General Methods for Optimized Design of Mueller Polarimeters

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Outline

- The Stokes formalism: What ellipsometers can do with?
- The use of the condition number as an objective criteria for optical design.
- Calibration of Mueller polarimeters.

Jones and Stokes Formalisms

Jones vector (2 complex components)

$$\begin{pmatrix} E_p \\ E_s \end{pmatrix} = \begin{pmatrix} A_p e^{i\varphi_p} \\ A_s e^{i\varphi_s} \end{pmatrix}$$

Stokes vector (4 real components)

$$S = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I_x + I_y \\ I_x - I_y \\ I_{+45} - I_{-45} \\ I_G - I_L \end{pmatrix} = \begin{pmatrix} < E_x E_x^* + E_y E_y^* > \\ < E_x E_x^* - E_y E_y^* > \\ < E_x E_y^* + E_y E_x^* > \\ < E_x E_y^* - E_y E_x^* > \end{pmatrix}$$

Jones matrix (2 x 2)

$$\begin{pmatrix} E_p \\ E_s \end{pmatrix} = \begin{pmatrix} J_{pp} & J_{ps} \\ J_{sp} & J_{ss} \end{pmatrix} \cdot \begin{pmatrix} E_p^0 \\ E_s^0 \end{pmatrix}$$

Mueller matrix (4 x 4)

$$\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}_{OUT} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \cdot \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}_{IN}$$

For non depolarizing samples both, the Mueller matrix and the Jones matrix have the same physical meaning.

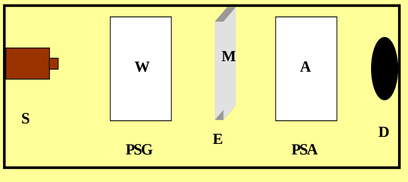
$$\mathbf{M} = \begin{bmatrix} \frac{1}{2}\left(|J_{11}|^2 + |J_{22}|^2 + |J_{12}|^2 + |J_{21}|^2\right) & \frac{1}{2}\left(|J_{11}|^2 - |J_{22}|^2 - |J_{12}|^2 + |J_{21}|^2\right) & \mathrm{Re}\left(J_{11}^* J_{12} + J_{21}^* J_{22}\right) & -\mathrm{Im}\left(J_{11}^* J_{12} + J_{21}^* J_{22}\right) \\ \frac{1}{2}\left(|J_{11}|^2 - |J_{22}|^2 + |J_{12}|^2 - |J_{21}|^2\right) & \frac{1}{2}\left(|J_{11}|^2 + |J_{22}|^2 - |J_{12}|^2 - |J_{21}|^2\right) & \mathrm{Re}\left(J_{11}^* J_{12} - J_{21}^* J_{22}\right) & \mathrm{Im}\left(-J_{11}^* J_{12} + J_{21}^* J_{22}\right) \\ \mathrm{Re}\left(J_{11}^* J_{21} + J_{12}^* J_{22}\right) & \mathrm{Re}\left(J_{11}^* J_{21} - J_{12}^* J_{22}\right) & \mathrm{Re}\left(J_{11}^* J_{22} + J_{12}^* J_{21}\right) & \mathrm{Im}\left(-J_{11}^* J_{22} + J_{12}^* J_{21}\right) \\ \mathrm{Im}\left(J_{11}^* J_{21} + J_{12}^* J_{22}\right) & \mathrm{Im}\left(J_{11}^* J_{21} - J_{12}^* J_{22}\right) & \mathrm{Im}\left(J_{11}^* J_{22} + J_{12}^* J_{21}\right) & \mathrm{Re}\left(J_{11}^* J_{22} - J_{12}^* J_{21}\right) \end{bmatrix}$$

If depolarization is important, only the Mueller matrix is physically meaningful.

Reflection of transmision by a planar isotropic surface can be expressed in terms of ellipsometric angles Ψ and Δ.

$$R(\tau,\Psi,\Delta) = \tau \begin{pmatrix} \tan\Psi e^{i\Delta} & 0 \\ 0 & 1 \end{pmatrix} \qquad R(\tau,\Psi,\Delta) = \begin{pmatrix} \tau & -\cos 2\Psi & 0 & 0 \\ -\cos 2\Psi & 1 & 0 & 0 \\ 0 & 0 & \sin 2\Psi\cos\Delta & \sin 2\Psi \sin\Delta \\ 0 & 0 & -\sin 2\Psi \sin\Delta & \sin 2\Psi\cos\Delta \end{pmatrix}$$

Measurement Principle and Experimental set-up



- **Modulation matrix W** : Stokes vectors generated by the polarization state generator (PSG)
- **Analysis matrix A** : signal vector on the polarization state analyser (PSA)
- **Complete measurement** **B = A.M.W**
- Obtention of the **Mueller Matrix** :**M = A^{-1}.B.W^{-1}**
- **M** can only be obtained if **W** and **A** are invertible.
- A and W must be as much « non-singular » as possible.

- **Singular value decomposition of any square matrix A**

  $\mathbf{A} = {}^t\mathbf{U}\,\mathbf{D}\,\mathbf{V}$ with U, V unitary, D diagonal (singular values)

- **Conditioning number**

  $$C(A) = \left|\frac{D_{ii}^{\min}}{D_{ii}^{\max}}\right|$$

  C(A) goes from 0 (A singular) to 1 (A unitary)

- **Error propagation in a linear transformation**

  X and **Y** : vectors related by **Y = A X**, with errors δX and δY

  Error propagation is proportional to the inverse of C(A)

  $$\left\|\frac{\delta Y}{Y}\right\| \leq \frac{1}{C(A)}\left\|\frac{\delta X}{X}\right\|$$

•To minimize error propagation, C(A) must be as much close to 1 as possible.

•The optical elements and configuration of the PSG and the PSA must be chosen in order to maximize the condition number.

Calibration: Theoretical principle

Ideal matrices

$$AM_1W = (am_1w)$$
$$AM_2W = (am_2w)$$

Experimental matrices

$$\Longrightarrow$$

$$\mathbf{W} \begin{cases} M_1^{-1}M_2W - W(am_1w)^{-1}(am_2w) = 0 \\ H^W_{M_1,M_2} : M_4(\Re^4) \rightarrow M_4(\Re^4) \\ H^W_{M_1,M_2}(X) = M_1^{-1}M_2X - X(am_1w)^{-1}(am_2w) \end{cases}$$

$$\mathbf{A} \begin{cases} AM_2M_1^{-1} - (am_2w)(am_1w)^{-1}A = 0 \\ H^A_{M_1,M_2} : M_4(\Re^4) \rightarrow M_4(\Re^4) \\ H^A_{M_1,M_2}(X) = XM_2M_1^{-1} - (am_2w)(am_1w)^{-1}X \end{cases}$$

Calibration: Practical determination of H^W and H^A

Calibration of the PSG and the PSA requires a set of calibration samples. The choice of the calibration set is not unique.

Example of calibration set:
- 1 metallic reflecting surface (M_0)
- 1 polarizer at 2 azimuths (M_1) and (M_2)
- 1 retarder or compensator (M_3)

1. Measurement of B_0 from the set of reference samples : $B_0 = A.M_0W$

2. Measurement of B_1, B_2, B_3 from the set of reference samples :

$$B_1 = A.M_0P_{\theta 1}(\tau)W, \quad B_2 = A.M_0P_{\theta 2}(\tau)W, \quad B_3 = A.M_0D(\tau_D,\psi,\Delta)W$$

3. Determination of the Mueller matrices of the ref. samples using Eigenvalue Method :

$$\left.\begin{aligned} B_0^{-1}B_1 &= W^{-1}P_{\theta 1}(\tau)W \\ B_0^{-1}B_2 &= W^{-1}P_{\theta 2}(\tau)W \end{aligned}\right\}$$ τ of the polarizer,

$$B_0^{-1}B_3 = W^{-1}D(\tau_D,\psi,\Delta)W \longrightarrow$$ τ, ψ et Δ of the retarder.

4. Calculation of the linear mappings

$$\begin{cases} H^W_{M_0P_{\theta 1}}X = P_{\theta 1}X - X\,B_0^{-1}B_1 \\ H^W_{M_0P_{\theta 2}}X = P_{\theta 2}X - X\,B_0^{-1}B_2 \\ H^W_{M_0D}X = DX - X\,B_0^{-1}B_3 \end{cases}$$

5. Calculation of the matrix

$$\begin{cases} K^W = H^W_{M_1}\cdot(H^W_{M_1})^T + H^W_{M_2}\cdot(H^W_{M_2})^T + H^W_{M_3}\cdot(H^W_{M_3})^T \\ K^A = H^A_{M_1}\cdot(H^A_{M_1})^T + H^A_{M_2}\cdot(H^A_{M_2})^T + H^A_{M_3}\cdot(H^A_{M_3})^T \end{cases}$$

6. Solution of the 2 linear systems:

$$\begin{cases} K^W W = 0 \\ K^A A = 0 \end{cases}$$

Main advantages:

Simplicity:
- Only 3 measurements allow to find 16 unknown coefficients.
- No need to model the properties of the optical elements of the PSG and the PSA.

Robustness:
- Some parasitic artifacts (multiple reflections, diverging beams) are automatically accounted for.

Flexibility:
- The sample calibration set is not unique.
- Sample calibration set can be optimized to current working conditions.

Conclusions

- The design of a polarimeter is based on an objective criterion: conditioning optimization in order to minimize error propagation.
- Multiple optical configurations optimizing the condition number can be considered.
- A robust calibration procedure is applied that does not require modeling of the PSG or PSA optical elements. Easy implementation!