^b{}_{dc}\cdot w^c\cdot dx^d - g_{ab}\cdot\Gamma^a{}_{dc}\cdot v^c\cdot dx^d\cdot w^b \quad /\cdot\frac{1}{dx^k}$$

$$0 = \frac{\partial g_{ab}}{\partial x^k}\cdot v^a\cdot w^b - g_{ab}\cdot v^a\cdot\Gamma^b{}_{kc}\cdot w^c - g_{ab}\cdot\Gamma^a{}_{kc}\cdot v^c\cdot w^b \quad /\cdot\frac{1}{v^i}$$

$$0 = \frac{\partial g_{ib}}{\partial x^k}\cdot w^b - g_{ib}\cdot\Gamma^b{}_{kc}\cdot w^c - g_{ab}\cdot\Gamma^a{}_{ki}\cdot w^b \quad /\cdot\frac{1}{w^j}$$

$$g_{ia}\cdot\Gamma^a{}_{kj} + g_{aj}\cdot\Gamma^a{}_{ki} = \frac{\partial g_{ij}}{\partial x^k} \tag{1.5.4}$$

Using this result it can be shown, that the parallel displacement of the metric tensor transforms it into the local metric tensor of the destination, thus we are recovering the Taylor series:

$$g_{ij}(y) = g_{ij}(x) + \left(\Gamma^a{}_{bi}\cdot g_{aj}(x) + \Gamma^a{}_{bj}\cdot g_{ia}(x)\right)\cdot dx^b = g_{ij}(x) + \frac{\partial g_{ij}(x)}{\partial x^b}\cdot dx^b \tag{1.5.5}$$

Permute the indices:

(1) $$g_{ia}\cdot\Gamma^{a}{}_{kj}+g_{aj}\cdot\Gamma^{a}{}_{ki}=\frac{\partial g_{ij}}{\partial x^{k}}$$

(2) $$g_{ja}\cdot\Gamma^{a}{}_{ik}+g_{ak}\cdot\Gamma^{a}{}_{ij}=\frac{\partial g_{jk}}{\partial x^{i}}$$

(3) $$g_{ka}\cdot\Gamma^{a}{}_{ji}+g_{ai}\cdot\Gamma^{a}{}_{jk}=\frac{\partial g_{ki}}{\partial x^{j}} \qquad (1.5.6)$$

Summarize the equations in the following way: (1) + (2) – (3):

$$g_{ia}\cdot\Gamma^{a}{}_{kj}+g_{aj}\cdot\Gamma^{a}{}_{ki}+g_{ja}\cdot\Gamma^{a}{}_{ik}+g_{ak}\cdot\Gamma^{a}{}_{ij}-g_{ka}\cdot\Gamma^{a}{}_{ji}-g_{ai}\cdot\Gamma^{a}{}_{jk}=\frac{\partial g_{ij}}{\partial x^{k}}+\frac{\partial g_{jk}}{\partial x^{i}}-\frac{\partial g_{ki}}{\partial x^{j}} \qquad (1.5.7)$$

Reorder the connections according to their lower indices:

$$g_{ia}\cdot(\Gamma^{a}{}_{kj}-\Gamma^{a}{}_{jk})+g_{aj}\cdot(\Gamma^{a}{}_{ki}+\Gamma^{a}{}_{ik})+g_{ak}\cdot(\Gamma^{a}{}_{ij}-\Gamma^{a}{}_{ji})=\frac{\partial g_{ij}}{\partial x^{k}}+\frac{\partial g_{jk}}{\partial x^{i}}-\frac{\partial g_{ki}}{\partial x^{j}} \quad /\cdot\frac{1}{2}$$

Arrange the antisymmetric terms to the right side:

$$g_{aj}\cdot C^{a}{}_{ki}=\frac{1}{2}\cdot\left(\frac{\partial g_{ij}}{\partial x^{k}}+\frac{\partial g_{jk}}{\partial x^{i}}-\frac{\partial g_{ki}}{\partial x^{j}}\right)-g_{ia}\cdot S^{a}{}_{kj}-g_{ak}\cdot S^{a}{}_{ij} \quad /\cdot g^{jb} \qquad (1.5.8)$$

Apply the Kronecker-delta on the left side:

$$C^{j}{}_{ki}=\frac{1}{2}\cdot g^{jb}\cdot\left(\frac{\partial g_{ib}}{\partial x^{k}}+\frac{\partial g_{bk}}{\partial x^{i}}-\frac{\partial g_{ki}}{\partial x^{b}}\right)-g^{jb}\cdot g_{ia}\cdot S^{a}{}_{kb}-g^{jb}\cdot g_{ak}\cdot S^{a}{}_{ib} \quad /+S^{j}{}_{ki} \qquad (1.5.9)$$

The general connection is the sum of the symmetric and antisymmetric parts:

$$\Gamma^{j}{}_{ki}=\frac{1}{2}\cdot g^{jb}\cdot\left(\frac{\partial g_{ib}}{\partial x^{k}}+\frac{\partial g_{bk}}{\partial x^{i}}-\frac{\partial g_{ki}}{\partial x^{b}}\right)-g^{jb}\cdot g_{ia}\cdot S^{a}{}_{kb}-g^{jb}\cdot g_{ak}\cdot S^{a}{}_{ib}+S^{j}{}_{ki} \qquad (1.5.10)$$

If we use a symmetric connection since the beginning, our formula gives the relationship between the connection and the metric tensor:

$$\Gamma^{j}{}_{ki}=\frac{1}{2}\cdot g^{ja}\cdot\left(\frac{\partial g_{ia}}{\partial x^{k}}+\frac{\partial g_{ak}}{\partial x^{i}}-\frac{\partial g_{ki}}{\partial x^{a}}\right) \qquad (1.5.11)$$

The infinitesimal surrounding of every point can be approximated with a flat space, where it is possible to set up a coordinate system, where the partial derivative of the metric tensor is zero. In other words, we approach the point with a Taylor series, where the first derivative is zero, but others are not. In this case the symmetric connection is zero, its partial derivative however is not:

$$\frac{\partial g_{ij}}{\partial x^k}=0 \quad , \quad \frac{\partial^2 g_{ij}}{\partial x^l \cdot \partial x^k}\neq 0 \qquad \rightarrow \qquad \Gamma^i{}_{jk}=0 \quad , \quad \frac{\partial \Gamma^i{}_{jk}}{\partial x^l}\neq 0 \tag{1.5.12}$$

1.6 Derivation

First in order to deduce the transformation rule of the second partial derivative, we differentiate both sides of the transformation formula of the partial derivative of the scalar function:

$$\frac{\partial \Phi}{{}_2\partial x^i}=\frac{\partial \Phi}{\partial x^a}\cdot\frac{\partial x^a}{{}_2\partial x^i} \qquad /\frac{\partial}{{}_2\partial x^j}$$

$$\frac{\partial^2 \Phi}{{}_2\partial x^j \cdot {}_2\partial x^i}=\frac{\partial^2 \Phi}{{}_2\partial x^j \cdot \partial x^a}\cdot\frac{\partial x^a}{{}_2\partial x^i}+\frac{\partial \Phi}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^j \cdot {}_2\partial x^i} \tag{1.6.1}$$

In the first term of the right side, we transform one of the denominator differential from the second to the first coordinate system:

$$\frac{\partial^2 \Phi}{{}_2\partial x^j \cdot {}_2\partial x^i}=\frac{\partial^2 \Phi}{\partial x^b \cdot \partial x^a}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}+\frac{\partial \Phi}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^j \cdot {}_2\partial x^i} \tag{1.6.2}$$

The second partial derivative of the scalar field, that is the partial derivative of the covariant vector, does not transform like a tensor. Substitute it to the formula above:

$$v_i=\frac{\partial \Phi}{\partial x^i}$$

$$\frac{{}_2\partial v_i}{{}_2\partial x^j}=\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}+v_a\cdot\frac{\partial^2 x^a}{{}_2\partial x^j \cdot {}_2\partial x^i} \tag{1.6.3}$$

$$\frac{{}_2\partial v_i}{{}_2\partial x^j}-v_a\cdot\frac{\partial^2 x^a}{{}_2\partial x^j \cdot {}_2\partial x^i}=\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i} \tag{1.6.4}$$

In the left side, we transform the vector in the second term, so it would be written in the same coordinate system as the partial derivative in the first term. The transformation rule of the derivative of the covariant vector in the general case:

$$\frac{{}_2\partial v_i}{{}_2\partial x^j}-{}_2v_b\cdot\frac{{}_2\partial x^b}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^j \cdot {}_2\partial x^i}=\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i} \tag{1.6.5}$$

This quantity does not transform like a tensor. But if we switch the indices in it and subtract it from the original expression, we get a tensor-like quantity called rotation. The right side of the equation:

$$\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}-\frac{\partial v_b}{\partial x^a}\cdot\frac{\partial x^a}{{}_2\partial x^i}\cdot\frac{\partial x^b}{{}_2\partial x^j}=\left(\frac{\partial v_a}{\partial x^b}-\frac{\partial v_b}{\partial x^a}\right)\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}=T_{ab}\cdot\Lambda_j^{\ b}\cdot\Lambda_i^{\ a}$$

The left side of the equation:

$$\left(\frac{{}_2\partial v_i}{{}_2\partial x^j}-{}_2v_b\cdot\frac{{}_2\partial x^b}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^j\cdot{}_2\partial x^i}\right)-\left(\frac{{}_2\partial v_j}{{}_2\partial x^i}-{}_2v_b\cdot\frac{{}_2\partial x^b}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^i\cdot{}_2\partial x^j}\right)=\frac{{}_2\partial v_i}{{}_2\partial x^j}-\frac{{}_2\partial v_j}{{}_2\partial x^i}={}_2T_{ij}$$

The transformation rule of the rotation corresponds to tensors of second rank:

$$T_{ab}\cdot\Lambda_j^{\ b}\cdot\Lambda_i^{\ a}={}_2T_{ij} \quad (1.6.6)$$

This tensor is antisymmetric:

$$T_{ij}=-T_{ji}=\frac{\partial v_i}{\partial x^j}-\frac{\partial v_j}{\partial x^i}=-\left(\frac{\partial v_j}{\partial x^i}-\frac{\partial v_i}{\partial x^j}\right) \quad (1.6.7)$$

Its diagonal elements are always zero: $$T_{ii}=\frac{\partial v_i}{\partial x^i}-\frac{\partial v_i}{\partial x^i}=0 \quad (1.6.8)$$

If we cyclic permute the indices of the partial derivative of the antisymmetric tensor and add them together, the result is zero, because the second derivatives of the vectors cancel out:

$$\frac{\partial T_{ij}}{\partial x^k}+\frac{\partial T_{jk}}{\partial x^i}+\frac{\partial T_{ki}}{\partial x^j}=\frac{\partial}{\partial x^k}\left(\frac{\partial v_i}{\partial x^j}-\frac{\partial v_j}{\partial x^i}\right)+\frac{\partial}{\partial x^i}\left(\frac{\partial v_j}{\partial x^k}-\frac{\partial v_k}{\partial x^j}\right)+\frac{\partial}{\partial x^j}\left(\frac{\partial v_k}{\partial x^i}-\frac{\partial v_i}{\partial x^k}\right)=$$
$$\frac{\partial^2 v_i}{\partial x^k\cdot\partial x^j}-\frac{\partial^2 v_j}{\partial x^k\cdot\partial x^i}+\frac{\partial^2 v_j}{\partial x^i\cdot\partial x^k}-\frac{\partial^2 v_k}{\partial x^i\cdot\partial x^j}+\frac{\partial^2 v_k}{\partial x^j\cdot\partial x^i}-\frac{\partial^2 v_i}{\partial x^j\cdot\partial x^k}=0 \quad (1.6.9)$$

1.7 Invariant derivative

Let us assume, that a covariant vector field is constant in a given coordinate system, its partial derivative is zero everywhere:

$$\frac{\partial v_i}{\partial x^j}=0 \quad (1.7.1)$$

Rewrite this formula in another coordinate system, utilizing the results of the previous chapter:

$$\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}=\frac{{}_2\partial v_i}{{}_2\partial x^j}-{}_2v_b\cdot\frac{{}_2\partial x^b}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^j\cdot{}_2\partial x^i}=0 \qquad (1.7.2)$$

Insert the connection in the second term:

$${}_2\Gamma^b{}_{ji}=\frac{{}_2\partial x^b}{\partial x^a}\cdot\frac{\partial^2 x^a}{{}_2\partial x^j\cdot{}_2\partial x^i} \qquad (1.7.3)$$

This formula describes in an arbitrary coordinate system, that the partial derivative of the vector field vanishes in the original coordinate system:

$$\frac{{}_2\partial v_i}{{}_2\partial x^j}-{}_2v_b\cdot{}_2\Gamma^b{}_{ji}=0 \qquad (1.7.4)$$

Let us define the invariant derivative of the covariant vector:

$$\nabla_j v_i=\frac{\partial v_i}{\partial x^j}-v_b\cdot\Gamma^b{}_{ji} \qquad (1.7.5)$$

Connection: a Γ quantity, that makes sure, that in the case of a coordinate transformation, the invariant derivative transforms like a tensor:

$$\nabla_j v_i=\frac{\partial v_i}{\partial x^j}-v_a\cdot\Gamma^a{}_{ji} \qquad {}_2\nabla_j\,{}_2v_i=\nabla_b v_a\cdot\Lambda_j{}^b\cdot\Lambda_i{}^a \qquad (1.7.6)$$

Transformation of the invariant derivative, substitute the transformation rules of the partial derivative of the covariant vector and the connection:

$$\frac{{}_2\partial v_i}{{}_2\partial x^j}-{}_2v_d\cdot{}_2\Gamma^d{}_{ji}=$$

$$\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}+v_a\cdot\frac{\partial^2 x^a}{{}_2\partial x^j\cdot{}_2\partial x^i}-v_c\cdot\frac{\partial x^c}{{}_2\partial x^d}\cdot\left(\frac{\partial^2 x^e}{{}_2\partial x^j\cdot{}_2\partial x^i}\cdot\frac{{}_2\partial x^d}{\partial x^e}+\Gamma^e{}_{ba}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}\cdot\frac{{}_2\partial x^d}{\partial x^e}\right)=$$

$$\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}+v_a\cdot\frac{\partial^2 x^a}{{}_2\partial x^j\cdot{}_2\partial x^i}-v_c\cdot\frac{\partial x^c}{{}_2\partial x^d}\cdot\frac{\partial^2 x^e}{{}_2\partial x^j\cdot{}_2\partial x^i}\cdot\frac{{}_2\partial x^d}{\partial x^e}-v_c\cdot\frac{\partial x^c}{{}_2\partial x^d}\cdot\Gamma^e{}_{ba}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}\cdot\frac{{}_2\partial x^d}{\partial x^e}=$$

We recognize the Kronecker-delta in the third and fourth terms:

$$\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}+v_a\cdot\frac{\partial^2 x^a}{{}_2\partial x^j\cdot{}_2\partial x^i}-v_c\cdot\delta^c_e\cdot\frac{\partial^2 x^e}{{}_2\partial x^j\cdot{}_2\partial x^i}-v_c\cdot\delta^c_e\cdot\Gamma^e{}_{ba}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}=$$

The second and the third terms cancel out, we pull out the transformation matrices from the first and fourth terms:

$$\frac{\partial v_a}{\partial x^b}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}-v_c\cdot\Gamma^c{}_{ba}\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}=\left(\frac{\partial v_a}{\partial x^b}-v_c\cdot\Gamma^c{}_{ba}\right)\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i}$$

$$\frac{{}_2\partial v_i}{{}_2\partial x^j}-{}_2v_d\cdot{}_2\Gamma^d{}_{ji}=\left(\frac{\partial v_a}{\partial x^b}-v_c\cdot\Gamma^c{}_{ba}\right)\cdot\frac{\partial x^b}{{}_2\partial x^j}\cdot\frac{\partial x^a}{{}_2\partial x^i} \tag{1.7.7}$$

The partial derivative of a scalar field is a tensor, therefore it coincides with the invariant derivative:

$$\nabla_i\Phi=\frac{\partial\Phi}{\partial x^i} \tag{1.7.9}$$

The scalar product of the covariant and the contravariant vector is a scalar:

$$\nabla_i(u^a\cdot v_a)=\frac{\partial(u^a\cdot v_a)}{\partial x^i} \tag{1.7.10}$$

$$\nabla_i u^a\cdot v_a+u^a\cdot\nabla_i v_a=\frac{\partial u^a}{\partial x^i}\cdot v_a+u^a\cdot\frac{\partial v_a}{\partial x^i}$$

$$\nabla_i u^a\cdot v_a+u^a\cdot\frac{\partial v_a}{\partial x^i}-u^a\cdot v_b\cdot\Gamma^b{}_{ia}=\frac{\partial u^a}{\partial x^i}\cdot v_a+u^a\cdot\frac{\partial v_a}{\partial x^i}$$

$$\nabla_i u^a\cdot v_a=\frac{\partial u^a}{\partial x^i}\cdot v_a+u^a\cdot v_b\cdot\Gamma^b{}_{ia}\quad /\cdot\frac{1}{v_a}$$

The invariant derivative of the contravariant vector:

$$\nabla_i u^j=\frac{\partial u^j}{\partial x^i}+u^a\cdot\Gamma^j{}_{ia} \tag{1.7.11}$$

Now we can determine the invariant derivative of arbitrary tensors, using the products of contravariant and covariant vectors:

$$\nabla_n T^{ijk\ldots}{}_{lm\ldots}=\nabla_n(v^i\cdot u^j\cdot w^k\cdot\ldots\cdot p_l\cdot q_m\cdot\ldots)$$

$$\nabla_n T^{ijk\ldots}{}_{lm\ldots}=\nabla_n v^i\cdot u^j\cdot w^k\cdot\ldots\cdot p_l\cdot q_m\cdot\ldots+v^i\cdot\nabla_n u^j\cdot w^k\cdot\ldots\cdot p_l\cdot q_m\cdot\ldots+\ldots$$

$$\nabla_n T^{ijk\ldots}{}_{lm\ldots}=\frac{\partial v^i}{\partial x^n}\cdot u^j\cdot w^k\cdot\ldots\cdot p_l\cdot q_m\cdot\ldots+v^a\cdot\Gamma^i{}_{na}\cdot u^j\cdot w^k\cdot\ldots\cdot p_l\cdot q_m\cdot\ldots+\ldots$$

Group together the vectors on the right side:

$$\nabla_n T^{ijk\dots}{}_{lm\dots}=$$
$$\frac{\partial T^{ijk\dots}{}_{lm\dots}}{\partial x^n}+\Gamma^i{}_{na}\cdot T^{ajk\dots}{}_{lm\dots}+\Gamma^j{}_{na}\cdot T^{iak\dots}{}_{lm\dots}+\Gamma^k{}_{na}\cdot T^{ija\dots}{}_{lm\dots}+\dots-\Gamma^a{}_{nl}\cdot T^{ijk\dots}{}_{am\dots}-\Gamma^a{}_{nm}\cdot T^{ijk\dots}{}_{la\dots}-\dots \quad (1.7.12)$$

It is possible for certain quantities to serve as the connection, that cannot be formulated like the previously defined version, therefore we have to spend some time with the properties of the general case. The rotation, using the invariant derivative:

$$\nabla_j v_i-\nabla_i v_j=\frac{\partial v_i}{\partial x^j}-\frac{\partial v_j}{\partial x^i}-v_b\cdot(\Gamma^b{}_{ji}-\Gamma^b{}_{ij})=\frac{\partial v_i}{\partial x^j}-\frac{\partial v_j}{\partial x^i} \quad (1.7.13)$$

This equation is satisfied only if the connection is symmetric:

$$\Gamma^k{}_{ij}=\Gamma^k{}_{ji} \quad (1.7.14)$$

The invariant derivative of the Kronecker-delta:

$$\nabla_k \delta^i_j=\frac{\partial \delta^i_j}{\partial x^k}+\Gamma^i_{ak}\cdot\delta^a_j-\Gamma^a_{jk}\cdot\delta^i_a=0+\Gamma^i_{jk}-\Gamma^i_{jk}=0 \quad (1.7.15)$$

The formula for the invariant derivative of the metric tensor is the same, that we permuted three times during the derivation of the formula for the relationship between the connection and the metric tensor, and its zero:

$$\nabla_k g_{ij}=\frac{\partial g_{ij}}{\partial x^k}-g_{ia}\cdot\Gamma^a_{kj}-g_{aj}\cdot\Gamma^a_{ki}=0 \quad (1.7.16)$$

The invariant derivative of the twice contravariant metric tensor:

$$\nabla_k(g^{ia}\cdot g_{aj})=\nabla_k\delta^i_j=0$$

$$\nabla_k g^{ia}\cdot g_{aj}+g^{ia}\cdot\nabla_n g_{aj}=0$$

$$\nabla_k g^{ia}\cdot g_{aj}+g^{ia}\cdot 0=0$$

Since the twice covariant metric tensor is not zero in the general case, the invariant derivative of the twice contravariant metric tensor has to be zero:

$$\nabla_k g^{ij}=0 \quad (1.7.17)$$

1.8 Derivative along a curve

1.8 Derivative along a curve

Let us push a vector of a vector field a bit along a curve. The curve is parametrized by an invariant quantity:

The vector field at a certain point of the curve: $v^i(\lambda)$

The vector field at an infinitesimally nearby point of the curve: $v^i(\lambda+\delta\lambda)$

The parallel translated vector along the curve into the second point:

$$w^i(\lambda+\delta\lambda)=v^i(\lambda)-\Gamma^i{}_{ba}\cdot v^a(\lambda)\cdot dx^b$$

The difference between two infinitesimally close points of a vector field is the total differential:

$$dv^i(\lambda)=v^i(\lambda+\delta\lambda)-v^i(\lambda) \qquad (1.8.1)$$

The differential along a curve is the difference between the parallel translated vector and the local vector of the vector field in that point:

$$Dv^i=v^i(\lambda+\delta\lambda)-w^i(\lambda+\delta\lambda) \qquad (1.8.2)$$

$$Dv^i=v^i(\lambda+\delta\lambda)-\left(v^i(\lambda)-\Gamma^i{}_{ba}\cdot v^a(\lambda)\cdot dx^b\right)$$

$$Dv^i=dv^i(\lambda)+\Gamma^i{}_{ba}\cdot v^a(\lambda)\cdot dx^b \quad /\cdot\frac{1}{d\lambda}$$

Substitute the tangent vector: $\frac{dx^i}{d\lambda}=u^i$

When the derivative of a contravariant vector field along a curve is zero, it has the same form as the geodesic equation with tangent vectors:

$$Đv^i=\frac{dv^i}{d\lambda}+\Gamma^i{}_{ba}\cdot v^a\cdot u^b \qquad (1.8.3)$$

We proceed the same way with covariant vector fields:

$$Dv_i=v_i(\lambda+\delta\lambda)-w_i(\lambda+\delta\lambda) \qquad (1.8.4)$$

$$Dv_i=v_i(\lambda+\delta\lambda)-\left(v_i(\lambda)+\Gamma^a{}_{bi}\cdot v_a(\lambda)\cdot dx^b\right)$$

$$Dv_i=dv_i(\lambda)-\Gamma^a{}_{bi}\cdot v_a(\lambda)\cdot dx^b \quad /\cdot\frac{1}{d\lambda}$$

The derivative along a curve of a covariant vector field:

$$Đv_i = \frac{dv_i}{d\lambda} - \Gamma^a{}_{bi} \cdot v_a \cdot u^b \tag{1.8.5}$$

Continue to example the two formulas. Since they have the same structure, we perform the following changes on both of them simultaneously. Let us expand the first term and reorder it:

$$Đv_i = \frac{dx^b}{dx^b} \cdot \frac{dv_i}{d\lambda} - \Gamma^a{}_{bi} \cdot v_a \cdot u^b = \frac{dx^b}{d\lambda} \cdot \frac{dv_i}{dx^b} - \Gamma^a{}_{bi} \cdot v_a \cdot u^b$$

$$Đv^i = \frac{dx^b}{dx^b} \cdot \frac{dv^i}{d\lambda} + \Gamma^i{}_{ba} \cdot v^a \cdot u^b = \frac{dx^a}{d\lambda} \cdot \frac{dv^i}{dx^a} + \Gamma^i{}_{ba} \cdot v^a \cdot u^b \tag{1.8.6}$$

Substitute the tangent vector:

$$Đv_i = u^b \cdot \frac{dv_i}{dx^b} - \Gamma^a{}_{bi} \cdot v_a \cdot u^b = u^b \cdot \left(\frac{dv_i}{dx^b} - \Gamma^a{}_{bi} \cdot v_a \right)$$

$$Đv^i = u^b \cdot \frac{dv^i}{dx^b} + \Gamma^i{}_{ba} \cdot v^a \cdot u^b = u^b \cdot \left(\frac{dv^i}{dx^b} + \Gamma^i{}_{ba} \cdot v^a \right) \tag{1.8.7}$$

We identify the invariant derivative inside the parentheses:

$$\nabla_j v_i = \frac{\partial v_i}{\partial x^j} - v_a \cdot \Gamma^a{}_{ji} \qquad \nabla_i u^j = \frac{\partial u^j}{\partial x^i} + u^a \cdot \Gamma^j{}_{ia}$$

the relationship between the derivative along a curve and the invariant derivative in the case of a symmetric connection:

$$Đv_i = u^b \cdot \nabla_b v_i \qquad Đv^i = u^b \cdot \nabla_b v^i \tag{1.8.8}$$

Now we can write down the derivative along a curve of arbitrary tensors:

$$ĐT^{ijk\ldots}{}_{lm\ldots} = u^a \cdot \nabla_a T^{ijk\ldots}{}_{lm\ldots} = u^a \cdot \nabla_a (v^i \cdot u^j \cdot w^k \cdot \ldots \cdot p_l \cdot q_m \cdot \ldots) \tag{1.8.9}$$

The derivative along a curve of the tangent vector of the curve is zero, because if it is displaced along the curve, it will coincide with the local tangent vector at the destination. Therefore the derivative of the tangent vector along a curve is the geodesic equation:

$$Đu^i = \frac{du^i}{d\lambda} + \Gamma^i{}_{ba} \cdot u^a \cdot u^b = \frac{\partial^2 x^i}{\partial \lambda^2} + \Gamma^i{}_{ab} \cdot \frac{\partial x^a}{\partial \lambda} \cdot \frac{\partial x^b}{\partial \lambda} = 0 \tag{1.8.10}$$

This condition is obviously satisfied only if the connection is symmetric. The derivative along a curve of the covariant tangent vector of the curve:

$$Đu_i = \frac{du_i}{d\lambda} - \Gamma^a{}_{bi} \cdot u_a \cdot u^b = 0$$

We rewrite the covariant vector to contravariant in the second term, using the metric tensor:

$$Đu_i = \frac{du_i}{d\lambda} - g_{ac} \cdot \Gamma^a{}_{bi} \cdot u^c \cdot u^b = 0 \qquad (1.8.11)$$

$$Đu_i = \frac{du_i}{d\lambda} - \frac{1}{2} \cdot \left(g_{ac} \cdot \Gamma^a{}_{bi} + g_{ab} \cdot \Gamma^a{}_{ci} \right) \cdot u^c \cdot u^b = 0$$

If the connection is symmetric, then in the second term we can identify the invariant derivative of the metric tensor inside the parentheses:

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - g_{ia} \cdot \Gamma^a{}_{kj} - g_{aj} \cdot \Gamma^a{}_{ki} = 0$$

$$\frac{\partial g_{ij}}{\partial x^k} = g_{ia} \cdot \Gamma^a{}_{kj} + g_{aj} \cdot \Gamma^a{}_{ki}$$

Substitute it:

$$Đu_i = \frac{du_i}{d\lambda} - \frac{1}{2} \cdot \frac{\partial g_{bc}}{\partial x^i} \cdot u^c \cdot u^b = 0 \qquad (1.8.12)$$

If the partial derivative of the twice covariant metric tensor is zero, the corresponding covariant tangent vector of the geodesics does not change:

$$\frac{\partial g_{ij}}{\partial x^k} = 0 \qquad \rightarrow \qquad \frac{du_k}{d\lambda} = 0 \qquad (1.8.13)$$

1.9 Curvature

We are going to examine the global properties of surfaces, that are by their nature independent of the coordinate systems. Our requirement is to be able to collect as many information about the structure of the surfaces as possible, using internally measurable quantities. The practical significance is, that we have to examine the shape of the spacetime using physical events and processes that happen inside, we do not have the option to observe them from somewhere outside.

The commutator of the invariant derivative of the contravariant vector:

$$\nabla_k \nabla_j v^i - \nabla_j \nabla_k v^i \qquad (1.9.1)$$

The invariant derivative is a tensor-like quantity by definition:

$$\nabla_j v^i = \frac{\partial v^i}{\partial x^j} + v^a \cdot \Gamma^i{}_{ja} = T^i{}_j \tag{1.9.2}$$

Expand the repeated invariant derivation:

$$\nabla_k \nabla_j v^i = \nabla_k T^i{}_j = \frac{\partial T^i{}_j}{\partial x^k} + \Gamma^i{}_{kb} \cdot T^b{}_j - \Gamma^b{}_{kj} \cdot T^i{}_b \tag{1.9.3}$$

And substitute the vector's invariant derivative:

$$\nabla_k \left(\frac{\partial v^i}{\partial x^j} + v^a \cdot \Gamma^i{}_{ja} \right) = \frac{\partial}{\partial x^k} \left(\frac{\partial v^i}{\partial x^j} + v^a \cdot \Gamma^i{}_{ja} \right) + \Gamma^i{}_{kb} \cdot \left(\frac{\partial v^b}{\partial x^j} + v^a \cdot \Gamma^b{}_{ja} \right) - \Gamma^b{}_{kj} \cdot \left(\frac{\partial v^i}{\partial x^b} + v^a \cdot \Gamma^i{}_{ba} \right)$$

Opening the parentheses:

$$\nabla_k \nabla_j v^i = \frac{\partial^2 v^i}{\partial x^k \cdot \partial x^j} + \frac{\partial v^a}{\partial x^k} \cdot \Gamma^i{}_{ja} + v^a \cdot \frac{\partial \Gamma^i{}_{ja}}{\partial x^k} + \Gamma^i{}_{kb} \cdot \frac{\partial v^b}{\partial x^j} + \Gamma^i{}_{kb} \cdot v^a \cdot \Gamma^b{}_{ja} - \Gamma^b{}_{kj} \cdot \frac{\partial v^i}{\partial x^b} - \Gamma^b{}_{kj} \cdot v^a \cdot \Gamma^i{}_{ba}$$

Doing the same with the opposite index order:

$$\nabla_j \nabla_k v^i = \frac{\partial^2 v^i}{\partial x^j \cdot \partial x^k} + \frac{\partial v^a}{\partial x^j} \cdot \Gamma^i{}_{ka} + v^a \cdot \frac{\partial \Gamma^i{}_{ka}}{\partial x^j} + \Gamma^i{}_{jb} \cdot \frac{\partial v^b}{\partial x^k} + \Gamma^i{}_{jb} \cdot v^a \cdot \Gamma^b{}_{ka} - \Gamma^b{}_{jk} \cdot \frac{\partial v^i}{\partial x^b} - \Gamma^b{}_{jk} \cdot v^a \cdot \Gamma^i{}_{ba}$$

Subtract one of the other:

$$\nabla_k \nabla_j v^i - \nabla_j \nabla_k v^i = \left(\frac{\partial \Gamma^i{}_{ja}}{\partial x^k} - \frac{\partial \Gamma^i{}_{ka}}{\partial x^j} + \Gamma^i{}_{kb} \cdot \Gamma^b{}_{ja} - \Gamma^i{}_{jb} \cdot \Gamma^b{}_{ka} + 2 \cdot S^b{}_{jk} \cdot \Gamma^i{}_{ba} \right) \cdot v^a + 2 \cdot S^b{}_{jk} \cdot \frac{\partial v^i}{\partial x^b} \tag{1.9.4}$$

Where we have substituted the antisymmetric expression with the torsion tensor:

$$S^b{}_{jk} = \frac{1}{2} \cdot (\Gamma^b{}_{jk} - \Gamma^b{}_{kj}) \tag{1.9.5}$$

Substitute the curvature tensor:

$$R^i{}_{jak} = \frac{\partial \Gamma^i{}_{ja}}{\partial x^k} - \frac{\partial \Gamma^i{}_{ka}}{\partial x^j} + \Gamma^i{}_{kb} \cdot \Gamma^b{}_{ja} - \Gamma^i{}_{jb} \cdot \Gamma^b{}_{ka} \tag{1.9.6}$$

$$\nabla_k \nabla_j v^i - \nabla_j \nabla_k v^i = R^i{}_{jak} \cdot v^a + 2 \cdot S^b{}_{jk} \cdot \left(\frac{\partial v^i}{\partial x^b} + \Gamma^i{}_{ba} \cdot v^a \right) = R^i{}_{jak} \cdot v^a + 2 \cdot S^b{}_{jk} \cdot \nabla_b v^i$$

If the connection is symmetric, then the commutator of the invariant derivative of the contravariant vector is the curvature tensor, where we can recognize the tensor property from the form of the expression:

$$\nabla_k \nabla_j v^i - \nabla_j \nabla_k v^i = R^i{}_{jak} \cdot v^a \tag{1.9.7}$$

We determine the commutator of the invariant derivative of the covariant vector using the same steps:

$$\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i \tag{1.9.8}$$

The invariant derivative is a tensor-like quantity:

$$\nabla_j v_i = \frac{\partial v_i}{\partial x^j} - v_a \cdot \Gamma^a{}_{ji} = T_{ij} \tag{1.9.9}$$

The repeated invariant derivation:

$$\nabla_k T_{ij} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^a{}_{kj} \cdot T_{ia} - \Gamma^a{}_{ki} \cdot T_{aj} \tag{1.9.10}$$

Substitute the invariant derivative of the vector:

$$\nabla_k \left(\frac{\partial v_i}{\partial x^j} - v_a \cdot \Gamma^a{}_{ji} \right) = \frac{\partial}{\partial x^k} \left(\frac{\partial v_i}{\partial x^j} - v_a \cdot \Gamma^a{}_{ji} \right) - \Gamma^b_{kj} \cdot \left(\frac{\partial v_i}{\partial x^b} - v_a \cdot \Gamma^a{}_{bi} \right) - \Gamma^b_{ki} \cdot \left(\frac{\partial v_b}{\partial x^j} - v_a \cdot \Gamma^a{}_{jb} \right)$$

Open the parentheses:

$$\nabla_k \nabla_j v_i = \frac{\partial^2 v_i}{\partial x^k \cdot \partial x^j} - \frac{\partial v_a}{\partial x^k} \cdot \Gamma^a{}_{ji} - v_a \cdot \frac{\partial \Gamma^a{}_{ji}}{\partial x^k} - \Gamma^b{}_{kj} \cdot \frac{\partial v_i}{\partial x^b} + \Gamma^b{}_{kj} \cdot v_a \cdot \Gamma^a{}_{bi} - \Gamma^b{}_{ki} \cdot \frac{\partial v_b}{\partial x^j} + \Gamma^b{}_{ki} \cdot v_a \cdot \Gamma^a{}_{jb}$$

Doing the same with the opposite index order:

$$\nabla_j \nabla_k v_i = \frac{\partial^2 v_i}{\partial x^j \cdot \partial x^k} - \frac{\partial v_a}{\partial x^j} \cdot \Gamma^a{}_{ki} - v_a \cdot \frac{\partial \Gamma^a{}_{ki}}{\partial x^j} - \Gamma^b{}_{jk} \cdot \frac{\partial v_i}{\partial x^b} + \Gamma^b{}_{jk} \cdot v_a \cdot \Gamma^a{}_{bi} - \Gamma^b{}_{ji} \cdot \frac{\partial v_b}{\partial x^k} + \Gamma^b{}_{ji} \cdot v_a \cdot \Gamma^a{}_{kb}$$

Subtract one of the other:

$$\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i = \left(\frac{\partial \Gamma^a{}_{ki}}{\partial x^j} - \frac{\partial \Gamma^a{}_{ji}}{\partial x^k} + \Gamma^b{}_{ki} \cdot \Gamma^a{}_{jb} - \Gamma^b{}_{ji} \cdot \Gamma^a{}_{kb} - 2 \cdot S^b{}_{jk} \cdot \Gamma^a{}_{bi} \right) \cdot v_a + 2 \cdot S^b{}_{jk} \cdot \frac{\partial v_i}{\partial x^b} \tag{1.9.11}$$

Where we have substituted again the antisymmetric expression with the torsion tensor:

$$S^b{}_{jk} = \frac{1}{2} \cdot (\Gamma^b{}_{jk} - \Gamma^b{}_{kj}) \tag{1.9.12}$$

Substitute the curvature tensor: $$R^a{}_{kij}=\frac{\partial\Gamma^a{}_{ki}}{\partial x^j}-\frac{\partial\Gamma^a{}_{ji}}{\partial x^k}+\Gamma^b{}_{ki}\cdot\Gamma^a{}_{jb}-\Gamma^b{}_{ji}\cdot\Gamma^a{}_{kb} \qquad (1.9.13)$$

$$\nabla_k\nabla_j v_i-\nabla_j\nabla_k v_i=R^a{}_{kij}\cdot v_a+2\cdot S^b{}_{jk}\cdot\left(\frac{\partial v_i}{\partial x^b}-\Gamma^a{}_{bi}\cdot v_a\right)=R^a{}_{kij}\cdot v_a+2\cdot S^b{}_{jk}\cdot\nabla_b v_i$$

If the connection is symmetric, then the commutator of the invariant derivative of the covariant vector is also the curvature tensor:

$$\nabla_k\nabla_j v_i-\nabla_j\nabla_k v_i=R^a_{kij}\cdot v_a \qquad (1.9.14)$$

If we set up a coordinate system in the immediate surrounding of a point, where the partial derivatives of the metric tensor and the connection are zeroes, the curvature tensor will not necessarily vanish, since the partial derivatives of the connection are not necessarily zeroes:

$$R^i{}_{jkl}=\frac{\partial\Gamma^i{}_{jk}}{\partial x^l}-\frac{\partial\Gamma^i{}_{lk}}{\partial x^j} \qquad (1.9.15)$$

The invariant derivative:

$$\nabla_m R^i{}_{jkl}=\nabla_m\left(\frac{\partial\Gamma^i{}_{jk}}{\partial x^l}-\frac{\partial\Gamma^i{}_{lk}}{\partial x^j}\right) \qquad (1.9.16)$$

Permute the first and the third lower indices and the index of the invariant derivation:

(1) $$\nabla_m R^i{}_{jkl}=\nabla_m\frac{\partial\Gamma^i{}_{jk}}{\partial x^l}-\nabla_m\frac{\partial\Gamma^i{}_{lk}}{\partial x^j}$$

(2) $$\nabla_j R^i{}_{lkm}=\nabla_j\frac{\partial\Gamma^i{}_{lk}}{\partial x^m}-\nabla_j\frac{\partial\Gamma^i{}_{mk}}{\partial x^l}$$

(3) $$\nabla_l R^i{}_{mkj}=\nabla_l\frac{\partial\Gamma^i{}_{mk}}{\partial x^j}-\nabla_l\frac{\partial\Gamma^i{}_{jk}}{\partial x^m} \qquad (1.9.17)$$

Add the three equations:

$$\nabla_m R^i{}_{jkl}+\nabla_j R^i{}_{lkm}+\nabla_l R^i{}_{mkj}=$$
$$\nabla_m\frac{\partial\Gamma^i{}_{jk}}{\partial x^l}-\nabla_m\frac{\partial\Gamma^i{}_{lk}}{\partial x^j}+\nabla_j\frac{\partial\Gamma^i{}_{lk}}{\partial x^m}-\nabla_j\frac{\partial\Gamma^i{}_{mk}}{\partial x^l}+\nabla_l\frac{\partial\Gamma^i{}_{mk}}{\partial x^j}-\nabla_l\frac{\partial\Gamma^i{}_{jk}}{\partial x^m} \qquad (1.9.18)$$

Since the connection coefficients are zeroes in this coordinate system, only the partial derivatives remain from the invariant derivatives, and they eventually cancel out:

$$\nabla_m R^i{}_{jkl} + \nabla_j R^i{}_{lkm} + \nabla_l R^i{}_{mkj} =$$
$$\frac{\partial^2 \Gamma^i{}_{jk}}{\partial x^l \cdot \partial x^m} - \frac{\partial^2 \Gamma^i{}_{lk}}{\partial x^m \cdot \partial x^j} + \frac{\partial^2 \Gamma^i{}_{lk}}{\partial x^j \cdot \partial x^m} - \frac{\partial^2 \Gamma^i{}_{mk}}{\partial x^j \cdot \partial x^l} + \frac{\partial^2 \Gamma^i{}_{mk}}{\partial x^l \cdot \partial x^j} - \frac{\partial^2 \Gamma^i{}_{jk}}{\partial x^l \cdot \partial x^m}$$

The following Bianchi identity is valid in every coordinate system:

$$\nabla_m R^i{}_{jkl} + \nabla_j R^i{}_{lkm} + \nabla_l R^i{}_{mkj} = 0 \qquad (1.9.19)$$

The curvature tensor is antisymmetric in its first and third lower indices:

$$R^l{}_{kij} = \frac{\partial \Gamma^l{}_{ki}}{\partial x^j} - \frac{\partial \Gamma^l{}_{ji}}{\partial x^k} + \Gamma^b{}_{ki} \cdot \Gamma^l{}_{jb} - \Gamma^b{}_{ji} \cdot \Gamma^l{}_{kb} \qquad R^l{}_{jik} = \frac{\partial \Gamma^l{}_{ji}}{\partial x^k} - \frac{\partial \Gamma^l{}_{ki}}{\partial x^j} + \Gamma^b{}_{ji} \cdot \Gamma^l{}_{kb} - \Gamma^b{}_{ki} \cdot \Gamma^l{}_{jb}$$

$$R^l{}_{kij} = -R^l{}_{jik} \qquad (1.9.20)$$

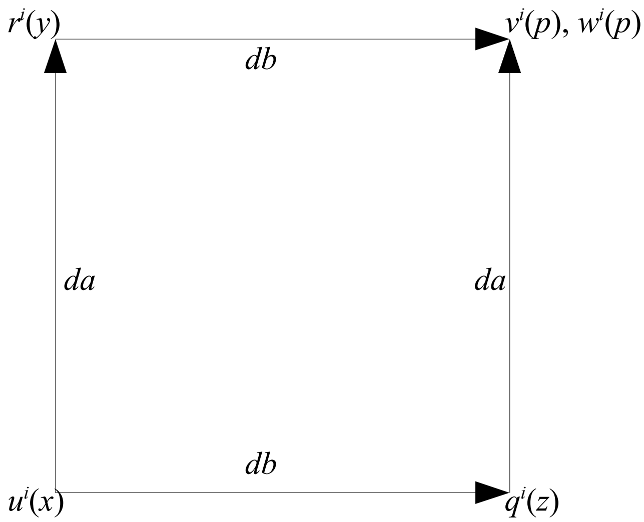
If the connection is symmetric, the sum of the cyclic permutations of the lower indices of the curvature tensor is zero:

$$R^l{}_{ijk} = \frac{\partial \Gamma^l{}_{ij}}{\partial x^k} - \frac{\partial \Gamma^l{}_{kj}}{\partial x^i} + \Gamma^b{}_{ij} \cdot \Gamma^l{}_{kb} - \Gamma^b{}_{kj} \cdot \Gamma^l{}_{ib} \qquad R^l{}_{jki} = \frac{\partial \Gamma^l{}_{jk}}{\partial x^i} - \frac{\partial \Gamma^l{}_{ik}}{\partial x^j} + \Gamma^b{}_{jk} \cdot \Gamma^l{}_{ib} - \Gamma^b{}_{ik} \cdot \Gamma^l{}_{jb}$$

$$R^l{}_{kij} + R^l{}_{ijk} + R^l{}_{jki} = 0 \qquad (1.9.21)$$

1.10 Parallel transport along a closed curve

Let us set up an infinitesimal parallelogram with edges *da* and *db* long. We parallel transport a vector along the edges from one corner (x) to the opposing corner (p), first through the y, later through the z intermediate points:



$u^i(x) \rightarrow r^i(y) \rightarrow v^i(p)$

$u^i(x) \rightarrow q^i(z) \rightarrow w^i(p)$

Parallel transfer from *x* to *y* to *da* distance:

$$r^i = u^i - \Gamma^i{}_{ba}(x) \cdot u^a \cdot da^b \tag{1.10.1}$$

Parallel transfer from *y* to *p* to *db* distance:

$$v^i = r^i - \Gamma^i{}_{dc}(y) \cdot r^c \cdot db^d \tag{1.10.2}$$

We approach the connection in *y* with a Taylor series from the connection in *x*, from *da* distance:

$$\Gamma^i{}_{jk}(y) = \Gamma^i{}_{jk}(x) + \frac{\partial \Gamma^i{}_{jk}(x)}{\partial a^e} \cdot da^e + \frac{1}{2} \cdot \frac{\partial^2 \Gamma^i{}_{jk}(x)}{\partial a^e \cdot \partial a^f} \cdot da^e \cdot da^f + \ldots \tag{1.10.3}$$

Substitute the formula of the first parallel transport into the second, and the approximation of the connection to the first degree:

$$v^i = (u^i - \Gamma^i{}_{ba}(x) \cdot u^a \cdot da^b) - \left(\Gamma^i{}_{dc}(x) + \frac{\partial \Gamma^i{}_{dc}(x)}{\partial a^e} \cdot da^e \right) \cdot (u^c - \Gamma^c{}_{ba}(x) \cdot u^a \cdot da^b) \cdot db^d$$

$$v^i = u^i - \Gamma^i{}_{ba} \cdot u^a \cdot da^b - \Gamma^i{}_{dc} \cdot u^c \cdot db^d + \Gamma^i{}_{dc} \cdot \Gamma^c{}_{ba} \cdot u^a \cdot da^b \cdot db^d$$
$$- \frac{\partial \Gamma^i{}_{dc}}{\partial a^e} \cdot da^e \cdot u^c \cdot db^d + \frac{\partial \Gamma^i{}_{dc}}{\partial a^e} \cdot da^e \cdot \Gamma^c{}_{ba} \cdot u^a \cdot da^b \cdot db^d$$

Neglect the last term with a differential in a higher order:

$$v^i = u^i - \Gamma^i{}_{ba} \cdot u^a \cdot da^b - \Gamma^i{}_{dc} \cdot u^c \cdot db^d + \Gamma^i{}_{dc} \cdot \Gamma^c{}_{ba} \cdot u^a \cdot da^b \cdot db^d - \frac{\partial \Gamma^i{}_{dc}}{\partial a^e} \cdot da^e \cdot u^c \cdot db^d \tag{1.10.4}$$

We perform these steps across the other corner point as well. Parallel transfer from *x* to *z* to *db* distance:

$$q^i = u^i - \Gamma^i{}_{ba}(x) \cdot u^a \cdot db^b \tag{1.10.5}$$

Parallel transfer from *z* to *p* to *da* distance:

$$w^i = q^i - \Gamma^i{}_{dc}(z) \cdot q^c \cdot da^d \tag{1.10.6}$$

We approach the connection in *z* with a Taylor series from the connection in *x*, from *db* distance:

$$\Gamma^i{}_{jk}(z) = \Gamma^i{}_{jk}(x) + \frac{\partial \Gamma^i{}_{jk}(x)}{\partial b^e} \cdot db^e + \frac{1}{2} \cdot \frac{\partial^2 \Gamma^i{}_{jk}(x)}{\partial b^e \cdot \partial b^f} \cdot db^e \cdot db^f + \ldots \tag{1.10.7}$$

Substitute the formula of the first parallel transport into the second, and the approximation of the connection to the first degree:

$$w^i=(u^i-\Gamma^i{}_{ba}(x)\cdot u^a\cdot db^b)-\left(\Gamma^i{}_{dc}(x)+\frac{\partial\Gamma^i{}_{dc}(x)}{\partial b^e}\cdot db^e\right)\cdot(u^c-\Gamma^c{}_{ba}(x)\cdot u^a\cdot db^b)\cdot da^d$$

$$w^i=u^i-\Gamma^i{}_{ba}\cdot u^a\cdot db^b-\Gamma^i{}_{dc}\cdot u^c\cdot da^d+\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}\cdot u^a\cdot db^b\cdot da^d$$
$$-\frac{\partial\Gamma^i{}_{dc}}{\partial a^e}\cdot db^e\cdot u^c\cdot da^d+\frac{\partial\Gamma^i{}_{dc}}{\partial a^e}\cdot db^e\cdot\Gamma^c{}_{ba}\cdot u^a\cdot db^b\cdot da^d$$

Neglect the last term with a differential in a higher order:

$$w^i=u^i-\Gamma^i{}_{ba}\cdot u^a\cdot db^b-\Gamma^i{}_{dc}\cdot u^c\cdot da^d+\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}\cdot u^a\cdot db^b\cdot da^d-\frac{\partial\Gamma^i{}_{dc}}{\partial a^e}\cdot db^e\cdot u^c\cdot da^d \qquad (1.10.8)$$

Subtract the two parallel transfer results from one another:

$$v^i-w^i=\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}\cdot u^a\cdot da^b\cdot db^d-\frac{\partial\Gamma^i{}_{dc}}{\partial a^e}\cdot da^e\cdot u^c\cdot db^d-\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}\cdot u^a\cdot db^b\cdot da^d+\frac{\partial\Gamma^i{}_{dc}}{\partial a^e}\cdot db^e\cdot u^c\cdot da^d$$

$c\rightarrow a$, $e\rightarrow b$:

$$v^i-w^i=-\left(\frac{\partial\Gamma^i{}_{da}}{\partial a^b}-\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}\right)\cdot u^a\cdot(da^b\cdot db^d-db^b\cdot da^d) \qquad (1.10.9)$$

The antisymmetric part of the tensor in the parentheses characterises the difference of the two vectors, it is the curvature tensor:

$$B^i{}_{dab}=\frac{\partial\Gamma^i{}_{da}}{\partial a^b}-\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}$$

$$\frac{1}{2}\cdot(B^i{}_{dab}-B^i{}_{bad})=-\frac{1}{2}\cdot\left(\frac{\partial\Gamma^i{}_{da}}{\partial a^b}-\frac{\partial\Gamma^i{}_{ba}}{\partial a^d}+\Gamma^i{}_{dc}\cdot\Gamma^c{}_{ba}-\Gamma^i{}_{bc}\cdot\Gamma^c{}_{da}\right)=\frac{1}{2}\cdot R^i{}_{dab}$$

$$v^i-w^i=\frac{1}{2}\cdot R^i{}_{dab}\cdot u^a\cdot(da^b\cdot db^d-db^b\cdot da^d) \qquad (1.10.10)$$

1.11 Geodesic deviation

In the immediate surrounding of any *y* point it is possible to set up a rectangular coordinate system, where the connection is zero. Therefore the equation of the geodesics crossing the point simplifies:

$$\frac{\partial^2 y^i}{\partial \lambda^2} = 0 \tag{1.11.1}$$

This is however true for the point only. In the immediate surrounding, the general equation of geodesics continues to apply:

$$\frac{\partial^2 x^i}{\partial \lambda^2} + \Gamma^i{}_{ab} \cdot \frac{\partial x^a}{\partial \lambda} \cdot \frac{\partial x^b}{\partial \lambda} = 0 \tag{1.11.2}$$

We approach the connection in the neighbouring *x* points with a Taylor series of the connection of the original point:

$$\Gamma^i{}_{jk}(x) = \Gamma^i{}_{jk}(y) + \frac{\partial \Gamma^i{}_{jk}(y)}{\partial x^a} \cdot dx^a + \frac{1}{2} \cdot \frac{\partial^2 \Gamma^i{}_{jk}(y)}{\partial x^a \cdot \partial x^b} \cdot dx^a \cdot dx^b + \dots \tag{1.11.3}$$

Substitute it into the equation of the neighbouring geodesics. Since the connection is zero in the centre, only the first derivative appears in the equation, the higher order derivatives are neglected:

$$\frac{\partial^2 x^i}{\partial \lambda^2} + \frac{\partial \Gamma^i{}_{ab}}{\partial x^c} \cdot dx^c \cdot \frac{\partial x^a}{\partial \lambda} \cdot \frac{\partial x^b}{\partial \lambda} = 0 \tag{1.11.4}$$

Keep one of the components of the distance vector from the centre zero, thus we move only in a subspace around the original geodesic. Therefore the partial derivative of the connection according to this coordinate will also be zero:

$$dx^i = (0 \quad dx^1 \quad \dots \quad dx^N) \qquad \rightarrow \qquad \frac{\partial \Gamma^i{}_{jk}}{\partial x^0} = 0 \tag{1.11.5}$$

The tangent vector is perpendicular to this subspace, thus the centre-crossing geodesic pierces this subspace perpendicularly:

$$\frac{\partial x^i}{\partial \lambda} = \left(\frac{\partial x^0}{\partial \lambda} \quad 0 \quad 0 \quad \dots \right) \tag{1.11.6}$$

The equation of the surrounding geodesics simplifies further:

$$\frac{\partial^2 x^i}{\partial \lambda^2} + \frac{\partial \Gamma^i{}_{00}}{\partial x^c} \cdot dx^c \cdot \frac{\partial x^0}{\partial \lambda} \cdot \frac{\partial x^0}{\partial \lambda} = 0 \tag{1.11.7}$$

Expand the connection derivative with a term that is known to be zero, and with them we produce the curvature tensor:

$$\frac{\partial^2 x^i}{\partial \lambda^2} + \left(\frac{\partial \Gamma^i{}_{00}}{\partial x^c} - \frac{\partial \Gamma^i{}_{c0}}{\partial x^0} \right) \cdot dx^c \cdot \frac{\partial x^0}{\partial \lambda} \cdot \frac{\partial x^0}{\partial \lambda} = 0 \qquad R^i{}_{0 0 l} = \frac{\partial \Gamma^i{}_{00}}{\partial x^l} - \frac{\partial \Gamma^i{}_{l0}}{\partial x^0}$$

$$\frac{\partial^2 x^i}{\partial \lambda^2} + R^i{}_{00c} \cdot dx^c \cdot \frac{\partial x^0}{\partial \lambda} \cdot \frac{\partial x^0}{\partial \lambda} = 0 \qquad (1.11.8)$$

This formula is valid in general, we have no reason to restrict it to the indices denoted with numbers. Therefore the deviation of geodesics in the immediate surrounding of a geodesic is:

$$\frac{\partial^2 x^i}{\partial \lambda^2} + R^i{}_{abc} \cdot dx^c \cdot \frac{\partial x^a}{\partial \lambda} \cdot \frac{\partial x^b}{\partial \lambda} = 0 \qquad (1.11.9)$$

1.12 Integration

We integrate with respect to all coordinate-variables in space, therefore we introduce a shorthand notation for the product of the differentials:

$$dx^N = \prod_{n=1}^{N} dx^n \qquad (1.12.1)$$

The simple multi-variable integral of the scalar function is not invariant in the general case, since the product of the differentials depends on the coordinate system:

$$\int A \cdot dx^N \neq \int A \cdot {}_2dx^N \qquad (1.12.2)$$

When we switch coordinate systems, the product of the differentials transforms with the determinant of the transformation matrix:

$$dx^N = \left| \frac{\partial x^i}{{}_2\partial x^j} \right| \cdot {}_2dx^N = |\Lambda_j{}^i| \cdot {}_2dx^N \qquad (1.12.3)$$

We insert this into the integral:

$$\int A \cdot dx^N = \int A \cdot \left| \frac{\partial x^i}{{}_2\partial x^j} \right| \cdot {}_2dx^N = \int A \cdot |\Lambda_j{}^i| \cdot {}_2dx^N \qquad (1.12.4)$$

The following expression is invariant, if we build in the determinant of the transformation matrix into the expression under the integral:

$$\int \tilde{A} \cdot dx^N = \int {}_2\tilde{A} \cdot {}_2dx^N \qquad (1.12.5)$$

The quantities that transform the following way are called scalar densities:

$${}_2\tilde{A} = \tilde{A} \cdot \left| \frac{\partial x^i}{{}_2\partial x^j} \right| = \tilde{A} \cdot |\Lambda_j{}^i| \qquad (1.12.6)$$

Let us examine the transformation rule of the metric tensor:

$$ {}_2g_{ij} = \Lambda_i^{\ a} \cdot \Lambda_j^{\ b} \cdot g_{ab} \tag{1.12.7}$$

Since by matrix multiplication the determinants are also multiplied:

$$ {}_2g = |\Lambda_i^{\ a}| \cdot |\Lambda_j^{\ b}| \cdot g \tag{1.12.8}$$

The determinant of the metric tensor transforms like a scalar quantity, this also shows, that the signature of this determinant is independent from the choice of coordinates:

$$\sqrt{{}_2g} = |\Lambda_j^{\ i}| \cdot \sqrt{g} \tag{1.12.9}$$

Therefore the following integral is invariant:

$$\int A \cdot \sqrt{g} \cdot dx^N = \int A \cdot \sqrt{{}_2g} \cdot {}_2dx^N \tag{1.12.10}$$

1.13 Variation and action principle

Integral of a scalar function that vanishes at the boundaries, and its integral has an extremity, thus by slightly changing the input parameters the value of the integral does not change:

$$S = \int s(x^i) \cdot \sqrt{g} \cdot dx^N \qquad\qquad \delta S = 0 \tag{1.13.1}$$

In order to apply the action principle, we need to choose a scalar that represents the space. For this we start with the curvature tensor and contract it to create the Ricci-tensor:

$$R_{ki} = R^a_{\ kia} = \frac{\partial \Gamma^a_{\ ki}}{\partial x^a} - \frac{\partial \Gamma^a_{\ ai}}{\partial x^k} + \Gamma^b_{\ ki} \cdot \Gamma^a_{\ ab} - \Gamma^b_{\ ai} \cdot \Gamma^a_{\ kb} \tag{1.13.2}$$

The trace of the Ricci-tensor is the curvature scalar, the simplest invariant scalar in the space:

$$R = g^{ab} \cdot R_{ab} \tag{1.13.3}$$

We use this as the scalar function:

$$S = \int R \cdot \sqrt{g} \cdot dx^N \tag{1.13.4}$$

Its variation:

$$\delta S = \delta \left(\int R \cdot \sqrt{g} \cdot dx^N \right) = \int \left(\delta R \cdot \sqrt{g} + R \cdot \delta \sqrt{g} \right) \cdot dx^N = 0 \tag{1.13.5}$$

$$\int \left(\delta\left(g^{ab}\cdot R_{ab}\right)\cdot\sqrt{g}+R\cdot\delta\sqrt{g}\right)\cdot dx^{N}=\int \left(\left(\delta g^{ab}\cdot R_{ab}+g^{ab}\cdot\delta R_{ab}\right)\cdot\sqrt{g}+R\cdot\delta\sqrt{g}\right)\cdot dx^{N}=0$$

Examine the second term of the inner parentheses in a coordinate system, where the connection is zero:

$$g^{ab}\cdot\delta R_{ab}=g^{ab}\cdot\delta\frac{\partial\Gamma^{c}{}_{ab}}{\partial x^{c}}-g^{ab}\cdot\delta\frac{\partial\Gamma^{c}{}_{cb}}{\partial x^{a}}=\frac{\partial}{\partial x^{c}}\left(g^{ab}\cdot\delta\Gamma^{c}{}_{ab}-g^{cb}\cdot\delta\Gamma^{a}{}_{ab}\right)=\frac{\partial v^{c}}{\partial x^{c}} \qquad (1.13.6)$$

where: $v^{i}=g^{ab}\cdot\delta\Gamma^{i}{}_{ab}-g^{ib}\cdot\delta\Gamma^{a}{}_{ab}$

Thus the expression inside the parentheses transforms like a vector. We rewrite the partial derivative into an invariant one, thus it will become valid in every coordinate system:

$$\int \nabla_{c} v^{c}\cdot\sqrt{g}\cdot dx^{N}=\oint v^{i}\cdot\sqrt{g}\cdot dx^{N-1}=0 \qquad (1.13.7)$$

Rewrite the integral into a surface integral using the divergence theorem, however our starting condition was that our scalar is zero on the boundaries, thus we succeeded in making this term disappear:

$$\int \left(\delta g^{ab}\cdot R_{ab}\cdot\sqrt{g}+R\cdot\delta\sqrt{g}\right)\cdot dx^{N}=0 \qquad (1.13.8)$$

Using differentiation rules we rewrite the variation of the square root of the determinant of the metric tensor (where *M* is the algebraic minor):

$$g\cdot\delta^{i}_{j}=M^{k\neq a,\, l\neq i}\cdot g_{aj}$$

$$dg=M^{k\neq a,\, l\neq i}\cdot dg_{ai}$$

$$\delta g=g\cdot g^{ab}\cdot\delta g_{ab}$$

$$\delta\sqrt{g}=-\frac{1}{2}\cdot\sqrt{g}\cdot g_{ab}\cdot\delta g^{ab} \qquad (1.13.9)$$

Substitute it:

$$\int \left(\delta g^{ab}\cdot R_{ab}\cdot\sqrt{g}-R\cdot\frac{1}{2}\cdot\sqrt{g}\cdot g_{ab}\cdot\delta g^{ab}\right)\cdot dx^{N}=0$$

$$\int \left(R_{ab}-\frac{1}{2}\cdot R\cdot g_{ab}\right)\cdot\delta g^{ab}\cdot\sqrt{g}\cdot dx^{N}=0 \qquad (1.13.10)$$

This expression is zero only if the expression inside the parentheses is zero, it is the Einstein equation:

$$R_{ij}-\frac{1}{2}\cdot R\cdot g_{ij}=0 \qquad (1.13.11)$$

The invariant derivative of the Einstein equation:

$$\nabla_k\left(R_{ij}-\frac{1}{2}\cdot R\cdot g_{ij}\right)=\nabla_k R_{ij}-\frac{1}{2}\cdot R\cdot\nabla_k g_{ij}=\nabla_k R_{ij} \qquad (1.13.12)$$

We already know, that the invariant derivative of the metric tensor is zero. We can determine the invariant derivative of the Ricci-tensor using the Bianchi identity:

$$\nabla_k R_{ij}=\frac{1}{3}\cdot(\nabla_k R^a{}_{ija}+\nabla_i R^a{}_{ajk}+\nabla_a R^a{}_{kji})=0 \qquad (1.13.13)$$

Thus our result is:

$$\nabla_k\left(R_{ij}-\frac{1}{2}\cdot R\cdot g_{ij}\right)=0 \qquad (1.13.14)$$

1.14 Runge-Kutta approximation method

On surfaces with known geometry, we can examine arbitrary geodesics using numerical methods. The geodesic is uniquely identified by the coordinates of a single point it crosses, and its tangent vector in that point. With this information it is possible to recover with small steps the coordinates of the other points the geodesic is crossing, the smaller the steps are, the greater the accuracy becomes.

Using the Runge-Kutta approximation method we can determine trajectories with high accuracy and by doing significantly less iteration steps, if we know the following variables at the *n.*-th step:

coordinates: ${}_n x^i$

coordinate-velocities: ${}_n v^i$

connection in a given point: $\Gamma^i{}_{jk}(x^i)$

change of the invariant parameter: $d\lambda$ (1.14.1)

It is important that the invariant parameter increases or decreases monotonically, because only in this case will it lead along the entire geodesic. The change in the parameter determines the the step size, that can be a conveniently chosen number.

We approach the trajectory of the moving body with four straight sections, determine the coordinate- and coordinate-velocity-changes along the sections, and then average them:

$$ {}_1a^i = -\Gamma^i{}_{ab}\left({}_nx^i\right)\cdot {}_nv^a\cdot {}_nv^b $$

$$ {}_1\Delta v^i = {}_1a^i\cdot d\lambda \qquad {}_1\Delta x^i = {}_nv^i\cdot d\lambda \tag{1.14.2} $$

$$ {}_2a^i = -\Gamma^i{}_{ab}\left({}_nx^i + \frac{{}_1\Delta x^i}{2}\right)\cdot\left({}_nv^a + \frac{{}_1\Delta v^a}{2}\right)\cdot\left({}_nv^b + \frac{{}_1\Delta v^b}{2}\right) $$

$$ {}_2\Delta v^i = {}_2a^i\cdot d\lambda \qquad {}_2\Delta x^i = \left({}_nv^i + \frac{{}_1\Delta v^i}{2}\right)\cdot d\lambda \tag{1.14.3} $$

$$ {}_3a^i = -\Gamma^i{}_{ab}\left({}_nx^i + \frac{{}_2\Delta x^i}{2}\right)\cdot\left({}_nv^a + \frac{{}_2\Delta v^a}{2}\right)\cdot\left({}_nv^b + \frac{{}_2\Delta v^b}{2}\right) $$

$$ {}_3\Delta v^i = {}_3a^i\cdot d\lambda \qquad {}_3\Delta x^i = \left({}_nv^i + \frac{{}_2\Delta v^i}{2}\right)\cdot d\lambda \tag{1.14.4} $$

$$ {}_4a^i = -\Gamma^i{}_{ab}\left({}_nx^i + {}_3\Delta x^i\right)\cdot\left({}_nv^a + {}_3\Delta v^a\right)\cdot\left({}_nv^b + {}_3\Delta v^b\right) $$

$$ {}_4\Delta v^i = {}_4a^i\cdot d\lambda \qquad {}_4\Delta x^i = \left({}_nv^i + {}_3\Delta v^i\right)\cdot d\lambda \tag{1.14.5} $$

We write down the weighted sum of the resulting coordinate- and coordinate-velocity-changes, thus we get the variables that determine the trajectory at the next step. The results calculated with this method deviate from the actual value only in the fifth order:

$$ {}_{n+1}x^i = {}_nx^i + \frac{{}_1\Delta x^i}{6} + \frac{{}_2\Delta x^i}{3} + \frac{{}_3\Delta x^i}{3} + \frac{{}_4\Delta x^i}{6} + \ldots $$

$$ {}_{n+1}v^i = {}_nv^i + \frac{{}_1\Delta v^i}{6} + \frac{{}_2\Delta v^i}{3} + \frac{{}_3\Delta v^i}{3} + \frac{{}_4\Delta v^i}{6} + \ldots \tag{1.14.6} $$

2. Examples

In this chapter we are going to visualize the quantities derived in the previous chapter with some easy examples. We examine various two dimensional surfaces, we determine the metric tensor, its derivatives, the connection and derivatives, the Ricci-tensor and the Ricci-scalar. By doing so we demonstrate their geometric meaning and we prepare ourselves to apply them in the real four dimensional spacetime.

2.1 Curvature on a two dimensional surface

The curvature of an arbitrary two dimensional surface in a given point is uniquely characterized by the reciprocal of the product of the radii of two, mutually perpendicular circles attached to the surface:

$$K=\frac{1}{r\cdot q} \tag{2.1.1}$$

We approach the point with a surface, where the parametric equation is written with rectangular coordinates, utilizing the curvature radii:

$$z=\frac{x^2}{2\cdot r}+\frac{y^2}{2\cdot q} \qquad dz=\frac{x}{r}\cdot dx+\frac{y}{q}\cdot dy \tag{2.1.2}$$

Substitute this into the three dimensional element of arc length squared:

$$ds^2=dx^2+dy^2+dz^2 \tag{2.1.3}$$

The arc length squared on the surface:

$$ds^2=\left(1+\frac{x^2}{r^2}\right)\cdot dx^2+\left(1+\frac{y^2}{q^2}\right)\cdot dy^2+\frac{2\cdot x\cdot y}{r\cdot q}\cdot dx\cdot dy \tag{2.1.4}$$

On this we can identify the metric tensor components:

$$g_{ij}=\begin{pmatrix} 1+\frac{x^2}{r^2} & \frac{x\cdot y}{r\cdot q} \\ \frac{x\cdot y}{r\cdot q} & 1+\frac{y^2}{q^2} \end{pmatrix} \tag{2.1.5}$$

The metric tensor determinant:

$$|g_{ij}| = g_{11} \cdot g_{22} - g_{12} \cdot g_{21} = \frac{r^2 \cdot y^2 + q^2 \cdot x^2 + r^2 \cdot q^2}{r^2 \cdot q^2} \tag{2.1.6}$$

The twice contravariant metric tensor:

$$g^{ij} = \frac{1}{r^2 \cdot y^2 + q^2 \cdot x^2 + r^2 \cdot q^2} \cdot \begin{pmatrix} r^2 \cdot y^2 + r^2 \cdot q^2 & -r \cdot q \cdot x \cdot y \\ -r \cdot q \cdot x \cdot y & q^2 \cdot x^2 + r^2 \cdot q^2 \end{pmatrix} \tag{2.1.7}$$

The metric tensor partial derivatives:

$$\frac{\partial g_{xx}}{\partial x} = \frac{2 \cdot x}{r^2} \qquad \frac{\partial g_{xy}}{\partial x} = \frac{\partial g_{yx}}{\partial x} = \frac{y}{r \cdot q}$$

$$\frac{\partial g_{xy}}{\partial y} = \frac{\partial g_{yx}}{\partial y} = \frac{x}{r \cdot q} \qquad \frac{\partial g_{yy}}{\partial y} = \frac{2 \cdot y}{q^2} \tag{2.1.8}$$

Calculating the connection:

$$\Gamma^j{}_{ki} = \frac{1}{2} \cdot g^{ja} \cdot \left(\frac{\partial g_{ia}}{\partial x^k} + \frac{\partial g_{ak}}{\partial x^i} - \frac{\partial g_{ki}}{\partial x^a} \right)$$

$$\Gamma^x{}_{xx} = \frac{q^2 \cdot x}{r^2 \cdot y^2 + q^2 \cdot x^2 + r^2 \cdot q^2} \qquad \Gamma^x{}_{yy} = \frac{r \cdot q \cdot x}{r^2 \cdot y^2 + q^2 \cdot x^2 + r^2 \cdot q^2}$$

$$\Gamma^y{}_{yy} = \frac{r^2 \cdot y}{r^2 \cdot y^2 + q^2 \cdot x^2 + r^2 \cdot q^2} \tag{2.1.9}$$

The partial derivatives of the connection, we put the common factor in a separate variable:

$$\frac{1}{a} = r^4 \cdot y^4 + q^4 \cdot x^4 + r^4 \cdot q^4 + 2 \cdot r^4 \cdot q^2 \cdot y^2 + 2 \cdot r^2 \cdot q^4 \cdot x^2 + 2 \cdot r^2 \cdot q^2 \cdot x^2 \cdot y^2$$

$$\frac{\partial \Gamma^x{}_{xx}}{\partial x} = a \cdot q^2 \cdot (r^2 \cdot y^2 - q^2 \cdot x^2 + r^2 \cdot q^2) \qquad \frac{\partial \Gamma^x{}_{yy}}{\partial x} = a \cdot r \cdot q \cdot (r^2 \cdot y^2 - q^2 \cdot x^2 + r^2 \cdot q^2)$$

$$\frac{\partial \Gamma^y{}_{yy}}{\partial x} = \frac{\partial \Gamma^x{}_{xx}}{\partial y} = -a \cdot 2 \cdot r^2 \cdot q^2 \cdot x \cdot y \qquad \frac{\partial \Gamma^x{}_{yy}}{\partial y} = -a \cdot 2 \cdot r^3 \cdot q \cdot x \cdot y$$

$$\frac{\partial \Gamma^y{}_{yy}}{\partial y} = -a \cdot r^2 \cdot (r^2 \cdot y^2 - q^2 \cdot x^2 - r^2 \cdot q^2) \tag{2.1.10}$$

The Ricci-tensor:

$$R_{ij}=R^{a}{}_{ija}=\frac{\partial \Gamma^{a}{}_{ij}}{\partial x^{a}}-\frac{\partial \Gamma^{a}{}_{aj}}{\partial x^{i}}+\Gamma^{b}{}_{ij}\cdot\Gamma^{a}{}_{ab}-\Gamma^{b}{}_{aj}\cdot\Gamma^{a}{}_{ib}$$

$$R_{ij}=a\cdot r\cdot q\cdot\begin{pmatrix} q^2\cdot x^2+r^2\cdot q^2 & r\cdot q\cdot x\cdot y \\ r\cdot q\cdot x\cdot y & r^2\cdot y^2+r^2\cdot q^2 \end{pmatrix} \qquad (2.1.11)$$

The Ricci-scalar:

$$R=g^{ab}\cdot R_{ab}=a\cdot 2\cdot r^3\cdot q^3$$

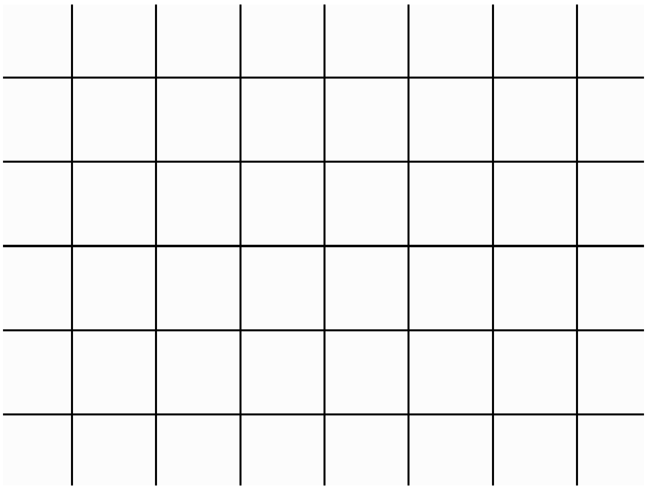
$$R=\frac{2\cdot r^3\cdot q^3}{r^4\cdot y^4+q^4\cdot x^4+r^4\cdot q^4+2\cdot r^4\cdot q^2\cdot y^2+2\cdot r^2\cdot q^4\cdot x^2+2\cdot r^2\cdot q^2\cdot x^2\cdot y^2} \qquad (2.1.12)$$

The centre of the coordinate system is in the point we are discussing, where the coordinates are zeroes:

$x = y = 0$

$$R=\frac{2\cdot r^3\cdot q^3}{r^4\cdot q^4}=\frac{2}{r\cdot q}=2\cdot K \qquad (2.1.13)$$

2.2 Plane



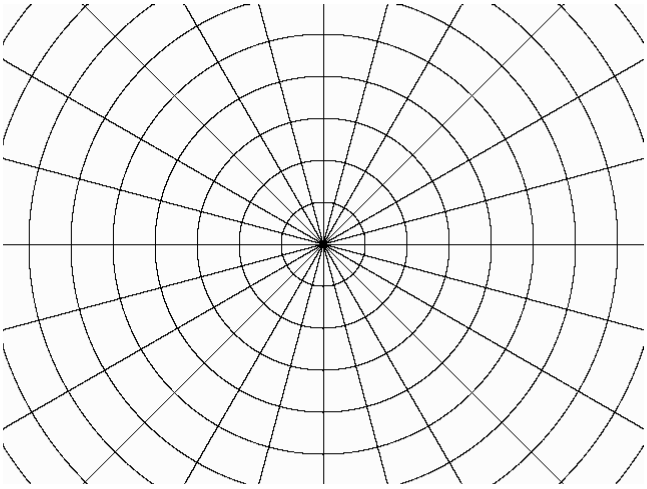
The element arc length squared and the metric tensor on the plane:

$$ds^2=dx^2+dy^2$$

$$g_{ij}=g^{ij}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (2.2.1)$$

There are no variables in the metric tensor, therefore all its derivatives and derivable quantities are zeroes.

The element arc length squared in polar coordinates and the other quantities:



$$ds^2 = dr^2 + r^2 \cdot d\varphi^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

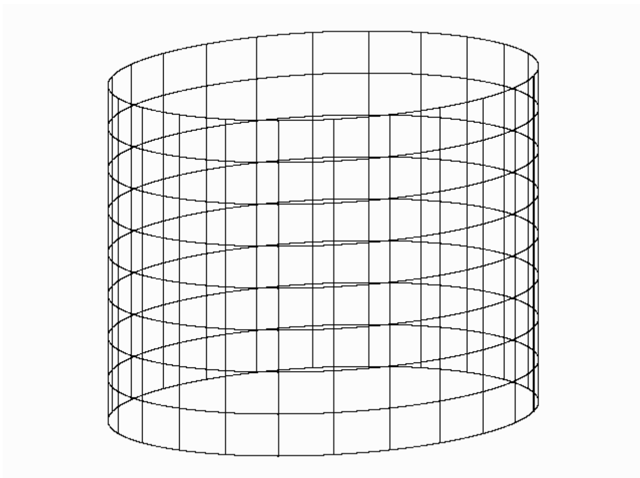
$$\frac{\partial g_{\varphi\varphi}}{\partial r} = r \qquad \frac{\partial g^{\varphi\varphi}}{\partial r} = -\frac{2}{r^3} \tag{2.2.2}$$

$$\Gamma^r{}_{\varphi\varphi} = -\frac{r}{2} \qquad \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{r}{2} \qquad \frac{\partial \Gamma^r{}_{\varphi\varphi}}{\partial r} = -\frac{1}{2} \qquad \frac{\partial \Gamma^\varphi{}_{r\varphi}}{\partial r} = \frac{\partial \Gamma^\varphi{}_{\varphi r}}{\partial r} = \frac{1}{2} \tag{2.2.3}$$

The Ricci-tensor and therefore the Ricci-scalar are zeroes.

2.3 Cylinder

We map the surface with a rectangular coordinate system, and the arc length squared depends on the constant radius of the cylinder:



$$ds^2 = \rho_c^2 \cdot d\varphi^2 + dz^2$$

$$g_{ij} = \begin{pmatrix} \rho_c^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \frac{1}{\rho_c^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.3.1}$$

The metric contains no variables again, therefore every derivative and derivable quantity is zero.

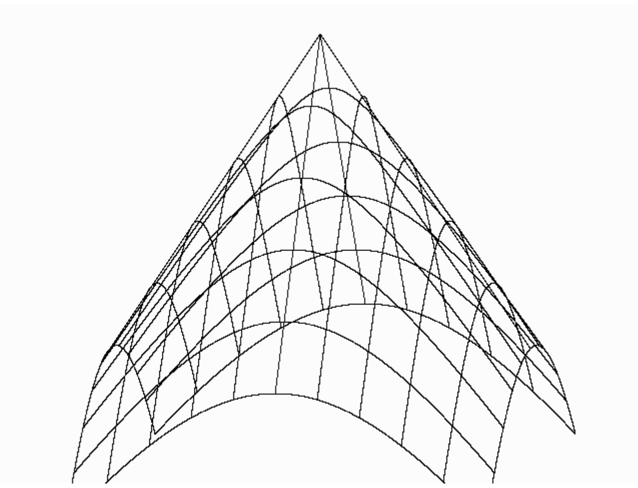
It is worth mentioning that it is possible to conceive a two dimensional surface with zero curvature, that cannot be embedded into three dimensional euclidean space. Let us imagine a cylinder where we are deforming the space it is embedded into. As the coordinate system on the cylinder differs from the plain case because one of the coordinates is made cyclic, it is possible to do so in three dimensions as well. While the *x* and *y* coordinates extend into infinity, the *z* coordinate returns into itself, its length is the circumference of a circle out in the fourth dimension. Since the curvature radius along the other coordinates is infinite, its easy to see that our space has zero curvature. If we define our cylinder on this manifold with the same orientation, and assume that the height of the cylinder is the same as the length of the z coordinate, we will notice that the top and bottom circles of the cylinder are touching. The newly formed surface is finite and homogeneous with both coordinates cyclic, cannot be embedded into common three dimensional space, still it has zero curvature (although it is not isotropic, circumnavigating the surface in various directions, the path taken would differ):

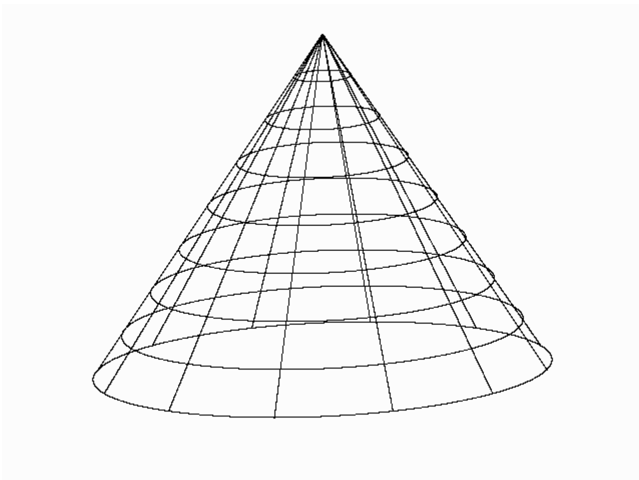
$$ds^2 = \rho_c^2 \cdot d\varphi^2 + \tau_c^2 \cdot dz^2$$

$$g_{ij} = \begin{pmatrix} \rho_c^2 & 0 \\ 0 & \tau_c^2 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \dfrac{1}{\rho_c^2} & 0 \\ 0 & \dfrac{1}{\tau_c^2} \end{pmatrix} \tag{2.3.2}$$

2.4 Cone

The cone is also a surface of zero curvature, because it can be unfolded to a plain. In rectangular and polar coordinates:





Some possible parametric equations:

$$x = r \cdot \sin(\vartheta) \cdot \cos(\varphi) \qquad y = r \cdot \sin(\vartheta) \cdot \sin(\varphi) \qquad z = r \cdot \sin(\varphi)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

$$x = \frac{h-u}{h} \cdot r \cdot \cos(\varphi) \qquad y = \frac{h-u}{h} \cdot r \cdot \sin(\varphi) \qquad z = u \tag{2.4.1}$$

Using the last equations, the arc length squared and the other quantities:

$$ds^2 = \left(1+\frac{r_c^2}{h_c^2}\right)\cdot du^2 + r_c^2\cdot\frac{(h_c-u)^2}{h_c^2}\cdot d\varphi^2$$

$$g_{ij} = \begin{pmatrix} 1+\frac{r_c^2}{h_c^2} & 0 \\ 0 & r_c^2\cdot\frac{(h_c-u)^2}{h_c^2} \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \frac{h_c^2}{h_c^2+r_c^2} & 0 \\ 0 & \frac{h_c^2}{r_c^2\cdot(h_c-u)^2} \end{pmatrix}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial u} = -2\cdot r_c^2\cdot\frac{h_c-u}{h_c^2} \qquad \frac{\partial g^{\varphi\varphi}}{\partial u} = \frac{2\cdot h_c^2}{r_c^2\cdot(h_c-u)^3} \qquad (2.4.2)$$

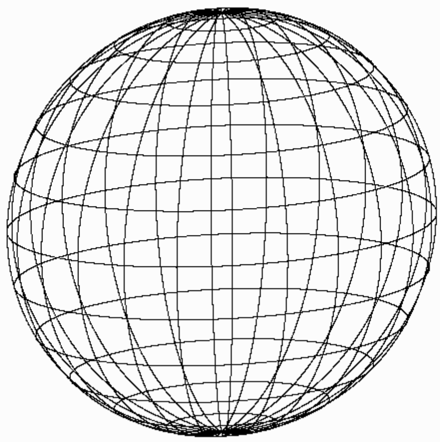
$$\Gamma^u{}_{\varphi\varphi} = \frac{r_c^2}{h_c^2}\cdot\frac{h_c-u}{1+\frac{r_c^2}{h_c^2}} \qquad \Gamma^\varphi{}_{u\varphi} = \Gamma^\varphi{}_{\varphi u} = -\frac{1}{h_c-u}$$

$$\frac{\partial \Gamma^u{}_{\varphi\varphi}}{\partial u} = -\frac{r_c^2}{h_c^2\cdot\left(1+\frac{r_c^2}{h_c^2}\right)} \qquad \frac{\partial \Gamma^\varphi{}_{u\varphi}}{\partial u} = \frac{\partial \Gamma^\varphi{}_{\varphi u}}{\partial u} = -\frac{1}{(h_c-u)^2} \qquad (2.4.3)$$

The Ricci-tensor, and so the Ricci-scalar are also zeroes, the cone can be unfolded to a plain.

2.5 Sphere

We introduce geographic latitudinal and longitudinal coordinates on a sphere, and calculate the geometric quantities from the arc length squared to the curvature:



$$ds^2 = r_c^2\cdot d\vartheta^2 + r_c^2\cdot\sin^2(\vartheta)\cdot d\varphi^2$$

$$g_{ij} = \begin{pmatrix} r_c^2 & 0 \\ 0 & r_c^2\cdot\sin^2(\vartheta) \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} \frac{1}{r_c^2} & 0 \\ 0 & \frac{1}{r_c^2\cdot\sin^2(\vartheta)} \end{pmatrix}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial \vartheta} = 2 \cdot r_c^2 \cdot \cos(\vartheta) \cdot \sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial \vartheta} = -\frac{2 \cdot \cos(\vartheta)}{r_c^2 \cdot \sin^3(\vartheta)} \tag{2.5.1}$$

$$\Gamma^{\vartheta}{}_{\varphi\varphi} = -\cos(\vartheta) \cdot \sin(\vartheta) \qquad \Gamma^{\varphi}{}_{\varphi\vartheta} = \Gamma^{\varphi}{}_{\vartheta\varphi} = \cot(\vartheta)$$

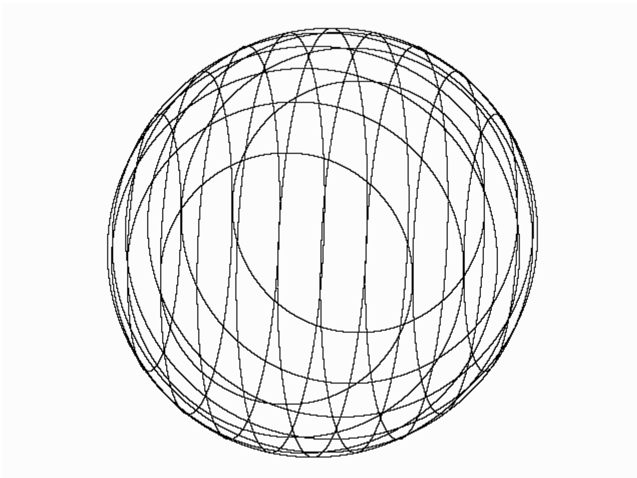
$$\frac{\partial \Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial \vartheta} = \sin^2(\vartheta) - \cos^2(\vartheta) \qquad \frac{\partial \Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial \vartheta} = \frac{\partial \Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial \vartheta} = -\frac{\cos^2(\vartheta)}{\sin^2(\vartheta)} - 1 \tag{2.5.2}$$

$$R_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\vartheta) \end{pmatrix} \qquad R = 2 \cdot K = \frac{2}{r_c^2} \tag{2.5.3}$$

We calculate the surface of the sphere by integrating the infinitesimal surface element:

$$dA = r_c \cdot d\vartheta \cdot r_c \cdot \sin(\vartheta) \cdot d\varphi$$

$$A = \int_0^{\pi} r_c \cdot d\vartheta \cdot \int_0^{2\cdot\pi} r_c \cdot \sin(\vartheta) \cdot d\varphi = r_c^2 \cdot 4 \cdot \pi \tag{2.5.4}$$



The sphere can be covered with rectangular coordinates also. Using this method we can map only one half of the sphere, thus it cannot be used to map the entire surface, just as polar coordinates cannot map the poles. The parametric equation:

$$z = \pm\sqrt{r_c^2 - x^2 - y^2}$$

$$dz = -\frac{x \cdot dx + y \cdot dy}{\sqrt{r_c^2 - x^2 - y^2}} \tag{2.5.5}$$

The characterizing quantities of the surface, from the arc length squared to the curvature:

$$ds^2 = \frac{1}{r_c^2 - x^2 - y^2} \cdot \left((r_c^2 - y^2) \cdot dx^2 + (r_c^2 - x^2) \cdot dy^2 + 2 \cdot x \cdot y \cdot dx \cdot dy \right)$$

$$g_{ij} = \frac{1}{r_c^2 - x^2 - y^2} \cdot \begin{pmatrix} r_c^2 - y^2 & x \cdot y \\ x \cdot y & r_c^2 - x^2 \end{pmatrix} \qquad g^{ij} = \frac{1}{r_c^2} \cdot \begin{pmatrix} r_c^2 - x^2 & -x \cdot y \\ -x \cdot y & r_c^2 - y^2 \end{pmatrix}$$

$$|g_{ij}| = \frac{r_c^2}{r_c^2 - x^2 - y^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}=\frac{2\cdot x\cdot(r_c^2-y^2)}{(r_c^2-x^2-y^2)^2} \qquad \frac{\partial g_{\varphi\varphi}}{\partial\varphi}=\frac{2\cdot y\cdot(r_c^2-x^2)}{(r_c^2-x^2-y^2)^2}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot x}{r_c^2-x^2-y^2}\cdot\left(\frac{r_c^2-x^2}{r_c^2-x^2-y^2}-1\right) \qquad \frac{\partial g_{\vartheta\vartheta}}{\partial\varphi}=\frac{2\cdot y}{r_c^2-x^2-y^2}\cdot\left(\frac{r_c^2-y^2}{r_c^2-x^2-y^2}-1\right)$$

$$\frac{\partial g_{\vartheta\varphi}}{\partial\varphi}=\frac{\partial g_{\varphi\vartheta}}{\partial\varphi}=\frac{x}{r_c^2-x^2-y^2}\cdot\left(1+\frac{2\cdot y^2}{r_c^2-x^2-y^2}\right) \qquad \frac{\partial g_{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial g_{\varphi\vartheta}}{\partial\vartheta}=\frac{y}{r_c^2-x^2-y^2}\cdot\left(1+\frac{2\cdot x^2}{r_c^2-x^2-y^2}\right)$$

$$\frac{\partial g^{\vartheta\vartheta}}{\partial\vartheta}=-\frac{2\cdot x}{r_c^2} \qquad \frac{\partial g^{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial g^{\varphi\vartheta}}{\partial\vartheta}=-\frac{y}{r_c^2} \qquad \frac{\partial g^{\vartheta\varphi}}{\partial\varphi}=\frac{\partial g^{\varphi\vartheta}}{\partial\varphi}=-\frac{x}{r_c^2} \qquad \frac{\partial g^{\varphi\varphi}}{\partial\varphi}=-\frac{2\cdot y}{r_c^2} \tag{2.5.6}$$

$$\Gamma^{\vartheta}{}_{\vartheta\vartheta}=\frac{x\cdot(r_c^2-y^2)}{r_c^2\cdot(r_c^2-x^2-y^2)} \qquad \Gamma^{\varphi}{}_{\vartheta\vartheta}=\frac{y\cdot(r_c^2-y^2)}{r_c^2\cdot(r_c^2-x^2-y^2)}$$

$$\Gamma^{\vartheta}{}_{\varphi\varphi}=\frac{x\cdot(r_c^2-x^2)}{r_c^2\cdot(r_c^2-x^2-y^2)} \qquad \Gamma^{\varphi}{}_{\varphi\varphi}=\frac{y\cdot(r_c^2-x^2)}{r_c^2\cdot(r_c^2-x^2-y^2)}$$

$$\Gamma^{\vartheta}{}_{\vartheta\varphi}=\Gamma^{\vartheta}{}_{\varphi\vartheta}=\frac{x^2\cdot y}{r_c^2\cdot(r_c^2-x^2-y^2)} \qquad \Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\frac{x\cdot y^2}{r_c^2\cdot(r_c^2-x^2-y^2)}$$

$$\frac{\partial\Gamma^{\vartheta}{}_{\vartheta\vartheta}}{\partial\vartheta}=\frac{y^4-x^2\cdot y^2+2\cdot r_c^2\cdot y^2+r_c^2\cdot x^2+r_c^4}{r_c^2\cdot(r_c^2-x^2-y^2)^2} \qquad \frac{\partial\Gamma^{\varphi}{}_{\varphi\varphi}}{\partial\varphi}=\frac{x^4-x^2\cdot y^2-2\cdot r_c^2\cdot x^2+r_c^2\cdot y^2+r_c^4}{r_c^2\cdot(r_c^2-x^2-y^2)^2}$$

$$\frac{\partial\Gamma^{\vartheta}{}_{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial\Gamma^{\vartheta}{}_{\varphi\vartheta}}{\partial\vartheta}=\frac{\partial\Gamma^{\varphi}{}_{\vartheta\vartheta}}{\partial\vartheta}=\frac{2\cdot x\cdot y\cdot(r_c^2-y^2)}{r_c^2\cdot(r_c^2-x^2-y^2)^2} \qquad \frac{\partial\Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial\varphi}=\frac{\partial\Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial\varphi}=\frac{\partial\Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial\varphi}=\frac{2\cdot x\cdot y\cdot(r_c^2-x^2)}{r_c^2\cdot(r_c^2-x^2-y^2)^2}$$

$$\frac{\partial\Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial\vartheta}=\frac{x^4+3\cdot x^2\cdot y^2-2\cdot r_c^2\cdot x^2-r_c^2\cdot y^2+r_c^4}{r_c^2\cdot(r_c^2-x^2-y^2)^2} \qquad \frac{\partial\Gamma^{\varphi}{}_{\vartheta\vartheta}}{\partial\varphi}=\frac{y^4+3\cdot x^2\cdot y^2-2\cdot r_c^2\cdot y^2-r_c^2\cdot x^2+r_c^4}{r_c^2\cdot(r_c^2-x^2-y^2)^2}$$

$$\frac{\partial\Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial\Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial\vartheta}=\frac{y^2\cdot(r_c^2+x^2-y^2)}{r_c^2\cdot(r_c^2-x^2-y^2)^2} \qquad \frac{\partial\Gamma^{\vartheta}{}_{\vartheta\varphi}}{\partial\varphi}=\frac{\partial\Gamma^{\vartheta}{}_{\varphi\vartheta}}{\partial\varphi}=\frac{x^2\cdot(r_c^2-x^2+y^2)}{r_c^2\cdot(r_c^2-x^2-y^2)^2}$$

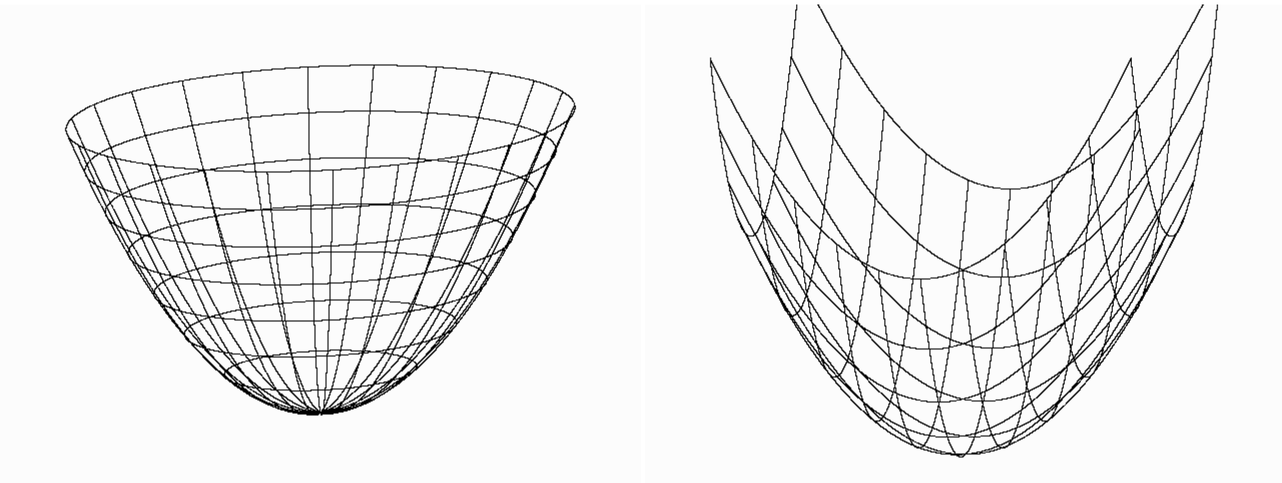
$$\frac{\partial\Gamma^{\varphi}{}_{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot x\cdot y^3}{r_c^2\cdot(r_c^2-x^2-y^2)^2} \qquad \frac{\partial\Gamma^{\vartheta}{}_{\vartheta\vartheta}}{\partial\varphi}=\frac{2\cdot x^3\cdot y}{r_c^2\cdot(r_c^2-x^2-y^2)^2} \tag{2.5.7}$$

$$R_{ij}=-g_{ij}=-\frac{1}{r_c^2-x^2-y^2}\cdot\begin{pmatrix} r_c^2-y^2 & x\cdot y \\ x\cdot y & r_c^2-x^2 \end{pmatrix} \qquad R=2\cdot K=\frac{2}{r_c^2} \tag{2.5.8}$$

Our result is of course the same as the result from calculating with polar coordinates.

2.6 Paraboloid

In polar and rectangular coordinates:



The equation:

$$z = b \cdot (x^2 + y^2)$$

The parametric equations:

$$x = a_c \cdot \sqrt{\frac{u}{h_c}} \cdot \cos(\varphi) \qquad y = a_c \cdot \sqrt{\frac{u}{h_c}} \cdot \sin(\varphi) \qquad z = u \qquad (2.6.1)$$

The characterizing geometric quantities of the surface, from the arc length squared to the curvature:

$$ds^2 = \left(1 + \frac{a_c^2}{4 \cdot h_c \cdot u}\right) \cdot du^2 + \frac{a_c^2 \cdot u}{h_c} \cdot d\varphi^2$$

$$g_{ij} = \begin{pmatrix} 1 + \frac{a_c^2}{4 \cdot h_c \cdot u} & 0 \\ 0 & \frac{a_c^2 \cdot u}{h_c} \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \frac{4 \cdot h_c \cdot u}{4 \cdot h_c \cdot u + a_c^2} & 0 \\ 0 & \frac{h_c}{a_c^2 \cdot u} \end{pmatrix}$$

$$\frac{\partial g_{uu}}{\partial u} = -\frac{a_c^2}{4 \cdot h_c \cdot u^2} \qquad \frac{\partial g^{uu}}{\partial u} = \frac{a_c^2}{4 \cdot h_c \cdot u^2 \cdot \left(1 + \frac{a_c^2}{4 \cdot h_c \cdot u}\right)^2}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial u}=\frac{a_c^2}{h_c} \qquad \frac{\partial g^{\varphi\varphi}}{\partial u}=-\frac{h_c}{a_c^2\cdot u^2} \tag{2.6.2}$$

$$\Gamma^u{}_{uu}=-\frac{a_c^2}{4\cdot h_c\cdot u^2\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)} \qquad \Gamma^u{}_{\varphi\varphi}=-\frac{a_c^2}{2\cdot h_c\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)} \qquad \Gamma^\varphi{}_{u\varphi}=\Gamma^\varphi{}_{\varphi u}=\frac{1}{2\cdot u}$$

$$\frac{\partial\Gamma^u{}_{uu}}{\partial u}=\frac{a_c^2}{4\cdot h_c\cdot u^2\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)}\cdot\left(\frac{1}{u}-\frac{a_c^2}{2\cdot h_c\cdot u^2\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)}\right)$$

$$\frac{\partial\Gamma^u{}_{\varphi\varphi}}{\partial u}=-2\cdot\left(\frac{a_c^2}{4\cdot h_c\cdot u\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)}\right)^2 \qquad \frac{\partial\Gamma^\varphi{}_{u\varphi}}{\partial u}=\frac{\partial\Gamma^\varphi{}_{\varphi u}}{\partial u}=-\frac{1}{2\cdot u^2} \tag{2.6.3}$$

$$R_{uu}=\frac{1}{4\cdot u^2}-\frac{1}{u}\left(\frac{a_c}{4\cdot h_c\cdot u\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)}\right)^2 \qquad R_{\varphi\varphi}=\frac{a_c}{4\cdot h_c\cdot u\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)}\cdot\left(1-\frac{a_c}{4\cdot h_c\cdot u\cdot\left(1+\frac{a_c^2}{4\cdot h_c\cdot u}\right)}\right)$$

$$R=2\cdot K=\frac{4\cdot h_c}{a_c^2+4\cdot h_c\cdot u} \tag{2.6.4}$$

Far away from the tip of the paraboloid, the curvature of the surface approaches zero:

$$\lim_{u\to\infty}\frac{4\cdot h_c}{a_c^2+4\cdot h_c\cdot u}=0 \tag{2.6.5}$$

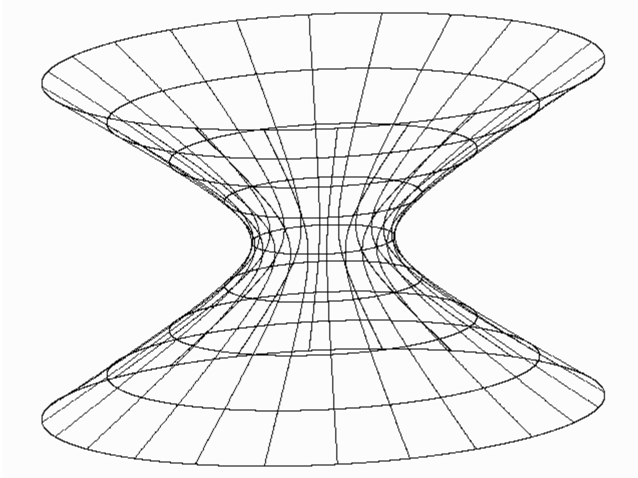
2.7 Hyperboloid

The parametric equation of a hyperboloid of one sheet:

$$x=a_c\cdot\sqrt{u^2+1}\cdot\cos(\varphi) \qquad y=a_c\cdot\sqrt{u^2+1}\cdot\sin(\varphi) \qquad z=b_c\cdot u$$

$$\frac{x^2+y^2}{a_c^2}-\frac{z^2}{b_c^2}=1 \tag{2.7.1}$$

From the arc length squared to the curvature:



$$ds^2=\left(\frac{a_c^2\cdot u^2}{u^2+1}+b_c^2\right)\cdot du^2+a_c^2\cdot(u^2+1)\cdot d\,\varphi^2$$

$$g_{ij}=\begin{pmatrix}\frac{a_c^2\cdot u^2}{u^2+1}+b_c^2 & 0\\ 0 & a_c^2\cdot(u^2+1)\end{pmatrix}$$

$$g^{ij}=\begin{pmatrix}\frac{u^2+1}{(a_c^2+b_c^2)\cdot u^2+b_c^2} & 0\\ 0 & \frac{1}{a_c^2\cdot(u^2+1)}\end{pmatrix}$$

$$\frac{\partial g_{uu}}{\partial u}=\frac{2\cdot a_c^2\cdot u}{u^2+1}\cdot\left(1-\frac{u}{u^2+1}\right) \qquad \frac{\partial g_{\varphi\varphi}}{\partial u}=2\cdot a_c^2\cdot u$$

$$\frac{\partial g^{uu}}{\partial u}=-\frac{2\cdot a_c^2\cdot u}{(u^2\cdot(a_c^2+b_c^2)+b_c^2)^2} \qquad \frac{\partial g^{\varphi\varphi}}{\partial u}=-\frac{2\cdot u}{a_c^2\cdot(u^2+1)^2} \tag{2.7.2}$$

$$\Gamma^u{}_{uu}=\frac{a_c^2\cdot u}{(a_c^2\cdot u^2+b_c^2\cdot(u^2+1))\cdot(u^2+1)} \qquad \Gamma^u{}_{\varphi\varphi}=-\frac{a_c^2\cdot u\cdot(u^2+1)}{a_c^2\cdot u^2+b_c^2\cdot(u^2+1)}$$

$$\Gamma^\varphi{}_{u\varphi}=\Gamma^\varphi{}_{\varphi u}=\frac{u}{u^2+1}$$

$$\frac{\partial\Gamma^u{}_{uu}}{\partial u}=-\frac{a_c^2\cdot(a_c^2\cdot u^2\cdot(3\cdot u^2+1)+b_c^2\cdot(3\cdot u^4+2\cdot u^2-1))}{((a_c^2\cdot u^2+b_c^2\cdot(u^2+1))\cdot(u^2+1))^2}$$

$$\frac{\partial\Gamma^u{}_{\varphi\varphi}}{\partial u}=-\frac{a_c^2\cdot(a_c^2\cdot u^2\cdot(u^2-1)+b_c^2\cdot(u^2+1)^2)}{(a_c^2\cdot u^2+b_c^2\cdot(u^2+1))^2}$$

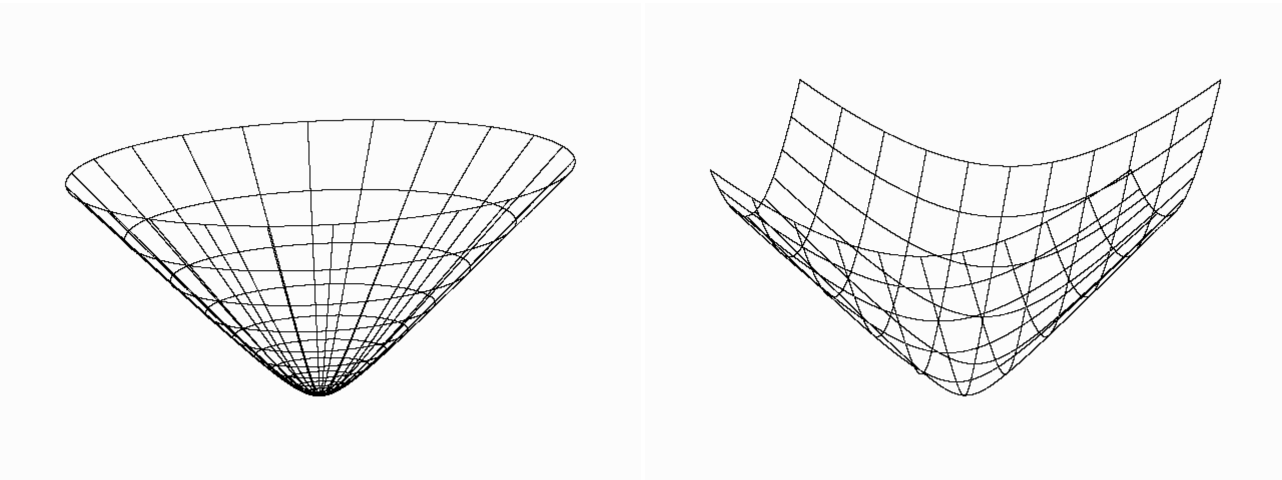
$$\frac{\partial\Gamma^\varphi{}_{u\varphi}}{\partial u}=\frac{\partial\Gamma^\varphi{}_{\varphi u}}{\partial u}=-\frac{u^2-1}{(u^2+1)^2} \tag{2.7.3}$$

$$R_{ij}=-\frac{b_c^2}{a_c^2\cdot u^2+b_c^2\cdot(u^2+1)}\cdot\begin{pmatrix}\frac{1}{u^2+1} & 0\\ 0 & \frac{a_c^2\cdot(u^2+1)}{a_c^2\cdot u^2+b_c^2\cdot(u^2+1)}\end{pmatrix}$$

$$R = 2 \cdot K = -\frac{2 \cdot b_c^2}{(a_c^2 \cdot u^2 + b_c^2 \cdot (u^2+1))^2} \tag{2.7.4}$$

2.8 Bolyai surface

The Bolyai surface is an infinite surface with constant negative curvature, it cannot be embedded into a three dimensional surface with a positive arc length squared. We start with the hyperboloid of two sheets. With polar and rectangular coordinates:



The equation of the surface:

$$z = \sqrt{a_c^2 + x^2 + y^2} \tag{2.8.1}$$

We set up a coordinate system in the three dimensional space, based on the hyperboloid:

$$x = r \cdot \sinh(\vartheta) \cdot \cos(\varphi) \qquad y = r \cdot \sinh(\vartheta) \cdot \sin(\varphi) \qquad z = r \cdot \cosh(\vartheta) \tag{2.8.2}$$

Substitute this into the equation of the surface, this shows that the hyperboloid is a coordinate surface in this coordinate system:

$$r \cdot \cosh(\vartheta) = \sqrt{a_c^2 + (r \cdot \sinh(\vartheta) \cdot \cos(\varphi))^2 + (r \cdot \sinh(\vartheta) \cdot \sin(\varphi))^2}$$

$$r = a_c \tag{2.8.3}$$

The arc length squared of the pseudo-euclidean three dimensional space, where we are going to embed the hyperboloid:

$$ds^2 = dx^2 + dy^2 - dz^2$$

Changes of coordinates on the embedded hyperboloid:

$$dr=0$$

$$dx=r\cdot\cosh(\vartheta)\cdot\cos(\varphi)\cdot d\,\vartheta-r\cdot\sinh(\vartheta)\cdot\sin(\varphi)\cdot d\,\varphi$$

$$dy=r\cdot\cosh(\vartheta)\cdot\sin(\varphi)\cdot d\,\vartheta-r\cdot\sinh(\vartheta)\cdot\cos(\varphi)\cdot d\,\varphi$$

$$dz=r\cdot\sinh(\vartheta)\cdot d\,\vartheta \qquad (2.8.4)$$

The arc length squared on the embedded hyperboloid and the other geometric quantities:

$$ds^2=r_c^2\cdot d\,\vartheta^2+r_c^2\cdot\sinh^2(\vartheta)\cdot d\,\varphi^2$$

$$g_{ij}=\begin{pmatrix} r_c^2 & 0 \\ 0 & r_c^2\cdot\sinh^2(\vartheta)\end{pmatrix} \qquad g^{ij}=\begin{pmatrix} \frac{1}{r_c^2} & 0 \\ 0 & \frac{1}{r_c^2\cdot\sinh^2(\vartheta)}\end{pmatrix}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=2\cdot r_c^2\cdot\cosh(\vartheta)\cdot\sinh(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=-\frac{2\cdot\cosh(\vartheta)}{r_c^2\cdot\sinh^3(\vartheta)} \qquad (2.8.5)$$

$$\Gamma^{\vartheta}{}_{\varphi\varphi}=-\cosh(\vartheta)\cdot\sinh(\vartheta) \qquad \Gamma^{\varphi}{}_{\varphi\vartheta}=\Gamma^{\varphi}{}_{\vartheta\varphi}=\coth(\vartheta)$$

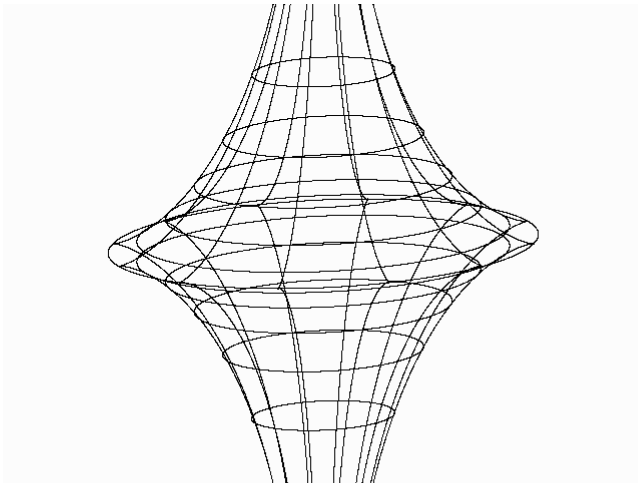
$$\frac{\partial\Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial\vartheta}=-\sinh^2(\vartheta)-\cosh^2(\vartheta) \qquad \frac{\partial\Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial\vartheta}=\frac{\partial\Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial\vartheta}=1-\frac{\cosh^2(\vartheta)}{\sinh^2(\vartheta)} \qquad (2.8.6)$$

$$R_{ij}=\begin{pmatrix} -1 & 0 \\ 0 & -\sinh^2(\vartheta)\end{pmatrix} \qquad R=2\cdot K=-\frac{2}{r_c^2} \qquad (2.8.7)$$

The Bolyai surface has infinite extension.

There is a two dimensional surface, that although lacks the symmetry features of the previous Bolyai surface, but it has the same constant negative curvature, and it can be embedded into three dimensional space: the tractroid. This surface is created by revolving a tractrix about its asymptote:

$$z=a\cdot\operatorname{arcosh}\left(\frac{a}{\rho}\right)-\sqrt{a^2-\rho^2}=a\cdot\ln\left(\frac{a+\sqrt{a^2-\rho^2}}{\rho}\right)-\sqrt{a^2-\rho^2} \qquad (2.8.8)$$



From the arc length to the curvature:

$$ds^2 = d\rho^2 + \rho^2 \cdot d\varphi^2 + dz^2$$

$$ds^2 = \frac{a_c^2}{\rho^2} d\rho^2 + \rho^2 \cdot d\varphi^2$$

$$g_{ij} = \begin{pmatrix} \frac{a_c^2}{\rho^2} & 0 \\ 0 & \rho^2 \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \frac{\rho^2}{a_c^2} & 0 \\ 0 & \frac{1}{\rho^2} \end{pmatrix}$$

$$\frac{\partial g_{\rho\rho}}{\partial \rho} = -\frac{2 \cdot a_c^2}{\rho^3} \qquad \frac{\partial g^{\rho\rho}}{\partial \rho} = \frac{2 \cdot \rho}{a_c^2} \qquad \frac{\partial g_{\varphi\varphi}}{\partial \rho} = 2 \cdot \rho \qquad \frac{\partial g^{\varphi\varphi}}{\partial \rho} = -\frac{2}{\rho^3} \tag{2.8.9}$$

$$\Gamma^{\rho}{}_{\rho\rho} = -\frac{1}{\rho} \qquad \Gamma^{\rho}{}_{\varphi\varphi} = -\frac{\rho^3}{a_c^2} \qquad \Gamma^{\varphi}{}_{\rho\varphi} = \Gamma^{\varphi}{}_{\varphi\rho} = \frac{1}{\rho}$$

$$\frac{\partial \Gamma^{\rho}{}_{\rho\rho}}{\partial \rho} = \frac{1}{\rho^2} \qquad \frac{\partial \Gamma^{\rho}{}_{\varphi\varphi}}{\partial \rho} = -\frac{3 \cdot \rho^2}{a_c^2} \qquad \frac{\partial \Gamma^{\varphi}{}_{\rho\varphi}}{\partial \rho} = \frac{\partial \Gamma^{\varphi}{}_{\varphi\rho}}{\partial \rho} = -\frac{1}{\rho^2} \tag{2.8.10}$$

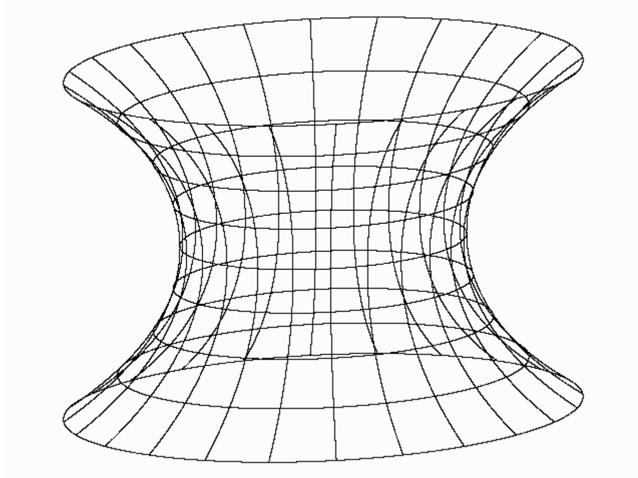
$$R_{ij} = \begin{pmatrix} -\frac{1}{\rho^2} & 0 \\ 0 & -\frac{\rho^2}{a_c^2} \end{pmatrix} \qquad R = 2 \cdot K = -\frac{2}{a_c^2} \tag{2.8.11}$$

2.9 Catenoid

Although the curvature of the catenoid is non-zero, but it is a minimal surface, its average curvature is zero, just like the plain. The parametric equation:

$$x = a_c \cdot \cosh\left(\frac{v}{a_c}\right) \cdot \cos(u) \qquad x = a_c \cdot \cosh\left(\frac{v}{a_c}\right) \cdot \sin(u) \qquad z = v \tag{2.9.1}$$

From the arc length squared to the curvature:



$$ds^2 = \cosh^2\left(\frac{v}{a_c}\right)\cdot dv^2 + \cosh^2\left(\frac{v}{a_c}\right)\cdot du^2$$

$$g_{ij} = \begin{pmatrix} \cosh\left(\frac{v}{a_c}\right) & 0 \\ 0 & \cosh\left(\frac{v}{a_c}\right) \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} \frac{1}{\cosh\left(\frac{v}{a_c}\right)} & 0 \\ 0 & \frac{1}{\cosh\left(\frac{v}{a_c}\right)} \end{pmatrix}$$

$$\frac{\partial g_{vv}}{\partial v} = \frac{\partial g_{uu}}{\partial v} = \frac{2\cdot\cosh\left(\frac{v}{a_c}\right)\cdot\sinh\left(\frac{v}{a_c}\right)}{a_c} \qquad \frac{\partial g^{vv}}{\partial v} = \frac{\partial g^{uu}}{\partial v} = -\frac{2\cdot\sinh\left(\frac{v}{a_c}\right)}{a_c\cdot\cosh^3\left(\frac{v}{a_c}\right)} \tag{2.9.2}$$

$$\Gamma^{v}{}_{vv} = \Gamma^{u}{}_{vu} = \Gamma^{u}{}_{uv} = \frac{1}{a_c}\cdot\coth\left(\frac{v}{a_c}\right) \qquad \Gamma^{v}{}_{uu} = -\frac{1}{a_c}\cdot\coth\left(\frac{v}{a_c}\right)$$

$$\frac{\partial \Gamma^{v}{}_{vv}}{\partial v} = \frac{\partial \Gamma^{u}{}_{vu}}{\partial v} = \frac{\partial \Gamma^{u}{}_{uv}}{\partial v} = \frac{1}{a_c^2\cdot\cosh^2\left(\frac{v}{a_c}\right)} \qquad \frac{\partial \Gamma^{v}{}_{uu}}{\partial v} = -\frac{1}{a_c^2\cdot\cosh^2\left(\frac{v}{a_c}\right)} \tag{2.9.3}$$

$$R_{ij} = \begin{pmatrix} -\frac{1}{a_c^2\cdot\cosh^2\left(\frac{v}{a_c}\right)} & 0 \\ 0 & -\frac{1}{a_c^2\cdot\cosh^2\left(\frac{v}{a_c}\right)} \end{pmatrix} \qquad R = 2\cdot K = -\frac{2}{a_c^2\cdot\cosh^4\left(\frac{v}{a_c}\right)} \tag{2.9.4}$$

Far away from the signature funnel of the catenoid, the curvature of the surface approaches zero:

$$\lim_{v\to\infty}\left(-\frac{2}{a_c^2\cdot\cosh^4\left(\frac{v}{a_c}\right)}\right)=0 \tag{2.9.5}$$

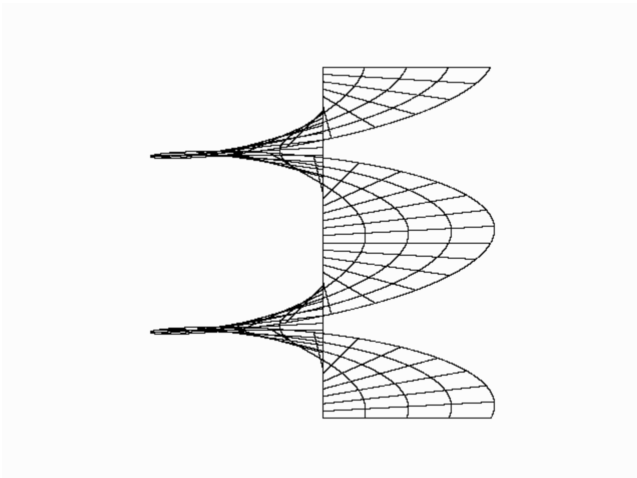
2.10 Helicoid

The helicoid is also a minimal surface, and it is isometric to the catenoid, thus it is possible to deform it into a catenoid without distortion and vice versa, just as the cylinder can also be flattened into a plain. The parametric equations:

$$x=v\cdot\cos(\varphi) \qquad y=v\cdot\sin(\varphi) \qquad z=a_c\cdot v$$

$$y=x\cdot\tan\left(\frac{z}{a_c}\right) \tag{2.10.1}$$

From the arc length squared to the curvature:



$$ds^2=dv^2+(a_c^2+v^2)\cdot du^2$$

$$g_{ij}=\begin{pmatrix}1 & 0\\ 0 & a_c^2+v^2\end{pmatrix}$$

$$g^{ij}=\begin{pmatrix}1 & 0\\ 0 & \frac{1}{a_c^2+v^2}\end{pmatrix}$$

$$\frac{\partial g_{uu}}{\partial v}=2\cdot v \qquad \frac{\partial g^{uu}}{\partial v}=-\frac{2\cdot v}{(a_c^2+v^2)^2} \tag{2.10.2}$$

$$\Gamma^v{}_{uu}=-v \qquad \Gamma^u{}_{vu}=\Gamma^u{}_{uv}=\frac{v}{a_c^2+v^2}$$

$$\frac{\partial\Gamma^v{}_{uu}}{\partial v}=-1 \qquad \frac{\partial\Gamma^u{}_{vu}}{\partial v}=\frac{\partial\Gamma^u{}_{uv}}{\partial v}=\frac{a_c^2-v^2}{(a_c^2+v^2)^2} \tag{2.10.3}$$

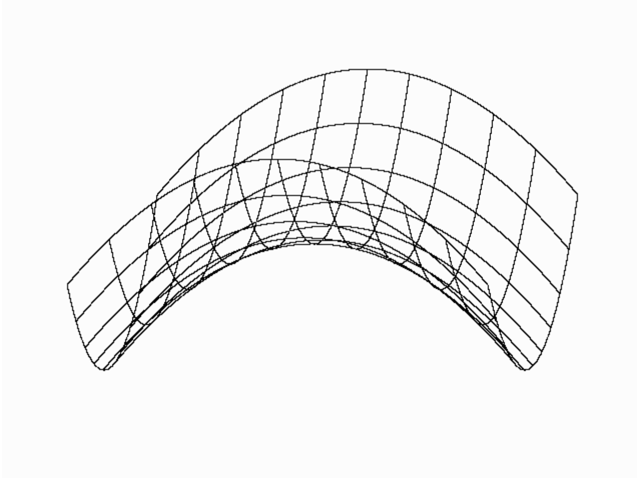
$$R_{ij}=\begin{pmatrix}-\frac{a_c^2}{(a_c^2+v^2)^2} & 0\\ 0 & \frac{v^2}{a_c^2+v^2}-1\end{pmatrix} \qquad R=2\cdot K=-\frac{2\cdot a_c^2}{(a_c^2+v^2)^2} \tag{2.10.4}$$

2.11 Hyperbolic paraboloid

The parametric equation of the surface with negative curvature in a possible coordinate system:

$$x=u \qquad y=v$$

$$z=x\cdot y \qquad (2.11.1)$$



From the arc length squared to the curvature:

$$ds^2=(1+v^2)\cdot du^2+(1+u^2)\cdot dv^2+2\cdot u\cdot v\cdot du\cdot dv$$

$$g_{ij}=\begin{pmatrix}1+v^2 & u\cdot v\\ u\cdot v & 1+u^2\end{pmatrix} \qquad g^{ij}=\frac{1}{1+u^2+v^2}\cdot\begin{pmatrix}1+u^2 & -u\cdot v\\ -u\cdot v & 1+v^2\end{pmatrix}$$

$$|g_{ij}|=1+u^2+v^2$$

$$\frac{\partial g_{uv}}{\partial u}=\frac{\partial g_{vu}}{\partial u}=v \qquad \frac{\partial g_{uv}}{\partial v}=\frac{\partial g_{vu}}{\partial v}=u \qquad \frac{\partial g_{vv}}{\partial u}=2\cdot u \qquad \frac{\partial g_{uu}}{\partial v}=2\cdot v$$

$$\frac{\partial g^{uu}}{\partial u}=\frac{2\cdot u\cdot v^2}{(1+u^2+v^2)^2} \qquad \frac{\partial g^{vv}}{\partial v}=\frac{2\cdot u^2\cdot v}{(1+u^2+v^2)^2}$$

$$\frac{\partial g^{uv}}{\partial u}=\frac{\partial g^{vu}}{\partial u}=-\frac{v\cdot(1-u^2+v^2)}{(1+u^2+v^2)^2} \qquad \frac{\partial g^{uv}}{\partial v}=\frac{\partial g^{vu}}{\partial v}=-\frac{u\cdot(1-u^2+v^2)}{(1+u^2+v^2)^2}$$

$$\frac{\partial g^{vv}}{\partial u}=-\frac{2\cdot u\cdot(1+v^2)}{(1+u^2+v^2)^2} \qquad \frac{\partial g^{uu}}{\partial v}=-\frac{2\cdot v\cdot(1+u^2)}{(1+u^2+v^2)^2} \qquad (2.11.2)$$

$$\Gamma^u{}_{uv}=\Gamma^u{}_{vu}=\frac{v}{1+u^2+v^2} \qquad \Gamma^v{}_{uv}=\Gamma^v{}_{vu}=\frac{u}{1+u^2+v^2}$$

$$\frac{\partial \Gamma^u{}_{uv}}{\partial u}=\frac{\partial \Gamma^u{}_{vu}}{\partial u}=\frac{\partial \Gamma^v{}_{uv}}{\partial v}=\frac{\partial \Gamma^v{}_{vu}}{\partial v}=-\frac{2\cdot u\cdot v}{(1+u^2+v^2)^2}$$

$$\frac{\partial \Gamma^v{}_{uv}}{\partial u}=\frac{\partial \Gamma^v{}_{vu}}{\partial u}=\frac{1-u^2+v^2}{(1+u^2+v^2)^2} \qquad \frac{\partial \Gamma^u{}_{uv}}{\partial v}=\frac{\partial \Gamma^u{}_{vu}}{\partial v}=\frac{1-u^2-v^2}{(1+u^2+v^2)^2} \qquad (2.11.3)$$

$$R_{ij}=-\frac{1}{(1+u^2+v^2)^2}\cdot\begin{pmatrix}1+u^2 & u\cdot v\\ u\cdot v & 1+v^2\end{pmatrix}\qquad R=2\cdot K=-\frac{2}{(1+u^2+v^2)^2}\tag{2.11.4}$$

Far away from the signature saddle of the hyperbolic paraboloid, the curvature of the surface approaches zero:

$$\lim_{u\to\infty,\,v\to\infty}\left(-\frac{2}{(1+u^2+v^2)^2}\right)=0\tag{2.11.5}$$

Parametric equation in another possible coordinate system:

$$z=\frac{y^2}{b_c^2}-\frac{x^2}{a_c^2}\tag{2.11.6}$$

From the arc length squared to the curvature:

$$ds^2=\left(\frac{4\cdot u^2}{a_c^4}+1\right)\cdot du^2+\left(\frac{4\cdot v^2}{b_c^4}+1\right)\cdot dv^2-\frac{8\cdot u\cdot v}{a_c^2\cdot b_c^2}\cdot du\cdot dv$$

$$g_{ij}=\begin{pmatrix}\frac{4\cdot u^2}{a_c^4}+1 & -\frac{4\cdot u\cdot v}{a_c^2\cdot b_c^2}\\ -\frac{4\cdot u\cdot v}{a_c^2\cdot b_c^2} & \frac{4\cdot v^2}{b_c^4}+1\end{pmatrix}\qquad |g_{ij}|=\left(\frac{4\cdot u^2}{a_c^4}+1\right)\cdot\left(\frac{4\cdot v^2}{b_c^4}+1\right)-\frac{16\cdot u^2\cdot v^2}{a_c^4\cdot b_c^4}$$

$$g^{ij}=\frac{1}{a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2}\cdot\begin{pmatrix}a_c^4\cdot(b_c^4+4\cdot v^2) & 4\cdot a_c^2\cdot b_c^2\cdot u\cdot v\\ 4\cdot a_c^2\cdot b_c^2\cdot u\cdot v & b_c^4\cdot(a_c^4+4\cdot u^2)\end{pmatrix}$$

$$\frac{\partial g_{uu}}{\partial u}=\frac{8\cdot u}{a_c^4}\qquad \frac{\partial g_{vv}}{\partial v}=\frac{8\cdot v}{b_c^4}$$

$$\frac{\partial g_{uv}}{\partial u}=\frac{\partial g_{vu}}{\partial u}=-\frac{4\cdot v}{a_c^2\cdot b_c^2}\qquad \frac{\partial g_{uv}}{\partial v}=\frac{\partial g_{vu}}{\partial v}=-\frac{4\cdot u}{a_c^2\cdot b_c^2}$$

$$\frac{\partial g^{uu}}{\partial u}=-\frac{8\cdot a_c^4\cdot b_c^4\cdot u\cdot(b_c^4+4\cdot v^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}\qquad \frac{\partial g^{vv}}{\partial v}=-\frac{8\cdot a_c^4\cdot b_c^4\cdot v\cdot(a_c^4+4\cdot u^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}$$

$$\frac{\partial g^{vv}}{\partial u}=\frac{32\cdot a_c^4\cdot b_c^4\cdot u\cdot v^2}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}\qquad \frac{\partial g^{uu}}{\partial v}=\frac{32\cdot a_c^4\cdot b_c^4\cdot u^2\cdot v}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}$$

$$\frac{\partial g^{uv}}{\partial u}=\frac{\partial g^{vu}}{\partial u}=\frac{4\cdot a_c^2\cdot b_c^2\cdot v\cdot(a_c^4\cdot(b_c^4+4\cdot v^2)-4\cdot b_c^4\cdot u^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}$$

$$\frac{\partial g^{uv}}{\partial v}=\frac{\partial g^{vu}}{\partial v}=\frac{4\cdot a_c^2\cdot b_c^2\cdot u\cdot(b_c^4\cdot(a_c^4+4\cdot u^2)-4\cdot a_c^4\cdot v^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2} \tag{2.11.7}$$

$$\Gamma^u{}_{uu}=\frac{4\cdot b^4\cdot u}{a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2} \qquad \Gamma^u{}_{vv}=-\frac{4\cdot a^2\cdot b^2\cdot u}{a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2}$$

$$\Gamma^v{}_{vv}=\frac{4\cdot b^4\cdot v}{a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2} \qquad \Gamma^v{}_{uu}=-\frac{4\cdot a^2\cdot b^2\cdot v}{a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2}$$

$$\frac{\partial \Gamma^u{}_{uu}}{\partial u}=\frac{4\cdot b_c^4\cdot(a_c^4\cdot(b_c^4+4\cdot v^2)-4\cdot b_c^4\cdot u^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2} \qquad \frac{\partial \Gamma^v{}_{vv}}{\partial v}=\frac{4\cdot a_c^4\cdot(b_c^4\cdot(a_c^4+4\cdot u^2)-4\cdot a_c^4\cdot v^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}$$

$$\frac{\partial \Gamma^v{}_{uu}}{\partial u}=\frac{32\cdot a_c^2\cdot b_c^6\cdot u\cdot v}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2} \qquad \frac{\partial \Gamma^u{}_{vv}}{\partial v}=\frac{32\cdot a_c^6\cdot b_c^2\cdot u\cdot v}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}$$

$$\frac{\partial \Gamma^u{}_{vv}}{\partial u}=\frac{4\cdot a_c^2\cdot b_c^2\cdot(a_c^4\cdot(4\cdot v^2-b_c^4)+4\cdot b^4\cdot u^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2} \qquad \frac{\partial \Gamma^v{}_{uu}}{\partial v}=\frac{4\cdot a_c^2\cdot b_c^2\cdot(a_c^4\cdot(4\cdot v^2-b_c^4)-4\cdot b^4\cdot u^2)}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}$$

$$\frac{\partial \Gamma^v{}_{vv}}{\partial u}=\frac{\partial \Gamma^u{}_{uu}}{\partial v}=-\frac{32\cdot a_c^4\cdot b_c^4\cdot u\cdot v}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2} \tag{2.11.8}$$

$$R_{ij}=\frac{4\cdot a_c^2\cdot b_c^2}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2}\cdot\begin{pmatrix} b_c^4\cdot(4\cdot u^2+a_c^4) & 4\cdot a_c^2\cdot b_c^2\cdot u\cdot v \\ 4\cdot a_c^2\cdot b_c^2\cdot u\cdot v & a_c^4\cdot(4\cdot v^2+b_c^4)\end{pmatrix}$$

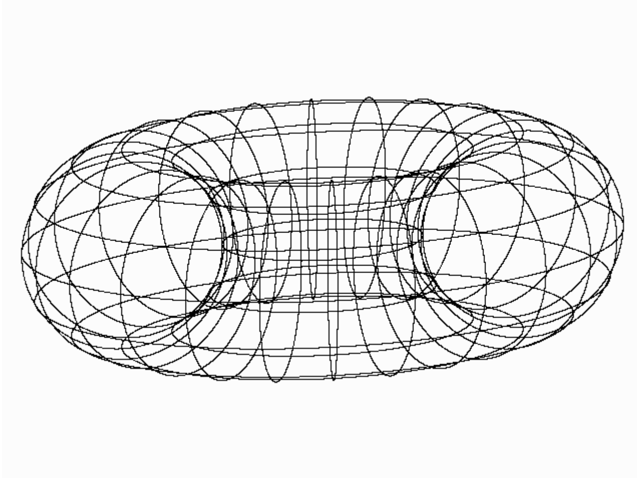
$$R=2\cdot K=-\frac{8\cdot a_c^6\cdot b_c^6}{(a_c^4\cdot(b_c^4+4\cdot v^2)+4\cdot b_c^4\cdot u^2)^2} \tag{2.11.9}$$

2.12 Torus

It is a variable curvature surface, negative on the inner side, and positive on the outer edge. The parametric equation, where *a* is the main radius, *b* is the secondary radius:

$$x=(a_c+b_c\cdot\sin(\vartheta))\cdot\cos(\varphi) \qquad y=(a_c+b_c\cdot\sin(\vartheta))\cdot\sin(\varphi) \qquad z=b_c\cdot\cos(\vartheta) \tag{2.12.1}$$

From the arc length squared to the curvature:



$$ds^2 = b_c^2 \cdot d\vartheta^2 + (a_c + b_c \cdot \sin(\vartheta))^2 \cdot d\varphi^2$$

$$g_{ij} = \begin{pmatrix} b_c^2 & 0 \\ 0 & (a_c + b_c \cdot \sin(\vartheta))^2 \end{pmatrix}$$

$$g^{ij} = \begin{pmatrix} \frac{1}{b_c^2} & 0 \\ 0 & \frac{1}{(a_c + b_c \cdot \sin(\vartheta))^2} \end{pmatrix}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial \vartheta} = 2 \cdot \cos(\vartheta) \cdot \sin(\vartheta) \cdot (2 \cdot b_c^2 \cdot \sin^2(\vartheta) + 3 \cdot a_c \cdot b_c \cdot \sin(\vartheta) + a_c^2)$$

$$\frac{\partial g^{\vartheta\vartheta}}{\partial \vartheta} = -\frac{2 \cdot \cos(\vartheta)}{(a_c + b_c \cdot \sin(\vartheta))^2 \cdot \sin^2(\vartheta)} \cdot \left(\frac{1}{\sin(\vartheta)} + \frac{b_c}{a_c + b_c \cdot \sin(\vartheta)} \right) \tag{2.12.2}$$

$$\Gamma^{\vartheta}{}_{\varphi\varphi} = -(a_c + b_c \cdot \sin(\vartheta)) \cdot \cos(\vartheta) \cdot \sin(\vartheta) \cdot \frac{a_c + b_c \cdot \sin(\vartheta) + \sin^2(\vartheta)}{b_c^2}$$

$$\Gamma^{\varphi}{}_{\vartheta\varphi} = \Gamma^{\varphi}{}_{\varphi\vartheta} = \cos(\vartheta) \cdot \frac{a_c + b_c \cdot \sin(\vartheta) \cdot (1 + \sin(\vartheta))}{(a_c + b_c \cdot \sin(\vartheta)) \cdot \sin(\vartheta)}$$

$$\frac{\partial \Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial \vartheta} = \frac{a_c + b_c \cdot \sin(\vartheta)}{b_c} \cdot \left(\frac{(\sin^2(\vartheta) - \cos^2(\vartheta)) \cdot (a_c + b_c \cdot \sin(\vartheta))}{b_c} + (\sin^2(\vartheta) - 4 \cdot \cos^2(\vartheta)) \cdot \sin(\vartheta) \right)$$
$$- \cos^2(\vartheta) \cdot \sin^2(\vartheta)$$

$$\frac{\partial \Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial \vartheta} = \frac{\partial \Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial \vartheta} = \frac{a_c^2 + 3 \cdot a_c \cdot b_c \cdot \sin(\vartheta) + 2 \cdot b_c + (a_c^2 + 2 \cdot a_c \cdot b_c \cdot \sin(\vartheta)) \cdot \cot^2(\vartheta)}{(a_c^2 + b_c^2 \cdot \sin^2(\vartheta))^2} \tag{2.12.3}$$

$$R_{ij} = \begin{pmatrix} \frac{(a_c + 2 \cdot b_c \cdot \sin(\vartheta)) \cdot \sin(\vartheta) - 2 \cdot b_c \cdot \cos^2(\vartheta)}{(a_c + b_c \cdot \sin(\vartheta)) \cdot \sin(\vartheta)} & 0 \\ 0 & R_{\varphi\varphi} \end{pmatrix}$$

$$R_{\varphi\varphi} = \frac{(3 \cdot a_c + 2 \cdot b_c \cdot \sin(\vartheta)) \cdot b_c \cdot \sin^2(\vartheta) + (a_c^2 - 2 \cdot b_c^2 \cdot \cos^2(\vartheta)) \cdot \sin(\vartheta) - 2 \cdot a_c \cdot b_c \cdot \cos^2(\vartheta)}{b_c^2} \cdot \sin(\vartheta)$$

$$R = 2 \cdot K = \frac{2}{b_c^2} \cdot \frac{(a_c + 2 \cdot b_c \cdot \sin(\vartheta)) \cdot \sin(\vartheta) - 2 \cdot b_c \cdot \cos^2(\vartheta)}{(a_c + b_c \cdot \sin(\vartheta)) \cdot \sin(\vartheta)} \tag{2.12.4}$$

The surface of the torus:

$$dA = b_c \cdot d\vartheta \cdot (a_c + b_c \cdot \sin(\vartheta)) \cdot d\varphi$$

$$A = \int_0^{2\cdot\pi} b_c \cdot d\vartheta \cdot \int_0^{2\cdot\pi} (a_c + b_c \cdot \sin(\vartheta)) \cdot d\varphi = 4 \cdot a_c \cdot b_c \cdot \pi^2 \qquad (2.12.5)$$

3. Flat and general spacetime

The simplest four dimensional solution of the Einstein equations is the matter-free flat spacetime. In it we can neglect the gravitation of test particles, and the geodesics they are following become everyday straight lines that can be described with simple partial derivatives. The theory of special relativity deals with this solution, described by Albert Einstein in 1905, not long before the birth of general relativity.

In the Kaluza model, electromagnetism also has a geometric origin, it is not part of the matter distribution, with this extension the theory describes a five dimensional spacetime. However in our experience, the gravitational interaction is more general than the electromagnetic. Thus gravitational interaction can act alone, while an electromagnetic field is not possible without gravitation, since the presence of electromagnetic energy in itself is already causing changes in the spacetime geometry. Thus we can discuss pure gravitational interaction without neglecting anything.

In order to understand what is going on, it is important to keep in mind, that motion happens in the entire four dimensional spacetime. Time is a direction of motion that can be used to measure distances, just as the other three directions of space.

3.1 Proper time

We use rectangular coordinates in flat space, and we write down the arc length squared in the following way:

$$ds^2 = c^2 \cdot dt^2 - dx^2 - dy^2 - dz^2$$

t: time coordinate
x, y, z: space coordinates
c: speed of light (3.1.1)

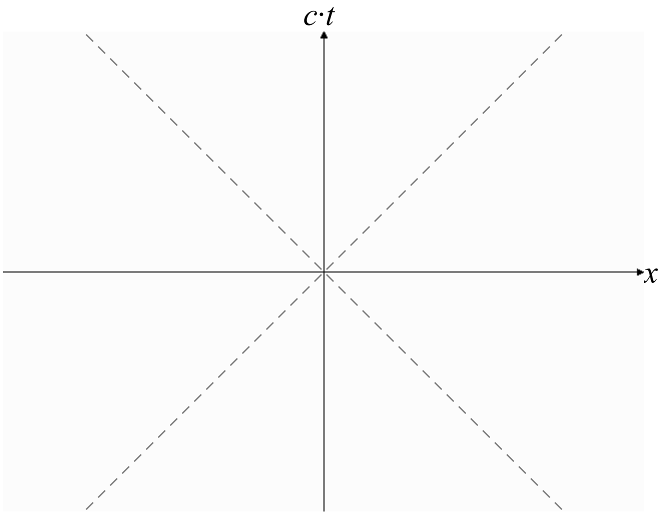
It is customary to use unique notation for the metric tensor:

$$\eta_{\eta\kappa} = \eta^{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.1.2}$$

It is easy to see, that this is a solution of the Einstein equations in four dimensions:

$$R_{\eta\kappa} - \frac{1}{2} \cdot R \cdot \eta_{\eta\kappa} = 0 \tag{3.1.3}$$

Two dotted lines denote the paths of light rays on the following graph, that lead away or into the event in the centre. The spacetime domains separated by the light cone have different significances. Under the lower cone sheet lay past events, that might have influenced the observer in the centre, these compose the time-like past. Events above the upper cone sheet might be influenced by the centre, therefore they compose the time-like future. Since information cannot propagate faster than the speed of light, the outside domain is not influenced by this, therefore the concepts of future and past cannot be interpreted there. At the graphical representation of the coordinate system, we neglected the y and z coordinates:



In the spacetime using different coordinate systems the arc length squared is the same, it is an invariant quantity. Let us write it on the left side of the equation in a coordinate system, where the arc is parallel to the time coordinate, and use a generally oriented coordinate system on the right side:

$$c^2\cdot d\tau^2=c^2\cdot dt^2-dl^2 \tag{3.1.4}$$

Where we have grouped together the space-like coordinate differentials:

$$dl^2=dx^2+dy^2+dz^2 \tag{3.1.5}$$

The space-like velocity squared:

$$v^2(t)=\frac{dl^2}{dt^2} \tag{3.1.6}$$

And the proper time:

$$d\tau=dt\cdot\sqrt{1-\frac{dl^2}{c^2\cdot dt^2}}=dt\cdot\sqrt{1-\frac{v(t)^2}{c^2}} \tag{3.1.7}$$

3.2 Lorentz-transformation

According to the relativity principle, reference frames in state of constant, rectilinear motion with respect to one another are equivalent. We determine the transformation matrix that corresponds to switching between them:

$$ {}_2x^i = \frac{{}_2\partial x^i}{\partial x^a} \cdot x^a = \Lambda^i{}_a \cdot x^a \qquad \begin{pmatrix} {}_2t \\ {}_2x \\ {}_2y \\ {}_2z \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{3.2.1} $$

Linear transformations can always be represented by matrices, and matrices always create linear transformations. The transformation matrices compose a group, because they obey the following rules:

Identity element: $\Lambda_{1\leftarrow 1} \cdot \vec{x} = \vec{x}$

Closure: $\Lambda_{3\leftarrow 2} \cdot \Lambda_{2\leftarrow 1} \cdot \vec{x} = \Lambda_{3\leftarrow 1} \cdot \vec{x}$

Associativity: $\Lambda_{4\leftarrow 3} \cdot (\Lambda_{3\leftarrow 2} \cdot \Lambda_{2\leftarrow 1}) \cdot \vec{x} = (\Lambda_{4\leftarrow 3} \cdot \Lambda_{3\leftarrow 2}) \cdot \Lambda_{2\leftarrow 1} \cdot \vec{x}$

Inverse element: $\Lambda_{2\leftarrow 1} \cdot \vec{x} = \Lambda^{-1}_{1\leftarrow 2} \cdot \vec{x}$ (3.2.2)

We examine first the identity element, we approach it with a transformation that changes the vector only slightly:

$$ \begin{pmatrix} t+\varepsilon \\ x+\varepsilon \\ y+\varepsilon \\ z+\varepsilon \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \cdot \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{3.2.3} $$

Write out the matrix operation in detail:

$$ t+\varepsilon = a\cdot t + b\cdot x + c\cdot y + d\cdot z \qquad \varepsilon = (a-1)\cdot t + b\cdot x + c\cdot y + d\cdot z $$

$$ x+\varepsilon = e\cdot t + f\cdot x + g\cdot y + h\cdot z \qquad \varepsilon = e\cdot t + (f-1)\cdot x + g\cdot y + h\cdot z $$

$$ y+\varepsilon = i\cdot t + j\cdot x + k\cdot y + l\cdot z \qquad \varepsilon = i\cdot t + j\cdot x + (k-1)\cdot y + l\cdot z $$

$$ z+\varepsilon = m\cdot t + n\cdot x + o\cdot y + p\cdot z \qquad \varepsilon = m\cdot t + n\cdot x + o\cdot y + (p-1)\cdot z \tag{3.2.4} $$

If the small deviation approaches zero, the transformation matrix approaches the identity matrix:

$$ \varepsilon \rightarrow 0 \qquad \rightarrow \qquad \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2.5} $$

Since the transformation is linear, the macroscopic translations will also have the same form. The small deviation is a contravariant vector, but we are going to write the macroscopic deviation as a covariant tangent vector:

$$\varepsilon^{\eta}=\eta^{\eta\alpha}\cdot u_{\alpha} \qquad u_{\eta}=\frac{v_{\eta}}{c}\cdot t \tag{3.2.6}$$

Investigate the transformation of the unit vectors, first in the time-like direction. We drop the speed of light in the formulas, to avoid confusion with one of the matrix elements:

$$\begin{pmatrix}\tau+\eta^{tt}\cdot u_t\\ \eta^{xx}\cdot u_x\\ \eta^{yy}\cdot u_y\\ \eta^{zz}\cdot u_z\end{pmatrix}=\begin{pmatrix}\tau+u_t\\ -u_x\\ -u_y\\ -u_z\end{pmatrix}=\begin{pmatrix}a & b & c & d\\ e & f & g & h\\ i & j & k & l\\ m & n & o & p\end{pmatrix}\cdot\begin{pmatrix}\tau\\0\\0\\0\end{pmatrix} \qquad \begin{matrix}\tau+v_t\cdot t=a\cdot\tau\\ -v_x\cdot t=e\cdot\tau\\ -v_y\cdot t=i\cdot\tau\\ -v_z\cdot t=m\cdot\tau\end{matrix} \tag{3.2.7}$$

from this the matrix elements are:

$$1+v_t\cdot\frac{t}{\tau}=a \qquad -\frac{v_x}{v_t}\cdot(a-1)=e \qquad -\frac{v_y}{v_t}\cdot(a-1)=i \qquad -\frac{v_z}{v_t}\cdot(a-1)=m \tag{3.2.8}$$

The unit vector in the x direction:

$$\begin{pmatrix}u_t\\ x-u_x\\ -u_y\\ -u_z\end{pmatrix}=\begin{pmatrix}a & b & c & d\\ e & f & g & h\\ i & j & k & l\\ m & n & o & p\end{pmatrix}\cdot\begin{pmatrix}0\\x\\0\\0\end{pmatrix} \qquad \begin{matrix}v_t\cdot t=b\cdot x\\ x-v_x\cdot t=f\cdot x\\ -v_y\cdot t=j\cdot x\\ -v_z\cdot t=n\cdot x\end{matrix} \tag{3.2.9}$$

the matrix elements:

$$v_t\cdot\frac{t}{x}=b \qquad 1-\frac{v_x}{v_t}\cdot b=f \qquad -\frac{v_y}{v_t}\cdot b=j \qquad -\frac{v_z}{v_t}\cdot b=n \tag{3.2.10}$$

Spacetime is the same in every direction, therefore we expect the same form in both other space-like directions:

$$v_t\cdot\frac{t}{y}=c \qquad -\frac{v_x}{v_t}\cdot c=g \qquad 1-\frac{v_y}{v_t}\cdot c=k \qquad -\frac{v_z}{v_t}\cdot c=o \tag{3.2.11}$$

$$v_t\cdot\frac{t}{z}=d \qquad -\frac{v_x}{v_t}\cdot d=h \qquad -\frac{v_y}{v_t}\cdot d=l \qquad 1-\frac{v_z}{v_t}\cdot d=p \tag{3.2.12}$$

The diagonally opposite elements look the same, and since the length of the unit vectors is the same, they coincide:

$$v_t \cdot \frac{t}{x} = b \qquad \leftrightarrow \qquad v_t \cdot \frac{t}{\tau} = e \qquad \rightarrow \qquad e = b$$

$$-\frac{v_x}{v_t} \cdot (a-1) = e = b \qquad -\frac{v_z}{v_t} \cdot (a-1) = i = c \qquad -\frac{v_z}{v_t} \cdot (a-1) = m = d \qquad (3.2.13)$$

Thus the matrix is symmetric:

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & f & g & h \\ c & g & k & l \\ d & h & l & p \end{pmatrix} \qquad (3.2.14)$$

The velocity squared in three dimensional space:

$$\eta^{\alpha\beta} \cdot v_\alpha \cdot v_\beta = v_t^2 - v_x^2 - v_y^2 - v_z^2 \qquad \rightarrow \qquad v_t^2 = v_x^2 + v_y^2 + v_z^2 = v^2 \qquad (3.2.15)$$

After all this the form of the matrix elements:

$$1 + v_t \cdot \frac{t}{\tau} = a \qquad v_t \cdot \frac{t}{x} = b \qquad v_t \cdot \frac{t}{y} = c \qquad v_t \cdot \frac{t}{z} = d$$

$$-\frac{v_x}{v_t} \cdot (a-1) = b \qquad 1 + \frac{v_x^2}{v^2} \cdot (a-1) = f \qquad \frac{v_x \cdot v_y}{v^2} \cdot (a-1) = g \qquad \frac{v_x \cdot v_z}{v^2} \cdot (a-1) = h$$

$$-\frac{v_y}{v_t} \cdot (a-1) = c \qquad \frac{v_y \cdot v_x}{v^2} \cdot (a-1) = g \qquad 1 + \frac{v_y^2}{v^2} \cdot (a-1) = k \qquad \frac{v_y \cdot v_z}{v^2} \cdot (a-1) = l$$

$$-\frac{v_z}{v_t} \cdot (a-1) = d \qquad \frac{v_z \cdot v_x}{v^2} \cdot (a-1) = h \qquad \frac{v_z \cdot v_y}{v^2} \cdot (a-1) = l \qquad 1 + \frac{v_z^2}{v^2} \cdot (a-1) = p$$

$$(3.2.16)$$

The coordinates of the centre of the standing coordinate system in the moving coordinate system:

$$\begin{pmatrix} t \\ -v_x \cdot t \\ -v_y \cdot t \\ -v_z \cdot t \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ b & f & g & h \\ c & g & k & l \\ d & h & l & p \end{pmatrix} \cdot \begin{pmatrix} \tau \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{matrix} \frac{t}{\tau} = a \\ -v_x \cdot a = b \\ -v_y \cdot a = c \\ -v_z \cdot a = d \end{matrix} \qquad (3.2.17)$$

The reverse transformation:

$$\begin{pmatrix}\tau\\0\\0\\0\end{pmatrix}=\begin{pmatrix}a & -v_x\cdot a & -v_y\cdot a & -v_z\cdot a\\ -v_x\cdot a & f & g & h\\ -v_y\cdot a & g & k & l\\ -v_z\cdot a & h & l & p\end{pmatrix}\cdot\begin{pmatrix}t\\v_x\cdot t\\v_y\cdot t\\v_z\cdot t\end{pmatrix} \tag{3.2.18}$$

Determine the „0,0" matrix element:

$$\begin{matrix}\tau=a\cdot t\cdot(1-v^2)\\ 0=(f-a)\cdot v_x+g\cdot v_y+h\cdot v_z\\ 0=g\cdot v_x+(k-a)\cdot v_y+l\cdot v_z\\ 0=h\cdot v_x+l\cdot v_y+(p-a)\cdot v_z\end{matrix} \quad\rightarrow\quad a=\frac{1}{\sqrt{1-v^2}} \tag{3.2.19}$$

The matrix of the Lorentz-transformation with SI units:

$$\Lambda=\begin{pmatrix}a & -\frac{v_x}{c}\cdot a & -\frac{v_y}{c}\cdot a & -\frac{v_z}{c}\cdot a\\ -\frac{v_x}{c}\cdot a & 1+\frac{v_x^2}{v^2}\cdot(a-1) & \frac{v_x\cdot v_y}{v^2}\cdot(a-1) & \frac{v_x\cdot v_z}{v^2}\cdot(a-1)\\ -\frac{v_y}{c}\cdot a & \frac{v_y\cdot v_x}{v^2}\cdot(a-1) & 1+\frac{v_y^2}{v^2}\cdot(a-1) & \frac{v_y\cdot v_z}{v^2}\cdot(a-1)\\ -\frac{v_z}{c}\cdot a & \frac{v_z\cdot v_x}{v^2}\cdot(a-1) & \frac{v_z\cdot v_y}{v^2}\cdot(a-1) & 1+\frac{v_z^2}{v^2}\cdot(a-1)\end{pmatrix} \qquad a=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.2.20}$$

Investigate the transformation along the *x* axis:

$$\begin{pmatrix}c\cdot{}_2t\\{}_2x\\0\\0\end{pmatrix}=\begin{pmatrix}a & b & c & d\\ b & f & g & h\\ c & g & k & l\\ d & h & l & p\end{pmatrix}\cdot\begin{pmatrix}c\cdot t\\x\\0\\0\end{pmatrix} \tag{3.2.21}$$

The transformation formulas:

$${}_2t=a\cdot t+\frac{b\cdot x}{c}=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot t-\frac{v_x}{c}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{x}{c}=\frac{t-\frac{v_x}{c^2}\cdot x}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.2.22}$$

$$ {}_2x = b\cdot c\cdot t + f\cdot x = -\frac{v_x}{c}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot c\cdot t + \left(1+\frac{v_x^2}{v^2}\cdot\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)\right)\cdot x = \frac{x-v_x\cdot t}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (3.2.23) $$

For the reverse transformation, all we have to do is change the sign of the velocity. Substitute them into the four-distance, that is the finite variant of the arc length squared:

$$ s^2 = c^2\cdot t^2 - x^2 - y^2 - z^2 \qquad (3.2.24) $$

$$ c^2\cdot{}_2t^2 - {}_2x^2 = c^2\cdot\left(\frac{t-\frac{v_x}{c^2}\cdot x}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - \left(\frac{x-v_x\cdot t}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 $$

$$ c^2\cdot{}_2t^2 - {}_2x^2 = \frac{1}{1-\frac{v^2}{c^2}}\cdot\left(c^2\cdot t^2 - 2\cdot t\cdot v_x\cdot x + \frac{v_x^2}{c^2}\cdot x^2 - x^2 + 2\cdot x\cdot v_x\cdot t - v_x^2\cdot t^2\right) $$

$$ c^2\cdot{}_2t^2 - {}_2x^2 = \frac{1}{1-\frac{v^2}{c^2}}\cdot\left(c^2\cdot t^2 - x^2\right)\cdot\left(1-\frac{v_x^2}{c^2}\right) $$

$$ c^2\cdot{}_2t^2 - {}_2x^2 = c^2\cdot t^2 - x^2 \qquad (3.2.25) $$

The coordinate transformation is therefore correct, because it preserves the invariant four-distance. By writing out the entire four-distance, we can conclude that the *y* and *z* coordinates do not transform:

$$ c^2\cdot{}_2t^2 - {}_2x^2 - {}_2y^2 - {}_2z^2 = c^2\cdot t^2 - x^2 - y^2 - z^2 $$

$$ {}_2y^2 + {}_2z^2 = y^2 + z^2 $$

$$ {}_2y = y \qquad {}_2z = z \qquad (3.2.26) $$

Length contraction of a moving rod; let the coordinates of the end points of the rod in each coordinate systems be x and y, and ${}_2x$ and ${}_2y$:

$$ {}_2x - {}_2y = \frac{x-v_x\cdot t}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{y-v_x\cdot t}{\sqrt{1-\frac{v^2}{c^2}}} $$

$${}_2x-{}_2y=\frac{x-y}{\sqrt{1-\frac{v^2}{c^2}}}$$ The length of the rod: $l=x-y$

$$l={}_2l\cdot\sqrt{1-\frac{v^2}{c^2}} \qquad (3.2.27)$$

3.3 Addition of velocity and acceleration

How much is the measured speed of an object that is moving with constant velocity, observed from a different moving reference frame? Let one of the coordinate systems move relatively to the other in the *x* direction, let their relative velocity be V_x. The moving test object also moves along the *x* direction, and its speed in each coordinate system is:

$$v_x=\frac{dx}{dt} \qquad {}_2v_x=\frac{{}_2dx}{{}_2dt} \qquad (3.3.1)$$

The transformation of the change in the coordinate:

$${}_2dx={}_2x_B-{}_2x_A=\frac{(x_B-x_A)-V_x\cdot(t_B-t_A)}{\sqrt{1-\frac{V^2}{c^2}}}=\frac{dx-V_x\cdot dt}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (3.3.2)$$

Substitute it:

$${}_2v_x=\frac{{}_2dx}{{}_2dt}=\frac{{}_2dx}{dt}\cdot\frac{dt}{{}_2dt}=\frac{dx-V_x\cdot dt}{\sqrt{1-\frac{V^2}{c^2}}}\cdot\frac{1}{dt}\cdot\frac{dt}{{}_2dt}=\frac{v_x-V_x}{\sqrt{1-\frac{V^2}{c^2}}}\cdot\frac{dt}{{}_2dt} \qquad (3.3.3)$$

The transformation of the change of time:

$${}_2dt={}_2t_B-{}_2t_A=\frac{(t_B-t_A)-\frac{V_x}{c^2}\cdot(x_B-x_A)}{\sqrt{1-\frac{V^2}{c^2}}}=\frac{dt-\frac{V_x}{c^2}\cdot dx}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (3.3.4)$$

The mutual ratios of the changes of time:

$$\frac{{}_2dt}{dt}=\frac{dt-\frac{V_x}{c^2}\cdot dx}{\sqrt{1-\frac{V^2}{c^2}}}\cdot\frac{1}{dt}=\frac{1-\frac{V_x\cdot v_x}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (3.3.5)$$

Reinsert is:

$${}_2v_x=\frac{v_x-V_x}{\sqrt{1-\frac{V^2}{c^2}}}\cdot\frac{dt}{{}_2dt}=\frac{v_x-V_x}{\sqrt{1-\frac{V^2}{c^2}}}\cdot\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V_x\cdot v_x}{c^2}}$$

The transformation formula of the velocity along the *x* direction:

$${}_2v_x=\frac{v_x-V_x}{1-\frac{V_x\cdot v_x}{c^2}} \qquad (3.3.6)$$

We can always find a coordinate system, where the velocity of the moving object is zero, this is the co-moving coordinate system, where:

$$V_x=v_x \qquad (3.3.7)$$

The perpendicular velocity components are also transforming:

$${}_2v_y=\frac{{}_2dy}{{}_2dt}=\frac{{}_2dy}{dt}\cdot\frac{dt}{{}_2dt}=\frac{dy}{dt}\cdot\frac{dt}{{}_2dt}=v_y\cdot\frac{dt}{{}_2dt}$$

$${}_2v_y=v_y\cdot\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V_x\cdot v_x}{c^2}} \qquad (3.3.8)$$

$${}_2v_z=\frac{{}_2dz}{{}_2dt}=\frac{{}_2dz}{dt}\cdot\frac{dt}{{}_2dt}=\frac{dz}{dt}\cdot\frac{dt}{{}_2dt}=v_z\cdot\frac{dt}{{}_2dt}$$

$${}_2v_z=v_z\cdot\frac{\sqrt{1-\frac{V^2}{c^2}}}{1-\frac{V_x\cdot v_x}{c^2}} \qquad (3.3.9)$$

For the reverse transformation formulas all we have to do is reversing the sign of the relative velocity between the two coordinate systems. Write down the transformation law of the change of the velocity:

$$ {}_2dv_x = d\left(\frac{v_x - V_x}{1 - \frac{V_x \cdot v_x}{c^2}}\right) = \frac{1 - \frac{V^2}{c^2}}{\left(1 - \frac{V_x \cdot v_x}{c^2}\right)^2} \cdot dv_x \qquad (3.3.10) $$

Divide it with the change of time:

$$ {}_2dt = \frac{1 - \frac{V_x \cdot v_x}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \cdot dt $$

The transformation of the acceleration in the *x* direction:

$$ {}_2a_x = \frac{{}_2dv_x}{{}_2dt} = \frac{\left(1 - \frac{V^2}{c^2}\right)^{\frac{3}{2}}}{\left(1 - \frac{V_x \cdot v_x}{c^2}\right)^3} \cdot a_x \qquad (3.3.11) $$

Let us examine the momentary state of an object with constant acceleration. In this case, its velocity in its own reference frame is zero, and the mutual velocities of the two coordinate systems coincides with the velocity of the object in the other coordinate system:

$$ a_0 = \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} \cdot a = \frac{d}{dt} \cdot \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (3.3.12) $$

Perform the integration:

$$ v = \frac{a_0 \cdot t}{\sqrt{1 + \left(\frac{a_0}{c} \cdot t\right)^2}} \qquad (3.3.13) $$

With further integration we get the dependence of the objects position from the coordinate-time:

$$ x = \frac{c^2}{a_0} \cdot \left(\sqrt{1 + \left(\frac{a_0}{c} \cdot t\right)^2} - 1\right) + x_0 \qquad (3.3.14) $$

We introduce the contravariant four-velocity, the derivative of the coordinate-change according to the proper time. Calculating the time-like component:

$$V^0 = \frac{dx^0}{d\tau} = \frac{c \cdot dt}{dt \cdot \sqrt{1 - \frac{v^2}{c^2}}} = \frac{c}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (3.3.15)$$

Determine the space-like components of the four-velocity, from the common three-velocity:

$$V^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \cdot \frac{dt}{d\tau} = \frac{v_i}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (3.3.16)$$

The reverse relationship:

$$v_i = c \cdot \frac{V^i}{V^0} \qquad (3.3.17)$$

The four-velocity is a vector, therefore its square is an invariant scalar:

$$V^2 = \eta_{\alpha\beta} \cdot \frac{dx^\alpha}{d\tau} \cdot \frac{dx^\beta}{d\tau} = \frac{ds^2}{d\tau^2} = c^2 \qquad (3.3.18)$$

The four-acceleration is the differential of the four-velocity according to the proper time:

$$A^\eta = \frac{dV^\eta}{d\tau} \qquad (3.3.19)$$

Every component of the four-acceleration of a non-moving object is zero:

$$A^\eta = \frac{dV^\eta}{d\tau} = d\,\frac{(c \quad 0 \quad 0 \quad 0)}{d\tau} = (0 \quad 0 \quad 0 \quad 0) \qquad (3.3.20)$$

In this case the scalar product of the two vectors can be easily written down. This is however an invariant formula, therefore its true for any moving body:

$$V^\alpha \cdot A_\alpha = 0 \qquad (3.3.21)$$

3.4 Aberration of light

The movement of an object in two different coordinate systems can be characterized by the following velocity components:

$$v_x = v \cdot \cos(\varphi) \qquad v_y = v \cdot \sin(\varphi)$$

$$_2v_x = {}_2v \cdot \cos({}_2\varphi) \qquad {}_2v_y = {}_2v \cdot \sin({}_2\varphi) \tag{3.4.1}$$

One of the coordinate systems moves in the *x* direction, with V_x velocity relatively to the other. The transformation laws in the *x* and *y* directions using the formulas containing angles above:

$$_2v_x = \frac{v_x - V_x}{1 - \frac{V_x \cdot v_x}{c^2}} = {}_2v \cdot \cos({}_2\varphi) = \frac{v \cdot \cos(\varphi) - V_x}{1 - \frac{V_x \cdot v \cdot \cos(\varphi)}{c^2}} \tag{3.4.2}$$

$$_2v_y = v_y \cdot \frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V_x \cdot v_x}{c^2}} = {}_2v \cdot \sin({}_2\varphi) = v \cdot \sin(\varphi) \cdot \frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{V_x \cdot v \cdot \cos(\varphi)}{c^2}} \tag{3.4.3}$$

Divide them with each other and we get the transformation law of the azimuth angle of the trajectory of the moving body:

$$\cot({}_2\varphi) = \frac{v \cdot \cos(\varphi) - V_x}{v \cdot \sin(\varphi) \cdot \sqrt{1 - \frac{V^2}{c^2}}} \tag{3.4.4}$$

In the case of light, the speed *v* is equal to the speed of light *c*, this is the aberration of light:

$$\cot({}_2\varphi) = \frac{\cos(\varphi) - \frac{V_x}{c}}{\sin(\varphi) \cdot \sqrt{1 - \frac{V^2}{c^2}}} \tag{3.4.5}$$

3.5 Doppler-effect

Since the length and the elapsed time are also coordinate dependent quantities, therefore, while the speed of light is constant, the wavelength of light is measured to be different by different observers. Wave-fronts leaving the light source with λ wavelength distance from each other, reach the observer moving away with *v* velocity under the following time intervals:

$$t = \frac{\lambda}{c - v} \tag{3.5.1}$$

This can be expressed also using the frequency:

$$t=\frac{1}{\left(1-\frac{v}{c}\right)\cdot\nu} \qquad \nu=\frac{c}{\lambda}$$

The elapsed time calculated in the proper time of the distancing observer:

$$_{2}t=t\cdot\sqrt{1-\frac{v^2}{c^2}}=\frac{\sqrt{1-\frac{v^2}{c^2}}}{\left(1-\frac{v}{c}\right)\cdot\nu}=\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\cdot\frac{1}{\nu} \qquad (3.5.2)$$

From this the measured frequency:

$$_{2}\nu=\frac{1}{_{2}t}=\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\cdot\nu \qquad (3.5.3)$$

If the light source and the observer pass by each other, for a short period of time their distance does not change, their relative velocity along the line connecting them is zero. From the point of view of the light source, wave-fronts leaving the light source with λ wavelength distance from each other, reach the observer with the following time intervals:

$$t=\frac{\lambda}{c}=\frac{1}{\nu} \qquad (3.5.4)$$

However the proper time of the observe differs from the light source, therefore it measures the arrival of the wave-fronts with different time intervals:

$$_{2}t=t\cdot\sqrt{1-\frac{v^2}{c^2}}=\frac{\sqrt{1-\frac{v^2}{c^2}}}{\nu} \qquad (3.5.5)$$

The frequency Doppler shift in the perpendicular direction:

$$_{2}\nu=\sqrt{1-\frac{v^2}{c^2}}\cdot\nu \qquad (3.5.6)$$

3.6 Sequence of events

The ordering between cause and effect can be secured only, if by observing from every possible reference frames, the moment the effect happens is later in time than the cause. We are going to investigate, under what circumstances this condition is fulfilled. The difference between moments in time transforms between coordinate systems the following way:

$$ {}_2t_B - {}_2t_A = \frac{(t_B - t_A) - \frac{V_x}{c^2}\cdot(x_B - x_A)}{\sqrt{1-\frac{V^2}{c^2}}} = (t_B - t_A)\cdot\frac{1-\frac{V_x}{c^2}\cdot\frac{(x_B - x_A)}{(t_B - t_A)}}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.6.1} $$

The velocity v_x is the speed the information travels with, from event *A* (the cause) to event *B* (the effect):

$$ v_x = \frac{(x_B - x_A)}{(t_B - t_A)} \tag{3.6.2} $$

Substitute it to the transformation formula:

$$ {}_2t_B - {}_2t_A = (t_B - t_A)\cdot\frac{1-\frac{V_x \cdot v_x}{c^2}}{\sqrt{1-\frac{V^2}{c^2}}} \tag{3.6.3} $$

It follows from our condition, that the time difference has to be positive:

$$ {}_2t_B - {}_2t_A > 0 \qquad t_B - t_A > 0 \tag{3.6.4} $$

This leads to the following inequality:

$$ 1-\frac{V_x \cdot v_x}{c^2} > 0 $$

$$ c^2 > V_x \cdot v_x \tag{3.6.5} $$

If $v_x = c$, that is, the information causing the second event comes from the first event with the greatest possible velocity, the speed of light, the mutual velocity of the reference frames cannot exceed the speed of light:

$$ c > V_x \tag{3.6.6} $$

Returning to our original transformation formula, let us examine what effects it has, if we demand that the time differences are positive:

$${}_2t_B - {}_2t_A = \frac{(t_B - t_A) - \frac{V_x}{c^2}\cdot(x_B - x_A)}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$${}_2t_B - {}_2t_A > 0$$

$$(t_B - t_A) - \frac{V_x}{c^2}\cdot(x_B - x_A) > 0 \qquad \frac{|V|}{c} < 0$$

$$c\cdot(t_B - t_A) - (x_B - x_A) > 0 \tag{3.6.7}$$

Squaring the terms does not change the direction of the inequality:

$$c^2\cdot(t_B - t_A)^2 - (x_B - x_A)^2 > 0 \tag{3.6.8}$$

Thus the four-distance between the two events is always bigger than zero, it is time-like.

3.7 Energy and momentum

Picture a weightless floating empty box. One of the internal walls emits a photon, that is absorbed by the opposite wall later. Because of the conservation of momentum, the box slightly moves in the opposite direction and then stops, as the photon is absorbed. The photon has no rest mass, but it has momentum:

$$p_f = \frac{E_0}{c} \tag{3.7.1}$$

We can write the momentum of the box using its mass and velocity:

$$p_d = M\cdot v \tag{3.7.2}$$

It takes Δt time for the photon to reach the other side of the box, while the box gets displaced for Δx distance, this is the velocity of the box:

$$v = \frac{\Delta x}{\Delta t} \tag{3.7.3}$$

Because of the conservation of momentum, in the centre of mass system the magnitude of the momentum of the photon and the box are equal:

$$M\cdot\frac{\Delta x}{\Delta t} = \frac{E_0}{c} \tag{3.7.4}$$

If we know the l width of the box, and the speed of the photon, we can determine the time that passed while it crossed the box:

$$\Delta t = \frac{l}{c} \tag{3.7.5}$$

Substitute it into the formula for the conservation of momentum:

$$M \cdot \Delta x = \frac{E_0 \cdot l}{c^2} \tag{3.7.6}$$

In order to determine the movement of the box relative to the centre of mass, let us write the position of the centre of mass, as if a particle with mass moving inside it has caused its displacement, it will be called the effective mass of the photon:

$$x = \frac{M \cdot x_d + m \cdot x_f}{M + m} \tag{3.7.7}$$

The position of the centre of mass is the same both at the emission and absorption of the photon:

$$\frac{M \cdot x_d + m \cdot x_f}{M + m} = \frac{M \cdot (x_d - \Delta x) + m \cdot l_d}{M + m} \tag{3.7.8}$$

If we consider the starting position of the photon ($x_f = 0$), and the box ($x_d = 0$) both zero, we can significantly simplify the relationship above:

$$m \cdot l_d = M \cdot \Delta x \tag{3.7.9}$$

Substitute the conservation of momentum formula:

$$m \cdot l_d = \frac{E_0 \cdot l_d}{c^2}$$

From this we can express the equivalence of the rest mass and energy:

$$\boxed{E_0 = m \cdot c^2} \tag{3.7.10}$$

The equation of motion of an object accelerating because of a constant force:

$$F = m \cdot a = \frac{dp}{dt} \tag{3.7.11}$$

Let us examine the situation is a given moment, when the velocity of the moving body is zero in one of the coordinate systems, in this case its velocity looks the same as the relative velocity of the two reference frames in the other coordinate system:

$$v=V \tag{3.7.12}$$

Express its acceleration:

$$_2a_x=\frac{\left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}}}{\left(1-\frac{V\cdot v}{c^2}\right)^3}\cdot a_x=\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\cdot a_x=\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\cdot\frac{dv}{dt}=\frac{d}{dt}\cdot\frac{v}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.7.13}$$

Substitute it into the equation of motion:

$$F=m\cdot a=\frac{d}{dt}\cdot\frac{m\cdot v}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{d}{dt}\cdot p \tag{3.7.14}$$

We can read from this the relativistic momentum:

$$p=\frac{m\cdot v}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.7.15}$$

Force is the negative gradient of the potential energy, we rewrite this:

$$F=-\frac{dE}{dx}$$

$$F\cdot v=-\frac{dE}{dx}\cdot\frac{dx}{dt}=-\frac{dE}{dt} \tag{3.7.16}$$

Substitute the momentum, and one version of the expression of the acceleration:

$$v\cdot F=v\cdot\frac{dp}{dt}=v\cdot\frac{d}{dt}\cdot\frac{m\cdot v}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m\cdot v}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\cdot\frac{dv}{dt}=\frac{d}{dt}\cdot\frac{m\cdot c^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.7.17}$$

From this the relativistic energy:

$$E=\frac{m\cdot c^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.7.18}$$

Relationship between energy and momentum:

$$E=\frac{p\cdot c^2}{v} \tag{3.7.19}$$

In order to investigate the relationship between energy and momentum, we examine the contravariant velocity. Divide the change in coordinate with the proper time:

$$dx^\eta=(c\cdot dt \quad dx \quad dy \quad dz) \qquad d\tau=dt\cdot\sqrt{1-\frac{v^2}{c^2}}$$

$$\frac{dx^\eta}{d\tau}=v^\eta=\left(\frac{c}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{v^x}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{v^y}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{v^z}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{3.7.20}$$

If we multiply this with the mass, we get the contravariant energy-momentum four-vector:

$$m\cdot v^\eta=\left(\frac{m\cdot c}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{m\cdot v^x}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{m\cdot v^y}{\sqrt{1-\frac{v^2}{c^2}}} \quad \frac{m\cdot v^z}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\left(\frac{E}{c} \quad p^x \quad p^y \quad p^z\right) \tag{3.7.21}$$

During changes between coordinate system, the four-vectors Lorentz-transform:

$${}_2x^i=\frac{{}_2\partial x^i}{\partial x^a}\cdot x^a=\Lambda^i{}_a\cdot x^a$$

The equation of transformation while moving along a line:

$$\begin{pmatrix}\frac{{}_2E}{c}\\ {}_2p_x\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}a & b & c & d\\ b & f & g & h\\ c & g & k & l\\ d & h & l & p\end{pmatrix}\cdot\begin{pmatrix}\frac{E}{c}\\ p_x\\ 0\\ 0\end{pmatrix}$$

$${}_2E=a\cdot E+b\cdot p_x\cdot c=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot E-\frac{v_x}{c}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot p_x\cdot c=\frac{E-v_x\cdot p_x}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.7.22}$$

$${}_2p_x=b\cdot\frac{E}{c}+f\cdot p_x=-\frac{v_x}{c}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot\frac{E}{c}+\left(1+\frac{v_x^2}{v^2}\cdot\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)\right)\cdot p_x=\frac{p_x-\frac{v_x}{c^2}\cdot E}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.7.23}$$

Using the arc length squared, we can once again conclude, that the y and z components remain unchanged:

$$s^2=\frac{E^2}{c^2}-p_x^2-p_y^2-p_z^2$$

$$\frac{{}_2E^2}{c^2}-{}_2p_x^2=\frac{E^2}{c^2}-p_x^2$$

$${}_2p_y=p_y \qquad {}_2p_z=p_z \tag{3.7.24}$$

Insert the transformation of the rest mass and the momentum into each other:

$${}_2E=\frac{E-v_x\cdot p_x}{\sqrt{1-\frac{v^2}{c^2}}} \qquad {}_2p_x=\frac{p_x-\frac{v_x}{c^2}\cdot E}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$p_x={}_2p_x\cdot\sqrt{1-\frac{v^2}{c^2}}+\frac{v_x}{c^2}\cdot E$$

$${}_2E=E\cdot\sqrt{1-\frac{v^2}{c^2}}-v_x\cdot{}_2p_x \tag{3.7.25}$$

If the particle moves at the speed of light (the sign shows the direction of the movement):

$${}_2E_0=-c\cdot{}_2p_x \tag{3.7.26}$$

Divide the arc length squared with the infinitesimal change in time:

$$ds^2=c^2\cdot dt^2-dx^2-dy^2-dz^2 \quad /\cdot\frac{1}{dt}$$

$$\frac{ds^2}{dt^2}=c^2-v_x^2-v_y^2-v_z^2 \tag{3.7.27}$$

In order to keep it simple, we calculate in one dimension. First we write an obvious identity, that we reorder:

$$c^2-v^2=c^2-v^2$$

$$\frac{c^2-v^2}{1-\frac{v^2}{c^2}}=c^2$$

$$\left(\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2=\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2+c^2$$

$$\frac{c}{\sqrt{1-\frac{v^2}{c^2}}}=\sqrt{\left(\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2+c^2} \quad / \cdot m \cdot c$$

$$\frac{m \cdot c^2}{\sqrt{1-\frac{v^2}{c^2}}}=\sqrt{\left(\frac{m \cdot v}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 \cdot c^2+m^2 \cdot c^4}$$

By substituting the energy and the momentum, we have the total energy of a moving body:

$$E=\sqrt{p^2 \cdot c^2+m^2 \cdot c^4} \qquad (3.7.28)$$

If its mass is zero:

$$E_0=p \cdot c \qquad (3.7.29)$$

$$\frac{m \cdot c^2}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{m \cdot v}{\sqrt{1-\frac{v^2}{c^2}}} \cdot c$$

Objects with zero mass move at the speed of light in every reference frame:

$$v=c \qquad (3.7.30)$$

3.8 Relativistic rocket

The rocket is a complex system that loses mass while constantly accelerating. The two main components are the payload and the fuel, that is exhausted with a constant velocity in the rocket's reference frame. The relativistic rocket equation establishes a relationship between the following quantities:

M: initial mass of the rocket

dm: mass of the fuel that is ejected in an infinitesimally small period of time
w: the constant exhaust velocity of the fuel relatively to the rocket
U: velocity of the rocket in the centre of mass system
u: velocity of the exhaust in the centre of mass system

During the movement of the rocket, the conservation of momentum is valid, the left side is the change of momentum of the rocket, the right side is of the exhausted fuel:

$$d\left(\frac{M}{\sqrt{1-\frac{U^2}{c^2}}}\cdot U\right)=u\cdot\frac{dm}{\sqrt{1-\frac{u^2}{c^2}}} \tag{3.8.1}$$

As well as the conservation of energy:

$$d\left(\frac{M}{\sqrt{1-\frac{U^2}{c^2}}}\cdot c^2\right)=-\frac{dm}{\sqrt{1-\frac{u^2}{c^2}}}\cdot c^2 \tag{3.8.2}$$

Observing from the centre of mass system, the velocity of the exhausted fuel is the sum of the exhaust velocity relative to the rocket and the rocket velocity:

$$u=\frac{w-U}{1-\frac{U\cdot w}{c^2}}$$

Substitute it to the conservation of momentum formula:

$$d\left(\frac{M}{\sqrt{1-\frac{U^2}{c^2}}}\cdot U\right)=\frac{U-w}{1-\frac{U\cdot w}{c^2}}\cdot d\left(\frac{M}{\sqrt{1-\frac{U^2}{c^2}}}\right) \tag{3.8.3}$$

Reorder the differential of the denominator with the square root:

$$d\left(\frac{1}{\sqrt{1-\frac{U^2}{c^2}}}\right)=\frac{U\cdot dU}{(c^2-U^2)\cdot\sqrt{1-\frac{U^2}{c^2}}}$$

Substitute:

$$M\cdot dU+U\cdot dM+\frac{M\cdot U^2\cdot dU}{c^2-U^2}=\frac{U-w}{1-\frac{U\cdot w}{c^2}}\cdot\left(dM+\frac{M\cdot U\cdot dU}{c^2-U^2}\right)$$

$$\left(U-\frac{U-w}{1-\frac{U\cdot w}{c^2}}\right)\cdot dM=-\left(M+\frac{M\cdot U}{c^2-U^2}\cdot\left(U-\frac{U-w}{1-\frac{U\cdot w}{c^2}}\right)\right)\cdot dU$$

Simplify:

$$\frac{dM}{M}=-\frac{dU}{w\cdot\left(1-\frac{U^2}{c^2}\right)} \tag{3.8.4}$$

Substitute the differential of the logarithm:

$$d(\log(x))=\frac{dx}{x}$$

$$\log\left(\frac{M}{M_{start}}\right)=-\frac{c}{2\cdot w}\cdot\log\left(\frac{1+\frac{U}{c}}{1-\frac{U}{c}}\right)$$

The traditional form of the relativistic rocket equation:

$$M=M_{start}\cdot\left(\frac{1+\frac{U}{c}}{1-\frac{U}{c}}\right)^{-\frac{c}{2\cdot w}} \tag{3.8.5}$$

The maximal exhaust velocity of the fuel:

$$w=\sqrt{e\cdot(2-e)}\cdot c \tag{3.8.6}$$

3.9 Faster than light particles

Theoretical particles that can move faster than light are called tachyons. If we calculate their relativistic energy, we get an imaginary result:

$$E=\frac{m\cdot c^2}{\sqrt{1-\frac{v^2}{c^2}}}=-i\cdot\frac{m\cdot c^2}{\sqrt{\frac{v^2}{c^2}-1}} \qquad v>c \tag{3.9.1}$$

Therefore we should define their mass imaginary, thus their energy and momentum will be real numbers. From an experimental point of view, we can do this, because there are no inertial frames faster than the speed of light where the tachyon could be at rest, thus its mass is not measurable.

$$m = i \cdot m' \qquad m' = -i \cdot m$$

$$E = \frac{m' \cdot c^2}{\sqrt{\frac{v^2}{c^2} - 1}} \qquad \vec{p} = \frac{m' \cdot \vec{v}}{\sqrt{\frac{v^2}{c^2} - 1}} \tag{3.9.2}$$

From this we can see, that in the case of tachyons, an increase in energy is followed by a decrease in velocity, and vice versa. The sign of the momentum four-vector changes:

$$\frac{E^2}{c^2} - p^2 = -m'^2 \cdot c^2 \tag{3.9.3}$$

The energy and momentum of an object slower than the speed of light, the energy can vary between the energy at rest and infinity, the momentum can take on any value:

$$0 \le v < c \qquad m \cdot c^2 \le E < \infty \qquad 0 \le p < \infty \tag{3.9.4}$$

The energy and momentum of an object faster than the speed of light, the energy can take on any value, however the momentum cannot decrease beyond a certain value, tachyons cannot slow down:

$$c < v \le \infty \qquad 0 \le E < \infty \qquad m' \cdot c \le p < \infty \tag{3.9.5}$$

Let us examine the velocity addition in the case of tachyons:

$$_2v_x = \frac{v_x - V_x}{1 - \frac{V_x \cdot v_x}{c^2}} \tag{3.9.6}$$

We can always find a reference frame, where the velocity of the faster-than-light object is infinite:

$$\frac{V_x \cdot v_x}{c^2} = 1$$

$$V_x = \frac{c^2}{v_x}$$ (since $v_x > c$, it is always true that $V_x < c$)

Reinserting this we can determine the limit of the velocity addition formula, where the velocity in it goes to infinity:

$$\lim_{v_x \to \infty} \frac{v_x - V_x}{1 - \frac{V_x \cdot v_x}{c^2}} = \frac{c^2}{V_x} \tag{3.9.7}$$

3.10 Circular motion and Thomas precession

We determine the proper acceleration of an object in constant circular motion, this is what it measures in its proper time, and also its coordinate acceleration, that is measured by the non-moving observer in coordinate time. Let us write down the arc length squared, metric tensor and its derivatives and the connection of the cylindrical coordinate system in flat spacetime.

$$ds^2 = c^2 \cdot dt^2 - dr^2 - r^2 \cdot d\varphi^2 - dz^2 \quad (3.10.1)$$

$$g_{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad g^{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (3.10.2)$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r} = -2 \cdot r \qquad \frac{\partial g^{\varphi\varphi}}{\partial r} = \frac{2}{r^3} \quad (3.10.3)$$

$$\Gamma^r{}_{\varphi\varphi} = -r \qquad \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r} \quad (3.10.4)$$

$$\frac{\partial \Gamma^r{}_{\varphi\varphi}}{\partial r} = -1 \qquad \frac{\partial \Gamma^\varphi{}_{r\varphi}}{\partial r} = \frac{\partial \Gamma^\varphi{}_{\varphi r}}{\partial r} = -\frac{1}{r^2} \quad (3.10.5)$$

Insert these values into the geodesic equation, where we partially differentiate, once according to proper time, and then according to coordinate time (we are allowed to do this, because both values increase monotonically).

$$\frac{\partial^2 x^j}{\partial \tau^2} + \Gamma^j{}_{ab} \cdot \frac{\partial x^a}{\partial \tau} \cdot \frac{\partial x^b}{\partial \tau} = 0 \qquad \frac{\partial^2 x^j}{\partial t^2} + \Gamma^j{}_{ab} \cdot \frac{\partial x^a}{\partial t} \cdot \frac{\partial x^b}{\partial t} = 0 \quad (3.10.6)$$

The equations have the same form in both cases, therefore we use the general dot notation for partial differentiation:

$$c \cdot \ddot{t} = 0$$

$$\ddot{r} + \Gamma^r{}_{\varphi\varphi} \cdot \dot{\varphi}^2 = \ddot{r} - r \cdot \dot{\varphi}^2 = 0$$

$$\ddot{\varphi} + \Gamma^\varphi{}_{r\varphi} \cdot \dot{r} \cdot \dot{\varphi} + \Gamma^\varphi{}_{\varphi r} \cdot \dot{\varphi} \cdot \dot{r} = \ddot{\varphi} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\varphi} = 0$$

$$\ddot{z} = 0 \quad (3.10.7)$$

The following coordinate conditions apply to objects in constant circular motion:

$$\dot{t}=0 \qquad \dot{r}=0 \qquad \dot{\varphi}=const. \qquad \dot{z}=0 \tag{3.10.8}$$

From this the centripetal acceleration:

$$\ddot{r}=r\cdot\dot{\varphi}^2 \tag{3.10.9}$$

Determine the relationship between the angular velocities according to the proper time and the coordinate time:

$$\frac{\partial^2 r}{\partial\tau^2}=r\cdot\left(\frac{\partial\varphi}{\partial\tau}\right)^2 \qquad \frac{\partial^2 r}{\partial t^2}=r\cdot\left(\frac{\partial\varphi}{\partial t}\right)^2 \tag{3.10.10}$$

Substitute the proper time into the second derivative of the radial coordinate:

$$d\,\tau=dt\cdot\sqrt{1-\frac{v^2}{c^2}}$$

$$\frac{\partial^2 r}{\partial t^2}\cdot\frac{1}{1-\frac{v^2}{c^2}}=r\cdot\left(\frac{\partial\varphi}{\partial\tau}\right)^2 \tag{3.10.11}$$

Reorder it, and make it equal to the formula written with the coordinate time:

$$\frac{\partial^2 r}{\partial t^2}=r\cdot\left(\frac{\partial\varphi}{\partial\tau}\right)^2\cdot\left(1-\frac{v^2}{c^2}\right)$$

$$r\cdot\left(\frac{\partial\varphi}{\partial t}\right)^2=r\cdot\left(\frac{\partial\varphi}{\partial\tau}\right)^2\cdot\left(1-\frac{v^2}{c^2}\right) \tag{3.10.12}$$

The relationship between the angular velocities using the proper time and the coordinate time:

$$\frac{\partial\varphi}{\partial\tau}=\frac{\partial\varphi}{\partial t}\cdot\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3.10.13}$$

The difference between the two quantities shows, that a non-rotating object moving on a circular orbit, after having completed a circle it will not face in the same direction as before, this phenomenon is called the Thomas precession:

$$\omega=\frac{\partial\varphi}{\partial\tau}-\frac{\partial\varphi}{\partial t}=\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1\right)\cdot\frac{\partial\varphi}{\partial t} \tag{3.10.14}$$

3.11 Gravitational redshift

On the world line of an object at rest relatively to the coordinate system only the coordinate time changes. The relationship with the proper time can be determined with the four-distance:

$$c^2\cdot\tau^2=c^2\cdot g_{tt}\cdot t^2 \tag{3.11.1}$$

The proper times of two different test objects:

$${}_1\tau=\sqrt{{}_1g_{tt}}\cdot t \qquad {}_2\tau=\sqrt{{}_2g_{tt}}\cdot t \tag{3.11.2}$$

The *tt* component of the metric tensor has to be positive, so that the direction of the proper time and the coordinate time coincides, and that the second assumption, the principle of the ordering between cause and effect is not violated. By substituting the coordinate time we can determine the relationship between the proper times:

$${}_1\tau=\sqrt{\frac{{}_1g_{tt}}{{}_2g_{tt}}}\cdot{}_2\tau \tag{3.11.3}$$

The frequency of light or any periodic phenomenon:

$$\nu=\frac{1}{\tau} \tag{3.11.4}$$

The gravitational redshift:

$${}_1\nu=\sqrt{\frac{{}_2g_{tt}}{{}_1g_{tt}}}\cdot{}_2\nu \tag{3.11.5}$$

4. Spherically symmetric spacetime

The most general spherically symmetric solution of the Einstein equations was derived by Karl Schwarzschild in 1916, not long after the initial discovery of those equations. This chapter deals with the matter free version of it. We map the spacetime of the Schwarzschild solution with various coordinate systems, and investigate the trajectories of moving bodies in it. We verify our results with observations from the Solar System.

Several well known phenomena get a new interpretation, once we use geometric methods to understand them, and unexpected new effects also occur. Since every phenomenon is a result of interplay between distances and angles, tampering with them has many, previously unknown impacts on the orbits of celestial bodies.

4.1 Spherically symmetric coordinate system

Let us set up a spherically symmetric coordinate system in flat spacetime, that we are going to use as a basis for the subsequent general derivation. The arc length squared is created by extending the arc length squared of the sphere with radial and time coordinates:

$$ds^2 = c^2 \cdot dt^2 - dr^2 - r^2 \cdot \left(d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2\right) \tag{4.1.1}$$

Determine the metric tensor, the connection, and their derivatives:

$$g_{\eta\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \cdot \sin^2(\vartheta) \end{pmatrix} \qquad g^{\eta\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2 \cdot \sin^2(\vartheta)} \end{pmatrix} \tag{4.1.2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r} = -2 \cdot r \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial r} = \frac{2}{r^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r} = -2 \cdot r \cdot \sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r} = \frac{2}{r^3 \cdot \sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial \vartheta} = -2 \cdot r^2 \cdot \cos(\vartheta) \cdot \sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial \vartheta} = \frac{2 \cdot \cos(\vartheta)}{r^2 \cdot \sin^3(\vartheta)} \tag{4.1.3}$$

$$\Gamma^r{}_{\vartheta\vartheta} = -r \qquad \Gamma^r{}_{\varphi\varphi} = -r \cdot \sin^2(\vartheta)$$

$$\Gamma^{\vartheta}{}_{r\vartheta}=\Gamma^{\vartheta}{}_{\vartheta r}=\Gamma^{\varphi}{}_{r\varphi}=\Gamma^{\varphi}{}_{\varphi r}=\frac{1}{r} \qquad \Gamma^{\vartheta}{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\cot(\vartheta) \tag{4.1.4}$$

$$\frac{\partial\Gamma^{r}{}_{\vartheta\vartheta}}{\partial r}=-1 \qquad \frac{\partial\Gamma^{r}{}_{\varphi\varphi}}{\partial r}=-\sin^2(\vartheta) \qquad \frac{\partial\Gamma^{r}{}_{\varphi\varphi}}{\partial\vartheta}=-2\cdot r\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial\Gamma^{\vartheta}{}_{r\vartheta}}{\partial r}=\frac{\partial\Gamma^{\vartheta}{}_{\vartheta r}}{\partial r}=\frac{\partial\Gamma^{\varphi}{}_{r\varphi}}{\partial r}=\frac{\partial\Gamma^{\varphi}{}_{\varphi r}}{\partial r}=-\frac{1}{r^2} \qquad \frac{\partial\Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial\vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta)$$

$$\frac{\partial\Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial\Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial\vartheta}=-\cot^2(\vartheta)-1 \tag{4.1.5}$$

Every component of the curvature tensor is zero, since the spacetime is flat. We write down the geodesic equations:

$$c\cdot\ddot{t}=0$$

$$\ddot{r}+\Gamma^{r}{}_{\vartheta\vartheta}\cdot\dot{\vartheta}^2+\Gamma^{r}{}_{\varphi\varphi}\cdot\dot{\varphi}^2=\ddot{r}-r\cdot\dot{\vartheta}^2-r\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2=0$$

$$\ddot{\vartheta}+\Gamma^{\vartheta}{}_{r\vartheta}\cdot\dot{r}\cdot\dot{\vartheta}+\Gamma^{\vartheta}{}_{\vartheta r}\cdot\dot{\vartheta}\cdot\dot{r}+\Gamma^{\vartheta}{}_{\varphi\varphi}\cdot\dot{\varphi}^2=\ddot{\vartheta}+\frac{2}{r}\cdot\dot{r}\cdot\dot{\vartheta}-\cos(\vartheta)\cdot\sin(\vartheta)\cdot\dot{\varphi}^2=0$$

$$\ddot{\varphi}+2\cdot\Gamma^{\varphi}{}_{r\varphi}\cdot\dot{r}\cdot\dot{\varphi}+2\cdot\Gamma^{\varphi}{}_{\vartheta\varphi}\cdot\dot{\vartheta}\cdot\dot{\varphi}=\ddot{\varphi}+\frac{2}{r}\cdot\dot{r}\cdot\dot{\varphi}+2\cdot\cot(\vartheta)\cdot\dot{\vartheta}\cdot\dot{\varphi}=0 \tag{4.1.6}$$

4.2 Schwarzschild coordinates

We start by simplifying the Einstein equations further:

$$R_{\eta\mu}-\frac{1}{2}\cdot R\cdot g_{\eta\mu}=0 \quad /\cdot g^{\eta\mu}$$

$$R-\frac{1}{2}\cdot R\cdot 4=0$$

$$R=0 \tag{4.2.1}$$

By reinserting this we get, that the Ricci-tensor is zero in vacuum:

$$R_{\eta\mu}=0 \tag{4.2.2}$$

We assume about the shape of the resulting spacetime, that at great distances from the source of gravitation, it approaches the flat spacetime. Since none of the coordinates change as a function of another, there are no mixed coordinate products. Therefore we use spherically symmetric coordinates to calculate the arc length squared, and we extend it with unknown functions, that depend only on the distance from the centre, and the elapsed time:

$$ds^2 = A(r,t)\cdot c^2\cdot dt^2 - B(r,t)\cdot dr^2 - C(r,t)\cdot r^2\cdot d\vartheta^2 - D(r,t)\cdot r^2\cdot \sin^2(\vartheta)\cdot d\varphi^2 \qquad (4.2.3)$$

The position of the axis of the coordinate system can be arbitrary, by adjusting it to our liking, we have one less unknown functions:

$$C(r,t) = D(r,t) \qquad (4.2.4)$$

Let us write an even more general form, where we allow the radial and time coordinates to mix:

$$ds^2 = f^2\cdot c^2\cdot_{\#} dt^2 + 2\cdot f\cdot g\cdot c\cdot dt\cdot dr - h^2\cdot dr^2 - C\cdot r^2\cdot(d\vartheta^2 + \sin^2(\vartheta)\cdot d\varphi^2) \qquad (4.2.5)$$

This however can be led back to the diagonal form, using substitutions where we rescale the time coordinate:

$$A\cdot c^2\cdot dt^2 = (f\cdot c\cdot_{\#} dt + g\cdot dr)^2 \qquad\qquad B = g^2 + h^2$$

$$ds^2 = A(r,t)\cdot c^2\cdot dt^2 - B(r,t)\cdot dr^2 - C(r,t)\cdot r^2\cdot(d\vartheta^2 + \sin^2(\vartheta)\cdot d\varphi^2) \qquad (4.2.6)$$

Since the radial coordinate can also be arbitrarily rescaled, arbitrary relationships can be established between the remaining functions:

Schwarzschild coordinates: $C(r,t) = 1$
displays distances perpendicular to the radial direction undistorted

Isotropic coordinates: $B(r,t) = C(r,t)$
displays directions undistorted

Gaussian polar coordinates: $B(r,t) = 1$
displays distances parallel to the radial direction undistorted

Co-moving coordinates: $A(r,t) = 1$
coordinates of radially falling bodies are constant (4.2.7)

The arc length squared in Schwarzschild coordinates, we determine the geometric quantities that characterize the surface, from the metric tensor to the Ricci-tensor:

$$ds^2 = A(r,t)\cdot c^2\cdot dt^2 - B(r,t)\cdot dr^2 - r^2\cdot d\vartheta^2 - r^2\cdot \sin^2(\vartheta)\cdot d\varphi^2 \qquad (4.2.8)$$

$$g_{\eta\mu} = \begin{pmatrix} A(r,t) & 0 & 0 & 0 \\ 0 & -B(r,t) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\cdot\sin^2(\vartheta) \end{pmatrix}$$

$$g^{\eta\mu}=\begin{pmatrix} \frac{1}{A(r,t)} & 0 & 0 & 0 \\ 0 & -\frac{1}{B(r,t)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\cdot\sin^2(\vartheta)} \end{pmatrix} \tag{4.2.9}$$

We symbolize partial differentiation with respect to time with an upper point, and with respect to space with an upper apostrophe:

$$\frac{\partial g_{tt}}{\partial t}=\dot{A} \qquad \frac{\partial g^{tt}}{\partial t}=-\frac{\dot{A}}{A^2}$$

$$\frac{\partial g_{rr}}{\partial t}=-\dot{B} \qquad \frac{\partial g^{rr}}{\partial t}=\frac{\dot{B}}{B^2}$$

$$\frac{\partial g_{tt}}{\partial r}=A' \qquad \frac{\partial g^{tt}}{\partial r}=-\frac{A'}{A^2}$$

$$\frac{\partial g_{rr}}{\partial r}=-B' \qquad \frac{\partial g^{rr}}{\partial r}=\frac{B'}{B^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r}=-2\cdot r \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial r}=\frac{2}{r^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=-2\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r}=\frac{2}{r^3\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-2\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot\cos(\vartheta)}{r^2\cdot\sin^3(\vartheta)} \tag{4.2.10}$$

$$\Gamma^t{}_{tt}=\frac{\dot{A}}{2\cdot A} \qquad \Gamma^t{}_{tr}=\Gamma^t{}_{rt}=\frac{A'}{2\cdot A} \qquad \Gamma^t{}_{rr}=\frac{\dot{B}}{2\cdot A}$$

$$\Gamma^r{}_{tt}=\frac{A'}{2\cdot B} \qquad \Gamma^r{}_{tr}=\Gamma^r{}_{rt}=\frac{\dot{B}}{2\cdot B} \qquad \Gamma^r{}_{rr}=\frac{B'}{2\cdot B}$$

$$\Gamma^r{}_{\vartheta\vartheta}=-\frac{r}{B} \qquad \Gamma^r{}_{\varphi\varphi}=-\frac{r\cdot\sin^2(\vartheta)}{B}$$

$$\Gamma^{\vartheta}{}_{r\vartheta}=\Gamma^{\vartheta}{}_{\vartheta r}=\Gamma^{\varphi}{}_{r\varphi}=\Gamma^{\varphi}{}_{\varphi r}=\frac{1}{r} \qquad \Gamma^{\vartheta}{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\cot(\vartheta) \tag{4.2.11}$$

$$\frac{\partial\Gamma^{t}{}_{tt}}{\partial t}=\frac{A\cdot\ddot{A}-\dot{A}^2}{2\cdot A^2} \qquad \frac{\partial\Gamma^{t}{}_{tr}}{\partial t}=\frac{\partial\Gamma^{t}{}_{rt}}{\partial t}=\frac{\partial\Gamma^{t}{}_{tt}}{\partial r}=\frac{A\cdot\dot{A}'-\dot{A}\cdot A'}{2\cdot A^2}$$

$$\frac{\partial\Gamma^{t}{}_{rr}}{\partial t}=\frac{A\cdot\ddot{B}-\dot{A}\cdot\dot{B}}{2\cdot A^2} \qquad \frac{\partial\Gamma^{r}{}_{tt}}{\partial t}=\frac{B\cdot\dot{A}'-A'\cdot\dot{B}}{2\cdot B^2}$$

$$\frac{\partial\Gamma^{r}{}_{tr}}{\partial t}=\frac{\partial\Gamma^{r}{}_{rt}}{\partial t}=\frac{B\cdot\ddot{B}-\dot{B}^2}{2\cdot B^2} \qquad \frac{\partial\Gamma^{r}{}_{rr}}{\partial t}=\frac{\partial\Gamma^{r}{}_{tr}}{\partial r}=\frac{\partial\Gamma^{r}{}_{rt}}{\partial r}=\frac{B\cdot\dot{B}'-\dot{B}\cdot B'}{2\cdot B^2}$$

$$\frac{\partial\Gamma^{r}{}_{\vartheta\vartheta}}{\partial t}=\frac{r\cdot\dot{B}}{B^2} \qquad \frac{\partial\Gamma^{r}{}_{\varphi\varphi}}{\partial t}=\frac{r\cdot\dot{B}\cdot\sin^2(\vartheta)}{B^2}$$

$$\frac{\partial\Gamma^{t}{}_{tr}}{\partial r}=\frac{\partial\Gamma^{t}{}_{rt}}{\partial r}=\frac{A\cdot A''-A'^2}{2\cdot A^2} \qquad \frac{\partial\Gamma^{t}{}_{rr}}{\partial r}=\frac{A\cdot\dot{B}'-A'\cdot\dot{B}}{2\cdot A^2}$$

$$\frac{\partial\Gamma^{r}{}_{tt}}{\partial r}=\frac{B\cdot A''-A'\cdot B'}{2\cdot B^2} \qquad \frac{\partial\Gamma^{r}{}_{rr}}{\partial r}=\frac{B\cdot B''-B'^2}{2\cdot B^2}$$

$$\frac{\partial\Gamma^{r}{}_{\vartheta\vartheta}}{\partial r}=\frac{r\cdot B'-B}{B^2} \qquad \frac{\partial\Gamma^{r}{}_{\varphi\varphi}}{\partial r}=\frac{(r\cdot B'-B)\cdot\sin^2(\vartheta)}{B^2}$$

$$\frac{\partial\Gamma^{r}{}_{\varphi\varphi}}{\partial\vartheta}=-\frac{2\cdot r}{B}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial\Gamma^{\vartheta}{}_{r\vartheta}}{\partial r}=\frac{\partial\Gamma^{\vartheta}{}_{\vartheta r}}{\partial r}=\frac{\partial\Gamma^{\varphi}{}_{r\varphi}}{\partial r}=\frac{\partial\Gamma^{\varphi}{}_{\varphi r}}{\partial r}=-\frac{1}{r^2}$$

$$\frac{\partial\Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial\vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta) \qquad \frac{\partial\Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial\Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial\vartheta}=-\cot^2(\vartheta)-1 \tag{4.2.12}$$

$$R^{t}{}_{rtr}=-R^{t}{}_{rrt}=\frac{\ddot{B}-A''}{2\cdot A}-\frac{\dot{B}^2-A'\cdot B'}{4\cdot A\cdot B}+\frac{A'^2-\dot{A}\cdot\dot{B}}{4\cdot A^2}$$

$$R^{t}{}_{\vartheta t\vartheta}=-R^{t}{}_{\vartheta\vartheta t}=-\frac{r\cdot A'}{2\cdot A\cdot B} \qquad R^{t}{}_{\vartheta r\vartheta}=-R^{t}{}_{\vartheta\vartheta r}=-\frac{r\cdot\dot{B}}{2\cdot A\cdot B}$$

$$R^{t}{}_{\varphi t\varphi}=-R^{t}{}_{\varphi\varphi t}=-\frac{r\cdot A'}{2\cdot A\cdot B}\cdot\sin^2(\vartheta) \qquad R^{t}{}_{\varphi r\varphi}=-R^{t}{}_{\varphi\varphi r}=-\frac{r\cdot\dot{B}}{2\cdot A\cdot B}\cdot\sin^2(\vartheta)$$

$$R^{r}{}_{ttr}=-R^{r}{}_{trt}=\frac{\ddot{B}-A''}{2\cdot B}-\frac{\dot{B}^2-A'\cdot B'}{4\cdot B^2}+\frac{A'^2-\dot{A}\cdot\dot{B}}{4\cdot A\cdot B}$$

$$R^{r}{}_{\vartheta t\vartheta}=-R^{r}{}_{\vartheta\vartheta t}=\frac{r\cdot\dot{B}}{2\cdot B^2} \qquad R^{r}{}_{\vartheta r\vartheta}=-R^{r}{}_{\vartheta\vartheta r}=\frac{r\cdot B'}{2\cdot B^2}$$

$$R^{r}{}_{\varphi t\varphi}=-R^{r}{}_{\varphi\varphi t}=\frac{r\cdot\dot{B}}{2\cdot B^2}\cdot\sin^2(\vartheta) \qquad R^{r}{}_{\varphi r\varphi}=-R^{r}{}_{\varphi\varphi r}=\frac{r\cdot B'}{2\cdot B^2}\cdot\sin^2(\vartheta)$$

$$R^{\vartheta}{}_{tt\vartheta}=-R^{\vartheta}{}_{t\vartheta t}=R^{\varphi}{}_{tt\varphi}=-R^{\varphi}{}_{t\varphi t}=-\frac{A'}{2\cdot r\cdot B} \qquad R^{\vartheta}{}_{rr\vartheta}=-R^{\vartheta}{}_{r\vartheta r}=R^{\varphi}{}_{rr\varphi}=-R^{\varphi}{}_{r\varphi r}=-\frac{B'}{2\cdot r\cdot B}$$

$$R^{\vartheta}{}_{tr\vartheta}=-R^{\vartheta}{}_{t\vartheta r}=R^{\vartheta}{}_{rt\vartheta}=-R^{\vartheta}{}_{r\vartheta t}=R^{\varphi}{}_{tr\varphi}=-R^{\varphi}{}_{t\varphi r}=R^{\varphi}{}_{rt\varphi}=-R^{\varphi}{}_{r\varphi t}=-\frac{\dot{B}}{2\cdot r\cdot B}$$

$$R^{\vartheta}{}_{\varphi\vartheta\varphi}=-R^{\vartheta}{}_{\varphi\varphi\vartheta}=\left(1-\frac{1}{B}\right)\cdot\sin^2(\vartheta) \qquad R^{\varphi}{}_{\vartheta\varphi\vartheta}=-R^{\varphi}{}_{\vartheta\vartheta\varphi}=\left(1-\frac{1}{B}\right) \qquad (4.2.13)$$

$$R_{tt}=\frac{\dot{B}}{4\cdot B}\cdot\left(\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)-\frac{A'}{4\cdot B}\cdot\left(\frac{B'}{B}+\frac{A'}{A}\right)+\frac{A''-\ddot{B}}{2\cdot B}+\frac{A'}{r\cdot B}$$

$$R_{rr}=-\frac{\dot{B}}{4\cdot A}\cdot\left(\frac{\dot{B}}{B}+\frac{\dot{A}}{A}\right)+\frac{A'}{4\cdot A}\cdot\left(\frac{B'}{B}+\frac{A'}{A}\right)-\frac{A''-\ddot{B}}{2\cdot A}+\frac{B'}{r\cdot B}$$

$$R_{\vartheta\vartheta}=\frac{r\cdot B'}{2\cdot B^2}-\frac{r\cdot A'}{2\cdot A\cdot B}-\frac{1}{B}-1$$

$$R_{\varphi\varphi}=\left(\frac{r\cdot B'}{2\cdot B^2}-\frac{r\cdot A'}{2\cdot A\cdot B}-\frac{1}{B}-1\right)\cdot\sin^2(\vartheta)$$

$$R_{tr}=R_{rt}=\frac{\dot{B}}{r\cdot B} \qquad (4.2.14)$$

We derived from the Einstein equations, that in vacuum the Ricci-tensor is zero. The non-diagonal term shows, that the derivative of the *B* function with respect to time is zero:

$$R_{tr}=R_{rt}=\frac{\dot{B}}{r\cdot B}=0$$

$$\dot{B}=0 \qquad (4.2.15)$$

Reinsert this into the *tt* and *rr* components of the Ricci-tensor. Simplify the *tt* component:

$$R_{tt}=-\frac{A'}{4\cdot B}\cdot\left(\frac{B'}{B}+\frac{A'}{A}\right)+\frac{A''}{2\cdot B}+\frac{A'}{r\cdot B}=0 \quad /\cdot B$$

$$-\frac{A'}{4}\cdot\left(\frac{B'}{B}+\frac{A'}{A}\right)+\frac{A''}{2}+\frac{A'}{r}=0 \qquad (4.2.16)$$

Simplify the *rr* component also:

$$R_{rr}=\frac{A'}{4\cdot A}\cdot\left(\frac{B'}{B}+\frac{A'}{A}\right)-\frac{A''}{2\cdot A}+\frac{B'}{r\cdot B}=0 \quad /\cdot A$$

$$\frac{A'}{4}\cdot\left(\frac{B'}{B}+\frac{A'}{A}\right)-\frac{A''}{2}+\frac{B'\cdot A}{r\cdot B}=0 \qquad (4.2.17)$$

Add the two equations together:

$$\frac{B'\cdot A}{r\cdot B}+\frac{A'}{r}=0$$

$$\frac{B'}{B}+\frac{A'}{A}=0 \qquad (4.2.18)$$

Reinsert the result into the equation coming from the *tt* component, where now we have just the derivatives of the *A* function. With substitution we reduce the degree of derivatives:

$$\frac{A''}{2}+\frac{A'}{r}=0 \qquad f(r)=A' \qquad (4.2.19)$$

$$\frac{df}{dr}+\frac{2\cdot f}{r}=0$$

$$\int\frac{df}{f}=-2\cdot\int\frac{dr}{r}$$

$$\log(f)=-2\cdot\log(r)\cdot c_1=\log\left(\frac{c_1}{r^2}\right)$$

We raise to natural power both sides of the equation, and reinsert the original function:

$$f=\frac{c_1}{r^2}=A'=\frac{dA}{dr} \qquad (4.2.20)$$

$$\int\frac{c_1}{r^2}\cdot dr=\int dA$$

Thus the *A* function is also time independent:

$$-\frac{c_1}{r}+c_2=A \qquad (4.2.21)$$

Let us examine the formula, that we got when we added the two equations we made from the Ricci-tensor components:

$$\frac{B'}{B}+\frac{A'}{A}=0$$

With a little reordering we can show, that the product of the two functions is a constant:

$$F'=A\cdot B'+A'\cdot B=0 \qquad \rightarrow \qquad F=A\cdot B=c_3 \tag{4.2.22}$$

From this the other function:

$$B=c_3\cdot\left(\frac{1}{c_2-\frac{c_1}{r}}\right)$$

We can reinterpret the unknown variables in a way, that the third is understood as part of the first two, in this case the two unknown functions are mutually reciprocals of each other:

$$A=c_2-\frac{c_1}{r} \qquad B=\frac{1}{c_2-\frac{c_1}{r}} \qquad c_3=1 \tag{4.2.23}$$

The first constant is a quantity characteristic for the spherically symmetric spacetime, it is the Schwarzschild radius. In this distance from the centre there is a coordinate singularity, and its physical meaning can be found with Newtonian approximation:

$$r_g=c_1 \tag{4.2.24}$$

The value for the second constant can be recovered from the condition, that the Schwarzschild solution at great distances shall approach the flat spacetime, in other words *A* and *B* in the arc length squared shall approach 1:

$$\lim_{r\rightarrow\infty} A=c_2-\lim_{r\rightarrow\infty}\frac{c_1}{r}=c_2-0=1 \tag{4.2.25}$$

Because of the time independence of the metric, if a celestial body suffers radial changes, but does not receive or lose mass, the geometry of the surrounding spacetime will not change. For example a spherically symmetric pulsating star does not create gravitational radiation, neither a symmetric supernova explosion nor a celestial body collapsing to a black hole. Because of the slow rotation of the Sun and the minimal contribution of the planets, the spacetime of the Solar System is Schwarzschild to a great accuracy. The arc length squared and the other geometric quantities in the Schwarzschild metric:

$$ds^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-\frac{dr^2}{1-\frac{r_g}{r}}-r^2\cdot\left(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2\right) \tag{4.2.26}$$

$$g_{\eta\mu}=\begin{pmatrix} 1-\frac{r_g}{r} & 0 & 0 & 0 \\ 0 & -\frac{1}{1-\frac{r_g}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\cdot\sin^2(\vartheta) \end{pmatrix}$$

$$g^{\eta\mu}=\begin{pmatrix} \frac{1}{1-\frac{r_g}{r}} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{r_g}{r}\right) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\cdot\sin^2(\vartheta)} \end{pmatrix} \qquad (4.2.27)$$

$$\frac{\partial g_{tt}}{\partial r}=-\frac{\partial g^{rr}}{\partial r}=\frac{r_g}{r^2} \qquad \frac{\partial g_{rr}}{\partial r}=-\frac{\partial g^{tt}}{\partial r}=\frac{r_g}{r^2\cdot\left(1-\frac{r_g}{r}\right)^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r}=-2\cdot r \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial r}=\frac{2}{r^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=-2\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r}=\frac{2}{r^3\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-2\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot\cos(\vartheta)}{r^2\cdot\sin^3(\vartheta)} \qquad (4.2.28)$$

$$\Gamma^t{}_{tr}=\Gamma^t{}_{rt}=-\Gamma^r{}_{rr}=\frac{r_g}{2\cdot r\cdot(r-r_g)} \qquad \Gamma^r{}_{tt}=\frac{r_g\cdot(r-r_g)}{2\cdot r^3}$$

$$\Gamma^r{}_{\vartheta\vartheta}=-(r-r_g) \qquad \Gamma^r{}_{\varphi\varphi}=-(r-r_g)\cdot\sin^2(\vartheta)$$

$$\Gamma^\vartheta{}_{r\vartheta}=\Gamma^\vartheta{}_{\vartheta r}=\Gamma^\varphi{}_{r\varphi}=\Gamma^\varphi{}_{\varphi r}=\frac{1}{r} \qquad \Gamma^\vartheta{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^\varphi{}_{\vartheta\varphi}=\Gamma^\varphi{}_{\varphi\vartheta}=\cot(\vartheta) \qquad (4.2.29)$$

$$\frac{\partial \Gamma^{t}{}_{tr}}{\partial r}=\frac{\partial \Gamma^{t}{}_{rt}}{\partial r}=-\frac{\partial \Gamma^{r}{}_{rr}}{\partial r}=-\frac{r_g\cdot(2\cdot r-r_g)}{2\cdot r^2\cdot(r-r_g)^2}$$

$$\frac{\partial \Gamma^{r}{}_{tt}}{\partial r}=-\frac{r_g\cdot(2\cdot r-3\cdot r_g)}{2\cdot r^4} \qquad \frac{\partial \Gamma^{r}{}_{\vartheta\vartheta}}{\partial r}=-1 \qquad \frac{\partial \Gamma^{r}{}_{\varphi\varphi}}{\partial r}=-\sin^2(\vartheta)$$

$$\frac{\partial \Gamma^{\vartheta}{}_{r\vartheta}}{\partial r}=\frac{\partial \Gamma^{\vartheta}{}_{\vartheta r}}{\partial r}=\frac{\partial \Gamma^{\varphi}{}_{r\varphi}}{\partial r}=\frac{\partial \Gamma^{\varphi}{}_{\varphi r}}{\partial r}=-\frac{1}{r^2}$$

$$\frac{\partial \Gamma^{r}{}_{\varphi\varphi}}{\partial \vartheta}=-2\cdot(r-r_g)\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial \Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial \vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta)$$

$$\frac{\partial \Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial \vartheta}=\frac{\partial \Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial \vartheta}=-\cot^2(\vartheta)-1 \tag{4.2.30}$$

$$R^{t}{}_{rtr}=-R^{t}{}_{rrt}=\frac{r_g}{r^2\cdot(r-r_g)}$$

$$R^{t}{}_{\vartheta t\vartheta}=-R^{t}{}_{\vartheta\vartheta t}=R^{r}{}_{\vartheta r\vartheta}=-R^{r}{}_{\vartheta\vartheta r}=-\frac{r_g}{2\cdot r}$$

$$R^{t}{}_{\varphi t\varphi}=-R^{t}{}_{\varphi\varphi t}=R^{r}{}_{\varphi r\varphi}=-R^{r}{}_{\varphi\varphi r}=-\frac{r_g}{2\cdot r}\cdot\sin^2(\vartheta)$$

$$R^{r}{}_{trt}=-R^{r}{}_{ttr}=\frac{r_g\cdot(r-r_g)}{r^4}$$

$$R^{\vartheta}{}_{t\vartheta t}=-R^{\vartheta}{}_{tt\vartheta}=R^{\varphi}{}_{t\varphi t}=-R^{\varphi}{}_{tt\varphi}=\frac{r_g\cdot(r-r_g)}{2\cdot r^4}$$

$$R^{\vartheta}{}_{r\vartheta r}=-R^{\vartheta}{}_{rr\vartheta}=R^{\varphi}{}_{r\varphi r}=-R^{\varphi}{}_{rr\varphi}=-\frac{r_g}{2\cdot r^2\cdot(r-r_g)}$$

$$R^{\vartheta}{}_{\varphi\vartheta\varphi}=-R^{\vartheta}{}_{\varphi\varphi\vartheta}=\frac{r_g}{r}\cdot\sin^2(\vartheta)$$

$$R^{\varphi}{}_{\vartheta\varphi\vartheta}=-R^{\varphi}{}_{\vartheta\vartheta\varphi}=\frac{r_g}{r} \tag{4.2.31}$$

This spacetime is a hypersurface of a six dimensional, flat, pseudo-euclidean space with a signature of (+ + – – – –). The parametric form:

$$x^1=\sqrt{1-\frac{r_g}{r}}\cdot\cos(c\cdot t) \qquad x^2=\sqrt{1-\frac{r_g}{r}}\cdot\sin(c\cdot t)$$

$$x^3 = \int \sqrt{\frac{r_g}{r-r_g} \cdot \left(\frac{r_g}{4 \cdot r^3} + 1 \right)} \cdot dr \qquad x^4 = r \cdot \sin(\vartheta) \cdot \cos(\varphi)$$

$$x^5 = r \cdot \sin(\vartheta) \cdot \sin(\varphi) \qquad x^6 = r \cdot \cos(\vartheta) \tag{4.2.32}$$

4.3 Geodesic equations

Substitute the connection coefficients of the Schwarzschild solution into the geodesic equations:

$$c \cdot \ddot{t} + 2 \cdot \Gamma^t{}_{tr} \cdot c \cdot \dot{t} \cdot \dot{r} = 0$$

$$\ddot{t} + \frac{r_g}{r \cdot (r-r_g)} \cdot \dot{t} \cdot \dot{r} = 0 \tag{4.3.1}$$

$$\ddot{r} + \Gamma^r{}_{tt} \cdot c^2 \cdot \dot{t}^2 + \Gamma^r{}_{rr} \cdot \dot{r}^2 + \Gamma^r{}_{\vartheta\vartheta} \cdot \dot{\vartheta}^2 + \Gamma^r{}_{\varphi\varphi} \cdot \dot{\varphi}^2 = 0$$

$$\ddot{r} + \frac{r_g \cdot (r-r_g)}{2 \cdot r^3} \cdot c^2 \cdot \dot{t}^2 - \frac{r_g}{2 \cdot r \cdot (r-r_g)} \cdot \dot{r}^2 - (r-r_g) \cdot \dot{\vartheta}^2 - (r-r_g) \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 = 0 \tag{4.3.2}$$

$$\ddot{\vartheta} + 2 \cdot \Gamma^\vartheta{}_{r\vartheta} \cdot \dot{r}\,\dot{\vartheta} + \Gamma^\vartheta{}_{\varphi\varphi} \cdot \dot{\varphi}^2 = 0$$

$$\ddot{\vartheta} + \frac{2}{r} \cdot \dot{r}\,\dot{\vartheta} - \cos(\vartheta) \cdot \sin(\vartheta) \cdot \dot{\varphi}^2 = 0 \tag{4.3.3}$$

$$\ddot{\varphi} + 2 \cdot \Gamma^\varphi{}_{r\varphi} \cdot \dot{r}\,\dot{\varphi} + 2 \cdot \Gamma^\varphi{}_{\vartheta\varphi} \cdot \dot{\vartheta} \cdot \dot{\varphi} = 0$$

$$\ddot{\varphi} + \frac{2}{r} \cdot \dot{r}\,\dot{\varphi} + 2 \cdot \cot(\vartheta) \cdot \dot{\vartheta} \cdot \dot{\varphi} = 0 \tag{4.3.4}$$

Let us investigate the third equation. If we orientate the coordinate system in such a way, that the test body is in the equatorial plane of the coordinate system, and the direction of the motion falls into this plane, then it will also stay in this plane. Substitute the longitudinal angle and the momentarily zero angular velocity along this coordinate into the third geodesic equation:

$$\vartheta = \frac{\pi}{2} \qquad \dot{\vartheta} = 0$$

$$\ddot{\vartheta} + \frac{2}{r} \cdot \dot{r}\,\dot{\vartheta} - \cos(\vartheta) \cdot \sin(\vartheta) \cdot \dot{\varphi}^2 = \ddot{\vartheta} + \frac{2}{r} \cdot \dot{r}\,0 - 0 \cdot 1 \cdot \dot{\varphi}^2 = 0$$

The longitudinal angular acceleration is zero, the test body stays in the equatorial plane:

$$\ddot{\vartheta}=0 \tag{4.3.5}$$

Substitute into the other geodesic equations as well:

$$\ddot{t}+\frac{r_g}{r\cdot(r-r_g)}\cdot\dot{t}\cdot\dot{r}=0 \tag{4.3.6}$$

$$\ddot{r}+\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2\cdot\dot{t}^2-\frac{r_g}{2\cdot r\cdot(r-r_g)}\cdot\dot{r}^2-(r-r_g)\cdot\dot{\varphi}^2=0 \tag{4.3.7}$$

$$\ddot{\vartheta}=0 \tag{4.3.8}$$

$$\ddot{\varphi}+\frac{2}{r}\cdot\dot{r}\,\dot{\varphi}=0 \tag{4.3.9}$$

4.4 Gravitational redshift

This phenomenon is one of the classical evidences for general relativity. Einstein described it already in 1907, based on the principle of equivalence, but initially he did not think it was possible to measure it experimentally. The experiment was eventually performed by R. V. Pound and G. A. Rebka, in 1959 in the laboratory of Harward University, in the United States. The measured red shift of the gamma rays emitted by radioactive iron atoms, and directed 22.5 meters upwards to the detectors confirmed Einstein's prediction within the 10% error margin. Later the error margin has been reduced to less than 1% using hydrogen masers.

We substitute the Schwarzschild metric tensor components into the earlier formula:

$$_1\nu=\sqrt{\frac{_2g_{tt}}{_1g_{tt}}}\cdot{}_2\nu=\sqrt{\frac{1-\frac{r_g}{_2r}}{1-\frac{r_g}{_1r}}}\cdot{}_2\nu \tag{4.4.1}$$

If the light source is closer to the source of the gravitational field than the observer, then:

$$_1r\geq{}_2r \qquad\rightarrow\qquad {}_1\nu\leq{}_2\nu \tag{4.4.2}$$

Thus the observed frequency is higher than the emitted, the radiation in the visible spectrum is shifted towards the red, this is where the name of the phenomenon comes from. Redshift observed by an observer at a great distance, where the light source is at *r* distance from the centre of gravitation (like the surface of a star):

$$z=\frac{1}{\sqrt{1-\frac{r_g}{r}}}-1 \tag{4.4.3}$$

4.5 Wormhole

In order to demonstrate the shape of the spacetime, we investigate the properties of a coordinate surface. The parameters of the surface:

$$t=const. \qquad \vartheta=\frac{\pi}{2}$$

$$dt=0 \qquad d\,\vartheta=0 \tag{4.5.1}$$

Substitute them into the Schwarzschild arc length squared:

$$ds^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot 0^2-\frac{dr^2}{1-\frac{r_g}{r}}-r^2\cdot\left(0^2+\sin^2\left(\frac{\pi}{2}\right)\cdot d\,\varphi^2\right)$$

$$-ds^2=dl^2=\frac{dr^2}{1-\frac{r_g}{r}}+r^2\cdot d\,\varphi^2 \tag{4.5.2}$$

The characteristic geometric quantities on the surface:

$$g_{ij}=\begin{pmatrix}\frac{1}{1-\frac{r_g}{r}} & 0\\ 0 & r^2\end{pmatrix} \qquad g^{ij}=\begin{pmatrix}1-\frac{r_g}{r} & 0\\ 0 & \frac{1}{r^2}\end{pmatrix} \tag{4.5.3}$$

$$\frac{\partial g_{rr}}{\partial r}=-\frac{r_g}{r^2\cdot\left(1-\frac{r_g}{r}\right)^2} \qquad \frac{\partial g^{rr}}{\partial r}=\frac{r_g}{r^2}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=2\cdot r \qquad \frac{\partial g^{\varphi\varphi}}{\partial r}=-\frac{2}{r^3} \tag{4.5.4}$$

$$\Gamma^r{}_{rr}=-\frac{r_g}{2\cdot r\cdot(r-r_g)} \qquad \Gamma^r{}_{\varphi\varphi}=-(r-r_g) \qquad \Gamma^\varphi{}_{r\varphi}=\Gamma^\varphi{}_{\varphi r}=\frac{1}{r} \tag{4.5.5}$$

$$\frac{\partial \Gamma^r{}_{rr}}{\partial r}=\frac{r_g\cdot(2\cdot r-r_g)}{2\cdot r\cdot(r-r_g)^2} \qquad \frac{\partial \Gamma^r{}_{\varphi\varphi}}{\partial r}=-1 \qquad \frac{\partial \Gamma^\varphi{}_{r\varphi}}{\partial r}=\frac{\partial \Gamma^\varphi{}_{\varphi r}}{\partial r}=-\frac{1}{r^2} \tag{4.5.6}$$

$$R_{ij}=\begin{pmatrix} -\frac{r_g}{2\cdot r^2\cdot(r-r_g)} & 0 \\ 0 & -\frac{r_g}{2\cdot r} \end{pmatrix} \tag{4.5.7}$$

$$R=2\cdot K=-\frac{r_g}{r^3} \tag{4.5.8}$$

This coordinate surface can be embedded into the flat three dimensional space, thus its easy to visualize. We set up a spherical coordinate system, and spread the surface in it. The arc length squared:

$$dl^2=dr^2+r^2\cdot d\,\varphi^2+dz^2$$

Make it equal with the arc length squared measured on the surface:

$$dr^2+r^2\cdot d\,\varphi^2+dz^2=\frac{dr^2}{1-\frac{r_g}{r}}+r^2\cdot d\,\varphi^2$$

$$dr^2+dz^2=\frac{dr^2}{1-\frac{r_g}{r}} \tag{4.5.9}$$

We do not know the relationship that describes the z coordinates of the surface, therefore we substitute it as an unknown function, and since the surface inherited circular symmetry from the Schwarzschild metric, we assume that it depends on the radius only:

$$z=f(r)$$

$$dz=f'(r)\cdot dr$$

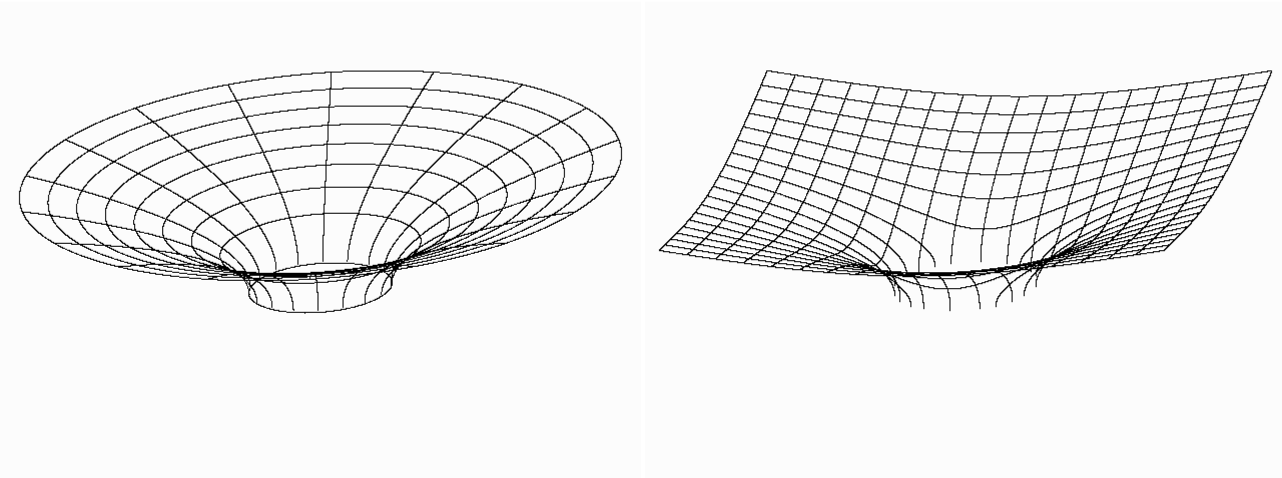
$$(1+f'^2(r))\cdot dr^2=\frac{dr^2}{1-\frac{r_g}{r}}$$

$$f'(r)=\sqrt{\frac{r_g}{r-r_g}} \tag{4.5.10}$$

Perform the integration. The shape of the entry of the wormhole:

$$f(r) = 2 \cdot \sqrt{r_g \cdot (r - r_g)} + c \qquad (4.5.11)$$

The surface with angular and rectangular coordinates:



4.6 Newtonian approximation

According to the correspondence principle, the new physical laws in limiting cases must approximate the old ones, this is no different in the case of relativity theory. In this case the previous model is the Newtonian absolute space and time, and the forces and emerging potentials acting in it. In order to establish the correspondence, we must formulate the two gravitational theories in the same language. Classical mechanics does not work with geometric methods, but with diverse terms instead, like the force, and other quantities derived from it. Using this set of tools, it is possible to describe only a limited set of gravitational phenomena, but within its limit, it provides correct results. Since this can be experimentally verified, the broadly valid geometry based theory has to approach the Newtonian model within the mentioned limits. The validity of the former is limited to those situations, where movements are much slower than the speed of light, and the proper time coincides with the coordinate time:

$$v \ll c \qquad d\tau \approx dt \qquad (4.6.1)$$

The Lagrange function summarizes the properties of a dynamical system. From it the equations of movement can be derived using the action principle. With the action functional in the non-relativistic case, the following relationship is satisfied:

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}, t) \cdot dt \qquad (4.6.2)$$

According to the action principle, the evolution of a mechanical system is characterized by the solution of the following functional equation:

$$\frac{\delta S}{\delta x(t)}=0 \tag{4.6.3}$$

Let $x(t)$ be the function describing the possible evolution of the system. In this case $\varepsilon(t)$ is an infinitesimally small variation on it, that is zero in the starting and ending points, this is our boundary condition:

$$\varepsilon(t_1)=\varepsilon(t_2)=0 \tag{4.6.4}$$

Use it to vary the action functional, we assume that the Lagrange function does not depend on time:

$$\delta S=\int_{t_1}^{t_2}(L(x+\varepsilon,\dot{x}+\dot{\varepsilon})-L(x,\dot{x}))\cdot dt \tag{4.6.5}$$

Write down the Taylor series of the Lagrange function, and we write down the variation of the action functional again, using the first order terms:

$$L(x+\varepsilon,\dot{x}+\dot{\varepsilon})=L(x,\dot{x})+\varepsilon_i\cdot\frac{\partial L}{\partial x^i}+\dot{\varepsilon}_i\cdot\frac{\partial L}{\partial \dot{x}^i}$$

$$\delta S_i=\int_{t_1}^{t_2}\left(\varepsilon_i\cdot\frac{\partial L}{\partial x^i}+\dot{\varepsilon}_i\cdot\frac{\partial L}{\partial \dot{x}^i}\right)\cdot dt \tag{4.6.6}$$

Partially integrate the second term:

$$\delta S_i=\int_{t_1}^{t_2}\dot{\varepsilon}_i\cdot\frac{\partial L}{\partial \dot{x}^i}\cdot dt=\left[\varepsilon_i(t)\cdot\frac{\partial L}{\partial \dot{x}^i}\right]_{t_1}^{t_2}-\int_{t_1}^{t_2}\varepsilon_i\cdot\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\cdot dt$$

Reinsert it into the variation equation:

$$\delta S_i=\left[\varepsilon_i(t)\cdot\frac{\partial L}{\partial \dot{x}^i}\right]_{t_1}^{t_2}+\int_{t_1}^{t_2}\left(\varepsilon_i\cdot\frac{\partial L}{\partial x^i}-\varepsilon_i\cdot\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\cdot dt \tag{4.6.7}$$

Because of the boundary condition, the first term is zero:

$$\delta S_i=\int_{t_1}^{t_2}\varepsilon_i\cdot\left(\frac{\partial L}{\partial x^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\cdot dt=0$$

According to the action principle, the variation of the action functional is zero. This is satisfied, if the expression in the parentheses is zero, that is the general equation of movement:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}-\frac{\partial L}{\partial x^i}=0 \tag{4.6.8}$$

4.6 Newtonian approximation

In a non-relativistic conservative force field the Lagrange function is the difference of the kinetic and the potential energy:

$$L = E_k - E_p \tag{4.6.9}$$

The total energy of the moving body:

$$E = \frac{m \cdot c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The kinetic energy is the part above the rest energy:

$$E_k = m \cdot c^2 \cdot \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \tag{4.6.10}$$

Write down the expression in the parentheses with a binomial series:

$$(a+b)^n = \sum_{i=0}^{n} \frac{n!}{(n-i)! \cdot i!} \cdot a^{n-i} \cdot b^i = a^n + n \cdot a^{n-1} \cdot b + \frac{n \cdot (n-1)}{2!} \cdot a^{n-2} \cdot b^2 + \dots$$

$$\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{1}{2} \cdot \frac{v^2}{c^2} + \frac{3}{8} \cdot \frac{v^4}{c^4} + \dots \tag{4.6.11}$$

Reinsert it into the formula for kinetic energy:

$$E_k = m \cdot c^2 \cdot \left(\frac{1}{2} \cdot \frac{v^2}{c^2} + \frac{3}{8} \cdot \frac{v^4}{c^4} + \dots \right)$$

At velocities slow compared to the speed of light, the kinetic energy is approximately the following:

$$E_k \approx \frac{1}{2} \cdot m \cdot v^2 + \dots \tag{4.6.12}$$

In a central force field the potential energy depends only on the mass of the test body and the distance from the centre:

$$E_p = m \cdot \phi(r) \tag{4.6.13}$$

In Newtonian mechanics space is flat. Write down the Lagrange function is a three dimensional spherical coordinate system. The velocity squared is calculated from the arc length squared:

$$ds^2 = dr^2 + r^2 \cdot \left(d\,\vartheta^2 + \sin^2(\vartheta) \cdot d\,\varphi^2\right) \quad / \cdot \frac{1}{\partial t^2}$$

$$v^2 = \dot{r}^2 + r^2 \cdot \left(\dot{\vartheta}^2 + \sin^2(\vartheta) \cdot \dot{\varphi}^2\right) \tag{4.6.14}$$

The Lagrange function:

$$L = E_k - E_p = m \cdot \left(\frac{1}{2} \cdot (\dot{r}^2 + r^2 \cdot (\dot{\vartheta}^2 + \sin^2(\vartheta) \cdot \dot{\varphi}^2)) - \phi(r)\right) \tag{4.6.15}$$

Substitute it into the equation of movement and divide with the mass:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = 0 \quad / \cdot \frac{1}{m}$$

The first term according to the radial coordinate:

$$\frac{d}{dt} \frac{\partial}{\partial \dot{r}} \left(\frac{1}{2} \cdot (\dot{r}^2 + r^2 \cdot (\dot{\vartheta}^2 + \sin^2(\vartheta) \cdot \dot{\varphi}^2)) - \phi(r)\right) = \frac{1}{2} \cdot \frac{d}{dt} \frac{\partial \dot{r}^2}{\partial \dot{r}} = \ddot{r}$$

The second term according to the radial coordinate:

$$-\frac{\partial}{\partial r} \left(\frac{1}{2} \cdot (\dot{r}^2 + r^2 \cdot (\dot{\vartheta}^2 + \sin^2(\vartheta) \cdot \dot{\varphi}^2)) - \phi(r)\right) = -r \cdot \dot{\vartheta}^2 - r \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 + \frac{d\,\phi(r)}{dr}$$

The equation of motion in the radial direction is their sum:

$$\boxed{\ddot{r} - r \cdot \dot{\vartheta}^2 - r \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 + \frac{d\,\phi(r)}{dr} = 0} \tag{4.6.16}$$

The first term of the equation of movement according to the latitude:

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\vartheta}} \left(\frac{1}{2} \cdot (\dot{r}^2 + r^2 \cdot (\dot{\vartheta}^2 + \sin^2(\vartheta) \cdot \dot{\varphi}^2)) - \phi(r)\right) = \frac{1}{2} \cdot \frac{d}{dt} \frac{\partial}{\partial \dot{\vartheta}} (r^2 \cdot \dot{\vartheta}^2) = \frac{d}{dt} r^2 \cdot \dot{\vartheta} + r^2 \cdot \ddot{\vartheta} = 2 \cdot r \cdot \dot{r} \cdot \dot{\vartheta} + r^2 \cdot \ddot{\vartheta}$$

The second term according to the latitude:

$$-\frac{\partial}{\partial \vartheta} \left(\frac{1}{2} \cdot (\dot{r}^2 + r^2 \cdot (\dot{\vartheta}^2 + \sin^2(\vartheta) \cdot \dot{\varphi}^2)) - \phi(r)\right) = -\frac{\partial}{\partial \vartheta} (r^2 \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2) = -r^2 \cdot \cos(\vartheta) \cdot \sin(\vartheta) \cdot \dot{\varphi}^2$$

The equation of motion in the latitude direction is their sum:

$$\boxed{\ddot{\vartheta} + \frac{2}{r} \cdot \dot{r} \cdot \dot{\vartheta} - \cos(\vartheta) \cdot \sin(\vartheta) \cdot \dot{\varphi}^2 = 0} \tag{4.6.17}$$

The first term of the equation of motion according to the longitude:

$$\frac{d}{dt}\frac{\partial}{\partial\dot{\varphi}}\left(\frac{1}{2}\cdot(\dot{r}^2+r^2\cdot(\dot{\vartheta}^2+\sin^2(\vartheta)\cdot\dot{\varphi}^2))-\phi(r)\right)=\frac{1}{2}\cdot\frac{d}{dt}\frac{\partial}{\partial\dot{\varphi}}(r^2\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2)=$$

$$\frac{d}{dt}(r^2\cdot\sin^2(\vartheta)\cdot\dot{\varphi})=\frac{d}{dt}r^2\cdot\sin^2(\vartheta)\cdot\dot{\varphi}+r^2\cdot\frac{d}{dt}\sin^2(\vartheta)\cdot\dot{\varphi}+r^2\cdot\sin^2(\vartheta)\cdot\ddot{\varphi}=$$

$$2\cdot r\cdot\dot{r}\cdot\sin^2(\vartheta)\cdot\dot{\varphi}+r^2\cdot 2\cdot\cos(\vartheta)\cdot\sin(\vartheta)\cdot\dot{\varphi}+r^2\cdot\sin^2(\vartheta)\cdot\ddot{\varphi}$$

The second term is zero, therefore by rewriting the first term we get the equation of motion according to the longitude:

$$\ddot{\varphi}+\frac{2}{r}\cdot\dot{r}\cdot\dot{\varphi}+2\cdot\cot(\vartheta)\cdot\dot{\vartheta}\cdot\dot{\varphi}=0 \qquad (4.6.18)$$

In flat space using a spherical coordinate system the radial geodesic equation:

$$\ddot{r}+\Gamma^r{}_{\vartheta\vartheta}\cdot\dot{\vartheta}^2+\Gamma^r{}_{\varphi\varphi}\cdot\dot{\varphi}^2=\ddot{r}-r\cdot\dot{\vartheta}^2-r\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2=0$$

In the presence of a central gravitational field we see a difference in the radial equation of motion:

$$\ddot{r}-r\cdot\dot{\vartheta}^2-r\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2+\frac{d\phi(r)}{dr}=0$$

Only the time-like coordinate velocity is constant, therefore the new term is an exactly identifiable connection coefficient:

$$\frac{d\phi}{dr}=\Gamma^r{}_{tt}\cdot\frac{c\cdot dt}{dt}\cdot\frac{c\cdot dt}{dt}=c^2\cdot\Gamma^r{}_{tt} \qquad (4.6.19)$$

We seek the metric that produces geodesics, like the Newtonian equations of movement. The calculation of the connection from the metric:

$$\Gamma^\kappa{}_{\mu\eta}=\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\left(\frac{\partial g_{\eta\alpha}}{\partial x^\mu}+\frac{\partial g_{\alpha\mu}}{\partial x^\eta}-\frac{\partial g_{\mu\eta}}{\partial x^\alpha}\right)$$

$$\Gamma^r{}_{tt}=\frac{1}{2}\cdot g^{r\alpha}\cdot\left(\frac{\partial g_{t\alpha}}{\partial t}+\frac{\partial g_{\alpha t}}{\partial t}-\frac{\partial g_{tt}}{\partial x^\alpha}\right) \qquad (4.6.20)$$

Assume that the metric does not depend on time:

$$\Gamma^r{}_{tt}=-\frac{1}{2}\cdot g^{r\alpha}\cdot\frac{\partial g_{tt}}{\partial x^\alpha}=-\frac{1}{2}\cdot\left(g^{rt}\cdot\frac{\partial g_{tt}}{\partial t}+g^{rr}\cdot\frac{\partial g_{tt}}{\partial r}+g^{r\vartheta}\cdot\frac{\partial g_{tt}}{\partial\vartheta}+g^{r\varphi}\cdot\frac{\partial g_{tt}}{\partial\varphi}\right)$$

In the spherically symmetric spacetime it also does not depend on the angular coordinates, thus the equation simplifies:

$$\Gamma^{r}{}_{tt}=-\frac{1}{2}\cdot g^{rr}\cdot\frac{\partial g_{tt}}{\partial r} \tag{4.6.21}$$

$$\frac{1}{c^2}\cdot\frac{d\,\phi(r)}{dr}=-\frac{1}{2}\cdot(-1)\cdot\frac{\partial g_{tt}}{\partial r}$$

$$\frac{2}{c^2}\cdot\frac{d\,\phi(r)}{dr}=\frac{\partial g_{tt}}{\partial r}$$

$$\frac{2}{c^2}\cdot\phi(r)+c_1=g_{tt} \tag{4.6.22}$$

At great distances from the source of gravity the shape of space approaches the plain, with this we can determine the integration constant:

$$\lim_{r\to\infty}\phi(r)=0$$

$$\lim_{\phi(r)\to 0} g_{tt}=\lim_{\phi(r)\to 0}\frac{2}{c^2}\cdot\phi(r)+c_1=0+c_1=1 \tag{4.6.23}$$

The form of the gravitational potential in the Newtonian theory of gravity:

$$\phi(r)=-\frac{\gamma\cdot M}{r} \tag{4.6.24}$$

Where γ is the gravitational constant, M is the central mass causing gravity, r is the distance of the test body from it. Substitute it into the metric tensor component, and write down the arc length squared of the spacetime, that causes the exact same geodesics, like the Newtonian equations of movement:

$$g_{tt}=1-\frac{2\cdot\gamma\cdot M}{r\cdot c^2} \tag{4.6.25}$$

$$ds^2=\left(1-\frac{2\cdot\gamma\cdot M}{r\cdot c^2}\right)\cdot c^2\cdot dt^2-dr^2-r^2\cdot\left(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2\right) \tag{4.6.26}$$

Compare the corresponding metric tensor components of the Schwarzschild metric and the just derived Newtonian limiting case, and with this we can determine the second integration constant of the Schwarzschild derivation, the Schwarzschild radius:

$$g_{tt}=1-\frac{2\cdot\gamma\cdot M}{r\cdot c^2}=1-\frac{r_g}{r}$$

$$r_g = \frac{2 \cdot \gamma \cdot M}{c^2} \quad (4.6.27)$$

The presently accepted value of the gravitational constant:

$$\gamma = 6.67428 \cdot 10^{-11} \frac{m^3}{s^2 \cdot kg}$$

This constant of nature is one of the hardest to measure, therefore using methods of celestial mechanics, the masses of celestial bodies can be measured only with the same error margin. Thus for precise orbit calculations the product of the two quantities is used, this is the standard gravitational parameter. With this it is possible to calculate the Schwarzschild radius of celestial bodies as well, with great accuracy. The Schwarzschild arc length squared, using pure SI units:

$$ds^2 = \left(1 - \frac{2 \cdot \gamma \cdot M}{r \cdot c^2}\right) \cdot c^2 \cdot dt^2 - \frac{dr^2}{1 - \frac{2 \cdot \gamma \cdot M}{r \cdot c^2}} - r^2 \cdot \left(d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2\right) \quad (4.6.28)$$

4.7 Circular orbit

All energy present in the system causes spacetime curvature, therefore we let go a test body in it with such a small mass, that has negligible influence on events. For the sake of simplicity, it will move on a circular orbit around the gravitational centre, at a distance far greater than the Schwarzschild radius, along a force-free local line, a geodesic. In order to determine the orbit parameters, we write down the coordinate conditions first, that come from the properties of the circular orbit:

$$t = t(\tau) \qquad \frac{\partial t}{\partial \tau} = const.$$

$$r = const. \qquad dr = 0 \qquad \frac{\partial r}{\partial \tau} = \frac{\partial^2 r}{\partial \tau^2} = 0$$

$$\vartheta = \frac{\pi}{2} \qquad d\vartheta = 0$$

$$\varphi = \varphi(\tau) \qquad \frac{\partial \varphi}{\partial \tau} = const. \quad (4.7.1)$$

These simplify the general geodesic equations:

$$\ddot{t} + \frac{r_g}{r \cdot (r - r_g)} \cdot \dot{t} \cdot \dot{r} = 0$$

$$\ddot{t}=0 \tag{4.7.2}$$

$$\ddot{r}+\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2\cdot\dot{t}^2-\frac{r_g}{2\cdot r\cdot(r-r_g)}\cdot\dot{r}^2-(r-r_g)\cdot\dot{\varphi}^2=0$$

$$\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2\cdot\dot{t}^2-(r-r_g)\cdot\dot{\varphi}^2=0 \tag{4.7.3}$$

$$\ddot{\vartheta}=0 \tag{4.7.4}$$

$$\ddot{\varphi}+\frac{2}{r}\cdot\dot{r}\,\dot{\varphi}=0$$

$$\ddot{\varphi}=0 \tag{4.7.5}$$

Utilize the coordinate conditions on the arc length squared as well:

$$ds^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-\frac{0^2}{1-\frac{r_g}{r}}-r^2\cdot\left(0^2+\sin^2\left(\frac{\pi}{2}\right)\cdot d\,\varphi^2\right)$$

$$ds^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-r^2\cdot d\,\varphi^2 \tag{4.7.6}$$

This is the point of view of the infinitely distant observer. The astronaut moving on the orbit however does not see this. He “feels” to be weightless, and by him the Minkowski arc length squared can be written down locally. We set it equal to the arc length squared seen by the distant observer:

$$c^2\cdot d\,\tau^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-r^2\cdot d\,\varphi^2$$

$$d\,\tau^2=\left(\left(1-\frac{r_g}{r}\right)-\frac{r^2}{c^2}\cdot\frac{d\,\varphi^2}{dt^2}\right)\cdot dt^2 \qquad \frac{d\,\varphi}{d\,t}=\omega$$

Substitute the angular velocity along the orbit, and we get the relationship between the proper time and the coordinate time:

$$d\,\tau=\sqrt{\left(1-\frac{r_g}{r}\right)-\frac{r^2\cdot\omega^2}{c^2}}\cdot dt \tag{4.7.7}$$

Since the ratio of the two quantities is constant, the coordinate time is also guaranteed to grow monotonically during the movement, therefore it can be used as parameter when solving the geodesic equation. The geodesic equation in the radial direction:

$$\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2\cdot\left(\frac{\partial t}{\partial t}\right)^2-(r-r_g)\cdot\left(\frac{\partial\varphi}{\partial t}\right)^2=0$$

$$\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2-(r-r_g)\cdot\omega^2=0$$

$$\frac{r_g}{2\cdot r^3}\cdot c^2-\omega^2=0$$

The angular frequency of a test object with negligible mass on a circular orbit:

$$\omega=c\cdot\sqrt{\frac{r_g}{2\cdot r^3}} \tag{4.7.8}$$

The orbital period calculated from it equals to the Newtonian limiting case at arbitrary orbital radii:

$$t_k=\frac{2\cdot\pi}{c}\cdot\sqrt{\frac{2\cdot r^3}{r_g}} \tag{4.7.9}$$

The Earth orbits on an approximately circular orbit, therefore it is a good example to demonstrate the relationship. The standard gravitational parameter of the Sun and its gravitational radius:

$$\gamma\cdot M=1.32712440018\cdot10^{20}\,\frac{m^3}{s^2}\qquad\rightarrow\qquad r_g=\frac{2\cdot\gamma\cdot M}{c^2}=2.9532500765\cdot10^3\,m$$

The semi-major axis of the Earth's orbit:

$$r=1.49598261\cdot10^{11}\,m$$

From these we can calculate the orbital period:

$$t_k=3.15583195\cdot10^7\,s=365.258328\,days \tag{4.7.10}$$

The difference between the actual and the calculated orbital period:

$$t_{k2}=365.256363004\,days$$

$$\frac{t_k}{t_{k2}}-1=5.37919714\cdot10^{-6} \tag{4.7.11}$$

The difference from the measured value is caused by ignoring that the Earth's orbit deviates from the ideal circle, the other planets also influence the Earth's movement, and the Sun's rotation also has an influence on spacetime. The ratio of the orbital periods and the radii of the circular orbits gives Kepler's third law:

$$\frac{{}_1t_k^2}{{}_2t_k^2}=\frac{{}_2r^3}{{}_1r^3} \qquad (4.7.12)$$

Substitute the angular frequency of the orbit into the relationship of the proper time and the coordinate time:

$$d\tau=\sqrt{\left(1-\frac{r_g}{r}\right)-\frac{r^2\cdot\left(c\cdot\sqrt{\frac{r_g}{2\cdot r^3}}\right)^2}{c^2}}\cdot dt$$

$$d\tau=\sqrt{1-\frac{3\cdot r_g}{2\cdot r}}\cdot dt \qquad (4.7.13)$$

If the change of the proper time is zero, it means a light-like geodesic, thus we are speaking about light on a circular orbit:

$$0=\sqrt{1-\frac{3\cdot r_g}{2\cdot r}}\cdot dt$$

The radius of the orbit:

$$r=\frac{3\cdot r_g}{2} \qquad (4.7.14)$$

Objects slower than the speed of light can orbit around the centre only at greater distances than this. The circular geodesics inside this radius are all space-like, that is shown in the fact that the number under the square root is negative, thus the proper time becomes an imaginary quantity.

4.8 Surface acceleration and hovering

If an object does not move in a Schwarzschild coordinate system, for example it is at rest on the surface of a spherical planet, how much is its acceleration? We perform the calculations with respect to the coordinate time. In the case of an every-day size planet with a solid surface it is not a significant discrepancy. The coordinate conditions in this case:

$$t=t(\tau) \qquad \frac{\partial t}{\partial\tau}=const.$$

$$r=const. \qquad dr=0$$

$$\vartheta=const.=\frac{\pi}{2} \qquad d\vartheta=0$$

$$\varphi = const. \qquad d\varphi = 0 \tag{4.8.1}$$

We are looking for the radial acceleration, we substitute into the corresponding equation of movement:

$$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 \cdot \dot{t}^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\varphi}^2 = 0$$

Surface acceleration in the Schwarzschild solution, when the rotation of the planet is negligible:

$$\ddot{r} = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 \cdot \dot{t}^2 \tag{4.8.2}$$

If we want to take the rotation of the planet into account, the coordinate conditions will expand, the observer will perform circular motion along a latitude. We continue to work with the same equation of movement (because of using the Schwarzschild metric, we neglect the effects of the rotation on the spacetime, but in the case of a small angular momentum, this is an adequate approximation):

$$t = t(\tau) \qquad \frac{\partial t}{\partial \tau} = const.$$

$$r = const. \qquad dr = 0$$

$$\vartheta = const. \qquad d\vartheta = 0$$

$$\varphi = const. \qquad \frac{d\varphi}{d\tau} = const. \tag{4.8.3}$$

We start with the most general radial equation of movement:

$$\ddot{r} + \frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 \cdot \dot{t}^2 - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot \dot{r}^2 - (r - r_g) \cdot \dot{\vartheta}^2 - (r - r_g) \cdot \sin^2(\vartheta) \cdot \dot{\varphi}^2 = 0$$

Surface acceleration in the Schwarzschild solution, when the planet rotates:

$$\ddot{r} = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 + (r - r_g) \cdot \sin^2(\vartheta) \cdot \omega^2 \tag{4.8.4}$$

The standard gravitational parameter of Earth, and the gravitational radius:

$$\gamma \cdot M = 3.986004418 \cdot 10^{14} \frac{m^3}{s^2} \qquad \rightarrow \qquad r_g = \frac{2 \cdot \gamma \cdot M}{c^2} = 8.870056078 \cdot 10^{-3}\, m$$

The equatorial radius of Earth and the angular frequency of the rotation:

$$r = 6.3781366 \cdot 10^6 m \qquad \omega = \frac{2 \cdot \pi}{t_k} = 7.292115 \cdot 10^{-5} \frac{1}{s}$$

Surface acceleration on the Earth's equator $\left(\vartheta = \frac{\pi}{2}\right)$:

$$\ddot{r} = -9.7982867 \frac{m}{s^2} + 0.0339157 \frac{m}{s^2} = -9.764371 \frac{m}{s^2} \tag{4.8.5}$$

The actual acceleration measured on the Earth's equator, and the difference from the calculated value:

$$\ddot{r}_2 = -9.780327 \frac{m}{s^2}$$

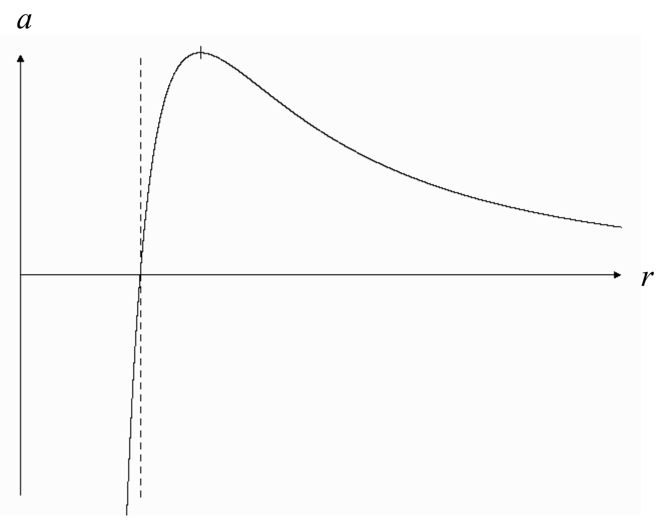
$$\frac{\ddot{r}_2}{\ddot{r}} - 1 = 1.634105 \cdot 10^{-3} \tag{4.8.6}$$

The discrepancy from the geographic value is caused by the Earth's not exactly spherical shape, this has mainly an impact on the term that takes the rotation into account.

Acceleration of a hovering object with respect to distance, from the point of view of the infinitely distant observer:

$$a = -\frac{r_g \cdot (r - r_g)}{2 \cdot r^3} \cdot c^2 \tag{4.8.7}$$

On the graphic the distance from the gravitational centre increases from left to right, the coordinate acceleration of the hovering body is displayed on the vertical axis, the dotted line shows the position of the event horizon:



Approaching the gravitational radius, the coordinate acceleration goes to zero:

$$a=\lim_{r\to r_g} -\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2=0 \qquad (4.8.8)$$

The derivative according to the radial coordinate is zero at the point of maximal acceleration:

$$\frac{\partial a}{\partial r}=r_g\cdot\frac{2\cdot r-3\cdot r_g}{2\cdot r^3}\cdot c^2=0$$

$$r=\frac{3}{2}\cdot r_g \qquad (4.8.9)$$

4.9 Geodesic precession

In 1916, while working on the relativistic correction of the Moon's orbit, Willem de Sitter Dutch astronomer pointed out this phenomenon. By analysing laser light reflected from prisms placed on the surface of the Moon during the Apollo program the phenomenon was confirmed to 0.7% accuracy. NASA launched Gravity Probe B in 2004 with the best mechanical gyroscopes on board ever created by mankind. The result of the experiment, that confirmed the accuracy of the theory of relativity within 1% was published in April of 2007 at the annual congress of the American Physical Society.

Maybe one of the best evidences for the curvature of spacetime is the parallel displacement along a geodesic, for example a circular orbit. The direction of the vector arriving at the starting point will differ from the original. Parallel displacement along a geodesic, where *v* is the vector in the original point, *u* is already displaced:

$$u^\eta=v^\eta-\Gamma^\eta{}_{\beta\alpha}\cdot v^\alpha\cdot dx^\beta \qquad (4.9.1)$$

Our geodesic of choice is the circular orbit. The infinitesimal displacement vector along the orbit:

$$dx^\eta=(c\cdot dt \quad 0 \quad 0 \quad d\varphi) \qquad (4.9.2)$$

The vector lays in the orbital plane, therefore its zeroth and second components remain zeroes after the displacement:

$$u^\eta=(0 \quad u^r \quad 0 \quad u^\varphi) \qquad\qquad v^\eta=(0 \quad v^r \quad 0 \quad v^\varphi) \qquad (4.9.3)$$

The orbiting happens in the equatorial plane:

$$\vartheta=\frac{\pi}{2} \qquad (4.9.4)$$

Substitute the connection of the Schwarzschild solution into the formula of parallel displacement:

$$u^t = v^t - \Gamma^t{}_{tr} \cdot v^r \cdot dt - \Gamma^t{}_{rt} \cdot v^t \cdot dr = v^t - \frac{r_g}{2 \cdot r \cdot (r - r_g)} \cdot v^r \cdot dt \tag{4.9.5}$$

$$u^r = v^r - \Gamma^r{}_{tt} \cdot v^t \cdot dt - \Gamma^r{}_{rr} \cdot v^r \cdot dr - \Gamma^r{}_{\vartheta\vartheta} \cdot v^\vartheta \cdot d\vartheta - \Gamma^r{}_{\varphi\varphi} \cdot v^\varphi \cdot d\varphi = v^r + (r - r_g) \cdot v^\varphi \cdot d\varphi \tag{4.9.6}$$

$$u^\vartheta = v^\vartheta - \Gamma^\vartheta{}_{r\vartheta} \cdot v^\vartheta \cdot dr - \Gamma^\vartheta{}_{\vartheta r} \cdot v^r \cdot d\vartheta - \Gamma^\vartheta{}_{\varphi\varphi} \cdot v^\varphi \cdot d\varphi = 0 \tag{4.9.7}$$

$$u^\varphi = v^\varphi - \Gamma^\varphi{}_{r\varphi} \cdot v^\varphi \cdot dr - \Gamma^\varphi{}_{\varphi r} \cdot v^r \cdot d\varphi - \Gamma^\varphi{}_{\vartheta\varphi} \cdot v^\varphi \cdot d\vartheta - \Gamma^\varphi{}_{\varphi\vartheta} \cdot v^\vartheta \cdot d\varphi = v^\varphi - \frac{1}{r} \cdot v^r \cdot d\varphi \tag{4.9.8}$$

Rearrange the second and fourth displacement equations, substitute the difference between the original and the displaced vector, as well as the angular velocity:

$$dv^r + (r - r_g) \cdot v^\varphi \cdot d\varphi = 0 \quad / \cdot \frac{1}{dt^2} \qquad \leftarrow \qquad v^\eta - u^\eta = dv^\eta$$

$$\ddot{v}^r + (r - r_g) \cdot \dot{v}^\varphi \cdot \omega = 0 \qquad \leftarrow \qquad \frac{d\varphi}{dt} = \omega \tag{4.9.9}$$

$$dv^\varphi - \frac{1}{r} \cdot v^r \cdot d\varphi = 0 \quad / \cdot \frac{1}{dt}$$

$$\dot{v}^\varphi = \frac{1}{r} \cdot v^r \cdot \omega \tag{4.9.10}$$

Express the change of the displaced vector, and substitute it into the radial displacement equation:

$$\ddot{v}^r + \frac{r - r_g}{r} \cdot \omega^2 \cdot v^r = 0 \tag{4.9.11}$$

$$\ddot{v}^r = -\frac{r - r_g}{r} \cdot \omega^2 \cdot v^r$$

This is the differential equation of the harmonic oscillator, that looks like this in the general case:

$$v^r = \sin(\Omega \cdot t) \qquad \frac{d^2}{dt^2} \sin(\Omega \cdot t) = -\Omega^2 \cdot \sin(\Omega \cdot t)$$

We can see its angular frequency:

$$\Omega = \sqrt{\frac{r - r_g}{r}} \cdot \omega \tag{4.9.12}$$

The geodesic precession is the difference between this and the angular frequency:

$$\varpi=\omega-\Omega=\left(1-\sqrt{\frac{r-r_g}{r}}\right)\cdot\omega \tag{4.9.13}$$

In the case of weak gravitational field this effect is small, but accumulates over several revolutions. The orbital period and frequency of Gravity Probe B, that was in Earth orbit for 50 weeks between 2004 and 2005 at an altitude of 642 km:

$$r=7013000\,m \qquad t_k=5850\,s=1\,h\,37\,min\,30\,s$$

$$\omega=\frac{2\cdot\pi}{t_k}=1.074\cdot10^{-3}\frac{1}{s}$$

The standard gravitational parameter of Earth, and the gravitational radius:

$$\gamma\cdot M=3.986004418\cdot10^{14}\frac{m^3}{s^2} \quad\rightarrow\quad r_g=\frac{2\cdot\gamma\cdot M}{c^2}=8.870056078\cdot10^{-3}\,m$$

From these the angular velocity of the geodesic precession:

$$\varpi=6.792\cdot10^{-13}\frac{1}{s} \tag{4.9.14}$$

Rotation in a year:

$$\Delta\varphi=\varpi\cdot t_{year}=2.143\cdot10^{-5}(rad)=4.421\,'' \tag{4.9.15}$$

The de Sitter effect is the precession of the lunar orbit in the gravitational field of the Sun. The orbital frequency of the Earth from the orbital period:

$$t_k=365.256363004\,days \quad\rightarrow\quad \omega=\frac{2\cdot\pi}{t_k}=1.99098659277\cdot10^{-7}\frac{1}{s}$$

We substitute the gravitational radius of the Sun (from the standard gravitational parameter) and the radius of the Earth orbit into our derived equation:

$$\gamma\cdot M=1.32712440018\cdot10^{20}\frac{m^3}{s^2} \quad\rightarrow\quad r_g=\frac{2\cdot\gamma\cdot M}{c^2}=2.9532500765\cdot10^{3}\,m$$

The semi-major axis of the Earth orbit:

$$r=1.49598261\cdot10^{11}\,m$$

The angular velocity of the lunar orbit's precession:

$$\varpi=1.96522383\cdot10^{-15}\frac{1}{s} \tag{4.9.16}$$

Rotation in a year:

$$\Delta\varphi=\varpi\cdot t_{year}=6.20188278\cdot 10^{-8}(rad)=0.0127923015'' \qquad (4.9.17)$$

4.10 Stability of circular orbits

In the spacetime of the Schwarzschild solution, circular orbits are obviously geodesics, but this is just a theoretical situation, the trajectories of real objects always deviate from this, if only a little bit. The question is, will the spacetime geometry generated by the central gravitating celestial body correct their movement if they stray from the ideal path? Will they move on stable orbits, or will the geometry increase the perturbation and make them leave the system forever, or turn in the wrong direction and increase the mass of the central celestial object?

Therefore we want to find out, what orbital radius belongs to what angular frequency, and in the vicinity of the orbit, in what direction will the geometric potential herd the orbiting bodies. Calculating the elapsed proper time on time-like geodesics in a single plane, around the gravitational centre (we use the metric functions as a shorthand):

$$\vartheta=\frac{\pi}{2} \qquad d\vartheta=0$$

$$c^2\cdot d\tau^2=A\cdot c^2\cdot dt^2-B\cdot dr^2-r^2\cdot d\varphi^2 \quad /\cdot\frac{1}{c^2\cdot d\tau^2} \qquad (4.10.1)$$

$$1=A\cdot\frac{dt^2}{d\tau^2}-\frac{B}{c^2}\cdot\frac{dr^2}{d\tau^2}-\frac{r^2}{c^2}\cdot\frac{d\varphi^2}{d\tau^2}$$

We determine the components of the covariant tangent vector from the arc length squared, in the time-like and horizontal direction they are the following:

$$ds_t^2=A\cdot c^2\cdot dt^2 \qquad ds_\varphi^2=r^2\cdot d\varphi^2 \qquad /\cdot\frac{1}{d\tau}$$

$$u_t=A\cdot c^2\cdot\frac{dt}{d\tau} \qquad u_\varphi=r^2\cdot\frac{d\varphi}{d\tau}=r^2\cdot\omega \qquad (4.10.2)$$

These quantities are constants of movement, because the metric tensor does not depend on the coordinates they are directed to (see chapter 1). Substitute the relationship between the tangent vectors and the two metric functions, and we determine the dependence of the radial velocity from the selected velocity vectors depending on the distance:

$$B=\frac{1}{A}$$

$$1=\frac{u_t^2}{A\cdot c^2}-\frac{1}{A\cdot c^2}\cdot\frac{dr^2}{d\tau^2}-\frac{u_\varphi^2}{r^2\cdot c^2}\quad /\cdot A\cdot c^2$$

$$\frac{dr^2}{d\tau^2}=u_t^2-A\cdot\left(\frac{u_\varphi^2}{r^2}+c^2\right) \tag{4.10.3}$$

Substitute the function from the Schwarzschild solution:

$$\frac{dr^2}{d\tau^2}=u_t^2-\left(1-\frac{r_g}{r}\right)\cdot\left(\frac{u_\varphi^2}{r^2}+c^2\right) \tag{4.10.4}$$

The second term on the right is the geometric potential:

$$U_{eff}=\left(1-\frac{r_g}{r}\right)\cdot\left(\frac{u_\varphi^2}{r^2}+c^2\right) \tag{4.10.5}$$

We are discussing the behaviour of this function. Where the derivative according to the radial coordinate is zero, the geometric potential is horizontal, this means a potential orbit around the gravitational centre:

$$\frac{dU_{eff}}{dr}=\frac{r_g}{r^2}\cdot\left(\frac{u_\varphi^2}{r^2}+c^2\right)-\left(1-\frac{r_g}{r}\right)\cdot\frac{2\cdot u_\varphi^2}{r^3}=0$$

$$r_g\cdot c^2\cdot r^2-2\cdot u_\varphi^2\cdot r+3\cdot u_\varphi^2\cdot r_g=0 \tag{4.10.6}$$

Solve the quadratic equation, the canonical form and the quadratic formula:

$$a\cdot x^2+b\cdot x+c=0$$

$$x_{1,2}=\frac{-b\pm\sqrt{b^2-4\cdot a\cdot c}}{2\cdot a}$$

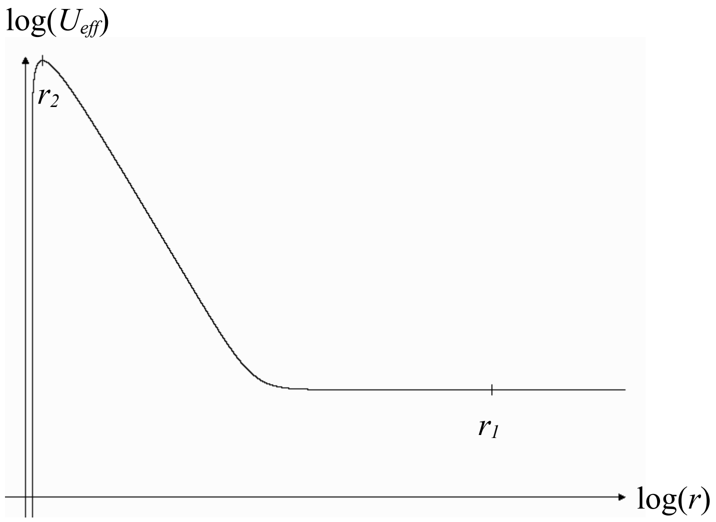
Substitute into the quadratic formula:

$$r_{1,2}=\frac{2\cdot u_\varphi^2\pm\sqrt{(2\cdot u_\varphi^2)^2-4\cdot r_g\cdot c^2\cdot 3\cdot u_\varphi^2\cdot r_g}}{2\cdot r_g\cdot c^2}$$

The geometric potential is horizontal at the following distances, the radii of possible orbits at a given horizontal velocity:

$$r_{1,2}=\frac{u_\varphi^2\pm u_\varphi\cdot\sqrt{u_\varphi^2-3\cdot r_g^2\cdot c^2}}{r_g\cdot c^2} \tag{4.10.7}$$

The piece of the geometric potential that is interesting to us, on a logarithmic diagram, with the maxima and minima noted:



We get circular orbits only if the quantity under the square root, the discriminant is not negative. In the other case, there can be no circular orbit, the test object falls on a spiral path, or leaves forever in the opposite direction. If the discriminant is zero:

$$u_\varphi^2 = 3 \cdot r_g^2 \cdot c^2 \qquad (4.10.8)$$

Reinsert into the quadratic formula:

$$r_{1,2} = \frac{3 \cdot r_g^2 \cdot c^2 \pm \sqrt{3 \cdot r_g^2 \cdot c^2} \cdot \sqrt{0}}{r_g \cdot c^2}$$

$$r_1 = r_2 = 3 \cdot r_g \qquad (4.10.9)$$

In the general case, r_1 is always greater and r_2 is always less than this limit separating the two kinds of circular orbits. Since the geometric radius of the Sun vastly exceeds this limiting case, only r_1 orbits occur in the Solar System. By substituting these two values into the second derivative of the geometric potential, it turns out that the r_1 orbits are stable, r_2 orbits are unstable geodesics, as we can see it on the graph.

4.11 Perihelion precession

After Urbain le Verrier – by examining the orbit of Uranus – discovered Neptune on paper, he did similar calculations in 1859 regarding the movement of Mercury. After he determined the contributions by every other planet, a discrepancy remained, greater than the error margin between the measured and calculated values of the perihelion precession. Contemporary explanations failed, until Einstein in 1915 was able to explain this anomaly easily using general relativity, this problem became one of the classical tests of his theory. At this time, none of the exact solutions of his equation were known (except for the flat spacetime), therefore Einstein used a different version than the one presented here.

We examine the movement of a body on an orbit that is slightly different from a circle. It turned out when we investigated the stability of the orbit, that the orbiting distance oscillates around a medium value with a definite period: T_r, and because of the revolution the angle of the position around the centre changes also of course, with the following period: T_φ. The shape of the resulting orbit approximates a rotating ellipse in the simplest case, where a angular turn with respect to the coordinate time is:

$$\Delta\varphi=\omega_{et}\cdot(T_r-T_\varphi)=\omega_{et}\cdot\left(\frac{2\cdot\pi}{\omega_e}-\frac{2\cdot\pi}{\omega}\right)=2\cdot\pi\cdot\left(1-\frac{\omega_{et}}{\omega}\right) \tag{4.11.1}$$

Arbitrary orbits around the Sun are characterized by the following relationship, we met previously:

$$\frac{dr^2}{d\tau^2}=u_t^2-U_{eff} \tag{4.11.2}$$

If we approximate the geometric potential with its second derivative around the equilibrium point, then we can ultimately replace it with the differential equation of the harmonic oscillator, where we can identify the angular frequency of the periodic movement:

$$U_{eff}(r)\approx\frac{1}{2}\cdot\frac{d^2}{dr^2}\cdot U(r_+)\cdot(r-r_+)^2$$

$$\frac{dr^2}{d\tau^2}=u_t^2-\frac{1}{2}\cdot\frac{d^2}{dr^2}\cdot U(r_+)\cdot(r-r_+)^2 \tag{4.11.3}$$

We identify the angular frequency of the angular turn, this time with respect to the proper time:

$$\omega_e^2=\frac{U''_{eff}}{2} \tag{4.11.4}$$

Calculate the second derivative of the geometric potential:

$$\frac{d^2U_{eff}}{dr^2}=-\frac{2\cdot r_g}{r^3}\cdot\left(\frac{u_\varphi^2}{r^2}+c^2\right)-\frac{r_g}{r^2}\cdot\frac{2\cdot u_\varphi^2}{r^3}-\frac{r_g}{r^2}\cdot\frac{2\cdot u_\varphi^2}{r^3}+\left(1-\frac{r_g}{r}\right)\cdot\frac{6\cdot u_\varphi^2}{r^4}$$

$$\frac{d^2U_{eff}}{dr^2}=-\frac{2\cdot r_g}{r^3}\cdot c^2+\left(\frac{6}{r^4}-\frac{12\cdot r_g}{r^5}\right)\cdot u_\varphi^2 \tag{4.11.5}$$

$$u_\varphi=r^2\cdot\frac{d\varphi}{d\tau}\approx r^2\cdot\omega$$

Determine the angular frequency of the rotation of the ellipse, we recognize in the first term the orbital frequency according to the coordinate time of the orbiting body, and we substitute the horizontal velocity in a form that is valid for circular movement:

$$\omega_e^2 = -\frac{r_g}{r^3}\cdot c^2 + \left(\frac{3}{r^4} - \frac{6\cdot r_g}{r^5}\right)\cdot u_\varphi^2$$

$$\omega_e^2 = -2\cdot\omega^2 + \left(3 - \frac{6\cdot r_g}{r}\right)\cdot\omega^2 = \left(1 - \frac{6\cdot r_g}{r}\right)\cdot\omega^2 \qquad (4.11.6)$$

The relationship between the angular frequencies according to the coordinate time and the proper time can be derived from the relationship between the coordinate time and the proper time:

$$d\,\tau = \sqrt{1 - \frac{3\cdot r_g}{2\cdot r}}\cdot dt$$

$$\frac{d\,\varphi}{d\,\tau} = \frac{d\,\varphi}{\sqrt{1 - \frac{3\cdot r_g}{2\cdot r}}\cdot dt}$$

$$\omega_e = \frac{\omega_{et}}{\sqrt{1 - \frac{3\cdot r_g}{2\cdot r}}} \quad \rightarrow \quad \omega_{et}^2 = \left(1 - \frac{3\cdot r_g}{2\cdot r}\right)\cdot\omega_e^2 \qquad (4.11.7)$$

We discuss distant orbits, where the ratio of the Schwarzschild radius and the distance is very small, therefore we can allow ourselves a small inaccuracy, that keeps us within the error margin, but we can arrange to a more comfortable form the relationship we are looking for:

$$\omega_e^2 = \left(1 - \frac{6\cdot r_g}{r}\right)\cdot\omega^2 \approx \left(1 - \frac{3\cdot r_g}{2\cdot r}\right)\cdot\omega^2$$

$$\omega_{et} = \left(1 - \frac{3\cdot r_g}{2\cdot r}\right)\cdot\omega \qquad (4.11.8)$$

By reinserting we get the perihelion precession of orbits:

$$\Delta\,\varphi = 2\cdot\pi\cdot\left(1 - \frac{\omega_{et}}{\omega}\right) = 3\cdot\pi\cdot\frac{r_g}{r} \qquad (4.11.9)$$

The explanation for the perihelion precession of Mercury is a famous confirmation of general relativity. The standard gravitational parameter of the Sun, and the gravitational radius:

$$\gamma\cdot M = 1.32712440018\cdot 10^{20}\,\frac{m^3}{s^2} \quad \rightarrow \quad r_g = \frac{2\cdot\gamma\cdot M}{c^2} = 2.9532500765\cdot 10^3\,m$$

From the semi-major axis and the orbital period of Mercury the angular turn in a single revolution and in a century:

$r = 5.79091\cdot 10^{10}\, m \qquad t_k = 7.60053024\cdot 10^{6}\, s$

$\Delta\varphi = 4.80645\cdot 10^{7} = 0.0991402\,''$

$$\Delta\varphi_{century} = 1.99565\cdot 10^{-4} = 41.1633\,'' \tag{4.11.10}$$

The total measured perihelion precession of Mercury in a century is 5599,7”, where 5028,83” is a coordinate effect due to the precession of the equinoxes, 530” is caused by the gravitational tug of the other planets, and 0,0254” is also caused by the oblateness of the Sun. The difference is:

$\Delta\varphi_{measured} = 40.8446\,''$

Based on this the difference is most likely caused by the curvature of spacetime.

4.12 Bending of light

Based on the Newtonian particle model of light, Johann Georg von Soldner suggested already in 1801, that light rays are deflected if influenced by gravitation, and simply considering the light particles as bodies moving on orbits, he determined their deflection near the Sun. His result was half of the actual value. Einstein used relativity theory, and after an unsuccessful attempt, he correctly predicted the angle of light deflection, as it was confirmed by the British expedition led by Arthur Eddington in 1919. They travelled to Brazil and Equatorial Guinea, and determined the coordinates of stars with known positions near the dark disk of the eclipsed Sun. Later during the 1960s, radio astronomical measurements confirmed the calculations with a few times of 0.01% error margin.

The method presented here differs from the traditional approach, we essentially search the shape of a geodesic from four dimensional spacetime in a subspace. In our case, in the three dimensional speacetime determined by the coordinate condition $\vartheta = \frac{\pi}{2}$, the path of the light rays are also geodesics, as we have already seen when we wrote down the general geodesics. This is the reason for the success of the following procedure.

We examine the paths of light rays for a general case in the gravitational field. Except in the case of the photon sphere, these will not be closed curves, they will either avoid the celestial body on an arched trajectory, or cross the event horizon while falling. Since the metric is spherically symmetric, we are not losing anything if we restrict ourselves to a coordinate surface. The Schwarzschild arc length squared:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)\cdot c^2\cdot dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2\cdot\left(d\vartheta^2 + \sin^2(\vartheta)\cdot d\varphi^2\right) \tag{4.12.1}$$

The arc length squared is zero along light-like geodesics. The coordinate conditions in the equatorial plane of the coordinate system:

$$ds^2 = 0 \qquad \vartheta = \frac{\pi}{2} \qquad d\vartheta = 0 \tag{4.12.2}$$

Substitute into the arc length squared. By rearranging it, the radial coordinate and the original vertical angular coordinate describes a two dimensional surface, where the coordinate time measures the distance:

$$0 = \left(1 - \frac{r_g}{r}\right) \cdot c^2 \cdot dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 \cdot d\varphi^2$$

$$c^2 \cdot dt^2 = \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)^2} + \frac{r^2 \cdot d\varphi^2}{1 - \frac{r_g}{r}} = \left(\frac{r}{r - r_g}\right)^2 \cdot dr^2 + \frac{r^3}{r - r_g} \cdot d\varphi^2 \tag{4.12.3}$$

This surface is a projection of the original spacetime, that however preserved the mutual dependence of the coordinates. If we consider the previous relationship an arc length squared, we can calculate the usual geometric quantities from the metric tensor to the connection:

$$g_{ij} = \begin{pmatrix} \left(\frac{r}{r - r_g}\right)^2 & 0 \\ 0 & \frac{r^3}{r - r_g} \end{pmatrix} \qquad g^{ij} = \begin{pmatrix} \left(\frac{r - r_g}{r}\right)^2 & 0 \\ 0 & \frac{r - r_g}{r^3} \end{pmatrix} \tag{4.12.4}$$

$$\frac{\partial g_{rr}}{\partial r} = \frac{2 \cdot r}{(r - r_g)^2} \cdot \left(1 - \frac{r}{r - r_g}\right) \qquad \frac{\partial g^{rr}}{\partial r} = \frac{2 \cdot (r - r_g)}{r^2} \cdot \left(1 - \frac{r - r_g}{r}\right)$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r} = \frac{r^2}{r - r_g} \cdot \left(3 - \frac{r}{r - r_g}\right) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r} = \frac{1}{r^3} \cdot \left(1 - \frac{3 \cdot (r - r_g)}{r}\right) \tag{4.12.5}$$

$$\Gamma^r{}_{rr} = -\frac{r_g}{r \cdot (r - r_g)} \qquad \Gamma^r{}_{\varphi\varphi} = -\frac{2 \cdot r - 3 \cdot r_g}{2}$$

$$\Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{2 \cdot r - 3 \cdot r_g}{2 \cdot r \cdot (r - r_g)} \tag{4.12.6}$$

The pictures of the light rays on this projection are geodesics, that can be parametrized by the distance that is valid on the surface:

(1) $$\frac{\partial^2 r}{\partial t^2} + \Gamma^r{}_{rr} \cdot \frac{\partial r}{\partial t} \cdot \frac{\partial r}{\partial t} + \Gamma^r{}_{\varphi\varphi} \cdot \frac{\partial \varphi}{\partial t} \cdot \frac{\partial \varphi}{\partial t} = 0$$

$$\frac{\partial^2 r}{\partial t^2}=\frac{r_g}{r\cdot(r-r_g)}\cdot\left(\frac{\partial r}{\partial t}\right)^2+\frac{2\cdot r-3\cdot r_g}{2}\cdot\left(\frac{\partial \varphi}{\partial t}\right)^2 \tag{4.12.7}$$

(2) $$\frac{\partial^2 \varphi}{\partial t^2}+2\cdot\Gamma^{\varphi}{}_{r\varphi}\cdot\frac{\partial r}{\partial t}\cdot\frac{\partial \varphi}{\partial t}=0$$

$$\frac{\partial^2 \varphi}{\partial t^2}=-2\cdot\frac{2\cdot r-3\cdot r_g}{2\cdot r\cdot(r-r_g)}\cdot\frac{\partial r}{\partial t}\cdot\frac{\partial \varphi}{\partial t} \tag{4.12.8}$$

Determine the coordinate changes, or in other words, the velocities:

$$v_\varphi=\frac{\partial \varphi}{\partial t}$$

$$\frac{\partial v_\varphi}{\partial t}=-\frac{2\cdot r-3\cdot r_g}{r\cdot(r-r_g)}\cdot\frac{\partial r}{\partial t}\cdot v_\varphi$$

$$\frac{1}{v_\varphi}\cdot dv_\varphi=-\frac{2\cdot r-3\cdot r_g}{r\cdot(r-r_g)}\cdot dr$$

$$\log(v_\varphi)=\log(r-r_g)-3\cdot\log(r)+C$$

$$v_\varphi=C\cdot\frac{r-r_g}{r^3} \tag{4.12.9}$$

The integration constant can be determined, if we rearrange the arc length squared of the surface, and determine the angular velocity at an extremal case:

$$c^2\cdot dt^2=\left(\frac{r}{r-r_g}\right)^2\cdot dr^2+\frac{r^3}{r-r_g}\cdot d\varphi^2 \quad /\cdot\frac{1}{dt^2}$$

$$c^2=\left(\frac{r}{r-r_g}\right)^2\cdot\frac{dr^2}{dt^2}+\frac{r^3}{r-r_g}\cdot\frac{d\varphi^2}{dt^2} \tag{4.12.10}$$

The change of the radial coordinate is zero at the closest proximity of orbits that avoid the celestial body on an arched trajectory:

$$\frac{dr_0^2}{dt^2}=0$$

$$\frac{d\varphi}{dt}=v_\vartheta=c\cdot\sqrt{\frac{r_0-r_g}{r_0^3}} \tag{4.12.11}$$

Substitute it into the angular velocity measured in this extremal position, and determine the integration constant:

$$v_\varphi = C \cdot \frac{r_0 - r_g}{r_0^3} = c \cdot \sqrt{\frac{r_0 - r_g}{r_0^3}}$$

$$C = c \cdot \sqrt{\frac{r_0^3}{r_0 - r_g}} \tag{4.12.12}$$

The angular velocity:

$$\frac{d\varphi}{dt} = v_\varphi = c \cdot \sqrt{\frac{r_0^3}{r_0 - r_g} \cdot \frac{r - r_g}{r^3}} \tag{4.12.13}$$

The radial velocity can also be determined, if we substitute the above formula into the arc length squared:

$$c^2 = \left(\frac{r}{r - r_g}\right)^2 \cdot \frac{dr^2}{dt^2} + \frac{r^3}{r - r_g} \cdot \frac{d\varphi^2}{dt^2}$$

$$\frac{dr}{dt} = \frac{r - r_g}{r} \cdot \sqrt{c^2 - \frac{r^3}{r - r_g} \cdot \frac{d\varphi^2}{dt^2}}$$

$$\frac{dr}{dt} = v_r = c \cdot \frac{r - r_g}{r} \cdot \sqrt{1 - \frac{r_0^3}{r_0 - r_g} \cdot \frac{r - r_g}{r^3}} \tag{4.12.14}$$

The ratio of the two velocities determines the change of the angular coordinate with respect to the distance. By integrating the relationship we can determine the total angular turn performed by the light ray in the proximity of the celestial body, between the closest approach and infinity:

$$\frac{v_\varphi}{v_r} = \frac{d\varphi}{dr} = \frac{1}{r^2} \cdot \sqrt{\frac{\frac{r_0^3}{r_0 - r_g}}{1 - \frac{r_0^3}{r_0 - r_g} \cdot \frac{r - r_g}{r^3}}}$$

$$\varphi = \int_{r_0}^{\infty} \frac{1}{r^2} \cdot \sqrt{\frac{\frac{r_0^3}{r_0 - r_g}}{1 - \frac{r_0^3}{r_0 - r_g} \cdot \frac{r - r_g}{r^3}}} \cdot dr \tag{4.12.15}$$

We get a formula that is easier to handle, if we introduce a new parameter, that changes between zero and one:

$$\rho = \frac{r_0}{r}$$

$$\varphi = \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{1}{\sqrt{1 - \frac{r_g}{r_0} \cdot \frac{1-\rho^3}{1-\rho^2}}} \cdot d\rho \tag{4.12.16}$$

This integral cannot be written in a closed form, but the integrand can be broken up to a sum of terms, that can be individually integrated:

$$\varphi = \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \cdot \left(1 + \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \frac{1-\rho^3}{1-\rho^2} + \frac{3}{8} \cdot \left(\frac{r_g}{r_0} \cdot \frac{1-\rho^3}{1-\rho^2}\right)^2 + \frac{5}{16} \cdot \left(\frac{r_g}{r_0} \cdot \frac{1-\rho^3}{1-\rho^2}\right)^3 + \ldots\right) \cdot d\rho \tag{4.12.17}$$

The first term characterizes the light ray that moves in flat spacetime:

$$\varphi_1 = \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \cdot d\rho = \arcsin(1) - \arcsin(0) = \frac{\pi}{2} \tag{4.12.18}$$

The deviation from this is called the bending of light in the presence of gravitation. This phenomenon could be observed during solar eclipses for the first time, when the coordinates of stars with known positions was determined near the dark disk of the Sun. These measurements have an error margin, that is greater than that of the second term in the formula:

$$\varphi_2 = \int_0^1 \frac{1}{\sqrt{1-\rho^2}} \cdot \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \frac{1-\rho^3}{1-\rho^2} \cdot d\rho$$

$$\varphi_2 = \frac{1}{2} \cdot \frac{r_g}{r_0} \cdot \left(-\sqrt{\frac{1-\rho}{1+\rho}} - \sqrt{(1-\rho)\cdot(1+\rho)}\right)\Bigg|_1^0 = \frac{r_g}{r_0} \tag{4.12.19}$$

Since this angle is valid only from the perihelion to infinity, the total turn is two times this value:

$$\Delta\varphi = 2 \cdot \frac{r_g}{r_0} \tag{4.12.20}$$

By evaluating the other terms, we get a more precise relationship for light bending in a gravitational field, verified by radio astronomical measurements:

$$\Delta\varphi = 2 \cdot \frac{r_g}{r_0} + \left(\frac{15}{16} \cdot \pi - 1\right) \cdot \left(\frac{r_g}{r_0}\right)^2 - \left(\frac{15}{16} \cdot \pi + \frac{61}{12}\right) \cdot \left(\frac{r_g}{r_0}\right)^3 + \ldots \tag{4.12.21}$$

We calculate the deflection of the light rays that graze the surface of the Sun. Since the error margin of the observation does not exceed that of the first term, we consider only this. The standard gravitational parameter of the Sun, and the gravitational radius:

$$\gamma \cdot M = 1.32712440018 \cdot 10^{20} \frac{m^3}{s^2} \quad \rightarrow \quad r_g = \frac{2 \cdot \gamma \cdot M}{c^2} = 2.9532500765 \cdot 10^3 \, m$$

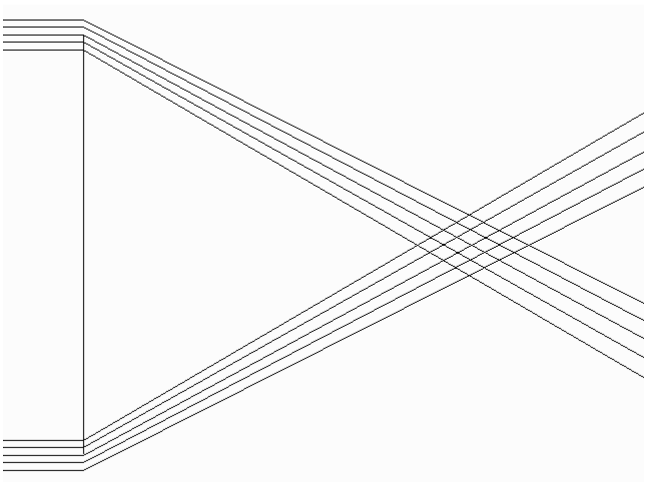
The radius of the Sun:

$$r_0 = 6.955 \cdot 10^8 \, m$$

The deflection of light rays near the Sun:

$$\Delta \varphi = 2 \cdot \frac{r_g}{r_0} = 8.492 \cdot 10^{-6} = 1.752'' \qquad (4.12.22)$$

Light rays coming from the same direction are focused by the Sun into an opposite area, thus it behaves like a gravitational lens:



The paths of the parallel light rays approaching the Sun (that is represented with a vertical line) hold together on the other side, however unlike in the case of optical lenses, they slightly diverge, not all of them are focused into the same point. Despite this, the pictures of objects on the other side are enlarged and brightened. This phenomenon can be utilized in practice by astronomers. The graph is strongly distorted by the way, the parallel Sun-grazing light rays incoming from infinity meet again at a great distance from the Sun, this is called the gravitational focal distance of the Sun:

$$f = r_0 \cdot \cot(\varphi) = 8.19 \cdot 10^{13} \, m \qquad (4.12.23)$$

4.13 Tides

Back in the year of 1616, Galilei considered it a superstition, that according to Johannes Kepler, the tides on Earth are caused by the Moon's gravitational pull, however history verified the latter scientist. Already in the Newtonian theory of gravitation, the Moon and the Sun are responsible for the appearance of the tidal bulges, according to Kepler's suspicion. The theory of relativity can describe this phenomenon to an even greater accuracy, that is the deviation of geodesics:

$$\frac{\partial^2 x^\eta}{\partial \lambda^2}+R^\eta{}_{\alpha\beta\gamma}\cdot dx^\gamma\cdot\frac{\partial x^\alpha}{\partial\lambda}\cdot\frac{\partial x^\beta}{\partial\lambda}=0 \tag{4.13.1}$$

Apply this in the Schwarzschild coordinate system. The question is, what kind of tides are caused by the central mass in the surrounding extended objects that orbit it? At a given point on the circular orbit, where according to our choice:

$$x^\eta=(0\quad x_r\quad x_\vartheta\quad 0)$$

$$\frac{\partial x^\eta}{\partial\lambda}=\left(c\cdot\frac{dt}{d\tau}\quad 0\quad 0\quad \frac{d\varphi}{dt}\right)=\left(c\cdot\frac{dt}{d\tau}\quad 0\quad 0\quad \omega\right) \tag{4.13.2}$$

Substitute into the general formula, and use the coordinate time as the parameter. Under these conditions only four components of the curvature tensor play a role:

(1) $$\frac{\partial^2 r}{\partial t^2}+R^r{}_{ttr}\cdot dr\cdot c\cdot\frac{\partial t}{\partial t}\cdot c\cdot\frac{\partial t}{\partial t}+R^r{}_{\varphi\varphi r}\cdot dr\cdot\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial t}=0$$

$$\frac{\partial^2 r}{\partial t^2}-\frac{r_g\cdot(r-r_g)}{r^4}\cdot c^2\cdot dr+\frac{r_g}{2\cdot r}\cdot\sin^2(\vartheta)\cdot dr\cdot\omega^2=0 \tag{4.13.3}$$

We are in the equatorial plane, and we substitute the angular frequency also:

$$\vartheta=\frac{\pi}{2}\qquad\qquad \omega=c\cdot\sqrt{\frac{r_g}{2\cdot r^3}}$$

$$\frac{\partial^2 r}{\partial t^2}-\frac{r_g\cdot(r-r_g)}{r^4}\cdot c^2\cdot dr+\frac{r_g}{2\cdot r}\cdot dr\cdot c^2\cdot\frac{r_g}{2\cdot r^3}=0$$

The coordinate acceleration along the orbital radius:

$$\boxed{\frac{\partial^2 r}{\partial t^2}=\frac{r_g\cdot(4\cdot r-5\cdot r_g)}{4\cdot r^4}\cdot c^2\cdot dr} \tag{4.13.4}$$

(2) $$\frac{\partial^2\vartheta}{\partial t^2}+R^\vartheta{}_{tt\vartheta}\cdot d\vartheta\cdot c\cdot\frac{\partial t}{\partial t}\cdot c\cdot\frac{\partial t}{\partial t}+R^\vartheta{}_{\varphi\varphi\vartheta}\cdot d\vartheta\cdot\frac{\partial\varphi}{\partial t}\cdot\frac{\partial\varphi}{\partial t}=0$$

$$\frac{\partial^2\vartheta}{\partial t^2}-\frac{r_g\cdot(r-r_g)}{2\cdot r^4}\cdot c^2\cdot d\vartheta-\frac{r_g}{r}\cdot\sin^2(\vartheta)\cdot d\vartheta\cdot\omega^2=0 \tag{4.13.5}$$

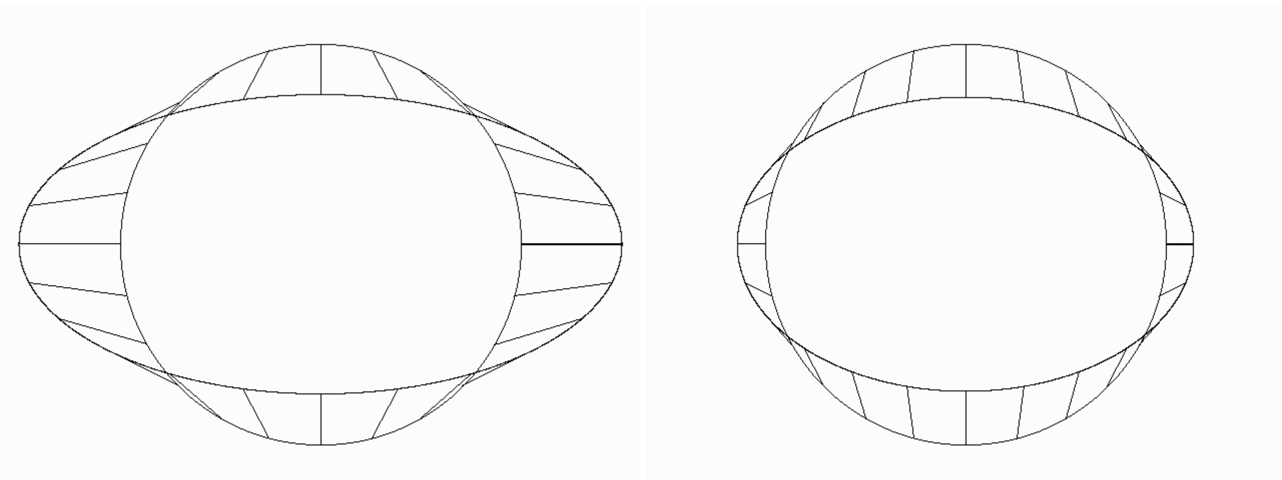
We perform the same substitutions:

$$\frac{\partial^2 \vartheta}{\partial t^2} - \frac{r_g \cdot (r - r_g)}{2 \cdot r^4} \cdot c^2 \cdot d\,\vartheta - \frac{r_g}{r} \cdot d\,\vartheta \cdot c^2 \cdot \frac{r_g}{2 \cdot r^3} = 0$$

The coordinate acceleration perpendicular to the orbital radius:

$$\frac{\partial^2 \vartheta}{\partial t^2} = \frac{r_g}{2 \cdot r^3} \cdot c^2 \cdot d\,\vartheta \qquad (4.13.6)$$

The two acceleration components above distort a spherical planet – like Earth – as seen on the graph on the left side, and it shows in what direction the points of the surface are accelerated. At distances with the same magnitude like the Schwarzschild radius, the normally weaker component becomes significant, the shape of the ellipse changes. The elongation of the falling body is called spaghettification. At the right side the graph displays the acceleration vectors distorting a sphere with a diameter of 5 meters, positioned 5000 meters away from a solar mass black hole, where time-like circular orbits are still possible:



When will the tides become destructive to the celestial body? Pieces start to detach from the surface, when the tidal acceleration exceeds the surface acceleration. When doing the comparison we must keep in mind, that the two formulas do not apply to the same spacetime curvature; the tidal acceleration is calculated in the spacetime of the central star, the surface acceleration is calculated in the spacetime of the orbiting planet. The approximation method has two limitations: first, the Einstein equation is non-linear, therefore gravitational acceleration of the two bodies is added with a certain error. Second, we did not take into account, that the celestial bodies under investigation have distorted each other, neither their shape, nor their gravitational field is spherically symmetric. Since the tidal acceleration on the surface grows linearly with the size of the object (if the mass does not change), this phenomenon creates an upper limit to the sizes of the celestial objects orbiting gravitational sources:

$$\frac{r_g \cdot (4 \cdot r - 5 \cdot r_g)}{4 \cdot r^4} \cdot c^2 \cdot {}_b r = \frac{\partial^2 r}{\partial t^2} = \frac{\partial^2\, {}_b r}{\partial t^2} = -\frac{{}_b r_g \cdot ({}_b r - {}_b r_g)}{2 \cdot {}_b r^3} \cdot c^2$$

$$\frac{r_g\cdot(4\cdot r-5\cdot r_g)}{2\cdot r^4}=-\frac{{}_b r_g\cdot({}_b r-{}_b r_g)}{{}_b r^4}$$

$$\frac{r_g\cdot(4\cdot r-5\cdot r_g)}{2\cdot r^4}\cdot{}_b r^4-{}_b r_g\cdot{}_b r+{}_b r_g^2=0 \qquad (4.13.7)$$

The solution of this equation of the fourth degree gives the upper limit for the size of the celestial body, that not yet gets peeled by the tides. We rewrite it first:

$${}_b r^4-\frac{{}_b r_g}{D_r}\cdot{}_b r+\frac{{}_b r_g^2}{D_r}=0 \qquad D_r=\frac{r_g\cdot(4\cdot r-5\cdot r_g)}{2\cdot r^4} \qquad (4.13.8)$$

The first step towards the solution of the equation of the fourth degree is to write down the resulting equation of the third degree:

$$x^4+b\cdot x^3+c\cdot x^2+d\cdot x+e=0$$

$$y^3+c\cdot y^2+(b\cdot d-4\cdot e)\cdot y+(4\cdot c\cdot e-d^2-b^2\cdot e)\cdot d=0 \qquad (4.13.9)$$

Substituting:

$$y^3-4\cdot e\cdot y-d^3=0$$

$$y^3-\frac{4\cdot{}_b r_g^2}{D_r}\cdot y+\left(\frac{{}_b r_g}{D_r}\right)^3=0 \qquad (4.13.10)$$

We have to solve this equation in the second step. The special form of the equation of the third degree, and the solution formula (during substitution, let us be careful about the signatures):

$$y^3-p\cdot y-q=0$$

$$y_{1,2,3}=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}} \qquad (4.13.11)$$

Finally, we substitute the result into the following expressions, and get the solutions of the equation of the fourth degree:

$$R=\sqrt{\frac{b^2}{4}-c+y}$$

$$R=0 \qquad \rightarrow \qquad D=\sqrt{\frac{3\cdot b^2}{4}-2\cdot c\pm 2\cdot\sqrt{y^2-4\cdot e}}$$

$$R \neq 0 \qquad \rightarrow \qquad D = \sqrt{\frac{3 \cdot b^2}{4} - R^2 - 2 \cdot c \pm \frac{b \cdot c - 8 \cdot d - b^3}{4 \cdot R}}$$

$$x_{1,2,3,4} = -\frac{b}{4} \pm \frac{R}{2} \pm \frac{D}{2} \tag{4.13.12}$$

Substitute from the Earth's standard gravitational parameter the gravitational radius:

$$\gamma \cdot M = 3.986004418 \cdot 10^{14} \frac{m^3}{s^2} \qquad \rightarrow \qquad r_g = \frac{2 \cdot \gamma \cdot M}{c^2} = 8.870056078 \cdot 10^{-3} m$$

The semi-major axis of the Moon's orbit:

$$r = 3.84399 \cdot 10^8 m$$

From the Moon's standard gravitational parameter its gravitational radius:

$$_b r_g = 1.091020268509284 \cdot 10^{-4} m$$

Among the solutions of the equation of the fourth degree, the first result can be physical, this is the greatest possible size the Moon can have. With a greater radius than this, the Earth attracts the rocks on the surface more than the Moon itself:

$$_b r_1 = 7.04273 \cdot 10^7 m \qquad _b r_2 = 1.09248 \cdot 10^{-4} m$$

$$_b r_3 = -3.52137 \cdot 10^7 m - i \cdot 6.09918 \cdot 10^7 m \qquad _b r_4 = -3.52137 \cdot 10^7 m + i \cdot 6.09918 \cdot 10^7 m \tag{4.13.13}$$

The distance within which a celestial body will disintegrate due to the tides caused by the central celestial body is called the Roche radius, and it is the solution of the following equation of the fourth order:

$$\frac{r_g \cdot (4 \cdot r - 5 \cdot r_g)}{2 \cdot r^4} = -\frac{_b r_g \cdot (_b r - _b r_g)}{_b r^4}$$

$$\frac{_b r_g \cdot (_b r - _b r_g)}{_b r^4} \cdot r^4 - 2 \cdot r_g \cdot r + \frac{5 \cdot r_g^2}{2} = 0 \tag{4.13.13}$$

Rewrite and write down the resulting equation of the third degree:

$$r^4 - \frac{2 \cdot r_g}{D_r} \cdot r + \frac{5 \cdot r_g^2}{2 \cdot D_r} = 0 \qquad D_r = \frac{_b r_g \cdot (_b r - _b r_g)}{_b r^4} \tag{4.13.14}$$

$$y^3 - \frac{10 \cdot r_g^2}{D_r} \cdot y + \left(\frac{2 \cdot r_g}{D_r}\right)^3 = 0 \tag{4.13.15}$$

Just like in the previous case, we write down the solution formulas, substitute the variable, among them the radius of the Moon:

$$1.73814\cdot 10^{7}\,m$$

Among the solutions of the equation of the fourth degree, the first result again can be physical, this is the smallest possible distance the Moon can approach the Earth. With a smaller distance than this, the Earth attracts the rocks on the surface more than the Moon itself:

$$r_1=9.48694\cdot 10^{7}\,m \qquad r_2=0.0110876\,m$$

$$r_3=-4.74347\cdot 10^{7}\,m-i\cdot 8.21593\cdot 10^{7}\,m \qquad r_4=-4.74347\cdot 10^{7}\,m+i\cdot 8.21593\cdot 10^{7}\,m \tag{4.13.16}$$

4.14 Falling orbit

A test body moves along a geodesic also when it falls into the black hole directly, this is characterized by the following coordinate conditions:

$$t=t(\tau) \qquad \tau=\tau(t)$$

$$r=r(\tau) \qquad r=r(t)$$

$$\vartheta=const.=\frac{\pi}{2} \qquad d\vartheta=0$$

$$\varphi=const. \qquad d\varphi=0 \tag{4.14.1}$$

The equations of movement of the trajectory:

$$c\cdot\ddot{t}+2\cdot\Gamma^{t}{}_{tr}\cdot c\cdot\dot{t}\cdot\dot{r}=0$$

$$\ddot{t}+\frac{r_g}{r\cdot(r-r_g)}\cdot\dot{t}\cdot\dot{r}=0 \tag{4.14.2}$$

$$\ddot{r}+\Gamma^{r}{}_{tt}\cdot c^2\cdot\dot{t}^2+\Gamma^{r}{}_{rr}\cdot\dot{r}^2+\Gamma^{r}{}_{\vartheta\vartheta}\cdot\dot{\vartheta}^2+\Gamma^{r}{}_{\varphi\varphi}\cdot\dot{\varphi}^2=0$$

$$\ddot{r}+\frac{r_g\cdot(r-r_g)}{2\cdot r^3}\cdot c^2\cdot\dot{t}^2-\frac{r_g}{2\cdot r\cdot(r-r_g)}\cdot\dot{r}^2=0 \tag{4.14.3}$$

$$\ddot{\vartheta}+2\cdot\Gamma^{\vartheta}{}_{r\vartheta}\cdot\dot{r}\,\dot{\vartheta}+\Gamma^{\vartheta}{}_{\varphi\varphi}\cdot\dot{\varphi}^2=0$$

$$\ddot{\vartheta}=0 \tag{4.14.4}$$

$$\ddot{\varphi}+2\cdot\Gamma^{\varphi}{}_{r\varphi}\cdot\dot{r}\,\dot{\varphi}+2\cdot\Gamma^{\varphi}{}_{\vartheta\varphi}\cdot\dot{\vartheta}\cdot\dot{\varphi}=0$$

$$\ddot{\varphi}=0 \tag{4.14.5}$$

Substitute the coordinate conditions into the arc length squared:

$$ds^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-\frac{dr^2}{1-\frac{r_g}{r}}-r^2\cdot\left(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2\right)$$

$$c^2\cdot d\,\tau^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-\frac{dr^2}{1-\frac{r_g}{r}}$$

The relationship between the proper time and the coordinate time is velocity dependent:

$$d\,\tau=\sqrt{1-\frac{r_g}{r}-\frac{v_r^2}{c^2\cdot\left(1-\frac{r_g}{r}\right)}}\cdot dt \qquad v_r=\frac{dr}{dt} \tag{4.14.6}$$

We make the arc length squared along a time-like infalling geodesic equal to the arc length squared of the co-moving coordinate system, then divide with the change in proper time, and write down the equation with the tangent vectors:

$$A\cdot c^2\cdot\frac{dt^2}{d\,\tau^2}-B\cdot\frac{dr^2}{d\,\tau^2}=\frac{c^2\cdot d\,\tau^2}{d\,\tau^2}=c^2$$

$$u^t=\frac{dt}{d\,\tau} \qquad u^r=\frac{dr}{d\,\tau}$$

$$A\cdot c^2\cdot(u^t)^2-B\cdot(u^r)^2=c^2 \tag{4.14.7}$$

We have derived in the mathematical introduction, that if the partial derivative of the metric tensor along a coordinate is zero, then the corresponding covariant tangent vector is a constant of movement:

$$\frac{\partial g_{\eta\kappa}}{\partial t}=0 \quad\rightarrow\quad \frac{\partial u_t}{\partial t}=0 \tag{4.14.8}$$

We calculate the time-like covariant tangent vector from the contravariant one with index lowering:

$$u_t=g_{t\alpha}\cdot u^{\alpha}=g_{tt}\cdot u^t=A\cdot u^t \tag{4.14.9}$$

Rearrange the arc length squared and express the square of the time-like covariant tangent vector:

$$c^2 \cdot (u_t)^2 = A^2 \cdot c^2 \cdot (u^t)^2 = A \cdot (c^2 + B \cdot (u^r)^2)$$

$$c^2 \cdot (u_t)^2 = A \cdot c^2 + (u^r)^2 \qquad \text{because:} \qquad B = \frac{1}{A} \tag{4.14.10}$$

At the beginning of the fall, the radial velocity is zero:

$$c^2 \cdot (u_t)^2 = A(r_0) \cdot c^2 \tag{4.14.11}$$

We make the two results equal, and express the radial velocity. We pick the negative root, because the numeric value of the radial coordinate has to decrease, we are looking for the infalling solution. r_0 is the radial coordinate of the starting point:

$$u^r = -c \cdot \sqrt{A(r_0) - A(r)}$$

$$\frac{dr}{d\tau} = -c \cdot \sqrt{\frac{r_g}{r_0} \cdot \frac{r_0 - r}{r}}$$

$$\int_0^\tau d\tau = -\frac{1}{c} \cdot \sqrt{\frac{r_0}{r_g}} \cdot \int_{r_0}^{r} \sqrt{\frac{r'}{r_0 - r'}} \cdot dr'$$

The time dependence of the fall:

$$\tau = \frac{1}{c} \cdot \sqrt{\frac{r_0}{r_g}} \cdot \left(\sqrt{r \cdot (r_0 - r)} + \frac{r_0}{2} \cdot \left(\frac{\pi}{2} + \arcsin\left(1 - \frac{2 \cdot r}{r_0}\right)\right)\right) \tag{4.14.12}$$

The proper time passes from the left to the right on the graph, the vertical axis is the radius. The horizontal dotted line is the event horizon. The trajectory of the infalling body apparently crosses it as if it were not there:

The test body falling into the black hole reaches the centre in a finite proper time:

$$\tau_{max}=\frac{\pi}{2}\cdot\sqrt{\frac{r_0}{r_g}}\cdot\frac{r_0}{c} \tag{4.14.13}$$

In order to form an idea of the magnitudes, let us replace the Sun with a black hole of the same mass, and we jump into the depth from a distance that corresponds to the surface of the Sun. The radius of the Sun:

$$r_0=6.955\cdot 10^8 m$$

The time that passes until the impact, from the point of view of the falling astronauts:

$$\tau_{max}=29\,min\,28.5\,s \tag{4.14.14}$$

Calculate the movement with respect to the coordinate time:

$$\frac{dr}{dt}=\frac{dr}{d\tau}\cdot\frac{d\tau}{dt}=\frac{dr}{d\tau}\cdot\frac{1}{u^t} \tag{4.14.15}$$

The contravariant time-oriented tangent vector changes during the movement, its covariant counterpart however does not, therefore we substitute the latter:

$$\frac{dr}{dt}=\frac{dr}{d\tau}\cdot\frac{A}{u_t} \qquad u^t=\frac{u_t}{A} \tag{4.14.16}$$

Substitute the time-oriented covariant tangent vector:

$$\frac{dr}{dt}=\frac{dr}{d\tau}\cdot\frac{A}{\sqrt{A(r_0)}} \qquad u_t=\sqrt{A(r_0)} \tag{4.14.17}$$

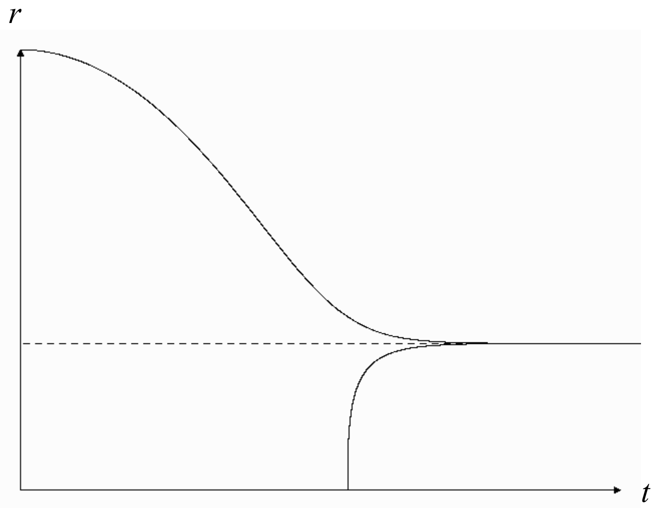
The complete expression cannot be integrated in a closed form:

$$dt=\frac{1}{c}\cdot\sqrt{\frac{r_0\cdot r_g}{r_0}}\cdot\sqrt{\frac{r_0}{r_g}}\cdot\frac{r}{r-r_g}\cdot\sqrt{\frac{r}{r_0-r}}\cdot dr$$

$$t=\frac{1}{c}\cdot\sqrt{\frac{r_0\cdot r_g}{r_g}}\cdot\int_{r_0}^{r}\frac{r'}{r'-r_g}\cdot\sqrt{\frac{r'}{r_0-r'}}\cdot dr' \tag{4.14.18}$$

This integral asymptotically approaches the event horizon, but it reaches it after an infinite time. The situation inside the event horizon is similar, if we track backwards the geodesics with respect to the coordinate time, we experience the same when approaching the Schwarzschild radius. The distant observer cheering the afore mentioned brave astronauts will see, that his peers get slower as they approach the black hole, and never reach it, or rather they do but in an infinite time.

The coordinate time passes from the left to the right on the graph, the vertical axis is the radius. The horizontal dotted line is the event horizon. The trajectory of the infalling body breaks on this graph:



The function describing the movement of the body asymptotically approaches the event horizon from both sides, but reaches it only at infinity.

4.15 Isotropic coordinates

The spherically symmetric spacetime can be mapped with other kind of coordinate systems as well. We look at the general form of the arc length squared again, and modify the arbitrary functions:

$$ds^2 = A_I(r)\cdot c^2\cdot dt^2 - B_I(r)\cdot(dr_I^2 + r_I^2\cdot(d\vartheta^2 + \sin^2(\vartheta)\cdot d\varphi^2)) \quad (4.15.1)$$

Compare it with the arc length squared in the Schwarzschild coordinate system:

$$ds^2 = A_S(r)\cdot c^2\cdot dt^2 - B_S(r)\cdot dr_S^2 - r_S^2\cdot(d\vartheta^2 + \sin^2(\vartheta)\cdot d\varphi^2) \quad (4.15.2)$$

The arc length squared is an invariant quantity, therefore the two are equal. This is also true, if we just measure it along one of the coordinates, that has always the same direction in each cases. Therefore the following equations can be written down with the arc length squares along the time-like, radial, and horizontal coordinates:

$$A_I = A_S$$

$$B_I\cdot dr_I^2 = B_S\cdot dr_S^2$$

$$B_I\cdot r_I^2 = r_S^2 \quad (4.15.3)$$

Divide the two equations with each other, and square root:

$$\frac{dr_I}{r_I} = \sqrt{B_S} \cdot \frac{dr_S}{r_S}$$

$$\frac{dr_I}{r_I} = \frac{1}{r_S \cdot \sqrt{1 - \frac{r_g}{r_S}}} \cdot dr_S$$

Integrate and resolve the logarithm:

$$\log(r_I) = \log\left(\sqrt{1 - \frac{r_g}{r_S}} + 1\right) - \log\left(\sqrt{1 - \frac{r_g}{r_S}} - 1\right) + C$$

$$r_I = C \cdot \frac{\sqrt{1 - \frac{r_g}{r_S}} + 1}{\sqrt{1 - \frac{r_g}{r_S}} - 1}$$

$$r_I = C \cdot \left(r_S - \frac{r_g}{2} + \sqrt{r_S \cdot (r_S - r_g)}\right) \tag{4.15.4}$$

The unknown multiplier can be determined from a geometric condition: infinitely distant from the gravitational centre, or by turning off the gravity completely, the two coordinate systems should coincide, in this case the extent of the event horizon is zero:

$$r_I = C \cdot \left(r_S - \frac{0}{2} + \sqrt{r_S \cdot (r_S - 0)}\right)$$

$$C = \frac{1}{2} \tag{4.15.5}$$

Express the radial coordinate of the Schwarzschild coordinate system, and determine with it the first unknown function of the isotropic coordinate system's arc length squared:

$$r_S = r_I \cdot \left(1 + \frac{r_g}{4 \cdot r_I}\right)^2 \tag{4.15.6}$$

$$A_I = A_S = 1 - \frac{r_g}{r_S} = 1 - \frac{r_g}{r_I \cdot \left(1 + \frac{r_g}{4 \cdot r_I}\right)^2} = \left(\frac{4 \cdot r_I - r_g}{4 \cdot r_I + r_g}\right)^2 \tag{4.15.7}$$

The other unknown function:

$$B_I=\frac{r_S^2}{r_I^2}=\frac{r_I\cdot\left(1+\frac{r_g}{4\cdot r_I}\right)^2}{r_I^2}=\left(\frac{4\cdot r_I+r_g}{4\cdot r_I}\right)^4 \tag{4.15.8}$$

The gravitational radius appearing in the equations continues to be measured in the Schwarzschild coordinate system, here it is a constant independently from the coordinate system, a quantity that characterizes the mass. The arc length squared in the isotropic coordinate system:

$$ds^2=\left(\frac{4\cdot r-r_g}{4\cdot r+r_g}\right)^2\cdot c^2\cdot dt^2-\left(\frac{4\cdot r+r_g}{4\cdot r}\right)^4\cdot(dr^2+r^2\cdot(d\vartheta^2+\sin^2(\vartheta)\cdot d\varphi^2)) \tag{4.15.9}$$

The geometric quantities from the metric tensor to the curvature tensor:

$$g_{ij}=\begin{pmatrix}\left(\frac{4\cdot r-r_g}{4\cdot r+r_g}\right)^2 & 0 & 0 & 0\\ 0 & -\left(\frac{4\cdot r+r_g}{4\cdot r}\right)^4 & 0 & 0\\ 0 & 0 & -\left(\frac{4\cdot r+r_g}{4\cdot r}\right)^4\cdot r^2 & 0\\ 0 & 0 & 0 & -\left(\frac{4\cdot r+r_g}{4\cdot r}\right)^4\cdot r^2\cdot\sin^2(\vartheta)\end{pmatrix}$$

$$g^{ij}=\begin{pmatrix}\left(\frac{4\cdot r+r_g}{4\cdot r-r_g}\right)^2 & 0 & 0 & 0\\ 0 & -\left(\frac{4\cdot r}{4\cdot r+r_g}\right)^4 & 0 & 0\\ 0 & 0 & -\left(\frac{4\cdot r}{4\cdot r+r_g}\right)^4\cdot\frac{1}{r^2} & 0\\ 0 & 0 & 0 & -\left(\frac{4\cdot r}{4\cdot r+r_g}\right)^4\cdot\frac{1}{r^2\cdot\sin^2(\vartheta)}\end{pmatrix} \tag{4.15.10}$$

$$\frac{\partial g_{tt}}{\partial r}=\frac{16\cdot r_g\cdot(4\cdot r-rg)}{(4\cdot r+r_g)^3} \qquad \frac{\partial g_{rr}}{\partial r}=\frac{r_g\cdot(4\cdot r+r_g)^3}{64\cdot r^5}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r}=-\left(2\cdot r\cdot\left(\frac{r_g}{4\cdot r}+1\right)-r_g\right)\cdot\left(\frac{r_g}{4\cdot r}+1\right)^3$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=-\left(2\cdot r\cdot\left(\frac{r_g}{4\cdot r}+1\right)-r_g\right)\cdot\left(\frac{r_g}{4\cdot r}+1\right)^3\cdot\sin^2(\vartheta)$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-\frac{(4\cdot r+r_g)^4}{128\cdot r^2}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial g^{tt}}{\partial r}=-\frac{16\cdot r_g\cdot(4\cdot r+r_g)}{(4\cdot r-r_g)^3}\qquad\frac{\partial g^{rr}}{\partial r}=-\frac{1024\cdot r^3\cdot r_g}{(4\cdot r+r_g)^5}$$

$$\frac{\partial g^{\vartheta\vartheta}}{\partial r}=\frac{512\cdot r\cdot(4\cdot r-r_g)}{(4\cdot r+r_g)^5}\qquad\frac{\partial g^{\varphi\varphi}}{\partial r}=\frac{512\cdot r\cdot(4\cdot r-r_g)}{(4\cdot r+r_g)^5\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=\frac{512\cdot r^2\cdot\cos(\vartheta)}{(4\cdot r+r_g)^4\cdot\sin^3(\vartheta)}\tag{4.15.11}$$

$$\Gamma^t{}_{tr}=\Gamma^t{}_{rt}=\frac{8\cdot r_g}{16\cdot r^2-r_g^2}$$

$$\Gamma^r{}_{tt}=\frac{2048\cdot r^4\cdot r_g\cdot(4\cdot r-r_g)}{(4\cdot r+r_g)^7}\qquad\Gamma^r{}_{rr}=-\frac{2\cdot r_g}{r\cdot(4\cdot r+r_g)}$$

$$\Gamma^r{}_{\vartheta\vartheta}=-\frac{r\cdot(4\cdot r-r_g)}{4\cdot r+r_g}\qquad\Gamma^r{}_{\varphi\varphi}=-\frac{r\cdot(4\cdot r-r_g)}{4\cdot r+r_g}\cdot\sin^2(\vartheta)$$

$$\Gamma^\vartheta{}_{r\vartheta}=\Gamma^\vartheta{}_{\vartheta r}=\Gamma^\varphi{}_{r\varphi}=\Gamma^\varphi{}_{\varphi r}=\frac{4\cdot r-r_g}{r\cdot(4\cdot r+r_g)}\qquad\Gamma^\vartheta{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^\varphi{}_{\vartheta\varphi}=\Gamma^\varphi{}_{\varphi\vartheta}=\cot(\vartheta)\tag{4.15.12}$$

$$\frac{\partial\Gamma^t{}_{tr}}{\partial r}=\frac{\partial\Gamma^t{}_{rt}}{\partial r}=-\frac{256\cdot r\cdot r_g}{(16\cdot r^2-r_g^2)^2}$$

$$\frac{\partial\Gamma^r{}_{tt}}{\partial r}=\frac{8192\cdot r^3\cdot r_g\cdot(8\cdot r^2-8\cdot r\cdot r_g+r_g^2)}{(4\cdot r+r_g)^8}\qquad\frac{\partial\Gamma^r{}_{rr}}{\partial r}=\frac{2\cdot r_g\cdot(8\cdot r+r_g)}{r^2\cdot(4\cdot r+r_g)^2}$$

$$\frac{\partial\Gamma^r{}_{\vartheta\vartheta}}{\partial r}=-\frac{16r^2+8\cdot r\cdot r_g-r_g^2}{(4\cdot r+r_g)^2}\qquad\frac{\partial\Gamma^r{}_{\vartheta\vartheta}}{\partial r}=-\frac{16r^2+8\cdot r\cdot r_g-r_g^2}{(4\cdot r+r_g)^2}\cdot\sin^2(\vartheta)$$

$$\frac{\partial\Gamma^\vartheta{}_{r\vartheta}}{\partial r}=\frac{\partial\Gamma^\vartheta{}_{\vartheta r}}{\partial r}=\frac{\partial\Gamma^\varphi{}_{r\varphi}}{\partial r}=\frac{\partial\Gamma^\varphi{}_{\varphi r}}{\partial r}=-\frac{16\cdot r^2-8\cdot r\cdot r_g-r_g^2}{r^2\cdot(4\cdot r+r_g)^2}$$

$$\frac{\partial \Gamma^{r}{}_{\varphi\varphi}}{\partial \vartheta}=-\frac{2\cdot r\cdot(4\cdot r-r_g)}{4\cdot r+r_g}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial \Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial \vartheta}=-\cos(2\cdot\vartheta) \qquad \frac{\partial \Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial \vartheta}=\frac{\partial \Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial \vartheta}=-\frac{1}{\sin^2(\vartheta)} \tag{4.15.13}$$

$$R^{t}{}_{rtr}=-R^{t}{}_{rrt}=\frac{16\cdot r_g}{r\cdot(4\cdot r+r_g)^2}$$

$$R^{t}{}_{\vartheta t\vartheta}=-R^{t}{}_{\vartheta\vartheta t}=R^{r}{}_{\vartheta r\vartheta}=-R^{r}{}_{\vartheta\vartheta r}=-\frac{8\cdot r\cdot r_g}{(4\cdot r+r_g)^2}$$

$$R^{t}{}_{\varphi t\varphi}=-R^{t}{}_{\varphi\varphi t}=R^{r}{}_{\varphi r\varphi}=-R^{r}{}_{\varphi\varphi r}=-\frac{8\cdot r\cdot r_g}{(4\cdot r+r_g)^2}\cdot\sin^2(\vartheta)$$

$$R^{r}{}_{ttr}=-R^{r}{}_{trt}=\frac{4096\cdot r^3\cdot r_g\cdot(4\cdot r-r_g)^2}{(4\cdot r+r_g)^8}$$

$$R^{\vartheta}{}_{tt\vartheta}=-R^{\vartheta}{}_{t\vartheta t}=R^{\varphi}{}_{tt\varphi}=-R^{\varphi}{}_{t\varphi t}=-\frac{2048\cdot r^3\cdot r_g\cdot(4\cdot r-r_g)^2}{(4\cdot r+r_g)^8}$$

$$R^{\vartheta}{}_{rr\vartheta}=-R^{\vartheta}{}_{r\vartheta r}=R^{\varphi}{}_{rr\varphi}=-R^{\varphi}{}_{r\varphi r}=\frac{8\cdot r_g}{r\cdot(4\cdot r+r_g)^2}$$

$$R^{\vartheta}{}_{\varphi\vartheta\varphi}=-R^{\vartheta}{}_{\varphi\varphi\vartheta}=\frac{16\cdot r\cdot r_g}{(4\cdot r+r_g)^2}\cdot\sin^2(\vartheta) \qquad R^{\varphi}{}_{\vartheta\vartheta\varphi}=-R^{\varphi}{}_{\vartheta\varphi\vartheta}=-\frac{16\cdot r\cdot r_g}{(4\cdot r+r_g)^2} \tag{4.15.14}$$

Substitute the connection coefficients of the isotropic coordinate system into the geodesic equations:

$$c\cdot\ddot{t}+2\cdot\Gamma^{t}{}_{tr}\cdot c\cdot\dot{t}\cdot\dot{r}=0$$

$$\ddot{t}+\frac{16\cdot r_g}{16\cdot r^2-r_g^2}\cdot\dot{t}\cdot\dot{r}=0 \tag{4.15.15}$$

$$\ddot{r}+\Gamma^{r}{}_{tt}\cdot c^2\cdot\dot{t}^2+\Gamma^{r}{}_{rr}\cdot\dot{r}^2+\Gamma^{r}{}_{\vartheta\vartheta}\cdot\dot{\vartheta}^2+\Gamma^{r}{}_{\varphi\varphi}\cdot\dot{\varphi}^2=0$$

$$\ddot{r}+\frac{2048\cdot r^4\cdot r_g\cdot(4\cdot r-r_g)}{(4\cdot r+r_g)^7}\cdot c^2\cdot\dot{t}^2-\frac{2\cdot r_g}{r\cdot(4\cdot r+r_g)}\cdot\dot{r}^2-\frac{r\cdot(4\cdot r-r_g)}{4\cdot r+r_g}\cdot\dot{\vartheta}^2-\frac{r\cdot(4\cdot r-r_g)}{4\cdot r+r_g}\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2=0 \tag{4.15.16}$$

$$\ddot{\vartheta}+2\cdot\Gamma^{\vartheta}{}_{r\vartheta}\cdot\dot{r}\,\dot{\vartheta}+\Gamma^{\vartheta}{}_{\varphi\varphi}\cdot\dot{\varphi}^2=0$$

$$\ddot{\vartheta}+\frac{8\cdot r-r_g}{r\cdot(4\cdot r+r_g)}\cdot\dot{r}\,\dot{\vartheta}-\cos(\vartheta)\cdot\sin(\vartheta)\cdot\dot{\varphi}^2=0 \tag{4.15.17}$$

$$\ddot{\varphi}+2\cdot\Gamma^{\varphi}{}_{r\varphi}\cdot\dot{r}\,\dot{\varphi}+2\cdot\Gamma^{\varphi}{}_{\vartheta\varphi}\cdot\dot{\vartheta}\cdot\dot{\varphi}=0$$

$$\ddot{\varphi}+\frac{8\cdot r-r_g}{r\cdot(4\cdot r+r_g)}\cdot\dot{r}\,\dot{\varphi}+2\cdot\cot(\vartheta)\cdot\dot{\vartheta}\cdot\dot{\varphi}=0 \tag{4.15.18}$$

Gravitational redshift in isotropic coordinates:

$${}_1\nu=\sqrt{\frac{{}_2g_{00}}{{}_1g_{00}}}\cdot{}_2\nu=\sqrt{\frac{\left(\frac{4\cdot{}_2r-r_g}{4\cdot{}_2r+r_g}\right)^2}{\left(\frac{4\cdot{}_1r-r_g}{4\cdot{}_1r+r_g}\right)^2}}\cdot{}_2\nu=\frac{4\cdot{}_2r-r_g}{4\cdot{}_1r-r_g}\cdot{}_2\nu \tag{4.15.19}$$

If the light source is closer to the source of the gravitational field than the observer, then the mutual ratios of the radii and the frequencies are the same as in the Schwarzschild case:

$${}_1r\geq{}_2r \qquad\rightarrow\qquad {}_1\nu\leq{}_2\nu \tag{4.15.20}$$

To get the orbital frequency of the test body on a circular orbit, insert the exchange formula between the coordinate systems into the result from the Schwarzschild coordinates:

$$\omega_I=c\cdot\sqrt{\frac{r_g}{2\cdot r_S^3}}=c\cdot\sqrt{\frac{2048\cdot r_I^3}{r_g^5}} \tag{4.15.21}$$

4.16 Gaussian polar coordinates

We try out further unknown functions in the general formula for the arc length squared:

$$ds^2=A_G(r)\cdot c^2\cdot dt^2-dr_G^2-C_G(r)\cdot r_G^2\cdot(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2) \tag{4.16.1}$$

Compare it with the arc length squared from the Schwarzschild coordinate system:

$$ds^2=A_S(r)\cdot c^2\cdot dt^2-B_S(r)\cdot dr_S^2-r_S^2\cdot(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2) \tag{4.16.2}$$

Arc length squared along the time-like, radial, and horizontal coordinates:

$$A_G=A_S$$

$$dr_G^2 = B_S \cdot dr_S^2$$

$$C_G \cdot r_G^2 = r_S^2 \tag{4.16.3}$$

The second and third equations are related, we perform the integration:

$$r_G = \int \sqrt{B_S} \cdot dr_S = \frac{r_S}{\sqrt{C_G}}$$

$$r_G = \int \frac{1}{\sqrt{1 - \frac{r_g}{r_S}}} \cdot dr_S = \frac{r_S}{\sqrt{C_G}}$$

Exchange between the radial coordinates of the Gaussian polar and the Schwarzschild coordinate systems:

$$r_G = \frac{r_g}{2} \cdot \left(\log\left(\sqrt{1 - \frac{r_g}{r_S}} + 1 \right) - \log\left(\sqrt{1 - \frac{r_g}{r_S}} - 1 \right) \right) + r_S \cdot \sqrt{1 - \frac{r_g}{r_S}} + K$$

$$r_g = 0 \qquad \rightarrow \qquad r_G = 0 + r_S \cdot \sqrt{1 - \frac{0}{r_S}} + K \qquad \rightarrow \qquad K = 0$$

$$r_G = \frac{r_g}{2} \cdot \left(\log\left(\sqrt{1 - \frac{r_g}{r_S}} + 1 \right) - \log\left(\sqrt{1 - \frac{r_g}{r_S}} - 1 \right) \right) + r_S \cdot \sqrt{1 - \frac{r_g}{r_S}} \tag{4.16.4}$$

Compare the Gaussian polar coordinates with the isotropic coordinates as well:

$$ds^2 = A_G(r) \cdot c^2 \cdot dt^2 - dr_G^2 - C_G(r) \cdot r_G^2 \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2)$$

$$ds^2 = A_I(r) \cdot c^2 \cdot dt^2 - B_I(r) \cdot (dr_I^2 + r_I^2 \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2)) \tag{4.16.6}$$

Arc length squared along the time-like, radial, and horizontal coordinates:

$$A_G = A_I$$

$$dr_G^2 = B_I \cdot dr_I^2$$

$$C_G \cdot r_G^2 = B_I \cdot r_I^2 \tag{4.16.7}$$

Integrate the second relationship:

$$dr_G^2 = B_I \cdot dr_I^2$$

$$dr_G = \sqrt{B_I} \cdot dr_I = \sqrt{\left(\frac{4 \cdot r_I + r_g}{4 \cdot r_I}\right)^4} \cdot dr_I$$

$$r_G = \int \left(\frac{4 \cdot r_I + r_g}{4 \cdot r_I}\right)^2 \cdot dr_I$$

$$r_G = \frac{r_g}{2} \cdot \left(\log(r_I) - \frac{r_g}{8 \cdot r_I}\right) + r_I + C$$

$$r_g = 0 \qquad \rightarrow \qquad r_G = 0 + r_I + C \qquad \rightarrow \qquad C = 0$$

Exchange between the radial coordinates of the Gaussian polar and the isotropic coordinate systems:

$$r_G = \frac{r_g}{2} \cdot \left(\log(r_I) - \frac{r_g}{8 \cdot r_I}\right) + r_I \tag{4.16.8}$$

We get transcendent equations in both cases that lack an analytic solution, therefore we are satisfied with the relationships we have found.

4.17 Rotating Schwarzschild coordinates

It is useful to discuss several problems in rotating coordinate systems. We transform from the usual Schwarzschild coordinate system the following way:

$$\varphi \rightarrow \varphi + \omega \cdot t \tag{4.17.1}$$

This is how the arch length squared changes:

$$\begin{aligned} ds^2 = \left(A(r) - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta)\right) \cdot c^2 \cdot dt^2 - 2 \cdot \omega \cdot r^2 \cdot \sin^2(\vartheta) \cdot dt \cdot d\varphi \\ - B(r) \cdot dr^2 - r^2 \cdot (d\varphi^2 + \sin^2(\vartheta) \cdot d\varphi^2) \end{aligned} \tag{4.17.2}$$

The domain of validity extends until it preserves the signature of the original metric. For example the sign of one of the metric tensor component changes, when its value is zero:

$$A(r) - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta) = 0$$

$$1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta) = 0$$

$$r^3 - \frac{c^2}{\omega^2 \cdot \sin^2(\vartheta)} \cdot r - \frac{r_g \cdot c^2}{\omega^2 \cdot \sin^2(\vartheta)} = 0 \qquad (4.17.3)$$

The special form of the equation of the third degree and the solution formula:

$$x^3 - p \cdot x - q = 0$$

$$x_{1,2,3} = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \qquad (4.17.4)$$

Since there is a negative number under the square root, the arithmetic rules of the complex numbers apply when using the solution formula. The equation of the third degree always has a real number solution. We set up a rotating coordinate system in the Solar System, with the same angular frequency like the orbit of the Earth:

$$t_{k2} = 365.256363004\, days \qquad \rightarrow \qquad \omega = \frac{2 \cdot \pi}{t_k} = 1.99098659277 \cdot 10^{-7} \frac{1}{s}$$

The standard gravitational parameter of the Sun and the gravitational radius:

$$\gamma \cdot M = 1.32712440018 \cdot 10^{20} \frac{m^3}{s^2} \qquad \rightarrow \qquad r_g = \frac{2 \cdot \gamma \cdot M}{c^2} = 2.9532500765 \cdot 10^3\, m$$

The result:

$$r = 1.5057509 \cdot 10^{15}\, m \qquad (4.17.5)$$

This is a circle with a radius of one lightyear. At greater distances than this the constant coordinate points of the coordinate system move with a speed greater than the speed of light, therefore they are not suitable to describe the time-like paths of moving bodies.

The condition for the validity of the solution is, that the value under the square root is negative. This stops to be the case, when the angular frequency becomes so big, that the mentioned expression become zero or positive:

$$\frac{q^2}{4} + \frac{p^3}{27} \geq 0$$

$$\frac{\left(-\frac{r_g \cdot c^2}{\omega^2 \cdot \sin^2(\vartheta)}\right)^2}{4} + \frac{\left(-\frac{c^2}{\omega^2 \cdot \sin^2(\vartheta)}\right)^3}{27} \geq 0$$

$$\frac{r_g^2}{4} - \frac{c^2}{27 \cdot \omega^2 \cdot \sin^2(\vartheta)} \geq 0$$

$$\omega \geq \frac{2 \cdot c}{\sqrt{27} \cdot \sin(\vartheta) \cdot r_g} \tag{4.17.6}$$

The twice covariant metric tensor of the rotating Schwarzschild coordinate system:

$$g_{\eta\kappa} = \begin{pmatrix} 1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta) & 0 & 0 & -\omega \cdot r^2 \cdot \sin^2(\vartheta) \\ 0 & -\frac{1}{1 - \frac{r_g}{r}} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ -\omega \cdot r^2 \cdot \sin^2(\vartheta) & 0 & 0 & -r^2 \cdot \sin^2(\vartheta) \end{pmatrix} \tag{4.17.7}$$

The metric tensor has non-zero non-diagonal components. We write down a partial matrix using the rows and columns where these components appear:

$$g_{ij} = \begin{pmatrix} 1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta) & -\omega \cdot r^2 \cdot \sin^2(\vartheta) \\ -\omega \cdot r^2 \cdot \sin^2(\vartheta) & -r^2 \cdot \sin^2(\vartheta) \end{pmatrix}$$

The determinant of the partial matrix:

$$g = g_{tt} \cdot g_{\varphi\varphi} - g_{t\varphi} \cdot g_{\varphi t}$$

$$g = -\left(1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta)\right) \cdot r^2 \cdot \sin^2(\vartheta) - \omega^2 \cdot r^4 \cdot \sin^4(\vartheta) \tag{4.17.8}$$

After this the components of the twice contravariant metric tensor:

$$g^{tt} = \frac{g_{\varphi\varphi}}{g} = \frac{1}{\left(1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta)\right) - \omega^2 \cdot r^2 \cdot \sin^2(\vartheta)}$$

$$g^{t\varphi} = \frac{g_{\varphi t}}{g} = g^{\varphi t} = \frac{g_{t\varphi}}{g} = -\frac{\omega}{\left(1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta)\right) - \omega^2 \cdot r^2 \cdot \sin^2(\vartheta)}$$

$$g^{\varphi\varphi} = \frac{g_{tt}}{g} = \frac{1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta)}{-\left(1 - \frac{r_g}{r} - \left(\frac{r \cdot \omega}{c}\right)^2 \cdot \sin^2(\vartheta)\right) \cdot r^2 \cdot \sin^2(\vartheta) - \omega^2 \cdot r^4 \cdot \sin^4(\vartheta)}$$

$$g^{rr}=\frac{1}{g_{rr}}=-\left(1-\frac{r_g}{r}\right)$$

$$g^{\vartheta\vartheta}=\frac{1}{g_{\vartheta\vartheta}}=-\frac{1}{r^2} \tag{4.17.9}$$

The derivatives of the metric tensor and the connection:

$$\frac{\partial g_{tt}}{\partial r}=-\frac{2\cdot\omega^2\cdot r\cdot\sin^2(\vartheta)}{c^2}-\frac{r_g}{r^2} \qquad \frac{\partial g_{tt}}{\partial\vartheta}=-\frac{2\cdot\omega^2\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta)}{c^2}$$

$$\frac{\partial g_{t\varphi}}{\partial r}=\frac{\partial g_{\varphi t}}{\partial r}=-2\cdot\omega\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g_{t\varphi}}{\partial\vartheta}=\frac{\partial g_{\varphi t}}{\partial\vartheta}=-2\cdot\omega\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial g_{rr}}{\partial r}=\frac{r_g}{(r-r_g)^2} \qquad \frac{\partial g_{\vartheta\vartheta}}{\partial r}=-2\cdot r$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=-2\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-2\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \tag{4.17.10}$$

$$\Gamma^t{}_{tr}=\Gamma^t{}_{rt}=\frac{2\cdot(c^2-1)\cdot\omega^2\cdot r^3\cdot\sin^2(\vartheta)+c^2\cdot r_g}{2\cdot(c^2-1)\cdot\omega^2\cdot r^4\cdot\sin^2(\vartheta)+2\cdot c^2\cdot r\cdot(r-r_g)}$$

$$\Gamma^t{}_{t\vartheta}=\Gamma^t{}_{\vartheta t}=\frac{(c^2-1)\cdot\omega^2\cdot r^3\cdot\cos(\vartheta)\cdot\sin(\vartheta)}{(c^2-1)\cdot\omega^2\cdot r^3\cdot\sin^2(\vartheta)+c^2\cdot(r-r_g)}$$

$$\Gamma^r{}_{tt}=\frac{(c^2\cdot r_g-2\cdot\omega^2\cdot r^3\cdot\sin^2(\vartheta))\cdot(r-r_g)}{2\cdot c^2\cdot r^3} \qquad \Gamma^r{}_{t\varphi}=\Gamma^r{}_{\varphi t}=-\omega\cdot(r-r_g)\cdot\sin^2(\vartheta)$$

$$\Gamma^r{}_{rr}=-\frac{r_g}{2\cdot r\cdot(r-r_g)} \qquad \Gamma^r{}_{\vartheta\vartheta}=-(r-r_g) \qquad \Gamma^r{}_{\varphi\varphi}=-(r-r_g)\cdot\sin^2(\vartheta)$$

$$\Gamma^\vartheta{}_{tt}=-\frac{\omega^2}{c^2}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \Gamma^\vartheta{}_{t\varphi}=\Gamma^\vartheta{}_{\varphi t}=-\omega\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^\vartheta{}_{r\vartheta}=\Gamma^\vartheta{}_{\vartheta r}=\Gamma^\varphi{}_{r\varphi}=\Gamma^\varphi{}_{\varphi r}=\frac{1}{r} \qquad \Gamma^\vartheta{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^\varphi{}_{tr}=\Gamma^\varphi{}_{rt}=\frac{c^2\cdot\omega\cdot(2\cdot r-3\cdot r_g)}{2\cdot(c^2-1)\cdot\omega^2\cdot r^4\cdot\sin^2(\vartheta)+2\cdot c^2\cdot r\cdot(r-r_g)}$$

$$\Gamma^{\varphi}{}_{t\vartheta}=\Gamma^{\varphi}{}_{\vartheta t}=\frac{c^2\cdot\omega\cdot(r-r_g)}{(c^2-1)\cdot\omega^2\cdot r^3\cdot\sin^2(\vartheta)+c^2\cdot(r-r_g)}\cdot\cot(\vartheta)$$

$$\Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\cot(\vartheta) \tag{4.17.11}$$

4.18 Kruskal-Szekeres coordinates

When we investigated the infalling path, it turned out that from the test body's point of view, the event horizon poses no obstacle. Our goal is to map the spacetime of the Schwarzschild black hole with such a coordinate system, that can be interpreted on the event horizon, and certainly covers the entire spacetime. For this we determine the trajectories of the infalling and outward heading light rays. We multiply the general spherically symmetric arc length squared on the falling light-like geodesics with a monotonic changing parameter:

$$A(r)\cdot c^2\cdot dt^2-B(r)\cdot dr^2=0 \quad /\cdot\frac{1}{d\lambda^2}$$

$$A(r)\cdot c^2\cdot\frac{dt^2}{d\lambda^2}-B(r)\cdot\frac{dr^2}{d\lambda^2}=0 \tag{4.18.1}$$

Substitute the time-oriented tangent vector, and that the second function is the reciprocal of the first:

$$u_t=A\cdot c^2\cdot\frac{dt}{d\lambda} \qquad\qquad B=\frac{1}{A}$$

$$\frac{u_t^2}{A\cdot c^2}-\frac{dr^2}{A\cdot d\lambda^2}=0 \quad /\cdot A$$

$$\frac{u_t}{c}=\pm\frac{dr}{d\lambda} \tag{4.18.2}$$

The left side of the equation is constant, therefore the change of the velocity relates linearly to the monotonic changing parameter, therefore it is also a monotonic changing parameter. Thus we can use the radius as a parameter as well:

$$d\lambda\cdot\pm\frac{u_t}{c}=dr \qquad\qquad d\lambda=\pm dr$$

$$\frac{u_t}{c}=\pm\frac{dr}{d\lambda}=\pm 1 \tag{4.18.3}$$

Substitute into the tangent vector and integrate:

$$\frac{u_t}{c} = A \cdot c \cdot \frac{dt}{dr} = \pm 1$$

$$\frac{c \cdot dt}{dr} = \pm \frac{1}{A} = \pm \frac{1}{1 - \frac{r_g}{r}}$$

$$c \cdot t = \pm \int \frac{1}{1 - \frac{r_g}{r}} \cdot dr$$

Because of the logarithm we have to differentiate between two cases, outside and inside the gravitational radius. The positive and negative sign makes a distinction between the outward and inward going light rays:

$$c \cdot t = \pm r \pm r_g \cdot \log\left(\frac{r}{r_g} - 1\right) + C \qquad r > r_g$$

$$c \cdot t = \pm r \pm r_g \cdot \log(r_g - r) + K \qquad r < r_g \tag{4.18.4}$$

Write down the two light paths separately outside the gravitational radius, and choose dimensionless integration constants:

$$c \cdot t = r + r_g \cdot \log\left(\frac{r}{r_g} - 1\right) + r_g \cdot u \tag{4.18.5}$$

$$c \cdot t = -r - r_g \cdot \log\left(\frac{r}{r_g} - 1\right) + r_g \cdot v \tag{4.18.6}$$

The u and v parameters, together with the angular coordinates describing the spherical coordinate surface, are suitable to represent the Schwarzschild solution, and eliminate the coordinate singularity. These are the Eddington-Finkelstein coordinates $(u \quad v \quad \vartheta \quad \varphi)$:

$$u = \frac{1}{r_g} \cdot \left(c \cdot t - r - r_g \cdot \log\left(\frac{r}{r_g} - 1\right)\right) \tag{4.18.7}$$

$$v = \frac{1}{r_g} \cdot \left(c \cdot t + r + r_g \cdot \log\left(\frac{r}{r_g} - 1\right)\right) \tag{4.18.8}$$

These coordinates are not mutually affine parameters, this also means for example, that if we move along the values of u from $-\infty$ and $+\infty$, we would not be able to explore the entire extent of the v geodesic:

$$u = v - 2\cdot\left(\frac{r}{r_g} + \log\left(\frac{r}{r_g} - 1\right)\right) \quad (4.18.9)$$

$$v = -u + 2\cdot\left(\frac{r}{r_g} + \log\left(\frac{r}{r_g} - 1\right)\right) \quad (4.18.10)$$

Therefore these geodesics are not complete, they leave the u and v coordinate plane. The Eddington-Finkelstein coordinates cover the same manifold as the Schwarzschild coordinates. However this is apparently not the entire spacetime, since we have found geodesics that leave the area we have mapped so far. In order to obtain the entire map, we need coordinates describing light-like geodesics, that are mutually affine parameters. Perform the following modification:

$$U = e^{-\frac{u}{2}} = e^{-\frac{v}{2}}\cdot\left(\frac{r}{r_g} - 1\right)\cdot e^{\frac{r}{r_g}} \qquad V = e^{\frac{v}{2}} = e^{\frac{u}{2}}\cdot\left(\frac{r}{r_g} - 1\right)\cdot e^{\frac{r}{r_g}} \quad (4.18.11)$$

By substitution we obtain the dependence from the Schwarzschild coordinates, these are the Kruskal-Szekeres coordinates $(U \quad V \quad \vartheta \quad \varphi)$:

$$U = \sqrt{\frac{r}{r_g} - 1}\cdot e^{\frac{r - c\cdot t}{2\cdot r_g}} \quad (4.18.12)$$

$$V = \sqrt{\frac{r}{r_g} - 1}\cdot e^{\frac{r + c\cdot t}{2\cdot r_g}} \quad (4.18.13)$$

Since they are monotonic increasing functions of the radial coordinate, that turned out to be an affine parameter earlier, these coordinates are mutually affine parameters from the point of view of the light-like geodesics they represent. Combinations of the two coordinates:

$$U\cdot V = \left(\frac{r}{r_g} - 1\right)\cdot e^{\frac{r}{r_g}} \qquad \frac{V}{U} = e^{\frac{c\cdot t}{r_g}} \quad (4.18.14)$$

We express from the product the Schwarzschild radial coordinate. Rearrange the exponential equation to a general form:

$$e^{-\frac{r}{r_g}} = \frac{1}{U\cdot V}\cdot\frac{r}{r_g} - \frac{1}{U\cdot V} \quad (4.18.15)$$

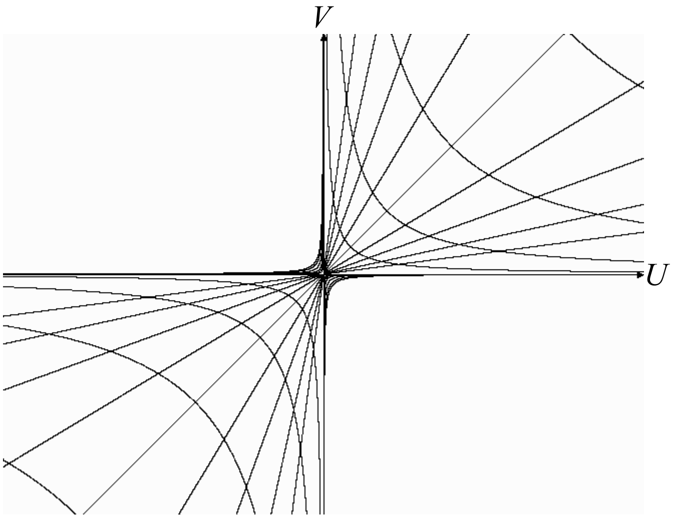
The general form and the solution formula, where $W(x)$ is the Lambert function:

$$p^{a\cdot x + b} = c\cdot x + d \qquad x = -\frac{W\left(-\frac{a\cdot\log(p)}{c}\cdot p^{b - \frac{a\cdot d}{c}}\right)}{a\cdot\log(p)} - \frac{d}{c}$$

Substitute into the solution formula:

$$\frac{r}{r_g}=W\left(\frac{U\cdot V}{e}\right)+1 \qquad r=r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g \tag{4.18.16}$$

If r = 0, then $U\cdot V=-1$, here is the true singularity of the surface, that is independent of the choice of coordinate system. The $U\cdot V>-1$ however is not only satisfied when both of them are sufficiently large positive numbers, but also when they are both negative numbers of the same magnitude. We have discovered a whole new portion of the spacetime of the Schwarzschild singularity, that was hidden from us because of the unfortunate choice of coordinate system:



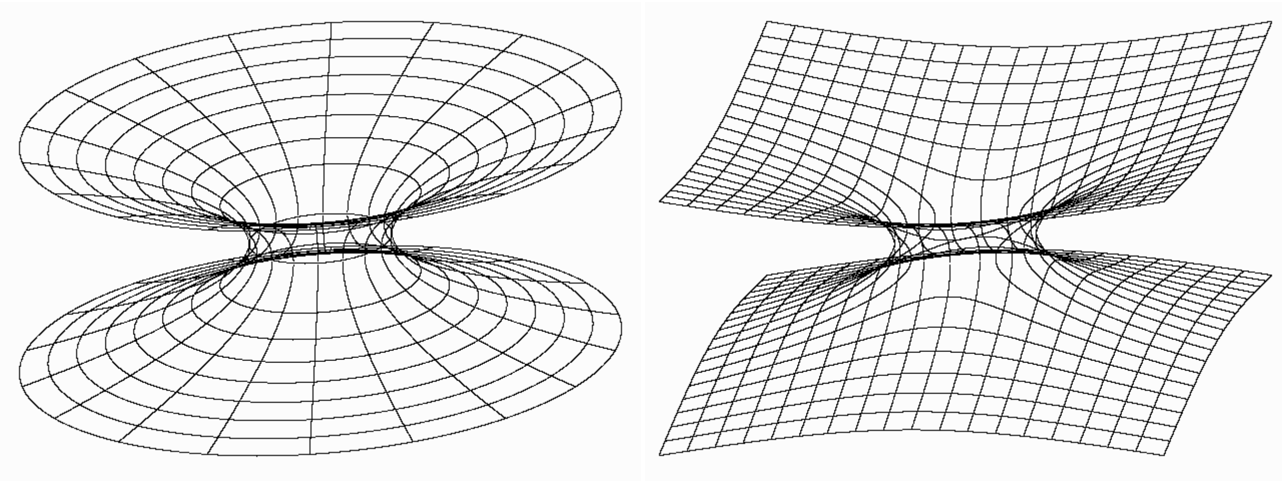
The hyperbolic surfaces connect points with the same distance from the centre, the linear surfaces connect points with the same time coordinate. The coordinate axes themselves satisfy these conditions, they connect points with -∞ and Schwarzschild radius distances, and with +∞ and -∞ time coordinates. According to this, the black hole has two “entrances”, in the lower left and the upper right quarters, separated from them by event horizons. The interior domains are represented in the upper left and lower right quarters. Negative *r* coordinates also can be defined (let us not forget that the meaning of *r* is not distance but coordinate), and its hyperboles fill the interior domain.

The coordinates that spanned the wormhole surface are also present here, the time coordinate is defined both in the first and third coordinate quarters, and the roles of the angular coordinates did not change. The curvature of the surface, thus its shape is a coordinate independent quantity, now we can depict the whole geometric shape. The parameters of the coordinate surface:

$$t=const. \qquad \vartheta=\frac{\pi}{2}$$

$$dt=0 \qquad d\vartheta=0 \tag{4.18.17}$$

The full wormhole surface in polar and rectangular coordinates:



The real significance of the wormhole is even more apparent on these graphs, that is seemingly connecting two asymptotically flat universes. However every world line that starts on one side and continues on the other, always has a space-like section at the neck of the funnel, therefore this wormhole is not traversable for massive objects slower than the speed of light.

We determine the arc length squared in the $U \cdot V > -1$ domain. Since U and V are light-like geodesics, therefore $g_{UU} = 0$ and $g_{VV} = 0$. Their product is however not zero, therefore we can expect that g_{UV} would not be either:

$$ds^2 = 2 \cdot g_{UV} \cdot dU \cdot dV - r^2 \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2) \tag{4.18.18}$$

The unknown component of the metric tensor can be calculated with the help of the Schwarzschild metric tensor, we apply the transformation formula:

$$g_{UV} = \frac{\partial U}{\partial t} \cdot \frac{\partial V}{\partial t} \cdot {}_S g_{tt} + \frac{\partial U}{\partial r} \cdot \frac{\partial V}{\partial r} \cdot {}_S g_{rr} \tag{4.18.19}$$

Substitute:

$$g_{UV} = \frac{\partial}{\partial t}\left(\sqrt{\frac{r}{r_g} - 1} \cdot e^{\frac{r - c \cdot t}{2 \cdot r_g}}\right) \cdot \frac{\partial}{\partial t}\left(\sqrt{\frac{r}{r_g} - 1} \cdot e^{\frac{r + c \cdot t}{2 \cdot r_g}}\right) \cdot \left(1 - \frac{r_g}{r}\right)$$
$$- \frac{\partial}{\partial r}\left(\sqrt{\frac{r}{r_g} - 1} \cdot e^{\frac{r - c \cdot t}{2 \cdot r_g}}\right) \cdot \frac{\partial}{\partial r}\left(\sqrt{\frac{r}{r_g} - 1} \cdot e^{\frac{r + c \cdot t}{2 \cdot r_g}}\right) \cdot \frac{1}{1 - \frac{r_g}{r}}$$

$$g_{UV}=-\frac{c\cdot\sqrt{\frac{r}{r_g}-1}\cdot e^{\frac{r-c\cdot t}{2\cdot r_g}}}{2\cdot r_g}\cdot\frac{c\cdot\sqrt{\frac{r}{r_g}-1}\cdot e^{\frac{r+c\cdot t}{2\cdot r_g}}}{2\cdot r_g}\cdot\left(1-\frac{r_g}{r}\right)$$
$$-\frac{r\cdot e^{\frac{r-c\cdot t}{2\cdot r_g}}}{2\cdot r_g^2\cdot\sqrt{\frac{r}{r_g}-1}}\cdot\frac{r\cdot e^{\frac{r+c\cdot t}{2\cdot r_g}}}{2\cdot r_g^2\cdot\sqrt{\frac{r}{r_g}-1}}\cdot\frac{1}{1-\frac{r_g}{r}}$$

$$g_{UV}=-\frac{(c^2\cdot r_g\cdot(r_g^3-4\cdot r\cdot r_g^2+6\cdot r^2\cdot r_g-4\cdot r^3)+r^2\cdot(c^2\cdot r^2-r_g^2))\cdot e^{\frac{r}{r_g}}}{4\cdot r_g^4\cdot(r-r_g)^2}$$

$$g_{UV}=-\frac{2\cdot r_g^3}{r}\cdot e^{-\frac{r}{r_g}} \tag{4.18.20}$$

The arc length squared of the Kruskal-Szekeres coordinate system:

$$ds^2=-\frac{4\cdot r_g^3}{r}\cdot e^{-\frac{r}{r_g}}\cdot dU\cdot dV-r^2\cdot(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2) \tag{4.18.21}$$

Substitute the transformation formula between the Schwarzschild and Kruskal-Szekeres coordinates into the arc length squared, that can be used already to calculate the usual geometric quantities:

$$ds^2=-\frac{4\cdot r_g^2}{W\left(\frac{U\cdot V}{e}\right)+1}\cdot e^{-W\left(\frac{U\cdot V}{e}\right)-1}\cdot dU\cdot dV-\left(r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g\right)^2\cdot(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2) \tag{4.18.23}$$

$$g_{UV}=g_{VU}=-\frac{2\cdot r_g^2}{W\left(\frac{U\cdot V}{e}\right)+1}\cdot e^{-W\left(\frac{U\cdot V}{e}\right)-1} \qquad g^{UV}=g^{VU}=-\frac{W\left(\frac{U\cdot V}{e}\right)+1}{2\cdot r_g^2}\cdot e^{W\left(\frac{U\cdot V}{e}\right)+1}$$

$$g_{\vartheta\vartheta}=-\left(r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g\right)^2 \qquad g^{\vartheta\vartheta}=-\frac{1}{\left(r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g\right)^2}$$

$$g_{\varphi\varphi}=-\left(r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g\right)^2\cdot\sin^2(\vartheta) \qquad g^{\varphi\varphi}=-\frac{1}{\left(r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g\right)^2\cdot\sin^2(\vartheta)} \tag{4.18.24}$$

$$\frac{\partial g_{UV}}{\partial U}=\frac{\partial g_{VU}}{\partial U}=\frac{2\cdot r_g^2\cdot V\cdot\left(W\left(\frac{U\cdot V}{e}\right)+2\right)}{e^{2\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial U}=-\frac{2\cdot r_g^2\cdot V}{e^{W\left(\frac{U\cdot V}{e}\right)+2}} \qquad \frac{\partial g_{\varphi\varphi}}{\partial U}=-\frac{2\cdot r_g^2\cdot V}{e^{W\left(\frac{U\cdot V}{e}\right)+2}}\cdot\sin^2(\vartheta)$$

$$\frac{\partial g_{UV}}{\partial V}=\frac{\partial g_{VU}}{\partial V}=\frac{2\cdot r_g^2\cdot U\cdot\left(W\left(\frac{U\cdot V}{e}\right)+2\right)}{e^{2\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial V}=-\frac{2\cdot r_g^2\cdot U}{e^{W\left(\frac{U\cdot V}{e}\right)+2}} \qquad \frac{\partial g_{\varphi\varphi}}{\partial V}=-\frac{2\cdot r_g^2\cdot U}{e^{W\left(\frac{U\cdot V}{e}\right)+2}}\cdot\sin^2(\vartheta)$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-\left(r_g\cdot W\left(\frac{U\cdot V}{e}\right)+r_g\right)^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad (4.18.25)$$

$$\Gamma^U{}_{UU}=-\frac{V}{e^{W\left(\frac{U\cdot V}{e}\right)+1}}\cdot\frac{W\left(\frac{U\cdot V}{e}\right)+2}{\left(W\left(\frac{U\cdot V}{e}\right)+1\right)^2}$$

$$\Gamma^U{}_{\vartheta\vartheta}=-\frac{U}{2}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right) \qquad \Gamma^V{}_{\vartheta\vartheta}=-\frac{V}{2}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)$$

$$\Gamma^U{}_{\varphi\varphi}=-\frac{U}{2}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)\cdot\sin^2(\vartheta) \qquad \Gamma^V{}_{\varphi\varphi}=-\frac{V}{2}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)\cdot\sin^2(\vartheta)$$

$$\Gamma^V{}_{VV}=-\frac{U}{e^{W\left(\frac{U\cdot V}{e}\right)+1}}\cdot\frac{W\left(\frac{U\cdot V}{e}\right)+2}{\left(W\left(\frac{U\cdot V}{e}\right)+1\right)^2}$$

$$\Gamma^\vartheta{}_{U\vartheta}=\Gamma^\vartheta{}_{\vartheta U}=\Gamma^\varphi{}_{U\varphi}=\Gamma^\varphi{}_{\varphi U}=\frac{V}{e^{W\left(\frac{U\cdot V}{e}\right)+1}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)^2}$$

$$\Gamma^{\vartheta}{}_{V\vartheta}=\Gamma^{\vartheta}{}_{\vartheta V}=\Gamma^{\varphi}{}_{V\varphi}=\Gamma^{\varphi}{}_{\varphi V}=\frac{U}{e^{W\left(\frac{U\cdot V}{e}\right)+1}\cdot\left(W\left(\frac{U\cdot V}{e}\right)+1\right)^2}$$

$$\Gamma^{\vartheta}{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta) \qquad \Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\cot(\vartheta) \tag{4.18.26}$$

4.19 Kruskal-Szekeres spacetime

We can introduce coordinates, where one of them is time-like and the other three are space-like, similar to the Schwarzschild coordinates, however they cover the entire spacetime of the singularity. This is the Kruskal-Szekeres spacetime $(c\cdot T \quad R \quad \vartheta \quad \varphi)$. The transformation formulas:

$$c\cdot T=r_g\cdot(V-U) \qquad R=r_g\cdot(V+U)$$

$$U=\frac{1}{2\cdot r_g}\cdot(R-c\cdot T) \qquad V=\frac{1}{2\cdot r_g}\cdot(R+c\cdot T) \tag{4.19.1}$$

Some useful combinations of the coordinates:

$$R^2-c^2\cdot T^2=4\cdot r_g\cdot U\cdot V=4\cdot r_g\cdot(r-r_g)\cdot e^{\frac{r}{r_g}} \qquad \frac{U}{V}=\frac{R-c\cdot T}{R+c\cdot T} \tag{4.19.2}$$

We express the Schwarzschild radial coordinate from the first. Bring the exponential equation to the general form:

$$e^{-\frac{r}{r_g}}=\frac{4\cdot r_g^2}{R^2-c^2\cdot T^2}\cdot\frac{r}{r_g}-\frac{4\cdot r_g^2}{R^2-c^2\cdot T^2} \tag{4.19.3}$$

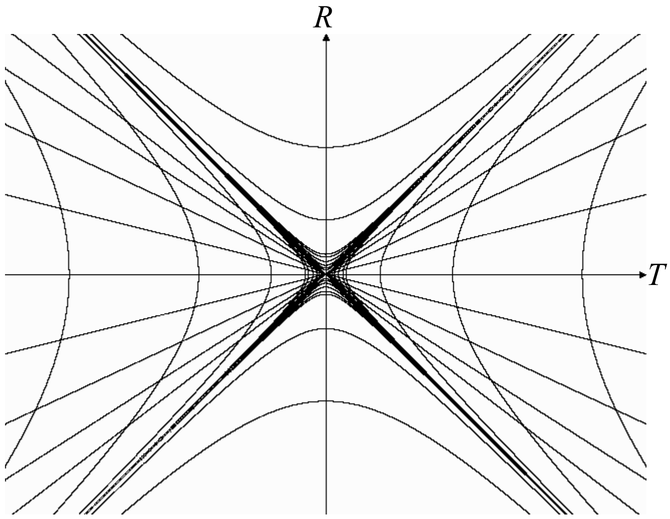
The general form and the solution formula, where *W*(*x*) is the Lambert function:

$$p^{a\cdot x+b}=c\cdot x+d \qquad x=-\frac{W\left(-\frac{a\cdot\log(p)}{c}\cdot p^{b-\frac{a\cdot d}{c}}\right)}{a\cdot\log(p)}-\frac{d}{c}$$

Substitute it into the solution formula:

$$\frac{r}{r_g}=W\left(\frac{R^2-c^2\cdot T^2}{4\cdot e\cdot r_g^2}\right)+1 \qquad r=r_g\cdot W\left(\frac{R^2-c^2\cdot T^2}{4\cdot e\cdot r_g^2}\right)+r_g \tag{4.19.4}$$

It is the same graph, but according to the coordinates of the Kruskal-Szekeres spacetime:



The arc length square of the Kruskal-Szekeres spacetime:

$$ds^2 = \frac{r_g}{r} \cdot e^{-\frac{r}{r_g}} \cdot (c^2 \cdot dT^2 - dR^2) - r^2 \cdot (d\,\vartheta^2 + \sin^2(\vartheta) \cdot d\,\varphi^2) \tag{4.19.5}$$

Substitute the transformation formula between the Schwarzschild and Kruskal-Szekeres spacetime into the arc length squared:

$$ds^2 = \frac{e^{-W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) - 1}}{W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + 1} \cdot (c^2 \cdot dT^2 - dR^2) - \left(r_g \cdot W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + r_g\right)^2 \cdot (d\,\vartheta^2 + \sin^2(\vartheta) \cdot d\,\varphi^2) \tag{4.19.6}$$

$$g_{TT} = -g_{RR} = \frac{e^{-W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) - 1}}{W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + 1} \qquad g^{TT} = -g^{RR} = \frac{W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + 1}{e^{-W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) - 1}}$$

$$g_{\vartheta\vartheta} = -\left(r_g \cdot W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + r_g\right)^2 \qquad g^{\vartheta\vartheta} = -\frac{1}{\left(r_g \cdot W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + r_g\right)^2}$$

$$g_{\varphi\varphi} = -\left(r_g \cdot W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + r_g\right)^2 \cdot \sin^2(\vartheta) \qquad g^{\varphi\varphi} = -\frac{1}{\left(r_g \cdot W\left(\frac{R^2 - c^2 \cdot T^2}{4 \cdot e \cdot r_g^2}\right) + r_g\right)^2 \cdot \sin^2(\vartheta)} \tag{4.19.7}$$

5. Spacetime of the rotating black hole

Celestial bodies influence spacetime not only with their mass, but also with their rotation. The structure of the surrounding spacetime around a rotating black hole and a rotating massive body does not coincide like in the spherically symmetric case. However those simpler phenomena that appear in the spacetime of a black hole also appear around a rotating body, although their effect depends slightly differently on the mass and rotation of the gravitational source.

We are going to encounter new phenomena, that are alien to the Newtonian theory of gravitation. The spacetime marks an axis in space, that breaks the symmetry between the two orbiting directions around the axis, and it also influences rotations pointing in the direction of the axis.

5.1 Axially symmetric spacetime

The spacetime around uniformly rotating bodies is called stationary. The metric does not change, therefore it does not depend on time, and when rotating it returns into itself, therefore neither on the longitudinal angular coordinates. The arc length squared of the axially symmetric metric in the general case, where the unknown functions depend only on *x* and *y*, that are arbitrary coordinates $(t \quad x \quad y \quad \varphi)$:

$$ds^2 = e^{2\cdot\nu}\cdot c^2\cdot dt^2 - e^{2\cdot\psi}\cdot(d\varphi - \omega\cdot c\cdot dt)^2 - e^{2\cdot\mu}\cdot dx^2 - e^{2\cdot\eta}\cdot dy^2 \tag{5.1.1}$$

We multiplied the coordinates with four unknown functions and used the angular frequency. The metric tensor:

$$g_{\eta\kappa} = \begin{pmatrix} e^{2\cdot\nu} - \omega^2\cdot e^{2\cdot\psi} & 0 & 0 & \omega\cdot e^{2\cdot\psi} \\ 0 & -e^{2\cdot\mu} & 0 & 0 \\ 0 & 0 & -e^{2\cdot\eta} & 0 \\ \omega\cdot e^{2\cdot\psi} & 0 & 0 & -e^{2\cdot\psi} \end{pmatrix} \tag{5.1.2}$$

The metric tensor has non-zero non-diagonal components. We write down a partial matrix using the rows and columns where these components appear:

$$g_{ij} = \begin{pmatrix} e^{2\cdot\nu} - \omega^2\cdot e^{2\cdot\psi} & \omega\cdot e^{2\cdot\psi} \\ \omega\cdot e^{2\cdot\psi} & -e^{2\cdot\psi} \end{pmatrix} \tag{5.1.3}$$

The determinant of the partial matrix:

$$g = g_{00}\cdot g_{\varphi\varphi} - g_{t\varphi}\cdot g_{\varphi t} = (e^{2\cdot\nu} - \omega^2\cdot e^{2\cdot\psi})\cdot(-e^{2\cdot\psi}) - (\omega\cdot e^{2\cdot\psi})\cdot(\omega\cdot e^{2\cdot\psi}) = -e^{2\cdot(\nu-\psi)} \tag{5.1.4}$$

Invert the partial matrix and extend with it the twice contravariant metric tensor:

$$g^{\eta\kappa}=\begin{pmatrix} e^{-2\cdot\nu} & 0 & 0 & \omega\cdot e^{-2\cdot\nu} \\ 0 & -e^{-2\cdot\mu} & 0 & 0 \\ 0 & 0 & -e^{-2\cdot\eta} & 0 \\ \omega\cdot e^{-2\cdot\nu} & 0 & 0 & -e^{-2\cdot\psi}+\omega^2\cdot e^{-2\cdot\nu} \end{pmatrix} \qquad (5.1.5)$$

Keep the angular frequency on a constant value for now, in this case for example we restrict ourselves to a single direction of revolution on an equatorial circular orbit. After we calculated the geometric quantities, we obtain the components of the simplified Ricci tensor:

$$P=\frac{\partial\eta}{\partial y}-\frac{\partial\mu}{\partial y}-\frac{\partial\psi}{\partial y}-\frac{\partial\nu}{\partial y} \qquad Q=\frac{\partial\mu}{\partial x}-\frac{\partial\eta}{\partial x}-\frac{\partial\psi}{\partial x}-\frac{\partial\nu}{\partial x}$$

$$R_{tt}=\omega^2\cdot e^{2\cdot(\psi-\eta)}\cdot\left(\frac{\partial\psi}{\partial y}\cdot P-\frac{\partial^2\psi}{\partial y^2}\right)+\omega^2\cdot e^{2\cdot(\psi-\mu)}\cdot\left(\frac{\partial\psi}{\partial x}\cdot Q-\frac{\partial^2\psi}{\partial x^2}\right)$$
$$+e^{2\cdot(\nu-\mu)}\cdot\left(\frac{\partial^2\nu}{\partial x^2}-\frac{\partial\nu}{\partial x}\cdot Q\right)+e^{2\cdot(\nu-\eta)}\cdot\left(\frac{\partial^2\nu}{\partial y^2}-\frac{\partial\nu}{\partial y}\cdot P\right)$$

$$R_{t\varphi}=R_{\varphi t}=\omega\cdot e^{2\cdot(\psi-\eta)}\cdot\left(\frac{\partial\psi}{\partial y}\cdot P-\frac{\partial^2\psi}{\partial y^2}\right)+\omega\cdot e^{2\cdot(\psi-\mu)}\cdot\left(\frac{\partial\psi}{\partial x}\cdot Q-\frac{\partial^2\psi}{\partial x^2}\right)$$

$$R_{xx}=e^{2\cdot(\mu-\eta)}\cdot\left(\frac{\partial\mu}{\partial y}\cdot P-\frac{\partial^2\mu}{\partial y^2}\right)+\left(\frac{\partial\mu}{\partial x}-Q\right)\cdot\frac{\partial\mu}{\partial x}-\frac{\partial^2\eta}{\partial x^2}-\left(\frac{\partial\eta}{\partial x}\right)^2-\frac{\partial^2\psi}{\partial x^2}-\left(\frac{\partial\psi}{\partial x}\right)^2-\frac{\partial^2\nu}{\partial x^2}-\left(\frac{\partial\nu}{\partial x}\right)^2$$

$$R_{yy}=+e^{2\cdot(\eta-\mu)}\cdot\left(\left(Q+\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial\eta}{\partial x}-\frac{\partial^2\eta}{\partial x^2}-\left(\frac{\partial\eta}{\partial x}\right)^2\right)$$
$$+\left(\frac{\partial\eta}{\partial y}-P\right)\cdot\frac{\partial\eta}{\partial y}-\frac{\partial^2\mu}{\partial y^2}-\left(\frac{\partial\mu}{\partial y}\right)^2-\frac{\partial^2\psi}{\partial y^2}-\left(\frac{\partial\psi}{\partial y}\right)^2-\frac{\partial^2\nu}{\partial y^2}-\left(\frac{\partial\nu}{\partial y}\right)^2$$

$$R_{xy}=R_{yx}=\left(\frac{\partial\psi}{\partial y}+\frac{\partial\nu}{\partial y}\right)\cdot\frac{\partial\eta}{\partial x}+\left(\frac{\partial\psi}{\partial x}+\frac{\partial\nu}{\partial x}\right)\cdot\frac{\partial\mu}{\partial y}-\frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi}{\partial y}-\frac{\partial^2\psi}{\partial x\cdot\partial y}-\frac{\partial\nu}{\partial x}\cdot\frac{\partial\nu}{\partial y}-\frac{\partial^2\nu}{\partial x\cdot\partial y}$$

$$R_{\varphi\varphi}=e^{2\cdot(\psi-\eta)}\cdot\left(\frac{\partial\psi}{\partial y}\cdot P-\frac{\partial^2\psi}{\partial y^2}\right)+e^{2\cdot(\psi-\mu)}\cdot\left(\frac{\partial\psi}{\partial x}\cdot Q-\frac{\partial^2\psi}{\partial x^2}\right) \qquad (5.1.6)$$

We set the general Ricci tensor and the Einstein tensor to zero, and we obtain the system of equations for the rotationally symmetric stationary vacuum solutions:

$R_{tt}=0$:

$$e^{-2\cdot\mu}\cdot\left(\frac{\partial^2\nu}{\partial x^2}+\frac{\partial\nu}{\partial x}\cdot\frac{\partial}{\partial x}(\psi+\nu-\mu+\eta)\right)+e^{-2\cdot\eta}\cdot\left(\frac{\partial^2\nu}{\partial y^2}+\frac{\partial\nu}{\partial y}\cdot\frac{\partial}{\partial y}(\psi+\nu+\mu-\eta)\right)=$$
$$\frac{1}{2}\cdot e^{2\cdot(\psi-\nu)}\cdot\left(e^{-2\cdot\mu}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2+e^{-2\cdot\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$

$R_{\varphi\varphi}=0$:
$$e^{-2\cdot\mu}\cdot\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial\psi}{\partial x}\cdot\frac{\partial}{\partial x}(\psi+\nu-\mu+\eta)\right)+e^{-2\cdot\eta}\cdot\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial\psi}{\partial y}\cdot\frac{\partial}{\partial y}(\psi+\nu+\mu-\eta)\right)=$$
$$-\frac{1}{2}\cdot e^{2\cdot(\psi-\nu)}\cdot\left(e^{-2\cdot\mu}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2+e^{-2\cdot\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$

$R_{t\varphi}=0$:
$$\frac{\partial}{\partial x}\left(e^{3\cdot\psi-\nu-\mu+\eta}\cdot\frac{\partial\omega}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{3\cdot\psi-\nu-\mu+\eta}\cdot\frac{\partial\omega}{\partial y}\right)=0$$

$R_{xy}=0$:
$$\frac{\partial^2}{\partial x\cdot\partial y}(\psi+\nu)-\frac{\partial}{\partial x}(\psi+\nu)\cdot\frac{\partial\mu}{\partial y}-\frac{\partial}{\partial y}(\psi+\nu)\cdot\frac{\partial\eta}{\partial x}+\frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi}{\partial y}+\frac{\partial\nu}{\partial x}\cdot\frac{\partial\nu}{\partial y}=$$
$$\frac{1}{2}\cdot e^{2\cdot(\psi-\nu)}\cdot\frac{\partial\omega}{\partial x}\cdot\frac{\partial\omega}{\partial y}$$

$G_{xx}=0$:
$$e^{-2\cdot\eta}\cdot\left(\frac{\partial^2}{\partial y^2}(\psi+\nu)+\frac{\partial}{\partial y}(\psi+\nu)\cdot\frac{\partial}{\partial y}(\nu-\eta)+\left(\frac{\partial\psi}{\partial y}\right)^2\right)$$
$$+e^{-2\cdot\mu}\cdot\left(\frac{\partial\nu}{\partial x}\cdot\frac{\partial}{\partial x}(\psi+\eta)+\frac{\partial\psi}{\partial x}\cdot\frac{\partial\eta}{\partial x}\right)=$$
$$\frac{1}{4}\cdot e^{2\cdot(\psi-\nu)}\cdot\left(e^{-2\cdot\mu}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2-e^{-2\cdot\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$

$G_{yy}=0$:
$$e^{-2\cdot\mu}\cdot\left(\frac{\partial^2}{\partial x^2}(\psi+\nu)+\frac{\partial}{\partial x}(\psi+\nu)\cdot\frac{\partial}{\partial x}(\nu-\mu)+\left(\frac{\partial\psi}{\partial x}\right)^2\right)$$
$$+e^{-2\cdot\eta}\cdot\left(\frac{\partial\nu}{\partial y}\cdot\frac{\partial}{\partial y}(\psi+\mu)+\frac{\partial\psi}{\partial y}\cdot\frac{\partial\mu}{\partial y}\right)= \tag{5.1.7}$$
$$\frac{1}{4}\cdot e^{2\cdot(\psi-\nu)}\cdot\left(e^{-2\cdot\mu}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2-e^{-2\cdot\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$

Introduce a new notation, that can be used to rewrite the equations into a symmetric form, and we rewrite the equations originating from the *tt* and *φφ* components of the Ricci tensor:

$$\beta=\nu+\psi$$

$$\frac{\partial}{\partial x}\left(e^{\beta-\mu+\eta}\cdot\frac{\partial\nu}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\beta-\mu+\eta}\cdot\frac{\partial\nu}{\partial y}\right)=\frac{1}{2}\cdot e^{3\cdot\psi-\nu}\cdot\left(e^{-\mu+\eta}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2+e^{\mu-\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$

$$\frac{\partial}{\partial x}\left(e^{\beta-\mu+\eta}\cdot\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\beta-\mu+\eta}\cdot\frac{\partial\psi}{\partial y}\right)=-\frac{1}{2}\cdot e^{3\cdot\psi-\nu}\cdot\left(e^{-\mu+\eta}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2+e^{\mu-\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$
$$(5.1.8)$$

The sums and differences of these and the Einstein tensor components:

$$R_{tt}+R_{\varphi\varphi}=G_{xx}+G_{yy}=\frac{\partial}{\partial x}\left(e^{-\mu+\eta}\cdot\frac{\partial e^{\beta}}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{\mu-\eta}\cdot\frac{\partial e^{\beta}}{\partial y}\right)=0$$

$$R_{tt}-R_{\varphi\varphi}=-e^{3\cdot\psi-\nu}\cdot\left(e^{-\mu+\eta}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2+e^{\mu-\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)$$

$$G_{x}-G_{yy}=-e^{2\cdot(\psi-\nu)}\cdot\left(e^{-\mu+\eta}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2-e^{\mu-\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right) \tag{5.1.9}$$

We can make a coordinate condition because of the gauge freedom:

$$e^{2\cdot(\eta-\mu)}=\Delta(x,y) \tag{5.1.10}$$

Rewrite the arc length squared by substituting new functions:

$$\Delta=e^{2\cdot(\eta-\mu)} \qquad \beta=\nu+\psi \qquad \chi=e^{\nu-\psi}$$

$$ds^2=e^{\beta}\cdot\left(\chi\cdot c^2\cdot dt^2-\frac{1}{\chi}(d\varphi-\omega\cdot c\cdot dt)^2\right)-\frac{e^{\mu+\eta}}{\sqrt{\Delta}}\cdot(dx^2+\Delta\cdot dy^2) \tag{5.1.11}$$

Insert them into the equations originating from the Ricci tensor:

$$\frac{\partial}{\partial x}\left(e^{3\cdot\psi-\nu-\mu+\eta}\cdot\frac{\partial(\chi^2-\omega^2)}{\partial x}\right)+\frac{\partial}{\partial y}\left(e^{3\cdot\psi-\nu-\mu+\eta}\cdot\frac{\partial(\chi^2-\omega^2)}{\partial y}\right)=0 \tag{5.1.12}$$

Thus ω and $\chi^2-\omega^2$ satisfy the same equation. With this new solutions can be created from the axially-symmetric stationary solutions. For example the conjugate metric – that will become important later for the derivation of the Kerr solution – with the following transformation:

$$t\rightarrow i\cdot\varphi \qquad \varphi\rightarrow -i\cdot t$$

The arc length squared changes with the substitution:

$$\chi\cdot c^2\cdot dt^2-\frac{1}{\chi}\cdot(d\varphi-\omega\cdot c\cdot dt)^2 \qquad\rightarrow\qquad \frac{1}{\chi}\cdot c^2\cdot dt^2+\frac{2\cdot\omega}{\chi}\cdot c\cdot dt\cdot d\varphi-\frac{\chi^2-\omega^2}{\chi}\cdot d\varphi^2 \tag{5.1.13}$$

That can also be interpreted as the result of the following transformation:

$$\tilde{\chi}\cdot c^2\cdot dt^2-\frac{1}{\chi}\cdot(d\varphi-\tilde{\omega}\cdot c\cdot dt)^2$$

$$\tilde{\omega}=\frac{\omega}{\chi^2-\omega^2} \qquad \tilde{\chi}=\frac{\chi}{\chi^2-\omega^2} \tag{5.1.14}$$

With the choice of a proper gauge, we can rewrite the arc length squared:

$$\mu = \eta \qquad \Delta = 1$$

$$ds^2 = e^\beta \cdot \left(\chi \cdot c^2 \cdot dt^2 - \frac{1}{\chi} \cdot (d\varphi - \omega \cdot c \cdot dt)^2 \right) - e^{2\cdot\mu} \cdot (dx^2 + dy^2) \tag{5.1.15}$$

Because of this:
$$\frac{\partial^2 e^\beta}{\partial x \cdot \partial y} = 0 \tag{5.1.16}$$

We perform a coordinate transformation, where we use the exponential expression as a coordinate:

$$e^\beta = \rho \qquad (x, y) \rightarrow (\rho, z)$$

$$\frac{\partial \rho}{\partial x} = \frac{\partial z}{\partial y} \qquad \frac{\partial \rho}{\partial y} = -\frac{\partial z}{\partial x} \tag{5.1.17}$$

Substitute it into the arc length squared, where now the unknown functions depend on ρ and z, this is the Papapetrou metric $(t \quad \rho \quad z \quad \varphi)$:

$$ds^2 = \rho \cdot \left(\chi \cdot c^2 \cdot dt^2 - \frac{1}{\chi} \cdot (d\varphi - \omega \cdot c \cdot dt)^2 \right) - e^{2\cdot\mu} \cdot (d\rho^2 + dz^2) \tag{5.1.18}$$

5.2 Ernst equation

It is possible to make the metric more clear without losing generality, and the equations can be reduced to standard form. We assume that there exist a light-like surface in the metric, this is the first crucial distinction between the flat spacetime and the rotating black hole. We introduce spherical polar coordinates $(t \quad r \quad \vartheta \quad \varphi)$. The equation of the event horizon:

$$N(x, y) = N(r, \vartheta) = 0$$

The condition for being light-like is that the four-distance is zero on it:

$$g^{\alpha\beta} \cdot \frac{\partial N}{x^\alpha} \cdot \frac{\partial N}{x^\beta} = 0 \tag{5.2.1}$$

In our choice of metric:

$$e^{2\cdot(\eta-\mu)} \cdot \left(\frac{\partial N}{\partial r} \right)^2 + \left(\frac{\partial N}{\partial \vartheta} \right)^2 = 0 \tag{5.2.2}$$

A choice of gauge, and then using it for the equation of the surface:

$$e^{2\cdot(\eta-\mu)} = \Delta(r) = 0 \tag{5.2.3}$$

On the surface, we can assume the general form of our exponential expression to be the following:

$$e^{\beta}=\sqrt{\Delta}\cdot f(r,\vartheta)=\sqrt{\Delta}\cdot f(\vartheta)=0 \tag{5.2.4}$$

Insert all this into the sum of the *tt* and *φφ* components of the Ricci tensor:

$$\frac{1}{2}\cdot e^{2\cdot(\psi-\nu)}\cdot\left(e^{-2\cdot\mu}\cdot\left(\frac{\partial\omega}{\partial x}\right)^2+e^{-2\cdot\eta}\cdot\left(\frac{\partial\omega}{\partial y}\right)^2\right)=0$$

$$\frac{\partial}{\partial r}\left(\sqrt{\Delta}\cdot\frac{\partial\sqrt{\Delta}}{\partial r}\right)+\frac{1}{f}\cdot\frac{\partial^2 f}{\partial\varphi^2}=0 \tag{5.2.5}$$

The solutions of the equation: $\frac{\partial^2\Delta}{\partial r^2}=2$ $\quad f=\sin(\vartheta)$

Solution for Δ: $\Delta=r^2-r_g\cdot r+a^2$ (5.2.6)

Where a and r_g are constants of integration, and our choice of symbols is not accidental of course. The following expressions are transformation invariant, here p and q are real constants, we will use these relationships later:

$${}_2r_g=\frac{2}{p}\cdot\sqrt{\left(\frac{r_g}{2}\right)^2-a^2} \qquad {}_2a=\frac{q}{p}\cdot\sqrt{\left(\frac{r_g}{2}\right)^2-a^2}$$

$$\frac{{}_2r_g}{2}-{}_2a=\frac{r_g}{2}-a \qquad p^2+q^2=1 \tag{5.2.7}$$

Return to the Papapetrou metric, write down the solutions of the metric functions, and from them the transformations between the coordinates:

$$e^{\eta-\mu}=\sqrt{\Delta} \qquad \rho=e^{\beta}=\sqrt{\Delta}\cdot\sin(\vartheta)$$

$$z=\left(r-\frac{r_g}{2}\right)\cdot\cos(\vartheta) \tag{5.2.8}$$

Introduce new coordinates, substitute them into the components of the Ricci tensor, and their combinations $(t \quad r \quad \sigma \quad \varphi)$:

$$\sigma=\cos(\vartheta) \qquad \delta=1-\sigma^2=\sin^2(\vartheta)$$

$$R_{tt}-R_{\varphi\varphi}=\frac{\partial}{\partial r}\left(\Delta\cdot\frac{\partial(\psi-\nu)}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial(\psi-\nu)}{\partial\sigma}\right)=-e^{2\cdot(\psi-\nu)}\cdot\left(\Delta\cdot\left(\frac{\partial\omega}{\partial r}\right)^2+\delta\cdot\left(\frac{\partial\omega}{\partial\sigma}\right)^2\right)$$

$$R_{t\varphi}=R_{\varphi t}=\frac{\partial}{\partial r}\left(\Delta\cdot e^{2\cdot(\psi-\nu)}\cdot\frac{\partial\omega}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\delta\cdot e^{2\cdot(\psi-\nu)}\cdot\frac{\partial\omega}{\partial\sigma}\right)=0 \qquad (5.2.9)$$

The same equations with the substitution of $\chi=e^{\nu-\psi}$:

$$R_{tt}-R_{\varphi\varphi}=\frac{\partial}{\partial r}\left(\frac{\Delta}{\chi}\cdot\frac{\partial\chi}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\delta}{\chi}\cdot\frac{\partial\chi}{\partial\sigma}\right)=\frac{\Delta\cdot\left(\frac{\partial\omega}{\partial r}\right)^2+\delta\cdot\left(\frac{\partial\omega}{\partial\sigma}\right)^2}{\chi^2}$$

$$R_{t\varphi}=R_{\varphi t}=\frac{\partial}{\partial r}\left(\frac{\Delta}{\chi^2}\cdot\frac{\partial\omega}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\delta}{\chi^2}\cdot\frac{\partial\omega}{\partial\sigma}\right)=0 \qquad (5.2.10)$$

Rearrange the equations:

$$\chi\cdot\left(\frac{\partial}{\partial r}\left(\Delta\cdot\frac{\partial\chi}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial\chi}{\partial\sigma}\right)\right)=\Delta\cdot\left(\left(\frac{\partial\chi}{\partial r}\right)^2+\left(\frac{\partial\omega}{\partial r}\right)^2\right)+\delta\cdot\left(\left(\frac{\partial\chi}{\partial\sigma}\right)^2+\left(\frac{\partial\omega}{\partial\sigma}\right)^2\right)$$

$$\chi\cdot\left(\frac{\partial}{\partial r}\left(\Delta\cdot\frac{\partial\omega}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial\omega}{\partial\sigma}\right)\right)=2\cdot\left(\Delta\cdot\frac{\partial\chi}{\partial r}\cdot\frac{\partial\omega}{\partial r}+\delta\cdot\frac{\partial\chi}{\partial\sigma}\cdot\frac{\partial\omega}{\partial\sigma}\right) \qquad (5.2.11)$$

If we perform these substitutions, we obtain two symmetric equations:

$$X=\chi+\omega \qquad\qquad Y=\chi-\omega$$

$$\frac{1}{2}\cdot(X+Y)\cdot\left(\frac{\partial}{\partial r}\left(\Delta\cdot\frac{\partial X}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial X}{\partial\sigma}\right)\right)=\Delta\cdot\left(\frac{\partial X}{\partial r}\right)^2+\delta\cdot\left(\frac{\partial X}{\partial\sigma}\right)^2$$

$$\frac{1}{2}\cdot(X+Y)\cdot\left(\frac{\partial}{\partial r}\left(\Delta\cdot\frac{\partial Y}{\partial r}\right)+\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial Y}{\partial\sigma}\right)\right)=\Delta\cdot\left(\frac{\partial Y}{\partial r}\right)^2+\delta\cdot\left(\frac{\partial Y}{\partial\sigma}\right)^2 \qquad (5.2.12)$$

The following equations will help to determine μ and η at the (5.3.13) relationship:

$$R_{xy}=R_{yx}=-\frac{\sigma}{\delta}\cdot\frac{\partial(\mu+\eta)}{\partial r}+\frac{r-\frac{r_g}{2}}{\Delta}\cdot\frac{\partial(\mu+\eta)}{\partial\sigma}=\frac{2}{(X+Y)^2}\cdot\left(\frac{\partial X}{\partial r}\cdot\frac{\partial Y}{\partial\sigma}+\frac{\partial X}{\partial\sigma}\cdot\frac{\partial Y}{\partial r}\right)$$

$$G_{xx}-G_{yy}=2\cdot\left(r-\frac{r_g}{2}\right)\cdot\frac{\partial(\mu+\eta)}{\partial r}+2\cdot\sigma\cdot\frac{\partial(\mu+\eta)}{\partial\sigma}=$$

$$\frac{4}{(X+Y)^2}\cdot\left(\Delta\cdot\frac{\partial X}{\partial r}\cdot\frac{\partial Y}{\partial r}-\delta\cdot\frac{\partial X}{\partial\sigma}\cdot\frac{\partial Y}{\partial\sigma}\right)-3\cdot\frac{\left(r-\frac{r_g}{2}\right)+\Delta}{\Delta}+\frac{\sigma^2+\delta}{\delta} \qquad (5.2.13)$$

We introduce new coordinates again, and use the transformation rules we have introduced earlier

$(t \quad \kappa \quad \sigma \quad \varphi)$:

$$\kappa = \frac{r - \frac{r_g}{2}}{\sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}} \qquad \Delta = \left(\left(\frac{r_g}{2}\right)^2 - a^2\right) \cdot (\kappa^2 - 1) \tag{5.2.14}$$

Write down all four previous equations with them:

$$\frac{1}{2} \cdot (X+Y) \cdot \left(\frac{\partial}{\partial \kappa}\left((\kappa^2-1) \cdot \frac{\partial X}{\partial \kappa}\right) + \frac{\partial}{\partial \sigma}\left((1-\sigma^2) \cdot \frac{\partial X}{\partial \sigma}\right)\right) = (\kappa^2-1) \cdot \left(\frac{\partial X}{\partial \kappa}\right)^2 + (1-\sigma^2) \cdot \left(\frac{\partial X}{\partial \sigma}\right)^2$$

$$\frac{1}{2} \cdot (X+Y) \cdot \left(\frac{\partial}{\partial \kappa}\left((\kappa^2-1) \cdot \frac{\partial Y}{\partial \kappa}\right) + \frac{\partial}{\partial \sigma}\left((1-\sigma^2) \cdot \frac{\partial Y}{\partial \sigma}\right)\right) = (\kappa^2-1) \cdot \left(\frac{\partial Y}{\partial \kappa}\right)^2 + (1-\sigma^2) \cdot \left(\frac{\partial Y}{\partial \sigma}\right)^2 \tag{5.2.15}$$

$$-\frac{\sigma}{1-\sigma^2} \cdot \frac{\partial(\mu+\eta)}{\partial \kappa} + \frac{\kappa}{\kappa^2-1} \cdot \frac{\partial(\mu+\eta)}{\partial \sigma} = \frac{2}{(X+Y)^2} \cdot \left(\frac{\partial X}{\partial \kappa} \cdot \frac{\partial Y}{\partial \sigma} + \frac{\partial X}{\partial \sigma} \cdot \frac{\partial Y}{\partial \kappa}\right)$$

$$2 \cdot \kappa \cdot \frac{\partial(\mu+\eta)}{\partial \kappa} + 2 \cdot \sigma \cdot \frac{\partial(\mu+\eta)}{\partial \sigma} =$$
$$\frac{4}{(X+Y)^2} \cdot \left((\kappa^2-1) \cdot \frac{\partial X}{\partial r} \cdot \frac{\partial Y}{\partial r} - (1-\sigma^2) \cdot \frac{\partial X}{\partial \sigma} \cdot \frac{\partial Y}{\partial \sigma}\right) - \frac{3}{\kappa^2-1} + \frac{1}{1-\sigma^2} \tag{5.2.16}$$

The following transformations are also solutions of the equations describing *X* and *Y*, where *c* is an arbitrary constant, this will also become useful later:

$${}_2X = \frac{X}{1+c \cdot X} \qquad {}_2Y = \frac{Y}{1-c \cdot Y} \tag{5.2.17}$$

We express new functions from the old ones and substitute them into the symmetric equations:

$$X = \frac{1+F}{1-F} \qquad Y = \frac{1+G}{1-G}$$

$$\chi = \frac{1-F \cdot G}{(1-F) \cdot (1-G)} \qquad \omega = \frac{F-G}{(1-F) \cdot (1-G)} \tag{5.2.18}$$

$$(1-F \cdot G) \cdot \left(\frac{\partial}{\partial \kappa}\left((\kappa^2-1) \cdot \frac{\partial F}{\partial \kappa}\right) + \frac{\partial}{\partial \sigma}\left((1-\sigma^2) \cdot \frac{\partial F}{\partial \sigma}\right)\right) = -2 \cdot G \cdot \left((\kappa^2-1) \cdot \left(\frac{\partial F}{\partial \kappa}\right)^2 + (1-\sigma^2) \cdot \left(\frac{\partial F}{\partial \sigma}\right)^2\right)$$

$$(1-F \cdot G) \cdot \left(\frac{\partial}{\partial \kappa}\left((\kappa^2-1) \cdot \frac{\partial G}{\partial \kappa}\right) + \frac{\partial}{\partial \sigma}\left((1-\sigma^2) \cdot \frac{\partial G}{\partial \sigma}\right)\right) = -2 \cdot F \cdot \left((\kappa^2-1) \cdot \left(\frac{\partial G}{\partial \kappa}\right)^2 + (1-\sigma^2) \cdot \left(\frac{\partial G}{\partial \sigma}\right)^2\right) \tag{5.2.19}$$

The solutions of the equations, where p and q are real constants:

$$F = -p\cdot\kappa - q\cdot\sigma \qquad G = -p\cdot\kappa + q\cdot\sigma \qquad p^2 - q^2 = 1 \tag{5.2.20}$$

The angular frequency can be derived from a coordinate potential in the following way:

$$R_{t\varphi} = R_{\varphi t} = \frac{\partial}{\partial r}\left(\frac{\Delta}{\chi^2}\cdot\frac{\partial\omega}{\partial r}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\delta}{\chi^2}\cdot\frac{\partial\omega}{\partial\sigma}\right) = 0$$

$$\frac{\partial\Phi}{\partial\kappa} = \frac{\delta}{\chi^2}\cdot\frac{\partial\omega}{\partial\sigma} \qquad \frac{\partial\Phi}{\partial\sigma} = \frac{\Delta}{\chi^2}\cdot\frac{\partial\omega}{\partial r} \tag{5.2.21}$$

The potential is determined by the following equation:

$$\frac{\partial}{\partial\kappa}\left(\frac{\chi^2}{\delta}\cdot\frac{\partial\Phi}{\partial\kappa}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\chi^2}{\Delta}\cdot\frac{\partial\Phi}{\partial\sigma}\right) = 0 \tag{5.2.22}$$

The other equation can also be written down with a potential:

$$R_{tt} - R_{\varphi\varphi} = \frac{\partial}{\partial\kappa}\left(\Delta\cdot\frac{\partial(\log(\chi))}{\partial\kappa}\right) + \frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial(\log(\chi))}{\partial\sigma}\right) = \frac{\chi^2}{\Delta}\cdot\left(\frac{\partial\Phi}{\partial\sigma}\right)^2 + \frac{\chi^2}{\delta}\cdot\left(\frac{\partial\Phi}{\partial\kappa}\right)^2 \tag{5.2.23}$$

We introduce yet another potential, with it we can write down equations with the same form like (5.2.11), here Ψ corresponds to χ, Φ to ω, and κ to r:

$$\Psi = \frac{\sqrt{\Delta\cdot\delta}}{\chi}$$

$$\Psi\cdot\left(\frac{\partial}{\partial\kappa}\left(\Delta\cdot\frac{\partial\Psi}{\partial\kappa}\right) + \frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial\Psi}{\partial\sigma}\right)\right) = \Delta\cdot\left(\left(\frac{\partial\Psi}{\partial\kappa}\right)^2 + \left(\frac{\partial\Phi}{\partial\kappa}\right)^2\right) + \delta\cdot\left(\left(\frac{\partial\Psi}{\partial\sigma}\right)^2 + \left(\frac{\partial\Phi}{\partial\sigma}\right)^2\right)$$

$$\Psi\cdot\left(\frac{\partial}{\partial\kappa}\left(\Delta\cdot\frac{\partial\Phi}{\partial\kappa}\right) + \frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial\Phi}{\partial\sigma}\right)\right) = 2\cdot\left(\Delta\cdot\frac{\partial\Psi}{\partial\kappa}\cdot\frac{\partial\Phi}{\partial\kappa} + \delta\cdot\frac{\partial\Psi}{\partial\sigma}\cdot\frac{\partial\Phi}{\partial\sigma}\right) \tag{5.2.24}$$

If we consider the potentials to be the components of a single complex quantity, we can write down an equation like (5.2.15):

$$Z = \Psi + i\cdot\Phi$$

$$\Re(Z)\cdot\left(\frac{\partial}{\partial\kappa}\left(\Delta\cdot\frac{\partial Z}{\partial\kappa}\right) + \frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial Z}{\partial\sigma}\right)\right) = \Delta\cdot\left(\frac{\partial Z}{\partial\kappa}\right)^2 + \delta\cdot\left(\frac{\partial Z}{\partial\sigma}\right)^2 \tag{5.2.25}$$

We can write down a transformation relationship with the same form like previous one here too:

$$ {}_2Z=\frac{Z}{1+i\cdot c\cdot Z} \tag{5.2.26}$$

Since the equation has the same form, we can use a function substitution of the same form, thus we obtain the Ernst equation:

$$Z=-\frac{1+E}{1-E}$$

$$(1-E\cdot E^*)\cdot\left(\frac{\partial}{\partial\kappa}\left(\Delta\cdot\frac{\partial E}{\partial\kappa}\right)\right)+\left(\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial E}{\partial\sigma}\right)\right)=-2\cdot E^*\cdot\left(\Delta\cdot\left(\frac{\partial E}{\partial\kappa}\right)^2+\delta\cdot\left(\frac{\partial E}{\partial\sigma}\right)^2\right) \tag{5.2.27}$$

Conjugate potentials, using the conjugate metric functions:

$$\tilde{\omega}=\frac{\omega}{\chi^2-\omega^2} \qquad \tilde{\chi}=\frac{\chi}{\chi^2-\omega^2}$$

$$\tilde{\Psi}=\frac{\sqrt{\Delta\cdot\delta}}{\tilde{\chi}}=e^{\nu+\psi}\cdot\frac{\chi^2-\omega^2}{\chi}=e^{2\cdot\nu}-\omega^2\cdot e^{2\cdot\psi}$$

$$\frac{\partial\tilde{\Phi}}{\partial\kappa}=\frac{\delta}{\tilde{\chi}^2}\cdot\frac{\partial\tilde{\omega}}{\partial\sigma}=\frac{\tilde{\Psi}^2}{\Delta}\cdot\frac{\partial\tilde{\omega}}{\partial\sigma} \qquad \frac{\partial\tilde{\Phi}}{\partial\sigma}=-\frac{\Delta}{\tilde{\chi}^2}\cdot\frac{\partial\tilde{\omega}}{\partial\kappa}=-\frac{\tilde{\Psi}^2}{\delta}\cdot\frac{\partial\tilde{\omega}}{\partial\kappa}$$

$$\tilde{Z}=\tilde{\Psi}+i\cdot\tilde{\Phi}=-\frac{1+\tilde{E}}{1-\tilde{E}} \tag{5.2.28}$$

Write down the conjugate Ernst equation, and notice two more relationships between the conjugate potentials:

$$(1-\tilde{E}\cdot\tilde{E}^*)\cdot\left(\frac{\partial}{\partial\kappa}\left(\Delta\cdot\frac{\partial\tilde{E}}{\partial\kappa}\right)\right)+\left(\frac{\partial}{\partial\sigma}\left(\delta\cdot\frac{\partial\tilde{E}}{\partial\sigma}\right)\right)=-2\cdot\tilde{E}^*\cdot\left(\Delta\cdot\left(\frac{\partial\tilde{E}}{\partial\kappa}\right)^2+\delta\cdot\left(\frac{\partial\tilde{E}}{\partial\sigma}\right)^2\right)$$

$$\tilde{\Psi}=\Re(\tilde{Z})=-\frac{1-\tilde{E}\cdot\tilde{E}^*}{|1-\tilde{E}|^2} \qquad \tilde{\Phi}=\Im(\tilde{Z})=-i\cdot\frac{\tilde{E}-\tilde{E}^*}{|1-\tilde{E}|^2} \tag{5.2.29}$$

5.3 The derivation of the Kerr solution

If we investigate every possible symmetry in the axially-symmetric vacuum spacetime, we will arrive at a single analytic expression about the metric, the first time found by Roy Kerr in 1963. The conjugate Ernst equation has a similar form to (5.2.19), therefore the following match is possible, and we directly obtain the solution:

$$F=\tilde{E} \qquad G=\tilde{E}^*$$

$$\tilde{E} = -p \cdot \kappa - i \cdot q \cdot \sigma \qquad p^2 + q^2 = 1 \tag{5.3.1}$$

Express the complex potential:

$$\tilde{Z} = \tilde{\Psi} + i \cdot \tilde{\Phi} = -\frac{1 - p \cdot \kappa - i \cdot q \cdot \sigma}{1 + p \cdot \kappa + i \cdot q \cdot \sigma}$$

$$\tilde{\Psi} = \frac{p^2 \cdot (\kappa^2 - 1) - q^2 \cdot (1 - \sigma^2)}{(p \cdot \kappa + 1)^2 + q^2 \cdot \sigma^2} \qquad \tilde{\Phi} = \frac{2 \cdot q \cdot \sigma}{(p \cdot \kappa + 1)^2 + q^2 \cdot \sigma^2} \tag{5.3.2}$$

Return to the r coordinate, substitute p and q, and also introduce $\rho \quad (t \quad r \quad \sigma \quad \varphi)$:

$$p = \frac{2}{r_g} \cdot \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2} \qquad q = \frac{2 \cdot a}{r_g}$$

$$\rho^2 = r^2 + a^2 \cdot \sigma^2 = r^2 + a^2 \cdot \cos^2(\vartheta) \tag{5.3.3}$$

Write down the two potentials with them:

$$\tilde{\Psi} = \frac{\Delta - a^2 \cdot \delta}{\rho^2} \qquad \tilde{\Phi} = \frac{a \cdot r_g \cdot \sigma}{\rho^2} \tag{5.3.4}$$

Determining unknown conjugate metric functions from the potentials:

$$\frac{\partial \tilde{\Phi}}{\partial r} = -\frac{2 \cdot a \cdot r \cdot r_g \cdot \sigma}{\rho^4} = \frac{\tilde{\Psi}^2}{\Delta} \cdot \frac{\partial \tilde{\omega}}{\partial \sigma} = \frac{(\Delta - a^2 \cdot \delta)^2}{\rho^4 \cdot \Delta} \cdot \frac{\partial \tilde{\omega}}{\partial \sigma}$$

$$\frac{\partial \tilde{\Phi}}{\partial \sigma} = \frac{a \cdot r_g}{\rho^4} \cdot (r^2 - a^2 \cdot \sigma^2) = -\frac{\tilde{\Psi}^2}{\delta} \cdot \frac{\partial \tilde{\omega}}{\partial r} = -\frac{(\Delta - a^2 \cdot \delta)^2}{\rho^4 \cdot \delta} \cdot \frac{\partial \tilde{\omega}}{\partial r}$$

$$\frac{\partial \tilde{\omega}}{\partial r} = -\frac{a \cdot r_g \cdot (r^2 - a^2 \cdot \sigma^2) \cdot \delta}{(\Delta - a^2 \cdot \delta)^2} \qquad \frac{\partial \tilde{\omega}}{\partial \sigma} = -\frac{2 \cdot a \cdot r \cdot r_g \cdot \sigma \cdot \Delta}{(\Delta - a^2 \cdot \delta)^2} \tag{5.3.5}$$

Conjugate angular frequency, and from it the angular frequency:

$$\tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2} = \frac{a \cdot r \cdot r_g \cdot \delta}{\Delta - a^2 \cdot \delta} \tag{5.3.6}$$

$$\tilde{\Psi} = e^{2 \cdot \psi} \cdot (\chi^2 - \omega^2) = e^{2 \cdot \nu} - \omega^2 \cdot e^{2 \cdot \psi} = \frac{\Delta - a^2 \cdot \delta}{\rho^2}$$

$$\omega = \frac{a \cdot r \cdot r_g \cdot \delta}{\Delta - a^2 \cdot \delta} \cdot (\chi^2 - \omega^2) = \frac{a \cdot r \cdot r_g \cdot \delta}{\rho^2} \cdot e^{-2 \cdot \psi} \tag{5.3.7}$$

Combine the two upper equations and substitute the product of the two deltas:

$$e^{2\cdot\beta}=\Delta\cdot\delta$$

$$\frac{\Delta-a^2\cdot\delta}{\rho^2}\cdot e^{2\cdot\psi}=e^{2\cdot\beta}-\omega^2\cdot e^{4\cdot\psi}=\frac{\delta}{\rho^4}\cdot(\Delta\cdot\rho^4-a^2\cdot r^2\cdot r_g^2\cdot\delta) \tag{5.3.8}$$

Write down a few algebraic identities, and introduce yet another metric function:

$$((r^2+a^2)\mp a\cdot\sqrt{\Delta\cdot\delta})\cdot(\sqrt{\Delta}\pm a\cdot\sqrt{\delta})=\rho^2\cdot\sqrt{\Delta}\pm a\cdot r\cdot r_g\cdot\sqrt{\delta}$$

$$\Sigma^2\cdot(\Delta-a^2\cdot\delta)=\rho^4\cdot\Delta-a^2\cdot r^2\cdot r_g^2\cdot\delta$$

$$\Sigma^2=(r^2+a^2)^2-a^2\cdot\Delta\cdot\delta \tag{5.3.9}$$

Using this the metric functions:

$$e^{2\cdot\psi}=\frac{\delta\cdot\Sigma^2}{\rho^2} \qquad \omega=\frac{a\cdot r\cdot r_g}{\Sigma^2}$$

$$e^{2\cdot\nu}=e^{2\cdot\beta}-e^{2\cdot\psi}=\frac{\rho^2\cdot\Delta}{\Sigma^2} \qquad \chi=e^{\nu-\psi}=\frac{\rho^2}{\Sigma^2}\cdot\sqrt{\frac{\Delta}{\delta}} \tag{5.3.10}$$

Using the identity we can express *X* and *Y*, and their derivatives:

$$X=\chi+\omega=\frac{\sqrt{\Delta}+a\cdot\sqrt{\delta}}{((r^2+a^2)+a\cdot\sqrt{\Delta\cdot\delta})\cdot\sqrt{\delta}}$$

$$X=\chi-\omega=\frac{\sqrt{\Delta}-a\cdot\sqrt{\delta}}{((r^2+a^2)-a\cdot\sqrt{\Delta\cdot\delta})\cdot\sqrt{\delta}} \tag{5.3.11}$$

$$\frac{\partial X}{\partial r}=\frac{\partial Y}{\partial r}=\frac{\rho^2\cdot\left(r-\frac{r_g}{2}\right)-2\cdot r\cdot(\sqrt{\Delta}+a\cdot\sqrt{\delta})\cdot\sqrt{\delta}}{((r^2+a^2)+a\cdot\sqrt{\Delta\cdot\delta})^2\cdot\sqrt{\Delta\cdot\delta}}$$

$$\frac{\partial X}{\partial\sigma}=\frac{\partial Y}{\partial\sigma}=\frac{\sigma\cdot\sqrt{\Delta}\cdot((r^2+a^2)+a^2\cdot\delta+2\cdot a\cdot\sqrt{\Delta\cdot\delta})}{((r^2+a^2)+a\cdot\sqrt{\Delta\cdot\delta})^2\cdot\sqrt{\delta^3}} \tag{5.3.12}$$

Substitute the results above into (5.2.13):

$$-\frac{\sigma}{\delta}\cdot\frac{\partial(\mu+\eta)}{\partial r}+\frac{r-\frac{r_g}{2}}{\Delta}\cdot\frac{\partial(\mu+\eta)}{\partial\sigma}=\frac{\sigma}{\rho^2\cdot\Delta\cdot\delta}\cdot\left(\left(r-\frac{r_g}{2}\right)\cdot(\rho^2+2\cdot a^2\cdot\delta)-2\cdot r\cdot\Delta\right)$$

$$\left(r-\frac{r_g}{2}\right)\cdot\frac{\partial(\mu+\eta)}{\partial r}+\sigma\cdot\frac{\partial(\mu+\eta)}{\partial\sigma}=2-\frac{\left(r-\frac{r_g}{2}\right)^2}{\Delta}+\frac{r\cdot r_g}{\rho^2} \qquad (5.3.13)$$

The solution of the system of equation:

$$e^{\mu+\eta}=\frac{\rho^2}{\sqrt{\Delta}} \qquad (5.3.14)$$

In our choice of notation the metric functions are the followings:

$$e^{\mu-\eta}=\Delta \qquad e^{2\cdot\mu}=\frac{\rho^2}{\Delta} \qquad e^{2\cdot\eta}=\rho^2 \qquad (5.3.15)$$

We have expressed every metric function, therefore the only thing left to do is to substitute them into the original arc length squared:

$$ds^2=\rho^2\cdot\frac{\Delta}{\Sigma^2}\cdot c^2\cdot dt^2-\frac{\Sigma^2}{\rho^2}\cdot\left(d\varphi-\frac{a\cdot r\cdot r_g}{\Sigma^2}\cdot c\cdot dt\right)^2\cdot\sin^2(\vartheta)-\frac{\rho^2}{\Delta}\cdot dr^2-\rho^2\cdot d\vartheta^2 \qquad (5.3.16)$$

Where: $\Sigma^2=(r^2+a^2)^2-a^2\cdot\Delta\cdot\delta=(r^2+a^2)^2-a^2\cdot(r^2-r_g\cdot r+a^2)\cdot\sin^2(\vartheta)$

Write down the spacetime of the black hole in rotational ellipsoid coordinates:

$$x=\sqrt{r^2+a^2}\cdot\sin(\vartheta)\cdot\cos(\vartheta)$$

$$y=\sqrt{r^2+a^2}\cdot\sin(\vartheta)\cdot\sin(\vartheta)$$

$$z=r\cdot\cos(\vartheta) \qquad (5.3.17)$$

The arc length squared of the Kerr solution in Boyer-Lindquist coordinates:

$$\begin{aligned}ds^2=&\left(1-\frac{r\cdot r_g}{\rho^2}\right)\cdot c^2\cdot dt^2+\frac{2\cdot r\cdot r_g\cdot a}{\rho^2}\cdot\sin^2(\vartheta)\cdot d\varphi\cdot c\cdot dt-\frac{\rho^2}{\Delta}\cdot dr^2-\rho^2\cdot d\vartheta^2\\&-\left(r^2+a^2+\frac{r\cdot r_g\cdot a}{\rho^2}\cdot\sin^2(\vartheta)\right)\cdot\sin^2(\vartheta)\cdot d\varphi^2\end{aligned} \qquad (5.3.18)$$

Where: $\Delta=r^2-r_g\cdot r+a^2$ $\qquad$ $\rho^2=r^2+a^2\cdot\cos^2(\vartheta)$

If *a* approaches zero, we get back the spherically symmetric Schwarzschild solution, with this we managed to identify this quantity, the geometric angular momentum:

$$a=0 \quad \rightarrow \qquad ds^2=\left(1-\frac{r_g}{r}\right)\cdot c^2\cdot dt^2-\frac{dr^2}{1-\frac{r_g}{r}}-r^2\cdot\left(d\,\vartheta^2+\sin^2(\vartheta)\cdot d\,\varphi^2\right)$$

Schwarzschild radius: $r_g=\frac{2\cdot\gamma\cdot M}{c^2}$

Kerr angular momentum: $a=\frac{J}{m\cdot c}=\frac{2\cdot\gamma\cdot J}{c^3\cdot r_g}$ (5.3.19)

If the mass of the black hole approaches zero, we get back the rotational ellipsoid coordinate system in flat spacetime:

$$ds^2=c^2\cdot dt^2-\frac{\rho^2}{r^2+a^2}\cdot dr^2-\rho^2\cdot d\,\vartheta^2-(r^2+a^2)\cdot\sin^2(\vartheta)\cdot d\,\varphi^2 \qquad (5.3.18)$$

The geometric quantities from the metric tensor to the connection:

$$g_{\eta\kappa}=\begin{pmatrix} 1-\frac{r\cdot r_g}{\rho^2} & 0 & 0 & \frac{a\cdot r\cdot r_g\cdot\sin^2(\vartheta)}{\rho^2} \\ 0 & -\frac{\rho^2}{\Delta} & 0 & 0 \\ 0 & 0 & -\rho^2 & 0 \\ \frac{a\cdot r\cdot r_g\cdot\sin^2(\vartheta)}{\rho^2} & 0 & 0 & -\left(r^2+a^2+\frac{r\cdot r_g\cdot a}{\rho^2}\cdot\sin^2(\vartheta)\right)\cdot\sin^2(\vartheta) \end{pmatrix} \qquad (5.3.19)$$

The metric tensor has non-zero non-diagonal components. We write down a partial matrix using the rows and columns where these components appear:

$$g_{ij}=\begin{pmatrix} 1-\frac{r\cdot r_g}{\rho^2} & \frac{a\cdot r\cdot r_g\cdot\sin^2(\vartheta)}{\rho^2} \\ \frac{a\cdot r\cdot r_g\cdot\sin^2(\vartheta)}{\rho^2} & -\left(r^2+a^2+\frac{r\cdot r_g\cdot a}{\rho^2}\cdot\sin^2(\vartheta)\right)\cdot\sin^2(\vartheta) \end{pmatrix} \qquad (5.3.20)$$

The determinant of the partial matrix:

$$g=g_{tt}\cdot g_{\varphi\varphi}-g_{t\varphi}\cdot g_{\varphi t}=-\frac{a\cdot r\cdot r_g\cdot(\rho^2+(a-1)\cdot r\cdot r_g)\cdot\sin^2(\vartheta)+(r^2+a^2)\cdot\rho^2\cdot(\rho^2-r^2\cdot r_g)}{\rho^4}\cdot\sin^2(\vartheta) \qquad (5.3.21)$$

Invert the partial matrix and extend with it the twice contravariant metric tensor:

$$g^{\eta\kappa}=\begin{pmatrix} \frac{(r^2+a^2)^2-a^2\cdot\Delta\cdot\sin^2(\vartheta)}{\Delta\cdot\rho^2} & 0 & 0 & \frac{a\cdot r\cdot r_g}{\Delta\cdot\rho^2} \\ 0 & -\frac{\Delta}{\rho^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{\rho^2} & 0 \\ \frac{a\cdot r\cdot r_g}{\Delta\cdot\rho^2} & 0 & 0 & -\frac{\Delta-a^2\cdot\sin^2(\vartheta)}{\Delta\cdot\rho^2} \end{pmatrix} \tag{5.3.22}$$

$$\frac{\partial g_{tt}}{\partial r}=-\frac{r_g\cdot(2\cdot r^2-\rho^2)}{\rho^2} \qquad \frac{\partial g_{t\varphi}}{\partial r}=\frac{\partial g_{\varphi t}}{\partial r}=-\frac{a\cdot r_g\cdot(2\cdot r^2-\rho^2)}{\rho^2}\cdot\sin^2(\vartheta)$$

$$\frac{\partial g_{rr}}{\partial r}=\frac{(2\cdot r-r_g)\cdot\rho^2-2\cdot\Delta\cdot r}{\Delta^2} \qquad \frac{\partial g_{\vartheta\vartheta}}{\partial r}=-2\cdot r$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=\frac{a\cdot r_g\cdot(2\cdot r^2-\rho^2)\cdot\sin^2(\vartheta)-2\cdot r^2\cdot\rho^4}{\rho^2}\cdot\sin^2(\vartheta)$$

$$\frac{\partial g_{tt}}{\partial\vartheta}=-\frac{2\cdot a^2\cdot r\cdot r_g}{\rho^4}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial g_{t\varphi}}{\partial\vartheta}=\frac{\partial g_{\varphi t}}{\partial\vartheta}=\frac{2\cdot a\cdot r\cdot r_g\cdot(a^2\cdot\sin^2(\vartheta)+\rho^2)}{\rho^4}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial g_{rr}}{\partial\vartheta}=\frac{2\cdot a^2}{\Delta}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g_{\vartheta\vartheta}}{\partial\vartheta}=2\cdot a^2\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot a\cdot r\cdot r_g\cdot(a^2\cdot\sin^2(\vartheta)-2\cdot\rho^2)\cdot\sin^2(\vartheta)-2\cdot(a^2+r^2)\cdot\rho^4}{\rho^4}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \tag{5.3.23}$$

$$\Gamma^t{}_{tr}=\Gamma^t{}_{rt}=r_g\cdot(2\cdot r^2-\rho^2)\cdot\frac{(r^2+a^2)^2-a^2\cdot(r\cdot r_g+\Delta)\cdot\sin^2(\vartheta)}{2\cdot\Delta\cdot\rho^6}$$

$$\Gamma^t{}_{t\vartheta}=\Gamma^t{}_{\vartheta t}=a^2\cdot r\cdot r_g\cdot\frac{a^2\cdot(r\cdot r_g+\Delta)\cdot\sin(\vartheta)+r\cdot r_g\cdot\rho^2-(r^2+a^2)^2}{\Delta\cdot\rho^6}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^t{}_{r\varphi}=\Gamma^t{}_{\varphi r}=-a\cdot r_g\cdot\frac{(2\cdot r^2-\rho^2)\cdot((r^2+a^2)^2-a\cdot(r\cdot r_g+a\cdot\Delta)\cdot\sin^2(\vartheta))+2\cdot r^2\cdot\rho^4}{2\cdot\Delta\cdot\rho^6}\cdot\sin^2(\vartheta)$$

$$\Gamma^t{}_{\vartheta\varphi}=\Gamma^t{}_{\varphi\vartheta}=-a\cdot r\cdot r_g\cdot\cos(\vartheta)\cdot\sin(\vartheta)\cdot$$
$$\frac{a^3\cdot(r\cdot r_g+a\cdot\Delta)\cdot\sin^4(\vartheta)+a\cdot((2\cdot r\cdot r_g+a\cdot\Delta)\cdot\rho^2-a\cdot(r^2+a^2)^2)\cdot\sin^2(\vartheta)+\rho^2\cdot(r^2+a^2)\cdot(\rho^2-r^2-a^2)}{\Delta\cdot\rho^6}$$

$$\Gamma^{r}{}_{tt}=\frac{\Delta\cdot r_g\cdot(2\cdot r^2-\rho^2)}{2\cdot\rho^6} \qquad \Gamma^{r}{}_{t\varphi}=\Gamma^{r}{}_{\varphi t}=\frac{a\cdot\Delta\cdot r_g\cdot(\rho^2-2\cdot r^2)}{2\cdot\rho^6}\cdot\sin^2(\vartheta)$$

$$\Gamma^{r}{}_{rr}=\frac{r}{\rho^2}-\frac{2\cdot r-r_g}{2\cdot\Delta} \qquad \Gamma^{r}{}_{r\vartheta}=\Gamma^{r}{}_{\vartheta r}=\Gamma^{r}{}_{\vartheta\vartheta}=-\frac{a^2}{\rho^2}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^{r}{}_{\vartheta\vartheta}=-\frac{r\cdot\Delta}{\rho^2} \qquad \Gamma^{r}{}_{\varphi\varphi}=\Delta\cdot\frac{a\cdot r_g\cdot(2\cdot r^2-\rho^2)\cdot\sin^2(\vartheta)-2\cdot r\cdot\rho^4}{2\cdot\rho^6}\cdot\sin^2(\vartheta)$$

$$\Gamma^{\vartheta}{}_{tt}=-\frac{a^2\cdot r\cdot r_g}{\rho^6}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \Gamma^{\vartheta}{}_{t\varphi}=\Gamma^{\vartheta}{}_{\varphi t}=\frac{a\cdot r\cdot r_g\cdot(a^2\cdot\sin^2(\vartheta)+\rho^2)}{\rho^6}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^{\vartheta}{}_{rr}=\frac{a^2}{\Delta\cdot\rho^2}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \Gamma^{\vartheta}{}_{r\vartheta}=\Gamma^{\vartheta}{}_{\vartheta r}=\frac{r}{\rho^2}$$

$$\Gamma^{\varphi}{}_{tr}=\Gamma^{\varphi}{}_{rt}=a\cdot r_g\cdot(2\cdot r^2-\rho^2)\cdot\frac{(\Delta-a^2\cdot\sin^2(\vartheta))\cdot\sin^2(\vartheta)+r\cdot r_g}{2\cdot\Delta\cdot\rho^6}$$

$$\Gamma^{\varphi}{}_{t\vartheta}=\Gamma^{\varphi}{}_{\vartheta t}=a\cdot r\cdot r_g\cdot\frac{a^2\cdot((a^2\cdot\sin^2(\vartheta)+\rho^2-\Delta)\cdot\sin^2(\vartheta)-r\cdot r_g)-\Delta\cdot\rho^2}{\Delta\cdot\rho^6}\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^{\varphi}{}_{r\varphi}=\Gamma^{\varphi}{}_{\varphi r}=\frac{a\cdot r_g\cdot((a^2\cdot\sin^2(\vartheta)-\Delta)\cdot\sin^2(\vartheta)-a\cdot r\cdot r_g)\cdot(2\cdot r-\rho^2)+2\cdot r\cdot\rho^4\cdot(\Delta-a^2\cdot\sin^2(\vartheta))}{2\cdot\Delta\cdot\rho^6}\cdot\sin^2(\vartheta)$$

$$\begin{aligned}\Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\frac{\cos(\vartheta)\cdot\sin(\vartheta)}{\Delta\cdot\rho^6}\cdot\big(&a^2\cdot r\cdot r_g\cdot(-a^3\cdot\sin^6(\vartheta)+a\cdot(\Delta-2\cdot\rho^2)\cdot\sin^4(\vartheta)+r\cdot r_g\cdot\rho^2)\\&+a\cdot(-a\cdot\rho^4\cdot(r^2+a^2)+r\cdot r_g\cdot(2\cdot\rho^2\cdot\Delta+a^3\cdot r\cdot r_g))\cdot\sin^2(\vartheta)+\rho^4\cdot\Delta\cdot(r^2+a^2)\big)\end{aligned} \tag{5.3.24}$$

The partial derivatives of the connection get very complicated, just like the other quantities that follow from them, therefore we do not write down all of them. The geodesics of the Kerr solution:

$$c\cdot\ddot{t}+2\cdot(\Gamma^{t}{}_{tr}\cdot c\cdot\dot{t}\cdot\dot{r}+\Gamma^{t}{}_{t\vartheta}\cdot c\cdot\dot{t}\cdot\dot{\vartheta}+\Gamma^{t}{}_{r\varphi}\cdot\dot{r}\cdot\dot{\varphi}+\Gamma^{t}{}_{\vartheta\varphi}\cdot\dot{\vartheta}\cdot\dot{\varphi})=0$$

$$\ddot{r}+\Gamma^{r}{}_{tt}\cdot c^2\cdot\dot{t}^2+\Gamma^{r}{}_{rr}\cdot\dot{r}^2+\Gamma^{r}{}_{\vartheta\vartheta}\cdot\dot{\vartheta}^2+\Gamma^{r}{}_{\varphi\varphi}\cdot\dot{\varphi}^2+2\cdot(\Gamma^{r}{}_{t\varphi}\cdot c\cdot\dot{t}\cdot\dot{\varphi}+\Gamma^{r}{}_{r\vartheta}\cdot\dot{r}\cdot\dot{\vartheta})=0$$

$$\ddot{\vartheta}+\Gamma^{\vartheta}{}_{tt}\cdot c^2\cdot\dot{t}^2+\Gamma^{\vartheta}{}_{rr}\cdot\dot{r}^2+2\cdot(\Gamma^{\vartheta}{}_{t\varphi}\cdot c\cdot\dot{t}\cdot\dot{\varphi}+\Gamma^{\vartheta}{}_{r\vartheta}\cdot\dot{r}\cdot\dot{\vartheta})=0$$

$$c\cdot\ddot{t}+2\cdot(\Gamma^{t}{}_{tr}\cdot c\cdot\dot{t}\cdot\dot{r}+\Gamma^{t}{}_{t\vartheta}\cdot c\cdot\dot{t}\cdot\dot{\vartheta}+\Gamma^{t}{}_{r\varphi}\cdot\dot{r}\cdot\dot{\varphi}+\Gamma^{t}{}_{\vartheta\varphi}\cdot\dot{\vartheta}\cdot\dot{\varphi})=0 \tag{5.3.25}$$

5.4 Coordinate singularities

The singularities in the Kerr metric are much more diverse than in the spherically symmetric case. We examine the *tt* component of the metric tensor:

$$g_{tt}=1-\frac{r\cdot r_g}{\rho^2} \tag{5.4.1}$$

It becomes meaningless in the following case:

$$r\cdot r_g=\rho^2=r^2+a^2\cdot\cos^2(\vartheta)$$

$$r^2-r\cdot r_g+a^2\cdot\cos^2(\vartheta)=0$$

The solution of the quadratic equation gives the place of the infinite redshift:

$$r_{1,2}=r_g\pm\frac{\sqrt{r_g^2-4\cdot a^2\cdot\cos^2(\vartheta)}}{2} \tag{5.4.2}$$

The *rr* component of the metric tensor:

$$g_{rr}=-\frac{\rho^2}{\Delta} \tag{5.4.3}$$

$$\Delta=r^2-r_g\cdot r+a^2=0$$

The solutions give the places of the event horizons, where the metric changes signature:

$$r_{1,2}=r_g\pm\frac{\sqrt{r_g^2-4\cdot a^2}}{2} \tag{5.4.4}$$

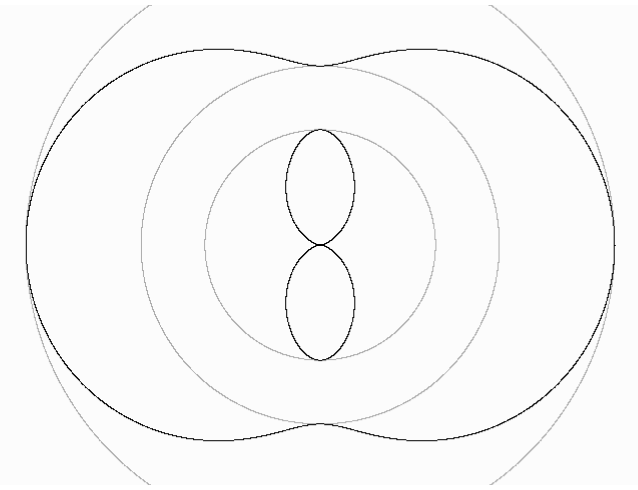
For real results, the discriminant of the quadratic formula has to be greater than zero, that creates a condition for the angular momentum:

$$r_g^2-4\cdot a^2\geq 0$$

$$r_g\geq 2\cdot|a|$$

$$\frac{\gamma\cdot M^2}{c}\geq J \tag{5.4.5}$$

For the sake of example, we examine these surfaces on the longitudinal section of a black hole with an extreme angular momentum – mass ratio:



$$\frac{a}{r_g}=\frac{84}{41}$$

Infinite redshift occurs on the black surfaces with variable shapes. The grey, spherically symmetric surfaces are the exterior and interior event horizons. The most outward, grey, unfinished spherical surface would be the Schwarzschild radius, if the black hole would not rotate. The domain between the external redshift limit and the exterior event horizon is the ergosphere.

5.5 Redshift

This time we substitute the components of the Kerr metric tensor into the previous equation:

$${}_1\nu=\sqrt{\frac{{}_2g_{tt}}{{}_1g_{tt}}}\cdot{}_2\nu=\sqrt{\frac{1-\frac{{}_2r\cdot r_g}{{}_2\rho^2}}{1-\frac{{}_1r\cdot r_g}{{}_1\rho^2}}}\cdot{}_2\nu=\sqrt{\frac{1-\frac{{}_2r\cdot r_g}{{}_2r^2+a^2\cdot\cos^2({}_2\vartheta)}}{1-\frac{{}_1r\cdot r_g}{{}_1r^2+a^2\cdot\cos^2({}_1\vartheta)}}}\cdot{}_2\nu \tag{5.5.1}$$

If the light source is closer to the source of the gravitational field than the observer, then:

$${}_1r\geq{}_2r \qquad \rightarrow \qquad {}_1\nu\leq{}_2\nu \tag{5.5.2}$$

There is a discrepancy also, when the source and the observer have the same distance from the gravitating centre, but they are on different latitudes:

$${}_1\vartheta\leq{}_2\vartheta \qquad \rightarrow \qquad {}_1\nu\leq{}_2\nu \qquad\qquad \vartheta<\frac{\pi}{2} \tag{5.5.3}$$

5.6 Frame dragging

NASA launched in 1976 and in 1992 a LAGEOS (Laser Geodynamics Satellite) each, that are passive metal spheres with diameters of 60 cm and masses of 411 kg, therefore the upper atmosphere of the Earth has practically no effect on them. By examining the laser light reflected from mirrors on their surface, it is possible to measure their highly regular orbits with great accuracy. The are used to accurately determine the shape of the Earth and the velocities of the tectonic plates, while relativistic effects cumulate in their orbital parameters. By analysing observations lasting for decades, it was possible to show the effects of spacetime dragged by the rotating Earth, with 20% accuracy.

The existence of non-diagonal tensor components has interesting consequences on the relationship between the contravariant and the covariant velocities:

$$v^t = g^{t\alpha} \cdot v_\alpha = g^{tt} \cdot c \cdot v_t + g^{t\varphi} \cdot v_\varphi \tag{5.6.1}$$

$$v^\varphi = g^{\varphi\alpha} \cdot v_\alpha = g^{\varphi t} \cdot v_t + g^{\varphi\varphi} \cdot v_\varphi \tag{5.6.2}$$

For example in the second case, if the horizontal momentum of the test body is zero, it can still have non-zero velocity and vice versa, and in the first case, it can have momentum without rest energy. Furthermore a test body at infinite distance with zero orbital velocity falling radially into the gravitational field of the rotating black hole is forced to orbit, it obtains the following angular frequency:

$$\omega_f = \frac{v^\varphi}{v^t} = \frac{g^{\varphi t} \cdot v_t + g^{\varphi\varphi} \cdot v_\varphi}{g^{tt} \cdot v_t + g^{t\varphi} \cdot v_\varphi} \qquad v_\varphi = 0 \tag{5.6.3}$$

Angular frequency caused by the frame dragging of the rotating black hole:

$$\omega_f = \frac{g^{\varphi t}}{g^{tt}} = \frac{c \cdot a \cdot r \cdot r_g}{(r^2 + a^2)^2 - a^2 \cdot \Delta \cdot \sin^2(\vartheta)} \tag{5.6.4}$$

We approach the Earth's spacetime with the Kerr metric, our instruments are not yet precise enough to distinguish between the effects of a rotating body and a rotating black hole. The angular momentum is a product of the angular frequency and the moment of inertia:

$$J = \omega \cdot \Theta \tag{5.6.5}$$

We approximate the Earth with a rigid rotating sphere:

$$\Theta = \frac{2}{5} \cdot M \cdot R^2 \qquad \rightarrow \qquad a = \frac{2}{5} \cdot \frac{\omega \cdot R^2}{c} \tag{5.6.6}$$

From the Earth's radius and rotational angular frequency, the geometric angular momentum:

$$R = 6.371 \cdot 10^6\, m \qquad t_f = 86164.1\, s \qquad \rightarrow \qquad \omega = 7.292115 \cdot 10^{-5} \frac{1}{s}$$

$$a = 3.949\, m \tag{5.6.7}$$

Earth's standard gravitational parameter and the gravitational radius:

$$\gamma \cdot M = 3.986004418 \cdot 10^{14} \frac{m^3}{s^2} \qquad \rightarrow \qquad r_g = \frac{2 \cdot \gamma \cdot M}{c^2} = 8.870056078 \cdot 10^{-3} \, m$$

We assume for simplicity, that the LAGEOS-1 satellite is on an equatorial orbit. The semi-major axis and orbital period of the orbit, and the frame dragging:

$$r = 1.227 \cdot 10^7 m \qquad t_k = 3.758\, h = 13528.8\, s$$

$$\omega_f = 5.685 \cdot 10^{-15} \frac{1}{s} \tag{5.6.8}$$

The accumulated displacement along the orbit in a year:

$$\Delta \varphi_{year} = 0.037'' \qquad \rightarrow \qquad \Delta s = 2.201\, m \tag{5.6.9}$$

We assume the same about the LAGEOS-2 satellite as well. The semi-major axis and orbital period of the orbit, and the frame dragging:

$$r = 1.2163 \cdot 10^7 m \qquad t_k = 223\, min = 13380\, s$$

$$\omega_f = 5.836 \cdot 10^{-15} \frac{1}{s} \tag{5.6.10}$$

The accumulated displacement along the orbit in a year:

$$\Delta \varphi_{year} = 0.038'' \qquad \rightarrow \qquad \Delta s = 2.24\, m \tag{5.6.9}$$

5.7 Equatorial circular orbit

The coordinate conditions coincide with the spherically symmetric case:

$$t = t(\tau) \qquad \frac{\partial t}{\partial \tau} = const.$$

$$r = const. \qquad dr = 0 \qquad \frac{\partial r}{\partial \tau} = \frac{\partial^2 r}{\partial \tau^2} = 0$$

$$\vartheta = \frac{\pi}{2} \qquad d\vartheta = 0$$

$$\varphi = \varphi(\tau) \qquad \frac{\partial \varphi}{\partial \tau} = const. \tag{5.7.1}$$

Because of the coordinate conditions, the geodesics simplify:

$$\Gamma^r{}_{tt} \cdot c^2 \cdot \dot{t}^2 + \Gamma^r{}_{\varphi\varphi} \cdot \dot{\varphi}^2 + 2 \cdot \Gamma^r{}_{t\varphi} \cdot c \cdot \dot{t} \cdot \dot{\varphi} = 0$$

$$\Gamma^\vartheta{}_{tt} \cdot c^2 \cdot \dot{t}^2 + 2 \cdot \Gamma^\vartheta{}_{t\varphi} \cdot c \cdot \dot{t} \cdot \dot{\varphi} = 0 \qquad (5.7.2)$$

The arc length squared also simplifies:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) \cdot c^2 \cdot dt^2 + \frac{2 \cdot r_g \cdot a}{r} \cdot d\varphi \cdot c \cdot dt - \left(r^2 + a^2 + \frac{r_g \cdot a}{r}\right) \cdot d\varphi^2 \qquad (5.7.3)$$

This of course is equal to the arc length measured in the coordinate system of the moving observer:

$$\omega = \frac{d\varphi}{dt}$$

$$c^2 \cdot d\tau^2 = \left(\left(1 - \frac{r_g}{r}\right) \cdot c^2 + \frac{2 \cdot r_g \cdot a}{r} \cdot c \cdot \omega - \left(r^2 + a^2 + \frac{r_g \cdot a}{r}\right) \cdot \omega^2\right) \cdot dt^2$$

The relationship between the proper time and the coordinate time:

$$d\tau = dt \cdot \sqrt{\left(1 - \frac{r_g}{r}\right) + \frac{2 \cdot r_g \cdot a}{c \cdot r} \cdot \omega - \frac{1}{c^2} \cdot \left(r^2 + a^2 + \frac{r_g \cdot a}{r}\right) \cdot \omega^2} \qquad (5.7.4)$$

This equation is satisfied by two different angular momenta, therefore it is valid on two different circular orbits.

Since the ratio of the two quantities is constant, the coordinate time also can be used as a parameter for the geodesic equations. In this case the tangent vectors can be identified, the equation can be solved, and the possible angular frequencies can be determined:

$$c \cdot \dot{t} = c \qquad \dot{\varphi} = \omega$$

$$\Gamma^r{}_{\varphi\varphi} \cdot \omega^2 + 2 \cdot \Gamma^r{}_{t\varphi} \cdot c \cdot \omega + \Gamma^r{}_{tt} \cdot c^2 = 0 \qquad (5.7.5)$$

Apply the quadratic formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4 \cdot a \cdot c}}{2 \cdot a}$$

$$\omega_{1,2} = \frac{-2 \cdot \Gamma^r{}_{t\varphi} \cdot c \pm \sqrt{(2 \cdot \Gamma^r{}_{t\varphi} \cdot c)^2 - 4 \cdot \Gamma^r{}_{\varphi\varphi} \cdot \Gamma^r{}_{tt} \cdot c^2}}{2 \cdot \Gamma^r{}_{\varphi\varphi}}$$

$$\omega_{1,2} = c \cdot \frac{-\Gamma^r{}_{t\varphi} \pm \sqrt{(\Gamma^r{}_{t\varphi})^2 - \Gamma^r{}_{\varphi\varphi} \cdot \Gamma^r{}_{tt}}}{\Gamma^r{}_{\varphi\varphi}} \qquad (5.7.6)$$

We obtain two possible orbital frequencies, each corresponding to an orbital direction The affected connection coefficients are simplified by the coordinate conditions:

$$\Gamma^{r}{}_{tt}=\frac{\Delta\cdot r_g\cdot(2\cdot r^2-\rho^2)}{2\cdot\rho^6}$$

$$\Gamma^{r}{}_{tt}=\frac{\Delta\cdot r_g}{2\cdot r^4} \tag{5.7.7}$$

$$\Gamma^{r}{}_{t\varphi}=\Gamma^{r}{}_{\varphi t}=\frac{a\cdot\Delta\cdot r_g\cdot(\rho^2-2\cdot r^2)}{2\cdot\rho^6}\cdot\sin^2(\vartheta)$$

$$\Gamma^{r}{}_{t\varphi}=\Gamma^{r}{}_{\varphi t}=-\frac{a\cdot\Delta\cdot r_g}{2\cdot r^4} \tag{5.7.8}$$

$$\Gamma^{r}{}_{\varphi\varphi}=\Delta\cdot\frac{a\cdot r_g\cdot(2\cdot r^2-\rho^2)\cdot\sin^2(\vartheta)-2\cdot r\cdot\rho^4}{2\cdot\rho^6}\cdot\sin^2(\vartheta)$$

$$\Gamma^{r}{}_{\varphi\varphi}=\Delta\cdot\frac{a\cdot r_g-2\cdot r^3}{2\cdot r^4} \tag{5.7.9}$$

Substituting:

$$\omega_{1,2}=c\cdot\frac{\frac{a\cdot\Delta\cdot r_g}{2\cdot r^4}\pm\sqrt{\left(-\frac{a\cdot\Delta\cdot r_g}{2\cdot r^4}\right)^2-\Delta\cdot\frac{a\cdot r_g-2\cdot r^3}{2\cdot r^4}\cdot\frac{\Delta\cdot r_g}{2\cdot r^4}}}{\Delta\cdot\frac{a\cdot r_g-2\cdot r^3}{2\cdot r^4}}$$

Two orbital frequencies are possible on equatorial circular orbits, depending on the orbital direction:

$$\omega_{1,2}=c\cdot\frac{a\cdot r_g\pm\sqrt{(a\cdot r_g)^2-(a\cdot r_g-2\cdot r^3)\cdot r_g}}{a\cdot r_g-2\cdot r^3} \tag{5.7.10}$$

5.8 Kerr-Schild metrics

We write down their general form. The Kerr spacetime also belongs to this group, actually Roy Kerr was also looking for the solution in this form, $\eta_{\eta\kappa}$ is the metric tensor of flat spacetime here, and l_η is a light-like vector:

$$g_{\eta\kappa}=\eta_{\eta\kappa}+l_\eta\cdot l_\kappa \qquad g^{\eta\kappa}=\eta^{\eta\kappa}-l^\eta\cdot l^\kappa$$

$$\eta_{\eta\alpha}\cdot l^{\alpha}=l_{\eta} \qquad l_{\alpha}\cdot l^{\alpha}=0 \tag{5.8.1}$$

Since the Schwarzschild spacetime is a special case of the Kerr spacetime, it is also of this form. Reorder the Kerr arc length squared:

$$ds^2=\frac{\Delta}{\rho^2}\cdot(dt-a\cdot\sin^2(\vartheta)\cdot d\varphi)^2-\frac{\sin^2(\vartheta)}{\rho^2}\cdot((r^2+a^2)\cdot d\varphi-a\cdot dt)^2-\frac{\Delta}{\rho^2}\cdot dr^2-\rho^2\cdot d\vartheta^2 \tag{5.8.2}$$

Introduce new coordinates and substitute them:

$$du=dt-\frac{r^2+a^2}{\Delta}\cdot dr \qquad d\phi=d\varphi-\frac{a}{\Delta}\cdot dr \tag{5.8.3}$$

$$ds^2=$$
$$\frac{\Delta}{\rho^2}\cdot(du-a\cdot\sin^2(\vartheta)\cdot d\phi)^2-\frac{\sin^2(\vartheta)}{\rho^2}\cdot((r^2+a^2)\cdot d\phi-a\cdot du)^2+2\cdot(du-a\cdot\sin^2(\vartheta)\cdot d\phi)\cdot dr-\rho^2\cdot d\vartheta^2 \tag{5.8.4}$$

Rearrange metric tensor and arc length squared:

$$g_{\eta\kappa}=\begin{pmatrix} 1-\frac{r\cdot r_g}{\rho^2} & 1 & 0 & \frac{a\cdot r\cdot r_g}{\rho^2}\cdot\sin^2(\vartheta) \\ 1 & 0 & 0 & -a\cdot\sin^2(\vartheta) \\ 0 & 0 & -\rho^2 & 0 \\ \frac{a\cdot r\cdot r_g}{\rho^2}\cdot\sin^2(\vartheta) & -a\cdot\sin^2(\vartheta) & 0 & -\frac{\Sigma^2}{\rho^2}\cdot\sin^2(\vartheta) \end{pmatrix}$$

$$ds^2=(du+dr)^2-dr^2-\rho^2\cdot d\vartheta^2-(r^2+a^2)\cdot\sin^2(\vartheta)\cdot d\phi^2-2\cdot a\cdot\sin^2(\vartheta)\cdot d\phi\cdot dr$$
$$\frac{-r\cdot r_g}{\rho^2}\cdot(du-a\cdot\sin^2(\vartheta)\cdot d\phi)^2 \tag{5.8.5}$$

Write down the light-like vector and substitute it:

$$l^{\eta}=(0 \quad 1 \quad 0 \quad 0) \qquad l_{\eta}=(1 \quad 0 \quad 0 \quad -a\cdot\sin^2(\vartheta)) \tag{5.8.6}$$

$$ds^2=(du+dr)^2-dr^2-\rho^2\cdot d\vartheta^2-(r^2+a^2)\cdot\sin^2(\vartheta)\cdot d\phi^2-2\cdot a\cdot\sin^2(\vartheta)\cdot d\phi\cdot dr$$
$$\frac{-r\cdot r_g}{\rho^2}\cdot l_{\alpha}\cdot l_{\beta}\cdot dx^{\alpha}\cdot dx^{\beta} \tag{5.8.7}$$

Substitute rectangular coordinates into it:

$$t=u+r \qquad x=(r\cdot\cos(\phi)+a\cdot\sin(\phi))\cdot\sin(\vartheta)$$

$$y=(r\cdot\sin(\phi)-a\cdot\cos(\phi))\cdot\sin(\vartheta) \qquad z=r\cdot\cos(\vartheta)$$

$$x^2+y^2=(r^2+a^2)\cdot\sin^2(\vartheta) \tag{5.8.8}$$

The original metric written down by Kerr:

$$ds_\eta^2=c^2\cdot dt^2-dx^2-dy^2-dz^2$$

$$ds^2=ds_\eta^2-\frac{r^3\cdot r_g}{r^4+a^2\cdot z^2}\cdot\left(c\cdot dt-\frac{r\cdot(x\cdot dx+y\cdot dy)+a\cdot(x\cdot dx-y\cdot dy)}{r^2+a^2}-\frac{z\cdot dz}{r}\right)^2 \tag{5.8.9}$$

The equation expressing the radial coordinate:

$$r^4-(x^2+y^2+z^2-a^2)\cdot r^2-a^2\cdot z^2=0 \tag{5.8.10}$$

The circular singularity is the true singularity of the Kerr spacetime:

$$x^2+y^2+z^2=a^2 \qquad z=0 \tag{5.8.11}$$

The new coordinates can be introduced with another sign as well:

$$du=dt+\frac{r^2+a^2}{\Delta}\cdot dr \qquad d\phi=d\varphi+\frac{a}{\Delta}\cdot dr \tag{5.8.12}$$

5.9 Tomimatsu-Sato spacetimes

They are the models of the axially symmetric spacetimes of rotating bodies, however they do not cover every possible solution. Write down the complex Ernst potential in the following form:

$$\xi=\frac{\alpha}{\beta} \tag{5.9.1}$$

Where α and β are polynomials in the x and y coordinates, and posses the following properties:

(a) always real: $\beta\cdot\frac{\partial\alpha}{\partial x}-\alpha\cdot\frac{\partial\beta}{\partial x}$

always imaginary: $\beta\cdot\frac{\partial\alpha}{\partial y}-\alpha\cdot\frac{\partial\beta}{\partial y}$ (5.9.2)

(b) the coefficients of the even powers of y are real, of the odd powers are imaginary.

(c) the degree of the α polynomial is δ^2, of the β is δ^2-1, where δ is a natural number called the deformation parameter.

(d) α and β are δ-degree polynomials in the p and q real parameters, where:

$$p^2+q^2=1 \tag{5.9.3}$$

(e) in the q = 1 case, let ξ be the potential of the static Zipoy-Voorhees spacetimes. Therefore in the case of $q \ll 1$, meaning slow rotation, the form of the potential:

$$\xi=\frac{(x+1)^\delta+(x-1)^\delta}{(x+1)^\delta-(x-1)^\delta}+i\cdot q\cdot\xi_1(x,y) \tag{5.9.4}$$

We obtain from the Ernst equation the term that is small in the first order, where the P functions are Legendre polynomials with a degree noted on the lower right corner:

$$\xi_1(x,y)=\frac{1}{(x+1)^\delta-(x-1)^\delta}\cdot\sum_{l=1}^{\delta} a_{2\cdot l-1}(x)\cdot P_{2\cdot l-1}(x) \tag{5.9.5}$$

The δ = 1 value determines the Kerr spacetime, the polynomials of the δ = 2 case:

$$\alpha=p^2\cdot x^4+q^2\cdot y^4-1-2\cdot i\cdot p\cdot q\cdot x\cdot y\cdot(x^2-y^2)$$

$$\beta=2\cdot p\cdot x\cdot(x^2-1)-2\cdot i\cdot q\cdot y\cdot(1-y^2) \tag{5.9.6}$$

For an arbitrary δ value, the potential gives the spacetime of a body with

$m=\frac{a}{q}$	mass,
$J=a\cdot m$	angular momentum,
$Q=m^3\cdot\left(\frac{\delta^2-1}{3\cdot\delta^2}\cdot p^2+q^2\right)$	quadrupole moment. (5.9.7)

The curvature of these empty spacetimes decreases far away from the source. The curvature singularities are positioned along concentric rings.

We introduce the real G, H and I functions in the following way:

$$\xi=\frac{\alpha}{\beta}=\frac{\alpha\cdot\bar{\beta}}{\beta\cdot\bar{\beta}}=\frac{H+i\cdot I}{G} \qquad G=\beta\cdot\bar{\beta}$$

$$E=\frac{\xi-1}{\xi+1}=\frac{\alpha-\beta}{\alpha+\beta}=\frac{(\alpha-\beta)\cdot\bar{\alpha}+(\alpha-\beta)\cdot\bar{\beta}}{(\alpha+\beta)\cdot(\bar{\alpha}+\bar{\beta})}=\frac{(\alpha\cdot\bar{\alpha}-\beta\cdot\bar{\beta})-(\alpha\cdot\bar{\beta}-\beta\cdot\bar{\alpha})}{B}=\frac{A+2\cdot i\cdot I}{B}$$

$$H^2+I^2=A\cdot G+G^2 \tag{5.9.8}$$

the left side: $H^2+I^2=|\alpha|^2\cdot|\beta|^2$

the right side: $A+G=\alpha\cdot\bar{\alpha}-\beta\cdot\bar{\beta}+\beta\cdot\bar{\beta}$

Introduce new notation:

$$a = x^2 - 1 \qquad b = y^2 - 1 \qquad f(r) = p^2 \cdot a^r + q^2 \cdot b^r \tag{5.9.9}$$

The δ-th Tomimatsu-Sato arc length squared is a function of the δ-th Hankel matrix, that has the determinant:

$$M_\delta(a,b) = \begin{vmatrix} f(1) & \frac{f(2)}{2} & \frac{f(3)}{3} & \dots & \frac{f(\delta)}{\delta} \\ \frac{f(2)}{2} & \frac{f(3)}{3} & \frac{f(4)}{4} & \dots & \frac{f(\delta+1)}{\delta+1} \\ \frac{f(3)}{3} & \frac{f(4)}{4} & \frac{f(5)}{5} & \dots & \frac{f(\delta+2)}{\delta+2} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{f(\delta)}{\delta} & \frac{f(\delta+1)}{\delta+1} & \frac{f(\delta+2)}{\delta+2} & \dots & \frac{f(2\cdot\delta-1)}{2\cdot\delta-1} \end{vmatrix} \tag{5.9.10}$$

Using this the metric functions:

$$A = \frac{M_\delta(a,b)}{M_\delta(1,1)} = F(\delta)$$

$$B = A + G + H$$

$$C = \frac{p}{2\cdot q\cdot b\cdot\delta}\cdot\left(Q + R - \frac{\delta}{p\cdot q}\cdot A\right) \tag{5.9.11}$$

The coefficients of the metric functions:

$$c(\delta,r) = \delta\cdot\frac{(\delta+r-1)!}{(\delta-r)!\cdot(2\cdot r)!}\cdot 2^{2\cdot r-1}$$

$$d(r) = (-1)^{r-1}\cdot\frac{(2\cdot r-2)!}{(2^{r-1}\cdot(r-1)!)^2}$$

$$e(r) = -2\cdot d\cdot(r+1)$$

$$g(\delta,r,r') = \frac{\Re(r)\cdot c(\delta,r)}{\delta^2}\cdot\sum_{t=r'}^{\delta}\frac{t\cdot d\cdot(t-r'-1)\cdot c(\delta,r)}{r+t-1}$$

$$h(\delta,r,r') = \frac{r\cdot r'\cdot e(r)\cdot c(\delta,r)\cdot c(\delta,r')}{\delta^2\cdot(r+r'-1)} \tag{5.9.12}$$

From these the metric functions:

$$G=2\cdot\sum_{r=1}^{\delta}c(\delta,r)\cdot F(\delta^2-r)$$

$$H=2\cdot p\cdot x\cdot\sum_{r=1}^{\delta}d(r)\cdot a^{r-1}\cdot\sum_{r'=r}^{\delta}c(\delta,r')\cdot F(\delta^2-r')$$

$$Q=-\frac{2\cdot x}{q}\cdot\delta\cdot\sum_{r=1}^{\delta}\sum_{r'=1}^{\delta}q^2\cdot a^r\cdot b^{1-r'}\cdot g(\delta,r,r')\cdot F(\delta^2-1)$$

$$R=\frac{\delta}{p\cdot q}\cdot\sum_{r=1}^{\delta}\sum_{r'=1}^{\delta}(p^2\cdot a^r\cdot b^{1-r'}-q^2\cdot b^r\cdot a^{1-r'})\cdot h(\delta,r,r')\cdot F(\delta^2-1) \tag{5.9.13}$$

The arc length squared of the δ-th Tomimatsu-Sato spacetime:

$$ds^2=\frac{B}{\delta^2\cdot p^{\delta^2-2}\cdot(a-b)^{\delta^2-1}}\cdot\left(\frac{dy^2}{b}-\frac{dx^2}{a}\right)+\frac{A}{B}\cdot dt^2+4\cdot q\cdot\frac{b\cdot C}{B}\cdot dt\cdot d\varphi+\frac{b\cdot D}{\delta^2\cdot B}\cdot d\varphi^2 \tag{5.9.14}$$

The equation determining the D polynomial:

$$A\cdot D-p^2\cdot a\cdot B^2-4\cdot\delta^2\cdot q^2\cdot b\cdot C^2=0 \tag{5.9.15}$$

These solutions describe only the spacetime outside rotating bodies. At distances from the centre where they loose validity, we find ring singularities not covered by event horizons.

6. Spacetime with matter

In the presence of matter, Einstein's equations show, how the given matter distribution curves spacetime, and the trajectories of moving bodies are determined by the geodesic equations. Thus spacetime and matter dynamically interact with each other. We have to point out, that the Einstein equation is a very simple tool, it does nothing else than to give the spacetime geometry caused by matter with arbitrary properties and distribution. It has nothing to say about the physical reality of that matter distribution.

We will see, that an arbitrarily high amount of matter cannot stay stable in a given volume of space, infinite internal pressure will build up even in the case of a finite quantity, and the celestial body will collapse into a black hole.

6.1 Energy-momentum tensor

We investigate gravitation in matter-filled space. The form of the Einstein equation in this case:

$$R_{\eta\kappa}-\frac{1}{2}\cdot R\cdot g_{\eta\kappa}=-k\cdot T_{\eta\kappa} \tag{6.1.1}$$

On the right side of the equation is the energy-momentum tensor, multiplied with a constant that is unknown for the moment. This tensor contains all information necessary to describe the gravitational effect of matter. In general relativity, we talk about the presence of matter, when at a given point, the value of this tensor is different from zero, this is fulfilled also in the case of radiations and interaction force fields.

The connection is symmetric in general relativity, therefore the Ricci tensor and the energy-momentum tensor will also be symmetric, thus the system of equations above contains ten unknowns. In the general case, $T_{\eta\kappa}$ means the flow of the η-th component of the energy-momentum vector across the κ-th coordinate surface. The meaning of the energy-momentum vector components:

$$E_{\eta}=\begin{pmatrix}\frac{E}{c} & p^{\kappa} & p^{\mu} & p^{\nu}\end{pmatrix} \tag{6.1.2}$$

Where E is energy, and the p-s are the three spatial components of momentum. Thus for example in the case of one time-like and three space-like coordinates, T_{00} means the flow of energy across the constant time coordinate surface, it means we are talking about the energy density:

$$T_{00}=\rho\cdot c^{2} \tag{6.1.3}$$

If one of the indices is non-zero, then – according to our reasoning above – it is either momentum density, or the flow of energy across surfaces determined by spatial coordinates, which means the same because of the symmetry of the tensor:

$$T_{0i} = T_{i0} = \pi_i \cdot c \tag{6.1.4}$$

If the indices are spatial and are the same, it means the crossing of momentum across perpendicular coordinate surfaces. They are in fact the three components of the pressure:

$$T_{ii} = p_i \tag{6.1.5}$$

If the indices above are different, then one of the momentum components changes direction, this means for example distortions in solid matter, that we call stress. The symmetry of the tensor applies:

$$T_{ij} = T_{ji} = s_{ij} = s_{ji} \tag{6.1.6}$$

The general energy-momentum tensor:

$$T_{\eta\kappa} = \begin{pmatrix} \rho \cdot c^2 & \pi_1 \cdot c & \pi_2 \cdot c & \pi_3 \cdot c \\ \pi_1 \cdot c & p_1 & s_{12} & s_{13} \\ \pi_2 \cdot c & s_{21} & p_2 & s_{23} \\ \pi_3 \cdot c & s_{31} & s_{32} & p_3 \end{pmatrix} \tag{6.1.7}$$

The great practical benefit of general relativity is, that by knowing the tensor above, the gravitational field can be determined, but no other theories about the origin of matter are necessary.

6.2 Einstein equation with matter

Generalize the Newtonian gravitational potential to arbitrary mass distributions. This is how our formula looked like in the vicinity of a spherically symmetric mass distribution:

$$\phi(r) = -\gamma \cdot \frac{m}{r} \tag{6.2.1}$$

The mass and gravitational potential of an arbitrary mass distribution:

$$m = \int \rho(\vec{r}) \cdot dr^3 \qquad \phi(r) = -\gamma \cdot \int \frac{\rho(\vec{r})}{r} \cdot dr^3 \tag{6.2.2}$$

Write down the equation of movement and the field equation with the divergence theorem:

$$\ddot{x}_i = \frac{\partial \phi}{\partial x^i} \qquad \frac{\partial^2 \phi}{\partial (x^i)^2} = 4 \cdot \pi \cdot \gamma \cdot \rho \tag{6.2.3}$$

Equation of movement and field equation in general relativity, rearrange the Einstein equation:

$$\ddot{x}^{\eta}=-\Gamma^{\eta}{}_{\alpha\beta}\cdot\dot{x}^{\alpha}\cdot\dot{x}^{\beta} \qquad R_{\eta\kappa}=k\cdot\left(T_{\eta\kappa}+\frac{1}{2}\cdot T\cdot g_{\eta\kappa}\right) \tag{6.2.4}$$

We determine the important geometric quantities of the Einsteinian model in the Newtonian approximation. This means small velocities, where the decisive origin of the gravitation of bodies is the rest mass. Coordinate time approaches proper time, the velocities and non-linear effects can be neglected:

$$\dot{x}^{\eta}=(c \quad 0 \quad 0 \quad 0) \qquad d\tau\approx dt$$

$$\ddot{x}^{i}=-\Gamma^{i}{}_{00}\cdot c^{2}\approx-\frac{\partial\Phi}{\partial x^{i}} \tag{6.2.5}$$

The non-zero components of the connection:

$$\Gamma^{i}{}_{00}=\frac{1}{2}\cdot g^{ia}\cdot\left(\frac{\partial g_{0a}}{\partial x^{0}}+\frac{\partial g_{a0}}{\partial x^{0}}-\frac{\partial g_{00}}{\partial x^{a}}\right)$$

$$\frac{2}{c^{2}}\cdot\frac{\partial\Phi}{\partial x^{i}}\approx g^{ia}\cdot\left(-\frac{\partial g_{00}}{\partial x^{a}}\right)\approx-\frac{\partial g_{00}}{\partial x^{i}} \tag{6.2.6}$$

We determine the metric tensor component with integration:

$$g_{00}\approx 1-\frac{2\cdot\Phi}{c^{2}}\approx 1 \tag{6.2.7}$$

Substitute it into the rearranged Einstein equation:

$$R_{00}=k\cdot\left(T_{00}+\frac{1}{2}\cdot T\cdot g_{00}\right) \tag{6.2.8}$$

We continue to identify the quantities in the equation. Because of the applied approximations, the energy-momentum tensor simplifies significantly, the origin of gravitation is the rest mass of the matter distribution:

$$T_{\eta\kappa}=\begin{pmatrix}\rho\cdot c^{2} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \tag{6.2.9}$$

The contraction of the energy-momentum tensor:

$$T=g^{\alpha\beta}\cdot T_{\alpha\beta}\approx g^{00}\cdot T_{00}\approx\rho\cdot c^{2} \tag{6.2.10}$$

Write down the right side of the equation:

$$k\cdot\left(T_{00}+\frac{1}{2}\cdot T\cdot g_{00}\right)=\frac{1}{2}\cdot k\cdot\rho\cdot c^2 \tag{6.2.11}$$

And the left side:

$$R_{00}\approx\frac{\partial\Gamma^{a}{}_{00}}{\partial x^a}-\frac{\partial\Gamma^{a}{}_{a0}}{\partial x^0}=\frac{\partial\Gamma^{a}{}_{00}}{\partial x^a}\approx\frac{1}{c^2}\cdot\frac{\partial^2\phi}{\partial(x^i)^2}=\frac{4\cdot\pi\cdot\gamma\cdot\rho}{c^2} \tag{6.2.12}$$

By making them equal, we express the physical constant:

$$\frac{1}{2}\cdot k\cdot\rho\cdot c^2=\frac{4\cdot\pi\cdot\gamma\cdot\rho}{c^2}$$

$$k=\frac{8\cdot\pi\cdot\gamma}{c^4} \tag{6.2.13}$$

Reinsert it into the Einstein equation:

$$R_{\eta\kappa}-\frac{1}{2}\cdot R\cdot g_{\eta\kappa}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\eta\kappa} \tag{6.2.14}$$

We will use two assumptions about the structure of matter in the following sections. It can be continuous like the electromagnetic field, or composed of particles, like the atomic matter. Galaxies fall into the latter category, that are composed of stars, or the Universe itself, that is composed by galaxies, discrete islands of matter.

6.3 Perfect fluid

The behaviour of matter composing particles follows quantum mechanical rules, however in relativity theory, the paths can be exactly determined. Therefore we investigate such a small volume in the fluid, that is on one hand negligibly small compared to the entire matter quantity, on the other hand it is big enough to make quantum effects in it negligible, thus it is composed of sufficiently many particles. These restrains demonstrate the limitations of the continuous fluid model, and are in the same time the limitations of the general theory of relativity as well.

We neglect internal friction and viscosity in our modelled medium. With this approach we can significantly simplify the form of the energy-momentum tensor, because only the density and pressure will determine it. On small volumes our medium is homogeneous and isotropic, therefore the magnitude of the three pressure components will be equal. In a local coordinate system, that is co-moving with the fluid particles, the form of the energy-momentum tensor and the four-velocity, because of our conditions for the perfect fluids:

$$T_{\eta\kappa}=\begin{pmatrix}\rho\cdot c^2 & 0 & 0 & 0\\ 0 & p & 0 & 0\\ 0 & 0 & p & 0\\ 0 & 0 & 0 & p\end{pmatrix} \qquad u_\eta=(c \quad 0 \quad 0 \quad 0) \tag{6.3.1}$$

In the most general case, the energy-momentum tensor is the linear combination of the four-velocity and the metric tensor, multiplied with unknown functions:

$$T_{\eta\kappa}=A\cdot u_\eta\cdot u_\kappa+B\cdot g_{\eta\kappa} \tag{6.3.2}$$

Write down the spatial components of the local variant with index notation, with the symbols of the general formula, in this case the spatial components of the four-velocity are zeroes, and we use the metric tensor of the flat spacetime:

$$T_{ii}=B\cdot\eta_{ii}=-B \tag{6.3.3}$$

By comparing with the general formula, we recognize the multiplier function:

$$B=-p \tag{6.3.4}$$

The purely time-like component of the energy-momentum tensor:

$$T_{00}=A\cdot u_0\cdot u_0+B\cdot\eta_{00}=A+B=A\cdot c^2-p=\rho\cdot c^2$$

$$A=\rho+\frac{p}{c^2} \tag{6.3.5}$$

The energy-momentum tensor of the relativistic perfect fluid in the general case:

$$T_{\eta\kappa}=\left(\rho+\frac{p}{c^2}\right)\cdot u_\eta\cdot u_\kappa+p\cdot g_{\eta\kappa} \tag{6.3.6}$$

6.4 Spherically symmetric celestial body

If the diameter of a celestial body is higher than approximately 500 km, the molecular binding forces do not dominate its shape any more. It is determined by its own gravitation and of neighbouring bodies, and its rotation instead. In a safe distance from its neighbours, the shape and internal distribution of slowly rotating celestial bodies is approximately spherically symmetric. The Schwarzschild metric describes the external spacetime of these objects with great accuracy. In order to determine the shape of the spacetime that is valid inside them, we have to take the derivation in empty spacetime a step further. As we shall see soon enough, the Ricci tensor will not be zero here, but the entire Einstein equation has to be solved instead. The general form of the arc length squared of the spherically symmetric spacetime in Schwarzschild coordinates:

$$ds^2 = A(r)\cdot c^2\cdot dt^2 - B(r)\cdot dr^2 - r^2\cdot d\vartheta^2 - r^2\cdot\sin^2(\vartheta)\cdot d\varphi^2 \qquad (6.4.1)$$

This time it is useful to write them down in a form, where the unknown functions are exponentials:

$$ds^2 = e^{2\cdot\Phi(r)}\cdot c^2\cdot dt^2 - e^{2\cdot\Lambda(r)}\cdot dr^2 - r^2\cdot d\vartheta^2 - r^2\cdot\sin^2(\vartheta)\cdot d\varphi^2 \qquad (6.4.2)$$

Determine the geometric quantities characterizing the surface, from the metric tensor to the Ricci scalar, in order to write down the Einstein equations. We note the derivative according to the radial coordinate with an upper comma:

$$g_{\eta\kappa} = \begin{pmatrix} e^{2\cdot\Phi} & 0 & 0 & 0 \\ 0 & -e^{2\cdot\Lambda} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\cdot\sin^2(\vartheta) \end{pmatrix} \quad g^{\eta\kappa} = \begin{pmatrix} \frac{1}{e^{2\cdot\Phi}} & 0 & 0 & 0 \\ 0 & -\frac{1}{e^{2\cdot\Lambda}} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\cdot\sin^2(\vartheta)} \end{pmatrix} \qquad (6.4.3)$$

$$\frac{\partial g_{tt}}{\partial r} = 2\cdot e^{2\cdot\Phi}\cdot\Phi' \qquad \frac{\partial g^{tt}}{\partial r} = -2\cdot e^{-2\cdot\Phi}\cdot\Phi'$$

$$\frac{\partial g_{rr}}{\partial r} = -2\cdot e^{2\cdot\Lambda}\cdot\Lambda' \qquad \frac{\partial g^{rr}}{\partial r} = 2\cdot e^{-2\cdot\Lambda}\cdot\Lambda'$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r} = -2\cdot r \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial r} = \frac{2}{r^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r} = -2\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r} = \frac{2}{r^3\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta} = -2\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta} = \frac{2\cdot\cos(\vartheta)}{r^2\cdot\sin^3(\vartheta)} \qquad (6.4.4)$$

$$\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \Phi' \qquad \Gamma^r_{tt} = e^{2\cdot(\Phi-\Lambda)}\cdot\Phi'$$

$$\Gamma^r{}_{rr} = \Lambda' \qquad \Gamma^r{}_{\vartheta\vartheta} = -r\cdot e^{-2\cdot\Lambda}$$

$$\Gamma^r{}_{\varphi\varphi} = -r\cdot e^{-2\cdot\Lambda}\cdot\sin^2(\vartheta) \qquad \Gamma^\vartheta{}_{r\vartheta} = \Gamma^\vartheta{}_{\vartheta r} = \Gamma^\varphi{}_{r\varphi} = \Gamma^\varphi{}_{\varphi r} = \frac{1}{r}$$

$$\Gamma^\vartheta{}_{\varphi\varphi} = -\cos(\vartheta)\cdot\sin(\vartheta) \qquad \Gamma^\varphi{}_{\vartheta\varphi} = \Gamma^\varphi{}_{\varphi\vartheta} = \cot(\vartheta) \qquad (6.4.5)$$

$$\frac{\partial \Gamma^t{}_{tr}}{\partial r}=\frac{\partial \Gamma^t{}_{rt}}{\partial r}=\Phi'' \qquad \frac{\partial \Gamma^r{}_{tt}}{\partial r}=e^{2\cdot(\Phi-\Lambda)}\cdot(\Phi'\cdot 2\cdot(\Phi-\Lambda)+\Phi'')$$

$$\frac{\partial \Gamma^r{}_{rr}}{\partial r}=\Lambda'' \qquad \frac{\partial \Gamma^r{}_{\vartheta\vartheta}}{\partial r}=e^{-2\cdot\Lambda}\cdot(2\cdot r\cdot\Lambda'-1)$$

$$\frac{\partial \Gamma^r{}_{\varphi\varphi}}{\partial r}=e^{-2\cdot\Lambda}\cdot(2\cdot r\cdot\Lambda'-1)\cdot\sin^2(\vartheta) \qquad \frac{\partial \Gamma^\vartheta{}_{r\vartheta}}{\partial r}=\frac{\partial \Gamma^\vartheta{}_{\vartheta r}}{\partial r}=\frac{\partial \Gamma^\varphi{}_{r\varphi}}{\partial r}=\frac{\partial \Gamma^\varphi{}_{\varphi r}}{\partial r}=-\frac{1}{r^2}$$

$$\frac{\partial \Gamma^r{}_{\varphi\varphi}}{\partial \vartheta}=-2\cdot r\cdot e^{-2\cdot\Lambda}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial \Gamma^\vartheta{}_{\varphi\varphi}}{\partial \vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta)$$

$$\frac{\partial \Gamma^\varphi{}_{\vartheta\varphi}}{\partial \vartheta}=\frac{\partial \Gamma^\varphi{}_{\varphi\vartheta}}{\partial \vartheta}=-\cot^2(\vartheta)-1 \qquad (6.4.6)$$

$$R^t{}_{rtr}=-R^t{}_{rrt}=\Phi'\cdot\Lambda'-\Phi''-\Phi'^2 \qquad R^t{}_{\vartheta t\vartheta}=-R^t{}_{\vartheta\vartheta t}=-r\cdot e^{-2\cdot\Lambda}\cdot\Phi'$$

$$R^t{}_{\varphi t\varphi}=-R^t{}_{\varphi\varphi t}=-r\cdot e^{-2\cdot\Lambda}\cdot\Phi'\cdot\sin^2(\vartheta) \qquad R^r{}_{ttr}=-R^r{}_{trt}=e^{2\cdot(\Phi-\Lambda)}\cdot(\Phi'\cdot\Lambda'-\Phi''-\Phi'^2)$$

$$R^r{}_{\vartheta r\vartheta}=-R^r{}_{\vartheta\vartheta r}=r\cdot e^{-2\cdot\Lambda}\cdot\Lambda' \qquad R^r{}_{\varphi r\varphi}=-R^r{}_{\varphi\varphi r}=r\cdot e^{-2\cdot\Lambda}\cdot\Lambda'\cdot\sin^2(\vartheta)$$

$$R^\vartheta{}_{tt\vartheta}=-R^\vartheta{}_{t\vartheta t}=R^\varphi{}_{tt\varphi}=-R^\varphi{}_{t\varphi t}=-\frac{e^{2\cdot(\Phi-\Lambda)}\cdot\Phi'}{r}$$

$$R^\vartheta{}_{rr\vartheta}=-R^\vartheta{}_{r\vartheta r}=R^\varphi{}_{rr\varphi}=-R^\varphi{}_{r\varphi r}=-\frac{\Lambda'}{r}$$

$$R^\vartheta{}_{\varphi\vartheta\varphi}=R^\vartheta{}_{\varphi\varphi\vartheta}=\left(1-e^{-2\cdot\Lambda}\right)\cdot\sin^2(\vartheta) \qquad R^\varphi{}_{\vartheta\varphi\vartheta}=R^\varphi{}_{\vartheta\vartheta\varphi}=1-e^{-2\cdot\Lambda} \qquad (6.4.7)$$

$$R_{tt}=-e^{2\cdot(\Phi-\Lambda)}\cdot\left(\Phi'\cdot\Lambda'-\Phi''-\Phi'^2-\frac{2\cdot\Phi'}{r}\right)$$

$$R_{rr}=\Phi'\cdot\Lambda'-\Phi''-\Phi'^2+\frac{2\cdot\Lambda'}{r}$$

$$R_{\vartheta\vartheta}=e^{-2\cdot\Lambda}\cdot(r\cdot(\Phi'-\Lambda')-1)+1$$

$$R_{\varphi\varphi}=(e^{-2\cdot\Lambda}\cdot(r\cdot(\Phi'-\Lambda')-1)+1)\cdot\sin^2(\vartheta) \qquad (6.4.8)$$

$$R=2\cdot e^{-2\cdot\Lambda}\cdot\left(\Phi'\cdot\Lambda'-\Phi''-\Phi'^2+\frac{2\cdot(\Lambda'-\Phi')}{r}+\frac{e^{2\cdot\Lambda}-1}{r^2}\right) \qquad (6.4.9)$$

The Einstein equations:

$$G_{\eta\kappa}=R_{\eta\kappa}-\frac{1}{2}\cdot R\cdot g_{\eta\kappa}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\eta\kappa} \tag{6.4.10}$$

The Einstein tensor is diagonal:

$$G_{tt}=e^{2\cdot(\Phi-\Lambda)}\cdot\frac{2\cdot r\cdot\Lambda'+e^{2\cdot\Lambda}-1}{r^2}$$

$$G_{rr}=\frac{2\cdot r\cdot\Phi'-e^{2\cdot\Lambda}+1}{r^2}$$

$$G_{\vartheta\vartheta}=-r\cdot e^{-2\cdot\Lambda}\cdot((r\cdot\Phi'+1)\cdot\Lambda'-r\cdot\Phi''-r\cdot\Phi'^2-\Phi')$$

$$G_{\varphi\varphi}=-r\cdot e^{-2\cdot\Lambda}\cdot((r\cdot\Phi'+1)\cdot\Lambda'-r\cdot\Phi''-r\cdot\Phi'^2-\Phi')\cdot\sin^2(\vartheta) \tag{6.4.11}$$

We make it equal to the diagonal variant of the energy-momentum tensor of the perfect fluids, where the four-velocity is zero, thus we assume that the internal currents of the celestial body are negligible. The fluid rests in the Schwarzschild coordinate system, therefore the form of the energy-momentum tensor:

$$T_{\eta\kappa}=\begin{pmatrix}\rho\cdot c^2\cdot e^{-2\cdot\Phi(r)} & 0 & 0 & 0\\ 0 & -p\cdot e^{-2\cdot\Lambda(r)} & 0 & 0\\ 0 & 0 & -\frac{p}{r^2} & 0\\ 0 & 0 & 0 & -\frac{p}{r^2\cdot\sin^2(\vartheta)}\end{pmatrix} \tag{6.4.12}$$

Write down the equations to be solved:

(1) $$e^{-2\cdot\Lambda}\cdot\frac{2\cdot r\cdot\Lambda'+e^{2\cdot\Lambda}-1}{r^2}=-\frac{8\cdot\pi\cdot\gamma}{c^2}\cdot\rho(r)$$

(2) $$e^{-2\cdot\Lambda}\cdot\frac{2\cdot r\cdot\Phi'-e^{2\cdot\Lambda}+1}{r^2}=\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot p(r)$$

$$-e^{-2\cdot\Lambda}\cdot\left(\left(\Phi'+\frac{1}{r}\right)\cdot\Lambda'-\Phi''-\Phi'^2-\frac{\Phi'}{r}\right)=\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot p(r) \tag{6.4.13}$$

This time the mass of the celestial body is not interpreted in a single point, but it is spread out from the centre to the surface. With taking the metric into account, we have to integrate the energy density function in the entire spherical volume, thus the mass of the body is given by the sum of the constituting mass and the gravitational potential energy:

$$M = 4\cdot\pi\cdot\int_0^R A(r)\cdot\rho(r)\cdot r^2\cdot dr \tag{6.4.14}$$

We can rewrite the first (1) equation to the following form:

$$\frac{\partial}{\partial r}\left(\frac{r}{e^{2\cdot\Lambda}}\right) = 1 - \frac{8\cdot\pi\cdot\gamma}{c^2}\cdot r^2\cdot\rho(r)$$

$$\frac{1}{e^{2\cdot\Lambda}} = 1 - \frac{2\cdot\gamma}{c^2\cdot r}\cdot 4\cdot\pi\cdot\int_0^r r^2\cdot\rho(r)\cdot dr$$

$$e^{2\cdot\Lambda} = \frac{1}{1 - \frac{2\cdot\gamma\cdot m(r)}{c^2\cdot r}} \tag{6.4.15}$$

Subtract the first (1) equation from the second (2):

$$\frac{1}{e^{2\cdot\Lambda}}\cdot\left(\frac{2\cdot\Phi'}{r} + \frac{2\cdot\Lambda'}{r}\right) = \frac{8\cdot\pi\cdot\gamma}{c^2}\cdot\left(\rho(r) - \frac{p(r)}{c^2}\right)$$

$$\left(1 - \frac{2\cdot\gamma\cdot m(r)}{c^2\cdot r}\right)\cdot(\Phi' + \Lambda') = \frac{4\cdot\pi\cdot\gamma}{c^2}\cdot r\cdot\left(\rho(r) - \frac{p(r)}{c^2}\right)$$

$$\Phi' = \frac{1}{1 - \frac{2\cdot\gamma\cdot m(r)}{c^2\cdot r}}\cdot\left(\frac{\gamma\cdot m(r)}{c^2\cdot r^2} + \frac{4\cdot\pi\cdot\gamma}{c^4}\cdot r\cdot p(r)\right) \tag{6.4.16}$$

Rearrange:

$$\Phi' = \frac{\gamma\cdot m(r)}{c^2\cdot r^2}\cdot\frac{1 + \frac{4\cdot\pi\cdot r^3\cdot p(r)}{c^4\cdot m(r)}}{1 - \frac{2\cdot\gamma\cdot m(r)}{c^2\cdot r}}$$

$$\Phi' = -\frac{1}{p(r) + \rho(r)\cdot c^2}\cdot\frac{dp(r)}{dr} \tag{6.4.17}$$

Express the metric functions from the Einstein equations:

$$-2\cdot r\cdot\Lambda' = \left(1 - \frac{8\cdot\pi\cdot\gamma}{c^2}\cdot r^2\cdot\rho(r)\right)\cdot e^{2\cdot\Lambda} - 1$$

$$2\cdot r\cdot\Phi' = \left(1 + \frac{8\cdot\pi\cdot\gamma}{c^4}\cdot r^2\cdot p(r)\right)\cdot e^{2\cdot\Lambda} - 1 \tag{6.4.18}$$

We differentiate the second again according to *r*, and then multiply it with *r*:

$$2\cdot r\cdot\Phi'+2\cdot r^2\cdot\Phi''=\left(2\cdot r\cdot\Lambda'\cdot\left(1+\frac{8\cdot\pi\cdot\gamma\cdot r^2}{c^4}\cdot p(r)\right)+\left(\frac{16\cdot\pi\cdot\gamma\cdot r^2}{c^4}\cdot(p(r)+r\cdot p'(r))\right)\right)\cdot e^{2\cdot\Lambda}$$

Express the second derivative, and substitute both metric functions:

$$2\cdot r^2\cdot\Phi''=1+\left(\frac{16\cdot\pi\cdot\gamma\cdot r^2}{c^4}\cdot(p(r)+r\cdot p'(r))\right)\cdot e^{2\cdot\Lambda}$$
$$-\left(1+\frac{8\cdot\pi\cdot\gamma\cdot r^2}{c^4}\cdot p(r)\right)\cdot\left(1-\frac{8\cdot\pi\cdot\gamma\cdot r^2}{c^4}\cdot p(r)\right)\cdot e^{4\cdot\Lambda}$$

Square the second metric function:

$$4\cdot r^2\cdot\Phi'^2=\left(1+\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot r^2\cdot p(r)\right)^2\cdot e^{4\cdot\Lambda}-2\cdot\left(1+\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot r^2\cdot p(r)\right)\cdot e^{2\cdot\Lambda}+1$$

With the substitution of the above results we obtain the hydrostatic equilibrium in a symmetric, isotropic, spherical celestial body:

$$\frac{dp(r)}{dr}=-\frac{\gamma\cdot\left(\rho(r)+\frac{p(r)}{c^2}\right)\cdot\left(m(r)+4\cdot\pi\cdot r^3\cdot\frac{p(r)}{c^2}\right)}{r^2\cdot\left(1-\frac{2\cdot\gamma\cdot m(r)}{r\cdot c^2}\right)} \quad (6.4.19)$$

This equation satisfies the following conditions:

$$m(0)=0$$

$$\frac{dm(r)}{dr}=4\cdot\pi\cdot\rho(r)\cdot r^2 \quad (6.4.20)$$

The pressure on the surface of the celestial body is zero (we neglect any atmosphere), and the metric continuously goes over to the vacuum Schwarzschild solution:

$$p(R)=0$$

$$e^{2\cdot\Phi(R)}=1-\frac{2\cdot\gamma\cdot M}{r\cdot c^2} \quad (6.4.21)$$

Inside the celestial body, in material medium the hydrostatic equation corresponds to the following general metric:

$$ds^2 = e^{2\cdot\Phi(r)}\cdot c^2\cdot dt^2 - \frac{dr^2}{1-\frac{2\cdot\gamma\cdot m(r)}{r\cdot c^2}} - r^2\cdot(d\vartheta^2 + \sin^2(\vartheta)\cdot d\varphi^2) \tag{6.4.22}$$

6.5 Sphere with constant density

The equations for spherically symmetric celestial bodies are solvable analytically, if we assume, that the density is the same at every point in the entire volume, this characterizes perfect homogeneous fluids. This is a heavily idealized model, however it is not far from reality, since in the case of many naturally occurring objects (like the Earth), the density varies from the surface to the centre much less, than the pressure.

$$\rho = const. \tag{6.5.1}$$

We can already write down the mass distribution function inside the body easily, and the total mass:

$$m(r) = \frac{4\cdot\pi}{3}\cdot\rho\cdot r^3 \qquad r < R$$

$$M = \frac{4\cdot\pi}{3}\cdot\rho\cdot R^3 \qquad r \le R \tag{6.5.2}$$

Their ratio:

$$\frac{m}{M} = \frac{r^3}{R^3} \tag{6.5.3}$$

This can be used when determining the valid arc length squared inside the celestial body, where in the spirit of changing to units of length, we substitute the gravitational radius:

$$e^{2\cdot\Lambda} = \frac{1}{1-\frac{2\cdot\gamma\cdot m(r)}{c^2\cdot r}} = \frac{1}{1-\frac{r_g}{r}\cdot\frac{m(r)}{M(R)}} = \frac{1}{1-\frac{r_g\cdot r^2}{R^3}} \tag{6.5.4}$$

We insert the constant density and the mass distribution function into the equation of hydrostatic equilibrium:

$$\frac{dp(r)}{dr} = -\frac{\gamma\cdot\left(\rho+\frac{p(r)}{c^2}\right)\cdot\left(\frac{4\cdot\pi}{3}\cdot\rho\cdot r^3 + 4\cdot\pi\cdot r^3\cdot\frac{p(r)}{c^2}\right)}{r^2\cdot\left(1-\frac{2\cdot\gamma}{r\cdot c^2}\cdot\frac{4\cdot\pi}{3}\cdot\rho\cdot r^3\right)}$$

$$\frac{dp(r)}{dr}=-\frac{4\cdot\pi\cdot\gamma\cdot\left(\rho+\frac{p(r)}{c^2}\right)\cdot\left(\frac{\rho}{3}+\frac{p(r)}{c^2}\right)\cdot r}{1-\frac{8\cdot\pi\cdot\gamma}{3\cdot c^2}\cdot\rho\cdot r^2} \tag{6.5.5}$$

The solution of the differential equation, by taking the boundary conditions into account:

$$p=\rho\cdot\frac{\sqrt{1-\frac{r_g\cdot r^2}{R^3}}-\sqrt{1-\frac{r_g}{R}}}{3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}} \qquad r<R \tag{6.5.6}$$

Pressure in the centre of the spherical celestial body:

$$p_c=\rho\cdot\frac{1-\sqrt{1-\frac{r_g}{R}}}{3\cdot\sqrt{1-\frac{r_g}{R}}-1} \qquad r=0 \tag{6.5.7}$$

If the denominator is zero, the central pressure becomes infinite:

$$3\cdot\sqrt{1-\frac{r_g}{R}}=1 \qquad \rightarrow \qquad \frac{r_g}{R}=\frac{8}{9} \tag{6.5.8}$$

As we can see, there is an upper limit to mass in a given volume, that is independent of the chemical composition of the celestial body. Therefore it cannot come up at the creation of black holes, that although the matter composition of white dwarfs and neutron stars cannot stop the collapse, but maybe there is some unknown matter with better endurance, that could stop the star from collapsing in time. With our result above we know with certainty, that such matter does not exist, by approaching the limit above, nothing can save the celestial body becoming a black hole.

We investigate the speed of sound in spherically symmetric celestial bodies of constant density. The speed of sound in perfect fluid, where α is the adiabatic index:

$$v=\sqrt{\alpha\cdot\frac{p}{\rho}} \tag{6.5.9}$$

If we substitute the relationship on pressure, we get the dependency of the speed of sound on the depth:

$$v=\sqrt{\alpha\cdot\frac{\sqrt{1-\frac{r_g\cdot r^2}{R^3}}-\sqrt{1-\frac{r_g}{R}}}{3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}}} \tag{6.5.10}$$

The dependence of the depth on the speed of sound:

$$(3\cdot v^2+\alpha)\cdot\sqrt{1-\frac{r_g}{R}}=(v^2+\alpha)\cdot\sqrt{1-\frac{r_g\cdot r^2}{R^3}}$$

$$r=\sqrt{\frac{R^3}{r_g}\cdot\left(1-\left(\frac{3\cdot v^2+\alpha}{v^2+\alpha}\right)^2\cdot\left(1-\frac{r_g}{R}\right)\right)} \qquad (6.5.11)$$

The condition of the result being a real number:

$$\frac{R^3}{r_g}\cdot\left(1-\left(\frac{3\cdot v^2+\alpha}{v^2+\alpha}\right)^2\cdot\left(1-\frac{r_g}{R}\right)\right)\geq 0$$

$$\frac{r_g}{R}\geq\frac{8}{9}\geq 1-\left(\frac{v^2+\alpha}{3\cdot v^2+\alpha}\right)^2 \qquad (6.5.12)$$

Independently from the value of the adiabatic index, the ratio of the gravitational radius and the geometric radius is always greater, than the previously determined limiting value for the infinite central pressure.

We insert the previous result also into the second metric function:

$$\Phi'=\frac{1}{1-\frac{2\cdot\gamma\cdot m(r)}{r\cdot c^2}}\cdot\left(\frac{\gamma}{r^2\cdot c^2}\cdot m(r)+\frac{4\cdot\pi\cdot\gamma}{c^4}\cdot r\cdot p(r)\right) \qquad (6.5.13)$$

We need the relationship between density and Schwarzschild radius, that we use to determine the dependence of mass and pressure from the geometric quantities:

$$\rho=\frac{3\cdot c^2\cdot r_g}{8\cdot\pi\cdot\gamma\cdot R^3} \qquad (6.5.14)$$

$$m(r)=\frac{c^2\cdot r_g}{2\cdot\gamma\cdot R^3}\cdot r^3 \qquad r<R \qquad (6.5.15)$$

$$p=\frac{3\cdot c^2\cdot r_g}{8\cdot\pi\cdot\gamma\cdot R^3}\cdot\frac{\sqrt{1-\frac{r_g\cdot r^2}{R^3}}-\sqrt{1-\frac{r_g}{R}}}{3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}} \qquad r<R \qquad (6.5.16)$$

Substitute them into the derivative of the metric function according to *r*, and integrate:

$$\Phi' = \frac{r_g \cdot r}{\left(3\cdot\sqrt{1-\frac{r_g}{R}} - \sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)\cdot\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\cdot R^3}$$

$$\Phi = \log\left(\frac{1}{2}\cdot\left(\sqrt{1-\frac{r_g\cdot r^2}{R^3}} - 3\cdot\sqrt{1-\frac{r_g}{R}}\right)\right) \qquad (6.5.17)$$

$$e^{2\cdot\Phi} = \frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}} - \sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2 \qquad (6.5.18)$$

The arc length squared in the interior of a spherically symmetric homogeneous celestial body:

$$ds^2 = \frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}} - \sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2\cdot c^2\cdot dt^2 - \frac{dr^2}{1-\frac{r_g\cdot r^2}{R^3}} - r^2\cdot(d\vartheta^2 + \sin^2(\vartheta)\cdot\varphi^2) \qquad (6.5.19)$$

The entire spacetime is free of singularities. Neither do they occur in the centre of the celestial body, because in this case the denominator of the second metric function does not become zero. The values of the square roots in the second metric function are always real numbers, because of our previous condition on the infinite pressure. Write down the usual geometric quantities from the metric tensor to the Einstein tensor:

$$P = \sqrt{1-\frac{r_g}{R}} \qquad Q = \sqrt{1-\frac{r_g\cdot r^2}{R^3}}$$

$$g_{\eta\kappa} = \begin{pmatrix} \frac{1}{4}\cdot(3\cdot P - Q)^2 & 0 & 0 & 0 \\ 0 & -\frac{1}{Q^2} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\cdot\sin^2(\vartheta) \end{pmatrix}$$

$$g^{\eta\kappa} = \begin{pmatrix} \frac{4}{(3\cdot P - Q)^2} & 0 & 0 & 0 \\ 0 & -Q^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\cdot\sin^2(\vartheta)} \end{pmatrix} \qquad (6.5.20)$$

$$\frac{\partial g_{tt}}{\partial r}=\frac{r\cdot r_g\cdot(3\cdot P-Q)}{2\cdot Q\cdot R^3} \qquad \frac{\partial g^{tt}}{\partial r}=-\frac{8\cdot r\cdot r_g}{(3\cdot P-Q)^3\cdot Q\cdot R^3}$$

$$\frac{\partial g_{rr}}{\partial r}=-\frac{2\cdot r\cdot r_g}{Q^4\cdot R^3} \qquad \frac{\partial g^{rr}}{\partial r}=\frac{2\cdot r\cdot r_g}{R^3}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r}=-2\cdot r \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial r}=\frac{2}{r^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=-2\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r}=\frac{2}{r^3\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-2\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot\cos(\vartheta)}{r^2\cdot\sin^3(\vartheta)} \tag{6.5.21}$$

$$\Gamma^t{}_{tr}=\Gamma^t{}_{rt}=\frac{r\cdot r_g}{(3\cdot P-Q)\cdot Q\cdot R^3}$$

$$\Gamma^r{}_{tt}=\frac{r\cdot r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3} \qquad \Gamma^r{}_{rr}=\frac{r\cdot r_g}{R^3-r^2\cdot r_g}$$

$$\Gamma^r{}_{\vartheta\vartheta}=-r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right) \qquad \Gamma^r{}_{\varphi\varphi}=-r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right)\cdot\sin^2(\vartheta)$$

$$\Gamma^\vartheta{}_{r\vartheta}=\Gamma^\vartheta{}_{\vartheta r}=\Gamma^\varphi{}_{r\varphi}=\Gamma^\varphi{}_{\varphi r}=\frac{1}{r} \qquad \Gamma^\vartheta{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^\varphi{}_{\vartheta\varphi}=\Gamma^\varphi{}_{\varphi\vartheta}=\cot(\vartheta) \tag{6.5.22}$$

$$\frac{\partial\Gamma^t{}_{tr}}{\partial r}=\frac{\partial\Gamma^t{}_{rt}}{\partial r}=\frac{r_g}{(3\cdot P-Q)\cdot Q\cdot R^3}\cdot\left(1-\frac{r^2\cdot r_g}{(3\cdot P-Q)\cdot Q\cdot R^3}+\frac{r^2\cdot r_g}{Q^2\cdot R^3}\right)$$

$$\frac{\partial\Gamma^r{}_{tt}}{\partial r}=\frac{r_g}{2\cdot R^3}\cdot\left((3\cdot P-Q)\cdot\left(\frac{Q}{2}+\frac{r^2\cdot r_g}{2\cdot Q\cdot R^3}-\frac{r^2\cdot r_g}{Q\cdot R^3}\right)+\frac{r^2\cdot r_g}{2\cdot R^3}\right)$$

$$\frac{\partial\Gamma^r{}_{rr}}{\partial r}=r_g\cdot\frac{R^3+r^2\cdot r_g}{(R^3-r^2\cdot r_g)^2} \qquad \frac{\partial\Gamma^r{}_{\vartheta\vartheta}}{\partial r}=-\left(1-\frac{3\cdot r^2\cdot r_g}{R^3}\right)$$

$$\frac{\partial\Gamma^r{}_{\varphi\varphi}}{\partial r}=-\left(1-\frac{3\cdot r^2\cdot r_g}{R^3}\right)\cdot\sin^2(\vartheta) \qquad \frac{\partial\Gamma^\vartheta{}_{r\vartheta}}{\partial r}=\frac{\partial\Gamma^\vartheta{}_{\vartheta r}}{\partial r}=\frac{\partial\Gamma^\varphi{}_{r\varphi}}{\partial r}=\frac{\partial\Gamma^\varphi{}_{\varphi r}}{\partial r}=-\frac{1}{r^2}$$

$$\frac{\partial \Gamma^{r}{}_{\varphi\varphi}}{\partial \vartheta}=-2\cdot r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right)\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial \Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial \vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta) \qquad \frac{\partial \Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial \vartheta}=\frac{\partial \Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial \vartheta}=-\cot^2(\vartheta)-1 \tag{6.5.23}$$

$$R^{t}{}_{rtr}=-R^{t}{}_{rrt}=-\frac{r_g}{(3\cdot P-Q)\cdot Q\cdot R^3} \qquad R^{t}{}_{\vartheta t\vartheta}=-R^{t}{}_{\vartheta\vartheta t}=-\frac{r^2\cdot r_g\cdot Q}{(3\cdot P-Q)\cdot R^3}$$

$$R^{t}{}_{\varphi t\varphi}=-R^{t}{}_{\varphi\varphi t}=-\frac{r^2\cdot r_g\cdot Q}{(3\cdot P-Q)\cdot R^3}\cdot\sin^2(\vartheta)$$

$$R^{r}{}_{ttr}=-R^{r}{}_{trt}=R^{\vartheta}{}_{tt\vartheta}=-R^{\vartheta}{}_{t\vartheta t}=R^{\varphi}{}_{tt\varphi}=-R^{\varphi}{}_{t\varphi t}=-\frac{r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3}$$

$$R^{r}{}_{\vartheta r\vartheta}=-R^{r}{}_{\vartheta\vartheta r}=-R^{\varphi}{}_{\vartheta\vartheta\varphi}=R^{\varphi}{}_{\vartheta\varphi\vartheta}=\frac{r^2\cdot r_g}{R^3}$$

$$R^{r}{}_{\varphi r\varphi}=-R^{r}{}_{\varphi\varphi r}=R^{\vartheta}{}_{\varphi\vartheta\varphi}=-R^{\vartheta}{}_{\varphi\varphi\vartheta}=\frac{r^2\cdot r_g}{R^3}\cdot\sin^2(\vartheta)$$

$$R^{\vartheta}{}_{rr\vartheta}=-R^{\vartheta}{}_{r\vartheta r}=R^{\varphi}{}_{rr\varphi}=-R^{\varphi}{}_{r\varphi r}=-\frac{r_g}{R^3-r^2\cdot r_g} \tag{6.5.24}$$

$$R_{tt}=\frac{3\cdot r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3} \qquad R_{rr}=\frac{r_g}{Q\cdot R^3}\cdot\left(\frac{2}{Q}-\frac{1}{3\cdot P-Q}\right)$$

$$R_{\vartheta\vartheta}=\frac{r^2\cdot r_g}{R^3}\cdot\left(2-\frac{Q}{3\cdot P-Q}\right) \qquad R_{\varphi\varphi}=\left(\frac{r^2\cdot r_g}{R^3}\cdot\left(1-\frac{Q}{3\cdot P-Q}\right)-Q^2+1\right)\cdot\sin^2(\vartheta) \tag{6.5.25}$$

$$R_{ricci}=\frac{r^2\cdot r_g\cdot(11\cdot Q-15\cdot P)-(3\cdot P-Q)\cdot(1-Q^2)\cdot R^3}{r^2\cdot(3\cdot P-Q)\cdot R^3} \tag{6.5.26}$$

$$G_{tt}=(3\cdot P-Q)^2\cdot\frac{(1-Q^2)\cdot R^3+5\cdot r^2\cdot r_g}{8\cdot r^2\cdot R^3}$$

$$G_{rr}=-\frac{(3\cdot P-Q)\cdot(1-Q^2)\cdot R^3+r^2\cdot r_g\cdot(3\cdot P-5\cdot Q)}{2\cdot r^2\cdot(3\cdot P-Q)\cdot Q^2\cdot R^3}$$

$$G_{\vartheta\vartheta}=-\frac{(3\cdot P-Q)\cdot(1-Q^2)\cdot R^3+r^2\cdot r_g\cdot(3\cdot P-5\cdot Q)}{2\cdot(3\cdot P-Q)\cdot R^3}$$

$$G_{\varphi\varphi}=\frac{(3\cdot P-Q)\cdot(1-Q^2)\cdot R^3-r^2\cdot r_g\cdot(9\cdot P-7\cdot Q)}{2\cdot(3\cdot P-Q)\cdot R^3} \tag{6.5.27}$$

6.6 Fall into the centre

Particle trajectories moving on shorter paths than the average mean path, and of objects falling in conveniently shaped tunnels are described by the following geodesic equations:

$$c\cdot\ddot{t}+2\cdot\Gamma^t{}_{tr}\cdot c\cdot\dot{t}\cdot\dot{r}=0$$

$$\ddot{t}+\frac{r\cdot r_g}{(3\cdot P-Q)\cdot Q\cdot R^3}\cdot\dot{t}\cdot\dot{r}=0 \tag{6.6.1}$$

$$\ddot{r}+\Gamma^r{}_{tt}\cdot c^2\cdot\dot{t}^2+\Gamma^r{}_{rr}\cdot\dot{r}^2+\Gamma^r{}_{\vartheta\vartheta}\cdot\dot{\vartheta}^2+\Gamma^r{}_{\varphi\varphi}\cdot\dot{\varphi}^2=0$$

$$\ddot{r}+\frac{r\cdot r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3}\cdot c^2\cdot\dot{t}^2+\frac{r\cdot r_g}{R^3-r^2\cdot r_g}\cdot\dot{r}^2-r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right)\cdot\dot{\vartheta}^2-r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right)\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2=0 \tag{6.6.2}$$

$$\ddot{\vartheta}+2\cdot\Gamma^\vartheta{}_{r\vartheta}\cdot\dot{r}\,\dot{\vartheta}+\Gamma^\vartheta{}_{\varphi\varphi}\cdot\dot{\varphi}^2=0$$

$$\ddot{\vartheta}+\frac{2}{r}\cdot\dot{r}\,\dot{\vartheta}-\cos(\vartheta)\cdot\sin(\vartheta)\cdot\dot{\varphi}^2=0 \tag{6.6.3}$$

$$\ddot{\varphi}+2\cdot\Gamma^\varphi{}_{r\varphi}\cdot\dot{r}\,\dot{\varphi}+2\cdot\Gamma^\varphi{}_{\vartheta\varphi}\cdot\dot{\vartheta}\cdot\dot{\varphi}=0$$

$$\ddot{\varphi}+\frac{2}{r}\cdot\dot{r}\,\dot{\varphi}+2\cdot\cot(\vartheta)\cdot\dot{\vartheta}\cdot\dot{\varphi}=0 \tag{6.6.4}$$

In the case of vertical fall, the coordinate conditions coincide with the case of the vacuum solution:

$$t=t(\tau) \qquad \tau=\tau(t)$$

$$r=r(\tau) \qquad r=r(t)$$

$$\vartheta=const.=\frac{\pi}{2} \qquad d\vartheta=0$$

$$\varphi=const. \qquad d\varphi=0 \tag{6.6.5}$$

By substituting them, the equations of movement of the trajectory:

$$\ddot{t}+\frac{r\cdot r_g}{(3\cdot P-Q)\cdot Q\cdot R^3}\cdot\dot{t}\cdot\dot{r}=0 \tag{6.6.6}$$

$$\ddot{r}+\frac{r\cdot r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3}\cdot c^2\cdot\dot{t}^2+\frac{r\cdot r_g}{R^3-r^2\cdot r_g}\cdot\dot{r}^2=0 \tag{6.6.7}$$

$$\ddot{\vartheta}=0 \tag{6.6.8}$$

$$\ddot{\varphi}=0 \tag{6.6.9}$$

Substitute the coordinate conditions into the arc length squared:

$$ds^2=\frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2\cdot c^2\cdot dt^2-\frac{dr^2}{1-\frac{r_g\cdot r^2}{R^3}}-r^2\cdot(d\,\vartheta^2+\sin^2(\vartheta)\cdot\varphi^2)$$

$$c^2\cdot d\,\tau^2=\frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2\cdot c^2\cdot dt^2-\frac{dr^2}{1-\frac{r_g\cdot r^2}{R^3}}$$

The relationship between proper time and coordinate time is velocity dependent:

$$d\,\tau=\sqrt{\frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2-\frac{v_r^2}{c^2\cdot\left(1-\frac{r_g\cdot r^2}{R^3}\right)}}\cdot dt \qquad v_r=\frac{dr}{dt} \tag{6.6.10}$$

We make the arc length squared along a time-like infalling geodesic equal to the arc length squared of the co-moving coordinate system, then divide with the change in proper time, and write down the equation with the tangent vectors:

$$A\cdot c^2\cdot\frac{dt^2}{d\,\tau^2}-B\cdot\frac{dr^2}{d\,\tau^2}=\frac{c^2\cdot d\,\tau^2}{d\,\tau^2}=c^2$$

$$u^t=\frac{dt}{d\,\tau} \qquad u^r=\frac{dr}{d\,\tau}$$

$$A\cdot c^2\cdot(u^t)^2-B\cdot(u^r)^2=c^2 \tag{6.6.11}$$

We have derived in the mathematical introduction, that if the partial derivative of the metric tensor along a coordinate is zero, then the corresponding covariant tangent vector is a constant of movement:

$$\frac{\partial g_{\eta\kappa}}{\partial t}=0 \qquad \rightarrow \qquad \frac{\partial u_t}{\partial t}=0 \tag{6.6.12}$$

We calculate the time-like covariant tangent vector from the contravariant one with index lowering:

$$u_t=g_{t\alpha}\cdot u^{\alpha}=g_{tt}\cdot u^t=A\cdot u^t \tag{6.6.13}$$

Rearrange the arc length squared and express the square of the time-like covariant tangent vector:

$$c^2\cdot(u_t)^2=A^2\cdot c^2\cdot(u^t)^2=A\cdot(c^2+B\cdot(u^r)^2)$$

At the beginning of the fall, the radial velocity is zero:

$$c^2\cdot(u_t)^2=A(r_0)\cdot c^2 \tag{6.6.14}$$

We make the two results equal, and express the radial velocity. We pick the negative root, because the numeric value of the radial coordinate has to decrease, we are looking for the infalling solution. r_0 is the radial coordinate of the starting point:

$$A\cdot(c^2+B\cdot(u^r)^2)=A(r_0)\cdot c^2$$

$$u^r=\sqrt{\frac{1}{B}\cdot\left(\frac{A(r_0)\cdot c^2}{A}-c^2\right)}$$

$$\frac{dr}{d\tau}=\sqrt{\left(1-\frac{r_g\cdot r^2}{R^3}\right)\cdot\left(\frac{\frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r_0^2}{R^3}}\right)^2\cdot c^2}{\frac{1}{4}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2}-c^2\right)}$$

$$H=\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r_0^2}{R^3}}\right)^2 \qquad\qquad K=3\cdot\sqrt{1-\frac{r_g}{R}}$$

The time dependence of the fall cannot be integrated in a closed form:

$$\tau=\frac{1}{c}\cdot\int_{r_0}^{r}\frac{1}{\sqrt{\left(1-\frac{r_g\cdot r'^2}{R^3}\right)\cdot\left(\frac{H}{\left(K-\sqrt{1-\frac{r_g\cdot r'^2}{R^3}}\right)^2}-1\right)}}\cdot dr' \tag{6.6.15}$$

Calculate the movement as a function of the coordinate time:

$$\frac{dr}{dt}=\frac{dr}{d\tau}\cdot\frac{d\tau}{dt}=\frac{dr}{d\tau}\cdot\frac{1}{u^t} \tag{6.6.16}$$

The contravariant time-like tangent vector changes during the movement, its covariant companion however does not, therefore we substitute the latter:

$$\frac{dr}{dt}=\frac{dr}{d\tau}\cdot\frac{A}{u_t} \qquad u^t=\frac{u_t}{A} \tag{6.6.17}$$

Substitute the time-oriented covariant tangent vector:

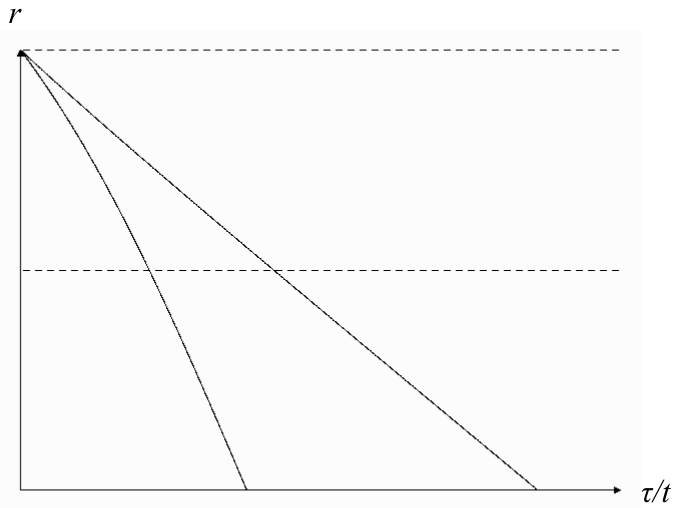
$$\frac{dr}{dt}=\frac{dr}{d\tau}\cdot\frac{A}{\sqrt{A(r_0)}} \qquad u_t=\sqrt{A(r_0)} \tag{6.6.18}$$

$$\frac{dr}{dt}=c\cdot\sqrt{\left(1-\frac{r_g\cdot r^2}{R^3}\right)\cdot\left(\frac{\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r_0^2}{R^3}}\right)^2}{\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2}-1\right)}\cdot\frac{1}{2}\cdot\frac{\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)^2}{3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r_0^2}{R^3}}}$$

The complete expression cannot be integrated in a closed form:

$$\tau=\frac{2}{c\cdot\sqrt{H}}\cdot\int_{r_0}^{r}\frac{\left(K-\sqrt{1-\frac{r_g\cdot r'^2}{R^3}}\right)^2}{\sqrt{\left(1-\frac{r_g\cdot r'^2}{R^3}\right)\cdot\left(\frac{H}{\left(K-\sqrt{1-\frac{r_g\cdot r'^2}{R^3}}\right)^2}-1\right)}}\cdot dr' \tag{6.6.19}$$

We display the two functions together on this graph. Time passes from left to right, that means proper time for the left curve, and coordinate time for the right curve, the vertical axis is the radius. The upper dotted line is the surface of the celestial object, the middle is the gravitational radius:



We can see on this graph, that the falling observer reaches the bottom of the pit faster according to his own watch, than according to the watch of the infinitely distant observer.

The acceleration of an observer at rest inside the celestial body can be calculated with the geodesic equation:

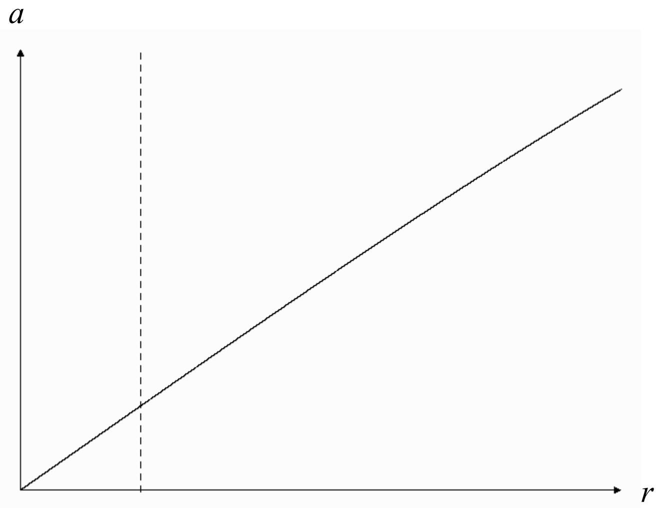
$$\ddot{r}+\frac{r\cdot r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3}\cdot c^2\cdot\dot{t}^2+\frac{r\cdot r_g}{R^3-r^2\cdot r_g}\cdot\dot{r}^2-r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right)\cdot\dot{\vartheta}^2-r\cdot\left(1-\frac{r^2\cdot r_g}{R^3}\right)\cdot\sin^2(\vartheta)\cdot\dot{\varphi}^2=0 \qquad (6.6.20)$$

$$\ddot{r}=-\frac{r\cdot r_g\cdot(3\cdot P-Q)\cdot Q}{4\cdot R^3}\cdot c^2\cdot\dot{t}^2$$

$$P=\sqrt{1-\frac{r_g}{R}} \qquad Q=\sqrt{1-\frac{r_g\cdot r^2}{R^3}}$$

$$\ddot{r}=-\frac{r\cdot r_g}{4\cdot R^3}\cdot\left(3\cdot\sqrt{1-\frac{r_g}{R}}-\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\right)\cdot\sqrt{1-\frac{r_g\cdot r^2}{R^3}}\cdot c^2\cdot\dot{t}^2 \qquad (6.6.21)$$

Distance from the gravitational centre increases from left to right, the coordinate acceleration of the observer inside the celestial body is on the vertical axis, the dotted line shows the place of the gravitational radius:



We can see on this graph, that as we approach the centre, the acceleration of observers sitting in caves is going to zero.

6.7 Relativistic dust

Dust is matter that is characterized only by density, with zero internal pressure, it is the limiting case of the perfect fluid. In this case the energy-momentum tensor is determined only by the density and the four-velocity:

$$T_{\eta\kappa}=\rho\cdot u_{\eta}\cdot u_{\kappa} \tag{6.7.1}$$

In the special case, when the observer co-moves with the dust particles, the four-velocity simplifies, and only one non-zero component of the energy-momentum tensor remains:

$$u_{\eta}=(c \quad 0 \quad 0 \quad 0)$$

$$T_{\eta\kappa}=\begin{pmatrix} \rho\cdot c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.7.2}$$

In the general case, the four-velocity transforms the following way from the point of view of a stationary observer:

$$u_\eta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \begin{pmatrix} c & v_x & v_y & v_z \end{pmatrix} \tag{6.7.3}$$

In this case the general energy-momentum tensor:

$$T_{\eta\kappa} = \rho \cdot \frac{1}{1-\frac{v^2}{c^2}} \begin{pmatrix} c^2 & v_x \cdot c & v_x \cdot c & v_z \cdot c \\ v_x \cdot c & v_x \cdot v_x & v_x \cdot v_y & v_x \cdot v_z \\ v_y \cdot c & v_y \cdot v_x & v_y \cdot v_y & v_y \cdot v_z \\ v_z \cdot c & v_z \cdot v_x & v_z \cdot v_y & v_z \cdot v_z \end{pmatrix} \tag{6.7.4}$$

6.8 Collapsing spherical dust cloud

Set up a co-moving coordinate system, write down the general arc length squared, and calculated the geometric quantities:

$$ds^2 = c^2 \cdot dt^2 - B(r,t) \cdot dr^2 - C(r,t) \cdot r^2 \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2) \tag{6.8.1}$$

$$g_{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & -C \cdot r^2 & 0 \\ 0 & 0 & 0 & -C \cdot r^2 \cdot \sin^2(\vartheta) \end{pmatrix}$$

$$g^{\eta\kappa} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{B} & 0 & 0 \\ 0 & 0 & -\frac{1}{C \cdot r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{C \cdot r^2 \cdot \sin^2(\vartheta)} \end{pmatrix} \tag{6.8.2}$$

$$\frac{\partial g_{rr}}{\partial t} = -\dot{B} \qquad \frac{\partial g^{rr}}{\partial t} = \frac{\dot{B}}{B^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial t} = -\dot{C} \cdot r^2 \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial t} = \frac{\dot{C}}{C^2 \cdot r^2}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial t} = -\dot{C} \cdot r^2 \cdot \sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial t} = \frac{\dot{C}}{C^2 \cdot r^2 \cdot \sin^2(\vartheta)}$$

$$\frac{\partial g_{rr}}{\partial r}=-B' \qquad \frac{\partial g^{rr}}{\partial r}=\frac{B'}{B^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial r}=-(C'\cdot r+2\cdot C)\cdot r \qquad \frac{\partial g_{\vartheta\vartheta}}{\partial r}=\frac{C'\cdot r+2\cdot C}{C^2\cdot r^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial r}=-(C'\cdot r+2\cdot C)\cdot r\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial r}=\frac{C'\cdot r+2\cdot C}{C^2\cdot r^3\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-2\cdot C\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot\cos(\vartheta)}{C\cdot r^2\cdot\sin^3(\vartheta)} \qquad (6.8.3)$$

$$\Gamma^t{}_{rr}=\frac{\dot{B}}{2} \qquad \Gamma^t{}_{\vartheta\vartheta}=\frac{\dot{C}\cdot r^2}{2} \qquad \Gamma^t{}_{\varphi\varphi}=\frac{\dot{C}\cdot r^2}{2}\cdot\sin^2(\vartheta)$$

$$\Gamma^r{}_{tr}=\Gamma^r{}_{rt}=\frac{\dot{B}}{2\cdot B} \qquad \Gamma^r{}_{rr}=\frac{B'}{2\cdot B} \qquad \Gamma^r{}_{\vartheta\vartheta}=-\frac{C'\cdot r+2\cdot C}{2\cdot B}\cdot r$$

$$\Gamma^r{}_{\varphi\varphi}=-\frac{C'\cdot r+2\cdot C}{2\cdot B}\cdot r\cdot\sin^2(\vartheta)$$

$$\Gamma^\vartheta{}_{t\vartheta}=\Gamma^\vartheta{}_{\vartheta t}=\Gamma^\varphi{}_{t\varphi}=\Gamma^\varphi{}_{\varphi t}=\frac{\dot{C}}{2\cdot C} \qquad \Gamma^\vartheta{}_{r\vartheta}=\Gamma^\vartheta{}_{\vartheta r}=\Gamma^\varphi{}_{r\varphi}=\Gamma^\varphi{}_{\varphi r}=\frac{C'\cdot r+2\cdot C}{2\cdot C\cdot r}$$

$$\Gamma^\vartheta{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta) \qquad \Gamma^\varphi{}_{\vartheta\varphi}=\Gamma^\varphi{}_{\varphi\vartheta}=\cot(\vartheta) \qquad (6.8.4)$$

$$\frac{\partial\Gamma^t{}_{rr}}{\partial t}=\frac{\ddot{B}}{2} \qquad \frac{\partial\Gamma^t{}_{\vartheta\vartheta}}{\partial t}=\frac{\ddot{C}\cdot r^2}{2} \qquad \frac{\partial\Gamma^t{}_{\varphi\varphi}}{\partial t}=\frac{\ddot{C}\cdot r^2}{2}\cdot\sin^2(\vartheta)$$

$$\frac{\partial\Gamma^r{}_{tr}}{\partial t}=\frac{\partial\Gamma^r{}_{rt}}{\partial t}=\frac{B\cdot\ddot{B}-\dot{B}^2}{2\cdot B^2} \qquad \frac{\partial\Gamma^r{}_{rr}}{\partial t}=\frac{\partial\Gamma^r{}_{tr}}{\partial r}=\frac{\partial\Gamma^r{}_{rt}}{\partial r}=\frac{B\cdot\dot{B}'-B'\cdot\dot{B}}{2\cdot B^2}$$

$$\frac{\partial\Gamma^r{}_{\vartheta\vartheta}}{\partial t}=-\frac{2\cdot(B\cdot\dot{C}-\dot{B}\cdot C)+(B\cdot\dot{C}'-\dot{B}\cdot C')\cdot r}{2\cdot B^2}\cdot r$$

$$\frac{\partial\Gamma^r{}_{\varphi\varphi}}{\partial t}=-\frac{2\cdot(B\cdot\dot{C}-\dot{B}\cdot C)+(B\cdot\dot{C}'-\dot{B}\cdot C')\cdot r}{2\cdot B^2}\cdot r\cdot\sin^2(\vartheta)$$

$$\frac{\partial\Gamma^\vartheta{}_{t\vartheta}}{\partial t}=\frac{\partial\Gamma^\vartheta{}_{\vartheta t}}{\partial t}=\frac{\partial\Gamma^\varphi{}_{t\varphi}}{\partial t}=\frac{\partial\Gamma^\varphi{}_{\varphi t}}{\partial t}=\frac{C\cdot\ddot{C}-\dot{C}^2}{2\cdot C^2}$$

$$\frac{\partial\Gamma^\vartheta{}_{r\vartheta}}{\partial t}=\frac{\partial\Gamma^\vartheta{}_{\vartheta r}}{\partial t}=\frac{\partial\Gamma^\varphi{}_{r\varphi}}{\partial t}=\frac{\partial\Gamma^\varphi{}_{\varphi r}}{\partial t}=\frac{\partial\Gamma^\vartheta{}_{t\vartheta}}{\partial r}=\frac{\partial\Gamma^\vartheta{}_{\vartheta t}}{\partial r}=\frac{\partial\Gamma^\varphi{}_{t\varphi}}{\partial r}=\frac{\partial\Gamma^\varphi{}_{\varphi t}}{\partial r}=\frac{C\cdot\dot{C}'-C'\cdot\dot{C}}{2\cdot C^2}$$

$$\frac{\partial \Gamma^{t}{}_{rr}}{\partial r}=\frac{\dot{B}'}{2} \qquad \frac{\partial \Gamma^{t}{}_{\vartheta\vartheta}}{\partial r}=\frac{2\cdot\dot{C}+\dot{C}'\cdot r}{2}\cdot r \qquad \frac{\partial \Gamma^{t}{}_{\varphi\varphi}}{\partial r}=\frac{2\cdot\dot{C}+\dot{C}'\cdot r}{2}\cdot r\cdot\sin^2(\vartheta)$$

$$\frac{\partial \Gamma^{r}{}_{rr}}{\partial r}=\frac{B\cdot B''-B'^2}{2\cdot B^2} \qquad \frac{\partial \Gamma^{r}{}_{\vartheta\vartheta}}{\partial r}=-\frac{B\cdot C''\cdot r^2+C'\cdot r\cdot(4\cdot B-B'\cdot r)+2\cdot C\cdot(B-B'\cdot r)}{2\cdot B^2}$$

$$\frac{\partial \Gamma^{r}{}_{\varphi\varphi}}{\partial r}=-\frac{B\cdot C''\cdot r^2+C'\cdot r\cdot(4\cdot B-B'\cdot r)+2\cdot C\cdot(B-B'\cdot r)}{2\cdot B^2}\cdot\sin^2(\vartheta)$$

$$\frac{\partial \Gamma^{\vartheta}{}_{r\vartheta}}{\partial r}=\frac{\partial \Gamma^{\vartheta}{}_{\vartheta r}}{\partial r}=\frac{\partial \Gamma^{\varphi}{}_{r\varphi}}{\partial r}=\frac{\partial \Gamma^{\varphi}{}_{\varphi r}}{\partial r}=\frac{(C\cdot C''-C'^2)\cdot r^2+2\cdot C^2}{2\cdot C^2\cdot r^2}$$

$$\frac{\partial \Gamma^{t}{}_{\varphi\varphi}}{\partial \vartheta}=\dot{C}\cdot r^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial \Gamma^{r}{}_{\varphi\varphi}}{\partial \vartheta}=-\frac{C'\cdot r+2\cdot C}{B}\cdot r\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial \Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial \vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta) \qquad \frac{\partial \Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial \vartheta}=\frac{\partial \Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial \vartheta}=-\cot^2(\vartheta)-1 \qquad (6.8.5)$$

$$R_{tt}=-\frac{\ddot{C}}{C}+\frac{\dot{C}^2}{2\cdot C^2}-\frac{\ddot{B}}{2\cdot B}+\frac{\dot{B}^2}{4\cdot B^2}$$

$$R_{tr}=R_{rt}=\frac{C'}{2\cdot C^2}+\frac{\dot{C}}{C\cdot r}-\frac{\dot{C}'}{C}+\frac{\dot{B}\cdot C'}{2\cdot B\cdot C}+\frac{\dot{B}}{B\cdot r}$$

$$R_{rr}=\frac{\dot{B}\cdot\dot{C}}{2\cdot C}-\frac{C''}{C}+\frac{C'^2}{2\cdot C^2}+\frac{B'\cdot C'}{2\cdot B\cdot C}-\frac{2\cdot C'}{C\cdot r}+\frac{\ddot{B}}{2}-\frac{\dot{B}^2}{4\cdot B}+\frac{B'}{B\cdot r}$$

$$R_{\vartheta\vartheta}=\frac{\ddot{C}\cdot r^2}{2}+\frac{\dot{B}\cdot\dot{C}\cdot r^2}{4\cdot B}-\frac{C''\cdot r^2}{2\cdot B}+\frac{B'\cdot C'\cdot r^2}{4\cdot B^2}-\frac{2\cdot C'\cdot r}{B}+\frac{B'\cdot C\cdot r}{2\cdot B^2}-\frac{C}{B}+1$$

$$R_{\varphi\varphi}=\left(\frac{\ddot{C}\cdot r^2}{2}+\frac{\dot{B}\cdot\dot{C}\cdot r^2}{4\cdot B}-\frac{C''\cdot r^2}{2\cdot B}+\frac{B'\cdot C'\cdot r^2}{4\cdot B^2}-\frac{2\cdot C'\cdot r}{B}+\frac{B'\cdot C\cdot r}{2\cdot B^2}-\frac{C}{B}+1\right)\cdot\sin^2(\vartheta)$$
(6.8.6)

Rearrange the Einstein equation:

$$R_{\eta\kappa}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot\left(T_{\eta\kappa}-\frac{1}{2}\cdot T\cdot g_{\eta\kappa}\right) \qquad (6.8.7)$$

The quantity between the parentheses in a medium with zero pressure:

$$T_{\eta\kappa}-\frac{1}{2}\cdot T\cdot g_{\eta\kappa}=\frac{1}{2}\cdot\rho\cdot c^2\cdot\begin{pmatrix}1&0&0&0\\0&B&0&0\\0&0&C&0\\0&0&0&C\cdot\sin^2(\vartheta)\end{pmatrix} \quad (6.8.8)$$

Substitute into the reorganized Einstein equations, and write down the system of equations we have to solve:

(1) $$-\frac{\ddot{C}}{C}+\frac{\dot{C}^2}{2\cdot C^2}-\frac{\ddot{B}}{2\cdot B}+\frac{\dot{B}^2}{4\cdot B^2}=-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho$$

(2) $$\frac{C'}{2\cdot C^2}+\frac{\dot{C}}{C\cdot r}-\frac{\dot{C}'}{C}+\frac{\dot{B}\cdot C'}{2\cdot B\cdot C}+\frac{\dot{B}}{B\cdot r}=0$$

(3) $$\frac{\dot{B}\cdot\dot{C}}{2\cdot C}-\frac{C''}{C}+\frac{C'^2}{2\cdot C^2}+\frac{B'\cdot C'}{2\cdot B\cdot C}-\frac{2\cdot C'}{C\cdot r}+\frac{\ddot{B}}{2}-\frac{\dot{B}^2}{4\cdot B}+\frac{B'}{B\cdot r}=-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho\cdot B$$

(4) $$\frac{\ddot{C}\cdot r^2}{2}+\frac{\dot{B}\cdot\dot{C}\cdot r^2}{4\cdot B}-\frac{C''\cdot r^2}{2\cdot B}+\frac{B'\cdot C'\cdot r^2}{4\cdot B^2}-\frac{2\cdot C'\cdot r}{B}+\frac{B'\cdot C\cdot r}{2\cdot B^2}-\frac{C}{B}+1=-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho\cdot C$$

(6.8.9)

We assume, that the unknown functions can be separated according to their variables, in the following form:

$$B(r,t)=R^2(t)\cdot f(r) \qquad C(r,t)=S^2(t)\cdot g(r) \quad (6.8.10)$$

Substitute them into the second (2) equation to solve:

$$\frac{\dot{S}}{S}=\frac{\dot{R}}{R} \qquad \rightarrow \qquad S=k\cdot R \quad (6.8.11)$$

We rescale the r coordinate, so the k constant becomes unit sized. We can freely choose one of the remaining functions, thus we bring our two unknown functions into the following forms:

$$B(r,t)=R^2(t)\cdot f(r) \qquad C(r,t)=R^2(t)\cdot r^2 \quad (6.8.12)$$

Substitute into (3) and (4) and separate the variables. Since the two sides of the equations depend on different variables, their value is a separation constant:

$$-\frac{f'}{f^2\cdot r}=\ddot{R}\cdot R+2\cdot\dot{R}^2-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho(t)\cdot R^2$$

$$-\frac{1}{r^2}+\frac{1}{f\cdot r^2}-\frac{f'}{2\cdot f^2\cdot r}=\ddot{R}\cdot R+2\cdot\dot{R}^2-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho(t)\cdot R^2 \quad (6.8.13)$$

The two left sides are equal with each other, and the separation constant in the form of our choice:

$$-\frac{f'}{f^2\cdot r}=\ddot{R}\cdot R+2\cdot\dot{R}^2-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho(t)\cdot R^2=-2\cdot k$$

$$f(r)=\frac{1}{1-k\cdot r^2} \tag{6.8.14}$$

The arc length squared of the spherically symmetric dust cloud:

$$ds^2=c^2\cdot dt^2-R^2(t)\cdot\left(\frac{dr^2}{1-k\cdot r^2}+r^2\cdot(d\vartheta^2+\sin^2(\vartheta)\cdot d\varphi^2)\right) \tag{6.8.15}$$

Substitute the functions and the separation constant into the first (1) equation we have to solve:

$$\ddot{R}\cdot R=-\frac{4\cdot\pi\cdot\gamma}{3\cdot c^2}\cdot\rho(t)\cdot R^2$$

$$-2\cdot k=-\frac{4\cdot\pi\cdot\gamma}{3\cdot c^2}\cdot\rho(t)\cdot R^2+2\cdot\dot{R}^2-\frac{4\cdot\pi\cdot\gamma}{c^2}\cdot\rho(t)\cdot R^2 \tag{6.8.16}$$

Determine the total mass of the cloud and substitute it:

$$M=\frac{4\cdot\pi}{3}\cdot\rho(t)\cdot R^3(t) \tag{6.8.17}$$

The separation constant:

$$\dot{R}^2-\frac{2\cdot\gamma\cdot M}{c^2}\cdot\frac{1}{R}=\dot{R}^2-\frac{r_g}{R}=-k \tag{6.8.18}$$

By agreement, $R(t)$ is the time dependent radius of the spherical dust cloud, its unit is length, r and k have no dimension. Examine a collapsing dust cloud, or a collapsing star with negligible internal pressure. The particles composing the celestial body are at rest at the beginning moment:

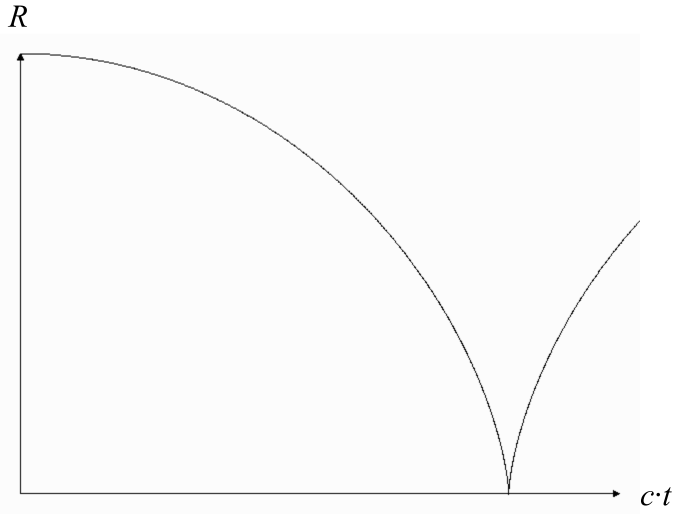
$$\dot{R}=0$$

In this case the separation constant and the change in R:

$$k=\frac{r_g}{R} \qquad \dot{R}^2=k\cdot\frac{R(0)-R(t)}{R(t)} \tag{6.8.19}$$

The solution of this differential equation is a cycloid curve, with parametric equations:

$$c\cdot t=\frac{R(0)}{2\cdot\sqrt{k}}\cdot(\psi+\sin(\psi)) \qquad R=\frac{R(0)}{2}\cdot(\psi+\sin(\psi)) \tag{6.8.20}$$



In the case of any radial change of the spherically symmetric mass distribution, the spacetime of the external vacuum is the Schwarzschild solution. Therefore the collapse time of the star is the same as the time of the observer falling at the height of the star surface.

6.9 Electromagnetic interaction

On the curved four dimensional spacetime of general relativity, gravitation is an inertial force, because it occurs only when the reference frame does not move on a straight line. The electromagnetic field however exerts a real force, that diverts charged bodies from the geodesics, and is detectable in every reference frame (it is coordinate system independent). The properties of the electromagnetic field are determined by a four-vector potential:

$$A_\eta = (\phi \quad A_x \quad A_y \quad A_z) \tag{6.9.1}$$

In flat spacetime the classical form of the action functional:

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}, t) \cdot dt \tag{6.9.2}$$

In the presence of a general conservative force law, the function can be determined easily, the contribution coming from the interactions between the particles has to be subtracted from the term depending on the movement:

$$L = E_{kinetic} - E_{potential} \tag{6.9.3}$$

In the case of the action describing the movement of the charged particle, the first term is the free movement, the second is the contribution of the electromagnetic field, where Q is the charge of the particle:

$$S=-\int m\cdot c\cdot ds+\frac{Q}{c}\cdot A_{\alpha}\cdot dx^{\alpha} \tag{6.9.4}$$

The principle of least action:

$$\delta S=-\delta\int m\cdot c\cdot\sqrt{dx_{\alpha}\cdot dx^{\alpha}}+\frac{Q}{c}\cdot A_{\alpha}\cdot dx^{\alpha}=0$$

$$\delta S=-\int m\cdot c\cdot\frac{dx_{\alpha}\cdot\delta\, dx^{\alpha}}{ds}+\frac{Q}{c}\cdot A_{\alpha}\cdot\delta\, dx^{\alpha}+\frac{Q}{c}\cdot\delta\, A_{\alpha}\cdot dx^{\alpha}=0 \tag{6.9.5}$$

The last term is zero, where u is the coordinate velocity:

$$\left(\int m\cdot c\cdot du_{\alpha}\cdot\delta\, x^{\alpha}+\frac{Q}{c}\cdot dA_{\alpha}\cdot\delta\, x^{\alpha}-\frac{Q}{c}\cdot\delta\, A_{\alpha}\cdot dx^{\alpha}\right)-\left(m\cdot c\cdot u_{\alpha}+\frac{Q}{c}\cdot A_{\alpha}\right)\cdot dx^{\alpha}=0$$

$$\delta\, A_{\eta}=\frac{\partial A_{\eta}}{\partial x^{\alpha}}\cdot\delta\, x^{\alpha} \qquad\qquad dA_{\eta}=\frac{\partial A_{\eta}}{\partial x^{\alpha}}\cdot dx^{\alpha}$$

$$\int m\cdot c\cdot du_{\alpha}\cdot\delta\, x^{\alpha}+\frac{Q}{c}\cdot\frac{\partial A_{\beta}}{\partial x^{\alpha}}\cdot dx^{\alpha}\cdot\delta\, x^{\beta}-\frac{Q}{c}\cdot\frac{\partial A_{\alpha}}{\partial x^{\beta}}\cdot\delta\, x^{\alpha}\cdot dx^{\beta}=0 \tag{6.9.6}$$

Substitution:

$$du_{\eta}=\frac{du_{\eta}}{ds}\cdot ds \qquad\qquad dx^{\eta}=u^{\eta}\cdot ds$$

$$\int\left(m\cdot c\cdot\frac{du_{\alpha}}{ds}-\frac{Q}{c}\cdot\left(\frac{\partial A_{\beta}}{\partial x^{\alpha}}-\frac{\partial A_{\alpha}}{\partial x^{\beta}}\right)\cdot u^{\beta}\right)\cdot\delta\, x^{\alpha}\cdot ds=0 \tag{6.9.7}$$

Equation of motion of a charged particle moving in electromagnetic field:

$$m\cdot c\cdot\frac{du_{\eta}}{ds}=\frac{Q}{c}\cdot\left(\frac{\partial A_{\beta}}{\partial x^{\eta}}-\frac{\partial A_{\eta}}{\partial x^{\beta}}\right)\cdot u^{\beta}=\frac{Q}{c}\cdot F_{\eta\beta}\cdot u^{\beta} \tag{6.9.8}$$

Where the form of the electromagnetic tensor is the same in arbitrarily curved spacetime as well, because the connections of the invariant derivatives cancel mutually:

$$F_{\eta\kappa}=\nabla_{\eta}A_{\kappa}-\nabla_{\kappa}A_{\eta}=\frac{\partial A_{\kappa}}{\partial x^{\eta}}-\frac{\partial A_{\eta}}{\partial x^{\kappa}} \tag{6.9.9}$$

Thus the F_{η} electromagnetic four-force depends on the Q charge, the $F_{\eta\kappa}$ electromagnetic tensor and the u four-velocity, or in other words the electromagnetic tensor and the j current density:

$$F_\eta = \frac{Q}{c} \cdot F_{\eta\alpha} \cdot v^\alpha = \frac{1}{c} \cdot F_{\eta\alpha} \cdot j^\alpha \tag{6.9.10}$$

$$j = (c \cdot \rho \quad j_x \quad j_y \quad j_z)$$

If we add to the vector potential the partial derivative of an arbitrary scalar function, it does not influence the result, this is the gauge invariance:

$$_2A_\eta = A_\eta + \frac{\partial \phi}{\partial x^\eta} \qquad \rightarrow \qquad \nabla_\alpha A^\alpha = 0 \tag{6.9.11}$$

The components of the electromagnetic tensor are the E electric field strength and the B magnetic induction:

$$F_{\eta\kappa} = \begin{pmatrix} 0 & \frac{1}{c} \cdot E_x & \frac{1}{c} \cdot E_y & \frac{1}{c} \cdot E_z \\ -\frac{1}{c} \cdot E_x & 0 & -B_z & B_y \\ -\frac{1}{c} \cdot E_y & B_z & 0 & -B_x \\ -\frac{1}{c} \cdot E_z & -B_y & B_x & 0 \end{pmatrix} \tag{6.9.12}$$

The Maxwell equations describe the electromagnetic interaction. According to one of them, the source of the electromagnetic tensor is the current density, the other is a simple identity:

$$\frac{\partial F^{\eta\alpha}}{\partial x^\alpha} = \mu_0 \cdot j^\eta \qquad \frac{\partial F_{\eta\kappa}}{\partial x^\mu} + \frac{\partial F_{\mu\eta}}{\partial x^\kappa} + \frac{\partial F_{\kappa\mu}}{\partial x^\eta} = 0 \tag{6.9.13}$$

In curved spacetime, the partial derivative changes to invariant derivative in the Maxwell equations:

$$\nabla_\alpha F^{\eta\alpha} = \mu_0 \cdot j^\eta \qquad \nabla_\mu F_{\eta\kappa} + \nabla_\kappa F_{\mu\eta} + \nabla_\eta F_{\kappa\mu} = 0 \tag{6.9.14}$$

The vacuum Einstein equations were derived for spacetimes, where bodies move on trajectories described by geodesic equations. Thus in every case where the moving bodies deviate from the geodesics, we can be sure, that an energy-momentum tensor is present, that describes the properties of the matter, that diverted the test bodies. The electromagnetic field also has energy, thus it exerts gravitational influence. Based on the definition of the energy-momentum tensor, it can be expressed from the equation of movement of the charged particle.

First we write down the electromagnetic force purely with the electromagnetic tensor. For this we substitute the first Maxwell equations into the formula for the electromagnetic force:

$$F_\eta = \frac{1}{c} \cdot F_{\eta\alpha} \cdot j^\alpha = \frac{1}{c \cdot \mu_0} \cdot F_{\eta\alpha} \cdot \nabla_\beta F^{\alpha\beta}$$

$$c\cdot\mu_0\cdot F_\eta = F_{\eta\alpha}\cdot\nabla_\beta F^{\alpha\beta} \quad (6.9.15)$$

We expand the invariant derivative on the right side using the product differentiation rule:

$$c\cdot\mu_0\cdot F_\eta = \nabla_\beta(F_{\eta\alpha}\cdot F^{\alpha\beta}) - F^{\alpha\beta}\cdot\nabla_\beta F_{\eta\alpha} \quad (6.9.16)$$

Substitute the second Maxwell equation into the right-side term:

$$F^{\alpha\beta}\cdot\nabla_\beta F_{\eta\alpha} = \frac{1}{2}\cdot F^{\alpha\beta}\cdot\nabla_\beta F_{\eta\alpha} + \frac{1}{2}\cdot F^{\beta\alpha}\cdot\nabla_\alpha F_{\eta\beta} =$$
$$\frac{1}{2}\cdot F^{\beta\alpha}\cdot(\nabla_\beta F_{\eta\alpha} + \nabla_\alpha F_{\beta\eta}) - \frac{1}{2}\cdot F^{\alpha\beta}\cdot\nabla_\eta F_{\alpha\beta} = -\frac{1}{4}\cdot\nabla_\eta(F^{\alpha\beta}\cdot F_{\alpha\beta}) = -\frac{1}{4}\cdot g^{\eta\alpha}\cdot\nabla_\eta(F^\beta_\alpha\cdot F_{\alpha\beta}) \quad (6.9.17)$$

Reinsert it into the electromagnetic force:

$$c\cdot\mu_0\cdot F_\eta = \nabla_\beta(F_{\eta\alpha}\cdot F^{\alpha\beta}) + \frac{1}{4}\cdot g^{\eta\alpha}\cdot\nabla_\eta(F^\beta_\alpha\cdot F_{\alpha\beta})$$

$$F_\eta = -\frac{1}{c\cdot\mu_0}\nabla_\eta\left(F^{\alpha\beta}\cdot F_{\alpha\beta} + \frac{1}{4}\cdot g^{\alpha\gamma}\cdot F^\beta_\alpha\cdot F_{\gamma\beta}\right) = \nabla_\alpha T^\alpha_\eta \quad (6.9.18)$$

The electromagnetic energy-momentum tensor:

$$T^{\eta\kappa} = -\frac{1}{c\cdot\mu_0}\cdot\left(F^{\eta\alpha}\cdot F^\kappa_\alpha + \frac{1}{4}\cdot g^{\eta\kappa}\cdot F_{\alpha\beta}\cdot F^{\alpha\beta}\right) \quad (6.9.19)$$

In matrix form:

$$T^{\eta\kappa} = \begin{pmatrix} \frac{1}{2}\cdot\left(\varepsilon_0\cdot E^2 + \frac{1}{\mu_0}\cdot B^2\right) & \frac{S_x}{c} & \frac{S_y}{c} & \frac{S_z}{c} \\ \frac{S_x}{c} & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ \frac{S_y}{c} & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ \frac{S_z}{c} & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{pmatrix} \quad (6.9.20)$$

The direction of movement of the electromagnetic wave is shown by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0}\cdot\vec{E}\times\vec{B} \quad (6.9.21)$$

The form of the Maxwell stress tensor composing the spatial terms in flat spacetime:

$$\sigma_{ij}=\varepsilon_0\cdot E_i\cdot E_j+\frac{1}{\mu_0}\cdot B_i\cdot B_j-\frac{1}{2}\cdot\left(\varepsilon_0\cdot E^2+\frac{1}{\mu_0}\cdot B^2\right)\cdot\delta_{ij} \qquad (6.9.22)$$

The divergence of the electromagnetic energy-momentum tensor:

$$\nabla_\alpha T^\alpha_\eta=F_\eta=\frac{1}{c}\cdot F_{\eta\alpha}\cdot j^\alpha \qquad (6.9.23)$$

6.10 Electromagnetic waves

In order to determine the phase equation describing the electromagnetic wave, we start with the Maxwell equations:

$$\nabla_\alpha F^{\eta\alpha}=\mu_0\cdot j^\eta$$

$$\nabla_\alpha(g^{\eta\beta}\cdot\nabla_\beta A^\alpha-g^{\alpha\beta}\cdot\nabla_\beta A^\eta)=\mu_0\cdot j^\eta$$

$$-g^{\alpha\beta}\nabla_\alpha\cdot\nabla_\beta A^\eta+g^{\eta\beta}\nabla_\alpha\cdot\nabla_\beta A^\alpha=\mu_0\cdot j^\eta \quad /+g^{\eta\beta}\nabla_\beta\cdot\nabla_\alpha A^\alpha-g^{\eta\beta}\nabla_\beta\cdot\nabla_\alpha A^\alpha$$

$$-g^{\alpha\beta}\nabla_\alpha\cdot\nabla_\beta A^\eta+g^{\eta\beta}(\nabla_\alpha\cdot\nabla_\beta-\nabla_\beta\cdot\nabla_\alpha)A^\alpha+g^{\eta\beta}\nabla_\beta\cdot\nabla_\alpha A^\alpha=\mu_0\cdot j^\eta \qquad (6.10.1)$$

The electromagnetic waves propagate in space detached from their source, therefore the current density is zero, and we also write down the gauge invariance:

$$j^\eta=0 \qquad \nabla_\alpha A^\alpha=0 \qquad (6.10.2)$$

Substitute the Ricci tensor as well, it is derived with index contraction from the commutator of the invariant derivative, the curvature tensor:

$$R^\eta_\kappa=g^{\eta\beta}\cdot(\nabla_\kappa\nabla_\beta-\nabla_\beta\nabla_\kappa) \qquad (6.10.3)$$

The result is the wave equation of the electromagnetic waves:

$$-g^{\alpha\beta}\cdot\nabla_\alpha\nabla_\beta A^\eta+R^\eta_\alpha\cdot A^\alpha=0 \qquad (6.10.4)$$

We are investigating the wave function in the following form:

$$A^\eta=a^\eta\cdot e^{i\cdot\psi} \qquad (6.10.5)$$

Where the wave function is proportional to the scalar product of the wave number vector and the position vector:

$$\psi \propto k_\alpha \cdot x^\alpha$$

Where the formulas for the wave number vector and the light ray are:

$$k_\eta = -\frac{\partial \psi}{\partial x^\eta} \qquad\qquad k^\eta(x) = \frac{dx^\eta}{d\lambda} \tag{6.10.6}$$

And the wave function changes faster than the wave number vector, the amplitude, or the metric tensor. By using these the original equation is simplified, this is the eikonal or phase equation:

$$g^{\alpha\beta} \cdot \frac{\partial \psi}{\partial x^\alpha} \cdot \frac{\partial \psi}{\partial x^\beta} = 0 \tag{6.10.7}$$

We can see that the wave number vector is light-like, thus we have shown, that the electromagnetic waves move in light-like directions:

$$g^{\alpha\beta} \cdot k_\alpha \cdot k_\beta = 0 \tag{6.10.8}$$

Examine the invariant derivative of the scalar product:

$$\nabla_\eta (k^\alpha \cdot k_\alpha) = 0$$

$$k_\alpha \cdot \nabla_\eta k^\alpha + k^\alpha \cdot \nabla_\eta k_\alpha = 0 \tag{6.10.9}$$

We can rearrange the first term, until it looks like the second term:

$$k_\alpha \cdot \nabla_\eta k^\alpha = k_\alpha \cdot \nabla_\eta (g^{\alpha\beta} \cdot k_\beta) = k_\alpha \cdot k_\beta \cdot \nabla_\eta g^{\alpha\beta} + k_\alpha \cdot g^{\alpha\beta} \cdot \nabla_\eta k_\beta$$

The invariant derivative of the metric tensor is zero, and we use up the second metric tensor to raise the index:

$$k_\alpha \cdot g^{\alpha\beta} \cdot \nabla_\eta k_\beta = k^\beta \cdot \nabla_\eta k_\beta$$

Reinsert the result into the original equation:

$$2 \cdot k^\alpha \cdot \nabla_\eta k_\alpha = 0 \tag{6.10.10}$$

If the connection is symmetric, we can replace the indices of the invariant derivative and the wave number vector:

$$\nabla_\eta k_\alpha = \frac{\partial^2 \psi}{\partial x^\eta \cdot \partial x^\alpha} - k_\beta \cdot \Gamma^\beta{}_{\eta\alpha} = \frac{\partial^2 \psi}{\partial x^\alpha \cdot \partial x^\eta} - k_\beta \cdot \Gamma^\beta{}_{\alpha\eta} = \nabla_\alpha k_\eta \qquad (\alpha\text{: free index})$$

Reinsert it and then substitute the equation of light rays replacing the contravariant wave number vector:

$$2 \cdot k^\alpha \cdot \nabla_\alpha k_\eta = 0 \tag{6.10.11}$$

$$2 \cdot \frac{dx^{\alpha}}{d\lambda} \cdot \nabla_{\alpha} k_{\eta} = 0 \quad / \cdot \frac{d\lambda}{2}$$

We recognize the relationship between the invariant derivative and the derivative along a curve. When the derivative along a curve is zero, we recover the geodesic equation, thus we have shown, that light rays propagate along geodesics:

$$dx^{\alpha} \cdot \nabla_{\alpha} k_{\eta} = Ðk_{\eta} = 0 \tag{6.10.12}$$

6.11 Unification of interactions

Three elementary interactions govern the macroscopic world in our experience: gravitational, electrical and magnetic. Scientific development gradually recognized them in the cavalcade of phenomena, and that there are common organizing principles behind them. The Kaluza theory builds on general relativity, and by using the generalized Maxwell equations in Riemannian geometry (that unify electricity and magnetism), it unifies and geometrizes all macroscopic interactions. The five dimensional Einstein equation in vacuum:

$$R_{PQ} - \frac{1}{2} \cdot R \cdot g_{PQ} = 0 \tag{6.11.1}$$

If electromagnetic fields are not present, in this limiting case the metric of the four dimensional spacetime is independent of the fifth coordinate. This assumption is the basis of the choice for the general metric. The original choice of Theodor Kaluza for the five dimensional metric tensor:

$$g_{PQ} = \begin{pmatrix} {}^{4}g_{\eta\kappa} & C \cdot A_{\eta} \\ C \cdot A_{\kappa} & 2 \cdot \phi \end{pmatrix} \tag{6.11.2}$$

Here A_{η} is the electromagnetic four-potential, ϕ is an unknown constant. We differentiate between the four dimensional quantities, and the components of five dimensional quantities in four coordinates with a dimension number in the upper left index. Another possible choice is by Oscar Klein, we will use this to calculate the geometric quantities. His arc length squared and the twice covariant metric tensor:

$$ds^2 = g_{\alpha\beta} \cdot dx^{\alpha} \cdot dx^{\beta} + \phi^2 \cdot (C \cdot A_{\alpha} \cdot dx^{\alpha} + dx^{(4)})^2$$

$$g_{PQ} = \begin{pmatrix} {}^{4}g_{\eta\kappa} + C^2 \cdot \phi^2 \cdot A_{\eta} \cdot A_{\kappa} & C \cdot \phi^2 \cdot A_{\eta} \\ C \cdot \phi^2 \cdot A_{\kappa} & \phi^2 \end{pmatrix} \tag{6.11.3}$$

We can see, that the four dimensional metric tensor and the four dimensional components of the five dimensional metric tensor are already not equal:

$$ {}^4g_{\eta\kappa} \neq {}^5g_{\eta\kappa} \tag{6.11.4}$$

We declare that the derivative of the metric tensor according to the fifth direction shall be zero, therefore the corresponding derivative of the connection will also be zero, this is the cylinder condition:

$$\frac{\partial g_{PQ}}{\partial x^4}=0 \qquad \rightarrow \qquad \frac{\partial \Gamma^Q{}_{PR}}{\partial x^4}=0 \tag{6.11.5}$$

The twice contravariant metric tensor:

$$g^{PQ}=\begin{pmatrix} {}^4g^{\eta\kappa} & -C\cdot A^\eta \\ -C\cdot A^\kappa & \dfrac{1}{\phi^2}+C^2\cdot A^2 \end{pmatrix} \tag{6.11.6}$$

The partial derivative of the metric tensor:

$$\frac{\partial g_{PQ}}{\partial x^R}=\begin{pmatrix} \dfrac{\partial^4 g_{\eta\kappa}}{\partial x^R}+C^2\cdot\phi^2\cdot\left(\dfrac{\partial A_\eta}{\partial x^R}\cdot A_\kappa+A_\eta\cdot\dfrac{\partial A_\kappa}{\partial x^R}\right) & C\cdot\phi^2\cdot\dfrac{\partial A_\eta}{\partial x^R} \\ C\cdot\phi^2\cdot\dfrac{\partial A_\kappa}{\partial x^R} & 0 \end{pmatrix} \tag{6.11.7}$$

During the derivation we utilize, that the electromagnetic tensor and the vector potential are quantities also defined in four dimensions, therefore we can perform index operations on them using the the four dimensional metric tensor as well. The connection in five dimensions:

$$\Gamma^Q{}_{PR}=\frac{1}{2}\cdot g^{QA}\cdot\left(\frac{\partial g_{RA}}{\partial x^P}+\frac{\partial g_{AP}}{\partial x^R}-\frac{\partial g_{PR}}{\partial x^A}\right) \tag{6.11.8}$$

We determine the various index combinations separately. First the components of the five dimensional connection with 0 – 3 coordinate indices:

$${}^5\Gamma^\kappa{}_{\eta\mu}=\frac{1}{2}\cdot g^{\kappa A}\cdot\left(\frac{\partial g_{\mu A}}{\partial x^\eta}+\frac{\partial g_{A\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^A}\right)$$

$${}^5\Gamma^\kappa{}_{\eta\mu}=\frac{1}{2}\cdot{}^5g^{\kappa\alpha}\cdot\left(\frac{\partial^5 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^5 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^\alpha}\right)+\frac{1}{2}\cdot g^{\kappa 4}\cdot\left(\frac{\partial g_{\mu 4}}{\partial x^\eta}+\frac{\partial g_{4\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^4}\right) \tag{6.11.9}$$

Substitute the metric tensor components. The last term in the second parentheses is zero because of the cylinder condition:

$${}^5\Gamma^\kappa{}_{\eta\mu}=\frac{1}{2}\cdot{}^4g^{\kappa\alpha}\cdot\left(\frac{\partial^5 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^5 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^\alpha}\right)-\frac{1}{2}\cdot C^2\cdot\phi^2\cdot A^\kappa\cdot\left(\frac{\partial A_\mu}{\partial x^\eta}+\frac{\partial A_\eta}{\partial x^\mu}\right) \tag{6.11.10}$$

We calculate the quantity in the first parentheses separately, because we will need this result later:

$$\frac{\partial^5 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^5 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^\alpha}=\frac{\partial^4 g_{\mu\alpha}}{\partial x^\eta}+C^2\cdot\phi^2\cdot\left(\frac{\partial A_\mu}{\partial x^\eta}\cdot A_\alpha+A_\mu\cdot\frac{\partial A_\alpha}{\partial x^\eta}\right)$$
$$+\frac{\partial^4 g_{\alpha\eta}}{\partial x^\mu}+C^2\cdot\phi^2\cdot\left(\frac{\partial A_\alpha}{\partial x^\mu}\cdot A_\eta+A_\alpha\cdot\frac{\partial A_\eta}{\partial x^\mu}\right)-\frac{\partial^4 g_{\eta\mu}}{\partial x^\alpha}-C^2\cdot\phi^2\cdot\left(\frac{\partial A_\eta}{\partial x^\alpha}\cdot A_\mu+A_\eta\cdot\frac{\partial A_\mu}{\partial x^\alpha}\right)$$

In this partial result α is a free index. Rearrange the formula and substitute the electromagnetic tensor:

$$\frac{\partial^5 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^5 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^\alpha}=\frac{\partial^4 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^4 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^4 g_{\eta\mu}}{\partial x^\alpha}$$
$$+C^2\cdot\phi^2\cdot\left(A_\alpha\cdot\left(\frac{\partial A_\mu}{\partial x^\eta}+\frac{\partial A_\eta}{\partial x^\mu}\right)+A_\mu\cdot\left(\frac{\partial A_\alpha}{\partial x^\eta}-\frac{\partial A_\eta}{\partial x^\alpha}\right)+A_\eta\cdot\left(\frac{\partial A_\alpha}{\partial x^\mu}-\frac{\partial A_\mu}{\partial x^\alpha}\right)\right)$$

$$\frac{\partial^5 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^5 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^5 g_{\eta\mu}}{\partial x^\alpha}=\frac{\partial^4 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^4 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^4 g_{\eta\mu}}{\partial x^\alpha}+C^2\cdot\phi^2\cdot\left(A_\alpha\cdot\left(\frac{\partial A_\mu}{\partial x^\eta}+\frac{\partial A_\eta}{\partial x^\mu}\right)+A_\mu\cdot F_{\eta\alpha}+A_\eta\cdot F_{\mu\alpha}\right)$$
(6.11.11)

Continue the derivation, substitute the partial result:

$${}^5\Gamma^\kappa{}_{\eta\mu}=\frac{1}{2}\cdot{}^4g^{\kappa\alpha}\cdot\left(\frac{\partial^4 g_{\mu\alpha}}{\partial x^\eta}+\frac{\partial^4 g_{\alpha\eta}}{\partial x^\mu}-\frac{\partial^4 g_{\eta\mu}}{\partial x^\alpha}\right)$$
$$+\frac{1}{2}\cdot C^2\cdot\phi^2\cdot{}^4g^{\kappa\alpha}\cdot\left(A_\alpha\cdot\left(\frac{\partial A_\mu}{\partial x^\eta}+\frac{\partial A_\eta}{\partial x^\mu}\right)+A_\mu\cdot F_{\eta\alpha}+A_\eta\cdot F_{\mu\alpha}\right)$$
$$-\frac{1}{2}\cdot C^2\cdot\phi^2\cdot A^\kappa\cdot\left(\frac{\partial A_\mu}{\partial x^\eta}+\frac{\partial A_\eta}{\partial x^\mu}\right)$$

We recognize the four dimensional connection in the first term, raise the index in the second term, this gives us a term that cancels out the third term, thus the result is:

(1) $${}^5\Gamma^\kappa{}_{\eta\mu}={}^4\Gamma^\kappa{}_{\eta\mu}+\frac{1}{2}\cdot C^2\cdot\phi^2\cdot(A_\mu\cdot F^\kappa_\eta+A_\eta\cdot F^\kappa_\mu)$$ (6.11.12)

Secondly we determine connection components with coordinate indices, where one of the lower index has 4th coordinate. In the absence of torsion the connection is symmetric, therefore it is enough to do it for just one case:

$$\Gamma^\kappa{}_{4\mu}=\frac{1}{2}\cdot g^{\kappa A}\cdot\left(\frac{\partial g_{\mu A}}{\partial x^4}+\frac{\partial g_{A4}}{\partial x^\mu}-\frac{\partial g_{4\mu}}{\partial x^A}\right)$$ (6.11.13)

We see immediately that the first term in the parentheses is zero because of the cylinder condition:

$$\Gamma^{\kappa}{}_{4\mu}=\frac{1}{2}\cdot{}^{5}g^{\kappa\alpha}\cdot\left(\frac{\partial g_{\alpha 4}}{\partial x^{\mu}}-\frac{\partial g_{4\mu}}{\partial x^{\alpha}}\right)+\frac{1}{2}\cdot g^{\kappa 4}\cdot\left(\frac{\partial g_{44}}{\partial x^{\mu}}-\frac{\partial g_{4\mu}}{\partial x^{4}}\right)$$

Applying the cylinder condition again, substituting:

$$\Gamma^{\kappa}{}_{4\mu}=\frac{1}{2}\cdot{}^{4}g^{\kappa\alpha}\cdot\left(C\cdot\phi^{2}\cdot\frac{\partial A_{\alpha}}{\partial x^{\mu}}-C\cdot\phi^{2}\cdot\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)$$

Substitute the electromagnetic tensor and raise the index. We obtained, that the electromagnetic tensor is the same as one of the five dimensional connection components up to constant multipliers, this justifies Oscar Klein's choice for the metric tensor:

(2) $$\Gamma^{\kappa}{}_{4\mu}=\Gamma^{\kappa}{}_{\mu 4}=\frac{1}{2}\cdot C\cdot\phi^{2}\cdot F^{\kappa}_{\mu} \tag{6.11.14}$$

Thirdly it is the turn of the connection components with 4th upper index:

$$\Gamma^{4}{}_{\eta\mu}=\frac{1}{2}\cdot g^{4A}\cdot\left(\frac{\partial g_{\mu A}}{\partial x^{\eta}}+\frac{\partial g_{A\eta}}{\partial x^{\mu}}-\frac{\partial^{5}g_{\eta\mu}}{\partial x^{A}}\right)$$

$$\Gamma^{4}{}_{\eta\mu}=\frac{1}{2}\cdot g^{4\alpha}\cdot\left(\frac{\partial^{5}g_{\mu\alpha}}{\partial x^{\eta}}+\frac{\partial^{5}g_{\alpha\eta}}{\partial x^{\mu}}-\frac{\partial^{5}g_{\eta\mu}}{\partial x^{\alpha}}\right)+\frac{1}{2}\cdot g^{44}\cdot\left(\frac{\partial g_{\mu 4}}{\partial x^{\eta}}+\frac{\partial g_{4\eta}}{\partial x^{\mu}}-\frac{\partial^{5}g_{\eta\mu}}{\partial x^{4}}\right) \tag{6.11.15}$$

Substitute the previous partial result into the first parentheses, apply the cylinder condition in the second, and also substitute:

$$\Gamma^{4}{}_{\eta\mu}=\frac{1}{2}\cdot\left(-C\cdot A^{\alpha}\right)\cdot\left(\frac{\partial^{4}g_{\mu\alpha}}{\partial x^{\eta}}+\frac{\partial^{4}g_{\alpha\eta}}{\partial x^{\mu}}-\frac{\partial^{4}g_{\eta\mu}}{\partial x^{\alpha}}+C^{2}\cdot\phi^{2}\cdot\left(A_{\alpha}\cdot\left(\frac{\partial A_{\mu}}{\partial x^{\eta}}+\frac{\partial A_{\eta}}{\partial x^{\mu}}\right)+A_{\mu}\cdot F_{\eta\alpha}+A_{\eta}\cdot F_{\mu\alpha}\right)\right)$$
$$+\frac{1}{2}\cdot\left(\frac{1}{\phi^{2}}+C^{2}\cdot A^{2}\right)\cdot\left(C\cdot\phi^{2}\cdot\frac{\partial A_{\mu}}{\partial x^{\eta}}+C\cdot\phi^{2}\cdot\frac{\partial A_{\eta}}{\partial x^{\mu}}\right)$$

$$\Gamma^{4}{}_{\eta\mu}=-\frac{1}{2}\cdot C\cdot A^{\alpha}\cdot\left(\frac{\partial^{4}g_{\mu\alpha}}{\partial x^{\eta}}+\frac{\partial^{4}g_{\alpha\eta}}{\partial x^{\mu}}-\frac{\partial^{4}g_{\eta\mu}}{\partial x^{\alpha}}\right)$$
$$-\frac{1}{2}\cdot C^{3}\cdot\phi^{2}\cdot A_{\alpha}\cdot A^{\alpha}\cdot\left(\frac{\partial A_{\mu}}{\partial x^{\eta}}+\frac{\partial A_{\eta}}{\partial x^{\mu}}\right)-\frac{1}{2}\cdot C^{3}\cdot\phi^{2}\cdot A^{\alpha}\cdot\left(A_{\mu}\cdot F_{\eta\alpha}+A_{\eta}\cdot F_{\mu\alpha}\right)$$
$$+\frac{1}{2}\cdot C\cdot\left(\frac{\partial A_{\mu}}{\partial x^{\eta}}+\frac{\partial A_{\eta}}{\partial x^{\mu}}\right)+\frac{1}{2}\cdot C^{3}\cdot\phi^{2}\cdot A^{2}\cdot\left(\frac{\partial A_{\mu}}{\partial x^{\eta}}+\frac{\partial A_{\eta}}{\partial x^{\mu}}\right)$$

The second and the last terms cancel:

(3)

$$\Gamma^{4}{}_{\eta\mu}=-\frac{1}{2}\cdot C\cdot\left(A^{\alpha}\cdot\left(\frac{\partial^{4}g_{\mu\alpha}}{\partial x^{\eta}}+\frac{\partial^{4}g_{\alpha\eta}}{\partial x^{\mu}}-\frac{\partial^{4}g_{\eta\mu}}{\partial x^{\alpha}}\right)+C^{2}\cdot\phi^{2}\cdot A^{\alpha}\cdot(A_{\mu}\cdot F_{\eta\alpha}+A_{\eta}\cdot F_{\mu\alpha})-\left(\frac{\partial A_{\mu}}{\partial x^{\eta}}+\frac{\partial A_{\eta}}{\partial x^{\mu}}\right)\right) \quad (6.11.16)$$

Fourthly both lower indices of the connection are 4, because of the cylinder condition several terms fall out immediately, finally it turns out about the last term too, that it is zero:

$$\Gamma^{\kappa}{}_{44}=\frac{1}{2}\cdot g^{\kappa A}\cdot\left(\frac{\partial g_{4A}}{\partial x^{4}}+\frac{\partial g_{A4}}{\partial x^{4}}-\frac{\partial g_{44}}{\partial x^{A}}\right)=-\frac{1}{2}\cdot g^{\kappa A}\cdot\frac{\partial g_{44}}{\partial x^{A}}$$

(4) $$\Gamma^{\kappa}{}_{44}=-\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\frac{\partial g_{44}}{\partial x^{\alpha}}-\frac{1}{2}\cdot g^{\kappa 4}\cdot\frac{\partial g_{44}}{\partial x^{4}}=-\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\frac{\partial g_{44}}{\partial x^{\alpha}}=0 \quad (6.11.17)$$

Fifthly we examine connection components, where one of the lower and the upper indices are 4. We start with the cylinder condition again:

$$\Gamma^{4}{}_{4\mu}=\frac{1}{2}\cdot g^{4A}\cdot\left(\frac{\partial g_{\mu A}}{\partial x^{4}}+\frac{\partial g_{A4}}{\partial x^{\mu}}-\frac{\partial g_{4\mu}}{\partial x^{A}}\right)=\frac{1}{2}\cdot g^{4A}\cdot\left(\frac{\partial g_{A4}}{\partial x^{\mu}}-\frac{\partial g_{4\mu}}{\partial x^{A}}\right)$$

$$\Gamma^{4}{}_{4\mu}=\frac{1}{2}\cdot g^{4\alpha}\cdot\left(\frac{\partial g_{\alpha 4}}{\partial x^{\mu}}-\frac{\partial g_{4\mu}}{\partial x^{\alpha}}\right)+\frac{1}{2}\cdot g^{44}\cdot\left(\frac{\partial g_{44}}{\partial x^{\mu}}-\frac{\partial g_{4\mu}}{\partial x^{4}}\right) \quad (6.11.18)$$

Only the first term is non-zero, substitute:

$$\Gamma^{4}{}_{4\mu}=-\frac{1}{2}\cdot C\cdot A^{\alpha}\cdot\left(C\cdot\phi^{2}\cdot\frac{\partial A_{\alpha}}{\partial x^{\mu}}-C\cdot\phi^{2}\cdot\frac{\partial A_{\mu}}{\partial x^{\alpha}}\right)$$

Express the electromagnetic tensor:

(5) $$\Gamma^{4}{}_{4\mu}=\Gamma^{4}{}_{\mu 4}=-\frac{1}{2}\cdot C^{2}\cdot\phi^{2}\cdot A^{\alpha}\cdot F_{\mu\alpha} \quad (6.11.19)$$

Sixthly the connection component with all indices 4[th] is zero because of the cylinder condition:

$$\Gamma^{4}{}_{44}=\frac{1}{2}\cdot g^{4A}\cdot\left(\frac{\partial g_{4A}}{\partial x^{4}}+\frac{\partial g_{A4}}{\partial x^{4}}-\frac{\partial g_{44}}{\partial x^{A}}\right)=-\frac{1}{2}\cdot g^{4A}\cdot\frac{\partial g_{44}}{\partial x^{A}}$$

(6) $$\Gamma^{4}{}_{44}=-\frac{1}{2}\cdot g^{4\alpha}\cdot\frac{\partial g_{44}}{\partial x^{\alpha}}-\frac{1}{2}\cdot g^{44}\cdot\frac{\partial g_{44}}{\partial x^{4}}=0 \quad (6.11.20)$$

Now we can write down the five dimensional geodesic equation. When we write out the terms in detail, the last one of them is zero:

$$\frac{\partial^2 x^P}{\partial \lambda^2}+\Gamma^P{}_{AB}\cdot\frac{\partial x^A}{\partial \lambda}\cdot\frac{\partial x^B}{\partial \lambda}=0$$

$$\frac{\partial^2 x^P}{\partial \lambda^2}+\Gamma^P{}_{\alpha\beta}\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}+2\cdot\Gamma^P{}_{4\alpha}\cdot\frac{\partial x^4}{\partial \lambda}\cdot\frac{\partial x^\alpha}{\partial \lambda}+\Gamma^P{}_{44}\cdot\frac{\partial x^4}{\partial \lambda}\cdot\frac{\partial x^4}{\partial \lambda}=0 \tag{6.11.21}$$

Its projection in the four dimensional spacetime:

$$\frac{\partial^2 x^\eta}{\partial \lambda^2}+{}^5\Gamma^\eta{}_{\alpha\beta}\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}+2\cdot\Gamma^\eta{}_{4\alpha}\cdot\frac{\partial x^4}{\partial \lambda}\cdot\frac{\partial x^\alpha}{\partial \lambda}=0 \tag{6.11.22}$$

Substitute and rearrange the formula:

$$\frac{\partial^2 x^\eta}{\partial \lambda^2}+\left({}^4\Gamma^\eta{}_{\alpha\beta}+\frac{1}{2}\cdot C^2\cdot\phi^2\cdot(A_\beta\cdot F^\eta_\alpha+A_\alpha\cdot F^\eta_\beta)\right)\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}+2\cdot\left(\frac{1}{2}\cdot C\cdot\phi^2\cdot F^\eta_\alpha\right)\cdot\frac{\partial x^4}{\partial \lambda}\cdot\frac{\partial x^\alpha}{\partial \lambda}=0$$

$$\frac{\partial^2 x^\eta}{\partial \lambda^2}+{}^4\Gamma^\eta{}_{\alpha\beta}\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}+C^2\cdot\phi^2\cdot A_\beta\cdot F^\eta_\alpha\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}+C\cdot\phi^2\cdot F^\eta_\alpha\cdot\frac{\partial x^4}{\partial \lambda}\cdot\frac{\partial x^\alpha}{\partial \lambda}=0$$

The equation of motion of a charged particle influenced by the gravitational and the electromagnetic interactions:

$$\frac{\partial^2 x^\eta}{\partial \lambda^2}+{}^4\Gamma^\eta{}_{\alpha\beta}\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}=-C\cdot\phi^2\cdot\left(C\cdot A_\beta\cdot\frac{\partial x^\beta}{\partial \lambda}+\frac{\partial x^4}{\partial \lambda}\right)\cdot F^\eta_\alpha\cdot\frac{\partial x^\alpha}{\partial \lambda} \tag{6.11.23}$$

The previously derived equation of movement of the charged particle under the influence of an electromagnetic field:

$$\frac{\partial^2 x^\eta}{\partial \lambda^2}+{}^4\Gamma^\eta{}_{\alpha\beta}\cdot\frac{\partial x^\alpha}{\partial \lambda}\cdot\frac{\partial x^\beta}{\partial \lambda}=\frac{Q}{c\cdot m}\cdot F^\eta_\alpha\cdot\frac{\partial x^\alpha}{\partial \lambda} \tag{6.11.24}$$

Compare it to our derived equation:

$$\frac{Q}{c\cdot m}\cdot F^\eta_\alpha\cdot\frac{\partial x^\alpha}{\partial \lambda}=-C\cdot\phi^2\cdot\left(C\cdot A_\beta\cdot\frac{\partial x^\beta}{\partial \lambda}+\frac{\partial x^4}{\partial \lambda}\right)\cdot F^\eta_\alpha\cdot\frac{\partial x^\alpha}{\partial \lambda}$$

$$\frac{Q}{c\cdot m}=-C\cdot\phi^2\cdot\left(C\cdot A_\beta\cdot\frac{\partial x^\beta}{\partial \lambda}+\frac{\partial x^4}{\partial \lambda}\right) \tag{6.11.25}$$

If the charge of the particle is zero, only the gravitation of the electromagnetic field will affect it:

$$-C\cdot\phi^2\cdot\left(C\cdot A_\beta\cdot\frac{\partial x^\beta}{\partial \lambda}+\frac{\partial x^4}{\partial \lambda}\right)=0$$

$$\frac{\partial x^4}{\partial \lambda} = -C \cdot A_\beta \cdot \frac{\partial x^\beta}{\partial \lambda} \qquad (6.11.26)$$

The entire electromagnetic field is the sum of the external field, and the field of the charged particle:

$$-C \cdot A_\beta \cdot \frac{\partial x^\beta}{\partial \lambda} = \frac{\partial x^4}{\partial \lambda} - \frac{Q}{c \cdot C \cdot \phi^2 \cdot m} \qquad (6.11.27)$$

6.12 Klein-Gordon equation

The spacetime of general relativity provides a stage for theories describing all the other interactions. Therefore it is worth to summarize the equations, that describe the behaviour of particles of matter on this background.

In 1925, this is how Erwin Schrödinger originally wrote down his famous equation of quantum mechanics, taking special relativity into account, that was already 20 years old at the time. However he did not manage to interpret the fine structure of the hydrogen spectrum with it, therefore he chose the non-relativistic, well known form. Later in 1927 Oscar Klein and Walter Gordon recommended the same formula for the relativistic equation of the electron, however it failed here too, because of the electron spin. It can however accurately describe spin-free particles, like the π-meson.

The equation of the conservation of energy:

$$-E^2 + p^2 \cdot c^2 + m^2 \cdot c^4 = 0 \qquad (6.12.1)$$

The method of first quantization is the following: substitute the quantum mechanical operators, and we consider every term as an operator on a complex function. The quantum mechanical operators of the energy and momentum:

$$\hat{E} = i \cdot \hbar \cdot \frac{\partial}{\partial t} \qquad \hat{p}_i = -i \cdot \hbar \cdot \frac{\partial}{\partial x^i}$$

$$-\hat{E}^2 \psi + c^2 \cdot \hat{p}^2 \psi + m^2 \cdot c^4 \cdot \psi = 0 \qquad (6.12.2)$$

$$-\left(i \cdot \hbar \cdot \frac{\partial}{\partial t}\right)^2 \psi + c^2 \cdot \left(-i \cdot \hbar \cdot \frac{\partial}{\partial x^i}\right)^2 \psi + m^2 \cdot c^4 \cdot \psi = 0$$

We amass the operators:

$$\hbar^2 \cdot \left(\frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial (x^i)^2}\right) \psi + m^2 \cdot c^2 \cdot \psi = 0 \qquad (6.12.3)$$

$$\hbar^2 \cdot \eta^{\alpha\beta} \cdot \frac{\partial^2 \psi}{\partial x^\alpha \cdot \partial x^\beta} + m^2 \cdot c^2 \cdot \psi = 0$$

We obtain the general relationship by substituting the geometric quantities of the general spacetime:

$$g^{\alpha\beta}\cdot\nabla^2_{\alpha\beta}\psi+\frac{m^2\cdot c^2}{\hbar^2}\cdot\psi=0 \tag{6.12.4}$$

The solution of the equation for a free particle:

$$\psi(x^\eta)=e^{i\cdot k_\alpha\cdot x^\alpha}=e^{i\cdot k_a\cdot x^a-\omega\cdot t} \tag{6.12.5}$$

The eigenvalue equation of the energy operator:

$$\hat{E}\,\psi=E\cdot\psi$$

$$i\cdot\hbar\cdot\frac{\partial\psi}{\partial t}=i\cdot\hbar\cdot\frac{\partial}{\partial t}e^{i\cdot k_a\cdot x^a-\omega\cdot t}=i\cdot\hbar\cdot(-i\cdot\omega)\cdot e^{i\cdot k_a\cdot x^a-\omega\cdot t}=\hbar\cdot\omega\cdot e^{i\cdot k_a\cdot x^a-\omega\cdot t}$$

$$E=\hbar\cdot\omega \tag{6.12.6}$$

The eigenvalue equation of the momentum operator:

$$\hat{p}_i\psi=p_i\cdot\psi$$

$$i\cdot\hbar\cdot\frac{\partial\psi}{\partial x^i}=i\cdot\hbar\cdot\frac{\partial}{\partial x^i}e^{i\cdot k_a\cdot x^a-\omega\cdot t}=i\cdot\hbar\cdot(-i\cdot k_i)\cdot e^{i\cdot k_a\cdot x^a-\omega\cdot t}=\hbar\cdot k_i\cdot e^{i\cdot k_a\cdot x^a-\omega\cdot t}$$

$$p_i=\hbar\cdot k_i \tag{6.12.7}$$

Reinsert the eigenvalues in the Klein-Gordon equation into the place of the operators:

$$-\hat{E}^2\psi+c^2\cdot\hat{p}^2\psi+m^2\cdot c^4\cdot\psi=0$$

$$-(\hbar\cdot\omega)^2\psi+c^2\cdot(\hbar\cdot k_i)^2\psi+m^2\cdot c^4\cdot\psi=0 \tag{6.12.8}$$

The energy eigenvalues can be negative as well, we interpret them as antiparticles with positive energy:

$$\hbar\cdot\omega=\pm c\cdot\sqrt{(\hbar\cdot k_i)^2+m^2\cdot c^2} \tag{6.12.9}$$

6.13 Proca equation

It is similar to the Klein-Gordon equation, but instead of a scalar, it applies to a four-component wave function, it describes particles with spin 1, like the photon and the mediators of the weak interaction, the W^+, W^- and Z bosons:

$$g^{\alpha\beta}\cdot\nabla^2_{\alpha\beta}\psi^\eta+\frac{m^2\cdot c^2}{\hbar^2}\cdot\psi^\eta=0 \tag{6.13.1}$$

This is usually supplemented with a continuity condition:

$$\frac{\partial\psi^\alpha}{\partial x^\alpha}=0 \tag{6.13.2}$$

The Maxwell equations represent the limiting case of zero mass, this is the previously derived eikonal, or phase equation:

$$g^{\alpha\beta}\cdot\nabla^2_{\alpha\beta}\psi^\eta=0 \tag{6.13.3}$$

6.14 Dirac equation

Paul Adrien Maurice Dirac derived the equation named after him in 1928, that correctly describes the relativistic particles with half spin, like the electron and the quarks. Unlike the Klein-Gordon equation, it is a first order differential equation. We start with the Klein-Gordon equation:

$$\hbar^2\cdot\eta^{\alpha\beta}\cdot\frac{\partial^2\psi}{\partial x^\alpha\cdot\partial x^\beta}+m^2\cdot c^2\cdot\psi=0 \tag{6.14.1}$$

It is actually a second order eigenvalue equation:

$$-\hbar^2\cdot\eta^{\alpha\beta}\cdot\frac{\partial^2\psi}{\partial x^\alpha\cdot\partial x^\beta}=m^2\cdot c^2\cdot\psi \tag{6.14.2}$$

This equation can be made first order, if we introduce factors with convenient algebraic properties. Take the square root of both sides:

$$i\cdot\hbar\cdot\gamma^\alpha\cdot\frac{\partial\psi}{\partial x^\alpha}=\pm m\cdot c\cdot\psi \tag{6.14.3}$$

This equation is valid only if the following condition is satisfied by γ, unknown for the moment:

$$\gamma^\eta\cdot\gamma^\kappa+\gamma^\kappa\cdot\gamma^\eta=2\cdot\eta^{\eta\kappa} \tag{6.14.4}$$

The condition is fulfilled, if the γ are at least four times four, specially chosen matrices:

$$\hat{\gamma}=\gamma_{\eta\kappa}=\begin{pmatrix}\gamma_{00} & \gamma_{01} & \gamma_{02} & \gamma_{03}\\ \gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13}\\ \gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23}\\ \gamma_{30} & \gamma_{31} & \gamma_{32} & \gamma_{33}\end{pmatrix} \qquad \hat{1}=e_{\eta\kappa}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\hat{\gamma}^{\eta}\cdot\hat{\gamma}^{\kappa}+\hat{\gamma}^{\kappa}\cdot\hat{\gamma}^{\eta}=2\cdot\eta^{\eta\kappa}\cdot\hat{1} \tag{6.14.5}$$

This wave function will also have four components. Substitute into the eigenvalue equation:

$$i\cdot\hbar\cdot(\gamma_{\eta\beta})^{\alpha}\cdot\frac{\partial\psi^{\beta}}{\partial x^{\alpha}}=\pm m\cdot c\cdot e_{\eta\beta}\cdot\psi^{\beta} \tag{6.14.6}$$

Rearrange it and write down the Dirac equation:

$$i\cdot\hbar\cdot(\gamma_{\eta\beta})^{\alpha}\cdot\frac{\partial\psi^{\beta}}{\partial x^{\alpha}}\pm m\cdot c\cdot e_{\eta\beta}\cdot\psi^{\beta}=0 \tag{6.14.7}$$

Since there are no criteria on the components of the γ-matrices, they can be written in several possible forms. To simplify the notation, we introduce the Pauli matrices:

$$\hat{\sigma}^{0}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \hat{\sigma}^{1}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\hat{\sigma}^{2}=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \hat{\sigma}^{3}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{6.14.8}$$

The Dirac representation of the γ-matrices:

$$\hat{\gamma}^{0}=\begin{pmatrix}\hat{1} & \hat{0}\\ \hat{0} & -\hat{1}\end{pmatrix} \qquad \hat{\gamma}^{i}=\begin{pmatrix}\hat{0} & \hat{\sigma}^{i}\\ -\hat{\sigma}^{i} & \hat{0}\end{pmatrix} \tag{6.14.9}$$

The Weyl representation of the γ-matrices:

$$\hat{\gamma}^{0}=\begin{pmatrix}\hat{0} & \hat{1}\\ \hat{1} & \hat{0}\end{pmatrix} \qquad \hat{\gamma}^{i}=\begin{pmatrix}\hat{0} & \hat{\sigma}^{i}\\ -\hat{\sigma}^{i} & \hat{0}\end{pmatrix} \tag{6.14.10}$$

The Majorana representation of the γ-matrices:

$$\hat{\gamma}^{0}=\begin{pmatrix}\hat{0} & \hat{\sigma}^{2}\\ \hat{\sigma}^{2} & \hat{0}\end{pmatrix} \qquad \hat{\gamma}^{1}=i\cdot\begin{pmatrix}\hat{\sigma}^{3} & \hat{0}\\ \hat{0} & \hat{\sigma}^{3}\end{pmatrix}$$

$$\hat{\gamma}^{2}=\begin{pmatrix}\hat{0} & -\hat{\sigma}^{2}\\ \hat{\sigma}^{2} & \hat{0}\end{pmatrix} \qquad \hat{\gamma}^{3}=-i\cdot\begin{pmatrix}\hat{\sigma}^{1} & \hat{0}\\ \hat{0} & \hat{\sigma}^{1}\end{pmatrix} \tag{6.14.11}$$

Total momentum of a charged particle in an electromagnetic field, where q is the charge of the particle, A_η is the four-potential of the external field:

$$p_\eta + \frac{q}{c} \cdot A_\eta$$

Dirac equation of a charged particle:

$$i \cdot \hbar \cdot (\gamma_{\eta\beta})^\alpha \cdot \frac{\partial \psi^\beta}{\partial x^\alpha} + (\gamma_{\eta\beta})^\alpha \cdot \frac{q}{c} \cdot A_\alpha \cdot \psi^\beta \pm m \cdot c \cdot e_{\eta\beta} \cdot \psi^\beta = 0 \qquad (6.14.7)$$

6.15 Weyl equation

The Weyl equation describes massless particles with half spin, like the neutrinos:

$$(\sigma_{ib})^\alpha \cdot \frac{\partial \psi^b}{\partial x^\alpha} = 0 \qquad (6.15.1)$$

The wave function has two components because of the Pauli matrix.

7. Gravitational waves

Changes in matter that are not spherically symmetric expansions or contractions cause propagating disturbances in spacetime. These waves become independent from their source and propagate at the speed of light. Far away from the originating celestial body they are probably very weak, therefore they can be approximated with small linear deviations from the flat background metric.

Albert Einstein derived the wave solution of the equations named after him for the first time in 1918. Several questions arose regarding the results. It was unclear for a long time, if those waves were coordinate effects, or real physical phenomena. The British astronomer Eddington had a major role in dispelling doubts. He confirmed light bending with observations during the famous solar eclipse of 1919. Later he concluded, that the transversal wave is a real phenomenon propagating at the speed of light, while – in his words – the "longitudinal gravitational waves propagate at the speed of thought". In 1938 Einstein and Rosen sent an article attempting to disprove the existence of gravitational waves to the Physical Review, but publication was not allowed by the anonymous peer review. Einstein took such a big offence, that he never published in the journal ever again, despite that it turned out later, that the proofreader was right.

Although waves in spacetime have not been directly detected by gravitational wave detectors, there is indirect evidence for the correctness of the equations. The 1993 Nobel Prize in Physics was awarded to Russel Alan Hulse and Joseph Hooton Taylor Jr. for measurements of the binary system containing the PSR B1913+16 pulsar in the constellation Aquila. Their observations confirmed several consequences of the theory of relativity, including the amount of energy carried away by gravitational waves, by measuring the decrease of the orbital period of the binary system.

7.1 Splitting the metric tensor

The first approach is to split the metric tensor to the background metric and the metric of the gravitational waves:

$$g_{\eta\kappa}=\eta_{\eta\kappa}+h_{\eta\kappa} \qquad |h_{\eta\kappa}|\ll 1$$

$$g^{\eta\kappa}=\eta^{\eta\kappa}-h^{\eta\kappa} \tag{7.1.1}$$

Where the metric tensor of flat spacetime is:

$$\eta_{\eta\kappa}=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \tag{7.1.2}$$

Let us give them a try:

$$g_{\eta\alpha}\cdot g^{\alpha\kappa}=(\eta_{\eta\alpha}+h_{\eta\alpha})\cdot(\eta^{\alpha\kappa}-h^{\alpha\kappa})$$

$$g_{\eta\alpha}\cdot g^{\alpha\kappa}=\eta_{\eta\alpha}\cdot\eta^{\alpha\kappa}-\eta_{\eta\alpha}\cdot h^{\alpha\kappa}+h_{\eta\alpha}\cdot\eta^{\alpha\kappa}-h_{\eta\alpha}\cdot h^{\alpha\kappa}$$

$$\delta_\eta^\kappa=\delta_\eta^\kappa-h_\eta^\kappa+h_\eta^\kappa-h_{\eta\alpha}\cdot h^{\alpha\kappa} \tag{7.1.3}$$

Because we chose h to be very small, its square is even smaller and negligible. The partial derivatives of the metric tensor:

$$\frac{\partial g_{\eta\kappa}}{\partial x^\mu}=\frac{\partial\eta_{\eta\kappa}}{\partial x^\mu}+\frac{h_{\eta\kappa}}{\partial x^\mu}=\frac{h_{\eta\kappa}}{\partial x^\mu}$$

$$\frac{\partial g^{\eta\kappa}}{\partial x^\mu}=\frac{\partial\eta^{\eta\kappa}}{\partial x^\mu}-\frac{h^{\eta\kappa}}{\partial x^\mu}=-\frac{h^{\eta\kappa}}{\partial x^\mu} \tag{7.1.4}$$

Therefore the connection:

$$\Gamma^\kappa{}_{\eta\mu}=\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\left(\frac{\partial h_{\mu\alpha}}{\partial x^\eta}+\frac{\partial h_{\alpha\eta}}{\partial x^\mu}-\frac{\partial h_{\eta\mu}}{\partial x^\alpha}\right) \tag{7.1.5}$$

The derivatives of the connection:

$$\frac{\partial\Gamma^\kappa{}_{\eta\mu}}{\partial x^\nu}=\frac{1}{2}\cdot\frac{\partial}{\partial x^\nu}\left(g^{\kappa\alpha}\cdot\left(\frac{\partial h_{\mu\alpha}}{\partial x^\eta}+\frac{\partial h_{\alpha\eta}}{\partial x^\mu}-\frac{\partial h_{\eta\mu}}{\partial x^\alpha}\right)\right)$$

$$\frac{\partial\Gamma^\kappa{}_{\eta\mu}}{\partial x^\nu}=\frac{1}{2}\cdot\frac{\partial h^{\kappa\alpha}}{\partial x^\nu}\cdot\left(\frac{\partial h_{\mu\alpha}}{\partial x^\eta}+\frac{\partial h_{\alpha\eta}}{\partial x^\mu}-\frac{\partial h_{\eta\mu}}{\partial x^\alpha}\right)+\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\left(\frac{\partial^2 h_{\mu\alpha}}{\partial x^\nu\partial x^\eta}+\frac{\partial^2 h_{\alpha\eta}}{\partial x^\nu\partial x^\mu}-\frac{\partial^2 h_{\eta\mu}}{\partial x^\nu\partial x^\alpha}\right) \tag{7.1.6}$$

We assume in our approximation, that we can easily separate the changes in spacetime to fast and slowly changing terms. Calculate the curvature tensor, neglect the slowly changing terms, keep only the second derivatives:

$$R^\kappa{}_{\eta\mu\nu}=\frac{\partial\Gamma^\kappa{}_{\eta\mu}}{\partial x^\nu}-\frac{\partial\Gamma^\kappa{}_{\nu\mu}}{\partial x^\eta}$$

$$R^\kappa{}_{\eta\mu\nu}=\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\left(\frac{\partial^2 h_{\alpha\eta}}{\partial x^\nu\partial x^\mu}-\frac{\partial^2 h_{\eta\mu}}{\partial x^\nu\partial x^\alpha}-\frac{\partial^2 h_{\alpha\nu}}{\partial x^\eta\partial x^\mu}+\frac{\partial^2 h_{\nu\mu}}{\partial x^\eta\partial x^\alpha}\right) \tag{7.1.7}$$

Select a case and calculate the completely covariant curvature tensor:

$$R^\kappa{}_{tt\nu}=\frac{\partial\Gamma^\kappa{}_{tt}}{\partial x^\nu}-\frac{\partial\Gamma^\kappa{}_{\nu t}}{\partial x^t}$$

$$R^{\kappa}{}_{tt\nu}=\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\left(\frac{\partial^2 h_{\alpha t}}{\partial x^{\nu}\partial x^{t}}-\frac{\partial^2 h_{tt}}{\partial x^{\nu}\partial x^{\alpha}}-\frac{\partial^2 h_{\alpha\nu}}{\partial x^{t}\partial x^{t}}+\frac{\partial^2 h_{\nu t}}{\partial x^{t}\partial x^{\alpha}}\right) \tag{7.1.8}$$

$$g_{\kappa\beta}\cdot R^{\beta}{}_{tt\nu}=\frac{1}{2}\cdot g_{\kappa\beta}\cdot g^{\beta\alpha}\cdot\left(\frac{\partial^2 h_{\alpha t}}{\partial x^{\nu}\partial x^{t}}-\frac{\partial^2 h_{tt}}{\partial x^{\nu}\partial x^{\alpha}}-\frac{\partial^2 h_{\alpha\nu}}{\partial x^{t}\partial x^{t}}+\frac{\partial^2 h_{\nu t}}{\partial x^{t}\partial x^{\alpha}}\right)$$

$$R_{\kappa tt\nu}=\frac{1}{2}\cdot\left(\frac{\partial^2 h_{\kappa t}}{\partial x^{\nu}\partial x^{t}}-\frac{\partial^2 h_{tt}}{\partial x^{\nu}\partial x^{\kappa}}-\frac{\partial^2 h_{\kappa\nu}}{\partial x^{t}\partial x^{t}}+\frac{\partial^2 h_{\nu t}}{\partial x^{t}\partial x^{\kappa}}\right) \tag{7.1.9}$$

Choose a part of the metric tensor of the gravitational radiation, that's change does not depend on any spatial coordinates, this is the transversal term:

$$R_{\kappa tt\nu}=-\frac{1}{2}\cdot\frac{\partial^2 h_{\kappa\nu}}{\partial t^2} \tag{7.1.10}$$

Substitute it into the formula for geodesic deviation:

$$\frac{\partial^2 x^{\kappa}}{\partial t^2}+R^{\kappa}{}_{\alpha\beta\gamma}\cdot dx^{\alpha}\cdot\frac{\partial x^{\beta}}{\partial t}\cdot\frac{\partial x^{\gamma}}{\partial t}=0$$

$$g_{\kappa\alpha}\cdot\frac{\partial^2 x^{\alpha}}{\partial t^2}=-g_{\kappa\delta}\cdot R^{\delta}{}_{tt\gamma}\cdot dt\cdot\frac{\partial t}{\partial t}\cdot\frac{\partial x^{\gamma}}{\partial t}$$

$$\frac{\partial^2 x_{\kappa}}{\partial t^2}=-R_{\kappa tt\gamma}\cdot dx^{\gamma}$$

$$\frac{\partial^2 x_{\kappa}}{\partial t^2}=\frac{1}{2}\cdot\frac{\partial^2 h_{\kappa\gamma}}{\partial t^2}\cdot dx^{\gamma} \tag{7.1.11}$$

The δx_{κ} oscillations caused by weak waves will be small, therefore we can consider the dx^{γ} distance from the centre constant. After integration the discrepancies caused by the gravitational waves:

$$\delta x_{\kappa}=\frac{1}{2}\cdot h_{\kappa\gamma}\cdot dx^{\gamma} \tag{7.1.12}$$

7.2 Examining the metric

Determine the spacetime of the gravitational waves. The linear connection:

$$\Gamma^{\kappa}{}_{\eta\mu}=\frac{1}{2}\cdot g^{\kappa\alpha}\cdot\left(\frac{\partial h_{\mu\alpha}}{\partial x^{\eta}}+\frac{\partial h_{\alpha\eta}}{\partial x^{\mu}}-\frac{\partial h_{\eta\mu}}{\partial x^{\alpha}}\right)$$

$$\Gamma^{\kappa}{}_{\eta\mu}=\frac{1}{2}\cdot\left(\frac{\partial h^{\kappa}_{\mu}}{\partial x^{\eta}}+\frac{\partial h^{\kappa}_{\eta}}{\partial x^{\mu}}-\frac{\partial h_{\eta\mu}}{\partial x_{\kappa}}\right) \tag{7.2.1}$$

Write down the Ricci tensor in a linear approximation, substitute the connection:

$$R_{\eta\mu}=R^{\alpha}{}_{\eta\mu\alpha}=\frac{\partial\Gamma^{\alpha}{}_{\eta\mu}}{\partial x^{\alpha}}-\frac{\partial\Gamma^{\alpha}{}_{\alpha\mu}}{\partial x^{\eta}}$$

$$R_{\eta\mu}=\frac{1}{2}\cdot\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial h^{\alpha}_{\mu}}{\partial x^{\eta}}+\frac{\partial h^{\alpha}_{\eta}}{\partial x^{\mu}}-\frac{\partial h_{\eta\mu}}{\partial x_{\alpha}}\right)-\frac{1}{2}\cdot\frac{\partial}{\partial x^{\eta}}\left(\frac{\partial h^{\alpha}_{\mu}}{\partial x^{\alpha}}+\frac{\partial h^{\alpha}_{\alpha}}{\partial x^{\mu}}-\frac{\partial h_{\alpha\mu}}{\partial x_{\alpha}}\right)$$

$$R_{\eta\mu}=\frac{1}{2}\cdot\left(\frac{\partial^2 h^{\alpha}_{\eta}}{\partial x^{\alpha}\cdot\partial x^{\mu}}-\frac{\partial^2 h_{\eta\mu}}{\partial x^{\alpha}\cdot\partial x_{\alpha}}-\frac{\partial^2 h^{\alpha}_{\alpha}}{\partial x^{\eta}\cdot\partial x^{\mu}}+\frac{\partial^2 h_{\alpha\mu}}{\partial x^{\eta}\cdot\partial x_{\alpha}}\right) \tag{7.2.2}$$

Introduce a new notation, the definition of the overline in the case of tensors with two indices:

$$\overline{M_{\eta\kappa}}=M_{\eta\kappa}-\frac{1}{2}\cdot M^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa} \tag{7.2.3}$$

Double overline recovers the original tensor:

$$\overline{\overline{M_{\eta\kappa}}}=\overline{M_{\eta\kappa}-\frac{1}{2}\cdot M^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa}}=\left(M_{\eta\kappa}-\frac{1}{2}\cdot M^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa}\right)-\frac{1}{2}\cdot\left(M^{\alpha}_{\alpha}-\frac{1}{2}\cdot M^{\beta}_{\beta}\cdot\eta^{\alpha}_{\alpha}\right)\cdot\eta_{\eta\kappa}$$

$$M_{\eta\kappa}-\frac{1}{2}\cdot M^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa}-\frac{1}{2}\cdot M^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa}+\frac{1}{4}\cdot M^{\beta}_{\beta}\cdot\eta^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa}=M_{\eta\kappa}-M^{\alpha}_{\alpha}\cdot\eta_{\eta\kappa}+M^{\beta}_{\beta}\cdot\eta_{\eta\kappa}$$

$$\overline{\overline{M_{\eta\kappa}}}=M_{\eta\kappa} \tag{7.2.4}$$

We used that $\eta^{\alpha}_{\alpha}=4$, therefore this relationship is valid only in four dimensions. Substitute the Ricci tensor into the Einstein equation, and obtain its linearised variant:

$$R_{\eta\kappa}-\frac{1}{2}\cdot R\cdot\eta_{\eta\kappa}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\eta\kappa}$$

$$\overline{R_{\eta\kappa}}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\eta\kappa}$$

$$\frac{1}{2}\cdot\left(\frac{\partial^2\overline{h^{\alpha}_{\eta}}}{\partial x^{\alpha}\cdot\partial x^{\mu}}-\frac{\partial^2\overline{h_{\eta\mu}}}{\partial x^{\alpha}\cdot\partial x_{\alpha}}-\frac{\partial^2\overline{h^{\alpha}_{\alpha}}}{\partial x^{\eta}\cdot\partial x^{\mu}}+\frac{\partial^2\overline{h_{\alpha\mu}}}{\partial x^{\eta}\cdot\partial x_{\alpha}}\right)=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\eta\kappa} \tag{7.2.5}$$

With a suitable choice of coordinate system, this equation can be simplified further. Examine a function describing an arbitrary coordinate transformation, similar to *h* in magnitude:

$$_2x^\eta = x^\eta + \xi^\eta(x^\eta)$$

$$x^\eta = {_2x^\eta} - \xi^\eta({_2x^\eta}) \tag{7.2.6}$$

The transformation of the metric tensor:

$$_2g_{\eta\kappa}({_2x^\eta}) = g_{\alpha\beta}(x^\eta)\cdot \Lambda_\eta{}^\alpha \cdot \Lambda_\kappa{}^\beta = g_{\alpha\beta}(x^\eta)\cdot \frac{\partial x^\alpha}{_2\partial x^\eta}\cdot\frac{\partial x^\beta}{_2\partial x^\kappa} \tag{7.2.7}$$

The metric tensor of the first coordinate system:

$$g_{\eta\kappa}(x^\eta) = g_{\eta\kappa}({_2x^\eta} - \xi^\eta) = g_{\eta\kappa}({_2x^\eta}) - \xi^\alpha\cdot\frac{\partial g_{\eta\kappa}({_2x^\eta})}{_2\partial x^\alpha} \tag{7.2.8}$$

The second metric tensor can be written down with the transformation law and the first metric tensor:

$$_2g_{\eta\kappa}({_2x^\eta}) = \left(\delta_\eta^\alpha - \frac{\partial \xi^\alpha}{_2\partial x^\eta}\right)\cdot\left(\delta_\kappa^\beta - \frac{\partial \xi^\beta}{_2\partial x^\kappa}\right)\cdot\left(g_{\alpha\beta}({_2x^\eta}) - \xi^\gamma\cdot\frac{\partial g_{\alpha\beta}({_2x^\eta})}{_2\partial x^\gamma}\right) \tag{7.2.9}$$

The same with linear accuracy:

$$_2g_{\eta\kappa}({_2x^\eta}) = g_{\eta\kappa}({_2x^\eta}) - \frac{\partial \xi^\alpha}{_2\partial x^\eta}\cdot g_{\alpha\kappa}({_2x^\eta}) - \frac{\partial \xi^\beta}{_2\partial x^\kappa}\cdot g_{\eta\beta}({_2x^\eta}) - \xi^\gamma\cdot\frac{\partial g_{\eta\kappa}({_2x^\eta})}{_2\partial x^\gamma}$$

Omit the coordinate system indices, and note with double crosses the transformed quantities we seek:

$$_\#g_{\eta\kappa}(x^\eta) = g_{\eta\kappa}(x^\eta) - \frac{\partial \xi^\alpha}{\partial x^\eta}\cdot g_{\alpha\kappa}(x^\eta) - \frac{\partial \xi^\beta}{\partial x^\kappa}\cdot g_{\eta\beta}(x^\eta) - \xi^\gamma\cdot\frac{\partial g_{\eta\kappa}(x^\eta)}{\partial x^\gamma} \tag{7.2.10}$$

$$_\#g_{\eta\kappa} = \eta_{\eta\kappa} + {_\#h_{\eta\kappa}}$$

With linear precision h can be determined in the following way:

$$_\#h_{\eta\kappa} = h_{\eta\kappa} - \frac{\partial \xi_\kappa}{\partial x^\eta} - \frac{\partial \xi_\eta}{\partial x^\kappa}$$

$$_\#h_\alpha^\alpha = h_\alpha^\alpha - 2\cdot\frac{\partial \xi^\alpha}{\partial x^\alpha} \tag{7.2.11}$$

Apply the rule of overline:

$$\overline{{}_{\#}h_{\eta\kappa}} = {}_{\#}h_{\eta\kappa} - \frac{1}{2} \cdot {}_{\#}h^{\alpha}_{\alpha} \cdot \eta_{\eta\kappa}$$

$$\overline{{}_{\#}h_{\eta\kappa}} = h_{\eta\kappa} - \frac{\partial \xi_{\kappa}}{\partial x^{\eta}} - \frac{\partial \xi_{\eta}}{\partial x^{\kappa}} - \frac{1}{2} \cdot \left(h^{\alpha}_{\alpha} - 2 \cdot \frac{\partial \xi^{\alpha}}{\partial x^{\alpha}} \right) \cdot \eta_{\eta\kappa}$$

$$\overline{{}_{\#}h_{\eta\kappa}} = h_{\eta\kappa} - \frac{\partial \xi_{\kappa}}{\partial x^{\eta}} - \frac{\partial \xi_{\eta}}{\partial x^{\kappa}} + \frac{\partial \xi^{\alpha}}{\partial x^{\alpha}} \cdot \eta_{\eta\kappa} \qquad (7.2.12)$$

Partial derivatives with respect to covariant coordinates:

$$\frac{\partial h_{\beta\kappa}}{\partial x_{\beta}} - \frac{\partial^2 \xi_{\kappa}}{\partial x_{\beta} \cdot \partial x^{\beta}} - \frac{\partial \xi_{\beta}}{\partial x_{\beta} \cdot \partial x^{\kappa}} + \frac{\partial^2 \xi^{\alpha}}{\partial x^{\kappa} \cdot \partial x^{\alpha}} = 0$$

$$\frac{\partial \overline{h_{\beta\kappa}}}{\partial x_{\beta}} - \frac{\partial^2 \xi_{\kappa}}{\partial x_{\beta} \cdot \partial x^{\beta}} = 0 \qquad (7.2.13)$$

This equation is satisfied for sure, if we assume that the first term is always zero, this is the harmonic condition of the coordinates:

$$\frac{\partial \overline{h_{\beta\kappa}}}{\partial x_{\beta}} = 0 \qquad (7.2.14)$$

In this case the linearised Einstein equation:

$$\frac{\partial^2 \overline{h_{\eta\kappa}}}{\partial x_{\alpha} \cdot \partial x^{\alpha}} = -\frac{16 \cdot \pi \cdot \gamma}{c^4} \cdot T_{\eta\kappa} \qquad (7.2.15)$$

7.3 Plane wave solutions

The linearised Einstein equation in empty space:

$$\frac{\partial^2 \overline{h_{\eta\kappa}}}{\partial x_{\alpha} \cdot \partial x^{\alpha}} = 0 \qquad (7.3.1)$$

That contains the coordinates always in the following combination, we can use it to simplify:

$$u = t - \frac{z}{c} \qquad (7.3.2)$$

We investigate the solutions of this. The equations of the harmonic condition are satisfied:

$$\frac{d}{du}(\overline{h_{t\kappa}}+\overline{h_{z\kappa}})=0$$

$$\overline{h_{t\kappa}}=\overline{h_{\kappa t}}=\overline{h_{z\kappa}}=\overline{h_{\kappa z}}=0 \quad (7.3.3)$$

Thus only transversal waves exist. The following general arc length squared satisfies this condition:

$$ds^2=c^2\cdot dt^2-(1-a)\cdot dx^2-(1+a)\cdot dy^2+2\cdot b\cdot dx\cdot dy-dz^2 \quad (7.3.4)$$

Where a and b are arbitrary functions, very small in magnitude, and are the components of h:

$$a\ll 1 \qquad\qquad b\ll 1$$

Metric tensor and traceless transverse h:

$$g_{\eta\kappa}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1+a & b & 0\\ 0 & b & -1-a & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \qquad h_{\eta\kappa}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & a & b & 0\\ 0 & b & -a & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \quad (7.3.5)$$

The two components of the gravitational wave moving in the z direction:

$$h_{+}=h_{xx}=-h_{yy} \qquad\qquad h_{\times}=h_{xy}=h_{yx} \quad (7.3.6)$$

Connection with linear precision:

$$-\Gamma^{t}{}_{xx}=\Gamma^{t}{}_{yy}=-\Gamma^{z}{}_{xx}=\Gamma^{z}{}_{yy}=-\Gamma^{x}{}_{tx}=\Gamma^{y}{}_{ty}=-\Gamma^{x}{}_{xz}=\Gamma^{y}{}_{yz}=\frac{1}{2\cdot c}\cdot\frac{\partial a}{\partial u}$$

$$-\Gamma^{t}{}_{xy}=-\Gamma^{z}{}_{xy}=-\Gamma^{x}{}_{ty}=-\Gamma^{y}{}_{tx}=-\Gamma^{x}{}_{tx}=\Gamma^{x}{}_{yz}=\Gamma^{y}{}_{xz}=\frac{1}{2\cdot c}\cdot\frac{\partial b}{\partial u} \quad (7.3.7)$$

The completely covariant curvature tensor:

$$-R_{txtx}=R_{tyty}=-R_{zxzx}=R_{zyzy}=R_{txzx}=-R_{tyzy}=\frac{1}{2\cdot c^2}\cdot\frac{\partial^2 a}{\partial u^2}$$

$$-R_{txty}=-R_{zxzy}=-R_{zxzx}=R_{txzy}=R_{tyzx}=\frac{1}{2\cdot c^2}\cdot\frac{\partial^b}{\partial u^2} \quad (7.3.8)$$

We can see already on this, that the gravitational waves propagate at the speed of light. The monochromatic planar wave special solutions:

$$a(u)=A\cdot\cos(k\cdot z-\omega\cdot t)$$

$$b(u)=B\cdot\cos(k\cdot z-\omega\cdot t-\varphi) \quad (7.3.9)$$

Where A, B, φ are constants and the wave number vector is:

$$k=\frac{\omega}{c} \qquad (7.3.10)$$

7.4 Second order approximation

Calculate the energy carried away by the gravitational radiation. We write down the gravitational equations in second order approximation. The Ricci tensor in second order:

$$R^{(2)}_{\eta\mu}=\frac{1}{2}\cdot h^{\alpha\beta}\cdot\left(\frac{\partial^2 h_{\alpha\beta}}{\partial x^\eta\cdot\partial x^\mu}-\frac{\partial^2 h_{\mu\beta}}{\partial x^\alpha\cdot\partial x^\eta}-\frac{\partial^2 h_{\eta\beta}}{\partial x^\alpha\cdot\partial x^\mu}+\frac{\partial^2 h_{\eta\mu}}{\partial x^\alpha\cdot\partial x^\beta}\right)+\frac{1}{4}\cdot\frac{\partial h^{\alpha\beta}}{\partial x^\eta}\cdot\frac{\partial h_{\alpha\beta}}{\partial x^\mu}$$
$$+\frac{1}{2}\cdot\frac{\partial h^\beta_\mu}{\partial x_\alpha}\cdot\left(\frac{\partial h_{\beta\eta}}{\partial x^\alpha}\cdot\frac{\partial h_{\alpha\eta}}{\partial x^\beta}\right)+\frac{1}{2}\cdot\frac{\partial h^{\alpha\beta}}{\partial x^\alpha}\cdot\left(\frac{\partial h_{\eta\mu}}{\partial x^\beta}-\frac{\partial h_{\mu\beta}}{\partial x^\eta}+\frac{\partial h_{\eta\beta}}{\partial x^\mu}\right)-\frac{1}{4}\cdot\frac{\partial h^\beta_\beta}{\partial x_\alpha}\cdot\left(\frac{\partial h_{\eta\mu}}{\partial x^\alpha}+\frac{\partial h_{\mu\alpha}}{\partial x^\eta}-\frac{\partial h_{\eta\alpha}}{\partial x^\mu}\right)$$
(7.4.1)

The metric tensor of the radiation is the sum of the first order and second order term:

$$h_{\eta\mu}=h^{(1)}_{\eta\mu}+h^{(2)}_{\eta\mu} \qquad (7.4.2)$$

The Einstein equation is constrained in vacuum by the following condition:

$$G^{(1)}_{\eta\kappa}(h^{(2)}_{\mu\nu})+G^{(2)}_{\eta\kappa}(h_{\mu\nu})=0$$

$$t_{\eta\kappa}=G^{(1)}_{\eta\kappa}(h^{(2)}_{\mu\nu})=-G^{(2)}_{\eta\kappa}(h_{\mu\nu}) \qquad (7.4.3)$$

Local energy-momentum tensor of gravitational waves:

$$t_{\eta\kappa}=\frac{c^4}{32\cdot\pi\cdot\gamma}\cdot\frac{\partial h^{\alpha\beta}}{\partial x^\eta}\cdot\frac{\partial h_{\alpha\beta}}{\partial x^\kappa} \qquad (7.4.4)$$

This approach to the energy is not valid in every coordinate systems, but this time it will do:

$$E=\int t_{00}\cdot d^3x \qquad (7.4.5)$$

Next we calculate the quadrupole formula. Solve the Einstein equation, the S hypersurface here is the past lightcone of the x point:

$$h_{\eta\kappa}(x)=\frac{\gamma}{2\cdot\pi}\cdot\int_S\frac{T_{\eta\kappa}(x')}{|x-x'|}\cdot dS(x') \qquad R=|x-x'| \qquad (7.4.6)$$

Volume element on the lightcone:

$$dS = r^2 \cdot dr \cdot d\,\Omega \tag{7.4.7}$$

We assume, that the velocity of the source is much less than the speed of light, and its size is smaller than the wavelength of the emitted gravitational radiation, this is the dipole approximation. At great distance from the source the denominator *R* barely changes, therefore it can be brought before the integral:

$$h_{\eta\kappa}(x) = \frac{4}{2\cdot\pi}\cdot\frac{\gamma}{R}\cdot\int_S T_{\eta\kappa}\cdot dV \tag{7.4.8}$$

We modify the remaining integral with the linearised conservation law:

$$\frac{\partial T^{\alpha}_{\eta}}{\partial x^{\alpha}} = 0$$

(1) $$\frac{\partial T_{\eta\kappa}}{\partial x^{\kappa}} - \frac{\partial T_{\eta 0}}{\partial x^{0}} = 0$$

(2) $$\frac{\partial T_{0\kappa}}{\partial x^{\kappa}} - \frac{\partial T_{00}}{\partial x^{0}} = 0 \tag{7.4.9}$$

The middle equation (1):

$$\frac{\partial T_{\eta\kappa}}{\partial x^{\kappa}} - \frac{\partial T_{\eta 0}}{\partial x^{0}} = 0 \quad / \cdot x^{\mu}$$

Integrate it to the hypersurface crossing the source and the future light-line infinity:

$$\frac{\partial}{\partial x^0}\int T_{\eta 0}\cdot x^{\mu}\cdot dV = \int \frac{\partial T_{\eta\kappa}}{\partial x^{\kappa}}\cdot x^{\mu}\cdot dV = \int \frac{\partial}{\partial x^{\kappa}}(T_{\eta\kappa}\cdot x^{\mu})\cdot dV - \int T^{\mu}_{\eta}\cdot dV$$

$$\int T_{\eta\kappa}\cdot dV = -\frac{1}{2}\cdot\frac{\partial}{\partial x^0}\int (T_{0\eta}\cdot x_{\mu} + T_{0\mu}\cdot x_{\eta})\cdot dV \tag{7.4.10}$$

The lower equation (2):

$$\frac{\partial T_{0\kappa}}{\partial x^{\kappa}} - \frac{\partial T_{00}}{\partial x^{0}} = 0 \quad / \cdot x_{\eta}\cdot x_{\kappa}$$

$$\frac{\partial}{\partial x^0}\int T_{00}\cdot x_{\eta}\cdot x_{\kappa}\cdot dV = -\int (T_{0\eta}\cdot x_{\kappa} + T_{0\kappa}\cdot x_{\eta})\cdot dV \tag{7.4.11}$$

Equating the two equations:

$$\int T_{\eta\kappa}\cdot dV = -\frac{1}{2}\cdot\frac{\partial}{\partial x^0}\int T_{00}\cdot x_{\eta}\cdot x_{\kappa}\cdot dV \tag{7.4.12}$$

Introduce the symbols for energy and time, and substitute it into the solution of the Einstein equation (7.4.6):

$$T_{00}=m\cdot c^2 \qquad t=\frac{x^0}{c}$$

$$h_{\eta\kappa}(\vec{x},t)=\frac{4\cdot\gamma}{c^4\cdot R}\cdot\frac{\partial^2}{\partial t^2}\int x_\eta\cdot x_\kappa\cdot dV \tag{7.4.13}$$

Introduce the three dimensional quadrupole moment tensor:

$$Q_{ij}=\int\rho\cdot x_i\cdot x_j\cdot d^3x \tag{7.4.14}$$

At great distance from the source it is a planar wave, with the following non-zero components:

cross polarized: $$h_{23}=\frac{2\cdot\gamma}{3\cdot c^4\cdot R}\cdot\ddot{Q}_{23}$$

plus polarized: $$h_{22}-h_{33}=-\frac{2\cdot\gamma}{3\cdot c^4\cdot R}\cdot(\ddot{Q}_{22}-\ddot{Q}_{33}) \tag{7.4.15}$$

Substitute it into *t* (the local energy-momentum tensor), and write down the energy current along the *x* axis:

$$c\cdot t^{10}=\frac{\gamma}{36\cdot\pi\cdot c^5\cdot R^2}\cdot\left(\frac{1}{4}\cdot(\ddot{Q}_{22}-\ddot{Q}_{33})^2+(\ddot{Q}_{23})^2\right) \tag{7.4.16}$$

Energy current radiated into the $d\Omega$ solid angle: $R^2\cdot c\cdot t^{10}\cdot d\,\Omega$

Introduce the *e* polarisation unit vector, and define its properties. Here *n* is the three-vector of the plane wave:

e_{ij}

$$e_{aa}=0 \qquad e_{ia}\cdot n_a=0 \qquad e_{ab}\cdot e_{ab}=1 \tag{7.4.17}$$

With it the intensity of radiation with a given polarisation:

$$dI=\frac{\gamma}{75\cdot\pi\cdot c^3}\cdot(\dddot{Q}_{ab}\cdot e_{ab})^2\cdot d\,\Omega \tag{7.4.18}$$

Average over every polarisation direction. Express the polarisation unit tensor:

$$\begin{aligned}\overline{e_{ij}\cdot e_{kl}}=\frac{1}{4}\cdot\big(&n_i\cdot n_j\cdot n_k\cdot n_l+n_i\cdot n_j\cdot\delta_{kl}+n_k\cdot n_l\cdot\delta_{ij}-n_i\cdot n_k\cdot\delta_{jl}-n_j\cdot n_k\cdot\delta_{il}\\&-n_i\cdot n_l\cdot\delta_{jk}-n_j\cdot n_l\cdot\delta_{ik}-\delta_{ij}\cdot\delta_{kl}+\delta_{ik}\cdot\delta_{jl}+\delta_{jk}\cdot\delta_{il}\big)\end{aligned} \tag{7.4.19}$$

Use the three-vector of the plane wave to express the intensity:

$$dl=\frac{\gamma}{36\cdot\pi\cdot c^5}\cdot((\dddot{Q}_{ab}\cdot n_a\cdot n_b)^2+(\dddot{Q}_{ab})^2-\dddot{Q}_{ab}\cdot\dddot{Q}_{ac}\cdot n_b\cdot n_c)\cdot d\,\Omega \qquad (7.4.20)$$

The average of the energy current along the $\frac{dl}{d\,\Omega}\cdot 4\cdot\pi$ direction, the quadrupole formula:

$$-\frac{dE}{dt}=\frac{\gamma}{45\cdot c^3}\cdot(\dddot{Q}_{ab})^2 \qquad (7.4.21)$$

The second time derivative of the quadrupole moment is approximately the kinetic energy of the non-spherically symmetric movements of the source. The amplitude of the generated waves:

$$h=\frac{\gamma}{c^4}\cdot\frac{E_k}{r} \qquad (7.4.22)$$

Calculate from the quadrupole formula the radiated performance into a unit solid angle:

$$L_g=\frac{1}{5}\cdot\frac{\gamma}{c^5}\cdot\left(\dddot{Q}_{ab}\cdot\dddot{Q}_{ab}-\frac{1}{3}\cdot(\dddot{Q})^2\right) \qquad (7.4.23)$$

7.5 Examples

Quadrupole moment of mass points connected with springs:

$$Q=m\cdot l^2 \qquad (7.5.1)$$

The length of the spring changes periodically:

$$l=l_0+a\cdot\sin(\omega\cdot t) \qquad (7.5.2)$$

Substitute into the equation of the quadrupole moment:

$$Q=m\cdot l_0^2+2\cdot m\cdot l_0\cdot a\cdot\sin(\omega\cdot t)+m\cdot a^2\cdot\sin^2(\omega\cdot t) \qquad (7.5.3)$$

In the case of small difference the last term can be neglected:

$$\dddot{Q}=-2\cdot m\cdot l_0\cdot a\cdot\omega^3\cdot\cos(\omega\cdot t) \qquad (7.5.4)$$

Substitute into the equation for luminosity:

$$L_g = \frac{4}{5} \cdot \frac{\gamma}{c^5} \cdot (m \cdot l_0 \cdot a \cdot \omega^3 \cdot \cos(\omega \cdot t))^2 \tag{7.5.5}$$

Calculate how much gravitational energy is emitted by a vibrating rod in every second. For the sake of simplicity, we use unit but realistic sizes:

$$m = 1\,kg \qquad l_0 = 1\,m$$

$$a = 10^{-3}\,m \qquad \omega = 10^2 \frac{1}{s}$$

$$L_g = 6.6488 \cdot 10^{-49} \frac{J}{s} \tag{7.5.6}$$

Gravitational luminosity of a rotating rod from its quadrupole moment:

$$Q = \frac{\sqrt{2}}{18} \cdot m \cdot l^2 \cdot \omega^3 \cdot t^3$$

$$\dddot{Q} = \frac{\sqrt{2}}{3} \cdot m \cdot l^2 \cdot \omega^3$$

$$L_g = \frac{2}{45} \cdot \frac{\gamma}{c^5} \cdot (m \cdot l^2 \cdot \omega^3)^2 \tag{7.5.7}$$

Let the size of the rotating rod become comparable to its Schwarzschild radius:

$$r = \frac{2 \cdot \gamma \cdot m}{c^2} \qquad v = \omega \cdot r$$

$$r = \frac{l}{2} \qquad \omega = \frac{v}{c} \cdot \frac{c}{r} \tag{7.5.8}$$

Substitute these values into the luminosity and evaluate the greatest possible luminosity (or at least its magnitude):

$$L_g = \frac{8}{45} \cdot \frac{c^5}{\gamma} \cdot r^4 \cdot \left(\frac{r_g}{r}\right)^2 \cdot \left(\frac{v}{c}\right)^6$$

$$L_g = 3{,}63 \cdot 10^{52} \frac{J}{s} \tag{7.5.9}$$

This value more or less corresponds to the combined radiation performance of all stars of the Universe.

The two-body problem:

reduced mass: $\mu=\frac{m_1 \cdot m_2}{m_1+m_2}$

relative coordinates: $z^i=x_1^i-x_2^i$

absolute coordinates: $x_1^i=\frac{\mu}{m_1}\cdot z^i$ $\qquad x_2^i=\frac{\mu}{m_2}\cdot z^i$ (7.5.10)

The reduced quadrupole moment:

$$Q^{ij}=\mu\cdot(3\cdot z^i\cdot z^j-\delta^{ij}\cdot|z|^2) \tag{7.5.11}$$

The parameters of the circular orbit:

$$z^1=R\cdot\sin(\omega\cdot t) \qquad z^2=R\cdot\cos(\omega\cdot t) \qquad z^3=0 \tag{7.5.12}$$

Quadrupole moment:

$$\dddot{Q}^{ij}\cdot\dddot{Q}_{ij}=18\cdot\mu^2\cdot(\dddot{z}^2\cdot z^2+6\cdot\dddot{z}\cdot\dot{z}\cdot\dot{z}^2+9\cdot\ddot{z}^2\cdot\dot{z}^2) \tag{7.5.13}$$

Radiation performance:

$$-\frac{dE}{dt}=\frac{32\cdot\gamma}{5\cdot c^5}\cdot\mu^2\cdot\omega^6\cdot R^4 \tag{7.5.14}$$

Substitution:

$$\omega^2\cdot R^3=m_1+m_2 \qquad t_k=2\cdot\pi\cdot\sqrt{\frac{R^3}{\gamma\cdot(m_1+m_2)}} \qquad \text{(Kepler's law)} \tag{7.5.15}$$

The radiation output in case of a circular orbit:

$$-\frac{dE}{dt}=\frac{32\cdot\gamma^4}{5\cdot c^5}\cdot\frac{m_1^2\cdot m_2^2\cdot(m_1+m_2)}{R^5} \tag{7.5.16}$$

The E energy and the L angular momentum on an elliptic orbit:

semi-major axis: $a=-\gamma\cdot\frac{m_1\cdot m_2}{2\cdot E}$

eccentricity: $e^2=1+\frac{2\cdot E\cdot L^2\cdot(m_1+m_2)}{\gamma^2\cdot(m_1\cdot m_2)^3}$ (7.5.17)

The radiation output in case of an elliptic orbit:

$$-\frac{dE}{dt}=\frac{32\cdot\gamma^4}{5\cdot c^5}\cdot\frac{m_1^2\cdot m_2^2\cdot(m_1+m_2)}{a^5\cdot(1-e^2)^{\frac{7}{2}}}\cdot\left(1+\frac{73}{24}\cdot e^2+\frac{37}{96}\cdot e^4\right) \tag{7.5.18}$$

The change in the orbital period:

$$\frac{\dot{t}_k}{t_k}=\frac{3\cdot\dot{a}}{2\cdot a}=\frac{3\cdot\dot{E}}{2\cdot|E|}$$

$$\frac{\dot{t}_k}{t_k}=-\frac{96\cdot\gamma^2}{5\cdot c^5}\cdot\frac{m_1\cdot m_2}{\sqrt[3]{\gamma\cdot(m_1+m_2)}}\cdot\left(\frac{t_k}{2\cdot\pi}\right)^{-\frac{8}{3}}\cdot\frac{1}{(1-e^2)^{\frac{7}{2}}}\cdot\left(1+\frac{73}{24}\cdot e^2+\frac{37}{96}\cdot e^4\right) \tag{7.5.19}$$

8. Spacetime of the Universe

Gravitation as a macroscopic interaction influences the structure and future of the entire Universe. Although the arrangement of stars and galaxies in it highly varies, we can do some simplifications nevertheless. According to the cosmological principle, the Universe on the $\sim 10^8$ lightyears scale is already homogeneous and isotropic, and already in much smaller volumes the electromagnetic effect of the particles mutually cancels out, therefore in the investigated size range they do not influence the structure of spacetime.

8.1 Assumptions

According to our current knowledge the most general Einstein equation is:

$$R_{\eta\kappa} - \frac{1}{2} \cdot R \cdot g_{\eta\kappa} - \Lambda \cdot g_{\eta\kappa} = -\frac{8 \cdot \pi \cdot \gamma}{c^4} \cdot T_{\eta\kappa} \tag{8.1.1}$$

Where Λ denotes the cosmological constant we assume to be responsible for the accelerated expansion of the Universe, that was observed for the first time in 1998. This equation is the most general relationship that contains the metric tensor and its first and second derivatives, therefore the cosmological constant is also part of the spacetime geometry, it does not describe matter distribution.

We secure homogeneity by averaging, but this should be accepted with reservations, since for example in the case of the occurring quantities the product of averages is not equal to the average of products. The averaging of the Ricci tensor:

$$\overline{R_{ij}} = \overline{\frac{\partial \Gamma^a{}_{ij}}{\partial x^a} - \frac{\partial \Gamma^a{}_{aj}}{\partial x^i} + \Gamma^b{}_{ij} \cdot \Gamma^a{}_{ab} - \Gamma^b{}_{aj} \cdot \Gamma^a{}_{ib}} \neq \frac{\partial \overline{\Gamma^a{}_{ij}}}{\partial x^a} - \frac{\partial \overline{\Gamma^a{}_{aj}}}{\partial x^i} + \overline{\Gamma^b{}_{ij}} \cdot \overline{\Gamma^a{}_{ab}} - \overline{\Gamma^b{}_{aj}} \cdot \overline{\Gamma^a{}_{ib}} \tag{8.1.2}$$

Furthermore, we split up the space to cells for the averaging, but for this we would have to know the exact metric. This method works, if there are no larger scale structures in the Universe than $\sim 10^8$ lightyears. The consequence of isotropy is constant curvature, in this case calculating the three dimensional Riemann tensor is easy because of the Schur theorem:

$$R_{ijkl} = \frac{1}{2} \cdot R \cdot (g_{ik} \cdot g_{jl} - g_{il} \cdot g_{jk}) \tag{8.1.3}$$

These conditions reduce the number of possible spacetime configurations considerably. Consequently the three dimensional space is a maximally symmetric manifold, with positive, negative or zero possible scalar curvature, and we embed it into a four dimensional flat manifold. These three cases determine the shape of the space around us, the density of matter, and the future of the Universe.

8.2 Positive curvature

The first possibility is a sphere with three dimensional surface, embedded into four dimensional flat spacetime. The equation of the surface with rectangular coordinates:

$$x_1^2+x_2^2+x_3^2+x_4^2=a^2$$

$$x_4^2=a^2-x_1^2-x_2^2-x_3^2 \qquad (8.2.1)$$

The arc length squared on this surface:

$$dl=dx_1^2+dx_2^2+dx_3^2+dx_4^2$$

$$dx^4=-\frac{x_1\cdot dx_1+x_2\cdot dx_2+x_3\cdot dx_3}{x_4} \qquad (8.2.2)$$

Let us introduce polar coordinates:

$$x_1=r\cdot\sin(\vartheta)\cdot\cos(\varphi)$$

$$x_2=r\cdot\sin(\vartheta)\cdot\sin(\varphi) \quad \leftrightarrow \quad r^2=x_1^2+x_2^2+x_3^2$$

$$r\cdot dr=x_1\cdot dx_1+x_2\cdot dx_2+x_3\cdot dx_3$$

$$x_3=r\cdot\cos(\vartheta) \qquad (8.2.3)$$

The arc length squared with polar coordinates:

$$dl^2=\frac{dr^2}{1-\frac{r^2}{a^2}}+r^2\cdot(d\vartheta^2+\sin^2(\vartheta)\cdot d\varphi^2) \qquad (8.2.4)$$

Rescale the *r* coordinate depending on the *a* radius of the sphere:

$$\tilde{r}=\frac{r}{a} \qquad (8.2.5)$$

Substitute into the arc length squared:

$$dl^2=a^2\cdot\left(\frac{dr^2}{1-\tilde{r}^2}+\tilde{r}^2\cdot(d\vartheta^2+\sin^2(\vartheta)\cdot d\varphi^2)\right) \qquad (8.2.6)$$

Introduce a new coordinate, thus in three dimensions every one of them will be angular coordinates:

$$r=a\cdot\sin(\chi)$$

$$dr=a\cdot\cos(\chi)\cdot d\chi \qquad (8.2.7)$$

Substitute into the arc length squared:

$$dl^2 = a^2 \cdot (d\chi^2 + \sin^2(\chi) \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2)) \tag{8.2.8}$$

By adding the time coordinate, we write down the four dimensional arc length squared. The Robertson-Walker metric in the case of positive curvature, we allow the radius to be time dependent:

$$ds^2 = c^2 \cdot dt^2 - a^2(t) \cdot (d\chi^2 + \sin^2(\chi) \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2)) \tag{8.2.9}$$

The positive curvature, closed Universe has finite volume:

$$V = a^3 \cdot \int_{\chi=0}^{\pi} \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\cdot\pi} \sin^2(\chi) \cdot \sin(\vartheta) \cdot d\chi \cdot d\vartheta \cdot d\varphi = 2 \cdot \pi^2 \cdot a^3 \tag{8.2.10}$$

The calculation of the surface of a two dimensional sphere of radius *r* in this universe:

$$A = 4 \cdot \pi \cdot r^2 \cdot \sin^2(\chi) = 4 \cdot \pi \cdot r^2 \cdot \sin^2\left(\frac{a}{r}\right) \tag{8.2.11}$$

8.3 Negative curvature

In this case the radius is negative:

$$a^2 \quad \rightarrow \quad -a^2 \tag{8.3.1}$$

This has the following consequences for the metric:

$$a \quad \rightarrow \quad i \cdot a$$

$$\chi \quad \rightarrow \quad i \cdot \chi$$

$$\sin(i \cdot \chi) = \sinh(\chi) \tag{8.3.2}$$

The Robertson-Walker metric for the negative curvature can be determined from the positive case by substituting the former, the radius can be time dependent here too:

$$ds^2 = c^2 \cdot dt^2 - a^2(t) \cdot (d\chi^2 + \sinh^2(\chi) \cdot (d\vartheta^2 + \sin^2(\vartheta) \cdot d\varphi^2)) \tag{8.3.3}$$

The negative curvature, open Universe has infinite volume:

$$V=a^3\cdot\int_{\chi=0}^{\infty}\int_{\vartheta=0}^{\pi}\int_{\varphi=0}^{2\cdot\pi}\sinh^2(\chi)\cdot\sin(\vartheta)\cdot d\chi\cdot d\vartheta\cdot d\varphi=\infty \qquad (8.3.4)$$

The calculation of the surface of a two dimensional sphere of radius *r* in this universe:

$$A=4\cdot\pi\cdot r^2\cdot\sinh^2(\chi)=4\cdot\pi\cdot r^2\cdot\sinh^2\left(\frac{a}{r}\right) \qquad (8.3.5)$$

8.4 Zero curvature

In this case *a* merely scales distances in the Universe and χ changes back to a distance coordinate:

$$\chi=\frac{r}{a} \qquad (8.4.1)$$

Robertson-Walker metric in the case of zero curvature, the time dependence remains of course:

$$ds^2=c^2\cdot dt^2-a^2(t)\cdot(d\chi^2+\chi^2\cdot(d\vartheta^2+\sin^2(\vartheta)\cdot d\varphi^2)) \qquad (8.4.2)$$

The negative curvature, open Universe also has infinite volume:

$$V=a^3\cdot\int_{\chi=0}^{\infty}\int_{\vartheta=0}^{\pi}\int_{\varphi=0}^{2\cdot\pi}\chi^2\cdot\sin(\vartheta)\cdot d\chi\cdot d\vartheta\cdot d\varphi=\infty \qquad (8.4.3)$$

The surface of the two dimensional sphere can be calculated with the usual formula:

$$A=4\cdot\pi\cdot r^2 \qquad (8.4.4)$$

8.5 Cosmological redshift

In every possible Universe the parameter *a* is allowed to change with time. Our observations show, that the dimmer the galaxies around us are, the greater redshift they have. We can interpret this phenomenon as the expansion of the Universe.

The observer is in the centre of our coordinate system. The constant coordinates of the light source:

$$(\chi_{1,}\vartheta_{1,}\varphi_1) \qquad (8.5.1)$$

In the t_1 moment a lightwave maximum starts from the source, it arrives at the t_0 moment to the observer in the centre:

$$t_0 > t_1 \tag{8.5.2}$$

It propagates along one of the coordinates, therefore the coordinate conditions:

$$\vartheta = const. \qquad \varphi = const. \tag{8.5.3}$$

Write down the arc length squared along the light-like geodesic:

$$c^2 \cdot dt^2 - a^2(t) \cdot d\chi^2 = 0 \tag{8.5.4}$$

The elapsed time:

$$dt = \pm \frac{a(t) \cdot d\chi}{c}$$

$$\frac{dt}{a(t)} = \pm \frac{d\chi}{c}$$

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \pm \frac{1}{c} \cdot \int_{\chi_1}^{0} d\chi = \pm \frac{\chi_1}{c} \tag{8.5.5}$$

The next light maximum:

departs: $t_1 + \delta t_1$

arrives: $t_0 + \delta t_0$ (8.5.6)

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \frac{\chi_1}{c} = \int_{t_1}^{t_0} \frac{dt}{a(t)}$$

$$\int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{t_1+\delta t_1}^{t_1} \frac{dt}{a(t)} + \int_{t_1}^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)} = \int_{t_1}^{t_0} \frac{dt}{a(t)}$$

$$\int_{t_1+\delta t_1}^{t_1} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)} = 0 \tag{8.5.7}$$

Since the δt change in time is small, therefore during this time the time dependent $a(t)$ function does not change significantly:

$$-\frac{\delta t_1}{a(t_1)} + \frac{\delta t_0}{a(t_0)} = 0$$

$$\frac{\nu_0}{\nu_1} = \frac{\lambda_1}{\lambda_0} = \frac{\delta t_0}{\delta t_1} = \frac{a(t_1)}{a(t_0)} \tag{8.5.8}$$

The z parameter characterizes the ratios of the distances of distancing objects from us, without knowing their absolute distance:

$$z=\frac{\lambda_0}{\lambda_1}-1=\frac{a(t_0)}{a(t_1)}-1 \tag{8.5.9}$$

We observe this value to be positive, and we interpret this as the expansion of the Universe.

8.6 Hubble law

Distance along one of the coordinates:

$$l(t)=a(t)\cdot\chi \tag{8.6.1}$$

Two celestial bodies with constant coordinates:

$$\frac{l(t_1)}{a(t_1)}=\frac{l(t)}{a(t)} \qquad l(t)=\frac{l}{a}\cdot a(t) \tag{8.6.2}$$

We examine with derivatives according to time, how fast the distance changes because of the expansion:

$$\dot{l}=\frac{\dot{a}}{a}\cdot l=H\cdot l \qquad H=\frac{\dot{a}}{a}\cdot c \tag{8.6.3}$$

Where H is the current value of the Hubble constant:

$$H_0=73{,}8\pm 2{,}4\frac{km}{s}\cdot\frac{1}{MPc}=2.39\cdot 10^{-18}\frac{1}{s} \tag{8.6.4}$$

The Hubble time is approximately in the same magnitude with the age of the Universe:

$$t_H=\frac{1}{H_0}=4.18\cdot 10^{17}\,s=1.32\cdot 10^{10}\,year \tag{8.6.5}$$

It is possible to determine by approximation from the cosmic redshift and the Hubble constant, how long time it took, until the light has reached us. Series expansion around the state of the observer:

$$\frac{1}{a(t)}=\sum_0^{\infty}\frac{1}{n!}\cdot\frac{d^n}{dt^n}\frac{1}{a(t)}\cdot(-\Delta t)^n$$

$$\frac{1}{a(t)}\approx\frac{1}{a}+\frac{\dot{a}}{a^2}\cdot\Delta t+\frac{1}{2}\cdot\left(\frac{2\cdot\dot{a}^2}{a^3}-\frac{\ddot{a}}{a^2}\right)\cdot\Delta t^2-\frac{1}{6}\cdot\left(-\frac{\dddot{a}}{a^2}+\frac{6\cdot\dot{a}\cdot\ddot{a}}{a^3}-\frac{6\cdot\dot{a}^3}{a^4}\right)\cdot\Delta t^3+\dots$$

$$\frac{a}{a(t)}\approx 1+\frac{\dot{a}}{a}\cdot\Delta t+\left(\frac{\dot{a}^2}{a^2}-\frac{\ddot{a}}{2\cdot a}\right)\cdot\Delta t^2+\left(\frac{\dot{a}^3}{a^3}-\frac{\dot{a}\cdot\ddot{a}}{a^2}+\frac{\ddot{a}}{6\cdot a}\right)\cdot\Delta t^3+\ldots \qquad (8.6.6)$$

Substitute the H Hubble constant and the Q deceleration parameter, and write down with them the Taylor series of the cosmological redshift:

$$\frac{H}{c}=\frac{\dot{a}}{a} \qquad\qquad Q=\frac{\ddot{a}}{a}$$

$$z=\frac{a}{a(t)}-1\approx H\cdot\Delta t+\left(\frac{H^2}{c^2}-\frac{Q}{2}\right)\cdot\Delta t^2-\left(\frac{H^3}{c^3}-\frac{H}{c}\cdot Q+\frac{\ddot{a}}{6\cdot a}\right)\cdot\Delta t^3+\ldots \qquad (8.6.7)$$

The distance of the light source in the case of a small z:

$$\Delta l\approx\frac{H}{z}\cdot c \qquad (8.6.8)$$

8.7 Flat geometry

According to our observations, the space in the Universe is not curved on the large scale. We determine the critical density, that characterizes this universe-model. To write down the flat arc length squared, we start with Minkowskian coordinates and extend them with the time dependent scale factor:

$$ds^2=c^2\cdot dt^2-a^2(t)\cdot(dx^2+dy^2+dz^2) \qquad (8.7.1)$$

The geometric quantities from the metric tensor to the Ricci scalar:

$$g_{\eta\kappa}=\begin{pmatrix}1&0&0&0\\0&-a^2&0&0\\0&0&-a^2&0\\0&0&0&-a^2\end{pmatrix} \qquad g^{\eta\kappa}=\begin{pmatrix}1&0&0&0\\0&-\frac{1}{a^2}&0&0\\0&0&-\frac{1}{a^2}&0\\0&0&0&-\frac{1}{a^2}\end{pmatrix} \qquad (8.7.2)$$

$$\frac{\partial g_{xx}}{\partial t}=\frac{\partial g_{yy}}{\partial t}=\frac{\partial g_{zz}}{\partial t}=-2\cdot a\cdot\dot{a} \qquad \frac{\partial g^{xx}}{\partial t}=\frac{\partial g^{yy}}{\partial t}=\frac{\partial g^{zz}}{\partial t}=\frac{2\cdot\dot{a}}{a^3} \qquad (8.7.3)$$

$$\Gamma^t{}_{xx}=\Gamma^t{}_{yy}=\Gamma^t{}_{zz}=a\cdot\dot{a} \qquad \Gamma^x{}_{tx}=\Gamma^x{}_{xt}=\Gamma^y{}_{ty}=\Gamma^y{}_{yt}=\Gamma^z{}_{tz}=\Gamma^z{}_{zt}=\frac{\dot{a}}{a} \qquad (8.7.4)$$

$$\frac{\partial \Gamma^{t}{}_{xx}}{\partial t} = \frac{\partial \Gamma^{t}{}_{yy}}{\partial t} = \frac{\partial \Gamma^{t}{}_{zz}}{\partial t} = a \cdot \ddot{a} + \dot{a}^{2}$$

$$\frac{\Gamma^{x}{}_{tx}}{\partial t} = \frac{\Gamma^{x}{}_{xt}}{\partial t} = \frac{\Gamma^{y}{}_{ty}}{\partial t} = \frac{\Gamma^{y}{}_{yt}}{\partial t} = \frac{\Gamma^{z}{}_{tz}}{\partial t} = \frac{\Gamma^{z}{}_{zt}}{\partial t} = \frac{\ddot{a}}{a} - \frac{\dot{a}^{2}}{a^{2}} \tag{8.7.5}$$

$$R^{t}{}_{xtx} = -R^{t}{}_{xxt} = R^{t}{}_{yty} = -R^{t}{}_{yyt} = R^{t}{}_{ztz} = -R^{t}{}_{zzt} = a \cdot \ddot{a}$$

$$R^{x}{}_{ttx} = -R^{x}{}_{txt} = R^{y}{}_{tty} = -R^{y}{}_{tyt} = R^{z}{}_{ttz} = -R^{z}{}_{tzt} = \frac{\ddot{a}}{a}$$

$$R^{x}{}_{yxy} = -R^{x}{}_{yyx} = R^{x}{}_{zxz} = -R^{x}{}_{zzx} = R^{y}{}_{xyx} = -R^{y}{}_{xxy} = R^{y}{}_{zyz} = -R^{y}{}_{zzy} = R^{z}{}_{xzx} = -R^{z}{}_{xxz} = R^{z}{}_{yzy} = -R^{z}{}_{yyz} = \dot{a}^{2} \tag{8.7.6}$$

$$R_{\eta\kappa} = \begin{pmatrix} -\frac{3 \cdot \ddot{a}}{a} & 0 & 0 & 0 \\ 0 & a \cdot \ddot{a} + \dot{a}^{2} & 0 & 0 \\ 0 & 0 & a \cdot \ddot{a} + \dot{a}^{2} & 0 \\ 0 & 0 & 0 & a \cdot \ddot{a} + \dot{a}^{2} \end{pmatrix} \tag{8.7.7}$$

$$R = -6 \cdot \frac{a \cdot \ddot{a} + \dot{a}^{2}}{a^{2}} \tag{8.7.8}$$

The normalized form of the energy-momentum tensor:

$$T_{\eta\kappa} = \begin{pmatrix} \rho \cdot c^{2} & 0 & 0 & 0 \\ 0 & -\frac{p}{a^{2}} & 0 & 0 \\ 0 & 0 & -\frac{p}{a^{2}} & 0 \\ 0 & 0 & 0 & -\frac{p}{a^{2}} \end{pmatrix} \tag{8.7.9}$$

Substitute them into the Einstein equations:

$$R_{\eta\kappa} - \frac{1}{2} \cdot R \cdot g_{\eta\kappa} = -\frac{8 \cdot \pi \cdot \gamma}{c^{4}} \cdot T_{\eta\kappa} \tag{8.7.10}$$

The time-like component:

$$R_{tt} - \frac{1}{2} \cdot R \cdot g_{tt} = -\frac{8 \cdot \pi \cdot \gamma}{c^{4}} \cdot T_{tt} \tag{8.7.11}$$

$$-\frac{3\cdot\ddot{a}}{a}-\frac{1}{2}\cdot\left(-6\cdot\frac{a\cdot\ddot{a}+\dot{a}^2}{a^2}\right)\cdot 1=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot\rho\cdot c^2$$

$$3\cdot\frac{\dot{a}^2}{a^2}=\frac{8\cdot\pi\cdot\gamma}{c^2}\cdot\rho$$

The first Friedmann equation gives the density of the Universe:

$$\frac{3\cdot H^2}{8\cdot\pi\cdot\gamma}=\rho \tag{8.7.12}$$

Substitute and determine the numerical value:

$$\rho=1.02\cdot 10^{-26}\frac{kg}{m^3} \tag{8.7.13}$$

This means an average 6.11 hydrogen atoms in every cubic meters.

8.8 General Friedmann equations

We express the three possible cases with a single equation:

$$k\cdot(x_1^2+x_2^2+x_3^2)+x_4^2=a^2 \qquad k\begin{cases}>0\\<0\\=0\end{cases} \tag{8.8.1}$$

Where we differentiate with *k* between the solutions that are either flat, or have positive or negative curvature. The spatial arc length squared:

$$dl^2=dx_1^2+dx_1^2+dx_1^2+k\cdot dx_4^2 \tag{8.8.2}$$

Switch to the usual coordinate system and write down the arc length squared:

$$d\chi=\frac{d\sigma}{\sqrt{1-k\cdot\sigma^2}}$$

$$ds^2=c^2\cdot dt^2-a^2(t)\cdot\left(\frac{d\sigma^2}{1-k\cdot\sigma^2}+\sigma^2\cdot(d\vartheta^2+\sin^2(\vartheta)\cdot d\varphi^2)\right) \tag{8.8.3}$$

The geometric quantities from the metric tensor to the Ricci scalar:

$$g_{\eta\kappa}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -\frac{a^2(t)}{1-k\cdot\sigma^2} & 0 & 0\\ 0 & 0 & -a^2(t)\cdot\sigma^2 & 0\\ 0 & 0 & 0 & -a^2(t)\cdot\sigma^2\cdot\sin^2(\vartheta)\end{pmatrix}$$

$$g^{\eta\kappa}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -\frac{1-k\cdot\sigma^2}{a^2(t)} & 0 & 0\\ 0 & 0 & -\frac{1}{a^2(t)\cdot\sigma^2} & 0\\ 0 & 0 & 0 & -\frac{1}{a^2(t)\cdot\sigma^2\cdot\sin^2(\vartheta)}\end{pmatrix} \qquad (8.8.4)$$

$$\frac{\partial g_{\sigma\sigma}}{\partial t}=-\frac{2\cdot a\cdot\dot{a}}{1-k\cdot\sigma^2} \qquad \frac{\partial g^{\sigma\sigma}}{\partial t}=-\frac{2\cdot(1-k\cdot\sigma^2)\cdot\dot{a}}{a^3}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial t}=-2\cdot\sigma^2\cdot a\cdot\dot{a} \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial t}=\frac{2\cdot\dot{a}}{\sigma^2\cdot a^3}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial t}=-2\cdot\sigma^2\cdot a\cdot\dot{a}\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial t}=\frac{2\cdot\dot{a}}{\sigma^2\cdot a^3\cdot\sin^{(\vartheta)}}$$

$$\frac{\partial g_{\sigma\sigma}}{\partial\sigma}=-\frac{2\cdot k\cdot\sigma\cdot a^2}{(1-k\cdot\sigma^2)^2} \qquad \frac{\partial g^{\sigma\sigma}}{\partial\sigma}=\frac{2\cdot k\cdot\sigma}{a^2}$$

$$\frac{\partial g_{\vartheta\vartheta}}{\partial\sigma}=-2\cdot\sigma\cdot a^2 \qquad \frac{\partial g^{\vartheta\vartheta}}{\partial\sigma}=\frac{2}{\sigma^3\cdot a^2}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\sigma}=-2\cdot\sigma\cdot a^2\cdot\sin^2(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\sigma}=\frac{2}{\sigma^3\cdot a^2\cdot\sin^2(\vartheta)}$$

$$\frac{\partial g_{\varphi\varphi}}{\partial\vartheta}=-2\cdot\sigma^2\cdot a^2\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial g^{\varphi\varphi}}{\partial\vartheta}=\frac{2\cdot\cos(\vartheta)}{\sigma^2\cdot a^2\cdot\sin^3(\vartheta)} \qquad (8.8.5)$$

$$\Gamma^t{}_{\sigma\sigma}=\frac{a\cdot\dot{a}}{1-k\cdot\sigma^2} \qquad \Gamma^t{}_{\vartheta\vartheta}=\sigma^2\cdot a\cdot\dot{a} \qquad \Gamma^t{}_{\varphi\varphi}=\sigma^2\cdot a\cdot\dot{a}\cdot\sin^2(\vartheta)$$

$$\Gamma^\sigma{}_{t\sigma}=\Gamma^\sigma{}_{\sigma t}=\Gamma^\vartheta{}_{t\vartheta}=\Gamma^\vartheta{}_{\vartheta t}=\Gamma^\varphi{}_{t\varphi}=\Gamma^\varphi{}_{\varphi t}=\frac{\dot{a}}{a}$$

$$\Gamma^{\sigma}{}_{\sigma\sigma}=\frac{k\cdot\sigma}{1-k\cdot\sigma^2} \qquad \Gamma^{\sigma}{}_{\vartheta\vartheta}=-\sigma\cdot(1-k\cdot\sigma^2) \qquad \Gamma^{\sigma}{}_{\varphi\varphi}=-\sigma\cdot(1-k\cdot\sigma^2)\cdot\sin^2(\vartheta)$$

$$\Gamma^{\vartheta}{}_{\sigma\vartheta}=\Gamma^{\vartheta}{}_{\vartheta\sigma}=\Gamma^{\varphi}{}_{\sigma\varphi}=\Gamma^{\varphi}{}_{\varphi\sigma}=\frac{1}{\sigma} \qquad \Gamma^{\vartheta}{}_{\varphi\varphi}=-\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\Gamma^{\varphi}{}_{\vartheta\varphi}=\Gamma^{\varphi}{}_{\varphi\vartheta}=\cot(\vartheta) \tag{8.8.6}$$

$$\frac{\Gamma^{t}{}_{\sigma\sigma}}{\partial t}=\frac{a\cdot\ddot{a}+\dot{a}^2}{1-k\cdot\sigma^2} \qquad \frac{\partial\Gamma^{t}{}_{\vartheta\vartheta}}{\partial t}=\sigma^2\cdot(a\cdot\ddot{a}+\dot{a}^2) \qquad \frac{\partial\Gamma^{t}{}_{\varphi\varphi}}{\partial t}=\sigma^2\cdot(a\cdot\ddot{a}+\dot{a}^2)\cdot\sin^2(\vartheta)$$

$$\frac{\partial\Gamma^{\sigma}{}_{t\sigma}}{\partial t}=\frac{\partial\Gamma^{\sigma}{}_{\sigma t}}{\partial t}=\frac{\partial\Gamma^{\vartheta}{}_{t\vartheta}}{\partial t}=\frac{\partial\Gamma^{\vartheta}{}_{\vartheta t}}{\partial t}=\frac{\partial\Gamma^{\varphi}{}_{t\varphi}}{\partial t}=\frac{\partial\Gamma^{\varphi}{}_{\varphi t}}{\partial t}=\frac{a\cdot\ddot{a}-\dot{a}^2}{a^2}$$

$$\frac{\partial\Gamma^{t}{}_{\sigma\sigma}}{\partial\sigma}=\frac{2\cdot k\cdot\sigma\cdot a\cdot\dot{a}}{(1-k\cdot\sigma^2)^2} \qquad \frac{\partial\Gamma^{t}{}_{\vartheta\vartheta}}{\partial\sigma}=2\cdot\sigma\cdot a\cdot\dot{a} \qquad \frac{\partial\Gamma^{t}{}_{\varphi\varphi}}{\partial\sigma}=2\cdot\sigma\cdot a\cdot\dot{a}\cdot\sin^2(\vartheta)$$

$$\frac{\partial\Gamma^{\sigma}{}_{\sigma\sigma}}{\partial\sigma}=\frac{k\cdot(1+k\cdot\sigma^2)}{(1-k\cdot\sigma^2)^2} \qquad \frac{\partial\Gamma^{\sigma}{}_{\vartheta\vartheta}}{\partial\sigma}=-(1-3\cdot k\cdot\sigma^2)$$

$$\frac{\partial\Gamma^{\sigma}{}_{\varphi\varphi}}{\partial\sigma}=-(1-3\cdot k\cdot\sigma^2)\cdot\sin^2(\vartheta) \qquad \frac{\partial\Gamma^{\vartheta}{}_{\sigma\vartheta}}{\partial\sigma}=\frac{\partial\Gamma^{\vartheta}{}_{\vartheta\sigma}}{\partial\sigma}=\frac{\partial\Gamma^{\varphi}{}_{\sigma\varphi}}{\partial\sigma}=\frac{\partial\Gamma^{\varphi}{}_{\varphi\sigma}}{\partial\sigma}=-\frac{1}{\sigma^2}$$

$$\frac{\partial\Gamma^{t}{}_{\varphi\varphi}}{\partial\vartheta}=2\cdot\sigma^2\cdot a\cdot\dot{a}\cdot\cos(\vartheta)\cdot\sin(\vartheta) \qquad \frac{\partial\Gamma^{\sigma}{}_{\varphi\varphi}}{\partial\vartheta}=-2\cdot\sigma\cdot(1-k\cdot\sigma^2)\cdot\cos(\vartheta)\cdot\sin(\vartheta)$$

$$\frac{\partial\Gamma^{\vartheta}{}_{\varphi\varphi}}{\partial\vartheta}=\sin^2(\vartheta)-\cos^2(\vartheta) \qquad \frac{\partial\Gamma^{\varphi}{}_{\vartheta\varphi}}{\partial\vartheta}=\frac{\partial\Gamma^{\varphi}{}_{\varphi\vartheta}}{\partial\vartheta}=-\frac{1}{\sin^2(\vartheta)} \tag{8.8.7}$$

$$R^{t}{}_{\sigma t\sigma}=-R^{t}{}_{\sigma\sigma t}=\frac{a\cdot\ddot{a}}{1-k\cdot\sigma^2}$$

$$R^{t}{}_{\vartheta t\vartheta}=-R^{t}{}_{\vartheta\vartheta t}=\sigma^2\cdot a\cdot\ddot{a}$$

$$R^{t}{}_{\varphi t\varphi}=-R^{t}{}_{\varphi\varphi t}=\sigma^2\cdot a\cdot\ddot{a}\cdot\sin^2(\vartheta)$$

$$R^{\sigma}{}_{tt\sigma}=-R^{\sigma}{}_{t\sigma t}=R^{\vartheta}{}_{tt\vartheta}=-R^{\vartheta}{}_{t\vartheta t}=R^{\varphi}{}_{tt\varphi}=-R^{\varphi}{}_{t\varphi t}=\frac{\ddot{a}}{a}$$

$$R^{\sigma}{}_{\vartheta\sigma\vartheta}=-R^{\sigma}{}_{\vartheta\vartheta\sigma}=R^{\varphi}{}_{\vartheta\varphi\vartheta}=-R^{\varphi}{}_{\vartheta\vartheta\varphi}=\sigma^2\cdot(\dot{a}^2+k)$$

$$R^{\sigma}{}_{\varphi\sigma\varphi}=-R^{\sigma}{}_{\varphi\varphi\sigma}=R^{\vartheta}{}_{\varphi\vartheta\varphi}=-R^{\vartheta}{}_{\varphi\varphi\vartheta}=\sigma^2\cdot(\dot{a}^2+k)\cdot\sin^2(\vartheta)$$

$$R^{\vartheta}{}_{\sigma\vartheta\sigma}=-R^{\vartheta}{}_{\sigma\sigma\vartheta}=R^{\varphi}{}_{\sigma\varphi\sigma}=-R^{\varphi}{}_{\sigma\sigma\varphi}=\frac{\dot{a}^2+k}{1-k\cdot\sigma^2} \tag{8.8.8}$$

$$R_{\eta\kappa}=\begin{pmatrix} -\frac{3\cdot\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{a\cdot\ddot{a}+2\cdot(\dot{a}^2+k)}{1-k\cdot\sigma^2} & 0 & 0 \\ 0 & 0 & \sigma^2\cdot(a\cdot\ddot{a}+2\cdot(\dot{a}^2+k)) & 0 \\ 0 & 0 & 0 & \sigma^2\cdot(a\cdot\ddot{a}+2\cdot(\dot{a}^2+k))\cdot\sin^2(\vartheta) \end{pmatrix} \tag{8.8.9}$$

$$R=-\frac{6}{a^2}\cdot(a\cdot\ddot{a}+\dot{a}^2+k) \tag{8.8.10}$$

The normalized form of the energy-momentum tensor:

$$T_{\eta\kappa}=\begin{pmatrix} \rho\cdot c^2 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-k\cdot\sigma^2}\cdot p & 0 & 0 \\ 0 & 0 & -a^2(t)\cdot\sigma^2\cdot p & 0 \\ 0 & 0 & 0 & -a^2(t)\cdot\sigma^2\cdot\sin^2(\vartheta)\cdot p \end{pmatrix} \tag{8.8.11}$$

Substitute them into the Einstein equation:

$$R_{\eta\kappa}-\frac{1}{2}\cdot R\cdot g_{\eta\kappa}-\Lambda\cdot g_{\eta\kappa}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\eta\kappa} \tag{8.8.12}$$

The time-like component:

$$R_{tt}-\frac{1}{2}\cdot R\cdot g_{tt}-\Lambda\cdot g_{tt}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{tt} \tag{8.8.13}$$

$$-\frac{3\cdot\ddot{a}}{a}-\frac{1}{2}\cdot\left(-\frac{6}{a^2}\cdot(a\cdot\ddot{a}+\dot{a}^2+k)\right)\cdot 1-\Lambda\cdot 1=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot\rho\cdot c^2$$

$$\frac{\dot{a}^2\cdot c^2}{a^2}+\frac{k\cdot c^2}{a^2}-\frac{\Lambda\cdot c^2}{3}=-\frac{8\cdot\pi\cdot\gamma}{3}\cdot\rho$$

The first general Friedmann equation:

$$H^2+\frac{k\cdot c^2}{a^2}-\frac{\Lambda\cdot c^2}{3}=-\frac{8\cdot\pi\cdot\gamma}{3}\cdot\rho \tag{8.8.14}$$

The spatial components:

$$R_{ii}-\frac{1}{2}\cdot R\cdot g_{ii}-\Lambda\cdot g_{ii}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{ii} \tag{8.8.15}$$

$$R_{\sigma\sigma}-\frac{1}{2}\cdot R\cdot g_{\sigma\sigma}-\Lambda\cdot g_{\sigma\sigma}=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot T_{\sigma\sigma}$$

$$\frac{a\cdot\ddot{a}+2\cdot(\dot{a}^2+k)}{1-k\cdot\sigma^2}-\frac{1}{2}\cdot\left(-\frac{6}{a^2}\cdot(a\cdot\ddot{a}+\dot{a}^2+k)\right)\cdot\left(-\frac{a^2}{1-k\cdot\sigma^2}\right)-\Lambda\cdot\left(-\frac{a^2}{1-k\cdot\sigma^2}\right)=-\frac{8\cdot\pi\cdot\gamma}{c^4}\cdot\left(-\frac{a^2}{1-k\cdot\sigma^2}\cdot p\right)$$

$$-2\cdot\frac{\ddot{a}\cdot c^2}{a}-\frac{\dot{a}^2\cdot c^2}{a^2}-\frac{k\cdot c^2}{a^2}+\Lambda\cdot c^2=\frac{8\cdot\pi\cdot\gamma}{c^2}\cdot p$$

We get the second general Friedmann equation in the two other cases as well:

$$-2\cdot\frac{\ddot{a}\cdot c^2}{a}-H^2-\frac{k\cdot c^2}{a^2}+\Lambda\cdot c^2=\frac{8\cdot\pi\cdot\gamma}{c^2}\cdot p \tag{8.8.16}$$

8.9 World models

Write down the relationship between the density and the scale factor in two theoretical scenarios, one of them corresponds to the matter dominated universe, and in the second case the energy is present mostly in the form of radiation. We introduce dimensionless quantities:

$$\rho_m(t)\cdot a^3(t)=const. \qquad p=0 \qquad K_m=\frac{8\cdot\pi\cdot\gamma}{3\cdot c^2}\cdot\rho_m\cdot a^3=const.$$

$$\rho_r(t)\cdot a^4(t)=const. \qquad p=\frac{\rho\cdot c^2}{3} \qquad K_r=\frac{8\cdot\pi\cdot\gamma}{3\cdot c^2}\cdot\rho_s\cdot a^4=const. \tag{8.9.1}$$

Write down the Friedmann equations of movement with them:

$$\dot{a}^2-\frac{K_m}{a}-\frac{K_r}{a^2}-\frac{\Lambda\cdot a^2}{3}=\dot{a}^2+V(a)=-k \tag{8.9.2}$$

In it the Friedmann potential:

$$V(a)=-\frac{K_m}{a}-\frac{K_r}{a^2}-\frac{\Lambda\cdot a^2}{3} \tag{8.9.3}$$

The time dependence of the scale factor while neglecting the cosmological constant:

$$\dot{a}^2 \approx \frac{K_m}{a} \rightarrow a(t) \propto \sqrt[3]{t^2} \qquad \dot{a}^2 \approx \frac{K_r}{a^2} \rightarrow a(t) \propto \sqrt{t} \tag{8.9.4}$$

In a static Universe the scale factor does not change, its derivatives according to time are zeroes. In the present state the energy of radiative origin can be neglected, therefore the cosmological constant necessary for a static situation can be determined from the equation of movement:

$$\frac{-K_m}{a} - \frac{\Lambda_k \cdot a^2}{3} = -1$$

$$\Lambda_k = \frac{3}{a^2} - \frac{3 \cdot K_m}{a^3} \tag{8.9.5}$$

Friedmannian world models:

	$k = -1$	$k = 0$	$k = 1$
$\Lambda < 0$	closed, periodic	closed, periodic	closed, periodic
$\Lambda = 0$	open, expanding	open, asymptotic	closed, periodic
$0 < \Lambda < \Lambda_k$	open, expanding	open, expanding	periodic / open
$\Lambda = \Lambda_k$	open, expanding	open, expanding	open, static, unstable
$\Lambda > \Lambda_k$	open, expanding	open, expanding	open, expanding

Introduce the following dimensionless variables:

$$x(\tau) = \frac{a(t)}{a_0} \qquad \tau = H_0 \cdot t$$

$$x(\tau_0) = 0 \tag{8.9.6}$$

Characterizing the present state:

$$\dot{a}^2 - \frac{K_m}{a} - \frac{\Lambda \cdot a^2}{3} = -k \quad / \cdot \frac{c^2}{H_0^2 \cdot a_0^2}$$

$$\dot{x}^2 - \frac{\Omega_m}{a} - \Omega_\Lambda \cdot \dot{x}^2 = \Omega_k \tag{8.9.7}$$

Where dimensionless constants characterize the state of the Universe:

$$\Omega_m = \frac{8 \cdot \pi \cdot \gamma}{3 \cdot H_0^2} \cdot \rho_m = \frac{\rho_m}{\rho_k} \qquad \Omega_\Lambda = \frac{\Lambda \cdot c^2}{3 \cdot H_0^2} \qquad \Omega_k = -\frac{k \cdot c^2}{H_0^2 \cdot a_0^2}$$

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1 \quad (8.9.8)$$

Their values according to our present knowledge:

$$\Omega_m = 0.273 \qquad \Omega_\Lambda = 0.727 \qquad \Omega_k \approx -0.023 \quad (8.9.9)$$

Distance of a point from the centre of the coordinate system:

$$D = \int_0^\sigma \sqrt{g_{\sigma\sigma}} \cdot d\sigma = R(t) \cdot \sigma \quad (8.9.10)$$

Distance of the particle horizon:

$$D_0 = R_0 \cdot \int_0^{t_0} d\sigma = R_0 \cdot \int_0^{t_0} \frac{c \cdot dt}{R(t)} \quad (8.9.11)$$

Cosmic escape velocity:

$$v_k = \frac{d}{dt}(R \cdot \sigma) = \frac{dR}{dt} \cdot \sigma = H_0 \cdot R_0 \cdot \sigma$$

$$\lim_{\sigma \to \infty} v_k = \infty \quad (8.9.12)$$

Calculating the dimensionless time:

$$\tau = H_0 \cdot t = \int_0^x \frac{dx}{\sqrt{\frac{\Omega_m}{x} + \Omega_\Lambda \cdot x^2 + \Omega_K}} \quad (8.9.13)$$

The age of the Universe:

$$t_0 = \int_0^{x=1} \frac{dx}{\dot{x}} = \frac{1}{H_0} \cdot \int_0^{x=1} \frac{dx}{\sqrt{\frac{\Omega_m}{x} + \Omega_\Lambda \cdot x^2 + \Omega_K}} = \frac{0.9897}{H_0} = 4.14 \cdot 10^{17} s = 1.31 \cdot 10^{10} \, years \quad (8.9.14)$$

The radius of the observable part of the Universe:

$$D_0 = a_0 \cdot \int_0^{t_0} \frac{c \cdot dt}{a(t)} = \frac{c}{H_0} \cdot \int_0^{x=1} \frac{dx}{\sqrt{\Omega_m \cdot x + \Omega_\Lambda \cdot x^4 + \Omega_K \cdot x^2}} = \frac{c}{H_0} \cdot 3.433$$

$$D_0 = 4.475 \cdot 10^{26} m = 4.73 \cdot 10^{10} \, lightyears \quad (8.9.15)$$

The distancing velocity of the border of the visible region:

$$v(D_0) = c \cdot \dot{a}_0 \cdot \int_0^{t_0} \frac{c \cdot dt}{a(t)} = H_0 \cdot D_0 = c \cdot \int_0^{x=1} \frac{dx}{\sqrt{\Omega_m \cdot x + \Omega_\Lambda \cdot x^4 + \Omega_K \cdot x^2}} = 3.433 \cdot [c] \frac{m}{s} \qquad (8.9.16)$$

This value exceeds the speed of light significantly. However this has no consequences from the point of view of the bodies moving in the spacetime.

Summary

We have peaked into the world of deterministic physics. It was apparently possible to geometrize this part of science, however we always have to keep in mind that like every model, this also has limits. Some of them follow from our conditions earlier, when we determined what phenomena interest us, and what do not; others occurred on the way, and it happens that we cannot satisfy certain expectations; in the worst case our conclusions can be rebutted by experimental data.

First and foremost in the beginning of the 20th century an old philosophical debate was concluded: the world is essentially indeterministic. It does not mean that its is unpredictable, it means only that we cannot predict events with arbitrary accuracy. The problem is not that we do not have enough information about the states, like many have thought initially. They assumed hidden variables, that we cannot measure, but they unambiguously determine the flow of events. It turned out that such variables do not exist, nature has been determined to be probabilistic. These phenomena under a certain size limit and above a certain energy density make the results described in this book useless.

Several philosophical expectations are not satisfied by the results in the book. One of the most famous of them is the vaguely defined Mach principle, that would mean that the Einstein equations cannot have a solution in empty space. Further complications arise because particles with spin cannot be properly discussed within the framework of general relativity, it has to be extended with torsion besides curvature to geometrize the effects of spin on spacetime (although this claim is disputed by some). In the resulting model additional effects manifest that were not yet confirmed experimentally. A more serious problem than these is the appearance of naked singularities, that have to be dealt with because of the insufficient definition of the appearing complex quantities in the model.

The validity of the model is questioned time to time, alternative theories predict different outcomes for various phenomena. We can however point out two things: within the current boundaries of measurement, considering the shortcomings of our astronomical knowledge (and if we stay within the postulated limits of validity), there is no result that would contradict the general theory of relativity. On the other hand, the competing models that predict with high accuracy in some problematic phenomena (like the problem of dark matter), gravely err in completely everyday situations.

The Kaluza theory is a generalization of general relativity as much as general relativity is an extension of special relativity. It addresses several gaps in Einstein's original theory, it finally gets rid of the idea of force, thus erases such shortcomings like the possibility of a force that could accelerate objects to become faster than the speed of light, and several solutions of the Einstein equations, that are although completely valid, are also completely unphysical. Their combined model can handle the most complete deterministic limiting case with solely mathematical tools.

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Index

This series presents modern physics through examples and derivations. The organization of topics does not follow the traditional historical approach, it was determined by practical considerations instead. We build up the mathematical framework of the models first, and then completely derive the most important consequences and compare them to recent experimental results. The volumes can be used in the specialized fields as reference materials, and are suitable for self-study in each topic. They may also serve the needs of university lectures as well.

The first volume deals with the general theory of relativity. This description has only historical relevance by now, since during the last century, much experimental evidence was found for the consequences of Einstein's theory. It is a classical field, where centuries-old scientific and philosophical ideas got mathematical formulation, and the experimental confirmation. It is important to point out, that the traditional mechanical worldview, that is often considered to be easier to grasp, is in fact an incomplete intellectual achievement. The basic assumptions of general relativity draw from everyday experiences, and the recognition of the curved nature of spacetime follows naturally. It is comparable to the understanding of Earth's spherical shape, and if we are familiar with the mathematical foundations, it does not even require too much imagination.

The Reader is assumed to have some basic knowledge in higher mathematics and among the more traditional subjects in physics, but there is only as much mathematical depth in this book, as absolutely necessary. We use the traditional symbols of differential calculus and index notation. The SI system of measurement is used in all physical derivations.