Phys 531Lecture 2517 November 2005Retarders and the Jones Calculus

Last time, discussed how polarizers work:

- transmit one polarization
- blocks orthogonal polarization

Described ways to implement Discussed birefringence

Today: Discuss retarders Develop computational tools

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Outline:

- Review polarizers
- Retarders
- Matrix technique
- Partially polarized light

Finish polarization

Next time: Eat turkey After that: Quantum Optics

## Polarizers

Ideal polarizer has transmission axis a transmits light with  $\hat{j} = a$  blocks light with  $\hat{j} \perp a$ 

- If a complex, transmit circ or ellip polarization no true "axis"
- If  $\hat{\jmath}$  neither  $|| \mathbf{a} \text{ nor } \perp \mathbf{a}$ use  $T = |\hat{\jmath}^* \cdot \mathbf{a}|^2$

Output light is polarized along  ${\bf a}$ 

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Retarders (Hecht 8.7)

Use polarizers to make linear polarized light

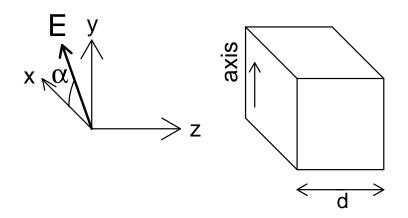
What about circular or elliptical?

Use retarder

Most common retarder = wave plate

Based on birefringence Light polarized along optic axis: index  $n_e$ Light polarized  $\perp$  axis: index  $n_o$  Suppose linear polarized light, angle  $\alpha$ 

Incident on uniaxial crystal with axis vertical Crystal thickness d



Coordinates as shown: z = 0 at front of crystal

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Incident wave has  $\hat{\jmath} = \cos \alpha \, \hat{\mathbf{x}} + \sin \alpha \, \hat{\mathbf{y}}$ 

$$\mathbf{E}_{\mathsf{inc}} = E_0 \hat{\jmath} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

In crystal  $k \rightarrow nk_0 = n\omega/c$ different for  $E_x$  and  $E_y$  components

## So

 $\mathbf{E}_{\text{crystal}} = E_0 \left( \cos \alpha \, e^{i n_o k_0 z} \, \hat{\mathbf{x}} + \sin \alpha \, e^{i n_e k_0 z} \, \hat{\mathbf{y}} \right) e^{-i \omega t}$ At output z = d:

$$\mathbf{E}_{d} = E_{0} \left( \cos \alpha \, e^{i n_{o} k_{0} d} \, \hat{\mathbf{x}} + \sin \alpha \, e^{i n_{e} k_{0} d} \, \hat{\mathbf{y}} \right) e^{-i \omega t}$$
$$= E_{0} e^{i n_{o} k_{0} d} \left[ \cos \alpha \, \hat{\mathbf{x}} + \sin \alpha \, e^{i (n_{e} - n_{o}) k_{0} d} \, \hat{\mathbf{y}} \right] e^{-i \omega t}$$

Define  $\hat{\jmath}_{out} = \cos \alpha \, \hat{\mathbf{x}} + \sin \alpha \, e^{i(n_e - n_o)k_0 d} \, \hat{\mathbf{y}}$ 

Then after crystal, have

$$E(z,t) = E_0 e^{in_o k_0 d} \,\hat{j}_{\text{out}} \, e^{i[k(z-d)-\omega t]}$$
$$= E_0 e^{i(n_o-1)k_0 d} \,\hat{j}_{\text{out}} \, e^{i(kz-\omega t)}$$
$$= E'_0 \,\hat{j}_{\text{out}} \, e^{i(kz-\omega t)}$$

Get plane wave out with

$$\hat{j} = \cos \alpha \, \hat{\mathbf{x}} + \sin \alpha \, e^{i\varepsilon} \, \hat{\mathbf{y}}$$
  
and  $\varepsilon = (n_e - n_o)k_0 d \equiv retardance$   
Set  $d$  to achieve desired  $\varepsilon$  value

By adjusting 
$$\alpha$$
 and  $\varepsilon$ ,  
make arbitrary polarization state  
Example:  $kd(n_e - n_o) = \pi/2$   
Then  $d(n_e - n_o) = \lambda/4$ : call *quarter-wave plate*  
Get  $\hat{j}_{out} = \cos \alpha \hat{x} + e^{i\pi/2} \sin \alpha \hat{y}$   
 $= \cos \alpha \hat{x} + i \sin \alpha \hat{y}$   
For  $\alpha = \pm 45^{\circ}$ ,  $\hat{j}_{out} = \frac{\hat{x} \pm i\hat{y}}{\sqrt{2}}$   
Make left and right circular polarizations  
Main use of quarter-wave plate:  
convert linear to circular polarization

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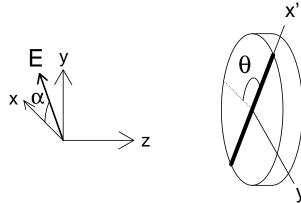
Other common configuration:  $\varepsilon = \pi$ Then  $d(n_e - n_o) = \lambda/2$ : half-wave plate Have  $e^{i\pi} = -1$  so  $\hat{j}_{out} = \cos \alpha \hat{\mathbf{x}} - \sin \alpha \hat{\mathbf{y}}$ Changes polarization angle  $\alpha \to -\alpha$ 

If  $\alpha_{in} = 45^{\circ}$ , then  $\alpha_{out} = -45^{\circ}$ Rotated by 90°: orthogonal to input

Generally, use half-wave plate at arb angle  $\theta$  how to calculate effect?

 $\Rightarrow$  need to express input in crystal basis

Define x' = axis of waveplate



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Then  

$$\hat{\mathbf{x}}' = \cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}}$$
  
 $\hat{\mathbf{y}}' = \cos\theta \hat{\mathbf{y}} - \sin\theta \hat{\mathbf{x}}$   
and  $\hat{\jmath} = \cos\alpha \hat{\mathbf{x}} + \sin\alpha \hat{\mathbf{y}}$   
Calculate  $j_{x'} = \hat{\mathbf{x}}' \cdot \hat{\jmath}$   
 $= \cos\alpha \cos\theta - \sin\alpha \sin\theta = \cos(\alpha - \theta)$   
and  $j_{y'} = \hat{\mathbf{y}}' \cdot \hat{\jmath}$   
 $= -\cos\alpha \sin\theta + \sin\alpha \cos\theta = \sin(\alpha - \theta)$   
So:  $\hat{\jmath} = \cos\alpha' \hat{\mathbf{x}}' + \sin\alpha' \hat{\mathbf{y}}'$   
for  $\alpha' = \alpha - \theta$   
(Or just use  $x' - y'$  coords from beginning)

Effect of half wave plate is  $\alpha' \rightarrow -\alpha'$ 

So 
$$\alpha_{out} - \theta = -(\alpha_{in} - \theta)$$

 $\alpha_{\rm out} = 2\theta - \alpha_{\rm in}$ 

Change  $\theta$  by  $\Delta \theta$ , output polarization rotates by  $2\Delta \theta$ 

Main use of half-wave plate: rotate linear polarization by arbitrary angle

**Question:** What happens when  $\theta = \alpha_{in}$ ? What is physically happening in this situation?

One problem: waveplates very thin Quarter wave plate:  $d = \lambda/4(n_e - n_o) \sim \lambda$ Often make  $\varepsilon = \pi/4 + 2\pi m$  for integer m: get same effect Called *multiple order waveplate* typical  $m \approx 10$ Most common waveplate material: quartz cost \$200 for 1-cm diameter plate Cheaper: plastic (\$5 for 5 cm square) -  $\varepsilon$  less accurate - distorts laser beams

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Terminology:

- Fast axis: axis of wave plate with lower n
- $\bullet$  Slow axis: axis of wave plate with higher n

Doesn't really matter which is optical axis depends on whether  $n_o > n_e$  ("negative" crystal) or  $n_o > n_e$  ("positive" crystal)

Other ways to make retarders (Hecht pg 356-357):

- Fresnel rhomb: use phase shift from TIR
- Babinet-Soliel compensator:

waveplate with variable thickness

Jones Calculus (Hecht 8.13.2-3)

Polarizers and retarders simple in proper basis Polarizer: pass one component, block other Retarder: add phase shift to one component

Always good to choose basis carefully

But with multiple elements, no one basis is simple

Half-wave plate example illustrated how to deal: change basis for each element

Easy way to implement: use matrix notation

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Basic idea:  $\hat{j}$  is a vector

write 
$$\hat{\jmath} = \begin{bmatrix} j_x \\ j_y \end{bmatrix}$$

Polarizers and retarders are linear elements

write as matrices  $\hat{j}_{out} = \mathcal{U}\hat{j}_{in}$  $\mathcal{U} = 2 \times 2$  matrix

Hecht writes matrix as  $\mathcal{A}$ 

I'll use  ${\cal U}$ 

Examples:

Matrix for polarizer with  $\mathbf{a}=\hat{\mathbf{x}}$  is

$$\mathcal{U} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right]$$

Arbitrary incident polarization:

$$\hat{\jmath}_{in} = \begin{bmatrix} \cos \alpha \\ e^{i\varepsilon} \sin \alpha \end{bmatrix}$$
  
So  $\hat{\jmath}_{out} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos \alpha \\ e^{i\varepsilon} \sin \alpha \end{bmatrix}$ 
$$= \begin{bmatrix} \cos \alpha \\ 0 \end{bmatrix}$$

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See 
$$\hat{j}_{out} \parallel \mathbf{a}$$
 and  $|\hat{j}_{out}|^2 = |\hat{j}_{in}^* \cdot \mathbf{a}|^2$   
as required

Not really any different than previous calcs Just written differently:

Matrix is taking  $\widehat{\jmath} \cdot \mathbf{a}$  for you

**Question:** What would  $\mathcal{U}$  be for polarizer with transmission axis along  $\hat{\mathbf{y}}$ ?

One issue:  $\hat{\jmath}$  is supposed to be unit vector

Really have  $t \,\hat{j}_{out} = \mathcal{U}\hat{j}_{in} = \cos \alpha \begin{bmatrix} 1\\ 0 \end{bmatrix}$ 

where t = amplitude transmission coeff  $E_{0out} = tE_{0in}$ 

Then intensity transmittance  $T = |t|^2$ 

Easier to write 
$$\hat{j}_{out} = \mathcal{U}\hat{j}_{in}$$
  
use  $T = |\hat{j}_{out}|^2$   
Just remember to make  $|\hat{j}_{in}|^2 = 1$ 

More examples:

Wave plate with optic axis along y

Saw 
$$\hat{j} \to j_x e^{in_o k_0 d} \hat{\mathbf{x}} + j_y e^{in_e k_0 d} \hat{\mathbf{y}}$$
  
So  $\mathcal{U} = \begin{bmatrix} e^{in_o k_0 d} & 0\\ 0 & e^{in_e k_0 d} \end{bmatrix} = e^{in_o k_0 d} \begin{bmatrix} 1 & 0\\ 0 & e^{i(n_e - n_o)k_0 d} \end{bmatrix}$ 

Say quarter wave plate,  $n_e > n_o$ 

- then slow axis along y

$$\mathcal{U} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & i \end{array} \right]$$

ignoring overall phase

Or, quarter wave plate with fast axis along y:

$$\mathcal{U} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -i \end{array} \right]$$

 $\equiv$  slow axis along x

(Doesn't matter which is optic axis)

Half wave plate, either axis along y:

$$\mathcal{U} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right]$$
 since  $e^{i\pi} = e^{-i\pi} = -1$ 

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Generally called Jones matrices

Easy when element axes aligned to x or y

What if axis at angle  $\theta$ ?

Define two coordinates: x-y = "lab" coordinates x'-y' = "element" coordinates

For instance, x' = transmission axis of polarizer

Express  $\hat{j}$  in x'-y' coords If x' axis at angle  $\theta$  to x, then  $\hat{x}' = \cos \theta \hat{x} + \sin \theta \hat{y}$   $\hat{y}' = \cos \theta \hat{y} - \sin \theta \hat{x}$ as before

So  

$$j_{x'} = \hat{\mathbf{x}}' \cdot \hat{\jmath} = \cos \theta j_x + \sin \theta j_y$$
  
 $j_{y'} = \hat{\mathbf{y}}' \cdot \hat{\jmath} = -\sin \theta j_x + \cos \theta j_y$ 

Write

$$\begin{bmatrix} j_{x'} \\ j_{y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} j_x \\ j_y \end{bmatrix}$$

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Define rotation matrix

$$\mathcal{R}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Then  $\hat{\jmath}' = \mathcal{R}(\theta)\hat{\jmath}$ 

In primed coordinates,  $\ensuremath{\mathcal{U}}$  is easy

For polarizer

$$\begin{aligned} \mathcal{U}' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \text{and } \hat{\jmath}_{\mathsf{out}}' = \mathcal{U}' \, \hat{\jmath}_{\mathsf{in}}' = \mathcal{U}' \, \mathcal{R}(\theta) \, \hat{\jmath} \end{aligned}$$

Still need to get  $\hat{j}_{out}$  in lab coords Generally  $\hat{j} = R(\theta)^{-1} \hat{j}'$   $\mathcal{R}(\theta)^{-1} = \text{matrix inverse of } \mathcal{R}(\theta)$ Find  $\mathcal{R}(\theta)^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \mathcal{R}(-\theta)$ So  $\hat{j}_{out} = \mathcal{R}(-\theta) \mathcal{U}' \mathcal{R}(\theta) \hat{j}_{in}$ 

Or, without specifying coordinates:  

$$\begin{array}{c}
\mathcal{U}(\theta) = \mathcal{R}(-\theta) \mathcal{U}_0 \mathcal{R}(\theta) \\
\text{for } \mathcal{U}_0 \equiv \mathcal{U}(\theta = 0)
\end{array}$$

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So polarizer with axis at angle  $\theta$ :

$$\mathcal{U}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$
If  $\hat{\jmath}_{\text{in}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \hat{x}$ get  $\hat{\jmath}_{\text{out}} = \begin{bmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ Get  $T = |\hat{\jmath}_{\text{out}}|^2 = \cos^2\theta$  as expected

Other examples:

QWP fast axis at angle  $\theta$  (measured from x axis)

$$\mathcal{U}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix}$$

HWP at angle  $\theta$ 

$$\mathcal{U}(\theta) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

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Get matrices for all elements in lab coordinates For multiple elements, multiply matrices

**Example:** Suppose light linearly polarized along y passes through a quarter-wave plate with slow axis at  $\theta_1 = 30^\circ$  from x, then a half-wave plate with axis at  $\theta_2 = 60^\circ$ , and finally a polarizer with axis at  $\theta_3 = 45^\circ$ . What is the total transmission?

## Solution:

Get matrix for each element.

$$\begin{aligned} \mathcal{U}_{1} &= \begin{bmatrix} 0.75 + 0.25i & 0.433 - 0.433i \\ 0.433 - 0.433i & 0.25 + 0.75i \end{bmatrix} \\ \mathcal{U}_{2} &= \begin{bmatrix} -0.5 & 0.866 \\ 0.866 & 0.5 \end{bmatrix} \quad \mathcal{U}_{3} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \\ \mathcal{U}_{tot} &= \mathcal{U}_{3}\mathcal{U}_{2}\mathcal{U}_{1} = \begin{bmatrix} 0.433 - 0.25i & 0.25 + 0.433i \\ 0.433 - 0.25i & 0.25 + 0.433i \end{bmatrix} \\ \text{Given } \hat{\jmath}_{\text{in}} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ so } \hat{\jmath}_{\text{out}} = \begin{bmatrix} 0.25 + 0.433i \\ 0.25 + 0.433i \end{bmatrix} \\ \text{So } T &= |\hat{\jmath}_{\text{out}}|^{2} = 2 \times |0.25 + 0.433|^{2} = \boxed{0.5} \end{aligned}$$

Question: What is the polarization state of the output?

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Often output state is "disguised" by extra phase factors

Example:

Light polarized along  $\widehat{\mathbf{y}}$  into QWP at 45°

Expect to get circular polarized light out

Have 
$$\mathcal{U} = \frac{1}{2} \begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$$
  
so  $\hat{j}_{out} = \frac{1}{2} \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$   
Doesn't look like circular polarization!

Note 
$$1 + i = \sqrt{2}e^{i\pi/4}$$
  
 $1 - i = \sqrt{2}e^{-i\pi/4}$   
So  $\hat{j}_{out} = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\pi/4} \\ e^{-i\pi/4} \end{bmatrix} = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 \\ e^{-i\pi/2} \end{bmatrix}$   
 $= \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix}$ 

= right circular polarization

Good approach:

always factor out phase to make  $j_x$  real Gives standard form  $\hat{j} = \cos \beta \hat{\mathbf{x}} + e^{i\varepsilon} \sin \beta \hat{\mathbf{y}}$ 

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Can generalize and work in any basis  $(\widehat{e}_1,\widehat{e}_2)$ 

Write 
$$\hat{j} = j_1 \hat{\mathbf{e}}_1 + j_2 \hat{\mathbf{e}}_2 = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}_{12}$$
  
then  $\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}_{12} = \begin{bmatrix} \hat{\mathbf{e}}_1^* \cdot \hat{\mathbf{x}} & \hat{\mathbf{e}}_1^* \cdot \hat{\mathbf{y}} \\ \hat{\mathbf{e}}_2^* \cdot \hat{\mathbf{x}} & \hat{\mathbf{e}}_2^* \cdot \hat{\mathbf{y}} \end{bmatrix} \begin{bmatrix} j_x \\ j_y \end{bmatrix}_{xy}$   
 $\equiv S_{(xy \to 12)} \begin{bmatrix} j_x \\ j_y \end{bmatrix}_{xy}$ 

Then in 1-2 basis, have

$$\mathcal{U}_{12} = \mathcal{S}_{(xy \to 12)} \, \mathcal{U}_{xy} \, \mathcal{S}^{\dagger}_{(xy \to 12)}$$

where  $S^{\dagger}$  = conjugate transposition of S

Partially Polarized Light

Sometimes light is not perfectly polarized also not unpolarized

Example: pass natural light through poor polarizer

No Jones vector in this case:

 $\hat{\jmath}$  is fluctuating

Instead define *coherency matrix* 

$$G = \left[ \begin{array}{cc} \langle j_x j_x^* \rangle & \langle j_x j_y^* \rangle \\ \langle j_y j_x^* \rangle & \langle j_y j_y^* \rangle \end{array} \right] = \left\langle \left[ \begin{array}{cc} j_x \\ j_y \end{array} \right] \left[ \begin{array}{cc} j_x^* & j_y^* \end{array} \right] \right\rangle$$

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For polarized light, G and  $\hat{j}$  have same information Unpolarized light =

mixture of x and y polarized light

Then 
$$G_{\text{unpolarized}} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Get G in other cases by propagation If system has Jones matrix  $\mathcal{U}$ , then have

$$G_{\mathsf{out}} = \mathcal{U} G_{\mathsf{in}} \mathcal{U}^{\dagger}$$

Example:

Suppose glass surface has  $t_{\parallel} = 0.95$ ,  $t_{\perp} = 0.85$  from Fresnel relations

Then 
$$\mathcal{U} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix}$$
 in  $\parallel - \perp$  basis

Illuminate with natural light:

$$G_{\text{out}} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix} \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.95 & 0 \\ 0 & 0.85 \end{bmatrix}$$
$$= \begin{bmatrix} 0.45 & 0 \\ 0 & 0.36 \end{bmatrix}$$

Total irradiance =  $|j_x|^2 + |j_y|^2$  = Trace G

= sum of diagonal elements

Here write

$$G_{\text{out}} = 0.81 \begin{bmatrix} 0.55 & 0\\ 0 & 0.45 \end{bmatrix}$$

Total transmission is 0.81

Transmitted light is partially polarized: 55%  $\parallel$  and 45%  $\perp$ 

Could then pass through a QWP at  $45^\circ$ 

$$\mathcal{U} = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$
  
Get  $G_{\text{out}} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \begin{bmatrix} 0.55 & 0 \\ 0 & 0.45 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ 
$$= \begin{bmatrix} 0.5 & 0.05i \\ -0.05i & 0.5 \end{bmatrix}$$

Describes partial circular polarization

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Coherency matrix is generally nice way to describe polarization

- Takes care of phase factors
- Has correct symmetry property: 180° rotation doesn't change state

Older way to describe partially polarized light: Stokes parameters (Hecht 8.13.1)

Coherency matrix is better Reference Klein and Furtak §9.3 Summary:

- Retarders based on birefringence
  - Quarter-wave plate: make circ polarization
  - Half-wave plate: rotate linear polarization
- Use Jones matrices for multiple elements
- Partially polarized light
  - $\rightarrow$  coherency matrix
  - Also uses Jones matrices

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