NONRESIDENT
TRAINING
COURSE

December 1988

Mathematics, Pre-Calculus and Introduction to Probability

NAVEDTRA 14141

Although the words “he,” “him,” and “his” are used sparingly in this course to enhance communication, they are not intended to be gender driven or to affront or discriminate against anyone.

PREFACE

By enrolling in this self-study course, you have demonstrated a desire to improve yourself and the Navy. Remember, however, this self-study course is only one part of the total Navy training program. Practical experience, schools, selected reading, and your desire to succeed are also necessary to successfully round out a fully meaningful training program.

COURSE OVERVIEW: The objective of this course is to enable the student to identify and perform calculations involving the equations to the various conic sections; recognize and work with concepts in calculus (limits, differentiation, and integration); and recognize the elements of introductory probability theory.

THE COURSE: This self-study course is organized into subject matter areas, each containing learning objectives to help you determine what you should learn along with text and illustrations to help you understand the information. The subject matter reflects day-to-day requirements and experiences of personnel in the rating or skill area. It also reflects guidance provided by Enlisted Community Managers (ECMs) and other senior personnel, technical references, instructions, etc., and either the occupational or naval standards, which are listed in the *Manual of Navy Enlisted Manpower Personnel Classifications and Occupational Standards*, NAVPERS 18068.

THE QUESTIONS: The questions that appear in this course are designed to help you understand the material in the text.

VALUE: In completing this course, you will improve your military and professional knowledge. Importantly, it can also help you study for the Navy-wide advancement in rate examination. If you are studying and discover a reference in the text to another publication for further information, look it up.

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Sailor’s Creed

“I am a United States Sailor.

I will support and defend the Constitution of the United States of America and I will obey the orders of those appointed over me.

I represent the fighting spirit of the Navy and those who have gone before me to defend freedom and democracy around the world.

I proudly serve my country’s Navy combat team with honor, courage and commitment.

I am committed to excellence and the fair treatment of all.”

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INSTRUCTIONS FOR TAKING THE COURSE

ASSIGNMENTS

The text pages that you are to study are listed at the beginning of each assignment. Study these pages carefully before attempting to answer the questions. Pay close attention to tables and illustrations and read the learning objectives. The learning objectives state what you should be able to do after studying the material. Answering the questions correctly helps you accomplish the objectives.

SELECTING YOUR ANSWERS

Read each question carefully, then select the BEST answer. You may refer freely to the text. The answers must be the result of your own work and decisions. You are prohibited from referring to or copying the answers of others and from giving answers to anyone else taking the course.

SUBMITTING YOUR ASSIGNMENTS

To have your assignments graded, you must be enrolled in the course with the Nonresident Training Course Administration Branch at the Naval Education and Training Professional Development and Technology Center (NETPDTC). Following enrollment, there are two ways of having your assignments graded: (1) use the Internet to submit your assignments as you complete them, or (2) send all the assignments at one time by mail to NETPDTC.

Grading on the Internet: Advantages to Internet grading are:

- you may submit your answers as soon as you complete an assignment, and
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Follow the instructions for marking your answers on the answer sheet. Be sure that blocks 1, 2, and 3 are filled in correctly. This information is necessary for your course to be properly processed and for you to receive credit for your work.

COMPLETION TIME

Courses must be completed within 12 months from the date of enrollment. This includes time required to resubmit failed assignments.

PASS/FAIL ASSIGNMENT PROCEDURES

If your overall course score is 3.2 or higher, you will pass the course and will not be required to resubmit assignments. Once your assignments have been graded you will receive course completion confirmation.

If you receive less than a 3.2 on any assignment and your overall course score is below 3.2, you will be given the opportunity to resubmit failed assignments. **You may resubmit failed assignments only once.** Internet students will receive notification when they have failed an assignment--they may then resubmit failed assignments on the web site. Internet students may view and print results for failed assignments from the web site. Students who submit by mail will receive a failing result letter and a new answer sheet for resubmission of each failed assignment.

COMPLETION CONFIRMATION

After successfully completing this course, you will receive a letter of completion.

ERRATA

Errata are used to correct minor errors or delete obsolete information in a course. Errata may also be used to provide instructions to the student. If a course has an errata, it will be included as the first page(s) after the front cover. Errata for all courses can be accessed and viewed/downloaded at:

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STUDENT FEEDBACK QUESTIONS

We value your suggestions, questions, and criticisms on our courses. If you would like to communicate with us regarding this course, we encourage you, if possible, to use e-mail. If you write or fax, please use a copy of the Student Comment form that follows this page.

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Student Comments

Course Title: *Mathematics, Pre-Calculus and Introduction to Probability*

NAVEDTRA: 14141 **Date**: ______________________

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NETPDTC 1550/41 (Rev 4-00

CHAPTER 1

STRAIGHT LINES

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Calculate the distance between two points.
2. Locate a point by dividing a line segment.
3. Define the inclination of a line and determine the line's slope.
4. Solve for the slopes of parallel and perpendicular lines.
5. Compute the angle between two lines.
6. Determine the equation of a straight line using the point-slope form, the slope-intercept form, and the normal form.
7. Determine the equations of parallel and perpendicular lines.
8. Calculate the distance from a point to a line.

INTRODUCTION

The study of straight lines provides an excellent introduction to analytic geometry. As its name implies, this branch of mathematics is concerned with geometrical relationships. However, in contrast to plane and solid geometry, the study of these relationships in analytic geometry is accomplished by algebraic analysis.

The invention of the rectangular coordinate system made algebraic analysis of geometrical relationships possible. Rene Descartes, a French mathematician, is credited with this invention; the coordinate system is often designated as the Cartesian coordinate system in his honor.

You should recall our study of the rectangular coordinate system in *Mathematics,* Volume 1, NAVEDTRA 10069-D1, in which we reviewed the following definitions and terms:

1. The values of x along, or parallel to, the X axis are *abscissas*. They are positive if measured to the right of the origin; they are negative if measured to the left of the origin. (See fig. 1-1.)

2. The values of y along, or parallel to, the Y axis are *ordinates*. They are positive if measured above the origin; they are negative if measured below the origin.

3. The abscissa and ordinate of a point are its *coordinates*.

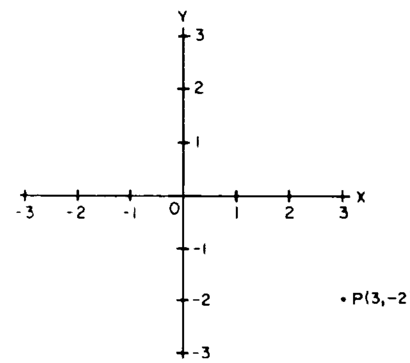


Figure 1-1.—Rectangular coordinate system.

Any point on the coordinate system is designated by naming its abscissa and ordinate. For example, the abscissa of point P (fig. 1-1) is 3 and the ordinate is -2. Therefore, the symbolic notation for P is

$$P(3, -2)$$

In using this symbol to designate a point, the abscissa is always written first, followed by a comma. The ordinate is written last. The general form of the symbol is

$$P(x,y)$$

DISTANCE BETWEEN TWO POINTS

The distance between two points, P_1 and P_2, can be expresssed in terms of their coordinates by using the *Pythagorean theorem.* From your study of *Mathematics,* Volume 1, you should recall that this theorem is stated as follows:

In a right triangle, the square of the length of the hypotenuse (longest side) is equal to the sum of the squares of the lengths of the other two sides.

Let the coordinates of P_1 be (x_1,y_1) and let those of P_2 be (x_2,y_2), as shown in figure 1-2. By the Pythagorean theorem,

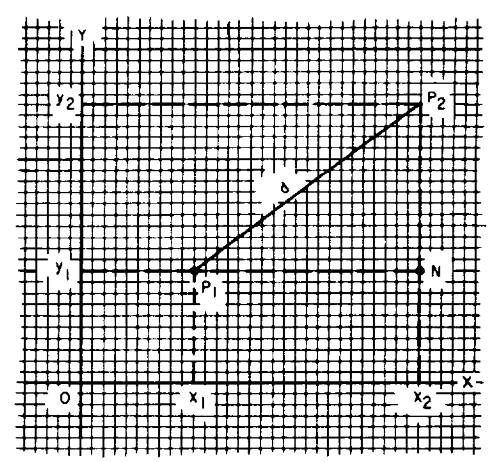


Figure 1-2.—Distance between two points.

$$d = \sqrt{(P_1N)^2 + (P_2N)^2}$$

where P_1N represents the distance between x_1 and x_2, P_2N represents the distance between y_1 and y_2, and d represents the distance from P_1 to P_2. We can express the length of P_1N in terms of x_1 and x_2 and the length of P_2N in terms of y_1 and y_2 as follows:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Although we have demonstrated the formula for the first quadrant only, it can be proven for all quadrants and all pairs of points.

EXAMPLE: In figure 1-2, $x_1 = 2$, $x_2 = 6$, $y_1 = 2$, and $y_2 = 5$. Find the length of d.

SOLUTION:

$$d = \sqrt{(6 - 2)^2 + (5 - 2)^2} \qquad \text{Step (1)}$$

$$= \sqrt{4^2 + 3^2} \qquad (2)$$

$$= \sqrt{16 + 9} \qquad (3)$$

$$= \sqrt{25} \qquad (4)$$

$$= 5 \qquad (5)$$

This result could have been foreseen by observing that triangle P_1NP_2 is a 3-4-5 triangle.

EXAMPLE: Find the distance between P_1 (4,6) and P_2 (10,4).

SOLUTION:

$$d = \sqrt{(10 - 4)^2 + (4 - 6)^2} \qquad \text{Step (1)}$$

$$= \sqrt{36 + 4} \qquad (2)$$

$$= \sqrt{40} \qquad (3)$$

$$= \sqrt{(4)(10)} \qquad (4)$$

$$= 2\sqrt{10} \qquad (5)$$

DIVISION OF A LINE SEGMENT

Many times you may need to find the coordinates of a point that is some known fraction of the distance between P_1 and P_2.

In figure 1-3, P is a point lying on the line joining P_1 and P_2 so that

$$\frac{P_1P}{P_1P_2} = k$$

If P should lie 1/4 of the way between P_1 and P_2, then k would equal 1/4.

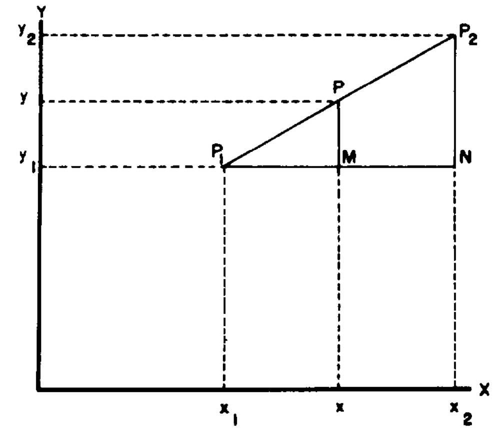


Figure 1-3.—Division of a line segment.

Triangles P_1MP and P_1NP_2 are similar. Therefore,

$$\frac{P_1M}{P_1N} = \frac{P_1P}{P_1P_2}$$

Since $\frac{P_1P}{P_1P_2}$ is the ratio that defines k, then

$$\frac{P_1M}{P_1N} = k$$

Therefore,

$$P_1M = k(P_1N)$$

Refer again to figure 1-3 and observe that P_1N is equal to $x_2 - x_1$. Likewise, P_1M is equal to $x - x_1$. When you replace P_1M and P_1N with their equivalents in terms of x, the preceding equation becomes

$$x - x_1 = k(x_2 - x_1) \qquad \text{Step (1)}$$

$$x = x_1 + k(x_2 - x_1) \qquad (2)$$

By similar reasoning,

$$y = y_1 + k(y_2 - y_1)$$

The x and y found as a result of the foregoing discussion are the coordinates of the desired point, whose distances from P_1 and P_2 are determined by the value of k.

EXAMPLE: Find the coordinates of a point 1/4 of the way from $P_1(2,3)$ to $P_2(4,1)$.

SOLUTION:

$$k = \frac{1}{4},\ x_2 - x_1 = 2,\ y_2 - y_1 = -2 \qquad \text{Step (1)}$$

$$x = 2 + \frac{1}{4}(2) = 2 + \frac{1}{2} = \frac{5}{2} \qquad (2)$$

$$y = 3 + \frac{1}{4}(-2) = 3 - \frac{1}{2} = \frac{5}{2} \qquad (3)$$

Therefore, point P is $\left(\frac{5}{2},\frac{5}{2}\right)$.

When the *midpoint* of a line segment is to be found, the value of (K) is 1/2. Therefore,

$$x = x_1 + \frac{1}{2}(x_2 - x_1)$$

$$= x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1$$

$$= \frac{1}{2}x_1 + \frac{1}{2}x_2$$

such that

$$x = \frac{1}{2}(x_1 + x_2)$$

By similar reasoning,

$$y = \frac{1}{2}(y_1 + y_2)$$

EXAMPLE: Find the midpoint of the line between $P_1(2,4)$ and $P_2(4,6)$.

SOLUTION:

$$k = \frac{1}{2} \qquad \text{Step (1)}$$

$$x = \frac{1}{2}(2 + 4) \qquad (2)$$

$$= 3 \qquad (3)$$

$$y = \frac{1}{2}(4 + 6) \qquad (4)$$

$$= 5 \qquad (5)$$

Therefore, the midpoint is (3,5).

INCLINATION AND SLOPE

The *angle of inclination*, denoted by the Greek letter alpha (α), as portrayed in figure 1-4, views A and B, is the angle formed by the line crossing the X axis and the positively directed portion of the X axis, such that $0° \leq \alpha < 180°$.

The *slope* of any line is equal to the tangent of its angle of inclination. Slope is denoted by the letter m. Therefore,

$$m = \tan \alpha$$

If the axes are in their conventional positions, a line, such as AB in figure 1-4, view A, that slopes upward and to the right will have a positive slope. A line, such as CD in figure 1-4, view B, that slopes downward and to the right will have a negative slope.

Since the tangent of α is the ratio of P_2M to P_1M, we can relate the slope of line AB to the points P_1 and P_2 as follows:

$$m = \tan \alpha = \frac{P_2M}{P_1\text{M}}$$

Designating the coordinates of P_1 as (x_1,y_1) and those of P_2 as (x_2,y_2), we recall that

$$P_2M = y_2 - y_1$$

$$P_1M = x_2 - x_1$$

so that

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

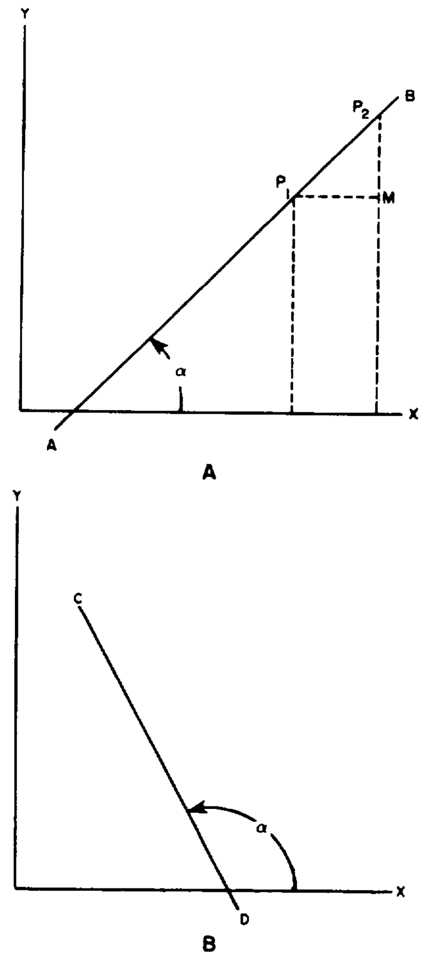


Figure 1-4.—Angles of inclination.

The quantities $(x_2 - x_1)$ and $(y_2 - y_1)$ represent changes that occur in the values of the x and y coordinates as a result of the change from P_2 to P_1 on line AB. The symbol used by mathematicians to represent an increment of change is the Greek letter delta (Δ). Therefore, Δx means "the change in x" and Δy means "the change in y." The amount of change in the x coordinate, as we change from P_2 to P_1, is $x_2 - x_1$. Therefore,

$$\Delta x = x_2 - x_1$$

and likewise,

$$\Delta y = y_2 - y_1$$

We use this notation to express the slope of line *AB* as follows:

$$m = \frac{\Delta y}{\Delta x}$$

EXAMPLE: Find the slope of the line connecting $P_2(7,6)$ and $P_1(-1,-4)$.

SOLUTION:

$$m = \frac{\Delta y}{\Delta x} \qquad \text{Step (1)}$$

$$\Delta y = y_2 - y_1 = 6 - (-4) = 10 \qquad (2)$$

$$\Delta x = x_2 - x_1 = 7 - (-1) = 8 \qquad (3)$$

$$m = \frac{10}{8} = \frac{5}{4} \qquad (4)$$

Note that the choice of labels for P_1 and P_2 is strictly arbitrary. In the previous example, if we had chosen the point (7,6) to be P_1 and the point $(-1,-4)$ to be P_2, the following results would have occurred:

$$m = \frac{\Delta y}{\Delta x} \qquad \text{Step (1)}$$

$$\Delta y = y_2 - y_1 = -4 - 6 = -10 \qquad (2)$$

$$\Delta x = x_2 - x_1 = -1 - 7 = -8 \qquad (3)$$

$$m = \frac{-10}{-8} = \frac{5}{4} \qquad (4)$$

Notice that this solution yields the same result as the last example.

A slope of 5/4 means that a point moving along this line would move vertically $+5$ units for every horizontal movement of $+4$ units. This result is consistent with our definition of positive slope; that is, sloping upward and to the right.

If line *AB* in figure 1-4, view A, was parallel to the *X* axis, y_1 and y_2 would be equal and the difference $(y_2 - y_1)$ would be 0. Therefore,

$$m = \frac{0}{x_2 - x_1} = 0$$

We conclude that the slope of a horizontal line is 0. We can also reach this conclusion by noting that angle α (fig. 1-4, view A) is $0°$ when the line is parallel to the X axis. Since the tangent of $0°$ is 0, then

$$m = \tan 0° = 0$$

The slope of a line that is parallel to the Y axis becomes meaningless. The tangent of the angle α increases indefinitely as α approaches $90°$. We sometimes say that $m \rightarrow \infty$ (m approaches infinity) when α approaches $90°$.

SLOPES OF PARALLEL AND PERPENDICULAR LINES

If two lines are *parallel*, their slopes must be equal. Each line will cut the X axis at the same angle α so that if

$$m_1 = \tan \alpha \text{ and } m_2 = \tan \alpha$$

then

$$m_1 = m_2$$

We conclude that two lines which are parallel have the same slope.

Suppose that two lines are perpendicular to each other, as lines L_1 and L_2 in figure 1-5. The slope and angle of inclination of L_1 are m_1 and α_1, respectively. The slope and angle of inclination of L_2 are m_2 and α_2, respectively. Then the following is true:

$$m_1 = \tan \alpha_1$$

$$m_2 = \tan \alpha_2$$

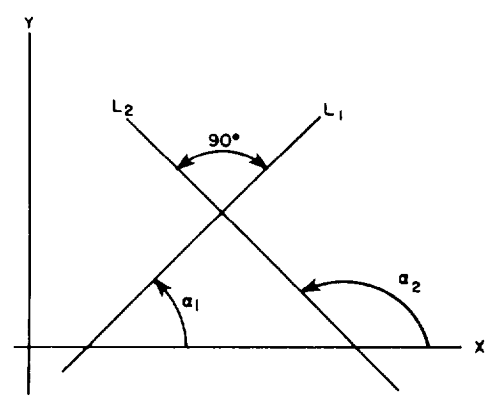


Figure 1-5.—Slopes of perpendicular lines.

Although not shown here, the fact that α_2 (fig. 1-5) is equal to α_1 plus $90°$ can be proven geometrically. Because of this relationship

$$\begin{aligned}\tan \alpha_2 &= \tan (\alpha_1 + 90°) \\ &= -\cot \alpha_1 \\ &= -\frac{1}{\tan \alpha_1}\end{aligned}$$

Replacing tan α_1 and tan α_2 by their equivalents in terms of slope, we have

$$m_2 = -\frac{1}{m_1}$$

We conclude that if two lines are *perpendicular*, the slope of one is the negative reciprocal of the slope of the other. Conversely, if the slopes of two lines are negative reciprocals of each other, the lines are perpendicular.

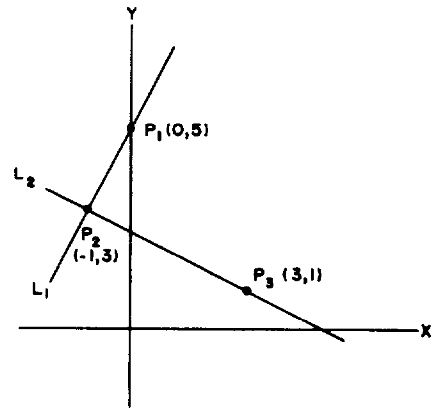


Figure 1-6.—Proving lines perpendicular.

EXAMPLE: In figure 1-6 show that line L_1 is perpendicular to line L_2. Line L_1 passes through points $P_1(0,5)$ and $P_2(-1,3)$. Line L_2 passes through points $P_2(-1,3)$ and $P_3(3,1)$.

SOLUTION: Let m_1 and m_2 represent the slope of lines L_1 and L_2, respectively. Then we have

$$m_1 = \frac{3-5}{-1-0} = \frac{-2}{-1} = 2$$

$$m_2 = \frac{1-3}{3-(-1)} = \frac{-2}{4} = -\frac{1}{2}$$

Since their slopes are negative reciprocals of each other, the lines are perpendicular.

PRACTICE PROBLEMS:

1. Find the distance between $P_1(5,3)$ and $P_2(6,7)$.

2. Find the distance between $P_1(1/2,1)$ and $P_2(3/2,5/3)$.

3. Find the midpoint of the line connecting $P_1(5,2)$ and $P_2(-1,-3)$.

4. Find the slope of the line joining $P_1(-2,-5)$ and $P_2(2,5)$.

5. Find the slope of the line perpendicular to the line joining $P_1(-3,6)$ and $P_2(-5,-2)$.

ANSWERS:

1. $\sqrt{17}$

2. $\frac{\sqrt{13}}{3}$

3. $(2, -\frac{1}{2})$

4. $\frac{5}{2}$

5. $\frac{-1}{4}$

ANGLE BETWEEN TWO LINES

When two lines intersect, the angle between them is defined as the angle through which one of the lines must be rotated to make it coincide with the other line. For example, the angle ϕ (the Greek letter phi) in figure 1-7 is the acute angle between lines L_1 and L_2.

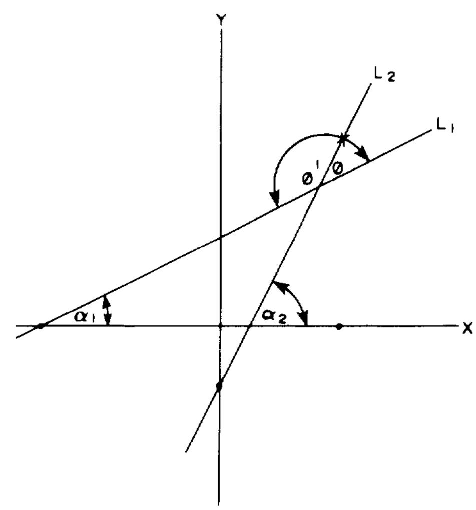


Figure 1-7.—Angle between two lines.

Referring to figure 1-7,

$$\alpha_2 = \alpha_1 + \phi$$

$$\therefore \phi = \alpha_2 - \alpha_1$$

We will determine the value of ϕ directly from the slopes of lines L_1 and L_2, as follows:

$$\tan \phi = \tan (\alpha_2 - \alpha_1)$$

$$= \frac{\tan \alpha_2 - \tan \alpha_1}{1 + \tan \alpha_1 \tan \alpha_2}$$

We obtain this result by using the trigonometric identity for the tangent of the difference between two angles. Trigonometric identities are discussed in chapter 6 of *Mathematics*, Volume 2-A, NAVEDTRA 10062.

Recalling that the tangent of the angle of inclination is the slope of the line, we have

$$\tan \alpha_1 = m_1 \text{ (the slope of } L_1)$$

$$\tan \alpha_2 = m_2 \text{ (the slope of } L_2)$$

such that

$$m_2 > m_1$$

Substituting these expressions in the tangent formula derived in the above discussion, we have

$$\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}$$

EXAMPLE: Referring to figure 1-7, find the acute angle between the two lines that have $m_1 = \frac{1}{2}$ and $m_2 = 2$ for their slopes.

SOLUTION:

$$\tan \phi = \frac{2 - \frac{1}{2}}{1 + \left(\frac{1}{2}\right)(2)}$$

$$= \frac{3}{4}$$

$$= .75$$

such that

$$\phi = \arctan (.75)$$

or, referring to Appendix I,

$$\phi = 36^\circ\ 52'$$

NOTE: To find the obtuse angle between lines L_1 and L_2, just subtract the acute angle between L_1 and L_2 from 180°. Referring to figure 1-7,

$$\phi' = 180^\circ - \phi$$

If the obtuse angle in the previous example was to be found, then

$$\begin{aligned}\phi' &= 180° - 36° 52' \\ &= 143° 8'\end{aligned}$$

If one of the lines was parallel to the Y axis, its slope would be infinite. This would render the slope formula for tan ϕ useless, because an infinite value in both the numerator and denominator of the fraction $\frac{m_2 - m_1}{1 + m_1 m_2}$ produces an indeterminate form. However, if only one of the lines is known to be parallel to the Y axis, the tangent of ϕ may be expressed by another method.

Suppose that L_2 (fig 1-7) was parallel to the Y axis. Then we would have

$$\alpha_2 = 90°$$

If L_1 has a positive slope, then the acute angle between L_1 and L_2 would be found by

$$\begin{aligned}\phi &= 90° - \alpha_1 \\ \tan \phi &= \cot \alpha_1 \\ &= \frac{1}{m_1}\end{aligned}$$

If L_1 has a negative slope, then

$$\begin{aligned}\phi &= \alpha_1 - 90° \\ &= -(90° - \alpha_1) \\ \tan \phi &= -\cot \alpha_1 \\ &= -\frac{1}{m_1}\end{aligned}$$

As before, the obtuse angle can be found by using

$$\phi' = 180° - \phi$$

PRACTICE PROBLEMS:

1. Find the acute angle between the two lines that have $m_1 = 3$ and $m_2 = 7$ for their slopes.
2. Find the acute angle between two lines whose slopes are $m_1 = 0$ and $m_2 = 1$. ($m_1 = 0$ signifies that line L_1 is horizontal and the formula still holds.)

3. Find the acute angle between the Y axis and a line with a slope of $m = -8$.

4. Find the obtuse angle between the X axis and a line with a slope of $m = -8$.

ANSWERS:

1. $10^\circ\ 18'$

2. 45°

3. $7^\circ\ 7'$

4. $97^\circ\ 7'$

EQUATION OF A STRAIGHT LINE

In *Mathematics*, Volume 1, equations such as

$$2x + y = 6$$

are designated as linear equations, and their graphs are shown to be straight lines. The purpose of this discussion is to study the relationship of slope to the equation of a straight line.

POINT-SLOPE FORM

Suppose that we want to find the equation of a straight line that passes through a known point and has a known slope. Let (x,y) represent the coordinates of any point on the line, and let (x_1,y_1) represent the coordinates of the known point. The slope is represented by m.

Recalling the formula defining slope in terms of the coordinates of two points, we have

$$m = \frac{y - y_1}{x - x_1}$$

$$\therefore y - y_1 = m(x - x_1)$$

EXAMPLE: Find the equation of a line passing through the point (2,3) and having a slope of 3.

SOLUTION:

$$x_1 = 2 \text{ and } y_1 = 3 \qquad \text{Step (1)}$$

$$y - y_1 = m(x - x_1) \qquad (2)$$

$$\therefore y - 3 = 3(x - 2) \qquad (3)$$

$$y - 3 = 3x - 6 \qquad (4)$$

$$\text{or } y - 3x = -3$$

The *point-slope* form may be used to find the equation of a line through two known points. The values of x_1, x_2, y_1, and y_2 are first used to find the slope of the line; then either known point is used with the slope in the point-slope form.

EXAMPLE: Find the equation of the line through the points $(-3,4)$ and $(4,-2)$.

SOLUTION:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \qquad \text{Step (1)}$$

$$= \frac{-2 - 4}{4 + 3} = \frac{-6}{7} \qquad (2)$$

Letting (x,y) represent any point on the line and using $(-3,4)$ as a known point, we have

$$y - 4 = \frac{-6}{7}[x - (-3)] \qquad (3)$$

$$7(y - 4) = -6(x + 3) \qquad (4)$$

$$7y - 28 = -6x - 18 \qquad (5)$$

$$7y + 6x = 10 \qquad (6)$$

Using $(4,-2)$ as the known point will also give $7y + 6x = 10$ as the linear equation.

SLOPE-INTERCEPT FORM

Any line that is not parallel to the Y axis intersects the Y axis at some point. The x coordinate of the point of intersection is 0, because the Y axis is vertical and passes through the origin. Let the y coordinate of the point of intersection be represented by b. Then the point of intersection is $(0,b)$, as shown in figure 1-8. The y coordinate, b, is called the y *intercept*.

The slope of the line in figure 1-8 is $\frac{\Delta y}{\Delta x}$. The value of Δy in this expression is $y - b$, where y represents the y coordinate of any point on the line. The value of Δx is $x - 0 = x$, so

$$m = \frac{\Delta y}{\Delta x} = \frac{y - b}{x}$$

$$mx = y - b$$

$$y = mx + b$$

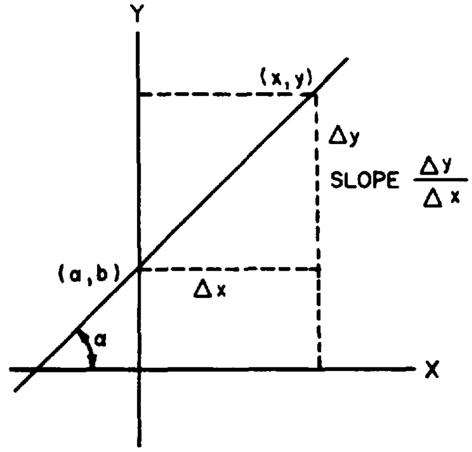


Figure 1-8.—Slope-intercept form.

This is the standard *slope-intercept* form of a straight line.

EXAMPLE: Find the equation of a line that intersects the Y axis at the point (0,3) and has a slope of 5/3.

SOLUTION:

$$y = mx + b \qquad \text{Step (1)}$$

$$y = \frac{5}{3}x + 3 \qquad (2)$$

$$3y = 5x + 9 \qquad (3)$$

$$\text{or } 3y - 5x = 9$$

PRACTICE PROBLEMS:

Write equations for lines having points and slopes as follows:

1. $P(3,5)$, $m = -2$

2. $P(-2,-1)$, $m = \frac{1}{3}$

3. $P_1(2,2)$ and $P_2(-4,-1)$

4. y intercept $= 2$, $m = 3$

ANSWERS:

1. $y = -2x + 11$ or $y + 2x = 11$
2. $3y = x - 1$ or $3y - x = -1$
3. $2y = x + 2$ or $2y - x = 2$
4. $y = 3x + 2$ or $y - 3x = 2$

NORMAL FORM

Methods for determining the equation of a line usually depend upon some knowledge of a point or points on the line. Let's now consider a method that does not require advance knowledge concerning any of the line's points. All that is known about the line is its perpendicular distance from the origin and the angle between the perpendicular and the X axis, where the angle is measured counterclockwise from the positive side of the X axis.

In figure 1-9, line AB is a distance p away from the origin, and line OM forms an angle θ (the Greek letter theta) with the X axis. We select any point $P(x,y)$ on line AB and develop the

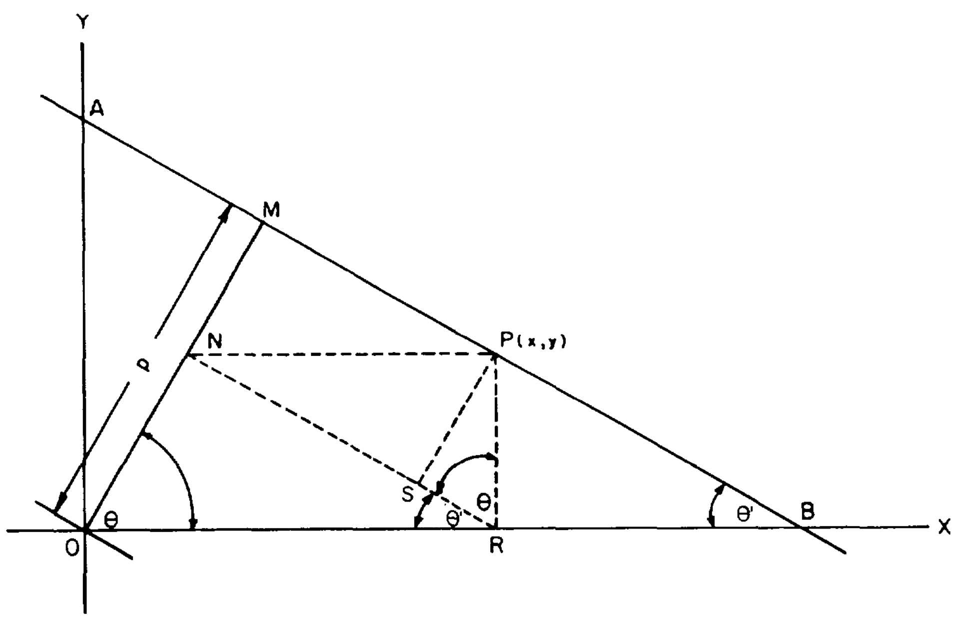


Figure 1-9.—Normal form.

equation of line AB in terms of the x and y of P. Since P represents ANY point on the line, the x and y of the equation will represent EVERY point on the line and therefore will represent the line itself.

PR is constructed perpendicular to OB at point R. NR is drawn parallel to AB, and PN is parallel to OB. PS is perpendicular to NR and to AB. A right angle is formed by angles NRO and PRN. Triangles ONR and OMB are similar right triangles. Therefore, angles NRO and MBO are equal and are designated as θ'. Since $\theta + \theta' = 90°$ in triangle OMB and angle NRO is equal to θ', then angle PRN equals θ. Finally, the x distance of point P is equal to OR, and the y distance of P is equal to PR.

To relate the distance p to x and y, we reason as follows:

$$ON = (OR)\ (\cos\theta)$$

$$= x \cos\theta$$

$$PS = (PR)\ (\sin\theta)$$

$$= y \sin\theta$$

$$OM = ON + PS$$

$$p = ON + PS$$

$$p = x\cos\theta + y\sin\theta$$

This final equation is the *normal form*. The word "normal" in this usage refers to the perpendicular relationship between OM and AB. "Normal" frequently means "perpendicular" in mathematical and scientific usage. The distance p is always considered to be positive, and θ is any angle between 0° and 360°.

EXAMPLE: Find the equation of the line that is 5 units away from the origin, if the perpendicular from the line to the origin forms an angle of 30° from the positive side of the X axis.

SOLUTION:

$$p = 5;\ \theta = 30° \qquad \text{Step (1)}$$

$$p = x\cos\theta + y\sin\theta \qquad (2)$$

$$5 = x\cos 30° + y\sin 30° \qquad (3)$$

$$5 = x\left(\frac{\sqrt{3}}{2}\right) + y\left(\frac{1}{2}\right) \qquad (4)$$

$$10 = x\sqrt{3} + y \qquad (5)$$

PARALLEL AND PERPENDICULAR LINES

The general equation of a straight line is often written with capital letters for coefficients, as follows:

$$Ax + By + C = 0$$

These literal coefficients, as they are called, represent the numerical coefficients encountered in a typical linear equation.

Suppose we are given two equations that are duplicates except for the constant term, as follows:

$$Ax + By + C = 0$$

$$Ax + By + D = 0$$

By placing these two equations in slope-intercept form, we can show that their slopes are equal, as follows:

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{D}{B}\right)$$

Thus, the slope of each line is $-A/B$.

Since lines having equal slopes are parallel, we reach the following conclusion: *In any two linear equations, if the coefficients of the x and y terms are identical in value and sign, then the lines represented by these equations are parallel.*

EXAMPLE: Write the equation of a line parallel to $3x - y - 2 = 0$ and passing through the point (5,2).

SOLUTION: The coefficients of x and y in the desired equation are the same as those in the given equation. Therefore, the equation is

$$3x - y + D = 0$$

Since the line passes through (5,2), the values $x = 5$ and $y = 2$ must satisfy the equation. Substituting these, we have

$$3(5) - (2) + D = 0$$

$$D = -13$$

Thus, the required equation is

$$3x - y - 13 = 0$$

A situation similar to that prevailing with parallel lines involves perpendicular lines. For example, consider the equations

$$Ax + By + C = 0$$

$$Bx - Ay + D = 0$$

Transposing these equations into the slope-intercept form, we have

$$y = \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right)$$

$$y = \left(\frac{B}{A}\right)x + \left(\frac{D}{A}\right)$$

Since the slopes of these two lines are negative reciprocals, the lines are perpendicular.

The conclusion derived from the foregoing discussion is as follows: *If a line is to be perpendicular to a given line, the coefficients of x and y in the required equation are found by interchanging the coefficients of x and y in the given equation and changing the sign of one of them.*

EXAMPLE: Write the equation of a line perpendicular to the line $x + 3y + 3 = 0$ and having a y intercept of 5.

SOLUTION: The required equation is

$$3x - y + D = 0$$

Notice the interchange of coefficients and the change of sign. At the point where the line crosses the Y axis, the value of x is 0 and the value of y is 5. Therefore, the equation is

$$3(0) - (5) + D = 0$$

$$D = 5$$

The required equation is

$$3x - y + 5 = 0$$

PRACTICE PROBLEMS:

1. Find the equation of the line whose perpendicular forms an angle of 135° from the positive side of the X axis and whose perpendicular distance is $\sqrt{2}$ units from the origin.

Find the equations of the following lines:

2. Through (1,1) and parallel to $5x - 3y = 9$.

3. Through $(-3,2)$ and perpendicular to $x + y = 5$.

ANSWERS:

1. $2 = -x + y$

2. $5x - 3y = 2$

3. $x - y = -5$

DISTANCE FROM A POINT TO A LINE

We must often express the distance from a point to a line in terms of the coefficients in the equation of the line. To do this, we compare the two forms of the equation of a straight line, as follows:

General equation: $Ax + By + C = 0$

Normal form: $x \cos \theta + y \sin \theta - p = 0$

The general equation and the normal form represent the same straight line. Therefore, A (the coefficient of x in the general form) is proportional to $\cos \theta$ (the coefficient of x in the normal form). By similar reasoning, B is proportional to $\sin \theta$, and C is proportional to $-p$. Recalling that quantities proportional to each other form ratios involving a constant of proportionality, let k be this constant. Thus, we have

$$\frac{\cos\theta}{A} = k$$

$$\frac{\sin\theta}{B} = k$$

$$\cos\theta = kA$$

$$\sin\theta = kB$$

Squaring both sides of these two expressions and then adding, we have

$$\cos^2\theta + \sin^2\theta = k^2(A^2 + B^2)$$

$$\therefore 1 = k^2(A^2 + B^2)$$

$$k^2 = \frac{1}{A^2 + B^2}$$

$$k = \frac{1}{\pm\sqrt{A^2 + B^2}}$$

The coefficients in the normal form, expressed in terms of A, B, and C, are as follows:

$$\cos\theta = \frac{A}{\pm\sqrt{A^2 + B^2}}$$

$$\sin\theta = \frac{B}{\pm\sqrt{A^2 + B^2}}$$

$$-p = \frac{C}{\pm\sqrt{A^2 + B^2}}$$

The sign of $\sqrt{A^2 + B^2}$ is chosen so as to make p (a distance) always positive.

The conversion formulas developed in the foregoing discussion are used in finding the distance from a point to a line. Let p represent the distance of line LK from the origin. (See fig. 1-10.) To find d, the distance from point P_1 to line LK, we construct a line

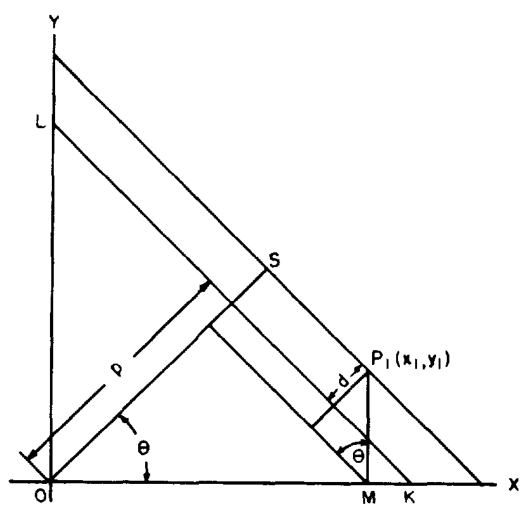


Figure 1-10.—Distance from a point to a line.

through P_1 parallel to LK. The distance of this line from the origin is OS, and the difference between OS and p is d.

We obtain an expression for d, based on the coordinates of P_1, as follows:

$$OS = x_1 \cos\theta + y_1 \sin\theta$$

and

$$d = OS - p$$

$$= x_1 \cos\theta + y_1 \sin\theta - p$$

Returning to the expressions for sin θ, cos θ, and $-p$ in terms of A, B, and C (the coefficients in the general equation), we have

$$d = x_1\left(\frac{A}{\pm\sqrt{A^2 + B^2}}\right) + y_1\left(\frac{B}{\pm\sqrt{A^2 + B^2}}\right) + \frac{C}{\pm\sqrt{A^2 + B^2}}$$

In the formula for d, the denominator in each of the expressions is the same. Therefore, we may combine terms as follows:

$$d = \left|\frac{x_1 A + y_1 B + C}{\sqrt{A^2 + B^2}}\right|$$

We use the absolute value, since d is a distance, and thus avoid any confusion arising from the $\pm$ radical.

Note that the absolute value, $|\ \ |$, of a number is defined as follows:

$$|b| = b \text{ for } b \geq 0$$

and

$$|b| = -b \text{ for } b < 0$$

That is, for the positive number 2,

$$|2| = 2$$

For the negative number -2,

$$|-2| = -(-2) = 2$$

The absolute value of $\frac{6-12}{3}$ is

$$\left|\frac{6-12}{3}\right| = \left|\frac{-6}{3}\right| = \left|-2\right| = -(-2) = 2$$

EXAMPLE: Find the distance from the point (2,1) to the line $4x + 2y + 7 = 0$.

SOLUTION:

$$d = \left|\frac{(4)(2) + (2)(1) + 7}{\sqrt{4^2 + 2^2}}\right| \qquad \text{Step (1)}$$

$$= \frac{8 + 2 + 7}{\sqrt{20}} \qquad (2)$$

$$= \frac{17}{2\sqrt{5}} \qquad (3)$$

$$= \frac{17\sqrt{5}}{10} \qquad (4)$$

PRACTICE PROBLEMS:

In each of the following problems, find the distance from the point to the line:

1. $(5,2)$, $3x - y + 6 = 0$
2. $(-2,5)$, $3x + 4y - 9 = 0$

ANSWERS:

1. $\frac{19\sqrt{10}}{10}$
2. 1

SUMMARY

The following are the major topics covered in this chapter:

1. **Distance between two points:**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

   where (x_1,y_1) and (x_2,y_2) are given points on a line.

2. **Division of a line segment:**

$$x = x_1 + k(x_2 - x_1)$$

$$y = y_1 + k(y_2 - y_1)$$

   where k is the desired proportion of the distance between points (x_1,y_1) and (x_2,y_2) and (x,y) is the desired point.

3. **Midpoint of a line segment:**

$$x = \frac{1}{2}(x_1 + x_2)$$

$$y = \frac{1}{2}(y_1 + y_2)$$

4. **Inclination**: The *angle of inclination* is the angle the line crossing the X axis makes with the positively directed portion of the X axis, such that $0° \leq \alpha < 180°$.

5. **Slope**:

$$m = \tan \alpha = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$$

   where Δ means "change in."

   The slope of a horizontal line is zero.

   The slope of a vertical line is meaningless.

6. **Slopes of parallel lines**: Slopes are equal or

$$m_1 = m_2$$

   where m_1 and m_2 are the slopes of the lines L_1 and L_2, respectively.

7. **Slopes of perpendicular lines**: Slopes are negative reciprocals or

$$m_2 = -\frac{1}{m_1}$$

8. **Acute angle between two lines:**

$$\phi = \arctan\left(\frac{m_2 - m_1}{1 + m_1 m_2}\right)$$

However, if one line, L_2, is parallel to the Y axis and the other, L_1, has a positive slope, then

$$\phi = \arctan\left(\frac{1}{m_1}\right)$$

If L_2 is parallel to the Y axis and L_1 has a negative slope, then

$$\phi = \arctan\left(-\frac{1}{m_1}\right)$$

9. **Obtuse angle between two lines:**

$$\phi' = 180° - \phi$$

where ϕ is the acute angle between the two lines.

10. **Point-slope form of a straight line:**

$$y - y_1 = m(x - x_1)$$

11. **Slope-intercept form of a straight line:**

$$y = mx + b$$

where b is the y intercept.

12. **Normal form of a straight line:**

$$p = x \cos\theta + y \sin\theta$$

where p is the line's perpendicular distance from the origin and θ is the angle between the perpendicular and the X axis.

13. **Parallel lines:**

In any two linear equations, if the coefficients of the x and y terms are identical in value and sign, then the lines represented by these equations are parallel; that is,

$$Ax + By + C = 0 \text{ and } Ax + By + D = 0$$

are parallel lines.

14. **Perpendicular lines:**

If a line is to be perpendicular to a given line, the coefficients of x and y in the required equations are found by interchanging the coefficients of x and y in the given equation and changing the sign of one of them; that is,

$$Ax + By + C = 0 \text{ and } Bx - Ay + D = 0$$

are perpendicular lines.

15. **Distance from a point to a line:**

$$d = \left| \frac{x_1A + y_1B + C}{\sqrt{A^2 + B^2}} \right|$$

where A, B, and C are the coefficients of the general equation of a line $Ax + By + C = 0$ and (x_1, y_1) are the coordinates of the point.

ADDITIONAL PRACTICE PROBLEMS

1. Find the distance between $P_1(-3,-2)$ and $P_2(-7,1)$.
2. Find the distance between $P_1(-3/4,-2)$ and $P_2(1,-1/2)$.
3. Find the coordinates of a point 1/5 of the way from $P_1(-2,0)$ to $P_2(3,-5)$.
4. Find the midpoint of the line between $P_1(-8/3,4/5)$ and $P_2(-4/3,6/5)$.
5. Find the slope of the line joining $P_1(4,6)$ and $P_2(-4,6)$.
6. Find the slope of the line parallel to the line joining $P_1(7,4)$ and $P_2(4,7)$.
7. Find the slope of the line perpendicular to the line joining $P_1(8,1)$ and $P_2(2,4)$.
8. Find the obtuse angle between the two lines which have $m_1 = 7$ and $m_2 = -3$ for slopes.
9. Find the obtuse angle between the Y axis and a line with a slope of $m = -1/4$.
10. Find the equation of the line through the points $(-6,5)$ and $(6,5)$.
11. Find the equation of the line whose y intercept is $(0,0)$ and whose slope is 4.
12. Find the slope and y intercept of the line whose equation is $4y + 8x = 7$.
13. Find the equation of the line that is 3/2 units away from the origin, if the perpendicular from the line to the origin forms an angle of 210° from the positive side of the X axis.
14. Find the equation of the line through (2,3) and perpendicular to $3x - 2y = 7$.
15. Find the equation of the line through (2,3) and parallel to $3x - 2y = 7$.
16. Find the distance from the point $(3,-5)$ to the line $2x + y + 4 = 0$.
17. Find the distance from the point $(3,-4)$ to the line $4x + 3y = 10$.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 5
2. $\sqrt{85}/4$
3. $(-1,-1)$
4. $(-2,1)$
5. 0
6. -1
7. 2
8. $153° 26'$
9. $104° 2'$
10. $y = 5$
11. $y = 4x$
12. $-2, 7/4$
13. $3 = -\sqrt{3}x - y$
14. $2x + 3y = 13$
15. $3x - 2y = 0$
16. $\sqrt{5}$
17. 2

CHAPTER 2

CONIC SECTIONS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Determine the equation of a curve using the locus of the equation.

2. Determine the equation and properties of a circle, a parabola, an ellipse, and a hyperbola.

3. Transform polar coordinates to Cartesian coordinates and vice-versa.

INTRODUCTION

This chapter is a continuation of the study of analytic geometry. The figures presented in this chapter are plane figures, which are included in the general class of conic sections or simply "conics."

Conic sections are so named because they are all plane sections of a right circular cone. A circle is formed when a cone is cut perpendicular to its axis. An ellipse is produced when the cone is cut obliquely to the axis and the surface. A hyperbola results when the cone is intersected by a plane parallel to the axis, and a parabola results when the intersecting plane is parallel to an element of the surface. These are illustrated in figure 2-1.

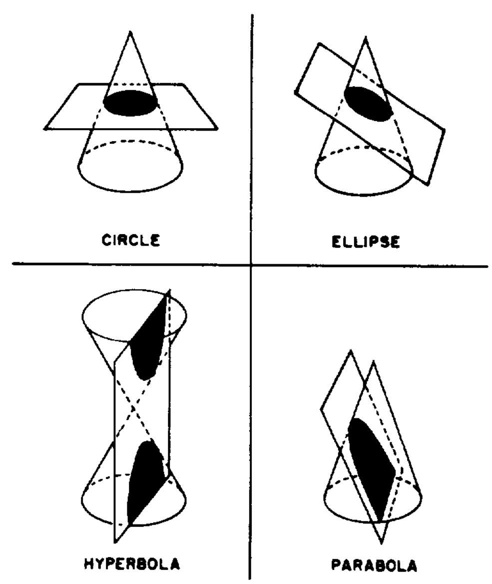


Figure 2-1.—Conic sections.

When such a curve is plotted on a coordinate system, it may be defined as follows:

A *conic section* is the locus of all points in a plane whose distance from a fixed point is a constant ratio to its distance from a fixed line. The fixed point is the *focus*, and the fixed line is the *directrix*.

The ratio referred to in the definition is called the *eccentricity* (e). If the eccentricity is greater than 0 and less than 1, the curve is an ellipse. If e is greater than 1, the curve is a hyperbola. If e is equal to 1, the curve is a parabola. A circle is a special case having an eccentricity equal to 0. It is actually a limiting case of an ellipse in which the eccentricity approaches 0. Thus, if

$0 < e < 1$, it is an *ellipse*;

$e > 1$, it is a *hyperbola*;

$e = 1$, it is a *parabola*;

$e = 0$, it is a *circle*.

The eccentricity, focus, and directrix are used in the algebraic analysis of conic sections and their corresponding equations. The concept of the locus of an equation also enters into analytic geometry; this concept is discussed before the individual conic sections are presented.

THE LOCUS OF AN EQUATION

In chapter 1 of this course, methods for analysis of linear equations are presented. If a group of x and y values [or ordered pairs, $P(x,y)$] that satisfy a given linear equation are plotted on a coordinate system, the resulting graph is a straight line.

When higher-ordered equations such as

$$x^2 + y^2 = 1 \text{ or } y = \sqrt{2x + 3}$$

are encountered, the resulting graph is not a straight line. However, the points whose coordinates satisfy most of the equations in x and y are normally not scattered in a random field. If the values are plotted, they will seem to follow a line or curve (or a combination of lines and curves). In many texts the plot of an equation is called a curve, even when it is a straight line. This curve is called the *locus* of the equation. The *locus* of an equation is a curve containing those points, and only those points, whose coordinates satisfy the equation.

At times the curve may be defined by a set of conditions rather than by an equation, though an equation may be derived from the given conditions. Then the curve in question would be the locus of all points that fit the conditions. For instance a circle may be said to be the locus of all points in a plane that is a fixed distance from a fixed point. A straight line may be defined as the locus of all points in a plane equidistant from two fixed points. The method of expressing a set of conditions in analytical form gives an equation. Let us draw up a set of conditions and translate them into an equation.

EXAMPLE: What is the equation of the curve that is the locus of all points equidistant from the two points (5,3) and (2,1)?

SOLUTION: First, as shown in figure 2-2, choose some point having coordinates (x,y). Recall from chapter 1 of this course that the distance between this point and (2,1) is given by

$$\sqrt{(x-2)^2+(y-1)^2}$$

The distance between point (x,y) and (5,3) is given by

$$\sqrt{(x-5)^2+(y-3)^2}$$

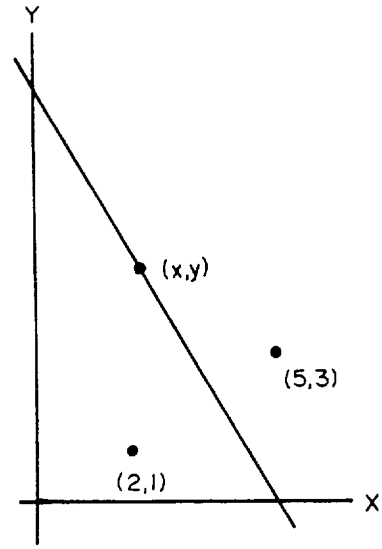


Figure 2-2.—Locus of points equidistant from two given points.

Equating these distances, since the point is to be equidistant from the two given points, we have

$$\sqrt{(x-2)^2+(y-1)^2}=\sqrt{(x-5)^2+(y-3)^2}$$

Squaring both sides, we have

$$(x-2)^2+(y-1)^2=(x-5)^2+(y-3)^2$$

Expanding, we have

$$x^2-4x+4+y^2-2y+1$$

$$=x^2-10x+25+y^2-6y+9$$

Canceling and collecting terms, we see that

$$4y+5=-6x+34$$

$$4y=-6x+29$$

$$y=-1.5x+7.25$$

This is the equation of a straight line with a slope of minus 1.5 and a y intercept of $+7.25$.

EXAMPLE: Find the equation of the curve that is the locus of all points equidistant from the line $x = -3$ and the point (3,0).

SOLUTION: As shown in figure 2-3, the distance from the point (x,y) on the curve to the line $x = -3$ is

$$\sqrt{[x - (-3)]^2 + (y - y)^2} = \sqrt{(x + 3)^2}$$

The distance from the point (x,y) to the point (3,0) is

$$\sqrt{(x - 3)^2 + (y - 0)^2}$$

Equating the two distances yields

$$\sqrt{(x + 3)^2} = \sqrt{(x - 3)^2 + y^2}$$

Squaring and expanding both sides yields

$$x^2 + 6x + 9 = x^2 - 6x + 9 + y^2$$

Canceling and collecting terms yields

$$y^2 = 12x$$

which is the equation of a parabola.

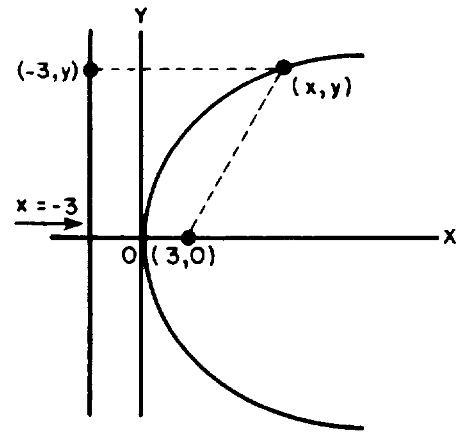


Figure 2-3.—Parabola.

EXAMPLE: What is the equation of the curve that is the locus of all points in which the ratio of its distance from the point (3,0) to its distance from the line $x = 25/3$ is equal to 3/5? Refer to figure 2-4.

SOLUTION: The distance from the point (x,y) to the point (3,0) is given by

$$d_1 = \sqrt{(x - 3)^2 + (y - 0)^2}$$

The distance from the point (x,y) to the line $x = 25/3$ is

$$d_2 = \frac{25}{3} - x$$

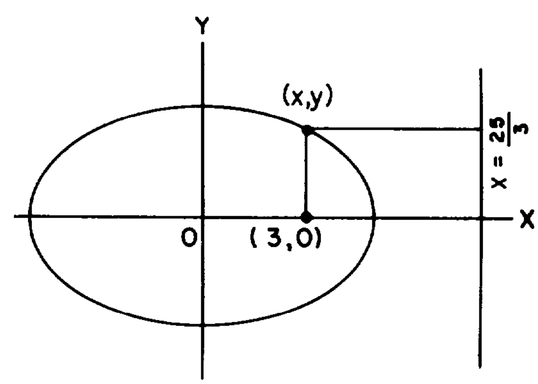


Figure 2-4.—Ellipse.

Since

$$\frac{d_1}{d_2} = \frac{3}{5} \text{ or } d_1 = \frac{3}{5}d_2$$

then

$$\sqrt{(x-3)^2 + y^2} = \frac{3}{5}\left(\frac{25}{3} - x\right)$$

Squaring both sides and expanding, we have

$$x^2 - 6x + 9 + y^2 = \frac{9}{25}\left(x^2 - \frac{50}{3}x + \frac{625}{9}\right)$$

$$x^2 - 6x + 9 + y^2 = \frac{9}{25}x^2 - 6x + 25$$

Collecting terms and transposing, we see that

$$\frac{16}{25}x^2 + y^2 = 16$$

Dividing both sides by 16, we have

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

This is the equation of an ellipse.

PRACTICE PROBLEMS:

Find the equation of the curve that is the locus of all points equidistant from the following:

1. The points (0,0) and (5,4).

2. The points (3, −2) and (−3,2).

3. The line $x = -4$ and the point (3,4).

4. The point (4,5) and the line $y = 5x - 4$.

   HINT: Use the standard distance formula to find the distance from the point $P(x,y)$ and the point $P(4,5)$. Then use the formula for finding the distance from a point to a line, given in chapter 1 of this course, to find the distance from $P(x,y)$ to the given line. Put the equation of the line in the form $Ax + By + C = 0$.

ANSWERS:

1. $10x + 8y - 41 = 0$ or $y = -1.25x + \frac{41}{8}$

2. $2y = 3x$ or $y = \frac{3}{2}x$

3. $y^2 - 8y = 14x - 9$ or $(y - 4)^2 = 14x + 7$

4. $x^2 + 10xy + 25y^2 - 168x - 268y + 1050 = 0$

THE CIRCLE

A *circle* is the locus of all points, in a plane that is a fixed distance from a fixed point, called the *center.*

The fixed distance spoken of here is the *radius* of the circle.

The equation of a circle with its center at the origin (fig. 2-5) is

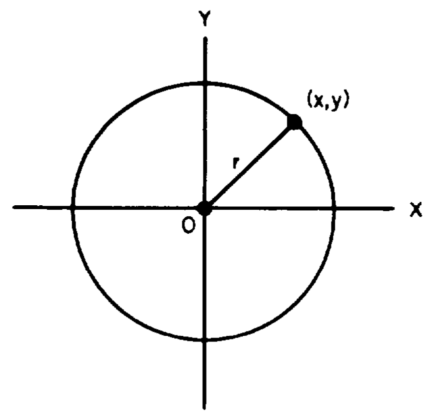


Figure 2-5.—Circle with center at the origin.

$$\sqrt{(x - 0)^2 + (y - 0)^2} = r$$

where (x,y) is a point on the circle and r is the radius (r replaces d in the standard distance formula). Then

$$\sqrt{x^2 + y^2} = r$$

or

$$x^2 + y^2 = r^2 \qquad (2.1)$$

If the center of a circle, figure 2-6, is at some point where $x = h$, $y = k$, then the distance of the point (x,y) from the center will be constant and equal to

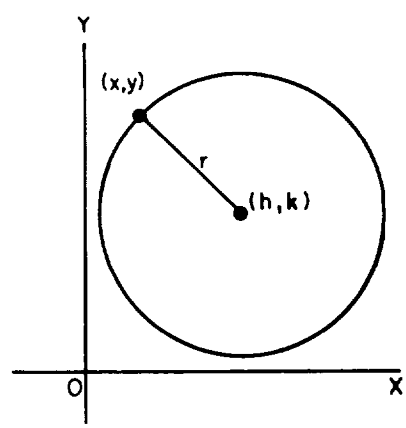


Figure 2-6.—Circle with center at (h,k).

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

or

$$(x - h)^2 + (y - k)^2 = r^2 \qquad (2.2)$$

Equations (2.1) and (2.2) are the standard forms for the equation of a circle. Equation (2.1) is merely a special case of equation (2.2) in which h and k are equal to zero.

The equation of a circle may also be expressed in the general form:

$$x^2 + y^2 + Bx + Cy + D = 0 \tag{2.3}$$

where B, C, and D are constants.

Theorem: *An equation of the second degree in which the coefficients of the x^2 and y^2 terms are equal and the xy term does not exist, represents a circle.*

Whenever we find an equation in the form of equation (2.3), we should convert it to the form of equation (2.2) so that we have the coordinates of the center of the circle and the radius as part of the equation. This may be done as shown in the following example problems:

EXAMPLE: Find the coordinates of the center and the radius of the circle described by the following equation:

$$x^2 + y^2 - 4x - 6y - 3 = 0$$

SOLUTION: First rearrange the terms

$$x^2 - 4x + y^2 - 6y - 3 = 0$$

and complete the square in both x and y. Completing the square is discussed in the chapter on quadratic solutions in *Mathematics*, Volume 1. The procedure consists basically of adding certain quantities to both sides of a second-degree equation to form the sum of two perfect squares. When both the first- and second-degree members are known, the square of one-half the coefficient of the first-degree term is added to both sides of the equation. This will allow the quadratic equation to be factored into the sum of two perfect squares. To complete the square in x in the given equation

$$x^2 - 4x + y^2 - 6y - 3 = 0$$

add the square of one-half the coefficient of x to both sides of the equation

$$x^2 - 4x + (2)^2 + y^2 - 6y - 3 = 0 + (2)^2$$

then

$$(x^2 - 4x + 4) + y^2 - 6y - 3 = 4$$

$$(x - 2)^2 + y^2 - 6y - 3 = 4$$

completes the square in x.

If we do the same for y,

$$(x - 2)^2 + y^2 - 6y + (3)^2 - 3 = 4 + (3)^2$$

$$(x - 2)^2 + (y^2 - 6y + 9) - 3 = 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 - 3 = 4 + 9$$

completes the square in y.

Transpose all constant terms to the right-hand side and simplify:

$$(x - 2)^2 + (y - 3)^2 = 4 + 9 + 3$$

$$(x - 2)^2 + (y - 3)^2 = 16$$

The equation is now in the standard form of equation (2.2). This equation represents a circle with the center at (2,3) and with a radius equal to $\sqrt{16}$ or 4.

EXAMPLE: Find the coordinates of the center and the radius of the circle given by the equation

$$x^2 + y^2 + \frac{1}{2}x - 3y - \frac{27}{16} = 0$$

SOLUTION: Rearrange and complete the square in both x and y:

$$x^2 + \frac{1}{2}x + y^2 - 3y - \frac{27}{16} = 0$$

$$\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 - 3y + \frac{9}{4}\right) - \frac{27}{16} = \frac{1}{16} + \frac{9}{4}$$

Transposing all constant terms to the right-hand side and adding, results in

$$\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) + \left(y^2 - 3y + \frac{9}{4}\right) = 4$$

Reducing to the equation in standard form results in

$$\left(x + \frac{1}{4}\right)^2 + \left(y - \frac{3}{2}\right)^2 = (2)^2$$

Thus, the equation represents a circle with its center at $(-1/4, 3/2)$ and a radius equal to 2.

PRACTICE PROBLEMS:

Find the coordinates of the center and the radius for the circles described by the following equations:

1. $x^2 - \frac{4}{5}x + y^2 - 4y + \frac{29}{25} = 0$

2. $x^2 + 6x + y^2 - 14y = 23$

3. $x^2 - 14x + y^2 + 22y = -26$

4. $x^2 + y^2 + \frac{2}{5}x + \frac{2}{3}y = \frac{2}{25}$

5. $x^2 + y^2 - 1 = 0$

ANSWERS:

1. Center $\left(\frac{2}{5}, 2\right)$, radius $\sqrt{3}$

2. Center $(-3, 7)$, radius 9

3. Center $(7, -11)$, radius 12

4. Center $\left(-\frac{1}{5}, -\frac{1}{3}\right)$, radius $\frac{2\sqrt{13}}{15}$

5. Center (0,0), radius 1

In certain situations you will want to consider the following general form of a circle

$$x^2 + y^2 + Bx + Cy + D = 0$$

as the equation of a circle in which the specific values of the constants B, C, and D are to be determined. In this problem the unknowns to be found are not x and y, but the values of the constants B, C, and D. The conditions that define the circle are used to form algebraic relationships between these constants. For example, if one of the conditions imposed on the circle is that it pass through the point (3,4), then the general form is written with x and y replaced by 3 and 4, respectively; thus,

$$x^2 + y^2 + Bx + Cy + D = 0$$

is rewritten as

$$(3)^2 + (4)^2 + B(3) + C(4) + D = 0$$

$$3B + 4C + D = -25$$

Three independent constants (B, C, and D) are in the equation of a circle; therefore, three conditions must be given to define a circle. Each of these conditions will yield an equation with B, C, and D as the unknowns. These three equations are then solved simultaneously to determine the values of the constants, which satisfy all of the equations. In an analysis, the number of independent constants in the general equation of a curve indicate how many conditions must be set before a curve can be completely defined. Also, the number of unknowns in an equation indicates the number of equations that must be solved simultaneously to find the values of the unknowns. For example, if B, C, and D are unknowns in an equation, three separate equations involving these variables are required for a solution.

A circle may be defined by three noncollinear points; that is, by three points not lying on a straight line. Only one circle is possible through any three noncollinear points. To find the *equation of the circle determined by three points, substitute the x and y values of each of the given points into the general equation to form three equations with B, C, and D as the unknowns. These equations are then solved simultaneously to find the values of B, C, and D in the equation which satisfies the three given conditions.*

The solution of simultaneous equations involving two variables is discussed in *Mathematics,* Volume 1. Systems involving three variables use an extension of the same principles, but with three equations instead of two. Step-by-step explanations of the solution are given in the example problems.

EXAMPLE: Write the equation of the circle that passes through the points (2,8), (5,7), and (6,6).

SOLUTION: The method used in this solution corresponds to the addition-subtraction method used for solution of equations involving two variables. However, the method or combination of methods used depends on the particular problem. No single method is best suited to all problems.

First, write the general form of a circle:

$$x^2 + y^2 + Bx + Cy + D = 0$$

For each of the given points, substitute the given values for x and y and rearrange the terms:

For (2,8) $\quad 4 + 64 + 2B + 8C + D = 0$

$$2B + 8C + D = -68$$

For (5,7) $\quad 25 + 49 + 5B + 7C + D = 0$

$$5B + 7C + D = -74$$

For (6,6) $\quad 36 + 36 + 6B + 6C + D = 0$

$$6B + 6C + D = -72$$

To aid in the explanation, we number the three resulting equations:

$$2B + 8C + D = -68 \qquad \text{Equation (1)}$$

$$5B + 7C + D = -74 \qquad (2)$$

$$6B + 6C + D = -72 \qquad (3)$$

The first step is to eliminate one of the unknowns and have two equations and two unknowns remaining. The coefficient of D is the same in all three equations and is, therefore, the one most easily eliminated by addition and subtraction. To eliminate D, subtract equation (2) from equation (1):

$$\begin{array}{rrl} 2B + 8C + D = -68 & & (1) \\ \underline{5B + 7C + D = -74} & & (-)\ (2) \\ -3B + C \quad\ = \quad 6 & & (4) \end{array}$$

Subtract equation (3) from equation (2):

$$\begin{array}{rrl} 5B + 7C + D = -74 & & (2) \\ \underline{6B + 6C + D = -72} & & (-)\ (3) \\ -B + C \quad\ = \ -2 & & (5) \end{array}$$

We now have two equations, (4) and (5), in two unknowns that can be solved simultaneously. Since the coefficient of C is the same in both equations, it is the most easily eliminated variable.

To eliminate C, subtract equation (4) from equation (5):

$$\begin{array}{rrl} -B + C = -2 & & (5) \\ \underline{-3B + C = \quad 6} & & (-)\ (4) \\ 2B \quad = -8 & & \\ B \quad = -4 & & (6) \end{array}$$

To find the value of C, substitute the value found for B in equation (6) in equation (4) or (5)

$$\begin{array}{rl} -B + C = -2 & \text{using (5)} \\ -(-4) + C = -2 & \\ C = -6 & (7) \end{array}$$

Now the values of B and C can be substituted in any one of the original equations to determine the value of D.

If the values are substituted in equation (1),

$$\begin{array}{rl} 2B + 8C + D = -68 & (1) \\ 2(-4) + 8(-6) + D = -68 & \\ -8 - 48 + D = -68 & \\ D = -68 + 56 & \\ D = -12 & (8) \end{array}$$

The solution of the system of equations gave values for three independent constants in the general equation

$$x^2 + y^2 + Bx + Cy + D = 0$$

When the constant values are substituted, the equation takes the form of

$$x^2 + y^2 - 4x - 6y - 12 = 0$$

Now rearrange and complete the square in both x and y:

$$(x^2 - 4x + 4) + (y^2 - 6y + 9) - 12 = 4 + 9$$

$$(x - 2)^2 + (y - 3)^2 = 25$$

The equation now corresponds to a circle with its center at (2,3) and a radius of 5. This is the circle passing through three given points, as shown in figure 2-7, view A.

The previous example problem showed one method we can use to determine the equation of a circle when three points are given. The next example shows another method we can use to solve the same problem. One of the most important things to keep in mind when you study analytic geometry is that many problems may be solved by more than one method. Each problem should be analyzed carefully to determine what relationships exist between the given data and the desired results of the problem. Relationships such as distance from one point to another, distance from a point to a line, slope of a line, and the Pythagorean theorem will be used to solve various problems.

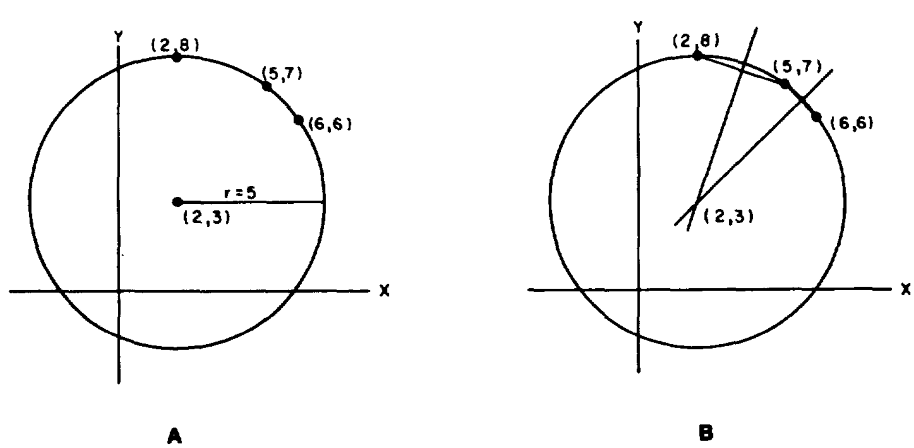


Figure 2-7.—Circle described by three points.

EXAMPLE: Find the equation of the circle that passes through the points (2,8), (5,7), and (6,6). Use a method other than that used in the previous example problem.

SOLUTION: A different method of solving this problem results from the reasoning in the following paragraphs:

The center of the desired circle will be the intersection of the perpendicular bisectors of the chords connecting points (2,8) with (5,7) and (5,7) with (6,6), as shown in figure 2-7, view B.

The perpendicular bisector of the line connecting two points is the locus of all points equidistant from the two points. Using this analysis, we can get the equations of the perpendicular bisectors of the two lines.

Equating the distance formulas that describe the distances from the center, point (x,y), which is equidistant from the points (2,8) and (5,7), gives

$$\sqrt{(x-2)^2 + (y-8)^2} = \sqrt{(x-5)^2 + (y-7)^2}$$

Squaring both sides gives

$$(x-2)^2 + (y-8)^2 = (x-5)^2 + (y-7)^2$$

or

$$x^2 - 4x + 4 + y^2 - 16y + 64 =$$
$$x^2 - 10x + 25 + y^2 - 14y + 49$$

Canceling and combining terms results in

$$6x - 2y = 6$$

or

$$3x - y = 3$$

Follow the same procedure for the points (5,7) and (6,6):

$$\sqrt{(x-5)^2 + (y-7)^2} = \sqrt{(x-6)^2 + (y-6)^2}$$

Squaring each side gives

$$(x-5)^2 + (y-7)^2 = (x-6)^2 + (y-6)^2$$

or

$$x^2 - 10x + 25 + y^2 - 14y + 49 =$$
$$x^2 - 12x + 36 + y^2 - 12y + 36$$

Canceling and combining terms gives a second equation in x and y:

$$2x - 2y = -2$$

or

$$x - y = -1$$

Solving the equations simultaneously gives the coordinates of the intersection of the two perpendicular bisectors; this intersection is the center of the circle.

$$\begin{array}{rcrl} 3x - y & = & 3 & \\ x - y & = & -1 & \text{(Subtract)} \\ \hline 2x & = & 4 & \\ x & = & 2 & \end{array}$$

Substitute the value $x = 2$ in one of the equations to find the value of y:

$$x - y = -1$$

$$2 - y = -1$$

$$-y = -3$$

$$y = 3$$

Thus, the center of the circle is the point (2,3).

The radius is the distance between the center (2,3) and one of the three given points. Using point (2,8), we obtain

$$r = \sqrt{(2-2)^2 + (8-3)^2} = \sqrt{25} = 5$$

The equation of this circle is

$$(x - 2)^2 + (y - 3)^2 = 25$$

as was found in the previous example.

If a circle is to be defined by three points, the points must be noncollinear. In some cases the three points are obviously noncollinear. Such is the case with the points (1,1), (−2,2), and (−1,−1), since these points cannot be connected by a straight line. However, in many cases you may find difficulty determining by inspection whether or not the points are collinear;

therefore, you need a method for determining this analytically. In the following example an attempt is made to find the circle determined by three points that are collinear.

EXAMPLE: Find the equation of the circle that passes through the points (1,1), (2,2), and (3,3).

SOLUTION: Substitute the given values of x and y in the general form of the equation of a circle to get three equations in three unknowns:

$$x^2 + y^2 + Bx + Cy + D = 0$$

For (1,1) $\quad 1 + 1 + B + C + D = 0$

$$B + C + D = -2 \qquad \text{Equation (1)}$$

For (2,2) $\quad 4 + 4 + 2B + 2C + D = 0$

$$2B + 2C + D = -8 \qquad (2)$$

For (3,3) $\quad 9 + 9 + 3B + 3C + D = 0$

$$3B + 3C + D = -18 \qquad (3)$$

To eliminate D, first subtract equation (1) from equation (2):

$$\begin{array}{rlr} 2B + 2C + D &= -8 & (2) \\ \underline{B + C + D} &\underline{= -2} & (-)\ (1) \\ B + C \quad &= -6 & (4) \end{array}$$

Next subtract equation (2) from equation (3):

$$\begin{array}{rlr} 3B + 3C + D &= -18 & (3) \\ \underline{2B + 2C + D} &\underline{= -\ 8} & (-)\ (2) \\ B + C \quad &= -10 & (5) \end{array}$$

Then subtract equation (5) from equation (4) to eliminate one of the unknowns:

$$\begin{array}{rlr} B + C &= -\ 6 & (4) \\ \underline{B + C} &\underline{= -10} & (-)\ (5) \\ 0 + 0 &= \quad 4 & \\ 0 &= \quad 4 & \end{array}$$

This solution is not valid, so no circle passes through the three given points. You should attempt to solve equations (4) and (5) by the substitution method. When the three given points are collinear, an inconsistent solution of some type will result.

If you try to solve the problem by eliminating both B and C at the same time (to find D), another type of inconsistent solution results. With the given coefficients you can easily eliminate both A and B at the same time. First, multiply equation (2) by 3 and equation (3) by -2 and add the resultant equations:

$$\begin{array}{rlr} 6B + 6C + 3D = & -24 & 3\ (\times)\ (2) \\ -6B - 6C - 2D = & 36 & (+)\ -2\ (\times)\ (3) \\ \hline D = & 12 & \end{array}$$

Then multiply equation (1) by -2 and add the resultant to equation (2):

$$\begin{array}{rlr} -2B - 2C - 2D = & 4 & -2\ (\times)\ (1) \\ 2B + 2C + D = & -8 & (+)\ (2) \\ \hline -D = & -4 & \\ D = & 4 & \end{array}$$

This gives two values for D, which is inconsistent since each of the constants must have a unique value consistent with the given conditions. The three points are on the straight line $y = x$.

PRACTICE PROBLEMS:

In each of the following problems, find the equation of the circle that passes through the three given points:

1. (14,0), (12,4), and (3,7)

2. (10,3), (11,8), and (7,14)

3. (1,1), (0,0), and $(-1,-1)$

4. $(12,-5)$, $(-9,-12)$, and $(-4,3)$

ANSWERS:

1. $x^2 + y^2 - 10x + 4y = 56$

2. $x^2 + y^2 - 6x - 14y = 7$

3. No solution; the given points describe the straight line $y = x$.

4. $x^2 + y^2 - 2x + 14y = 75$

THE PARABOLA

A *parabola* is the locus of all points in a plane equidistant from a fixed point, called the *focus*, and a fixed line, called the *directrix*. In the parabola shown in figure 2-8, point V, which lies halfway between the focus and the directrix, is called the *vertex* of the parabola. In this figure and in many of the parabolas discussed in the first portion of this section, the vertex of the parabola falls at the origin; however, the vertex of the parabola, like the center of the circle, can fall at any point in the plane.

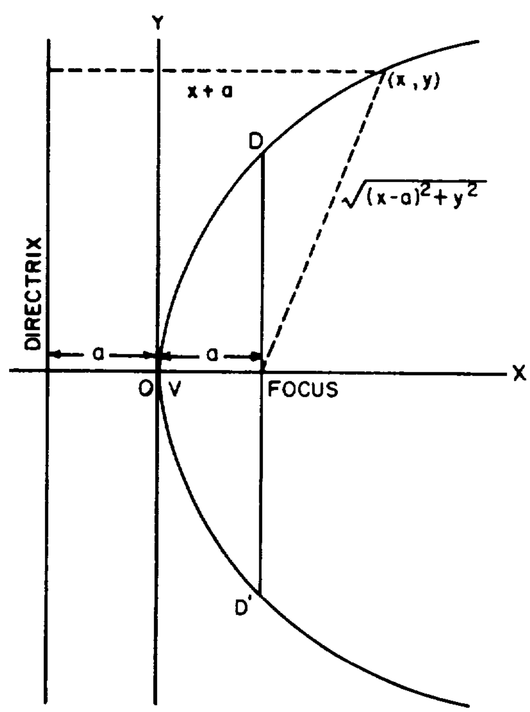


Figure 2-8.—The parabola.

The distance from the point (x,y) on the curve to the focus $(a,0)$ is

$$\sqrt{(x - a)^2 + y^2}$$

The distance from the point (x,y) to the directrix $x = -a$ is

$$x + a$$

Since by definition these two distances are equal, we may set them equal:

$$\sqrt{(x - a)^2 + y^2} = x + a$$

Squaring both sides, we have

$$(x - a)^2 + y^2 = (x + a)^2$$

Expanding, we have

$$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$$

Canceling and combining terms, we have an equation for the parabola:

$$y^2 = 4ax$$

For every positive value of x in the equation of the parabola, we have two values of y. But when x becomes negative, the values of y are imaginary. Thus, the curve must be entirely to the right of the Y axis when the equation is in this form. If the equation is

$$y^2 = -4ax$$

the curve lies entirely to the left of the Y axis.

If the form of the equation is

$$x^2 = 4ay$$

the curve opens upward and the focus is a point on the Y axis. For every positive value of y, you will have two values of x, and the curve will be entirely above the X axis. When the equation is in the form

$$x^2 = -4ay$$

the curve opens downward, is entirely below the X axis, and has as its focus a point on the negative Y axis. Parabolas that are representative of the four cases given here are shown in figure 2-9.

When x is equal to a in the equation

$$y^2 = 4ax$$

then

$$y^2 = 4a^2$$

and

$$y = 2a$$

This value of y is the height of the curve at the focus or the distance from the focus to point D in figure 2-8. The width of the curve at the focus, which is the distance from point D to point D' in

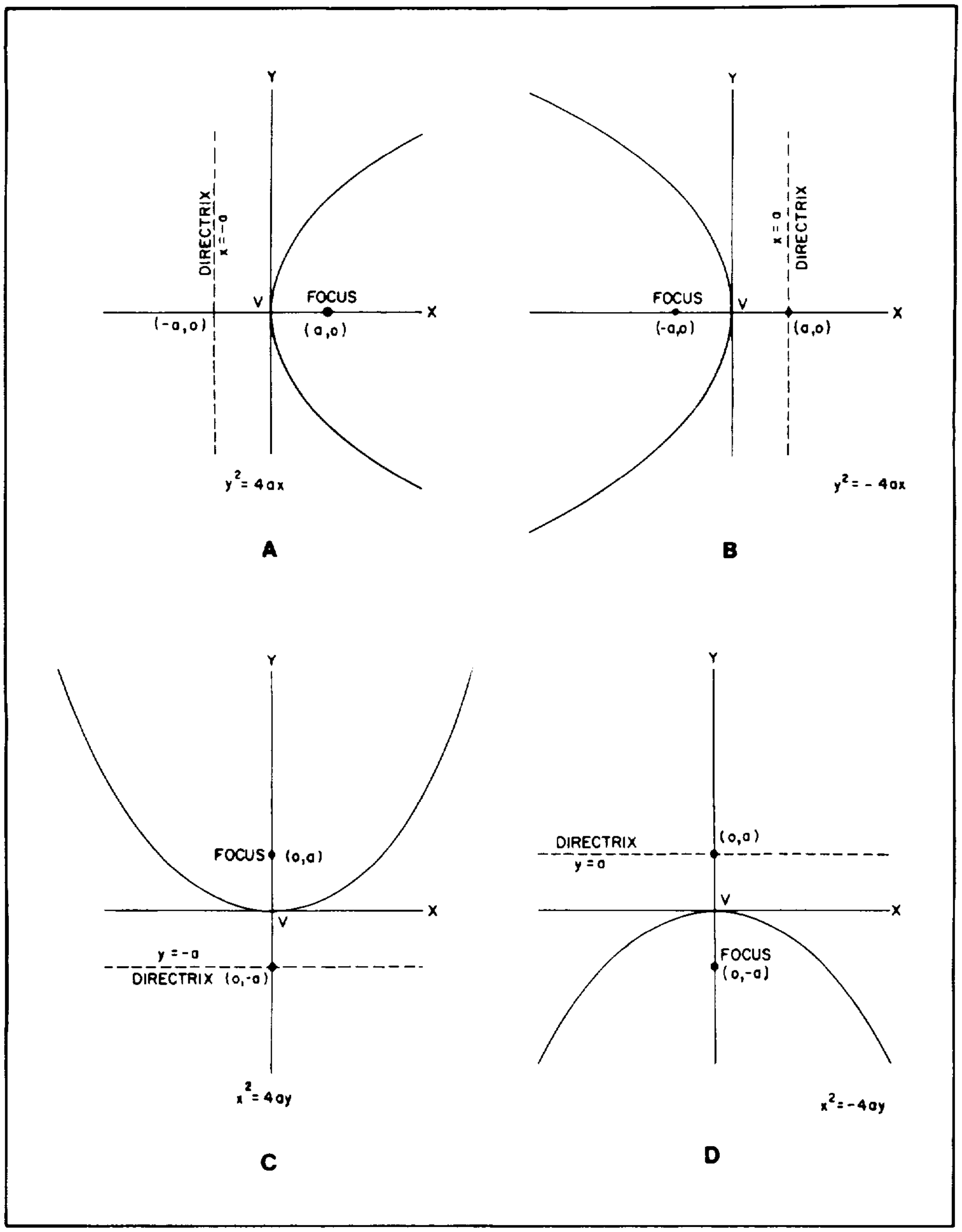


Figure 2-9.—Parabolas corresponding to four forms of the equation.

the figure, is equal to $4a$. This width is called the *focal chord*. The focal chord is one of the properties of a parabola used in the analysis of a parabola or in the sketching of a parabola.

EXAMPLE: Give the length of a; the length of the focal chord; and the equation of the parabola, which is the locus of all points equidistant from the point (3,0) and the line $x = -3$.

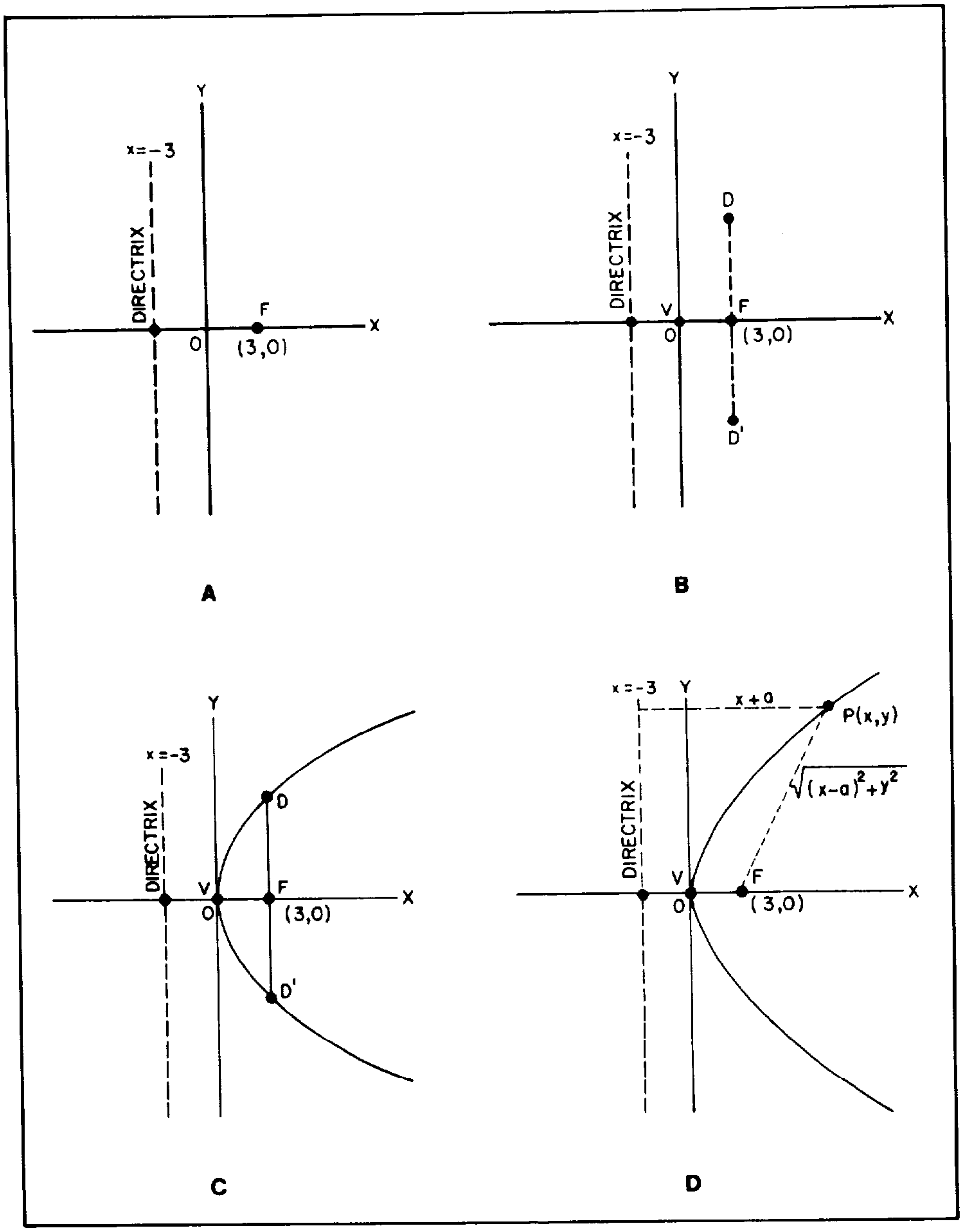


Figure 2-10.—Sketch of a parabola.

SOLUTION: First plot the given information on a coordinate system as shown in figure 2-10, view A. Figure 2-8 shows you that the point (3,0) corresponds to the position of the focus and that the line $x = -3$ is the directrix of the parabola. Figure 2-8 also shows you that the length of a is equal to one half the distance from the focus to the directrix or, in this problem, one half the distance from $x = -3$ to $x = 3$. Thus, the length of a is 3.

The second value required by the problem is the length of the focal chord. As stated previously, the focal chord length is equal to $4a$. The length of a was found to be 3, so the length of the focal chord is 12. Figure 2-8 shows that one extremity of the focal chord is a point on the curve $2a$ or 6 units above the focus, and the other extremity is a second point $2a$ or 6 units below the focus. Using this information and recalling that the vertex is one-half the distance from the focus to the directrix, plot three more points as shown in figure 2-10, view B.

Now a smooth curve through the vertex and the two points that are the extremities of the focal chord provide a sketch of the parabola in this problem. (See fig. 2-10, view C.)

To find the equation of the parabola, refer to figure 2-10, view D, and use the procedure used earlier. We know by definition that any point $P(x,y)$ on the parabola is equidistant from the focus and directrix. Thus, we equate these two distances:

$$\sqrt{(x-a)^2+y^2} = x + a$$

However, we have found distance a to be equal to 3, so we substitute:

$$\sqrt{(x-3)^2+y^2} = x + 3$$

We square both sides:

$$(x-3)^2 + y^2 = (x+3)^2$$

Then we expand:

$$x^2 - 6x + 9 + y^2 = x^2 + 6x + 9$$

We cancel and combine terms to obtain the equation of the parabola:

$$y^2 = 12x$$

If we check the consistency of our findings, we see that the form of the equation and the sketch agree with figure 2-9, view A. Also, the 12 in the right side of the equation corresponds to the $4a$ in the standard form, which is correct since we determined that the value of a was 3. Or, since the curve is entirely to the right of the Y axis, then we can apply the formula $y^2 = 4ax$ by substituting $a = 3$ to give

$$y^2 = 4(3)x$$

$$= 12x$$

NOTE: When the focus of a parabola lies on the Y axis, the equated distance equation is

$$\sqrt{(y-a)^2 + x^2} = y + a$$

PRACTICE PROBLEMS:

Give the equation; the length of a; and the length of the focal chord for the parabola, which is the locus of all points equidistant from the point and the line, given in the following problems:

1. The point $(-2,0)$ and the line $x = 2$
2. The point $(0,4)$ and the line $y = -4$
3. The point $(0,-1)$ and the line $y = 1$
4. The point $(1,0)$ and the line $x = -1$

ANSWERS:

1. $y^2 = -8x$, $a = 2$, $f.c. = 8$
2. $x^2 = 16y$, $a = 4$, $f.c. = 16$
3. $x^2 = -4y$, $a = 1$, $f.c. = 4$
4. $y^2 = 4x$, $a = 1$, $f.c. = 4$

Up to now, all of the parabolas we have dealt with have had a vertex at the origin and a corresponding equation in one of the four following forms:

1. $y^2 = 4ax$
2. $y^2 = -4ax$

3. $x^2 = 4ay$

4. $x^2 = -4ay$

We will now present four more forms of the equation of a parabola. Each one is a standardized parabola with its vertex at point $V(h,k)$. When the vertex is moved from the origin to the point $V(h,k)$, the x and y terms of the equation are replaced by $(x - h)$ and $(y - k)$. Then the standard equation for the parabola that opens to the right (fig. 2-9, view A) is

$$(y - k)^2 = 4a(x - h)$$

The four standard forms of the equations for parabolas with vertices at the point $V(h,k)$ are as follows:

1. $(y - k)^2 = 4a(x - h)$, corresponding to $y^2 = 4ax$, parabola opens to the right

2. $(y - k)^2 = -4a(x - h)$, corresponding to $y^2 = -4ax$, parabola opens to the left

3. $(x - h)^2 = 4a(y - k)$, corresponding to $x^2 = 4ay$, parabola opens upward

4. $(x - h)^2 = -4a(y - k)$, corresponding to $x^2 = -4ay$, parabola opens downward

The method for reducing an equation to one of these standard forms is similar to the method used for reducing the equation of a circle.

EXAMPLE: Reduce the equation

$$y^2 - 6y - 8x + 1 = 0$$

to standard form.

SOLUTION: Rearrange the equation so that the second-degree term and any first-degree terms of the same unknown are on the left side. Then group the unknown term appearing only in the first degree and all constants on the right:

$$y^2 - 6y = 8x - 1$$

Then complete the square in y:

$$y^2 - 6y + 9 = 8x - 1 + 9$$

$$(y - 3)^2 = 8x + 8$$

To get the equation in the form

$$(y - k)^2 = 4a(x - h)$$

factor an 8 out of the right side. Thus,

$$(y - 3)^2 = 8(x + 1)$$

is the equation of the parabola with its vertex at $(-1,3)$.

PRACTICE PROBLEMS:

Reduce the equations given in the following problems to standard form:

1. $x^2 + 4 = 4y$
2. $y^2 - 4x = 6y - 9$
3. $4x + 8y + y^2 + 20 = 0$
4. $4x - 12y + 40 + x^2 = 0$

ANSWERS:

1. $x^2 = 4(y - 1)$
2. $(y - 3)^2 = 4x$
3. $(y + 4)^2 = -4(x + 1)$
4. $(x + 2)^2 = 12(y - 3)$

THE ELLIPSE

An *ellipse* is a conic section with an eccentricity greater than 0 and less than 1.

Referring to figure 2-11, let

$$PO = a$$

$$FO = c$$

$$OM = d$$

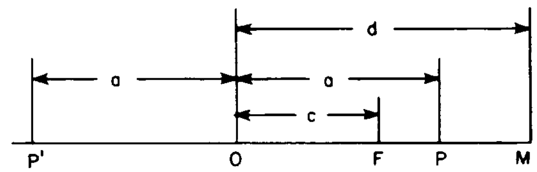


Figure 2-11.—Development of focus and directrix.

where F is the focus, O is the center, and P and P' are points on the ellipse. Then from the definition of eccentricity,

$$\frac{a - c}{d - a} = e \text{ or } a - c = ed - ea$$

and,

$$\frac{a + c}{d + a} = e \text{ or } a + c = ed + ea$$

Subtraction and addition of the two equations give

$$2c = 2ae \text{ or } c = ae \qquad (2.4)$$

$$2a = 2de \text{ or } d = \frac{a}{e}$$

Place the center of the ellipse at the origin so that one focus lies at $(-ae,0)$ and one directrix is the line $x = -a/e$.

Figure 2-12 shows a point on the Y axis that satisfies the conditions for an ellipse. If

$$P''O = b$$

and

$$FO = c$$

then

$$P''F = b^2 + c^2$$

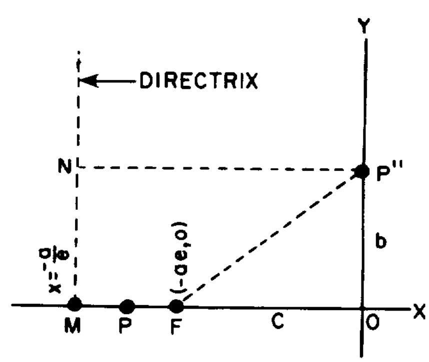


Figure 2-12.—Focus, directrix, and point P''.

By definition, e is the ratio of the distance of P'' from the focus and the directrix, so

$$e = \frac{P''F}{P''N}$$

or

$$e = \frac{\sqrt{b^2 + c^2}}{\frac{a}{e}}$$

Multiplying both sides by a/e gives

$$\sqrt{b^2 + c^2} = a$$

or

$$b^2 + c^2 = a^2$$

so

$$b^2 = a^2 - c^2 \qquad (2.5)$$

Now combining equations (2.4) and (2.5) gives

$$b^2 = a^2 - a^2e^2$$

or

$$b^2 = a^2(1 - e^2) \qquad (2.6)$$

Refer to figure 2-13. If the point (x,y) is on the ellipse, then the ratio of its distance from F to its distance from the directrix is e. The distance from (x,y) to the focus $(-ae,0)$ is

$$\sqrt{(x + ae)^2 + y^2}$$

and the distance from (x,y) to the directrix $x = -\frac{a}{e}$ is

$$x + \frac{a}{e}$$

The ratio of these two distances is equal to e, so

$$\frac{\sqrt{(x + ae)^2 + y^2}}{x + \frac{a}{e}} = e$$

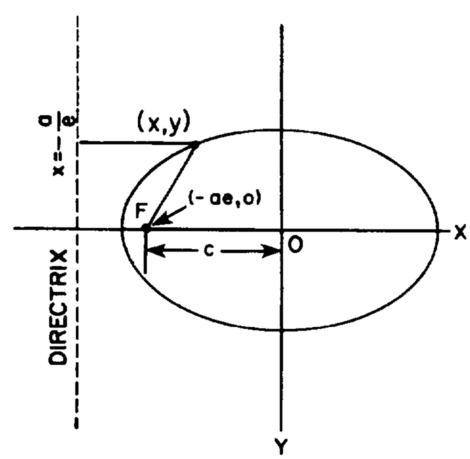


Figure 2-13.—The ellipse.

or

$$\sqrt{(x + ae)^2 + y^2} = e\left(x + \frac{a}{e}\right)$$

$$= ex + a$$

Squaring and expanding both sides gives

$$x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

Canceling like terms and transposing terms in x to the left-hand side of the equation gives

$$x^2 - e^2x^2 + y^2 = a^2 - a^2e^2$$

Removing a common factor gives

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2) \qquad (2.7)$$

Dividing both sides of equation (2.7) by the right-hand member gives

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

From equation (2.6) we obtain

$$b^2 = a^2(1 - e^2)$$

so that the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is the equation of an ellipse in standard form. In figure 2-14, views A and B, a is the length of the semimajor axis and b is the length of the semiminor axis.

The curve is symmetrical with respect to the X and Y axes, so you can easily see that figure 2-14, view A, has another focus at $(ae,0)$ and a corresponding directrix, $x = a/e$. The curve also has vertices at $(\pm a,0)$.

The distance from the center through the focus to the curve is always designated a and is called the *semimajor axis*. This axis may be in either the x or y direction. When it is in the y direction, the directrix is a line denoted by the equation

$$y = k$$

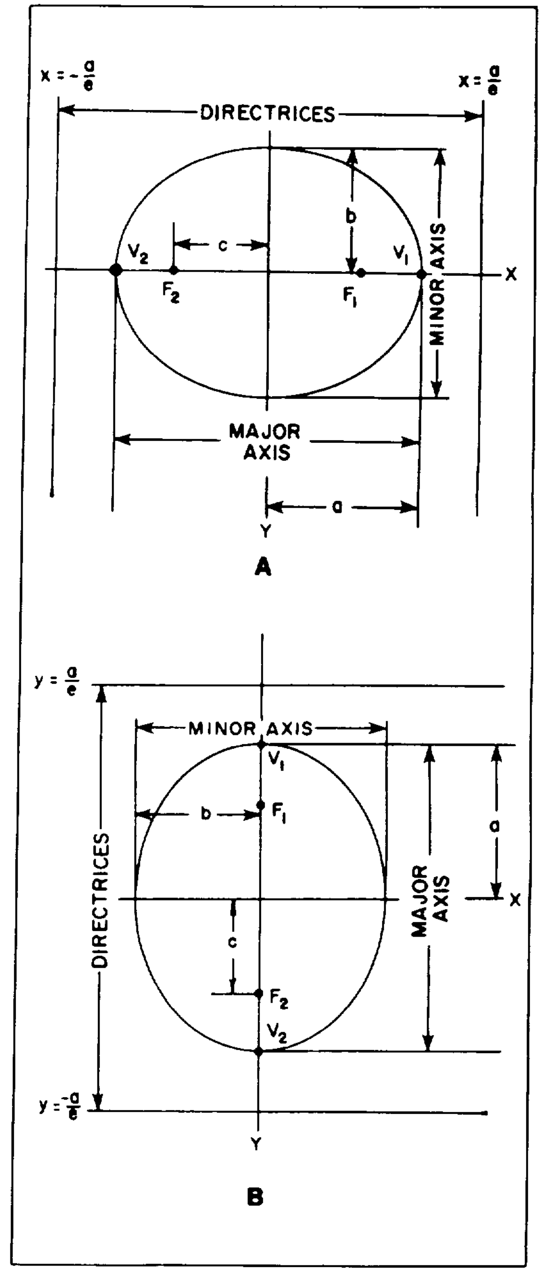


Figure 2-14.—Ellipse showing axes.

In the case we have studied, the directrix was denoted by the formula

$$x = k$$

where k is a constant equal to $\pm a/e$.

The perpendicular distance from the midpoint of the major axis to the curve is called the *semiminor axis* and is always signified by b.

The distance from the center of the ellipse to the focus is called c. In any ellipse the following relations are true for a, b, and c:

$$c = \sqrt{a^2 - b^2} \text{ or } c^2 = a^2 - b^2$$

$$b = \sqrt{a^2 - c^2} \text{ or } b^2 = a^2 - c^2$$

$$a = \sqrt{b^2 + c^2} \text{ or } a^2 = b^2 + c^2$$

Whenever the directrix is a line denoted by the equation $y = k$, the major axis is in the y direction and the equation of the ellipse is as follows:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Refer to figure 2-14, view B. This curve has foci at $(0, \pm c)$ and vertices at $(0, \pm a)$.

In an ellipse the position of the a^2 and b^2 terms indicates the orientation of the ellipse axis. As shown in figure 2-14, views A and B, value a is the semimajor or longer axis.

In the previous paragraphs formulas were given showing the relationship between a, b, and c. In the first portion of this discussion, a formula showing the relationship between a, c, and the eccentricity was given. These relationships are used to find the equation of an ellipse in the following example:

EXAMPLE: Find the equation of the ellipse with its center at the origin and having foci at $(\pm 2\sqrt{6}, 0)$ and an eccentricity equal to $\frac{2\sqrt{6}}{7}$.

SOLUTION: With the focal points on the X axis, the ellipse is oriented as in figure 2-14, view A, and the standard form of the equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With the center at the origin, the numerators of the fractions on the left are x^2 and y^2. The problem is to find the values of a and b.

The distance from the center to either of the foci is equal to c (fig. 2-14, view A), so in this problem

$$c = 2\sqrt{6}$$

from the given coordinates of the foci.

The values of a, c, and e (eccentricity) are related by

$$c = ae$$

or

$$a = \frac{c}{e}$$

From the known information, substitute the values of c and e,

$$a = \frac{2\sqrt{6}}{\frac{2\sqrt{6}}{7}}$$

$$a = 2\sqrt{6}\left(\frac{7}{2\sqrt{6}}\right)$$

and

$$a = 7$$

$$a^2 = 49$$

Then, using the formula

$$b = \sqrt{a^2 - c^2}$$

or

$$b^2 = a^2 - c^2$$

and substituting for a^2 and c^2,

$$b^2 = 49 - (2\sqrt{6})^2$$

$$b^2 = 49 - (4)(6)$$

$$b^2 = 49 - 24$$

gives the final required value of

$$b^2 = 25$$

Then the equation of the ellipse is

$$\frac{x^2}{49} + \frac{y^2}{25} = 1$$

PRACTICE PROBLEMS:

Find the equation of the ellipse with its center at the origin and for which the following properties are given:

1. Foci at $(\pm\sqrt{7},0)$ and an eccentricity of $\frac{\sqrt{7}}{4}$.
2. Length of the semiminor axis is 5 along the X axis and $e = \sqrt{11}/6$.
3. Vertices at $(\pm 4,0)$ and directrices $x = \pm 8\sqrt{3}/3$.
4. Foci at $(0, \pm 4)$ and vertices at $(0, \pm 5)$.

ANSWERS:

1. $\frac{x^2}{4^2} + \frac{y^2}{3^2} = 1$ or $\frac{x^2}{16} + \frac{y^2}{9} = 1$
2. $\frac{x^2}{5^2} + \frac{y^2}{6^2} = 1$ or $\frac{x^2}{25} + \frac{y^2}{36} = 1$
3. $\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$ or $\frac{x^2}{16} + \frac{y^2}{4} = 1$
4. $\frac{x^2}{3^2} + \frac{y^2}{5^2} = 1$ or $\frac{x^2}{9} + \frac{y^2}{25} = 1$

An *ellipse* may be defined as the locus of all points in a plane, the sum of whose distances from two fixed points, called the foci, is a constant equal to $2a$. This is shown as follows:

Let the foci be F_1 and F_2 at $(\pm c,0)$, as shown in figure 2-15. Using the standard form of an ellipse,

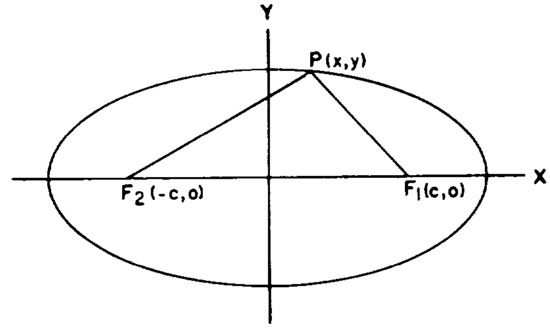


Figure 2-15.—Ellipse, center at origin.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

solve for y^2:

$$y^2 = \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}$$

Referring to figure 2-15, we see that

$$F_1P = \sqrt{(x - c)^2 + y^2}$$

and

$$F_2P = \sqrt{(x + c)^2 + y^2}$$

Substitute y^2 into both equations above and simplify

$$F_1P = \sqrt{(x - c)^2 + \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}}$$

$$= a - \frac{cx}{a}$$

and

$$F_2P = \sqrt{(x + c)^2 + \frac{(a^2 - c^2)(a^2 - x^2)}{a^2}}$$

$$= a + \frac{cx}{a}$$

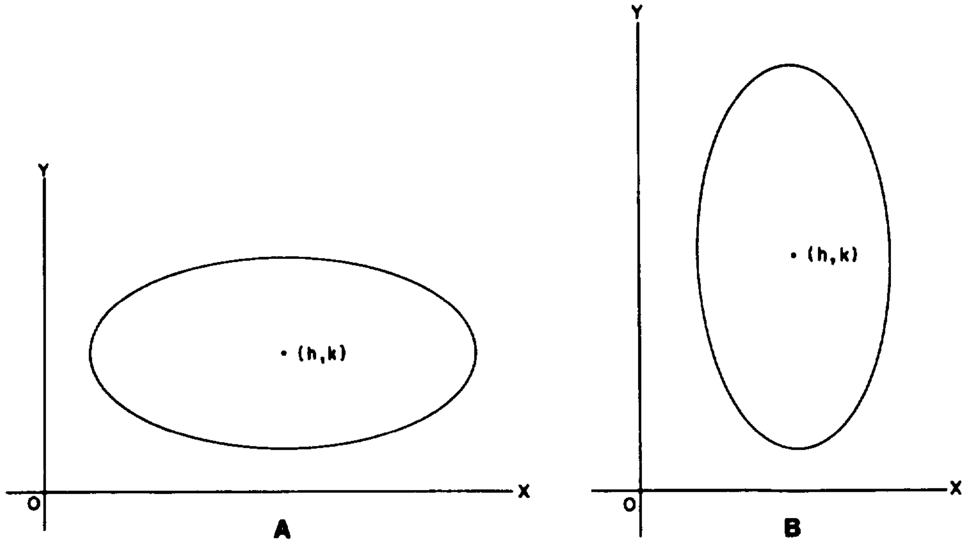


Figure 2-16.—Ellipse, center at (h,k).

so

$$F_1P + F_2P = a - \frac{cx}{a} + a + \frac{cx}{a}$$

$$= 2a$$

Whenever the center of the ellipse is at some point other than (0,0), such as the point (h,k) in figure 2-16, views A and B, the equation of the ellipse must be modified to the following standard forms:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \qquad (2.8)$$

$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1 \qquad (2.9)$$

Subtracting h from the value of x reduces the value of the term $(x - h)$ to the value x would have if the center were at the origin. The term $(y - k)$ is identical in value to the value of y if the center were at the origin.

Whenever we have an equation in the general form, such as

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have the same sign, we can reduce the equation to the standard form for an ellipse. Completing the square in both x and y and performing a few simple algebraic transformations will change the form to that of equations (2.8) and (2.9).

Theorem: *An equation of the second degree, in which the xy term does not exist and the coefficients of x^2 and y^2 are different but have the same sign, represents an ellipse with axes parallel to the coordinate axes.*

EXAMPLE: Reduce the equation

$$4x^2 + 9y^2 - 40x - 54y + 145 = 0$$

to an ellipse in standard form.

SOLUTION: Collect terms in x and y and remove the common factors of these terms:

$$4x^2 - 40x + 9y^2 - 54y + 145 = 0$$

$$4(x^2 - 10x) + 9(y^2 - 6y) + 145 = 0$$

Transpose the constant terms and complete the square in both x and y. When factored terms are involved in completing the square, as in this example, an error is frequently made. The factored value operates on the term added inside the parentheses as well as the original terms. Therefore, the values added to the right side of the equation are the products of the factored values and the terms added to complete the square:

$$\begin{aligned} 4(x^2 - 10x + 25) + 9(y^2 - 6y + 9) \\ &= -145 + 4(25) + 9(9) \\ &= -145 + 100 + 81 \\ &= 36 \end{aligned}$$

$$4(x - 5)^2 + 9(y - 3)^2 = 36$$

Divide both sides by the right-hand (constant) term. This reduces the right member to 1 as required by the standard form:

$$\frac{4(x-5)^2}{36} + \frac{9(y-3)^2}{36} = 1$$

$$\frac{(x-5)^2}{9} + \frac{(y-3)^2}{4} = 1$$

This reduces to the standard form

$$\frac{(x-5)^2}{(3)^2} + \frac{(y-3)^2}{(2)^2} = 1$$

corresponding to equation (2.8). This equation represents an ellipse with the center at (5,3); its semimajor axis, a, equal to 3; and its semiminor axis, b, equal to 2.

EXAMPLE: Reduce the equation

$$3x^2 + y^2 + 20x + 32 = 0$$

to an ellipse in standard form.

SOLUTION: First, collect terms in x and y. As in the previous example, the coefficients of x^2 and y^2 must be reduced to 1 to complete the square in both x and y. Thus the coefficient of the x^2 term is divided out of the two terms containing x, as follows:

$$3x^2 + 20x + y^2 + 32 = 0$$

$$3\left(x^2 + \frac{20x}{3}\right) + y^2 = -32$$

Complete the square in x, noting that a product is added to the right side:

$$3\left(x^2 + \frac{20x}{3} + \frac{100}{9}\right) + y^2 = -32 + 3\left(\frac{100}{9}\right)$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = \frac{-288 + 300}{9}$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = \frac{12}{9}$$

$$3\left(x + \frac{10}{3}\right)^2 + y^2 = \frac{4}{3}$$

Divide both sides by the right-hand term:

$$\frac{3\left(x + \frac{10}{3}\right)^2}{\frac{4}{3}} + \frac{y^2}{\frac{4}{3}} = 1$$

$$\frac{\left(x + \frac{10}{3}\right)^2}{\left(\frac{4}{9}\right)} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1$$

This equation reduces to the standard form,

$$\frac{\left(x+\frac{10}{3}\right)^2}{\left(\frac{2}{3}\right)^2}+\frac{y^2}{\left(\frac{2}{\sqrt{3}}\right)^2}=1$$

$$\frac{\left(x+\frac{10}{3}\right)^2}{\left(\frac{2}{3}\right)^2}+\frac{y^2}{\left(\frac{2\sqrt{3}}{3}\right)^2}=1$$

corresponding to equation (2.9), and represents an ellipse with the center at $\left(-\frac{10}{3},0\right)$.

PRACTICE PROBLEMS:

Express the following equations as an ellipse in standard form:

1. $5x^2 - 110x + 4y^2 + 425 = 0$

2. $x^2 - 14x + 36y^2 - 216y + 337 = 0$

3. $9x^2 - 54x + 4y^2 + 16y + 61 = 0$

4. $3x^2 - 14x + 4y^2 + 11 = 0$

ANSWERS:

1. $\frac{(x-11)^2}{(6)^2}+\frac{y^2}{(3\sqrt{5})^2}=1$

2. $\frac{(x-7)^2}{(6)^2}+\frac{(y-3)^2}{(1)^2}=1$

3. $\dfrac{(x-3)^2}{(2)^2} + \dfrac{(y+2)^2}{(3)^2} = 1$

4. $\dfrac{\left(x-\frac{7}{3}\right)^2}{\left(\frac{4}{3}\right)^2} + \dfrac{y^2}{\left(\frac{2\sqrt{3}}{3}\right)^2} = 1$

THE HYPERBOLA

A *hyperbola* is a conic section with an eccentricity greater than 1.

The formulas

$$c = ae$$

and

$$d = \frac{a}{e}$$

developed in the section concerning the ellipse were derived so that they are true for any value of eccentricity. Thus, they are true for the hyperbola as well as for an ellipse. Since e is greater than 1 for a hyperbola, then

$$c = ae \text{ and } c > a$$

$$d = \frac{a}{e} \text{ and } d < a$$

Therefore $c > a > d$.

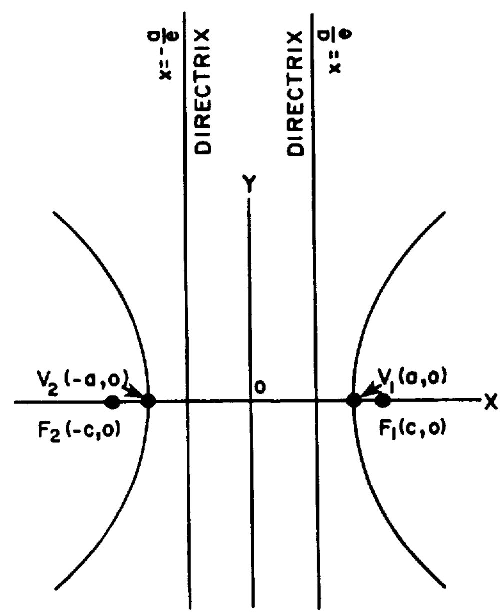


Figure 2-17.—The hyperbola.

According to this analysis, if the center of symmetry of a hyperbola is the origin, then the foci lies farther from the origin than the directrices. An inspection of figure 2-17 shows that the curve never crosses the Y axis. Thus the solution for the value of b, the semiminor axis of the ellipse, yields no real value for b. In other words, b is an imaginary number. This can easily be seen from the equation

$$b = \sqrt{a^2 - c^2}$$

since $c > a$ for a hyperbola.

However, we can square both sides of the the above equation, and since the square of an imaginary number is a negative real number we write

$$-b^2 = a^2 - c^2$$

or

$$b^2 = c^2 - a^2$$

and, since $c = ae$,

$$b^2 = a^2e^2 - a^2 = a^2(e^2 - 1)$$

Now we can use this equation to obtain the equation of a hyperbola from the following equation, which was developed in the section on the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$$

and since

$$a^2(1 - e^2) = -a^2(e^2 - 1) = -b^2$$

we have

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is a standard form for the equation of a hyperbola with its center, O, at the origin. The solution of this equation for y gives

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

which shows that y is imaginary only when $x^2 < a^2$. The curve, therefore, lies entirely beyond the two lines $x = \pm a$ and crosses the X axis at $V_1(a,0)$ and $V_2(-a,0)$, the vertices of the hyperbola.

The two straight lines

$$bx + ay = 0 \text{ and } bx - ay = 0 \tag{2.10}$$

can be used to illustrate an interesting property of a hyperbola. The distance from the line $bx - ay = 0$ to the point (x_1,y_1) on the curve is given by

$$d = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}} \tag{2.11}$$

Since (x_1,y_1) is on the curve, its coordinates satisfy the equation

$$b^2x_1^2 - a^2y_1^2 = a^2b^2$$

which may be written

$$(bx_1 - ay_1)(bx_1 + ay_1) = a^2b^2$$

or

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}$$

Now substituting this value into equation (2.11) gives us

$$d = \frac{a^2b^2}{\sqrt{a^2 + b^2}}\left(\frac{1}{bx_1 + ay_1}\right)$$

As the point (x_1,y_1) is chosen farther and farther from the center of the hyperbola, the absolute values for x_1 and y_1 will increase and the distance, d, will approach zero. A similar result can easily be derived for the line $bx + ay = 0$.

The lines of equation (2.10), which are usually written

$$y = -\frac{b}{a}x \text{ and } y = +\frac{b}{a}x$$

are called the *asymptotes* of the hyperbola. They are very important in tracing a curve and studying its properties. The

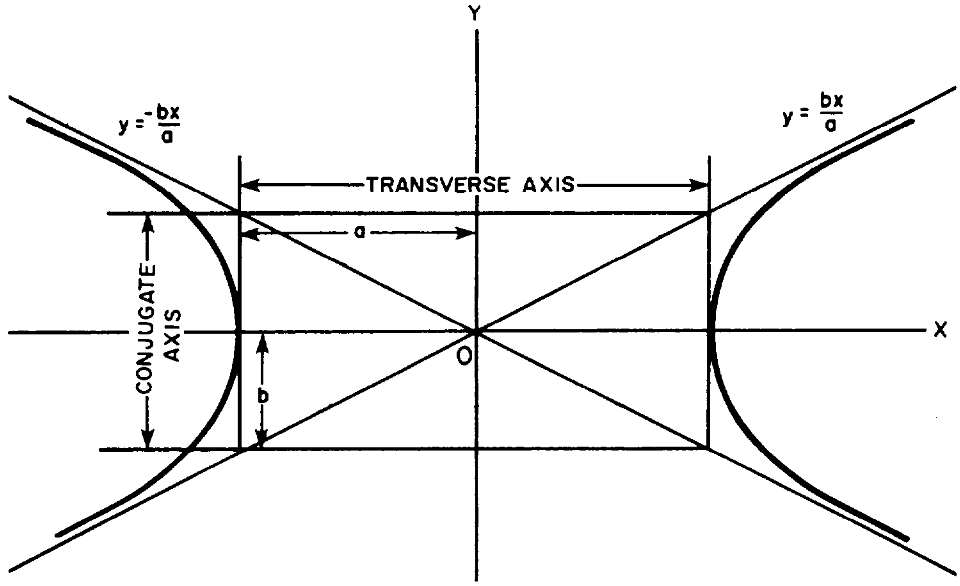


Figure 2-18.—Using asymptotes to sketch a hyperbola.

asymptotes of a hyperbola, figure 2-18, are the diagonals of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve. The *focal chord* of a hyperbola is equal to $\frac{2b^2}{a}$.

Another definition of a *hyperbola* is the locus of all points in a plane such that the difference of their distances from two fixed points is constant. The fixed points are the *foci*, and the constant difference is $2a$.

The nomenclature of the hyperbola is slightly different from that of an ellipse. The *transverse axis* is of length $2a$ and is the distance between the intersections (vertices) of the hyperbola with its focal axis. The *conjugate axis* is of length $2b$ and is perpendicular to the transverse axis.

Whenever the foci are on the Y axis and the directrices are lines of the form $y = \pm k$, where k is a constant, the equation of the hyperbola will read

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

This equation represents a hyperbola with its transverse axis on the Y axis. Its asymptotes are the lines $by - ax = 0$ and $by + ax = 0$ or

$$x = \frac{b}{a}y \text{ and } x = -\frac{b}{a}y$$

The properties of the hyperbola most often used in analysis of the curve are the foci, directrices, length of the focal chord, and the equations of the asymptotes.

Figure 2-17 shows that the foci are given by the points $F_1(c,0)$ and $F_2(-c,0)$ when the equation of the hyperbola is in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

If the equation were

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

the foci would be the points $(0,c)$ and $(0,-c)$. The value of c is either determined from the formula

$$c^2 = a^2 + b^2$$

or the formula

$$c = ae$$

Figure 2-17 also shows that the directrices are the lines $x = \pm\frac{a}{e}$ or, in the case where the hyperbolas open upward and downward, $y = \pm\frac{a}{e}$. This was also given earlier in this discussion as $d = \frac{a}{e}$.

The equations of the asymptotes were given earlier as

$$bx + ay = 0 \text{ and } bx - ay = 0$$

or

$$y = -\frac{b}{a}x \text{ and } y = +\frac{b}{a}x$$

The earlier reference also pointed out that the length of the focal chord is equal to $\frac{2b^2}{a}$.

Note that you have no restriction of $a > b$ for the hyperbola as you have for the ellipse. Instead, the direction in which the hyperbola opens corresponds to the transverse axis on which the foci and vertices lie.

The properties of a hyperbola can be determined from the equation of a hyperbola or the equation can be written given certain properties, as shown in the following examples. In these examples and in the practice problems immediately following, all of the hyperbolas considered have their centers at the origin.

EXAMPLE: Find the equation of the hyperbola with an eccentricity of 3/2, directrices $x = \pm 4/3$, and foci at $(\pm 3,0)$.

SOLUTION: The foci lie on the X axis at the points (3,0) and $(-3,0)$, so the equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This fact is also shown by the equation of the directrices.

Since we have determined the form of the equation and since the center of the curve in this section is restricted to the origin, the problem is reduced to finding the values of a^2 and b^2.

First, the foci are given as $(\pm 3,0)$; and since the foci are also the points $(\pm c,0)$, then

$$c = 3$$

The eccentricity is given and the value of a^2 can be determined from the formula

$$c = ae$$

$$a = \frac{c}{e}$$

$$a = \frac{3}{\frac{3}{2}}$$

$$a = \frac{6}{3}$$

$$a = 2$$

$$a^2 = 4$$

The relationship of a, b, and c for the hyperbola is

$$b^2 = c^2 - a^2$$

and

$$b^2 = (3)^2 - (2)^2$$

$$b^2 = 9 - 4$$

$$b^2 = 5$$

When these values are substituted in the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

the equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

results and is the equation of the hyperbola.

The equation could also be found by the use of other relationships using the given information.

The directrices are given as

$$x = \pm\frac{4}{3}$$

and, since

$$d = \frac{a}{e}$$

or

$$a = de$$

substituting the values given for d and e results in

$$a = \frac{4}{3}\left(\frac{3}{2}\right)$$

therefore,

$$a = 2$$

and

$$a^2 = 4$$

While the value of c can be determined by the given information in this problem, it could also be computed since

$$c = ae$$

and a has been found to equal 2 and e is given as $\frac{3}{2}$; therefore,

$$c = 2\left(\frac{3}{2}\right)$$

$$= 3$$

With values for a and c computed, the value of b is found as before and the equation can be written.

EXAMPLE: Find the foci, directrices, eccentricity, length of the focal chord, and equations of the asymptotes of the hyperbola described by the equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

SOLUTION: This equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and the values for a and b are determined by inspection to be

$$a^2 = 9$$

$$a = 3$$

and

$$b^2 = 16$$

$$b = 4$$

With a and b known, we find c by using the formula

$$c^2 = a^2 + b^2$$

$$c = \sqrt{a^2 + b^2}$$

$$c = \sqrt{9 + 16}$$

$$c = \sqrt{25}$$

$$c = 5$$

From the form of the equation, we know that the foci are at the points

$$F_1(c,0)$$

and

$$F_2(-c,0)$$

so the foci = $(\pm 5,0)$.

The eccentricity is found by the formula

$$e = \frac{c}{a}$$

$$e = \frac{5}{3}$$

Figure 2-17 shows that with the center at the origin, c and a will have the same sign.

The directrix is found by the formula

$$d = \frac{a}{e}$$

or, since this equation will have directrices parallel to the Y axis, by the formula

$$x = \pm\frac{a}{e}$$

Then

$$x = \pm\frac{3}{\frac{5}{3}}$$

$$x = \pm 3\left(\frac{3}{5}\right)$$

So the directrices are the lines

$$x = \pm\frac{9}{5}$$

The focal chord (*f.c.*) is found by

$$f.c. = \frac{2b^2}{\text{a}}$$

$$f.c. = \frac{2(16)}{3}$$

$$f.c. = \frac{32}{3}$$

Finally, the equations of the asymptotes are the equations of the two straight lines:

$$bx + ay = 0$$

and

$$bx - ay = 0$$

In this problem, substituting the values of a and b in each equation gives

$$4x + 3y = 0$$

and

$$4x - 3y = 0$$

or

$$4x \pm 3y = 0$$

The equations of the lines asymptotic to the curve can also be written in the form

$$y = \frac{b}{a}x$$

and

$$y = -\frac{b}{a}x$$

In this form the lines are

$$y = \frac{4}{3}x$$

and

$$y = -\frac{4}{3}x$$

or

$$y = \pm\frac{4}{3}x$$

If we think of this equation as a form of the slope-intercept formula

$$y = mx + b$$

from chapter 1, the lines would have slopes of $\pm\frac{b}{a}$ and each would have its y intercept at the origin as shown in figure 2-18.

PRACTICE PROBLEMS:

1. Find the equation of the hyperbola with an eccentricity of $\sqrt{2}$, directrices $y = \pm\frac{\sqrt{2}}{2}$, and foci at $(0, \pm\sqrt{2})$.

2. Find the equation of the hyperbola with an eccentricity of 5/3, foci at $(\pm 5,0)$, and directrices $x = \pm 9/5$.

Find the foci, directrices, eccentricity, equations of the asymptotes, and length of the focal chord of the hyperbolas given in problems 3 and 4.

3. $\frac{x^2}{9} - \frac{y^2}{9} = 1$

4. $\frac{y^2}{9} - \frac{x^2}{4} = 1$

ANSWERS:

1. $y^2 - x^2 = 1$

2. $\frac{x^2}{9} - \frac{y^2}{16} = 1$

3. foci $= (\pm 3\sqrt{2}, 0)$; directrices $x = \frac{\pm 3}{\sqrt{2}}$;
   eccentricity $= \sqrt{2}$; $f.c. = 6$; asymptotes $y = \pm x$

4. foci $= (0, \pm\sqrt{13})$; directrices $y = \frac{\pm 9}{\sqrt{13}}$;
   eccentricity $= \frac{\sqrt{13}}{3}$; $f.c. = \frac{8}{3}$; asymptotes $x = \pm\frac{2}{3}y$

The hyperbola can be represented by an equation in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have different signs. These equations can be reduced to standard form in the same manner in which similar equations for the ellipse were reduced to standard form. The standard forms with the center at (h,k) are given by the equations

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

and

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

POLAR COORDINATES

So far we have located a point in a plane by giving the distances of the point from two perpendicular lines. We can define the location of a point equally well by noting its distance and bearing. This method is commonly used aboard ship to show the position of another ship or target. Thus, 3 miles at 35° locates the position of a ship relative to the course of the ship making the reading. We can use this method to develop curves and bring out their properties. Assume a fixed direction OX and a fixed point O on the line in figure 2-19. The position of any point, P, is fully determined if we know the directed distance from O to P and the angle that the line OP makes with reference line OX. The line OP is called the *radius vector* and the angle POX is called the *polar angle*. The radius vector is denoted by ϱ, while θ denotes the polar angle.

Point O is the *pole* or *origin*. As in conventional trigonometry, *the polar angle is positive when measured counterclockwise and negative when measured clockwise*. However, unlike the convention established in trigonometry, *the radius vector for polar coordinates is positive only when it is laid off on the terminal side of the angle. When the radius vector is laid off on the terminal side of the ray produced beyond the pole (the given angle plus 180°), a negative value is assigned the radius vector*. For this reason, more than one equation may be used in polar coordinates to describe a given locus. It is sufficient that you remember that the radius vector can be negative. In this course, however, the radius vector, ϱ, will always be positive.

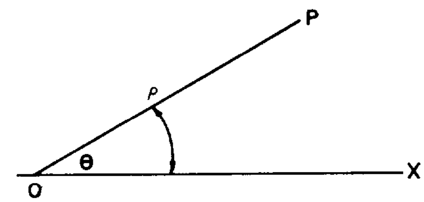


Figure 2-19.—Defining the polar coordinates.

TRANSFORMATION FROM CARTESIAN TO POLAR COORDINATES

At times you will find working with the equation of a curve in polar coordinates will be easier than working in Cartesian coordinates. Therefore, you need to know how to change from one system to the other. Sometimes both forms are useful, for some properties of the curve may be more apparent from one form of the equation.

We can make transformations by applying the following equations, which can be derived from figure 2-20:

$$x = \varrho \cos \theta \qquad (2.12)$$

$$y = \varrho \sin \theta$$

$$\varrho^2 = x^2 + y^2 \qquad (2.13)$$

$$\tan \theta = \frac{y}{x}$$

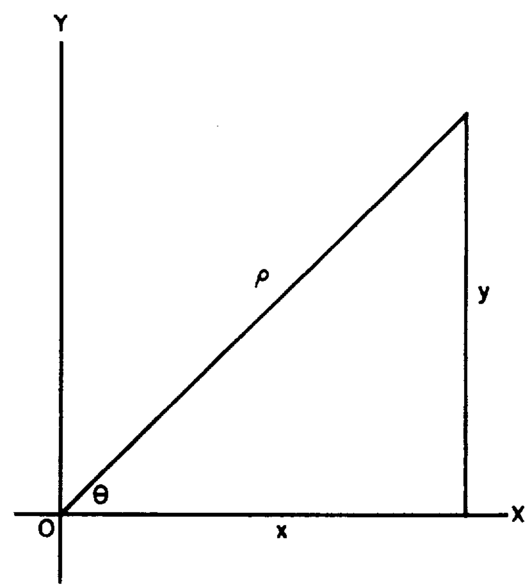


Figure 2-20.—Cartesian and polar relationship.

EXAMPLE: Change the equation

$$y = x^2$$

from rectangular to polar coordinates.

SOLUTION: Substitute $\varrho \cos \theta$ for x and $\varrho \sin \theta$ for y so that we have

$$\varrho \sin \theta = \varrho^2 \cos^2\theta$$

$$\sin \theta = \varrho \cos^2\theta$$

or

$$\varrho = \frac{\sin \theta}{\cos^2\theta}$$

$$\varrho = \tan \theta \sec \theta$$

EXAMPLE: Express the equation of the following circle with its center at $(a,0)$ and with radius a, as shown in figure 2-21, in polar coordinates:

$$(x - a)^2 + y^2 = a^2$$

SOLUTION: First, expanding this equation gives us

$$x^2 - 2ax + a^2 + y^2 = a^2$$

Rearranging terms, we have

$$x^2 + y^2 = 2ax$$

The use of equation (2.13) gives us

$$\varrho^2 = 2ax$$

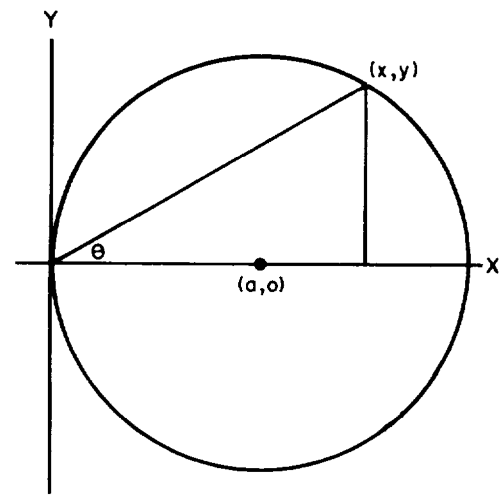


Figure 2-21.—Circle with center $(a,0)$.

and applying the value of x given by equation (2.12), results in

$$\varrho^2 = 2a\varrho \cos \theta$$

Dividing both sides by ϱ, we have the equation of a circle with its center at $(a,0)$ and radius a in polar coordinates

$$\varrho = 2a \cos \theta$$

TRANSFORMATION FROM POLAR TO CARTESIAN COORDINATES

To transform to an equation in Cartesian or rectangular coordinates from an equation in polar coordinates, use the following equations, which can be derived from figure 2-22:

$$\rho = \sqrt{x^2 + y^2} \qquad (2.14)$$

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\tan \theta = \frac{y}{x} \qquad (2.15)$$

$$\sec \theta = \frac{\sqrt{x^2 + y^2}}{x} \qquad (2.16)$$

$$\csc \theta = \frac{\sqrt{x^2 + y^2}}{y}$$

$$\cot \theta = \frac{x}{y}$$

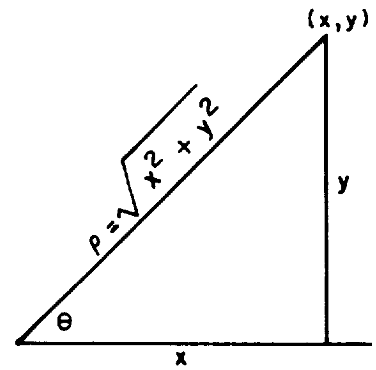


Figure 2-22.—Polar to cartesian relationship.

EXAMPLE: Change the equation

$$\rho = \sec \theta \tan \theta$$

to an equation in rectangular coordinates.

SOLUTION: Applying relations (2.14), (2.15), and (2.16) to the above equation gives

$$\sqrt{x^2 + y^2} = \frac{\sqrt{x^2 + y^2}}{x}\left(\frac{y}{x}\right)$$

Dividing both sides by $\sqrt{x^2 + y^2}$, we obtain

$$1 = \left(\frac{y}{x^2}\right)$$

or

$$y = x^2$$

which is the equation we set out to find.

EXAMPLE: Change the following equation to an equation in rectangular coordinates:

$$\varrho = \frac{3}{\sin\theta - 3\cos\theta}$$

SOLUTION: Written without a denominator, the polar equation is

$$\varrho \sin\theta - 3\varrho \cos\theta = 3$$

Using the transformations

$$\varrho \sin\theta = y$$

$$\varrho \cos\theta = x$$

we have

$$y - 3x = 3$$

as the equation in rectangular coordinates.

PRACTICE PROBLEMS:

Change the equations in problems 1 through 4 to equations having polar coordinates.

1. $x^2 + y^2 = 4$

2. $\left(x^2 + y^2\right) = 9\left(\frac{y}{x}\right)^2$

3. $3y - 7x = 10$

4. $y = 2x - 3$

Change the equations in the following problems to equations having Cartesian coordinates.

5. $\varrho = 4\sin\theta$

6. $\varrho = \sin\theta + \cos\theta$

7. $\varrho = a^2$

ANSWERS:

1. $\varrho = \pm 2$

2. $\varrho = 3 \tan \theta$

3. $\varrho = \dfrac{10}{3 \sin \theta - 7 \cos \theta}$

4. $\varrho = \dfrac{-3}{\sin \theta - 2 \cos \theta}$

5. $x^2 + y^2 - 4y = 0$

6. $x^2 + y^2 = y + x$

7. $x^2 + y^2 = a^4$

SUMMARY

The following are the major topics covered in this chapter:

1. **Conic section:** A *conic section* is the locus of all points in a plane whose distance from a fixed point is a constant ratio to its distance from a fixed line. The fixed point is the *focus,* and the fixed line is the *directrix*. The ratio referred to is called the *eccentricity*.

2. **Eccentricity:**

   If $0 < e < 1$, then the curve is an *ellipse.*

   If $e > 1$, then the curve is a *hyperbola.*

   If $e = 1$, then the curve is a *parabola.*

   If $e = 0$, then the curve is a *circle.*

3. **Locus of an equation:** The *locus* of an equation is a curve containing those points, and only those points, whose coordinates satisfy the equation.

4. **Circle:** A *circle* is the locus of all points, in a plane that is always a fixed distance, called the *radius*, from a fixed point, called the *center.*

   Theorem: *An equation of the second degree, in which the coefficients of the x^2 and y^2 terms are equal and the xy term does not exist represents a circle.*

5. **Standard equation of a circle:**

   $$(x - h)^2 + (y - k)^2 = r^2$$

   where (x,y) is a point on the circle, (h,k) is the center, and r is the radius of the circle.

6. **General equation of a circle:**

   $$x^2 + y^2 + Bx + Cy + D = 0$$

   where B, C, and D are constants.

7. **Circle defined by three points:** A circle may be defined by three noncollinear points; that is, by three points not lying on

a straight line. Only one circle is possible through any three noncollinear points. To find the equation of the circle determined by three points, substitute the x and y values of each of the given points into the general equation to form three equations with B, C, and D as the unknowns. These equations are then solved simultaneously to find the values of B, C, and D in the equation that satisfies the three given conditions.

8. **Parabola:** A *parabola* is the locus of all points in a plane equidistant from a fixed point, called the *focus*, and a fixed line, called the *directrix*.

   The point which lies halfway between the focus and the directrix is called the *vertex*.

   The *focal chord* is equal to $4a$, where a is the distance from the vertex to the focus.

   A parabola with its vertex at the origin and opening to the right has its focus at $(a,0)$ and its directrix at $x = -a$; its corresponding equation is $y^2 = 4ax$.

   A parabola with its vertex at the origin and opening to the left has its focus at $(-a,0)$ and its directrix at $x = a$; its corresponding equation is $y^2 = -4ax$.

   A parabola with its vertex at the origin and opening upward has its focus at $(0,a)$ and its directrix at $y = -a$; its corresponding equation is $x^2 = 4ay$.

   A parabola with its vertex at the origin and opening downward has its focus at $(0,-a)$ and its directrix at $y = a$; its corresponding equation is $x^2 = -4ay$.

9. **Standard equations for parabolas:**

   1. $(y - k)^2 = 4a(x - h)$ (parabola opening to the right)
   2. $(y - k)^2 = -4a(x - h)$ (parabola opening to the left)
   3. $(x - h)^2 = 4a(y - k)$ (parabola opening upward)
   4. $(x - h)^2 = -4a(y - k)$ (parabola opening downward)

   where (h,k) is the coordinate of the vertex and a is the distance from the vertex to the focus.

10. **Ellipse:** An *ellipse* is the locus of all points, in a plane the sum of whose distances from two fixed points (the foci) is a constant equal to $2a$.

The ellipse is symmetrical with respect to the X and Y axes, so an ellipse with its center at the origin and its major axis along the X axis has foci at $(\pm ae,0)$ or $(\pm c,0)$, vertices at $(\pm a,0)$, and directrices at $x = \pm a/e$; its corresponding equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

An ellipse with its center at the origin and its major axis along the Y axis has foci at $(0, \pm ae)$ or $(0, \pm c)$, vertices at $(0, \pm a)$, and directrices at $y = \pm a/e$; its corresponding equation is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

The distance from the center through the focus to the curve is always designated by a and is called the *semimajor axis*. The perpendicular distance from the midpoint of the major axis to the curve is called the *semiminor axis* and is always signified by b. The distance from the center of the ellipse to the focus is called c. The eccentricity is designated by e, which is equal to c/a.

The following relationships are true for a, b, and c in an ellipse:

$$c = \sqrt{a^2 - b^2} \quad \text{or } c^2 = a^2 - b^2$$

$$b = \sqrt{a^2 - c^2} \quad \text{or } b^2 = a^2 - c^2$$

$$a = \sqrt{b^2 + c^2} \quad \text{or } a^2 = b^2 + c^2$$

$$c < a \qquad \text{and} \qquad b < a$$

Theorem: *An equation of the second degree, in which the xy term does not exist and the coeficients of x^2 and y^2 are different but have the same sign, represents an ellipse with axes parallel to the coordinate axes.*

11. **Standard equations for ellipses:**

1. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

2. $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$

where (h,k) is the center of the ellipse, a is the length of the semimajor axis, and b is the length of the semiminor axis.

12. **General equation of an ellipse:**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have the same sign.

13. **Hyperbola:** A *hyperbola* is the locus of all points in a plane such that the difference of their distances from two fixed points is constant. The fixed points are the *foci* and the constant difference is $2a$.

The *transverse axis* is of length $2a$ and is the distance between the intersections (vertices) of the hyperbola with its focal axis. The *conjugate axis* is of length $2b$ and is perpendicular to the transverse axis.

The *focal chord* of a hyperbola is equal to $2b^2/a$.

A hyperbola with its center at the origin and transverse axis along the X axis has foci at $(\pm ae,0)$ or $(\pm c,0)$, vertices at $(\pm a,0)$, directrices at $x = \pm a/e$, and asymptotes at $y = \pm(b/a)x$; its corresponding equation is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

A hyperbola with its center at the origin and transverse axis along the Y axis has foci at $(0, \pm ae)$ or $(0, \pm c)$, vertices at $(0, \pm a)$, directrices at $y = \pm a/e$, and asymptotes at $x = \pm(b/a)y$; its corresponding equation is $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$.

The following relationships are true for a, b, and c in a hyperbola:

$$c^2 = a^2 + b^2$$

$$b^2 = c^2 - a^2$$

$$a^2 = c^2 - b^2$$

$$c > a$$

14. **Standard equations for hyperbolas:**

1. $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

2. $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

where (h,k) is the center of the hyperbola, a is half the length of the transverse axis, and b is half the length of the conjugate axis.

15. **General equation of a hyperbola:**

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

where the capital letters refer to independent constants and A and C have different signs.

16. **Polar coordinates:** The position of any point, P, is fully determined if we know the directed distance, called the *radius vector*, and the angle that the radius vector makes with the reference line, called the *polar angle*. The radius vector is denoted by ϱ, while θ denotes the polar angle. The origin is also called the *pole*.

The polar angle is positive when measured counterclockwise and negative when measured clockwise. The radius vector is positive only when it is laid off on the terminal side of the angle. When the radius vector is laid off on the terminal side of the ray produced beyond the pole (the given angle plus 180°), a negative value is assigned the radius vector.

17. **Transformation from Cartesian to polar coordinates:**

$$x = \varrho \cos \theta \qquad y = \varrho \sin \theta$$

$$\varrho^2 = x^2 + y^2 \qquad \tan \theta = \frac{y}{x}$$

18. **Transformation from polar to Cartesian coordinates:**

$$\varrho = \sqrt{x^2 + y^2}$$

$$\cos \theta = \frac{x}{\sqrt{x^2 + y^2}} \qquad \sec \theta = \frac{\sqrt{x^2 + y^2}}{x}$$

$$\sin \theta = \frac{y}{\sqrt{x^2 + y^2}} \qquad \csc \theta = \frac{\sqrt{x^2 + y^2}}{y}$$

$$\tan \theta = \frac{y}{x} \qquad \cot \theta = \frac{x}{y}$$

ADDITIONAL PRACTICE PROBLEMS

1. Find the equation of the curve that is the locus of all points equidistant from the point $(-3,-4)$ and the line $6x - 8y = -2$.

2. Find the coordinates of the center and the radius of a circle for the equation $x^2 + y^2 - 10x = -9$.

3. Find the equation of the circle that passes through points $(-4,3)$, $(0,-5)$, and $(3,-4)$.

4. Give the equation; the length of a; and the length of the focal chord for the parabola, which is the locus of all points equidistant from the point $(0,-23/4)$ and the line $y = 23/4$.

5. Reduce the equation $3x^2 - 30x + 24y + 99 = 0$ to a parabola in standard form.

6. Find the equation of the ellipse with its center at the origin, semimajor axis of length 14, and directrices $y = \pm 28$.

7. Reduce the equation $4x^2 + y^2 - 16x - 16y = 64$ to an ellipse in standard form.

8. Find the equation of the hyperbola with asymptotes at $y = \pm(4/3)x$ and vertices at $(\pm 6,0)$.

9. Find the foci, directrices, eccentricity, equations of the asymptotes, and length of the focal chord of the hyperbola $\frac{y^2}{25} - \frac{x^2}{100} = 1$.

10. Change the equation $x^2 + 2x + y^2 = 0$ from rectangular to polar coordinates.

11. Change the equation $\varrho = \tan\theta \cos\theta$ to an equation in rectangular coordinates.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. $64x^2 + 96xy + 36y^2 + 576x + 832y + 2496 = 0$

   or

   $16x^2 + 24xy + 9y^2 + 144x + 208y + 624 = 0$

2. center (5,0), radius 4

3. $x^2 + y^2 = 25$

4. $x^2 = -23y$, $a = 23/4$, $f.c. = 23$

5. $(x - 5)^2 = -8(y + 1)$

6. $\dfrac{x^2}{(7\sqrt{3})^2} + \dfrac{y^2}{(14)^2} = 1$ or $\dfrac{x^2}{147} + \dfrac{y^2}{196} = 1$

7. $\dfrac{(x - 2)^2}{(6)^2} + \dfrac{(y - 8)^2}{(12)^2} = 1$

8. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$

9. foci $= (0, \pm 5\sqrt{5})$; directrices $y = \pm\sqrt{5}$; eccentricity $= \sqrt{5}$; asymptotes $x = \pm 2y$; $f.c. = 40$

10. $\varrho = -2 \cos \theta$

11. $x^2 + y^2 - y = 0$

CHAPTER 3

TANGENTS, NORMALS, AND SLOPES OF CURVES

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Find the slope and equation of the tangent line for a standard parabola and for other curves at a given point.

2. Find equations of the tangent line and the normal line and lengths of the tangent and the normal between a point on a curve and the X axis for various curves.

3. Apply parametric equations to motion in a straight line and in a circle; and also, find equations and lengths of tangents and normals of curves using parametric equations.

INTRODUCTION

In chapter 1, the notation $\frac{\Delta y}{\Delta x}$ was introduced to represent the slope of a line. The straight line discussed had a constant slope and the symbol Δy was defined as $(y_2 - y_1)$ and Δx was defined as $(x_2 - x_1)$. In this chapter we will discuss the slope of curves at specific points on the curves. We will do this with as little calculus as possible, but our discussion will be directed toward the study of calculus.

SLOPE OF A CURVE AT A POINT

In figure 3-1 the slope of the curve is represented at two different places by $\frac{\Delta y}{\Delta x}$. The value of $\frac{\Delta y}{\Delta x}$ on the lower part of the curve is extremely close to the actual slope at P_1 because P_1 lies on a nearly straight portion of the curve. The value of the slope at P_2 is less accurate than the slope near P_1 because P_2 lies on a portion of the curve that has more curvature than the portion of the curve near P_1. To obtain an accurate measure of the slope of the curve at each point, as small a portion of the curve as possible should be used. When the curve is nearly a straight line, a very small error will occur when you are finding the slope, regardless of the value of the increments Δy and Δx. If the curvature is great and large increments are used when you are finding the slope of a curve, the error will become very large.

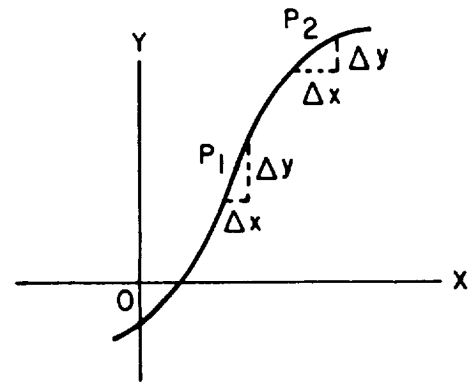


Figure 3-1.—Curve with increments Δy and Δx.

Thus, the error can be reduced to as small an amount as you want provided you choose your increments to be sufficiently small. Whenever the slope of a curve at a given point is desired, the increments Δy and Δx should be extremely small. Consequently, the arc of the curve can be replaced by a straight line, which determines the slope of the line tangent to the curve at that point.

You must understand that when we speak of the line tangent to a curve at a specific point, we are really considering the secant line, which cuts a curve in at least two points. Refer to figure 3-2. As point P_2 moves closer to point P_1, the slope of the secant line varies by smaller and smaller amounts and causes the secant line between P_1 and P_2 to approach P_1. P_1 is then extended to form the line tangent to the curve at that specific point.

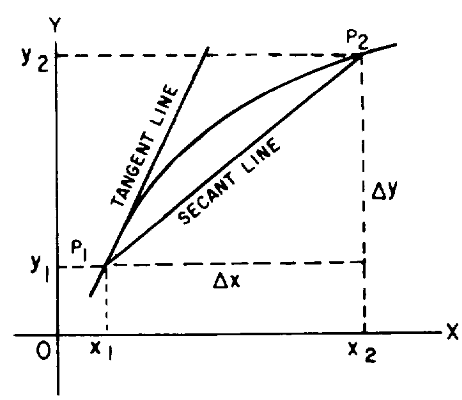


Figure 3-2.—Curve with secant line and tangent line.

If we allow

$$y = f(x)$$

to represent the equation of a curve, then $\frac{\Delta y}{\Delta x}$ is the slope of the line tangent to the curve at $P(x,y)$.

The direction of a curve is defined as the direction of the tangent line at any point on the curve. Let θ equal the inclination of the tangent line; then the slope equals tan θ and

$$\frac{\Delta y}{\Delta x} = \tan \theta$$

is the slope of the curve at any point $P(x,y)$. The angle θ_1 is the inclination of the tangent line at P_1 in figure 3-3. This angle is acute and the value of tan θ_1 is positive. Hence, the slope of the tangent line is positive at point P_1. The angle θ_2 is an obtuse angle, tan θ_2 is negative, and the slope of the tangent line at point P_2 is negative. All lines leaning to the right have positive slopes, and all lines leaning to the left have negative slopes. At point P_3 the tangent line to the curve is horizontal and θ equals 0. This means that

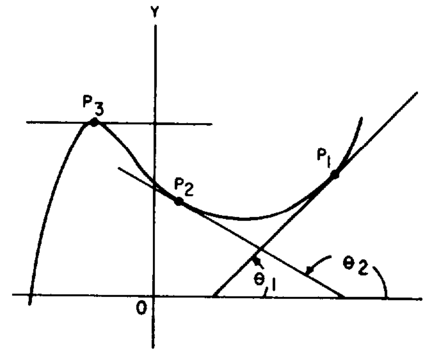


Figure 3-3.—Curve with tangent lines.

$$\frac{\Delta y}{\Delta x} = \tan 0° = 0$$

The fact that the slope of a curve is zero when the tangent line to the curve at that point is horizontal is of great importance in calculus when you are determining the maximum or minimum points of a curve. *Whenever the slope of a curve is zero, the curve may be at either a maximum or a minimum.*

Whenever the inclination of the tangent line to a curve at a point is 90°, the tangent line is vertical and parallel to the Y axis. This results in an infinitely large slope where

$$\frac{\Delta y}{\Delta x} = \tan 90° = \infty$$

TANGENT AT A GIVEN POINT ON THE STANDARD PARABOLA

The standard parabola is represented by the equation

$$y^2 = 4ax$$

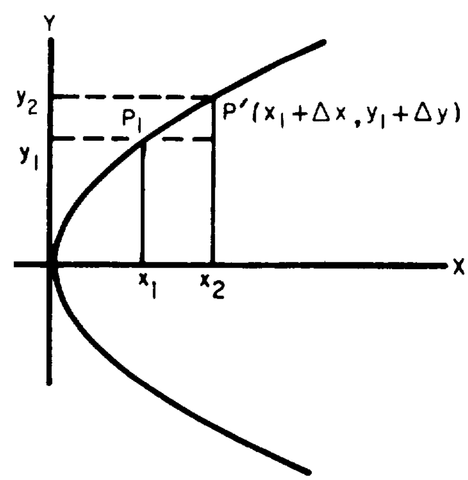


Figure 3-4.—Parabola.

Let P_1 with coordinates (x_1,y_1) be a point on the curve. Choose P' on the curve, figure 3-4, near the given point so that the coordinates of P' are (x_2,y_2). As previously stated

$$\Delta x = x_2 - x_1$$

and

$$\Delta y = y_2 - y_1$$

so that by rearranging terms, the coordinates of P' may be written as

$$(x_1 + \Delta x, y_1 + \Delta y)$$

Since P' is a point on the curve,

$$y^2 = 4ax$$

the values of its coordinates may be substituted for x and y. This gives

$$(y_1 + \Delta y)^2 = 4a(x_1 + \Delta x)$$

or

$$y_1^2 + 2y_1\Delta y + (\Delta y)^2 = 4ax_1 + 4a\Delta x \tag{3.1}$$

The point $P_1(x_1,y_1)$ also lies on the curve, so we have

$$y_1^2 = 4ax_1$$

Substituting this value for y_1^2 into equation (3.1) transforms it into

$$4ax_1 + 2y_1\Delta y + (\Delta y)^2 = 4ax_1 + 4a\Delta x$$

Simplifying, we obtain

$$2y_1\Delta y + (\Delta y)^2 = 4a\Delta x \tag{3.2}$$

Divide both sides by Δx, obtaining

$$\frac{2y_1\Delta y}{\Delta x} + \frac{(\Delta y)^2}{\Delta x} = \frac{4a\Delta x}{\Delta x}$$

which gives

$$(2y_1)\frac{\Delta y}{\Delta x} = 4a - \frac{(\Delta y)^2}{\Delta x}$$

Solving for $\frac{\Delta y}{\Delta x}$, we find

$$\frac{\Delta y}{\Delta x} = \frac{4a}{2y_1} - \frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1}$$

$$= \frac{2a}{y_1} - \frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1} \qquad (3.3)$$

Before proceeding, we need to discuss the term

$$\frac{\frac{(\Delta y)^2}{\Delta x}}{2y_1}$$

in equation (3.3). If we solve equation (3.2) for Δy, we find

$$2y_1\Delta y + (\Delta y)^2 = 4a\Delta x$$

then

$$\Delta y(2y_1 + \Delta y) = 4a\Delta x$$

and

$$\Delta y = \frac{4a\Delta x}{2y_1 + \Delta y}$$

Since the denominator contains a term not dependent upon Δy or Δx, as we let Δx approach zero, Δy will also approach zero.

NOTE: We may find a value for Δx that will make Δy less than 1; then when Δy is squared, it will approach zero at least as rapidly as Δx does.

We now refer to equation (3.3) again and make the statement

so that we may disregard $\dfrac{\frac{(\Delta y)^2}{\Delta x}}{2y_1}$,

since it approaches zero when Δx approaches zero.

Then

$$\frac{\Delta y}{\Delta x} = \frac{2a}{y_1} \tag{3.4}$$

The quantity $\frac{\Delta y}{\Delta x}$ is the slope of the line connecting P_1 and P'. From figure 3-4, the slope of the curve at P_1 is obviously different from the slope of the line connecting P_1 and P'.

As Δx and Δy approach zero, the ratio $\frac{\Delta y}{\Delta x}$ will approach more and more closely the true slope of the curve at P_1. We designate the slope by m. Thus, as Δx approaches zero, equation (3.4) becomes

$$m = \frac{2a}{y_1}$$

The equation for a straight line in the point-slope form is

$$y - y_1 = m(x - x_1)$$

Substituting $\frac{2a}{y_1}$ for m gives

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$

Clearing fractions, we have

$$yy_1 - y_1^2 = 2ax - 2ax_1 \tag{3.5}$$

but

$$y_1^2 = 4ax_1 \tag{3.6}$$

Adding equations (3.5) and (3.6) yields

$$yy_1 = 2ax + 2ax_1$$

Dividing by y_1 gives

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

which is an equation of a straight line in the slope-intercept form. This is the equation of the tangent line to the parabola

$$y^2 = 4ax$$

at the point (x_1,y_1).

EXAMPLE: Given the equation

$$y^2 = 8x$$

find the slope of the curve and the equation of the tangent line at the point (2,4).

SOLUTION:

$$y^2 = 8x$$

has the form

$$y^2 = 4ax$$

where

$$4a = 8$$

$$a = 2$$

and

$$2a = 4$$

The slope, m, at point (2,4) becomes

$$m = \frac{2a}{y_1}$$

$$= \frac{4}{4} = 1$$

Since the slope of the line is 1, then the equation of the tangent to the curve at the point (2,4) is

$$y = \frac{2ax}{y_1} + \frac{2ax_1}{y_1}$$

$$= \frac{(2)\,(2)\,(x)}{4} + \frac{(2)\,(2)\,(2)}{4}$$

$$= x + 2$$

TANGENT AT A GIVEN POINT ON OTHER CURVES

The technique used to find the slope and equation of the tangent line for a standard parabola can be used to find the slope and equation of the tangent line to a curve at any point regardless of the type of curve. The method can be used to find these relationships for circles, hyperbolas, ellipses, and general algebraic curves.

This general method is outlined as follows: *To find the slope, m, of a given curve at the point $P_1(x_1,y_1)$, choose a second point, P', on the curve so that it has coordinates $(x_1 + \Delta x, y_1 + \Delta y)$; then substitute each of the coordinates of P' and P_1 in the equation of the curve and simplify. Divide both sides by Δx and eliminate terms that contain powers of Δy higher than the first power, as previously discussed. Solve for $\frac{\Delta y}{\Delta x}$. Let Δx approach zero and $\frac{\Delta y}{\Delta x}$ will approach the slope of the tangent line, m, at point P_1.*

When the slope and coordinates of a point on the curve are known, you can find the equation of the tangent line by using the point-slope method.

EXAMPLE: Using the method outlined, find the slope and equation of the tangent line to the curve

$$x^2 + y^2 = r^2 \text{ at } (x_1,y_1) \qquad \text{Equation (1)}$$

SOLUTION: Choose a second point such that it has coordinates

$$(x_1 + \Delta x, y_1 + \Delta y)$$

Substitute into equation (1)

$$(x_1 + \Delta x)^2 + (y_1 + \Delta y)^2 = r^2$$

Thus

$$x_1^2 + 2x_1\Delta x + (\Delta x)^2 + y_1^2 + 2y_1\Delta y + (\Delta y)^2 = r^2$$

Then

$$\begin{aligned} 2x_1\Delta x + (\Delta x)^2 + 2y_1\Delta y + (\Delta y)^2 &= r^2 - x_1^2 - y_1^2 \\ &= r^2 - (x_1^2 + y_1^2) \\ &= 0 \end{aligned}$$

Divide both sides by Δx

$$\frac{2x_1\Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} + \frac{2y_1\Delta y}{\Delta x} + \frac{(\Delta y)^2}{\Delta x} = 0$$

and eliminating $(\Delta y)^2$ results in

$$2x_1 + \Delta x + 2y_1\frac{\Delta y}{\Delta x} = 0$$

Solve for $\frac{\Delta y}{\Delta x}$:

$$\frac{\Delta y}{\Delta x} = \frac{-2x_1 - \Delta x}{2y_1}$$

Let Δx approach zero, so that

$$m = \frac{-x_1}{y_1}$$

Now using the point-slope form of a straight line, substitute $\frac{-x_1}{y_1}$ for m:

$$y - y_1 = m(x - x_1)$$

$$= \frac{-x_1}{y_1}(x - x_1)$$

Multiply both sides by y_1:

$$yy_1 - y_1^2 = -x_1x + x_1^2$$

Rearrange:

$$yy_1 = -x_1x + x_1^2 + y_1^2$$

but

$$x_1^2 + y_1^2 = r^2$$

Then, by substitution

$$yy_1 = -x_1x + r^2$$

and

$$y = \frac{-x_1x}{y_1} + \frac{r^2}{y_1}$$

which is the general equation of the tangent line to the curve

$$x^2 + y^2 = r^2 \text{ at } (x_1,y_1)$$

EXAMPLE: Using the given method, with minor changes, find the slope and equation of the tangent line to the curve

$$x^2 - y^2 = k^2 \text{ at } (x_1,y_1) \qquad \text{Equation (1)}$$

SOLUTION: Choose a second point such that it has coordinates

$$(x_1 + \Delta x, y_1 + \Delta y)$$

Substitute into equation (1):

$$(x_1 + \Delta x)^2 - (y_1 + \Delta y)^2 = k^2$$

$$x_1^2 + 2x_1\Delta x + (\Delta x)^2 - y_1^2 - 2y_1\Delta y - (\Delta y)^2 = k^2 \qquad (2)$$

Since $x_1^2 - y_1^2 = k^2$, then

$$2x_1\Delta x + (\Delta x)^2 - 2y_1\Delta y - (\Delta y)^2 = 0$$

Then divide by Δx and eliminate $(\Delta y)^2$

$$2x_1 + \Delta x - 2y_1\frac{\Delta y}{\Delta x} = 0$$

Solve for $\frac{\Delta y}{\Delta x}$

$$\frac{\Delta y}{\Delta x} = \frac{2x_1 + \Delta x}{2y_1}$$

Let Δx approach zero, so that

$$m = \frac{x_1}{y_1}$$

which is the slope desired.

Use the point-slope form of a straight line to find the equation of the tangent line to the curve at point (x_1,y_1) as shown in the following:

$$y - y_1 = m(x - x_1)$$

Substitute $\frac{x_1}{y_1}$ for m:

$$y - y_1 = \frac{x_1}{y_1}(x - x_1)$$

Multiply both sides by y_1:

$$yy_1 - y_1^2 = x_1x - x_1^2$$

Rearrange to obtain

$$yy_1 = x_1x - x_1^2 + y_1^2$$

$$= x_1x - (x_1^2 - y_1^2)$$

Substitute k^2 for $x_1^2 - y_1^2$:

$$yy_1 = x_1x - k^2$$

Divide both sides by y_1 to obtain

$$y = \frac{x_1x}{y_1} - \frac{k^2}{y_1}$$

which is the equation desired.

PRACTICE PROBLEMS:

Find the slope and equation of the tangent line to the curve, in problems 1 through 6, at the given points.

1. $y^2 = \frac{4}{3}x$	at (3,2)
2. $y^2 = 12x$	at (3,6)
3. $x^2 + y^2 = 25$	at $(-3,4)$
4. $x^2 + y^2 = 100$	at (6,8)
5. $x^2 - y^2 = 9$	at (5,4)
6. $x^2 - y^2 = 3$	at (2,1)
7. Find the slope of $y = x^2$	at (2,4)
8. Find the slope of $y = 2x^2 - 3x + 2$	at (2,4)

ANSWERS:

1. $y = \frac{x}{3} + 1,\ m = \frac{1}{3}$
2. $y = x + 3,\ m = 1$
3. $y = \frac{3x}{4} + \frac{25}{4},\ m = \frac{3}{4}$
4. $y = \frac{-3x}{4} + \frac{25}{2},\ m = \frac{-3}{4}$
5. $y = \frac{5x}{4} - \frac{9}{4},\ m = \frac{5}{4}$
6. $y = 2x - 3,\ m = 2$
7. $m = 4$
8. $m = 5$

EQUATIONS AND LENGTHS OF TANGENTS AND NORMALS

In figure 3-5, the coordinates of point P_1 on the curve are (x_1,y_1). Let the slope of the tangent line to the curve at point P_1 be denoted by m_1. If you know the slope and a point through which the tangent line passes, you can determine the equation of that tangent line by using the point-slope form.

Thus, the *equation of the tangent line, MP_1,* is

$$y - y_1 = m_1(x - x_1)$$

The normal to a curve at a point (x_1,y_1) is the line perpendicular to the tangent line at that point. The slope of the normal line, m_2, is

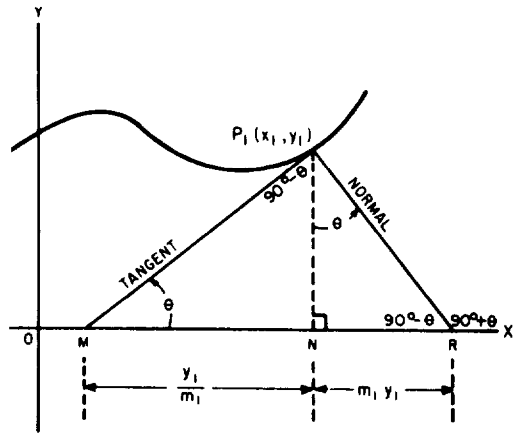


Figure 3-5.—Curve with tangent and normal lines.

$$m_2 = -\frac{1}{m_1}$$

where, as before, the slope of the tangent line is m_1. This is shown in the following:

If

$$m_1 = \tan\theta$$

then

$$\begin{aligned} m_2 &= \tan(\theta + 90°) \\ &= -\tan[180° - (\theta + 90°)] \\ &= -\tan(90° - \theta) \\ &= -\cot\theta \\ &= -\frac{1}{\tan\theta} \\ &= -\frac{1}{m_1} \end{aligned}$$

therefore,

$$m_2 = -\frac{1}{m_1}$$

The *equation of the normal line* through P_1 is

$$y - y_1 = -\frac{1}{m_1}(x - x_1)$$

Notice that since the slope of the tangent line is m_1 and the slope of the line which is normal to the tangent is m_2 and

$$m_2 = -\frac{1}{m_1}$$

then the product of the slopes of the tangent and normal lines equals -1. The relationship between the slopes of the tangent and normal lines is stated more formally as follows: *The slope of the normal line is the negative reciprocal of the slope of the tangent line.*

Another approach to show the relationship between the slopes of the tangent and normal lines follows: *The inclination of one line must be 90° greater than the other.* Then

$$\theta_2 = \theta_1 + 90°$$

If

$$\tan \theta_2 = m_2$$

then

$$\tan \theta_2 = \tan (\theta_1 + 90°)$$

We know from trigonometry that

$$\begin{aligned}
\tan (\theta_1 + 90°) &= \frac{\sin (\theta_1 + 90°)}{\cos (\theta_1 + 90°)} \\
&= \frac{\sin \theta_1 \cos 90° + \cos \theta_1 \sin 90°}{\cos \theta_1 \cos 90° - \sin \theta_1 \sin 90°} \\
&= \frac{(\sin \theta_1)(0) + (\cos \theta_1)(1)}{(\cos \theta_1)(0) - (\sin \theta_1)(1)} \\
&= -\frac{\cos \theta_1}{\sin \theta_1} \\
&= -\cot \theta_1 \\
&= -\frac{1}{\tan \theta_1}
\end{aligned}$$

Therefore,

$$\tan \theta_2 = -\frac{1}{\tan \theta_1}$$

The *length of the tangent* is defined as that portion of the tangent line between the point $P_1(x_1,y_1)$ and the point where the tangent line crosses the X axis. In figure 3-5, the length of the tangent is the line MP_1.

The *length of the normal* is defined as that portion of the normal line between the point P_1 and the X axis; that is the line, P_1R, which is perpendicular to the tangent line.

As shown in figure 3-5, the lengths of the tangent and normal may be found by using the Pythagorean theorem.

From triangle MP_1N, in figure 3-5,

$$\tan \theta = m_1 = \frac{P_1N}{MN}$$

and

$$MN = \frac{P_1N}{m_1}$$

Since

$$P_1N = y_1$$

then

$$MN = \frac{y_1}{m_1}$$

In triangle NP_1R,

$$\tan \theta = m_1 = \frac{NR}{NP_1}$$

and

$$NR = m_1 NP_1$$

$$= m_1 y_1$$

Thus, the length of the tangent is equal to

$$\sqrt{\left(\frac{y_1}{m_1}\right)^2 + (y_1)^2}$$

and the length of the normal is equal to

$$\sqrt{(y_1 m_1)^2 + (y_1)^2}$$

EXAMPLE: Find the equation of the tangent line, the equation of the normal line, and the lengths of the tangent and the normal of

$$y^2 = \frac{4}{3}x \text{ at } (3,2)$$

SOLUTION: Find the value of $2a$ from

$$y^2 = 4ax$$

Since

$$y^2 = \frac{4}{3}x$$

then

$$4a = \frac{4}{3}$$

$$a = \frac{1}{3}$$

$$2a = \frac{2}{3}$$

The slope is

$$m_1 = \frac{2a}{y_1} = \frac{\frac{2}{3}}{2} = \frac{1}{3}$$

Using the point-slope form of a straight line, we have

$$y - y_1 = m_1(x - x_1)$$

then, at point (3,2)

$$y - 2 = \frac{1}{3}(x - 3)$$

$$= \frac{x}{3} - 1$$

and

$$y = \frac{x}{3} + 1$$

which is the equation of the tangent line.

Use the negative reciprocal of the slope to find the equation of the normal line as follows:

$$y - 2 = -3(x - 3)$$
$$= -3x + 9$$

then

$$y = -3x + 11$$

To find the length of the tangent, we use the Pythagorean theorem. Thus, the length of the tangent is

$$\sqrt{\left(\frac{y_1}{m_1}\right)^2 + (y_1)^2} = \sqrt{(6)^2 + (2)^2}$$
$$= \sqrt{40}$$
$$= 2\sqrt{10}$$

The length of the normal is equal to

$$\sqrt{(y_1 m_1)^2 + (y_1)^2} = \sqrt{\left(\frac{2}{3}\right)^2 + (2)^2}$$
$$= \sqrt{\frac{40}{9}}$$
$$= \frac{2\sqrt{10}}{3}$$

PRACTICE PROBLEMS:

Find the equations of the tangent line and the normal line, and the lengths of the tangent and the normal for the following:

1. $y^2 = 12x$ at (3,6)
2. $x^2 + y^2 = 25$ at $(-3,4)$
3. $x^2 - y^2 = 9$ at (5,4)
4. $y = 2x^2 - 3x + 2$ at (1,1)

ANSWERS:

1. Equation of tangent line	$y = x + 3$
Equation of normal line	$y = -x + 9$
Length of tangent	$6\sqrt{2}$
Length of normal	$6\sqrt{2}$
2. Equation of tangent line	$y = \frac{3x}{4} + \frac{25}{4}$
Equation of normal line	$y = \frac{-4x}{3}$
Length of tangent	$\frac{20}{3}$
Length of normal	5
3. Equation of tangent line	$y = \frac{5x}{4} - \frac{9}{4}$
Equation of normal line	$y = \frac{-4x}{5} + 8$
Length of tangent	$\frac{4\sqrt{41}}{5}$
Length of normal	$\sqrt{41}$
4. Equation of tangent line	$y = x$
Equation of normal line	$y = -x + 2$
Length of tangent	$\sqrt{2}$
Length of normal	$\sqrt{2}$

PARAMETRIC EQUATIONS

If the variables x and y of the Cartesian coordinate system are expressed in terms of a third variable, say t (or θ), then the variable

t (or θ) is called a *parameter*. The two equations $x = x(t)$ and $y = y(t)$ [or $x = x(\theta)$ and $y = y(\theta)$] are called *parametric equations*.

MOTION IN A STRAIGHT LINE

To illustrate the application of a parameter, we will assume that an aircraft takes off from a field, which we will call the origin. Figure 3-6 shows the diagram we will use. The aircraft is flying on a compass heading of due north. There is a wind blowing from the west at 20 miles per hour, and the airspeed of the aircraft is 400 miles per hour. Let the direction of the positive Y axis be due north and the positive X axis be due east, as shown in figure 3-6. Use the scales as shown.

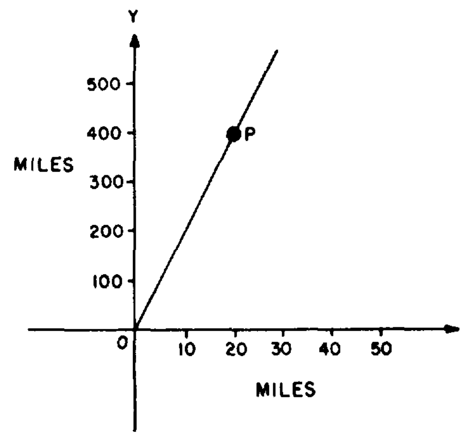


Figure 3-6.—Aircraft position.

One hour after takeoff the position of the aircraft, represented by point P, is 400 miles north and 20 miles east of the origin. If we use t as the parameter, then at any time, t, the aircraft's position (x,y) will be given by x equals $20t$ and y equals $400t$. The equations are

$$x = 20t$$

and

$$y = 400t$$

and are called parametric equations. Notice that time is not plotted on the graph of figure 3-6. The parameter t is used only to plot the position (x,y) of the aircraft.

We may eliminate the parameter t to obtain a direct relationship between x and y as follows:

If

$$t = \frac{x}{20}$$

then

$$y = 400\left(\frac{x}{20}\right)$$

$$y = 20x$$

and we find the graph to be a straight line. When we eliminated the parameter, the result was the rectangular coordinate equation of the line.

MOTION IN A CIRCLE

Consider the parametric equations

$$x = r \cos t$$

and

$$y = r \sin t$$

These equations describe the position of a point (x,y) at any time, t. They can be transposed into a single equation by squaring both sides of each equation to obtain

$$x^2 = r^2 \cos^2 t$$

$$y^2 = r^2 \sin^2 t$$

and adding

$$x^2 + y^2 = r^2 \cos^2 t + r^2 \sin^2 t$$

Rearranging, we have

$$x^2 + y^2 = r^2(\cos^2 t + \sin^2 t)$$

but

$$\cos^2 t + \sin^2 t = 1$$

so that

$$x^2 + y^2 = r^2$$

which is the equation of a circle.

This means that if various values were assigned to t and the corresponding values of x and y were calculated and plotted, the result would be a circle. In other words, the point (x,y) moves in a circular path.

Using this example again, that is

$$x = r \cos t$$

and

$$y = r \sin t$$

and given that

$$m_2 = \frac{\Delta x}{\Delta t} = -r \sin t$$

and

$$m_1 = \frac{\Delta y}{\Delta t} = r \cos t$$

we are able to express the slope at any point on the circle in terms of t.

NOTE: We may find the expressions for $\frac{\Delta x}{\Delta t}$ and $\frac{\Delta y}{\Delta t}$ by using calculus, but we will accept them for the present without proof.

If we know $\frac{\Delta y}{\Delta t}$ and $\frac{\Delta x}{\Delta t}$, we may find $\frac{\Delta y}{\Delta x}$, which is the slope of a curve at any point.

That is,

$$m = \frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta x}{\Delta t}}$$

By substituting, we find

$$\frac{\Delta y}{\Delta x} = \frac{r \cos t}{-r \sin t} = -\cot t$$

In terms of a parameter, we see that

$$m = -\cot t$$

while in terms of rectangular coordinates, we know from trigonometry that

$$m = -\frac{x_1}{y_1}$$

OTHER PARAMETRIC EQUATIONS

EXAMPLE: Find the equations of the tangent line and the normal line and the lengths of the tangent and the normal for the curve represented by

$$x = t^2$$

and

$$y = 2t + 1$$

at

$$t = 1$$

given that

$$\frac{\Delta x}{\Delta t} = 2t$$

and

$$\frac{\Delta y}{\Delta t} = 2$$

SOLUTION: Since t equals 1 we write

$$x = 1$$

and

$$y = 3$$

and

$$\frac{\Delta y}{\Delta x} = \frac{2}{2t} = \frac{1}{t}$$

so that

$$m = \frac{1}{t} = 1$$

The equation of the tangent line when t is equal to 1 is

$$y - 3 = 1(x - 1)$$

$$y = x + 2$$

The equation of the normal line is

$$y - 3 = -1(x - 1)$$

$$y = -x + 4$$

The length of the tangent is

$$\sqrt{\left(\frac{3}{1}\right)^2 + (3)^2} = 3\sqrt{2}$$

The length of the normal is

$$\sqrt{(3 \cdot 1)^2 + (3)^2} = 3\sqrt{2}$$

EXAMPLE: Find the equations of the tangent line and the normal line and the lengths of the tangent and the normal to the curve represented by the parametric equations

$$x = 2 \cos \theta$$

and

$$y = 2 \sin \theta$$

at the point where

$$\theta = 45°$$

given that

$$\frac{\Delta x}{\Delta \theta} = -2 \sin \theta$$

and

$$\frac{\Delta y}{\Delta \theta} = 2 \cos \theta$$

SOLUTION: We know that

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta \theta}}{\frac{\Delta x}{\Delta \theta}} = \frac{2 \cos \theta}{-2 \sin \theta} = -\cot \theta$$

Then at the point where $\theta = 45°$, we have

$$m = \frac{\Delta y}{\Delta x} = -\cot 45° = -1$$

If $\theta = 45°$ is substituted in the parametric equations, then

$$x = 2 \cos 45° = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

and

$$y = 2 \sin 45° = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}$$

The equation of the tangent line when $\theta = 45°$ is

$$y - \sqrt{2} = -1(x - \sqrt{2})$$

or

$$x + y = 2\sqrt{2}$$

The equation of the normal is

$$y - \sqrt{2} = 1(x - \sqrt{2})$$

or

$$x - y = 0$$

The length of the tangent is

$$\sqrt{\left(\frac{\sqrt{2}}{-1}\right)^2 + (\sqrt{2})^2} = 2$$

The length of the normal is

$$\sqrt{(\sqrt{2} \cdot -1)^2 + (\sqrt{2})^2} = 2$$

The horizontal and vertical tangents of a curve can be found very easily when the curve is represented by parametric equations. The slope of a curve at any point equals zero when the tangent line is parallel to the X axis. In parametric equations, if $x = x(t)$ and $y = y(t)$, then the horizontal and vertical tangents can be found easily by setting

$$\frac{\Delta y}{\Delta t} = 0$$

and

$$\frac{\Delta x}{\Delta t} = 0$$

For the horizontal tangent solve $\frac{\Delta y}{\Delta t}$ equals zero for t and for the vertical tangent solve $\frac{\Delta x}{\Delta t}$ equals zero for t.

EXAMPLE: Find the points of contact of the horizontal and the vertical tangents to the curve represented by the parametric equations

$$x = 3 - 4 \sin \theta$$

and

$$y = 4 + 3 \cos \theta$$

Plot the graph of the curve by taking θ from 0° to 360° in increments of 30°, given that

$$\frac{\Delta x}{\Delta \theta} = -4 \cos \theta$$

and

$$\frac{\Delta y}{\Delta \theta} = -3 \sin \theta$$

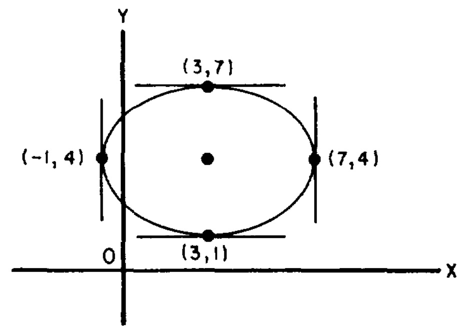


Figure 3-7.—Ellipse.

SOLUTION: The graph of the curve shows that the figure is an ellipse, figure 3-7; consequently, it will have two horizontal and two vertical tangents. The coordinates of the horizontal tangent points are found by first setting

$$\frac{\Delta y}{\Delta \theta} = 0$$

This gives

$$-3 \sin \theta = 0$$

so that

$$\sin \theta = 0$$

and

$$\theta = 0° \text{ or } 180°$$

Substituting 0°, we have

$$\begin{aligned} x &= 3 - 4 \sin 0° \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} y &= 4 + 3 \cos 0^\circ \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Substituting 180°, we obtain

$$\begin{aligned} x &= 3 - 4 \sin 180^\circ \\ &= 3 - 0 \\ &= 3 \end{aligned}$$

and

$$\begin{aligned} y &= 4 + 3 \cos 180^\circ \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

The coordinates of the points of contact of the horizontal tangents to the ellipse are (3,1) and (3,7).

The coordinates of the vertical tangent points of contact are found by setting

$$\frac{\Delta x}{\Delta \theta} = 0$$

We find

$$-4 \cos \theta = 0$$

from which

$$\theta = 90^\circ \text{ or } 270^\circ$$

Substituting 90°, we obtain

$$\begin{aligned} x &= 3 - 4 \sin 90^\circ \\ &= 3 - 4 \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} y &= 4 + 3 \cos 90^\circ \\ &= 4 + 0 \\ &= 4 \end{aligned}$$

Substituting 270° gives

$$x = 3 - 4 \sin 270°$$
$$= 3 + 4$$
$$= 7$$

and

$$y = 4 + 3 \cos 270°$$
$$= 4 + 0$$
$$= 4$$

The coordinates of the points of contact of the vertical tangents to the ellipse are $(-1,4)$ and $(7,4)$.

PRACTICE PROBLEMS:

Find the equations of the tangent line and the normal line and the lengths of the tangent and the normal for each of the following curves at the point indicated:

1. $x = t^3$

   $y = 3t$

   at $t = -1$

   given $\frac{\Delta x}{\Delta t} = 3t^2$ and $\frac{\Delta y}{\Delta t} = 3$

2. $x = t^2 + 8$

   $y = t^2 + 1$

   at $t = 2$

   given $\frac{\Delta x}{\Delta t} = 2t$ and $\frac{\Delta y}{\Delta t} = 2t$

3. $x = t$

$y = t^2$

at $t = 1$

given $\frac{\Delta x}{\Delta t} = 1$ and $\frac{\Delta y}{\Delta t} = 2t$

4. Find the points of contact of the horizontal and the vertical tangents to the curve

$$x = 2 \cos \theta$$

$$y = 3 \sin \theta$$

given

$$\frac{\Delta x}{\Delta \theta} = -2 \sin \theta$$

$$\frac{\Delta y}{\Delta \theta} = 3 \cos \theta$$

ANSWERS:

1. Equation of tangent line	$y = x - 2$
Equation of normal line	$y = -x - 4$
Length of tangent	$3\sqrt{2}$
Length of normal	$3\sqrt{2}$
2. Equation of tangent line	$y = x - 7$
Equation of normal line	$y = -x + 17$
Length of tangent	$5\sqrt{2}$
Length of normal	$5\sqrt{2}$

3. Equation of tangent line $y = 2x - 1$

Equation of normal line $y = \frac{-x}{2} + \frac{3}{2}$

Length of tangent $\frac{\sqrt{5}}{2}$

Length of normal $\sqrt{5}$

4. Coordinates of the points of contact of the horizontal tangent to the ellipse are $(0,3)$ and $(0,-3)$ and the vertical tangent to the ellipse are $(2,0)$ and $(-2,0)$.

SUMMARY

The following are the major topics covered in this chapter:

1. **Slope of a curve at a point:**

$$\frac{\Delta y}{\Delta x} = \tan\,\theta$$

where θ equals the inclination of the tangent line. If the line tangent to the curve is horizontal, then

$$\frac{\Delta y}{\Delta x} = \tan\,0^\circ = 0$$

If the line tangent to the curve is vertical, then

$$\frac{\Delta y}{\Delta x} = \tan\,90^\circ = \infty$$

When the slope of a curve is zero, the curve may be at either a maximum or a minimum.

2. **Tangent at a given point on the standard parabola** $y^2 = 4ax$:

$$m = \frac{2a}{y_1}$$

where a is the same as in the standard equation for parabolas, and y_1 is the y coordinate of the given point (x_1,y_1).

3. **Tangent at a given point on other curves:** To find the slope, m, of a given curve at point $P_1(x_1,y_1)$, choose a second point, P', on the curve so that it has coordinates $(x_1 + \Delta x, y_1 + \Delta y)$; then substitute each of the coordinates of P' and P_1 in the equation of the curve and simplify. Divide both sides by Δx and eliminate terms that contain powers of Δy higher than the first power. Solve for $\frac{\Delta y}{\Delta x}$. Let Δx approach zero and $\frac{\Delta y}{\Delta x}$ will approach the slope of the tangent line, m, at point P_1.

4. **Equation of the tangent line:**

$$y - y_1 = m_1(x - x_1)$$

5. **Equation of the normal line:**

$$y - y_1 = \frac{-1}{m_1}(x - x_1)$$

6. **Relationships between the slopes of the tangent and normal lines:** The slope of the normal line is the negative reciprocal of the slope of the tangent line.

   The inclination of one line must be 90° greater than the other.

7. **Length of the tangent:** The *length of the tangent* is defined as that portion of the tangent line between the point $P_1(x_1,y_1)$ and the point where the tangent line crosses the X axis.

$$\text{length of the tangent} = \sqrt{\left(\frac{y_1}{m_1}\right)^2 + (y_1)^2}$$

8. **Length of the normal:** The *length of the normal* is defined as that portion of the normal line between the point $P_1(x_1,y_1)$ and the X axis.

$$\text{length of the normal} = \sqrt{(y_1 m_1)^2 + (y_1)^2}$$

9. **Parametric equations:** If the variables x and y of the Cartesian coordinate system are expressed in terms of a third variable, say t (or θ), then the variable t (or θ) is called a *parameter*. The two equations $x = x(t)$ and $y = y(t)$ [or $x = x(\theta)$ and $y = y(\theta)$] are called *parametric equations*.

ADDITIONAL PRACTICE PROBLEMS

Find a) the slope, b) equation of the tangent line, c) equation of the normal line, d) length of the tangent, and e) length of the normal, in problems 1 through 6, at the given points.

1. $x^2 = \frac{-16}{5}y$ at $(4, -5)$

2. $x^2 + y^2 = 64$ at $(\sqrt{32}, \sqrt{32})$

3. $\frac{y^2}{8} - \frac{x^2}{9} = 1$ at $(-3,4)$

4. $\frac{x^2}{4} + \frac{y^2}{12} = 1$ at $(-1,3)$

5. $x = 2t - 3t^2$

   $y = 2t^3 - t^2$

   at $t = 2$

   given $\frac{\Delta x}{\Delta t} = 2 - 6t$ and

   $\frac{\Delta y}{\Delta t} = 6t^2 - 2t$

6. $x = \cos^2\theta - \sin^2\theta$

   $y = \sin^2\theta - \cos^2\theta$

   at $\theta = 30°$

   given $\frac{\Delta x}{\Delta \theta} = -4 \cos \theta \sin \theta$ and

   $\frac{\Delta y}{\Delta \theta} = 4 \sin \theta \cos \theta$

7. Find the points of contact of a) the horizontal and b) the vertical tangents to the curve

$$x = t^2 + t$$

$$y = t^2 - t$$

   given $\frac{\Delta x}{\Delta t} = 2t + 1$ and

   $\frac{\Delta y}{\Delta t} = 2t - 1$

ANSWERS TO ADDITONAL PRACTICE PROBLEMS

1a. $m = -5/2$
b. $y = (-5/2)x + 5$
c. $y = (2/5)x - 33/5$
d. $\sqrt{29}$
e. $5\sqrt{29}/2$

2a. $m = -1$
b. $y = -x + 8\sqrt{2}$
c. $y = x$
d. 8
e. 8

3a. $m = -2/3$
b. $y = (-2/3)x + 2$
c. $y = (3/2)x + 17/2$
d. $2\sqrt{13}$
e. $4\sqrt{13}/3$

4a. $m = 1$
b. $y = x + 4$
c. $y = -x + 2$
d. $3\sqrt{2}$
e. $3\sqrt{2}$

5a. $m = -2$
b. $y = -2x - 4$
c. $y = x/2 + 16$
d. $6\sqrt{5}$
e. $12\sqrt{5}$

6a. $m = -1$
b. $y = -x$
c. $y = x - 1$
d. $1/\sqrt{2}$ or $\sqrt{2}/2$
e. $1/\sqrt{2}$ or $\sqrt{2}/2$

7a. $(3/4, -1/4)$
b. $(-1/4, 3/4)$

CHAPTER 4

LIMITS AND DIFFERENTIATION

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define a limit, find the limit of indeterminate forms, and apply limit formulas.
2. Define an infinitesimal, determine the sum and product of infinitesimals, and restate the concept of infinitesimals.
3. Identify discontinuities in a function.
4. Relate increments to differentiation, apply the general formula for differentiation, and find the derivative of a function using the general formula.

INTRODUCTION

Limits and differentiation are the beginning of the study of calculus, which is an important and powerful method of computation.

LIMIT CONCEPT

The study of the limit concept is very important, for it is the very heart of the theory and operation of calculus. We will include in this section the definition of limit, some of the indeterminate forms of limits, and some limit formulas, along with example problems.

DEFINITION OF LIMIT

Before we start differentiation, we must understand certain concepts. One of these concepts deals with the limit of a function. Many times you will need to find the value of the limit of a function.

The discussion of limits will begin with an intuitive point of view.

We will work with the equation

$$y = f(x) = x^2$$

which is shown in figure 4-1. Point P represents the point corresponding to

$$y = 16$$

and

$$x = 4$$

The behavior of y for given values of x near the point

$$x = 4$$

is the center of the discussion. For the present we will exclude point P, which is encircled on the graph.

We will start with values lying between and including

$$x = 2$$

and

$$x = 6$$

indicated by interval AB in figure 4-1, view A. This interval may be written as

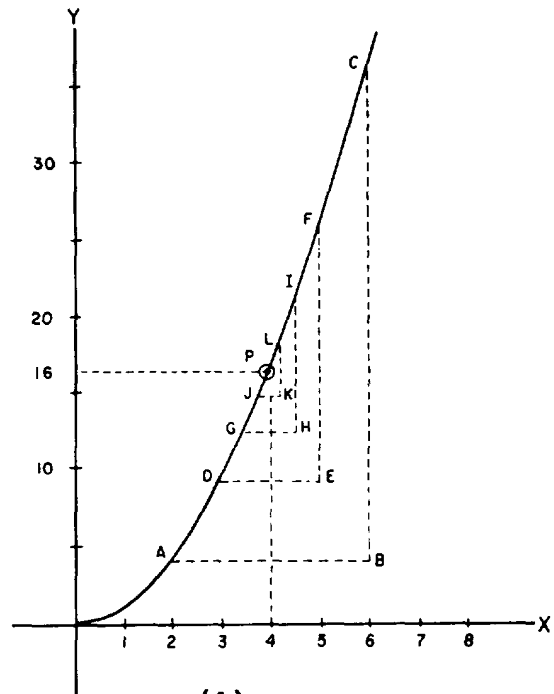
$$2 \leq x \leq 6 \qquad \text{or} \qquad 0 \leq |x - 4| \leq 2$$

The corresponding interval for y is between and includes

$$y = 4$$

and

$$y = 36$$



(A)

Interval of x		Interval of $f(x)$	
2 - 6	AB	4 - 36	BC
3 - 5	DE	9 - 25	EF
3.5 - 4.5	GH	12.25 - 20.25	HI
3.9 - 4.1	JK	15.21 - 16.81	KL

(B)

Figure 4-1.—(A) Graph of $y = x^2$; (B) value chart.

We now take a smaller interval, DE, about $x = 4$ by using values of

$$x = 3$$

and

$$x = 5$$

and find the corresponding interval for y to be between

$$y = 9$$

and

$$y = 25$$

inclusively.

These intervals for x and y are written as

$$0 \leq |x - 4| \leq 1$$

and

$$9 \leq y \leq 25$$

As we diminish the interval of x around

$$x = 4 \text{ (intervals } GH \text{ and } JK)$$

we find the values of

$$y = x^2$$

to be grouped more and more closely around

$$y = 16$$

This is shown by the chart in figure 4-1, view B.

Although we have used only a few intervals of x in the discussion, you should easily see that we can make the values about y group as closely as we desire by merely limiting the values assigned to x about

$$x = 4$$

Because the foregoing is true, we may now say that the limit of x^2, as x approaches 4, results in the value 16 for y, and we write

$$\lim_{x \to 4} x^2 = 16$$

In the general form we may write

$$\lim_{x \to a} f(x) = L \tag{4.1}$$

Equation (4.1) means that as x approaches a, the limit of $f(x)$ will approach L, where L is the limit of $f(x)$ as x approaches a. No statement is made about $f(a)$, for it may or may not exist, although the limit of $f(x)$, as x approaches a, is defined.

We are now ready to define a limit.

Let $f(x)$ be defined for all x in the interval near

$$x = a$$

but not necessarily at

$$x = a$$

Then there exists a number, L, such that for every positive number ε (epsilon), however small,

$$| f(x) - L | < \varepsilon$$

provided that we may find a positive number δ (delta) such that

$$0 < | x - a | < \delta$$

Then we say L is the limit of $f(x)$ as x approaches a, and we write

$$\lim_{x \to a} f(x) = L$$

This means that for every given number $\varepsilon > 0$, we must find a number δ such that the difference between $f(x)$ and L is smaller than the number ε whenever

$$0 < | x - a | < \delta$$

EXAMPLE: Suppose we are given $\varepsilon = 0.1$ and

$$\lim_{x \to 1} \frac{x^2 + x - 2}{3(x - 1)} = 1$$

find a $\delta > 0$.

SOLUTION: We must find a number δ such that for all points except

$$x = 1$$

we have the difference between $f(x)$ and 1 smaller than 0.1.

We write

$$\left| \frac{x^2 + x - 2}{3(x - 1)} - 1 \right| < 0.1$$

and

$$\frac{x^2 + x - 2}{3(x - 1)} - 1$$

$$= \frac{(x + 2)(x - 1)}{3(x - 1)} - 1$$

and we consider only values where

$$x \neq 1$$

Simplifying the first term, we have

$$\frac{(x + 2)(x - 1)}{3(x - 1)} = \frac{x + 2}{3}$$

Finally, combine terms as follows:

$$\frac{x + 2}{3} - 1 = \frac{x + 2 - 3}{3} = \frac{x - 1}{3}$$

so that

$$\left| \frac{x - 1}{3} \right| < 0.1$$

or

$$| x - 1 | < 0.3$$

Therefore, $\delta = 0.3$ and we have fulfilled the definition of the limit.

If the limit of a function exists, then

$$\lim_{x \to a} f(x) = f(a)$$

So we can often evaluate the limit by substitution.

For instance, to find the limit of the function $x^2 - 3x + 2$ as x approaches 3, we substitute 3 for x in the function. Then

$$\begin{aligned} f(3) &= 3^2 - 3(3) + 2 \\ &= 9 - 9 + 2 \\ &= 2 \end{aligned}$$

Since x is a variable, it may assume a value as close to 3 as we wish; and the closer we choose the value of x to 3, the closer $f(x)$ will approach the value of 2. Therefore, 2 is called the limit of $f(x)$ as x approaches 3, and we write

$$\lim_{x \to 3} (x^2 - 3x + 2) = 2$$

PRACTICE PROBLEMS:

Find the limit of each of the following functions:

1. $\lim_{x \to 1} \dfrac{2x^2 - 1}{2x - 1}$

2. $\lim_{x \to 2} (x^2 - 2x + 3)$

3. $\lim\limits_{x \to a} \dfrac{x^2 - a}{a}$

4. $\lim\limits_{t \to 0} (5t^2 - 3t + 2)$

5. $\lim\limits_{E \to 6} \dfrac{E^3 - E}{E - 1}$

6. $\lim\limits_{Z \to 0} \dfrac{Z^2 - 3Z + 2}{Z - 4}$

ANSWERS:

1. 1

2. 3

3. $a - 1$

4. 2

5. 42

6. $\dfrac{-1}{2}$

INDETERMINATE FORMS

When the value of a limit is obtained by substitution and it assumes any of the following forms, another method for finding the limit must be used:

$$\frac{0}{0}, \frac{\infty}{\infty}, (\infty)0, 0^0, \infty^0, 1^\infty$$

These are called *indeterminate forms.*

There are many methods of evaluating indeterminate forms. *Two methods of evaluating indeterminate forms* *are (1) factoring and (2) division of the numerator and denominator by powers of the variable.*

Sometimes factoring will resolve an indeterminate form.

EXAMPLE: Find the limit of

$$\frac{x^2 - 9}{x - 3} \text{ as } x \text{ approaches } 3$$

SOLUTION: By substitution we find

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{0}{0}$$

which is an indeterminate form and is therefore excluded as a possible limit. We must now search for a method to find the limit. Factoring is attempted, which results in

$$\frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{x - 3}$$

$$= x + 3$$

so that

$$\lim_{x \to 3} (x + 3) = 6$$

and we have a determinate limit of 6.

Another indeterminate form is often met when we try to find the limit of a function as the independent variable approaches infinity.

EXAMPLE: Find the limit of

$$\frac{x^4 + 2x^3 - 3x^2 + 2x}{3x^4 - 2x^2 + 1}$$

as $x \to \infty$.

SOLUTION: If we let x approach infinity in the original expression, the result will be

$$\lim_{x \to \infty} \frac{x^4 + 2x^3 - 3x^2 + 2x}{3x^4 - 2x^2 + 1} = \frac{\infty}{\infty}$$

which must be excluded as an indeterminate form. However, if we divide both numerator and denominator by x^4, we obtain

$$\lim_{x \to \infty} \frac{1 + \frac{2}{x} - \frac{3}{x^2} + \frac{2}{x^3}}{3 - \frac{2}{x^2} + \frac{1}{x^4}}$$

$$= \frac{1 + 0 - 0 + 0}{3 - 0 + 0}$$

$$= \frac{1}{3}$$

and we have a determinate limit of $\frac{1}{3}$.

PRACTICE PROBLEMS:

Find the limit of the following:

1. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

2. $\lim\limits_{x \to \infty} \dfrac{2x + 3}{7x - 6}$

3. $\lim\limits_{a \to 0} \dfrac{2a^2b - 3ab^2 + 2ab}{5ab - a^3b^2}$

4. $\lim\limits_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

5. $\lim\limits_{x \to a} \dfrac{x^4 - a^4}{x - a}$

6. $\lim\limits_{a \to 0} \dfrac{(x - a)^2 - x^2}{a}$

ANSWERS:

1. 4
2. $\frac{2}{7}$
3. $\frac{2 - 3b}{5}$
4. 5
5. $4a^3$
6. $-2x$

LIMIT THEOREMS

To obtain results in calculus, we will frequently operate with limits. The proofs of theorems shown in this section will be omitted in the interest of brevity. The theorems will be stated and examples will be given.

Assume that we have three simple functions of x. Further, let these functions [$f(x)$, $g(x)$, and $h(x)$] have separate limits such that

$$\lim_{x \to a} f(x) = A$$

$$\lim_{x \to a} g(x) = B$$

$$\lim_{x \to a} h(x) = C$$

Theorem 1. *The limit of the sum of two functions is equal to the sum of the limits*:

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= A + B$$

This theorem may be extended to include any number of functions, such as

$$\lim_{x \to a} [f(x) + g(x) + h(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)$$

$$= A + B + C$$

EXAMPLE: Find the limit of

$$(x - 3)^2 \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} (x - 3)^2 = \lim_{x \to 3} (x^2 - 6x + 9)$$

$$= \lim_{x \to 3} x^2 - \lim_{x \to 3} 6x + \lim_{x \to 3} 9$$

$$= 9 - 18 + 9$$

$$= 0$$

Theorem 2. *The limit of a constant, c, times a function, f(x), is equal to the constant, c, times the limit of the function:*

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) = cA$$

EXAMPLE: Find the limit of

$$2x^2 \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} 2x^2 = 2 \lim_{x \to 3} x^2$$

$$= (2)(9)$$

$$= 18$$

Theorem 3. *The limit of the product of two functions is equal to the product of their limits*:

$$\lim_{x \to a} f(x)g(x) = \left[\lim_{x \to a} f(x)\right]\left[\lim_{x \to a} g(x)\right] = AB$$

EXAMPLE: Find the limit of

$$(x^2 - x)(\sqrt{2x}) \text{ as } x \to 2$$

SOLUTION:

$$\lim_{x \to 2} (x^2 - x)(\sqrt{2x}) = \left[\lim_{x \to 2} (x^2 - x)\right]\left[\lim_{x \to 2} \sqrt{2x}\right]$$

$$= (4 - 2)(\sqrt{4})$$

$$= 4$$

Theorem 4. *The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the divisor is not equal to zero:*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}, \text{ if } B \neq 0$$

EXAMPLE: Find the limit of

$$\frac{3x^2 + x - 6}{2x - 5} \text{ as } x \to 3$$

SOLUTION:

$$\lim_{x \to 3} \frac{3x^2 + x - 6}{2x - 5}$$

$$= \frac{\lim_{x \to 3} 3x^2 + x - 6}{\lim_{x \to 3} 2x - 5}$$

$$= 24$$

PRACTICE PROBLEMS

Find the limits of the following, using the theorem indicated:

1. $x^2 + x + 2$ as $x \to 1$ (Theorem 1)
2. $7(x^2 - 13)$ as $x \to 4$ (Theorem 2)
3. $(5x^4)(x - 1)$ as $x \to 2$ (Theorem 3)
4. $\dfrac{2x^2 + x - 4}{3x - 7}$ as $x \to 3$ (Theorem 4)

ANSWERS:

1. 4
2. 21
3. 80
4. $\frac{17}{2}$

INFINITESIMALS

In chapter 3, we found the slope of a curve at a given point by taking very small increments of Δy and Δx, and the slope was said to be equal to $\frac{\Delta y}{\Delta x}$. This section will be a continuation of this concept.

DEFINITION

A variable that approaches 0 as a limit is called an *infinitesimal.* This may be written as

$$\lim V = 0$$

or

$$V \to 0$$

and means, as recalled from a previous section of this chapter, that the numerical value of V becomes and remains less than any positive number ε.

If the

$$\lim V = L$$

then

$$\lim V - L = 0$$

which indicates the difference between a variable and its limit is an *infinitesimal.* Conversely, if the difference between a variable and a constant is an infinitesimal, then the variable approaches the constant as a limit.

EXAMPLE: As x becomes increasingly large, is the term $\frac{1}{x^2}$ an infinitesimal?

SOLUTION: By the definition of infinitesimal, if $\frac{1}{x^2}$ approaches 0 as x increases in value, then $\frac{1}{x^2}$ is an infinitesimal. We see that $\frac{1}{x^2} \rightarrow 0$ and is therefore an infinitesimal.

EXAMPLE: As x approaches 2, is the expression $\frac{x^2 - 4}{x - 2} - 4$ an infinitesimal?

SOLUTION: By the converse of the definition of infinitesimal, if the difference between $\frac{x^2 - 4}{x - 2}$ and 4 approaches 0, as x approaches 2, the expression $\frac{x^2 - 4}{x - 2} - 4$ is an infinitesimal. By direct substitution we find an indeterminate form; therefore, we make use of our knowledge of indeterminates and write

$$\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$$

and

$$\lim_{x \to 2} (x + 2) = 4$$

The difference between 4 and 4 is 0, so the expression $\frac{x^2 - 4}{x - 2} - 4$ is an infinitesimal as x approaches 2.

SUMS

An infinitesimal is a variable that approaches 0 as a limit. We state that ε and δ, in figure 4-2, are infinitesimals because they both approach 0 as shown.

Theorem 1. *The algebraic sum of any number of infinitesimals is an infinitesimal.*

In figure 4-2, as ε and δ approach 0, notice that their sum approaches 0; by definition this sum is an infinitesimal. This approach may be used for the sum of any number of infinitesimals.

δ \ ε	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\to 0$
1	2	$\frac{5}{4}$	$\frac{17}{16}$	$\frac{65}{64}$	$\frac{257}{256}$	
$\frac{1}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	$\frac{5}{16}$	$\frac{17}{64}$	$\frac{65}{256}$	
$\frac{1}{16}$	$\frac{17}{16}$	$\frac{5}{16}$	$\frac{1}{8}$	$\frac{5}{64}$	$\frac{17}{256}$	
$\frac{1}{64}$	$\frac{65}{64}$	$\frac{17}{64}$	$\frac{5}{64}$	$\frac{1}{32}$	$\frac{5}{256}$	
$\frac{1}{256}$	$\frac{257}{256}$	$\frac{65}{256}$	$\frac{17}{256}$	$\frac{5}{256}$	$\frac{1}{128}$	
$\downarrow 0$						$\searrow 0$

Figure 4-2.—Sums of infinitesimals.

PRODUCTS

Theorem 2. *The product of any number of infinitesimals is an infinitesimal.*

In figure 4-3, the product of two infinitesimals, ε and δ, is an infinitesimal as shown. The product of any number of infinitesimals is also an infinitesimal by the same approach as shown for two numbers.

Theorem 3. *The product of a constant and an infinitesimal is an infinitesimal.*

This may be shown, in figure 4-3, by holding either ε or δ constant and noticing their product as the variable approaches 0.

δ \ ε	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\to 0$
1	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	
$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	
$\frac{1}{64}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	$\frac{1}{16384}$	
$\frac{1}{256}$	$\frac{1}{256}$	$\frac{1}{1024}$	$\frac{1}{4096}$	$\frac{1}{16384}$	$\frac{1}{65536}$	
$\downarrow 0$						$\searrow 0$

Figure 4-3.—Products of infinitesimals.

CONCLUSIONS

The term infinitesimal was used to describe the term Δx as it approaches zero. The quantity Δx was called an increment of x, where an increment was used to imply that we made a change in x. Thus $x + \Delta x$ indicates that we are holding x constant and changing x by a variable amount which we will call Δx.

A very small increment is sometimes called a *differential*. A small Δx is indicated by dx. The differential of θ is $d\theta$ and that of y is dy. The limit of Δx as it approaches zero is, of course, zero; but that does not mean the ratio of two infinitesimals cannot be a real number or a real function of x. For instance, no matter how small Δx is chosen, the ratio $\frac{dx}{dx}$ will still be equal to 1.

In the section on indeterminate forms, a method for evaluating the form $\frac{0}{0}$ was shown. This form results whenever the limit takes the form of one infinitesimal over another. In every case the limit was a real number.

DISCONTINUITIES

The discussion of discontinuties will be based on a comparison to continuity.

A function, $f(x)$, is *continuous* at $x = a$ if the following three conditions are met:

1. *$f(x)$ is defined at $x = a$.*
2. *The limit of $f(x)$ exists as x approaches a or $x \rightarrow a$.*
3. *The value of $f(x)$ at $x = a$ is equal to the limit of $f(x)$ at $x = a$ or $\lim_{x \rightarrow a} f(x) = f(a)$.*

If a function $f(x)$ is not continuous at

$$x = a$$

then it is said to be *discontinuous* at

$$x = a$$

We will use examples to show the above statements.

EXAMPLE: In figure 4-4, is the function

$$f(x) = x^2 + x - 4$$

continuous at $f(2)$?

SOLUTION:

$$f(2) = 4 + 2 - 4$$

$$= 2$$

and

$$\lim_{x \to 2} x^2 + x - 4 = 2$$

and

$$\lim_{x \to 2} f(x) = f(2)$$

Therefore, the curve is continuous at

$$x = 2$$

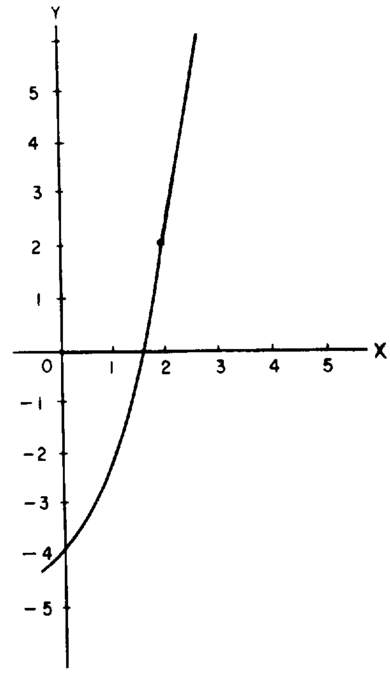


Figure 4-4.—Function $f(x) = x^2 + x - 4$.

EXAMPLE: In figure 4-5, is the function

$$f(x) = \frac{x^2 - 4}{x - 2}$$

continuous at $f(2)$?

SOLUTION:

$f(2)$ is undefined at

$$x = 2$$

and the function is therefore discontinuous at

$$x = 2$$

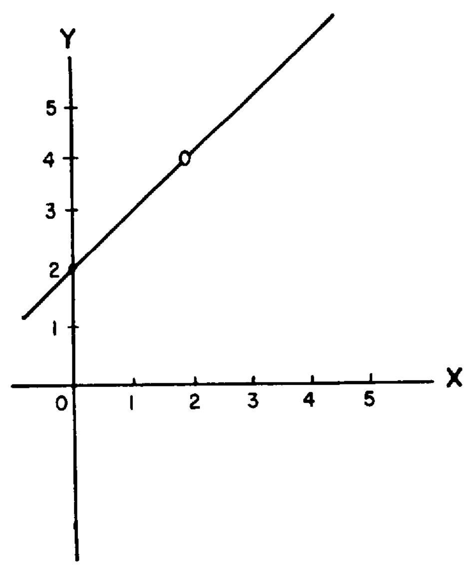


Figure 4-5.—Function $f(x) = \frac{x^2 - 4}{x - 2}$.

However, by extending the original equation of $f(x)$ to read

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, x \neq 2 \\ \\ 4, x = 2 \end{cases}$$

we will have a continuous function at

$$x = 2$$

NOTE: The value of 4 at $x = 2$ was found by factoring the numerator of $f(x)$ and then simplifying.

A common kind of discontinuity occurs when we are dealing with the tangent function of an angle. Figure 4-6 is the graph of the tangent as the angle varies from 0° to 90°; that is, from 0 to $\frac{\pi}{2}$. The value of the tangent at $\frac{\pi}{2}$ is undefined.

Thus the function is said to be discontinuous at $\frac{\pi}{2}$.

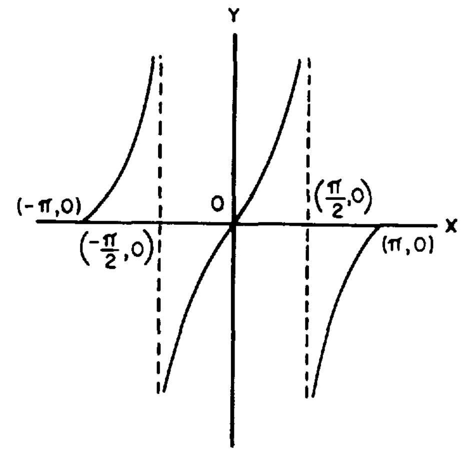


Figure 4-6.—Graph of tangent function.

PRACTICE PROBLEMS:

In the following definitions of the functions, find where the functions are discontinuous and then extend the definitions so that the functions are continuous:

1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

2. $f(x) = \dfrac{x^2 + 2x - 3}{x + 3}$

3. $f(x) = \dfrac{x^2 + x - 12}{3x - 9}$

ANSWERS:

1. $x = 2, f(2) = 3$
2. $x = -3, f(-3) = -4$
3. $x = 3, f(3) = \frac{7}{3}$

INCREMENTS AND DIFFERENTIATION

In this section we will extend our discussion of limits and examine the idea of the derivative, the basis of differential calculus.

We will assume we have a particular function of x, such that

$$y = x^2$$

If x is assigned the value 10, the corresponding value of y will be $(10)^2$ or 100. Now, if we increase the value of x by 2, making it 12, we may call this increase of 2 an increment or Δx. This results in an increase in the value of y, and we may call this increase an increment or Δy. From this we write

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ &= (10 + 2)^2 \\ &= 144 \end{aligned}$$

As x increases from 10 to 12, y increases from 100 to 144 so that

$$\Delta x = 2$$

$$\Delta y = 44$$

and

$$\frac{\Delta y}{\Delta x} = \frac{44}{2} = 22$$

We are interested in the ratio $\frac{\Delta y}{\Delta x}$ because the limit of this ratio as Δx approaches zero is the derivative of

$$y = f(x)$$

As you recall from the discussion of limits, as Δx is made smaller, Δy gets smaller also. For our problem, the ratio $\frac{\Delta y}{\Delta x}$ approaches 20. This is shown in table 4-1.

Table 4-1.—Slope Values

Variable	Values of the variable						
Δx	2	1	0.5	0.2	0.1	0.01	0.0001
Δy	44	21	10.25	4.04	2.01	0.2001	0.00200001
$\frac{\Delta y}{\Delta x}$	22	21	20.5	20.2	20.1	20.01	20.0001

We may use a much simpler way to find that the limit of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero is, in this case, equal to 20. We have two equations

$$y + \Delta y = (x + \Delta x)^2$$

and

$$y = x^2$$

By expanding the first equation so that

$$y + \Delta y = x^2 + 2x\Delta x + (\Delta x)^2$$

and subtracting the second from this, we have

$$\Delta y = 2x\Delta x + (\Delta x)^2$$

Dividing both sides of the equation by Δx gives

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x$$

Now, taking the limit as Δx approaches zero, gives

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

Thus,

$$\frac{dy}{dx} = 2x \qquad (1)$$

NOTE: Equation (1) is one way of expressing the derivative of y with respect to x. Other ways are

$$\frac{dy}{dx} = y' = f'(x) = D(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Equation (1) has the advantage that it is exact and true for all values of x. Thus if

$$x = 10$$

then

$$\frac{dy}{dx} = 2(10) = 20$$

and if

$$x = 3$$

then

$$\frac{dy}{dx} = 2(3) = 6$$

This method for obtaining the derivative of y with respect to x is general and may be formulated as follows:

1. *Set up the function of x as a function of* $(x + \Delta x)$ *and expand this function.*

2. *Subtract the original function of x from the new function of* $(x + \Delta x)$.

3. *Divide both sides of the equation by* Δx.

4. *Take the limit of all the terms in the equation as* Δx *approaches zero. The resulting equation is the derivative of* $f(x)$ *with respect to x.*

GENERAL FORMULA

To obtain a formula for the derivative of any expression in x, assume the function

$$y = f(x) \tag{4.2}$$

so that

$$y + \Delta y = f(x + \Delta x) \tag{4.3}$$

Subtracting equation (4.2) from equation (4.3) gives

$$\Delta y = f(x + \Delta x) - f(x)$$

and dividing both sides of the equation by Δx, we have

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The desired formula is obtained by taking the limit of both sides as Δx approaches zero so that

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

NOTE: The notation $\frac{dy}{dx}$ is not to be considered as a fraction in which dy is the numerator and dx is the denominator. The expression $\frac{\Delta y}{\Delta x}$ is a fraction with Δy as its numerator and Δx as its denominator. Whereas, $\frac{dy}{dx}$ is a symbol representing the limit approached by $\frac{\Delta y}{\Delta x}$ as Δx approaches zero.

EXAMPLES OF DIFFERENTIATION

In this last section of the chapter, we will use several examples of differentiation to obtain a firm understanding of the general formula.

EXAMPLE: Find the derivative, $\frac{dy}{dx}$, for the function

$$y = 5x^3 - 3x + 2$$

determine the slopes of the tangent lines to the curve at

$$x = -1, \frac{-1}{\sqrt{5}}, 0, \frac{1}{\sqrt{5}}, 1$$

and draw the graph of the function.

SOLUTION: Finding the derivative by formula, we have

$$f(x + \Delta x)$$

$$= 5(x + \Delta x)^3 - 3(x + \Delta x) + 2 \qquad (1)$$

and

$$f(x) = 5x^3 - 3x + 2 \qquad (2)$$

Expand equation (1), then subtract equation (2) from equation (1), and simplify to obtain

$$f(x + \Delta x) - f(x)$$

$$= 5[3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3] - 3\Delta x$$

Dividing both sides by Δx, we have

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= 5[3x^2 + 3x\Delta x + (\Delta x)^2] - 3$$

Take the limit of both sides as $\Delta x \rightarrow 0$:

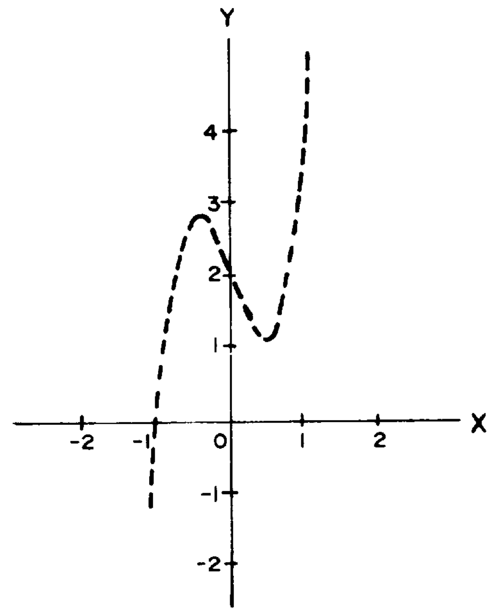
$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 15x^2 - 3$$

Then

$$\frac{dy}{dx} = 15x^2 - 3 \qquad (3)$$

The slopes of the tangent lines to the curve at the points given, using this derivative, are shown in figure 4-7, view B.

Thus we have a new method of graphing an equation. By substituting different values of x in equation (3), we can find the slope of the tangent line to the curve at the point corresponding to the value of x. The graph of the curve is shown in figure 4-7, view A.



(A)

x	-1	$-\frac{1}{\sqrt{5}}$	0	$\frac{1}{\sqrt{5}}$	1
$\frac{dy}{dx}$	12	0	-3	0	12
y	0	2.89	2	1.1	4

(B)

Figure 4-7.—(A) Graph of $f(x) = 5x^3 - 3x + 2$; (B) chart of values.

EXAMPLE: Differentiate the function; that is, find $\frac{dy}{dx}$ of

$$y = \frac{1}{x}$$

and then find the slope of the tangent line to the curve at

$$x = 2$$

SOLUTION: Apply the formula for the derivative, and simplify as follows:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$$

$$= \frac{\frac{x - (x + \Delta x)}{x(x + \Delta x)}}{\Delta x}$$

$$= \frac{-1}{x(x + \Delta x)}$$

Now take the limit of both sides as $\Delta x \to 0$ so that

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

To find the slope of the tangent line to the curve at the point where x has the value 2, substitute 2 for x in the expression for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{-1}{2^2}$$

$$= \frac{-1}{4}$$

EXAMPLE: Find the slope of the tangent line to the curve

$$f(x) = x^2 + 4$$

at

$$x = 3$$

SOLUTION: We need to find $\frac{dy}{dx}$, which is the slope of the tangent line at a given point. Apply the formula for the derivative as follows:

$$f(x + \Delta x) = (x + \Delta x)^2 + 4 \tag{1}$$

and

$$f(x) = x^2 + 4 \tag{2}$$

Expand equation (1) so that

$$f(x + \Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 + 4$$

Then subtract equation (2) from equation (1):

$$f(x + \Delta x) - f(x) = 2x\Delta x + (\Delta x)^2$$

Now, divide both sides by Δx:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = 2x + \Delta x$$

Then take the limit of both sides as $\Delta x \to 0$:

$$\frac{dy}{dx} = 2x$$

Substitute 3 for x in the expression for the derivative to find the slope of the tangent line at

$$x = 3$$

so that

$$\text{slope} = 6$$

In this last example we will set the derivative of the function, $f(x)$, equal to zero and determine the values of the independent variable that will make the derivative equal to zero to find a *maximum* or *minimum point on the curve.* By maximum or minimum of a curve, we mean the point or points through which the slope of the tangent line to the curve changes from positive to negative or from negative to positive.

NOTE: *When the derivative of a function is set equal to zero, that does not mean in all cases we will have found a maximum or minimum point on the curve.* A complete discussion of maxima or minima may be found in most calculus texts.

To set the derivative equal to zero, we will require that the following conditions be met:

1. We have a maximum or minimum point.

2. The derivative exists.

3. We are dealing with an interior point on the curve.

When these conditions are met, the derivative of the function will be equal to zero.

EXAMPLE: Find the derivative of the function

$$y = 5x^3 - 6x^2 - 3x + 3$$

set the derivative equal to zero, and find the points of maximum and minimum on the curve. Then verify this by drawing the graph of the curve.

SOLUTION: Apply the formula for $\frac{dy}{dx}$ as follows:

$$f(x + \Delta x) = 5(x + \Delta x)^3 - 6(x + \Delta x)^2 - 3(x + \Delta x) + 3 \qquad (1)$$

and

$$f(x) = 5x^3 - 6x^2 - 3x + 3 \qquad (2)$$

Expand equation (1) and subtract equation (2), obtaining

$$f(x + \Delta x) - f(x) =$$

$$5(3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) - 6(2x\Delta x + \Delta x^2) - 3\Delta x$$

Now, divide both sides by Δx, and take the limit as $\Delta x \to 0$ so that

$$\frac{dy}{dx} = 5(3x^2) - 6(2x) - 3$$

$$= 15x^2 - 12x - 3$$

Set $\frac{dy}{dx}$ equal to zero; thus

$$15x^2 - 12x - 3 = 0$$

Then

$$3(5x^2 - 4x - 1) = 0$$

and

$$(5x + 1)(x - 1) = 0$$

Set each factor equal to zero and find the points of maximum or minimum; that is,

$$5x = -1$$

$$x = \frac{-1}{5}$$

and

$$x = 1$$

The graph of the function is shown in figure 4-8.

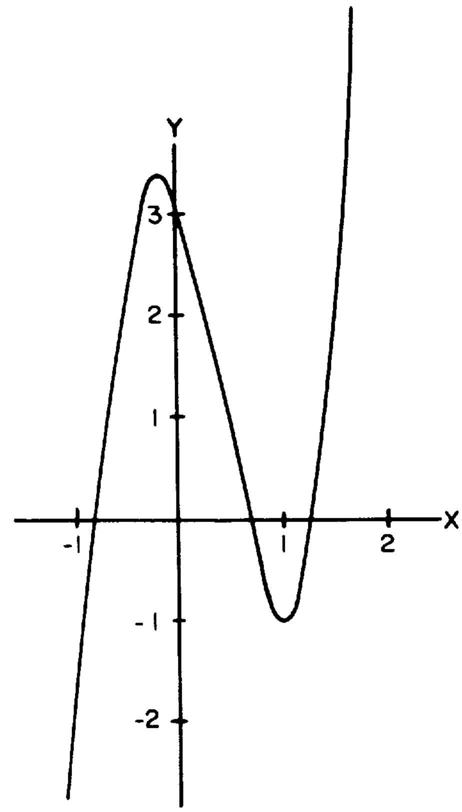


Figure 4-8.—Graph of $f(x) = 5x^3 - 6x^2 - 3x + 3$.

PRACTICE PROBLEMS:

Differentiate the functions in problems 1 through 3.

1. $f(x) = x^2 - 3$

2. $f(x) = x^2 - 5x$

3. $f(x) = 3x^2 - 2x + 3$

4. Find the slope of the tangent line to the curve

$$y = x^3 - 3x + 2$$

at the points

$$x = -2, 0, \text{ and } 3$$

5. Find the values of x where the function

$$f(x) = 2x^3 - 9x^2 - 60x + 12$$

has a maximum or a minimum.

ANSWERS:

1. $2x$
2. $2x - 5$
3. $6x - 2$
4. $m = 9, -3$, and 24
5. $x = -2$, $x = 5$

SUMMARY

The following are the major topics covered in this chapter:

1. **Definition of a limit**: Let $f(x)$ be defined for all x in the interval near $x = a$. Then there exists a number, L, such that for every positive number ε, however small, $|f(x) - L| < \varepsilon$, provided that we may find a positive number δ such that $0 < |x - a| < \delta$. Then we say L is the limit of $f(x)$ as x approaches a, and we write $\lim_{x \to a} f(x) = L$. This means that for every given number $\varepsilon > 0$, we must find a number δ such that the difference between $f(x)$ and L is smaller than the number ε whenever $0 < |x - a| < \delta$.

2. **Indeterminate forms**:

$$0/0, \infty/\infty,\ (\infty)0,\ 0^0, \infty^0,\ 1^\infty$$

   Two methods of evaluating indeterminate forms are (1) factoring and (2) division of the numerator and denominator by powers of the variable.

3. **Limit theorems**:

   Theorem 1. *The limit of the sum of two functions is equal to the sum of the limits*:

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

   This theorem can be extended to include any number of functions, such as

$$\lim_{x \to a} [f(x) + g(x) + h(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) + \lim_{x \to a} h(x)$$

   Theorem 2. *The limit of a constant, c, times a function, f(x), is equal to the constant, c, times the limit of the function:*

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$$

   Theorem 3. *The limit of the product of two functions is equal to the product of their limits:*

$$\lim_{x \to a} f(x)\, g(x) = \left[\lim_{x \to a} f(x)\right] \left[\lim_{x \to a} g(x)\right]$$

Theorem 4. *The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the divisor is not equal to zero:*

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)},\ \text{if}\ \lim_{x\to a} g(x)\neq 0$$

4. **Infinitesimals**: A variable that approaches 0 as a limit is called an *infinitesimal*:

$$\lim V = 0 \text{ or } V\to 0$$

The difference between a variable and its limit is an infinitesimal:

$$\text{If } \lim V = L, \text{ then } \lim V - L = 0$$

5. **Sum and product of infinitesimals:**

Theorem 1. *The algebraic sum of any number of infinitesimals is an infinitesimal.*

Theorem 2. *The product of any number of infinitesimals is an infinitesimal.*

Theorem 3. *The product of a constant and an infinitesimal is an infinitesimal.*

6. **Continuity**: A function, $f(x)$, is *continuous* at $x = a$ if the following three conditions are met:

   1. $f(x)$ is defined at $x = a$.

   2. The limit of $f(x)$ exists as x approaches a or $x\to a$.

   3. The value of $f(x)$ at $x = a$ is equal to the limit of $f(x)$ at $x = a$ or $\lim\limits_{x\to a} f(x) = f(a)$.

7. **Discontinuity**: If a function is not continuous at $x = a$, then it is said to be *discontinuous* at $x = a$.

8. **Ways of expressing the derivative of y with respect to x:**

$$\frac{dy}{dx} = y' = f'(x) = D(x) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$$

9. **Increment method for obtaining the derivative of *y* with respect to *x*:**

   1. Set up the function of x as a function of $(x + \Delta x)$ and expand this function.

   2. Subtract the original function of x from the new function of $(x + \Delta x)$.

   3. Divide both sides of the equation by Δx.

   4. Take the limit of all the terms in the equation as Δx approaches zero. The resulting equation is the derivative of $f(x)$ with respect to x.

10. **General formula for the derivative of any expression in *x*:**

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

11. **Maximum or minimum points on a curve:** Set the derivative of the function, $f(x)$, equal to zero and determine the values of the independent variable that will make the derivative equal to zero. (Note: When the derivative of a function is set equal to zero, that does not mean in all cases the curve will have a maximum or minimum point.)

ADDITIONAL PRACTICE PROBLEMS

Find the limit of each of the following:

1. $\lim\limits_{x \to 0} \sqrt{\dfrac{x + 16}{3x + 4}}$

2. $\lim\limits_{t \to -2} \dfrac{3t^2 + t}{t - 3}$

3. $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x^2 + 5x + 6}$

4. $\lim\limits_{h \to 0} \dfrac{(2 + h)^2 - 4}{h}$

5. $\lim\limits_{x \to \infty} \dfrac{3 + 2x + 10x^2}{2x^2 + 8}$

6. $\lim\limits_{x \to 9} \sqrt{x} + (x - 6)^2$

   (using Limit Theorem 1.)

7. $\lim\limits_{x \to 4} 6\sqrt{(x + 5)}$

   (using Limit Theorem 2.)

8. $\lim\limits_{x \to 3} \left(\dfrac{15}{x + 2}\right)\left(\dfrac{8x}{x^2 - 3}\right)$

   (using Limit Theorem 3.)

9. $\lim\limits_{x \to 5} \dfrac{60/(1 + x) + 2}{16/(x^2 - 21)}$

   (using Limit Theorem 4.)

10. Find where

$$f(x) = \frac{(x^2 + 6x + 8)(x - 3)}{x^2 - x - 6}$$

is discontinuous and then extend the equation so that the function is continuous.

11. Differentiate $f(x) = 6/x - 1$.

12. Find the slope of the tangent line to the curve $y = 3x^2 - 9x$ at the points $x = 0$ and 3.

13. Find the values of x where the function $f(x) = x(4 - 2x)$ has a maximum or a minimum.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 2
2. -2
3. 6
4. 4
5. 5
6. 12
7. 18
8. 12
9. 3
10. $x = -2, 3$

    $f(-2) = 2$

    $f(3) = 7$
11. $-6/x^2$
12. $m = -9$ and 9
13. $x = 1$

CHAPTER 5

DERIVATIVES

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Compute the derivative of a constant.

2. Compute the derivative of a variable raised to a power.

3. Compute the derivative of the sum and product of two or more functions and the quotient of two functions.

4. Compute the derivative of a function raised to a power, in radical form, and by using the chain rule.

5. Compute the derivative of an inverse function, an implicit function, a trigonometric function, and a natural logarithmic function.

6. Compute the derivative of a constant raised to a variable power.

INTRODUCTION

In the previous chapter on limits, we used the delta process to find the limit of a function as Δx approached zero. We called the result of this tedious and, in some cases, lengthy process the derivative. In this chapter we will examine some rules used to find the derivative of a function without using the delta process.

To find how y changes as x changes, we take the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$ and write

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

which is called the derivative of y with respect to x; we use the symbol $\frac{dy}{dx}$ to indicate the derivative and write

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$$

In this section we will learn a number of rules that will enable us to easily obtain the derivative of many algebraic functions. In the derivation of these rules, which will be called theorems, we will assume that

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = f'(x)$$

or

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

exists and is finite.

DERIVATIVE OF A CONSTANT

The method we will use to find the derivative of a constant is similar to the delta process used in the previous chapter but includes an analytical proof. A diagram is used to give a geometrical meaning of the function.

Theorem 1. *The derivative of a constant is zero*. Expressed as a formula, this may be written as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

where $y = c$.

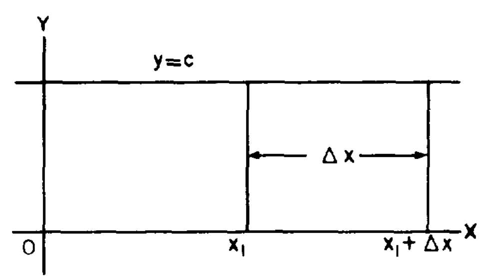


Figure 5-1.—Graph of y = c, where c is a constant.

PROOF: In figure 5-1, the graph of

$$y = c$$

where c is a constant, the value of y is the same for all values of x, and any change in x (that is, Δx) does not affect y; then

$$\Delta y = c - c = 0$$

$$\frac{\Delta y}{\Delta x} = 0$$

and

$$\frac{dy}{dx} = 0$$

Another way of stating this is that when x is equal to x_1 and when x is equal to $x_1 + \Delta x$, y has the same value. Therefore,

$$y = c$$

and

$$y + \Delta y = c$$

so that

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{c - c}{\Delta x}$$

and

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0$$

Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c - c}{\Delta x} = 0$$

The equation

$$y = c$$

represents a straight line parallel to the X axis. The slope of this line will be zero for all values of x. Therefore, the derivative is zero for all values of x.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$y = 6$$

SOLUTION:

$$y = 6$$

and

$$y + \Delta y = 6$$

Therefore,

$$\begin{aligned}\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{6 - 6}{\Delta x} \\ &= 0\end{aligned}$$

DERIVATIVES OF VARIABLES

In this section of variables, we will extend the theorems of limits covered previously. Recall that a derivative is actually a limit. The proof of the theorems presented here involve the delta process.

POWER FORM

Theorem 2. *The derivative of the function*

$$y = x^n$$

is given by

$$\frac{dy}{dx} = nx^{n-1}$$

if n is any real number.

PROOF: By definition

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - (x)^n}{\Delta x}$$

The expression $(x + \Delta x)^n$ may be expanded by the binomial theorem into

$$x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \cdots + \Delta x^n$$

Substituting in the expression for the derivative, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + \frac{n(n-1)}{2!}x^{n-2}\Delta x^2 + \cdots + \Delta x^n}{\Delta x}$$

Simplifying, this becomes

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-2}\Delta x + \cdots + \Delta x^{n-1}\right]$$

Letting Δx approach zero, we have

$$\frac{dy}{dx} = nx^{n-1}$$

Thus, the proof is complete.

EXAMPLE: Find the derivative of

$$f(x) = x^5$$

SOLUTION: Apply Theorem 2, such that,

$$x^5 = x^n$$

Therefore,

$$n = 5$$

and

$$n - 1 = 4$$

so that given

$$\frac{dy}{dx} = nx^{n-1}$$

and substituting values for n, find that

$$\frac{dy}{dx} = 5x^4$$

EXAMPLE: Find the derivative of

$$f(x) = x$$

SOLUTION: Apply Theorem 2, such that,

$$x^n = x$$

Therefore,

$$n = 1$$

and

$$n - 1 = 0$$

so that

$$\frac{dy}{dx} = x^0$$

$$= 1$$

The previous example is a special case of the power form and indicates that the derivative of a function with respect to itself is 1.

EXAMPLE: Find the derivative of

$$f(x) = ax$$

where a is a constant.

SOLUTION:

$$f(x) = ax$$

and

$$f(x + \Delta x) = a(x + \Delta x)$$

$$= ax + a\Delta x$$

so that

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= (ax + a\Delta x) - ax$$

$$= a\Delta x$$

Therefore,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{a\Delta x}{\Delta x}$$

$$= a$$

Table 5-1.—Derivatives of Functions

$f(x)$	3	x	x^2	x^3	x^4	$3x^2$	$9x^3$	x^{-1}	x^{-2}	$3x^{-4}$
$\frac{dy}{dx}$	0	1	$2x$	$3x^2$	$4x^3$	$6x$	$27x^2$	$-x^{-2}$	$-2x^{-3}$	$-12x^{-5}$

The previous example is a continuation of the derivative of a function with respect to itself and indicates that the derivative of a function with respect to itself, times a constant, is that constant.

EXAMPLE: Find the derivative of

$$f(x) = 6x$$

SOLUTION:

$$\frac{dy}{dx} = 6$$

A study of the functions and their derivatives in table 5-1 should further the understanding of this section.

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = 21$

2. $f(x) = x$

3. $f(x) = 21x$

4. $f(x) = 7x^3$

5. $f(x) = 4x^2$

6. $f(x) = 3x^{-2}$

ANSWERS:

1. 0
2. 1
3. 21
4. $21x^2$
5. $8x$
6. $-6x^{-3}$

SUMS

Theorem 3. *The derivative of the sum of two or more differentiable functions of x is equal to the sum of their derivatives.*

If two functions of x are given, such that

$$u = g(x)$$

and

$$v = h(x)$$

and also

$$\begin{aligned} y &= u + v \\ &= g(x) + h(x) \end{aligned}$$

then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

PROOF:

$$y = g(x) + h(x) \tag{5.1}$$

and

$$y + \Delta y = g(x + \Delta x) + h(x + \Delta x) \tag{5.2}$$

Subtract equation (5.1) from equation (5.2):

$$\Delta y = g(x + \Delta x) + h(x + \Delta x) - g(x) - h(x)$$

Rearrange this equation such that

$$\Delta y = g(x + \Delta x) - g(x) + h(x + \Delta x) - h(x)$$

Divide both sides of the equation by Δx and then take the limit as $\Delta x \to 0$:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$

$$+ \lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x}$$

But, by definition

$$\lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} = \frac{du}{dx}$$

and

$$\lim_{\Delta x \to 0} \frac{h(x + \Delta x) - h(x)}{\Delta x} = \frac{dv}{dx}$$

Then by substitution,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = x^3 - 8x^2 + 7x - 5$$

SOLUTION: Theorem 3 indicates that we should find the derivative of each term and then show them as a sum; that is, if

$$y = x^3, \frac{dy}{dx} = 3x^2$$

$$y = -8x^2, \frac{dy}{dx} = -16x$$

$$y = 7x, \frac{dy}{dx} = 7$$

$$y = -5, \frac{dy}{dx} = 0$$

and

$$y = x^3 - 8x^2 + 7x - 5$$

then

$$\frac{dy}{dx} = 3x^2 - 16x + 7 + 0$$

$$= 3x^2 - 16x + 7$$

PRACTICE PROBLEMS:

Find the derivative of the following:

1. $f(x) = x^2 + x - 1$
2. $f(x) = 2x^4 + 3x + 16$
3. $f(x) = 2x^3 + 3x^2 + x - 3$
4. $f(x) = 3x^3 + 2x^2 - 4x + 2 + 2x^{-1} - 3x^{-3}$

ANSWERS:

1. $2x + 1$
2. $8x^3 + 3$
3. $6x^2 + 6x + 1$
4. $9x^2 + 4x - 4 - 2x^{-2} + 9x^{-4}$

PRODUCTS

Theorem 4. *The derivative of the product of two differentiable functions of x is equal to the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function.*

If

$$y = uv$$

then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This theorem may be extended to include the product of three differentiable functions or more. The result for three functions would be as follows:

If

$$y = uvw$$

then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}$$

EXAMPLE: Find the derivative of

$$f(x) = (x^2 - 2)\ (x^4 + 5)$$

SOLUTION: The derivative of the first factor is $2x$, and the derivative of the second factor is $4x^3$. Therefore,

$$f'(x) = (x^2 - 2)\ (4x^3) + (x^4 + 5)\ (2x)$$

$$= 4x^5 - 8x^3 + 2x^5 + 10x$$

$$= 6x^5 - 8x^3 + 10x$$

EXAMPLE: Find the derivative of

$$f(x) = (x^3 - 3)\ (x^2 + 2)\ (x^4 - 5)$$

SOLUTION: The derivatives of the three factors, in the order given, are $3x^2$, $2x$, and $4x^3$.

Therefore,

$$f'(x) = (x^3 - 3)\ (x^2 + 2)\ (4x^3)$$

$$+ (x^2 + 2)\ (x^4 - 5)\ (3x^2)$$

$$+ (x^3 - 3)\ (x^4 - 5)\ (2x)$$

Expanding, we get

$$f'(x) = 4x^8 + 8x^6 - 12x^5 - 24x^3$$
$$+ 3x^8 + 6x^6 - 15x^4 - 30x^2$$
$$+ 2x^8 - 6x^5 - 10x^4 + 30x$$
$$= 9x^8 + 14x^6 - 18x^5 - 25x^4 - 24x^3 - 30x^2 + 30x$$

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = x^3(x^2 - 4)$
2. $f(x) = (x^3 - 3)\ (x^2 + 2x)$
3. $f(x) = (x^2 - 7x)\ (x^5 - 4x^2)$
4. $f(x) = (x - 2)\ (x^2 - 3)\ (x^3 - 4)$

ANSWERS:

1. $5x^4 - 12x^2$
2. $5x^4 + 8x^3 - 6x - 6$
3. $7x^6 - 42x^5 - 16x^3 + 84x^2$
4. $6x^5 - 10x^4 - 12x^3 + 6x^2 + 16x + 12$

QUOTIENTS

Theorem 5. *At a point where the denominator is not equal to zero, the derivative of the quotient of two differentiable functions of x is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

If

$$y = \frac{u}{v}$$

then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

EXAMPLE: Find the derivative of the function

$$f(x) = \frac{x^2 - 7}{2x + 8}$$

SOLUTION: The derivative of the numerator is $2x$, and the derivative of the denominator is 2. Therefore,

$$f'(x) = \frac{(2x + 8)\,(2x) - (x^2 - 7)\,(2)}{(2x + 8)^2}$$

$$= \frac{4x^2 + 16x - 2x^2 + 14}{(2x + 8)^2}$$

$$= \frac{2x^2 + 16x + 14}{4(x + 4)^2}$$

$$= \frac{x^2 + 8x + 7}{2(x + 4)^2}$$

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = \dfrac{x^4}{x^2 - 2}$

2. $f(x) = \dfrac{x^2 - 3}{x + 7}$

3. $f(x) = \dfrac{x^2 + 3x + 5}{x^3 - 4}$

ANSWERS:

1. $\dfrac{2x^5 - 8x^3}{(x^2 - 2)^2}$

2. $\dfrac{x^2 + 14x + 3}{(x + 7)^2}$

3. $\dfrac{-(x^4 + 6x^3 + 15x^2 + 8x + 12)}{(x^3 - 4)^2}$

POWERS OF FUNCTIONS

Theorem 6. *The derivative of any differentiable function of x raised to the power n, where n is any real number, is equal to n times the polynomial function of x to the (n − 1) power times the derivative of the polynomial itself.*

If

$$y = u^n$$

where u is any differentiable function of x, then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = (x^3 - 3x^2 + 2x)^7$$

SOLUTION: Apply Theorem 6 and find

$$\frac{dy}{dx} = 7(x^3 - 3x^2 + 2x)^6\,(3x^2 - 6x + 2)$$

EXAMPLE: Find the derivative of the function

$$f(x) = \frac{(x^2 + 2)^3}{x - 1}$$

SOLUTION: This problem involves Theorem 5 and Theorem 6. Theorem 6 is used to find the derivative of the numerator; then Theorem 5 is used to find the derivative of the resulting quotient.

The derivative of the numerator is

$$3(x^2 + 2)^2 (2x)$$

and the derivative of the denominator is 1. Then, by Theorem 5

$$\frac{dy}{dx} = \frac{(x - 1)[3(x^2 + 2)^2 (2x)] - (1)(x^2 + 2)^3}{(x - 1)^2}$$

$$= \frac{6x(x^2 + 2)^2 (x - 1) - (x^2 + 2)^3}{(x - 1)^2}$$

$$= \frac{(x^2 + 2)^2 [6x(x - 1) - (x^2 + 2)]}{(x - 1)^2}$$

$$= \frac{(x^2 + 2)^2 (5x^2 - 6x - 2)}{(x - 1)^2}$$

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = (x^3 + 2x - 6)^2$

2. $f(x) = 5(x^2 + x + 7)^4$

3. $f(x) = \dfrac{2(x + 3)^3}{3x}$

ANSWERS:

1. $2(x^3 + 2x - 6)(3x^2 + 2)$

2. $20(x^2 + x + 7)^3 (2x + 1)$

3. $\dfrac{18x(x + 3)^2 - 6(x + 3)^3}{9x^2}$

RADICALS

To differentiate a function containing a radical, replace the radical by a fractional exponent; then find the derivative by applying the appropriate theorems.

EXAMPLE: Find the derivative of

$$f(x) = \sqrt{2x^2 - 5}$$

SOLUTION: Replace the radical by the proper fractional exponent, such that

$$f(x) = (2x^2 - 5)^{1/2}$$

and by Theorem 6

$$\frac{dy}{dx} = \frac{1}{2}(2x^2 - 5)^{(1/2)-1}(4x)$$

$$= \frac{1}{2}(2x^2 - 5)^{-1/2}(4x)$$

$$= 2x(2x^2 - 5)^{-1/2}$$

$$= \frac{2x}{\sqrt{2x^2 - 5}}$$

$$= \frac{2x\sqrt{2x^2 - 5}}{2x^2 - 5}$$

EXAMPLE: Find the derivative of

$$f(x) = \frac{2x + 1}{\sqrt{3x^2 + 2}}$$

SOLUTION: Replace the radical by the proper fractional exponent, thus

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}}$$

At this point a decision is in order. This problem may be solved by either writing

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}} \tag{1}$$

and applying Theorem 6 in the denominator and then applying Theorem 5 for the quotient or writing

$$f(x) = (2x + 1)\ (3x^2 + 2)^{-1/2} \tag{2}$$

and applying Theorem 6 for the second factor and then applying Theorem 4 for the product.

The two methods of solution are completed individually as follows:

Use equation (1):

$$f(x) = \frac{2x + 1}{(3x^2 + 2)^{1/2}}$$

Find the derivative of the denominator

$$\frac{d}{dx}(3x^2 + 2)^{1/2}$$

by applying the power theorem

$$\frac{d}{dx}(3x^2 + 2)^{1/2} = \frac{1}{2}(3x^2 + 2)^{(1/2)-1}(6x)$$

$$= 3x(3x^2 + 2)^{-1/2}$$

The derivative of the numerator is

$$\frac{d}{dx}(2x + 1) = 2$$

Now apply Theorem 5:

$$f'(x) = \frac{(3x^2 + 2)^{1/2}\ (2) - (2x + 1)\ [3x(3x^2 + 2)^{-1/2}]}{(3x^2 + 2)}$$

Multiply both numerator and denominator by

$$(3x^2 + 2)^{1/2}$$

and simplify:

$$f'(x) = \frac{2(3x^2 + 2) - 3x(2x + 1)}{(3x^2 + 2)^{3/2}}$$

$$= \frac{6x^2 + 4 - 6x^2 - 3x}{(3x^2 + 2)^{3/2}}$$

$$= \frac{4 - 3x}{(3x^2 + 2)^{3/2}}$$

To find the same solution by a different method, use equation (2):

$$f(x) = (2x + 1)\ (3x^2 + 2)^{-1/2}$$

Find the derivative of each factor:

$$\frac{d}{dx}\frac{(2x + 1)}{} = 2$$

and

$$\frac{d}{dx}(3x^2 + 2)^{-1/2} = -1/2(3x^2 + 2)^{(-1/2)-1}(6x)$$

$$= -3x(3x^2 + 2)^{-3/2}$$

Now apply Theorem 4:

$$f'(x) = (2x + 1)\ \left[-3x(3x^2 + 2)^{-3/2}\right] + (3x^2 + 2)^{-1/2}(2)$$

Multiply both numerator and denominator by

$$(3x^2 + 2)^{3/2}$$

such that,

$$f'(x) = \frac{-3x(2x + 1) + 2(3x^2 + 2)}{(3x^2 + 2)^{3/2}}$$

$$= \frac{-6x^2 - 3x + 6x^2 + 4}{(3x^2 + 2)^{3/2}}$$

$$= \frac{4 - 3x}{(3x^2 + 2)^{3/2}}$$

which agrees with the solution of the first method used.

PRACTICE PROBLEMS:

Find the derivatives of the following:

1. $f(x) = \sqrt{x}$

2. $f(x) = \dfrac{1}{\sqrt{x}}$

3. $f(x) = \sqrt{3x - 4}$

4. $f(x) = \sqrt[3]{4x^2 - 3x + 2}$

ANSWERS:

1. $\dfrac{1}{2\sqrt{x}}$ or $\dfrac{\sqrt{x}}{2x}$

2. $-\dfrac{1}{2\sqrt{x^3}}$ or $-\dfrac{\sqrt{x^3}}{2x^3}$

3. $\dfrac{3}{2\sqrt{3x - 4}}$ or $\dfrac{3\sqrt{3x - 4}}{2(3x - 4)}$

4. $\dfrac{8x - 3}{3\sqrt[3]{(4x^2 - 3x + 2)^2}}$ or $\dfrac{(8x - 3)\sqrt[3]{4x^2 - 3x + 2}}{3(4x^2 - 3x + 2)}$

CHAIN RULE

A frequently used rule in differential calculus is the chain rule. This rule links together derivatives that have related variables. The *chain rule* is

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where the variable $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x.

EXAMPLE: Find the derivative of

$$y = (x + x^2)^2$$

SOLUTION: Let

$$u = (x + x^2)$$

and

$$y = u^2$$

Then,

$$\frac{dy}{du} = 2u$$

and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u\frac{du}{dx} \qquad (1)$$

Now,

$$\frac{du}{dx} = 1 + 2x$$

and substituting into equation (1) gives

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 2u(1 + 2x)$$

but,

$$u = (x + x^2)$$

Therefore,

$$\frac{dy}{dx} = 2(x + x^2)\,(1 + 2x)$$

EXAMPLE: Find $\frac{dy}{dx}$ where

$$y = 12t^4 + 7t$$

and

$$t = x^2 + 4$$

SOLUTION: By the chain rule

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$$

$$\frac{dy}{dt} = 48t^3 + 7$$

and

$$\frac{dt}{dx} = 2x$$

Then,

$$\frac{dy}{dx} = (48t^3 + 7)\ (2x)$$

and by substitution

$$\frac{dy}{dx} = \left[48(x^2 + 4)^3 + 7\right]\ (2x)$$

PRACTICE PROBLEMS:

Find $\frac{dy}{dx}$ in the following:

1. $y = 3t^3 + 8t$ and

   $t = x^3 + 2$

2. $y = 7n^2 + 8n + 3$ and

   $n = 2x^3 + 4x^2 + x$

ANSWERS:

1. $\left[9(x^3 + 2)^2 + 8\right]\ (3x^2)$

2. $\left[14(2x^3 + 4x^2 + x) + 8\right]\ (6x^2 + 8x + 1)$

INVERSE FUNCTIONS

Theorem 7. *The derivative of an inverse function is equal to the reciprocal of the derivative of the direct function.*

In the equations to this point, x has been the independent variable and y has been the dependent variable. The equations have been in a form such as

$$y = x^2 + 3x + 2$$

Suppose that we have a function like

$$x = \frac{1}{y^2} - \frac{1}{y}$$

and we wish to find the derivative $\frac{dy}{dx}$. Notice that if we solve for y in terms of x, using the quadratic formula, we get the more complicated function:

$$y = \frac{-1 \pm \sqrt{1 + 4x}}{2x}$$

If we call this function the direct function, then

$$x = \frac{1}{y^2} - \frac{1}{y}$$

is the inverse function. To determine $\frac{dy}{dx}$ from the inverse function is easy.

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$x = \frac{1}{y^2} - \frac{1}{y}$$

SOLUTION: The derivative $\frac{dx}{dy}$ is

$$\frac{dx}{dy} = -2y^{-3} + y^{-2}$$

$$= \frac{-2}{y^3} + \frac{1}{y^2}$$

$$= \frac{-2 + y}{y^3}$$

The reciprocal of $\frac{dx}{dy}$ is the derivative $\frac{dy}{dx}$ of the direct function, and we find

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{y^3}{y - 2}$$

EXAMPLE: Find the derivative $\frac{dy}{dx}$ of the function

$$x = y^2$$

SOLUTION: Find $\frac{dx}{dy}$ to be

$$\frac{dx}{dy} = 2y$$

Then

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{2y}$$

PRACTICE PROBLEMS:

Find the derivative $\frac{dy}{dx}$ of the following functions:

1. $x = 4 - y^2$

2. $x = y^2 + 9y$

ANSWERS:

1. $-\frac{1}{2y}$

2. $\frac{1}{2y + 9}$

IMPLICIT FUNCTIONS

In equations containing x and y, separating the variables is not always easy. If we do not solve an equation for y, we call y an *implicit function* of x. In the equation

$$x^2 - 4y = 0$$

y is an implicit function of x, and x is also called an implicit function of y. If we solve this equation for y, that is

$$y = \frac{x^2}{4}$$

then y would be called an explicit function of x. In many cases such a solution would be far too complicated to handle conveniently.

When y is given by an equation such as

$$y^2 + xy^2 = 2$$

y is an implicit function of x.

Whenever we have an equation of this type in which y is an implicit function of x, we can differentiate the function in a straightforward manner. The derivative of each term containing y will be followed by $\frac{dy}{dx}$. Refer to Theorem 6.

EXAMPLE: Obtain the derivative $\frac{dy}{dx}$ of

$$y^2 + xy^2 = 2$$

SOLUTION: Find the derivative of y^2:

$$\frac{d\,(y^2)}{dx} = 2y\,\frac{dy}{dx}$$

the derivative of xy^2:

$$\frac{d\,(xy^2)}{dx} = x(2y)\,\frac{dy}{dx} + (y^2)\,(1)$$

and the derivative of 2:

$$\frac{d\,(2)}{dx} = 0$$

such that,

$$2y\frac{dy}{dx} + 2xy\frac{dy}{dx} + y^2 = 0$$

Solving for $\frac{dy}{dx}$ we find that

$$(2y + 2xy)\frac{dy}{dx} = -y^2$$

and

$$\frac{dy}{dx} = \frac{-y^2}{2y + 2xy}$$

$$= \frac{-y}{2 + 2x}$$

Thus, whenever we differentiate an implicit function, the derivative will usually contain terms in both x and y.

PRACTICE PROBLEMS:

Find the derivative $\frac{dy}{dx}$ of the following:

1. $x^5 + 4xy^3 - 3y^5 = 0$
2. $x^3y^2 = 3xy$
3. $x^2y + y^3 = 4$

ANSWERS:

1. $\dfrac{-5x^4 - 4y^3}{12xy^2 - 15y^4}$
2. $\dfrac{3x^2y^2 - 3y}{3x - 2x^3y}$
3. $\dfrac{-2xy}{x^2 + 3y^2}$

TRIGONOMETRIC FUNCTIONS

If we are given

$$y = \sin u$$

we may state that, from the general formula,

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\sin (u + \Delta u) - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin u \cos \Delta u + \cos u \sin \Delta u - \sin u}{\Delta u}$$

$$= \lim_{\Delta u \to 0} \frac{\sin u(\cos \Delta u - 1)}{\Delta u} + \lim_{\Delta u \to 0} \frac{\sin \Delta u \cos u}{\Delta u} \tag{5.3}$$

Since

$$\lim_{\Delta u \to 0} \frac{\cos \Delta u - 1}{\Delta u} = 0 \tag{5.4}$$

and

$$\lim_{\Delta u \to 0} \frac{\sin \Delta u}{\Delta u} = 1 \tag{5.5}$$

Then by substituting equations (5.4) and (5.5) into equation (5.3),

$$\frac{dy}{du} = \cos u \tag{5.6}$$

Now we are interested in finding the derivative $\frac{dy}{dx}$ of the function sin u, so we apply the chain rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

From the chain rule and equation (5.6), we find

$$\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$$

In other words, to find the derivative of the sine of a function, we use the cosine of the function times the derivative of the function.

By a similar process we find the derivative of the cosine function to be

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$$

The derivatives of the other trigonometric functions may be found by expressing them in terms of the sine and cosine. That is,

$$\frac{d(\tan u)}{dx} = \frac{d}{dx}\left(\frac{\sin u}{\cos u}\right)$$

and by substituting sin u for u, cos u for v, and du for dx in the expression of the quotient theorem

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

we have

$$\frac{dy}{du} = \frac{d}{du}\left(\frac{\sin u}{\cos u}\right)$$

$$= \frac{\cos u \frac{d(\sin u)}{du} - \sin u \frac{d(\cos u)}{du}}{\cos^2 u} \qquad (5.7)$$

Taking

$$\frac{d(\sin u)}{du} = \cos u$$

and

$$\frac{d(\cos u)}{du} = -\sin u$$

and substituting into equation (5.7), we find that

$$\frac{dy}{du} = \frac{\cos^2 u + \sin^2 u}{\cos^2 u}$$

$$= \frac{1}{\cos^2 u}$$

$$= \sec^2 u \tag{5.8}$$

Now using the chain rule and equation (5.8), we find

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \sec^2 u \frac{du}{dx}$$

By stating the other trigonometric functions in terms of the sine and cosine and using similar processes, we may find the following derivatives:

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = \sin 3x$$

SOLUTION:

$$\frac{dy}{dx} = \cos 3x \frac{d(3x)}{dx}$$

$$= 3 \cos 3x$$

EXAMPLE: Find the derivative of the function

$$y = \tan^2(3x)$$

SOLUTION: Use the power theorem to find

$$\frac{dy}{dx} = 2 \tan 3x \frac{d(\tan 3x)}{dx}$$

Then find

$$\frac{d}{dx}(\tan 3x) = \sec^2 3x \, \frac{d}{dx}(3x)$$

and

$$\frac{d}{dx}(3x) = 3$$

Combining all of these, we find that

$$\begin{aligned}\frac{dy}{dx} &= (2 \tan 3x)\,(\sec^2 3x)\,(3)\\ &= 6 \tan 3x \sec^2 3x\end{aligned}$$

PRACTICE PROBLEMS:

Find the derivative of the following:

1. $y = \sin 2x$
2. $y = (\cos x^2)^2$

ANSWERS:

1. $2 \cos 2x$
2. $-4x \cos x^2 \sin x^2$

NATURAL LOGARITHMIC FUNCTIONS

Theorem 8. *The natural logarithm*

$$y = \ln x$$

has the derivative

$$\frac{dy}{dx} = \frac{1}{x} \tag{5.9}$$

for $x > 0$.

If u is a positive differentiable function of x, then by (5.9) and the chain rule,

$$\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$$

EXAMPLE: Find the derivative of the function

$$y = \ln (6 - x^2)$$

SOLUTION:

$$\frac{dy}{dx} = \left(\frac{1}{6 - x^2}\right)\frac{d(6 - x^2)}{dx}$$

$$= \left(\frac{1}{6 - x^2}\right)(-2x)$$

$$= \frac{-2x}{6 - x^2} \text{ or } \frac{2x}{x^2 - 6}$$

DERIVATIVE OF CONSTANTS TO VARIABLE POWERS

In this section two forms of a constant to a variable power will be presented. The two exponential functions will be e^x and a^x, where x is the variable, a is any constant, and e is equal to 2.71828. . . .

Recalling our study of logarithms in *Mathematics,* Volume 2-A, since ln and e are inverse functions, then

$$\ln (e) = 1$$

$$\ln (e^x) = x \qquad (5.10)$$

and

$$\ln (a^x) = x \ln a$$

If

$$y = e^x$$

then

$$\frac{dy}{dx} = e^x \qquad (5.11)$$

PROOF: Since $y = \ln x$ is differentiable, so is its inverse, $y = e^x$. To obtain the derivative of $y = e^x$, we differentiate both sides of equation (5.10) with respect to x, which gives

$$\frac{1}{e^x}\frac{d(e^x)}{dx} = 1 \tag{5.12}$$

Multiplying both sides of equation (5.12) by e^x gives

$$\frac{d(e^x)}{dx} = e^x$$

Chain rule differentiation and equation (5.11) give

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$$

EXAMPLE: Find $\frac{dy}{dx}$ for $y = e^{6x}$.

SOLUTION:

$$\frac{dy}{dx} = e^{6x}\frac{d(6x)}{dx}$$

$$= e^{6x}(6) \text{ or } 6e^{6x}$$

EXAMPLE: Find $\frac{dy}{dx}$ for $y = e^{\sin x}$.

SOLUTION:

$$\frac{dy}{dx} = e^{\sin x}\frac{d(\sin x)}{dx}$$

$$= e^{\sin x} \cos x$$

If

$$y = a^x$$

then

$$\frac{dy}{dx} = (\ln a)a^x \tag{5.13}$$

PROOF: Applying logarithmic rules,

$$a^x = e^{\ln (a^x)}$$

$$= e^{x \ln a} \tag{5.14}$$

Differentiating both sides of equation (5.14) gives

$$\frac{d\,(a^x)}{dx} = \frac{d\,(e^{x \ln a})}{dx}$$

$$= e^{x \ln a}\frac{d\,(x \ln a)}{dx}$$

$$= e^{x \ln a}(\ln a)$$

$$= e^{\ln (a^x)}(\ln a)$$

$$= a^x(\ln a) \text{ or } (\ln a)a^x$$

NOTE: $\ln a$ is a constant.

EXAMPLE: Find $\frac{dy}{dx}$ of $y = 2^x$.

SOLUTION:

$$\frac{dy}{dx} = (\ln 2)2^x$$

Chain rule differentiation and equation (5.13) give

$$\frac{d\,(a^u)}{dx} = (\ln a)a^u \frac{du}{dx}$$

EXAMPLE: Find $\frac{dy}{dx}$ of $y = 3^{(x^2 - 1)}$.

SOLUTION:

$$\frac{dy}{dx} = (\ln 3)3^{(x^2 - 1)}\frac{d\,(x^2 - 1)}{dx}$$

$$= (\ln 3)3^{(x^2 - 1)}(2x)$$

PRACTICE PROBLEMS:

Find $\frac{dy}{dx}$ of the following:

1. $y = \ln (2 - e^x)$

2. $y = e^{\sin (5x)}$

3. $y = 4^{(-x)}$

ANSWERS:

1. $e^x/(e^x - 2)$

2. $5e^{\sin (5x)} \cos (5x)$

3. $-(\ln 4)4^{(-x)}$

SUMMARY

The following are the major topics covered in this chapter:

1. **Derivative of a constant:**

   Theorem 1. *The derivative of a constant is zero.*

$$\text{If } y = c, \text{ then } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 0.$$

2. **Derivative of a variable raised to a power:**

   Theorem 2. *The derivative of the function $y = x^n$ is given by $\frac{dy}{dx} = nx^{n-1}$, if n is any real number.*

3. **Derivative of the sum of two or more functions:**

   Theorem 3. *The derivative of the sum of two or more differentiable functions of x is equal to the sum of their derivatives.*

   If two functions of x are given, such that $u = g(x)$ and $v = h(x)$, and also $y = u + v = g(x) + h(x)$, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

4. **Derivative of the product of two or more functions:**

   Theorem 4. *The derivative of the product of two differentiable functions of x is equal to the first function multiplied by the derivative of the second function, plus the second function multiplied by the derivative of the first function.*

$$\text{If } y = uv, \text{ then } \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

   This theorem can be extended to three or more functions.

$$\text{If } y = uvw, \text{ then } \frac{dy}{dx} = uv\frac{dw}{dx} + vw\frac{du}{dx} + uw\frac{dv}{dx}.$$

5. **Derivative of the quotient of two functions:**

Theorem 5. *At a point where the denominator is not equal to zero, the derivative of the quotient of two differentiable functions of x is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

$$\text{If } y = \frac{u}{v}, \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

6. **Derivative of a function raised to a power:**

Theorem 6. *The derivative of any differentiable function of x raised to the power n, where n is any real number, is equal to n times the polynomial function of x to the (n − 1) power times the derivative of the polynomial itself.*

If $y = u^n$, where u is any differentiable function of x, then

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}$$

7. **Derivative of a function in radical form:** To differentiate a function containing a radical, replace the radical by a fractional exponent; then find the derivative by applying the appropriate theorems.

8. **Derivative of a function using the chain rule:**

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

where the variable $y = f(u)$ is a differentiable function of u and $u = g(x)$ is differentiable function of x.

9. **Derivative of an inverse function:**

Theorem 7. *The derivative of an inverse function is equal to the reciprocal of the derivative of the direct function.*

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \text{ or } \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

10. **Derivative of an implicit function:** In equations containing x and y, if an equation of y is not solved for, then y is called an *implicit function* of x. The derivative of each term containing y will be followed by $\frac{dy}{dx}$.

11. **Derivative of trigonometric functions:**

$$\frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

$$\frac{d(\cos u)}{dx} = -\sin u \frac{du}{dx}$$

$$\frac{d(\tan u)}{dx} = \sec^2 u \frac{du}{dx}$$

$$\frac{d(\cot u)}{dx} = -\csc^2 u \frac{du}{dx}$$

$$\frac{d(\sec u)}{dx} = \sec u \tan u \frac{du}{dx}$$

$$\frac{d(\csc u)}{dx} = -\csc u \cot u \frac{du}{dx}$$

12. **Derivative of natural logarithmic functions:**

Theorem 8. *The natural logarithm* $y = \ln x$ *has the derivative*

$$\frac{dy}{dx} = \frac{1}{x} \text{ for } x > 0$$

If u is a positive differentiable function of x, then

$$\frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$$

13. **Derivative of a constant to a variable power:**

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d(e^u)}{dx} = e^u \frac{du}{dx}$$

$$\frac{d(a^x)}{dx} = (\ln a)a^x$$

$$\frac{d(a^u)}{dx} = (\ln a)a^u \frac{du}{dx}$$

where x is a variable, u is a function of x, a is a constant, and e is equal to 2.71828. . . .

ADDITIONAL PRACTICE PROBLEMS

Find the derivative $\frac{dy}{dx}$ of the following:

1. $f(x) = 3/4$
2. $f(x) = 3x^{-2/3}$
3. $f(x) = 6x^{-1/3} - 8x^{1/4}$
4. $f(x) = (2x^3 + 5)\ (4x^{-2} - 3)$
5. $f(x) = (x - 2)\ (5x - 2)\ (4x + 3)$
6. $f(x) = \dfrac{x^2 + 1}{1 - x^2}$
7. $f(x) = (x^5 - 3x^{-2} + 1)^{-13}$
8. $f(x) = 6\sqrt[3]{x} + 8\sqrt[4]{x}$
9. $y = 2t^5 + 1/t^5$ and $t = x^4$
10. $x = 1 + (y + 4)^{3/4}$
11. $2x + yx - xy^3 = 4$
12. $f(x) = \sqrt{\cot\,(x^2)}$
13. $f(x) = 2\sec\,(\sqrt{x}) + 2\sqrt{\csc x}$
14. $f(x) = (x^3 + 4)^{1/3}\tan\,(3x)$
15. $y = [\ln\,(x^3)]^4$
16. $y = e^{-1/x} + x^e$
17. $y = 7^{3(\ln\ x)}$

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 0

2. $-2x^{-5/3}$

3. $-2x^{-4/3} - 2x^{-3/4}$

4. $-18x^2 + 8 - 40x^{-3}$

5. $60x^2 - 66x - 20$

6. $\dfrac{4x}{(1 - x^2)^2}$

7. $-13(x^5 - 3x^{-2} + 1)^{-14}(5x^4 + 6x^{-3})$

8. $2x^{-2/3} + 2x^{-3/4}$

9. $40x^{19} - 20x^{-21}$

10. $(4/3)\ (y + 4)^{1/4}$

11. $\dfrac{y^3 - y - 2}{x - 3xy^2}$

12. $-x[\cot\ (x^2)]^{-1/2}[\csc^2(x^2)]$

13. $x^{-1/2} \sec\ (\sqrt{x})\ \tan\ (\sqrt{x}) - \sqrt{\csc x}\ \cot\ \mathrm{x}$

14. $3(x^3 + 4)^{1/3} \sec^2(3x) + \tan\ (3x)\ (x^3 + 4)^{-2/3}\ x^2$

15. $\dfrac{12[\ln\ (x^3)]^3}{x}$

16. $e^{-1/x}\ \dfrac{1}{x^2} + ex^{e-1}$

17. $(\ln\ 7)\ 7^{3(\ln\ x)}\ \dfrac{3}{x}$

CHAPTER 6

INTEGRATION

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define integration.

2. Find the area under a curve and interpret indefinite integrals.

3. Apply rules of integration.

4. Apply definite integrals to problem solving.

INTRODUCTION

The two main branches of calculus are differential calculus and integral calculus. Having studied differential calculus in previous chapters, we now turn our attention to integral calculus. Basically, integration is the inverse of differentiation just as division is the inverse of multiplication, and as subtraction is the inverse of addition.

DEFINITIONS

Integration is defined as the inverse of differentiation. When we were dealing with differentiation, we were given a function $F(x)$ and were required to find the derivative of this function. In integration we will be given the derivative of a function and will be required to find the function. That is, when we are given the function $f(x)$, we will find another function, $F(x)$, such that

$$\frac{d\,F(x)}{dx} = f(x) \tag{6.1}$$

In other words, when we have the function $f(x)$, we must find the function $F(x)$ whose derivative is the function $f(x)$.

If we change equation (6.1) to read

$$d\,F(x) = f(x)\,dx \tag{6.2}$$

we have used dx as a *differential*. An equivalent statement for equation (6.2) is

$$F(x) = \int f(x)\,dx$$

We call $f(x)$ the *integrand*, and we say $F(x)$ is equal to the indefinite integral of $f(x)$. The elongated $S, \int$, is the *integral sign*. This symbol is used because integration may be shown to be the limit of a sum.

INTERPRETATION OF AN INTEGRAL

We will use the area under a curve for the interpretation of an integral. You should realize, however, that an integral may represent many things, and it may be real or abstract. It may represent the plane area, volume, or surface area of some figure.

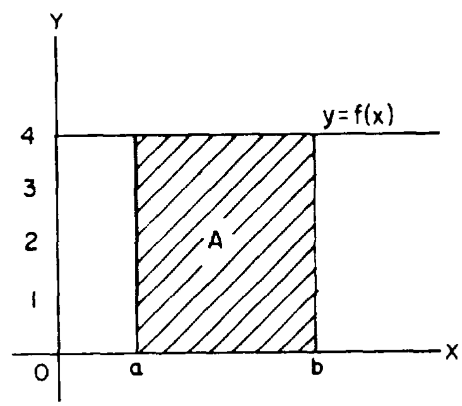


Figure 6-1.—Area of a rectangle.

AREA UNDER A CURVE

To find the area under a curve, we must agree on what is desired. In figure 6-1, where $f(x)$ is equal to the constant 4 and the "curve" is the straight line

$$y = 4$$

the area of the rectangle is found by multiplying the height times the width. Thus, the area under the curve is

$$A = 4(b - a)$$

The next problem is to find a method for determining the area under any curve, provided that the curve is continuous. In figure 6-2, the area under the curve

$$y = f(x)$$

between points x and $x + \Delta x$ is approximately $f(x)\Delta x$. We consider that Δx is small and the area given is ΔA.

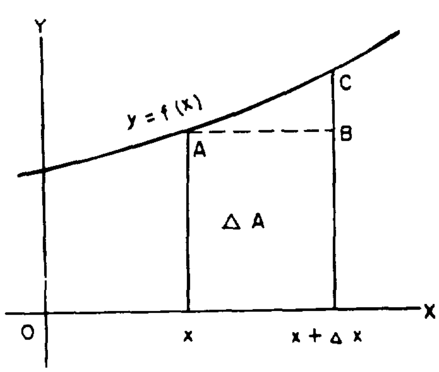


Figure 6-2.—Area ΔA.

This area under the curve is nearly a rectangle. The area ΔA, under the curve, would differ from the area of the rectangle by the area of the triangle ABC if AC were a straight line.

When Δx becomes smaller and smaller, the area of ABC becomes smaller at a faster rate, and ABC finally becomes indistinguishable from a triangle. The area of this triangle becomes negilible when Δx is sufficiently small. Therefore, for sufficiently small values of Δx, we can say that

$$\Delta A \approx f(x)\Delta x$$

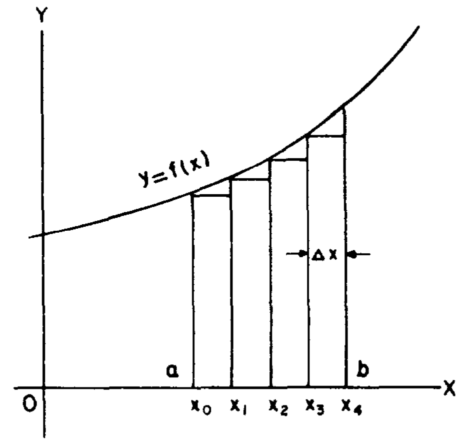


Figure 6-3.—Area of strips.

Now, if we have the curve in figure 6-3, the sum of all the rectangles will be approximately equal to the area under the curve and bounded by the lines at a and b. The difference between the actual area under the curve and the sum of the areas of the rectangles will be the sum of the areas of the triangles above each rectangle.

As Δx is made smaller and smaller, the sum of the rectangular areas will approach the value of the area under the curve. The sum of the areas of the rectangles may be indicated by

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x \tag{6.3}$$

where Σ (sigma) is the symbol for sum, n is the number of rectangles, $f(x)\Delta x$ is the area of each rectangle, and k is the designation number of each rectangle. In the particular example just discussed, where we have four rectangles, we would write

$$A = \sum_{k=1}^{4} f(x_k)\Delta x$$

and we would have only the sum of four rectangles and not the limiting area under the curve.

When using the limit of a sum, as in equation (6.3), we are required to use extensive algebraic techniques to find the actual area under the curve.

To this point we have been given a choice of using arithmetic and finding only an approximation of the area under a curve or using extensive algebra to find the actual area.

We will now use calculus to find the area under a curve fairly easily.

In figure 6-4, the areas under the curve, from a to b, is shown as the sum of the areas of ${}_aA_c$ and ${}_cA_b$. The notation ${}_aA_c$ means the area under the curve from a to c.

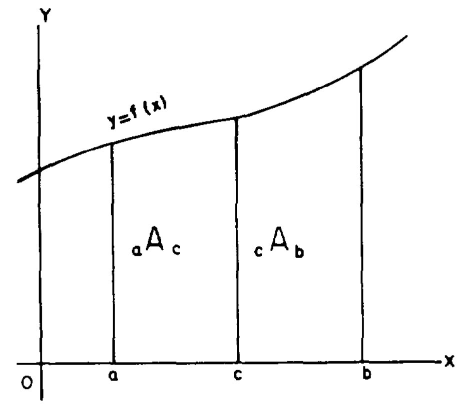


Figure 6-4.—Designation of limits.

The *Intermediate Value Theorem* states that

$$ {}_aA_b = f(c)\ (b - a) $$

where $f(c)$ in figure 6-4 is the value of the function at an intermediate point between a and b.

We now modify figure 6-4 as shown in figure 6-5.

When

$$ x = a $$

then

$$ {}_aA_a = 0 $$

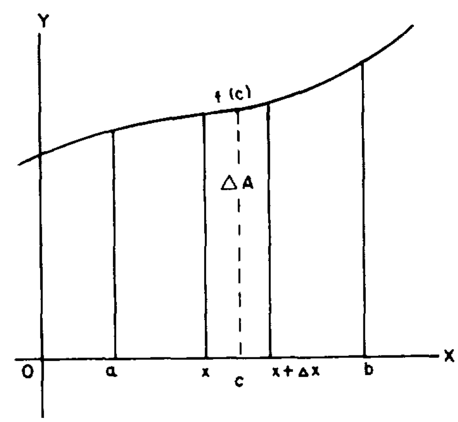


Figure 6-5.—Increments of area at $f(c)$.

We see in figure 6-5 that

$$ {}_aA_x + {}_xA_{(x+\Delta x)} = {}_aA_{(x+\Delta x)} $$

therefore, the increase in area, as shown, is

$$ \Delta A = {}_xA_{(x+\Delta x)} $$

Reference to figure 6-5 shows

$$ {}_xA_{(x+\Delta x)} = f(c)\ (x + \Delta x - x) = f(c)\Delta x $$

where c is a point between a and b. Then by substitution

$$ \Delta A = f(c)\Delta x $$

or

$$ \frac{\Delta A}{\Delta x} = f(c) $$

and as Δx approaches zero, we have

$$\frac{d(A)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x}$$

$$= \lim_{c \to x} f(c)$$

$$= f(x)$$

Now, from the definition of integration

$$_aA_x = \int f(x)\, dx \tag{6.4}$$
$$= F(x) + C$$

where C is the constant of integration, and

$$_aA_a = F(a) + C$$

but

$$_aA_a = 0$$

therefore,

$$F(a) + C = 0$$

By solving for C, we have

$$C = -F(a)$$

and by substituting $-F(a)$ into equation (6.4), we find

$$_aA_x = F(x) - F(a)$$

If we let

$$x = b$$

then

$$_aA_b = F(b) - F(a) \tag{6.5}$$

where $F(b)$ and $F(a)$ are the integrals of the function of the curve at the values b and a.

The constant of integration C is omitted in equation (6.5) because when the function of the curve at b and a is integrated, C will occur with both $F(a)$ and $F(b)$ and will therefore be subtracted from itself.

NOTE: The concept of the constant of integration is more fully explained later in this chapter.

EXAMPLE: Find the area under the curve

$$y = 2x - 1$$

in figure 6-6, bounded by the vertical lines at a and b and the X axis.

SOLUTION: We know that

$$ {}_aA_b = F(b) - F(a) $$

and we find that

$$ F(x) = \int f(x)\,dx $$

$$ = \int (2x - 1)\,dx $$

$$ = x^2 - x \text{ (This step will be justified later.)} $$

Then, substituting the values for a and b into $F(x) = x^2 - x$, we find that when

$$ x = a $$

$$ = 1 $$

$$ F(a) = 1 - 1 $$

$$ = 0 $$

and when

$$ x = b $$

$$ = 5 $$

$$ F(b) = 25 - 5 $$

$$ = 20 $$

Then by substituting these values in

$$ {}_aA_b = F(b) - F(a) $$

we find that

$$ {}_aA_b = 20 - 0 $$

$$ = 20 $$

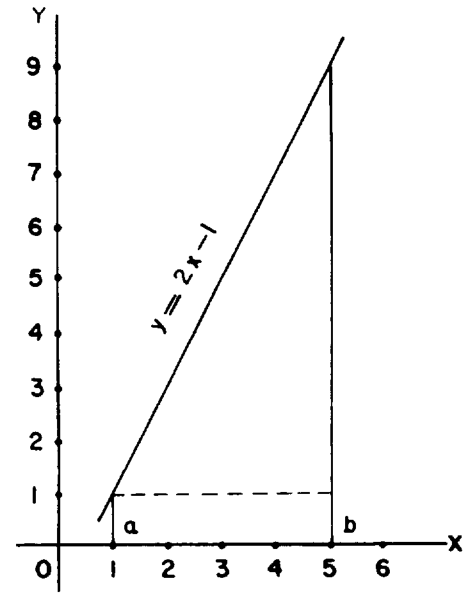


Figure 6-6.—Area of triangle and rectangle.

We may verify this by considering figure 6-6 to be a triangle with base 4 and height 8 sitting on a rectangle of height 1 and base 4. By known formulas, we find the area under the curve to be 20.

INDEFINITE INTEGRALS

When we were finding the derivative of a function, we wrote

$$\frac{dF(x)}{dx} = f(x)$$

where the derivative of $F(x)$ is $f(x)$. Our problem is to find $F(x)$ when we are given $f(x)$.

We know that the symbol $\int \ldots dx$ is the inverse of $\frac{d}{dx}$, or when dealing with differentials, the operator symbols d and $\int$ are the inverse of each other; that is,

$$F(x) = \int f(x)\, dx$$

and when the derivative of each side is taken, d annulling $\int$, we have

$$d\, F(x) = f(x)\, dx$$

or where $\int \ldots dx$ annuals $\frac{d}{dx}$, we have

$$\frac{dF(x)}{dx} = \frac{d}{dx}\int f(x)\, dx$$

$$= f(x)$$

From this, we find that

$$d\,(x^3) = 3x^2\, dx$$

so that,

$$\int 3x^2\, dx = x^3 + C$$

Also we find that

$$d\,(x^3 + 3) = 3x^2\, dx$$

so that,

$$\int 3x^2\,dx = x^3 + 3$$

Again, we find that

$$d\,(x^3 - 9) = 3x^2\,dx$$

so that,

$$\int 3x^2\,dx = x^3 - 9$$

This is to say that

$$d\,(x^3 + C) = 3x^2\,dx$$

and

$$\int 3x^2\,dx = x^3 + C$$

where C is any *constant of integration.*

A number that is independent of the variable of integration is called a *constant of integration*. Since C may have infinitely many values, then a differential expression may have infinitely many integrals differing only by the constant. This is to say that two integrals of the same function may differ by the constant of integration. We assume the differential expression has at least one integral. Because the integral contains C and C is indefinite, we call

$$F(x) + C$$

an *indefinite integral* of $f(x)\,dx$. In the general form we say

$$\int f(x)\,dx = F(x) + C$$

With regard to the constant of integration, a theorem and its converse are stated as follows:

Theorem 1. *If two functions differ by a constant, they have the same derivative.*

Theorem 2. *If two functions have the same derivative, their difference is a constant.*

RULES FOR INTEGRATION

Although integration is the inverse of differentiation and we were given rules for differentiation, we are required to determine the answers in integration by trial and error. However, there are some rules to aid us in the determination of the answer.

In this section we will discuss four of these rules and how they are used to integrate standard elementary forms. In the rules we will let u and v denote a differentiable function of a variable such as x. We will let C, n, and a denote constants.

Our proofs will involve searching for a function $F(x)$ whose derivative is $f(x)\, dx$.

Rule 1. $\int du = u + C$

The integral of a differential of a function is the function plus a constant.

PROOF: If

$$\frac{d\,(u + C)}{du} = 1$$

then

$$d\ (u + C) = du$$

and

$$\int du = u + C$$

EXAMPLE: Evaluate the integral

$$\int dx$$

SOLUTION: By Rule 1, we have

$$\int dx = x + C$$

Rule 2. $\int a\ du = a \int du = au + C$

A constant may be moved across the integral sign. NOTE: A variable may NOT be moved across the integral sign.

PROOF: If

$$\frac{d\,(au + C)}{du} = (a)\frac{d\,(u + C)}{du} = a$$

then

$$d\,(au + C) = a\,d\,(u + C) = a\,du$$

and

$$\int a\,du = a\int du = au + C$$

EXAMPLE: Evaluate the integral

$$\int 4\,dx$$

SOLUTION: By Rule 2,

$$\int 4\,dx = 4\int dx$$

and by Rule 1,

$$\int dx = x + C$$

therefore,

$$\int 4\,dx = 4x + C$$

Rule 3. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C$

The integral of $u^n\,du$ may be obtained by adding 1 to the exponent and then dividing by this new exponent. NOTE: If n is minus 1, this rule is not valid and another method must be used.

PROOF: If

$$d\left(\frac{u^{n+1}}{n+1} + C\right) = \frac{(n+1)u^n}{n+1}\,du$$

$$= u^n\,du$$

then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C$$

EXAMPLE: Evaluate the integral

$$\int x^3\,dx$$

SOLUTION: By Rule 3,

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + C$$

$$= \frac{x^4}{4} + C$$

EXAMPLE: Evaluate the integral

$$\int \frac{7}{x^3}\,dx$$

SOLUTION: First write the integral

$$\int \frac{7}{x^3}\,dx$$

as

$$\int 7x^{-3}\,dx$$

Then, by Rule 2,

$$7\int x^{-3}\,dx$$

and by Rule 3,

$$7\int x^{-3}\,dx = 7\left(\frac{x^{-2}}{-2}\right) + C = -\frac{7}{2x^2} + C$$

Rule 4. $\int (du + dv + dw) = \int du + \int dv + \int dw$

$$= u + v + w + C$$

The integral of a sum is equal to the sum of the integrals.

PROOF: If

$$d\,(u + v + w + C) = du + dv + dw$$

then

$$\int (du + dv + dw) = (u + C_1) + (v + C_2) + (w + C_3)$$

such that

$$\int (du + dv + dw) = u + v + w + C$$

where

$$C = C_1 + C_2 + C_3$$

EXAMPLE: Evaluate the integral

$$\int (2x - 5x + 4)\, dx$$

SOLUTION: We will not combine $2x$ and $-5x$.

$$\int (2x - 5x + 4)\, dx$$

$$= \int 2x\, dx - \int 5x\, dx + \int 4\, dx$$

$$= 2 \int x\, dx - 5 \int x\, dx + 4 \int dx$$

$$= \frac{2x^2}{2} + C_1 - \frac{5x^2}{2} + C_2 + 4x + C_3$$

$$= x^2 - \frac{5}{2} x^2 + 4x + C$$

where C is the sum of C_1, C_2, and C_3.

EXAMPLE: Evaluate the integral

$$\int (x^{1/2} + x^{2/3})\, dx$$

SOLUTION:

$$\int (x^{1/2} + x^{2/3})\, dx$$

$$= \int x^{1/2}\, dx + \int x^{2/3}\, dx$$

$$= \frac{x^{3/2}}{3/2} + C_1 + \frac{x^{5/3}}{5/3} + C_2$$

$$= \frac{2x^{3/2}}{3} + \frac{3x^{5/3}}{5} + C$$

Now we will discuss the evaluation of the constant of integration.

If we are to find the equation of a curve whose first derivative is 2 times the independent variable x, we may write

$$\frac{dy}{dx} = 2x$$

or

$$dy = 2x\, dx \qquad (1)$$

We may obtain the desired equation for the curve by integrating the expression for dy; that is, by integrating both sides of equation (1). If

$$dy = 2x\,dx$$

then,

$$\int dy = \int 2x\,dx$$

But, since

$$\int dy = y$$

and

$$\int 2x\,dx = x^2 + C$$

then

$$y = x^2 + C$$

We have obtained only a general equation of the curve because a different curve results for each value we assign to C. This is shown in figure 6-7. If we specify that

$$x = 0$$

and

$$y = 6$$

we may obtain a specific value for C and hence a particular curve.

Suppose that

$$y = x^2 + C,\ x = 0, \text{ and } y = 6$$

then,

$$6 = 0^2 + C$$

or

$$C = 6$$

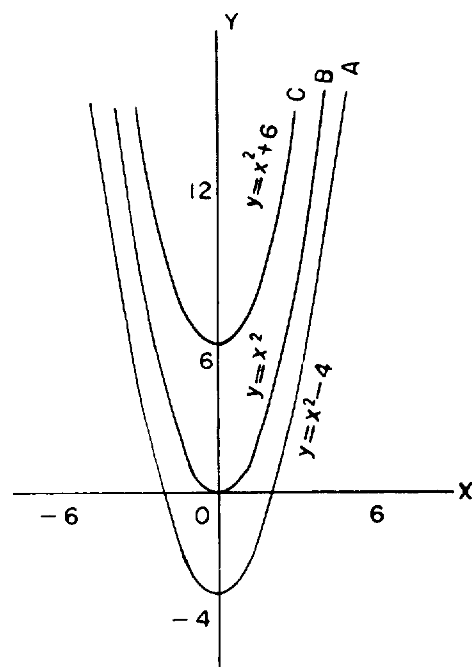


Figure 6-7.—Family of curves.

By substituting the value 6 into the general equation, we find that the equation for the particular curve is

$$y = x^2 + 6$$

which is curve C of figure 6-7.

The values for x and y will determine the value for C and also determine the particular curve of the family of curves.

In figure 6-7, curve A has a constant equal to -4, curve B has a constant equal to 0, and curve C has a constant equal to 6.

EXAMPLE: Find the equation of the curve if its first derivative is 6 times the independent variable, y equals 2, and x equals 0.

SOLUTION: We may write

$$\frac{dy}{dx} = 6x$$

or

$$\int dy = \int 6x \, dx$$

such that,

$$y = 3x^2 + C$$

Solving for C when

$$x = 0$$

and

$$y = 2$$

we have

$$2 = 3(0^2) + C$$

or

$$C = 2$$

so that the equation of the curve is

$$y = 3x^2 + 2$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int x^2\,dx$

2. $\int 4x\,dx$

3. $\int (x^3 + x^2 + x)\,dx$

4. $\int 6\,dx$

5. $\displaystyle\int \frac{5}{x^2}\,dx$

6. Find the equation of the curve if its first derivative is 9 times the independent variable squared, y equals 5, and x equals -1.

ANSWERS:

1. $\dfrac{x^3}{3} + C$

2. $2x^2 + C$

3. $\dfrac{x^4}{4} + \dfrac{x^3}{3} + \dfrac{x^2}{2} + C$

4. $6x + C$

5. $-\dfrac{5}{x} + C$

6. $y = 3x^3 + 8$

DEFINITE INTEGRALS

The general form of the indefinite integral is

$$\int f(x)\,dx = F(x) + C$$

and has two identifying characteristics. First, the constant of integration must be added to each integration. Second, the result of integration is a function of a variable and has no definite value, even after the constant of integration is determined, until the variable is asigned a numerical value.

The definite integral eliminates these two characteristics. The form of the *definite integral* is

$$\int_a^b f(x)\,dx = [F(b) + C] - [F(a) + C] \tag{6.6}$$

$$= F(b) - F(a)$$

where a and b are given values. Notice that the constant of integration does not appear in the final expression of equation (6.6). In words, this equation states that the difference of the values of

$$\int_a^b f(x)\,dx$$

for

$$x = a$$

and

$$x = b$$

gives the area under the curve defined by $f(x)$, the X axis, and the ordinates where

$$x = a$$

and

$$x = b$$

In figure 6-8, the value of b is the upper limit and the value of a is the lower limit. These upper and lower limits may be any assigned values in the range of the curve. The upper limit is positive with respect to the lower limit in that it is located to the right (positive in our case) of the lower limit.

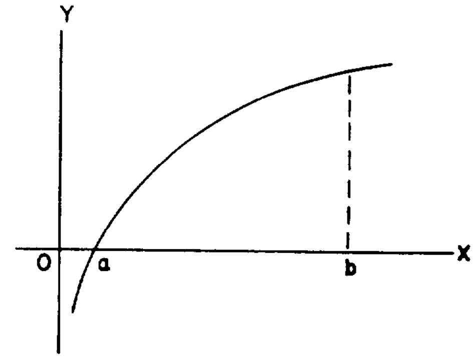


Figure 6-8.—Upper and lower limits.

Equation (6.6) is the limit of the sum of all the strips between a and b, having areas of $f(x)\Delta x$; that is,

$$\lim \sum_{x=a}^{x=b} f(x)\Delta x = \int_a^b f(x)\, dx$$

The definite integral evaluated from a to b is

$$\int_a^b f(x)\, dx = F(x)\Big|_a^b \qquad (6.7)$$

$$= F(b) - F(a)$$

The notation $F(x)\Big|_a^b$ in equation (6.7) means we first substitute the upper limit, b, into the function $F(x)$ to obtain $F(b)$; and from $F(b)$ we subtract $F(a)$, the value obtained by substituting the lower limit, a, into $F(x)$.

EXAMPLE: Find the area bounded by the curve

$$y = x^2$$

the X axis, and the ordinates where

$$x = 2$$

and

$$x = 3$$

as shown in figure 6-9.

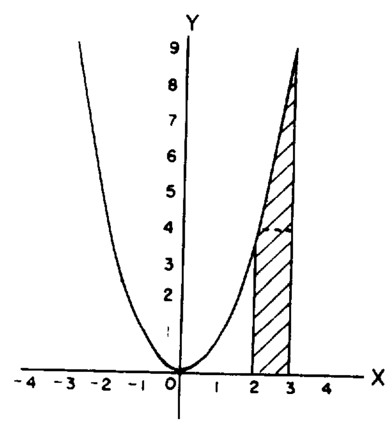


Figure 6-9.—Area from $x = 2$ to $x = 3$.

SOLUTION: Substituting into equation (6.7), we have

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

$$= \int_2^3 x^2\,dx$$

$$= \frac{x^3}{3}\Big|_2^3$$

$$= \frac{3^3}{3} - \frac{2^3}{3}$$

$$= \frac{27}{3} - \frac{8}{3}$$

$$= \frac{19}{3}$$

$$= 6\,\frac{1}{3}$$

We may make an estimate of this solution by considering the area desired in figure 6-9 as being a right triangle resting on a rectangle. The triangle has an approximate area of

$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}(1)(5)$$

$$= \frac{5}{2}$$

and the area of the rectangle is

$$A = bh$$

$$= (1)(4)$$

$$= 4$$

so that the total area is

$$4 + \frac{5}{2} = \frac{13}{2} = 6\,\frac{1}{2}$$

which is a close approximation of the area found by the process of integration.

EXAMPLE: Find the area bounded by the curve

$$y = x^2$$

the X axis, and the ordinates where

$$x = -2$$

and

$$x = 2$$

as shown in figure 6-10.

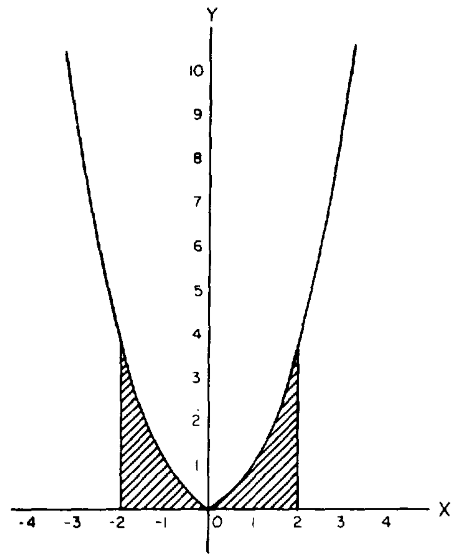


Figure 6-10.—Area under a curve.

SOLUTION: Substituting into equation (6.7), we have

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

$$= F(b) - F(a)$$

$$= \int_{-2}^{2} x^2\,dx$$

$$= \frac{x^3}{3}\Big|_{-2}^{2}$$

$$= \frac{8}{3} - \left(-\frac{8}{3}\right)$$

$$= \frac{16}{3}$$

$$= 5\,\frac{1}{3}$$

The area above a curve and below the X axis, as shown in figure 6-11, will, through integration, furnish a negatvie answer.

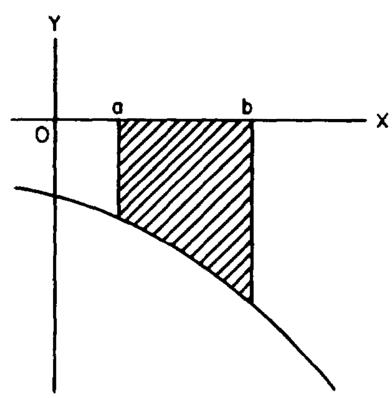


Figure 6-11.—Area above a curve.

If the graph of $y = f(x)$, between $x = a$ and $x = b$, has portions above and portions below the X axis, as shown in figure 6-12, *then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

is the sum of the absolute values of the positive areas above the X axis and the negative areas below the X axis, such that

$$\int_a^b f(x)\,dx = |A_1| + |A_2| + |A_3| + |A_4|$$

where

$$A_1 = \int_a^c f(x)\,dx \qquad A_2 = \int_c^d f(x)\,dx$$

$$A_3 = \int_d^e f(x)\,dx \qquad A_4 = \int_e^b f(x)\,dx$$

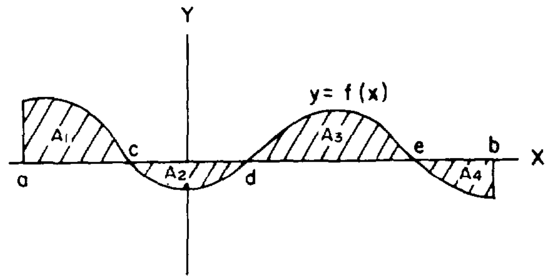


Figure 6-12.—Areas above and below a curve.

EXAMPLE: Find the areas between the curve

$$y = x$$

and the X axis bounded by the lines

$$x = -2$$

and

$$x = 2$$

as shown in figure 6-13.

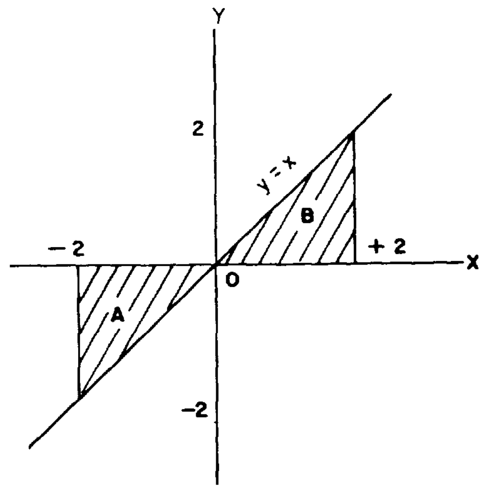


Figure 6-13.—Negative and positive value areas.

SOLUTION: These areas must be computed separately; therefore, we write

$$\begin{aligned}\text{Area A} &= \int_{-2}^{0} f(x)\,dx \\ &= \int_{-2}^{0} x\,dx \\ &= \left.\frac{x^2}{2}\right|_{-2}^{0} \\ &= 0 - \frac{4}{2} \\ &= -2\end{aligned}$$

and the absolute value of -2 is

$$|-2| = 2$$

Then,

$$\text{Area } B = \int_0^2 f(x)\,dx$$

$$= \frac{x^2}{2}\Bigg|_0^2$$

$$= \frac{4}{2} - 0$$

$$= 2$$

Adding the two areas, $|A|$ and $|B|$, we find

$$|A| + |B| = 2 + 2$$

$$= 4$$

NOTE: If the function is integrated from -2 to 2, the following INCORRECT result will occur:

$$\text{Area} = \int_{-2}^{2} f(x)\,dx$$

$$= \int_{-2}^{2} x\,dx$$

$$= \frac{x^2}{2}\Bigg|_{-2}^{2}$$

$$= \frac{4}{2} - \frac{4}{2}$$

$$= 0 \qquad \text{(INCORRECT SOLUTION)}$$

This is obviously not the area shown in figure 6-13. Such an example emphasizes the value of making a commonsense check on every solution. A sketch of the function will aid this commonsense judgement.

EXAMPLE: Find the total area bounded by the curve

$$y = x^3 - 3x^2 - 6x + 8$$

the X axis, and the lines

$$x = -2$$

and

$$x = 4$$

as shown in figure 6-14.

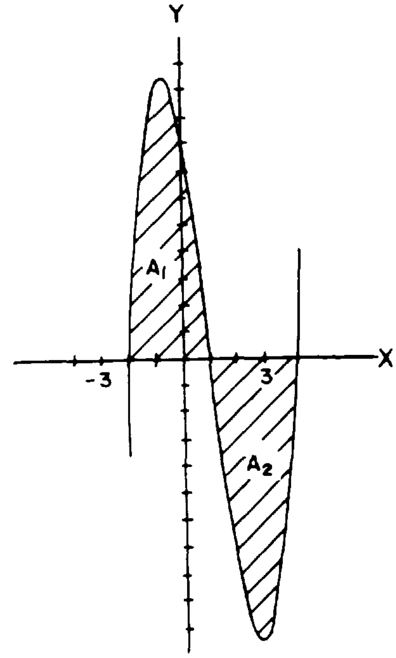


Figure 6-14.—Positive and negative value areas.

SOLUTION: The area desired is both above and below the X axis; therefore, we need to find the areas separately and then add them together using their absolute values.

Therefore,

$$A_1 = \int_{-2}^{1} (x^3 - 3x^2 - 6x + 8)\, dx$$

$$= \frac{x^4}{4} - x^3 - 3x^2 + 8x \Big|_{-2}^{1}$$

$$= \left(\frac{1}{4} - 1 - 3 + 8\right) - \left(4 + 8 - 12 - 16\right)$$

$$= 4\,\frac{1}{4} + 16$$

$$= 20\,\frac{1}{4}$$

and

$$A_2 = \int_1^4 (x^3 - 3x^2 - 6x + 8)\, dx$$

$$= \frac{x^4}{4} - x^3 - 3x^2 + 8x \Big|_1^4$$

$$= \left(64 - 64 - 48 + 32\right) - \left(\frac{1}{4} - 1 - 3 + 8\right)$$

$$= -16 - 4\frac{1}{4}$$

$$= -20\frac{1}{4}$$

Then, the total area is

$$|A_1| + |A_2| = 20\frac{1}{4} + 20\frac{1}{4}$$

$$= 40\frac{1}{2}$$

PRACTICE PROBLEMS:

1. Find, by integration, the area under the curve

$$y = x + 4$$

bounded by the X axis and the lines

$$x = 2$$

and

$$x = 7$$

verify this by a geometric process.

2. Find the area under the curve

$$y = 3x^2 + 2$$

bounded by the X axis and the lines

$$x = 0$$

and

$$x = 2$$

3. Find the area between the curve

$$y = x^3 - 12x$$

and the X axis, from

$$x = -1$$

to

$$x = 3$$

ANSWERS:

1. 42 1/2
2. 12
3. 39 1/2

SUMMARY

The following are the major topics covered in this chapter:

1. **Definition of integration:** *Integration* is defined as the inverse of differentiation.

$$F(x) = \int f(x)\, dx$$

where $F(x)$ is the function whose derivative is the function $f(x)$; $\int$ is the *integral sign*; $f(x)$ is the *integrand*; and dx is the *differential.*

2. **Area under a curve:**

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where Σ (sigma) is the symbol for sum, n is the number of rectangles, $f(x)\Delta x$ is the area of each rectangle, and k is the designation number of each rectangle.

Intermediate Value Theorem:

$${}_aA_b = f(c)(b - a)$$

where $f(c)$ is the function at an intermediate point between a and b.

$${}_aA_b = F(b) - F(a)$$

where $F(b) - F(a)$ are the integrals of the function of the curve at the values b and a.

3. **Indefinite integrals:**

$$\int f(x)\, dx = F(x) + C$$

where C is called a *constant of integration*, a number which is independent of the variable of integration.

Theorem 1. *If two functions differ by a constant, they have the same derivative.*

Theorem 2. *If two functions have the same derivative, their difference is a constant.*

4. **Rules for integration:**

Rule 1. $\int du = u + C$

The integral of a differential of a function is the function plus a constant.

Rule 2. $\int a\, du = a \int du = au + C$

A constant may be moved across the integral sign. NOTE: A variable may NOT be moved across the integral sign.

Rule 3. $\displaystyle\int u^n\, du = \frac{u^{n+1}}{n+1} + C$

The integral of u^n du may be obtained by adding 1 to the exponent and then dividing by this new exponent. NOTE: If n is minus 1, this rule is not valid.

Rule 4. $\int (du + dv + dw) = \int du + \int dv + \int dw$

$$= u + v + w + C$$

The integral of a sum is equal to the sum of the integrals.

5. **Definite integrals:**

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

where b, the upper limit, and a, the lower limit, are given values.

6. **Areas above and below a curve:**

If the graph of $y = f(x)$, between $x = a$ and $x = b$, has portions above and portions below the X axis, then

$\int_a^b f(x)\, dx = F(b) - F(a)$ is the sum of the absolute values of the positive areas above the X axis and the negative areas below the X axis.

ADDITIONAL PRACTICE PROBLEMS

Evaluate the following integrals:

1. $\int (3/4)\, dx$

2. $\int \frac{-8}{x^3}\, dx$

3. $\int (7x^6 - 5x^{-6} - 10x^4 + 9x^{-4})\, dx$

4. $\int 18x^{5/4}\, dx$

5. Find the equation of the curve if its first derivative is 4 plus 8 times the independent variable plus 12 times the independent variable cubed, y equals 12, and x equals 2.

6. Find the area under the curve $y = 2x + 3$ bounded by the X axis and the lines $x = -3/2$ and $x = 1$.

7. Find the area between the curve $y = x^2 + 4x$ and the X axis, from $x = -4$ to $x = 0$.

8. Find the total area bounded by the curve $y = 2 + x - x^2$, the X axis, and the lines $x = -2$ and $x = 3$. (Hint: Be sure to sketch the graph.)

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. $(3/4)x + C$

2. $4/x^2 + C$

3. $x^7 + x^{-5} - 2x^5 - 3x^{-3} + C$

4. $8x^{9/4} + C$

5. $y = 3x^4 + 4x^2 + 4x - 60$

6. 25/4

7. 10 2/3

8. 49/6

CHAPTER 7

INTEGRATION FORMULAS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Integrate a variable to a power, a constant, a function raised to a power, and a constant to a variable power.

2. Integrate the sum of differentiable functions, quotients, and trigonometric functions.

INTRODUCTION

In this chapter several of the integration formulas and proofs are discussed and examples are given. Some of the formulas from the previous chapter are repeated because they are considered essential for the understanding of integration. The formulas in this chapter are basic and should not be considered a complete collection of integration formulas. Integration is so complex that tables of integrals have been published as reference sources.

In the following formulas and proofs, u, v, and w are considered functions of a single variable.

POWER OF A VARIABLE

The integral of a variable to a power is the variable to a power increased by one and divided by the new power.

Formula.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$$

PROOF:

$$d\left(\frac{x^{n+1}}{n+1} + C\right) = \frac{(n+1)x^{n+1-1}}{(n+1)} \, dx$$

$$= \frac{(n+1)x^n}{(n+1)} \, dx$$

$$= x^n \, dx$$

Therefore,

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

EXAMPLE: Evaluate

$$\int x^5\, dx$$

SOLUTION:

$$\int x^5\, dx = \frac{x^{5+1}}{5+1} + C$$

$$= \frac{x^6}{6} + C$$

EXAMPLE: Evaluate

$$\int x^{-5}\, dx$$

SOLUTION:

$$\int x^{-5}\, dx = \frac{x^{-4}}{-4} + C$$

CONSTANTS

A constant may be written either before or after the integral sign.

Formula.

$$\int a\, du = a \int du = au + C$$

PROOF:

$$d\,(au + C) = a\, d\left(u + \frac{C}{a}\right)$$

$$= a\, du$$

Therefore,

$$\int a\, du = a \int du = au + C$$

EXAMPLE: Evaluate

$$\int 17\,dx$$

SOLUTION:

$$\int 17\,dx = 17 \int dx$$

$$= 17x + C$$

EXAMPLE: Evaluate

$$\int 3x^4\,dx$$

SOLUTION:

$$\int 3x^4\,dx = 3 \int x^4\,dx$$

$$= (3)\frac{x^5}{5} + C$$

$$= \frac{3x^5}{5} + C$$

SUMS

The integral of an algebraic sum of differentiable functions is the same as the algebraic sum of the integrals of these functions taken separately; that is, the integral of a sum is the sum of the integrals.

Formula.

$$\int (du + dv + dw) = \int du + \int dv + \int dw$$

PROOF:

$$d\,(u + v + w + C) = du + dv + dw$$

Therefore,

$$\int du + \int dv + \int dw = u + C_1 + v + C_2 + w + C_3$$

where

$$C_1 + C_2 + C_3 = C$$

Then

$$\int du + \int dv + \int dw = u + v + w + C$$

and

$$\int (du + dv + dw) = \int du + \int dv + \int dw$$

$$= u + v + w + C$$

EXAMPLE: Evaluate

$$\int (3x^2 + x)\, dx$$

SOLUTION:

$$\int (3x^2 + x)\, dx = \int 3x^2\, dx + \int x\, dx$$

$$= x^3 + C_1 + \frac{x^2}{2} + C_2$$

$$= x^3 + \frac{x^2}{2} + C$$

EXAMPLE: Evaluate

$$\int (x^5 + x^{-3})\, dx$$

SOLUTION:

$$\int (x^5 + x^{-3})\, dx = \int x^5\, dx + \int x^{-3}\, dx$$

$$= \frac{x^6}{6} + C_1 + \frac{x^{-2}}{-2} + C_2$$

$$= \frac{x^6}{6} - \frac{x^{-2}}{2} + C$$

$$= \frac{x^6}{6} - \frac{1}{2x^2} + C$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int x^6 \, dx$

2. $\int x^{-4} \, dx$

3. $\int 17x^2 \, dx$

4. $\int \pi r \, dr$

5. $\int 7x^{1/2} \, dx$

6. $\int (x^7 + x^6 + 3x^3) \, dx$

7. $\int (6 - x^3) \, dx$

ANSWERS:

1. $\frac{x^7}{7} + C$

2. $-\frac{1}{3x^3} + C$

3. $\frac{17}{3}x^3 + C$

4. $\frac{\pi r^2}{2} + C$

5. $\frac{14}{3}x^{3/2} + C$

6. $\frac{x^8}{8} + \frac{x^7}{7} + \frac{3}{4}x^4 + C$

7. $6x - \frac{x^4}{4} + C$

POWER OF A FUNCTION

The integral of a function raised to a power is found by the following steps:

1. *Increase the power of the function by 1.*
2. *Divide the result of step 1 by this increased power.*
3. *Add the constant of integration.*

Formula.

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C, \; n \neq -1$$

PROOF:

$$d\left(\frac{u^{n+1}}{n+1} + C\right) = \frac{(n+1)u^n}{n+1} \, du$$

$$= u^n \, du$$

Therefore,

$$\int u^n \, du = \frac{u^{n+1}}{n+1} + C$$

NOTE: Recall that

$$\frac{d}{dx}\left[\frac{1}{3}(2x-3)^3\right] = (3)\left(\frac{1}{3}\right)(2x-3)^2(2)$$

$$= 2(2x-3)^2$$

EXAMPLE: Evaluate

$$\int (2x-3)^2(2) \, dx$$

SOLUTION: Let

$$u = (2x-3)$$

so that

$$\frac{du}{dx} = 2$$

or

$$du = 2 \, dx$$

Then by substitution,

$$\int (2x - 3)^2(2)\, dx = \int u^2\, du$$

$$= \frac{u^3}{3} + C$$

Therefore,

$$\int (2x - 3)^2(2)\, dx = \frac{(2x - 3)^3}{3} + C$$

When you use this formula the integral must contain precisely *du*. If the required constant in *du* is not present, it must be placed in the integral and then compensation must be made.

EXAMPLE: Evaluate

$$\int (3x + 5)^6\, dx$$

SOLUTION: Let

$$u = (3x + 5)$$

so that

$$du = 3\, dx$$

We find *dx* in the integral but not 3 *dx*. A 3 must be included in the integral to fulfill the requirements of *du*.

In words, this means the integral

$$\int (3x + 5)^6\, dx$$

needs *du* so that the formula may be used.

Therefore, we write

$$\frac{3}{3} \int (3x + 5)^6\, dx$$

and recalling that a constant may be carried across the integral sign, we write

$$\frac{3}{3} \int (3x + 5)^6\, dx = \frac{1}{3} \int (3x + 5)^6(3)\, dx$$

Notice that we needed 3 in the integral for du, and we included 3 in the integral; we then compensated for the 3 by multiplying the integral by 1/3.

Then

$$\frac{1}{3}\int (3x + 5)^6(3)\, dx = \frac{1}{3}\int u^6\, du$$

$$= \left(\frac{1}{3}\right)\frac{u^7}{7} + C$$

$$= \frac{(3x + 5)^7}{21} + C$$

EXAMPLE: Evaluate

$$\int 3x(2 + x^2)^2\, dx$$

SOLUTION: Let

$$u = (2 + x^2)$$

so that

$$du = 2x\, dx$$

Then

$$\int 3x(2 + x^2)^2\, dx = \frac{2}{2}\int 3x(2 + x^2)^2\, dx$$

$$= \frac{3}{2}\int 2x(2 + x^2)^2\, dx$$

$$= \frac{3}{2}\int u^2\, du$$

$$= \left(\frac{3}{2}\right)\frac{u^3}{3} + C$$

$$= \frac{(2 + x^2)^3}{2} + C$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int (x^2 + 6)\,(2x)\,dx$

2. $\int x^2(7 + x^3)^2\,dx$

3. $\int (3x^2 + 2x)^2(6x + 2)\,dx$

4. $\int (6x^3 + 2x)^{1/2}(9x^2 + 1)\,dx$

5. $\int 4(x^2 + 7)^{-7}(x)\,dx$

ANSWERS:

1. $\dfrac{(x^2 + 6)^2}{2} + C$

2. $\dfrac{(7 + x^3)^3}{9} + C$

3. $\dfrac{(3x^2 + 2x)^3}{3} + C$

4. $\dfrac{(6x^3 + 2x)^{3/2}}{3} + C$

5. $-\dfrac{1}{3(x^2 + 7)^6} + C$

QUOTIENT

In this section three methods of integrating quotients are discussed, but only the second method will be proven.

The first method is to put the quotient into the form of the power of a function. The second method results in operations with logarithms. The third method is a special case in which the quotient must be simplified to use the sum rule.

METHOD 1

If we are given the integral

$$\int \frac{2x}{(9 - 4x^2)^{1/2}}\, dx$$

we observe that this integral may be written as

$$\int 2x(9 - 4x^2)^{-1/2}\, dx$$

By letting

$$u = (9 - 4x^2)$$

then

$$du = -8x\, dx$$

The only requirement for this to fit the form

$$\int u^n\, du$$

is the factor for du of -4. We accomplish this by multiplying $2x\, dx$ by -4, giving $-8x\, dx$, which is du. We then compensate for the factor -4 by multiplying the integral by $-1/4$.

Therefore,

$$\int \frac{2x}{(9 - 4x^2)^{1/2}}\, dx = \int 2x(9 - 4x^2)^{-1/2}\, dx$$

$$= -\frac{1}{4} \int (-4)\,(2x)\,(9 - 4x^2)^{-1/2}\, dx$$

$$= -\frac{1}{4} \int -8x(9 - 4x^2)^{-1/2}\, dx$$

$$= -\frac{1}{4} \int u^{-1/2}\, du$$

$$= \left(-\frac{1}{4}\right) \frac{u^{1/2}}{1/2} + C$$

$$= -\frac{(9 - 4x^2)^{1/2}}{2} + C$$

EXAMPLE: Evaluate

$$\int \frac{x}{\sqrt{(3 + x^2)}}\, dx$$

SOLUTION:

$$\int \frac{x}{\sqrt{(3 + x^2)}}\, dx = \int x(3 + x^2)^{-1/2}\, dx$$

Let

$$u = (3 + x^2)$$

so that

$$du = 2x\, dx$$

The factor 2 is used in the integral to give *du* and is compensated for by multiplying the integral by 1/2.
Therefore,

$$\int x(3 + x^2)^{-1/2}\, dx = \frac{1}{2} \int 2x(3 + x^2)^{-1/2}\, dx$$

$$= \frac{1}{2} \int u^{-1/2}\, du$$

$$= \left(\frac{1}{2}\right) \frac{u^{1/2}}{1/2} + C$$

$$= (3 + x^2)^{1/2} + C$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int \frac{x}{(2 + x^2)^{1/2}}\, dx$

2. $\int \frac{dx}{\sqrt[3]{3x + 1}}$

3. $\int \frac{dx}{(3x + 2)^5}$

ANSWERS:

1. $(2 + x^2)^{1/2} + C$

2. $\dfrac{(3x + 1)^{2/3}}{2} + C$

3. $\dfrac{-1}{12(3x + 2)^4} + C$

METHOD 2

In the previous formulas for integration of a function, the exponent was not allowed to be -1. In the special case of

$$\int u^n \, du$$

where

$$n = -1$$

we would have applied the following formula:

Formula.

$$\int \frac{du}{u} = \ln |u| + C, \ u \neq 0$$

PROOF:

$$d (\ln |u| + C) = \frac{1}{\text{u}} \, du$$

Therefore,

$$\int \frac{du}{u} = \ln |u| + C$$

NOTE: The derivative formula $\frac{d}{dx}(\ln \ u) = \frac{1}{u}$, $u > 0$, presented in chapter 5, can be extended for negative functions also. Hence, $\frac{d}{dx}(\ln |u|) = \frac{1}{u}$, $u \neq 0$.

EXAMPLE: Evaluate the integral

$$\int \frac{1}{x}\, dx$$

SOLUTION: If we write

$$\int \frac{1}{x}\, dx = \int x^{-1}\, dx$$

we find we are unable to evaluate

$$\int x^{-1}\, dx$$

by use of the power of a variable rule, so we write

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

because the 1 dx in the numerator is precisely du and we have fulfilled the requirements for

$$\int \frac{du}{u} = \ln|u| + C$$

EXAMPLE: Evaluate

$$\int \frac{2}{2x+1}\, dx$$

SOLUTION: Let

$$u = 2x + 1$$

so that

$$du = 2\, dx$$

We have the form

$$\int \frac{du}{u} = \ln|u| + C$$

therefore,

$$\int \frac{2}{2x+1}\, dx = \ln|2x+1| + C$$

EXAMPLE: Evaluate

$$\int \frac{2}{3x+1}\, dx$$

SOLUTION: Let

$$u = 3x + 1$$

so that

$$du = 3\ dx$$

We find we need 3 dx but we have 2 dx. We compensate as follows:

$$\int \frac{2}{3x+1}\,dx = 2\int \frac{dx}{3x+1}$$

$$= 2\left(\frac{1}{3}\right)\int \frac{3}{3x+1}\,dx$$

$$= \frac{2}{3}\int \frac{du}{u}$$

$$= \frac{2}{3}\ln|u| + C$$

Therefore,

$$\int \frac{2}{3x+1}\,dx = \frac{2}{3}\ln|3x+1| + C$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\displaystyle\int \frac{dx}{3x+2}$

2. $\displaystyle\int \frac{dx}{5-2x}$

3. $\displaystyle\int \frac{x}{2-3x^2}\,dx$

4. $\displaystyle\int \frac{2x^3}{3+2x^4}\,dx$

ANSWERS:

1. $\frac{1}{3} \ln |3x + 2| + C$

2. $-\frac{1}{2} \ln |5 - 2x| + C$

3. $-\frac{1}{6} \ln |2 - 3x^2| + C$

4. $\frac{1}{4} \ln |3 + 2x^4| + C$

METHOD 3

In the third method for solving integrals of quotients, we find that to integrate an algebraic function with a numerator that is not of lower degree than its denominator, we proceed as follows: *Change the integrand into a polynomial plus a fraction by dividing the denominator into the numerator.* After this is accomplished, apply the rules available.

EXAMPLE: Evaluate

$$\int \frac{16x^2 - 4x - 8}{2x + 1}\, dx$$

SOLUTION: Divide the denominator into the numerator so that

$$\int \frac{16x^2 - 4x - 8}{2x + 1}\, dx = \int \left(8x - 6 - \frac{2}{2x + 1}\right) dx$$

$$= \int 8x\, dx - \int 6\, dx - \int \frac{2}{2x + 1}\, dx$$

Integrating each term separately, we have

$$\int 8x\, dx = 4x^2 + C_1$$

and

$$-\int 6\, dx = -6x + C_2$$

and

$$-\int \frac{2}{2x + 1}\, dx = -\ln |2x + 1| + C_3$$

Then, by substitution, we find that

$$\int \frac{16x^2 - 4x - 8}{2x + 1}\, dx = 4x^2 - 6x - \ln|2x + 1| + C$$

where

$$C = C_1 + C_2 + C_3$$

EXAMPLE: Evaluate

$$\int \frac{x}{x + 1}\, dx$$

SOLUTION: The numerator in not of lower degree than the denominator; therefore, we divide and find that

$$\int \frac{x}{x + 1}\, dx = \int \left(1 - \frac{1}{x + 1}\right) dx$$

$$= \int dx - \int \frac{1}{x + 1}\, dx$$

Integrating each term separately, we find that

$$\int dx = x + C_1$$

and

$$-\int \frac{1}{x + 1}\, dx = -\ln|x + 1| + C_2$$

Therefore,

$$\int \frac{x}{x + 1}\, dx = x - \ln|x + 1| + C$$

where

$$C = C_1 + C_2$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int \frac{2x^2 + 6x + 5}{x + 1}\, dx$

2. $\int \frac{3x - 8}{x}\, dx$

3. $\int \frac{6x^3 + 13x^2 + 20x + 23}{2x + 3}\, dx$

4. $\int \frac{21x^2 + 16x + 4}{3x + 1}\, dx$

ANSWERS:

1. $x^2 + 4x + \ln|x + 1| + C$
2. $3x - 8 \ln|x| + C$
3. $x^3 + x^2 + 7x + \ln|2x + 3| + C$
4. $\frac{7}{2}x^2 + 3x + \frac{1}{3} \ln|3x + 1| + C$

CONSTANT TO A VARIABLE POWER

In this section a discussion of two forms of a constant to a variable power is presented. *The two forms are e^u and a^u, where u is a variable, a is any constant, and e is a defined constant.*

Formula.

$$\int e^u\, du = e^u + C$$

PROOF:

$$d\,(e^u + C) = e^u\, du$$

Therefore,

$$\int e^u\, du = e^u + C$$

EXAMPLE: Evaluate

$$\int e^x \, dx$$

SOLUTION: Let

$$u = x$$

so that

$$du = 1 \, dx$$

The integral is in the correct form to use

$$\int e^u \, du = e^u + C$$

therefore, using substitution, we find

$$\int e^x \, dx = e^x + C$$

EXAMPLE: Evaluate

$$\int e^{2x} dx$$

SOLUTION: Let

$$u = 2x$$

so that

$$du = 2 \, dx$$

We need a factor of 2 in the integral so that

$$\int e^{2x} \, dx = \frac{1}{2} \int e^{2x} 2 \, dx$$

$$= \frac{1}{2} \int e^u \, du$$

$$= \frac{1}{2} e^u + C$$

$$= \frac{1}{2} e^{2x} + C$$

EXAMPLE: Evaluate

$$\int xe^{2x^2}\,dx$$

SOLUTION: Let

$$u = 2x^2$$

so that

$$du = 4x\,dx$$

Here a factor of 4 is needed in the integral; therefore,

$$\begin{aligned}
\int xe^{2x^2}\,dx &= \frac{1}{4}\int 4xe^{2x^2}\,dx \\
&= \frac{1}{4}\int e^u\,du \\
&= \frac{1}{4}e^u + C \\
&= \frac{1}{4}e^{2x^2} + C
\end{aligned}$$

EXAMPLE: Evaluate

$$\int \frac{x^2}{e^{x^3}}\,dx$$

SOLUTION:

$$\int \frac{x^2}{e^{x^3}}\,dx = \int x^2e^{-x^3}\,dx$$

Let

$$u = -x^3$$

so that

$$du = -3x^2\,dx$$

Therefore,

$$\begin{aligned}
\int x^2e^{-x^3}\,dx &= -\frac{1}{3}\int -3x^2e^{-x^3}\,dx \\
&= -\frac{1}{3}\int e^u\,du \\
&= -\frac{1}{3}e^u + C \\
&= -\frac{1}{3}e^{-x^3} + C
\end{aligned}$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int -2xe^{-x^2}\,dx$
2. $\int e^{4x}\,dx$
3. $\int e^{(2x-1)}\,dx$
4. $\displaystyle\int \frac{2x}{e^{x^2}}\,dx$

ANSWERS:

1. $e^{-x^2} + C$
2. $\frac{1}{4}e^{4x} + C$
3. $\frac{1}{2}e^{(2x-1)} + C$
4. $-e^{-x^2} + C$

We will now discuss the second form of the integral of a constant to a variable power.

Formula.

$$\int a^u\,du = \frac{a^u}{\ln a} + C,\ a > 0$$

PROOF:

$$\begin{aligned} d\,(a^u + C_1) &= d\,(e^{\ln\,(a^u)} + C_1) \\ &= d\,(e^{u\,\ln\,a} + C_1) \\ &= e^{u\,\ln\,a}\ln a\,du \\ &= a^u \ln a\,du \end{aligned}$$

so that

$$\int a^u \ln a\,du = a^u + C_1$$

But ln a is a constant, so

$$\ln a \int a^u \, du = a^u + C_1$$

Then, by dividing both sides by ln a:

$$\int a^u \, du = \frac{a^u}{\ln a} + \frac{C_1}{\ln a}$$

or

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

where

$$C = \frac{C_1}{\ln a}$$

EXAMPLE: Evaluate

$$\int 3^x \, dx$$

SOLUTION: Let

$$u = x$$

so that

$$du = 1 \, dx$$

Therefore, by knowing that

$$\int a^u \, du = \frac{a^u}{\ln a} + C$$

and using substitution, we find that

$$\int 3^x \, dx = \frac{3^x}{\ln 3} + C$$

EXAMPLE: Evaluate

$$\int 3^{2x} \, dx$$

SOLUTION: Let

$$u = 2x$$

so that

$$du = 2\ dx$$

The integral should contain a factor of 2. Thus we insert a factor of 2 in the integral and compensate by multiplying the integral by 1/2.

Then,

$$\int 3^{2x}\ dx = \frac{1}{2}\int (2)3^{2x}\ dx$$

$$= \frac{1}{2}\int 3^{2x}2\ dx$$

Therefore,

$$\frac{1}{2}\int 3^{2x}2\ dx = \frac{1}{2}\int 3^{u}\ du$$

$$= \left(\frac{1}{2}\right)\frac{3^{u}}{\ln 3} + C$$

$$= \frac{3^{2x}}{2\ln 3} + C$$

EXAMPLE: Evaluate

$$\int 7xb^{x^2}\ dx$$

SOLUTION: Let

$$u = x^2$$

so that

$$du = 2x\ dx$$

We find we need $2x\,dx$; therefore, we remove the 7 and insert a 2 by writing

$$\begin{aligned}\int 7xb^{x^2}\,dx &= 7\int xb^{x^2}\,dx\\ &= 7\left(\frac{1}{2}\right)\int 2xb^{x^2}\,dx\\ &= \frac{7}{2}\int b^u\,du\\ &= \frac{7b^u}{2\ln b} + C\\ &= \frac{7b^{x^2}}{2\ln b} + C\end{aligned}$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int 7^{3x}\,dx$
2. $\int (3x)\,9^{x^2}\,dx$
3. $\int 2^{(3x^2+1)}x\,dx$

ANSWERS:

1. $\dfrac{7^{3x}}{3\ln 7} + C$
2. $\left(\dfrac{3}{2}\right)\dfrac{9^{x^2}}{\ln 9} + C$
3. $\dfrac{2^{(3x^2+1)}}{6\ln 2} + C$

TRIGONOMETRIC FUNCTIONS

Trigonometric functions, which comprise one group of transcendental functions, may be differentiated and integrated in the same fashion as the other functions. We will limit our proofs to the sine, cosine, and secant functions but will list several others.

Formula.

$$\int \sin u \, du = -\cos u + C$$

PROOF:

$$d\,(\cos u + C) = -\sin u \, du$$

and

$$d\,(-\cos u + C) = \sin u \, du$$

Therefore,

$$\int \sin u \, du = -\cos u + C$$

Formula.

$$\int \cos u \, du = \sin u + C$$

PROOF:

$$d\,(\sin u + C) = \cos u \, du$$

Therefore,

$$\int \cos u \, du = \sin u + C$$

Formula.

$$\int \sec^2 u \, du = \tan u + C$$

PROOF:

$$d\,(\tan u + C) = d\left(\frac{\sin u}{\cos u} + C\right)$$

and by the quotient rule

$$d\left(\frac{\sin u}{\cos u} + C\right) = \frac{(\cos u)(\cos u) - (\sin u)(-\sin u)}{\cos^2 u}\, du$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u}\, du$$

$$= \frac{1}{\cos^2 u}\, du$$

$$= \sec^2 u\, du$$

Therefore,

$$\int \sec^2 u\, du = \tan u + C$$

To this point we have considered integrals of trigonometric functions that result in functions of the sine, cosine, and tangent. Those integrals that result in functions of the cotangent, secant, and cosecant are included in the following list of elementary integrals:

$$\int \sin u\, du = -\cos u + C$$

$$\int \cos u\, du = \sin u + C$$

$$\int \sec^2 u\, du = \tan u + C$$

$$\int \csc^2 u\, du = -\cot u + C$$

$$\int \sec u \tan u\, du = \sec u + C$$

$$\int \csc u \cot u\, du = -\csc u + C$$

EXAMPLE: Evaluate

$$\int \sin 3x\, dx$$

SOLUTION: We need the integral in the form of

$$\int \sin u\, du = -\cos u + C$$

We let

$$u = 3x$$

so that

$$du = 3\, dx$$

but we do not have 3 dx. Therefore, we multiply the integral by 3/3 and rearrange as follows:

$$\int \sin 3x\, dx = \frac{1}{3}\int 3 \sin 3x\, dx$$

$$= \frac{1}{3}\int \sin u\, du$$

$$= \frac{1}{3}(-\cos u) + C$$

$$= -\frac{1}{3}\cos 3x + C$$

EXAMPLE: Evaluate

$$\int \cos(2x + 4)\, dx$$

SOLUTION: Let

$$u = (2x + 4)$$

so that

$$du = 2\, dx$$

Therefore,

$$\int \cos(2x + 4)\, dx = \frac{1}{2}\int \cos(2x + 4)\, 2\, dx$$

$$= \frac{1}{2}\int \cos u\, \mathrm{d}u$$

$$= \frac{1}{2}\sin u + C$$

$$= \frac{1}{2}\sin(2x + 4) + C$$

EXAMPLE: Evaluate

$$\int (3 \sin 2x + 4 \cos 3x)\, dx$$

SOLUTION: We use the rule for sums and write

$$\int (3 \sin 2x + 4 \cos 3x)\, dx$$

$$= \int 3 \sin 2x\, dx + \int 4 \cos 3x\, dx$$

Then, in the integral

$$\int 3 \sin 2x\, dx$$

let

$$u = 2x$$

so that

$$du = 2\, dx$$

but we have

$$3\, dx$$

Hence, proper compensation has to be made as follows:

$$\int 3 \sin 2x\, dx = 3 \int \sin 2x\, dx$$

$$= 3\left(\frac{1}{2}\right) \int 2 \sin 2x\, dx$$

$$= \frac{3}{2} \int \sin u\, du$$

$$= \frac{3}{2}(-\cos u) + C_1$$

$$= -\frac{3}{2} \cos 2x + C_1$$

The second integral

$$\int 4 \cos 3x \, dx$$

with

$$u = 3x$$

and

$$du = 3 \, dx$$

is evaluated as follows:

$$\int 4 \cos 3x \, dx = \frac{4}{3} \int 3 \cos 3x \, dx$$

$$= \frac{4}{3} \int \cos u \, du$$

$$= \frac{4}{3}(\sin u) + C_2$$

$$= \frac{4}{3}(\sin 3x) + C_2$$

Then, by combining the two solutions, we have

$$\int (3 \sin 2x + 4 \cos 3x) \, dx$$

$$= -\frac{3}{2} \cos 2x + C_1 + \frac{4}{3} \sin 3x + C_2$$

$$= -\frac{3}{2} \cos 2x + \frac{4}{3} \sin 3x + C$$

where

$$C_1 + C_2 = C$$

EXAMPLE: Evaluate

$$\int \sec^2 5x \, dx$$

SOLUTION: Let

$$u = 5x$$

so that

$$du = 5 \, dx$$

We need 5 dx so we write

$$\int \sec^2 5x \, dx = \frac{1}{5} \int 5 \sec^2 5x \, dx$$

$$= \frac{1}{5} \int \sec^2 u \, du$$

$$= \frac{1}{5}(\tan u) + C$$

$$= \frac{1}{5}(\tan 5x) + C$$

EXAMPLE: Evaluate

$$\int \csc 2x \cot 2x \, dx$$

SOLUTION: Let

$$u = 2x$$

so that

$$du = 2 \, dx$$

We require du equal to 2 dx, so we write

$$\int \csc 2x \cot 2x \, dx = \frac{1}{2} \int 2 \csc 2x \cot 2x \, dx$$

$$= \frac{1}{2} \int \csc u \cot u \, du$$

$$= \frac{1}{2}(-\csc u) + C$$

$$= -\frac{1}{2} \csc 2x + C$$

EXAMPLE: Evaluate

$$\int \csc^2 6x \, dx$$

SOLUTION: Let

$$u = 6x$$

so that

$$du = 6\ dx$$

then,

$$\int \csc^2 6x\ dx = \frac{1}{6}\int 6 \csc^2 6x\ dx$$

$$= -\frac{1}{6} \cot 6x + C$$

EXAMPLE: Evaluate

$$\int \sec \frac{x}{2} \tan \frac{x}{2}\ dx$$

SOLUTION: Let

$$u = \frac{x}{2}$$

so that

$$du = \frac{1}{2}\ dx$$

then,

$$\int \sec \frac{x}{2} \tan \frac{x}{2}\ dx = 2 \int \frac{1}{2} \sec \frac{x}{2} \tan \frac{x}{2}\ dx$$

$$= 2 \sec \frac{x}{2} + C$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int \cos 4x\ dx$
2. $\int \sin 5x\ dx$
3. $\int \sec^2 6x\ dx$
4. $\int 3 \cos (6x + 2)\ dx$
5. $\int x \sin (2x^2)\ dx$
6. $\int 2 \csc^2 5x\ dx$
7. $\int 3 \sec \frac{x}{3} \tan \frac{x}{3}\ dx$

ANSWERS:

1. $\frac{1}{4}\sin 4x + C$

2. $-\frac{1}{5}\cos 5x + C$

3. $\frac{1}{6}\tan 6x + C$

4. $\frac{1}{2}\sin (6x + 2) + C$

5. $-\frac{1}{4}\cos (2x^2) + C$

6. $-\frac{2}{5}\cot 5x + C$

7. $9 \sec \frac{x}{3} + C$

POWERS OF TRIGONOMETRIC FUNCTIONS

The integrals of powers of trigonometric functions will be limited to those which may, by substitution, be written in the form

$$\int u^n \, du$$

EXAMPLE: Evaluate

$$\int \sin^4 x \cos x \, dx$$

SOLUTION: Let

$$u = \sin x$$

so that

$$du = \cos x \, dx$$

By substitution,

$$\int \sin^4 x \cos x \, dx = \int u^4 \, du$$

$$= \frac{u^5}{5} + C$$

Then, by substitution again, find that

$$\frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C$$

Therefore,

$$\int \sin^4 x \cos x \, dx = \frac{\sin^5 x}{5} + C$$

EXAMPLE: Evaluate

$$\int (\cos^3 x)(-\sin x) \, dx$$

SOLUTION: Let

$$u = \cos x$$

so that

$$du = -\sin x \, dx$$

We know that

$$\int u^3 \, du = \frac{u^4}{4} + C$$

so by substitution

$$\frac{u^4}{4} + C = \frac{\cos^4 x}{4} + C$$

PRACTICE PROBLEMS:

Evaluate the following integrals:

1. $\int \sin^2 x \cos x \, dx$
2. $\int \sin^4\left(\frac{x}{2}\right) \cos \frac{x}{2} \, dx$
3. $\int 6 \sin x \cos^5 x \, dx$
4. $\int \frac{\cos 2x}{\sin^3 2x} \, dx$
5. $\int \cos^3\left(\frac{x}{8}\right) \sin \frac{x}{8} \, dx$
6. $\int (\sin x \cos x)(\sin x + \cos x) \, dx$

ANSWERS:

1. $\frac{1}{3} \sin^3 x + C$
2. $\frac{2}{5} \sin^5\left(\frac{x}{2}\right) + C$
3. $-\cos^6 x + C$
4. $\frac{-1}{4 \sin^2 2x} + C$
5. $-2 \cos^4\left(\frac{x}{8}\right) + C$
6. $\frac{\sin^3 x - \cos^3 x}{3} + C$

SUMMARY

The following are the major topics covered in this chapter:

1. **Integral of a variable to a power:** The integral of a variable to a power is the variable to a power increased by one and divided by the new power.

   Formula. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

2. **Integral of a constant:** A constant may be written either before or after the integral sign.

   Formula. $\int a \, du = a \int du = au + C$

3. **Integral of the sum of differentiable functions:** The integral of an algebraic sum of differentiable functions is the same as the algebraic sum of the integrals of these functions taken separately.

   Formula. $\int (du + dv + dw) = \int du + \int dv + \int dw$

4. **Integral of a function raised to a power:** The integral of a function raised to a power is found by the following steps:

   1. Increase the power of the function by 1.

   2. Divide the result of step 1 by this increased power.

   3. Add the constant of integration.

   Formula. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$

5. **Integral of quotients:**

   Method 1. Integrate by putting the quotient into the form of the power of a function.

   Method 2. Integrate quotients by use of operations of logarithms.

   Formula. $\int \frac{du}{u} = \ln |u| + C,\ u \neq 0$

Method 3. Integrate quotients by changing the integrand into a polynomial plus a fraction by dividing the denominator into the numerator.

6. **Integral of a constant to a variable power:**

Formula. $\int e^u \, du = e^u + C$ and $\int a^u \, du = \frac{a^u}{\ln a} + C,\ a > 0$

where u is a variable, a is any constant, and e is a defined constant.

7. **Integral of trigonometric functions:**

Formula. $\int \sin u \, du = -\cos u + C$

Formula. $\int \cos u \, du = \sin u + C$

Formula. $\int \sec^2 u \, du = \tan u + c$

Formula. $\int \csc^2 u \, du = -\cot u + C$

Formula. $\int \sec u \tan u \, du = \sec u + C$

Formula. $\int \csc u \cot u \, du = -\csc u + C$

8. **Integral of powers of trigonometric functions:** The integrals of powers of trigonometric functions will be limited to those which may, by substitution, be written in the form $\int u^n \, du$.

ADDITIONAL PRACTICE PROBLEMS

Evaluate the following integrals:

1. $\int (\pi^2x^2 - \pi x + \pi)\, dx$

2. $\int (6x + 3)\sqrt{x^2 + x}\, dx$

3. $\int \dfrac{4x^3 + 1}{(1 - x - x^4)^2}\, dx$

4. $\int \dfrac{4x^{1/3}}{3 + x^{4/3}}\, dx$

5. $\int \dfrac{\pi x^2 + 2\pi x + 3\pi}{\pi x + \pi}\, dx$

6. $\int 6e^{3x}(6 + e^{3x})^7\, dx$

7. $\int (x^\pi - \pi^x)\, dx$

8. $\int e^{-x} \sin (e^{-x})\, dx$

9. $\int 12x^2 \csc (2x^3) \cot (2x^3)\, dx$

10. $\int \dfrac{\sin (6x)}{1 + \cos (6x)}\, dx$

11. $\int \sec^2(x - 1)\, e^{\tan\,(x-1)}\, dx$

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. $\frac{\pi^2}{3}x^3 - \frac{\pi}{2}x^2 + \pi x + C$

2. $2(x^2 + x)^{3/2} + C$

3. $\frac{1}{(1 - x - x^4)} + C$

4. $3 \ln |3 + x^{4/3}| + C$

5. $x^2/2 + x + 2 \ln |x + 1| + C$

6. $\frac{1}{4}(6 + e^{3x})^8 + C$

7. $\frac{x^{\pi + 1}}{\pi + 1} - \frac{\pi^x}{\ln \pi} + C$

8. $\cos (e^{-x}) + C$

9. $-2 \csc (2x^3) + C$

10. $-\frac{1}{6} \ln |1 + \cos (6x)| + C$

11. $e^{\tan (x - 1)} + C$

CHAPTER 8

COMBINATIONS AND PERMUTATIONS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Define combinations and permutations.

2. Apply the concept of combinations to problem solving.

3. Apply the concept of principle of choice to problem solving.

4. Apply the concept of permutations to problem solving.

INTRODUCTION

This chapter deals with concepts required for the study of probability and statistics. Statistics is a branch of science that is an outgrowth of the theory of probability. Combinations and permutations are used in both statistics and probability; and they, in turn, involve operations with factorial notation. Therefore, combinations, permutations, and factorial notation are discussed in this chapter.

DEFINITIONS

A *combination* is defined as a possible selection of a certain number of objects taken from a group without regard to order. For instance, suppose we were to choose two letters from a group of three letters. If the group of three letters were A, B, and C, we could choose the letters in combinations of two as follows:

$$AB, AC, BC$$

The order in which we wrote the letters is of no concern; that is, AB could be written BA, but we would still have only one combination of the letters A and B.

A permutation is defined as a possible selection of a certain number of objects taken from a group with regard to order. The permutations of two letters from the group of three letters would be as follows:

$$AB, AC, BC, BA, CA, CB$$

The symbol used to indicate the foregoing combination will be ${}_3C_2$, meaning a group of three objects taken two at a time. For the previous permutation we will use ${}_3P_2$, meaning a group of three objects taken two at a time and ordered.

You will need an understanding of factorial notation before we begin a detailed discussion of combinations and permutations. We define the product of the integers n through 1 as *n factorial* and use the symbol $n!$ to denote this; that is,

$$3! = 3 \cdot 2 \cdot 1$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n - 1) \cdot \cdot \cdot 3 \cdot 2 \cdot 1$$

EXAMPLE: Find the value of 5!

SOLUTION:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 120$$

EXAMPLE: Find the value of

$$\frac{5!}{3!}$$

SOLUTION:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and

$$3! = 3 \cdot 2 \cdot 1$$

Then

$$\frac{5!}{3!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1}$$

and by simplification

$$\frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1} = 5\cdot 4$$

$$= 20$$

The previous example could have been solved by writing

$$\frac{5!}{3!} = \frac{5\cdot 4\cdot 3!}{3!}$$

$$= 5\cdot 4$$

Notice that we wrote

$$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1$$

and combine the factors

$$3\cdot 2\cdot 1$$

as

$$3!$$

so that

$$5! = 5\cdot 4\cdot 3!$$

EXAMPLE: Find the value of

$$\frac{6! - 4!}{4!}$$

SOLUTION:

$$6! = 6\cdot 5\cdot 4!$$

and

$$4! = 4!\cdot 1$$

Then

$$\frac{6! - 4!}{4!} = \frac{(6\cdot 5 - 1)\ 4!}{4!}$$

$$= (6\cdot 5 - 1)$$

$$= 29$$

Notice that 4! was factored from the expression

$$6! - 4!$$

Theorem. *If n and r are positive integers, with n greater than r, then*

$$n! = (n)(n-1)\cdots(r+2)(r+1)r!$$

This theorem allows us to simplify an expression as follows:

$$\begin{aligned} 5! &= 5\cdot 4! \\ &= 5\cdot 4\cdot 3! \\ &= 5\cdot 4\cdot 3\cdot 2! \\ &= 5\cdot 4\cdot 3\cdot 2\cdot 1 \end{aligned}$$

Another example is

$$\begin{aligned} (n+2)! &= (n+2)(n+1)! \\ &= (n+2)(n+1)(n!) \\ &= (n+2)(n+1)(n)\cdots 1 \end{aligned}$$

EXAMPLE: Simplify

$$\frac{(n+3)!}{n!}$$

SOLUTION:

$$(n+3)! = (n+3)(n+2)(n+1)n!$$

then

$$\begin{aligned} \frac{(n+3)!}{n!} &= \frac{(n+3)(n+2)(n+1)n!}{n!} \\ &= (n+3)(n+2)(n+1) \end{aligned}$$

PRACTICE PROBLEMS:

Find the value of problems 1 through 4 and simplify problems 5 and 6.

1. $6!$
2. $3!4!$
3. $\frac{8!}{11!}$
4. $\frac{5! - 3!}{3!}$
5. $\frac{n!}{(n - 1)!}$
6. $\frac{(n + 2)!}{n!}$

ANSWERS:

1. 720
2. 144
3. $\frac{1}{990}$
4. 19
5. n
6. $(n + 1)(n + 2)$

COMBINATIONS

As indicated previously, a combination is the selection of a certain number of objects taken from a group of objects without regard to order. We use the symbol ${}_5C_3$ to indicate that we have five objects taken three at a time, without regard to order. Using the letters A, B, C, D, and E to designate the five objects, we list the combinations as follows:

$$ABC \; ABD \; ABE \; ACD \; ACE$$

$$ADE \; BCD \; BCE \; BDE \; CDE$$

We find 10 combinations of 5 objects are taken 3 at a time. The word *combinations* implies that order is not considered.

EXAMPLE: Suppose we wish to know how many color combinations can be made from four different colored marbles if we use only three marbles at a time. The marbles are colored red, green, white, and yellow.

SOLUTION: We let the first letter in each word indicate the color; then we list the possible combinations as follows:

$$RGW \quad RGY \quad RWY \quad GWY$$

If we rearrange the groups, for example *RGW*, to form *GWR* or *RWG*, we still have the same color combination within each group; therefore, order is not important.

The previous examples are relatively easy to understand; but suppose we have 20 boys and wish to know how many different basketball teams we could form, one at a time, from these boys. Our listing would be quite lengthy, and we would have a difficult task to determine we had all of the possible combinations. In fact, we would have to list over 15,000 combinations. This indicates the need for a formula.

The general formula for possible combinations of r objects from a group of n objects is

$$_nC_r = \frac{n(n-1)\cdots(n-r+1)}{r\cdots 3\cdot 2\cdot 1}$$

The denominator may be written as

$$r\cdots 3\cdot 2\cdot 1 = r!$$

and if we multiply both numerator and denominator by

$$(n-r)\cdots 2\cdot 1$$

which is

$$(n-r)!$$

we have

$$_nC_r = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1}{r!(n-r)\cdots 2\cdot 1}$$

The numerator

$$n(n-1)\cdots(n-r+1)(n-r)\cdots 2\cdot 1$$

is

$$n!$$

Therefore,

$$_nC_r = \frac{n!}{r!(n-r)!}$$

This formula is read: *The number of combinations of n objects taken r at a time is equal to n factorial divided by r factorial times n minus r factorial.*

EXAMPLE: In the previous problem where 20 boys were available, how many different basketball teams could be formed?

SOLUTION: If the choice of which boy played center, guard, or forward is not considered, we find the number of combinations of 20 boys taken 5 at a time by writing

$$_nC_r = \frac{n!}{r!(n-r)!}$$

where

$$n = 20$$

and

$$r = 5$$

Then, by substitution, we have

$$\begin{aligned} _nC_r = {_{20}C_5} &= \frac{20!}{5!(20-5)!} \\ &= \frac{20!}{5!15!} \\ &= \frac{20\cdot 19\cdot 18\cdot 17\cdot 16\cdot 15!}{5!15!} \\ &= \frac{20\cdot 19\cdot 18\cdot 17\cdot 16}{5\cdot 4\cdot 3\cdot 2\cdot 1} \\ &= \frac{1\cdot 19\cdot 3\cdot 17\cdot 16}{1} \\ &= 15{,}504 \end{aligned}$$

EXAMPLE: A man has , in his pocket, a silver dollar, a half-dollar, a quarter, a dime, a nickel, and a penny. If he reaches into his pocket and pulls out three coins, how many different sums may he have?

SOLUTION: The order is not important; therefore, the number of combinations of coins possible is

$$
\begin{aligned}
{}_6C_3 &= \frac{6!}{3!(6-3)!} \\
&= \frac{6!}{3!3!} \\
&= \frac{6\cdot 5\cdot 4\cdot 3!}{3!3!} \\
&= \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} \\
&= \frac{5\cdot 4}{1} \\
&= 20
\end{aligned}
$$

EXAMPLE: Find the value of

$$ {}_3C_3 $$

SOLUTION: We use the formula given and find that

$$
\begin{aligned}
{}_3C_3 &= \frac{3!}{3!(3-3)!} \\
&= \frac{3!}{3!0!}
\end{aligned}
$$

This seems to violate the rule, *division by zero is not allowed*, but we *define 0! to equal 1.*

Then

$$ \frac{3!}{3!0!} = \frac{3!}{3!} = 1 $$

which is obvious if we list the combinations of three things taken three at a time.

PRACTICE PROBLEMS:

Find the value of problems 1 through 6 and solve problems 7, 8, and 9.

1. ${}_6C_2$
2. ${}_6C_4$
3. ${}_{15}C_5$
4. ${}_7C_7$
5. $\dfrac{{}_6C_3 + {}_7C_3}{{}_{13}C_6}$
6. $\dfrac{{}_7C_3 \cdot {}_6C_3}{{}_{14}C_4}$
7. We want to paint three rooms in a house, each a different color, and we may choose from seven different colors of paint. How many color combinations are possible for the three rooms?
8. If 20 boys go out for the football team, how many different teams may be formed, one at a time?
9. Two girls and their dates go to the drive-in, and each wants a different flavored ice cream cone. The drive-in has 24 flavors of ice cream. How many combinations of flavors may be chosen among the four of them if each one selects one flavor?

ANSWERS:

1. 15
2. 15
3. 3,003
4. 1
5. $\frac{5}{156}$
6. $\frac{100}{143}$
7. 35
8. 167,960
9. 10,626

PRINCIPLE OF CHOICE

The *principle of choice* is discussed in relation to combinations, although it is also discussed later in this chapter in relation to permutations. It is stated as follows:

If a selection can be made in n_1 ways; and after this selection is made, a second selection can be made in n_2 ways; and after this selection is made, a third selection can be made in n_3 ways; and so forth for r selections, then the r selections can be made together in

$$n_1 \cdot n_2 \cdot n_3 \cdot \cdot \cdot n_r \text{ ways}$$

EXAMPLE: In how many ways can a coach choose first a football team and then a basketball team from 18 boys?

SOLUTION: First let the coach choose a football team; that is,

$$
\begin{aligned}
{}_{18}C_{11} &= \frac{18!}{11!(18-11)!} \\
&= \frac{18!}{11!7!} \\
&= \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11!}{11!7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= 31{,}824
\end{aligned}
$$

The coach now must choose a basketball team from the remaining seven boys; that is,

$$
\begin{aligned}
{}_7C_5 &= \frac{7!}{5!(7-5)!} \\
&= \frac{7!}{5!2!} \\
&= \frac{7 \cdot 6 \cdot 5!}{5!2!} \\
&= \frac{7 \cdot 6}{2} \\
&= 21
\end{aligned}
$$

Then, together, the two teams can be chosen in

$$(31{,}824)(21) = 668{,}304 \text{ ways}$$

The same answer would be achieved if the coach chose the basketball team first and then the football team; that is,

$$
\begin{aligned}
{}_{18}C_5 \cdot {}_{13}C_{11} &= \frac{18!}{5!13!} \cdot \frac{13!}{11!2!} \\
&= (8{,}568)(78) \\
&= 668{,}304
\end{aligned}
$$

which is the same number as before.

EXAMPLE: A woman ordering dinner has a choice of one meat dish from four, four vegetables from seven, one salad from three, and one dessert from four. How many different menus are possible?

SOLUTION: The individual combinations are as follows:

meat	${}_4C_1$
vegetable	${}_7C_4$
salad	${}_3C_1$
dessert	${}_4C_1$

The values of these combinations are

$$
\begin{aligned}
{}_4C_1 &= \frac{4!}{1!(4-1)!} \\
&= \frac{4!}{3!} \\
&= 4
\end{aligned}
$$

and

$$
\begin{aligned}
{}_7C_4 &= \frac{7!}{4!(7-4)!} \\
&= \frac{7!}{4!3!} \\
&= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} \\
&= 35
\end{aligned}
$$

and

$$
\begin{aligned}
{}_3C_1 &= \frac{3!}{1!(3-1)!} \\
&= \frac{3!}{2!} \\
&= 3
\end{aligned}
$$

Therefore, the woman has a choice of

$$(4)(35)(3)(4) = 1{,}680$$

different menus.

PRACTICE PROBLEMS:

Solve the following problems:

1. A man has 12 different colored shirts and 20 different ties. How many shirt and tie combinations can he select to take on a trip if he takes 3 shirts and 5 ties?

2. A petty officer in charge of posting the watch has 12 men in his duty section. He must post 3 different fire watches and then post 4 aircraft guards on different aircraft. How many different assignments of men can he make?

3. If 10 third class and 14 second class petty officers are in a division that must furnish 2 second class and 6 third class petty officers for shore patrol, how many different shore patrol parties can be made?

ANSWERS:

1. 3,410,880

2. 27,720

3. 19,110

PERMUTATIONS

Permutations are similar to combinations but extend the requirements of combinations by considering order.

Suppose we have two letters, A and B, and wish to know how many arrangements of these letters can be made. Obviously the answer is two; that is,

$$AB \text{ and } BA$$

If we extend this to the three letters A, B, and C, we find the answer to be

$$ABC,\ ACB,\ BAC,\ BCA,\ CAB,\ CBA$$

We had three choices for the first letter; after we chose the first letter, we had only two choices for the second letter; and after the second letter, we had only one choice. This is shown in the "tree" diagram in figure 8-1. Notice that a total of six different paths lead to the ends of the "branches" of the "tree" diagram.

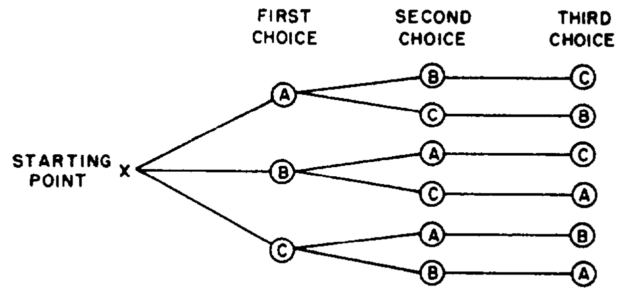


Figure 8-1.—"Tree" diagram.

If the number of objects is large, the tree would become very complicated; therefore, we approach the problem in another manner, using parentheses to show the possible choices. Suppose we were to arrange six objects in as many different orders as possible. For the first choice we have six objects:

$$(6)(\ )(\ )(\ )(\ )(\ )$$

For the second choice we have only five choices:

$$(6)(5)(\ )(\ )(\ )(\ )$$

For the third choice we have only four choices:

$$(6)(5)(4)(\ )(\ )(\ )$$

This may be continued as follows:

$$(6)(5)(4)(3)(2)(1)$$

By applying the principle of choice, we find the total possible ways of arranging the objects to be the product of the individiual choices; that is,

$$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

and this may be written as

$$6!$$

This leads to the statement: *The number of permutations of n objects taken all together is equal to n*!.

EXAMPLE: How many permutations of seven different letters may be made?

SOLUTION: We could use a "tree" diagram, but this would become complicated. (Try it to see why.) We could use the parentheses as follows:

$$(7)(6)(5)(4)(3)(2)(1) = 5040$$

The easiest solution is to use the previous statement and write

$$_7P_7 = 7!$$

that is, the number of possible arrangements of seven objects taken seven at a time is 7!.

NOTE: Compare this with the number of combinations of seven objects taken seven at a time.

If we are faced with finding the number of permutations of seven objects taken three at a time, we use three parentheses as follows:

In the first position we have a choice of seven objects:

$$(7)(\ )(\ )$$

In the second position we have a choice of six objects:

$$(7)(6)(\ )$$

In the last position we have a choice of five objects:

$$(7)(6)(5)$$

Therefore by principle of choice, the solution is

$$7 \cdot 6 \cdot 5 = 210$$

At this point we will use our knowledge of combinations to develop a formula for the number of permutations of n objects taken r at a time.

Suppose we wish to find the number of permutations of five things taken three at a time. We first determine the number of groups of three as follows:

$$\begin{aligned} {}_5C_3 &= \frac{5!}{3!(5-3)!} \\ &= \frac{5!}{3!2!} \\ &= 10 \end{aligned}$$

Thus, we have 10 groups of 3 objects each.

We are now asked to arrange each of these 10 groups in as many orders as possible. We know that the number of permutations of three objects taken together is 3!. We may arrange each of the 10 groups in 3! or 6 ways. The total number of possible permutations of ${}_5C_3$ then is

$${}_5C_3 \cdot 3! = 10 \cdot 6$$

which can be written as

$${}_5C_3 \cdot 3! = {}_5P_3$$

The corresponding general form is

$${}_nC_r \cdot r! = {}_nP_r$$

Knowing that

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

then

$$\begin{aligned} {}_nC_r \cdot r! &= \frac{n!}{r!(n-r)!} \cdot r! \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

But

$${}_nC_r \cdot r! = {}_nP_r$$

therefore,

$${}_nP_r = \frac{n!}{(n-r)!}$$

This formula is read: *The number of permutations of n objects taken r at a time is equal to n factorial divided by n minus r factorial.*

EXAMPLE: How many permutation of six objects taken two at a time can be made?

SOLUTION: The number of permutations of six objects taken two at a time is written

$$
\begin{aligned}
{}_6P_2 &= \frac{6!}{(6-2)!} \\
&= \frac{6!}{4!} \\
&= \frac{6 \cdot 5 \cdot 4!}{4!} \\
&= 6 \cdot 5 \\
&= 30
\end{aligned}
$$

EXAMPLE: In how many ways can eight people be arranged in a row?

SOLUTION: All eight people must be in the row; therefore, we want the number of permutations of eight people taken eight at a time, which is

$$
\begin{aligned}
{}_8P_8 &= \frac{8!}{(8-8)!} \\
&= \frac{8!}{0!}
\end{aligned}
$$

(remember that 0! was defined as equal to 1)

then

$$
\begin{aligned}
\frac{8!}{0!} &= \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1} \\
&= 40{,}320
\end{aligned}
$$

Problems dealing with combinations and permutations often require the use of both formulas to solve one problem.

EXAMPLE: Eight first class and six second class petty officers are on the board of the 56 club. In how many ways can the members elect, from the board, a president, a vice-president, a secretary, and a treasurer if the president and secretary must be first class petty officers and the vice-president and treasurer must be second class petty officers?

SOLUTION: Since two of the eight first class petty officers are to fill two different offices, we write

$$
\begin{aligned}
{}_8P_2 &= \frac{8!}{(8-2)!} \\
&= \frac{8!}{6!} \\
&= 8 \cdot 7 \\
&= 56
\end{aligned}
$$

Then, two of the six second class petty officers are to fill two different offices; thus, we write

$$
\begin{aligned}
{}_6P_2 &= \frac{6!}{(6-2)!} \\
&= \frac{6!}{4!} \\
&= 6 \cdot 5 \\
&= 30
\end{aligned}
$$

The principle of choice holds in this case; therefore, the members have

$$56 \cdot 30 = 1{,}680$$

ways to select the required office holders.

EXAMPLE: For the preceding example, suppose we are asked the following: In how many ways can the members elect the office holders from the board if two of the office holders must be first class petty officers and two of the office holders must be second class petty officers?

SOLUTION: We have already determined how many ways eight things may be taken two at a time, how many ways six things may be taken two at a time, and how many ways they may be taken together; that is,

$${}_8P_2 = 56$$

and

$${}_6P_2 = 30$$

then

$${}_8P_2 \cdot {}_6P_2 = 1{,}680$$

Our problem now is to find how many ways the members can combine the four offices two at a time. Therefore, we write

$$
\begin{aligned}
{}_4C_2 &= \frac{4!}{2!(4-2)!} \\
&= \frac{4!}{2!2!} \\
&= \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2} \\
&= 6
\end{aligned}
$$

Then, in answer to the problem, we write

$$ {}_8P_2 \cdot {}_6P_2 \cdot {}_4C_2 = 10{,}080 $$

In other words, if the members have ${}_4C_2$ ways of combining the four offices and then for every one of these ways, the members have ${}_8P_2 \cdot {}_6P_2$ ways of arranging the office holders, then they have

$$ {}_8P_2 \cdot {}_6P_2 \cdot {}_4C_2 $$

ways of electing the petty officers.

PRACTICE PROBLEMS:

Find the answers to the following:

1. ${}_6P_3$
2. ${}_4P_3$
3. ${}_7P_2 \cdot {}_5P_2$
4. In how many ways can six people be seated in a row?
5. Seven boys and nine girls are in a club. In how many ways can they elect four different officers designated by A, B, C, and D if

   a. A and B must be boys and C and D must be girls?

   b. two of the officers must be boys and two of the officers must be girls?

ANSWERS:

1. 120
2. 24
3. 840
4. 720
5. a. 3,024

   b. 18,144

If we were asked how many different arrangements of the letters in the word *STOP* can be made, we would write

$$
\begin{aligned}
{}_4P_4 &= \frac{4!}{(4-4)!} \\
&= \frac{4!}{0!} \\
&= 24
\end{aligned}
$$

We would be correct since all letters are different. If some of the letters were the same, we would reason as given in the following problem.

EXAMPLE: How many different arrangements of the letters in the word *ROOM* can be made?

SOLUTION: We have two letters alike. If we list the possible arrangements, using subscripts to make a distinction between the *O*'s, we have

RO_1O_2M	O_1O_2MR	O_1MO_2R	MO_1O_2R
RO_2O_1M	O_2O_1MR	O_2MO_1R	MO_2O_1R
RO_1MO_2	O_1O_2RM	O_1RMO_2	MO_1RO_2
RO_2MO_1	O_2O_1RM	O_2RMO_1	MO_2RO_1
RMO_1O_2	O_1MRO_2	O_1RO_2M	MRO_1O_2
RMO_2O_1	O_2MRO_1	O_2RO_1M	MRO_2O_1

but we cannot distinguish between the O's; RO_1O_2M and RO_2O_1M would be the same arrangement without the subscript. Only half as many arrangements are possible without the use of subscripts (a total of 12 arrangements). This leads to the statement: *The number of arrangements of n items, where there are k groups of like items of size* $r_1, r_2, \ldots r_k$, *respectively, is given by*

$$\frac{n!}{r_1!r_2! \cdot \cdot \cdot r_k!}$$

In the previous example n was equal to 4 and two letters were alike; therefore, we would write

$$\frac{4!}{2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1}$$

$$= 12$$

EXAMPLE: How many arrangements can be made using the letters in the word *ADAPTATION*?

SOLUTION: We use

$$\frac{n!}{r_1!r_2! \cdot \cdot \cdot r_k!}$$

where

$$n = 10$$

and

$$r_1 = 2 \text{ (two } T\text{'s)}$$

and

$$r_2 = 3 \text{ (three } A\text{'s)}$$

Then

$$\frac{n!}{r_1!r_2! \cdot \cdot \cdot r_k!} = \frac{10!}{2!3!}$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 2}{1}$$

$$= 302{,}400$$

PRACTICE PROBLEMS:

Find the number of possible arrangements of the letters in the following words:

1. *WRITE*

2. *STRUCTURE*

3. *HERE*

4. *MILLIAMPERE*

5. *TENNESSEE*

ANSWERS:

1. 120

2. 45,360

3. 12

4. 2,494,800

5. 3,780

Although the previous discussions have been associated with formulas, problems dealing with combinations and permutations may be analyzed and solved in a more meaningful way without complete reliance upon the formulas.

EXAMPLE: How many four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, and 7

a. without repetition?

b. with repetition?

SOLUTION: The (a) part of the question is a straightforward permutation problem, and we reason that we want the number of permutations of six items taken four at a time. Therefore,

$$
\begin{aligned}
{}_6P_4 &= \frac{6!}{(6-4)!} \\
&= \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} \\
&= 360
\end{aligned}
$$

The (b) part of the question would become quite complicated if we tried to use the formulas; therefore, we reason as follows:

We desire a four-digit number and find we have six choices for the first position; that is, we may use any of the digits 2, 3, 4, 5, 6, or 7 in the first position:

$$(6)(\ )(\ )(\ )$$

Since we are allowed repetition of numbers, then we have six choices for the second position. In other words, any of the digits 2, 3, 4, 5, 6, or 7 can be used in the second position:

$$(6)(6)(\ )(\ )$$

Continuing this reasoning, we also have six choices each for the third and fourth positions:

$$(6)(6)(6)(6)$$

Therefore, the total number of four-digit numbers formed from the digits 2, 3, 4, 5, 6, and 7 with repetition is

$$6\cdot6\cdot6\cdot6 = 1{,}296$$

EXAMPLE: Suppose, in the previous example, we were to find how many three-digit odd numbers could be formed from the given digits without repetition.

SOLUTION: We would be required to start in the units column because an odd number is determined by the units column digit. Therefore, we have only three choices for the units position; that is, either 3, 5, or 7:

$$(3)(\ )(\ )$$

For the ten's position, we have only five choices, since we are not allowed repetition of numbers:

$$(3)(5)(\ )$$

Using the same reasoning of no repetition, we have only four choices for the hundred's position:

$$(3)(5)(4)$$

Therefore, we can form

$$3 \cdot 5 \cdot 4 = 60$$

three-digit odd numbers from the digits 2, 3, 4, 5, 6, and 7 without repetition.

PRACTICE PROBLEMS:

Solve the following problems:

1. Using the digits 4, 5, 6, and 7, how many two-digit numbers can be formed
   a. without repetition?
   b. with repetition?
2. Using the digits 4, 5, 6, 7, 8, and 9, how many five-digit numbers can be formed
   a. without repetition?
   b. with repetition?
3. Using the digits of problem 2, how many four-digit odd numbers can be formed without repetition?

ANSWERS:

1. a. 12
   b. 16
2. a. 720
   b. 7,776
3. 180

SUMMARY

The following are the major topics covered in this chapter:

1. **Definitions:**

   A *combination* is defined as a possible selection of a certain number of objects taken from a group without regard to order.

   A *permutation* is defined as a possible selection of a certain number of objects taken from a group with regard to order.

   The product of the integers n through 1 is defined as *n factorial*, and the symbol $n!$ is used to denote this.

2. **Factorial:** Theorem. *If n and r are positive integers, with n greater than r, then*

$$n! = (n)(n-1)\cdots(r+2)(r+1)r!$$

3. **Combination formula:**

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

   for the number of combinations of n objects taken r at a time.

4. **Principle of Choice:** If a selection can be made in n_1 ways; and after this selection is made, a second selection can be made in n_2 ways; and after this selection is made, a third selection can be made in n_3 ways; and so forth for r selections, then the sequence of r selections can be made together in $n_1 \cdot n_2 \cdot n_3 \cdots n_r$ ways.

5. **Permutation formula:**

$${}_nP_r = \frac{n!}{(n-r)!}$$

   for the number of permutations of n objects taken r at a time.

6. **Arrangements:** The number of arrangements of n items, where there are k groups of like items of size $r_1, r_2, \cdots, r_k$, respectively, is given by

$$\frac{n!}{r_1!r_2! \cdots r_k!}$$

7. **Repetition**: Combinations and permutation problems, with or without repetition, may be solved for using position notation instead of formulas.

ADDITIONAL PRACTICE PROBLEMS

1. Find the value of $\dfrac{6!7! - 6!5!}{5!4!}$.

2. Simplify $\dfrac{(n+2)!(n-2)! - (n+1)!(n-1)!}{(n)!(n-3)!}$.

3. Find the value of $\dfrac{{}_7C_5 + {}_7C_6}{{}_7C_5 - {}_7C_6}$.

4. On each trip, a salesman visits 4 of the 12 cities in his territory. In how many different ways can he schedule his route?

5. From six men and five women, find the number of groups of four that can be formed consisting of two men and two women.

6. Find the value of $\dfrac{{}_7P_6 + {}_7P_5}{{}_7P_6 - {}_7P_5}$.

7. In how many ways can the 18 members of a boy scout troop elect a president, a vice-president, and a secretary, assuming that no member can hold more than one office?

8. How many different ways can 4 red, 3 blue, 4 yellow, and 2 green bulbs be arranged on a string of Christmas tree lights with 13 sockets?

9. How many car tags can be made if the first three positions are letters and the last three positions are numbers (Hint: Twenty-six letters and ten distinct single-digit numbers are possible)

   a. with repetition?

   b. without repetition?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 1,230

2. $3(n + 1)(n - 2)$

3. 2

4. 495

5. 150

6. 3

7. 4,896

8. 900,900

9. a. 17,576,000

   b. 11,232,000

CHAPTER 9

PROBABILITY

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Apply the basic concepts of probability.

2. Solve for probabilities of success and failure.

3. Interpret numerical and mathematical expectation.

4. Apply the concept of compound probabilities to independent, dependent, and mutually exclusive events.

5. Apply the concept of empirical events.

INTRODUCTION

The history of probability theory dates back to the 17th century and at that time was related to games of chance. In the 18th century the probability theory was known to have applications beyond the scope of games of chance. Some of the applications in which probability theory is applied are situations with outcomes such as life or death and boy or girl. Statistics and probability are currently applied to insurance, annuities, biology, and social investigations.

The treatment of probability in this chapter is limited to simple applications. These applications will be, to a large extent, based on games of chance, which lend themselves to an understanding of basic ideas of probability.

BASIC CONCEPTS

If a coin were tossed, the chance it would land heads up is just as likely as the chance it would land tails up; that is, the coin has no more reason to land heads up than it has to land tails up. Every toss of the coin is called a *trial.*

We define *probability* as the ratio of the different number of ways a trial can succeed (or fail) to the total number of ways in which it may result. We will let p represent the *probability of success* and q represent the *probability of failure.*

One commonly misunderstood concept of probability is the effect prior trials have on a single trial. That is, after a coin has been tossed many times and every trial resulted in the coin falling heads up, will the next toss of the coin result in tails up? The answer is "not necessarily" and will be explained later in this chapter.

PROBABILITY OF SUCCESS

If a trial must result in any of n equally likely ways, and if s is the number of successful ways and f is the number of failing ways, the probability of success is

$$p = \frac{s}{s + f}$$

where

$$s + f = n$$

EXAMPLE: What is the probability that a coin will land heads up?

SOLUTION: There is only one way the coin can land heads up; therefore, s equals 1. There is also only one way the coin can land other than heads up; therefore, f equals 1. Since

$$s = 1$$

and

$$f = 1$$

then the probability of success is

$$p = \frac{s}{s + f}$$

$$= \frac{1}{1 + 1}$$

$$= \frac{1}{2}$$

This, then, is the ratio of successful ways in which the trial can succeed to the total number of ways the trial can result.

EXAMPLE: What is the probability that a die (singular of dice) will land with a 3 showing on the upper face?

SOLUTION: A die has a total of 6 sides. Therefore the die can land with a 3 face up 1 favorable way and 5 unfavorable ways.

Since

$$s = 1$$

and

$$f = 5$$

then

$$p = \frac{s}{s + f}$$

$$= \frac{1}{1 + 5}$$

$$= \frac{1}{6}$$

EXAMPLE: What is the probability of drawing a black marble from a box of marbles if all 6 of the marbles in the box are white?

SOLUTION: There are no favorable ways of success and there are 6 total ways. Therefore,

$$s = 0$$

and

$$f = 6$$

so that

$$p = \frac{0}{0 + 6}$$

$$= \frac{0}{6}$$

$$= 0$$

EXAMPLE: What is the probability of drawing a black marble from a box of 6 black marbles?

SOLUTION: There are 6 successful ways and no unsuccessful ways of drawing the marble. Therefore,

$$s = 6$$

and

$$f = 0$$

so that

$$p = \frac{6}{6 + 0}$$

$$= \frac{6}{6}$$

$$= 1$$

The previous two examples are the extremes of probabilities and intuitively demonstrate that the probability of an event ranges from 0 to 1 inclusively.

EXAMPLE: A box contains 6 hard lead pencils and 12 soft lead pencils. What is the probability of drawing a soft lead pencil from the box?

SOLUTION: We are given

$$s = 12$$

and

$$f = 6$$

therefore,

$$p = \frac{12}{12 + 6}$$

$$= \frac{12}{18}$$

$$= \frac{2}{3}$$

PRACTICE PROBLEMS:

1. What is the probability of drawing an ace from a standard deck of 52 playing cards?

2. What is the probability of drawing a black ace from a standard deck of playing cards?

3. If a die is rolled, what is the probability of an odd number showing on the upper face?

4. A man has 3 nickels, 2 dimes, and 4 quarters in his pocket. If he draws a single coin from his pocket, what is the probability that

   a. he will draw a nickel?

   b. he will draw a half-dollar?

   c. he will draw a quarter?

ANSWERS:

1. $\frac{1}{13}$

2. $\frac{1}{26}$

3. $\frac{1}{2}$

4. a. $\frac{1}{3}$

   b. 0

   c. $\frac{4}{9}$

PROBABILITY OF FAILURE

As before, if a trial results in any of n equally likely ways, and s is the number of successful ways and f is the number of failures, the probability of failure is

$$q = \frac{f}{s + f}$$

or

$$= \frac{n - s}{n}$$

where

$$s + f = n$$

or

$$n - s = f$$

A trial must result in either success or failure. If success is certain then p equals 1 and q equals 0. If success is impossible then p equals 0 and q equals 1. Combining both events, for either case, makes the probability of success plus the probability of failure equal to 1. If

$$p = \frac{s}{s + f}$$

and

$$q = \frac{f}{s + f}$$

then

$$p + q = \frac{s}{s + f} + \frac{f}{s + f}$$

$$= 1$$

If, in any event

$$p + q = 1$$

then

$$q = 1 - p$$

In the case of tossing a coin, the probability of success is

$$p = \frac{s}{s+f}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

and the probability of failure is

$$q = 1 - p$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2}$$

EXAMPLE: What is the probability of not drawing a black marble from a box containing 6 white, 3 red, and 2 black marbles?

SOLUTION: The probability of drawing a black marble from the box is

$$p = \frac{s}{s+f}$$

$$= \frac{2}{2+9}$$

$$= \frac{2}{11}$$

Since the probability of drawing a marble is 1, then the probability of not drawing a black marble is

$$q = 1 - p$$

$$= 1 - \frac{2}{11}$$

$$= \frac{9}{11}$$

PRACTICE PROBLEMS:

Compare the following problems and answers with the preceding problems dealing with the probability of success:

1. What is the probability of not drawing an ace from a standard deck of 52 playing cards?
2. What is the probability of not drawing a black ace from a standard deck of playing cards?
3. If a die is rolled, what is the probability of an odd number not showing on the upper face?
4. A man has 3 nickels, 2 dimes, and 4 quarters in his pocket. If he draws a single coin from his pocket, what is the probability that
   a. he will not draw a nickel?
   b. he will not draw a half-dollar?
   c. he will not draw a quarter?

ANSWERS:

1. $\frac{12}{13}$
2. $\frac{25}{26}$
3. $\frac{1}{2}$
4. a. $\frac{2}{3}$
   b. 1
   c. $\frac{5}{9}$

EXPECTATION

Expectation is the average of the values you would get in conducting an experiment or trial exactly the same way many times. In this discussion of expectation, we will consider two types. One is a numerical expectation and the other is a mathematical expectation.

Numerical Expectation

If you tossed a coin 50 times, you would expect the coin to fall heads (on the average) about 25 times. Your assumption is explained by the following definition of *numerical expectation: If the probability of success in one trial is p, and k is the total number of trials, then kp is the expected number of successes in the k trials.*

In the above example of tossing the coin 50 times, the probability of heads (successes) is

$$E_n = kp$$

where

$$E_n = \text{expected number}$$

$$k = \text{number of tosses}$$

$$p = \text{probability of heads (successes)}$$

Substituting values in the equation, we find that

$$E_n = 50\left(\frac{1}{2}\right)$$

$$= 25$$

EXAMPLE: A die is rolled by a player. What is the expectation of rolling a 6 in 30 trials?

SOLUTION: The probability of rolling a 6 in 1 trial is

$$p = \frac{1}{6}$$

and the number of rolls is

$$k = 30$$

therefore,

$$E_n = kp$$

$$= 30\left(\frac{1}{6}\right)$$

$$= 5$$

In words, the player would expect (on the average) to roll a 6 five times in 30 rolls.

Mathematical Expectation

We will define *mathematical expectation* as follows: *If, in the event of a successful result, amount a, is to be received and the probability of success of that event is p, then ap is the mathematical expectation.*

If you were to buy 1 of 500 raffle tickets for a video recorder worth \$325.00, what would be your mathematical expectation?

In this case, the product of the amount you stand to win and the probability of winning is

$$E_m = ap$$

where

$$a = \text{amount you stand to win}$$

$$p = \text{probability of success}$$

and

$$E_m = \text{expected amount}$$

Then, by substitution

$$\begin{aligned} E_m &= ap \\ &= \$325.00\left(\frac{1}{500}\right) \\ &= \$0.65 \end{aligned}$$

Thus, you would not want to pay more than 65 cents for the ticket, unless, of course the raffle were for a worthy cause.

EXAMPLE: To entice the public to invest in their development, Sunshine Condominiums has offered a prize of \$2,000 to 1 randomly selected family out of the first 1,000 families that participate in the condominium's tour.

1. What would be each family's mathematical expectation?

2. Would it be worthwhile for the Jones family to spend \$3.00 in gasoline to drive to Sunshine Condominiums to take the tour?

SOLUTION:

1. $E_m = ap = \$2{,}000\left(\frac{1}{1{,}000}\right) = \$2.00$

2. No; since \$3.00 is \$1.00 over their expectation of \$2.00, it would not be worthwhile for the Jones family to take the tour.

PRACTICE PROBLEMS:

1. A box contains 7 slips of paper, each numbered differently. A girl makes a total of 50 draws, returning the drawn slip after each draw.

   a. What is the probability of drawing a selected numbered slip in 1 drawing?

   b. How many times would the girl expect to draw the single selected numbered slip in the 50 draws?

2. In a winner-take-all tournament among four professional tennis players, the prize money is \$500,000. Joe Conners, one of the tennis players, figures his probability of winning is 0.20.

   a. What is his mathematical expectation?

   b. Would he be better off if he made a secret agreement with the other tennis players to divide the prize money evenly regardless of who wins?

ANSWERS:

1. a. $\frac{1}{7}$

   b. $7\frac{1}{7}$

2. a. \$100,000

   b. Yes; he would be better off, since he would make \$125,000, which is greater than his expectation of \$100,000.

COMPOUND PROBABILITIES

The probabilities to this point have been single events. In the discussion on compound probabilities, events that may affect others will be covered. The word *may* is used because independent events are included with dependent events and mutually exclusive events.

INDEPENDENT EVENTS

Two or more events are *independent* if the occurrence or nonoccurrence of one of the events has no affect on the probability of occurrence of any of the others.

When two coins are tossed at the same time or one after the other, whether one falls heads or tails has no affect on the way the second coin falls. Suppose we call the coins A and B. The coins may fall in the following four ways:

1. A and B may fall heads.
2. A and B may fall tails.
3. A may fall heads and B may fall tails.
4. A may fall tails and B may fall heads.

The probability of any one way for the coins to fall is calculated as follows:

$$s = 1$$

and

$$n = 4$$

therefore,

$$p = \frac{1}{4}$$

This probability may be determined by considering the product of the separate probabilities; that is,

the probability that A will fall heads is $\frac{1}{2}$

the probability that B will fall heads is $\frac{1}{2}$

and the probability that both will fall heads is

$$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

In other words, when two events are independent, the probability that one and then the other will occur is the product of their separate probabilities.

EXAMPLE: A box contains 3 red marbles and 7 green marbles. If a marble is drawn, then replaced, and another marble is drawn, what is the probability that both marbles are red?

SOLUTION: Two solutions are offered. First, by the principle of choice, 2 marbles can be selected in $10 \cdot 10$ ways. The red marble may be selected on the first draw in three ways and on the second draw in three ways; and by the principle of choice, a red marble may be drawn on both trials in $3 \cdot 3$ ways. Then the required probability is

$$p = \frac{9}{100}$$

The second solution, using the product of independent events, follows: The probability of drawing a red marble on the first draw is $\frac{3}{10}$, and the probability of drawing a red marble on the second draw is $\frac{3}{10}$. Therefore, the probability of drawing a red marble on both draws is the product of the separate probabilities

or

$$p = \frac{3}{10} \cdot \frac{3}{10} = \frac{9}{100}$$

PRACTICE PROBLEMS:

1. If a die is tossed twice, what is the probability of rolling a 2 followed by a 3?

2. A box contains 2 white, 3 red, and 4 blue marbles. If after each selection the marble is replaced, what is the probability of drawing, in order

   a. a white then a blue marble?

   b. a blue then a red marble?

   c. a white, a red, then a blue marble?

ANSWERS:

1. $\frac{1}{36}$

2. a. $\frac{8}{81}$

   b. $\frac{4}{27}$

   c. $\frac{8}{243}$

DEPENDENT EVENTS

In some cases one event is dependent on another; that is, two or more events are said to be *dependent* if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others.

Consider that two or more events are dependent. If p_1 is the probability of a first event; p_2 the probability that after the first happens, the second will occur; p_3 the probability that after the first and second have happened, the third will occur; etc., then the probability that all events will happen in the given order is the product $p_1 \cdot p_2 \cdot p_3 \cdots$.

EXAMPLE: A box contains 3 white marbles and 4 black marbles. What is the probability of drawing 2 black marbles and 1 white marble in succession without replacement?

SOLUTION: On the first draw the probability of drawing a black marble is

$$p_1 = \frac{4}{7}$$

On the second draw the probability of drawing a black marble is

$$p_2 = \frac{3}{6}$$

$$= \frac{1}{2}$$

On the third draw the probability of drawing a white marble is

$$p_3 = \frac{3}{5}$$

Therefore, the probability of drawing 2 black marbles and 1 white marble is

$$\begin{aligned} p &= p_1 \cdot p_2 \cdot p_3 \\ &= \frac{4}{7} \cdot \frac{1}{2} \cdot \frac{3}{5} \\ &= \frac{6}{35} \end{aligned}$$

EXAMPLE: Slips numbered 1 through 9 are placed in a box. If 2 slips are drawn, without replacement, what is the probability that

1. both are odd?

2. both are even?

SOLUTION:

1. The probability that the first is odd is

$$p_1 = \frac{5}{9}$$

and the probability that the second is odd is

$$p_2 = \frac{4}{8}$$

Therefore, the probability that both are odd is

$$\begin{aligned} p &= p_1 \cdot p_2 \\ &= \frac{5}{9} \cdot \frac{4}{8} \\ &= \frac{5}{18} \end{aligned}$$

2. The probability that the first is even is

$$p_1 = \frac{4}{9}$$

and the probability that the second is even is

$$p_2 = \frac{3}{8}$$

Therefore, the probability that both are even is

$$\begin{aligned} p &= p_1 \cdot p_2 \\ &= \frac{4}{9} \cdot \frac{3}{8} \\ &= \frac{1}{6} \end{aligned}$$

A second method of solution involves the use of combinations.

1. A total of 9 slips are taken 2 at a time and 5 odd slips are taken 2 at a time; therefore,

$$\begin{aligned} p &= \frac{{}_5C_2}{{}_9C_2} \\ &= \frac{5}{18} \end{aligned}$$

2. A total of ${}_9C_2$ choices and 4 even slips are taken 2 at a time; therefore,

$$\begin{aligned} p &= \frac{{}_4C_2}{{}_9C_2} \\ &= \frac{1}{6} \end{aligned}$$

PRACTICE PROBLEMS:

In the following problems assume that no replacement is made after each selection:

1. A box contains 5 white and 6 red marbles. What is the probability of successfully drawing, in order, a red marble and then a white marble?

2. A bag contains 3 red, 2 white, and 6 blue marbles. What is the probability of drawing, in order, 2 red, 1 blue, and 2 white marbles?

3. Fifteen airmen are in the line crew. They must take care of the coffee mess and line shack cleanup. They put slips numbered 1 through 15 in a hat and decide that anyone who draws a number divisible by 5 will be assigned the coffee mess and anyone who draws a number divisible by 4 will be assigned cleanup. The first person draws a 4, the second a 3, and the third an 11. What is the probability that the fourth person to draw will be assigned

   a. the coffee mess?

   b. the cleanup?

ANSWERS:

1. $\frac{3}{11}$

2. $\frac{1}{770}$

3. a. $\frac{1}{4}$

   b. $\frac{1}{6}$

MUTUALLY EXCLUSIVE EVENTS

Two or more events are called *mutually exclusive* if the occurrence of any one of them precludes the occurrence of any of the others. The probability of occurrence of two or more mutually exclusive events is the sum of the probabilities of the individual events.

Sometimes when one event has occurred, the probability of another event is excluded (referring to the same given occasion or trial).

For example, throwing a die once can yield a 5 or 6, but not both, in the same toss. The probability that either a 5 or 6 occurs is the sum of their individual probabilities.

$$p = p_1 + p_2$$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$

EXAMPLE: From a bag containing 5 white balls, 2 black balls, and 11 red balls, 1 ball is drawn. What is the probability that it is either black or red?

SOLUTION: The draw can be made in 18 ways. The choices are 2 black balls and 11 red balls, which are favorable, or a total of 13 favorable choices. Then, the probability of success is

$$p = \frac{13}{18}$$

Since drawing a red ball excludes the drawing of a black ball, and vice versa, the two events are mutually exclusive; so the probability of drawing a black ball is

$$p_1 = \frac{2}{18}$$

and the probability of drawing a red ball is

$$p_2 = \frac{11}{18}$$

Therefore, the probability of success is

$$p = p_1 + p_2$$

$$= \frac{2}{18} + \frac{11}{18} = \frac{13}{18}$$

EXAMPLE: What is the probability of drawing either a king, a queen, or a jack from a deck of playing cards?

SOLUTION: The individual probabilities are

$$\text{king} = \frac{4}{52}$$

$$\text{queen} = \frac{4}{52}$$

$$\text{jack} = \frac{4}{52}$$

Therefore, the probability of success is

$$\begin{aligned} p &= \frac{4}{52} + \frac{4}{52} + \frac{4}{52} \\ &= \frac{12}{52} \\ &= \frac{3}{13} \end{aligned}$$

EXAMPLE: What is the probability of rolling a die twice and having a 5 and then a 3 show or having a 2 and then a 4 show?

SOLUTION: The probability of having a 5 and then a 3 show is

$$\begin{aligned} p_1 &= \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

and the probability of having a 2 and then a 4 show is

$$\begin{aligned} p_2 &= \frac{1}{6} \cdot \frac{1}{6} \\ &= \frac{1}{36} \end{aligned}$$

Then, the probability of either p_1 or p_2 is

$$\begin{aligned} p &= p_1 + p_2 \\ &= \frac{1}{36} + \frac{1}{36} \\ &= \frac{1}{18} \end{aligned}$$

PRACTICE PROBLEMS:

1. When tossing a coin, you have what probability of getting either a head or a tail?

2. A bag contains 12 blue, 3 red, and 4 white marbles. What is the probability of drawing

   a. in 1 draw, either a red or a white marble?

   b. in 1 draw, either a red, white, or blue marble?

   c. in 2 draws, either a red marble followed by a blue marble or a red marble followed by a red marble?

3. What is the probability of getting a total of at least 10 points in rolling two dice? (HINT: You want either a total of 10, 11, or 12.)

ANSWERS:

1. 1

2. a. $\frac{7}{19}$

   b. 1

   c. $\frac{7}{57}$

3. $\frac{1}{6}$

EMPIRICAL PROBABILITIES

Among the most important applications of probability are those situations where we cannot list all possible outcomes. To this point, we have considered problems in which the probabilities could be obtained from situations of equally likely results.

Because some problems are so complicated for analysis, we can only estimate probabilities from experience and observation. This is *empirical probability.*

In modern industry probability now plays an important role in many activities. Quality control and reliability of a manufactured article have become extremely important considerations in which probability is used.

Experience has shown that empirical probabilities, if carefully determined on the basis of adequate statistical samples, can be applied to large groups with the result that probability and relative frequency are approximately equal. By adequate samples we mean a large enough sample so that accidental runs of "luck," both good and bad, cancel each other. With enough trials, predicted results and actual results agree quite closely. On the other hand, applying a probability ratio to a single individual event is virtually meaningless.

Table 9-1.—Weather Forecast

Date	Forecast	Actual weather	Did the actual forecasted event occur?
1	Rain	Rain	Yes
2	Light showers	Sunny	No
3	Cloudy	Cloudy	Yes
4	Clear	Clear	Yes
5	Scattered showers	Warm and sunny	No
6	Scattered showers	Scattered showers	Yes
7	Windy and cloudy	Windy and cloudy	Yes
8	Thundershowers	Thundershowers	Yes
9	Clear	Cloudy and rain	No
10	Clear	Clear	Yes

We define *relative frequency of success* as follows: *After N trials of an event have been made, of which S trials are successes, the relative frequency of success is*

$$P = \frac{S}{N}$$

For example, table 9-1 shows a small number of weather forecasts from April 1st to April 10th. The actual weather on the dates is also given.

Observe that the forecasts on April 1, 3, 4, 6, 7, 8, and 10 were correct. We have observed 10 outcomes. The event of a correct forecast has occurred 7 times. Based on this information we might say that the probability for future forecasts being true is 7/10. This number is the best estimate we can make from the given information. In this case, since we have observed such a small number of outcomes, we would be incorrect to say that the estimate of P is dependable. A great many more cases should be used if we expect to make a good estimate of the probability that a weather forecast will be accurate. A great many factors affect the accuracy of a weather forecast. This example merely indicates something about how successful a particular weather office has been in making weather forecasts.

Another example may be drawn from industry. Many thousands of articles of a certain type are manufactured. The company selects 100 of these articles at random and subjects them to very careful tests. In these tests 98 of the articles are found to meet all measurement requirements and perform satisfactorily. This suggests that 98/100 is a measure of the reliability of the article.

One might expect that about 98% of all of the articles manufactured by this process will be satisfactory. The probability (measure of chance) that one of these articles will be satisfactory might be said to be 0.98.

This second example of empirical probability is different from the first example in one very important respect. In the first example we could list all of the possibilities, and in the second example we could not do so. The selection of a sample and its size is a problem of statistics.

Considered from another point of view, statistical probability can be regarded as relative frequency.

EXAMPLE: In a dart game, a player hit the bull's eye 3 times out of 25 trials. What is the statistical probability that he will hit the bull's eye on the next throw?

SOLUTION:

$$N = 25$$

and

$$S = 3$$

hence

$$P = \frac{3}{25}$$

EXAMPLE: Using table 9-2, what is the probability that a person 20 years old will live to be 50 years old?

SOLUTION: Of 95,148 persons at age 20, 81,090 survived to age 50. Hence

$$P = \frac{81{,}090}{95{,}148}$$

$$= 0.852 \text{ (rounded)}$$

EXAMPLE: How many times would a die be expected to land with a 5 or 6 showing in 20 trials?

Table 9-2.—Mortality Table (Based on 100,000 Individuals 1 Year of Age)

Age	Number of people
5	98,382
10	97,180
15	96,227
20	95,148
25	93,920
30	92,461
35	90,655
40	88,334
45	85,255
50	81,090
55	75,419
60	67,777
65	57,778
70	45,455
75	31,598
80	18,177
85	7,822
90	2,158

SOLUTION: The probability of a 5 or 6 showing is

$$p = \frac{1}{3}$$

The relative frequency is approximately equal to the probability

$$P \approx p$$

Therefore, since

$$P = \frac{S}{N}$$

where

$$P = \frac{1}{3}$$

$$N = 20$$

$$S = ?$$

then rearranging and substituting, we find that

$$\begin{aligned} S &= NP \\ &= 20\left(\frac{1}{3}\right) \\ &= \frac{20}{3} \\ &= 6.67 \text{ (rounded)} \end{aligned}$$

This says that the expected number of times a die would land with a 5 or 6 showing in 20 trials is 6.67; that is, on the average a die will land with a 5 or 6 showing 6.67 times per 20 trials.

PRACTICE PROBLEMS:

1. A construction crew consists of 6 electricians and 38 other workers. How many electricians would you expect to choose if you choose 1 person each day of a workweek for your

helper? (Sunday will not be considered part of the workweek.)

2. How many times would a tossed die be expected to turn up a 3 or less in 30 tosses?

3. Using table 9-2, find the probability that a person whose age is 30 will live to age 60.

ANSWERS:

1. 0.82

2. 15

3. 0.733

SUMMARY

The following are the major topics covered in this chapter:

1. **Probability:** *Probability* is the ratio of the different number of ways a trial can succeed (or fail) to the total number of ways in which it may result.

2. **Probabilities of success and failure:** If a trial must result in any of n equally likely ways, and if s is the number of successful ways and f is the number of failing ways, the *probability of success* is

$$p = \frac{s}{s+f}$$

   and the *probability of failure* is

$$q = \frac{f}{s+f} \text{ or } q = \frac{n-s}{n}$$

   where $s + f = n$ or $n - s = f$

3. **Expectation:** *Expectation* is the average of the values you would get in conducting an experiment or trial exactly the same way many times.

4. **Numerical expectation:** If the probability of success in one trial is p, and k is the total number of trials, then kp is the expected number of successes in the k trials or

$$E_n = kp$$

5. **Mathematical expectation:** If, in the event of a successful result, amount a is to be received, and p is the probability of success of that event, then ap is the mathematical expectation or

$$E_m = ap$$

6. **Independent events:** Two or more events are *independent* if the occurrence or nonoccurrence of one of the events has no effect on the probability of occurrence of any of the others.

7. **Dependent events:** Two or more events are *dependent* if the occurrence or nonoccurrence of one of the events affects the probabilities of occurrence of any of the others.

8. **Mutually exclusive events:** Two or more events are called *mutually exclusive* if the occurrence of any one of them precludes the occurrence of any of the others.

9. **Empirical probability:** *Empirical probability* is an estimated probability from experience and observation.

10. **Relative frequency of success:** After N trials of an event have been made, of which S trials are successes, the relative frequency of success is

$$P = \frac{S}{N}$$

ADDITIONAL PRACTICE PROBLEMS

1. A box contains 5 red marbles, 6 blue marbles, and 7 green marbles. If 1 marble is to be drawn, what is the probability that it is

   a. green?

   b. red?

   c. yellow?

2. A box contains 5 red marbles, 6 blue marbles, and 7 green marbles. If 1 marble is to be drawn, what is the probability that it is not

   a. green?

   b. red?

   c. yellow?

3. A child is to pick a letter of the alphabet from a box.

   a. What is the child's probability of picking a vowel in 1 draw? (*Y* will not be considered as a vowel.)

   b. What is the child's numerical expectation of picking a vowel in 20 draws? (The letter will be replaced after each draw.)

4. A concert promoter agrees to pay a band $5,600 in case the concert has to be cancelled because of rain. The promoter's actuary figures expected loss for this risk to be $717. What probability is assigned to the possibility that the concert will have to be cancelled because of rain?

5. A basket contains 3 apples, 5 pears, and 7 oranges. If after each selection the fruit is replaced, what is the probability of drawing, in order,

   a. an orange, then a pear?

   b. 2 apples?

   c. an apple, an orange, then a pear?

6. A car has 8 spark plugs, of which 3 are defective. Find the probability of locating all 3 defective spark plugs in 3 selections, without replacement.

7. In a convention of 120 politicians, 52 are Democrats and 33 are Republicans. Find the probability that a politician selected is a Democrat or a Republican.

8. Routine medical examinations are given to 44 smokers and 62 nonsmokers. If one of the subjects is selected for more detailed tests, what is the probability that the selected subject smokes?

9. In a classroom of 33 girls and 22 boys, how many girls would you expect to choose in 12 trials?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. a. 7/18

   b. 5/18

   c. 0

2. a. 11/18

   b. 13/18

   c. 1

3. a. 5/26

   b. 3.85

4. 0.128

5. a. 7/45

   b. 1/25

   c. 7/225

6. 1/56

7. 17/24

8. 22/53

9. 7.2

APPENDIX 1

NATURAL TANGENTS AND COTANGENTS

MIN	0° TAN	0° COT	1° TAN	1° COT	2° TAN	2° COT	3° TAN	3° COT	4° TAN	4° COT	
0	0000000	0000000	0.01746	57.2900	0.03492	28.6363	0.05241	19.0811	0.06993	14.3007	60
1	0.00029	3437.75	0.01775	56.3506	0.03521	28.3994	0.05270	18.9755	0.07022	14.2411	59
2	0.00058	1718.87	0.01804	55.4415	0.03550	28.1664	0.05299	18.8711	0.07051	14.1821	58
3	0.00087	1145.92	0.01833	54.5613	0.03579	27.9372	0.05328	18.7678	0.07080	14.1235	57
4	0.00116	859.436	0.01862	53.7086	0.03609	27.7117	0.05357	18.6656	0.07110	14.0655	56
5	0.00145	687.549	0.01891	52.8821	0.03638	27.4899	0.05387	18.5645	0.07139	14.0079	55
6	0.00175	572.957	0.01920	52.0807	0.03667	27.2715	0.05416	18.4645	0.07168	13.9507	54
7	0.00204	491.106	0.01949	51.3032	0.03696	27.0566	0.05445	18.3655	0.07197	13.8940	53
8	0.00233	429.718	0.01978	50.5485	0.03725	26.8450	0.05474	18.2677	0.07227	13.8378	52
9	0.00262	381.971	0.02007	49.8157	0.03754	26.6367	0.05503	18.1708	0.07256	13.7821	51
10	0.00291	343.774	0.02036	49.1039	0.03783	26.4316	0.05533	18.0750	0.07285	13.7267	50
11	0.00320	312.521	0.02066	48.4121	0.03812	26.2296	0.05562	17.9802	0.07314	13.6719	49
12	0.00349	286.478	0.02095	47.7395	0.03842	26.0307	0.05591	17.8863	0.07344	13.6174	48
13	0.00378	264.441	0.02124	47.0853	0.03871	25.8348	0.05620	17.7934	0.07373	13.5634	47
14	0.00407	245.552	0.02153	46.4489	0.03900	25.6418	0.05649	17.7015	0.07402	13.5098	46
15	0.00436	229.182	0.02182	45.8294	0.03929	25.4517	0.05678	17.6106	0.07431	13.4566	45
16	0.00465	214.858	0.02211	45.2261	0.03958	25.2644	0.05708	17.5205	0.07461	13.4039	44
17	0.00495	202.219	0.02240	44.6386	0.03987	25.0798	0.05737	17.4314	0.07490	13.3515	43
18	0.00524	190.984	0.02269	44.0661	0.04016	24.8978	0.05766	17.3432	0.07519	13.2996	42
19	0.00553	180.932	0.02298	43.5081	0.04046	24.7185	0.05795	17.2558	0.07548	13.2480	41
20	0.00582	171.885	0.02328	42.9641	0.04075	24.5418	0.05824	17.1693	0.07578	13.1969	40
21	0.00611	163.700	0.02357	42.4335	0.04104	24.3675	0.05854	17.0837	0.07607	13.1461	39
22	0.00640	156.259	0.02386	41.9158	0.04133	24.1957	0.05883	16.9990	0.07636	13.0958	38
23	0.00669	149.465	0.02415	41.4106	0.04162	24.0263	0.05912	16.9150	0.07665	13.0458	37
24	0.00698	143.237	0.02444	40.9174	0.04191	23.8593	0.05941	16.8319	0.07695	12.9962	36
25	0.00727	137.507	0.02473	40.4358	0.04220	23.6945	0.05970	16.7496	0.07724	12.9469	35
26	0.00756	132.219	0.02502	39.9655	0.04250	23.5321	0.05999	16.6681	0.07753	12.8981	34
27	0.00785	127.321	0.02531	39.5059	0.04279	23.3718	0.06029	16.5874	0.07782	12.8496	33
28	0.00815	122.774	0.02560	39.0568	0.04308	23.2137	0.06058	16.5075	0.07812	12.8014	32
29	0.00844	118.540	0.02589	38.6177	0.04337	23.0577	0.06087	16.4283	0.07841	12.7536	31
30	0.00873	114.589	0.02619	38.1885	0.04366	22.9038	0.06116	16.3499	0.07870	12.7062	30
31	0.00902	110.892	0.02648	37.7686	0.04395	22.7519	0.06145	16.2722	0.07899	12.6591	29
32	0.00931	107.426	0.02677	37.3579	0.04424	22.6020	0.06175	16.1952	0.07929	12.6124	28
33	0.00960	104.171	0.02706	36.9560	0.04454	22.4541	0.06204	16.1190	0.07958	12.5660	27
34	0.00989	101.107	0.02735	36.5627	0.04483	22.3081	0.06233	16.0435	0.07987	12.5199	26
35	0.01018	98.2179	0.02764	36.1776	0.04512	22.1640	0.06262	15.9687	0.08017	12.4742	25
36	0.01047	95.4895	0.02793	35.8006	0.04541	22.0217	0.06291	15.8945	0.08046	12.4288	24
37	0.01076	92.9085	0.02822	35.4313	0.04570	21.8813	0.06321	15.8211	0.08075	12.3838	23
38	0.01105	90.4633	0.02851	35.0695	0.04599	21.7426	0.06350	15.7483	0.08104	12.3390	22
39	0.01135	88.1436	0.02881	34.7151	0.04628	21.6056	0.06379	15.6762	0.08134	12.2946	21
40	0.01164	85.9398	0.02910	34.3678	0.04658	21.4704	0.06408	15.6048	0.08163	12.2505	20
41	0.01193	83.8435	0.02939	34.0273	0.04687	21.3369	0.06438	15.5340	0.08192	12.2067	19
42	0.01222	81.8470	0.02968	33.6935	0.04716	21.2049	0.06467	15.4638	0.08221	12.1632	18
43	0.01251	79.9434	0.02997	33.3662	0.04745	21.0747	0.06496	15.3943	0.08251	12.1201	17
44	0.01280	78.1263	0.03026	33.0452	0.04774	20.9460	0.06525	15.3254	0.08280	12.0772	16
45	0.01309	76.3900	0.03055	32.7303	0.04803	20.8188	0.06554	15.2571	0.08309	12.0346	15
46	0.01338	74.7292	0.03084	32.4213	0.04833	20.6932	0.06584	15.1893	0.08339	11.9923	14
47	0.01367	73.1390	0.03114	32.1181	0.04862	20.5691	0.06613	15.1222	0.08368	11.9504	13
48	0.01396	71.6151	0.03143	31.8205	0.04891	20.4465	0.06642	15.0557	0.08397	11.9087	12
49	0.01425	70.1533	0.03172	31.5284	0.04920	20.3253	0.06671	14.9898	0.08427	11.8673	11
50	0.01455	68.7501	0.03201	31.2416	0.04949	20.2056	0.06700	14.9244	0.08456	11.8262	10
51	0.01484	67.4019	0.03230	30.9599	0.04978	20.0872	0.06730	14.8596	0.08485	11.7853	9
52	0.01513	66.1055	0.03259	30.6833	0.05007	19.9702	0.06759	14.7954	0.08514	11.7448	8
53	0.01542	64.8580	0.03288	30.4116	0.05037	19.8546	0.06788	14.7317	0.08544	11.7045	7
54	0.01571	63.6567	0.03317	30.1446	0.05066	19.7403	0.06817	14.6685	0.08573	11.6645	6
55	0.01600	62.4992	0.03346	29.8823	0.05095	19.6273	0.06847	14.6059	0.08602	11.6248	5
56	0.01629	61.3829	0.03376	29.6245	0.05124	19.5156	0.06876	14.5438	0.08632	11.5853	4
57	0.01658	60.3058	0.03405	29.3711	0.05153	19.4051	0.06905	14.4823	0.08661	11.5461	3
58	0.01687	59.2659	0.03434	29.1220	0.05182	19.2959	0.06934	14.4212	0.08690	11.5072	2
59	0.01716	58.2612	0.03463	28.8771	0.05212	19.1879	0.06963	14.3607	0.08720	11.4685	1
60	0.01746	57.2900	0.03492	28.6363	0.05241	19.0811	0.06993	14.3007	0.08749	11.4301	0
	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	MIN
	89°		88°		87°		86°		85°		

MIN	5°		6°		7°		8°		9°		
	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	
0	0.08749	11.4301	0.10510	9.51436	0.12278	8.14435	0.14054	7.11537	0.15838	6.31375	60
1	0.08778	11.3919	0.10540	9.48781	0.12308	8.12481	0.14084	7.10038	0.15868	6.30189	59
2	0.08807	11.3540	0.10569	9.46141	0.12338	8.10536	0.14113	7.08546	0.15898	6.29007	58
3	0.08837	11.3163	0.10599	9.43515	0.12367	8.08600	0.14143	7.07059	0.15928	6.27829	57
4	0.08866	11.2789	0.10628	9.40904	0.12397	8.06674	0.14173	7.05579	0.15958	6.26655	56
5	0.08895	11.2417	0.10657	9.38307	0.12426	8.04756	0.14202	7.04105	0.15988	6.25486	55
6	0.08925	11.2048	0.10687	9.35724	0.12456	8.02848	0.14232	7.02637	0.16017	6.24321	54
7	0.08954	11.1681	0.10716	9.33155	0.12485	8.00948	0.14262	7.01174	0.16047	6.23160	53
8	0.08983	11.1316	0.10746	9.30599	0.12515	7.99058	0.14291	6.99718	0.16077	6.22003	52
9	0.09013	11.0954	0.10775	9.28058	0.12544	7.97176	0.14321	6.98268	0.16107	6.20851	51
10	0.09042	11.0594	0.10805	9.25530	0.12574	7.95302	0.14351	6.96823	0.16137	6.19703	50
11	0.09071	11.0237	0.10834	9.23016	0.12603	7.93438	0.14381	6.95385	0.16167	6.18559	49
12	0.09101	10.9882	0.10863	9.20516	0.12633	7.91582	0.14410	6.93952	0.16196	6.17419	48
13	0.09130	10.9529	0.10893	9.18028	0.12662	7.89734	0.14440	6.92525	0.16226	6.16283	47
14	0.09159	10.9178	0.10922	9.15554	0.12692	7.87895	0.14470	6.91104	0.16256	6.15151	46
15	0.09189	10.8829	0.10952	9.13093	0.12722	7.86064	0.14499	6.89688	0.16286	6.14023	45
16	0.09218	10.8483	0.10981	9.10646	0.12751	7.84242	0.14529	6.88278	0.16316	6.12899	44
17	0.09247	10.8139	0.11011	9.08211	0.12781	7.82428	0.14559	6.86874	0.16346	6.11779	43
18	0.09277	10.7797	0.11040	9.05789	0.12810	7.80622	0.14588	6.85475	0.16376	6.10664	42
19	0.09306	10.7457	0.11070	9.03379	0.12840	7.78825	0.14618	6.84082	0.16405	6.09552	41
20	0.09335	10.7119	0.11099	9.00983	0.12869	7.77035	0.14648	6.82694	0.16435	6.08444	40
21	0.09365	10.6783	0.11128	8.98598	0.12899	7.75254	0.14678	6.81312	0.16465	6.07340	39
22	0.09394	10.6450	0.11158	8.96227	0.12929	7.73480	0.14707	6.79936	0.16495	6.06240	38
23	0.09423	10.6118	0.11187	8.93867	0.12958	7.71715	0.14737	6.78564	0.16525	6.05143	37
24	0.09453	10.5789	0.11217	8.91520	0.12988	7.69957	0.14767	6.77199	0.16555	6.04051	36
25	0.09482	10.5462	0.11246	8.89185	0.13017	7.68208	0.14796	6.75838	0.16585	6.02962	35
26	0.09511	10.5136	0.11276	8.86862	0.13047	7.66466	0.14826	6.74483	0.16615	6.01878	34
27	0.09541	10.4813	0.11305	8.84551	0.13076	7.64732	0.14856	6.73133	0.16645	6.00797	33
28	0.09570	10.4491	0.11335	8.82252	0.13106	7.63005	0.14886	6.71789	0.16674	5.99720	32
29	0.09600	10.4172	0.11364	8.79964	0.13136	7.61287	0.14915	6.70450	0.16704	5.98646	31
30	0.09629	10.3854	0.11394	8.77689	0.13165	7.59575	0.14945	6.69116	0.16734	5.97576	30
31	0.09658	10.3538	0.11423	8.75425	0.13195	7.57872	0.14975	6.67787	0.16764	5.96510	29
32	0.09688	10.3224	0.11452	8.73172	0.13224	7.56176	0.15005	6.66463	0.16794	5.95448	28
33	0.09717	10.2913	0.11482	8.70931	0.13254	7.54487	0.15034	6.65144	0.16824	5.94390	27
34	0.09746	10.2602	0.11511	8.68701	0.13284	7.52806	0.15064	6.63831	0.16854	5.93335	26
35	0.09776	10.2294	0.11541	8.66482	0.13313	7.51132	0.15094	6.62523	0.16884	5.92283	25
36	0.09805	10.1988	0.11570	8.64275	0.13343	7.49465	0.15124	6.61219	0.16914	5.91236	24
37	0.09834	10.1683	0.11600	8.62078	0.13372	7.47806	0.15153	6.59921	0.16944	5.90191	23
38	0.09864	10.1381	0.11629	8.59893	0.13402	7.46154	0.15183	6.58627	0.16974	5.89151	22
39	0.09893	10.1080	0.11659	8.57718	0.13432	7.44509	0.15213	6.57339	0.17004	5.88114	21
40	0.09923	10.0780	0.11688	8.55555	0.13461	7.42871	0.15243	6.56055	0.17033	5.87080	20
41	0.09952	10.0483	0.11718	8.53402	0.13491	7.41240	0.15272	6.54777	0.17063	5.86051	19
42	0.09981	10.0187	0.11747	8.51259	0.13521	7.39616	0.15302	6.53503	0.17093	5.85024	18
43	0.10011	9.98931	0.11777	8.49128	0.13550	7.37999	0.15332	6.52234	0.17123	5.84001	17
44	0.10040	9.96007	0.11806	8.47007	0.13580	7.36389	0.15362	6.50970	0.17153	5.82982	16
45	0.10069	9.93101	0.11836	8.44896	0.13609	7.34786	0.15391	6.49710	0.17183	5.81966	15
46	0.10099	9.90211	0.11865	8.42795	0.13639	7.33190	0.15421	6.48456	0.17213	5.80953	14
47	0.10128	9.87338	0.11895	8.40705	0.13669	7.31600	0.15451	6.47206	0.17243	5.79944	13
48	0.10158	9.84482	0.11924	8.38625	0.13698	7.30018	0.15481	6.45961	0.17273	5.78938	12
49	0.10187	9.81641	0.11954	8.36555	0.13728	7.28442	0.15511	6.44720	0.17303	5.77936	11
50	0.10216	9.78817	0.11983	8.34496	0.13758	7.26873	0.15540	6.43484	0.17333	5.76937	10
51	0.10246	9.76009	0.12013	8.32446	0.13787	7.25310	0.15570	6.42253	0.17363	5.75941	9
52	0.10275	9.73217	0.12042	8.30406	0.13817	7.23754	0.15600	6.41026	0.17393	5.74949	8
53	0.10305	9.70441	0.12072	8.28376	0.13846	7.22204	0.15630	6.39804	0.17423	5.73960	7
54	0.10334	9.67680	0.12101	8.26355	0.13876	7.20661	0.15660	6.38587	0.17453	5.72974	6
55	0.10363	9.64935	0.12131	8.24345	0.13906	7.19125	0.15689	6.37374	0.17483	5.71992	5
56	0.10393	9.62205	0.12160	8.22344	0.13935	7.17594	0.15719	6.36165	0.17513	5.71013	4
57	0.10422	9.59490	0.12190	8.20352	0.13965	7.16071	0.15749	6.34961	0.17543	5.70037	3
58	0.10452	9.56791	0.12219	8.18370	0.13995	7.14553	0.15779	6.33761	0.17573	5.69064	2
59	0.10481	9.54106	0.12249	8.16398	0.14024	7.13042	0.15809	6.32566	0.17603	5.68094	1
60	0.10510	9.51436	0.12278	8.14435	0.14054	7.11537	0.15838	6.31375	0.17633	5.67128	0
	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	MIN
	84°		83°		82°		81°		80°		

MIN	10° TAN	10° COT	11° TAN	11° COT	12° TAN	12° COT	13° TAN	13° COT	14° TAN	14° COT	
0	0.17633	5.67128	0.19438	5.14455	0.21256	4.70463	0.23087	4.33148	0.24933	4.01078	60
1	0.17663	5.66165	0.19468	5.13658	0.21286	4.69791	0.23117	4.32573	0.24964	4.00582	59
2	0.17693	5.65205	0.19498	5.12862	0.21316	4.69121	0.23148	4.32001	0.24995	4.00086	58
3	0.17723	5.64248	0.19529	5.12069	0.21347	4.68452	0.23179	4.31430	0.25026	3.99592	57
4	0.17753	5.63295	0.19559	5.11279	0.21377	4.67786	0.23209	4.30860	0.25056	3.99099	56
5	0.17783	5.62344	0.19589	5.10490	0.21408	4.67121	0.23240	4.30291	0.25087	3.98607	55
6	0.17813	5.61397	0.19619	5.09704	0.21438	4.66458	0.23271	4.29724	0.25118	3.98117	54
7	0.17843	5.60452	0.19649	5.08921	0.21469	4.65797	0.23301	4.29159	0.25149	3.97627	53
8	0.17873	5.59511	0.19680	5.08139	0.21499	4.65138	0.23332	4.28595	0.25180	3.97139	52
9	0.17903	5.58573	0.19710	5.07360	0.21529	4.64480	0.23363	4.28032	0.25211	3.96651	51
10	0.17933	5.57638	0.19740	5.06584	0.21560	4.63825	0.23393	4.27471	0.25242	3.96165	50
11	0.17963	5.56706	0.19770	5.05809	0.21590	4.63171	0.23424	4.26911	0.25273	3.95680	49
12	0.17993	5.55777	0.19801	5.05037	0.21621	4.62518	0.23455	4.26352	0.25304	3.95196	48
13	0.18023	5.54851	0.19831	5.04267	0.21651	4.61868	0.23485	4.25795	0.25335	3.94713	47
14	0.18053	5.53927	0.19861	5.03499	0.21682	4.61219	0.23516	4.25239	0.25366	3.94232	46
15	0.18083	5.53007	0.19891	5.02734	0.21712	4.60572	0.23547	4.24685	0.25397	3.93751	45
16	0.18113	5.52090	0.19921	5.01971	0.21743	4.59927	0.23578	4.24132	0.25428	3.93271	44
17	0.18143	5.51176	0.19952	5.01210	0.21773	4.59283	0.23608	4.23580	0.25459	3.92793	43
18	0.18173	5.50264	0.19982	5.00451	0.21804	4.58641	0.23639	4.23030	0.25490	3.92316	42
19	0.18203	5.49356	0.20012	4.99695	0.21834	4.58001	0.23670	4.22481	0.25521	3.91839	41
20	0.18233	5.48451	0.20042	4.98940	0.21864	4.57363	0.23700	4.21933	0.25552	3.91364	40
21	0.18263	5.47548	0.20073	4.98188	0.21895	4.56726	0.23731	4.21387	0.25583	3.90890	39
22	0.18293	5.46648	0.20103	4.97438	0.21925	4.56091	0.23762	4.20842	0.25614	3.90417	38
23	0.18323	5.45751	0.20133	4.96690	0.21956	4.55458	0.23793	4.20298	0.25645	3.89945	37
24	0.18353	5.44857	0.20164	4.95945	0.21986	4.54826	0.23823	4.19756	0.25676	3.89474	36
25	0.18384	5.43966	0.20194	4.95201	0.22017	4.54196	0.23854	4.19215	0.25707	3.89004	35
26	0.18414	5.43077	0.20224	4.94460	0.22047	4.53568	0.23885	4.18675	0.25738	3.88536	34
27	0.18444	5.42192	0.20254	4.93721	0.22078	4.52941	0.23916	4.18137	0.25769	3.88068	33
28	0.18474	5.41309	0.20285	4.92984	0.22108	4.52316	0.23946	4.17600	0.25800	3.87601	32
29	0.18504	5.40429	0.20315	4.92249	0.22139	4.51693	0.23977	4.17064	0.25831	3.87136	31
30	0.18534	5.39552	0.20345	4.91516	0.22169	4.51071	0.24008	4.16530	0.25862	3.86671	30
31	0.18564	5.38677	0.20376	4.90785	0.22200	4.50451	0.24039	4.15997	0.25893	3.86208	29
32	0.18594	5.37805	0.20406	4.90056	0.22231	4.49832	0.24069	4.15465	0.25924	3.85745	28
33	0.18624	5.36936	0.20436	4.89330	0.22261	4.49215	0.24100	4.14934	0.25955	3.85284	27
34	0.18654	5.36070	0.20466	4.88605	0.22292	4.48600	0.24131	4.14405	0.25986	3.84824	26
35	0.18684	5.35206	0.20497	4.87882	0.22322	4.47986	0.24162	4.13877	0.26017	3.84364	25
36	0.18714	5.34345	0.20527	4.87162	0.22353	4.47374	0.24193	4.13350	0.26048	3.83906	24
37	0.18745	5.33487	0.20557	4.86444	0.22383	4.46764	0.24223	4.12825	0.26079	3.83449	23
38	0.18775	5.32631	0.20588	4.85727	0.22414	4.46155	0.24254	4.12301	0.26110	3.82992	22
39	0.18805	5.31778	0.20618	4.85013	0.22444	4.45548	0.24285	4.11778	0.26141	3.82537	21
40	0.18835	5.30928	0.20648	4.84300	0.22475	4.44942	0.24316	4.11256	0.26172	3.82083	20
41	0.18865	5.30080	0.20679	4.83590	0.22505	4.44338	0.24347	4.10736	0.26203	3.81630	19
42	0.18895	5.29235	0.20709	4.82882	0.22536	4.43735	0.24377	4.10216	0.26235	3.81177	18
43	0.18925	5.28393	0.20739	4.82175	0.22567	4.43134	0.24408	4.09699	0.26266	3.80726	17
44	0.18955	5.27553	0.20770	4.81471	0.22597	4.42534	0.24439	4.09182	0.26297	3.80276	16
45	0.18986	5.26715	0.20800	4.80769	0.22628	4.41936	0.24470	4.08666	0.26328	3.79827	15
46	0.19016	5.25880	0.20830	4.80068	0.22658	4.41340	0.24501	4.08152	0.26359	3.79378	14
47	0.19046	5.25048	0.20861	4.79370	0.22689	4.40745	0.24532	4.07639	0.26390	3.78931	13
48	0.19076	5.24218	0.20891	4.78673	0.22719	4.40152	0.24562	4.07127	0.26421	3.78485	12
49	0.19106	5.23391	0.20921	4.77978	0.22750	4.39560	0.24593	4.06616	0.26452	3.78040	11
50	0.19136	5.22566	0.20952	4.77286	0.22781	4.38969	0.24624	4.06107	0.26483	3.77595	10
51	0.19166	5.21744	0.20982	4.76595	0.22811	4.38381	0.24655	4.05599	0.26515	3.77152	9
52	0.19197	5.20925	0.21013	4.75906	0.22842	4.37793	0.24686	4.05092	0.26546	3.76709	8
53	0.19227	5.20107	0.21043	4.75219	0.22872	4.37207	0.24717	4.04586	0.26577	3.76268	7
54	0.19257	5.19293	0.21073	4.74534	0.22903	4.36623	0.24747	4.04081	0.26608	3.75828	6
55	0.19287	5.18480	0.21104	4.73851	0.22934	4.36040	0.24778	4.03578	0.26639	3.75388	5
56	0.19317	5.17671	0.21134	4.73170	0.22964	4.35459	0.24809	4.03076	0.26670	3.74950	4
57	0.19347	5.16863	0.21164	4.72490	0.22995	4.34879	0.24840	4.02574	0.26701	3.74512	3
58	0.19378	5.16058	0.21195	4.71813	0.23026	4.34300	0.24871	4.02074	0.26733	3.74075	2
59	0.19408	5.15256	0.21225	4.71137	0.23056	4.33723	0.24902	4.01576	0.26764	3.73640	1
60	0.19438	5.14455	0.21256	4.70463	0.23087	4.33148	0.24933	4.01078	0.26795	3.73205	0
	79° COT	79° TAN	78° COT	78° TAN	77° COT	77° TAN	76° COT	76° TAN	75° COT	75° TAN	MIN

MIN	15° TAN	15° COT	16° TAN	16° COT	17° TAN	17° COT	18° TAN	18° COT	19° TAN	19° COT	
0	0.26795	3.73205	0.28675	3.48741	0.30573	3.27085	0.32492	3.07768	0.34433	2.90421	60
1	0.26826	3.72771	0.28706	3.48359	0.30605	3.26745	0.32524	3.07464	0.34465	2.90147	59
2	0.26857	3.72338	0.28738	3.47977	0.30637	3.26406	0.32556	3.07160	0.34498	2.89873	58
3	0.26888	3.71907	0.28769	3.47596	0.30669	3.26067	0.32588	3.06857	0.34530	2.89600	57
4	0.26920	3.71476	0.28801	3.47216	0.30700	3.25729	0.32621	3.06554	0.34563	2.89327	56
5	0.26951	3.71046	0.28832	3.46837	0.30732	3.25392	0.32653	3.06252	0.34596	2.89055	55
6	0.26982	3.70616	0.28864	3.46458	0.30764	3.25055	0.32685	3.05950	0.34628	2.88783	54
7	0.27013	3.70188	0.28895	3.46080	0.30796	3.24719	0.32717	3.05649	0.34661	2.88511	53
8	0.27044	3.69761	0.28927	3.45703	0.30828	3.24383	0.32749	3.05349	0.34693	2.88240	52
9	0.27076	3.69335	0.28958	3.45327	0.30860	3.24049	0.32782	3.05049	0.34726	2.87970	51
10	0.27107	3.68909	0.28990	3.44951	0.30891	3.23714	0.32814	3.04749	0.34758	2.87700	50
11	0.27138	3.68485	0.29021	3.44576	0.30923	3.23381	0.32846	3.04450	0.34791	2.87430	49
12	0.27169	3.68061	0.29053	3.44202	0.30955	3.23048	0.32878	3.04152	0.34824	2.87161	48
13	0.27201	3.67638	0.29084	3.43829	0.30987	3.22715	0.32911	3.03854	0.34856	2.86892	47
14	0.27232	3.67217	0.29116	3.43456	0.31019	3.22384	0.32943	3.03556	0.34889	2.86624	46
15	0.27263	3.66796	0.29147	3.43084	0.31051	3.22053	0.32975	3.03260	0.34922	2.86356	45
16	0.27294	3.66376	0.29179	3.42713	0.31083	3.21722	0.33007	3.02963	0.34954	2.86089	44
17	0.27326	3.65957	0.29210	3.42343	0.31115	3.21392	0.33040	3.02667	0.34987	2.85822	43
18	0.27357	3.65538	0.29242	3.41973	0.31147	3.21063	0.33072	3.02372	0.35020	2.85555	42
19	0.27388	3.65121	0.29274	3.41604	0.31178	3.20734	0.33104	3.02077	0.35052	2.85289	41
20	0.27419	3.64705	0.29305	3.41236	0.31210	3.20406	0.33136	3.01783	0.35085	2.85023	40
21	0.27451	3.64289	0.29337	3.40869	0.31242	3.20079	0.33169	3.01489	0.35118	2.84758	39
22	0.27482	3.63874	0.29368	3.40502	0.31274	3.19752	0.33201	3.01196	0.35150	2.84494	38
23	0.27513	3.63461	0.29400	3.40136	0.31306	3.19426	0.33233	3.00903	0.35183	2.84229	37
24	0.27545	3.63048	0.29432	3.39771	0.31338	3.19100	0.33266	3.00611	0.35216	2.83965	36
25	0.27576	3.62636	0.29463	3.39406	0.31370	3.18775	0.33298	3.00319	0.35248	2.83702	35
26	0.27607	3.62224	0.29495	3.39042	0.31402	3.18451	0.33330	3.00028	0.35281	2.83439	34
27	0.27638	3.61814	0.29526	3.38679	0.31434	3.18127	0.33363	2.99738	0.35314	2.83176	33
28	0.27670	3.61405	0.29558	3.38317	0.31466	3.17804	0.33395	2.99447	0.35346	2.82914	32
29	0.27701	3.60996	0.29590	3.37955	0.31498	3.17481	0.33427	2.99158	0.35379	2.82653	31
30	0.27732	3.60588	0.29621	3.37594	0.31530	3.17159	0.33460	2.98868	0.35412	2.82391	30
31	0.27764	3.60181	0,29653	3.37234	0.31562	3.16838	0.33492	2.98580	0.35445	2.82130	29
32	0.27795	3.59775	0.29685	3.36875	0.31594	3.16517	0.33524	2.98292	0.35477	2.81870	28
33	0.27826	3.59370	0.29716	3.36516	0.31626	3.16197	0.33557	2.98004	0.35510	2.81610	27
34	0.27858	3.58966	0.29748	3.36158	0.31658	3.15877	0.33589	2.97717	0.35543	2.81350	26
35	0.27889	3.58562	0.29780	3.35800	0.31690	3.15558	0.33621	2.97430	0.35576	2.81091	25
36	0.27921	3.58160	0.29811	3.35443	0.31722	3.15240	0.33654	2.97144	0.35608	2.80833	24
37	0.27952	3.57758	0.29843	3.35087	0.31754	3.14922	0.33686	2.96858	0.35641	2.80574	23
38	0.27983	3.57357	0.29875	3.34732	0.31786	3.14605	0.33718	2.96573	0.35674	2.80316	22
39	0.28015	3.56957	0.29906	3.34377	0.31818	3.14288	0.33751	2.96288	0.35707	2.80059	21
40	0.28046	3.56557	0.29938	3.34023	0.31850	3.13972	0.33783	2.96004	0.35740	2.79802	20
41	0.28077	3.56159	0.29970	3.33670	0.31882	3.13656	0.33816	2.95721	0.35772	2.79545	19
42	0.28109	3.55761	0.30001	3.33317	0.31914	3.13341	0.33848	2.95437	0.35805	2.79289	18
43	0.28140	3.55364	0.30033	3.32965	0.31946	3.13027	0.33881	2.95155	0.35838	2.79033	17
44	0.28172	3.54968	0.30065	3.32614	0.31978	3.12713	0.33913	2.94872	0.35871	2.78778	16
45	0.28203	3.54573	0.30097	3.32264	0.32010	3.12400	0.33945	2.94591	0.35904	2.78523	15
46	0.28234	3.54179	0.30128	3.31914	0.32042	3.12087	0.33978	2.94309	0.35937	2.78269	14
47	0.28266	3.53785	0.30160	3.31565	0.32074	3.11775	0.34010	2.94028	0.35969	2.78014	13
48	0.28297	3.53393	0.30192	3.31216	0.32106	3.11464	0.34043	2.93748	0.36002	2.77761	12
49	0.28329	3.53001	0.30224	3.30868	0.32139	3.11153	0.34075	2.93468	0.36035	2.77507	11
50	0.28360	3.52609	0.30255	3.30521	0.32171	3.10842	0.34108	2.93189	0.36068	2.77254	10
51	0.28391	3.52219	0.30287	3.30174	0.32203	3.10532	0.34140	2.92910	0.36101	2.77002	9
52	0.28423	3.51829	0.30319	3.29829	0.32235	3.10223	0.34173	2.92632	0.36134	2.76750	8
53	0.28454	3.51441	0.30351	3.29483	0.32267	3.09914	0.34205	2.92354	0.36167	2.76498	7
54	0.28486	3.51053	0.30382	3.29139	0.32299	3.09606	0.34238	2.92076	0.36199	2.76247	6
55	0.28517	3.50666	0.30414	3.28795	0.32331	3.09298	0.34270	2.91799	0.36232	2.75996	5
56	0.28549	3.50279	0.30446	3.28452	0.32363	3.08991	0.34303	2.91523	0.36265	2.75746	4
57	0.28580	3.49894	0.30478	3.28109	0.32396	3.08685	0.34335	2.91246	0.36298	2.75496	3
58	0.28612	3.49509	0.30509	3.27767	0.32428	3.08379	0.34368	2.90971	0.36331	2.75246	2
59	0.28643	3.49125	0.30541	3.27426	0.32460	3.08073	0.34400	2.90696	0.36364	2.74997	1
60	0.28675	3.48741	0.30573	3.27085	0.32492	3.07768	0.34433	2.90421	0.36397	2.74748	0
	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	MIN
	74°		73°		72°		71°		70°		

MIN	20°		21°		22°		23°		24°		
	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	
0	0.36397	2.74748	0.38386	2.60509	0.40403	2.47509	0.42447	2.35585	0.44523	2.24604	60
1	0.36430	2.74499	0.38420	2.60283	0.40436	2.47302	0.42482	2.35395	0.44558	2.24428	59
2	0.36463	2.74251	0.38453	2.60057	0.40470	2.47095	0.42516	2.35205	0.44593	2.24252	58
3	0.36496	2.74004	0.38487	2.59831	0.40504	2.46888	0.42551	2.35015	0.44627	2.24077	57
4	0.36529	2.73756	0.38520	2.59606	0.40538	2.46682	0.42585	2.34825	0.44662	2.23902	56
5	0.36562	2.73509	0.38553	2.59381	0.40572	2.46476	0.42619	2.34636	0.44697	2.23727	55
6	0.36595	2.73263	0.38587	2.59156	0.40606	2.46270	0.42654	2.34447	0.44732	2.23553	54
7	0.36628	2.73017	0.38620	2.58932	0.40640	2.46065	0.42688	2.34258	0.44767	2.23378	53
8	0.36661	2.72771	0.38654	2.58708	0.40674	2.45860	0.42722	2.34069	0.44802	2.23204	52
9	0.36694	2.72526	0.38687	2.58484	0.40707	2.45655	0.42757	2.33881	0.44837	2.23030	51
10	0.36727	2.72281	0.38721	2.58261	0.40741	2.45451	0.42791	2.33693	0.44872	2.22857	50
11	0.36760	2.72036	0.38754	2.58038	0.40775	2.45246	0.42826	2.33505	0.44907	2.22683	49
12	0.36793	2.71792	0.38787	2.57815	0.40809	2.45043	0.42860	2.33317	0.44942	2.22510	48
13	0.36826	2.71548	0.38821	2.57593	0.40843	2.44839	0.42894	2.33130	0.44977	2.22337	47
14	0.36859	2.71305	0.38854	2.57371	0.40877	2.44636	0.42929	2.32943	0.45012	2.22164	46
15	0.36892	2.71062	0.38888	2.57150	0.40911	2.44433	0.42963	2.32756	0.45047	2.21992	45
16	0.36925	2.70819	0.38921	2.56928	0.40945	2.44230	0.42998	2.32570	0.45082	2.21819	44
17	0.36958	2.70577	0.38955	2.56707	0.40979	2.44027	0.43032	2.32383	0.45117	2.21647	43
18	0.36991	2.70335	0.38988	2.56487	0.41013	2.43825	0.43067	2.32197	0.45152	2.21475	42
19	0.37024	2.70094	0.39022	2.56266	0.41047	2.43623	0.43101	2.32012	0.45187	2.21304	41
20	0.37057	2.69853	0.39055	2.56046	0.41081	2.43422	0.43136	2.31826	0.45222	2.21132	40
21	0.37090	2.69612	0.39089	2.55827	0.41115	2.43220	0.43170	2.31641	0.45257	2.20961	39
22	0.37123	2.69371	0.39122	2.55608	0.41149	2.43019	0.43205	2.31456	0.45292	2.20790	38
23	0.37157	2.69131	0.39156	2.55389	0.41183	2.42819	0.43239	2.31271	0.45327	2.20619	37
24	0.37190	2.68892	0.39190	2.55170	0.41217	2.42618	0.43274	2.31086	0.45362	2.20449	36
25	0.37223	2.68653	0.39223	2.54952	0.41251	2.42418	0.43308	2.30902	0.45397	2.20278	35
26	0.37256	2.68414	0.39257	2.54734	0.41285	2.42218	0.43343	2.30718	0.45432	2.20108	34
27	0.37289	2.68175	0.39290	2.54516	0.41319	2.42019	0.43378	2.30534	0.45467	2.19938	33
28	0.37322	2.67937	0.39324	2.54299	0.41353	2.41819	0.43412	2.30351	0.45502	2.19769	32
29	0.37355	2.67700	0.39357	2.54082	0.41387	2.41620	0.43447	2.30167	0.45538	2.19599	31
30	0.37388	2.67462	0.39391	2.53865	0.41421	2.41421	0.43481	2.29984	0.45573	2.19430	30
31	0.37422	2.67225	0.39425	2.53648	0.41455	2.41223	0.43516	2.29801	0.45608	2.19261	29
32	0.37455	2.66989	0.39458	2.53432	0.41490	2.41025	0.43550	2.29619	0.45643	2.19092	28
33	0.37488	2.66752	0.39492	2.53217	0.41524	2.40827	0.43585	2.29437	0.45678	2.18923	27
34	0.37521	2.66516	0.39526	2.53001	0.41558	2.40629	0.43620	2.29254	0.45713	2.18755	26
35	0.37554	2.66281	0.39559	2.52786	0.41592	2.40432	0.43654	2.29073	0.45748	2.18587	25
36	0.37588	2.66046	0.39593	2.52571	0.41626	2.40235	0.43689	2.28891	0.45784	2.18419	24
37	0.37621	2.65811	0.39626	2.52357	0.41660	2.40038	0.43724	2.28710	0.45819	2.18251	23
38	0.37654	2.65576	0.39660	2.52142	0.41694	2.39841	0.43758	2.28528	0.45854	2.18084	22
39	0.37687	2.65342	0.39694	2.51929	0.41728	2.39645	0.43793	2.28348	0.45889	2.17916	21
40	0.37720	2.65109	0.39727	2.51715	0.41763	2.39449	0.43828	2.28167	0.45924	2.17749	20
41	0.37754	2.64875	0.39761	2.51502	0.41797	2.39253	0.43862	2.27987	0.45960	2.17582	19
42	0.37787	2.64642	0.39795	2.51289	0.41831	2.39058	0.43897	2.27806	0.45995	2.17416	18
43	0.37820	2.64410	0.39829	2.51076	0.41865	2.38863	0.43932	2.27626	0.46030	2.17249	17
44	0.37853	2.64177	0.39862	2.50864	0.41899	2.38668	0.43966	2.27447	0.46065	2.17083	16
45	0.37887	2.63945	0.39896	2.50652	0.41933	2.38473	0.44001	2.27267	0.46101	2.16917	15
46	0.37920	2.63714	0.39930	2.50440	0.41968	2.38279	0.44036	2.27088	0.46136	2.16751	14
47	0.37953	2.63483	0.39963	2.50229	0.42002	2.38084	0.44071	2.26909	0.46171	2.16585	13
48	0.37986	2.63252	0.39997	2.50018	0.42036	2.37891	0.44105	2.26730	0.46206	2.16420	12
49	0.38020	2.63021	0.40031	2.49807	0.42070	2.37697	0.44140	2.26552	0.46242	2.16255	11
50	0.38053	2.62791	0.40065	2.49597	0.42105	2.37504	0.44175	2.26374	0.46277	2.16090	10
51	0.38086	2.62561	0.40098	2.49386	0.42139	2.37311	0.44210	2.26196	0.46312	2.15925	9
52	0.38120	2.62332	0.40132	2.49177	0.42173	2.37118	0.44244	2.26018	0.46348	2.15760	8
53	0.38153	2.62103	0.40166	2.48967	0.42207	2.36925	0.44279	2.25840	0.46383	2.15596	7
54	0.38186	2.61874	0.40200	2.48758	0.42242	2.36733	0.44314	2.25663	0.46418	2.15432	6
55	0.38220	2.61646	0.40234	2.48549	0.42276	2.36541	0.44349	2.25486	0.46454	2.15268	5
56	0.38253	2.61418	0.40267	2.48340	0.42310	2.36349	0.44384	2.25309	0.46489	2.15104	4
57	0.38286	2.61190	0.40301	2.48132	0.42345	2.36158	0.44418	2.25132	0.46525	2.14940	3
58	0.38320	2.60963	0.40335	2.47924	0.42379	2.35967	0.44453	2.24956	0.46560	2.14777	2
59	0.38353	2.60736	0.40369	2.47716	0.42413	2.35776	0.44488	2.24780	0.46595	2.14614	1
60	0.38386	2.60509	0.40403	2.47509	0.42447	2.35585	0.44523	2.24604	0.46631	2.14451	0
	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	MIN
	69°		68°		67°		66°		65°		

MIN	25° TAN	25° COT	26° TAN	26° COT	27° TAN	27° COT	28° TAN	28° COT	29° TAN	29° COT	
0	0.46631	2.14451	0.48773	2.05030	0.50953	1.96261	0.53171	1.88073	0.55431	1.80405	60
1	0.46666	2.14288	0.48809	2.04879	0.50989	1.96120	0.53208	1.87941	0.55469	1.80281	59
2	0.46702	2.14125	0.48845	2.04728	0.51026	1.95979	0.53246	1.87809	0.55507	1.80158	58
3	0.46737	2.13963	0.48881	2.04577	0.51063	1.95838	0.53283	1.87677	0.55545	1.80034	57
4	0.46772	2.13801	0.48917	2.04426	0.51099	1.95698	0.53320	1.87546	0.55583	1.79911	56
5	0.46808	2.13639	0.48953	2.04276	0.51136	1.95557	0.53358	1.87415	0.55621	1.79788	55
6	0.46843	2.13477	0.48989	2.04125	0.51173	1.95417	0.53395	1.87283	0.55659	1.79665	54
7	0.46879	2.13316	0.49026	2.03975	0.51209	1.95277	0.53432	1.87152	0.55697	1.79542	53
8	0.46914	2.13154	0.49062	2.03825	0.51246	1.95137	0.53470	1.87021	0.55736	1.79419	52
9	0.46950	2.12993	0.49098	2.03675	0.51283	1.94997	0.53507	1.86891	0.55774	1.79296	51
10	0.46985	2.12832	0.49134	2.03526	0.51319	1.94858	0.53545	1.86760	0.55812	1.79174	50
11	0.47021	2.12671	0.49170	2.03376	0.51356	1.94718	0.53582	1.86630	0.55850	1.79051	49
12	0.47056	2.12511	0.49206	2.03227	0.51393	1.94579	0.53620	1.86499	0.55888	1.78929	48
13	0.47092	2.12350	0.49242	2.03078	0.51430	1.94440	0.53657	1.86369	0.55926	1.78807	47
14	0.47128	2.12190	0.49278	2.02929	0.51467	1.94301	0.53694	1.86239	0.55964	1.78685	46
15	0.47163	2.12030	0.49315	2.02780	0.51503	1.94162	0.53732	1.86109	0.56003	1.78563	45
16	0.47199	2.11871	0.49351	2.02631	0.51540	1.94023	0.53769	1.85979	0.56041	1.78441	44
17	0.47234	2.11711	0.49387	2.02483	0.51577	1.93885	0.53807	1.85850	0.56079	1.78319	43
18	0.47270	2.11552	0.49423	2.02335	0.51614	1.93746	0.53844	1.85720	0.56117	1.78198	42
19	0.47305	2.11392	0.49459	2.02187	0.51651	1.93608	0.53882	1.85591	0.56156	1.78077	41
20	0.47341	2.11233	0.49495	2.02039	0.51688	1.93470	0.53920	1.85462	0.56194	1.77955	40
21	0.47377	2.11075	0.49532	2.01891	0.51724	1.93332	0.53957	1.85333	0.56232	1.77834	39
22	0.47412	2.10916	0.49568	2.01743	0.51761	1.93195	0.53995	1.85204	0.56270	1.77713	38
23	0.47448	2.10758	0.49604	2.01596	0.51798	1.93057	0.54032	1.85075	0.56309	1.77592	37
24	0.47483	2.10600	0.49640	2.01449	0.51835	1.92920	0.54070	1.84946	0.56347	1.77471	36
25	0.47519	2.10442	0.49677	2.01302	0.51872	1.92782	0.54107	1.84818	0.56385	1.77351	35
26	0.47555	2.10284	0.49713	2.01155	0.51909	1.92645	0.54145	1.84689	0.56424	1.77230	34
27	0.47590	2.10126	0.49749	2.01008	0.51946	1.92508	0.54183	1.84561	0.56462	1.77110	33
28	0.47626	2.09969	0.49786	2.00862	0.51983	1.92371	0.54220	1.84433	0.56501	1.76990	32
29	0.47662	2.09811	0.49822	2.00715	0.52020	1.92235	0.54258	1.84305	0.56539	1.76869	31
30	0.47698	2.09654	0.49858	2.00569	0.52057	1.92098	0.54296	1.84177	0.56577	1.76749	30
31	0.47733	2.09498	0.49894	2.00423	0.52094	1.91962	0.54333	1.84049	0.56616	1.76629	29
32	0.47769	2.09341	0.49931	2.00277	0.52131	1.91826	0.54371	1.83922	0.56654	1.76510	28
33	0.47805	2.09184	0.49967	2.00131	0.52168	1.91690	0.54409	1.83794	0.56693	1.76390	27
34	0.47840	2.09028	0.50004	1.99986	0.52205	1.91554	0.54446	1.83667	0.56731	1.76271	26
35	0.47876	2.08872	0.50040	1.99841	0.52242	1.91418	0.54484	1.83540	0.56769	1.76151	25
36	0.47912	2.08716	0.50076	1.99695	0.52279	1.91282	0.54522	1.83413	0.56808	1.76032	24
37	0.47948	2.08560	0.50113	1.99550	0.52316	1.91147	0.54560	1.83286	0.56846	1.75913	23
38	0.47984	2.08405	0.50149	1.99406	0.52353	1.91012	0.54597	1.83159	0.56885	1.75794	22
39	0.48019	2.08250	0.50185	1.99261	0.52390	1.90876	0.54635	1.83033	0.56923	1.75675	21
40	0.48055	2.08094	0.50222	1.99116	0.52427	1.90741	0.54673	1.82906	0.56962	1.75556	20
41	0.48091	2.07939	0.50258	1.98972	0.52464	1.90607	0.54711	1.82780	0.57000	1.75437	19
42	0.48127	2.07785	0.50295	1.98828	0.52501	1.90472	0.54748	1.82654	0.57039	1.75319	18
43	0.48163	2.07630	0.50331	1.98684	0.52538	1.90337	0.54786	1.82528	0.57078	1.75200	17
44	0.48198	2.07476	0.50368	1.98540	0.52575	1.90203	0.54824	1.82402	0.57116	1.75082	16
45	0.48234	2.07321	0.50404	1.98396	0.52613	1.90069	0.54862	1.82276	0.57155	1.74964	15
46	0.48270	2.07167	0.50441	1.98253	0.52650	1.89935	0.54900	1.82150	0.57193	1.74846	14
47	0.48306	2.07014	0.50477	1.98110	0.52687	1.89801	0.54938	1.82025	0.57232	1.74728	13
48	0.48342	2.06860	0.50514	1.97966	0.52724	1.89667	0.54975	1.81899	0.57271	1.74610	12
49	0.48378	2.06706	0.50550	1.97823	0.52761	1.89533	0.55013	1.81774	0.57309	1.74492	11
50	0.48414	2.06553	0.50587	1.97681	0.52798	1.89400	0.55051	1.81649	0.57348	1.74375	10
51	0.48450	2.06400	0.50623	1.97538	0.52836	1.89266	0.55089	1.81524	0.57386	1.74257	9
52	0.48486	2.06247	0.50660	1.97395	0.52873	1.89133	0.55127	1.81399	0.57425	1.74140	8
53	0.48521	2.06094	0.50696	1.97253	0.52910	1.89000	0.55165	1.81274	0.57464	1.74022	7
54	0.48557	2.05942	0.50733	1.97111	0.52947	1.88867	0.55203	1.81150	0.57503	1.73905	6
55	0.48593	2.05790	0.50769	1.96969	0.52985	1.88734	0.55241	1.81025	0.57541	1.73788	5
56	0.48629	2.05637	0.50806	1.96827	0.53022	1.88602	0.55279	1.80901	0.57580	1.73671	4
57	0.48665	2.05485	0.50843	1.96685	0.53059	1.88469	0.55317	1.80777	0.57619	1.73555	3
58	0.48701	2.05333	0.50879	1.96544	0.53096	1.88337	0.55355	1.80653	0.57657	1.73438	2
59	0.48737	2.05182	0.50916	1.96402	0.53134	1.88205	0.55393	1.80529	0.57696	1.73321	1
60	0.48773	2.05030	0.50953	1.96261	0.53171	1.88073	0.55431	1.80405	0.57735	1.73205	0
	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	MIN
	64°		63°		62°		61°		60°		

MIN	30° TAN	30° COT	31° TAN	31° COT	32° TAN	32° COT	33° TAN	33° COT	34° TAN	34° COT	
0	0.57735	1.73205	0.60086	1.66428	0.62487	1.60033	0.64941	1.53986	0.67451	1.48256	60
1	0.57774	1.73089	0.60126	1.66318	0.62527	1.59930	0.64982	1.53888	0.67493	1.48163	59
2	0.57813	1.72973	0.60165	1.66209	0.62568	1.59826	0.65024	1.53791	0.67536	1.48070	58
3	0.57851	1.72857	0.60205	1.66099	0.62608	1.59723	0.65065	1.53693	0.67578	1.47977	57
4	0.57890	1.72741	0.60245	1.65990	0.62649	1.59620	0.65106	1.53595	0.67620	1.47885	56
5	0.57929	1.72625	0.60284	1.65881	0.62689	1.59517	0.65148	1.53497	0.67663	1.47792	55
6	0.57968	1.72509	0.60324	1.65772	0.62730	1.59414	0.65189	1.53400	0.67705	1.47699	54
7	0.58007	1.72393	0.60364	1.65663	0.62770	1.59311	0.65231	1.53302	0.67748	1.47607	53
8	0.58046	1.72278	0.60403	1.65554	0.62811	1.59208	0.65272	1.53205	0.67790	1.47514	52
9	0.58085	1.72163	0.60443	1.65445	0.62852	1.59105	0.65314	1.53107	0.67832	1.47422	51
10	0.58124	1.72047	0.60483	1.65337	0.62892	1.59002	0.65355	1.53010	0.67875	1.47330	50
11	0.58162	1.71932	0.60522	1.65228	0.62933	1.58900	0.65397	1.52913	0.67917	1.47238	49
12	0.58201	1.71817	0.60562	1.65120	0.62973	1.58797	0.65438	1.52816	0.67960	1.47146	48
13	0.58240	1.71702	0.60602	1.65011	0.63014	1.58695	0.65480	1.52719	0.68002	1.47053	47
14	0.58279	1.71588	0.60642	1.64903	0.63055	1.58593	0.65521	1.52622	0.68045	1.46962	46
15	0.58318	1.71473	0.60681	1.64795	0.63095	1.58490	0.65563	1.52525	0.68088	1.46870	45
16	0.58357	1.71358	0.60721	1.64687	0.63136	1.58388	0.65604	1.52429	0.68130	1.46778	44
17	0.58396	1.71244	0.60761	1.64579	0.63177	1.58286	0.65646	1.52332	0.68173	1.46686	43
18	0.58435	1.71129	0.60801	1.64471	0.63217	1.58184	0.65688	1.52235	0.68215	1.46595	42
19	0.58474	1.71015	0.60841	1.64363	0.63258	1.58083	0.65729	1.52139	0.68258	1.46503	41
20	0.58513	1.70901	0.60881	1.64256	0.63299	1.57981	0.65771	1.52043	0.68301	1.46411	40
21	0.58552	1.70787	0.60921	1.64148	0.63340	1.57879	0.65813	1.51946	0.68343	1.46320	39
22	0.58591	1.70673	0.60960	1.64041	0.63380	1.57778	0.65854	1.51850	0.68386	1.46229	38
23	0.58631	1.70560	0.61000	1.63934	0.63421	1.57676	0.65896	1.51754	0.68429	1.46137	37
24	0.58670	1.70446	0.61040	1.63826	0.63462	1.57575	0.65938	1.51658	0.68471	1.46046	36
25	0.58709	1.70332	0.61080	1.63719	0.63503	1.57474	0.65980	1.51562	0.68514	1.45955	35
26	0.58748	1.70219	0.61120	1.63612	0.63544	1.57372	0.66021	1.51466	0.68557	1.45864	34
27	0.58787	1.70106	0.61160	1.63505	0.63584	1.57271	0.66063	1.51370	0.68600	1.45773	33
28	0.58826	1.69992	0.61200	1.63398	0.63625	1.57170	0.66105	1.51275	0.68642	1.45682	32
29	0.58865	1.69879	0.61240	1.63292	0.63666	1.57069	0.66147	1.51179	0.68685	1.45592	31
30	0.58905	1.69766	0.61280	1.63185	0.63707	1.56969	0.66189	1.51084	0.68728	1.45501	30
31	0.58944	1.69653	0.61320	1.63079	0.63748	1.56868	0.66230	1.50988	0.68771	1.45410	29
32	0.58983	1.69541	0.61360	1.62972	0.63789	1.56767	0.66272	1.50893	0.68814	1.45320	28
33	0.59022	1.69428	0.61400	1.62866	0.63830	1.56667	0.66314	1.50797	0.68857	1.45229	27
34	0.59061	1.69316	0.61440	1.62760	0.63871	1.56566	0.66356	1.50702	0.68900	1.45139	26
35	0.59101	1.69203	0.61480	1.62654	0.63912	1.56466	0.66398	1.50607	0.68942	1.45049	25
36	0.59140	1.69091	0.61520	1.62548	0.63953	1.56366	0.66440	1.50512	0.68985	1.44958	24
37	0.59179	1.68979	0.61561	1.62442	0.63994	1.56265	0.66482	1.50417	0.69028	1.44868	23
38	0.59218	1.68866	0.61601	1.62336	0.64035	1.56165	0.66524	1.50322	0.69071	1.44778	22
39	0.59258	1.68754	0.61641	1.62230	0.64076	1.56065	0.66566	1.50228	0.69114	1.44688	21
40	0.59297	1.68643	0.61681	1.62125	0.64117	1.55966	0.66608	1.50133	0.69157	1.44598	20
41	0.59336	1.68531	0.61721	1.62019	0.64158	1.55866	0.66650	1.50038	0.69200	1.44508	19
42	0.59376	1.68419	0.61761	1.61914	0.64199	1.55766	0.66692	1.49944	0.69243	1.44418	18
43	0.59415	1.68308	0.61801	1.61808	0.64240	1.55666	0.66734	1.49849	0.69286	1.44329	17
44	0.59454	1.68196	0.61842	1.61703	0.64281	1.55567	0.66776	1.49755	0.69329	1.44239	16
45	0.59494	1.68085	0.61882	1.61598	0.64322	1.55467	0.66818	1.49661	0.69372	1.44149	15
46	0.59533	1.67974	0.61922	1.61493	0.64363	1.55368	0.66860	1.49566	0.69416	1.44060	14
47	0.59573	1.67863	0.61962	1.61388	0.64404	1.55269	0.66902	1.49472	0.69459	1.43970	13
48	0.59612	1.67752	0.62003	1.61283	0.64446	1.55170	0.66944	1.49378	0.69502	1.43881	12
49	0.59651	1.67641	0.62043	1.61179	0.64487	1.55071	0.66986	1.49284	0.69545	1.43792	11
50	0.59691	1.67530	0.62083	1.61074	0.64528	1.54972	0.67028	1.49190	0.69588	1.43703	10
51	0.59730	1.67419	0.62124	1.60970	0.64569	1.54873	0.67071	1.49097	0.69631	1.43614	9
52	0.59770	1.67309	0.62164	1.60865	0.64610	1.54774	0.67113	1.49003	0.69675	1.43525	8
53	0.59809	1.67198	0.62204	1.60761	0.64652	1.54675	0.67155	1.48909	0.69718	1.43436	7
54	0.59849	1.67088	0.62245	1.60657	0.64693	1.54576	0.67197	1.48816	0.69761	1.43347	6
55	0.59888	1.66978	0.62285	1.60553	0.64734	1.54478	0.67239	1.48722	0.69804	1.43258	5
56	0.59928	1.66867	0.62325	1.60449	0.64775	1.54379	0.67282	1.48629	0.69847	1.43169	4
57	0.59967	1.66757	0.62366	1.60345	0.64817	1.54281	0.67324	1.48536	0.69891	1.43080	3
58	0.60007	1.66647	0.62406	1.60241	0.64858	1.54183	0.67366	1.48442	0.69934	1.42992	2
59	0.60046	1.66538	0.62446	1.60137	0.64899	1.54085	0.67409	1.48349	0.69977	1.42903	1
60	0.60086	1.66428	0.62487	1.60033	0.64941	1.53986	0.67451	1.48256	0.70021	1.42815	0
	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	MIN
	59°		58°		57°		56°		55°		

MIN	35°		36°		37°		38°		39°		
	TAN	COT	TAN	COT	TAN	COT	TAN	COT	TAN	COT	
0	0.70021	1.42815	0.72654	1.37638	0.75355	1.32704	0.78129	1.27994	0.80978	1.23490	60
1	0.70064	1.42726	0.72699	1.37554	0.75401	1.32624	0.78175	1.27917	0.81027	1.23416	59
2	0.70107	1.42638	0.72743	1.37470	0.75447	1.32544	0.78222	1.27841	0.81075	1.23343	58
3	0.70151	1.42550	0.72788	1.37386	0.75492	1.32464	0.78269	1.27764	0.81123	1.23270	57
4	0.70194	1.42462	0.72832	1.37302	0.75538	1.32384	0.78316	1.27688	0.81171	1.23196	56
5	0.70238	1.42374	0.72877	1.37218	0.75584	1.32304	0.78363	1.27611	0.81220	1.23123	55
6	0.70281	1.42286	0.72921	1.37134	0.75629	1.32224	0.78410	1.27535	0.81268	1.23050	54
7	0.70325	1.42198	0.72966	1.37050	0.75675	1.32144	0.78457	1.27458	0.81316	1.22977	53
8	0.70368	1.42110	0.73010	1.36967	0.75721	1.32064	0.78504	1.27382	0.81364	1.22904	52
9	0.70412	1.42022	0.73055	1.36883	0.75767	1.31984	0.78551	1.27306	0.81413	1.22831	51
10	0.70455	1.41934	0.73100	1.36800	0.75812	1.31904	0.78598	1.27230	0.81461	1.22758	50
11	0.70499	1.41847	0.73144	1.36716	0.75858	1.31825	0.78645	1.27153	0.81510	1.22685	49
12	0.70542	1.41759	0.73189	1.36633	0.75904	1.31745	0.78692	1.27077	0.81558	1.22612	48
13	0.70586	1.41672	0.73234	1.36549	0.75950	1.31666	0.78739	1.27001	0.81606	1.22539	47
14	0.70629	1.41584	0.73278	1.36466	0.75996	1.31586	0.78786	1.26925	0.81655	1.22467	46
15	0.70673	1.41497	0.73323	1.36383	0.76042	1.31507	0.78834	1.26849	0.81703	1.22394	45
16	0.70717	1.41409	0.73368	1.36300	0.76088	1.31427	0.78881	1.26774	0.81752	1.22321	44
17	0.70760	1.41322	0.73413	1.36217	0.76134	1.31348	0.78928	1.26698	0.81800	1.22249	43
18	0.70804	1.41235	0.73457	1.36134	0.76180	1.31269	0.78975	1.26622	0.81849	1.22176	42
19	0.70848	1.41148	0.73502	1.36051	0.76226	1.31190	0.79022	1.26546	0.81898	1.22104	41
20	0.70891	1.41061	0.73547	1.35968	0.76272	1.31110	0.79070	1.26471	0.81946	1.22031	40
21	0.70935	1.40974	0.73592	1.35885	0.76318	1.31031	0.79117	1.26395	0.81995	1.21959	39
22	0.70979	1.40887	0.73637	1.35802	0.76364	1.30952	0.79164	1.26319	0.82044	1.21886	38
23	0.71023	1.40800	0.73681	1.35719	0.76410	1.30873	0.79212	1.26244	0.82092	1.21814	37
24	0.71066	1.40714	0.73726	1.35637	0.76456	1.30795	0.79259	1.26169	0.82141	1.21742	36
25	0.71110	1.40627	0.73771	1.35554	0.76502	1.30716	0.79306	1.26093	0.82190	1.21670	35
26	0.71154	1.40540	0.73816	1.35472	0.76548	1.30637	0.79354	1.26018	0.82238	1.21598	34
27	0.71198	1.40454	0.73861	1.35389	0.76594	1.30558	0.79401	1.25943	0.82287	1.21526	33
28	0.71242	1.40367	0.73906	1.35307	0.76640	1.30480	0.79449	1.25867	0.82336	1.21454	32
29	0.71285	1.40281	0.73951	1.35224	0.76686	1.30401	0.79496	1.25792	0.82385	1.21382	31
30	0.71329	1.40195	0.73996	1.35142	0.76733	1.30323	0.79544	1.25717	0.82434	1.21310	30
31	0.71373	1.40109	0.74041	1.35060	0.76779	1.30244	0.79591	1.25642	0.82483	1.21238	29
32	0.71417	1.40022	0.74086	1.34978	0.76825	1.30166	0.79639	1.25567	0.82531	1.21166	28
33	0.71461	1.39936	0.74131	1.34896	0.76871	1.30087	0.79686	1.25492	0.82580	1.21094	27
34	0.71505	1.39850	0.74176	1.34814	0.76918	1.30009	0.79734	1.25417	0.82629	1.21023	26
35	0.71549	1.39764	0.74221	1.34732	0.76964	1.29931	0.79781	1.25343	0.82678	1.20951	25
36	0.71593	1.39679	0.74267	1.34650	0.77010	1.29853	0.79829	1.25268	0.82727	1.20879	24
37	0.71637	1.39593	0.74312	1.34568	0.77057	1.29775	0.79877	1.25193	0.82776	1.20808	23
38	0.71681	1.39507	0.74357	1.34487	0.77103	1.29696	0.79924	1.25118	0.82825	1.20736	22
39	0.71725	1.39421	0.74402	1.34405	0.77149	1.29618	0.79972	1.25044	0.82874	1.20665	21

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Assignment Questions

Information: The text pages that you are to study are provided at the beginning of the assignment questions.

Assignment 1

Textbook assignment: Chapter 1, "Straight Lines," pages 1-1 through 1-28.

Learning Objective:

Determine distances, divisions, slopes, inclinations, angles, and equations of straight lines.

1-1. Analytic geometry is the branch of mathematics that is concerned with the algebraic analysis of geometrical relationships.

1. True
2. False

1-2. In the rectangular coordinate system, an abscissa is a

1. number that describes a point
2. distance measured parallel to the Y axis
3. distance measured parallel to the X axis
4. distance measured perpendicular to the X axis

1-3. An ordinate is positive if it is a measurement in what direction from the origin?

1. To the right
2. To the left
3. Above
4. Below

1-4. The point M whose abscissa is 4 and ordinate is -3 is designated by

1. 4,-3
2. M(4,-3)
3. P(-3,4)
4. M(-3,4)

1-5. The abscissa and ordinate of a point are called the

1. intersection
2. mantissa of the point
3. coordinates of the point
4. Cartesian symbols of the point

1-6. The length of a line segment parallel to the X axis can be measured in the same manner as a line segment on the X axis.

1. True
2. False

1-7. Using the formula for finding the distance from one point to another in a coordinate system, what is the distance from point P to point Q if P has the coordinates P(0,2) and Q has the coordinates Q(6,10)?

1. 10
2. 14
3. $\sqrt{20}$
4. 100

1-8. What is the distance from the point T(8,3) to the point U(-3,5)?

1. $5\sqrt{5}$
2. $\sqrt{27}$
3. $\sqrt{117}$
4. $\sqrt{185}$

1-9. What is the initial error in the following step-by-step calculation of the distance from M(6,4) to N(-3,8) which resulted in the incorrect answer 5?

A. $\sqrt{(6 - 3)^2 + (4 - 8)^2}$
B. $\sqrt{3^2 + (-4)^2}$
C. $\sqrt{9 + 16}$
D. $\sqrt{25}$
E. 5

1. The square of -4 in C
2. The addition result in D
3. The extraction of the square root in E
4. The sign before the 3 in A

1-10. If point Q is located $\frac{1}{6}$ of the distance from point R to point S, what is the ratio of RQ to RS?

1. $\frac{6}{1}$
2. $\frac{1}{2}$
3. $\frac{1}{3}$
4. $\frac{1}{6}$

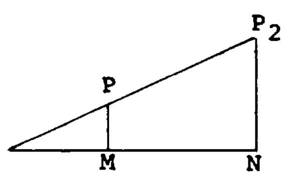

Figure 1A.--Similar triangles.

IN ANSWERING ITEM 1-11, REFER TO FIGURE 1A.

1-11. If the ratio of P_1M to P_1N is $\frac{1}{3}$, the ratio of P_1P to P_1P_2 is

1. $\frac{1}{6}$
2. $\frac{1}{3}$
3. $\frac{1}{2}$
4. $\frac{6}{1}$

1-12. Find the coordinates of point M one-third of the distance from P(3,8) to Q(3,5).

1. M(3,7)
2. M(4,7)
3. M(7,3)
4. M(7,4)

1-13. What are the coordinates of the midpoint M of the segment joining A(-4,-6) and B(8,10)?

1. M(0,0)
2. M(2,2)
3. M(-2,-2)
4. M(3,3)

1-14. In the rectangular coordinate system, the angle (α) the line crossing the X axis makes with the positively directed portion of the X axis, such that $0^\circ \leq \alpha < 180^\circ$, is called the

1. tangent
2. acute angle
3. reference angle
4. angle of inclination

1-15. The slope of any line is equal to the

1. angle of inclination
2. coordinate where it crosses the X axis
3. tangent of its angle of inclination
4. angle at the intersection of the line and the X axis

1-16. A line sloping downward and to the right has what kind of slope?

1. Zero
2. Negative
3. Positive
4. Infinite

1-17. The slope of a line is a ratio.

1. True
2. False

1-18. What is the slope m of a line connecting A(4,7) and B(-2,3)?

1. 0
2. $\frac{3}{5}$
3. $-\frac{3}{4}$
4. $\frac{2}{3}$

1-19. What change, if any, would occur in the answer to the previous item if A had the coordinates (-2,3) and B had the coordinates (4,7)?

1. There would be no change
2. The sign would be opposite
3. The slope of the line would be zero
4. The reciprocal of the answer would be true

1-20. The slope of a line parallel to the X axis is

1. 1
2. $\frac{1}{2}$
3. ∞
4. 0

1-21. The slope of a vertical line is not defined since the tangent of the line's angle of inclination increases without limit as the angle approaches

1. 90°
2. 0°
3. 180°
4. 45°

1-22. Which statement is true of two parallel lines in a rectangular coordinate system? They have the same

1. x intercept
2. y intercept
3. slope
4. coordinates

1-23. What is the relationship of the line between A(-1,-2) and B(1,-4) and the line between C(9,-1) and D(2,6)?

1. The lines are vertical
2. The lines are collinear
3. The lines are parallel
4. The lines are perpendicular

1-24. Points A, B, and C have the coordinates (3,6), (-89,0), and (2,13), respectively. What is the relationship of line segments AC and BC?

1. They are parallel
2. They are perpendicular
3. They are equal in length
4. They form an angle of 45°

In items 1-25 through 1-28, select from column B the set of lines that is appropriate to each description listed in column A.

A. DESCRIPTIONS	B. SETS
1-25. Lines having the same slope	1. Parallel lines
1-26. Slopes that are negative reciprocals	2. Perpendicular lines
1-27. Lines having the same angle of inclination	3. Lines having a positive slope
1-28. Lines that slope up and to the right	

1-29. If ϕ is the acute angle and ϕ' is the obtuse angle between lines L_1 and L_2, then

1. $\phi' = \phi$
2. $\phi' - \phi = 180^\circ$
3. $\phi' + \phi = 180^\circ$
4. $\phi' + \phi = 90^\circ$

1-30. If lines L_1 and L_2 have slopes of 2 and 6, respectively, what is the size of the acute angle between them?

1. $6^\circ\ 17'$
2. $17^\circ\ 6'$
3. $28^\circ\ 13'$
4. 176°

1-31. The size of the acute angle between L_1, which has a slope of 8, and L_2, which is parallel to the X axis is

1. $82^\circ\ 53'$
2. $97^\circ\ 7'$
3. 108°
4. 118°

1-32. What is the size of the obtuse angle formed by line L_1, whose slope is $\frac{1}{3}$, and line L_2, whose slope is undefined or ∞?

1. $8^\circ\ 26'$
2. $98^\circ\ 6'$
3. 100°
4. $108^\circ\ 26'$

1-33. The equation $y - y_1 = m(x - x_1)$ is called the

1. slope formula
2. quadratic formula
3. equation of a line whose slope is x_1
4. point-slope form of the equation of a line

1-34. The equation of a line through (3,8) which has a slope of 2 is

1. $3x + 8y = 0$
2. $2x - y + 2 = 0$
3. $2x + 2y - 2 = 0$
4. $2y - x + 2 = 0$

1-35. Using the point-slope form of the equation of a line, under which of the following conditions can you determine a linear equation?

1. One point on the line and the slope of the line are known
2. Two points on the line and the slope of the line are known
3. Two points on the line are known
4. All of the above

1-36. What is the equation of a line through points (3,3) and (-4,2)?

1. $x + y - 18 = 0$
2. $x + 7y - 18 = 0$
3. $x - 7y + 18 = 0$
4. $7x - y + 18 = 0$

1-37. Any line that is not perpendicular to the X axis intersects the Y axis at

1. no point
2. some point
3. all points
4. two points

1-38. The x coordinate of the point at which any line crosses the Y axis is always

1. zero
2. positive
3. negative
4. the same as the y coordinate

1-39. The equation $y = mx + b$ is called the slope-intercept form of the equation of a line.

1. True
2. False

1-40. Which of the following data, if known, is sufficient to permit the general form $y = mx + b$ to be used in finding the equation of a line?

1. One point on the line
2. The slope of the line
3. The y intercept of the line
4. The slope of the line and its y intercept

1-41. What is the slope of the line whose equation is $5y = 15x + 11$?

1. $\frac{11}{5}$
2. 11
3. 3
4. 15

1-42. The line whose y intercept is (0,3) and whose slope is $\frac{1}{6}$ has what equation?

1. $x = 6y + 18$
2. $6y = x + 18$
3. $6x = y - 18$
4. $6x + 6y - 1 = 0$

1-43. In using the normal form to write the equation of a line, what information must you know about the desired line?

1. The slope and the y intercept
2. One point on the line and the slope
3. Two points on the line and the perpendicular distance from the origin to the line
4. The perpendicular distance of the line from the origin and the angle the perpendicular makes with the positive side of the X axis

1-44. Which of the following is the equation of a line if the perpendicular distance from the origin is 4 units and the angle between the perpendicular and the positive side of the X axis is 60°?

1. $\sqrt{3} = 8x - y$
2. $8 = x\sqrt{3} + y$
3. $8 = x + y\sqrt{3}$
4. $8x - y = 2$

1-45. What is the equation of a line parallel to $2x + y - 6 = 0$ and passing through (4,6)?

1. $y = x + 8$
2. $y = 2x + 16$
3. $2y - x - 8 = 0$
4. $2x + y - 14 = 0$

1-46. Which of the following is the equation of a line perpendicular to $2x + y - 6 = 0$ and passing through (4,6)?

1. $2x + y - 14 = 0$
2. $2y - x - 8 = 0$
3. $y = x + 8$
4. $y = 2x + 16$

1-47. The formula $d = \left| \frac{x_1A + y_1B + C}{\sqrt{A^2 + B^2}} \right|$ is used to find the

1. normal to a line
2. general equation of a line
3. distance between two points
4. distance from a point to a line

1-48. What is the distance from (-2,0) to the line whose equation is $8x - 3y + 2 = 0$?

1. $\frac{14\sqrt{73}}{73}$
2. $-\frac{14\sqrt{73}}{73}$
3. $\frac{\sqrt{73}}{14}$
4. 14

1-49. What is the distance from (-3,2) to the line whose equation is $x + 2y - 8 = 0$?

1. $\frac{\sqrt{5}}{5}$
2. $\sqrt{5}$
3. $\frac{7\sqrt{5}}{5}$
4. $\frac{7}{5}$

1-50. What is the distance from $(5,\sqrt{7})$ to the line whose equation is $\sqrt{7}y = 3x$?

1. -2
2. 2
3. $5\frac{1}{2}$
4. $\frac{\sqrt{7}}{2}$

Assignment 2

Textbook assignment: Chapter 2, "Conic Sections," pages 2-1 through 2-60 and Chapter 3, "Tangents, Normals, and Slopes of Curves," pages 3-1 through 3-33.

Learning Objective:

Recognize elements of conic sections and analyze equations of conic sections.

2-1. Conic sections are formed by passing planes at varying angles through a right circular cone.

1. True
2. False

2-2. All of the following are examples of conic sections except the

1. ellipse
2. circle
3. sphere
4. parabola

2-3. A conic section is the locus of all points in a plane whose distance from a fixed point is a constant ratio to its distance from a fixed line. This constant ratio is known as the

1. focus
2. directrix
3. coordinate
4. eccentricity

2-4. Which points satisfy the locus of the equation $x^2 + y^2 = 4$?

1. $(\pm 2,0),(0,\pm 2),(\pm\sqrt{2},\pm\sqrt{2})$
2. $(4,0),(0,4),(3,1)$
3. $(\pm 2,0),(0,\pm 2),(2,2)$
4. $(0,0)$

2-5. The distance from a point (x,y) to a point (3,1) is indicated by which of the following expressions?

1. $\sqrt{(x + 3)^2 + (y + 1)^2}$
2. $\sqrt{(x - 3)^2 + (y - 1)^2}$
3. $(x - 1)^2 + (y - 3)^2$
4. $(x + 1)^2 + (y + 3)^2$

2-6. If a point on a circle with its center at the origin has coordinates (4,3), the radius of the circle is

1. $\frac{4}{3}$
2. 3.5
3. 5
4. 7

2-7. The locus of the equation $(x - 3)^2 + (y - 5)^2 = 64$ is a circle. Where is its center located and what is its radius?

1. (5,3); 8 units
2. (3,5); 8 units
3. (3,5); 64 units
4. (5,3); 64 units

2-8. Three noncollinear points can determine the definition of

1. one circle
2. two circles of different diameters
3. three circles of different diameters
4. an infinite number of circles of different diameters

2-9. The equation of the circle passing through points (8,9), (-3,7), and (-4,-5) is

1. $x^2 + y^2 - 23x - 17y - 77 = 0$
2. $11x^2 + 11y^2 - 87x - 18y - 179 = 0$
3. $13x^2 + 13y^2 - 101x - 10y - 987 = 0$
4. $15x^2 + 15y^2 - 120x - 107y - 225 = 0$

2-10. The locus of all points equidistant from two given points is

1. a circle whose radius cannot be determined from the information given
2. a circle with a radius equal to the distance between the two given points
3. a circle with a radius equal to the square root of the distance between the two given points
4. the perpendicular bisector of the line segment connecting the two given points

2-11. A possible explanation for an inconsistent solution in attempting to determine the equation of a circle passing through three points is that the three points

1. are collinear
2. are noncollinear
3. form three equations with three unknowns
4. are not equidistant from each other

2-12. The parabola with its vertex at the origin in the Cartesian coordinate system, with its focus at (0,a) on the Y axis, and with the line $y = -a$ as its directrix has the equation

1. $x^2 = -4ay$
2. $x^2 = 4ay$
3. $y^2 = -4ax$
4. $y^2 = 4ax$

2-13. The point that lies midway between the focus and directrix is known as the

1. vertex of the parabola
2. center of the parabola
3. origin of the parabola
4. midpoint of the parabola

2-14. The equation of the parabola $x^2 = 8y$ opens

1. to the right
2. to the left
3. downward
4. upward

2-15. The width of a parabolic curve at its focus is called the

1. vertex
2. center
3. directrix
4. focal chord

2-16. A parabola described by $y^2 = 16x$ has a horizontal distance between its focus and vertex of

1. 1
2. 4
3. 8
4. 16

2-17. $(y + 3)^2 = 16(x - 5)$ represents a parabola with its vertex at point

1. (-5,3)
2. (5,-3)
3. (-3,5)
4. (3,5)

2-18. Which of the following is true of an ellipse?

1. $e = 0$
2. $e = 1$
3. $e > 1$
4. $0 < e < 1$

2-19. The equation $\frac{x^2}{144} + \frac{y^2}{64} = 1$ describes an ellipse with a directrix parallel to the Y axis. Its semimajor and semiminor axes, respectively, are equal to

1. 6,4
2. 12,8
3. 18,8
4. 24,16

2-20. If an ellipse has its center at the origin, foci of (4,0) and (-4,0), and eccentricity of 0.8, what is the value of a in the formula $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$?

1. ±3.2
2. ±5
3. 25
4. ±25

2-21. If the center of an ellipse is moved from the origin to (-3,-2), the numerators of the fractions on the left side of the general equation for this ellipse are now

1. $(x + 3)^2$, $(y - 2)^2$
2. $(x - 3)^2$, $(y - 2)^2$
3. $(x + 3)^2$, $(y + 2)^2$
4. $(x - 3)^2$, $(y + 2)^2$

2-22. When completing the square of $3(A^2 - 18A)$ on the left side of an equation, the value to be added to the right side of the equation is

1. 36
2. 54
3. 81
4. 243

2-23. Reduce the equation

$$3x^2 + 4y^2 + 6x + 32y + 31 = 0$$

to the standard form of an ellipse. What are the lengths of the semimajor and semiminor axes, respectively?

1. $2\sqrt{3}, 3$
2. $\sqrt{3}, 2$
3. 1,4
4. 3,4

- The diagram in figure 2A refers to a hyperbola of form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

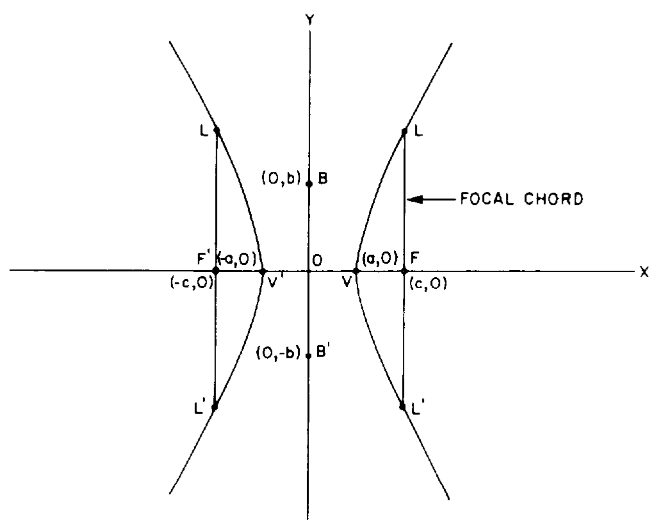

Figure 2A.--Hyperbola.

V and V' are the vertices of the hyperbola. V and V' may also be described as the intersections of the hyperbola with its focal axis. The focal axis, also called the principal axis, is the line segment FF' connecting the two foci of the hyperbola. That portion of the principal axis between V and V', that is, line segment VV', is called the transverse axis of the hyperbola and is equal to 2a.

Line segment BB', where BO = OB' = b, is called the conjugate axis of the hyperbola. Note that the conjugate axis is perpendicular to the transverse axis and its total length is equal to 2b.

The line segment LL' through either focus perpendicular to the principal axis is called the focal chord and is equal to $2b^2/a$.

2-24. The relationship between the foci and directrices of a hyperbola with respect to the origin as the center of symmetry is that

1. each focus is nearer the origin than its corresponding directrix
2. the focal axis is parallel to each directrix at the origin
3. each directrix is coincident with its corresponding focus at the origin
4. each focus is farther away from the origin than its corresponding directrix

2-25. The lines $y = -\frac{b}{a}x$ and $y = \frac{b}{a}x$ used in tracing a hyperbola are called

1. asymptotes
2. diagonals
3. center lines
4. directrices

2-26. $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is described verbally with respect to the rectangular coordinate system as a hyperbola which opens

1. to the left
2. to the right
3. upward and downward
4. to the left and to the right

2-27. A hyperbola, with its center at the origin, has foci at (±5,0) and eccentricity of 1.5. In the standard form, this hyperbola would have a^2 equal to

1. 3 1/3
2. 10
3. 11 1/9
4. 13

2-28. The equation $\frac{x^2}{49} - \frac{y^2}{25} = 1$ is a hyperbola with its center at the origin and foci at

1. $\pm\sqrt{74}$ along the X axis
2. $\pm\sqrt{74}$ along the Y axis
3. ± 9 along the X axis
4. ± 9 along the Y axis

2-29. The focal chord of $\frac{x^2}{25} - \frac{y^2}{9} = 1$ is

1. 3 3/5
2. -21/5
3. 50/3
4. -16 2/3

2-30. What are the asymptotes of $\frac{x^2}{64} - \frac{y^2}{49} = 1$?

1. $y = \pm \frac{7}{8}x$
2. $8x \pm 7y = 0$
3. Both 1 and 2 above
4. $x = \pm \frac{7}{8}y$

2-31. $9x^2 + 4y^2 - 18x + 16y - 11 = 0$ is an equation of

1. a parabola
2. a hyperbola
3. an ellipse
4. a circle

Learning Objective:

Transform Cartesian coordinates to Polar coordinates and vice versa.

2-32. What must be known in order to locate a point in a plane by means of polar coordinates?

1. The horizontal and vertical components of the point
2. The radius vector and its polar angle
3. The radius vector and its chord
4. The polar angle and focal chord

2-33. The origin of a polar coordinate system is also called the

1. apex
2. vertex
3. center
4. pole

2-34. Polar angles are

1. positive if measured clockwise and negative if measured counterclockwise
2. negative if measured clockwise and positive if measured counterclockwise
3. always positive
4. positive for $0^\circ \leq \theta \leq 180^\circ$ and negative for $180^\circ < \theta < 360^\circ$

2-35. The equation $x^2 + y^2 = 2a$ expressed in polar coordinates is

1. $\rho^2 = 2a$
2. $\rho = 2a$
3. $\rho^2(\sin\theta + \cos\theta) = 2a$
4. $\rho(\sin^2\theta \cos^2\theta) = a$

2-36. Cartesian coordinates are the same as polar coordinates.

1. True
2. False

2-37. The equation $\rho = 4 \sin\theta \cot\theta$ is equivalent to

1. $4(x^2 + y^2) = x$
2. $x^2 + y^2 = 4x$
3. $\sqrt{x^2 + y^2} = 4x + \sqrt{x^2 + y^2}$
4. $\left(\frac{x^2}{\sqrt{x^2 + y^2}}\right)\left(\frac{y^2}{\sqrt{x^2 + y^2}}\right) = 4$

Learning Objective:

Calculate the slopes of curves and solve for the equation of the tangent line and normal line to a curve.

2-38. The slope of a curve at a point, P(x,y), can be represented by $\frac{\Delta x}{\Delta y}$.

1. True
2. False

2-39. Which of the following statements is TRUE regarding the error incurred in calculating the slope of a curve at a given point?

1. Calculation error is small if the curvature is great and large increments of Δy and Δx are used
2. Calculation error is large if the curve is nearly flat, regardless of the value of the Δx and Δy increments
3. If the curvature is great, calculation error increases as the lengths of Δy and Δx increase
4. If the curvature is great, calculation error decreases as the lengths of Δy and Δx increase

2-40. In finding the slope of a curve at a given point using sufficiently small increments, you are actually determining the slope of the tangent to the curve at that point.

1. True
2. False

2-41. A tangent to a curve at P(x,y) has a slope equal to 0. Relative to maximum and minimum, the curve at this point may be

1. at a minimum
2. at a maximum
3. at either a maximum or a minimum
4. at both a maximum and minimum

2-42. If the tangent to a curve at point P(x,y) has an infinite slope, this tangent is described as being perpendicular to

1. the curve at point P
2. the X axis
3. the Y axis
4. Δy

2-43. If $\Delta y = \frac{4\Delta x}{2y_1 + \Delta y}$ and y_1 is independent of both Δx and Δy, then as Δx approaches zero Δy approaches a value of

1. zero
2. $\frac{1}{2y_1}$
3. $\frac{4}{2y_1 + \Delta y}$
4. $\frac{4}{2y_1}$

2-44. If the slope of the tangent line to a curve at the point (5,10) is 3/2, the equation of the tangent line is

1. $2x - 3y + 10 = 0$
2. $3x - 2y + 5 = 0$
3. $5x - 10y + 12 = 0$
4. $10x - 5y + 24 = 0$

2-45. In using the general method to find the equation of the tangent to a curve at (x_1,y_1), pick a second point to the curve, $(x_1 + \Delta x, y_1 + \Delta y)$, and substitute these values into the equation of the curve and simplify. When simplifying, the term containing (Δy) to the second power can be eliminated because this term

1. is independent of Δx
2. is eliminated through subtraction of $(\Delta y)^2$
3. approaches zero as Δx approaches zero
4. is eliminated through division by $(\Delta y)^2$

2-46. If $x^2 + y^2 = r^2$ and (x_1,y_1) is a point on the curve, then the expression $r^2 - x_1^2 - y_1^2$ is equivalent to

1. zero
2. $r^2 - (x^2 - y^2)$
3. $r^2 + (x^2 - y^2)$
4. $r^2 - (-x^2 - y^2)$

2-47. The equation of the tangent to the curve $8x^2 - y^2 = 8$ at the point (3,8) is

1. $y = 3x + 5$
2. $y = -3x + 17$
3. $y = 3x - 17$
4. $y = 3x - 1$

2-48. The normal to a curve at a given point is

1. parallel to the tangent line at that point
2. perpendicular to the tangent line at that point
3. perpendicular to the X axis at that point
4. perpendicular to the Y axis at that point

2-49. The slope of a tangent to a curve at a given point is m_p. Therefore, the slope of the normal to the curve at this same point is the

1. inverse of m_p
2. negative of m_p
3. negative reciprocal of m_p
4. same as m_p

2-50. The length of the normal is defined as that portion of the normal line between the point P(x,y) and the

1. X axis
2. Y axis
3. origin
4. point P(x,0)

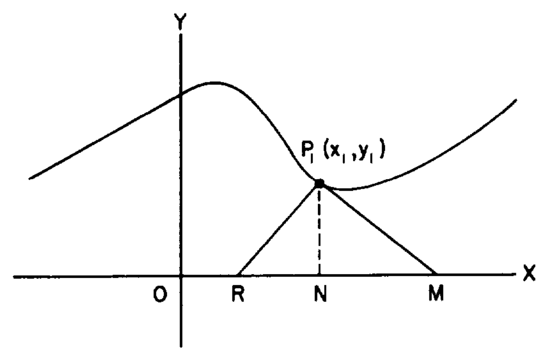

Figure 2B.--Curve with tangent and normal lines.

IN ANSWERING ITEM 2-51, REFER TO FIGURE 2B.

2-51. The lengths of the tangent and normal are represented, respectively, by

1. NM and RN
2. P_1M and P_1N
3. P_1N and P_1R
4. P_1M and P_1R

2-52. If $x^2 + y^2 = 4$, then the length of the tangent at $(1,\sqrt{3})$ is

1. $\sqrt{3}$
2. 2
3. 12
4. $\sqrt{12}$

Learning Objective:

Recognize parametric equations and determine coordinates of a tangent to a curve at a point when parametric equations are involved.

- In answering items 2-53 through 2-56, refer to the formulas $x = \pi r^2$ and $y = 2\pi r$.

2-53. The letter r is known as a

1. base
2. constant
3. parameter
4. parametric equation

2-54. If r is eliminated, the relationship between x and y is

1. $x = \frac{y^2}{2\pi}$
2. $x = \frac{y^2}{4\pi}$
3. $x = \frac{y}{2}$
4. $y^2 = \frac{x}{4\pi}$

2-55. If y = 2, what is the value of x?

1. π^2
2. 4π
3. $\frac{1}{\pi}$
4. $\frac{1}{4\pi^2}$

2-56. If $\frac{\Delta y}{\Delta r} = 2\pi$ and $\frac{\Delta x}{\Delta r} = 2\pi r$, the slope of the tangent to the curve at r = 3 is

1. $\frac{1}{\pi}$
2. $\frac{1}{3}$
3. 3
4. -3

- In answering items 2-57 and 2-58, refer to the following equations:

$x = 2 \sin \theta$, $y = 4 \cos \theta$
$\frac{\Delta x}{\Delta \theta} = 2 \cos \theta$, $\frac{\Delta y}{\Delta \theta} = -4 \sin \theta$

2-57. The general expression for the slope $\frac{\Delta y}{\Delta x}$ of the curve at (x_1,y_1) is

1. $-2 \tan \theta$
2. $-\frac{\tan \theta}{2}$
3. $-\frac{\cot \theta}{2}$
4. $2 \cot \theta$

2-58. At the point where $\theta = 30^\circ$ the equation of the normal to the curve is

1. $2x - y\sqrt{3} + 5 = 0$
2. $2y - x\sqrt{3} - 3\sqrt{3} = 0$
3. $6y - 3x - 9 = 0$
4. $6x - 3y + 8 = 0$

2-59. When using parametric equations, we find the vertical tangent to a curve by setting

1. $\frac{\Delta x}{\Delta t} = 0$
2. $\frac{\Delta y}{\Delta t} = 0$
3. $\frac{\Delta x}{\Delta y} = 0$
4. $\frac{\Delta y}{\Delta x} = 0$

2-60. If $x = t^2$, $y = 3t$, $\frac{\Delta x}{\Delta t} = 2t$, and $\frac{\Delta y}{\Delta t} = 3$, at what coordinates on the curve is the tangent parallel to the Y axis?

1. (4,6)
2. (1,1)
3. (0,1)
4. (0,0)

Assignment 3

Textbook assignment: Chapter 4, "Limits and Differentiation," pages 4-1 through 4-34 and Chapter 5, "Derivatives," pages 5-1 through 5-38.

A thorough understanding of functions and functional notation is necessary at this point. In mathematics a variable, y, is said to be a function of another variable, x. It is not enough to say that a change in x causes a corresponding change in y, because this is not always true; for example, when y equals a constant. To allow for certain exceptions, we have adopted the following definition: If for each value of variable x there is only one corresponding value y, then y is said to be a function of x. Consider the equation $y = 3$. This is a straight line parallel to the X axis and intersecting the Y axis at 3. No matter what value is assigned to x, there is a corresponding value for y; in this example 3.

Functional notation is used as a brief and convenient way of symbolizing that variable y is a function of variable x and is written $y = f(x)$ where y is the dependent variable and x is the independent variable. This is read, "y equals f of x." This means y is some function of x; that is, y has a value for each value of x. This symbolism does not mean that some quantity f is to be multiplied by some quantity x. It also does not indicate what the function of x is. It could be $1 - x^2$, $3 \sin x$, $x \log_{10} x$, or any function involving x as the independent variable.

If more than one function of x must be considered in a single discussion, a symbol other than f is used, such as: $g(x)$, $q(x)$, or $h(x)$. For example, if it is given that y is a function of $x \cos x$, $0 \leq x < 5$, and $x^2 + 2$, $5 \leq x \leq 9$, this could be stated symbolically as

$$y = \begin{cases} f(x), & 0 \leq x < 5 \\ g(x), & 5 \leq x \leq 9 \end{cases}$$

where $f(x) = x \cos x$ and $g(x) = x^2 + 2$.

When using functional notation, we should remember that throughout a single problem or discussion, a given functional symbol such as $f(x)$ will always represent the same function, and a symbol such as $f(a)$ means that the function $f(x)$ is to be evaluated at $x = a$. Stated in symbols, $f(a)$ represents the value of $f(x)$ at $x = a$.

As an example, find $f(2)$ when

$$y = f(x) = x^2 - x + 2$$

To find $f(2)$, we merely substitute a 2 every place an x occurs in $f(x)$.

$$f(2) = 2^2 - 2 + 2 = 4$$

Likewise, $f(x)$ may be evaluated for any constant value assigned to x and symbolized in functional notation by placing in parenthesis the value x will have for that particular problem. In the above example find $f(-2)$, $f(3)$, and $f(a)$. Hence,

$$f(-2) = (-2)^2 - (-2) + 2 = 8$$

$$f(3) = 3^2 - 3 + 2 = 8$$

$$f(a) = a^2 - a + 2$$

Learning Objective:

Define "limit" and recognize techniques useful in evaluating a limit.

- In answering items 3-1 and 3-2, refer to the curve $y = x^2$.

3-1. y = 16 on the graph is considered to be the limit of

1. L as x is allowed to approach the value 4
2. f(x) as y is allowed to approach the value 4
3. f(x) as x is allowed to approach the value 4
4. $y = x^2$ as x is allowed to approach the value 16

3-2. If the curve is defined at x equals 4, it can then be said that the

1. limit of x = 4
2. $\lim\limits_{x \to 4} f(x) = f(4)$
3. limit of $x^2 = 16$
4. limit f(4) = 4

3-3. Which interval would NOT contain $\lim\limits_{x \to 5} x^2$?

1. 1 - 81
2. 16 - 36
3. 20.25 - 30.25
4. 4 - 6

3-4. Which statement below explains why L is the limit of f(x) as x approaches a?

1. The value of f(a) equals L
2. The function f(x), when evaluated at x equal to a, is such that f(x) - f(a) is equal to zero
3. The absolute value of the sum of f(x) and L is made larger than any test number δ by choosing an appropriate value for x very near, but not equal to, ε
4. The absolute value of the difference between f(x) and L can be made smaller than any other positive number named by choosing an appropriate value for x very near, but not equal to, a

3-5. If $\lim\limits_{x \to a} f(x) = L$, then f(a) exists.

1. True
2. False

- For all the functions given in items 3-6 through 3-8, the $\lim\limits_{x \to a} f(x)$ may be found by substituting a for x; that is, by evaluating f(a).

3-6. The limit of $3x^2 - 5 - x$ as x approaches -3 is

1. 10
2. 18
3. 25
4. 35

3-7. $\lim\limits_{x \to 2} (\sqrt{2x^2 - 2x} + 3x)$ is

1. 12
2. 8
3. $2(\sqrt{3} + 3)$
4. 4

3-8. Find $\lim\limits_{z \to a} f(z)$ when f(z) equals $\dfrac{3z^2 - z}{z - 1}$ and a equals 0.

1. 1
2. -1
3. 0
4. 5

- In items 3-9 through 3-12, find the limit of the functions and, where necessary, change the function by division or factoring so that an indeterminate form does not occur.

3-9. $\lim\limits_{t \to 2} 16t^2 + 20t - 30$ is

1. 0
2. 6
3. 36
4. 74

3-10. $\lim_{x\to 0} \frac{x^2 - 1}{x - 1}$ is

1. 1
2. 2
3. 0
4. $x + 1$

3-11. $\lim_{x\to\infty} \frac{x^4 - 2x^2 + 3}{2x^4 + x^3 - 3x^2 + x}$ is

1. 0
2. ∞
3. 3
4. $\frac{1}{2}$

3-12. $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$ is

1. 1
2. 2
3. 0
4. $x + 1$

3-13. Which of the following statements correctly describes the outcome of $\frac{0}{0}$ produced when you attempt to determine $\lim_{x\to a} f(x)$ by evaluating $f(a)$?

1. The limit is zero
2. The limit, if it exists, cannot be determined by this method because the outcome is in an indeterminate form
3. The limit is one
4. The limit is infinity

3-14. What is $\lim_{x\to 0} \frac{\sin x + x}{2}$, where x is in radians?

1. +1
2. $\frac{1}{2}$
3. -1
4. 0

● Items 3-15 through 3-19 relate to the limit theorems.

In answering items 3-15 through 3-18, select from column B the word statement of each of the theorems in column A.

A. THEOREMS

3-15. $\lim_{x\to a} f(x)\, g(x) = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]$

3-16. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

3-17. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

3-18. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$

B. STATEMENTS

1. The limit of the quotient of two functions is equal to the quotient of their limits, provided the limit of the divisor is not equal to zero
2. The limit of the product of two functions is equal to the product of their limits
3. The limit of the sum of two functions is equal to the sum of the limits
4. The limit of a constant, c, times a function, f(x), is equal to the constant, c, times the limit of the function

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3-19. Using theorems 2 and 3 given in your text, find the limit of

$3(x - 2)(x^2 + 4)$ as x approaches 5.

1. 38
2. 81
3. 261
4. 340

Learning Objective:

Define and apply the concept of infinitesimals.

3-20. An infinitesimal results when a variable

1. approaches 0 as a limit
2. equals its limit
3. approaches ∞ as a limit
4. can be assigned a definite value

3-21. A variable approaches a constant as a limit if the difference between the variable and the constant becomes an infinitesimal.

1. True
2. False

● In items 3-22 through 3-26, mark your answer sheet True if the function becomes an infinitesimal in approaching its limit and False if it does not.

3-22. $\lim_{x \to 3} \frac{x^2 + 5x - 6}{x + 6} - 2$

3-23. $\lim_{x \to 4} (x^2 - 4x + 3)$

3-24. $\lim_{x \to 2} (5x - x^3)$

3-25. $\lim_{x \to -4} \frac{x^2 + 8x + 16}{x + 4}$

3-26. $\lim_{x \to 0} \frac{x^4 + 2x^2 - 1}{x^2 - 3x + 2}$

3-27. Which statement concerning infinitesimals is FALSE?

1. The product of a constant and an infinitesimal is always an infinitesimal
2. The product of two or more infinitesimals is always an infinitesimal
3. The sum of two or more infinitesimals is always an infinitesimal
4. The ratio of two infinitesimals is always an infinitesimal

● A function is discontinuous at x equals a if the evaluation of f(a) causes division by zero. Since a point has no physical dimensions, the graph of a curve with only one missing point would appear to be a continuous curve, but mathematically it is considered to be discontinuous. The graph of the function $\frac{x^2 - 4}{x - 2}$ is discontinuous at x equals 2 because division by 0 occurs. This point is represented by a small circle drawn on the curve at x equals 2. Even though the curve does not exist at x equals 2, no gap would be left that could be physically seen. The small circle would actually contain an infinite number of points and is therefore only symbolic of the discontinuity.

Other forms of discontinuity can be graphically shown. The function, y = tan x, is continuous in the interval $-\frac{\pi}{2} < \tan x < \frac{\pi}{2}$, or $\frac{\pi}{2} < \tan x < \frac{3\pi}{2}$, but is discontinuous at these limits. This curve continues to repeat itself at π radian intervals and is discontinuous at $\pi/2 + n\pi$, where n is any integer.

Generally speaking, a function is continuous in a given interval if the curve is unbroken; that is, if it can be drawn in that interval without lifting the pen or pencil from the paper.

Learning Objective:

Define and recognize continuous functions.

3-28. At which one of the following values of x is the function $\frac{2}{x + 2} + \frac{4}{x^2 + x - 12}$ continuous?

1. -4
2. -2
3. 3
4. 4

3-29. Which of the following functions is defined at x equals 2?

1. $\frac{x}{x - 2}$
2. $\frac{\sqrt{13} - x}{x^3 - 8}$
3. $\frac{x^3 + 3x - 5}{x + 2}$
4. $\frac{1}{x^2 - 6x + 8}$

3-30. Which of the following functions is continuous at x equals c?

1. $\frac{a}{cx - c^2}$, $c \neq 0$
2. $\frac{2c - cx}{x - c}$, $c \neq 0$
3. $\frac{x^2 - c^2}{x + c}$, $c \neq 0$
4. $\frac{x^2 + c^2}{x^2 - c^2}$, $c \neq 0$

3-31. Which of the following functions of x is continuous for all values of x?

1. $h(x) = \frac{x^2 - 4}{x - 2}$
2. $g(x) = \begin{cases} \frac{x^2 - x - 6}{x - 3}, & x \neq 3 \\ 5, & x = 3 \end{cases}$
3. $q(x) = \begin{cases} \frac{x^2 - 3x - 4}{x + 1}, & x \neq 4 \\ 0, & x = 4 \end{cases}$
4. $f(x) = \begin{cases} \frac{x^3 - 8}{x - 2}, & x \neq -2 \\ 4, & x = -2 \end{cases}$

Learning Objective:

Use the limit concept in differentiating functions.

3-32. In the function y = f(x), the limit of the ratio of Δy to Δx as Δx approaches zero is called the

1. integral of Δy
2. derivative of Δx
3. differential of y
4. derivative of f(x)

● Remember, the limit of an increment as it approaches zero is an infinitesimal and is considered to be zero. This does not, however, apply to the ratio of two infinitesimals. As pointed out in your text, the ratio may have any numerical value, including zero. Any term that includes an infinitesimal as a factor can be dropped because the whole term is an infinitesimal, except when two infinitesimals occur as a ratio. Note in the example below that when the limit is taken, terms containing increments that become infinitesimals drop out.

$$\lim_{\Delta x \to 0} (4x - 3x\Delta x + 1000\Delta x - \Delta x^3) = 4x$$

Here is an additional example using the four-step delta method given in your text. The derivative of

$y = 2x^2 - x + 8$ is found as follows:

If x were to increase by an increment Δx, the corresponding increase in y would be Δy. This is illustrated in the next equation where $y + \Delta y$ replaces the y and $x + \Delta x$ replaces all x's in the original equation.

$$y + \Delta y = 2(x + \Delta x)^2 - (x + \Delta x) + 8$$

Performing the indicated expansion of the squared binominal gives

$$y + \Delta y = 2(x^2 + 2x\Delta x + \Delta x^2) - (x + \Delta x) + 8$$

$$y + \Delta y = 2x^2 + 4x\Delta x + 2\Delta x^2 - x - \Delta x + 8$$

Now subtracting the original equation gives

$$\begin{array}{l} y + \Delta y = 2x^2 + 4x\Delta x + 2\Delta x^2 - x - \Delta x + 8 \\ y \qquad\quad = 2x^2 \qquad\qquad\qquad\quad - x \qquad + 8 \\ \hline \Delta y = \qquad + 4x\Delta x + 2\Delta x^2 \qquad - \Delta x \end{array}$$

This step is allowable because equals are subtracted from equals. Next divide both sides of the equation by Δx.

$$\frac{\Delta y}{\Delta x} = \frac{4x\Delta x}{\Delta x} + \frac{2\Delta x^2}{\Delta x} - \frac{\Delta x}{\Delta x}$$

which reduces to

$$\frac{\Delta y}{\Delta x} = 4x + 2\Delta x - 1$$

Take the limit of both sides of the equation.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4x + 2\Delta x - 1)$$

The left-hand member is symbolized $\frac{dy}{dx}$ and the right-hand member becomes $4x - 1$; thus $\frac{dy}{dx} = 4x - 1$.

● In items 3-33 through 3-35, find the derivative of the given function using the four-step delta method given in your text.

3-33. The derivative of $y = 3x^2 + 100$ is

1. $6x$
2. $6x^2$
3. $5x + 10$
4. $3x^2 + 6x\Delta x + 100$

3-34. The derivative of $y = 2x^3$ is

1. $2x^6$
2. $5x^3$
3. $6x^2$
4. $6x^2 + 6x$

3-35. The derivative of $y = \frac{1}{x^2}$ is

1. $2x$
2. $\frac{1}{2x}$
3. $-\frac{2}{x^3}$
4. $-\frac{1}{x} + 2$

● In answering items 3-36 and 3-37, find the slope of the tangent line to the given curve at the point indicated.

3-36. The slope of $y = 5x^2$ at $x = 4$ is

1. 5
2. 15
3. 30
4. 40

3-37. The slope of $y = x^2 - 4x + 4$ at $x = 1$ is

1. -2
2. 2
3. 0
4. 4

● Consider the graph of the curve $y = 5x^3 - 6x^2 - 3x + 3$.

The maximum or minimum values for this curve obviously are undefined since the curve grows without bounds. But in the interval between x equals -1 and x equals 2, the curve reaches a maximum at x equals $-\frac{1}{5}$ and a minimum at x equals 1.

It is these points which have physical significance and about which the text discussion is centered. These significant maxima and minima are usually referred to as local maxima and mimima to distinguish them from other kinds.

The text has defined the derivative to be equal to the slope of the tangent line on the curve at any point x in the interval under discussion. When the tangent line to the curve is parallel to the X axis, the value of the slope is zero. The tangent line can be parallel to the X axis (have a slope of zero) at only two points. These points are at x equals $-\frac{1}{5}$, where the curve is maximum, and at x equals 1, where the curve is minimum.

Because the derivative is equal to the slope of the tangent to the curve, maxima and minima can easily be found by determining the derivative, setting the derivative equal to zero, and determining the values of the independent variable that will make the derivative equal to zero. Since,

$$\frac{dy}{dx} = 15x^2 - 12x - 3$$

and

$$15x^2 - 12x - 3 = 0$$

then

$$3(5x^2 - 4x - 1) = 0$$

or

$$(5x^2 - 4x - 1) = 0$$

By factoring,

$$(5x + 1)(x - 1) = 0$$

such that

$$x = -\frac{1}{5} \text{ and } 1$$

The coordinates on the curve where a maximum or a minimum occurs can be found by substituting the determined values of x back into the original equation of the curve to find corresponding values of y. Thus when x = -1/5, then

$$y = 5\left(-\frac{1}{5}\right)^3 - 6\left(-\frac{1}{5}\right)^2 - 3\left(-\frac{1}{5}\right) + 3$$

$$= 3\frac{8}{25}$$

Therefore, the coordinates of the maxima are (-1/5,3 8/25). When x = 1

$$y = 5(1)^3 - 6(1)^2 - 3(1) + 3$$

$$= -1$$

Therefore, the coordinates of the minima are (1,-1).

● In answering items 3-38 and 3-39, determine the values of the independent variable for all maxima and minima of the given function.

3-38. There exists either a maximum or a minimum point on the curve

$y = 2x^2 - 8x$ at x equals

1. 1
2. 2
3. 8
4. 4

3-39. On the curve

$$s = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 12t + 4,$$

where $-7 \leq t \leq 6$, there are either maximum or minimum points at t equals

1. 2 and -5
2. 3 and -4
3. 3 and 6
4. 3 and -2

Learning Objective:

Differentiate explicit and implicit functions through the use of rules and formulas.

● The formulas on the following page may be used for reference. In these formulas a, n, and C always represent a constant and u and v always represent a function of x.

Table 3A.--Formulas Involving Common Derivative Forms

1. $\frac{d}{dx}(a) = 0$

2. $\frac{d}{dx}(x) = 1$

3. $\frac{d}{dx}(x^n) = nx^{n-1}$

4. $\frac{d}{dx}(ax^n) = anx^{n-1}$

5. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$

6. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

7. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

8. $\frac{d}{dx}(au^n) = anu^{n-1}\frac{du}{dx}$

9. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ (chain rule)

10. $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

11. $\frac{d}{dx}(\sin u) = \cos u \frac{du}{dx}$

12. $\frac{d}{dx}(\cos u) = -\sin u \frac{du}{dx}$

13. $\frac{d}{dx}(\tan u) = \sec^2 u \frac{du}{dx}$

14. $\frac{d}{dx}(\cot u) = -\csc^2 u \frac{du}{dx}$

15. $\frac{d}{dx}(\sec u) = \sec u \tan u \frac{du}{dx}$

16. $\frac{d}{dx}(\csc u) = -\csc u \cot u \frac{du}{dx}$

17. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

18. $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

19. $\frac{d}{dx}(e^x) = e^x$

20. $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

21. $\frac{d}{dx}(a^x) = (\ln a)a^x$

22. $\frac{d}{dx}(a^u) = (\ln a)a^u\frac{du}{dx}$

3-40. The derivative of a constant is

1. one
2. zero
3. an infinitesimal
4. the constant

3-41. Given the function $y = 12$, as you move from point x to $x + \Delta x$, what happens to the value of $y + \Delta y$?

1. It increases
2. It decreases
3. It varies in proportion to Δx
4. It remains the same

3-42. The derivative of $f(x) = x^4$ is

1. $\frac{x^3}{4}$
2. x^3
3. $4x^2$
4. $4x^3$

3-43. If $f(x) = x^n$ and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x},$$

then as Δx approaches zero, $f'(x)$ equals

1. $\frac{x^{n-1}}{0}$
2. $\frac{x^{n-1}}{\Delta x}$
3. nx^{n-1}
4. $\frac{x + \Delta x}{\Delta x}$

3-44. $\frac{dy}{dx}$ of $y = kx^2$ is

1. $2x$
2. $2kx$
3. kx
4. $\frac{k}{x}$

3-45. The derivative of the sum of two or more differentiable functions of x is equal to the

1. product of their derivatives
2. product of their limits
3. sum of their limits
4. sum of their derivatives

3-46. Find $\frac{dy}{dx}$ of $y = tx^4 + sx^3 + x^2 + 6$.

1. $4x^3 + 3x^2 + 2x + 1$
2. $4tx^3 + 3sx^2 + 2x$
3. $4tx^3 + 3sc^2 + 2x + 1$
4. $4x^3 + 3x^2 + 2x + 6$

● In answering items 3-47 and 3-48, refer to the following information:

The relationship between the time, t, of flight in seconds and the altitude, a, in feet of a projectile is given approximately by the formula $a = vt - 16t^2$, where v is the muzzle velocity of the projectile. Assuming that a gun having a muzzle velocity of 384 feet per second is fired, the formula for the projectile's altitude becomes $a = 384t - 16t^2$.

3-47. How long after the gun is fired will the altitude be maximum? (Hint: Solve for $\frac{da}{dt}$ and use the procedure for determining a maximum value.)

1. 14 seconds
2. 12 seconds
3. 10 seconds
4. 8 seconds

3-48. What will be the maximum altitude reached by the projectile?

1. 1,152 feet
2. 2,304 feet
3. 3,456 feet
4. 4,608 feet

3-49. The derivative of the product of two differentiable functions of x equals the product of the derivatives of the functions.

1. True
2. False

● In answering items 3-50 through 3-52, employ the theorem for finding the derivative of a product.

3-50. Find $\frac{dy}{dx}$ of $y = x^2(3x)$.

1. $6x^2$
2. $6x^2 + 3x$
3. $3x^2 + 6x$
4. $9x^2$

3-51. If $f(x) = (x^2 + 3x)(x^3 + 4)$, what is $\frac{dy}{dx}$?

1. $6x^3 + 9x^2$
2. $5x^4 + 12x^3 + 8x + 12$
3. $(x^2 + 3x)(3x^2 + 1) + (x^3 + 4)(2x)$
4. $(x^2 + 3x)(3x^2 + 1) + (x^3 + 4)(2x + 3)$

3-52. $\frac{dy}{dx}$ of $3x^2(2x + 4)(tx^3)$ is

1. $(2x + 4)(tx^3)(6x) + (3x^2)(tx^3)(6) + (3x^2)(2x + 4)(3tx^2)$
2. $(3x^2)(2x + 4)(3x^2) + (2x + 4)(3tx^2)(2) + (3x^2)(tx^3)(6x)$
3. $(3x^2)(2x + 4)(3tx^2) + (2x + 4)(tx^3)(2) + (3x^2)(tx^3)(2)$
4. $(3x^2)(2x + 4)(3tx^2) + (2x + 4)(tx^3)(6x) + (3x^2)(tx^3)(2)$

3-53. If $w = g(x)$ and $z = h(x)$, then the derivative of $\frac{z}{w}$ equals

1. $\dfrac{wg'(x) - zh'(x)}{z^2}$
2. $\dfrac{wh'(x) - zg'(x)}{w^2}$
3. $\dfrac{zg'(x) - wh'(x)}{z^2}$
4. $\dfrac{zh'(x) - wg'(x)}{w^2}$

3-54. The derivative of

$f(x) = \dfrac{x^2 - 3x + 3}{x^3 + 2}$ is

1. $\dfrac{-x^4 + 6x^3 + 9x^2 + 6x + 3}{(x^2 + 2)^2}$
2. $\dfrac{x^4 + 6x^3 + 9x^2 + 6x - 3}{(x^3 + 2)^2}$
3. $\dfrac{-x^4 + 6x^3 - 9x^2 + 4x - 6}{(x^3 + 2)^2}$
4. $\dfrac{x^4 + 9x^3 - 9x^2 + 4x - 6}{(x^3 + 2)^2}$

3-55. If $y = [u(x)]^t$, then $\frac{dy}{dx}$ equals

1. $t[u(x)]^{t-1}\ [u'(x)]$
2. $t[u(x)]^{t-1}\ \dfrac{u'(x)}{t'(x)}$
3. $t[u(x)]^{t-1}$
4. $\dfrac{tu'(x)}{[u(x)]^{t-1}}$

3-56. What is the derivative of $g(x) = (x^4 + 3x^3 + 2x)^5$?

1. $5(x^4 + 3x^3 + 2x)(4x^3 + 9x + 2)$
2. $5(x^4 + 3x^3 + 2x)(4x^3 + 9x^2 + 2)$
3. $5(x^4 + 3x^3 + 2x)^4(4x^3 + 9x + 2)$
4. $5(x^4 + 3x^3 + 2x)^4(4x^3 + 9x^2 + 2)$

3-57. Express the function

$f(x) = \dfrac{1}{\sqrt{(x + 3)^3}}$

using fractional exponents.

1. $(x + 3)^{-3/2}$
2. $(x + 3)^{3/2}$
3. $\dfrac{1}{(x + 3)^{-1/2}}$
4. $\dfrac{1}{(x + 3)^{-3/2}}$

3-58. $\frac{dy}{dx}$ of $f(x) = \dfrac{4x - 6}{\sqrt{x^2 + 3x + 2}}$ equals

1. $(4x^2 - 9)\sqrt{x^2 + 3x + 2} + 4\sqrt{x^2 + 3x + 2}$
2. $\dfrac{9 - 4x^2}{\sqrt{x^2 + 3x + 2}} + \dfrac{4}{\sqrt{x^2 + 3x + 2}}$
3. $\dfrac{4}{\sqrt{x^2 + 3x + 2}} - \dfrac{4x^2 - 9}{\sqrt{(x^2 + 3x + 2)^3}}$
4. $\dfrac{4x^2 - 9}{\sqrt{x^2 + 3x + 2}} + 4\sqrt{x^2 + 3x + 2}$

3-59. If $y = (z + 3)^2$ and $z = x + 3$, then according to the chain rule, $\frac{dy}{dx}$ equals

1. $\frac{dy}{du}\frac{du}{dx}$
2. $\frac{dy}{dx}\frac{dx}{dz}$
3. $\frac{dy}{dz}\frac{dz}{dx}$
4. $\frac{dz}{dx}\frac{dy}{dx}$

3-60. If $y = (t^2 + 1)^3$ and $t = (x^2 + 5)$, then $\frac{dy}{dx}$ equals

1. $12x(t^2 + 1)^2$
2. $12(x^2 + 5)[(x^2 + 5)^2 + 1]^2$
3. $3[(x^2 + 5)^2 + 1]^2$
4. $12x(x^2 + 5)[(x^2 + 5)^2 + 1]^2$

3-61. The derivative of an inverse function is equal to

1. the negative reciprocal of the derivative of the direct function
2. the reciprocal of the derivative of the direct function
3. the derivative of the direct function
4. the negative of the derivative of the direct function

3-62. If $x = \frac{5}{y} + y^2$, then $\frac{dy}{dx}$ equals

1. $y^2(2y^3 - 5y)^{-1}$
2. $y^2(2u^3 - 5)^{-2}$
3. $\frac{y^2}{2y^3 - 5}$
4. $\frac{y^2}{2y^3 - 5y}$

3-63. Which of the following statements is TRUE regarding the expression $ax^2 + xy + y^2 = 0$?

1. x is an explicit function of y
2. y is independent of x
3. y is an explicit function of x
4. y is an implicit function of x

3-64. The derivative $\frac{dy}{dx}$ of the function $x^2 + xy^2 + y = 0$ is

1. $\frac{-2x - y^2}{1 + 2x}$
2. $\frac{-2x - y^2}{1 + 2xy}$
3. $\frac{2x + y^2}{1 + 2x}$
4. $\frac{2x + y^2}{1 + 2xy}$

● The expression $\frac{d}{dx}(x^2)$ means to take the derivative of the quantity within the parentheses with respect to x; in this case $2x \frac{dx}{dx}$ or simply 2x because $\frac{dx}{dx}$ is 1.

Another example would be $\frac{d}{dz}(x^2 + 2z + 5)$, which means to take the derivative of the term $x^2 + 2z + 5$ with respect to z, or $2x \frac{dx}{dz} + 2$. Since the ratio $\frac{dx}{dz}$ is unknown, it is necessary to retain it as a factor.

3-65. Find $\frac{d}{dx}(\cos\theta)$.

1. $\sin\theta \frac{d\theta}{dx}$
2. $-\sin\theta \frac{d\theta}{dx}$
3. $-\cos\theta \frac{d\theta}{dx}$
4. $\cos\theta \frac{d\theta}{dx}$

● When differentiating trigonometric functions, it is important to remember to find the derivative of the angle in addition to the derivative of the function. The following example illustrates this point:

$\frac{dy}{d\theta}$ of $y = \cos 4\theta$ equals

$$\begin{aligned} \frac{d}{d\theta}(\cos 4\theta) &= -\sin 4\theta \frac{d}{d\theta}(4\theta) \\ &= -\sin 4\theta\ (4) \frac{d\theta}{d\theta} \\ &= -4 \sin 4\theta \end{aligned}$$

When using trigonometric functions raised to powers, you first apply the power rule for differentiating, then you apply the rules for differentiating the function, and finally you differentiate the angle. (PFA--power, function, angle --in that order.)

For example, if $y = (\tan 4\theta)^2$, then you determine $\frac{dy}{d\theta}$ by first applying the power rule,

$$\frac{dy}{d\theta} = 2(\tan 4\theta)\ \frac{d}{d\theta}(\tan 4\theta)$$

Next you apply the rule for differentiating $\tan 4\theta$,

$$\frac{dy}{d\theta} = 2(\tan 4\theta)(\sec^2 4\theta)\ \frac{d}{d\theta}(4\theta)$$

Finally, you differentiate the angle 4θ,

$$\frac{dy}{d\theta} = 2(\tan 4\theta)(\sec^2 4\theta)(4)$$

or

$$\frac{dy}{d\theta} = 8 \tan 4\theta \sec^2 4\theta$$

3-66. $\frac{d}{dx}\left(\frac{\sin u}{\cos u}\right)$ is equivalent to

1. $\frac{1}{\sec^2 u}\ \frac{du}{dx}$

2. $\frac{(\cos^2 u + \sin^2 u)}{\sec^2 u}\ \frac{du}{dx}$

3. $\frac{\cos^2 u \frac{du}{dx} - \sin^2 u \frac{du}{dx}}{\cos^2 u}$

4. $\frac{(\cos u)(\cos u)\ \frac{du}{dx}}{\cos^2 u} - \frac{(\sin u)(-\sin u)\ \frac{du}{dx}}{\cos^2 u}$

3-67. The function $y = \sin 2x \cos x$ has a derivative of

1. $-\sin^2 x + 2 \cos^2 x$
2. $-\sin 2x \sin x + \cos x \cos 2x$
3. $-\sin 2x \sin x + 2 \cos x \cos 2x$
4. $\sin 2x \sin x + 2 \cos x \cos 2x$

3-68. If $y = (\tan 2\theta)^2(\sin 2\theta)$, then $\frac{dy}{d\theta}$ equals

1. $(\tan^2 2\theta)(\cos 2\theta)$
   $+ (\sin 2\theta)(2)(\tan 2\theta)(2)(\sec^2 2\theta)$
2. $(\tan^2 2\theta)(2)(\cos 2\theta)$
   $+ (\sin 2\theta)(2 \tan 2\theta)(\sec^2 2\theta)$
3. $(\tan 2\theta)^2(\cos 2\theta)$
   $+ (\sin 2\theta)(2)(\tan 2\theta)(\sec^2 2\theta)$
4. $(\tan^2 2\theta)(\cos 2\theta)(2)$
   $+ (\sin 2\theta)(2)(\tan 2\theta)(\sec^2 2\theta)(2)$

3-69. Find $\frac{dy}{dx}$ of $y = e^{2x} + 1$.

1. $2e^{2x} + 1$
2. $e^{2x} + 1$
3. $2e^{2x}$
4. $2xe^{2x}$

3-70. Find $\frac{dy}{dx}$ of $y = 8^{4x} + \ln (x^4)$.

1. $8^{4x} + \frac{1}{x^4}$
2. $4(\ln 8)8^{4x} + \frac{4}{x}$
3. $(\ln 8)8^{4x} + \frac{4}{x}$
4. $4(\ln 8)8^{4x} + 4x^3$

Assignment 4

Textbook assignment: Chapter 6, "Integration," pages 6-1 through 6-28 and Chapter 7, "Integration Formulas," pages 7-1 through 7-37.

Learning Objective:

Determine areas under curves through integration and apply integration to work problems.

4-1. Which of the following statements regarding integration is FALSE?

1. It is the inverse of addition
2. It is the direct opposite of differentiation
3. $F(x) = \int f(x)\, dx$
4. The derivative of a function is given and the function must be found

4-2. In the integral $\int f(x)\, dx$, dx is known as the

1. limit of integration
2. integral sign
3. differential
4. integrand

4-3. If $y = \int x\, dx$, then x is known as the

1. integrand
2. differential
3. integral sign
4. limit of integration

4-4. An integral is used only to represent an area under a curve.

1. True
2. False

● Another way of viewing the expression

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

is to substitute the value of the x coordinate into the function $f(x_k)$ to find the corresponding y coordinate. Hence, for k = 1 to n, $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$, ..., $y_n = f(x_n)$.

Therefore, the sum of the areas of the rectangles with width Δx and heights y_0, y_1, y_2, ..., y_n may be written as

$$A = \lim_{n\to\infty} (y_0\Delta x + y_1\Delta x + y_2\Delta x + \ldots + y_n\Delta x)$$

The limit of this sum of products as $n\to\infty$ equals the integral from a to b of y dx, or

$$\int_a^b y\, dx$$

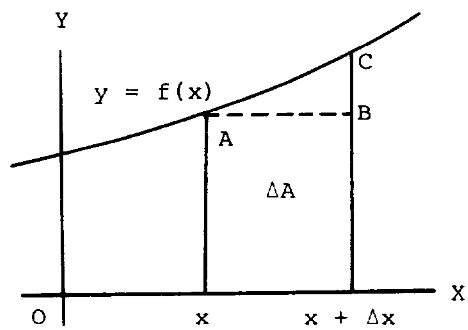

Figure 4A.--Area ΔA.

IN ANSWERING ITEM 4-5, REFER TO FIGURE 4A.

4-5. Under what condition can ΔA closely approximate the area under the curve y = f(x) between points x and x + Δx?

1. When Δx is extremely small
2. When f(x) equals f(x + Δx)
3. When point B moves to the right
4. When Δx increase without limit

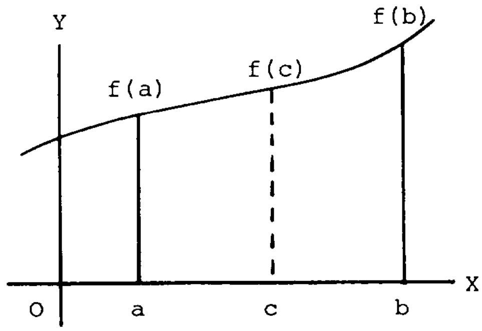

Figure 4B.--Intermediate value between a and b.

IN ANSWERING ITEM 4-6, REFER TO FIGURE 4B.

4-6. The Intermediate Value Theorem actually measures the area of a rectangle with width b - a and intermediate height

1. $\frac{f(b) - f(a)}{2}$
2. $f(b) - f(a)$
3. $f(a) + f(b)$
4. $f(c)$

● When calculating the area under a curve, you must ensure the curve, for example $y = f(x)$, is continuous at all points over which the area is being computed. If the curve is not continuous, the area cannot be calculated by direct methods.

4-7. If ${}_cA_x = F(x) - F(c)$ and $x = n$, then the area between points c and n on a curve equals

1. $\frac{F(n) + F(c)}{2}$
2. $F(n) - F(c)$
3. $F(n) + F(c)$
4. $\frac{F(n) - F(c)}{2}$

4-8. Which of the following is equivalent to $\frac{d}{dx} \int g(x)\ dx$?

1. $\frac{d}{dx} g(x)$
2. $g(x)$
3. $\frac{d}{dx} G(x)$
4. Both 2 and 3 above

4-9. $\frac{d}{dx}(x^4)$, $\frac{d}{dx}(x^4 + 5)$, and $\frac{d}{dx}(x^4 - 8)$ are equal to

1. $4x^3$
2. $4x^3$, $4x^3 + 5$, $4x^3 + 8$
3. $4x^3$, $4x^3 + 5$, $4x^3 - 8$
4. $4x^3$, $4x^3 - 5$, $4x^3 + 8$

4-10. All integrals of a given function have the same constant of integration.

1. True
2. False

4-11. If $\int f(x)\ dx = F(x) + C$, which statement concerning C is TRUE?

1. It has only one value
2. It has only positive values
3. It has only negative values
4. It has an infinite number of values

4-12. The statement "The integral of a differential of a function is the function plus a constant" is illustrated by

1. $\int 2x\ dx = x^2 + C$
2. $\int dy = y + C$
3. $\int ax\ dx = ax^2 + C$
4. $\int a\ dx = ax + C$

4-13. Which expression is TRUE if g is a constant and y is a variable?

1. $\int g\ dy = \int g \int dy$
2. $\int g\ dy = gy$
3. $\int g\ dy = g \int dy$
4. $\int g\ dy = y \int g$

4-14. If $\int (dx + dy + dz)$
$= \int dx + \int dy + \int dz$, then
$\int dx + \int dy + \int dz$ is equal to

1. $x - y + z$
2. $x + y + z + C$
3. $x + y + z$
4. $x + y - z$

4-15. Evaluate the integral $\int x^2\, dx$.

1. $2x$
2. $\frac{x^3}{3}$
3. $x^3 + C$
4. $\frac{x^3}{3} + C$

4-16. Which of the following integrals CANNOT be determined by applying the rule, $\int u^n\, du = \frac{u^{n+1}}{n + 1} + C$?

1. $\int x^{-3}\, dx$
2. $\int x^{-1}\, dx$
3. $\int x^5\, dx$
4. $\int x^{2/5}\, dx$

4-17. Evaluate the integral $\int \frac{x^7}{x^3}\, dx$.

1. $\frac{x^4}{4}$
2. $\frac{x^5}{5} + C$
3. $\frac{x^4}{4} + C$
4. $4x^2 + C$

4-18. Evaluate the integral
$\int 7(x^{1/2} + x)\, dx$.

1. $\frac{2}{3}x^{3/2} + \frac{x^2}{2} + C$
2. $\frac{14}{3}x^{3/2} + \frac{x^2}{2} + C$
3. $\frac{14}{3}x^{3/2} + \frac{7}{2}x^2 + C$
4. $\frac{2}{3}x^{3/2} + \frac{x^2}{2} + 7$

4-19. Given $\int dy = \int x^2\, dx = \frac{x^3}{3} + C$, what is the value of the constant of integration when $x = 3$ and $y = 5$?

1. It has an infinite number of values
2. $-110/3$
3. 14
4. -4

4-20. How do definite integrals differ from indefinite integrals?

1. The variable must be assigned a numerical value before the constant of integration can be found
2. The result of integration has a definite value
3. No constant of integration is needed
4. Both 2 and 3 above

4-21. $\int_a^b f(x)\, dx$ is an integral with which one of the following characteristics?

1. b is the lower limit
2. a is the upper limit
3. b is more positive than a
4. a and b must both be positive

4-22. $\int_5^{10} f(x)\, dx = F(x) \Big|_5^{10}$ is the same as

1. F(10) - F(5)
2. F(10) + F(5)
3. F5 - F(10)
4. F(5) - F(10)

4-23. The area bounded by the curve $y = x^2 + 3$, the X axis, and $x = 1$ and $x = 3$ is

1. 8 2/3
2. 10 2/3
3. 12
4. 14 2/3

4-24. The area above the curve $y = -x^2$ but below the X axis and between $x = 0$ and $x = 3$ equals

1. -9
2. $|-9|$
3. 3
4. -3

4-25. The area bounded by the curve $y = -x + 3$, the X axis, $x = 0$, and $x = 6$ equals (Hint: Be sure to sketch the graph of the curve)

1. 0
2. 9
3. 18
4. 36

● The work, W, done in moving an object from coordinate a to coordinate b is given by

$$W = \int_a^b F(x)\,dx$$

where F(x) is the variable force applied to the object.

This expression for work has application to the force required in stretching a spring. The force, F(x), required to stretch an elastic spring is directly proportional to the extension of the spring (Hooke's Law) or

$$F(x) = kx$$

where k is the constant of proportionality.

EXAMPLE: If the natural length of a spring is 18 inches and a force of 20 pounds will stretch the spring to 21 inches, find the amount of work done in stretching the spring from a length of 20 inches to a length of 24 inches.

SOLUTION: Since it is given that a force of 20 pounds will stretch the spring 3 inches (21 inches - 18 inches) or 1/4 foot, then solving $F(x) = kx$ for k gives

20 pounds = k(1/4 foot)

or

k = 80 pounds per foot

and

$$F(x) = 80x$$

The work done in stretching the spring from 20 inches (20 - 18 = 2-inch extension or 1/6-foot extension) to 24 inches (24 - 18 = 6-inch extension or 1/2-foot extension) is then given by

$$W = \int_a^b F(x)\,dx$$

$$= \int_{1/6}^{1/2} 80x\,dx$$

$$= \left.\frac{80x^2}{2}\right|_{1/6}^{1/2}$$

$$= \left.40x^2\right|_{1/6}^{1/2}$$

$$= 10 - 10/9$$

= 8 8/9 foot-pounds

4-26. Find the work done in stretching a 21-inch spring to 24 inches if a force of 12 lbs is necessary to stretch the spring from its natural length of 21 inches to 21 1/2 inches.

1. 7 ft-lbs
2. 9 ft-lbs
3. 12 ft-lbs
4. 14 ft-lbs

● You may also use the expression $W = \int_a^b F(x)\,dx$ in determining the work done in pumping water out of the top of a tank. For a vertical cylindrical tank with radius r feet and height h feet, the weight of the water remaining in the tank is equal to

$$F(x) = k\pi r^2(h - x)$$

where $h - x$ is the height of the water remaining in the tank and $k = 62.5$ (since water weighs about 62.5 pounds per cubic foot). Therefore, the work required to pump all of the water out of the top of the cylindrical tank is given by

$$W = \int_0^h 62.5\pi r^2(h - x)\,dx$$

4-27. Calculate the work required to completely pump all of the water out of the top of a vertical cylindrical tank whose radius is 2 feet and height is 10 feet.

1. $8{,}000\pi$ ft-lbs
2. $9{,}500\pi$ ft-lbs
3. $11{,}000\pi$ ft-lbs
4. $12{,}500\pi$ ft-lbs

Learning Objective:

Simplify and solve integrals using formulas.

● The formulas below may be used for reference. In these formulas a, n, and C always represent a constant and u and v always represent a function of x.

Table 4A.--Formulas Involving Common Integral Forms

1. $\int du = u + C$
2. $\int a\ du = a \int du = au + C$
3. $\int (du + dv) = \int du + \int dv = u + v + C$
4. $\int u^n\ du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1$
5. $\int u^{-1}\ du = \int \frac{1}{u}\ du = \ln |u| + C,\ u \neq 0$
6. $\int e^u\ du = e^u + C$
7. $\int a^u\ du = \frac{a^u}{\ln a} + C,\ a > 0$
8. $\int \sin u\ du = -\cos u + C$
9. $\int \cos u\ du = \sin u + C$
10. $\int \sec^2 u\ du = \tan u + C$
11. $\int \csc^2 u\ du = -\cot u + C$
12. $\int \sec u \tan u\ du = \sec u + C$
13. $\int \csc u \cot u\ du = -\csc u + C$

● In answering items 4-28 through 4-38, evaluate the given integral.

4-28. $\int 4x^3\ dx$ is

1. $x^4 + C$
2. $\frac{4}{3}x^4 + C$
3. $12x^2 + C$
4. $12x^4 + C$

4-29. $\int (5x^{3/2} - \pi x + 2x^{-5})\ dx$ is

1. $\frac{25}{2}x^{5/2} - \frac{\pi}{2}x^2 - \frac{x^{-4}}{2} + C$
2. $2x^{5/2} - \frac{\pi}{2}x^2 - \frac{x^{-6}}{3} + C$
3. $2x^{5/2} - \frac{\pi}{2}x^2 - \frac{x^{-4}}{2} + C$
4. $2x^{5/2} - \pi - \frac{x^{-4}}{2} + C$

4-30. $\int (x + 2)^2\ dx$ is

1. $\frac{1}{3}(x + 2)^3 + C$
2. $x^2 + 4x + 4$
3. $2x + 4 + C$
4. $2(x + 2)^3 + C$

4-31. $\int (x^3 - 3)^{1/2}x^2\ dx$ is

1. $\frac{1}{2}(x^3 - 3)^{1/2} + C$
2. $\frac{2}{9}(x^3 - 3)^{3/2} + C$
3. $(x^3 - 3)^{1/2}x + C$
4. $\frac{(x^4 - 3)^{3/2}}{4} + C$

4-32. $\int (x^2 - 6x + 2)^2(2x - 6)\ dx$ is

1. $5x^5 - 2x^3 + x + C$
2. $2(x^2 - 6x + 2) + C$
3. $\frac{1}{3}(x^2 - 6x + 2)^3 + C$
4. $\frac{1}{6}(x^2 - 6x + 2)^3(2x - 6)^2 + C$

4-33. $\int (x^2 - 8)(x^3 - 24x)^{-1/2}\ dx$ is

1. $\frac{1}{2}(x^2 - 8)^2 + C$
2. $-\frac{1}{2}(3x^2 - 24)^{-3/2} + C$
3. $x^{-3/2} - 24^{-1/2} + C$
4. $\frac{2}{3}(x^3 - 24x)^{1/2} + C$

4-34. $\int \frac{2x}{(1 + x^2)^3}\ dx$ is

1. $-1(1 + x^2)^2 + C$
2. $-\frac{1}{2(1 + x^2)^2} + C$
3. $\frac{1}{4(1 + x^2)^4} + C$
4. $\frac{1}{(1 + x^2)^3} + C$

4-35. $\int \frac{x - 1}{(2x - x^2)^{1/3}}\ dx$ is

1. $\frac{3(x - 1)^2}{8(2x - x^2)^{4/3}} + C$
2. $\frac{3}{2}(2x - x^2)^{2/3} + C$
3. $-\frac{3}{4}(2x - x^2)^{2/3} + C$
4. $-\frac{4}{3}(2x - x^2)^{4/3} + C$

4-36. $\int \frac{1}{x + 1}\ dx$ is

1. $\ln |x + 1| + C$
2. $\frac{1}{x + 1} + C$
3. $\frac{2}{(x + 1)^2} + C$
4. $2(x + 1)^2 + C$

4-37. $\int \frac{4}{x - 8}\ dx$ is

1. $\frac{1}{4} \ln |x - 8| + C$
2. $\frac{1}{2}(x - 8)^2 + C$
3. $\frac{8}{(x - 8)^2} + C$
4. $4 \ln |x - 8| + C$

4-38. $\int \frac{x}{2x^2 - 5}\ dx$ is

1. $4 \ln |2x^2 - 5| + C$
2. $\frac{1}{4} \ln |2x^2 - 5| + C$
3. $\frac{(2x^2 - 5)^2}{2} + C$
4. $\frac{2}{(2x^2 - 5)^2} + C$

4-39. When it is necessary to integrate a fractional function whose numerator contains a term to a higher power than any term in the denominator, the first step is to

1. apply the form $\int \frac{1}{u}\ du$
2. divide the denominator into the numerator
3. integrate each term in both numerator and denominator separately
4. divide each term of both numerator and denominator by the independent variable raised to the highest power contained in the numerator

- When the degree of the polynomial in the numerator is greater than or equal to the degree of the polynomial in the denominator, then division of a polynomial by a polynomial can be performed.

For example,

$(x^4 + x^2 + 2x - 80) \div (x + 3)$

could be determined as follows:

$$\begin{array}{r|l} & x^3 - 3x^2 + 10x - 28 \\ \hline x + 3 & x^4 + 0x^3 + x^2 + 2x - 80 \\ & \underline{x^4 + 3x^3} \\ & \quad -3x^3 + x^2 \\ & \quad \underline{-3x^3 - 9x^2} \\ & \qquad 10x^2 + 2x \\ & \qquad \underline{10x^2 + 30x} \\ & \qquad\quad -28x - 80 \\ & \qquad\quad \underline{-28x - 84} \\ & \qquad\qquad 4 \end{array}$$

So,

$$\frac{x^4 + x^2 + 2x - 80}{(x + 3)} = x^3 - 3x^2 + 10x - 28 + \frac{4}{x + 3}$$

- In answering items 4-40 through 4-54, evaluate the given integral.

4-40. $\int \frac{x^2}{x - 1}\, dx$ is

1. $2x + C$
2. $-\dfrac{x^3 - x^2 + 2x}{x^2 - 2x - 1} + C$
3. $\dfrac{(x + 1)^3}{3} + \ln |x - 1| + C$
4. $\frac{1}{2}x^2 + x + \ln |x - 1| + C$

4-41. $\int \frac{x^4 - 3x^2 - 4}{x + 2}\, dx$ is

1. $4x^3 - 6x + \ln |x + 2| + C$
2. $\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 + \ln \left|\dfrac{1}{(x + 2)^2}\right| + C$
3. $\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - 2 \ln |x + 2| + C$
4. $\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2 - 2x + C$

4-42. $\int e^x\, dx$ is

1. $e^x + C$
2. $\frac{1}{2}e^{x^2} + C$
3. $xe^{x-1} + C$
4. $\dfrac{e^x}{\ln x} + C$

4-43. $\int (x^2 - 1)e^{(3x-x^3)}\, dx$ is

1. $-\frac{1}{3} \ln |3x - x^3| + C$
2. $-\frac{1}{3}e^{(3x-x^3)} + C$
3. $e^{(3x-x^3)} + C$
4. $3e^{(3x-x^3)} + C$

4-44. $\int 2^x\, dx$ is

1. $2x + C$
2. $2^x \ln 2 + C$
3. $\dfrac{2^x}{\ln 2} + C$
4. $\dfrac{2^{x+1}}{x + 1} + C$

4-45. $\int 5^{(x/2-3)}\,dx$ is

1. $\dfrac{2[5^{(x/2-3)}]}{\ln 5} + C$

2. $\frac{5}{2}x - 15 + C$

3. $\dfrac{5^{(x/2-3)2}}{\ln 5} + C$

4. $\frac{1}{2}x - 3 \ln 5 + C$

4-46. $\int \sin \frac{x}{2}\,dx$ is

1. $-\frac{1}{2} \cos \frac{x}{2} + C$

2. $-2 \sin \frac{x}{2} + C$

3. $-2 \cos \frac{x}{2} + C$

4. $2 \tan \frac{x}{2} + C$

4-47. $\int x \cos 2x^2\,dx$ is

1. $-\sin 2x^2 + C$

2. $\frac{1}{4} \sin 2x^2 + C$

3. $\frac{x^2}{2} \cos \frac{2}{3}x^3 + C$

4. $4 \sin 4x + C$

4-48. $\int (\cos 2x - \sin \frac{x}{3})\,dx$ is

1. $\sin 2x + \cos \frac{x}{3} + C$

2. $6 \cos 2x \sin \frac{x}{3} + C$

3. $\frac{1}{2} \sin 2x + 3 \cos \frac{x}{3} + C$

4. $2 \sin 2x + \frac{x}{3} + C$

4-49. $\int \csc \frac{\theta}{2} \cot \frac{\theta}{2}\,d\theta$ is

1. $2 \cos \frac{\theta}{2} \tan \frac{\theta}{2} + C$

2. $2 \sec^2\left(\frac{\theta}{2}\right) + C$

3. $-\frac{1}{2} \cot \frac{\theta}{2} + C$

4. $-2 \csc \frac{\theta}{2} + C$

4-50. $\int \dfrac{\csc^2\sqrt{x}}{\sqrt{x}}\,dx$ is

1. $-\frac{1}{3} \cot^3(x^{1/2}) + C$

2. $\frac{1}{2} \cot \sqrt{x} + C$

3. $-\cot x^{-1/2} + C$

4. $-2 \cot \sqrt{x} + C$

4-51. $\int x \sec (1 - x^2) \tan (1 - x^2)\,dx$ is

1. $-\frac{1}{2} \sec (1 - x^2) + C$

2. $-\sec 2x \tan 2x + C$

3. $-\csc (1 - x^2) + C$

4. $\tan (1 - x^2) + C$

4-52. $\int \sec^2 ax\,dx$ is

1. $\frac{1}{3a} \sec^3 ax + C$

2. $\frac{1}{a} \tan^2 ax + C$

3. $\frac{1}{a} \tan ax + C$

4. $a \tan ax + C$

4-53. $-6 \int \cos^2 x \sin x \, dx$ is

1. $-6 \cos x + C$

2. $-2 \cos x^2 + C$

3. $2 \cos^3 x + C$

4. $2 \sin^3 x + C$

4-54. $\int \sin^3\left(\frac{x}{3}\right) \cos \frac{x}{3} \, dx$ is

1. $\frac{1}{12} \sin^4\left(\frac{x}{3}\right) + C$

2. $\frac{3}{4} \sin^4\left(\frac{x}{3}\right) + C$

3. $\frac{3}{4} \cos^4\left(\frac{x}{3}\right) + C$

4. $\frac{1}{12} \cos^4\left(\frac{x}{3}\right) + C$

Assignment 5

Textbook assignment: Chapter 8, "Combinations and Permutations," pages 8-1 through 8-27 and Chapter 9, "Probability," pages 9-1 through 9-29.

Learning Objective:

Apply the concepts of combinations, principle of choice, and permutations to problem solving.

5-1. Permutations and combinations are different in that permutations are ordered and combinations are not ordered.

1. True
2. False

5-2. The symbol ${}_7C_3$ means combinations of

1. three groups with seven items each
2. three items grouped in seven different ways
3. seven groups arranged in three different ways
4. a group of seven items taken three at a time

5-3. Find the value of 7!.

1. 7
2. 28
3. 720
4. 5,040

5-4. Subtracting 2! from 4! gives

1. 2!
2. 2!(2! - 1!)
3. 11
4. 22

5-5. $\frac{(n + 4)!}{(n + 2)}$ is equivalent to

1. (n + 3)(n + 4)
2. (n + 3)!(n + 4)
3. (n + 4)(n + 3)(n + 1)!
4. (n + 1)!(n + 2)(n + 3)(n + 4)

5-6. The value of ${}_5C_0$ is

1. 1
2. 5
3. 0
4. undefined

5-7. The formula ${}_nC_r = \frac{n!}{r!(n - r)!}$ is equivalent to which of the following results when n = r?

1. 1
2. 0
3. ∞
4. An undefined quantity

5-8. ${}_8C_5$ equals

1. 56
2. 336
3. 40
4. 6,720

5-9. In how many ways can a committee of four people be chosen out of seven people?

1. 28
2. 35
3. 840
4. 5,040

5-10. If a division consists of eight ships, how many combinations of two ships each can be made up from the division?

1. 16
2. 28
3. 1,230
4. 1,680

5-11. If 10 children are on a baseball team's roster, how many different teams of 9 players may be formed?

1. 10
2. 11
3. 19
4. 90

5-12. Find the value of ${}_5C_3 \cdot {}_3C_2$.

1. 10
2. 13
3. 21
4. 30

5-13. A person buying a pizza has a choice of one size from three and two toppings from six. How many different pizzas are possible?

1. 18
2. 45
3. 3
4. 15

5-14. S division has 5 workers, A division has 7 workers, and K division has 8 workers. How many possible working parties composed of 2 workers from S division, 3 workers from A division, and 3 workers from K division can be selected?

1. 101
2. 15,423
3. 17,647
4. 19,600

5-15. The number of permutations of four items taken together is equal to

1. 9
2. 10
3. 20
4. 24

5-16. The number of permutations of three items taken together exceeds the number of combinations of these three items taken together by

1. 0
2. 5
3. 6
4. 4

5-17. ${}_5P_3$ equals

1. 48
2. 60
3. 64
4. 72

5-18. How many five-digit student identification numbers can you form using the digits 1 through 8 if each digit can be used only once?

1. 40
2. 56
3. 6,720
4. 32,768

5-19. A jeweler is presented with 10 rings and asked to select the 4 she likes best and to rank them from first to fourth. What is the number of possible outcomes?

1. 40
2. 210
3. 5,040
4. 151,200

5-20. Find the value of ${}_9P_4 \cdot {}_6P_0$.

1. 9!
2. 126
3. 216
4. 3,024

5-21. The letters of the word THE can be arranged in several ways. If two of these arrangements are THE and HTE, the remaining arrangements are

1. HET, ETH, EHT, TEH
2. HET, ETH, EHT
3. HET, ETH, TEH
4. TEH, ETH, EHT

5-22. How many arrangements can you make using the letters in the word AIRCRAFT?

1. 1,680
2. 9,760
3. 10,080
4. 20,160

5-23. Using the digits 1 through 7, how many more three-digit numbers can you form <u>with</u> repetition than <u>without</u> repetition?

1. 64
2. 118
3. 133
4. 154

5-24. Using the digits 1 through 8, how many even four-digit numbers can you form without repetition?

1. 24
2. 840
3. 1,250
4. 1,680

5-25. How many possible outcomes are there in answering at random four true-false questions?

1. 16
2. 8
3. 32
4. 4

Learning Objective:

Apply the basic concepts of probability to problem solving.

5-26. If a trial must result in any of n equally likely outcomes, and if s is the number of successful outcomes and u is the number of unsuccessful outcomes, then the probability of succeeding is represented by which of the following equations?

1. $\frac{s}{u + n}$
2. $\frac{s}{s + u}$
3. $\frac{s}{n}$
4. Both 2 and 3 above

5-27. A bookshelf contains four chemistry books, three physics books, and five mathematics books. What is the probability of selecting a mathematics book?

1. 1
2. 5/12
3. 7/12
4. 1/5

5-28. If a pair of fair dice are tossed, what is the probability of getting a sum of 12?

1. 1/2
2. 1/3
3. 1/18
4. 1/36

5-29. The probability of an event always lies in the range from 0 and 1, inclusively.

1. True
2. False

5-30. If the probability of success is equal to $\frac{s}{s + f}$, then the expression $1 - \frac{s}{s + f}$ represents

1. a problematic outcome
2. the probability of success
3. the probability of failure
4. the number of possible outcomes

5-31. A box contains 3 red balls and 2 green balls. If drawing a red ball is considered successful, what is the probability of failing?

1. 0.2
2. 0.3
3. 0.4
4. 0.6

5-32. A shipment of 500 light bulbs contains 95 defective bulbs. If you purchase 1 of these 500 bulbs, what is the probability that the bulb will work?

1. 0.81
2. 0.19
3. 0.99
4. 0.01

5-33. If a man draws cards at random from a complete deck (52 cards) and replaces each card prior to the next draw, how many spades can he expect to draw out of 80 attempts?

1. 13
2. 20
3. 39
4. 40

5-34. A contractor bids on a job to construct a building. There's a 0.7 probability of making a $175,000 profit. What is the contractor's mathematical expectation?

1. $52,500
2. $250,000
3. $175,000
4. $122,500

5-35. If the outcome of one event has no influence on the outcome of another event, the events are said to be

1. separate
2. independent
3. unattached
4. disjointed

5-36. When dealing with independent events, you calculate the probability that one and then the other event will occur by combining their separate probabilities through

1. subtraction
2. division
3. addition
4. multiplication

5-37. If a coin is flipped three times, what is the probability of heads showing up all three times?

1. 1/8
2. 1/6
3. 1/3
4. 1/2

5-38. What is the probability of answering four true-false questions correctly?

1. 1/18
2. 1/16
3. 3/16
4. 1/2

5-39. If the outcome of one event influences the outcome of another event, the events are said to be

1. equivalent
2. additive
3. dependent
4. exclusive

5-40. Assuming no replacement after each drawing, what is the probability that you will draw first a nickel and then a dime out of a jar containing three dimes, four nickels, and five pennies?

1. 1/5
2. 1/6
3. 1/11
4. 1/12

5-41. A shipment of 40 items contains 6 defectives. What is the probability of randomly selecting 3 defective items in a row?

1. 9/20
2. 3/20
3. 27/8,000
4. 1/494

5-42. A workshop is attended by 10 men and 12 women. If the instructor randomly selects 4 attendees for a special project, what is the probability that they are all women?

1. 1/3
2. 9/133
3. 2/11
4. 1,296/14,641

5-43. Mutually exclusive events differ from dependent events in that with the former the occurrence of one event

1. precludes the other event from occurring
2. makes it possible for the other event to occur
3. increases the probability of the other event occurring
4. reduces but does not eliminate the probability of the other event occurring

5-44. What is the probability of drawing a red ace or a red king from a standard deck of 52 playing cards?

1. 1/26
2. 1/13
3. 2/13
4. 4/13

5-45. If each of eight horses in a race has an equally likely chance of winning, what is the probability of choosing a horse that finishes first or second?

1. 1/2
2. 1/8
3. 3/8
4. 1/4

5-46. A class consists of 7 physics majors, 10 chemistry majors, 5 biology majors, and 14 mathematics majors. What is the probability that you will select one student who is either a mathematics or a chemistry major?

1. 1/24
2. 35/324
3. 2/3
4. 1/9

5-47. Which of the following is NOT a characteristic of empirical probability?

1. All possible outcomes can always be listed
2. It is used in industrial quality control
3. Probabilities are estimated from experience and observation
4. Probabilities are obtained from situations that are not equally likely to occur

5-48. An adequate statistical sample is one that is large enough that accidental runs of luck offset each other.

1. True
2. False

5-49. If a batter hits 5 home runs in 200 times at bat, what is the statistical probability that he will hit a home run the next time at bat?

1. 0.4
2. 0.04
3. 0.05
4. 0.025

5-50. A grocery store is planning to sell unmarked boxes of candy, cookies, and chips. The unmarked assortment contains 200 boxes of candy, 250 boxes of cookies, and 300 boxes of chips. If one box is selected, what is the probability that it is a box of cookies?

1. 1
2. 1/3
3. 1/250
4. 1/2

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