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Arrays of Cylindrical Dipoles


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To
Tai Tsun Wu

Good reasons must, of force, give place to better.
Shakespeare

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## PREFACE

Studies of coupled antennas in arrays may be separated into two groups: those which postulate a single convenient distribution of current along all structurally identical elements regardless of their relative locations in the array and those which seek to determine the actual currents in the several elements. Virtually all of the early and most of the more recent analyses are in the first group in which both field patterns and impedances have been obtained for elements with assumed currents. Pioneer work in the determination of field patterns of arrays of elements with sinusoidally distributed currents was carried out for uniform arrays by Bontsch-Bruewitsch [1] in 1926, by Southworth [2] in 1930, by Sterba [3] and by Carter et al. [4] in 1931. Early studies of non-uniform arrays are by Schelkunoff [5] in 1943, by Dolph [6] in 1946, and by Taylor and Whinnery [7] in 1951. The self- and mutual impedances of arrays of elements with sinusoidally distributed currents were studied especially by Carter [8] in 1932, by Brown [9] in 1937, by Walkinshaw [10] in 1946, by Cox [11] in 1947, by Barzilai [12] in 1948, and by Starnecki and Fitch [13] in 1948. A thorough presentation of the basic theory of antennas with sinusoidal currents was given by Brückmann [14] in 1939. Actually, the current in any cylindrical antenna of length $2 h$ and finite radius $a$ is accurately sinusoidal only when it is driven by a continuous distribution of electromotive forces of proper amplitude and phase along its entire length. It is approximately sinusoidal in an isolated very thin antenna ( $a \ll h$ ) driven by a single lumped generator primarily when the antenna is near resonance. When antennas are coupled in an array with each driven by a single generator or excited parasitically, it is generally assumed that (1) the phase of the current along each element is the same as at the driving point and (2) the amplitude is distributed sinusoidally. Both of these assumptions are reasonably well satisfied only for very thin antennas $(a \ll \lambda)$ that are not too long $(h \leq \lambda / 4)$. Nevertheless, a very extensive theory of arrays has been developed based implicitly on one or both of these assumptions. Evidently it is correspondingly restricted in its generality.

The analysis of coupled antennas from the point of view of determining the actual distributions of current was studied for two antennas by Tai [15] in 1948 and extended to the $N$-element circular array by King [16] in 1950. A general analysis of arrays of coupled antennas has been given by King [17]. Unfortunately, the rigorous solution of the simultaneous integral equations for the distributions of current in the elements of an array of parallel elements is very complicated and no simple and practically useful set of formulas was obtained. As a consequence, the extensive study of the electromagnetic fields of antennas and arrays in this earlier work (chapters 5 and 6 in King [17]) was limited to arrays with currents in the elements that satisfied the assumptions of constant phase angle and sinusoidal amplitude. Similar restrictions are implicit in the fields calculated, for example, by Aharoni [18], Stratton [19], Hansen [20] and many others.

A practical method for obtaining solutions of the simultaneous integral equation for the distributions of current in the elements of a parallel array in a form that combines simplicity with quantitative accuracy was proposed by King [21] in 1959. In this analysis an approximate procedure was developed which provided simple, two-term trigonometric formulas for the currents in all of the arbitrarily driven or parasitic elements in a circular array of $N$ elements in a manner that took full account of the effects of mutual interactions on the distributions of current. These formulas applied to elements up to one and one-quarter wavelengths long. The application of this new procedure to actual arrays and the experimental verification of the results were carried out in an extensive series of investigations by Mack [22]. The generalization of the method to curtain arrays was developed by King and Sandler [23,24] in 1962. The extension of the method to parasitic elements in arrays of the Yagi type was verified experimentally by Mailloux [25] in 1964. A modification of the theory and its application to the optimization of Yagi arrays by the use of a high-speed computer were devised by Morris [26] in 1964. In 1967 Cheong [27] extended the theory to unequal and unequally spaced elements. (The several researches were supported in part by Joint Services Contract Nonr 1866(32), Air Force Contract AF19(604)-4118 and National Science Foundation Grants NSF-GP-851 and GK-273.)

A further improvement in the simplified trigonometric representation of the current in an isolated antenna was introduced by King and Wu [28] in 1964 and extended to arrays in the present work.

This book begins with an introductory chapter that reviews the foundations and limitations of conventional antenna theory. It then proceeds to derive the new two- and three-term formulas for the isolated antenna in chapter 2 and for two coupled antennas in chapter 3. Chapter 4 provides the complete formulation of the new theory for the $N$-element circular array; chapter 5 for the $N$-element curtain array of identical elements. The more difficult problem of treating elements of different lengths-notably in the Yagi array and the log-periodic antenna-is treated in chapter 6. Chapter 7 is devoted to planar and three-dimensional arrays that include staggered and collinear elements. Chapter 8 is concerned with the broad problems of measurement-currents, impedances, field patterns and the correlation of theory with experiment. In the appendices summaries of programmes are given for the computational analysis of circular, curtain, and Yagi arrays.

In the preparation of the manuscript, S. S. Sandler was responsible for chapters 1 and 5, R. B. Mack for chapters 4 and 8 , and R. W. P. King for chapters 2, 3, 6, and 7 and for the co-ordination of the several parts.

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S.S.S.

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## CHAPTER 1

## INTRODUCTION

### 1.1 Fundamentals-field vectors and potential functions

Radio communication depends upon the interaction of oscillating electric currents in specially designed, often widely separated configurations of conductors known as antennas. Those considered in this book consist of thin metal wires, rods or tubes arranged in parallel arrays of circular or planar form. Electric charges in the conductors of a transmitting array are maintained in systematic accelerated motion by suitable generators that are connected to one or more of the elements by transmission lines. These oscillating charges exert forces on other charges located in the distant conductors of a receiving array of elements of which at least one is connected by a transmission line to a receiver. Fundamental quantities upon which such an interaction depends are the electromagnetic field and the driving-point admittance. But these are completely determined by the distribution of current in the elements of an array. In this first chapter the basic electromagnetic equations are formulated and applied to simple antennas and arrays in the conventional manner which is based on assumed rather than actual currents. The limitations of this approach are pointed out as an introduction to the more accurate formulation of the theory of antennas and arrays that is presented in subsequent chapters.

Consider first the very simple, physically realizable transmitting antenna shown in Fig. 1.1. It consists of a thin conductor extending from $z=-h$ to $z=h$ that is centre driven by a generator which maintains a periodically varying potential difference across its terminals at $z= \pm \frac{1}{2} b$. The transmission line consists of two wires that are separated by a distance $b$ that is small compared to the wavelength $\lambda$ so that $b \ll \lambda$. Its output end is connected to the adjacent terminals of the antenna. Owing to the complications involved in a small region comparable in extent with the line spacing $b$, where antenna and line are coupled, it is convenient in an introductory and elementary analysis of the field properties of linear antennas to replace the actual generator-transmission-line
with an idealized so-called delta-function generator. This maintains the electric field $E_{z}=-V_{0} \delta(z)$ on the surface of the antenna. The properties of the delta function are:
and

$$
\begin{gather*}
\delta(z)= \begin{cases}0, & z \neq 0 \\
\infty, & z=0\end{cases}  \tag{1.1a}\\
\int_{-h}^{h} V(z) \delta(z) d z=V(0) \tag{1.1b}
\end{gather*}
$$



Fig. 1.1. Practical antenna system.
The problem of relating the impedance obtained for an idealized delta-function source to that actually measured with a transmission line is discussed later. A simplified linear antenna is shown in Fig. 1.2. For this introductory study the conventional approach is followed and a sinusoidally distributed current is assumed along the antenna. Measurements of the current along very thin cylindrical antennas indicate that the current is distributed approximately sinusoidally especially when $h \leqslant \lambda / 4$. Since the general shape of the field pattern of an isolated linear antenna does not depend critically on the distribution of current along the antenna, this approximation involves less error in the calculation of the major lobe of the far field pattern than in the evaluation of the minor lobe structure or the driving-point impedance. The assumption of a sinusoidal current implies that the distribution of current (but not its amplitude) is independent of the radius $a$ of the antenna. Measurements show that the assumed sinusoidal current is a fairly good approximation near the first one or two resonant lengths ( $h \sim n \lambda / 4, n=1,3, \ldots$ ) of very thin antennas; it
is not satisfactory near anti-resonant lengths ( $h \sim n \hat{\lambda} / 4, n=2,4, \ldots$ ). The sinusoidal assumption is critically involved in the accuracy of the calculation of the driving-point impedance of an isolated antenna. When coupled antennas are considered, an assumed sinusoidal distribution of current proves to be a major source of inaccuracy in the calculations of both the driving-point impedance and the radiation pattern.


Fig. 1.2. Linear antenna with cylindrical cross-section.
The interaction of charges and currents on conductors in space is governed by the well-known Maxwell-Lorentz equations which define the electromagnetic field. With an assumed time dependence $e^{j \omega t}$, they are

$$
\begin{array}{ll}
\nabla \times \mathbf{B}=\mu_{0}\left(\mathbf{J}+j \omega \varepsilon_{0} \mathbf{E}\right), & \nabla \cdot \mathbf{B}=0 \\
\nabla \times \mathbf{E}=-j \omega \mathbf{B}, & \nabla \cdot \mathbf{E}=\rho / \varepsilon_{0}=0 \tag{1.2b}
\end{array}
$$

where the electric vector $\mathbf{E}$ is in volts per meter, the magnetic vector B in webers per square meter. Rationalized MKS units are used throughout this book. The volume density of current $\mathbf{J}$ in amperes per square meter is the charge crossing unit area per second. In a perfect conductor $\mathbf{J}=0$. The volume density of charge $\rho$ in coulombs per cubic meter is zero in the interior of all conductors. The universal electric and magnetic constants are, respectively, $\varepsilon_{0}$ and $\mu_{0}$. They have the numerical values $\varepsilon_{0}=8.854 \times 10^{-12}$ farads per meter and $\mu_{0}=4 \pi \times 10^{-7}$ henrys per meter. The relevant boundary conditions at an interface between a conductor
(subscript 1) and air (subscript 2) are expressed in terms of the tangential and normal components of the electric and magnetic fields. Thus
and

$$
\begin{align*}
\hat{\mathbf{n}}_{1} \times \mathbf{E}_{1}+\hat{\mathbf{n}}_{2} \times \mathbf{E}_{2} & =0  \tag{1.3a}\\
\hat{\mathbf{n}}_{1} \times \mathbf{B}_{1}+\hat{\mathbf{n}}_{2} \times \mathbf{B}_{2} & =-\mu_{0} \mathbf{K}_{1}  \tag{1.3b}\\
\hat{\mathbf{n}}_{1} \cdot \mathbf{E}_{1}+\hat{\mathbf{n}}_{2} \cdot \mathbf{E}_{2} & =\frac{-\eta_{1}}{\varepsilon_{0}}  \tag{1.3c}\\
\hat{\mathbf{n}}_{1} \cdot \mathbf{B}_{1}+\hat{\mathbf{n}}_{2} \cdot \mathbf{B}_{2} & =0 \tag{1.3~d}
\end{align*}
$$

where $\hat{\mathbf{n}}$ is the unit outward normal to the region indicated by the subscript, $\mathbf{K}_{1}$ is the density of surface current in amperes per meter and $\eta_{1}$ is the density of surface charge in coulombs per square meter on the conductor. If medium 1 is a perfect conductor in which $\mathbf{E}_{1}=\mathbf{B}_{1}=0$ and medium 2 is air, then (1.3) simplifies to

$$
\begin{align*}
\hat{\mathbf{n}}_{2} \times \mathbf{E}_{2} & =0  \tag{1.4a}\\
\hat{\mathbf{n}}_{2} \times \mathbf{B}_{2} & =-\mu_{0} \mathbf{K}_{1}  \tag{1.4b}\\
\hat{\mathbf{n}}_{2} \cdot \mathbf{E}_{2} & =\frac{-\eta_{1}}{\varepsilon_{0}}  \tag{1.4c}\\
\hat{\mathbf{n}}_{2} \cdot \mathbf{B}_{2} & =0 . \tag{1.4d}
\end{align*}
$$

Equation (1.4a) states that the component of the electric field in air tangent to the surface of a perfect conductor must be zero. Equation (1.4b) states that at the surface of a perfect conductor the tangential magnetic field in air is proportional to the surface density of current in the conductor.

A convenient method of solving the vector partial differential equations (1.2) is with the use of scalar and vector potentials, $\phi$, A. The defining relationships between the potentials and the electromagnetic field vectors are obtained with the aid of Maxwell's equations. With the vector identity $\nabla \cdot(\nabla \times \mathbf{C})=0$ (where $\mathbf{C}$ is any vector) and the Maxwell equation $\nabla . \mathbf{B}=0$, the magnetic field may be expressed in the form

$$
\begin{equation*}
\mathbf{B}=\nabla \times \mathbf{A} . \tag{1.5a}
\end{equation*}
$$

If (1.5a) is substituted in (1.2b) it follows that

$$
\begin{equation*}
\nabla \times(\mathbf{E}+j \omega \mathbf{A})=0 . \tag{1.5b}
\end{equation*}
$$

The identity $\nabla \times(\nabla \psi)=0$, where $\psi$ is a scalar function, then permits the definition of $\phi$ in the form

$$
\begin{equation*}
-\nabla \phi=\mathbf{E}+j \omega \mathbf{A} . \tag{1.5c}
\end{equation*}
$$

The substitution of (1.5a) and (1.5c) into the remaining Maxwell equations leads to mixed vector equations for $\mathbf{A}$ and $\phi$. The variables can be separated if the following condition relating $\mathbf{A}$ and $\phi$ is imposed:

$$
\begin{equation*}
\nabla \cdot \mathbf{A}=\frac{-j \beta_{0}^{2}}{\omega} \phi \tag{1.6}
\end{equation*}
$$

This is known as the Lorentz condition. The resulting vector Helmholtz equations for $\mathbf{A}$ and $\phi$ in air are
and

$$
\begin{align*}
& \left(\nabla^{2}+\beta_{0}^{2}\right) \mathbf{A}=-\mu_{0} \mathbf{J}  \tag{1.7a}\\
& \left(\nabla^{2}+\beta_{0}^{2}\right) \phi=-\rho / \varepsilon_{0}=0 . \tag{1.7b}
\end{align*}
$$

The antenna theory developed in this book is concerned exclusively with thin cylindrical conductors all aligned in the $z$-direction in air so that it suffices to use only the axial component of the vector potential. With $\mathbf{A}=\hat{\mathbf{z}} A_{z}$ the vector Laplacian in (1.7a) is reduced to a scalar as in (1.7b). The simplified form of (1.7a) is

$$
\begin{equation*}
\left(\nabla^{2}+\beta_{0}^{2}\right) A_{z}=-\mu_{0} J_{z} . \tag{1.7c}
\end{equation*}
$$

Particular integrals of (1.7b) and (1.7c) are directly derivable with the use of the free-space Green's function. The integrals for $A_{z}$ and $\phi$ for a thin cylindrical conductor of length $2 h$ and radius $a$ with its centre at $z=0$ are

$$
\begin{equation*}
A_{z}=\frac{\mu_{0}}{4 \pi} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) \frac{e^{-j \beta_{0} R}}{R} d z^{\prime} \tag{1.8a}
\end{equation*}
$$

and

$$
\begin{equation*}
\phi=\frac{1}{4 \pi \varepsilon_{0}} \int_{-h}^{h} q\left(z^{\prime}\right) \frac{e^{-j \beta_{0} R}}{R} d z^{\prime} \tag{1.8b}
\end{equation*}
$$

where, with $\Sigma=\pi a^{2}$ the area of cross-section, the total axial current is

$$
\begin{equation*}
I_{z}(z)=\int_{\Sigma} J_{z} d \Sigma+2 \pi a K_{z} \tag{1.8c}
\end{equation*}
$$

and the charge per unit length is

$$
\begin{equation*}
q(z)=2 \pi a \eta \tag{1.8d}
\end{equation*}
$$

The wave number is $\beta_{0}=2 \pi / \lambda_{0}$, where $\lambda_{0}$ is the free-space wavelength; $R=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}}$. For a perfect conductor $J_{z}=0$. The one-dimensional Lorentz condition is

$$
\begin{equation*}
\frac{\partial A_{z}}{\partial z}=\frac{-j \beta_{0}^{2}}{\omega} \phi . \tag{1.9}
\end{equation*}
$$

The continuity equation expresses the condition of conservation of charge. For a thin cylindrical antenna it has the form

$$
\begin{equation*}
\frac{d I_{z}(z)}{d z}=-j \omega q(z) \tag{1.10}
\end{equation*}
$$

The $\mathbf{E}$ and $\mathbf{B}$ fields for a finite cylindrical conductor are obtained from (1.5a) and (1.5c) with (1.8a) and (1.9). In the cylindrical coordinates $\rho, \boldsymbol{\Phi}, z$, they are $\mathbf{B}=\boldsymbol{\Phi} B_{\Phi}$ and $\mathbf{E}=\hat{\boldsymbol{\rho}} E_{\rho}+\hat{\mathbf{z}} E_{z}$ where

$$
\begin{align*}
B_{\Phi} & =\frac{-\partial A_{z}}{\partial \rho}  \tag{1.11a}\\
E_{\rho} & =\frac{-j \omega}{\beta_{0}^{2}} \frac{\partial^{2} A_{z}}{\partial \rho \partial z}  \tag{1.11b}\\
E_{z} & =\frac{-j \omega}{\beta_{0}^{2}}\left(\frac{\partial^{2} A_{z}}{\partial z^{2}}+\beta_{0}^{2} A_{z}\right) . \tag{1.11c}
\end{align*}
$$

In the spherical coordinates $r, \Theta, \Phi$ with origin at the centre of the antenna, the field is given by

$$
\begin{align*}
E_{r} & =E_{z} \cos \Theta+E_{\rho} \sin \Theta  \tag{1.12a}\\
E_{\Theta} & =-E_{z} \sin \Theta+E_{\rho} \cos \Theta \tag{1.12b}
\end{align*}
$$

At sufficiently great distances from the antenna ( $r^{2} \gg h^{2}$ and $\left.\left(\beta_{0} r\right)^{2} \gg 1\right)$, the field reduces to a simple form known as the radiation or far field. It is given by

$$
\begin{equation*}
B_{\Phi}^{r}=E_{\oplus}^{r} / c \tag{1.13a}
\end{equation*}
$$

where $c$ is the velocity of light and

$$
\begin{equation*}
\mathbf{E}^{r} \doteq E_{\Theta}^{r} \hat{\boldsymbol{\Theta}}, \quad E_{\Theta}^{r}=\frac{j \omega \mu_{0}}{4 \pi} \sin \Theta \int_{-h}^{h} I_{z}\left(z^{\prime}\right) \frac{e^{-j \beta_{0} R}}{R} d z^{\prime} \tag{1.13b}
\end{equation*}
$$

The distance $R$ from an arbitrary point on the antenna to the field point is given in terms of $r$ and $z^{\prime}$ by the cosine law, viz., (Fig. 1.3)

$$
\begin{equation*}
R^{2}=r^{2}+\left(z^{\prime}\right)^{2}-2 r z^{\prime} \cos \Theta \tag{1.14a}
\end{equation*}
$$

In the radiation zone $r^{2} \gg\left(z^{\prime}\right)^{2}$. If the binomial expansion is applied to (1.14a) and only the linear term in $z^{\prime}$ is retained, the following approximate form is obtained for $R$ :

$$
\begin{equation*}
R \doteq r-z^{\prime} \cos \Theta, \quad\left(\beta_{0} r\right)^{2} \gg 1 \tag{1.14b}
\end{equation*}
$$

The phase variation of $\exp \left(-j \beta_{0} R\right) / R$ is replaced with the linear phase variation given by (1.14b), i.e. by $\exp \left(-j \beta_{0} r+j \beta_{0} z^{\prime} \cos \Theta\right)$. The amplitude $1 / R$ of $\exp \left(-j \beta_{0} R\right) / R$ is a slowly varying function of $z^{\prime}$ and is replaced by $1 / r$, where $r$ is the distance to the centre of
the antenna. Since $r$ is independent of $z^{\prime}$, all functions of $r$ may be removed from the integral in (1.13b) and the final form for $\mathbf{E}^{r}$ is

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{z}(0)}{2 \pi} \frac{e^{-j \beta_{0} r}}{r} F_{0}\left(\Theta, \beta_{0} h\right) \tag{1.15a}
\end{equation*}
$$

where $\zeta_{0}=\sqrt{\mu_{0} / \varepsilon_{0}} \doteq 120 \pi$ ohms and

$$
\begin{equation*}
F_{0}\left(\Theta, \beta_{0} h\right)=\frac{\beta_{0} \sin \Theta}{2 I_{z}(0)} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) e^{j \beta_{0} z^{\prime} \cos \Theta} d z^{\prime} \tag{1.15b}
\end{equation*}
$$



Fig. 1.3. Coordinate system for calculations in the far zone.
The term $F_{0}\left(\Theta, \beta_{0} h\right)$ contains all the directional properties of a linear radiator of length $2 h$. It is called the field characteristic or field factor and will be computed for some commonly used current distributions. The magnetic field $\mathbf{B}^{r}$ in the far zone is at right angles to $\mathbf{E}^{r}$ and also perpendicular to the direction of propagation $\mathbf{r}$. It is given by (1.13a). Thus

$$
\begin{equation*}
\mathbf{B}^{r}=\hat{\boldsymbol{\Phi}} B_{\Phi}^{r}, \quad B_{\Phi}^{r}=\frac{j \mu_{0} I_{z}(0)}{2 \pi} \frac{e^{-j \beta_{0} r}}{r} F_{0}\left(\Theta, \beta_{0} h\right) . \tag{1.15c}
\end{equation*}
$$

Note that the field in the far zone depends on $F_{0}\left(\Theta, \beta_{0} h\right)$ which is a function of the particular distribution of current in the antenna.

It is instructive to consider the instantaneous value of the field in ( 1.15 a ), which is obtained by multiplication with $e^{j \omega t}$ and selection
of the real part. If the phase of the field is referred to that of the current

$$
\begin{equation*}
E_{\Theta}^{r}(\mathbf{r}, t)=\operatorname{Re} E_{\Theta}(\mathbf{r}) e^{j \omega t} \sim \frac{\sin \left(\omega t-\beta_{0} r\right)}{r}=\frac{\sin \omega(t-r / c)}{r} \tag{1.16a}
\end{equation*}
$$

Note that the field at the point $r$ at the instant $t$ is computed from the current at $r=0$ at the earlier time $(t-r / c)$. This is a consequence of the finite velocity of propagation $c$.

The equiphase and equipotential surfaces of $\mathbf{E}$ and $\mathbf{B}$ are spherical shells on which $r$ is equal to a constant. There are an infinite number of such shells that have the same phase (differ by an integral multiple of $2 \pi$ ) but only one that has both the same amplitude and the same phase. The velocity of propagation is the outward radial velocity of the surfaces of constant phase where the phase is represented by the argument of the sine term in (1.16a), that is,

$$
\begin{equation*}
\text { phase }=\Psi=\omega t-\beta_{0} r . \tag{1.16b}
\end{equation*}
$$

For a constant phase

$$
\begin{equation*}
\frac{d \Psi}{d t}=0=\omega-\frac{\beta_{0} d r}{d t} \tag{1.16c}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
\frac{d r}{d t}=\frac{\omega}{\beta_{0}}=c=3 \times 10^{8} \mathrm{~m} / \mathrm{sec} . \tag{1.16d}
\end{equation*}
$$

Since the phase repeats itself every $2 \pi$ radians, a wavelength is the distance between two adjacent equiphase surfaces. For example, if one surface is defined by $r=r_{1}$ and the other by $r=r_{2}$ then

$$
\begin{equation*}
\omega t-\beta_{0} r_{1}=2 \pi \quad \text { and } \quad \omega t-\beta_{0} r_{2}=4 \pi \tag{1.17a}
\end{equation*}
$$

or

$$
\begin{equation*}
r_{2}-r_{1}=\frac{2 \pi}{\beta_{0}}=\lambda_{0} \tag{1.17b}
\end{equation*}
$$

where $\lambda_{0}$ is the wavelength in air. The physical picture of the fields in the far zone is quite simple. The electric and magnetic vectors are mutually orthogonal and tangent to an outward travelling spherical shell. Thus, both components of the field are transverse to the radius vector $\mathbf{r}$; they have the same phase velocity $c=3 \times 10^{8}$ $\mathrm{m} / \mathrm{sec}$, the velocity of light.

### 1.2 Power and the Poynting vector

In the conventional approach to antenna theory, the power radiated is usually determined by an application of the Poyntingvector theorem. An equation for the time-average power associated with a radiating antenna or array is readily derived from

$$
\begin{equation*}
\operatorname{Re}_{\frac{1}{2}}\left\{\int_{\tau} \mathbf{J}^{*} \cdot \mathbf{E} d \tau+\int_{\Sigma} \mathbf{K}^{*} \cdot \mathbf{E} d \Sigma\right\} \tag{1.18}
\end{equation*}
$$

where Re indicates the real part. $\tau$ is the volume occupied by the currents $\mathbf{J}$ when imperfect conductors are considered; $\Sigma$ is the surface of perfect conductors on which are the currents $\mathbf{K}$. The asterisk denotes the complex conjugate. It is, of course, clear that since $\mathbf{J}$ vanishes in air, the volume of integration may be enlarged to any desired size so long as the only contributions come from the antennas under study. When attention is directed to a single antenna isolated in space, $\tau$ may be extended to infinity.

The next step in the derivation of the desired power equation is the elimination of $\mathbf{J}^{*}$ and $\mathbf{K}^{*}$ from (1.18) by substitution from (1.2a) and (1.4b). Note first that with the vector identity

$$
\begin{equation*}
\nabla .\left(\mathbf{E} \times \mathbf{B}^{*}\right)=\mathbf{B}^{*} .(\nabla \times \mathbf{E})-\mathbf{E} .\left(\nabla \times \mathbf{B}^{*}\right) \tag{1.19a}
\end{equation*}
$$

and the complex conjugate of (1.2b), the following equation can be obtained:

$$
\begin{equation*}
\mathbf{E} \cdot\left(\nabla \times \mathbf{B}^{*}\right)=\nabla \cdot\left(\mathbf{E} \times \mathbf{B}^{*}\right)+j \omega \mathbf{B} \cdot \mathbf{B}^{*} . \tag{1.19b}
\end{equation*}
$$

With (1.19b) and (1.2a),

$$
\begin{align*}
\mathbf{J}^{*} \cdot \mathbf{E} & =\mu_{0}^{-1} \mathbf{E} \cdot\left(\nabla \times \mathbf{B}^{*}\right)+j \omega \varepsilon_{0} \mathbf{E} \cdot \mathbf{E}^{*} \\
& =\mu_{0}^{-1} \nabla \cdot\left(\mathbf{E} \times \mathbf{B}^{*}\right)+j \omega\left(\varepsilon_{0} \mathbf{E} \cdot \mathbf{E}^{*}+\mu_{0}^{-1} \mathbf{B} \cdot \mathbf{B}^{*}\right) . \tag{1.20a}
\end{align*}
$$

From (1.4b),

$$
\begin{equation*}
\mathbf{K}^{*} \cdot \mathbf{E}=-\mu_{0}^{-1}\left(\hat{\mathbf{n}} \times \mathbf{B}^{*}\right) \cdot \mathbf{E}=\mu_{0}^{-1} \hat{\mathbf{n}} .\left(\mathbf{E} \times \mathbf{B}^{*}\right) . \tag{1.20b}
\end{equation*}
$$

Since E. $\mathbf{E}^{*}=E^{2}$ is real, it follows that

$$
\begin{equation*}
\operatorname{Re}_{\frac{1}{2}}\left\{\int_{\tau} \mathbf{J}^{*} \cdot \mathbf{E} d \tau+\int_{\Sigma} \mathbf{K}^{*} \cdot \mathbf{E} d \Sigma\right\}=\operatorname{Re}\left\{\int_{\tau} \nabla \cdot \mathbf{S} d \tau+\int_{\Sigma} \hat{\mathbf{n}} \cdot \mathbf{S} d \Sigma\right\} \tag{1.21}
\end{equation*}
$$

where

$$
\mathbf{S}=\left(\mathbf{E} \times \mathbf{B}^{*}\right) / 2 \mu_{0}
$$

is the complex Poynting vector. In (1.21) $\tau$ is any region sufficiently large to contain the currents $\mathbf{J}$ and $\mathbf{K}$. Let the enclosing surface for $\tau$ be $\Sigma_{\tau}$. Note that $\Sigma$ represents surfaces across which $\mathbf{B}$ is discontinuous.

If the divergence theorem

$$
\begin{equation*}
\int_{\tau} \nabla \cdot \mathbf{S} d \tau=\int_{\Sigma_{\tau}} \hat{\mathbf{n}} \cdot \mathbf{S} d \Sigma \tag{1.22}
\end{equation*}
$$

where $\hat{\mathbf{n}}$ is the external normal to $\Sigma_{\tau}$, is applied to the first integral in (1.21), all surfaces of discontinuity $\Sigma$ in $S$ must be excluded by an enclosing boundary since (1.22) is valid only for continuous functions. If this is done, the integrals over these surfaces of discontinuity $\Sigma$ exactly cancel the last integral in (1.21). The result is

$$
\begin{equation*}
\operatorname{Re} \frac{1}{2}\left\{\int_{\tau} \mathbf{J}^{*} \cdot \mathbf{E} d \tau+\int_{\Sigma} \mathbf{K}^{*} \cdot \mathbf{E} d \Sigma\right\}=\operatorname{Re} \int_{\Sigma_{\mathrm{\varepsilon}}} \hat{\mathbf{n}} \cdot \mathbf{S} d \Sigma . \tag{1.23}
\end{equation*}
$$

The left side in (1.23) is now readily specialized to thin, currentcarrying antennas. For simplicity, let these be perfect conductors in the interior of which $\mathbf{J}$ and $\mathbf{E}$ vanish. With rotational symmetry, the complex conjugate of the total axial current is

$$
\begin{equation*}
I_{z}^{*}(z)=2 \pi a K_{z}^{*}(z) . \tag{1.2}
\end{equation*}
$$

In a tabular conductor there are no radial currents; if the ends are capped, the small radial currents on these may be neglected. It follows that (1.23) reduces to

$$
\begin{equation*}
\operatorname{Re}^{\frac{1}{2}} \int_{-h}^{h} I_{z}^{*}(z) E_{z}(z) d z=\operatorname{Re} \int_{\Sigma_{\mathrm{r}}} \hat{\mathbf{n}} \cdot \mathbf{S} d \Sigma . \tag{1.25}
\end{equation*}
$$

For an antenna driven by a delta-function generator at $z=0$, the boundary condition at $\rho=a$ is $E_{z}(z)=-V_{0} \delta(z)$ so that the left side of (1.25) gives simply
where $P$ is the time-average power supplied to the antenna at its terminals by the generator. Since there is no dissipation in the perfectly conducting antenna or the surrounding air, the integral on the right is the total radiated power. It is independent of the shape or size of the surface $\Sigma_{\tau}$ so long as this completely encloses the entire transmitting system consisting of antenna and generator and any connecting transmission line. It is important to note that the Poynting-vector theorem cannot be used logically to determine a path for the hypothetical flow of energy from a generator to a distant receiver. For example, let the surface $\Sigma_{\tau}$ in (1.26) be the surface of a pill box of infinitesimal thickness and radius $a$ that encloses the entire delta-function generator at $z=0$ but otherwise none of the centre-driven perfectly conducting antenna, that extends from
$z=-h$ to $z=h$. The same total radiated power is still transferred across $\Sigma_{\tau}$ and, since $\mathbf{E}=0$ in the interior of the conductor, the entire contribution to the integral comes from the ring bounding the delta function at $z=0$ and $\rho=a$. This might be interpreted naively to mean that energy is transferred directly from the generator to the rest of the universe. However, since it is the currents in the antenna that maintain the electromagnetic field that exerts forces on charges in a distant receiving antenna and so do work, they cannot logically be excluded from the radiation process.

It is readily shown with reference to the simple transmitting system in Fig. 1.1 that, when $\Sigma_{\tau}$ encloses only the transmitter oscillator, the total radiated power is obtained. On the other hand, if $\Sigma_{\tau}$ is a closed surface around the antenna-of course crossed by the transmission line-the integral and, hence, the total power transferred outward is zero. Again, this might be interpreted naively as an indication that the generator radiates power and the antenna has nothing to do with it.

The real part of the integral of the normal component of the complex Poynting vector over any surface that completely encloses a complete transmitting system-antennas, transmission line (if there is one), and generator-correctly gives the total power transferred from the region inside the surface to the region outside. The conclusion that therefore the Poynting vector itself specifies the rate of flow of energy across each unit of area is without foundation. Nevertheless, it is a common assumption.

### 1.3 The field of the electrically short antenna; directivity

Consider first the radiation field of an electrically short linear antenna, where $\left(\beta_{0} h\right)^{2} \ll 1$, with a triangular current distribution which vanishes at $z= \pm h$. Actually, this is a special case of a sinusoidal distribution which is discussed later. A diagram of the triangular distribution is shown in Fig. 1.4, where the magnitude of the current is plotted along an axis perpendicular to the antenna. Since $\left(\beta_{0} z^{\prime}\right)^{2} \leqslant\left(\beta_{0} h\right)^{2} \ll 1$, the exponent in $F_{0}\left(\Theta, \beta_{0} h\right)$ may be expanded through the linear term. Thus,

$$
\begin{align*}
& F_{0}\left(\Theta, \beta_{0} h\right) \doteq \frac{\beta_{0} \sin \Theta}{2} \int_{-h}^{h}\left(1-\frac{z^{\prime}}{h}\right)\left(1+j \beta_{0} z^{\prime} \cos \Theta+\ldots\right) d z^{\prime}  \tag{1.27a}\\
& F_{0}\left(\Theta, \beta_{0} h\right) \doteq \frac{\beta_{0} h \sin \Theta}{2}, \quad\left(\beta_{0} h\right)^{2} \ll 1 . \tag{1.27b}
\end{align*}
$$



Fig. 1.4. Linear antenna with triangular distribution of current.

Equation (1.27b) shows that the radiation field of a short linear antenna is proportional to $\sin \Theta$. Polar and rectangular graphs of the field are shown in Figs. $1.5 a$ and $1.5 b$, normalized with respect to the maximum at $\Theta=90^{\circ}$.

The field quite near an electrically short antenna is readily evaluated from (1.8a) with $I(z)=I(0)(1-|z| / h)$ and $R \doteq r$. This gives

$$
\begin{equation*}
A_{z} \doteq \frac{\mu_{0} h I(0)}{4 \pi} \frac{e^{-j \beta_{0} r}}{r} . \tag{1.28}
\end{equation*}
$$

The components of the field can be evaluated in the spherical coordinates $r, \Theta, \Phi$ from (1.5a) and (1.5c) with (1.9). The results are

$$
\begin{align*}
& B_{\Phi} \doteq \frac{\mu_{0} h I(0)}{4 \pi}\left(\frac{j \beta_{0}}{r}+\frac{1}{r^{2}}\right) e^{-j \beta_{0} r} \sin \Theta  \tag{1.29a}\\
& E_{r} \doteq \frac{\zeta_{0} h I(0)}{4 \pi}\left(\frac{2}{r^{2}}-\frac{j 2}{\beta_{0} r^{3}}\right) e^{-j \beta_{0} r} \cos \Theta  \tag{1.29b}\\
& E_{\Theta} \doteq \frac{j \zeta_{0} h I(0)}{4 \pi}\left(\frac{\beta_{0}}{r}-\frac{j}{r^{2}}-\frac{1}{\beta_{0} r^{3}}\right) e^{-j \beta_{0} r} \sin \Theta . \tag{1.29c}
\end{align*}
$$

These expressions are valid subject to the conditions

$$
\begin{equation*}
\left(\beta_{0} h\right)^{4} \ll 1, \quad(h / r)^{3} \ll 1, \quad(a / h)^{2} \ll 1 . \tag{1.29~d}
\end{equation*}
$$

They may be expressed in terms of the dipole moment $p_{z}=I(0) h / j \omega$ if desired. The electromagnetic power transferred across a closed surface is given by the integral of the normal component of the complex Poynting vector $\mathbf{S}=\frac{1}{2} \mu_{0}^{-1} \mathbf{E} \times \mathbf{B}^{*}$ over the surface. (The asterisk denotes the complex conjugate.) For an electrically small antenna ( $\hat{\mathbf{n}} . \mathbf{S}$ ) $\sim \sin ^{2} \Theta$. An angular graph of $\mathbf{S}$ is called a power pattern. Polar and rectangular graphs of the power pattern are shown in Figs. 1.5c and 1.5d. Note that because of symmetry both the field and power patterns are independent of the coordinate $\Phi$.


Fig. 1.5. (a) Field pattern, polar plot. (b) Field pattern, rectangular plot. (c) Power pattern. polar plot. (d) Power pattern. rectangular plot.

The half-power beamwidth $\Theta_{\mathrm{hp}}$ is defined as the angular distance between half-power points on the radiation pattern referred to the principal lobe. The value of $\Theta_{\mathrm{hp}}$ for the short linear antenna is $90^{\circ}$ since the field is down by a factor of $\sqrt{2} / 2$ at $\Theta= \pm 45^{\circ}$. Another parameter useful in defining the directive properties of an antenna is the absolute directivity $D$. This parameter is a measure of the total time-average power transferred across a closed surface in the direction of the principal lobe. The time-average power transferred across a closed surface $\Sigma$ is the integral of the normal component of S. Thus

$$
\begin{equation*}
P=\int_{\Sigma} \hat{\mathbf{n}} \cdot \mathbf{S} d \Sigma \tag{1.30}
\end{equation*}
$$

where $\hat{\mathbf{n}}$ is the unit external normal to the surface. The directivity $D$
is the ratio of $P$ with $\mathbf{S}$ set at its maximum value $\mathbf{S}_{m}$ to the actual value of $P$. For a short dipole with $|S| \sim \sin ^{2} \Theta$, the value of $D$ is

$$
\begin{equation*}
D=\frac{4 \pi}{\int_{0}^{2 \pi} \int_{0}^{\pi} \sin ^{2} \Theta \sin \Theta d \Theta}=\frac{3}{2}, \tag{1.31}
\end{equation*}
$$

where the integration has been carried out over the surface of a great sphere. A nearly omnidirectional pattern requires a large value of $\Theta_{\mathrm{hp}}$ and a nearly unity value of $D$. A more directional pattern requires a smaller value of $\Theta_{\mathrm{hp}}$ and a larger value of $D$.

### 1.4 The field of antennas with sinusoidally distributed currents; radiation resistance

Conventional antenna theory applies specifically to antennas along which a sinusoidally distributed current is maintained. That is

$$
\begin{equation*}
I_{z}(z)=\frac{I_{z}(0) \sin \beta_{0}(h-|z|)}{\sin \beta_{0} h}=I_{m} \sin \beta_{0}(h-|z|) \tag{1.32}
\end{equation*}
$$

For this current the field characteristic $F_{0}\left(\Theta, \beta_{0} h\right)$ is given by (1.15b) with (1.32),

$$
\begin{equation*}
F_{0}\left(\Theta, \beta_{0} h\right)=\frac{\cos \left(\beta_{0} h \cos \Theta\right)-\cos \beta_{0} h}{\sin \beta_{0} h \sin \Theta} \tag{1.33a}
\end{equation*}
$$

An alternative field characteristic $F_{m}\left(\Theta, \beta_{0} h\right)$ is referred to the maximum value of the sinusoid, viz., $I_{m}=I_{z}(0) / \sin \beta_{0} h$ which occurs at $h-\lambda_{0} / 4$ when $\beta_{0} h \geqslant \pi / 2$.

$$
\begin{equation*}
F_{m}\left(\Theta, \beta_{0} h\right)=\frac{\cos \left(\beta_{0} h \cos \Theta\right)-\cos \beta_{0} h}{\sin \Theta} \tag{1.33b}
\end{equation*}
$$

The function $F_{m}\left(\Theta, \beta_{0} h\right)$ is shown graphically in Fig. 1.6 for several values of $h$. It is seen that the pattern corresponding to $\beta_{0} h=\pi / 2$ ( $h=\lambda_{0} / 4$ ) is only slightly narrower than the pattern for $\left(\beta_{0} h\right)^{2} \ll 1$ which is shown in Fig. 1.5. Note that as $\beta_{0} h$ is increased beyond $\pi$, minor lobes appear which successively become the major lobe and point in directions other than $\Theta=\pi / 2$.

For an antenna with the simple sinusoidal distribution of current, the complete electromagnetic field can be evaluated in cylindrical coordinates for points near the antenna. This is accomplished with the substitution of the current (1.32) in the general integral for the vector potential (1.8a) and the subsequent use of this expression in


Fig. 1.6. Field factor of linear antenna.
(1.11a)-(1.11c). The indicated differentiations can be carried out directly without evaluating the integral. The results are:

$$
\begin{align*}
& B_{\Phi}=\frac{j I_{m} \mu_{0}}{4 \pi \rho}\left[e^{-j \beta_{0} R_{1 h}}+e^{-j \beta_{0} R_{2 h}}-2 \cos \beta_{0} h e^{-j \beta_{0} R_{0}}\right]  \tag{1.34a}\\
& E_{\rho}=\frac{j I_{m} \zeta_{0}}{4 \pi \rho}\left[\frac{z-h}{R_{1 h}} e^{-j \beta_{0} R_{1 h}}+\frac{z+h}{R_{2 h}} e^{-j \beta_{0} R_{2 h}}-\frac{2 z}{R_{0}} \cos \beta_{0} h e^{-j \beta_{0} R_{0}}\right]  \tag{1.34b}\\
& E_{z}=\frac{-j I_{m} \zeta_{0}}{4 \pi}\left[\frac{e^{-j \beta_{0} R_{1 h}}}{R_{1 h}}+\frac{e^{-j \beta_{0} R_{2 h}}}{R_{2 h}}-2 \cos \beta_{0} h \frac{e^{-j \beta_{0} R_{0}}}{R_{0}}\right] \tag{1.34c}
\end{align*}
$$

where $R_{0}=\sqrt{z^{2}+\rho^{2}}, R_{1 h}=\sqrt{(z-h)^{2}+\rho^{2}}, R_{2 h}=\sqrt{(z+h)^{2}+\rho^{2}}$ are, respectively, the distances from the point where the field is evaluated to the centre and the two ends of the antenna. Their interpretation in terms of spheroidal waves is available elsewhere. $\dagger$

It is often useful to relate the total power radiated by an antenna to the current at an arbitrary reference point. For a sinusoidally distributed current, the maximum value at $z=0$ when $h<\lambda / 4$ and at $z=h-\lambda / 4$ when $h \geqslant \lambda / 4$ is convenient. Since the total power radiated is given by (1.30), the desired relation is

$$
\begin{equation*}
\frac{1}{2}\left|I_{m}\right|^{2} R_{m}^{e}=P \tag{1.35}
\end{equation*}
$$

where the coefficient $R_{m}^{e}$ is the so-called radiation resistance referred to $I_{m}$. When $\beta_{0} h=\pi / 2, R_{m}^{e}=73.1$ ohms. The value of $R_{m}^{e}$ determined from (1.35) is not, in general, the driving-point resistance $R_{0}$ of a centre-driven antenna, although when $\beta_{0} h=\pi / 2$ so that $I_{m}=I(0), R_{m}^{e}$ does approximate $R_{0}$ when the antenna is sufficiently thin $\left(a / \lambda<10^{-5}\right.$ for an error of $5 \%$ or less). For very thin dipoles of resonant length the numerical values of $R_{0}^{e}$ determined from (1.35) resemble the experimental results. As defined in (1.35) and with (1.30) evaluated over the surface of a great sphere, the radiation resistance strictly is a characteristic of the far field and only approximately and under special circumstances a circuit property of the antenna at its terminals. This is considered in greater detail in section 1.7.

### 1.5 The field of a two-element array

Variations in the directional pattern of a single antenna obtained by changes in its length are of very limited practical value. Much

[^0]more useful directional properties are made available when additional antennas are arranged physically displaced and parallel to the original element in a configuration called an array. The twoelement array of Fig. 1.7a is an elementary example which illustrates important properties common to all arrays. The number of elements is denoted by $N=2$, the distance between the elements is $b$, and the relative phases and magnitudes of the driving-point currents $I_{1}(0)$ and $I_{2}(0)$ can be adjusted. It is tacitly assumed in this elementary theory that the interaction of the currents in the coupled antennas does not affect the distributions of current along the elements. These are taken to be identical for all elements in the array. (A more correct theory must, of course, consider the actual distributions of current, which are not alike except in special cases. Note that in general it is not possible to specify completely the current along an element in an array by assigning the current at the driving point.)


Fig. 1.7. (a) Two-element linear array. (b) Two-element array in $H$-plane.
It is required to find the total electric field for the array of Fig. 1.7 by superposition of the individual contributions from the elements. The formula for the field is simplest in the plane $\Theta=\pi / 2$, called the equatorial or $H$-plane. The field characteristic $F_{0}\left(\pi / 2, \beta_{0} h\right)$ is a constant in this plane and the total field is given by (1.15a). For two geometrically identical elements the result is

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0}}{2 \pi} F_{0}\left(\frac{\pi}{2}, \beta_{0} h\right)\left[I_{1}(0) \frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+I_{2}(0) \frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] \tag{1.36}
\end{equation*}
$$

where $R_{1}$ and $R_{2}$ are shown in Fig. 1.7b. The length $b$ of the array is assumed small in comparison to the radial distance $R_{0}$ to the
field point. The application of the law of cosines and the binomial expansion to the expressions for $R_{1}$ and $R_{2}$ yields the following simplified expressions:

$$
\begin{align*}
R_{1,2}^{2}= & R_{0}^{2}+\left(\frac{b}{2}\right)^{2} \pm b R_{0} \cos \Phi=R_{0}^{2}\left[1 \pm \frac{b}{R_{0}} \cos \Phi+\left(\frac{b}{a}\right)^{2} \frac{1}{R_{0}^{2}}\right] \\
& \text { for } b^{2} \ll R_{0}^{2}  \tag{1.37a}\\
R_{1,2}= & R_{0} \pm \frac{b}{2} \cos \Phi . \tag{1.37b}
\end{align*}
$$

In the exponents (phases) in (1.36) use is made of (1.37b); in the amplitudes let $R_{1} \doteq R_{2} \doteq R_{0}$. The result is
where $\quad C=\frac{j \zeta_{0}}{2 \pi} F_{0}\left(\frac{\pi}{2}, \beta_{0} h\right) \frac{e^{-j \beta_{0} R_{0}}}{R_{0}}$.

$$
\begin{equation*}
E_{\Theta}^{r}=C\left[I_{1}(0) e^{j\left(\beta_{0} b / 2\right) \cos \Phi}+I_{2}(0) e^{-j\left(\beta_{0} b / 2\right) \cos \Phi}\right] \tag{1.38}
\end{equation*}
$$

The term in brackets in (1.38) is the array factor $A(\Theta, \Phi)$. It is instructive to examine some special cases of $(1.38)$ which correspond to different driving conditions and spacings.

When the elements have identical driving-point currents $I_{0}=I_{1}(0)=I_{2}(0)$, the total electric field is

$$
\begin{equation*}
E_{\Theta}^{r}=2 C \cos \left(\frac{\beta_{0} b}{2} \cos \Phi\right) \tag{1.40}
\end{equation*}
$$

Since the electric field of an isolated antenna that is not too long ( $\beta_{0} h \leqslant \pi$ ) always has a rotationally symmetrical single maximum in the equatorial plane, $\Theta=\pi / 2$, the two-element array with equal driving-point currents will have principal maxima in this plane in the directions at right angles or broadside to the array. For this reason it is called a broadside array. Some representative patterns for such arrays are shown in Fig. 1.8. Due to symmetry in the H -plane the radiation pattern is always bilateral.

Now consider a two-element array in which the driving-point currents differ in phase by $\delta$. That is, let

$$
\begin{equation*}
I_{1}(0)=I_{01} e^{-j \delta / 2}, \quad I_{2}(0)=I_{02} e^{j \delta / 2} \tag{1.41}
\end{equation*}
$$

When the magnitudes of the driving-point currents are equal, $I_{01}=I_{02}=I_{0}$, the electric field is

$$
\begin{equation*}
E_{\Theta}^{r}=2 C I_{0} \cos \frac{1}{2}\left(\beta_{0} b \cos \Phi-\delta\right) . \tag{1.42}
\end{equation*}
$$

The main lobe may now be located at a desired angle $\Phi_{n}$ by an
adjustment in the phase difference $\delta$ to make the argument of the cosine in (1.42) an integral multiple of $\pi$. That is, let

$$
\begin{gather*}
\frac{1}{2}\left(\beta_{0} b \cos \Phi_{n}-\delta\right)=n \pi \quad \text { or } \quad \delta=2 n \pi-\beta_{0} b \cos \Phi_{n} \\
n=0,1,2, \ldots \tag{1.43}
\end{gather*}
$$




$$
\begin{aligned}
& N=2 \\
& n=\frac{1}{4}
\end{aligned}
$$


$N=2$
$n=\frac{1}{2}$

${ }^{N}=2$
$n=1$

Fig. 1.8. Representative broadside array factors $(N=2)$.
The higher multiplicities of $\pi$ indicate that an array with widely separated elements may have many maxima and, alternatively, an array with closely spaced elements can have only one maximum. An interesting example of an array with a specified phase difference between the driving-point currents is the endfire array, for which the major lobe in the field pattern is located at $\Phi=0$. An important endfire array is the unilateral couplet for which $\beta_{0} b=\pi / 2$ and $\delta=\pi / 2$. The $H$-plane field pattern shown in Fig. 1.9 is the familiar cardioid.

### 1.6 Fields of arrays of $\boldsymbol{N}$ elements; array factor

The directional properties of a two-element array are inadequate for many applications. Although the directivity of an array is roughly proportional to its overall length, it is not sufficient merely to increase the distance between elements in order to increase the directivity. A greater separation of the elements also assures more lobes in the field pattern. A greater directivity is best achieved by increasing the number of elements in the array. An array of $2 N+1$ elements is shown in Fig. 1.10. The electric field at a point $R, \Theta, \Phi$


Fig. 1.9. Field pattern of endfire couplet with $n=1 / 4$.
in the far zone is a superposition of the contributions by all of the individual elements. Thus, if the total field is $E_{\Theta}^{r}$ and the contribution due to the $i$ th element is $E_{\Theta i}^{r}$, and it is assumed that the distributions of current along all elements are identical, i.e. $I_{z i}(z)=I_{z 0}(z)$,

$$
\begin{align*}
E_{\Theta}^{r}= & E_{\Theta 0}^{r}+\sum_{i=1}^{N} E_{\Theta i}^{r}+\sum_{i=1}^{N} E_{\theta^{\prime}}^{r} \\
= & {\left[\frac{j \omega \mu_{0}}{4 \pi} \frac{e^{-j \beta_{0} R}}{R} \int_{-h}^{h} I_{z 0}\left(z^{\prime}\right) e^{j \beta_{0 z^{\prime}} \cos \Theta} \sin \Theta d \Theta\right] } \\
& {\left[1+\sum_{i=1}^{N} \frac{I_{z i}(0)}{I_{z 0}(0)} e^{j \beta_{0} S_{i}}+\sum_{i^{\prime}=1}^{N} \frac{I_{z i^{\prime}}(0)}{I_{z 0}(0)} e^{-j \beta_{0} S_{i}}\right] } \tag{1.44}
\end{align*}
$$

where $S_{i}=i b \sin \Theta \cos \Phi$ and $b$ is the distance between adjacent elements. This can be expressed in the simple form

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{20}(0)}{2 \pi} \frac{e^{-j \beta_{0} R}}{R} F\left(\Theta, \beta_{0} h\right) A(\Theta, \Phi) \tag{1.45}
\end{equation*}
$$

where $F_{0}\left(\Theta, \beta_{0} h\right)$ is the vertical field function of an isolated element


Fig. 1.10. Geometry for curtain array.
and $A(\Theta, \Phi)$ is the array factor. The conventional study of linear arrays is concerned primarily with the nature of the array factor $A(\Theta, \Phi)$ since $F_{0}\left(\Theta, \beta_{0} h\right)$ is a simple known function of $\Theta$.
A so-called uniform array with equally spaced elements and with $\left|I_{z i}(0)\right|=\left|I_{z 0}(0)\right|$ for all $i$ 's has the array factor

$$
\begin{equation*}
A(\Theta, \Phi)=\frac{\sin N x}{\sin x} \tag{1.46}
\end{equation*}
$$

where $x=\pi(n \sin \Theta \cos \Phi-t), n$ is the distance between elements in fractions of a wavelength and $t$ is the time delay from element to element in fractions of a cycle. The normalized array factor $A(x) / N$ is shown as a function of $x$ for different values of $N$ in Fig. 1.11. The curves for each value of $N$ consist of major and minor maxima and minima. The major extreme values occur at $x=q \pi$, $q=0,1,2,3, \ldots$ and the minor ones at $x \doteq(p+1 / 2) \pi / N$, $p=1,2,3, \ldots$. Between each pair of extremes is a sharp null which indicates a perfect cancellation of the electric field in a definite direction. The mathematically simple result in (1.45) is seldom obtained in actual practice. The differences between the ideal
array factor and an experimentally observed field are usually ascribed to 'mutual coupling effects' without further clarification.

Note that the array factor in (1.46) has a periodicity of $\pi$ in the variable $x$. The half-power beamwidth of an array with this array factor decreases with increasing $N$ but the level of the first side lobe is limited to a minimum of about $21 \%$ of the main beam. The extension of (1.45) to more than one dimension is straightforward. For example, a two-dimensional array of parallel curtain arrays has an array factor which is the product of two array factors.


Fig. 1.11. Normalized array factor $(N=2,5,12)$.
The conventional approach to the circuit properties of arrays follows by analogy with a low-frequency-circuit theory. Thus, an antenna is viewed as a circuit in which a driving voltage is impressed across a pair of terminals by a generator or transmission line and induced voltages are maintained at the same terminals due to coupling with other antennas. Self- and mutual impedances are required to relate the various driving and induced voltages and currents in the array. These impedances are usually computed from an energy-transfer formulation similar to (1.35).

### 1.7 Impedance of antenna; EMF method

The power radiated by an antenna is expressed in (1.30) in terms of the integral of the normal component of the complex Poynting vector over any surface $\Sigma$ that completely encloses a transmitting system. In the evaluation of the radiation resistance $R_{m}^{e}$ a great sphere was used since the formulas for the components of the radiation field are much simpler than those of the near field.

However, if the medium in which the antenna is immersed is nondissipative, any other surface that encloses the transmitting system must yield the same result.

It is customary in determining the circuit properties of linear antennas to choose the cylindrical surface of the antenna as the surface of integration. Thus, if the small ends are neglected, (1.30) gives

$$
\begin{equation*}
P=2 \pi a \int_{-h}^{h} S_{\rho} d z=-\frac{\pi a}{\mu_{0}} \int_{-h}^{h} E_{z} B_{\Phi}^{*} d z \tag{1.47}
\end{equation*}
$$

where $E_{z}$ and $B_{\Phi}^{*}$ are the values on the cylindrical surface of the antenna. Since
it follows that

$$
\begin{gather*}
B_{\Phi}^{*}=\frac{I_{2}^{*}}{\mu_{0} 2 \pi a}  \tag{1.48}\\
P=-\frac{1}{2} \int_{-h}^{h} E_{z} I_{z}^{*} d z . \tag{1.49}
\end{gather*}
$$

The power supplied to the antenna by a delta-function generator at its centre is

$$
\begin{equation*}
P=\frac{1}{2} V_{0} I_{0}^{*}=-\frac{1}{2} \int_{-h}^{h} E_{z} I_{z}^{*} d z . \tag{1.50}
\end{equation*}
$$

In this form the Poynting vector theorem is known as the EMF method for determining the radiated power from the electromagnetic field and for defining the impedance of the antenna. Thus, since $V_{0}=I_{0} Z_{0}$, it follows that

$$
\begin{equation*}
Z_{0}=-\frac{1}{\left|I_{0}\right|^{2}} \int_{-h}^{h} E_{z} I_{z}^{*} d z \tag{1.51}
\end{equation*}
$$

When this formula is applied to a perfectly conducting cylinder which is centre-driven by a delta-function generator, the electric field is given by

$$
\begin{equation*}
E_{z}=-V_{0} \delta(z) \tag{1.52}
\end{equation*}
$$

where $\delta(z)=0$ except at $z=0$. With (1.52) and the properties (1.1a, b) of the delta function, (1.51) becomes

$$
\begin{equation*}
Z_{0}=\frac{1}{\left|I_{0}\right|^{2}} \int_{-h}^{h} V_{0} I_{z}^{*} \delta(z) d z=\frac{1}{\left|I_{0}\right|^{2}} V_{0} I_{0}^{*}=Z_{0} . \tag{1.53}
\end{equation*}
$$

Thus, (1.51) is simply an identity and not a means for determining $Z_{0}$.

In the usual application of (1.51) the boundary condition (1.52) for the tangential electric field is ignored and a sinusoidally distributed current is assumed. $E_{z}$ due to the sinusoidally distributed current (1.32) is given by $\dagger$ ( 1.34 c ). It is

$$
\begin{equation*}
E_{z}=\frac{-j I_{m} \zeta_{0}}{4 \pi}\left(\frac{e^{-j \beta_{0} R_{1 h}}}{R_{1 h}}+\frac{e^{-j \beta_{0} R_{2 h}}}{R_{2 h}}-2 \cos \beta_{0} h \frac{e^{-j \beta_{0} R_{0}}}{R_{0}}\right) \tag{1.54}
\end{equation*}
$$

where, for a point on the surface of the antenna, $R_{0}=\sqrt{z^{\prime 2}+a^{2}}$, $R_{1 h}=\sqrt{\left(h-z^{\prime}\right)^{2}+a^{2}}, R_{2 h}=\sqrt{\left(h+z^{\prime}\right)^{2}+a^{2}}$. If (1.54) and (1.32) are used in (1.51), with $I_{0}=I_{m} \sin \beta_{0} h$, the result is

$$
\begin{align*}
Z_{0}= & \frac{j \zeta_{0}}{4 \pi} \frac{1}{\sin ^{2} \beta_{0} h}\left\{\operatorname { s i n } \beta _ { 0 } h \int _ { - h } ^ { h } \operatorname { c o s } \beta _ { 0 } z ^ { \prime } \left[\frac{e^{-j \beta_{0} R_{1 h}}}{R_{1 h}}+\frac{e^{-j \beta_{0} R_{2 h}}}{R_{2 h}}\right.\right. \\
& \left.-2 \cos \beta_{0} h \frac{e^{-j \beta_{0} R_{0}}}{R_{0}}\right] d z^{\prime}-\cos \beta_{0} h \int_{-h}^{h} \sin \beta_{0}\left|z^{\prime}\right| \\
& +\frac{e^{-j \beta_{0} R_{1 h}}}{R_{1 h}}  \tag{1.55}\\
& \left.\left.e_{2 h}^{-j \beta_{0} R_{2 h}}-2 \cos \beta_{0} h \frac{e^{-j \beta_{0} R_{0}}}{R_{0}}\right] d z^{\prime}\right\} .
\end{align*}
$$

The integrals

$$
\begin{align*}
C_{a}(h, z) & =\int_{-h}^{h} \cos \beta_{0} z^{\prime} \frac{e^{-j \beta_{0} R_{1}}}{R_{1}} d z^{\prime} \\
& =\int_{0}^{h} \cos \beta_{0} z^{\prime}\left[\frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+\frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] d z^{\prime}  \tag{1.56a}\\
S_{a}(h, z) & =\int_{-h}^{h} \sin \beta_{0}\left|z^{\prime}\right| \frac{e^{-j \beta_{0} R_{1}}}{R_{1}} d z^{\prime} \\
& =\int_{0}^{h} \sin \beta_{0} z^{\prime}\left[\frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+\frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] d z^{\prime} \tag{1.56b}
\end{align*}
$$

have been tabulated with $z=h$ and 0 . In (1.56a, b),

$$
R_{1}=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}}, R_{2}=\sqrt{\left(z+z^{\prime}\right)^{2}+a^{2}}
$$

With the tabulated integrals

$$
\begin{align*}
Z_{0}= & \frac{j \zeta_{0}}{2 \pi} \frac{1}{\sin ^{2} \beta_{0} h}\left\{\sin \beta_{0} h\left[C_{a}(h, h)-\cos \beta_{0} h C_{a}(h, 0)\right]\right. \\
& \left.-\cos \beta_{0} h\left[S_{a}(h, h)-\cos \beta_{0} h S_{a}(h, 0)\right]\right\} \tag{1.57}
\end{align*}
$$

In particular, when $\beta_{0} h=\pi / 2$,

$$
\begin{equation*}
Z_{0}=\frac{j \zeta_{0}}{2 \pi} C_{a}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right) . \tag{1.58}
\end{equation*}
$$

$\dagger$ See, for example, [1] p. 528.

The functions $C_{a}(h, h), C_{a}(h, 0), S_{a}(h, h)$ and $S_{a}(h, 0)$ may be expressed in terms of the generalized sine and cosine integral functions. When the radius $a$ of the antenna is small $(a \rightarrow 0)$,

$$
\begin{equation*}
Z_{0} \doteq \frac{\zeta_{0}}{4 \pi}[\operatorname{Cin} 2 \pi+j S i 2 \pi]=73 \cdot 1+j 42 \cdot 5 \text { ohms } \tag{1.59}
\end{equation*}
$$

The real part, $R_{0}=73 \cdot 1 \mathrm{ohms}$, is the value of $R_{m}^{e}$ given in section 1.3. When $\beta_{0} h=\pi, Z_{0}=\infty$. $\operatorname{Cin} x=\int_{0}^{x} u^{-1}(1-\cos u) d u$.

It is now necessary to inquire more deeply into the mechanism by which a sinusoidally distributed current can be maintained along an antenna. Since it is associated with a non-vanishing tangential electric field $E_{z}$ along the surface of the antenna, it cannot be maintained solely by a single generator at $z=0$ along a perfectly conducting cylinder. The boundary condition $E_{z}=0$ applies to the total electric field and must be satisfied on the surface of a perfect conductor. Since $E_{z}$ due to the currents in the conductor is not zero on the surface, it cannot be the total field. There must be an externally maintained field $E_{z}^{e}$ of such magnitude and phase that

$$
\begin{equation*}
E_{z \text { total }}=E_{z}+E_{z}^{e}=0, \quad E_{z}^{e}=-E_{z} . \tag{1.60}
\end{equation*}
$$

In other words, the existence of a sinusoidally distributed current along a perfectly conducting cylinder implies the simultaneous existence of a continuous distribution of generators or their equivalent along the antenna. These maintain an impressed field that exactly cancels the field maintained by the currents and charges in the antenna.

If there is a continuous distribution of generators along the antenna, there can be no single pair of terminals at its centre through which all the power is supplied and across which a driving-point impedance can be defined. It follows that the impedance $Z_{0}$ given by (1.57) or (1.58) is not the impedance of a centre-driven antenna but simply the total complex power radiated by an antenna (in which a continuous distribution of generators maintains a sinusoidal current), divided by the square of the magnitude of the current at the centre of the antenna.

A centre-driven antenna with $E_{z}$ on its surface given by (1.52) and an antenna with its distribution of current given by (1.32) and, therefore, with $E_{z}$ on its surface given by (1.54), are two quite different models with different currents, different fields and different power-supplying devices. It is a common mistake to assume that some of the properties of each can be combined as though the two
were, in fact, the same. Nevertheless, under special circumstances certain characteristics of the two models are comparable and no serious error is made if they are used interchangeably. But this is not true in general or of all significant quantities as is easily shown.

In Figs. 1.12 and 1.13 are shown the measured amplitude and phase of the current in a thin highly conducting base-driven monopole over a large metal ground screen together with the sinusoidally distributed current, respectively, for $\beta_{0} h=\pi / 2$ and $\pi$. When


$$
\frac{|I(z)|}{V_{0}} \mathrm{~mA} / \mathrm{V}
$$

Fig. 1.12. Distribution of amplitude and phase of current in half-wave dipole.


Fig. 1.13. Distribution of amplitude and phase of current in full-wave dipole.
$\beta_{0} h=\pi / 2$ both the measured and sinusoidal values are reasonably alike so that quantities that depend directly on $I_{z}$, viz. $I_{0}$ (and hence $Y_{0}$ and $Z_{0}$ ) and the magnetic field $B_{\Phi}$, must also be comparable. Since the electric field at great distances is linearly related to the magnetic field, it follows that the entire far field of the actual and the sinusoidal currents should be generally alike. On the other hand, when $\beta_{0} h=\pi$ the measured distribution and the sinusoidal distribution differ greatly in both amplitude and phase near the centre of the antenna. It is clear that $I_{0}$ (and hence $Z_{0}$ and $Y_{0}$ ) have nothing in common-the one is quite large, the other is zero. Moreover, since the measured phase reverses at some distance from the centre, whereas all currents remain exactly in phase with the sinusoidal distribution, even the radiation field must differ significantly. The measured current in the centre-driven antenna must have a small minor lobe, whereas the sinusoidal current has none. Thus, when $\beta_{0} h=\pi$, the only properties of the antenna with a sinusoidal current and the centre-driven antenna that are roughly comparable are the general nature of the major lobe in the far field.

Even though the currents, the near-zone magnetic fields, and the distant electric and magnetic fields are reasonably alike for the two differently driven antennas when $\beta_{0} h=\pi / 2$, this does not mean that all significant quantities are. That the associated distributions of charges differ significantly is shown in Fig. 1.14 where both


Fig. 1.14. Normalized distribution of charge in amplitude and phase for a half-wave dipole.
measured and cosinusoidally distributed charges per unit length are shown. It is seen that the charges near the centre and the ends have quite different magnitudes and that the phase reverses at some distance from $z=0$ when the antenna is centre-driven but exactly at $z=0$ when it has a sinusoidal current. Since the radial electric field $E_{\rho}$ near the antenna is proportional to the charge per unit length, it must be quite different for the two models. Finally, it can be shown that $E_{z}$ near the antenna depends not only on $I_{z}$ but on $\partial q / \partial z$, i.e. on the slope of the charge curve. Since the values of $\partial q / \partial z$ are entirely different for the centre-driven antenna and the antenna with a distribution of generators that maintains a sinusoidal current, no correspondence in $E_{z}$ for the two models can obtain. This explains how it is possible that for the centre-driven antenna $E_{z}=0$ at $\rho=a$ except at $z=0$, whereas for the antenna with sinusoidal currents $E_{z}$ has a maximum at $\rho=a$ along its entire length.

These and other studies lead to the conclusion that the hypothetical antenna with a sinusoidally distributed current (and the implied continuous distribution of generators along its length) may be substituted for a centre-driven antenna in determining roughly corresponding values of $I_{z}$ for all $z$ including 0 , the magnetic field at all points, and the radiation electric field provided the antenna is sufficiently thin ( $a / h<10^{-5}$ ) and its length is near resonance, i.e. $\beta_{0} h \sim \pi / 2$. However, no such correspondence exists for the charge distribution or the electric field near the antenna.

Within these limitations, $Z_{0}$ as given by (1.57), (1.58) and (1.59) may be used to obtain approximate values of the impedance of a centre-driven antenna. Specifically, when $h / a \sim 11,000$, the error in $R_{0}$ is of the order of $5 \%$, when $h / a \sim 27$ the error is about $34 \%$.

The correlation of linear antenna theory with experiment involves some important theoretical approximations and a knowledge of the particular driving conditions. Due to the difficulty in solving a single equation which includes both the transmission line and the antenna, the determination of the actual antenna current is based on the simplified model of Fig. 1.2. The effect of the inhomogeneous properties of the transmission line and the coupling of the transmission line to the antenna is taken into account by a lumped constant corrective network. Since the spacing between driving terminals of the antenna is finite, a correction must also be
made for the missing section of conductor. This is discussed in chapter 7.

### 1.8 Coupled antennas; self- and mutual impedances

When an antenna is an element in an array, the Poynting-vector integration over the surface of the antenna in the form (1.47) includes contributions to $E_{z}$ from other members in the array. By arranging the power terms in a certain sequence, the resulting expressions for the integrals resemble the standard equations for coupled circuits so that self- and mutual impedances can be defined in a manner analogous to that used in coupled-circuit equations.

Neglecting the ohmic loss in each antenna of Fig. 1.10, the total time-average power transferred across $\Sigma_{k}$, the surface of the $k^{\text {th }}$ antenna, is

$$
\begin{gather*}
P_{k}=\int_{\Sigma_{k}} \hat{\mathbf{n}} \cdot \mathbf{S}_{k} d \Sigma_{k}  \tag{1.61}\\
\mathbf{S}_{k}=\sum_{i} \mathbf{S}_{k i} . \tag{1.62}
\end{gather*}
$$

In (1.62) $\mathbf{S}_{k i}$ is the contribution of the $i^{\text {th }}$ element to the total Poynting vector at the surface of the $k^{\text {th }}$ element with the $k^{\text {th }}$ element as reference. The time-average Poynting vector for dipoles aligned in the $z$-direction is

$$
\begin{equation*}
\mathbf{S}_{k i}=\frac{1}{2 \mu_{0}}\left(\mathbf{E}_{k i} \times \mathbf{B}_{k i}^{*}\right)=\frac{1}{2 \mu_{0}}\left(\hat{\boldsymbol{\rho}} E_{z k i} B_{\Phi k i}^{*}\right) \tag{1.63}
\end{equation*}
$$

since on the cylindrical surface of antenna $k$

$$
\begin{align*}
& \mathbf{E}_{k i}=\hat{\mathbf{z}} E_{z k i}  \tag{1.64}\\
& \mathbf{B}_{k i}^{*}=\hat{\boldsymbol{\Phi}} B_{\Phi k i} . \tag{1.65}
\end{align*}
$$

$\mathbf{S}_{k i}$ on the small ends is neglected.
At the surface of the $k^{\text {th }}$ element the only magnetic field encircling that element is due to the current in the element itself. Therefore,

$$
\begin{equation*}
B_{\Phi k}^{*}=I_{z k}^{*} / 2 \pi a \mu_{0} . \tag{1.66}
\end{equation*}
$$

The Poynting vector equations (1.61) may be expanded with (1.62)-(1.65):

$$
\begin{align*}
P_{k}= & \frac{1}{2} \int_{-h}^{h} E_{z k 1} I_{z k}^{*} d z+\frac{1}{2} \int_{-h}^{h} E_{z k 2} I_{z k}^{*} d z+\ldots+\frac{1}{2} \int_{-h}^{h} E_{z k k} I_{z k}^{*} d z+\ldots \\
& +\frac{1}{2} \int_{-h}^{h} E_{z k N} I_{z k}^{*} d z, \quad k=1,2,3, \ldots, N \tag{1.67}
\end{align*}
$$

where $d \Sigma_{k}$ has been replaced by ( $2 \pi a$ ) $d z$ and $N$ is the total number of elements. Equation (1.67) is used to define the complex drivingpoint impedance $Z_{01}$ and the self- and mutual impedances $Z_{k i}$. Thus,

$$
\begin{align*}
& P_{k}=\frac{1}{2} I_{0}^{2} Z_{0}=\frac{1}{2} I_{1} I_{k}^{*} Z_{1 k}+\frac{1}{2} I_{2} I_{k}^{*} Z_{2 k}+\ldots+\frac{1}{2} I_{k}^{2} Z_{k k}+\ldots+\frac{1}{2} I_{N} I_{k}^{*} Z_{N k}, \\
& k=1,2,3, \ldots, N  \tag{1.68}\\
& \text { where } \quad Z_{i k}=-\frac{1}{I_{k}^{*} I_{i}} \int_{-h}^{h} E_{z k i} I_{z k}^{*} d z . \tag{1.69}
\end{align*}
$$

This is a generalization of (1.51) to include mutual impedances.
If the elements in an array are centre-driven (base-driven over a ground screen) the electric field along any element $k$ is the superposition of the fields $E_{z k i}$ with $i=1,2, \ldots k, \ldots N$. The fields $E_{z k i}$ must be computed from the actual currents in the $N$ elements and the resultant field must satisfy the boundary condition

$$
\begin{equation*}
\sum_{i=1}^{N} E_{z k i}=-V_{0 k} \delta(z) . \tag{1.70}
\end{equation*}
$$

Since these currents are unknown, this procedure for defining selfand mutual impedances is not directly useful.

On the other hand, if it is stipulated that the distribution of current along each element is sinusoidal irrespective of its location in the array, the impedances defined in (1.69) are readily evaluated. However, each antenna must then be driven by a distribution of generators with EMF's so disposed in amplitudes and phases that the postulated currents actually obtain. The impedances evaluated from (1.69) under these conditions are radiation impedances referred to a particular set of currents, those at $z=0$. However, these are not driving-point currents, since the generators are not localized at $z=0$.

As in the case of the isolated antenna, it is to be expected that there are certain circumstances under which some of the quantities that characterize an array are quantitatively similar for the two quite different sets of boundary and driving conditions: (1) Each element is centre-driven by a single generator, and (2) Each element is driven by a distribution of generators that maintains a sinusoidal current. Since it has been shown that the current maintained in a sufficiently thin antenna near resonance ( $\beta_{0} h \sim \pi / 2$ ) by a single generator at $z=0$ does not differ greatly in amplitude and phase from the cosinusoidal current, and since it is well known that the
current induced in a parasitic antenna by an incident plane wave is approximately cosinusoidal when $\beta_{0} h=\pi / 2, \dagger$ it is reasonable to expect the resultant current in any very thin element in an array of parallel elements to be approximately cosinusoidal. The actual impedances for such arrays should then be approximated by those defined in (1.69) and the far fields should be comparable in an approximate quantitative sense. When the elements are electrically short, the current has a maximum at $z=0$ and vanishes at $z=h$. It is to be expected that the far field and the impedances are not sensitive to the precise distributions of current in the short lengths of the elements.

As a simple example, the mutual impedance between two short Hertzian dipoles with a large separation will be computed. A triangular current is assumed. The current distribution and the relevant term of the electric field are
where

$$
\begin{gather*}
I_{z k}(z)=I_{k}\left(1-\frac{|z|}{h}\right), \quad|z| \leqslant h \ll \beta_{0}^{-1}  \tag{1.71}\\
E_{z k i}=\frac{-j I_{0} h \beta_{0}^{2}}{4 \pi \omega \varepsilon_{0} r^{3}}\left(r^{2}-z^{2}\right) e^{-j \beta_{0} r}, \quad\left(\beta_{0} b\right)^{2} \gg 1 . \tag{1.72}
\end{gather*}
$$

$$
\begin{equation*}
r=\sqrt{b^{2}+z^{2}} \doteq b, \quad b=b_{k i} \tag{1.73}
\end{equation*}
$$

When (1.71) and (1.72) are substituted in (1.69) the result is

$$
\begin{equation*}
Z_{i k} \doteq \frac{j \zeta_{0}}{2 \pi} \frac{\beta_{0}^{2} h^{2}}{\beta_{0} b} e^{-j \beta_{0} b}, \quad\left(\beta_{0} b\right)^{2} \gg 1 . \tag{1.74}
\end{equation*}
$$

As another example of the computation of mutual impedance, consider the case of half-wavelength elements ( $\beta_{0} h=\pi / 2$ ) with an assumed sinusoidal current given by

$$
\begin{equation*}
I_{z}(z)=I_{0} \cos \beta_{0} z, \quad\left|\beta_{0} z\right| \leqslant \pi / 2 \tag{1.75}
\end{equation*}
$$

The electric field $E_{z}$ due to this assumed current is given by (1.54) with $\beta_{0} h=\pi / 2$ and $I_{m}=I_{0}$. It is

$$
\begin{equation*}
E_{z}=\frac{-j I_{0} \zeta_{0}}{4 \pi}\left(\frac{e^{-j \beta_{0} R_{1 h}}}{R_{1 h}}+\frac{e^{-j \beta_{0} R_{2 h}}}{R_{2 h}}\right) \tag{1.76}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{1 h}=\sqrt{(h-z)^{2}+a^{2}}, \quad R_{2 h}=\sqrt{(h+z)^{2}+a^{2}} . \tag{1.77}
\end{equation*}
$$

When (1.75) and (1.76) are substituted in (1.69), the result for the mutual impedance is

$$
\begin{equation*}
Z_{i k}=\frac{j \zeta_{0}}{2 \pi} \int_{0}^{\lambda / 4} \cos \beta_{0} z^{\prime}\left(\frac{e^{-j \beta_{0} R_{1 h}}}{R_{1 h}}+\frac{e^{-j \beta_{0} R_{2 h}}}{R_{2 h}}\right) d z^{\prime} . \tag{1.78}
\end{equation*}
$$

The integral in (1.78) is one of the integral trigonometric functions tabulated by Mack and Mack $\dagger$. It is $C_{a}(h, z)$ defined in (1.56a). When $a$ is sufficiently small,

$$
C_{a}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right) \doteq C_{0}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)=\frac{1}{2}[\operatorname{Si} 2 \pi-j \operatorname{Cin} 2 \pi]
$$

so that $Z_{k k}=Z_{0}=73 \cdot 1+j 42 \cdot 5 \mathrm{ohms}$ as given in (1.59).
The mutual-impedance formula given by (1.78) with (1.56a) reduces to (1.58) for the self-impedance when the radius $a$ is set equal to the radius of the cylindrical antenna. The final formula for $Z_{i k}$ is

$$
\begin{equation*}
Z_{i k}=\frac{j \zeta_{0}}{2 \pi} C_{b}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right) \tag{1.79}
\end{equation*}
$$

where

$$
\begin{align*}
b & \equiv b_{i k} \text { is the distance between the } i^{\text {th }} \text { and } k^{\text {th }} \text { elements } \\
b_{i i} & =a \text { is the radius of each antenna. } \tag{1.80}
\end{align*}
$$

The mutual impedance between two half-wavelength elements in parallel is shown in Fig. 1.15. The calculations are based on (1.79). A comparison of the asymptotic formula (1.74) with assumed triangular current with the results of Fig. 1.15 shows that for large separation the mutual impedance is approximately independent of the current distribution.

### 1.9 Radiation-pattern synthesis

The far-field pattern represents the distribution of the electric field in space. Some applications, particularly point to point communication, require a large field directed within a small angular region. A rearrangement of the spatial distribution of the field is called shaping. The shaping of the radiation pattern of an antenna or array is accomplished by changes in the geometry and the relative currents in the antennas. A discussion of this problem lends insight into the effects of the actually unequal distributions of current in real arrays.

Consider first a single linear element aligned in the $z$-direction.

[^1]

Fig. 1.15. Mutual impedance between two half-wavelength dipoles with sinusoidal current (EMF method).

Three different distributions of current will be used to calculate the far field. They are

$$
\left.\begin{array}{l}
\text { (1) } I_{z 1}(z)=\cos \beta_{0} z \\
\text { (2) } I_{z 2}(z)=\cos \beta_{0} z+\frac{1}{2} \cos 3 \beta_{0} z  \tag{1.82}\\
\text { (3) } I_{z 3}(z)=\cos \beta_{0} z-\frac{1}{2} \cos 3 \beta_{0} z
\end{array}\right\} \beta_{0} h=\frac{\pi}{2} \text {. }
$$

The distributions are shown in Fig. 1.16a. The conventional distribution is $I_{z}(z)=\cos \beta_{0} z$ with a maximum at $z=0$ and a monotonic decrease in magnitude to $z= \pm h$. The second and third distributions (1.82) and (1.83) have, respectively, a greater concentration of current near $z=0$ and a maximum which is moved toward $z=h$. These distributions could be achieved with suitable generators placed along the element. The corresponding
electric fields in the far zone by (1.15a) with (1.80)-(1.83) are

$$
\begin{gather*}
\text { (1) } E_{\Theta}^{r}=K_{1}\left[\frac{\cos \left(\frac{\pi}{2} \cos \Theta\right)}{\sin \Theta}\right]  \tag{1.84}\\
\text { (2) } E_{\Theta}^{r}=K_{1}\left[\frac{\cos \left(\frac{\pi}{2} \cos \Theta\right)}{\sin \Theta}-\frac{3 \sin \Theta}{9-\cos ^{2} \Theta} \cos \left(\frac{\pi}{2} \cos \Theta\right)\right]  \tag{1.85}\\
\text { (3) } E_{\Theta}^{r}=K_{1}\left[\frac{\cos \left(\frac{\pi}{2} \cos \Theta\right)}{\sin \Theta}+\frac{3 \sin \Theta}{9-\cos ^{2} \Theta} \cos \left(\frac{\pi}{2} \cos \Theta\right)\right] . \tag{1.86}
\end{gather*}
$$


(a)

Distance along antenna ( $\beta_{0} z$ ) in radians

(b)
$\Theta$ in degrees
Fig. 1.16. (a) Assumed current distributions on half-wave dipole. (1) $I_{z}(z)=\cos \beta_{0} z$. (2) $I_{z}(z)=\cos \beta_{0} z+\frac{1}{2} \cos 3 \beta_{0} z$. (3) $I_{z}(z)=\cos \beta_{0} z+\frac{1}{2} \cos 3 \beta_{0} z$. (b) Radiation patterns for half-wave dipole with different current distributions.

The radiation patterns (1.84)-(1.86) are shown in Fig. 1.16b. The radiation pattern (1.85) has a half-power beamwidth that is increased when compared with the normal pattern for (1.81). Thus, when the current in an element is increased at the centre the beamwidth is increased. The opposite behaviour is shown by the radiation pattern for (1.86), which corresponds to (1.80). An increase in the magnitude of the current toward the ends of the element decreases the beamwidth of the radiation pattern. Although a single generator could not produce the currents shown in Fig. 1.16a, such distributions of current could be produced by coupling with other elements.

Consider again the half-wavelength element with a sinusoidally distributed current. This time let the current change linearly in phase along the element. That is, let

$$
\begin{equation*}
I_{z}(z)=\cos \beta_{0} z e^{-j \delta z} . \tag{1.87}
\end{equation*}
$$

The radiation field that corresponds to the current (1.87) is

$$
\begin{equation*}
E_{\Theta}^{r}=K \frac{\cos \left(\frac{\pi}{2} \cos \Theta-\delta / 4\right)}{\sin \Theta} . \tag{1.88}
\end{equation*}
$$

The maximum value of $E_{\Theta}^{r}(\Theta)$ now appears at a value of $\Theta$ given by

$$
\begin{equation*}
\Theta=\cos ^{-1} \frac{\delta}{2 \pi} . \tag{1.89}
\end{equation*}
$$

It may be concluded that both the phase and the magnitude of the current in an element affect the radiation pattern.

A single half-wave element can be approximated by a number of collinear infinitesimal dipoles. Each infinitesimal dipole has a prescribed current equal to that in the original full-length dipole at this location. Fig. 1.17 shows such an arrangement for a half-wave


Fig. 1.17. Replacement of continuous current distribution with discrete sources.
dipole with a sinusoidally distributed current. The field factor $F_{0}\left(\Theta, \beta_{0} h\right)$ is given by a sum instead of by an integral as in (1.15b):

$$
\begin{align*}
F_{0}\left(\Theta, \beta_{0} h\right)= & \sin \Theta\left[I_{0}+2 \sum_{i=1}^{2} I_{i} \delta\left(z-d_{i}\right) e^{j \beta_{0} d_{i} \cos \Theta}\right] / I_{0}  \tag{1.90}\\
= & \sin \Theta\left[I_{0}+2 I_{1}\left(e^{j \beta_{0} d_{1} \cos \Theta}+e^{-j \beta_{0} d_{1} \cos \Theta}\right)\right. \\
& \left.+2 I_{2}\left(e^{j \beta_{0} d_{2} \cos \Theta}+e^{-j \beta_{0} d_{2} \cos \Theta}\right)\right] / I_{0}  \tag{1.91}\\
F_{0}\left(\Theta, \beta_{0} h\right)= & \sin \Theta \sum_{i=1}^{2} 1+I_{i} \cos \left(\beta_{0} d_{i} \cos \Theta\right) \tag{1.92}
\end{align*}
$$

where for the currents of Fig. 1.17

$$
\begin{equation*}
I_{1}=I_{0} \sqrt{3} / 2 \quad \text { and } \quad I_{2}=I_{0} / 2 \tag{1.93}
\end{equation*}
$$

The field factor computed with (1.92) and (1.93) is compared to that for a continuous line source in Fig. 1.18. The agreement indicates that a sufficient number of discrete sources can approximate the radiation pattern of a continuous line source.


Fig. 1.18. Comparison of radiation patterns of half-wave dipole from continuous and discrete sources.

If the infinitesimal dipoles of Fig. 1.17 are each rotated $90^{\circ}$ clockwise, the result is a crude picture of a curtain array similar to that shown in Fig. 1.10. The problem of synthesis is to determine how the currents in the array must be adjusted to generate a desired radiation pattern. The current distributions on all elements are assumed to be identical. Actually, the distributions of current are not the same and their differences affect the radiation pattern
of the array. Differences in the amplitudes and phases of the currents along the elements in an array affect its field pattern in much the same way in which they affect the pattern of a single element.

Under the assumption of identical distributions of current on all elements, the far-zone electric field for the symmetrical array of Fig. 1.10 is

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta)=\frac{j \zeta_{0} I_{0}(0)}{2 \pi} F_{0}\left(\Theta, \beta_{0} h\right) \sum_{i=1}^{N}\left[1+2 \frac{I_{i}(0)}{I_{0}(0)} \cos \left(\beta_{0} i b \sin \Theta \cos \Phi\right)\right] \tag{1.94}
\end{equation*}
$$

where it is assumed that $t_{1}$, the time delay between elements, is zero.

In the $H$-plane $(\Theta=\pi / 2)$ the field factor $F_{0}\left(\Theta, \beta_{0} h\right)$ is a constant and (1.94) reduces to

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta)=E_{0} \sum_{i=1}^{N}\left[1+2 \frac{I_{i}(0)}{I_{0}(0)} \cos (i u)\right] \tag{1.95}
\end{equation*}
$$

where

$$
\begin{equation*}
u=2 \pi \frac{b}{\lambda} \cos \Phi \tag{1.96}
\end{equation*}
$$

and

$$
\begin{equation*}
E_{0}=\frac{j \zeta_{0} I_{0}(0)}{2 \pi} F_{0}\left(\frac{\pi}{2}, \beta_{0} h\right) . \tag{1.97}
\end{equation*}
$$

The synthesis problem for this array may be phrased as follows: Given a prescribed radiation pattern $E_{\Theta p}^{r}(u)$, what are the values of the parameters of an array which will best produce this pattern? Usually all but one of the parameters are fixed, and a single parameter is adjusted. For example, the number of elements $N$ and the spacing $b / \lambda$ might be fixed. The driving currents $I_{i}(0) / I_{0}(0)$ are then to be determined so as to produce the best approximation of the field. If this is defined in terms of the least mean-square error between $E_{\Theta p}^{r}(u)$ and $E_{\Theta}^{r}(u)$, the following integral must be minimized:

$$
\begin{equation*}
\min \int_{u_{1}}^{u_{2}}\left\{\left[E_{\Theta p}^{r}(u)\right]-\left[E_{\Theta}^{r}(u)\right]\right\}^{2} d u \tag{1.98}
\end{equation*}
$$

where $E_{\Theta}^{r}(\Theta)$ is given by (1.95). The integral (1.98) is minimized with respect to the coefficients $I_{1}(0) / I_{0}(0)$. Note that, since trigonometric functions are involved, the optimum range of $u$ is $|u| \leqslant \pi$. Note also that the integral is written in terms of the variable $u$ and not $\cos \Phi$. This serves to reduce the complexity of the results. With the interval in (1.98) equal to the range $-\pi \leqslant u \leqslant \pi$, the coefficients for the
current may be related to the Fourier trigonometric coefficients for the representation of $E_{\boldsymbol{\theta}_{p}}^{r}(u)$, given by
where

$$
\begin{align*}
& E_{\Theta p}^{r}(u)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos n u  \tag{1.99}\\
& a_{n}=\frac{2}{\pi} \int_{0}^{\pi} E_{\ominus p}^{r}(u) \cos n u d u \tag{1.100}
\end{align*}
$$

When (1.95) and (1.99) are equated term by term, the following values of the coefficients for the current are obtained:

$$
\begin{align*}
& 2 I_{0}(0) E_{0}=\frac{a_{0}}{2}  \tag{1.101}\\
& \frac{2 I_{n}(0)}{I_{0}(0)} E_{0}=a_{n}, \quad n=1,2,3, \ldots \tag{1.102}
\end{align*}
$$

With the driving-point current on element No. 1 normalized to unity, the relative currents on the elements are given by (1.101) and (1.102). It follows that

$$
\begin{equation*}
I_{n}(0)=\left\langle\frac{a_{n}}{a_{0}}\right\rangle \tag{1.103}
\end{equation*}
$$

Thus, the relative driving-point currents are proportional to the ratio of the Fourier coefficients for the prescribed radiation pattern.

Consider the particular field pattern shown in Fig. 1.19. It is desired to build a three-element array to produce this prescribed pattern. The inter-element spacing must also be specified. A value of $n=b / \lambda=1 / 2$ is chosen for this example. Note that the synthesis


Fig. 1.19. Prescribed field pattern in $H$-plane.
is limited to the range $-\pi \leqslant u \leqslant \pi$. This means that for certain values of the inter-element spacing $n$, the range $-\pi \leqslant u \leqslant \pi$ does not correspond to the visable range of the angle $\Phi$. For example, with $b / \lambda=1$, the pattern may only be synthesized for the range $|\Phi| \leqslant \cos ^{-1} \frac{1}{2}=60^{\circ}$. The mathematical formulation of the desired pattern is given below:

$$
\frac{E_{\Theta p p}^{r}(u)}{E_{0}}=\left\{\begin{array}{cl}
1-\frac{2}{\pi}|u|, & |u| \leqslant \frac{\pi}{2}  \tag{1.104}\\
0, & \frac{\pi}{2} \leqslant|u| \leqslant \pi
\end{array}\right.
$$

The coefficients for the current are found from (1.104) and (1.101) to be

$$
\left.\begin{array}{c}
a_{0}=\frac{2}{\pi} \int_{0}^{\pi / 2}\left(1-\frac{2 u}{\pi}\right) d u=\frac{1}{4}, \quad \frac{1}{2} a_{0}=\frac{1}{8}  \tag{1.106}\\
a_{n}=\frac{2}{\pi} \int_{0}^{\pi / 2}\left(1-\frac{2 u}{\pi}\right) \cos n u d u=\frac{8}{n^{2} \pi^{2}} \sin ^{2} \frac{n \pi}{4} \\
n=1,2,3, \ldots
\end{array}\right\}
$$

The ratios of the driving-point currents are given by (1.103):

$$
\begin{equation*}
\frac{I_{n}}{I_{0}}=\frac{32}{n^{2} \pi^{2}} \sin ^{2} \frac{n \pi}{4}, \quad n \neq 0 \tag{1.107}
\end{equation*}
$$

The realizable radiation pattern for the three-element array is compared to the prescribed pattern in Fig. 1.20. A better agreement


Fig. 1.20. Prescribed and actual field patterns.
between the desired and actual pattern may be achieved by increasing the number of elements. This is analogous to increasing the number of terms used in approximating a function by a Fourier series. The pattern realized with $N=5$ is also shown in Fig. 1.20 and is a very close approximation of the triangular distribution of Fig. 1.19. Note that these patterns have been obtained for the $H$-plane; in other planes the field factor must be taken into account.

The synthesis of the radiation pattern is not always successful if the number of elements is limited. Consider the pattern of Fig. 1.21 for which the coefficients for the current are

$$
\begin{equation*}
\frac{I_{n}(0)}{I_{0}}=\frac{8}{\pi^{2}}\left[\frac{\pi}{2 n} \sin \frac{n \pi}{2}+\frac{1}{n^{2}}\left(\cos \frac{n \pi}{2}-1\right)\right], \quad n \neq 0 \tag{1.108}
\end{equation*}
$$



Fig. 1.21. Prescribed field pattern in $H$-plane.
The array pattern realized for $N=3$ is compared to the prescribed pattern in Fig. 1.22. Here the agreement is very crude and many more terms are required.

### 1.10 The circular array

The locations of the elements in an array are arbitrary. Certain configurations of antennas lend themselves better to mathematical analysis and construction than others. An example of an analytically convenient simple array is the circular array generated by wrapping a curtain array around the surface of a right cylinder. To simplify the analysis, only circular arrays with elements that are located at


Fig. 1.22. Prescribed and actual field patterns.
the vertices of a polygon which is inscribed in a circle are considered. The case $N=2$ is the two-element array discussed in section 1.5 ; the case $N=3$ consists of three parallel elements located at the vertices of an equilateral triangle. The centres of the elements are located on a circle of radius $a$ in the $x-y$ plane as shown in Figs. 1.23a and 1.23b. The far-zone electric field is related to the $z$-component of the vector potential by

$$
\begin{equation*}
E_{\Theta}^{r} \doteq j \omega \sin \Theta A_{z} \tag{1.109}
\end{equation*}
$$

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \omega \mu_{0}}{4 \pi} \sin \Theta \int_{-h}^{h} I\left(z^{\prime}\right) \frac{e^{-j \beta_{0} R_{i}}}{R_{i}} d z^{\prime} . \tag{1.110}
\end{equation*}
$$

The distance $R_{1}$ to the centre of the element located at $\left(a, \pi / 2, \phi_{i}\right)$ is given by

$$
\begin{equation*}
R_{0 i}^{2}=z^{2}+r_{i}^{2}=z^{2}+r^{2}+a^{2}-2 a r \cos \left(\Phi-\phi_{i}\right) \tag{1.111}
\end{equation*}
$$

but

$$
\begin{equation*}
R_{0}^{2}=z^{2}+r^{2} \quad \text { and } \quad r=R_{0} \sin \Theta \tag{1.112}
\end{equation*}
$$

thus

$$
\begin{equation*}
R_{0 i}^{2}=R_{0}^{2}\left[1-\frac{2 a}{R_{0}} \sin \Theta \cos \left(\Phi-\phi_{i}\right)\right] . \tag{1.113}
\end{equation*}
$$

When higher-order terms in ( $a / R_{0}$ ) are neglected, an application of the binomial expansion for (1.113) yields

$$
\begin{equation*}
R_{0 i} \doteq R_{0}-a \sin \Theta \cos \left(\Phi-\phi_{i}\right) \tag{1.114}
\end{equation*}
$$


(a)

(b)

Fig. 1.23. (a) Geometry for the circular array. (b) Single-element geometry in a circular array.
Since it is the distance $R_{i}$ which appears in (1.:10), it follows from Fig. $1.23 b$ and (1.114) that

$$
\left.\begin{array}{l}
R_{i} \doteq R_{0 i}-z \cos \Theta \text { (phase) }  \tag{1.115}\\
R_{i} \doteq R_{0 i} \doteq R_{0} \quad \text { (amplitude). }
\end{array}\right\}
$$

If the array has $N$ elements of equal length and the current distributions are identical, the total field is

$$
\begin{equation*}
E_{\Theta}^{r}=j \frac{\zeta_{0}}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} F_{0}\left(\Theta, \beta_{0} h\right) \sum_{i=1}^{N} I_{0 i} e^{-j \beta_{0} S_{i}} \tag{1.116}
\end{equation*}
$$

where

$$
S_{i}=a \sin \Theta \cos \left(\Phi-\phi_{i}\right)-\delta_{0 i}
$$

$\delta$ is the phase delay between elements in fractions of a wavelength. The driving-point currents $I_{0 i}, i=1,2,3, \ldots, N$, and relative phases $\delta_{0 i}$ are arbitrary. The positions of the elements are fixed by their number. Thus

$$
\begin{equation*}
\phi_{i}=\frac{2 \pi}{N}(i-1), \quad i=1,2,3, \ldots, N . \tag{1.117}
\end{equation*}
$$

Two special cases provide examples of the far-field patterns of typical circular arrays.

Consider an electrically small array with equal driving-point currents and all phase differences $\delta_{0 i}$ set equal to zero. Thus

$$
\begin{equation*}
I_{0 i}=I_{0}, \quad \delta_{0 i}=0 \quad \text { and } \quad\left(\beta_{0} a\right)^{2} \ll 1 . \tag{1.118}
\end{equation*}
$$

The far-zone electric field reduces to

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta) \doteq j \frac{\zeta_{0}}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} F_{0}\left(\Theta, \beta_{0} h\right) N I_{0} . \tag{1.119}
\end{equation*}
$$

The field of (1.119) is identical to that of a single element. In this case the array factor is omnidirectional. Another case of interest is when $N$ is large. Here the finite sum may be approximated by an integral. The Euler-Maclaurin sum formula provides the necessary representation:

$$
\begin{equation*}
\sum_{i=1}^{N} E_{\boldsymbol{\theta} i}=\int_{1}^{N} E_{\boldsymbol{\theta} i} d i+\frac{1}{2} E_{\theta 1}+\frac{1}{2} E_{\Theta N}+O\left(\frac{1}{N^{2}}\right) . \tag{1.120}
\end{equation*}
$$

Since $E_{\boldsymbol{\theta} 1}$ and $E_{\boldsymbol{\theta N}}$ are of the order $N^{-1}$, equation (1.120) may be applied to (1.116) to give the following simplified form:

$$
\begin{align*}
E_{\Theta}^{r} \doteq j \frac{\zeta_{0}}{2 \pi} & \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} F_{0}\left(\Theta, \beta_{0} h\right)\left(\frac{N}{2 \pi}\right) \int_{1 / N \doteq 0}^{2 \pi} I_{0 i} e^{j \beta_{0} a \sin \theta \cos (\Phi-u)} d u \\
& + \text { terms of } O\left(\frac{1}{N}\right) \text { and higher } \tag{1.121}
\end{align*}
$$

where

$$
\begin{equation*}
u=\frac{2 \pi}{N} i \quad \text { and } \quad \delta_{0 i}=0 \tag{1.122}
\end{equation*}
$$

The integral appearing in (1.121) is the Bessel function of the first
kind and order zero. Hence,

$$
\begin{equation*}
E_{\Theta}^{r} \doteq N F_{0}\left(\Theta, \beta_{0} h\right) J_{0}\left(\beta_{0} a \sin \Theta\right) \tag{1.123}
\end{equation*}
$$

Since $J_{0}\left(\beta_{0} a \sin \Theta\right)$ has a maximum at $\Theta=0$, it represents an endfire array. However, since the array factor is multiplied by the field factor $F_{0}\left(\Theta, \beta_{0} h\right)$ which has a null at $\Theta=0$, the resultant maximum of the total field is shifted. The field representations (1.120) and (1.123) show that a wide range of array factors are obtainable from a circular array.

### 1.11 Limitations of conventional array theory

The validity of the conventional analysis of arrays, which is outlined in this chapter beginning with section 1.5, depends on the substitution of the simple approximation

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{z 0}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{1}}}{R_{1}} F_{0}\left(\Theta, \beta_{0} h\right) A(\Theta, \Phi) \tag{1.124}
\end{equation*}
$$

for the rigorous expression

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \omega \mu_{0}}{4 \pi} \frac{e^{-j \beta_{0} R_{1}}}{R_{1}} \sum_{i=1}^{N} e^{j \beta_{0} S_{i}} \int_{-h}^{h} I_{z i}\left(z^{\prime}\right) e^{j \beta_{0} z^{\prime} \cos \theta} \sin \Theta d z^{\prime} \tag{1.125}
\end{equation*}
$$

where $S_{i}=R_{1}-R_{i}$ and $R_{i}$ is the distance from the centre of element $i$ to the point of calculation. The first step in the derivation of (1.124) from (1.125) is the normalization of the currents in the form

$$
\begin{equation*}
I_{z i}\left(z^{\prime}\right)=I_{z i}(0) f_{i}\left(z^{\prime}\right)=I_{z 1}(0) k_{1 i} e^{-j \delta_{1 i}} f_{i}\left(z^{\prime}\right) \tag{1.126}
\end{equation*}
$$

where $k_{1 i}$ and $\delta_{1 i}$ are the normalized real amplitude and relative phase angle of the current at the centre of element $i$ as referred to the current at the centre of the reference element 1 . The next step is the definition of the field factor for element 1 ,

$$
\begin{equation*}
F_{1}\left(\Theta, \beta_{0} h\right)=\int_{-h}^{h} f_{1}\left(z^{\prime}\right) e^{j \beta_{0} z^{\prime} \cos \Theta} \sin \Theta \beta_{0} d z^{\prime} \tag{1.127}
\end{equation*}
$$

and the array factor

$$
\begin{equation*}
A(\Theta, \Phi)=\sum_{i=1}^{N} k_{1 i} e^{j\left(\beta_{0} s_{i}-\delta_{i i}\right)} \tag{1.128}
\end{equation*}
$$

With (1.126)-(1.128), (1.124) can be expressed as follows;

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{z 1}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{1}}}{R_{1}}\left[F_{1}\left(\Theta, \beta_{0} h\right) A(\Theta, \Phi)+D_{1}\left(\Theta, \Phi ; \beta_{0} h\right)\right] \tag{1.129}
\end{equation*}
$$

where

$$
\begin{align*}
& D_{1}\left(\Theta, \Phi ; \beta_{0} h\right) \\
& =\sum_{i=2}^{N} k_{1 i} e^{j\left(\beta_{0} S_{i}-\delta_{1 i}\right)} \int_{-h}^{h}\left[f_{i}\left(z^{\prime}\right)-f_{1}\left(z^{\prime}\right)\right] e^{j \beta_{0} z^{\prime} \cos \Theta} \sin \Theta \beta_{0} d z^{\prime} . \tag{1.130}
\end{align*}
$$

Conventional array theory always neglects $D_{1}\left(\Theta, \Phi ; \beta_{0} h\right)$ as defined in (1.130). This is usually justified by statements such as 'Because of the identical configuration and orientation of all the elements, this contribution (viz. $F_{1}\left(\Theta, \beta_{0} h\right)$ ) is the same for all other elements. These latter differences are accounted for by the array factor (1.128)'. Unfortunately, when applied to arrays of centredriven antennas, this statement is correct only for circular arrays of identical, equally spaced, non-staggered elements driven by equal voltages with constant and progressive phase differences from element to element around the circle. For such an array the geometrical and electrical environments of all elements are indeed the same, so that $f_{i}(z)=f_{1}(z), \quad F_{i}\left(\Theta, \beta_{0} h\right)=F_{1}\left(\Theta, \beta_{0} h\right)$ for $i=1,2, \ldots, N$ and $D_{1}\left(\Theta, \Phi ; \beta_{0} h\right)=0$. In all other arrays, $f_{i}(z)$ is a complex function of $z$ that differs both in its magnitude and phase from $f_{1}(z)$ for all $|z|>0$. It cannot be assumed without explicit verification that $D_{1}\left(\Theta, \Phi ; \beta_{0} h\right)$ is negligible.

A major purpose of this book is to develop a theory of arrays that does not neglect $D_{1}\left(\Theta, \Phi ; \beta_{0} h\right)$ in (1.129).

## CHAPTER 2

## AN APPROXIMATE ANALYSIS OF THE CYLINDRICAL ANTENNA

### 2.1 The sinusoidal current

The distribution of current along a thin centre-driven antenna of length $2 h$ (or along a base-driven antenna of length $h$ over an ideal ground plane) is assumed to have the sinusoidal form

$$
\begin{equation*}
I_{z}(z)=I_{z}(0) \frac{\sin \beta_{0}(h-|z|)}{\sin \beta_{0} h} \tag{2.1}
\end{equation*}
$$

throughout chapter 1. Actually, this is the correct distribution along a section of lossless coaxial line of length $h$ that is shortcircuited at $z=0$ and terminated at $z=h$ in an infinite impedance. This is illustrated in Fig. $2.1 a$ where the infinite impedance is obtained by means of an additional short-circuited quarter-wave section of coaxial line. In this case the current is entirely reactive, the electromagnetic field is completely confined within the coaxial shield in the form of axial standing waves and there is no radiation. When the ideal 'open' end at $z=h$ is replaced by an actual one as shown in Fig. 2.1b, the distribution of current and charges are changed in a manner that resembles a crowding of the entire pattern toward the open end. In addition to a large reactive component, the current now also includes a very small resistive part. The associated electromagnetic field is still primarily a standing wave within the coaxial sleeve, but it does extend outside especially near the open end and there is some radiation. From the point of view of the transmission line the differences between currents and fields for Figs. $2.1 a$ and $2.1 b$ are interpreted as end-effects. If the outside shield is removed as in Fig. 2.1c these 'end-effects' extend all the way to the generator and the distributions of current and charges are significantly changed over the entire length. The resistive component is now comparable in magnitude to the reactive part and the associated electromagnetic field includes a large radiation field that extends to infinity in the form of outward travelling waves. It is, of course, not at all surprising that the distributions of current along the conductors of radius $a$ and length $h$ are
not the same in the three quite different situations represented in Figs. 2.1a, b, c. The boundary conditions are not alike except at $r=a, 0 \leqslant z \leqslant h$, where the tangential electric field vanishes.


Fig. 2.1. (a) Coaxial line terminated in $Z_{\infty}$ at $z=h$. (b) Coaxial line terminated in open end. (c) Base-driven monopole over perfectly conducting ground screen.

### 2.2 The equation for the current

Instead of simply assuming a convenient current along the antenna in Fig. 2.1c, a more scientific and more difficult procedure is actually to determine the distribution of current by setting up and solving the appropriate integral equation. This is readily obtained from the boundary condition $E_{z}(z)=0$ on the surface $\rho=a$ of the perfectly conducting antenna. With (1.5c) and (1.6) the vector
potential is seen to satisfy the equation

$$
\begin{equation*}
\left(\frac{d^{2}}{d z^{2}}+\beta_{0}^{2}\right) A_{z}(z)=0 \tag{2.2}
\end{equation*}
$$

which has the general solution

$$
\begin{equation*}
A_{z}(z)=\frac{-j}{c}\left(C_{1} \cos \beta_{0} z+C_{2} \sin \beta_{0}|z|\right) \tag{2.3}
\end{equation*}
$$

if the symmetry conditions, $I_{z}(-z)=I_{z}(z), A_{z}(-z)=A_{z}(z)$ are imposed. $C_{1}$ and $C_{2}$ are arbitrary constants of integration. With (1.8a) the integral equation for the current is

$$
\begin{equation*}
\frac{4 \pi}{\mu_{0}} A_{z}(z)=\int_{-h}^{h} I_{z}\left(z^{\prime}\right) \frac{e^{-j \beta_{0} R}}{R} d z^{\prime}=\frac{-j 4 \pi}{\zeta_{0}}\left[C_{1} \cos \beta_{0} z+\frac{1}{2} V_{0}^{e} \sin \beta_{0}|z|\right] \tag{2.4}
\end{equation*}
$$

where $V_{0}^{e}$ is the EMF of the delta-function generator, $\zeta_{0}=\sqrt{\mu_{0} / \varepsilon_{0}} \doteq 120 \pi \mathrm{ohms}$, and $R=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}}$. The value $C_{2}=\frac{1}{2} V_{0}^{e}$ is obtained from (2.3) with (1.9) in the form

$$
\phi(z)=j\left(\omega / \beta_{0}^{2}\right)\left(\partial A_{z} / \partial z\right)=-\beta_{0} C_{1} \sin \beta_{0} z+\beta_{0} C_{2} \cos \beta_{0} z
$$

By definition the driving voltage of the delta-function generator is $\lim _{z \rightarrow 0}[\phi(z)-\phi(-z)]=V_{0}^{e}$. The second constant $C_{1}$ must be evaluated from the condition $I_{z}( \pm h)=0$.

Although it is not difficult to derive the integral equation (2.4), the problem of solving it for the current is very complicated. It has been carried out approximately in a variety of ways, $\dagger$ but the solutions so obtained are a very cumbersome series of terms that have proved invaluable in the accurate determination of the selfimpedance but are not very useful for calculating electromagnetic fields or for determining the mutual interaction of the currents in coupled antennas. What is needed is an approximate solution that is both sufficiently simple to be useful in the evaluation of the electromagnetic field and sufficiently accurate to provide quantitatively acceptable values not only of the details of the field but of the driving-point impedance. (In anticipation, it is well to note that a generalization of the method in order to make it useful in the solution of the simultaneous integral equations that occur in the analysis of arrays is also going to be required.)

The procedure to be followed in obtaining a useful approximate solution of (2.4) is straightforward and simple. It involves the

[^2]replacement of the integral equation (2.4) by an approximately equivalent algebraic equation. In order to accomplish this a careful study must be made of the integral in (2.4).

### 2.3 Properties of integrals

The integrand in (2.4) consists of two parts: (1) the current $I_{z}(z)$ which is to be determined and about which nothing is known except that it vanishes at the ends $z= \pm h$, is continuous through the generator at $z=0$, and satisfies the symmetry condition $I_{z}(-z)=I_{z}(z) ;(2)$ the kernel

$$
\begin{equation*}
K\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R}}{R}, \quad R=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}} \tag{2.5}
\end{equation*}
$$

which may be separated into its real and imaginary parts,

$$
\begin{equation*}
K_{R}\left(z, z^{\prime}\right)=\frac{\cos \beta_{0} R}{R} ; \quad K_{I}\left(z, z^{\prime}\right)=-\frac{\sin \beta_{0} R}{R} . \tag{2.6}
\end{equation*}
$$

The dimensionless quantities $K_{R}\left(z, z^{\prime}\right) / \beta_{0}$ and $K_{I}\left(z, z^{\prime}\right) / \beta_{0}$ are shown graphically in Fig. 2.2 as functions of $\beta_{0}\left|z-z^{\prime}\right|$. A comparison in the lower figure shows that their behaviours are quite different. $K_{R}\left(z, z^{\prime}\right) / \beta_{0}$ has a sharp high peak precisely at $z^{\prime}=z$; its magnitude $1 / \beta_{0} a$ is very large compared with 1 since it has been postulated that $\beta_{0} a \ll 1$. On the other hand, $K_{I}\left(z, z^{\prime}\right) / \beta_{0}$ varies only slowly with $\beta_{0}\left|z-z^{\prime}\right|$ and never exceeds the value 1 . It is seen in the upper part of Fig. 2.2 that $\sin \beta_{0} R / \beta_{0} R$ is very well approximated by $\cos \left(\beta_{0} R / 2\right)$ in the range $0 \leqslant \beta_{0}\left|z-z^{\prime}\right| \leqslant \pi$. Moreover, the value of $\cos \left(\beta_{0} R / 2\right)$ is hardly affected if the small quantity $k_{0} a$ is neglected and $\beta_{0} R$ is approximated by $\beta_{0}\left|z-z^{\prime}\right|$.

These facts suggest the following approximations for the two parts of the integral in (2.4):
$J_{R}(h, z)=\int_{-h}^{h} I_{z}\left(z^{\prime}\right) \frac{\cos \beta_{0} R}{R} d z^{\prime}=\Psi_{1}(z) I(z) \doteq \Psi_{1} I(z)$
$J_{I}(h, z)=-\int_{-h}^{h} I_{z}\left(z^{\prime}\right) \frac{\sin \beta_{0} R}{R} d z^{\prime}=-\beta_{0} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) \cos \frac{1}{2} \beta_{0}\left(z-z^{\prime}\right) d z^{\prime}$.
The reasoning behind the approximation in (2.7) is simple. Since the kernel is quite small except at and very near $z^{\prime}=z$, where it rises to a very large value, it is clear that the current near $z^{\prime}=z$ is primarily significant in determining the value of the integral at $z$. In other words, the integral is approximately proportional to $I(z)$.


Fig. 2.2. The functions $\frac{\sin \beta_{0} R}{\beta_{0} R}, \frac{\cos \beta_{0} R}{\beta_{0} R}$, and $\cos \frac{\beta_{0} R}{2}$.
The proportionality constant $\Psi_{1}$ is best determined where $I_{z}(z)$ is a maximum.

The integral in (2.8) may be transformed as follows:

$$
\begin{aligned}
J_{I}(h, z) & =-\beta_{0} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) \cos \frac{1}{2} \beta_{0}\left(z-z^{\prime}\right) d z^{\prime} \\
& =-\beta_{0} \int_{0}^{h} I_{z}\left(z^{\prime}\right)\left[\cos \frac{1}{2} \beta_{0}\left(z-z^{\prime}\right)+\cos \frac{1}{2} \beta_{0}\left(z+z^{\prime}\right)\right] d z^{\prime} \\
& =-2 \beta_{0} \cos \frac{1}{2} \beta_{0} z \int_{0}^{h} I_{z}\left(z^{\prime}\right) \cos \frac{1}{2} \beta_{0} z^{\prime} d z^{\prime} .
\end{aligned}
$$

It follows that for antennas that do not greatly exceed $\beta_{0} h=\pi$ in electrical half-length, specifically, $\beta_{0} h \leqslant 5 \pi / 4$,

$$
J_{I}(h, z) \doteq J_{I}(h, 0) \cos \frac{1}{2} \beta_{0} z ; \quad J_{I}(h, 0)=-2 \beta_{0} \int_{0}^{h} I_{z}\left(z^{\prime}\right) \cos \frac{1}{2} \beta_{0} z^{\prime} d z^{\prime}
$$

A further refinement in the approximation (2.7) is suggested by the fact that, while the integral on the left becomes quite small at the ends of the antenna where $z= \pm h$, the right-hand side vanishes identically at these points since $I_{z}( \pm h)=0$. Evidently a better approximation than (2.7) is the following:
$4 \pi \mu_{0}^{-1}\left[A_{z}(z)-A_{z}(h)\right]=\int_{-h}^{h} I_{z}\left(z^{\prime}\right)\left[K_{R}\left(z, z^{\prime}\right)-K_{R}\left(h, z^{\prime}\right)\right] d z^{\prime} \doteq \Psi_{2} I_{z}(z)$
where the left side is simply the vector potential difference between the point $(a, z)$ and the end $(a, h)$ of the antenna $; \Psi_{2}$ is a new constant.

### 2.4 Rearranged equation for the current

In order to make use of (2.10), the integral equation (2.4) may be modified by subtracting $4 \pi \mu_{0}^{-1} A_{z}(h)$ from both sides. The result is

$$
\begin{align*}
4 \pi \mu_{0}^{-1}\left[A_{z}(z)-A_{z}(h)\right] & =\int_{-h}^{h} I_{z}\left(z^{\prime}\right) K_{d}\left(z, z^{\prime}\right) d z^{\prime} \\
& =\frac{-j 4 \pi}{\zeta_{0}}\left[C_{1} \cos \beta_{0} z+\frac{1}{2} V_{0}^{e} \sin \beta_{0}|z|+U\right] \tag{2.11}
\end{align*}
$$

where

$$
\begin{equation*}
U=\frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) K\left(h, z^{\prime}\right) d z^{\prime} \tag{2.12}
\end{equation*}
$$

and the difference kernel is

$$
\begin{equation*}
K_{d}\left(z, z^{\prime}\right)=K\left(z, z^{\prime}\right)-K\left(h, z^{\prime}\right) . \tag{2.13}
\end{equation*}
$$

The constant $C_{1}$ can now be expressed in terms of $U$ and $V_{0}^{e}$ by setting $z=h$. Since the left side of (2.11) then vanishes, the right side can be solved for $C_{1}$ to give

$$
\begin{equation*}
C_{1}=-\frac{\frac{1}{2} V_{0}^{e} \sin \beta_{0} h+U}{\cos \beta_{0} h} \tag{2.14}
\end{equation*}
$$

If this value of $C_{1}$ is substituted into (2.11) the following equation is obtained:

$$
\begin{align*}
& \int_{-h}^{h} I_{z}\left(z^{\prime}\right) K_{d}\left(z, z^{\prime}\right) d z^{\prime} \\
& \quad=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V_{0}^{e} \sin \beta_{0}(h-|z|)+U\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] . \tag{2.15}
\end{align*}
$$

The integral equation (2.15) with (2.12) is a rearrangement of the original equation (2.4). No approximations are involved.

### 2.5 Reduction of integral equation to algebraic equation

The next and most important step is to make use of the information contained in (2.9) and (2.10) in order to reduce (2.15) to an approximately equivalent algebraic equation. The procedure is simple and straightforward. With (2.9) and (2.10) it is clear that the integral in (2.15) may be approximated as follows:

$$
\begin{equation*}
\int_{-h}^{h} I_{z}\left(z^{\prime}\right) K_{d}\left(z, z^{\prime}\right) d z^{\prime} \doteq I_{z}(z) \Psi_{2}+j J_{I}(h, 0)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right) . \tag{2.16}
\end{equation*}
$$

If this is substituted in (2.15), the resulting equation can be solved explicitly for $I_{z}(z)$. It is seen to have the following zero-order form :

$$
\begin{align*}
& I_{z}(z) \doteq\left[I_{z}(z)\right]_{0}=I_{V}\left[\sin \beta_{0}(h-|z|)+T_{U}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
&\left.+T_{D}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{2.17}
\end{align*}
$$

where $I_{V}, T_{U}$ and $T_{D}$ are complex coefficients.
This is a very significant result. It shows that an approximation of the current consists of three terms of which each represents a different distribution. One of the terms is the simple sinusoid. As for the completely shielded transmission line, the sinusoidal component of the current is maintained directly by the generator; it does not include the components that are induced by coupling between different parts of the antenna. The currents induced by the interaction between charges moving in the more or less widely separated sections of the antenna appear in two parts. One of these, the shifted cosine, is maintained by that part of the interaction that is equivalent to a constant field acting in phase at all points along the antenna. The other part, the shifted cosine with half-angle arguments, is the correction that takes account of the phase lag introduced by the retarded instead of instantaneous interaction.

Thus, the new three-term approximation augments the conventionally assumed sinusoidal distribution with components represented by a shifted cosine and a shifted cosine with halfangle arguments, each with a complex coefficient.

It is quite possible to evaluate the coefficients $\Psi_{2}, J_{I}(h, 0)$ and $U$ that are involved in $I_{V}, T_{U}$, and $T_{D}$-obtained when (2.16) is substituted in (2.15). However, it is preferable to use the arguments and approximations introduced up to this point merely to determine the form of the distribution of current. The three new coefficients, $I_{V}, T_{U}$ and $T_{D}$, may be evaluated directly if (2.17) is substituted in
the integral equation (2.15) and the principles involved in (2.9) and (2.10) are invoked.

The substitution of (2.17) in the integral in (2.15) involves the following parts obtained from the real part $K_{d R}\left(z, z^{\prime}\right)$ of the difference kernel $K_{d}\left(z, z^{\prime}\right)$ defined in (2.13):

$$
\begin{align*}
& \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d R}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d R} \sin \beta_{0}(h-|z|)  \tag{2.18a}\\
& \int_{-h}^{h}\left[\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right] K_{d R}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d U R}\left(\cos \beta_{0} z-\cos \beta_{0} h\right) \tag{2.18b}
\end{align*}
$$

$$
\begin{equation*}
\int_{-h}^{h}\left[\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right] K_{d R}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d D R}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right) . \tag{2.18c}
\end{equation*}
$$

These expressions follow from (2.10). In order to enhance the accuracy, each part of the current is separately treated and supplied with its own coefficient. The evaluation of these coefficients is considered below.

The integrals obtained with the imaginary part $K_{d I}\left(z, z^{\prime}\right)$ of the difference kernel are easily appproximated by the application of (2.9). Thus,

$$
\begin{align*}
& \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d I}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d I}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right) \\
& \int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K_{d I}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d U I}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)  \tag{2.19b}\\
& \int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) K_{d I}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d D I}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right) . \tag{2.19c}
\end{align*}
$$

The three constants $\Psi_{d I}, \Psi_{d U I}$ and $\Psi_{d D I}$ are evaluated later. Finally, if the distribution (2.17) is substituted in (2.12), the result is

$$
\begin{align*}
U & =\frac{-j \zeta_{0} I_{V}}{4 \pi}\left[\Psi_{V}(h)+T_{U} \Psi_{U}(h)+T_{D} \Psi_{D}(h)\right]  \tag{2.20}\\
\text { where } \quad \Psi_{V}(h) & =\int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K\left(h, z^{\prime}\right) d z^{\prime}  \tag{2.21a}\\
\Psi_{U}(h) & =\int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K\left(h, z^{\prime}\right) d z^{\prime} \\
\Psi_{D}(h) & =\int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) K\left(h, z^{\prime}\right) d z^{\prime} \tag{2.21b}
\end{align*}
$$

With (2.18a-c) and (2.19a-c) the integral on the left is reduced to a mere sum of terms with suitable coefficients. And the integral equation as a whole has been replaced by an algebraic equation that involves the three distributions $\sin \beta_{0}(h-|z|), \cos \beta_{0} z-\cos \beta_{0} h$, and $\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h$. It is

$$
\begin{align*}
& \left(I_{V} \Psi_{d R}-\frac{j 2 \pi V_{0}^{e}}{\zeta_{0} \cos \beta_{0} h}\right) \sin \beta_{0}(h-|z|) \\
& \quad+\left(I_{V} T_{U} \Psi_{d U R}-\frac{j 4 \pi U}{\zeta_{0} \cos \beta_{0} h}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right) \\
& \quad+I_{V}\left(j \Psi_{d I}+j \Psi_{d U I} T_{U}+\Psi_{d D} T_{D}\right)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)=0 \tag{2.22}
\end{align*}
$$

where $\Psi_{d D}=\Psi_{d D R}+j \Psi_{d D I}$.

### 2.6 Evaluation of coefficients

The algebraic equation (2.22) is satisfied for all values of $z$ when the coefficient of each of the three distributions vanishes. This step yields three equations for the determination of the coefficients $I_{V}$, $T_{U}$ and $T_{D}$ in (2.17). They are:

$$
\begin{gather*}
I_{V}=\frac{j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}  \tag{2.23a}\\
T_{U}\left[\Psi_{d U R} \cos \beta_{0} h-\Psi_{U}(h)\right]-T_{D} \Psi_{D}(h)=\Psi_{V}(h)  \tag{2.23b}\\
T_{U} \Psi_{d U I}-j T_{D} \Psi_{d D}=-\Psi_{d I} . \tag{2.23c}
\end{gather*}
$$

The last two equations are easily solved for $T_{U}$ and $T_{D}$. The results are

$$
\begin{align*}
T_{U} & =Q^{-1}\left[\Psi_{V}(h) \Psi_{d D}-j \Psi_{D}(h) \Psi_{d I}\right]  \tag{2.24a}\\
T_{D} & =-j Q^{-1}\left\{\Psi_{d I}\left[\Psi_{d U R} \cos \beta_{0} h-\Psi_{U}(h)\right]+\Psi_{V}(h) \Psi_{d U I}\right\}  \tag{2.24b}\\
Q & =\Psi_{d D}\left[\Psi_{d U R} \cos \beta_{0} h-\Psi_{U}(h)\right]+j \Psi_{D}(h) \Psi_{d U I} \tag{2.25}
\end{align*}
$$

The several $\Psi$ functions in (2.24)-(2.25) are defined with (2.18a-c) and ( $2.19 \mathrm{a}-\mathrm{c}$ ) at the value of $z$ that gives the maximum of the current distribution function. Since, in the range of interest, $k_{0} h<3 \pi / 2$, the maximum of $\sin \beta_{0}(h-|z|)$ is at $z=0$ when $\beta_{0} h \leqslant \pi / 2$ but at $z=h-\lambda / 4$ when $\beta_{0} h \geqslant \pi / 2$, whereas the maxima of $\left(\cos \beta_{0} z-\cos \beta_{0} h\right)$ and $\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)$ are at $z=0$, the following definitions are appropriate:

$$
\Psi_{d R}=\Psi_{d R}\left(z_{m}\right),\left\{\begin{array}{l}
z_{m}=0, \quad \beta_{0} h \leqslant \pi / 2  \tag{2.26}\\
z_{m}=h-\lambda / 4, \quad \beta_{0} h>\pi / 2
\end{array}\right.
$$

$$
\begin{align*}
\Psi_{d R}(z)= & \csc \beta_{0}(h-|z|) \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right)\left[K_{R}\left(z, z^{\prime}\right)-K_{R}\left(h, z^{\prime}\right)\right] d z^{\prime}  \tag{2.27}\\
\Psi_{d U R}= & \left(1-\cos \beta_{0} h\right)^{-1} \int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) \\
& \times\left[K_{R}\left(0, z^{\prime}\right)-K_{R}\left(h, z^{\prime}\right)\right] d z^{\prime}  \tag{2.28}\\
\Psi_{d D}= & \left(1-\cos \frac{1}{2} \beta_{0} h\right)^{-1} \int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) \\
& \times\left[K\left(0, z^{\prime}\right)-K\left(h, z^{\prime}\right)\right] d z^{\prime}  \tag{2.29}\\
\Psi_{d I}= & \left(1-\cos \frac{1}{2} \beta_{0} h\right)^{-1} \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) \\
& \times\left[K_{I}\left(0, z^{\prime}\right)-K_{I}\left(h, z^{\prime}\right)\right] d z^{\prime}  \tag{2.30}\\
\Psi_{d U I}= & \left(1-\cos \frac{1}{2} \beta_{0} h\right)^{-1} \int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) \\
& \times\left[K_{I}\left(0, z^{\prime}\right)-K_{I}\left(h, z^{\prime}\right)\right] d z^{\prime} . \tag{2.31}
\end{align*}
$$

These integrals may be evaluated directly by high-speed computer or reduced to the tabulated generalized sine and cosine integral functions given by ( $1.56 \mathrm{a}, \mathrm{b}$ ) and the exponential integral

$$
\begin{equation*}
E_{a}(h, z)=\int_{-h}^{h} \frac{e^{-j \beta_{0} R_{1}}}{R_{1}} d z^{\prime}=\int_{0}^{h}\left[\frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+\frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] d z^{\prime} . \tag{2.32}
\end{equation*}
$$

### 2.7 The approximate current and admittance

The final approximate expression for the current in an isolated cylindrical antenna for which $\beta_{0} h<3 \pi / 2$ and $\beta_{0} a \ll 1$ is

$$
\begin{align*}
I_{z}(z)= & \frac{j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T_{U}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] . \tag{2.33}
\end{align*}
$$

The associated driving-point admittance is

$$
\begin{equation*}
Y_{0}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0} h+T_{U}\left(1-\cos \beta_{0} h\right)+T_{D}\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] . \tag{2.34}
\end{equation*}
$$

When $\beta_{0} h=\pi / 2$, these formulas become indeterminate so that alternative expressions are needed. They are readily obtained from (2.33) and (2.34) by a simple rearrangement. The following new forms are useful near or at $\beta_{0} h=\pi / 2$ where both numerators and denominators in (2.33) and (2.34) are very small or zero:

$$
\begin{align*}
I_{z}(z)= & \frac{-j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+T_{U}^{\prime}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-T_{D}^{\prime}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{2.35}\\
Y_{0}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\sin \beta_{0} h-T_{U}^{\prime}\left(1-\cos \beta_{0} h\right)+T_{D}^{\prime}\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{2.36}
\end{align*}
$$

where $\quad T_{U}^{\prime}=-\frac{T_{U}+\sin \beta_{0} h}{\cos \beta_{0} h}, \quad T_{D}^{\prime}=\frac{T_{D}}{\cos \beta_{0} h}$.
$T_{U}^{\prime}$ and $T_{D}^{\prime}$ are both finite when $\beta_{0} h=\pi / 2$.
When the antenna is electrically short, so that $\beta_{0} h<1$, the trigonometric functions can be expanded in series and the leading terms retained. The current is then given by

$$
\begin{equation*}
I_{z}(z)=\frac{j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R}}\left[\beta_{0} h\left(1-\frac{|z|}{h}\right)+\frac{1}{2} \beta_{0}^{2} h^{2} T\left(1-\frac{z^{2}}{h^{2}}\right)\right] . \tag{2.38}
\end{equation*}
$$

This distribution includes triangular and parabolic components. The admittance is

$$
\begin{equation*}
Y_{0}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\beta_{0} h+\frac{1}{2} \beta_{0}^{2} h^{2} T\right] \tag{2.39}
\end{equation*}
$$

where $T=T_{U}+T_{D} / 4$.

### 2.8 Numerical examples; comparison with experiment

Numerical computations have been made for typical antennas for which extensive measurements are available. For these antennas $a / \lambda=7.022 \times 10^{-3}$. The parameters for the two critical lengths, $\beta_{0} h=\pi / 2$ with $\Omega=2 \ln 2 h / a=8.54$ and $\beta_{0} h=\pi$ with $\Omega=9.92$ are listed below:

$$
\begin{array}{ll}
\beta_{0} h=\frac{\pi}{2}, & \Psi_{d R}=6 \cdot 218, \quad T_{U}^{\prime}=3.085+j 3 \cdot 581 \\
T_{D}^{\prime}=1.061+j 0.025 \\
\beta_{0} h=\pi, & \Psi_{d R}=5 \cdot 737, \quad T_{U}=-0.117+j 0.114 \\
& T_{D}=-0.106+j 0 \cdot 108 \tag{2.40b}
\end{array}
$$

The corresponding currents in amperes per volt, admittances in mhos and impedances in ohms are as follows:

For $\beta_{0} h=\pi / 2$,

$$
\begin{align*}
\frac{I_{z}(z)}{V_{0}^{e}}= & \left\{9.597 \cos \beta_{0} z-0.067 \cos \frac{1}{2} \beta_{0} z+0.047\right. \\
& -j\left[2.680\left(\sin \beta_{0}|z|-1\right)+8.269 \cos \beta_{0} z-2.843 \cos \frac{1}{2} \beta_{0} z\right. \\
& +2.010]\} \times 10^{-3}  \tag{2.41a}\\
Y_{0}= & (9.577-j 4.756) \times 10^{-3}, \quad Z_{0}=83.76+j 41.60 . \tag{2.41b}
\end{align*}
$$

For $\beta_{0} h=\pi$,

$$
\begin{align*}
& \frac{I_{z}(z)}{V_{0}^{e}}=\left\{0.331\left(\cos \beta_{0} z+1\right)+0.314 \cos \frac{1}{2} \beta_{0} z\right. \\
&-j\left[2.905 \sin \beta_{0}|z|-0.340\left(\cos \beta_{0} z+1\right)\right. \\
&\left.\left.-0.308 \cos \frac{1}{2} \beta_{0} z\right]\right\} \times 10^{-3}  \tag{2.42a}\\
& Y_{0}=(0.976+j 0 \cdot 988) \times 10^{-3}, \quad Z_{0}=506 \cdot 0-j 512 \cdot 2 . \tag{2.42b}
\end{align*}
$$

Note that when a sinusoidal distribution of current is assumed the corresponding impedances are for $\beta_{0} h=\pi / 2, Z_{0}=73 \cdot 1+j 42 \cdot 5$ (see eq. 1.59); and for $\beta_{0} h=\pi, Z_{0}=\infty$.

Graphs of $I_{z}(z) / V_{0}^{e}=\left[I_{z}^{\prime \prime}(z)+j I_{z}^{\prime}(z)\right] / V_{0}^{e}$ are in Figs. 2.3 and 2.4 for $\beta_{0} h=\pi / 2$ and $\pi$ together with measured values. The approximate theoretical curves are seen to agree very well with measured values not only for $\beta_{0} h=\pi / 2$, but also for $\beta_{0} h=\pi$.

As can be seen from Figs. 2.3 and 2.4, and especially from the latter, the theoretical currents at the driving point and, hence, the admittances differ somewhat from the measured values. In order to achieve a more accurate admittance, higher-order terms are required in the expressions for the current. Simple trigonometric functions cannot take adequate account of the rapid change in the current near the driving point when the antenna is not near resonance. Since higher-order terms are necessarily complicated, their introduction would defeat the primary purpose of this formulation, namely, to maintain a reasonably simple representation. Fortunately, there is a useful alternative. Since the only large error in the current occurs in the quadrature component of the current very near the driving point, it is possible to introduce a lumped susceptance $B_{c}$ across the terminals which will correct the driving-point current and the susceptance while leaving the otherwise wellapproximated current unchanged. Actually, since the use of a lumped corrective network is required in any case to take account of the local geometry of the junction between the feeding line and
the antenna if quantitative accuracy is desired, the addition of $B_{c}$ to the susceptance $B_{T}$ of the terminal-zone network is no significant complication. In practice, it may be convenient to measure the apparent driving-point susceptance at $\beta_{0} h=\pi$ and use the difference between this and the approximate theoretical value as the total lumped susceptance $B_{T}+B_{c}$ to be used with all theoretical values based on the approximate theory for any given ratio of $a / \lambda$.


Fig. 2.3. Current in upper half of half-wave dipole.

### 2.9 The radiation field

The electric field in the radiation zone of an antenna with a distribution of current $I_{z}(z)$ is given by the integral

$$
\begin{equation*}
\mathbf{E}^{r}=\hat{\Theta} E_{\Theta}^{r} ; \quad E_{\Theta}^{r}=\frac{j \omega \mu}{4 \pi} \sin \Theta \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) e^{j \beta_{0} z^{\prime} \cos \theta} d z^{\prime} . \tag{2.43}
\end{equation*}
$$



Fig. 2.4. Current in upper half of full-wave dipole.

The far field maintained by the distribution (2.33) is obtained when

$$
\begin{align*}
I_{z}(z)= & \frac{j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T_{U}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{2.44}
\end{align*}
$$

is substituted in (2.43). The result may be expressed as follows:

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{-V_{0}^{e}}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} f\left(\Theta, \beta_{0} h\right) \tag{2.45a}
\end{equation*}
$$

where

$$
\begin{equation*}
f\left(\Theta, \beta_{0} h\right)=\left[F_{m}\left(\Theta, \beta_{0} h\right)+T_{U} G_{m}\left(\Theta, \beta_{0} h\right)+T_{D} D_{m}\left(\Theta, \beta_{0} h\right)\right] \sec \beta_{0} h . \tag{2.45b}
\end{equation*}
$$

The several field functions are

$$
\begin{align*}
& \begin{aligned}
& F_{m}\left(\Theta, \beta_{0} h\right)=\frac{\beta_{0}}{2} \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) e^{j \beta_{0} z^{\prime} \cos \Theta} \sin \Theta d z^{\prime} \\
&=\frac{\cos \left(\beta_{0} h \cos \Theta\right)-\cos \beta_{0} h}{\sin \Theta} \\
& \begin{aligned}
G_{m}\left(\Theta, \beta_{0} h\right) & =\frac{\beta_{0}}{2} \int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) e^{j \beta_{0} z^{\prime} \cos \Theta} \sin \Theta d z^{\prime} \\
& =\frac{\sin \beta_{0} h \cos \left(\beta_{0} h \cos \Theta\right) \cos \Theta-\cos \beta_{0} h \sin \left(\beta_{0} h \cos \Theta\right)}{\sin \Theta \cos \Theta} \\
& =\frac{\beta_{0}}{2} \int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) e^{j \beta_{0} z^{\prime} \cos \Theta} \sin \Theta d z^{\prime} \\
& =\left[\frac{2 \cos \left(\beta_{0} h \cos \Theta\right) \sin \frac{1}{2} \beta_{0} h-4 \sin \left(\beta_{0} h \cos \Theta\right) \cos \frac{1}{2} \beta_{0} h \cos \Theta}{1-4 \cos ^{2} \Theta}\right. \\
& \left.-\frac{\sin \left(\beta_{0} h \cos \Theta\right) \cos \frac{1}{2} \beta_{0} h}{\cos \Theta}\right] \sin \Theta .
\end{aligned}
\end{aligned} .
\end{align*}
$$

For the alternative current

$$
\begin{align*}
I_{z}(z)= & \frac{-j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+T_{U}^{\prime}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-T_{D}^{\prime}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{2.49}
\end{align*}
$$

which is useful when $\beta_{0} h$ is at and near $\pi / 2$, the far field is

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{V_{0}^{e}}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} f^{\prime}\left(\Theta, \beta_{0} h\right) \tag{2.50a}
\end{equation*}
$$

where

$$
\begin{equation*}
f^{\prime}\left(\Theta, \beta_{0} h\right)=H_{m}\left(\Theta, \beta_{0} h\right)+T_{U}^{\prime} G_{m}\left(\Theta, \beta_{0} h\right)-T_{D}^{\prime} D_{m}\left(\Theta, \beta_{0} h\right) \tag{2.50b}
\end{equation*}
$$

The new field function is

$$
\begin{align*}
& H_{m}\left(\Theta, \beta_{0} h\right)=\frac{\beta_{0}}{2} \int_{-h}^{h}\left(\sin \beta_{0}\left|z^{\prime}\right|-\sin \beta_{0} h\right) e^{j \beta_{0} z^{\prime} \cos \Theta} \sin \Theta d z^{\prime} \\
& =\frac{\left[1-\cos \beta_{0} h \cos \left(\beta_{0} h \cos \Theta\right)\right] \cos \Theta-\sin \beta_{0} h \sin \left(\beta_{0} h \cos \Theta\right)}{\sin \Theta \cos \Theta} \tag{2.51}
\end{align*}
$$

$G_{m}\left(\Theta, \beta_{0} h\right)$ and $D_{m}\left(\Theta, \beta_{0} h\right)$ are as in (2.47) and (2.48).
For the specific cases considered above, the coefficients are:

For $\beta_{0} h=\pi / 2$,

$$
\begin{align*}
H_{m}\left(\Theta, \frac{\pi}{2}\right)= & \frac{\cos \Theta-\sin \left(\frac{\pi}{2} \cos \Theta\right)}{\sin \Theta \cos \Theta}  \tag{2.52a}\\
G_{m}\left(\Theta, \frac{\pi}{2}\right)= & \frac{\cos \left(\frac{\pi}{2} \cos \Theta\right)}{\sin \Theta}=F_{m}\left(\Theta, \frac{\pi}{2}\right)  \tag{2.52b}\\
D_{m}\left(\Theta, \frac{\pi}{2}\right)= & \frac{\sqrt{2}}{2}\left\{\frac{2 \cos \left(\frac{\pi}{2} \cos \Theta\right)-4 \sin \left(\frac{\pi}{2} \cos \Theta\right) \cos \Theta}{1-4 \cos ^{2} \Theta}\right. \\
& \left.-\frac{\sin \left(\frac{\pi}{2} \cos \Theta\right)}{\cos \Theta}\right\} \sin \Theta . \tag{2.52c}
\end{align*}
$$

For $\beta_{0} h=\pi$,

$$
\begin{align*}
& F_{m}(\Theta, \pi)=\frac{\cos (\pi \cos \Theta)+1}{\sin \Theta}  \tag{2.53a}\\
& G_{m}(\Theta, \pi)=\frac{\sin (\pi \cos \Theta)}{\sin \Theta \cos \Theta}  \tag{2.53b}\\
& D_{m}(\Theta, \pi)=\frac{2 \cos (\pi \cos \Theta) \sin \Theta}{1-4 \cos ^{2} \Theta} . \tag{2.53c}
\end{align*}
$$

In the formulas (2.45a) and (2.50a), the field is referred to the driving voltage $V_{0}^{e}$. It can be referred to the current $I_{z}(0)$ at the driving point with the simple substitution of $I_{z}(0) / Y_{0}$ for $V_{0}^{e}$ where $Y_{0}$ is the admittance given by (2.34) or (2.36). The field in (2.45a) is then expressed as follows:

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{z}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} f_{I}\left(\Theta, \beta_{0} h\right) \tag{2.54a}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{I}\left(\Theta, \beta_{0} h\right)=\frac{F_{m}\left(\Theta, \beta_{0} h\right)+T_{V} G_{m}\left(\Theta, \beta_{0} h\right)+T_{D} D_{m}\left(\Theta, \beta_{0} h\right)}{\sin \beta_{0} h+T_{U}\left(1-\cos \beta_{0} h\right)+T_{D}\left(1-\cos \frac{1}{2} \beta_{0} h\right)} . \tag{2.54b}
\end{equation*}
$$

The alternative form (2.50a) becomes

$$
\begin{equation*}
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{z}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} f_{I}^{\prime}\left(\Theta, \beta_{0} h\right) \tag{2.55a}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{I}^{\prime}\left(\Theta, \beta_{0} h\right)=-\frac{H_{m}\left(\Theta, \beta_{0} h\right)+T_{U}^{\prime} G_{m}\left(\Theta, \beta_{0} h\right)-T_{D}^{\prime} D_{m}\left(\Theta, \beta_{0} h\right)}{\sin \beta_{0} h-T_{U}^{\prime}\left(1-\cos \beta_{0} h\right)+T_{D}^{\prime}\left(1-\cos \frac{1}{2} \beta_{0} h\right)} . \tag{2.55b}
\end{equation*}
$$

As a numerical illustration, the three functions $F_{m}(\Theta, \pi), G_{m}(\Theta, \pi)$ and $D_{m}(\Theta, \pi)$ are shown graphically in Fig. 2.5 for a full-wave antenna. They all have nulls at $\Theta=0$ and maxima at $\Theta=90^{\circ}$. However, $G_{m}(\Theta, \pi)$ and $D_{m}(\Theta, \pi)$ have relatively much greater values at small values of $\Theta$ than $F_{m}(\Theta, \pi)$.

If use is made of the numerical values of $T_{U}$ and $T_{D}$ given in (2.40b) for a cylindrical antenna with $a / \lambda=7.022 \times 10^{-3}$ (for which the distribution of current is given in (2.42a) and the admittance and impedance in (2.42b)) the field factor

$$
\begin{equation*}
f(\Theta, \pi)=f_{r}(\Theta, \pi)+j f_{i}(\Theta, \pi) \tag{2.56}
\end{equation*}
$$

may be evaluated. The real and imaginary parts $f_{r}(\Theta, \pi)$ and $f_{i}(\Theta, \pi)$ are shown in Fig. 2.5 together with the magnitude $|f(\Theta, \pi)|$. This last is seen to resemble $F_{m}(\Theta, \pi)$ quite closely except for $\Theta<30^{\circ}$ where it is significantly greater. However, since the field is quite small when $\Theta<30^{\circ}$, no serious error is made in calculating the far field if the following approximations are used when $\beta_{0} h \leqslant \pi$ :

$$
\begin{align*}
G_{m}\left(\Theta, \beta_{0} h\right) & \doteq \frac{G_{m}\left(\frac{\pi}{2}, \beta_{0} h\right)}{F_{m}\left(\frac{\pi}{2}, \beta_{0} h\right)} F_{m}\left(\Theta, \beta_{0} h\right) \\
& =\left(\frac{\sin \beta_{0} h-\beta_{0} h \cos \beta_{0} h}{1-\cos \beta_{0} h}\right) F_{m}\left(\Theta, \beta_{0} h\right)  \tag{2.57a}\\
D_{m}\left(\Theta, \beta_{0} h\right) & \doteq \frac{D_{m}\left(\frac{\pi}{2}, \beta_{0} h\right)}{F_{m}\left(\frac{\pi}{2}, \beta_{0} h\right)} F_{m}\left(\Theta, \beta_{0} h\right) \\
& =\left(\frac{2 \sin \frac{1}{2} \beta_{0} h-\beta_{0} h \cos \frac{1}{2} \beta_{0} h}{1-\cos \beta_{0} h}\right) F_{m}\left(\Theta, \beta_{0} h\right)  \tag{2.57b}\\
H_{m}\left(\Theta, \beta_{0} h\right) & \doteq \frac{H_{m}\left(\frac{\pi}{2}, \beta_{0} h\right)}{F_{m}\left(\frac{\pi}{2}, \beta_{0} h\right)} F_{m}\left(\Theta, \beta_{0} h\right)
\end{align*}
$$

$$
\begin{equation*}
=\left(\frac{1-\cos \beta_{0} h-\beta_{0} h \sin \beta_{0} h}{1-\cos \beta_{0} h}\right) F_{m}\left(\Theta, \beta_{0} h\right) . \tag{2.57c}
\end{equation*}
$$

These approximations are equivalent to the use of the far-field distribution associated with a sinusoidal current, but normalizing this to the value at $\Theta=\pi / 2$ obtained from the three-term form of the current.


Fig. 2.5. The functions $F_{m}(\Theta, \pi), G_{m}(\Theta, \pi), D_{m}(\Theta, \pi)$ and the field components $f(\Theta, \pi)=f_{r}(\Theta, \pi)+j f_{i}(\Theta, \pi)$ when $a / \lambda=7.022 \times 10^{-3}$.

### 2.10 An approximate two-term theory

When interest is entirely in the far-field and in the driving-point impedances, the difference between the distribution functions $F_{0 z}=\cos \beta_{0} z-\cos \beta_{0} h$ and $H_{0 z}=\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h$ is small and the formulation may be simplified further by consolidating the two terms. If $F_{0 z}$ is substituted everywhere for $H_{0 z}$, the current is well approximated as follows when $\beta_{0} h \leqslant 5 \pi / 4$ :
$I_{z}(z)=\frac{j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]$
or, in the form useful near $\beta_{0} h=\pi / 2$,

$$
\begin{equation*}
I_{z}(z)=\frac{-j 2 \pi V_{0}^{e}}{\zeta_{0} \Psi_{d R}}\left[\sin \beta_{0}|z|-\sin \beta_{0} h+T^{\prime}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \tag{2.59}
\end{equation*}
$$

where $T$ and $T^{\prime}$ are obtained by forming $T_{U}+T_{D}$ and $T_{U}^{\prime}-T_{D}^{\prime}$ but with the substitution $\Psi_{d D}=\Psi_{d U}, \Psi_{D}(h)=\Psi_{U}(h)$. The function $T$
is simply

$$
\begin{equation*}
T=\frac{\Psi_{V}(h)-j \Psi_{d I} \cos \beta_{0} h}{\Psi_{d U} \cos \beta_{0} h-\Psi_{U}(h)} \tag{2.60}
\end{equation*}
$$

$T^{\prime}$ is given by

$$
\begin{align*}
T^{\prime} & =-\frac{T+\sin \beta_{0} h}{\cos \beta_{0} h} \\
& =\frac{\left[\Psi_{V}(h)-\Psi_{U}(h) \sin \beta_{0} h\right] \sec \beta_{0} h+\Psi_{d U} \sin \beta_{0} h-j \Psi_{d I}}{\Psi_{U}(h)-\Psi_{d U} \cos \beta_{0} h}  \tag{2.61a}\\
& =\frac{\left[\Psi_{d U}+E_{a}(h, h)\right] \sin \beta_{0} h-j \Psi_{d I}-S_{a}(h, h)}{C_{a}(h, h)-\left[\Psi_{d U}+E_{a}(h, h)\right] \cos \beta_{0} h} \tag{2.61b}
\end{align*}
$$

Since $\Psi_{U}(h)=\Psi_{V}(h)=C_{a}(h, h)$ when $\beta_{0} h=\pi / 2$, this reduces simply to

$$
\begin{equation*}
T^{\prime}=\frac{\Psi_{d U}-j \Psi_{d I}-S_{a}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)+E_{a}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)}{C_{a}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)} \tag{2.62}
\end{equation*}
$$

when $\beta_{0} h=\pi / 2$.
For the numerical cases considered in section 2.8 for $a / \lambda=7.022 \times 10^{-3}$, the results for the two-term theory are:

$$
\left.\begin{array}{ll}
\beta_{0} h=\pi / 2, & \Psi_{d R}=6.218, \\
& Y_{0}=(10.17-j 4.43) \times 10^{-3} \mathrm{mhos} \tag{2.64}
\end{array}\right\}
$$

These are seen to be in good agreement with the values obtained with the more accurate three-term theory. A more extensive list of numerical values of $\Psi_{d R}, T, T^{\prime}$ and $Y_{0}=G_{0}+j B_{0}$ is in Table 1 of appendix 1 .

As with the three-term theory, the quadrature component of the current near the driving point is not adequately represented by simple trigonometric functions so that the same expedient previously described must be used in order to obtain quantitative agreement with measured values of the susceptance. The lumped value of $B_{c}$ to be used with the two-term theory differs only slightly from that for the three-term theory. For $a / \lambda=7.022 \times 10^{-3}$, it is $B_{c}=0.72$ millimhos. This value must be added to the two-term susceptance $B_{0}+B_{T}$ (where $B_{T}$ is the terminal-zone correction for
a particular transmission-line-antenna junction) in order to obtain the measurable apparent susceptance $B_{s a}=B_{0}+0.72+B_{T}$. It is seen in Fig. 2.6 that $B_{0}+0.72$ is in good agreement with the KingMiddleton second-order values of $B_{0}$ and the apparent measured values corrected for the terminal-zone effects, $B_{s a}-B_{T}$.


Fig. 2.6. King-Middleton second-order admittance $Y=G+j B$. Two-term zero-order admittance $Y_{0}=G_{0}+j B_{0}$, and measured.

### 2.11 The receiving antenna

The general method of analysis introduced in this chapter as a means of analyzing the centre-driven cylindrical antenna can be extended readily to the centre-loaded receiving antenna in an
incident plane-wave field. For the purposes of this book $\dagger$-which includes the properties of receiving arrays-it is sufficient to treat only the simple case of normal incidence with the electric vector parallel to the $z$-axis which is the axis of the antenna. The antenna is, therefore, in the plane wave front of the incident wave which may be assumed to travel in the positive $x$ direction. That is $E_{z}^{\text {inc }}(x)=E_{z}^{\text {inc }} e^{-j \beta_{0} x}$ where $E_{z}^{\text {inc }}$ is the constant amplitude. The boundary condition that requires the total tangential electric field to vanish on the surface of the antenna gives

$$
\begin{equation*}
\left(\frac{d^{2}}{d z^{2}}+\beta_{0}^{2}\right) A_{z}(z)=-\frac{j \beta_{0}^{2}}{\omega} E_{z}^{\mathrm{inc}}=-\beta_{0}^{2} A_{z}^{\mathrm{inc}} \tag{2.65}
\end{equation*}
$$

instead of (2.2). In (2.65), $A_{z}(z)$ is the vector potential due to the currents in the receiving antenna itself. $A_{z}^{\text {inc }}$ is the constant amplitude of the vector potential maintained on the surface of the antenna by the distant transmitter. Note that $E_{z}^{\text {inc }}=-j \omega A_{z}^{\text {inc }}$. Since the axis of the antenna lies in the wave front, even symmetry obtains with respect to $z$ for both the current and the associated vector potential so that $A_{z}(-z)=A_{z}(z), I_{z}(-z)=I_{z}(z)$. It follows that, on an unloaded receiving or scattering antenna, the vector potential on the surface of the antenna due to the currents in the antenna satisfies the equations
$4 \pi \mu_{0}^{-1} A_{z}(z)=\int_{-h}^{h} I\left(z^{\prime}\right) \frac{e^{-j \beta_{0} R}}{R} d z^{\prime}=\frac{-j 4 \pi}{\zeta_{0}}\left[C_{1} \cos \beta_{0} z+U^{\text {inc }}\right]$
where $C_{1}$ is an arbitrary constant to be evaluated from the condition $I_{z}(h)=0$ and

$$
\begin{equation*}
U^{\mathrm{inc}}=\frac{E_{z}^{\mathrm{inc}}}{\beta_{0}}=-\frac{j \omega A_{z}^{\mathrm{inc}}}{\beta_{0}} . \tag{2.67}
\end{equation*}
$$

This integral equation is like (2.4) with an added constant term on the right and with $V_{0}^{e}=0$. If the same rearrangement is carried out as for (2.11), the result is

$$
\begin{align*}
4 \pi \mu_{0}^{-1}\left[A_{z}(z)-A_{z}(h)\right] & =\int_{-h}^{h} I_{z}\left(z^{\prime}\right) K_{d}\left(z, z^{\prime}\right) d z^{\prime} \\
& =\frac{-j 4 \pi}{\zeta_{0}}\left[C_{1} \cos \beta_{0} z+U+U^{\mathrm{inc}}\right] \tag{2.68}
\end{align*}
$$

where $U$, as defined in (2.12), is proportional to the vector potential

[^3]at $z=h, \rho=a$ due to the currents in the antenna; $U^{\mathrm{inc}}$, as defined in (2.67), is proportional to the vector potential maintained on the surface of the antenna by the distant transmitter. The sum $U+U^{\text {inc }}$ is proportional to the total vector potential on the surface of the antenna.

Since the integral equation (2.68) is just like (2.11) with $V_{0}^{e}=0$, it follows that the rearranged equation corresponding to (2.15) is

$$
\begin{equation*}
\int_{-h}^{h} I_{z}\left(z^{\prime}\right) K_{d}\left(z, z^{\prime}\right) d z^{\prime}=\frac{j 4 \pi\left(U+U^{\mathrm{inc}}\right)}{\zeta_{0} \cos \beta_{0} h}\left(\cos \beta_{0} z-\cos \beta_{0} h\right) \tag{2.69}
\end{equation*}
$$

The approximate solution of this equation is like (2.33) with $V_{0}^{e}=0$. It is

$$
\begin{align*}
I_{z}(z)= & \frac{j 4 \pi U^{\mathrm{inc}}}{\zeta_{0} Q}\left[\Psi_{d D}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-j \Psi_{d U I}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{2.70}
\end{align*}
$$

where $Q$ is defined in (2.25), $\Psi_{d D}$ in (2.29) and $\Psi_{d U I}$ in (2.31). This solution for the unloaded receiving antenna corresponds to the three-term form (2.33) for the driven antenna. Corresponding to the simpler two-term approximation (2.58) for the driven antenna, is the expression

$$
\begin{equation*}
I_{z}(z) \doteq \frac{j 4 \pi U^{\mathrm{inc}}\left(\Psi_{d D}-j \Psi_{d U I}\right)}{Q}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)=U^{\mathrm{inc}} u(z) \tag{2.71}
\end{equation*}
$$

for the unloaded receiving antenna. Note, in particular, that the shifted cosine distribution $\left(\cos \beta_{0} z-\cos \beta_{0} h\right)$ is characteristic of the unloaded receiving antenna when its axis is parallel to the electric vector in an incident plane-wave field. When the axis of the antenna is oriented at an arbitrary angle with respect to the incident $E$-vector, the distribution of current is much more complicated. In particular, if the antenna does not lie in the plane wave front (surface of constant phase) of the incident field, the current and the vector potential have components with both even and odd symmetries with respect to $z$.

If the antenna is cut at $z=0$ and a load $Z_{L}$ is connected in series with the halves of the antenna, the current in the antenna is readily obtained. Note first that, if a generator with voltage $V_{0}^{e}$ is connected across the terminals at $z=0$ instead of the load, the resulting current in the antenna is

$$
\begin{equation*}
I_{z}(z)=V_{0}^{e} v(z)+U^{\mathrm{inc}} u(z) \tag{2.72}
\end{equation*}
$$

where $v(z)$ is $I_{z}(z) / V_{0}^{e}$ as obtained from (2.44) and $u(z)$ is $I_{z}(z) / U^{\text {inc }}$ as obtained from (2.71). If now $V_{0}^{e}$ is replaced by the negative of the voltage drop across a load $Z_{L}$ that is connected across the terminals of the antenna, that is,

$$
\begin{equation*}
V_{0}^{e}=-I_{z}(0) Z_{L} \tag{2.73}
\end{equation*}
$$

$V_{0}^{e}$ is readily eliminated between (2.73) and (2.72). With $Z_{0}$, the driving-point impedance of the same antenna when driven, the result can be expressed as follows:

$$
\begin{equation*}
I_{z}(z)=U^{\mathrm{inc}}\left[u(z)-v(z) u(0) \frac{Z_{L} Z_{0}}{Z_{L}+Z_{0}}\right] . \tag{2.74}
\end{equation*}
$$

This is the current at any point $z$ in the centre-loaded receiving antenna. The current in the load at $z=0$ is simply

$$
\begin{equation*}
I_{z}(0)=U^{\mathrm{inc}} u(0) \frac{Z_{0}}{Z_{0}+Z_{L}} \tag{2.75}
\end{equation*}
$$

since $v(0)=1 / Z_{0}$. When $Z_{L}=0$, this gives $I_{z}(0)=U^{\text {inc }} u(0)$. The voltage drop across the load is

$$
\begin{equation*}
I_{z}(0) Z_{L}=U^{\mathrm{inc}} u(0) \frac{Z_{0} Z_{L}}{Z_{0}+Z_{L}} \tag{2.76}
\end{equation*}
$$

When $Z_{L} \rightarrow \infty$, this is the open-circuit voltage across the terminals at $z=0$. That is

$$
\begin{equation*}
V\left(Z_{L} \rightarrow \infty\right)=\lim _{Z_{L} \rightarrow \infty} I_{z}(0) Z_{L}=U^{\mathrm{inc}} u(0) Z_{0}=\left[I_{z}(0) Z_{0}\right]_{Z_{L}=0} \tag{2.77}
\end{equation*}
$$

It is now clear that with (2.67) and (2.75) the current in the load is given by

$$
\begin{equation*}
I_{z}(0)=\frac{V\left(Z_{L} \rightarrow \infty\right)}{Z_{0}+Z_{L}}=\frac{E_{z}^{\mathrm{inc}} u(0) Z_{0}}{\beta_{0}\left(Z_{0}+Z_{L}\right)} . \tag{2.78}
\end{equation*}
$$

This is the current in a simple series circuit that consists of a generator with EMF equal to the open-circuit voltage across the terminals of the receiving antenna in series with the impedance of the antenna and the impedance of the load. The same conclusion is readily obtained by the application of Thévenin's theorem.

The quantity

$$
\begin{equation*}
u(0) Z_{0} / \beta_{0}=2 h_{e}\left(\frac{\pi}{2}\right) \tag{2.79}
\end{equation*}
$$

which occurs in (2.78) and is dimensionally a length, is known as the complex effective length of the receiving antenna with actual
length $2 h$. With (2.79), the current in the load is

$$
\begin{equation*}
I_{z}(0)=\frac{2 h_{e}\left(\frac{\pi}{2}\right) E_{z}^{\mathrm{inc}}}{Z_{0}+Z_{L}} \tag{2.80}
\end{equation*}
$$

Note that (2.78), (2.79) and (2.80) apply only when the axis of the receiving antenna is parallel to the incident electric vector and, therefore, also perpendicular to the direction of propagation of the incident wave. Similar but more general expressions that involve the orientation of the antenna relative to the incident wave and the direction of the electric vector in the plane wave front are available in the literature. $\dagger$
$\dagger$ See, for example, [1], chapter 4, section 4.

## CHAPTER 3

## THE TWO-ELEMENT ARRAY

### 3.1 The method of symmetrical components

An array is a configuration of two or more antennas so arranged that the superposition of the electromagnetic fields maintained at distant points by the currents in the individual elements yields a resultant field that fulfils certain desirable directional properties. Since the individual elements in an array are quite close togetherthe distance between adjacent elements is usually a half-wavelength or less-the currents in them necessarily interact. It follows that the distributions of both the amplitude and the phase of the current along each element depend not only on the length, radius, and driving voltage of that element, but also on the distributions in amplitude and phase of the currents along all elements in the array. Since these currents are the primary unknowns from which the radiation field is computed, a highly complicated situation arises if they are to be determined analytically, and not arbitrarily assumed known, as in conventional array theory.

In order to introduce the properties of arrays in a simple and direct manner, it is advantageous to study first the two-element array in some detail. The integral equation (2.11) for the current in a single isolated antenna is readily generalized to apply to the two identical parallel and non-staggered elements shown in Fig. 3.1. It is merely necessary to add to the vector potential on the surface of each element the contributions by the current in the other element. Thus, for element 1 , the vector potential difference is

$$
\begin{align*}
4 \pi \mu_{0}^{-1} & {\left[A_{1 z}(z)-A_{1 z}(h)\right] } \\
& =\int_{-h}^{h}\left[I_{1 z}\left(z^{\prime}\right) K_{11 d}\left(z, z^{\prime}\right)+I_{2 z}\left(z^{\prime}\right) K_{12 d}\left(z, z^{\prime}\right)\right] d z^{\prime} \\
& =\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h^{[ }}\left[\frac{1}{2} V_{10} \sin \beta_{0}(h-|z|)+U_{1}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] . \tag{3.1}
\end{align*}
$$

Similarly, for element 2:

$$
\begin{align*}
4 \pi \mu_{0}^{-1} & {\left[A_{2 z}(z)-A_{2 z}(h)\right] } \\
& =\int_{-h}^{h}\left[I_{1 z}\left(z^{\prime}\right) K_{21 d}\left(z, z^{\prime}\right)+I_{2 z}\left(z^{\prime}\right) K_{22 d}\left(z, z^{\prime}\right)\right] d z^{\prime} \\
& =\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V_{20} \sin \beta_{0}(h-|z|)+U_{2}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] . \tag{3.2}
\end{align*}
$$



Fig. 3.1. Two identical parallel antennas.
In these expressions

$$
\begin{align*}
& K_{11 d}\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{11}}}{R_{11}}-\frac{e^{-j \beta_{0} R_{11 h}}}{R_{11 h}}=K_{11}\left(z, z^{\prime}\right)-K_{11}\left(h, z^{\prime}\right)  \tag{3.3a}\\
& K_{12 d}\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{12}}}{R_{12}}-\frac{e^{-j \beta_{0} R_{12 h}}}{R_{12 h}}=K_{12}\left(z, z^{\prime}\right)-K_{12}\left(h, z^{\prime}\right) \tag{3.3b}
\end{align*}
$$

with

$$
\begin{array}{ll}
R_{11}=\sqrt{\left(z-z^{\prime}\right)^{2}+a^{2}}, & R_{11 h}=\sqrt{\left(h-z^{\prime}\right)^{2}+a^{2}} \\
R_{12}=\sqrt{\left(z-z^{\prime}\right)^{2}+b^{2}}, & R_{12 h}=\sqrt{\left(h-z^{\prime}\right)^{2}+b^{2}} \tag{3.4b}
\end{array}
$$

$K_{22 d}\left(z, z^{\prime}\right)$ and $K_{21 d}\left(z, z^{\prime}\right)$ are obtained from the above formulas when 1 is substituted for 2 and 2 for 1 in the subscripts.

The two simultaneous integral equations (3.1) and (3.2) can be reduced to a single equation in two special cases. These are (a) the so-called zero-phase sequence when the two driving voltages are identical so that the two currents are the same and (b) the firstphase sequence when the two driving voltages and the resulting two currents are equal in magnitude but $180^{\circ}$ out of phase. Specifically, for the zero-phase sequence,

$$
\begin{equation*}
V_{10}=V_{20}=V^{(0)}, \quad I_{1 z}(z)=I_{2 z}(z)=I_{z}^{(0)}(z), \tag{3.5}
\end{equation*}
$$

so that the equations (3.1) and (3.2) both become
$\int_{-h}^{h} I_{z}^{(0)}\left(z^{\prime}\right) K_{d}^{(0)}\left(z, z^{\prime}\right) d z^{\prime}$

$$
\begin{equation*}
=\frac{j 4 \pi}{\left.\zeta_{0} \cos \beta_{0} h^{\left[\frac{1}{2}\right.} V^{(0)} \sin \beta_{0}(h-|z|)+U^{(0)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]} \tag{3.6}
\end{equation*}
$$

where

$$
\begin{equation*}
U^{(0)}=\frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) K^{(0)}\left(h, z^{\prime}\right) d z^{\prime} \tag{3.7}
\end{equation*}
$$

and

$$
\left.\begin{array}{l}
K^{(0)}\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{11}}}{R_{11}}+\frac{e^{-j \beta_{0} R_{12}}}{R_{12}},  \tag{3.8}\\
K_{d}^{(0)}\left(z, z^{\prime}\right)=K^{(0)}\left(z, z^{\prime}\right)-K^{(0)}\left(h, z^{\prime}\right) .
\end{array}\right\}
$$

Similarly, for the first phase sequence,

$$
\begin{equation*}
V_{10}=-V_{20}=V^{(1)}, \quad I_{1 z}(z)=-I_{2 z}(z)=I_{z}^{(1)}(z) \tag{3.9}
\end{equation*}
$$

so that the two equations again become alike and equal to

$$
\begin{align*}
& \int_{-h}^{h} I_{z}^{(1)}\left(z^{\prime}\right) K_{d}^{(1)}\left(z, z^{\prime}\right) d z^{\prime} \\
& \quad=\frac{j 4 \pi}{\left.\zeta_{0} \cos \beta_{0} h^{\left[\frac{1}{2}\right.} V^{(1)} \sin \beta_{0}(h-|z|)+U^{(1)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]} \tag{3.10}
\end{align*}
$$

where

$$
\begin{equation*}
U^{(1)}=\frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h} I_{z}\left(z^{\prime}\right) K^{(1)}\left(h, z^{\prime}\right) d z^{\prime} \tag{3.11}
\end{equation*}
$$

and

$$
\left.\begin{array}{l}
K^{(1)}\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{11}}}{R_{11}}-\frac{e^{-j \beta_{0} R_{12}}}{R_{12}},  \tag{3.12}\\
K_{d}^{(1)}\left(z, z^{\prime}\right)=K^{(1)}\left(z, z^{\prime}\right)-K^{(1)}\left(h, z^{\prime}\right) .
\end{array}\right\}
$$

Note that the two phase sequences differ only in the $\operatorname{sign}$ in $K^{(0)}\left(z, z^{\prime}\right)$ and $K^{(1)}\left(z, z^{\prime}\right)$.

If (3.6) can be solved for the zero-phase-sequence current $I_{z}^{(0)}(z)$ and (3.10) for the first-phase-sequence current $I_{z}^{(1)}(z)$, the currents $I_{1 z}(z)$ and $I_{2 z}(z)$ maintained by the arbitrary voltages $V_{10}$ and $V_{20}$ can be obtained simply by superposition. This follows directly
if $V^{(0)}$ and $V^{(1)}$ are so chosen that

$$
\begin{equation*}
V^{(0)}=\frac{1}{2}\left[V_{10}+V_{20}\right], \quad V^{(1)}=\frac{1}{2}\left[V_{10}-V_{20}\right] \tag{3.13}
\end{equation*}
$$

In this case,

$$
\begin{equation*}
V_{10}=V^{(0)}+V^{(1)}, \quad V_{20}=V^{(0)}-V^{(1)} \tag{3.14}
\end{equation*}
$$

so that

$$
\begin{equation*}
I_{1 z}(z)=I_{z}^{(0)}(z)+I_{z}^{(1)}(z), \quad I_{2 z}(z)=I_{z}^{(0)}(z)-I_{z}^{(1)}(z) . \tag{3.15}
\end{equation*}
$$

### 3.2 Properties of the integrals

The two integral equations (3.6) and (3.10) for the phase-sequence currents are formally exactly like the equation (2.15) for the isolated antenna. They differ only in the kernels of the integrals on the left and in the definitions (3.7) and (3.11) of the functions $U$. Each of these is now the algebraic sum of two terms that are identical except that the radius $a$ appears in the first term, the distance $b$ between the elements in the second term. In order to determine the effect of this difference on the current it is convenient to consider first the two extreme cases when the elements are very close together and when they are very far apart.

The two elements may be considered close together when $\beta_{0} b<1$ and $b<h$. In this case, since $b$ satisfies substantially the same conditions as $a$, the behaviour of the integrals that contain $b$ corresponds closely to that of the integrals that contain $a$. These are discussed in the preceding chapter. When the antennas are so far apart $\left(\beta_{0} b \gg 1, b \gg h\right)$ that $\left(\beta_{0} \sqrt{b^{2}+h^{2}}-\beta_{0} b\right) \ll 1$, the contribution to the difference kernels $K_{d}^{(0)}\left(z, z^{\prime}\right)$ and $K_{d}^{(1)}\left(z, z^{\prime}\right)$ by the term $K_{12 d}\left(z, z^{\prime}\right)=\left(e^{\left.-j \beta_{0} R_{12} / R_{12}\right)-\left(e^{-j \beta_{0} R_{12 h}} / R_{12 h}\right) \text { is very small since } R_{12}}\right.$ and $R_{12 h}$ differ only slightly. In this case, the principal part of the interaction between the currents in the two antennas is included in the function $U^{(0)}$ or $U^{(1)}$ and the integrals on the left in (3.6) and (3.10) are only slightly different from the corresponding integral for the single antenna. The interaction between the currents in the two antennas is approximately as if each maintained along the other a vector potential that is uniform in amplitude and phase. Accordingly, the current induced in each element by the other is distributed in a first approximation as a shifted cosine. This conclusion follows directly from the fact that the component of current associated with the constant part of the vector potential along the surface of the isolated antenna is distributed in this manner.

When the separation of the two elements is such that $\beta_{0} b>1$ but not so great that $\beta_{0} \sqrt{b^{2}+h^{2}}$ differs negligibly from $\beta_{0} b$, the vector potentials maintained by the currents on the one antenna at points along the surface of the other differ significantly from one another in phase due to retarded action. The induced currents should then have two components, the one distributed approximately as the shifted cosine with half-angle arguments, the other as the shifted cosine.

In order to verify the correctness of these conclusions the difference integral

$$
\begin{equation*}
S_{b}(h, z)-S_{b}(h, h)=\int_{-h}^{h} \sin \beta_{0}\left|z^{\prime}\right| K_{d}\left(z, z^{\prime}\right) d z^{\prime} \tag{3.16}
\end{equation*}
$$

has been evaluated for $\beta_{0} h=\pi$ over a range of values of $\beta_{0} b$ extending from 0.04 to $4 \cdot 5$. The real and imaginary parts are shown in Fig. 3.2 together with the three trigonometric functions, $\sin \beta_{0} z$, $\left(\cos \beta_{0} z+1\right)$ and $\cos \frac{1}{2} \beta_{0} z$, to which the sine, shifted cosine and shifted cosine with half-angle arguments reduce when $\beta_{0} h=\pi$. For convenience in the graphical comparison, - $\left(\cos \beta_{0} z+1\right)$ and $-\cos \frac{1}{2} \beta_{0} z$ are shown. It is evident from Fig. 3.2 that the real part of the difference integral approximates $\sin \beta_{0} z$ when $\beta_{0} b<1$,


Fig. 3.2. The functions $S_{b}(h, z)-S_{b}(h, h)$ compared with three trigonometric functions.
$1+\cos \beta_{0} z$ when $\beta_{0} b \geqslant 1$. On the other hand, the imaginary part resembles the shifted cosine with half-angle arguments, in this case $\cos \frac{1}{2} \beta_{0} z$, for all values of $\beta_{0} b$.

As a consequence of these observations, the following approximate representation of the integrals in (3.6) and (3.10) is indicated: For $\beta_{0} b<1$,

$$
\begin{equation*}
\int_{-h}^{h} I_{z}\left(z^{\prime}\right)\left(\frac{\cos \beta_{0} R_{12}}{R_{12}}-\frac{\cos \beta_{0} R_{12 h}}{R_{12 h}}\right) d z^{\prime}=\Psi_{12}(z) I_{z}(z) \doteq \Psi_{12} I_{z}(z) \tag{3.17a}
\end{equation*}
$$

where $\Psi_{12}$ is a constant.
For $\beta_{0} b \geqslant 1$,
$\int_{-h}^{h} I_{z}\left(z^{\prime}\right)\left(\frac{\cos \beta_{0} R_{12}}{R_{12}}-\frac{\cos \beta_{0} R_{12 h}}{R_{12 h}}\right) d z^{\prime} \sim \cos \beta_{0} z-\cos \beta_{0} h$.
For all values of $\beta_{0} b$

$$
\begin{equation*}
\int_{-h}^{h} I_{z}\left(z^{\prime}\right)\left(\frac{\sin \beta_{0} R_{12}}{R_{12}}-\frac{\sin \beta_{0} R_{12 h}}{R_{12 h}}\right) d z^{\prime} \sim \cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h . \tag{3.17c}
\end{equation*}
$$

### 3.3 Reduction of integral equations for phase sequences to algebraic equations

The relations ( $3.17 \mathrm{a}, \mathrm{b}, \mathrm{c}$ ), combined with the results of chapter 2 , indicate that the current in each of the two coupled elements in both phase sequences must have leading terms that are well approximated by the following zero-order, three-term formula :

$$
\begin{align*}
I_{z}^{(m)}(z) \doteq\left[I_{z}^{(m)}(z)\right]_{0}= & I_{V}^{(m)}\left[\sin \beta_{0}(h-|z|)+T_{U}^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{3.18}
\end{align*}
$$

where $m=0$ or 1 and $I_{V}^{(m)}, T_{U}^{(m)}$ and $T_{D}^{(m)}$ are complex coefficients that must be determined.

The substitution of (3.18) into the integral in (3.6) and (3.10) involves the following parts obtained from the real part $K_{d R}^{(m)}\left(z, z^{\prime}\right)$ of the difference kernel $K_{d}^{(m)}\left(z, z^{\prime}\right)$ defined in (3.8) and (3.12) for $m=0$ and 1 :
$\beta_{0} b<1, \quad \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d R}^{(m)} \sin \beta_{0}(h-|z|)$

$$
\begin{gather*}
\beta_{0} b \geqslant 1, \quad \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d R} \sin \beta_{0}(h-|z|) \\
+\Psi_{d \Sigma R}^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right) \tag{3.19b}
\end{gather*}
$$

For all values of $\beta_{0} b$, $\dagger$

$$
\begin{equation*}
\int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K_{d R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d U R}^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right) \tag{3.20a}
\end{equation*}
$$

$$
\begin{equation*}
\int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) K_{d R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d D R}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right) \tag{3.20b}
\end{equation*}
$$

The corresponding integrals obtained with the imaginary part, $K_{d I}^{(m)}\left(z, z^{\prime}\right)$ of the difference kernel $K_{d}^{(m)}\left(z, z^{\prime}\right)$, are valid for all values of $\beta_{0} b$. They are:

$$
\begin{gather*}
\int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d I}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d I}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)  \tag{3.21}\\
\int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K_{d I}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d U I}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)  \tag{3.22}\\
\int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) K_{d I}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d D I}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right) . \tag{3.23}
\end{gather*}
$$

The several $\Psi$ functions introduced in the above expressions are defined as follows:

$$
\beta_{0} b<1:\left\{\begin{array}{l}
\Psi_{d R}^{(m)}=\Psi_{d R}^{(m)}\left(z_{m}\right) ;\left\{\begin{array}{l}
z_{m}=0, \quad \beta_{0} h \leqslant \pi / 2 \\
z_{m}=h-\lambda / 4, \quad \beta_{0} h>\pi / 2
\end{array}\right.  \tag{3.24a}\\
\Psi_{d R}^{(m)}(z)=\csc \beta_{0}(h-|z|) \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime}
\end{array}\right.
$$

$\dagger$ Strictly according to (3.17b) the integral in (3.20b) should be treated separately with different behaviours when $\beta_{0} b<1$ and $\beta_{0} b \geqslant 1$. However, since the distributions $\cos \beta_{0} z-\cos \beta_{0} h$ and $\cos \left(\beta_{0} z / 2\right)-\cos \left(\beta_{0} h / 2\right)$ are quite similar when $\beta_{0} h \leqslant 5 \pi / 4$, and since considerable complication is avoided by not making this separation, the relation (3.20b) is used for both real and imaginary parts of the kernel and for all spacings.


The following apply for all values of $\beta_{0} b$ :

$$
\begin{align*}
& \Psi_{d I}^{(m)}=\left(1-\cos \frac{1}{2} \beta_{0} h\right)^{-1} \int_{-h}^{h} \sin \beta_{0}\left(h-\mid z^{\prime}\right) K_{d I}^{(m)}\left(0, z^{\prime}\right) d z^{\prime}  \tag{3.26}\\
& \Psi_{d U R}^{(m)}=\left(1-\cos \beta_{0} h\right)^{-1} \int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K_{d R}^{(m)}\left(0, z^{\prime}\right) d z^{\prime}(3.27 \mathrm{a})  \tag{3.27a}\\
& \Psi_{d U I}^{(m)}=\left(1-\cos \frac{1}{2} \beta_{0} h\right)^{-1} \int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K_{d I}^{(m)}\left(0, z^{\prime}\right) d z^{\prime}(3.27 \mathrm{~b}) \\
& \Psi_{d D}^{(m)}=\left(1-\cos \frac{1}{2} \beta_{0} h\right)^{-1} \int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) K_{d}^{(m)}\left(0, z^{\prime}\right) d z^{\prime} \tag{3.28}
\end{align*}
$$

For each pair of real and imaginary parts, the notation $\Psi_{d}=\Psi_{d R}+\Psi_{d I}$ will be used.

When (3.18) is substituted in $U^{(m)}$ as defined in (3.7) and (3.11), the notation of $(2.20)-(2.21 \mathrm{c})$ applies in the form

$$
\begin{equation*}
U^{(m)}=\frac{-j \zeta_{0}}{4 \pi} I_{V}^{(m)}\left[\Psi_{V}^{(m)}(h)+T_{U}^{(m)} \Psi_{U}^{(m)}(h)+T_{D}^{(m)} \Psi_{D}^{(m)}(h)\right] \tag{3.29}
\end{equation*}
$$

where $\quad \Psi_{V}^{(m)}(h)=\int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K^{(m)}\left(h, z^{\prime}\right) d z^{\prime}$

$$
\begin{align*}
& \Psi_{U}^{(m)}(h)=\int_{-h}^{h}\left(\cos \beta_{0} z^{\prime}-\cos \beta_{0} h\right) K^{(m)}\left(h, z^{\prime}\right) d z^{\prime}  \tag{3.31}\\
& \Psi_{D}^{(m)}(h)=\int_{-h}^{h}\left(\cos \frac{1}{2} \beta_{0} z^{\prime}-\cos \frac{1}{2} \beta_{0} h\right) K^{(m)}\left(h, z^{\prime}\right) d z^{\prime}
\end{align*}
$$

with $m=0,1$.
If the approximate formulas for the several parts of the integrals when $\beta_{0} b<1$ are substituted in (3.6) and (3.11), an algebraic equation is obtained that is just like (2.22) for the single antenna but with superscripts $m$ on $I, T, \Psi, V$ and $U$. It follows that (2.23a), (2.24), (2.25) and (2.26) give the solutions for $I_{V}^{(m)}, T_{V}^{(m)}$ and $T_{D}^{(m)}$ if superscripts $m$ are affixed to $V_{0}$ and to all $\Psi$ 's.

When $\beta_{0} b \geqslant 1$, the equation corresponding to (2.22) has the
following slightly different form:

$$
\begin{align*}
\left(I_{V}^{(m)} \Psi_{d R}\right. & \left.-\frac{j 2 \pi V^{(m)}}{\zeta_{0} \cos \beta_{0} h}\right) \sin \beta_{0}(h-|z|) \\
& +\left(I_{V}^{(m)} \Psi_{d \Sigma R}^{(m)}+I_{V}^{(m)} T_{U}^{(m)} \Psi_{d U R}^{(m)}-\frac{j 4 \pi U^{(m)}}{\zeta_{0} \cos \beta_{0} h}\right) \\
& \times\left(\cos \beta_{0} z-\cos \beta_{0} h\right)+I_{V}^{(m)}\left(j \Psi_{d I}^{(m)}+j \Psi_{d U I}^{(m)} T_{U}^{(m)}\right. \\
& \left.+\Psi_{d D}^{(m)} T_{D}^{(m)}\right)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)=0 \tag{3.33}
\end{align*}
$$

If the coefficients of the trigonometric functions are individually equated to zero and (3.29) is substituted for $U^{(m)}$, three relations corresponding to ( $2.23 \mathrm{a}-\mathrm{c}$ ) are obtained. They are readily solved to give

$$
\begin{align*}
I_{V}^{(m)}= & \frac{j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}  \tag{3.34}\\
T_{V}^{(m)}= & \left\{\Psi_{d D}^{(m)}\left[\Psi_{V}^{(m)}(h)-\Psi_{d 2 R}^{(m)} \cos \beta_{0} h\right]-j \Psi_{D}^{(m)}(h) \Psi_{d l}^{(m)}\right\rangle / Q^{(m)}  \tag{3.35}\\
T_{D}^{(m)}= & -j\left\{\Psi_{d I}^{(m)}\left\lceil\Psi_{d U R}^{(m)} \cos \beta_{0} h-\Psi_{V}^{(m)}(h)\right]\right. \\
& \left.\quad+\Psi_{d U I}^{(m)}\left[\Psi_{V}^{(m)}(h)-\Psi_{d \Sigma R}^{(m)} \cos \beta_{0} h\right]\right\} / Q^{(m)}  \tag{3.36}\\
&  \tag{3.37}\\
Q^{(m)}= & \Psi_{d D}^{(m)}\left[\Psi_{d U R}^{(m)} \cos \beta_{0} h-\Psi_{V}^{(m)}(h)\right]+j \Psi_{D}^{(m)}(h) \Psi_{d U I}^{(m)} .
\end{align*}
$$

As throughout this chapter, $m=0,1$.

### 3.4 The phase-sequence currents and admittances

With the three coefficients $I_{V}^{(m)}, T_{U}^{(m)}$ and $T_{D}^{(m)}$ determined, the phase-sequence currents and the admittances may be written down directly. When $\beta_{0} b<1$, they are

$$
\begin{align*}
I_{z}^{(m)}(z)= & \frac{j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R}^{(m)} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T_{U}^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.38}\\
Y^{(m)}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R}^{(m)} \cos \beta_{0} h}\left[\sin \beta_{0} h+T_{U}^{(m)}\left(1-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}^{(m)}\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{3.39}
\end{align*}
$$

with

$$
\begin{align*}
& T_{V}^{(m)}=\left[\Psi_{V}^{(m)}(h) \Psi_{d D}^{(m)}-j \Psi_{D}^{(m)}(h) \Psi_{d I}^{(m)}\right] / Q^{(m)}  \tag{3.40}\\
& T_{D}^{(m)}=-j\left\{\Psi_{d}^{(m)}\left[\Psi_{d U R}^{(m)} \cos \beta_{0} h-\Psi_{V}^{(m)}(h)\right]+\Psi_{V}^{(m)}(h) \Psi_{d U I}^{(m)} ; / Q^{(m)}\right.  \tag{3.41}\\
& Q^{(m)}=\Psi_{d D}^{(m)}\left[\Psi_{d U R}^{(m)} \cos \beta_{0} h-\Psi_{U}^{(m)}(h)\right]+j \Psi_{D}^{(m)}(h) \Psi_{d U I}^{(m)} . \tag{3.42}
\end{align*}
$$

The functions $\Psi^{(m)}$ are defined in (3.24), (3.26)-(3.28) and (3.30)(3.32). For the zero-phase sequence, $m=0$; for the first-phase sequence, $m=1$.

As for the single antenna, these formulas for $I_{z}^{(m)}(z)$ and $Y_{0}^{(m)}$ become indeterminate when $\beta_{0} h=\pi / 2$. Convenient alternative forms when $\beta_{0} h$ is at or near $\pi / 2$ are

$$
\begin{align*}
I_{z}^{(m)}(z)= & \frac{-j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R}^{(m)}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+T_{U}^{\prime(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-T_{D}^{\prime(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.43}\\
Y^{(m)}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R}^{(m)}}\left[\sin \beta_{0} h-T_{U}^{(m)}\left(1-\cos \beta_{0} h\right)+T_{D}^{\prime(m)}\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{3.44}
\end{align*}
$$

where, as in (2.37),

$$
\begin{equation*}
T_{U}^{\prime(m)}=-\frac{T_{U}^{(m)}+\sin \beta_{0} h}{\cos \beta_{0} h}, \quad T_{D}^{(m)}=\frac{T_{D}^{(m)}}{\cos \beta_{0} h} \tag{3.45}
\end{equation*}
$$

$T_{U}^{(m)}$ and $T_{D}^{(m)}$ are in (3.40) and (3.41).
When $\beta_{0} b \geqslant 1$, the general form of the expressions for the phasesequence current and admittance are similar to those for $\beta_{0} b<1$. They are

$$
\begin{align*}
I_{z}^{(m)}(z)= & \frac{j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T_{U}^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}^{(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.46}\\
Y^{(m)}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0} h+T_{U}^{(m)}\left(1-\cos \beta_{0} h\right)\right. \\
& \left.+T_{D}^{(m)}\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{3.47}
\end{align*}
$$

with $T_{U}^{(m)}$ and $T_{D}^{(m)}$ given by (3.35) and (3.36) with (3.37). Similarly, when $\beta_{0} h$ is near $\pi / 2$ and $\beta_{0} b \geqslant 1$.

$$
\begin{align*}
I_{z}^{(m)}(z)= & \frac{-j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+T_{U}^{\prime(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-T_{D}^{\prime(m)}\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.48}\\
Y^{(m)}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\sin \beta_{0} h-T_{U}^{\prime(m)}\left(1-\cos \beta_{0} h\right)+T_{D}^{\prime(m)}\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] . \tag{3.49}
\end{align*}
$$

The parameters $T_{U}^{(m)}$ and $T_{D}^{\prime(m)}$ are defined as in (3.45); $T_{U}^{(m)}$ and $T_{D}^{(m)}$ are given by (3.35) and (3.36).

Note that the currents and admittances when $\beta_{0} b \geqslant 1$ differ from those when $\beta_{0} b<1$ not only in the $T$ (or $T^{\prime}$ ) parameters but also in the appearance of $\Psi_{d R}$ for the isolated antenna instead of $\Psi_{d R}^{(m)}$ for the coupled pair.

### 3.5 Currents for arbitrarily driven antennas; self- and mutual admittances and impedances

With the phase-sequence currents $I_{z}^{(0)}(z)$ and $I_{z}^{(1)}(z)$ determined, it is straightforward to obtain the expressions for the currents $I_{1 z}(z)$ and $I_{2 z}(z)$ in the two antennas when they are driven by the arbitrary voltages $V_{10}$ and $V_{20}$. If

$$
\begin{equation*}
V_{0}^{(0)}=\frac{1}{2}\left(V_{10}+V_{20}\right), \quad V_{0}^{(1)}=\frac{1}{2}\left(V_{10}-V_{20}\right) \tag{3.50}
\end{equation*}
$$

it follows that, when $\beta_{0} b \geqslant 1$,

$$
\begin{align*}
& I_{1 z}(z)=I_{z}^{(0)}(z)+I_{z}^{(1)}(z)=V_{10} v(z)+V_{20} w(z)  \tag{3.51a}\\
& I_{2 z}(z)=I_{z}^{(0)}(z)-I_{z}^{(1)}(z)=V_{10} w(z)+V_{20} v(z) \tag{3.51b}
\end{align*}
$$

where

$$
\begin{align*}
v(z)= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+\frac{1}{2}\left(T_{U}^{(0)}+T_{U}^{(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+\frac{1}{2}\left(T_{D}^{(0)}+T_{D}^{(1)}\right)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.52a}\\
w(z)= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\frac{1}{2}\left(T_{U}^{(0)}-T_{U}^{(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.+\frac{1}{2}\left(T_{D}^{(0)}-T_{D}^{(1)}\right)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] . \tag{3.52b}
\end{align*}
$$

Alternatively, when $\beta_{0} h$ is near $\pi / 2$,

$$
\begin{align*}
v(z)= & \frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+\frac{1}{2}\left(T_{U}^{\prime(0)}+T_{U}^{\prime(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-\frac{1}{2}\left(T_{D}^{\prime(0)}+T_{D}^{\prime(1)}\right)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.52c}\\
w(z)= & \frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\frac{1}{2}\left(T_{U}^{(0)}-T_{U}^{\prime(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right. \\
& \left.-\frac{1}{2}\left(T_{D}^{\prime(0)}-T_{D}^{\prime(1)}\right)\left(\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h\right)\right] . \tag{3.52d}
\end{align*}
$$

The corresponding expressions when $\beta_{0} b<1$ are easily obtained from (3.38). The driving-point currents may be expressed in the form

$$
\begin{align*}
& I_{1 z}(0)=V_{10} Y_{s 1}+V_{20} Y_{12}  \tag{3.53a}\\
& I_{2 z}(0)=V_{10} Y_{21}+V_{20} Y_{s 2} \tag{3.53b}
\end{align*}
$$

where $Y_{s 1}$ and $Y_{s 2}$ are the self-admittances, $Y_{12}$ and $Y_{21}$ are the
mutual admittances. They are given by

$$
\begin{align*}
& Y_{s 1}=Y_{s 2}=v(0)=\frac{1}{2}\left(Y^{(0)}+Y^{(1)}\right)  \tag{3.54}\\
& Y_{21}=Y_{12}=w(0)=\frac{1}{2}\left(Y^{(0)}-Y^{(1)}\right) . \tag{3.55}
\end{align*}
$$

Specifically,

$$
\begin{align*}
Y_{s 1}=Y_{s 2}=v(0)= & \frac{j \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[2 \sin \beta_{0} h+\left(T_{U}^{(0)}+T_{U}^{(1)}\right)\left(1-\cos \beta_{0} h\right)\right. \\
& \left.+\left(T_{D}^{(0)}+T_{D}^{(1)}\right)\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.56}\\
Y_{12}=Y_{21}=w(0)= & \frac{j \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\left(T_{U}^{(0)}-T_{U}^{(1)}\right)\left(1-\cos \beta_{0} h\right)\right. \\
& \left.+\left(T_{D}^{(0)}-T_{D}^{(1)}\right)\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] \tag{3.57}
\end{align*}
$$

When $\beta_{0} h=\pi / 2$, the self- and mutual admittances are

$$
\begin{align*}
Y_{s 2}=Y_{s 1}= & \frac{j \pi}{\zeta_{0} \Psi_{d R}}\left[2 \sin \beta_{0} h-\left(T_{U}^{\prime(0)}+T_{U}^{\prime(1)}\right)\left(1-\cos \beta_{0} h\right)\right. \\
& \left.+\left(T_{D}^{\prime(0)}+T_{D}^{\prime(1)}\right)\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right]  \tag{3.58}\\
Y_{21}=Y_{12}= & \frac{-j \pi}{\zeta_{0} \Psi_{d R}}\left[\left(T_{U}^{\prime(0)}-T_{U}^{\prime(1)}\right)\left(1-\cos \beta_{0} h\right)\right. \\
& \left.-\left(T_{D}^{\prime(0)}-T_{D}^{\prime(1)}\right)\left(1-\cos \frac{1}{2} \beta_{0} h\right)\right] . \tag{3.59}
\end{align*}
$$

The associated self- and mutual impedances are the coefficients of the currents in the equations

$$
\begin{align*}
& V_{10}=I_{1 z}(0) Z_{s 1}+I_{2 z}(0) Z_{12}  \tag{3.60a}\\
& V_{20}=I_{1 z}(0) Z_{21}+I_{2 z}(0) Z_{s 2} . \tag{3.60b}
\end{align*}
$$

It is readily shown that

$$
\begin{gather*}
Z_{s 1}=Y_{s 2} / D=\frac{1}{2}\left(Z^{(0)}+Z^{(1)}\right) ; Z_{s 2}=Y_{s 1} / D=\frac{1}{2}\left(Z^{(0)}+Z^{(1)}\right) \\
Z_{12}=-Y_{21} / D=\frac{1}{2}\left(Z^{(0)}-Z^{(1)}\right) ; Z_{21}=-Y_{12} / D=\frac{1}{2}\left(Z^{(0)}-Z^{(1)}\right) \tag{3.61b}
\end{gather*}
$$

where $D=Y_{s 1} Y_{s 2}-Y_{12} Y_{21}=\left(Z_{s 1} Z_{s 2}-Z_{12} Z_{21}\right)^{-1}$. If lumped impedances $Z_{1}$ and $Z_{2}$ are connected in series with $V_{10}$ and $V_{20}$, $Z_{s 1}$ and $Z_{s 2}$ in (3.53a, b) are replaced by $Z_{11}=Z_{s 1}+Z_{1}$ and $Z_{22}=Z_{s 2}+Z_{2}$.

This completes the general formulation for the currents and admittances of two parallel antennas driven by the arbitrary voltages $V_{10}$ and $V_{20}$.

### 3.6 Currents for one driven, one parasitic antenna

If antenna 2 is parasitic instead of driven and is centre-loaded by an arbitrary impedance $Z_{2}$, the driving voltage $V_{20}$ may be replaced by the negative of the voltage drop across the load. Thus,

$$
\begin{equation*}
V_{20}=-I_{2 z}(0) Z_{2}=-I_{2 z}(0) / Y_{2} \tag{3.62}
\end{equation*}
$$

If (3.62) is substituted in (3.53b), the result is

$$
\begin{equation*}
I_{2 z}(0)=V_{10} \frac{Y_{21}}{1+Y_{s 2} Z_{2}}=-V_{10} \frac{Z_{21}}{Z_{s 2}\left(Z_{s 1}+Z_{2}\right)+Z_{12} Z_{21}} \tag{3.63}
\end{equation*}
$$

so that

$$
\begin{equation*}
V_{20}=-V_{10}\left(\frac{Y_{21}}{Y_{2}+Y_{s 2}}\right)=V_{10} \frac{Z_{21} Z_{2}}{Z_{s 2}\left(Z_{s 1}+Z_{2}\right)+Z_{12} Z_{21}} . \tag{3.64}
\end{equation*}
$$

It follows from (3.51a, b) that

$$
\begin{align*}
& I_{1 z}(z)=V_{10}\left[v(z)-\left(\frac{Y_{21}}{Y_{2}+Y_{s 2}}\right) w(z)\right]  \tag{3.65a}\\
& I_{2 z}(z)=V_{10}\left[w(z)-\left(\frac{Y_{21}}{Y_{2}+Y_{s 2}}\right) v(z)\right] . \tag{3.65b}
\end{align*}
$$

The driving-point admittance and impedance are

$$
\begin{gather*}
Y_{1 i n}=\frac{I_{1 z}(0)}{V_{10}}=Y_{s 1}-\frac{Y_{21} Y_{12}}{Y_{2}+Y_{s 2}}  \tag{3.66a}\\
Z_{1 i n}=\frac{1}{Y_{1 i n}}=\frac{Z_{s 1}\left(Z_{s 2}+Z_{2}\right)+Z_{12} Z_{21}}{Z_{s 2}+Z_{2}} . \tag{3.66b}
\end{gather*}
$$

Note that when $Z_{2}=0$ or $Y_{2}=\infty$,

$$
\begin{equation*}
Y_{1 i n}=Y_{s 1} ; \quad I_{1 z}(z)=V_{10} v(z) ; \quad I_{2 z}(z)=V_{10} w(z) \tag{3.67a}
\end{equation*}
$$

Alternatively, when $Z_{2}=\infty$ or $Y_{2}=0$,

$$
\begin{gather*}
Z_{1 i n}=Z_{s 1} ; \quad I_{1 z}(z)=V_{10}\left[v(z)-\frac{Y_{21}}{Y_{s 2}} w(z)\right] ; \\
I_{2 z}(z)=V_{10}\left[w(z)-\frac{Y_{21}}{Y_{s 2}} v(z)\right] . \tag{3.67b}
\end{gather*}
$$

The parasitic element is tuned to resonance when $Y_{2}=j B_{2}$ and $B_{2}=-B_{s 2}$ in $Y_{s 2}=G_{s 2}+j B_{s 2}$. With this choice, $Y_{21} /\left(Y_{2}+Y_{s 2}\right)$ is maximized so that

$$
\begin{align*}
& I_{1 z}(z)=V_{10}\left[v(z)-\frac{Y_{21}}{G_{s 2}} w(z)\right]  \tag{3.68a}\\
& I_{2 z}(z)=V_{10}\left[w(z)-\frac{Y_{21}}{G_{s 2}} v(z)\right] . \tag{3.68b}
\end{align*}
$$

Since the coefficient $Y_{21} / G_{s 2}$ is of the order of magnitude of one, the coefficients of $v(z)$ and $w(z)$ are comparable. It follows that the distributions of $I_{1 z}(z)$ and $I_{2 z}(z)$ are roughly similar, whereas when $Z_{2}=0$ as in (3.67a), they are quite different unless $\beta_{0} h$ is near $\pi / 2$.

### 3.7 The couplet

Perhaps the most interesting two-element array is the couplet in which the distance between the elements is $\lambda / 4$ and the currents at the driving points are equal in amplitude but differ in phase by a quarter period. That is

$$
\begin{equation*}
I_{2 z}(0)=j I_{1 z}(0) \tag{3.69}
\end{equation*}
$$

It follows from (3.60a, b) that with $Z_{12}=Z_{21}$ and $Z_{s 2}=Z_{s 1}$,

$$
\begin{gather*}
V_{10}=I_{1 z}(0)\left[Z_{s 1}+j Z_{12}\right]  \tag{3.70a}\\
V_{20}=I_{2 z}(0)\left[Z_{s 1}-j Z_{12}\right]=I_{1 z}(0)\left[Z_{12}+j Z_{s 1}\right] . \tag{3.70b}
\end{gather*}
$$

Hence, $\quad Z_{1 i n}=Z_{s 1}+j Z_{12}, \quad Z_{2 i n}=Z_{s 1}-j Z_{12}$.
The distributions of current are obtained from (3.51a, b). Thus

$$
\begin{align*}
& I_{1 z}(z)=V_{10}\left[v(z)+\frac{Z_{12}+j Z_{s 1}}{Z_{s 1}+j Z_{12}} w(z)\right]  \tag{3.71a}\\
& I_{2 z}(z)=V_{10}\left[w(z)+\frac{Z_{12}+j Z_{s 1}}{Z_{s 1}+j Z_{12}} v(z)\right] . \tag{3.71b}
\end{align*}
$$

Instead of specifying the driving-point currents $I_{1 z}(0)$ and $I_{2 z}(0)$ as in (3.69), the driving voltages may be assigned so that

$$
\begin{equation*}
V_{20}=j V_{10} . \tag{3.72}
\end{equation*}
$$

It then follows from $(3.53 \mathrm{a}, \mathrm{b})$ that

$$
\begin{align*}
& I_{1 z}(0)=V_{10}\left(Y_{s 1}+j Y_{12}\right)  \tag{3.73a}\\
& I_{2 z}(0)=V_{20}\left(Y_{s 1}-j Y_{12}\right)=V_{10}\left(Y_{12}+j Y_{s 1}\right) . \tag{3.73b}
\end{align*}
$$

The driving-point admittances are

$$
\begin{equation*}
Y_{1 i n}=Y_{s 1}+j Y_{12}, \quad Y_{2 i n}=Y_{s 1}-j Y_{12} . \tag{3.74}
\end{equation*}
$$

The currents are obtained from (3.51a, b) with (3.61a, b). Thus,

$$
\begin{align*}
I_{1 z}(z) & =V_{10}\left[v(z)+\frac{Y_{12}+j Y_{s 1}}{Y_{s 1}-j Y_{12}} w(z)\right] \\
& =V_{10}\left[v(z)-\frac{Z_{12}-j Z_{s 1}}{Z_{s 1}+j Z_{12}} w(z)\right] \tag{3.75a}
\end{align*}
$$

$$
\begin{align*}
I_{2 z}(z) & =V_{10}\left[w(z)+\frac{Y_{12}+j Y_{s 1}}{Y_{s 1}-j Y_{12}} v(z)\right] \\
& =V_{10}\left[w(z)-\frac{Z_{12}-j Z_{s 1}}{Z_{s 1}+j Z_{12}} v(z)\right] \tag{3.75b}
\end{align*}
$$

The currents are not the same when $I_{1 z}(0)$ and $I_{2 z}(0)$ are specified as when $V_{10}$ and $V_{20}$ are assigned. Note that

$$
\begin{align*}
& {\left[I_{1 z}(z)\right]_{I}-\left[I_{1 z}(z)\right]_{V}=2 V_{10} Z_{12} w(z)}  \tag{3.76a}\\
& {\left[I_{2 z}(z)\right]_{I}-\left[I_{2 z}(z)\right]_{V}=2 V_{10} Z_{12} v(z) .} \tag{3.76b}
\end{align*}
$$

If the currents differ significantly, the field patterns cannot be the same.

### 3.8 Field patterns

The radiation field of an array of two parallel elements is the vector sum of the fields maintained by the currents in the individual elements. In terms of the spherical coordinates $R, \Theta, \Phi$, that have their origin midway between the centres of the two elements, the individual electric fields are readily expressed in the form $(2.45 \mathrm{a}, \mathrm{b})$ for the currents $(3.51 \mathrm{a}, \mathrm{b})$. Thus

$$
\begin{align*}
& E_{\Theta 1}^{r}=-\frac{1}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{1}}}{R}\left[V_{10} f\left(\Theta, \beta_{0} h\right)+V_{20} g\left(\Theta, \beta_{0} h\right)\right]  \tag{3.77a}\\
& E_{\Theta 2}^{r}=-\frac{1}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{2}}}{R}\left[V_{10} g\left(\Theta, \beta_{0} h\right)+V_{20} f\left(\Theta, \beta_{0} h\right)\right] \tag{3.77b}
\end{align*}
$$

The field functions $F_{m}\left(\Theta, \beta_{0} h\right), G_{m}\left(\Theta, \beta_{0} h\right)$ and $D_{m}\left(\Theta, \beta_{0} h\right)$ are defined in (2.46), (2.47) and (2.48). Alternatively, when $\beta_{0} h$ is near $\pi / 2$, the fields for the currents ( $3.52 \mathrm{c}, \mathrm{d}$ ) are:

$$
\begin{equation*}
E_{\Theta 1}^{r}=\frac{1}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{1}}}{R}\left[V_{10} f^{\prime}\left(\Theta, \beta_{0} h\right)+V_{20} g^{\prime}\left(\Theta, \beta_{0} h\right)\right] \tag{3.81a}
\end{equation*}
$$

$$
\begin{align*}
& E_{\Theta 2}^{r}=\frac{1}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{2}}}{R}\left[V_{10} g^{\prime}\left(\Theta, \beta_{0} h\right)+V_{20} f^{\prime}\left(\Theta, \beta_{0} h\right)\right]  \tag{3.81b}\\
& f^{\prime}\left(\Theta, \beta_{0} h\right)= H_{m}\left(\Theta, \beta_{0} h\right)+\frac{1}{2}\left(T_{U}^{(0)}+T_{U}^{(1)}\right) G_{m}\left(\Theta, \beta_{0} h\right) \\
&-\frac{1}{2}\left(T_{D}^{\prime(0)}+T_{D}^{\prime(1)}\right) D_{m}\left(\Theta, \beta_{0} h\right)  \tag{3.82}\\
& g^{\prime}\left(\Theta, \beta_{0} h\right)= \frac{1}{2}\left(T_{U}^{(0)}-T_{U}^{(1)}\right) G_{m}\left(\Theta, \beta_{0} h\right) \\
&-\frac{1}{2}\left(T_{D}^{\prime(0)}-T_{D}^{\prime(1)}\right) D_{m}\left(\Theta, \beta_{0} h\right) . \tag{3.83}
\end{align*}
$$

The function $H_{m}\left(\Theta, \beta_{0} h\right)$ is defined in (2.51).
The resultant radiation field of the arbitrarily driven two-element array is

$$
\begin{align*}
E_{\Theta}^{r}=E_{\Theta 1}^{r} & +E_{\Theta 2}^{r}=\frac{-1}{\Psi_{d R}} \frac{e^{-j \beta_{0} R}}{R}\left\{\left[V_{10} f\left(\Theta, \beta_{0} h\right)\right.\right. \\
& \left.+V_{20} g\left(\Theta, \beta_{0} h\right)\right] e^{-j\left(\beta_{0} b / 2\right) \cos \Phi \sin \Theta} \\
& \left.+\left[V_{10} g\left(\Theta, \beta_{0} h\right)+V_{20} f\left(\Theta, \beta_{0} h\right)\right] e^{j\left(\beta_{0} b / 2\right) \cos \Phi \sin \Theta}\right\} . \tag{3.84}
\end{align*}
$$

When $\beta_{0} h$ is near $\pi / 2,-f^{\prime}\left(\Theta, \beta_{0} h\right)$ and $-g^{\prime}\left(\Theta, \beta_{0} h\right)$ may be substituted, respectively, for $f\left(\Theta, \beta_{0} h\right)$ and $g\left(\Theta, \beta_{0} h\right)$.

### 3.9 The two-term approximation

As pointed out in section 10 of the preceding chapter, the difference between the distribution functions $F_{0 z}=\cos \beta_{0} z-\cos \beta_{0} h$ and $H_{0 z}=\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h$ is relatively unimportant in the determination of the far-field and the driving-point admittance of an isolated antenna when $\beta_{0} h \leqslant 5 \pi / 4$. This is also true of the far-field and driving-point admittances of two coupled antennas provided the interaction between them is not sensitive to small changes in the current distributions. When both elements are driven by comparable voltages and when the distance between them is sufficiently great so that $\beta_{0} b \geqslant 1$, it may be assumed that the substitution of $\cos \beta_{0} z-\cos \beta_{0} h$ for $\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h \operatorname{can}$ produce no important change in the admittances or the far-field. When one element is parasitic and unloaded, the three-term approximation is automatically reduced to two terms since the distribution $\sin \beta_{0}(h-|z|)$ is excited only by a generator or an equivalent voltage drop across a load. Correspondingly, the two-term approximation is reduced to a single term. However, this is quite adequate for many purposes. In anticipation, it may be added at this point that when an array consists of one driven antenna and many parasitic
elements, at least two terms are desirable in the representation of the current distributions. This is considered in a later chapter.

As for the single antenna, the two-term approximations are quickly obtained from the three-term formulas by the simple substitution of $\cos \beta_{0} z-\cos \beta_{0} h$ for $\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h$ and the representation of the resulting coefficient $\left(T_{U}+T_{D}\right)$ by $T$. It is implicit that $\Psi_{d D} \rightarrow \Psi_{d U}, \Psi_{D}(h) \rightarrow \Psi_{U}(h)$. Thus, the phase-sequence currents and admittances (3.46) and (3.47) become, for $\beta_{0} b \geqslant 1$,

$$
\begin{gather*}
I_{z}^{(m)}(z)=\frac{j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \\
Y^{(m)}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0} h+T^{(m)}\left(1-\cos \beta_{0} h\right)\right] \tag{3.85}
\end{gather*}
$$

where

$$
\begin{equation*}
T^{(m)}=T_{U}^{(m)}+T_{D}^{(m)}=-\frac{\Psi_{V}^{(m)}(h)-\left(\Psi_{d 2 R}^{(m)}+j \Psi_{d I}^{(m)}\right) \cos \beta_{0} h}{\Psi_{U}^{(m)}(h)-\Psi_{d U}^{(m)} \cos \beta_{0} h} \tag{3.87}
\end{equation*}
$$

Similarly, when $\beta_{0} h$ is near $\pi / 2,(3.48)$ and (3.49) reduce to

$$
\begin{equation*}
I_{z}^{(m)}(z)=\frac{-j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+T^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \tag{3.88}
\end{equation*}
$$

$$
\begin{equation*}
Y^{(m)}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\sin \beta_{0} h-T^{\prime(m)}\left(1-\cos \beta_{0} h\right)\right] \tag{3.89}
\end{equation*}
$$

where

$$
\begin{align*}
T^{\prime(m)} & =-\frac{T^{(m)}+\sin \beta_{0} h}{\cos \beta_{0} h} \\
& =\frac{\left[\Psi_{V}^{(m)}(h)-\Psi_{U}^{(m)}(h) \sin \beta_{0} h\right] \sec \beta_{0} h+\Psi_{d U}^{(m)} \sin \beta_{0} h-j \Psi_{d I}^{(m)}-\Psi_{d \Sigma R}^{(m)}}{\Psi_{U}^{(m)}(h)-\Psi_{d U}^{(m)} \cos \beta_{0} h} \\
& =\frac{\left[\Psi_{d U}^{(m)}+E_{a}(h, h)\right] \sin \beta_{0} h-j \Psi_{d I}^{(m)}-S_{a}(h, h)-\Psi_{d \Sigma R}^{(m)}}{C_{a}(h, h)-\left[\Psi_{d U}+E_{a}(h, h)\right] \cos \beta_{0} h} . \tag{3.90}
\end{align*}
$$

Note that when $\beta_{0} h=\pi / 2, \Psi_{U}^{(m)}\left(\frac{\lambda}{4}\right)=\Psi_{V}^{(m)}\left(\frac{\lambda}{4}\right)$.
As an example, the phase-sequence currents have been evaluated specifically for two antennas for which $\Omega=2 \ln (2 h / a)=10$, $\beta_{0} h=\pi$ and $\beta_{0} b=1 \cdot 5$. For these

$$
\left.\begin{array}{c}
\Psi_{d R}=5.834, \quad \Psi_{d \Sigma R}^{(0)}=-0.245, \quad \Psi_{d \Sigma R}^{(1)}=0.245 \\
\Psi_{d I}^{(0)}=-0.633-0.524=-1.157 ;  \tag{3.91b}\\
\Psi_{d I}^{(1)}=-0.633+0.524=-0.109
\end{array}\right\}
$$

$$
\begin{equation*}
\Psi_{d U}^{(0)}=7.848-j 3.939, \quad \Psi_{d U}^{(1)}=7.352-j 0.661 . \tag{3.91c}
\end{equation*}
$$

The amplitude functions are

$$
T^{(0)}\left(\frac{\lambda}{2}\right)=-0.216+j 0.274, \quad T^{(1)}\left(\frac{\lambda}{2}\right)=-0.177+j 0.066
$$

With these values the two-term zero-phase-sequence and first-phase-sequence currents (in amperes when $V_{0}$ is in volts) in the two antennas are

$$
\begin{align*}
I_{2 z}^{(0)}(z)= & I_{1 z}^{(0)}(z)=V^{(0)}\left\{0 \cdot 783\left(\cos \beta_{0} z+1\right)\right. \\
& \left.-j\left[2 \cdot 805 \sin \beta_{0}|z|-0 \cdot 617\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3}  \tag{3.92a}\\
-I_{2 z}^{(1)}(z)= & I_{1 z}^{(1)}(z)=V^{(1)}\left\{0 \cdot 189\left(\cos \beta_{0} z+1\right)\right. \\
& \left.-j\left[2 \cdot 805 \sin \beta_{0}|z|-0 \cdot 506\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3} . \tag{3.92b}
\end{align*}
$$

These currents are shown graphically in Fig. 3.3 in the form $I_{z}=I_{z}^{\prime \prime}+j I_{z}^{\prime}$, where $I_{z}^{\prime \prime}$ is in phase, $I_{z}^{\prime}$ in phase quadrature with $V_{0}$. The corresponding driving-point admittances and impedances are

$$
\left.\begin{array}{l}
Y^{(0)}=(1 \cdot 566+j 1 \cdot 234) \text { millimhos, } \\
Y^{(1)}=(0.378+j 1.012) \text { millimhos },
\end{array}\right\}
$$

$$
Z^{(0)}=394-j 310 \text { ohms }, \quad Z^{(1)}=324-j 867 \text { ohms. (3.93b) }
$$



Fig. 3.3. Zero- and first-phase-sequence currents on two-element array. $\Omega=10, \beta_{0} b=1.5$.

The two-term approximation of the general formulas (3.51a, b) are

$$
\begin{align*}
& I_{1 z}(z)=V_{10} v(z)+V_{20} w(z)  \tag{3.94a}\\
& I_{2 z}(z)=V_{10} w(z)+V_{20} v(z) \tag{3.94b}
\end{align*}
$$

where now

$$
\begin{align*}
v(z)= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)\right. \\
& \left.\quad+\frac{1}{2}\left(T^{(0)}+T^{(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]  \tag{3.95a}\\
w(z)= & \frac{j \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left(T^{(0)}-T^{(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right) \tag{3.95b}
\end{align*}
$$

When $\beta_{0} h$ is near $\pi / 2$,
$v(z)=\frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}}\left[\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)+\frac{1}{2}\left(T^{\prime(0)}+T^{\prime(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]$
$w(z)=\frac{-j \pi}{\zeta_{0} \Psi_{d R}}\left(T^{\prime(0)}-T^{\prime(1)}\right)\left(\cos \beta_{0} z-\cos \beta_{0} h\right)$.
The self- and mutual admittances (3.55) and (3.56) become

$$
\begin{align*}
& Y_{s 1}=Y_{s 2}=\frac{j \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[2 \sin \beta_{0} h+\left(T^{(0)}+T^{(1)}\right)\left(1-\cos \beta_{0} h\right)\right]  \tag{3.96}\\
& Y_{21}=Y_{12}=\frac{j \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left(T^{(0)}-T^{(1)}\right)\left(1-\cos \beta_{0} h\right) \tag{3.97}
\end{align*}
$$

Similarly, when $\beta_{0} h$ is near $\pi / 2,(3.57)$ and (3.58) reduce to

$$
\begin{align*}
Y_{s 1} & =Y_{s 2}=\frac{j \pi}{\zeta_{0} \Psi_{d R}}\left[2 \sin \beta_{0} h-\left(T^{\prime(0)}+T^{\prime(1)}\right)\left(1-\cos \beta_{0} h\right)\right]  \tag{3.98}\\
Y_{21} & =Y_{12}=\frac{-j \pi}{\zeta_{0} \Psi_{d R}}\left(T^{(0)}-T^{\prime(1)}\right)\left(1-\cos \beta_{0} h\right) \tag{3.99}
\end{align*}
$$

The two-term self- and mutual admittances for the special case $a / \lambda=7.022 \times 10^{-3}, \beta_{0} h=\pi$ are shown in Fig. 3.4 as a function of $b / \lambda$. The self-susceptance is expressed in the corrected form $B_{11}+0.72$. Agreement with measured values is seen to be very good. Numerical values of $\Psi_{d R}, T^{(m)}, T^{(m)}, Y^{(m)}, Y_{s i}=Y_{11}$ and $Y_{12}$ are in Tables 2-4 of appendix I for three values of $\beta_{0} h$ and a range of $b / \lambda=d / \lambda$.


Fig. 3.4. Self- and mutual admittances of two-element array; $b$ is the distance between elements; $\beta_{0} h=\pi$.

For the special case $\Omega=10, \beta_{0} h=\pi, \beta_{0} b=1 \cdot 5$, the two-term self- and mutual impedances defined in (3.61) with the two-term expressions (3.96) and (3.97) are

$$
\begin{align*}
Z_{s 2}=Z_{s 1} & =\frac{1}{2}\left(Z^{(0)}+Z^{(1)}\right)=359-j 588 \text { ohms }  \tag{3.100a}\\
Z_{21} & =Z_{12}=\frac{1}{2}\left(Z^{(0)}-Z^{(1)}\right)=35+j 278 \mathrm{ohms} . \tag{3.100b}
\end{align*}
$$

If antenna 1 is driven and antenna 2 is an unloaded parasitic element, (3.67a) applies. The two-term formulas for the currents may be obtained directly from ( $3.94 \mathrm{a}, \mathrm{b}$ ) with $V_{20}=0$. Then, in the special case $\Omega=10, \beta_{0} h=\pi, \beta_{0} b=1 \cdot 5$,

$$
\begin{align*}
I_{1 z}(z)= & V_{10}\left\{0.486\left(\cos \beta_{0} z+1\right)\right. \\
& \left.-j\left[2.805 \sin \beta_{0}|z|-0.566\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3} \tag{3.101a}
\end{align*}
$$

$$
\begin{equation*}
I_{2 z}(z)=V_{10}(0.287+j 0.055)\left(\cos \beta_{0} z+1\right) \times 10^{-3} . \tag{3.101b}
\end{equation*}
$$

The corresponding driving-point admittance and impedance are $Y_{1 \text { in }}=(0.972+j 1.33)$ millimhos, $Z_{1 \text { in }}=436-j 508$ ohms. (3.102)
The currents in the driven and parasitic antennas are shown in Fig. 3.5a. They differ from each other greatly in both distribution and amplitude. Indeed, contributions to the far-field by the currents in the parasitic element are insignificant and the horizontal field pattern is almost circular. Note that this behaviour is entirely different from what it would be if the two elements were half-wave instead of full-wave dipoles. In the former, the current in the parasitic element is comparable and essentially similar in distribution to that in the driven element. The reason for this difference is that the half-wave elements are near resonance, the full-wave elements near anti-resonance. This condition can be changed by inserting a lumped susceptance $B_{2}$ (or an equivalent transmission-line) in series with the full-wave parasitic element at its centre and tuning this susceptance to make the entire circuit resonant. When this is done the distribution functions $v(z)$ and $w(z)$ given by (3.95a, b) give

$$
\begin{align*}
I_{1 z}(z)= & V_{10}\left\{0 \cdot 369\left(\cos \beta_{0} z+1\right)\right. \\
& \left.-j\left[2.805 \sin \beta_{0}|z|-0.494\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3}  \tag{3.103a}\\
I_{2 z}(z)= & V_{10}\left\{\left[0.064\left(\cos \beta_{0} z+1\right)-0.320 \sin \beta_{0}|z|\right]\right. \\
& \left.+j\left[1.712 \sin \beta_{0}|z|-0.343\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3} . \tag{3.103b}
\end{align*}
$$

These currents are shown in Fig. 3.5b. They are very nearly alike in both distribution and amplitude, so that the horizontal field pattern of the tuned full-wave parasitic array must correspond closely to that of the half-wave array with an unloaded parasitic element.

The two-term formulas for the currents in the couplet are given by (3.71a, b) with $v(z)$ and $w(z)$ as in (3.95a, b). For the special case $\Omega=10, \beta_{0} h=\pi, \beta_{0} b=1 \cdot 5,(3.70 \mathrm{a}, \mathrm{b})$ give
$V_{20}=V_{10}\left(\frac{Z_{12}+j Z_{s 1}}{Z_{s 1}+j Z_{12}}\right)=(-0.966+j 1.267) V_{10}=1.59 V_{10} e^{-j 146.3^{\circ}}$.

With this value, the explicit formulas for the current in an array are

$$
\begin{align*}
I_{1 z}= & V_{10}\left\{0 \cdot 129\left(\cos \beta_{0} z+1\right)\right. \\
& \left.-j\left[2 \cdot 805 \sin \beta_{0}|z|-0 \cdot 884\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3} \tag{3.105a}
\end{align*}
$$

Antenna 1
(driven)

$$
\frac{I_{1 z}^{\prime \prime}}{V_{10}}=0.486\left(\cos \beta_{0} z+1\right)
$$ (parasitic)

(a)

(b)

Fig. 3.5. Currents on full-wave antenna with (a) $a / \lambda=7.022 \times 10^{-3}$ untuned parasite, $\Omega=10 ;(b)$ tuned parasite, $\Omega=10$.

$$
\begin{align*}
I_{2 z}= & V_{20}\left\{0 \cdot 400\left(\cos \beta_{0} z+1\right)\right. \\
& \left.-j\left[2 \cdot 805 \sin \beta_{0}|z|-0.397\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3} \tag{3.105b}
\end{align*}
$$

In order to obtain expressions for the current that are comparable from the point of view of maintaining an electromagnetic field, it is necessary to use the same reference for amplitude and phase. If $I_{2 z}$ is referred to $V_{10}$ instead of $V_{20}$, the following formula is obtained in place of (3.105b):

$$
\begin{align*}
I_{2 z}= & V_{10}\left\{\left[3 \cdot 554 \sin \beta_{0}|z|-0.884\left(\cos \beta_{0} z+1\right)\right]\right. \\
& \left.+j\left[2 \cdot 710 \sin \beta_{0}|z|+0.129\left(\cos \beta_{0} z+1\right)\right]\right\} \times 10^{-3} \tag{3.105c}
\end{align*}
$$

The corresponding driving-point admittances and impedances are $Y_{10}=(0.258+j 1.768)$ millimhos, $\quad Y_{20}=(0.801+j 0.784)$ millimhos

$$
\begin{equation*}
Z_{10}=80 \cdot 8-j 554 \text { ohms }, \quad Z_{20}=638-j 624 \text { ohms } \tag{3.106a}
\end{equation*}
$$

The ratio of the power supplied to antenna 1 to that supplied to antenna 2 is $\left|V_{20}\right|^{2} G_{20} /\left|V_{10}\right|^{2} G_{10}=7 \cdot 9$. The currents represented by (3.105a) and (3.105b) are shown in the upper diagram in Fig. 3.6 in the form $I_{1 z} / V_{10}$ and $I_{2 z} / V_{20}$. The distribution of $I_{2 z} / V_{10}$ is shown in the bottom diagram in Fig. 3.6. It differs greatly from $I_{1 z} / V_{10}$ (shown in the upper graph) even though the input currents at $z=0$ satisfy the assigned relation, $I_{20}=j I_{10}$.

The radiation field of the full-wave couplet may be expressed as follows:

$$
\begin{equation*}
E_{\Theta}^{r}=E_{\Theta 1}^{r}+E_{\Theta 2}^{r}=K\left[A_{1} e^{j\left(\beta_{0} b / 2\right) \cos \Phi}+A_{2} e^{-j\left(\beta_{0} b / 2\right) \cos \Phi}\right] \tag{3.107}
\end{equation*}
$$

where

$$
\begin{align*}
& A_{1}=V_{10}\left[(0.129+j 0.884) G_{m}(\Theta, \pi)-j 2 \cdot 805 F_{m}(\Theta, \pi)\right]  \tag{3.108a}\\
& A_{2}=V_{10}\left[(-0.884+j 0.129) G_{m}(\Theta, \pi)+(3.554+j 2.710) F_{m}(\Theta, \pi)\right] \tag{3.108b}
\end{align*}
$$

and where $K$ is a constant. Note that in the equatorial plane, $\Theta=\pi / 2$, and $G_{m}(\pi / 2, \pi)=\pi, F_{m}(\pi / 2, \pi)=2$. The field pattern calculated from the magnitude of (3.107) with (3.108a, b) for the couplet of full-wave elements is shown in Fig. 3.7 together with the corresponding pattern for the ideal couplet with identical distributions of current in the two elements. (This latter is quite closely approximated by the pattern of a couplet of half-wave elements.) Both patterns are normalized to unity at $\Phi=0$. It is seen that the deep minimum at $\Phi=180^{\circ}$ in the ideal pattern (this would be a null if
$\frac{I_{z}}{V_{0}}=\frac{I_{z}^{\prime \prime}}{V_{0}}+j \frac{I_{z}^{\prime}}{V_{0}}$


Fig. 3.6. Currents in full-wave couplet ; $I_{20}=j I_{10} ; \beta_{0} b=2 \pi b / \lambda=1 \cdot 5 ; \Omega=10$.
$\beta_{0} b=\pi / 2$ had been used instead of $\beta_{0} b=1 \cdot 5$ ) is replaced by a minor maximum with an amplitude that is about one-half that of the principal maximum at $\Phi=0$. Thus, the characteristic property of the ideal couplet of providing a null in one direction does not exist in actual couplets when $\beta_{0} h=\pi$ or, in fact, for any other


Fig. 3.7. Horizontal pattern of full-wave couplet with $I_{20}=j I_{10} ; \Omega=2 \ln (2 h / a)=10$, $\beta_{0} h=\pi, \beta_{0} b=1.5$.
value of $\beta_{0} h$ that is not near $\pi / 2$ or that is not an odd multiple thereof. Significantly, this makes the cardioid pattern of the halfwave couplet a relatively narrow-band property!

## CHAPTER 4

## THE CIRCULAR ARRAY

The two-element array, which is investigated in the preceding chapter, may be regarded as the special case $N=2$ of an array of $N$ elements arranged either at the vertices of a regular polygon inscribed in a circle, or along a straight line to form a curtain. Owing to its greater geometrical symmetry, the circular array is advantageously treated next. Indeed, the basic assumptions which underlie the subsequent study of the curtain array (chapter 5) depend for their justification on the prior analysis of the circular array.

The real difficulty in analysing an array of $N$ arbitrarily located elements is that the solution of $N$ simultaneous integral equations for $N$ unknown distributions of current is involved. Although the same set of equations applies to the circular array, they may be replaced by an equivalent set of $N$ independent integral equations in the manner illustrated in chapter 3 for the two-element array. Since the $N$ elements are geometrically indistinguishable, it is only necessary to make them electrically identical as well. One way is to drive them all with generators that maintain voltages that are equal in amplitude and in phase. When this is done all $N$ currents must also be equal in amplitude and in phase at corresponding points. But this is only one of $N$ possibilities. If the $N$ voltages are all equal in magnitude but made to increase equally and progressively in phase from element 1 to element $N$, a condition may be achieved such that each element is in exactly the same environment as every other element. There are $N$ such possibilities since the phase sequence closes around the circle when the phase shift from element to element is an integral multiple of $2 \pi / N$. Any increment in phase given by $2 \pi m / N$ with $m=0,1,2, \ldots N-1$ may be used. Specifically, when $N=2$, the two possibilities are 0 and $\pi$. This means that the two driving voltages and the two currents may be equal in magnitude and in phase ( 0,0 ) or equal in magnitude and $180^{\circ}$ out of phase $\left(0,180^{\circ}\right)$. Similarly, when $N=3$, there are three possibilities, $0,2 \pi / 3$ and $4 \pi / 3$. The voltages and currents around
the circle may now be equal in magnitude with phases, $(0,0,0)$, $\left(0,120^{\circ}, 240^{\circ}\right)$ or ( $0,240^{\circ}, 480^{\circ}$ ).

The analysis of the circular array involves the solution of $N$ simultaneous equations similar in form to (2.15). The case $N=2$ is solved in chapter 3 by rearranging the two simultaneous equations for the currents $I_{1 z}(z)$ and $I_{2 z}(z)$ into two independent equations. These were derived by adding and subtracting the two original equations. When the elements were driven by voltages which were equal in magnitude and either in phase or $180^{\circ}$ out of phase, the resulting currents were independent and named, respectively, the zero and first-phase-sequence currents. The solution for the phasesequence currents was then carried out and, after a simple algebraic transformation, the actual currents in the elements were derived. The solution for arbitrary driving conditions could also be obtained from the two phase-sequence solutions. A generalization of this procedure is followed in the analysis of the circular array.

The arrays considered here consist of $N$ identical, parallel, non-staggered, centre-driven elements that are equally spaced about the circumference of a circle. This means that the elements are at the vertices of an $N$-sided regular polygon. Arrays of this type are frequently called single-ring arrays in the literature. Their analysis formally parallels step-by-step the analysis of the two-element array in chapter 3. However, contributions to the vector potential on the surface of each antenna by the currents in all of the elements must be included and this leads to a set of $N$ coupled integral equations for the $N$ currents in the elements. The complete geometrical symmetry of the array permits the use of the method of symmetrical components to reduce the coupled set of integral equations to a single integral equation for each of $N$ possible phasesequence currents. All other quantities that are required to design and describe the array can be calculated from the solution of essentially one equation with $N$ somewhat different kernels.

The coordinate system and parameters that are used to specify an array are shown in Fig. 4.1 for five elements. The diameter of each element is $2 a$, its length is $2 h$, the distance between the $k^{\text {th }}$ and the $i^{\text {th }}$ elements is $b_{k i}$, the distance between adjacent elementsthe length of the side of a regular polygon with the elements at its vertices-is $d$, and the radius of the circle is $\rho$.

As indicated in sections 3.9 and 2.10, the two-term approximation is generally adequate when $h \leqslant 5 \lambda / 8$ and $b \geqslant \lambda / 2 \pi$. Since it is
much simpler and has been used to compute the theoretical results discussed in this chapter, the currents, admittances, and fields in the following sections are determined in the two-term form. Later when matrix notation is introduced, both the two- and the threeterm forms of the theory are presented in compact form. This serves both as a summary of the theory of circular arrays and as an introduction to the analysis of more general arrays in chapters 5 and 6.


Fig. 4.1. Coordinate system for circular arrays.

### 4.1 Integral equations for the sequence currents

The vector potential difference at the surface of each element in a circular array of $N$ elements is easily obtained as a generalization of (3.1). Since all elements are thin and parallel to the $z$-axis, only $z$-components of the current and the associated vector potential at the surface of each element are significant. Thus, the vector potential
difference on the surface of element 1 is

$$
\begin{align*}
& 4 \pi \mu_{0}^{-1}\left[A_{1 z}(z)-A_{1 z}(h)\right] \\
& =\int_{-h}^{h}\left[I_{1 z}\left(z^{\prime}\right) K_{11 d}\left(z, z^{\prime}\right)+I_{2 z}\left(z^{\prime}\right) K_{12 d}\left(z, z^{\prime}\right)+\ldots+I_{N z}\left(z^{\prime}\right) K_{1 N d}\left(z, z^{\prime}\right)\right] d z^{\prime} \\
& =\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V_{10} \sin \beta_{0}(h-|z|)+U_{1}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \tag{4.1a}
\end{align*}
$$

where

$$
\begin{align*}
U_{1}= & \frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h}\left[I_{1 z}\left(z^{\prime}\right) K_{11}\left(h, z^{\prime}\right)+I_{2 z}\left(z^{\prime}\right) K_{12}\left(h, z^{\prime}\right)+\ldots\right. \\
& \left.+I_{N z}\left(z^{\prime}\right) K_{1 N}\left(h, z^{\prime}\right)\right] d z^{\prime} \tag{4.1b}
\end{align*}
$$

Similarly, the vector potential difference on the surface of element 2 is

$$
\begin{align*}
& 4 \pi \mu_{0}^{-1}\left[A_{2 z}(z)-A_{2 z}(h)\right] \\
= & \int_{-h}^{h}\left[I_{1 z}\left(z^{\prime}\right) K_{21 d}\left(z, z^{\prime}\right)+I_{2 z}\left(z^{\prime}\right) K_{22 d}\left(z, z^{\prime}\right)+\ldots+I_{N z}\left(z^{\prime}\right) K_{2 N d}\left(z, z^{\prime}\right)\right] d z^{\prime} \\
= & \frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V_{20} \sin \beta_{0}(h-|z|)+U_{2}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \tag{4.1c}
\end{align*}
$$

where

$$
\begin{align*}
U_{2}= & \frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h}\left[I_{1 z}\left(z^{\prime}\right) K_{21}\left(h, z^{\prime}\right)+I_{2 z}\left(z^{\prime}\right) K_{22}\left(h, z^{\prime}\right)+\ldots\right. \\
& \left.+I_{N z}\left(z^{\prime}\right) K_{2 N}\left(h, z^{\prime}\right)\right] d z^{\prime} \tag{4.1~d}
\end{align*}
$$

The vector potential difference on the $k^{\text {th }}$ element is

$$
\begin{gather*}
4 \pi \mu_{0}^{-1}\left[A_{k z}(z)-A_{k z}(h)\right]=\int_{-h}^{h} \sum_{i=1}^{N} I_{i z}\left(z^{\prime}\right) K_{k i d}\left(z, z^{\prime}\right) d z^{\prime} \\
=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V_{k 0} \sin \beta_{0}(h-|z|)+U_{k}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]  \tag{4.1e}\\
k=1,2, \ldots, N
\end{gather*}
$$

where

$$
\begin{equation*}
U_{k}=\frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h} \sum_{i=1}^{N} I_{i z}\left(z^{\prime}\right) K_{k i}\left(h, z^{\prime}\right) d z^{\prime} \tag{4.1f}
\end{equation*}
$$

In these expressions the kernels are

$$
\begin{equation*}
K_{k i d}\left(z, z^{\prime}\right)=K_{k i}\left(z, z^{\prime}\right)-K_{k i}\left(h, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{k i}}}{R_{k i}}-\frac{e^{-j \beta_{0} R_{k i h}}}{R_{k i h}} \tag{4.1g}
\end{equation*}
$$

with

$$
\begin{gather*}
R_{k i}=\sqrt{\left(z_{k}-z_{i}^{\prime}\right)^{2}+b_{k i}^{2}}  \tag{4.1h}\\
R_{k i h}=\sqrt{\left(h-z_{i}^{\prime}\right)^{2}+b_{k i}^{2}}, \quad b_{k k}=a . \tag{4.1i}
\end{gather*}
$$

$V_{k 0}$ is the applied driving voltage at the centre of element $k$ (or the voltage of an equivalent generator if the element is parasitic with an impedance connected across its terminals), and $U_{k}$ is the effective driving function characteristic of that part of all the currents that maintains a vector potential of constant amplitude equal to that at $z=h$ along the entire length of the antenna. To reduce the set of $N$ simultaneous equations in (4.1e) to $N$ independent equations, the symmetry conditions characteristic of a circular array must be imposed and the phase-sequence voltages and currents introduced.

Assume that all of the driving voltages are equal in magnitude and have a uniformly progressive phase such that the total phase change around the circle is an integral multiple of $2 \pi$. Each multiple of $2 \pi$ is one of the $N$ phase sequences designated by a superscript $(m)$; these range from zero to $N-1$. In the zero phase sequence, all driving voltages are the same; in the first phase sequence, the driving voltages of adjacent elements differ by $\exp (j 2 \pi / N)$; in the $m^{\text {th }}$ phase sequence, the driving voltages of adjacent elements differ by $\exp (j 2 \pi m / N)$, and the voltages of the $k^{\text {th }}$ and the $i^{\text {th }}$ elements are related by

$$
\begin{equation*}
V_{i}=V_{k} e^{j 2 \pi(i-k) m / N} \tag{4.2a}
\end{equation*}
$$

Because of the symmetry of a circular array, the sequence currents in the elements must be related in the same manner as are the sequence driving voltages. That is,

$$
\begin{equation*}
I_{i}\left(z^{\prime}\right)=I_{k}\left(z^{\prime}\right) e^{j 2 \pi(i-k) m / N} \tag{4.2b}
\end{equation*}
$$

Note that with these driving voltages both the geometric and the electrical environments of each element in the array are identical. Therefore, when (4.2a) and (4.2b) are substituted into the set of coupled integral equations, $I_{i}\left(z^{\prime}\right)$ can be removed from the summation, the remaining kernel is the same regardless of the element
to which it is referred, and each equation in the set reduces to

$$
\begin{align*}
& \int_{-h}^{h} I^{(m)}\left(z^{\prime}\right) K_{d}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \\
& \quad=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V_{0}^{(m)} \sin \beta_{0}(h-|z|)+U^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \tag{4.3}
\end{align*}
$$

where $m=0,1, \ldots N-1$ and

$$
\begin{align*}
U^{(m)} & =\frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h} I^{(m)}\left(z^{\prime}\right) K^{(m)}\left(h, z^{\prime}\right) d z^{\prime}  \tag{4.3a}\\
K^{(m)}\left(h, z^{\prime}\right) & =\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N} \frac{e^{-j \beta_{0} R_{1 i h}}}{R_{1 i h}}  \tag{4.3~b}\\
K_{d}^{(m)}\left(z, z^{\prime}\right) & =\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N}\left[\frac{e^{-j \beta_{0} R_{1 i}}}{R_{1 i}}-\frac{e^{-j \beta_{0} R_{1 i h}}}{R_{1 i h}}\right] . \tag{4.3c}
\end{align*}
$$

For later use, it is convenient to separate this difference kernel into two parts that depend, respectively, on the real and imaginary parts of the exponential functions. That is

$$
\begin{align*}
K_{d}^{(m)}\left(z, z^{\prime}\right) & =K_{d R}^{(m)}\left(z, z^{\prime}\right)+j K_{d I}^{(m)}\left(z, z^{\prime}\right)  \tag{4.3d}\\
\text { where } K_{d R}^{(m)}\left(z, z^{\prime}\right) & =\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N} \operatorname{Re}\left[\frac{e^{-j \beta_{0} R_{1 i}}}{R_{1 i}}-\frac{e^{-j \beta_{0} R_{1 i h}}}{R_{1 i h}}\right]  \tag{4.3e}\\
K_{d I}^{(m)}\left(z, z^{\prime}\right) & =\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N} \operatorname{Im}\left[\frac{e^{-j \beta_{0} R_{1 i}}}{R_{1 i}}-\frac{e^{-j \beta_{0} R_{1 i h}}}{R_{1 i h}}\right] . \tag{4.3f}
\end{align*}
$$

The method of solution for (4.3) parallels that of (3.6) and (3.10); the discussion of section 3.2 and the steps of section 3.3 are applicable if note is taken of section 3.9 which relates the two-term to the three-term theory. In fact, the solution is formally given by (3.85) and (3.86) with $m=0,1,2, \ldots, N-1$. This is discussed in somewhat greater detail in a later section (section 4.6) rather than at this point in order to avoid complications in these initial stages of the analysis. Thus, the $m^{\text {th }}$ phase-sequence current in the two-term form is given by

$$
\begin{gather*}
I^{(m)}(z)=\frac{j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right], \\
\beta_{0} h \neq \frac{\pi}{2} \\
I^{(m)}(z)=\frac{j 2 \pi V_{0}^{(m)}}{\zeta_{0} \Psi_{d R}}\left[1-\sin \beta_{0}|z|-T^{\prime(m)} \cos \beta_{0} z\right], \quad \beta_{0} h=\frac{\pi}{2} . \tag{4.4b}
\end{gather*}
$$

The $\Psi$ and $T$ functions which occur in (4.4a, b) are defined as follows when $\beta_{0} d \geqslant 1$ :

$$
\begin{align*}
& T^{(m)}=\frac{\Psi_{V}^{(m)}(h)-\left[\Psi_{d \Sigma}^{(m)}+j \Psi_{d I}^{(m)}\right] \cos \beta_{0} h}{\Psi_{d U}^{(m)} \cos \beta_{0} h-\Psi_{U}^{(m)}(h)}  \tag{4.5a}\\
& T^{\prime(m)}=\frac{\Psi_{d K}+E_{\Sigma}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)-S_{\Sigma}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)}{C_{\Sigma}\left(\frac{\lambda}{4}, \frac{\lambda}{4}\right)}  \tag{4.5b}\\
& \Psi_{d R}=\operatorname{Re}\left[\sin \beta_{0} h C_{d \Sigma 1}\left(h, h-\frac{\lambda}{4}\right)\right. \\
& \left.-\cos \beta_{0} h S_{d \Sigma 1}\left(h, h-\frac{\lambda}{4}\right)\right], \quad h \geqslant \frac{\lambda}{4}  \tag{4.6a}\\
& =\operatorname{Re}\left[C_{d \Sigma 1}(h, 0)-\cot \beta_{0} h S_{d \Sigma 1}(h, 0)\right], \quad h<\frac{\lambda}{4}  \tag{4.6b}\\
& \Psi_{V}^{(m)}(h)=\sin \beta_{0} h C_{\Sigma}^{(m)}(h, h)-\cos \beta_{0} h S_{\Sigma}^{(m)}(h, h)  \tag{4.7}\\
& \Psi_{U}^{(m)}(h)=C_{\Sigma}^{(m)}(h, h)-\cos \beta_{0} h E_{\Sigma}^{(m)}(h, h)  \tag{4.8}\\
& \Psi_{d \Sigma}^{(m)}=\left(1-\cos \beta_{0} h\right)^{-1}\left[\sin \beta_{0} h C_{d \Sigma 2}^{(m)}(h, 0)-\cos \beta_{0} h S_{d \Sigma 2}^{(m)}(h, 0)\right]  \tag{4.9}\\
& \Psi_{d I}^{(m)}=\operatorname{Im}\left\{\left(1-\cos \beta_{0} h\right)^{-1}\left[\sin \beta_{0} h C_{d \Sigma 1}^{(m)}(h, 0)-\cos \beta_{0} h S_{d \Sigma 1}^{(m)}(h, 0)\right]\right\}  \tag{4.10}\\
& \Psi_{d U}^{(m)}=\left(1-\cos \beta_{0} h\right)^{-1}\left[C_{d \Sigma}^{(m)}(h, 0)-\cos \beta_{0} h E_{d \Sigma}^{(m)}(h, 0)\right]  \tag{4.11}\\
& C_{d \Sigma}^{(m)}(h, z)=C_{\Sigma}^{(m)}(h, z)-C_{\Sigma}^{(m)}(h, h), \quad S_{d \Sigma}^{(m)}(h, z)=S_{\Sigma}^{(m)}(h, z)-S_{\Sigma}^{(m)}(h, h) \\
& E_{d \Sigma}^{(m)}(h, z)=E_{\Sigma}^{(m)}(h, z)-E_{\Sigma}^{(m)}(h, h)  \tag{4.12}\\
& C_{\Sigma}^{(m)}(h, z)=\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N} C_{b i}, \quad C_{b i}=\int_{-h}^{h} \cos \beta_{0} z^{\prime} \frac{e^{-j \beta_{0} R_{b i}}}{R_{b i}} d z^{\prime}  \tag{4.13a}\\
& S_{\Sigma}^{(m)}(h, z)=\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N} S_{b i}, \quad S_{b i}=\int_{-h}^{h} \sin \beta_{0}\left|z^{\prime}\right| \frac{e^{-j \beta_{0} R_{b i}}}{R_{b i}} d z^{\prime} \\
& E_{\Sigma}^{(m)}(h, z)=\sum_{i=1}^{N} e^{j 2 \pi(i-1) m / N} E_{b i}, \quad E_{b i}=\int_{-h}^{h} \frac{e^{-j \beta_{0} R_{b i}}}{R_{b i}} d z^{\prime}  \tag{4.13b}\\
& R_{b i}=\sqrt{\left(z-z^{\prime}\right)^{2}+b_{i}^{2}}, \quad b_{i}=a \text { for } i=1 . \tag{4.14}
\end{align*}
$$

The subscript $d \Sigma 1$ indicates that only element number $1(i=1)$ is to be included and effects of all other elements are ignored; the
subscript $d \Sigma 2$ indicates that only the effects of elements other than element number 1 are to be included ( $i=1, \ldots, N$ ).

In order to evaluate ( $4.4 \mathrm{a}, \mathrm{b}$ ) it is convenient to lump the various coefficients into new parameters defined as follows:

$$
\begin{gather*}
s^{(m)}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}, \quad c^{(m)}=s^{(m)} T^{(m)}  \tag{4.15a}\\
s^{(m)}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R}}, \quad c^{\prime(m)}=s^{\prime(m)} T^{\prime(m)} \tag{4.15b}
\end{gather*}
$$

so that (4.4a, b) become

$$
\begin{equation*}
\frac{I^{(m)}(z)}{V^{(m)}}=s^{(m)} \sin \beta_{0}(h-|z|)+c^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right), \quad \beta_{0} h \neq \frac{\pi}{2} \tag{4.16a}
\end{equation*}
$$

$$
\begin{equation*}
=s^{\prime(m)}\left(1-\sin \beta_{0}|z|\right)-c^{\prime(m)} \cos \beta_{0} z, \quad \beta_{0} h=\frac{\pi}{2} . \tag{4.16b}
\end{equation*}
$$

The sequence admittances are given by the normalized sequence currents in amperes per volt evaluated at $z=0$. Thus,

$$
\begin{align*}
Y^{(m)}=\frac{I^{(m)}(0)}{V^{(m)}} & =s^{(m)} \sin \beta_{0} h+c^{(m)}\left(1-\cos \beta_{0} h\right), \quad \beta_{0} h \neq \frac{\pi}{2}  \tag{4.17a}\\
& =s^{\prime(m)}-c^{\prime(m)}, \quad \beta_{0} h=\frac{\pi}{2} . \tag{4.17b}
\end{align*}
$$

For a circular array of $N$ elements there are $N$ sequences but only $(N+1) / 2$ are different if $N$ is odd or $(N / 2)+1$ if $N$ is even. This is the same as the number of different self- and mutual admittances.

The sequence currents form a set of functions that are characteristic of the geometrical and electrical properties of the array. Thus, $\Psi_{d R}$ and the $T^{(m)}$ or $T^{\prime(m)}$ function depend upon the number of elements in the array, their spacing, and the length and thickness of the elements. Once these parameters have been specified, the set of sequence currents can be calculated. Distributions of current in the elements, their driving-point admittances, and the far-zone fields of the arrays with arbitrary driving conditions can be determined from the set of sequence currents with the relations given in section 4.2. Short tables of $\Psi_{d R}$ and $T^{(m)}$ or $T^{(m)}$ are given in appendix I; additional values are available [1]. It may be noted parenthetically that in the notation of [1], the terms 'quasi-zerothorder' and 'zeroth-order admittances' refer identically to what is called the 'two-term approximation'. in this book.

### 4.2 Sequence functions and array properties

Imagine the array to be excited simultaneously with currents in all of the $N$ possible phase sequences. Then the driving voltage and current for the $k^{\text {th }}$ element are

$$
\begin{align*}
V_{k} & =\sum_{m=0}^{N-1} V^{(m)} e^{j 2 \pi(k-1) m / N}  \tag{4.18a}\\
I_{k}(z) & =\sum_{m=0}^{N-1} I^{(m)}(z) e^{j 2 \pi(k-1) m / N} \tag{4.18b}
\end{align*}
$$

where $V^{(m)}$ is the $m^{\text {th }}$ phase-sequence voltage and $I^{(m)}(z)$ is the corresponding phase-sequence current. Similarly, from (4.18b) and (4.17), the self- and mutual admittances are

$$
\begin{equation*}
Y_{1 k}=\frac{1}{N} \sum_{m=0}^{N-1} Y^{(m)} e^{j 2 \pi(k-1) m / N} \tag{4.19}
\end{equation*}
$$

If the elements of the array are driven by arbitrary voltages $V_{i}$ which produce corresponding currents $I_{i}(z)$ along the elements, the sequence voltages and currents are readily obtained from the relations

$$
\begin{align*}
V^{(m)} & =\frac{1}{N} \sum_{i=1}^{N} V_{i} e^{-j 2 \pi(i-1) m / N}  \tag{4.20a}\\
I^{(m)}(z) & =\frac{1}{N} \sum_{i=1}^{N} I_{i}(z) e^{-j 2 \pi(i-1) m / N} \tag{4.20b}
\end{align*}
$$

With (4.18b) and (4.16) the normalized current distribution along the $k^{\text {th }}$ element can conveniently be expressed as follows:

$$
\begin{align*}
\frac{I_{k}(z)}{V_{1}} & =s_{k} \sin \beta_{0}(h-|z|)+c_{k}\left(\cos \beta_{0} z-\cos \beta_{0} h\right), \quad \beta_{0} h \neq \frac{\pi}{2}  \tag{4.21a}\\
& =s_{k}^{\prime}\left[1-\sin \beta_{0}|z|\right]-c_{k}^{\prime} \cos \beta_{0} z, \quad \beta_{0} h=\frac{\pi}{2} \tag{4.21b}
\end{align*}
$$

where the complex amplitude functions $s_{k}$ and $c_{k}$ are

$$
\begin{align*}
& s_{k}=\sum_{m=0}^{N-1} \frac{V^{(m)}}{V_{1}} s^{(m)} e^{j 2 \pi(k-1) m / N}  \tag{4.22a}\\
& c_{k}=\sum_{m=0}^{N-1} \frac{V^{(m)}}{V_{1}} c^{(m)} e^{j 2 \pi(k-1) m / N} . \tag{4.22b}
\end{align*}
$$

The corresponding expressions for $s_{k}^{\prime}$ and $c_{k}^{\prime}$ are similar. The radiation-zone electric field for each element is given by (2.43); the total field is a superposition of the fields maintained by each
element. When the currents in the form $(4.4 a, b)$ are substituted in (2.43), the resulting expressions for the field are

$$
\begin{align*}
\frac{E_{\Theta}^{r}}{K K_{1} V}= & F\left(\Theta, \beta_{0} h\right) \sum_{i=1}^{N} s_{i} e^{j \rho \cos \left(\phi_{i}-\Phi\right)} \\
& +G\left(\Theta, \beta_{0} h\right) \sum_{i=1}^{N} c_{i} e^{j \rho \cos \left(\phi_{i}-\Phi\right)}, \quad \beta_{0} h \neq \frac{\pi}{2}  \tag{4.23a}\\
= & H\left(\Theta, \frac{\pi}{2}\right) \sum_{i=1}^{N} s_{i}^{\prime} e^{j \rho \cos \left(\phi_{i}-\Phi\right)} \\
& +F\left(\Theta, \frac{\pi}{2}\right) \sum_{i=1}^{N} c_{i}^{\prime} e^{j \rho \cos \left(\phi_{i}-\Phi\right)}, \quad \beta_{0} h=\frac{\pi}{2} \tag{4.23b}
\end{align*}
$$

with

$$
K_{1}=\frac{e^{-j \beta_{0} R}}{R}, \quad K=\frac{j \zeta_{0}}{2 \pi}, \quad \rho=\frac{\pi d / \lambda}{\sin (\pi / N)}, \quad \phi_{i}=(i-1) 2 \pi / N .
$$

$F\left(\Theta, \beta_{0} h\right), G\left(\Theta, \beta_{0} h\right)$ and $H(\Theta, \pi / 2)$ are given by (2.46), (2.47), and (2.52a), respectively. These are the so-called element factors and there is one for each type of current distribution. The sums in $(4.23 \mathrm{a}, \mathrm{b})$ are the array factors. The complex amplitude coefficients are not simply related to one another and the array factors generally cannot be summed in a closed form to yield something equivalent to the familiar $\sin N x / \sin x$ patterns. In (4.23) the driving voltage $V_{1}$ appears since the other driving voltages have been referred to the voltage of element 1 . Any other element could have been used for this normalization.

The steps required to make use of this theory in the analysis of a particular array can now be summarized. If the driving voltages are specified, sequence voltages are computed from (4.20a), $s_{k}$ and $c_{k}$ from (4.22a, b), the current distributions from (4.21a), and farzone fields from (4.23a, b). Driving-point admittances are found either from the current evaluated at $z=0$, viz.,

$$
\begin{equation*}
Y_{k i n}=\frac{I_{k}(0)}{V_{k}}=\frac{I_{k}(0)}{V_{1}} \frac{V_{1}}{V_{k}} \tag{4.24a}
\end{equation*}
$$

or from the coupled circuit equations and the self- and mutual admittances

$$
\begin{equation*}
Y_{k i n}=\sum_{i=1}^{N} \frac{V_{i}}{V_{k}} Y_{k i} . \tag{4.24b}
\end{equation*}
$$

If the driving-point currents are specified, sequence currents can
be found from (4.20b), (4.16a) or (4.16b) solved for $V^{(m)}$, and the remaining steps carried out as when the driving voltages are specified. Numerical results for a particular array can be obtained from the tables of appendix I or [1], or from the program outlined in appendix II.

### 4.3 Self- and mutual admittances

For a circular array with uniformly-spaced elements, self- and mutual admittances are defined in terms of the sequence admittances by (4.19). The more general definition (discussed in chapter 8) of self- and mutual admittances as the coefficients of the drivingpoint voltages in the coupled circuit equations also applies. For the $p^{\text {th }}$ element,

$$
\begin{equation*}
I_{p}(0)=\sum_{i=1}^{N} V_{i} Y_{p i} \tag{4.25}
\end{equation*}
$$

from which it follows that the self-admittance $Y_{p p}$ of the $p^{\text {th }}$ element is the driving-point admittance of that element when all other elements are present and short-circuited at their driving points. The mutual admittance $Y_{p k}(p \neq k)$ between element $p$ and element $k$ is the driving-point current of element $p$ per unit driving voltage of element $k$ with all other elements present and short-circuited at their driving points. Thus, the mutual admittances characterize the degree in which power that is fed to one element of the array is transferred to the remaining elements.

Among the properties of circular arrays that are revealed by a study of their self- and mutual admittances are resonant spacings at which all of the elements interact vigorously and, in larger arrays, spacings at which at least some of the mutual admittances are very small compared with the self-admittance. In arrays containing only a few elements, the resonant spacings are most important for elements with lengths near $h=\lambda / 4$; in larger arrays they are most important for elements with somewhat greater lengths. When the elements in an array are at the resonant spacings, their currents are essentially all in phase and their properties are very sensitive to small changes in frequency. Although calculations of the driving-point admittances generally must include all of the mutual admittances when the array consists of only a few elements, there are ranges of spacings in larger arrays over which at least some of the mutual admittances are much smaller than the selfadmittance. In larger arrays there is also a range of spacings over
which many of the mutual admittances are nearly the same in magnitude and phase. These properties are illustrated in Figs. 4.2-4.7, which show graphically examples of self- and mutual admittances in millimhos for a range of values of $d / \lambda$, the distance between adjacent elements.


Fig. 4.2a. Measured and theoretical self- and mutual conductances for circular array: $N=4, h / \lambda=1 / 4, a / \lambda=0.007022$.

Except for the self-susceptance shown in Fig. $4.3 b$, the theoretical results are all evaluated from the two-term theory and were computed from (4.19), (4.17b), and the functions in (4.5)-(4.14) with the program outlined in appendix II. The theoretical self-susceptance


Fig. 4.2b. Like Fig. $4.2 a$ but for susceptances.
in Fig. $4.3 b$ is shown in the corrected form $B_{11}+j 1 \cdot 16$ with $B_{11}$ calculated from the two-term theory. The correcting susceptance 1.16 includes the term 0.72 needed to correct the two-term susceptance and an additional susceptance that takes account of the particular end-effects of the coaxial measuring line. The measured results in Figs. 4.2, 4.3 and 4.4 were obtained from load admittances apparently terminating the coaxial line. They were measured by the distribution-curve method discussed in chapter 8 . The experimental
apparatus consisted of combined slotted measuring lines and monopoles driven over a ground-screen as shown in Fig. 8.27. The actual measured results have been divided by two and an approximate terminal-zone correction of $Y_{T}=j 0.286$ millimhos as obtained from Fig. $8.3 b$ has been combined with $B_{11}$ so that the final results apply to an ideal centre-driven dipole with all contributions to the admittance by an associated driving mechanism eliminated.


Fig. 4.3a. Measured and theoretical self- and mutual conductances for circular array $N=4, h / \lambda=3 / 8, a / \lambda=0.007022$.


Fig. 4.3b. Like Fig. 4.3a but for the susceptances.
An array of four elements of length $h=\lambda / 4$ (Fig. 4.2) has a resonant spacing near $d / \lambda=0 \cdot 54$. At this spacing all conductances have sharp positive maxima while the susceptances are all essentially zero. If the length of the elements is increased to $h=3 \lambda / 8$, a similar resonance occurs in the range between $d / \lambda=0.37$ and $0 \cdot 40$, but the maxima are not as sharp. With eight elements (Fig. 4.5) there are several resonances but only the first two, which occur near $d / \lambda=0.35$ and 0.50 , are sharply defined. Also, from Fig. 4.5 it is seen that the conductances all have the same sign at the first resonance but not at the second. For twenty elements with length $h=\lambda / 4$ it is seen from Fig. 4.6 that a number of resonances occur, but that they no longer have large amplitudes. On the other hand, when the length of the elements is near $h=3 \lambda / 8$, the resonances are sharply defined and a small change in spacing (or frequency) produces large changes in the admittances as shown in Fig. 4.7.

Note also that, whereas the four- and eight-element arrays have only one spacing each at which some of the mutual conductances or susceptances are small compared to the self-conductance or susceptance, there is a considerable range of spacings for a twenty-
element array over which only $Y_{12}$ is important and all other mutual admittances are small compared to $Y_{11}$. For close spacings, many of the mutual admittances have essentially the same value in Figs. 4.5 and 4.6. Also, at small spacings the self-susceptance and the mutual susceptance between adjacent elements become very large compared to either the remaining susceptances or the


Fig. 4.4. Measured and theoretical self- and mutual admittances, 5-element circular array, $h=\lambda / 2$.
conductances. This indicates that it is these quantities which cause difficulties in matching arrays of closely-spaced elements. These susceptances can be controlled at least partially by an adjustment of the lengths of the elements. Additional, more extensive graphs and tables of self- and mutual admittances are in the literature [2]. All of the results discussed here are for elements with the radius $a / \lambda=0.007$, but since the parameters of an array change quite slowly with the thickness of the elements, the qualitative behaviour should be the same for thicknesses that do not violate the requirement of 'thin', i.e. $\beta_{0} a \leqslant 0.10$. Note, however, that the selfimpedances of the individual elements change significantly with their radius-especially when $h$ is not near $\lambda / 4$.

Figs. 4.2 and 4.3 indicate that, except near the sharp resonances, the results from the two-term theory are in good agreement with the measured values for all conductances and mutual susceptances. Part of the differences between measured and computed conductances at the sharp resonant maxima for $h=\lambda / 4$ may be due to the difficulty encountered in obtaining accurate measurements over this region. The self-susceptance and its correction has been discussed in chapter 2. In Fig. 4.2, no correction has been applied to $B_{11}$; the use of the correction 0.72 millimhos that was indicated in chapter 2 would yield better agreement for $d / \lambda<0 \cdot 40$. In Fig. 4.3, the correction applied to $B_{11}$ is 1.16 millimhos. As previously discussed, this includes both the term 0.72 and an additional empirically determined susceptance that takes account of the endcorrection for the coaxial measuring line actually used. It was determined from a comparison of theoretical and measured results for a single element (Fig. 2.6). Since the correction to $B_{11}$ is a constant, it is evident that the correct variation of $B_{11}$ with $d / \lambda$ is given by the theory. In a practical application, the characteristics of a given array are determined from the theory, a single model of the elements of the array is constructed, and its driving-point admittance measured. The difference between theoretical and measured driving-point susceptances for the single element may be used as a correction for the computed driving-point admittances in the array.

It is sometimes convenient to characterize element intercoupling by self- and mutual impedances instead of admittances. For a general array, the conversion from an admittance basis to an impedance basis requires an inversion of the admittance matrix.

For a circular array, the sequence admittances and impedances are reciprocals, that is,

$$
\begin{gather*}
Z^{(m)}=1 / Y^{(m)}  \tag{4.26}\\
Z_{1 i}=\frac{1}{N} \sum_{m=0}^{N-1} e^{j 2 \pi(i-1) m / N} Z^{(m)} \tag{4.27}
\end{gather*}
$$

so that the reciprocal of only one complex number is required for each sequence.


Fig. 4.5a. Theoretical self- and mutual conductances for circular array; $N=8, h=\lambda / 4$, $a / \lambda=0.007022$.

### 4.4 Currents and fields; arrays with one driven element

One of the simplest examples of the application of the two-term theory is provided by ring arrays with one element driven and the remaining elements short-circuited at their driving points. In the following examples, the radius of the elements is taken to be
$a / \lambda=0.007$ and the radiation patterns are all measured or computed in the equatorial plane, $\Theta=\pi / 2$. The relative radiation patterns are computed from

$$
\begin{equation*}
P_{d B}=10 \log _{10}\left|\frac{E_{\Theta}^{r}(\Theta, \Phi) \cdot E_{\odot}^{r *}(\Theta, \Phi)}{E_{\Theta m}^{r} \cdot E_{\Theta m}^{r *}}\right| \tag{4.28}
\end{equation*}
$$

when $E_{\Theta m}^{r}$ is the maximum value of the field in the plane $\Theta=\pi / 2$. An asterisk indicates the complex conjugate, $E_{\Theta}^{r}(\Theta, \Phi)$ is given by (4.23), and $P_{d B}$ is the relative magnitude of the Poynting vector in decibels.


Fig. 4.5b. Like Fig. $4.5 a$ but for the susceptances.

Fig. $4.8 a$ contains two examples of the radiation patterns of five-element arrays. One pattern is for $d=\lambda / 4$ and $h=\lambda / 4$ and has a back-to-front ratio of about -14 db with half-power beam widths of about $100^{\circ}$. The second pattern is also for $d=\lambda / 4$ but $h=3 \lambda / 8$; it has a very smooth angular variation with a back-to-front ratio
of about -20 db and beam widths of about $140^{\circ}$. Agreement between the theoretical and measured results is well within 1 db except near the deeper minimum in the backward direction near $\Phi=180^{\circ}$. Similar patterns for $h / \lambda=0.5$ and two values of $d / \lambda$ are in Fig. $4.8 b$.



Fig. 4.6a. Theoretical self- and mutual conductances for circular array; $N=20, h=\lambda / 4$, $a / \lambda=0.007022$.


Fig. 4.6b. Like Fig. $4.6 a$ but for the susceptances.


Fig. 4.7a. Theoretical self- and mutual conductances for circular array; $N=20, h=3 \lambda / 8$, $a / \lambda=0.007022$.
Corresponding currents in the elements of the two arrays with the patterns given in Fig. $4.8 a$ are shown in Figs. 4.9 and 4.10. As a consequence of the symmetry, only three of the currents are different for each five-element array. The radiation patterns depend only on the relative distributions of current. If the currents in Figs. 4.9 and 4.10 were simply normalized to their maximum values, it is evident that agreement between theoretical and measured results would be very good and, therefore, measured patterns well


Fig. 4.7b. Like Fig. 4.7a but for the susceptances.
represented by the theory. In order to permit detailed comparison of the experimental and theoretical models, the relative amplitude and phase of the current along each element were measured and normalized to the measured self- and mutual admittances. Thus,

$$
\begin{align*}
& \frac{I_{k}(z)}{V_{1}}=\frac{\left|I_{k}(z)\right|}{V_{1}} e^{j \Psi_{k}(z)} \\
& \quad=\frac{\left|I_{k}(z)\right|}{V_{1}}\left[\cos \Psi_{k}(z)+j \sin \Psi_{k}(z)\right]=\frac{\operatorname{Re} I_{k}(z)+j \operatorname{Im} I_{k}(z)}{V_{1}} \tag{4.29}
\end{align*}
$$

where the real and imaginary parts are, respectively, in phase and in phase quadrature with the driving voltage. The relatively small amplitude of the current $\left|I_{3}(z)\right| / V_{1}$ in Fig. 4.9 prevented an accurate measurement of phase in this case.


Fig. 4.8a. Radiation patterns for 5-element arrays with one driven element; $h / \lambda=0.25$ and 0.375 .

The experimental model that was used for the measurement of both field patterns and currents consisted of five monopoles over a ground plane combined with a measuring line for each. Use was made of the image-plane technique described in section 8.9. The equipment and procedures for measuring amplitude and phase are discussed in chapter 8 . The $s_{k}$ and $c_{k}$ coefficients for use in (4.21) and (4.23) can be computed from the $\Psi_{d R}$ 's and $T$ 's in the
tables of appendix 1 with the use of $(4.15 \mathrm{a}, \mathrm{b})$ and (4.22a, b). Numerical data for the two five-element arrays under discussion are

$$
\left.\begin{array}{c}
N=5, \quad d=\lambda / 4, \quad h=\lambda / 4: \\
s_{1}^{\prime}=j 2.6824 ; \quad s_{2}^{\prime}=s_{3}^{\prime}=s_{4}^{\prime}=s_{5}^{\prime}=0 \\
c_{1}^{\prime}=-4.2084+j 9.2159 ; \quad c_{2}^{\prime}=c_{5}^{\prime}=-0.8072-j 4.6906 ; \\
c_{3}^{\prime}=c_{4}^{\prime}=0.6835-j 0.3656 \\
N=5, \quad d=\lambda / 4, \quad h=3 \lambda / 8: \\
s_{1}=-j 3.6571 ; \quad s_{2}=s_{3}=s_{4}=s_{5}=0  \tag{4.31}\\
c_{1}=0.7504+j 1 \cdot 1391 ; \quad c_{2}=c_{5}=0.4492+j 0.4419 \\
c_{3}=c_{4}=0.1890+j 0.1678 .
\end{array}\right\}
$$




Fig. 4.8b. Horizontal pattern of parasitic arrays of 2 and 5 elements in a circle; $h / \lambda=0.50$; $d$ is the distance between adjacent elements.

Note that the currents in the parasitic elements are represented by shifted cosine components only.

The radiation patterns in Fig. $4.8 a$ suggest that spacings can be found at which the pattern is a smooth function of $\Phi$ and has a deep minimum near $\Phi=180^{\circ}$. Examples of such patterns are


Fig. 4.9. Element currents; $N=5, h=\lambda / 4, d=\lambda / 4$.
shown in Fig. 4.11 for $N=4,5,10$ and 20 and $h=3 \lambda / 8$. As $N$ increases, such patterns occur when the circumference of the circle containing the array approaches $2 \lambda$. For them, the phase of the electric field is also a smooth slowly changing function of the azimuth angle $\phi$ as shown in Fig. 4.11. The phase was computed from (4.23a) in the form

$$
\begin{align*}
\frac{E_{\Theta}^{r}(\pi / 2, \Phi)}{K K_{1} V_{1}} & =\operatorname{Re}\left[\frac{E_{\Theta}^{r}(\pi / 2, \Phi)}{K K_{1} V_{1}}\right]+j \operatorname{Im}\left[\frac{E_{\Theta}^{r}(\pi / 2, \Phi)}{K K_{1} V_{1}}\right] \\
& =\left|\frac{E_{\Theta}^{r}(\pi / 2, \Phi)}{K K_{1} V_{1}}\right| e^{j \Psi(\pi / 2, \Phi)} \tag{4.32a}
\end{align*}
$$

$$
\begin{equation*}
\Psi(\pi / 2, \Phi)=\tan ^{-1} \frac{\operatorname{Im} E_{\Theta}^{r}(\pi / 2, \Phi)}{\operatorname{Re} E_{\Theta}^{r}(\pi / 2, \Phi)} \tag{4.32b}
\end{equation*}
$$



Fig. 4.10. Element currents; $N=5, h=3 \lambda / 8, d=\lambda / 4$.

### 4.5 Matrix notation and the method of symmetrical components

In the preceding sections the $N$ simultaneous integral equations for the $N$ currents in a circular array were replaced by $N$ independent integral equations by a procedure known as the method of symmetrical components. This procedure was introduced as a generalization of the corresponding treatment of the two coupled equations analysed in chapter 3 . It is now appropriate to systematize the general formulation with the compact notation of matrices.

The general method of symmetrical components became well known in its application to problems in multi-phase electric circuits. Loads on three-phase power systems, for example, must generally be balanced to give equal currents in all three branches. Under some conditions unequal loads are placed across the supply lines. The calculation of the resulting branch currents is usually made in terms of three phase-sequence components. The zero phase-sequence currents are all in phase. The first sequence contains three equal phasors which have $120^{\circ}$ progressive phase
shifts. These phasors rotate in the counter-clockwise direction in the complex plane as time increases. The angular velocity is $\omega$. The second phase sequence has three phasors with equal magnitude and a progressive $-120^{\circ}$ phase shift. Since the currents generated by the three sets of phase-sequence voltages do not interact with one another, they may be calculated separately and later combined to give the actual currents. A similar procedure applies to an $N$-phase system.


Fig. 4.11. Power pattern of circular arrays with one driven element; $h=3 \lambda / 8$, $a / \lambda=0.007022$.

The equations which relate the currents and voltages in $N$ coupled circuits have the following matrix form :
where

$$
\begin{gather*}
{[Z]\{I\}=\{V\}}  \tag{4.33}\\
\{I\}=\left\{\begin{array}{c}
I_{1} \\
I_{2} \\
\vdots \\
I_{N}
\end{array}\right\}, \quad\{V\}=\left\{\begin{array}{c}
V_{1} \\
V_{2} \\
\vdots \\
V_{N}
\end{array}\right\} \tag{4.34}
\end{gather*}
$$

4.5] METHOD OF SYMMETRICAL COMPONENTS

$$
[Z]=\left[\begin{array}{lll}
Z_{11} Z_{12} Z_{13} & \ldots Z_{1 N}  \tag{4.35}\\
Z_{21} Z_{22} & \ldots Z_{2 N} \\
\vdots & & \\
Z_{N 1} & \ldots Z_{N N}
\end{array}\right]
$$

The usual reciprocity of off-diagonal impedances is assumed, i.e., $Z_{i j}=Z_{j i}$.

In order to illustrate the application of the method of symmetrical components, this set of equations will not be solved in the usual manner by setting $\{I\}=[Z]^{-1}\{V\}$. Instead, the phasesequence voltages and impedances will be calculated first by means of the following transformation matrices:
$\left\{\begin{array}{l}V^{(0)} \\ V^{(1)} \\ \vdots \\ \vdots \\ V^{(N-1)}\end{array}\right\}=\frac{1}{N}\left[\begin{array}{lllll}1 & 1 & 1 & \ldots & 1 \\ 1 & p^{-1} & p^{-2} & \ldots & p^{-(N-1)} \\ 1 & p^{-2} & p^{-4} & \ldots & p^{-2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & p^{-(N-1)} & p^{-2(N-1)} & \ldots p^{-(N-1)(N-1)}\end{array}\right]\left\{\begin{array}{l}V_{1} \\ V_{2} \\ V_{3} \\ \vdots \\ V_{N}\end{array}\right\}$
where $p=e^{j 2 \pi / N}$, or $\quad\left\{V^{(m)}\right\}=[P]^{-1}\{V\}$.
Similarly, for the impedances
where

$$
\begin{align*}
& \left\{Z^{(m)}\right\}=[P]^{-1}\{Z\}  \tag{4.38}\\
& \left\{Z^{(m)}\right\}=\left\{\begin{array}{l}
Z^{(0)} \\
Z^{(1)} \\
\vdots \\
Z^{(N-1)}
\end{array}\right\} \tag{4.39}
\end{align*}
$$

$$
\{Z\}=\left\{\begin{array}{l}
Z_{11}  \tag{4.40}\\
Z_{12} \\
Z_{13} \\
\vdots \\
Z_{1 N}
\end{array}\right\}
$$

$$
[P]=\left[\begin{array}{lllll}
1 & 1 & 1 & \ldots & 1  \tag{4.41}\\
1 & p & p^{2} & \ldots & p^{(N-1)} \\
1 & p^{2} & p^{4} & \ldots & p^{2(N-1)} \\
\vdots & \vdots & & & \\
1 & p^{(N-1)} & p^{2(N-1)} \ldots p^{(N-1)(N-1)}
\end{array}\right]
$$

The phase-sequence currents are given by the algebraic equations

$$
\begin{equation*}
I^{(m)}=V^{(m)} / Z^{(m)}, \quad m=0,1, \ldots(N-1) \tag{4.42}
\end{equation*}
$$

The original currents $I_{i}, i=1,2,3, \ldots N$ are given by
where

$$
\begin{align*}
\{I\} & =[P]\left\{I^{(m)}\right\}  \tag{4.43}\\
\{I\} & =\left\{\begin{array}{l}
I_{1} \\
I_{2} \\
\vdots \\
I_{N}
\end{array}\right\} \tag{4.44}
\end{align*}
$$

As a trivial example of the method, consider two coupled circuits with the same self-impedances. The matrix equation is

$$
\left[\begin{array}{ll}
Z_{11} & Z_{12}  \tag{4.45}\\
Z_{12} & Z_{11}
\end{array}\right]\left\{\begin{array}{l}
I_{1} \\
I_{2}
\end{array}\right\}=\left\{\begin{array}{l}
V_{1} \\
V_{2}
\end{array}\right\} .
$$

With $p=e^{j 2 \pi / N}=e^{j \pi}=-1$, the matrix $P^{-1}$ defined by (4.36) is

$$
[P]^{-1}=\frac{1}{2}\left[\begin{array}{rr}
1 & 1  \tag{4.46}\\
1 & -1
\end{array}\right]
$$

The phase-sequence voltages and impedances as obtained from (4.37) and (4.38) are

$$
\begin{align*}
& \left\{\begin{array}{l}
V^{(0)} \\
V^{(1)}
\end{array}\right\}=\frac{1}{2}\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{l}
V_{1} \\
V_{2}
\end{array}\right\}=\frac{1}{2}\left\{\begin{array}{l}
V_{1}+V_{2} \\
V_{1}-V_{2}
\end{array}\right\}  \tag{4.47}\\
& \left\{\begin{array}{l}
Z^{(0)} \\
Z^{(1)}
\end{array}\right\}=\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{l}
Z_{11} \\
Z_{12}
\end{array}\right\}=\left\{\begin{array}{l}
Z_{11}+Z_{12} \\
Z_{11}-Z_{12}
\end{array}\right\} . \tag{4.48}
\end{align*}
$$

The resulting phase-sequence currents $I^{(m)}, m=1,2$, are given by
and

$$
\begin{equation*}
I^{(0)}=\frac{1}{2}\left(V_{1}+V_{2}\right) /\left(Z_{11}+Z_{12}\right) \tag{4.49}
\end{equation*}
$$

The desired currents $I_{i}, i=1,2$ (which are generated by the actual driving voltages $V_{1}$ and $V_{2}$ ) are given by (4.43) with (4.49) and (4.50). They are

$$
\left\{\begin{array}{l}
I_{1}  \tag{4.51}\\
I_{2}
\end{array}\right\}=\left[\begin{array}{rr}
1 & 1 \\
1 & -1
\end{array}\right]\left\{\begin{array}{l}
I^{(0)} \\
I^{(1)}
\end{array}\right\}=\left\{\begin{array}{l}
\left(V_{1} Z_{11}-V_{2} Z_{12}\right) /\left(Z_{11}^{2}-Z_{12}^{2}\right) \\
\left(V_{1} Z_{11}+V_{2} Z_{12}\right) /\left(Z_{11}^{2}-Z_{12}^{2}\right)
\end{array}\right\} .
$$

These equations are, of course, the same as those obtained directly from (4.45). Note that in the method of symmetrical components the matrix inversion is performed in a number of straightforward steps. In the analysis of circular arrays it allows a large matrix to be inverted for each phase sequence by obtaining the reciprocal of one complex number.

### 4.6 General formulation and solution

In section 4.1 the solutions for the $N$ independent integral equations for the phase-sequence currents in a circular array of identical elements were obtained by a logical generalization of the parallel analysis for the two-element array in chapter 3. A more complete formulation and solution with special reference to the complications of the $N$-element array is now in order.

With the matrix notation introduced in section 4.5 , the integral equations (4.3) for the $N$ phase-sequence currents may be expressed as follows

$$
\begin{align*}
& \int_{-h}^{h} I_{z}^{(m)}\left(z^{\prime}\right) K_{d}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \\
& \quad=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h}\left[\frac{1}{2} V^{(m)} \sin \beta_{0}(h-|z|)+U^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right] \tag{4.52}
\end{align*}
$$

where

$$
\begin{align*}
\left\{I_{z}^{(m)}\right\} & =[P]^{-1}\left\{I_{z}\right\}  \tag{4.53a}\\
\left\{V^{(m)}\right\} & =[P]^{-1}\{V\}  \tag{4.53b}\\
\left\{K_{d}^{(m)}\left(z, z^{\prime}\right)\right\} & =[P]^{-1}\left\{K_{d}\left(z_{i}, z^{\prime}\right)\right\} \tag{4.54}
\end{align*}
$$

and

$$
\begin{equation*}
U^{(m)}=\frac{-j \zeta_{0}}{4 \pi} \int_{-h}^{h} I_{z}^{(m)}\left(z^{\prime}\right) K^{(m)}\left(h, z^{\prime}\right) d z^{\prime} \tag{4.55}
\end{equation*}
$$

In order to reduce the integral equation (4.52) to an approximately equivalent algebraic equation in the manner described in chapter 3 , it is necessary to introduce approximate expressions for the several parts of the integral. The procedure and the reasoning behind it is in principle the same as described in sections 3.2 and 3.3 for two elements. However, for $N$ elements in a circle the kernel consists of a sum of $N$ instead of two terms. In the interest of simplicity, the introductory discussion in section 4.1 assumed that all elements are separated by distances sufficiently great so that $\beta_{0} b_{k i} \geqslant 1$ for all values of $k$ and $i$. Although this condition is satisfied in most circular arrays, there are exceptions. One is the cage antenna in which the $N$ parallel elements are distributed around an electrically small circle so that the condition $\beta_{0} b_{k i}<1$
is satisfied for all $k$ and $i$. An intermediate case arises when the circle is electrically large, but the elements are quite closely spaced so that one or more on each side of every element satisfies the inequality $\beta_{0} b_{k i}<1$, but all of the others are far enough away so that $\beta_{0} b_{k i} \geqslant 1$. Since the behaviour of the parts of the integrals that relate closely spaced elements is different from the parts that represent widely spaced ones, it is necessary to treat them separately. Since for each phase sequence all elements are in identical environments, element no. 1 is conveniently selected for reference. Let it be assumed that $n$ elements on each side of element 1 are sufficiently near so that for them $\beta_{0} b_{1 i}<1,1 \leqslant i \leqslant n$, $N-n+1 \leqslant i \leqslant N$ and that for all other elements, $\beta_{0} b_{1 i} \geqslant 1$, $n+1<i<N-n+1$. Let the sum over all the $2 n+1$ elements for which $\beta_{0} b_{1 i}<1$ be denoted by $\Sigma 1$, the sum over all other elements in the circle by $\Sigma 2$. Similarly, let $K_{d \Sigma 1}^{(m)}\left(z, z^{\prime}\right)$ be the part of the sum in (4.3c) which includes the $2 n+1$ elements for which $\beta_{0} b_{1 i}<1$, $K_{d 22}^{(m)}\left(z, z^{\prime}\right)$ the rest of the sum. It now follows by analogy with (3.19a, b) that

$$
\begin{gather*}
\int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d \Sigma 1 R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d \Sigma 1 R}^{(m)} \sin \beta_{0}(h-|z|)  \tag{4.56a}\\
\int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d \Sigma 2 R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime} \doteq \Psi_{d \Sigma 2 R}^{(m)}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)
\end{gather*}
$$

where $K_{d \Sigma 1 R}^{(m)}\left(z, z^{\prime}\right)$ and $K_{d \Sigma 2 R}^{(m)}\left(z, z^{\prime}\right)$ are the appropriate parts of $K_{d R}^{(m)}\left(z, z^{\prime}\right)$ as defined in (4.3e). On the other hand, all remaining parts of the integral are independent of $\beta_{0} b_{1 i}$, so that they are the same as in (3.20a)-(3.23) but with $K_{d R}^{(m)}\left(z, z^{\prime}\right)$ and $K_{d I}^{(m)}\left(z, z^{\prime}\right)$ as given in (4.3e) and (4.3f).

The $\Psi$-functions introduced in $(4.56 \mathrm{a}, \mathrm{b})$ are defined as follows:

$$
\begin{gather*}
\Psi_{d R}^{(m)} \equiv \Psi_{d \Sigma 1 R}^{(m)}=\Psi_{d \Sigma 1 R}^{(m)}\left(z_{m}\right) ;\left\{\begin{array}{ll}
z_{m}=0 & \beta_{0} h \leqslant \pi / 2 \\
z_{m}=h-\lambda / 4, & \beta_{0} h>\pi / 2
\end{array}\right\}(4.57 \mathrm{a})  \tag{4.57a}\\
\Psi_{d \Sigma 1 R^{(m)}(z)=\csc \beta_{0}(h-|z|) \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d \Sigma 1 R}^{(m)}\left(z, z^{\prime}\right) d z^{\prime}(4.57 \mathrm{~b})}^{\Psi_{d \Sigma R}^{(m)} \equiv \Psi_{d \Sigma 2 R}^{(m)}=\left(1-\cos \beta_{0} h\right)^{-1} \int_{-h}^{h} \sin \beta_{0}\left(h-\left|z^{\prime}\right|\right) K_{d \Sigma 2 R}^{(m)}\left(0, z^{\prime}\right) d z^{\prime} .} \tag{4.57b}
\end{gather*}
$$

These are generalizations of (3.24a, b) and (3.25a, b). The other $\Psi$-functions, specifically $\Psi_{d U}^{(m)}=\Psi_{d V R}^{(m)}+j \Psi_{d U I}^{(m)}, \Psi_{d D}^{(m)}=\Psi_{d D R}^{(m)}+j \Psi_{d D I}^{(m)}$ and $\Psi_{d I}^{(m)}$ are the same as defined in (3.26)-(3.28) but with the $N$ -
term kernel given in (4.3d). Note that when all elements are sufficiently far apart to satisfy the inequality $\beta_{0} b_{1 i}>1,1<i \leqslant N$, only $i=1$, with $b_{11}=a$ contributes to $\Psi_{d R}^{(m)}$ which is then equal to $\Psi_{d R}$ for the isolated element.

With the notation introduced in (4.57a) and (4.58), the equation (3.33) applies directly to the $N$-element array. The same equation with $\Psi_{d R}^{(m)}$ substituted for $\Psi_{d R}$ is correct when some elements are sufficiently close together so that $\beta_{0} b_{1 i}<1, i>1$. It follows that the entire formal solution in sections 3.3 and 3.4 is valid for the phase sequences of the $N$-element array. The $N$ independent phasesequence currents $I^{(m)}(z), m=0,1, \ldots N$, may be expressed as the solution of a column matrix equation.

A summary of the relevant equations is given below.
Phase sequence currents

$$
\begin{align*}
\left\{I_{z}^{(m)}(z)\right\}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h} \\
& \times\left[\left\{V_{0}^{(m)} M_{0 z}\right\}+\left\{V_{0}^{(m)} T_{U}^{(m)} F_{0 z}\right\}+\left\{V_{0}^{(m)} T_{D}^{(m)} H_{0 z}\right\}\right] \tag{4.59}
\end{align*}
$$

where $\quad M_{0 z}=\sin \beta_{0}(h-|z|), \quad F_{0 z}=\cos \beta_{0} z-\cos \beta_{0} h, \quad$ and $H_{0 z}=\cos \left(\beta_{0} z / 2\right)-\cos \left(\beta_{0} h / 2\right)$.

$$
\left.\begin{array}{c}
\left\{\begin{array}{l}
T_{V}^{(m)} \\
T_{D}^{(m)}
\end{array}\right\}=\left[\Phi_{T}^{(m)}\right]-1
\end{array} \begin{array}{l}
\Psi_{V}^{(m)}(h)-\Psi_{d \Sigma R}^{(m)} \cos \beta_{0} h \\
-j \Psi_{d I}^{(m)}
\end{array}\right\}, ~\left[\Phi_{T}^{(m)}\right]=\left[\begin{array}{ll}
\Phi_{T(1)}^{(m)} & \Phi_{T(12}^{(m)}  \tag{4.60b}\\
\Phi_{T 21}^{(m)} & \Phi_{T 22}^{(m)}
\end{array}\right] .
$$

$\left[\Phi_{T}^{(m)}\right]^{-1}$ is the reciprocal of $\left[\Phi_{T}^{(m)}\right]$.

$$
\left.\begin{array}{l}
\Phi_{T 11}^{(m)}=\Psi_{d U R}^{(m)} \cos \beta_{0} h-\Psi_{U}^{(m)}(h)  \tag{4.61}\\
\Phi_{T 12}^{(m)}=-\Psi_{D}^{(m)}(h) \\
\Phi_{T 21}^{(m)}=j \Psi_{d U I}^{(m)} \\
\Phi_{T 22}^{(m)}=\Psi_{d D}^{(m)} .
\end{array}\right\} .
$$

The phase-sequence admittance is given by setting $z=0$ in (4.59), thus:

$$
\begin{equation*}
Y_{0}^{(m)}=\frac{I_{z}^{(m)}(0)}{V_{0}^{(m)}} \tag{4.62}
\end{equation*}
$$

The phase-sequence impedance is the reciprocal of the phasesequence admittance,

$$
\begin{equation*}
Z^{(m)}=\frac{1}{Y^{(m)}} . \tag{4.63}
\end{equation*}
$$

The mutual impedances may be calculated from the phase-sequence
admittances by multiplying by the inverse (4.36) of the phasesequence matrix $P$. Thus,

$$
\begin{equation*}
Z_{1 i}=[P]^{-1} Z^{(m)}, \quad 1<i \leqslant N . \tag{4.64}
\end{equation*}
$$

When the identical elements of a circular array are equally spaced around a circle, symmetry reduces the number of different admittances or impedances to $(N+1) / 2$ if $N$ is odd and $(N / 2)+1$ if $N$ is even. For example,

$$
\begin{equation*}
Z_{12}=Z_{1 N} ; \quad Z_{13}=Z_{1(N-1)} ; \quad Z_{14}=Z_{1(N-2)} \text { etc. } \tag{4.65}
\end{equation*}
$$

When the expression for the phase-sequence currents becomes indeterminate for $\beta_{0} h=\pi / 2$ and for a range near this value, the alternative form given in (3.43)-(3.45) is useful. It is

$$
\begin{align*}
\left\{I_{z}^{(m)}(z)\right\} & =\frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}}\left\{V_{0}^{(m)} S_{0 z}+V_{0}^{(m)} T_{U}^{\prime(m)} F_{0 z}\right. \\
& \left.-V_{0}^{(m)} T_{D}^{\prime(m)} H_{0 z}\right\}, \quad \beta_{0} h \sim \frac{\pi}{2} \tag{4.66}
\end{align*}
$$

where

$$
\begin{align*}
S_{0 z} & =\sin \beta_{0}|z|-\sin \beta_{0} h \\
& =\sin \beta_{0}|z|-1 \quad \text { when } \beta_{0} h=\frac{\pi}{2} \tag{4.67}
\end{align*}
$$

and

$$
\left.\begin{array}{l}
\left\{T_{U}^{\prime(m)}\right\}=-\left\{\left(T_{U}^{(m)}+\sin \beta_{0} h\right) / \cos \beta_{0} h\right\}  \tag{4.68}\\
\left\{T_{\boldsymbol{D}}^{\prime(m)}\right\}=\left\{T_{D}^{(m)} / \cos \beta_{0} h\right\} .
\end{array}\right\}
$$

The two-term approximation used earlier in this chapter is quickly obtained from the three-term formulas. As stated in section 3.9, the procedure involves the substitution of $F_{0 z}$ for $H_{0 z}$ and $T^{(m)}$ for $T_{U}^{(m)}+T_{D}^{(m)}$. This implies that $\Psi_{d D} \rightarrow \Psi_{d U}, \Psi_{D}(h) \rightarrow \Psi_{U}(h)$. The two-term forms for the phase-sequence currents and admittances (cf. (3.85)-(3.87)) are
and

$$
\begin{equation*}
\left\{I_{z}^{(m)}(z)\right\}=\frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left\{V_{0}^{(m)} M_{0 z}+V_{0}^{(m)} T^{(m)} F_{0 z}\right\} \tag{4.69}
\end{equation*}
$$

where $\quad\left\{T^{(m)}\right\}=-\left\{\frac{\Psi_{V}^{(m)}(h)-\left(\Psi_{d \Sigma R}^{(m)}+j \Psi_{d I}^{(m)}\right) \cos \beta_{0} h}{\Psi_{U}^{(m)}(h)-\Psi_{d U}^{(m)} \cos \beta_{0} h}\right\}$.
When $\beta_{0} h$ is at or near $\pi / 2$ the alternative formulas (3.88)-(3.90) are applicable. They are

$$
\begin{equation*}
\left\{I_{z}^{(m)}(z)\right\}=\frac{-j 2 \pi}{\zeta} \frac{\Psi_{0}}{\Psi_{d R}}\left\{V_{0}^{(m)} S_{0 z}+V_{0}^{(m)} T^{\prime(m)} F_{0 z}\right\} \tag{4.72}
\end{equation*}
$$

$$
\begin{align*}
\left\{Y^{(m)}\right\} & =\left\{\frac{I_{z}^{(m)}(0)}{V_{0}^{(m)}}\right\}  \tag{4.73}\\
\text { where } \quad\left\{T^{\prime(m)}\right\} & =-\left\{\frac{T^{(m)}+\sin \beta_{0} h}{\cos \beta_{0} h}\right\}, \quad \beta_{0} h \sim \frac{\pi}{2} \tag{4.74}
\end{align*}
$$

Note that (4.69) and (4.72) are the same as (4.4a) and (4.4b).

## CHAPTER 5

## THE CIRCUIT AND RADIATING PROPERTIES OF CURTAIN ARRAYS

In chapter 1 the conventional approach to antenna theory is reviewed and the radiation and circuit properties of the single antenna and array of antennas are presented under the conventional assumptions. Possible sources of error are also pointed out. In chapters 2 and 3 a new and more accurate theory is presented for a single antenna and for a two-element array. The present chapter is concerned with the analysis and synthesis of the general N element curtain array. This is a linear array with the centres of all elements along a straight line but with their axes all perpendicular to and in a plane containing the line.

### 5.1 General comparison of conventional and new theories

The analysis of arrays is conventionally formulated under the implicit assumption that distributions of current along all elements are identical. It follows that self- and mutual impedances depend only on the geometry of the elements. Circuit equations can then be written to relate the driving-point voltages and currents through an impedance matrix. Thus,

$$
\begin{gather*}
\{V\}=[Z]\{I\}  \tag{5.1}\\
\text { where }\{V\}=\left\{\begin{array}{c}
V_{01} \\
V_{02} \\
\vdots \\
V_{0 N}
\end{array}\right\}, \quad\{I\}=\left\{\begin{array}{c}
I_{01} \\
I_{02} \\
\vdots \\
I_{0 N}
\end{array}\right\}  \tag{5.2}\\
\text { and } \quad[Z]=\left[\begin{array}{ccccc}
Z_{11} & Z_{12} & Z_{13} & \ldots & Z_{1 N} \\
Z_{21} & Z_{22} & Z_{23} & \ldots & Z_{2 N} \\
\vdots & & & & \\
Z_{N 1} & Z_{N 2} & Z_{N 3} & \ldots & Z_{N N}
\end{array}\right] \tag{5.3}
\end{gather*}
$$

The bracket terms are $N \times N$ matrices; the terms in braces are column matrices. The usual reciprocity of off-diagonal impedances
in (5.3) holds (i.e. $Z_{12}=Z_{21}$ etc.). Equation (5.1) relates the quantities that can be assigned at the driving point of the antenna, namely the voltages and currents. The simple matrix relation between $V$ 's and $I$ 's shows that it is immaterial whether the voltages or the currents are specified, since the ratio between each voltage and current is unchanged. The phase and magnitude of the currents in the individual elements are normally specified so as to produce a particular radiation pattern. The assumption of identical distributions of current on all elements involves the tacit assumption that the phase and amplitude of the current at all points in each element are completely determined by their values at the driving point.
The preceding remarks may, at first glance, seem like a repetition of well-known facts. However, the assumptions implied in the conventional formulation are not satisfactory approximations for actual arrays except when the elements are very thin and have lengths near $\lambda / 2$. Even for this special case difficulties arise when the elements are very closely spaced. Fortunately, a more realistic theory can be developed that is generally applicable to arrays with elements that are less than $3 \lambda / 4$ in half-length. The new theory is somewhat more complicated than the conventional approach. However, for engineering purposes it is more important that a theory agree with experiment than that it be mathematically simple. As with most new approaches, much of the complexity disappears with continued use and understanding. At the outset the fundamental processes will be explained without reference to the details of the theory.

An example of the notation of a three-element array is shown in Fig. 5.1. The conventional assumption is that regardless of the driving conditions each element has the same distribution of current. For example,

$$
\begin{equation*}
I_{i}(z)=I_{0 i} \frac{\sin \beta_{0}(h-|z|)}{\sin \beta_{0} h}, \quad i=1,2,3 . \tag{5.4}
\end{equation*}
$$

Equation (5.4) shows that once the currents are assigned at any point, e.g. at $z=0$, the entire current is completely specified. The new theory does not require the individual currents to have the same distributions. They may have the following much more general trigonometric representation:

$$
\begin{equation*}
I_{i}(z)=j A_{i} \sin \beta_{0}\left(h-\left|z_{i}\right|\right)+B_{i}\left(\cos \beta_{0} z_{i}-\cos \beta_{0} h\right) \tag{5.5}
\end{equation*}
$$



Fig. 5.1. Three-element array.
where $i=1,2,3$, and $A_{i}$ is real and $B_{i}$ is complex. In (5.5) the $A$ coefficients are directly proportional to the respective driving voltages. That is,

$$
\begin{equation*}
A_{i}=C V_{0 i}, \quad i=1,2,3 \tag{5.6}
\end{equation*}
$$

where $C$ is a constant. On the other hand, the complex $B$ coefficients depend on contributions not only from the individual element but also from all of the remaining elements. For example, there are contributions to $B_{1}$ from $V_{01}$ and also from $V_{02}$ and $V_{03}$.

Consider now the simplest circuit problem for an array of $N$ elements. Given the $N$ driving-point currents, what must be the driving voltages? In addition, the individual and generally different distributions of the $N$ currents along the elements are required. By the methods of the conventional theory the specified problem is solved directly from (5.1). If the $I_{0 i}$ 's are given, the corresponding $V_{0 i}$ 's are obtained from the solution of (5.1). For the inverse problem, the $I_{0 i}$ 's that correspond to specifying the $V_{0 i}$ 's are given by the solution of the matrix equation

$$
\begin{equation*}
\{I\}=[Z]^{-1}\{V\}=[Y]\{V\} \tag{5.7}
\end{equation*}
$$

where $[Y]=[Z]^{-1}$ is the inverse of $Z$. The solution of (5.1) or (5.7) gives the relative values of the complex driving-point currents, hence also the entire current from (5.4). Clearly, the solution of this problem is more complicated in the new theory since the three quantities, $A_{i}$, and the real and imaginary parts of $B_{i}$ must be determined for each element from the assigned real and imaginary parts of the driving-point currents $I_{0 i}=I_{i}(0)$ and expressed in terms of the complex voltages $V_{0 i}$.

### 5.2 New theory of curtain arrays

The theoretical solution of the general problem of the curtain array will now be examined in detail. The essential basis for this theory was given in chapter 2 . Since the two-term theory described in section 2.10 yields results of sufficient accuracy, it will be used for the curtain array to reduce the complexity of the formulation. However, the more accurate three-term representation involves only added algebraic complications.

The integral equation (2.4) may be written as follows for the $k^{\text {th }}$ antenna of an $N$-element array:

$$
\begin{align*}
& 4 \pi \mu_{0}^{-1} A_{z k}(z)=\int_{-h}^{h} \sum_{i=1}^{N} I_{z i}\left(z^{\prime}\right) K_{k i}\left(z, z^{\prime}\right) d z^{\prime} \\
&=-j \frac{4 \pi}{\zeta_{0}}\left(C_{k} \cos \beta_{0} z+\frac{1}{2} V_{0 k} \sin \beta_{0}|z|\right)  \tag{5.8}\\
& K_{k i}\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{k i}}}{R_{k i}}, \tag{5.9a}
\end{align*}
$$

where

$$
\begin{equation*}
R_{k i}=\sqrt{\left(z_{k}-z_{i}^{\prime}\right)^{2}+b_{k i}^{2}}, \quad b_{k k}=a, \quad \zeta_{0} \doteq 120 \pi \text { ohms } \tag{5.9b}
\end{equation*}
$$

The notation is illustrated in Fig. 5.1 for a three-element array. If the array has $N$ elements, it is necessary to solve $N$ simultaneous integral equations of the form (5.8), where $k=1,2,3, \ldots N$. Following the procedure used in (2.3)-(2.6), an approximate zero-order solution will be obtained for the general linear array. That is, given the $N$ driving voltages, a solution will be obtained for the currents in the $N$ elements. Alternatively, given the $N$ driving-point currents, the $N$ driving voltages will be determined.

As a first step in the solution, the constant part of the vector potential is removed from the right-hand side of (5.8) by the introduction of the vector potential difference

$$
W_{z k}(z)=A_{z k}(z)-A_{z k}(h) .
$$

The result is

$$
\begin{align*}
4 \pi \mu_{0}^{-1} W_{z k}(z)= & \int_{-h}^{h} \sum_{i=1}^{N} I_{z i}\left(z^{\prime}\right) K_{k i d}\left(z, z^{\prime}\right) d z^{\prime}  \tag{5.10}\\
= & -j \frac{4 \pi}{\zeta_{0}}\left[C_{k} \cos \beta_{0} z+\frac{1}{2} V_{0 k} \sin \beta_{0}|z|\right] \\
& -\int_{-h}^{h} \sum_{i=1}^{N} I_{z i}\left(z^{\prime}\right) K_{k i}\left(h, z^{\prime}\right) d z^{\prime}  \tag{5.11}\\
\text { where } \quad & K_{k i d}\left(z, z^{\prime}\right)=\frac{e^{-j \beta_{0} R_{k i}}}{R_{k i}}-\frac{e^{-j \beta_{0} R_{k i h}}}{R_{k i h}} . \tag{5.12}
\end{align*}
$$

The constants of integration $C_{k}$ are expressed in terms of quantities $U_{k}$ that are proportional to the $A_{z k}(h)$ by means of the relation $W_{z k}(h)=0$. The final form of the integral (5.8) is [cf. (2.15)]

$$
\begin{equation*}
\int_{-h}^{h} \sum_{i=1}^{N} I_{z i}\left(z^{\prime}\right) K_{k i d}\left(z, z^{\prime}\right) d z^{\prime}=j \frac{4 \pi}{\zeta_{0} F_{0}(h)}\left(U_{k} F_{0 z}+\frac{1}{2} V_{0 k} M_{0 z}\right) \tag{5.13}
\end{equation*}
$$

where

$$
\begin{gather*}
F_{0 z}=F_{0}(z)-F_{0}(h)=\cos \beta_{0} z-\cos \beta_{0} h  \tag{5.14}\\
M_{0 z}=F_{0}(z) G_{0}(h)-G_{0}(z) F_{0}(h)=\sin \beta_{0}(h-|z|)  \tag{5.15}\\
U_{k}=\sum_{i=1}^{N} U_{k i}=-j \frac{\zeta_{0}}{4 \pi} \int_{-h}^{h} \sum_{i=1}^{N} I_{z i}\left(z^{\prime}\right) K_{k i}\left(h, z^{\prime}\right) d z^{\prime} . \tag{5.16}
\end{gather*}
$$

The difference kernel (5.12) may be separated into its real and imaginary parts as follows [cf. (2.5) et seq.]:

$$
\left.\begin{array}{c}
K_{k i d R}\left(z, z^{\prime}\right)+j K_{k i d I}\left(z, z^{\prime}\right)=K_{k i d}\left(z, z^{\prime}\right) \\
K_{k i d R}=K_{k i R}\left(z, z^{\prime}\right)-K_{k i R}\left(h, z^{\prime}\right)  \tag{5.18}\\
K_{k i d I}=K_{k i l}\left(z, z^{\prime}\right)-K_{k i I}\left(h, z^{\prime}\right)
\end{array}\right\} .
$$

where

For the single element, the integrals corresponding to those in (5.13) were separated into two groups depending on the manner in which their leading terms varied as functions of $z$. The same principle of separation may be applied to the integrals for the curtain array. As before, one group varies approximately as $M_{0 z}$, the other as $F_{0 z}$. The following functional forms for the integrals in (5.13) are important general criteria for the separation:

$$
\begin{array}{ll}
\int_{-h}^{h} I_{z i}\left(z^{\prime}\right) K_{k i R}\left(z, z^{\prime}\right) d z^{\prime} \sim I_{z i}(z) & \text { when } \beta_{0} b_{k i}<1 \\
\int_{-h}^{h} I_{z i}\left(z^{\prime}\right) K_{k i R}\left(z, z^{\prime}\right) d z^{\prime} \sim F_{0 z} & \text { when } \beta_{0} b_{k i} \geqslant 1 \tag{5.20}
\end{array}
$$

$\int_{-h}^{h} I_{z i}\left(z^{\prime}\right) K_{k i I}\left(z, z^{\prime}\right) d z^{\prime} \sim F_{0 z} \quad$ for any $I\left(z^{\prime}\right)$ and all $\beta_{0} b_{k i}$.
The current in each element can now be expressed in two parts in the form

$$
\begin{equation*}
I_{z i}(z)=I_{u i}(z)+I_{v i}(z) \tag{5.22}
\end{equation*}
$$

where, by definition, the leading terms behave approximately as follows:

$$
\begin{equation*}
I_{v i}(z) \sim M_{0 z}, \quad I_{u i}(z) \sim F_{0 z} \tag{5.23}
\end{equation*}
$$

Some appreciation of the importance of the general functional forms in (5.23) may be obtained from an investigation of the integral equation (5.13). If attention is directed to the right-hand side of (5.13), it is seen that the equation contains two apparent sources, the coefficients of $F_{0 z}$ and $M_{0 z}$. The function $U_{k}$ has a constant amplitude over the entire length of the $k^{\text {th }}$ element and is generated primarily by the distributed currents on each element in the array. The other source function is the potential difference $V_{0 k}$, as in a transmission line or in an isolated antenna; it is localized at $z=0$. The form of the integral equation (5.13) suggests that the current on each element can be separated into two parts, the one apparently generated by the $U_{k}$, the other by the $V_{0 k}$. The part of the current due to $U_{k}$ is closely related to the current in an unloaded receiving antenna that is located in the wave front of an incident plane-wave field that has the same amplitude and phase over the entire length of the element. For this the leading term varies as $F_{0 z}$. Except when the elements are very closely spaced (as in an open-wire line), the sinusoidal parts of the currents (i.e. $M_{0 z}$ ) are maintained primarily by the individual driving potentials $V_{0 k}$. Thus, the current due to each of the $V_{0 k}$ is essentially the same as in an isolated antenna.

When (5.22) is substituted in (5.13), groups of integrals occur that may be expressed as follows for all $k$ and $i$ in the ranges 1 to $N$ :

$$
\begin{align*}
& \int_{-h}^{h} I_{u i}\left(z^{\prime}\right) K_{k i d}\left(z, z^{\prime}\right) d z^{\prime}=\left(\frac{B_{i}}{B_{k}}\right) \Psi_{k i d u} I_{u k}(z)-D_{k i d u}(z)  \tag{5.24}\\
& \int_{-h}^{h} I_{v i}\left(z^{\prime}\right) K_{k i d}\left(z, z^{\prime}\right) d z^{\prime} \\
& \quad=\left(\frac{j A_{i}}{B_{k}}\right) \Psi_{k i d v} I_{v k}(z)-D_{k i d v}(z) ; \quad \beta_{0} b_{k i} \geqslant 1 \tag{5.25}
\end{align*}
$$

$$
\begin{align*}
& \int_{-h}^{h} I_{v i}\left(z^{\prime}\right) \\
& \quad \begin{aligned}
& K_{k i d R}\left(z, z^{\prime}\right) d z^{\prime} \\
& =\left(\frac{A_{i}}{A_{k}}\right) \Psi_{k i d R} I_{v k}(z)-D_{k i d R}(z) ; \quad \beta_{0} b_{k i}<1 \\
& =\left(\frac{j A_{i}}{B_{k}}\right) \Psi_{k i d I} I_{u k}(z)-D_{k i d I}(z) ; \quad \beta_{0} b_{k i}<1
\end{aligned} \tag{5.26}
\end{align*}
$$

It is assumed that the functions $\Psi_{k i}$ are defined so that the difference terms $D_{k i}(z)$ are small enough to be negligible in a solution of zero order. The coefficient $\left(j A_{i} / B_{k}\right)$ in (5.27) is the ratio of the amplitude of $I_{v i}(z)$ to that of $I_{u k}(z)$. When (5.24)-(5.27) are substituted in (5.13) and only the leading terms are retained, the following separation into two groups of equations may be carried out:

$$
\begin{array}{r}
\sum_{i=k-m}^{k+m}\left(\frac{A_{i}}{A_{k}}\right) \Psi_{k i d R} I_{v k}(z)=j \frac{2 \pi}{\zeta_{0} F_{0}(h)} V_{0 k} M_{0 z} \\
\sum_{i=1}^{N}\left(\frac{B_{i}}{B_{k}}\right) \Psi_{k i d u} I_{v k}(z)+\left[\sum_{i=1}^{k-m-1}+\sum_{k+m+1}^{N}\right]\left(\frac{j A_{i}}{B_{k}}\right) \Psi_{k i d v} I_{u k}(z) \\
+j \sum_{i=k-m}^{k+m}\left(\frac{j A_{i}}{B_{k}}\right) \Psi_{k i d I} I_{u k}(z)=j \frac{4 \pi}{\zeta_{0} F_{0}(h)} U_{k} F_{0 z} \tag{5.29}
\end{array}
$$

The index $m$ in the sums is defined by

$$
\begin{equation*}
\beta_{0} b_{k m}<1, \quad \beta_{0} b_{k, m+1} \geqslant 1 \tag{5.30}
\end{equation*}
$$

where $b_{k m}$ is the distance between the centres of the elements $m$ and $k$. In most curtain arrays the spacing of the elements is sufficiently great so that all elements with $m \neq k$ satisfy the righthand inequality in (5.30) and only $\beta_{0} b_{k k}=\beta_{0} a<1$. When this is true, (5.28) and (5.29) reduce to

$$
\begin{equation*}
I_{v k}(z)=j_{\frac{2 \pi}{\zeta_{0}} \frac{2 \pi V_{0 k}}{\Psi_{d R} F_{0}(h)} M_{0 z}} \tag{5.31}
\end{equation*}
$$

and
$\sum_{i=1}^{N}\left\{\left(\frac{B_{i}}{B_{k}}\right) \Psi_{k i d u}+\left(\frac{j A_{i}}{B_{k}}\right)\left[\Psi_{k i d v}\left(1-\delta_{i k}\right)+j \Psi_{k i d I} \delta_{i k}\right]\right\} I_{u k}(z)=\frac{j 4 \pi U_{k}}{\zeta_{0} F_{0}(h)} F_{0 z}$
where

$$
\delta_{i k}= \begin{cases}0, & i \neq k  \tag{5.32}\\ 1, & i=k .\end{cases}
$$

The notation $\Psi_{d R}=\Psi_{k k d R}$ is used, since with identical elements all the $\Psi_{k k d R}$ are identical and equal to $\Psi_{d R}$ for the isolated antenna.

It follows directly from (5.31) that the leading term in $I_{v k}(z)$ is always $M_{0 z}$ for each value of $k$. Similarly from (5.32) the leading term in $I_{u k}(z)$ is of the form $F_{0 z}$. Hence, it is possible to set
or

$$
\begin{gather*}
I_{v i}(z)=j A_{i} M_{0 z}, \quad I_{u i}(z)=B_{i} F_{0 z}  \tag{5.33}\\
I_{z i}(z)=j A_{i} M_{0 z}+B_{i} F_{0 z}
\end{gather*}
$$

Since $\Psi_{d R}$ is real, it is clear from (5.31) that $A_{i}$ is real when $V_{0 k}$ is real and from (5.32) that $B_{i}$ is in general complex, or

$$
\begin{equation*}
B_{i}=B_{i R}+j B_{i I} . \tag{5.35}
\end{equation*}
$$

Note that the constant $\left(j A_{i} / B_{k}\right)$, introduced in (5.25) and (5.27), is the ratio of the coefficients of the two terms in (5.34).

With the zero-order current formally determined, the constant $U_{k}$ may be obtained from the substitution of (5.34) in (5.16). It is given by

$$
\begin{equation*}
U_{k}=-j \frac{\zeta_{0}}{4 \pi} \sum_{i=1}^{N}\left[j A_{i} \Psi_{k i v}(h)+B_{i} \Psi_{k i u}(h)\right] \tag{5.36}
\end{equation*}
$$

where

$$
\begin{align*}
& \Psi_{k i v}(h)=\int_{-h}^{h} M_{0 z^{\prime}} K_{k i}\left(h, z^{\prime}\right) d z^{\prime}  \tag{5.37}\\
& \Psi_{k i u}(h)=\int_{-h}^{h} F_{0 z^{\prime}} K_{k i}\left(h, z^{\prime}\right) d z^{\prime} . \tag{5.38}
\end{align*}
$$

If (5.36), (5.33), and (5.34) are substituted in (5.31) and (5.32) the result is

$$
\begin{gather*}
A_{k}=\frac{2 \pi}{\zeta_{0} \Psi_{d R} F_{0}(h)} V_{0 k}  \tag{5.39}\\
\sum_{i=1}^{N} B_{i}\left[\Psi_{k i d u} F_{0}(h)-\Psi_{k i u}(h)\right] \\
=j \sum_{i=1}^{N} A_{i}\left\{\Psi_{k i v}(h)-\left[\Psi_{k i d v}\left(1-\delta_{i k}\right)+j \Psi_{k i d I} \delta_{i k}\right] F_{0}(h)\right\} \tag{5.40}
\end{gather*}
$$

where $k=1,2,3, \ldots N$. The physical significance of the zero-order solution is evident from (5.39) and (5.40). The coefficients of the 'transmitting part' of the current are given by (5.39). The $N$ driving voltages generate the expected sinusoidal distribution of current on each element. The coefficients of the 'receiving part' of the current are given by (5.40). The $N$ currents act as distributed sources to generate distributions of the receiving type which are present in all the elements of the array. Equation (5.40) permits the prediction
in each driven element of the shifted-cosine component of the current that is due to coupling between currents distributed along the element itself and along all other elements in the array. Conventional array theory is concerned only with (5.39), since all currents are assumed to have the same sinusoidal distribution. In the special case of an array with thin half-wave elements, the real and imaginary parts of the current in each element do have approximately the same distribution. It follows that conventional array theory should work quite well for an array of very thin half-wave elements. On the other hand, in the more general case, the real and imaginary parts of the current in each element have different distributions so that (5.40) is needed along with (5.39) to determine the actual currents.

An important case to which conventional theory has no application is the array of full-wave elements in which the currents are near anti-resonance, and their real and imaginary parts have quite different distributions. Before some particular parallel arrays are analysed, (5.40) is best expressed in matrix form. A general expression will be given for the $\Psi_{k i}(z)$ functions, and rigorous expressions will be derived for the radiation field.

Equation (5.40) is a set of linear algebraic equations with $N$ unknowns that may be solved for the $B_{i}$ in terms of the $A_{i}$. The $N$ values of the $A_{i}$ are expressed in terms of the $N$ driving voltages $V_{0 k}$ by (5.39). In order to express (5.40) in matrix form, let the following quantities be defined:

$$
\begin{gather*}
\Phi_{k i u}=\Psi_{k i d u} F_{0}(h)-\Psi_{k i u}(h)  \tag{5.41}\\
\Phi_{k i v}=\Psi_{k i v}(h)-\Psi_{k i d v} F_{0}(h)\left(1-\delta_{i k}\right)-j \Psi_{k i d I} F_{0}(h) \delta_{i k} . \tag{5.42}
\end{gather*}
$$

Also let $\quad\left[\Phi_{u}\right]=\left[\begin{array}{cccc}\Phi_{11 u} & \Phi_{12 u} & \ldots & \Phi_{1 N u} \\ \Phi_{21 u} & \Phi_{22 u} & \ldots & \Phi_{2 N u} \\ \vdots & & & \\ \Phi_{N 1 u} & \Phi_{N 2 u} & \ldots & \Phi_{N N u}\end{array}\right]$

$$
\left[\Phi_{v}\right]=\left[\begin{array}{cccc}
\Phi_{11 v} & \Phi_{12 v} & \ldots & \Phi_{1 N v}  \tag{5.44}\\
\Phi_{21 v} & \Phi_{22 v} & \ldots & \Phi_{2 N v} \\
\vdots & & & \\
\Phi_{N 1 v} & \Phi_{N 2 v} & \ldots & \Phi_{N N v}
\end{array}\right]
$$

$$
\{A\}=\left\{\begin{array}{c}
A_{1}  \tag{5.45}\\
A_{2} \\
\vdots \\
A_{N}
\end{array}\right\}, \quad\{B\}=\left\{\begin{array}{c}
B_{1} \\
B_{2} \\
\vdots \\
B_{N}
\end{array}\right\} .
$$

The bracket terms are $N \times N$ matrices; the terms in braces are column matrices. From the substitution of $(5.41)-(5.45)$ in (5.40), it follows that

$$
\begin{equation*}
\left[\Phi_{u}\right]\{B\}=\left[\Phi_{v}\right]\{j A\} \tag{5.46}
\end{equation*}
$$

and from (5.39)

$$
\begin{equation*}
A_{k}=\frac{2 \pi}{\zeta_{0} \Psi_{d R} F_{0}(h)} V_{0 k} . \tag{5.47}
\end{equation*}
$$

The solutions of two important problems in linear array theory are readily obtained from (5.46) and (5.47). Case $I$ is concerned with specifying the driving-point $\dagger$ currents and determining the $N$ potentials $V_{0 k}$ required to maintain these currents. In case II the $N$ potentials $V_{0 k}$ are specified and the corresponding driving-point currents are determined.

In the zero-order current distribution (5.34), the coefficients $B_{i}$ are the amplitudes of the shifted cosine currents due to the distributed interaction of all elements of current in the array. The $A_{i}$ coefficients are determined completely by the voltages of the individual generators. The distribution of the current in the $i^{\text {th }}$ element (5.34) may be separated into its real and imaginary parts as follows:

$$
\begin{align*}
I_{z i}(z)= & j\left\{A_{i} \sin \beta_{0}(h-|z|)+B_{i I}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right\} \\
& +B_{i R}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)  \tag{5.48}\\
= & I_{z i}^{\prime \prime}(z)+j I_{z i}^{\prime}(z) . \tag{5.49}
\end{align*}
$$

At $z=0$, the real and imaginary parts of the driving-point current are

$$
\begin{gather*}
I_{z i}^{\prime \prime}(0)=B_{i R}\left(1-\cos \beta_{0} h\right)  \tag{5.50a}\\
I_{z i}^{\prime}(0)=A_{i} \sin \beta_{0} h+B_{i I}\left(1-\cos \beta_{0} h\right) . \tag{5.50b}
\end{gather*}
$$

The driving-point impedance and admittance under the two driving conditions can be computed from the following general formulas obtained by combining (5.46)-(5.50). (Note: A special form is convenient when $\beta_{0} h$ is at or near $\pi / 2$.)
$\dagger$ The term 'base current' is also used for driving-point current.

Case I Input currents specified

$$
\begin{equation*}
\left\{V_{0}\right\}=\frac{1}{c_{1}\left(1-\cos \beta_{0} h\right)}\left[\Phi_{u}\right]\left[\Phi_{w}\right]^{-1}\left\{I_{z}(0)\right\} \tag{5.51}
\end{equation*}
$$

where

$$
c_{1}=j 2 \pi /\left(\zeta_{0} \Psi_{d R} \cos \beta_{0} h\right)
$$

$$
\begin{equation*}
\left[\Phi_{w}\right]=\left[\Phi_{v}+\Phi_{u} \sin \beta_{0} h /\left(1-\cos \beta_{0} h\right)\right] . \tag{5.52}
\end{equation*}
$$

Case II Driving voltages specified

$$
\begin{equation*}
\left\{I_{z}(0)\right\}=c_{1}\left(1-\cos \beta_{0} h\right)\left[\Phi_{w}\right]\left[\Phi_{u}\right]^{-1}\left\{V_{0}\right\} . \tag{5.53}
\end{equation*}
$$

The matrix components in (5.51) and (5.53) as well as numerical values of the driving-point impedances and admittances under different driving conditions are given in tables in the appendix. These tables were extracted from a more complete table [1]. The forms of the current for specified driving-point voltages and currents are not generally the same since the $A_{i}$ and $B_{i}$ coefficients differ for the two cases.

The symmetry properties of the impedance matrix in (5.1) and its counterpart in (5.51) are not identical. The assumption of identical current distributions implies that the mutual impedance between any two elements in an array is only a function of the size and spacing of the elements. Thus, with identical elements in an array, elements with the same centre-to-centre spacing have the same value of mutual impedance. For example, in an array with elements equally spaced, $Z_{12}=Z_{23}=Z_{34}$ and $Z_{13}=Z_{24}=$ $Z_{46}$. The mutual impedance for elements near the centre of the array is then the same as for corresponding elements near the ends of the array. The new theory correctly shows that the coupling properties of an element in the array depend on the distribution of current and the location of every element in the array. Elements near the edges of an array are coupled differently from elements near the centre.

It is important to emphasize the significance of the results in (5.51) for Case I, and in (5.53) for Case II. Here, for the first time in the theory of linear arrays it is possible to solve for the actual driving potentials in a parallel array without requiring the currents in the elements to have identical distributions. Moreover, the interaction between the currents in all of the elements has been included in a zero-order approximation. Similarly, (5.53) makes it possible for the first time to specify the $N$ driving potentials and
determine the $N$ zero-order currents in a manner that includes the effect of their mutual interactions on their distributions.

The general $\Psi(z)$ functions, obtained from the defining integrals (5.24)-(5.27), are

$$
\begin{align*}
\Psi_{k i d u}(z)= & \frac{1}{\cos \beta_{0} z-\cos \beta_{0} h}\left\{\left[C_{b}(h, z)-C_{b}(h, h)\right]\right. \\
& \left.-\cos \beta_{0} h\left[E_{b}(h, z)-E_{b}(h, h)\right]\right\}  \tag{5.54}\\
\Psi_{k i d R}(z)= & \frac{1}{\sin \beta_{0}(h-|z|)} \operatorname{Re}\left\{\left[C_{b}(h, z)-C_{b}(h, h)\right] \sin \beta_{0} h\right. \\
& \left.-\left[S_{b}(h, z)-S_{b}(h, h)\right] \cos \beta_{0} h\right\} \\
\Psi_{k i d I}(z)= & \frac{1}{\cos \beta_{0} z-\cos \beta_{0} h} \operatorname{Im}\left\{\left[C_{b}(h, z)-C_{b}(h, h)\right] \sin \beta_{0} h\right. \\
& \left.-\left[S_{b}(h, z)-S_{b}(h, h)\right] \cos \beta_{0} h\right\}  \tag{5.56}\\
\Psi_{k i d v}(z)= & \frac{1}{\cos \beta_{0} z-\cos \beta_{0} h}\left\{\left[C_{b}(h, z)-C_{b}(h, h)\right] \sin \beta_{0} h\right. \\
& \left.-\left[S_{b}(h, z)-S_{b}(h, h)\right] \cos \beta_{0} h\right\}  \tag{5.57}\\
\Psi_{k i v}(h)= & C_{b}(h, h) \sin \beta_{0} h-S_{b}(h, h) \cos \beta_{0} h  \tag{5.58}\\
\Psi_{k i u}(h)= & C_{b}(h, h)-E_{b}(h, h) \cos \beta_{0} h \tag{5.59}
\end{align*}
$$

where in subscripts $b=b_{k i}$, and

$$
\begin{gather*}
S_{b}(h, z)=\int_{0}^{h} \sin \beta_{0} z^{\prime}\left[\frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+\frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] d z^{\prime}  \tag{5.60}\\
C_{b}(h, z)=\int_{0}^{h} \cos \beta_{0} z^{\prime}\left[\frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+\frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] d z^{\prime}  \tag{5.61}\\
E_{b}(h, z)=\int_{0}^{h}\left[\frac{e^{-j \beta_{0} R_{1}}}{R_{1}}+\frac{e^{-j \beta_{0} R_{2}}}{R_{2}}\right] d z^{\prime}  \tag{5.62}\\
R_{1}=\sqrt{\left(z-z^{\prime}\right)^{2}+b_{k i}^{2}}, \quad R_{2}=\sqrt{\left(z+z^{\prime}\right)^{2}+b_{k i}^{2}} \tag{5.63}
\end{gather*}
$$

The functions $S_{b}, C_{b}$ and $E_{b}$ are found in King $\dagger$ and are tabulated for a wide range of values of $h, z$ and $b$ by Mack [3]. In order to obtain satisfactory overall agreement, the $\Psi$ functions are evaluated at the point of maximum current. This ensures a good approximation for the determination of both the far field and the input power. However, the input susceptance may be somewhat in error. This does not present any practical difficulty since appropriate corrections may be applied at the driving point (cf. section 2.8).

From the results of chapter 1, the far-zone electric field depends upon the location of each element and its current distribution. Thus, for the geometry of Fig. 5.2,

$$
\begin{equation*}
E_{\Theta}(\Theta, \Phi)=j \frac{\omega \mu_{0} \sin \Theta}{4 \pi} \sum_{i=1}^{N} \frac{e^{-j \beta_{0} R_{i}}}{R_{i}} \int_{-h}^{h} I_{i}\left(z^{\prime}\right) e^{j \beta_{0} z^{\prime} \cos \Theta} d z^{\prime} \tag{5.64}
\end{equation*}
$$



Fig. 5.2. Coordinate system locating one element with respect to the centre 0 of a parallel array.

For the conventional sinusoidal distribution of current, the electric field is given by

$$
\begin{equation*}
E_{\Theta}(\Theta, \Phi)=j \frac{\zeta_{0}}{2 \pi} \sum_{i=1}^{N} \frac{e^{-j p_{0} R_{i}}}{R_{i}} I_{i}(0) F_{m}\left(\Theta, \beta_{0} h\right) \tag{5.65}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{m}\left(\Theta, \beta_{0} h\right)=\frac{\cos \left(\beta_{0} h \cos \Theta\right)-\cos \beta_{0} h}{\sin \Theta} \tag{5.66}
\end{equation*}
$$

In contrast, the new theory yields a far-zone electric field given by

$$
\begin{equation*}
E_{\Theta}(\Theta, \Phi)=j \frac{\zeta_{0}}{2 \pi} \sum_{i=1}^{N} \frac{e^{-j \beta_{0} R_{i}}}{R_{i}}\left[j A_{i} F_{m}\left(\Theta, \beta_{0} h\right)+B_{i} G_{m}\left(\Theta, \beta_{0} h\right)\right] \tag{5.67}
\end{equation*}
$$

where
$G_{m}\left(\Theta, \beta_{0} h\right)=\frac{\sin \beta_{0} h \cos \left(\beta_{0} h \cos \Theta\right) \cos \Theta-\cos \beta_{0} h \sin \left(\beta_{0} h \cos \Theta\right)}{\sin \Theta \cos \Theta}$.
A comparison of (5.65) with (5.67) shows the difference between the far-field pattern for the conventional and for the new theory. It is now of interest to study the differences between the radiation patterns determined by the two theories under different driving conditions.

### 5.3 Examples

Consider a three-element array with elements that are a full wavelength long $(2 h=\lambda)$ and separated by a quarter wavelength ( $b_{i, i+1}=\lambda / 4$ ). The conventional approach to this problem is doomed to failure when $\beta_{0} h=\pi$, since an assumed sinusoidal current (i.e. $I_{z}(z) \sim \sin \beta_{0}|z|$ ) is zero at the driving point. This gives rise to a zero admittance or an infinite impedance for each element in the array. This difficulty does not exist in the new theory. When $\beta_{0} h=\pi$, (5.5) gives the following form for the current:

$$
\begin{equation*}
I_{i}=j A_{i} \sin \beta_{0}\left|z_{i}\right|+B_{i}\left(\cos \beta_{0} z_{i}+1\right) \tag{5.69}
\end{equation*}
$$

At $z=0$, the current is finite and is given by the coefficient $B_{i}$ for each element in the array. In order to demonstrate the difference between the two antenna theories, the conventional approach will be used for $\beta_{0} h=3.157$ and compared to the results of the new theory for $\beta_{0} h=\pi$.

Consider now the three-element array shown in Fig. 5.1. Either the driving-point voltages or the driving-point currents may be specified. Conventionally the phases of the equal driving-point currents are specified to produce a radiation pattern. The electric field $E_{\Theta}$ in the far zone is given by (1.45) with (1.46). Thus,

$$
\begin{equation*}
E_{\Theta}=C \frac{\sin N x}{\sin x} \tag{5.70}
\end{equation*}
$$

where

$$
\begin{equation*}
x=n \sin \Theta \cos \Phi-t . \tag{5.71}
\end{equation*}
$$

For the array of Fig. 5.1, the number of elements is $N=3$. The distance between the elements in fractions of a wavelength, $n$, is chosen to be $\frac{1}{4}$. Now let attention be directed to the horizontal pattern in the equatorial or $H$-plane defined by $\Theta=\pi / 2$. The three driving-point currents are equal in magnitude but the phase delay $t$
between elements in fractions of a period may be varied to produce a particular pattern. For example, a value of $t=\frac{1}{4}$ will produce an endfire radiation pattern with the maximum value of the directivity $D$ toward the right in Fig. 5.1.

The actual radiation pattern of the three-element endfire array with specified base currents differs from the ideal pattern shown on the left of Fig. 5.3. The several components of current on the elements which are discussed later in this section, are equivalent to separate sources producing different radiation patterns. The additional patterns in the middle and right of Fig. 5.3 fill in the deep nulls and reduce the back-to-front ratio of the ideal pattern. Specifically, it can be shown that the actual pattern for the 3-element array is proportional to the sum


$$
+(-0.07+j 0.50) a_{p}\left(\pi / 2, \Phi ; 2, \frac{1}{2}, \frac{1}{2}\right)
$$



Fig. 5.3. Array factors which comprise actual three-element array factor.
The ideal radiation pattern as determined from (5.70) depends on the vertical field factor $F_{0}\left(\Theta, \beta_{0} h\right)$ of an isolated element. This is contained in $C$ in (5.70). Consider now an array with full-wave elements ( $\beta_{0} h=\pi$ ). The particular value of $F_{0}\left(\Theta, \beta_{0} h\right)$ is given by (5.66). Thus,

$$
\begin{gather*}
F_{0}(\Theta, \pi)=\frac{\cos (\pi \cos \Theta)+1}{\sin \Theta}  \tag{5.72a}\\
F_{0}\left(\frac{\pi}{2}, \pi\right)=2 \tag{5.72b}
\end{gather*}
$$

If the driving-point impedance for $\beta_{0} h$ is computed from the

EMF method, the numerical value is infinity. This follows because the sinusoidal current is zero at the driving point. As an illustrative comparison of the results of the present theory with other methods, an examination of the driving-point impedances for the threeelement array with $\beta_{0} h=\pi / 2$ and $\beta_{0} h=\pi$ is given. The self- and mutual impedances for $\beta_{0} h=\pi / 2$ by the EMF method and an assumed sinusoidal current are:

$$
\left.\begin{array}{rlrl}
Z_{11}=71 \cdot 13+j 42.54 \mathrm{ohms} & \\
Z_{12}=41.79-j 28.35 \mathrm{ohms}, & & \beta_{0} b=\pi / 2 \\
Z_{13}=-12.53-j 29.94 \mathrm{ohms}, & \beta_{0} b=\pi
\end{array}\right\},
$$

The driving-point currents are specified in the following way to produce an endfire pattern:

$$
\{I\}=\left\{\begin{array}{l}
I_{01}  \tag{5.74}\\
I_{02} \\
I_{03}
\end{array}\right\}=I_{01}\left\{\begin{array}{r}
1 \\
-j \\
-1
\end{array}\right\} .
$$

The substitution of (5.74) and (5.73a) into (5.1) yields the following driving-point impedances for the three elements:

$$
\left.\begin{array}{rl}
Z_{01}=Z_{11}-j Z_{12}-Z_{13} & =55 \cdot 31+j 30 \cdot 69 \mathrm{ohms}=V_{01} / I_{01}  \tag{5.75}\\
Z_{02}=Z_{11} & =71 \cdot 13+j 42 \cdot 54 \mathrm{ohms}=V_{02} / I_{02} \\
Z_{03}=Z_{11}+j Z_{12}-Z_{13} & =112 \cdot 0+j 114 \cdot 3 \mathrm{ohms}=V_{03} / I_{03}
\end{array}\right\}
$$

The same results are, of course, obtained when the driving voltages are assigned instead of the currents by the substitution of $V$ for $I$ in (5.74), since no changes are possible in the assumed distributions of current and, hence, in the mutual coupling.

An apparently obvious method for improving the accuracy of the conventional theory for the three-element endfire array is to make use of the accurate second-order iterated results for two coupled cylindrical antennas $\dagger$ by the simple expedient of applying them to the three elements treated as quasi-independent pairs. To be sure, this procedure implies that the distributions of current along elements 1 and 2 are the same when element 3 is present as when it is absent in the evaluation of $Z_{12}$, and that the currents along elements 1 and 3 are unaffected by the presence of element 2 .

[^4]Moreover, since $Z_{11}$ is not the same for the two distances $b_{12}$ and $b_{13}$, it is assumed that the value for the closer spacing is to be used. It is interesting to evaluate the driving-point impedances under these conditions. By interpolation from Fig. 8.13 on page 306 of King, Theory of Linear Antennas, the following values are obtained :

$$
\begin{array}{ll}
Z_{11}=81 \cdot 7+j 39 \cdot 3 \text { ohms }, & \beta_{0} b=\frac{\pi}{2}, \quad \Omega=2 \ln \frac{2 h}{a}=10 \\
Z_{12}=42 \cdot 6+j 42 \cdot 7 \text { ohms }, & \beta_{0} b=\frac{\pi}{2}
\end{array}
$$

(5.76a)

$$
\left.\begin{array}{lll}
Z_{11}=86 \cdot 1+j 42 \cdot 6 \text { ohms }, & \beta_{0} b=\pi, & \Omega=2 \ln \frac{2 h}{a}=10  \tag{5.76b}\\
Z_{13}=-15 \cdot 7-j 29 \cdot 0 \text { ohms }, & \beta_{0} b=\pi &
\end{array}\right\} .
$$

Note that the two values of $Z_{11}$ for $\beta_{0} b=\pi / 2$ and $\pi$ are not very different when $\beta_{0} h=\pi / 2$. The corresponding driving-point impedances, when $Z_{11}$ for $\beta_{0} b=\pi / 2$ is used, are

$$
\left.\begin{array}{l}
Z_{01}=54 \cdot 7+j 25 \cdot 7 \mathrm{ohms}, \quad \Omega=2 \ln \frac{2 h}{a}=10  \tag{5.76c}\\
Z_{02}=81 \cdot 7+j 39 \cdot 3 \mathrm{ohms} \\
Z_{03}=141 \cdot 0+j 111 \cdot 0 \mathrm{ohms}
\end{array}\right\} .
$$

These values do not differ from those obtained from the conventional theory by more than about $25 \%$. This is a consequence of the fact that the actual distributions of current on half-wave dipoles do not vary greatly from the sinusoidal so long as they are moderately thin.

It is now in order to examine the results obtained by the new two-term theory which takes full account of the changes in the distributions of current due to the presence of any number of coupled elements. The driving-point impedances for the three elements are readily computed. $\dagger$ They are

$$
\left.\begin{array}{l}
Z_{01}=67 \cdot 51+j 24 \cdot 14 \text { ohms, } \quad \Omega=2 \ln \frac{2 h}{a}=10  \tag{5.77}\\
Z_{02}=78 \cdot 47+j 31 \cdot 23 \mathrm{ohms} \\
Z_{03}=145 \cdot 61+j 96 \cdot 91 \mathrm{ohms}
\end{array}\right\}
$$

$+[1]$, p. 84.

These values are comparable with those in both (5.73a) and (5.75) (with differences not exceeding about $30 \%$ ) simply because the current in half-wave dipoles is predominantly sinusoidal with only relatively small changes due to finite radius and mutual coupling.

The situation is quite different when the elements are not near resonance. This is well illustrated with the same three-element array but now with $\beta_{0} h$ near $\pi$ instead of $\pi / 2$.

The conventional application of the EMF method with assumed sinusoidal currents on all elements yields meaningless results. Since the currents at all three driving points are identically zero all driving-point impedances are infinite-which is, of course, absurd.

If an attempt is made to use the second-order theory for coupled pairs, the following values are obtained from pages 297 and 298 of King, Theory of Linear Antennas, with $\beta_{0} h=3 \cdot 157$ :
$\left.\begin{array}{ll}Z_{11}=199-j 351 \text { ohms, } & \beta_{0} b=\frac{\pi}{2}, \quad \Omega=2 \ln \frac{2 h}{a}=10 \\ Z_{12}=116+j 187 \mathrm{ohms} & \\ Z_{11}=288-j 441 \text { ohms, } & \beta_{0} b=\pi, \Omega=2 \ln \frac{2 h}{a}=10 \\ Z_{13}=122-j 15 \text { ohms }\end{array}\right\}$.
Note that the difference in the self-impedances $Z_{11}$ when the spacing is changed from $\beta_{0} b=\pi / 2$ to $\beta_{0} b=\pi$ is very much greater when $\beta_{0} h=3.157$ than when $\beta_{0} h=\pi / 2$. If it is again assumed in this rough approximation that the value for the closer elements ( $\beta_{0} b=\pi / 2$ ) is to be used, the three driving-point impedances become

$$
\left.\begin{array}{l}
Z_{01}=264-j 452 \mathrm{ohms}, \quad \Omega=2 \ln \frac{2 h}{a}=10  \tag{5.79}\\
Z_{02}=199-j 351 \text { ohms } \\
Z_{03}=-110-j 220 \mathrm{ohms}
\end{array}\right\}
$$

The negative input resistance for element 3 indicates that its generator supplies no power to the array but acts as a load to absorb power received by coupling from the other two elements.

Once again it is in order to introduce the results from the new two-term theory which actually determines the distributions of the currents on all three elements and the associated driving-point impedances. The following values are readily calculated $\dagger$ for the $\dagger[1]$ p. 203.
driving-point currents specified in (5.74):
$\left.\begin{array}{ll}Z_{01}=612-j 591 \mathrm{ohms} & Y_{01}=(0.845+j 0 \cdot 817) \times 10^{-3} \mathrm{mhos} \\ Z_{02}=160-j 590 \mathrm{ohms} & Y_{02}=(0.429+j 1 \cdot 578) \times 10^{-3} \mathrm{mhos} \\ Z_{03}=61 \cdot 5-j 435 \mathrm{ohms} & Y_{03}=(0.318+j 2 \cdot 252) \times 10^{-3} \mathrm{mhos}\end{array}\right\}$.

A comparison of these values with those in (5.79) shows no agreement. Clearly, the notion that coupled antennas in an array of more than two elements may be treated as independent pairs in determining mutual impedance is unreliable. The previously shown rough agreement for half-wave dipoles is a special case that must not be assumed to have any general significance for other lengths. The coupling between any two elements depends on their relative positions in an array and only the results in (5.77) and (5.80) include this correctly.

The voltages required to maintain the currents specified in (5.74) when $\beta_{0} h=\pi$ and $\beta_{0} b=\pi / 2$, i.e. $I_{02}=-j I_{01}, I_{03}=-I_{01}$, are $V_{01}=612-j 591, \quad V_{02}=-590-j 160, \quad V_{03}=-61 \cdot 5+j 435$. The power supplied each element $k$ by its generator is

$$
P_{k}=\left|I_{0 k}\right|^{2} R_{0 k}=\left|V_{0 k}\right|^{2} G_{0 k} .
$$

The ratios of the powers supplied to the three-element array are

$$
\begin{equation*}
P_{1} / P_{3}=9.82, \quad P_{2} / P_{3}=2.51 \tag{5.81}
\end{equation*}
$$

Evidently element 1 receives almost ten times the power that is supplied to the terminals of element 3.

The new theory gives the following values $\dagger$ of $Z_{0 i}$ and $Y_{0 i}$, $i=1,2,3$ when the driving-point voltages are specified ( $V_{02}=-j V_{01}, V_{03}=-V_{01}$ ) instead of the currents:
$\left.Z_{01}=675-j 484 \mathrm{ohms} \quad Y_{01}=(0.979+j 0.701) \times 10^{-3} \mathrm{mhos}\right)$
$\left.Z_{02}=359-j 479 \mathrm{ohms} \quad Y_{02}=(1.003+j 1.336) \times 10^{-3} \mathrm{mhos}\right\}$.
$Z_{03}=170-j 426$ ohms $\quad Y_{03}=(0.808+j 2 \cdot 024) \times 10^{-3}$ mhos $)$
Clearly, the results for Cases I and II are not the same as seen from a comparison of (5.80) and (5.82). This difference is due to the unequal distributions of the currents in the elements which cause non-uniform coupling. This effect will become clearer when the currents in the individual elements are examined.

The conventional currents in the three-element endfire array †[1], p. 221.
with $\beta_{0} h=3.157$ are

$$
\begin{align*}
& I_{1}(z)=I_{2}(z)=I_{3}(z)=I_{0 i} \sin \left(3 \cdot 157-\beta_{0}|z|\right) \\
& \quad \text { driving-point currents specified }  \tag{5.83}\\
& I_{i}(z)=V_{i} Y_{i} \sin \left(3 \cdot 157-\beta_{0}|z|\right), \quad i=1,2,3 \\
& \quad \text { driving voltages specified. } \tag{5.84}
\end{align*}
$$

The form of the currents in (5.83) and (5.84) is identical for each element. Both the real and imaginary parts have the same distribution. The currents in the new theory are given by (5.69) with ( 5.46 ) and (5.47). They are shown in Figs. 5.4 and 5.5 for the two different driving conditions. When the currents at $z=0$ are specified, the distributions differ widely in form from element to element. Note that the currents are shown both with respect to the individual driving voltages and with respect to $V_{02}$. In the computation of radiation patterns the currents must all be normalized with respect to a single driving voltage. The large differences in the real and


Fig. 5.4 Three-element endfire array; driving-point currents specified. Drawn with respect to (a) individual driving voltages, (b) $V_{02},\left(\lambda / 4\right.$ spacing, $\left.\beta_{0} h=\pi, \Omega=10\right)$.
imaginary parts of the currents in Fig. 5.4 practically disappear when the driving voltages are specified in Fig. 5.5.


Fig. 5.5. Three-element endfire array; driving voltages specified. Drawn with respect to (a) individual driving voltages, (b) $V_{02}$, ( $\lambda / 4$ spacing, $\left.\beta_{0} h=\pi, \Omega=10\right)$.

### 5.4 Electronically scanned arrays

Previous sections of this book have demonstrated the general invalidity of the assumption of equal current distributions in the elements of an array. A most significant result of the new theory is that the expected conventional radiation pattern is not achieved since the contributions by the individual elements to the radiation pattern are different. The results of the new theory for the broadside and endfire arrays show an appreciable difference not only between the driving-point impedances for the broadside and endfire arrays, but between the conventional and new theories. The experimental determination of the individual driving-point impedances is a complicated problem and a theoretical prediction of the individual circuit properties would certainly be an aid in the efficient operation of an array.

A comparison of the corresponding expressions for the far fields based on the conventional method and the new, more accurate approach, helps to illustrate some of the problems in the theory of scanned arrays. Consider an array in which the currents at the driving points of the elements are specified in both amplitude and phase. For the present, let the amplitudes be equal and the phases required to change linearly from element to element across the array. For example, the base current might increase in phase by $30^{\circ}$ toward one end of the array. Expressed in general terms

$$
\begin{equation*}
I_{i}=I_{0} e^{-j \delta_{i}}=I_{0} e^{-j 2 \pi i t} \tag{5.85}
\end{equation*}
$$

where $t$ is the time delay between elements in fractions of a period.
With the currents at $z=0$ specified in (5.85) and under the assumption of identical distributions of current, the far-zone electric field from (1.44) has the form
$E_{\Theta}^{r}=\left\{\frac{j \zeta_{0} I_{0}}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} F_{0}\left(\Theta, \beta_{0} h\right)\right\}\left\{1+\sum_{i=1}^{\frac{1}{2}(N-1)}\left[e^{-j\left(\delta_{i}-\beta_{0} S_{i}\right)}+e^{j\left(\delta_{i}-\beta_{0} S_{i}\right)}\right]\right\}$.

The second term in braces in (5.86) is the familiar array factor given by ( 5.70 ). If the distance between the elements is small enough, the radiation pattern has only one principal maximum in the visual ranges of $\Theta$ and $\Phi$. The first maximum of (5.70) occurs when $x=0$. Thus, to direct the main beam in a specific direction $\left(\Theta_{m}, \Phi_{m}\right)$ in space, the time delay between the currents in the elements must be set equal to a particular value $t_{m}$ such that

$$
n \sin \Theta_{m} \cos \Phi_{m}-t_{m}=0
$$

or

$$
\begin{equation*}
t_{m}=n \sin \Theta_{m} \cos \Theta_{m} . \tag{5.87}
\end{equation*}
$$

For example, in an array with half-wave spacing ( $n=\frac{1}{2}$ ) for which the main beam is to point in the direction $\Theta_{m}=\pi / 2$ ( $H$-plane) and $\Phi_{m}=60^{\circ}$, the required phase shift given by (5.87) is $t_{m}=\pi / 4$ radians. In a single curtain array it is not possible with ordinary elements to have any control over the beam pointing in the $\Theta$ direction. The control of the main beam in the $\Theta$ direction could be achieved by a planar array formed by an array of collinear elements.

Now let the conventional requirement, that the distributions of current be equal, be removed in (1.44). Let the currents at $z=0$
again be specified so that, on the basis of the conventional theory, the main beam will point in a desired direction. However, and this obvious fact is often overlooked, the specification of the currents at each driving point usually does not determine the entire current along each element. A variety of distributions of current may be associated with any given value at $z=0$. In general, the radiation pattern can be considered the superposition of two parts. One part is the pattern of an array of elements with equal distributions of current ; the other the pattern of the same array with dissimilar distributions of current. Conventional theories assume that the first part is the entire pattern.

The beam-pointing properties of a scanning array are affected by the interaction between the currents in the elements. The simple array factor in (5.70) characterizes an ideal array in which the exact phase and amplitude of the current are specified for each element. This specification applies not only at $z=0$ but all along each element. The phase of the current is of primary importance in the determination of the direction of the main beam. In an actual array the variation in phase along the length of each individual element differs from element to element. In practice, this variation is responsible for a beam-pointing error of non-negligible value. Furthermore, with this phase variation perfect phase cancellation and addition is impossible. Perfect nulls in the radiation pattern will disappear and side-lobe levels will be modified significantly. The side-lobe level and the angular width of the main beam are also affected by changes in the magnitudes of the currents from element to element across the array.

As a specific example, consider the three-element array with fullwave elements ( $\beta_{0} h=\pi$ ) and half-wave spacing ( $\beta_{0} b=\pi / 2$ ). The driving-point currents are specified to produce a maximum field in the direction indicated by the conventional theory. The drivingpoint impedance is to be calculated for each element as a function of the scanning angle. The actual position of the maximum, as given by the new theory, is to be compared with the corresponding angular position predicted by the conventional theory. The difference is the beam-pointing error $\Delta$.

The general matrix relation (5.51) between the driving-point voltages and currents may be reduced to the following symbolic form:

$$
\begin{equation*}
\left\{V_{0}\right\}=\left[\Phi_{T}\right]\left\{I_{0}\right\} \tag{5.88}
\end{equation*}
$$

where $\quad\left\{V_{0}\right\}=\left\{\begin{array}{c}V_{01} \\ V_{02} \\ \vdots \\ V_{0 N}\end{array}\right\}, \quad\left\{I_{0}\right\}=\left\{\begin{array}{c}I_{01} \\ I_{02} \\ \vdots \\ I_{0 N}\end{array}\right\}$
and $\quad\left[\Phi_{T}\right]=\left[\begin{array}{cccc}\Phi_{T 11} & \Phi_{T 12} & \ldots & \Phi_{T 1 N} \\ \Phi_{T 21} & \Phi_{T 22} & \ldots & \Phi_{T 2 N} \\ \vdots & & & \\ \Phi_{T N 1} & \Phi_{T N 2} & \ldots & \Phi_{T N N}\end{array}\right]=c_{1}\left[\Phi_{u}\right]\left[\Phi_{w}\right]^{-1}(5.90)$
[cf. (5.52)] where

$$
\left[\Phi_{w}\right]=\left[\Phi_{v}+\Phi_{u} \sin \beta_{0} h /\left(1-\cos \beta_{0} h\right)\right]
$$

and

$$
c_{1}=j 2 \pi / \zeta_{0} \Psi_{d R} \cos \beta_{0} h
$$

The elements of $\Phi_{T}$ may be derived from the machine-tabulated values of $\Phi_{u}$ and $\Phi_{v}$ in the tables [appendix III]. From the tabulated values for the two different sets of driving conditions [appendix IV] and a knowledge of the symmetry properties of the $\Phi_{u}$ and $\Phi_{v}$ values, the $\Phi_{T}$ values may also be calculated. In the present example calculations similar to those given in appendix IV yield the following information for the case $\beta_{0} h=\pi, \beta_{0} b=\pi / 2$ :

When

$$
\left\{I_{0}\right\}=\left\{\begin{array}{r}
1  \tag{5.91}\\
-j \\
-1
\end{array}\right\}
$$

then

$$
\left\{V_{0}\right\}=\left\{\begin{array}{c}
612-j 591  \tag{5.92}\\
-590-j 160 \\
-61 \cdot 5+j 435
\end{array}\right\}
$$

Also when

$$
\begin{align*}
& \left\{I_{0}\right\}=\left\{\begin{array}{l}
1 \\
1 \\
1
\end{array}\right\}  \tag{5.93}\\
& \left\{V_{0}\right\}=\left\{\begin{array}{l}
435-j 346 \\
309-j 37 \cdot 9 \\
435-j 346
\end{array}\right\} . \tag{5.94}
\end{align*}
$$

The specifications in (5.91) and (5.93) are the conventional ones for the endfire and broadside arrays. For $\beta_{0} h=\pi, \beta_{0} b=\pi / 2$, the
time delay between elements as given by (5.87) is

$$
\begin{equation*}
t_{m}=n \cos \Phi_{m}=\frac{1}{4} \cos \Phi_{m} \tag{5.95}
\end{equation*}
$$

The driving-point currents in the $N$ elements are expressed in terms of the angle $\Phi_{0}$ by the substitution of (5.95) in (5.85). The result with $I_{0}=1$ is

$$
I_{i}=e^{-j 2 \pi i n \cos \Phi_{m}}=e^{-j(\pi / 2) i \cos \Phi_{m}}, \quad n=\frac{1}{4}, \quad i=1,2,3 . \quad \text { (5.96) }
$$

The following table is useful for the computation of the drivingpoint impedances for different values of the angle $\Phi_{m}$.

Table 5.1

| $\Phi_{m}$ | 0 | $30^{\circ}$ | $60^{\circ}$ | $75^{\circ}$ | $90^{\circ}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $I_{1}$ | 1 | $1+j 0$ | $1+j 0$ | $1+j 0$ | 1 |
| $I_{2} / I_{1}$ | $-j$ | $0.209-j 0 \cdot 978$ | $0.707-j 0.707$ | $0.919-j 0 \cdot 395$ | 1 |
| $I_{3} / I_{1}$ | -1 | $-0.913-j 0 \cdot 409$ | $-0.707-j 0.707$ | $0.687-j 0.726$ | 1 |

Before the driving-point impedances can be computed the elements of the matrix $\Phi_{T}$ must be found. They can be computed directly from the basic matrix equations in terms of $\Phi_{u}$ and $\Phi_{v}$, or they may be computed from the tables of driving-point impedances for different driving conditions. For example, from the two sets of information contained in (5.92) and (5.94), the symbolic matrix multiplication (5.88) yields

$$
\left.\begin{array}{rlr}
\Phi_{T 11}+\Phi_{T 12}+\Phi_{T 13} & =435-j 346 \\
2 \Phi_{T 21}+\Phi_{T 22} & =309-j 37 \cdot 9 \\
\Phi_{T 11}-j \Phi_{T 12}-\Phi_{T 13} & & =612-j 591 \\
& -j \Phi_{T 22} & =-590-j 160 \\
-\Phi_{T 11}-j \Phi_{T 12}+\Phi_{T 13} & & =-61 \cdot 5+j 435
\end{array}\right\}
$$

where

The symmetry properties of the elements of (5.98) were deduced from those of the component matrices involved in (5.88). The elements of (5.98) may be compared to the impedance matrix whose elements are the self- and mutual impedances computed under the conventional assumptions. For example, $\Phi_{T 11}$ could be compared to $Z_{11}$, the self-impedance of the first antenna. The result shown symbolically in (5.98) indicates that the off-diagonal terms are not necessarily equal (e.g. $\Phi_{T 12} \neq \Phi_{T 21}$ ) and that the diagonal
terms may differ (e.g. $\Phi_{T 11} \neq \Phi_{T 22}$ ). The numerical values of the matrix elements of $\Phi_{T}$ are

$$
\left.\begin{array}{l}
\Phi_{T 11}=347-j 567 \\
\Phi_{T 22}=160-j 590 \\
\Phi_{T 12}=77 \cdot 9+j 275  \tag{5.99}\\
\Phi_{T 21}=74 \cdot 3+j 276 \\
\Phi_{T 13}=10 \cdot 4-j 53 \cdot 7 .
\end{array}\right\}
$$

Consider the specific case $\Phi_{m}=75^{\circ}$ where the driving-point currents are given in Table 5.1 and the elements of the $\Phi_{T}$ matrix are given by (5.99). Thus,

$$
\left\{\begin{array}{l}
V_{01}  \tag{5.100}\\
V_{02} \\
V_{03}
\end{array}\right\}=\left[\begin{array}{lll}
\Phi_{T 11} & \Phi_{T 12} & \Phi_{T 13} \\
\Phi_{T 21} & \Phi_{T 22} & \Phi_{T 21} \\
\Phi_{T 13} & \Phi_{T 12} & \Phi_{T 11}
\end{array}\right]\left\{\begin{array}{c}
1+j 0 \\
0.919-j 0 \cdot 395 \\
0 \cdot 687-j 0 \cdot 726
\end{array}\right\} I_{1}
$$

To compute, for example, $Z_{02}=\left(V_{02} / I_{2}\right)$, the quantity ( $V_{02} / I_{1}$ ) is computed from (5.100) or $\left(V_{02} / I_{1}\right)=\Phi_{T 21}(1+j 0)+\Phi_{T 22}(0.919-$ $j 0 \cdot 395)+\Phi_{T 21}(0 \cdot 687-j 0 \cdot 726)=240-j 193$. The driving-point impedance $Z_{02}$ is found from the substitution of the relation $I_{2}=(0.919-j 0.395) I_{1}$ in this expression with the result, $Z_{02}=$ 297-j82.8. The variation of the driving-point resistance and reactance with the beam-pointing angle $\Phi_{m}$ is shown in Fig. 5.6.


Fig. 5.6. Variation of driving-point resistance and reactance with beam-pointing angle $\boldsymbol{\Phi}_{m}$ for 3 -element array ( $\lambda / 4$ spacing, $\beta_{0} h=\pi, \Omega=10$ ).

It is seen that even if the beam-pointing angle is restricted to moderate departures from a normal position, significant changes in the impedance function occur. These will be apparent in the mismatch between the generator and the antenna. Note also that from symmetry, the continuation of $R_{01}$ and $X_{01}$ in the range $90^{\circ} \leqslant \Phi_{m} \leqslant 180^{\circ}$ is the mirror image of $R_{03}$ and $X_{03}$ about $\Phi_{m}=90^{\circ}$.

The driving-point currents have been specified according to the criteria of the conventional theory. This specification does not control the distribution of either the phase or the amplitude of the current away from the driving point. As a result, the location of the maximum of the main beam may differ from that predicted by the ideal angle in (5.87). This difference is the beam-pointing error $\Delta$ and represents the difference between the ideal scanning angle $\Phi_{m}$ and the actual angle $\Phi_{a}$.

The far-zone electric field is given by (5.67). The computation of this field requires all currents to be normalized with respect to a single driving voltage. Thus, with the $k^{\text {th }}$ element as a reference, (5.67) may be rearranged to give

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta)=\frac{j \zeta_{0}}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} \sin \Theta \sum_{i=1}^{N} C_{i} e^{j \beta_{0} b[(N-2 i+1) / 2] \cos \Phi \sin \Theta} \tag{5.101}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{i}=\xi_{i}\left[\frac{-j}{60 \Psi_{d R}} F_{m}\left(\Theta, \beta_{0} h\right)+\frac{Y_{i}}{2} G_{m}\left(\Theta, \beta_{0} h\right)\right] \tag{5.102}
\end{equation*}
$$

and

$$
\begin{equation*}
\xi_{i}=V_{0 i} / V_{0 k} . \tag{5.103}
\end{equation*}
$$

The conventional theory equates the $C$ coefficient in (5.101) to the driving-point currents [cf. (5.65)]. These, in turn, are chosen to produce a given radiation pattern. The new theory has shown that the currents in the elements as well as the radiation pattern cannot be specified merely by adjusting the currents at the driving point. Moreover, the direction of the main beam may differ considerably from the value predicted by the conventional theory.

The true location of the principal lobe is found from the location of the major maximum of $\left|E_{\Theta}^{r}(\Theta)\right|$ or of $\left|E_{\Theta}^{r}(\Theta)\right|^{2}$. For the special case $\Theta=\pi / 2, \beta_{0} b=\pi / 2, N=3$, and $\beta_{0} h=\pi / 2$, the electric field in the far zone is
where

$$
\begin{gather*}
E_{\Theta}^{r}(\Theta)=K\left(C_{1} e^{-j u}+C_{2}+C_{3} e^{j u}\right)  \tag{5.104}\\
K=\frac{j \zeta_{0}}{2 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}}
\end{gather*}
$$

$$
\begin{aligned}
u & =\beta_{0} b \cos \Phi=\frac{\pi}{2} \cos \Phi \\
\Theta & \left.=\frac{\pi}{2} \text { (H-plane }\right) .
\end{aligned}
$$

The square of the absolute value of $E_{\Theta}^{r}(\Theta)$ is formed from (5.104) with the result

$$
\begin{align*}
E_{\boldsymbol{\Theta}}^{r} E_{\boldsymbol{\Theta}}^{r *}= & K K^{*} C_{2} C_{2}^{*}\left[C_{12}^{*} C_{32} e^{j 2 u}+\left(C_{12}^{*}+C_{32}\right) e^{j u}\right. \\
& +\left(1+C_{12} C_{12}^{*}+C_{32} C_{32}^{*}\right) \\
& +\left[\left(C_{12}+C_{32}^{*}\right) e^{-j u}+C_{12} C_{32}^{*} e^{-j 2 u}\right] \tag{5.105}
\end{align*}
$$

where

$$
\begin{equation*}
C_{12}=\frac{C_{1}}{C_{2}} \quad \text { and } \quad C_{32}=\frac{C_{3}}{C_{2}} . \tag{5.106}
\end{equation*}
$$

With the substitution $x=e^{j u}$, (5.105) is seen to be an algebraic equation of fourth degree. Thus,

$$
\begin{equation*}
\left|E_{\boldsymbol{\theta}}^{r}(x)\right|^{2}=C_{a} x^{4}+C_{b} x^{3}+C_{c} x^{2}+C_{d} x+C_{e} . \tag{5.107}
\end{equation*}
$$

The true location of the principal lobe is determined from the equation obtained when (5.107) is differentiated with respect to $x$ and equated to zero. The computed beam-pointing error for the


Fig. 5.7. Variation of beam-pointing error with beam-pointing angle for 3 -element array ( $\lambda / 4$ spacing, $\beta_{0} h=\pi, \Omega=10$ ).
three-element array as determined from the conventional theory is shown in Fig. 5.7. This graph shows an appreciable plus and minus variation over most of the visible range of $\Phi$.

The expression for the square of the absolute value of the farfield for the $N$-element array is

$$
\begin{equation*}
\left|E_{\Theta}^{r}(x)\right|^{2}=K K^{*} \sum_{i} \sum_{n} C_{i} C_{n}^{*} x^{(i-n)} . \tag{5.108}
\end{equation*}
$$

The location of the extremes of (5.108) is given by

$$
\begin{equation*}
\frac{\partial}{\partial x}\left|E_{\theta}^{r}(x)\right|^{2}=K K^{*} j \sum_{i} \sum_{n}(i-n) C_{i} C_{n}^{*} x^{(i-n)}=0 \quad x=x_{0}, x_{1}, x_{2} \ldots \tag{5.109}
\end{equation*}
$$

### 5.5 Examples of the general theory for large arrays

Thus far the simple array with $N=3$ and $\beta_{0} b=\pi / 2$ has been examined for a variety of driving-point conditions. Calculations have also been made for arrays with a larger number of elements. For these the lengths $2 h$ of the elements were varied from a quarter to a full wavelength. The driving-point voltages or currents were specified according to conventional array theory to produce a broadside or endfire radiation pattern.

The driving-point currents required for an ideal broadside array are

$$
\begin{equation*}
I_{z 1}(0)=I_{z 2}(0)=I_{z 3}(0), \quad \text { etc. } \tag{5.110}
\end{equation*}
$$

or, in matrix form,

$$
\left\{I_{z}(0)\right\}=I_{z 1}(0)\left\{\begin{array}{l}
1  \tag{5.111}\\
1 \\
\vdots
\end{array}\right\} .
$$

Alternatively, the driving voltages may be assigned as follows:

$$
\{V\}=V_{1}\left\{\begin{array}{c}
1  \tag{5.112}\\
1 \\
1 \\
\vdots
\end{array}\right\}
$$

The relatively large sinusoidal parts of the currents on the antennas are determined directly from (5.47) by the specification of the
voltages. However, the relations (5.50a) and (5.50b) between the coefficients $A, B_{R}, B_{I}$ and the currents at $z=0$ do not in general suffice to determine the distributions of current along the elements.

The driving-point impedances for broadside arrays are shown in Figs. 5.8 to 5.10 . Driving-point currents and voltages are specified for arrays of up to 25 elements ( $N \leqslant 25$ ) for quarter and halfwavelength spacings ( $\beta_{0} b=\pi / 2$ and $\beta_{0} b=\pi$ ). In Figs. $5.8 a-d$ are shown graphs of the resistances and reactances of the individual elements of a broadside array when the driving-point currents are specified. In Figs. $5.8 a$ and $5.8 b$ the distance between the adjacent antennas is one-quarter wavelength; the lengths of the elements are, respectively, a quarter and a half wavelength. In Figs. 5.8c and $5.8 d$ the spacing of the elements has been increased to a half wavelength.

Since the main beam of a broadside array is at right angles to the curtain of antennas, it is to be expected that the effect of mutual coupling will be much less than for an endfire array. However, when the elements are separated by only a quarter wavelength, differences in the interactions between the currents in differently situated elements are sufficient to produce small but significant changes in the resistances even when the elements are as short as a quarter wavelength (Fig. 5.8a). In this case there is only a very small variation in the reactance. When the length of the elements is increased to a half wavelength with the same quarter wavelength spacing, both resistance and reactance vary greatly from element to element (Fig. 5.8b). Note that the change in the reactance from the central element in the array to one at the extremities may be as large as from near 100 ohms to near zero.

As is to be expected, an increase in the spacing of the elements to a half wavelength substantially reduces the changes in resistance and reactance due to differences in mutual interaction. When $2 h=\lambda / 4$ both resistance and reactance are substantially constant across the array (Fig. $5.8 c$ ). When $2 h=\lambda / 2$ significant differences in both resistance and reactance exist, but they are much smaller than for the more closely spaced array (Fig. 5.8b). In all cases, the obviously different environment of elements at the extremities of the array is responsible for the largest differences in the impedances. For the two lengths, $2 h=\lambda / 4$ and $2 h=\lambda / 2$, there is little difference between the results obtained with specified voltages and with specified driving-point currents.


(a)



(b)


Fig. 5.8. Driving-point impedances for broadside arrays, current specified; $\beta_{0} h=\pi / 4$ and $\pi / 2, N=5,9,15$ and 25


Fig. 5.9. Driving-point impedances for broadside arrays, current specified : $\beta_{0} h=3 \pi / 4$
and $\pi, N=5,9,15$ and 25 .


Fig. 5.10. Driving-point impedances for broadside arrays, voltage specified: $\beta_{0} h=3 \pi / 4$
and $\pi, N=5,9,15$ and 25 .

Graphs of the resistances of the individual antennas in a broadside array of three-quarter and full wavelength elements are shown in Figs. $5.9 a-d$ when the driving-point currents are specified. Similar curves for the same array with the voltages specified are in Figs. $5.10 a-d$. Especially noteworthy when $2 h=3 \lambda / 4$ are the large differences between the resistances and reactances of the elements when the driving-point voltages are specified instead of the drivingpoint currents (Figs. 5.9a, c and 5.10a, c). When $2 h=\lambda$ the reactance and to a lesser extent the resistance of the elements at the extremities of the array differ greatly from the others (Figs. 5.9b, $d$ and $5.10 b, d)$. As an example of typical digital results prepared for this study, a table of impedances is given in appendix IV.

The radiation patterns in the equatorial or $H$-plane are shown in Fig. 5.11 for a broadside array of 15 elements. The ideal patterns are fairly well approximated when the amplitude and phase of the current along each antenna are specified near the point of maximum amplitude. For the array of half-wave dipoles this occurs essentially


Fig. 5.11. Field patterns in the $H$-plane for a 15 -element broadside array.


Fig. 5.12. Driving-point impedances for endfire arrays, current specified; $\beta_{0} h=\pi / 4$


Fig. 5.13. Driving-point impedances for endfire arrays, current specified; $\beta_{0} h=3 \pi / 4$
and $\pi, N=5,9,15$ and 25 .
when the driving-point currents are specified, for the full-wave dipoles when the voltages are specified. On the other hand, when the current is not specified at the maximum, the actual pattern differs considerably from the ideal especially in the region of the minima (nulls). This is true when the driving-point currents are specified for the full-wave elements and when the voltages are specified for the half-wave elements.

In endfire arrays the currents are adjusted to produce the main beam of the radiation pattern along the line of the elements. For the unilateral endfire array there is a single major lobe in the direction $\Phi=0$; for the bilateral endfire array there are two major lobes, one in the direction $\Phi=0$, the other in the opposite direction, $\Phi=180^{\circ}$. Whereas in the broadside array the interaction between all but the next adjacent elements is quite small owing to extensive cancellation of the fields of the several elements in both directions along the line of the array; exactly the opposite is true for the endfire array. In the unilateral endfire array there is a cumulative reenforcement of the fields due to the several elements in one direction from one end of the array to the other, a more or less complete cancellation in the opposite direction. In the bilateral array the cumulative re-enforcement is in both directions. It is to be expected, therefore, that mutual coupling between neighbouring and even quite widely separated elements must play a major role in determining the amplitude, phase, and distribution of each current.

In an ideal endfire array the currents must all be equal in amplitude and vary progressively in phase by an amount equal to the electrical distance between the elements. The specifications for an unilateral endfire array are

$$
\left\{I_{z}(0)\right\}=I_{z 1}(0)\left\{\begin{array}{c}
1  \tag{5.113}\\
-j \\
-1 \\
\vdots
\end{array}\right\}, \quad \beta_{0} b=\frac{\pi}{2}
$$

For the bilateral array,

$$
\left\{I_{z}(0)\right\}=I_{z 1}(0)\left\{\begin{array}{r}
1  \tag{5.114}\\
-1 \\
1 \\
\vdots
\end{array}\right\}, \quad \beta_{0} b=\pi
$$

Alternatively, the voltages may be specified in the same manner. Thus, for the unilateral array

$$
\{V\}=V_{1}\left\{\begin{array}{c}
1  \tag{5.115}\\
-j \\
-1 \\
\vdots
\end{array}\right\}, \quad \beta_{0} b=\frac{\pi}{2}
$$

For the bilateral array,

$$
\{V\}=V_{1}\left\{\begin{array}{r}
1  \tag{5.116}\\
-1 \\
1 \\
\vdots
\end{array}\right\}, \quad \beta_{0} b=\pi
$$

The resistances and reactances of the individual elements in a unilateral endfire array are shown in Figs. $5.12 a$ and $5.12 b$, respectively with $2 h=\lambda / 4$ and $2 h=\lambda / 2$. The driving-point currents were specified according to (5.113). Corresponding values for the bilateral array are in Figs. $5.12 c$ and $5.12 d$. Note that these are symmetrical with respect to the centre of the array. For the shorter elements ( $2 h=\lambda / 4$ ), the reactances of all elements are reasonably alike; the resistances also vary little except for the two elements at the ends of the unilateral array. When the elements are a halfwavelength long, the resistances and reactances both vary greatly along the unilateral array (Fig. 5.12b), moderately along the bilateral array (Fig. 5.12d). It is interesting to note that in the unilateral array the impedance of the forward element (in the direction of the beam) is greatest, that of the rear element smallest. Since the amplitudes of the driving-point currents are all the same, the power supplied to each element is proportional to its resistance. It follows from Fig. $5.12 b$ that the power supplied to and radiated from the forward element is approximately five times that supplied to and radiated from the rear element. Note that the resistance and the reactance of all but the last two elements in each array are significantly greater than for an isolated antenna. In effect, each element after the forward one acts partly as a driven element, partly as a parasitic reflector for the element in front of it.
(a)


Fig. 5.14. Driving-point impedances for endfire arrays, voltage specified: $\beta_{0} h=3 \pi / 4$ and

The resistances of the antennas in a unilateral endfire array with elements of length $2 h=3 \lambda / 4$ (Fig. 5.13a) decrease continually from the forward element to that in the rear in a manner resembling that for the half-wave elements (Fig. 5.12b). However, the range of magnitudes is much greater. The corresponding values for the same array but constructed of full-wave elements $(2 h=\lambda)$ are in Fig. 5.13b. They are startlingly different. The resistances of all elements are now reasonably alike except for that of the rear element which is much greater. Evidently, the rear element is supplied and radiates the most power-approximately four to six times as much as any other element. This suggests that all but the rear element act in part as driven radiators and in part as parasitic directors for the elements behind them, especially the rear one. Note that for the bilateral array of full-wave elements (Fig. 5.13d) the resistances of the elements increase from the centre outward, whereas for the corresponding array of half-wave elements (Fig. $5.12 d$ ) the resistances decrease from the centre outward. If the voltages are specified according to (5.115) and (5.116) instead of the


Fig. 5.15. Field patterns in the $H$-plane for a 15 -element unilateral endfire array.
driving-point currents, the graphs of Figs. $5.13 a-d$ are replaced by those of Figs. 5.14a-d. The two sets are seen to differ considerably.

The radiation patterns in the equatorial or $H$-plane are shown in Figs. 5.15 and 5.16 respectively for the unilateral and bilateral endfire arrays. The ideal pattern is fairly well approximated when the current along each antenna is specified near its point of maximum according to the criteria for an ideal array. For the half-wave dipoles this is true essentially when the driving-point currents $I(0)$ are specified, for the full-wave dipoles when the voltages are assigned. On the other hand, when the current is not specified at its maximum value, the actual pattern differs considerably from the ideal, especially in its minor lobe structure and the region of the minima (nulls). This is true when the driving-point currents are specified for the full-wave dipoles, when the voltages are specified for the half-wave dipoles. In general, the departure from the ideal patterns is greater for the unilateral endfire array (Fig. 5.15) than for the broadside array (Fig. 5.11a,b) since the effect of mutual interaction is greater.


Fig. 5.16. Radiation patterns in the $H$-plane for a 15 -element bilateral endfire array.

### 5.6 The special case when $\beta_{0} h=\pi / 2$

The general functional form for the currents in the elements given by (5.34) with (5.46) and (5.47) presents some difficulties when $\beta_{0} h=\pi / 2$. For both circular and curtain arrays the expression for the currents becomes indeterminate in the form $0 / 0$ when $\beta_{0} h=\pi / 2$. This behaviour is illustrated for the curtain array in the following matrix equation for the currents:

$$
\begin{align*}
\left\{I_{z}(z)\right\}= & \frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left\{V_{0}\right\} \sin \beta_{0}(h-|z|) \\
& +\frac{j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\Phi_{u}\right]^{-1}\left[\Phi_{v}\right]\left\{V_{0}\right\}\left(\cos \beta_{0} z-\cos \beta_{0} h\right) . \tag{5.117}
\end{align*}
$$

From the form of the $\Psi$ functions at $\beta_{0} h=\pi / 2$ it follows that

$$
\begin{equation*}
\lim _{\beta_{0} h \rightarrow \pi / 2}\left[\Phi_{u}\right]^{-1}\left[\Phi_{v}\right]=-[\mathscr{I}] \tag{5.118}
\end{equation*}
$$

where $[\mathscr{I}]$ is the identity matrix. The indeterminate form for the currents in the elements follows directly when (5.118) is used in (5.117). It is

$$
\begin{equation*}
\left\{I_{z}(z)\right\}=\frac{j 2 \pi \cos \beta_{0} z}{\zeta_{0} \Psi_{d R} \cdot 0}\left\{V_{0}\right\}-j \frac{2 \pi \cos \beta_{0} z}{\zeta_{0} \Psi_{d R} \cdot 0}\left\{V_{0}\right\}=\frac{0}{0}, \quad \beta_{0} h=\frac{\pi}{2} . \tag{5.119}
\end{equation*}
$$

Two alternatives are available for avoiding this difficulty: (a) The formula for the currents may be rearranged as in section 2.7 or (b) a special formulation for $\beta_{0} h=\pi / 2$ may be used. The former method has the advantage that it is applicable over a range near $\beta_{0} h=\pi / 2$, whereas the latter method is valid only at $\beta_{0} h=\pi / 2$. Both methods are presented here, although the numerical results were calculated based on the special form for $\beta_{0} h=\pi / 2$. Numerical calculations have shown the results of the two approximate forms useful when $\beta_{0} h=\pi / 2$ to be approximately the same.

The expression for the currents when $\beta_{0} h$ is near $\pi / 2$ follows directly from the results of section 2.7. In matrix form

$$
\begin{align*}
\left\{I_{z}(z)\right\}= & \frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}}\left(\left\{V_{0}\right\}\left(\sin \beta_{0}|z|-\sin \beta_{0} h\right)\right. \\
& \left.+\left[\Phi_{u}^{\prime}\right]^{-1}\left[\Phi_{v}^{\prime}\right]\left\{V_{0}\right\}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right) \tag{5.120a}
\end{align*}
$$

where the elements of the matrices are

$$
\begin{equation*}
\boldsymbol{\Phi}_{k i u}^{\prime}=-\Phi_{k i u} \cos \beta_{0} h \tag{5.120b}
\end{equation*}
$$

$$
\begin{equation*}
\Phi_{k i v}^{\prime}=\Phi_{k i v}+\Phi_{k i u} \sin \beta_{0} h . \tag{5.120c}
\end{equation*}
$$

When $\beta_{0} h=\pi / 2$,

$$
\begin{equation*}
\left\{I_{z}(z)\right\}=\frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}}\left(\left\{V_{0}\right\}\left(\sin \beta_{0}|z|-1\right)+\left[\Phi_{u}^{\prime}\right]^{-1}\left[\Phi_{v}^{\prime}\right]\left\{V_{0}\right\} \cos \beta_{0} z\right) . \tag{5.121}
\end{equation*}
$$

For $\beta_{0} h=\pi / 2$, the elements of the $\Phi_{u}^{\prime}$ and $\Phi_{v}^{\prime}$ matrices are

$$
\begin{equation*}
\Phi_{k i u}^{\prime}=\Psi_{k i u}(h) \tag{5.122a}
\end{equation*}
$$

and

$$
\begin{equation*}
\Phi_{k i v}^{\prime}=\Psi_{k i d u}-\Psi_{k i d v}\left(1-\delta_{i k}\right)-j \Psi_{k i d I} \delta_{i k} \tag{5.122b}
\end{equation*}
$$

The alternative approach begins with the special form for the integral equation valid at $\beta_{0} h=\pi / 2$. It was this latter method which was used for the original curtain-array calculations. The final form is similar to ( 5.120 b ) with slightly different values for the constant $\Psi_{d R}$ and the $\Phi_{u}^{\prime}$ and $\Phi_{v}^{\prime}$ matrices. In this method the $\Psi$ functions are computed with the following cosine and shifted-sine currents:

$$
\begin{equation*}
I_{z i}(z)=-j A_{i} S_{0 z}+B_{i} F_{0 z}, \tag{5.123}
\end{equation*}
$$

where $\quad S_{0 z}=\sin \beta_{0}|z|-\sin \beta_{0} h \quad$ and $\quad F_{0 z}=\cos \beta_{0} z-\cos \beta_{0} h$ The final expression for the current with $\beta_{0} h=\pi / 2$ is

$$
\begin{equation*}
\left\{I_{z}(z)\right\}=\frac{-j 2 \pi}{\zeta_{0} \Psi_{d R}^{h}}\left\{V_{0}\right\}\left(\sin \beta_{0}|z|-1\right)-j \frac{2 \pi}{\zeta_{0} \Psi_{d R}^{h}}\left[\Phi_{u}\right]^{-1}\left[\Phi_{v}^{h}\right]\left\{V_{0}\right\} \cos \beta_{0} z \tag{5.124}
\end{equation*}
$$

where

$$
\begin{align*}
\Phi_{k i v}^{h} & =\Psi_{k i d v}^{h}\left(1-\delta_{i k}\right)+j \Psi_{k i d I}^{h} \delta_{i k}-\Psi_{k i v}^{h}(0)  \tag{5.125a}\\
\Phi_{k i d u} & =-\Psi_{k i d u}+\Psi_{k i u}(0) \tag{5.125b}
\end{align*}
$$

$$
\begin{align*}
\Psi_{d R}^{h} & =-\operatorname{Re}\left\{\left[S_{b}(h, 0)-E_{b}(h, 0)\right]-\left[S_{b}(h, h)-E_{b}(h, h)\right]\right\}  \tag{5.126a}\\
\Psi_{k i d I}^{h} & =\operatorname{Im}\left\{\left[S_{b}(h, 0)-E_{b}(h, 0)\right]-\left[S_{b}(h, h)-E_{b}(h, h)\right]\right\} \\
\Psi_{k i d v}^{h} & =\left[S_{b}(h, 0)-E_{b}(h, 0)\right]-\left[S_{b}(h, h)-E_{b}(h, h)\right]  \tag{5.126b}\\
\Psi_{k i d u} & =C_{b}(h, 0)-C_{b}(h, h)  \tag{5.126c}\\
\Psi_{k i d v}(0) & =S_{b}(h, 0)-E_{b}(h, 0)  \tag{5.126d}\\
\Psi_{k i u}(0) & =C_{b}(h, 0) \tag{5.126e}
\end{align*}
$$

and

$$
\beta_{0} h=\pi / 2, \quad b \equiv b_{k i}, \quad b_{k k}=a .
$$

Numerical calculations show that the results obtained with (5.124) are comparable with those obtained with ( 5.120 b ).

### 5.7 Design of curtain arrays

The antenna designer is generally confronted with both a geometrical and an electrical problem. First, the proper overall length and spacing must be chosen for a prescribed beamwidth and electrical scanning range. The currents or voltages at the driving point must then be selected for a prescribed radiation pattern. A knowledge of the driving-point impedance for each element is also necessary for properly matching the antennas to the transmission lines. If the array is operated over a large frequency range, the radiation pattern and driving-point impedances must be computed for representative points. Furthermore, these calculations must be repeated if the driving conditions are changed. With appropriate theoretical calculations available, the designer is better able to select the necessary auxiliary equipment and interpret any measurements made on the array.

The design procedure is illustrated in terms of the following somewhat simplified problem: An array is to be constructed to operate over a two-to-one frequency range. At the lowest frequency the spacing is a quarter wavelength and the elements are each a half wavelength long. The array is to have fifteen elements. Both broadside and endfire radiation patterns are desired. The problem is to find the actual impedances and radiation patterns with the currents specified at the driving point. The problem may be summarized as follows:

1. Geometry of Array
(Number of elements, $N=15)(\Omega=2 \ln 2 h / a=10)$
(a) at lowest frequency,
spacing $\beta_{0} b=\pi / 2$
element half length $\beta_{0} h=\pi / 2$
(b) at highest frequency,
spacing $\beta_{0} b=\pi$ element half length $\beta_{0} h=\pi$
2. Electronic Scanning Range (in $H$-plane only)
(a) at lowest frequency, $\Phi_{0}=0$ and $\Phi_{0}=90^{\circ}$
(b) at highest frequency, $\Phi_{0}=0$ and $\Phi_{0}=90^{\circ}$
3. Specification of Base Currents
(a) at lowest frequency ( $\beta_{0} h=\pi / 2, \beta_{0} b=\pi / 2$ )

$$
\begin{array}{ll}
\left\{I_{z}(0)\right\}=I_{1}(0)\left\{\begin{array}{l}
1 \\
1 \\
1 \\
\vdots
\end{array}\right\}, & \Phi_{0}=\pi / 2 \text { (broadside) } \\
\left\{I_{z}(0)\right\}=I_{1}(0)\left\{\begin{array}{c}
1 \\
-j \\
-1 \\
\vdots
\end{array}\right\}, & \Phi_{0}=0 \text { (endfire) } \tag{5.128}
\end{array}
$$

(b) at highest frequency ( $\beta_{0} h=\pi, \beta_{0} b=\pi$ )

$$
\begin{align*}
& \left\{I_{z}(0)\right\}=I_{1}(0)\left\{\begin{array}{c}
1 \\
1 \\
1 \\
\vdots
\end{array}\right\}, \quad \Phi_{0}=\pi / 2  \tag{5.129}\\
& \left\{I_{z}(0)\right\}=I_{1}(0)\left\{\begin{array}{r}
1 \\
-1 \\
1 \\
\vdots
\end{array}\right\}, \quad \Phi_{0}=0 . \tag{5.130}
\end{align*}
$$

Calculations at lowest frequency
The driving-point impedances are calculated from (5.124) with ( $5.125 \mathrm{a}, \mathrm{b}$ ) and (5.127) with the relevant values of the $\Phi$ functions. Tables of the $\Phi$ functions along with values of $\Psi_{d R}$ are given in appendix III based on an improved Romberg integration method. Since most of the calculations in this chapter are based on the tables of Mack [3], the calculations which follow are also based on the Mack tables.

$$
\begin{align*}
& \beta_{0} h=1.5708 \quad \beta_{0} b=1.5708 \quad \Psi_{d R}=6.9087 \quad H_{m}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)=-0.5708 \quad G_{m}\left(\frac{\pi}{2}, \frac{\pi}{2}\right)=1.0000 \\
& \Phi_{k i u}=\Phi_{u m} \text { with }|k-i|=m \\
& \Phi_{u 1}=0.6880-j 1.2187 \quad \Phi_{k 2}=-0.4725-j 0.6798 \\
& \Phi_{u 4}=0.1105+j 0.3750 \quad \Phi_{u 5}=0.2957-j 0.0669 \\
& \Phi_{u 8}=0.0234+j 0.1773 \\
& \Phi_{u 11}=-0.1257+j 0.0117 \\
& \Phi_{u 14}=-0.0070-j 0.0972  \tag{5.131}\\
& \Phi_{u 3}=-0.4988+j 0.2089 \\
& \Phi_{u 6}=-0.0444-j 0.2426 \\
& \Phi_{u 9}=0.1561-j 0.0181 \\
& \Phi_{\mathrm{w} 12}=0.0097+j 0 \cdot 1146 \\
& \Phi_{w 15}=-0.0904+j 0.0060
\end{align*}
$$

$$
\begin{array}{rlrlr}
\Phi_{k i r}=\Phi_{r m} \text { with }|k-i|=m . & & \\
\Phi_{v 1} & =7.0756-j 0.2036 & \Phi_{v 2}=-0.2864-j 0.3970 & & \Phi_{v 3}=-0.2925+j 0.1186 \\
\Phi_{v 4} & =0.0609+j 0.2176 & \Phi_{v 5}=0.1706-j 0.0363 & \Phi_{v 6}=-0.0239-j 0.1395 \\
\Phi_{v 7}=-0.1177+j 0.0169 & \Phi_{v 8}=0.0125+j 0.1016 & \Phi_{v 9}=0.0894-j 0.0097 \\
\Phi_{v 10}=-0.0077-j 0.0797 & \Phi_{v 11}=-0.0719+j 0.0062 & \Phi_{v 12}=0.0052+j 0.0655 \\
\Phi_{v 13}=0.0601-j 0.0044 & \Phi_{v 14}=-0.0037-j 0.0556 & \Phi_{v 15}=-0.0516+j 0.0032 \tag{5.132}
\end{array}
$$

The driving-point currents for $\Phi_{0}=0$ are given by (5.128).

## Driving-Point Currents

$I_{z}(0)$

$$
\begin{array}{rrrrr}
I_{1}(0)=1 \cdot 00+j 0 & I_{2}(0)=1 \cdot 00+j 0 & I_{3}(0)=1 \cdot 00+j 0 & I_{4}(0)=1 \cdot 00+j 0 \\
I_{5}(0)=1 \cdot 00+j 0 & I_{6}(0)=1 \cdot 00+j 0 & I_{7}(0)=1 \cdot 00+j 0 & I_{8}(0)=1 \cdot 00+j 0 \\
I_{9}(0)=1 \cdot 00+j 0 & I_{10}(0)=1 \cdot 00+j 0 & I_{11}(0)=1 \cdot 00+j 0 & I_{12}(0)=1 \cdot 00+j 0 \\
I_{13}(0)=1 \cdot 00+j 0 & I_{14}(0)=1 \cdot 00+j 0 & I_{15}(0)=1 \cdot 00+j 0 & \tag{5.133}
\end{array}
$$

The driving-point impedances are computed from (5.124) with (5.131)-(5.133). The results are:

## Driving-Point Impedances

$$
\begin{array}{rlrlrl}
Z_{k}=V_{k} / I_{k}(0) & & & \\
Z_{1} & =90 \cdot 816-j 21 \cdot 298 & Z_{2} & =129 \cdot 267-j 52 \cdot 450 & Z_{3} & =99 \cdot 602-j 75 \cdot 308 \\
Z_{4} & =73 \cdot 401-j 67 \cdot 853 & Z_{5} & =93 \cdot 734-j 54.840 & Z_{6} & =114 \cdot 393-j 55 \cdot 375 \\
Z_{7} & =96 \cdot 241-j 63 \cdot 012 & Z_{8} & =77 \cdot 829-j 66 \cdot 659 & Z_{9} & =96 \cdot 241-j 63 \cdot 012 \\
Z_{10} & =114 \cdot 393-j 55 \cdot 375 & Z_{11} & =93 \cdot 734-j 54.840 & Z_{12} & =73 \cdot 401-j 67.853 \\
Z_{13} & =99 \cdot 602-j 75 \cdot 308 & Z_{14} & =129 \cdot 267-j 52 \cdot 450 & Z_{15}=90.816-j 21 \cdot 298 \tag{5.134}
\end{array}
$$

The driving-point admittances are obtained by inverting $Z_{k k}$ in (5.134).

## Driving-Point Admittances

$\mathrm{Y}_{\mathrm{k}}$

| $Y_{1}=0.010437+j 0.002448$ | $Y_{2}=0.006642+j 0.002695$ | $Y_{3}=0.006388+j 0.004830$ |
| :---: | :---: | :---: |
| $Y_{4}=0.007346+j 0.006791$ | $Y_{5}=0.007948+j 0.004650$ | $Y_{6}=0.007082+j 0.003428$ |
| $Y_{7}=0.007273+j 0.004762$ | $Y_{8}=0.007412+j 0.006348$ | $Y_{9}=0.007273+j 0.004762$ |
| $Y_{10}=0.007082+j 0.003428$ | $Y_{11}=0.007948+j 0.004650$ | $Y_{12}=0.007346+j 0.006791$ |
| $Y_{13}=0.006388+j 0.004830$ | $Y_{14}=0.006642+j 0.002695$ | $Y_{15}=0.010437+j 0.002448$ |

The electric field may be computed from (5.67) which is given in the following form for ease of computation:

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta, \Phi)=\frac{j \zeta_{0}}{4 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} \sum_{i=1}^{N} C_{i} e^{j \beta_{0} b[(N-2 i+1) / 2] \sin \Theta \cos \Phi} \tag{5.136}
\end{equation*}
$$

where

$$
\begin{gather*}
C_{i}=\xi_{i}\left\{j A_{i} F_{m}\left(\Theta, \beta_{0} h\right)+\left(Y_{i}-j A_{i} \sin \beta_{0} h\right) G_{m}\left(\Theta, \beta_{0} h\right)\right\} \quad \begin{array}{r}
\beta_{0} h \neq \pi / 2 \\
(5.137) \\
C_{i}=\xi_{i}\left\{H_{m}\left(\Theta, \frac{\pi}{2}\right)+\left(Y_{i}-j A_{i}\right) G_{m}\left(\Theta, \frac{\pi}{2}\right)\right\} \quad \beta_{0} h=\pi / 2 \\
\xi_{i}=V_{0 i} / V_{01} .
\end{array} \tag{5.137}
\end{gather*}
$$

For $\beta_{0} h=\pi / 2, \beta_{0} b=\pi / 2, \Theta=\pi / 2$ and $\Phi_{0}=0^{\circ}$, the constants $C_{i}$ are:

## Field-Pattern Constants

| $C_{i}$ |  |  |  |
| ---: | ---: | ---: | ---: |
| $C_{1}$ | $=0.01147+j 0.0014123$ | $C_{2}=0.01173+j 0.0006786$ | $C_{3}=0.01114+j 0.0006191$ |
| $C_{4}$ | $=0.01086+j 0.0009353$ | $C_{5}=0.01123+j 0.0009407$ | $C_{6}=0.01151+j 0.0007627$ |
| $C_{7}$ | $=0.01120+j 0.0008109$ | $C_{8}=0.01092+j 0.0009146$ | $C_{9}=0.01120+j 0.0008109$ |
| $C_{10}$ | $=0.01151+j 0.0007627$ | $C_{11}=0.01123+j 0.0009407$ | $C_{12}=0.01086+j 0.0009353$ |
| $C_{13}$ | $=0.01114+j 0.0006191$ | $C_{14}=0.01173+j 0.0006786$ | $C_{15}=0.01147+j 0.0001412$ |

The field patterns computed from (5.136) with (5.138)-(5.140) are shown in Fig. 5.15.

The driving-point admittances for the endfire case ( $\Phi_{0}=\pi / 2$ ) are computed with (5.124) and (5.128). Thus, with $I_{1}(0)=1$, the driving-point currents are specified to be

Driving-Point Currents
$I_{z}(0)$

$$
\begin{array}{rrrrr}
I_{1}=1 \cdot 000+j 0 & I_{2}=0-j 1 \cdot 000 & I_{3}=-1 \cdot 000+j 0 & I_{4}=0+j 1 \cdot 000 \\
I_{5}=1 \cdot 000+j 0 & I_{6}=0-j 1.000 & I_{7}=-1.000+j 0 & I_{8}=0+j 1.000 \\
I_{9}=1 \cdot 000+j 0 & I_{10}=0-j 1.000 & I_{11}=-1.000+j 0 & I_{12}=0+j 1.000 \\
I_{13}=1.000+j 0 & I_{14}=0-j 1.000 & I_{15}=-1.000+j 0 & \tag{5.141}
\end{array}
$$

The corresponding driving-point impedances and constants $C_{i}$ for computing the field pattern are:

## Driving-Point Impedances

$Z_{k}$

| $Z_{1}=59.432+j 15.260$ | $Z_{2}=91 \cdot 243+j 43 \cdot 280$ | $Z_{3}=119.668+j 81.021$ |
| :---: | :---: | :---: |
| $Z_{4}=127.775+j 94$ | $Z_{5}=146 \cdot 811+j 123 \cdot 305$ | $Z_{6}=148.136+j 128.287$ |
| $Z_{7}=164 \cdot 970+j 154 \cdot 466$ | $Z_{8}=162 \cdot 206+j 153 \cdot 871$ | $Z_{9}=179 \cdot 341+j 179 \cdot 810$ |
| $Z_{10}=172 \cdot 518+j 173 \cdot 925$ | $Z_{11}=192 \cdot 236+j 202 \cdot 053$ | $Z_{12}=179 \cdot 290+j 189 \cdot 402$ |
| $Z_{13}=206 \cdot 483+j 223 \cdot 550$ | $Z_{14}=178 \cdot 456+j 1$ |  |

## Field-Pattern Constants

$C_{i}$

$$
\begin{array}{rlrlr}
C_{1}=0.01682-j 0.00509 & C_{2}=-0.00540-j 0.01778 & & C_{3}=-0.01890+j 0.00553 \\
C_{4} & =0.00553+j 0.01927 & C_{5}=0.02010-j 0.00557 & & C_{6}=-0.00554-j 0.02022 \\
C_{7} & =-0.02096+j 0.00557 & C_{8}=0.00552+j 0.02092 & C_{9}=0.02166-j 0.00555 \\
C_{10} & =-0.00548+j 0.02145 & C_{11}=-0.02227+j 0.00555 & C_{12}=0.00544+j 0.02185 \\
C_{13}=0.02289-j 0.00558 & C_{14}=-0.00530-j 0.02205 & C_{15}=-0.02371+j 0.00590
\end{array}
$$

The endfire field pattern determined by (5.143) is shown in Fig. 5.11. The impedances for the endfire and broadside arrays at the lowest frequency are shown in Figs. 5.8 and 5.12.

Calculations at the highest frequency $\left(\beta_{0} h=\pi, \beta_{0} b=\pi\right)$
The relevant $\Phi$ functions from appendix III are:

| $\beta_{0} h=3.1416$ | $\beta_{0} b=3.1416$ |  | $\Psi_{d R}=5.8340$ | $F_{m}(\pi, \pi)=2.000$ |
| :--- | :--- | :--- | :--- | :--- |$\quad G_{m}(\pi, \pi)=3.1416$

$$
\begin{align*}
\Phi_{k i v}=\Phi_{v m} \text { with }|k-i|=m & & \\
\Phi_{v 1}=0.6721-j 1.6605 & \Phi_{v 2}=-0.6296+j 0.4070 & \Phi_{v 3}=0.4441-j 0.2174 \\
\Phi_{v 4}=-0.3470+j 0.1287 & \Phi_{v 5}=0.2810-j 0.0822 & \Phi_{v 5}=-0.2342+j 0.0561 \\
\Phi_{v 7}=0.1999-j 0.0405 & \Phi_{v 8}=-0.1740+j 0.0304 & \Phi_{v 9}=0.1538-j 0.0237 \\
\Phi_{v 10}=-0.1377+j 0.0100 & \Phi_{v 11}=0.1245-j 0.0154 & \Phi_{v 12}=-0.1136+j 0.0128 \\
\Phi_{v 13}=0.1045-j 0.0108 & \Phi_{v 14}=-0.0967+j 0.0093 & \Phi_{v 15}=0.0899-j 0.0080
\end{align*}
$$

The specifications of the driving-point currents for the broadside array are:

## Driving-Point Currents

$I_{z}(0)$

$$
\begin{array}{rlrrr}
I_{1}(0)=1 \cdot 000+j 0 & I_{2}(0)=1 \cdot 000+j 0 & I_{3}(0)=1 \cdot 000+j 0 & I_{4}(0)=1 \cdot 000+j 0 \\
I_{5}(0)=1 \cdot 000+j 0 & I_{6}(0)=1 \cdot 000+j 0 & I_{7}(0)=1 \cdot 000+j 0 & I_{8}(0)=1 \cdot 000+j 0 \\
I_{9}(0)=1 \cdot 000+j 0 & I_{10}(0)=1 \cdot 000+j 0 & I_{11}(0)=1 \cdot 000+j 0 & I_{12}(0)=1 \cdot 000+j 0 \\
I_{13}(0)=1 \cdot 000+j 0 & I_{14}(0)=1 \cdot 000+j 0 & I_{15}(0)=1 \cdot 000+j 0 & & \tag{5.146}
\end{array}
$$

The driving-point impedances from (5.51) are:

## Driving-Point Impedances

$Z_{k}$

$$
\begin{array}{rlrr}
Z_{1}=702 \cdot 588-j 350 \cdot 682 & Z_{2}=915 \cdot 158-j 239 \cdot 232 & Z_{3}=828 \cdot 211-j 245 \cdot 278 \\
Z_{4}=872 \cdot 733-j 246 \cdot 841 & Z_{5}=843 \cdot 055-j 244 \cdot 341 & Z_{6}=866 \cdot 245-j 246 \cdot 898 \\
Z_{7}=846 \cdot 108-j 244 \cdot 293 & Z_{8}=864 \cdot 939-j 246 \cdot 801 & Z_{9}=846 \cdot 108-j 244 \cdot 293 \\
Z_{10}=866 \cdot 245-j 246 \cdot 898 & Z_{11}=843 \cdot 055-j 244 \cdot 341 & Z_{12}=872 \cdot 733-j 246 \cdot 841 \\
Z_{13}=828 \cdot 211-j 245 \cdot 278 & Z_{14}=915 \cdot 159-j 239 \cdot 232 & Z_{15}=702 \cdot 589-j 350 \cdot 682
\end{array}
$$

The constants $C_{i}$ computed from (5.136) with (5.137) are:

## Field-Pattern Constants

$C_{i}$

| $C_{1}=0.00179-j 0.00482$ | $C_{2}=0.00321-j 0.00584$ | $C_{3}=0.00288-j 0.00530$ |
| ---: | ---: | ---: | ---: |
| $C_{4}=0.00302-j 0.00559$ | $C_{5}=0.00294-j 0.00539$ | $C_{6}=0.00300-j 0.00555$ |
| $C_{7}=0.00295-j 0.00541$ | $C_{8}=0.00299-j 0.00554$ | $C_{9}=0.00295-j 0.00541$ |
| $C_{10}=0.00300-j 0.00555$ | $C_{11}=0.00294-j 0.00539$ | $C_{12}=0.00302-j 0.00559$ |
| $C_{13}=0.00288-j 0.00530$ | $C_{14}=0.00321-j 0.00584$ | $C_{15}=0.00179-j 0.00482$ |
|  | $\Theta=\pi / 2$. |  |

The field pattern determined by (5.148) is shown in Fig. 5.15. Finally, the driving-point currents, impedances, and field-pattern constants are given below for the endfire case, $\Phi_{0}=\pi / 2$ :

## Driving-Point Currents

$$
\begin{align*}
& I_{z}(0) \\
& \begin{array}{rlrrr}
I_{1} & =1 \cdot 000+j 0 & I_{2} & =-1 \cdot 000+j 0 & I_{3}
\end{array}=1 \cdot 000+j 0 \quad r o r l o-1 \cdot 000+j 0 \tag{5.149}
\end{align*}
$$

## Driving-Point Impedances

## Z

$$
\begin{align*}
& Z_{1}=196 \cdot 512-j 454 \cdot 406 \quad Z_{2}=119 \cdot 475-j 446 \cdot 232 \quad Z_{3}=97 \cdot 429-j 431 \cdot 258 \\
& Z_{4}=87.725-j 423.369 \quad Z_{5}=82.423-j 418.986 \quad Z_{6}=79 \cdot 400-j 416.431 \\
& Z_{7}=77.782-j 415.071 \quad Z_{8}=77.300-j 414.663 \quad Z_{9}=77.782-j 415.071 \\
& Z_{10}=79.400-j 416.431 \quad Z_{11}=82.423-j 418.987 \quad Z_{12}=87.725-j 423.369 \\
& Z_{13}=97.429-j 431.258 \quad Z_{14}=119.475-j 446.232 \quad Z_{15}=196 \cdot 512-j 454.406 \tag{5.150}
\end{align*}
$$

## Field-Pattern Constants

$$
\begin{array}{rlrl}
C_{i} & & \\
C_{1} & =0.001259-j 0.002801 & C_{2} & =-0.000481+j 0.002362  \tag{5.151}\\
C_{3} & =0.000316-j 0.002102 & C_{4} & =-0.000249+j 0.001974 \\
C_{5} & =0.000213-j 0.001904 & C_{6} & =-0.000193+j 0.001863 \\
C_{1} & =0.000182-j 0.001841 & C_{8} & =-0.000179+j 0.001834 \\
C_{9} & =0.000182-j 0.001841 & C_{10} & =-0.000193+j 0.001863 \\
C_{11} & =0.000213-j 0.001904 & C_{12} & =-0.000249+j 0.001974 \\
C_{13} & =0.000316-j 0.002102 & C_{14} & =-0.000481+j 0.002362 \\
C_{15} & =0.001259-j 0.002801 & &
\end{array}
$$

### 5.8 Summary

In this chapter a complete theory of curtain arrays of practical antennas has been presented. Mutual coupling among all elements is included in a manner that takes account of changes in the amplitudes and the phases of the currents along all elements as determined by their locations in an array. The theory is quantitatively useful for cylindrical elements with electrical half-lengths in the range $\beta_{0} h \leqslant 5 \pi / 4$ and electrical radii with values $\beta_{0} a \leqslant 0.02$. This includes lengths over the full range in which the principal lobe in the vertical field pattern is in the equatorial plane; it provides a 5 to 1 frequency band for electrical half-lengths included in the range $\pi / 4 \leqslant \beta_{0} h \leqslant 5 \pi / 4$.

In this chapter no measurements have been cited to verify the quantitative correctness of the new theory in determining distributions of current, driving-point impedances or admittances, and field patterns of typical curtain arrays. This is due in part to the relative difficulty in carrying out accurate measurements of the self- and mutual impedances for curtain arrays owing to the lack of the symmetry which underlies the corresponding measurements
with the circular array. It is due primarily to the adequacy of the experimental verification of all phases of the theory as applied to the two-element array-the simplest curtain array (chapter 3), general circular arrays (chapter 4) and to curtain arrays of parasitic elements (chapter 6). As in the case of the circular array, the most sensitive and, at the same time, the most convenient experimental verification of the theory is in its application to an array in which only one element is driven while all others are parasitic. The first section in the next chapter is concerned specifically with the application of the theory developed in this chapter to a curtain array of twenty elements of which only one is driven and a comparison of theoretically and experimentally determined currents, admittances, and field patterns.

## CHAPTER 6

## ARRAYS WITH UNEQUAL ELEMENTS; PARASITIC AND LOG-PERIODIC ANTENNAS

The general theory of curtain arrays which is developed in the preceding chapter requires all $N$ elements to be identical geometrically, but allows them to be driven by arbitrary voltages or loaded by arbitrary reactors at their centres. If some of these voltages are zero, the corresponding elements are parasitic and their currents are maintained entirely by mutual interaction. In arrays of the well-known Yagi-Uda type, only one element is driven so that the importance of an accurate analytical treatment of the inter-element coupling is increased. In a long array the possible cumulative effect of a small error in the interaction between the currents in adjacent elements must not be overlooked. As an added complication, the tuning of the individual parasitic elements is accomplished by adjustments in their lengths and spacings. This introduces the important problem of arrays with elements that are different in length and that are separated by different distances. In the YagiUda array the range of these differences is relatively small. On the other hand, in frequency-independent arrays of the log-periodic type the range of lengths and distances between adjacent elements is very great.

In this chapter the analytical treatment of arrays with elements that are different in length and unequally spaced is carried out successively for parasitic arrays of the conventional Yagi-Uda type and for driven log-periodic arrays. However, the formulation is sufficiently general to permit its extension to arrays of other types, both parasitic and driven, that involve geometrically different elements.

### 6.1 Application of the two-term theory to a simple parasitic array

The simplest parasitic array consists of $N$ geometrically identical antennas each of length $2 h$ and radius $a$ arranged in a curtain of parallel non-staggered elements with spacing $b$. Element 1 is driven, all others are parasitic. Such an array is illustrated in Fig.
6.1. The directional properties of the electromagnetic field maintained by the array depend on the relative amplitudes and phases of the currents in all of the elements. The currents in the parasitic elements are all induced by their mutual interaction. The current in the driven antenna is determined in part by the driving generator, in part by the mutual interaction with the currents in the other elements. The coupling between the currents in any pair of elements of given length depends primarily on the distance between them.


Fig. 6.1. Parasitic array of identical elements.
The general theory of curtain arrays formulated in the preceding chapter may be applied directly by setting $V_{0 i}=0,1<i \leqslant N$. The currents in the $N$ elements are given by (5.34). They are

$$
\begin{align*}
I_{z 1}(z) & =j A_{1} \sin \beta_{0}(h-|z|)+B_{1}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)  \tag{6.1}\\
I_{z i}(z) & =B_{i}\left(\cos \beta_{0} z-\cos \beta_{0} h\right), \quad i=2,3, \ldots N \tag{6.2}
\end{align*}
$$

where from (5.47)

$$
\begin{equation*}
A_{1}=\frac{2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h} V_{01} \tag{6.3}
\end{equation*}
$$

and the $B_{i}$ are obtained from (5.46). With $V_{01}$ specified the currents at the centres of the elements are obtained from (5.53). The drivingpoint admittance of element 1 is

$$
\begin{equation*}
Y_{01}=I_{1 z}(0) / V_{01} . \tag{6.4}
\end{equation*}
$$

The field pattern of the array is obtained from (5.67) with the appropriate values of $A_{i}$ and $B_{i}$. Since only $A_{1}$ differs from zero the applicable formula is

$$
\begin{align*}
E_{\Theta}(\Theta, \Phi)= & j \frac{\omega \mu \sin \Theta}{4 \pi}\left\{j A_{1} \frac{e^{-j \beta_{0} R_{1}}}{R_{1}} F_{m}\left(\Theta, \beta_{0} h\right)\right. \\
& \left.+\sum_{i=2}^{N} B_{i} \frac{e^{-j \beta_{0} R_{i}}}{R_{i}} G_{m}\left(\Theta, \beta_{0} h\right)\right\} \tag{6.5}
\end{align*}
$$

where $F_{m}\left(\Theta, \beta_{0} h\right)$ and $G_{m}\left(\Theta, \beta_{0} h\right)$ are defined in (5.66) and (5.68). In (6.5) the field is evaluated in the far zone of each element so that the distances $R_{i}$ are measured to the centres of the elements. The far field of the array implies in addition that $R_{i} \doteq R_{1}$ in amplitudes and $R_{i}=R_{1}-(i-1) b \sin \Theta \cos \Phi$ in phase angles.

Numerical computations have been made by Mailloux [1] for an array of 20 elements with $a / \lambda=0.00635$ and $b / \lambda=0 \cdot 20$. Several


Fig. 6.2. Components of current on driven dipole in 20-element parasitic array (Mailloux), (a) in phase with driving voltage ( mA ), (b) in phase quadrature with driving voltage ( mA ). $b / \lambda=0 \cdot 20, a / \lambda=0.00635$.
values of $h / \lambda$ were chosen in the range for endfire operation between 0.16 and 0.204 .

The calculated distributions of current along the driven element are shown in Fig. 6.2 together with measured values. The agreement is excellent for $h / \lambda=0.16$ and 0.18 . The agreement when $h / \lambda=0.20$ is not so close. However, the theoretical curves for antennas with $h / \lambda$ increased by only 0.004 -a distance of less than $a / \lambda=0.00635$ -are in excellent agreement with the experimental data for $h / \lambda=0 \cdot 20$. Evidently, as resonance is approached the current amplitude becomes increasingly sensitive to small changes in length. The theoretical and experimental driving-point admittances are shown in Fig. 6.3. As for the current distribution in general, the agreement is very good for $h / \lambda=0.16$ and $0 \cdot 18$, but the theoretical value at $h / \lambda=0.204$ is in better agreement with the measured value for $h / \lambda=0.20$ than is the theoretical value for $h / \lambda=0.20$.


Fig. 6.3. Driving-point admittance of 20 -element parasitic array (Mailloux). $b / \lambda=0.20$, $a / \lambda=0.00635$.

The normalized theoretical distributions of current along all parasitic dipoles are the same. The experimental values were also found to be remarkably alike. Theoretical and experimental distributions of the magnitude of the current along a typical parasitic element are shown in Fig. 6.4. It is seen that the theoretical currents differ somewhat from the measured values. Measured changes in the phase of the current along the parasitic elements were very small.


Fig. 6.4. Normalized current amplitudes on a typical parasitic element in a 20 -element array (Mailloux). $b / \lambda=0 \cdot 20, a / \lambda=0.00635$.

The amplitudes of the currents at $z=0$ along each of the twenty elements are shown in Fig. 6.5. The agreement with measured values is again excellent for $h / \lambda=0.16$ and 0.18 . As before, the theoretical curve for $h / \lambda=0.204$ is in much better agreement


Fig. 6.5. Amplitudes of currents at centres of the elements in a 20 -element array with element No. 1 driven; comparison of King-Sandler theory with experiment (Mailloux). $b / i=0 \cdot 20, a / \lambda=0.00635$, frequency 600 Mc .
with the measured curve for $h / \lambda=0.20$ than is the theoretical curve for $h / \lambda=0 \cdot 20$. The corresponding phases are shown in Fig. 6.6.


Fig. 6.6. Same as Fig. 6.5 but for phases of currents (Mailloux).
It is interesting to note that when $h / \lambda=0.16$ and 0.18 the amplitudes of the currents in all of the parasitic elements except those nearest the driven antenna are quite small and substantially equal and the phase shift from element to element is linear. On the other hand, as $h / \lambda$ approaches resonance the amplitudes of the currents increase greatly and they oscillate in magnitude from element to element. The small constant amplitude and linear phase shift that is characteristic of the shorter elements suggests a travelling wave along the array; the large oscillating amplitudes near resonance are characteristic of a standing wave.

The theoretical and experimental field patterns are shown in Fig. 6.7 for the three values of $h / \lambda$. Although the measurements were made in the far zone of each element ( $E$ far zone), the length $L$ of the 20 -element array was such that the true far-zone approximations $R_{i} \doteq R_{1}$ in amplitudes and $R_{i} \doteq R_{1}-(i-1) b \sin \Theta \cos \Phi$ in phases were not sufficiently well satisfied. Accordingly, the field was evaluated from (6.5) with the actual distances to the elements for comparison with the measured values. The true far-field was also computed for comparison. The former is designated ' $E$ far zone' in the figures, the latter is labelled 'far zone'. The agreement


Fig. 6.7. Field patterns of 20-element parasitic array ; comparison of King-Sandler theory with measurements (Mailloux). (Far zone is far zone referred to length of array, $E$ far zone is far zone referred to length of the elements.)
between theory and experiment is seen to be quite good even in the details of the minor lobe structure.

It may be concluded that the two-term theory of curtain arrays developed in chapter 5 provides remarkably accurate results even for parasitic arrays for which one of the terms vanishes for each of the $N-1$ parasitic elements. This is somewhat surprising since the single term provides no flexibility in the representation of the distribution of the currents in the parasitic elements. They are all assumed to be the same and given by $I(z) \sim \cos \beta_{0} z-\cos \beta_{0} h$. Moreover, the phase of the current $I(z)$ along each element is assumed to be the same as that of the current $I(0)$ at the centre. This means that the current distribution function $f(z)$ in $I(z)=I(0) f(z)$ is assumed to be real for all parasitic elements.

It is unreasonable to suppose that these implied assumptions are generally valid when longer elements are involved. After all, the investigation in this section has been limited to relatively short elements with $h / \lambda \leqslant 0 \cdot 2$. It would appear that a more accurate representation of the currents in the parasitic elements is required this is suggested in Fig. 6.4 where the actual distributions of current even on the relatively short elements were not very accurately represented by the single shifted cosine term.

### 6.2 The problem of arrays with parasitic elements of unequal lengths

In order to provide a more accurate representation of the current in the parasitic elements of an array, use may be made of the threeterm approximation given in (3.18). This is known to be an improvement over the two-term theory used in chapter 5 and, when applied to parasitic elements, it provides two terms with complex coefficients instead of only a single term. Specifically let

$$
\begin{equation*}
I_{z k}\left(z_{k}\right)=A_{k} M_{0 z k}+B_{k} F_{0 z k}+D_{k} H_{0 z k} \tag{6.6}
\end{equation*}
$$

where

$$
\begin{align*}
M_{0 z k} & =\sin \beta_{0}\left(h_{k}-\left|z_{k}\right|\right)  \tag{6.7a}\\
F_{0 z k} & =\cos \beta_{0} z_{k}-\cos \beta_{0} h_{k}  \tag{6.7b}\\
H_{0 z k} & =\cos \frac{1}{2} \beta_{0} z_{k}-\cos \frac{1}{2} \beta_{0} h_{k} . \tag{6.7c}
\end{align*}
$$

In parasitic elements the coefficient $A_{k}$ is zero, but the two terms $B_{k} F_{0 z k}+D_{k} H_{0 z k}$ remain.

It is anticipated that the distribution (6.6) provides sufficient flexibility to represent the currents in elements of different lengths when each element is allowed to have its own length $2 h_{k}$.

When the several antennas in an array are not all equal in length so that the $h_{i}$ differ, the problem of solving the $N$ simultaneous integral equations

$$
\begin{equation*}
\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{k}}\left[\frac{1}{2} V_{0 k} M_{0 z k}+U_{k} F_{0 z k}\right] \tag{6.8}
\end{equation*}
$$

with $k=1,2, \ldots N$, is more complicated. The kernel has the form

$$
\begin{equation*}
K_{k i d}\left(z_{k}, z_{i}^{\prime}\right)=K_{k i}\left(z_{k}, z_{i}^{\prime}\right)-K_{k i}\left(h_{k}, z_{i}^{\prime}\right)=\frac{e^{-j \beta_{0} R_{k i}}}{R_{k i}}-\frac{e^{-j \beta_{0} R_{k i h}}}{R_{k i h}} \tag{6.9}
\end{equation*}
$$

where

$$
\begin{equation*}
R_{k i}=\sqrt{\left(z_{k}-z_{i}^{\prime}\right)^{2}+b_{k i}^{2}}, \quad R_{k i h}=\sqrt{\left(h_{k}-z_{i}^{\prime}\right)^{2}+b_{k i}^{2}} \tag{6.10}
\end{equation*}
$$

Note that $b_{k k}=a$. The function $U_{k}$ is

$$
\begin{equation*}
U_{k}=\frac{-j \zeta_{0}}{4 \pi} \sum_{i=1}^{N} \int_{-h_{k}}^{h_{k}} I_{z i}\left(z_{i}^{\prime}\right) K_{k i}\left(h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} . \tag{6.11}
\end{equation*}
$$

In a parasitic antenna $l$ the driving voltage $V_{0 l}=0$, so that

$$
\begin{equation*}
\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) K_{k i d}\left(z_{l}, z_{i}^{\prime}\right) d z_{i}^{\prime}=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{l}} U_{l} F_{0 z l} \tag{6.12}
\end{equation*}
$$

In order to obtain approximate solutions of the $N$ simultaneous integral equations (6.8) by the procedure developed in the earlier chapters, use may be made of the properties of the real and imaginary parts of the kernel. As shown in chapter 2,

$$
\begin{equation*}
\int_{-h_{k}}^{h_{k}} G_{0 z^{\prime} k} K_{k k d R}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \sim G_{0 z k} \tag{6.13}
\end{equation*}
$$

where $G_{0 z^{\prime} k}$ stands for $M_{0 z^{\prime} k}, F_{0 z^{\prime} k}$ or $H_{0 z^{\prime} k}$ and $K_{k k d R}\left(z_{k}, z_{k}^{\prime}\right)$ is the real part of the kernel. On the other hand,

$$
\begin{equation*}
\int_{-h_{k}}^{h_{k}} G_{0 z^{\prime} k} K_{k k d I}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \sim H_{0 z k} \tag{6.14}
\end{equation*}
$$

for any distribution $G_{0 z k}$. It follows that

$$
\begin{align*}
& W_{k k V}\left(z_{k}\right) \equiv \int_{-h_{k}}^{h_{k}} M_{0 z^{\prime} k} K_{k k d}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \doteq \Psi_{k k d V}^{m} M_{0 z k}+\Psi_{k k d V}^{h} H_{0 z k}  \tag{6.15}\\
& W_{k k U}\left(z_{k}\right) \equiv \int_{-h_{k}}^{h_{k}} F_{0 z^{\prime} k} K_{k k d}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \doteq \Psi_{k k d U} F_{0 z k}+\Psi_{k k d U}^{h} H_{0 z k} \tag{6.16}
\end{align*}
$$

$$
\begin{equation*}
W_{k k D}\left(z_{k}\right) \equiv \int_{-h_{k}}^{h_{k}} H_{0 z^{\prime} k} K_{k k d}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \doteq \Psi_{k k d D}^{f} F_{0 z k}+\Psi_{k k d D}^{h} H_{0 z k} \tag{6.17}
\end{equation*}
$$

where the $\Psi ' s$ are complex coefficients yet to be determined. Actually, (6.13) with $G=H$ and (6.14) suggest that the term $\Psi_{k k d D}^{h} H_{0 z k}$ should be an adequate approximation. The term $\Psi_{k k d D}^{f} F_{0 z k}$ is added in order to provide greater flexibility and symmetry.

When $i \neq k$ and $\beta_{0} b \geqslant 1$, it has been shown by direct comparison in chapter 3 that

$$
\begin{align*}
& \int_{-h_{i}}^{h_{i}} G_{0 z_{i}^{\prime}} K_{k i d R}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \sim F_{0 z k}  \tag{6.18}\\
& \int_{-h_{i}}^{h_{i}} G_{0 z^{\prime} i} K_{k i d I}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \sim H_{0 z k} \tag{6.19}
\end{align*}
$$

where $G_{0 z^{\prime} i}$ stands for $M_{0 z^{\prime} i}, F_{0 z^{\prime} i}$ or $H_{0 z^{\prime} i}$. It follows that with $i \neq k$,

$$
\begin{align*}
& W_{k i V}\left(z_{k}\right) \equiv \int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \doteq \Psi_{k i d V}^{f} F_{0 z k}+\Psi_{k i d V}^{h} H_{0 z k}  \tag{6.20}\\
& W_{k i U}\left(z_{k}\right) \equiv \int_{-h_{i}}^{h_{i}} F_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \doteq \Psi_{k i d U}^{f} F_{0 z k}+\Psi_{k i d U}^{h} H_{0 z k} \\
& W_{k i D}\left(z_{k}\right) \equiv \int_{-h_{i}}^{h_{i}} H_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \doteq \Psi_{k i d D}^{f} F_{0 z k}+\Psi_{k i d D}^{h} H_{0 z k} \tag{6.21}
\end{align*}
$$

where the $\Psi$ 's are complex coefficients yet to be determined.
In the formulation developed in the earlier chapters for driven elements of equal lengths, the coefficients $\Psi$ were defined individually in terms of the two integrals obtained from the real and imaginary parts of the kernel. In order to take account of the more varied distributions that may be obtained when the elements are neither all driven nor all equal in length, the separation into two parts is not made. Instead the entire integral is represented by a linear combination of the two distributions that best represent the parts of the integral. The complex coefficients of these distributions are to be determined by matching the integral and its approximation at two points along the antenna, instead of at only one such point.

It is anticipated that by fitting the trigonometric approximations to the integrals at $z=0, h_{k} / 2$, and $h_{k}$ (where both must vanish) a good representation may be achieved in reasonably simple form of all of the different distributions which may occur along antennas of unequal lengths. It is, of course, assumed that $\beta_{0} h_{i} \leqslant 5 \pi / 4$ for all $h_{i}$.

### 6.3 Application to the Yagi-Uda array

In order to clarify the description of the procedure used to solve the $N$ simultaneous integral equations for a parasitic array, it will be carried out in detail for the specific and practically useful Yagi-Uda array. In general, this consists of a curtain of $N$ antennas of which No. 1 is parasitic and adjusted in length to function as a reflector, No. 2 is driven by a voltage $V_{02}$ and Nos. 3 to $N$ are also parasitic and adjusted to act as directors. Such an array is shown in Fig. 6.8 for the special case (treated later) with $2 h_{1}=0.51 \lambda$; $2 h_{2}=0.50 \lambda ; 2 h_{i}=2 h, i>2 ; b_{21}=0.25 \lambda ; b_{i, i+1}=b, i>2$. The details of these adjustments are examined later.


Fig. 6.8. A Yagi array with directors of constant length, radius and spacing.
On the basis of the three-term approximation, the current in the single driven element has the form

$$
\begin{equation*}
I_{z 2}\left(z_{2}\right)=A_{2} M_{0 z 2}+B_{2} F_{0 z 2}+D_{2} H_{0 z 2} . \tag{6.23}
\end{equation*}
$$

The currents in the parasitic elements are

$$
\begin{equation*}
I_{z i}\left(z_{i}\right)=B_{i} F_{0 z i}+D_{i} H_{0 z i}, \quad i=1,3,4, \ldots N \tag{6.24}
\end{equation*}
$$

where the constants $A_{2}, B_{i}$ and $D_{i}$ must be evaluated ultimately in terms of $V_{02}$. The integral equation for the driven element is

$$
\begin{align*}
& A_{2} \int_{-h_{2}}^{h_{2}} M_{0 z^{\prime} 2} K_{22 d}\left(z_{2}, z_{2}^{\prime}\right) d z_{2}^{\prime}+\sum_{i=1}^{N} B_{i} F_{0 z^{\prime} i} K_{2 i d}\left(z_{2}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \quad+\sum_{i=1}^{N} D_{i} H_{0 z^{\prime} i} K_{2 i d}\left(z_{2}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
&=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{2}}\left[\frac{1}{2} V_{02} M_{0 z 2}+U_{2} F_{0 z 2}\right] . \tag{6.25}
\end{align*}
$$

The remaining $N-1$ integral equations are

$$
\begin{align*}
& A_{2} \int_{-h_{2}}^{h_{2}} M_{0 z^{\prime} 2} K_{k 2 d}\left(z_{k}, z_{2}^{\prime}\right) d z_{2}^{\prime}+\sum_{i=1}^{N} B_{i} F_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
&+\sum_{i=1}^{N} D_{i} H_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
&=\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{k}} U_{k} F_{0 z k}, \quad k=1,3,4, \ldots N . \tag{6.26}
\end{align*}
$$

With (6.15)-(6.17) and (6.20)-(6.22) these may be expressed in terms of the parameters $\Psi$. Thus, for (6.25)

$$
\begin{align*}
A_{2}\left[\Psi_{22 d V}^{m} M_{0 z 2}\right. & \left.+\Psi_{22 d V}^{h} H_{0 z 2}\right]+\sum_{i=1}^{N} B_{i}\left[\Psi_{2 i d U}^{f} F_{0 z 2}+\Psi_{2 i d U}^{h} H_{0 z 2}\right] \\
& +\sum_{i=1}^{N} D_{i}\left[\Psi_{2 i d D}^{f} F_{0 z 2}+\Psi_{2 i d D}^{h} H_{0 z 2}\right] \\
= & \frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{2}}\left[\frac{1}{2} V_{02} M_{0 z 2}+U_{2} F_{0 z 2}\right] \tag{6.27}
\end{align*}
$$

For (6.26), the $N-1$ equations are

$$
\begin{align*}
A_{2}\left[\Psi_{k 2 d V}^{f} F_{0 z k}\right. & \left.+\Psi_{k 2 d V}^{h} H_{0 z k}\right]+\sum_{i=1}^{N} B_{i}\left[\Psi \Psi_{k i d U}^{f} F_{0 z k}+\Psi_{k i d U}^{h} H_{0 z k}\right] \\
& +\sum_{i=1}^{N} D_{i}\left[\Psi \Psi_{k i d D} F_{0 z k}+\Psi_{k i d D}^{h} H_{0 z k}\right] \\
& =\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{k}} U_{k} F_{0 z k}, \quad k=1,3,4, \ldots N . \tag{6.28}
\end{align*}
$$

These equations will be satisfied if the coefficient of each of the three distribution functions is individually required to vanish. That is, in (6.27):

$$
\begin{equation*}
A_{2}=\frac{j 2 \pi V_{02}}{\zeta_{0} \Psi_{22 d V}^{m} \cos \beta_{0} h_{2}} \tag{6.29}
\end{equation*}
$$

$$
\begin{gather*}
\sum_{i=1}^{N}\left[B_{i} \Psi_{2 i d U}^{f}+D_{i} \Psi_{2 i d D}\right] \cos \beta_{0} h_{2}-\frac{j 4 \pi}{\zeta_{0}} U_{2}=0  \tag{6.30a}\\
A_{2} \Psi_{22 d V}^{h}+\sum_{i=1}^{N}\left[B_{i} \Psi_{2 i d U}^{h}+D_{i} \Psi_{2 i d D}^{h}\right]=0 \tag{6.30b}
\end{gather*}
$$

Similarly in (6.28) with $k=1,3, \ldots N$

$$
\begin{gather*}
\left\{A_{2} \Psi_{k 2 d V}^{f}+\sum_{i=1}^{N}\left[B_{i} \Psi_{k i d U}+D_{i} \Psi_{k i d D}^{f}\right]\right\} \cos \beta_{0} h_{k}-\frac{j 4 \pi}{\zeta_{0}} U_{k}=0  \tag{6.30c}\\
A_{2} \Psi_{k 2 d V}^{h}+\sum_{i=1}^{N}\left[B_{i} \Psi_{k i d U}^{h}+D_{i} \Psi_{k i d D}^{h}\right]=0 \tag{6.30d}
\end{gather*}
$$

Actually, the single equations in (6.30a) and (6.30b) may be combined with the $N-1$ equations in ( 6.30 c ) and ( 6.30 d ) with the aid of the Kronecker $\delta$ defined by

$$
\delta_{i k}= \begin{cases}0 & i \neq k \\ 1 & i=k\end{cases}
$$

The $2 N$ equations are

$$
\begin{array}{r}
\left\{A_{2}\left(1-\delta_{k 2}\right) \Psi_{k 2 d V}^{f}+\sum_{i=1}^{N}\left[B_{i} \Psi_{k i d U}^{f}+D_{i} \Psi_{k i d D}^{f}\right]\right\} \cos \beta_{0} h_{k}-\frac{j 4 \pi}{\zeta_{0}} U_{k}=0  \tag{6.31a}\\
A_{2} \Psi_{k 2 d V}^{h}+\sum_{i=1}^{N}\left[B_{i} \Psi_{k i d U}^{h}+D_{i} \Psi_{k i d D}^{h}\right]=0
\end{array}
$$

with $k=1,2, \ldots N$. These equations, together with (6.29), determine the $2 N+1$ constants $A_{2}, B_{i}$ and $D_{i}, i=1,2, \ldots N$.

Before these two sets of equations can be solved, it is necessary to evaluate the functions $U_{k}$. This is readily done in terms of the following integrals:

$$
\begin{align*}
& \Psi_{k i V}\left(h_{k}\right)=\int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i}\left(h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{6.32}\\
& \Psi_{k i v}\left(h_{k}\right)=\int_{-h_{i}}^{h_{i}} F_{0 z^{\prime} i} K_{k i}\left(h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{6.33}\\
& \Psi_{k i D}\left(h_{k}\right)=\int_{-h_{i}}^{h_{i}} H_{0 z^{\prime} i} K_{k i}\left(h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \tag{6.34}
\end{align*}
$$

where

$$
\begin{equation*}
K_{k i}\left(h_{k}, z_{i}^{\prime}\right)=\frac{e^{-j \beta_{0} R_{k i h}}}{R_{k i h}}, \quad R_{k i h}=\sqrt{\left(h_{k}-z_{i}^{\prime}\right)^{2}+b_{i k}} \tag{6.35}
\end{equation*}
$$

It follows from the definition in (6.11) that

$$
U_{k}=\frac{-j \zeta_{0}}{4 \pi} \sum_{i=1}^{N}\left[A_{i} \Psi_{k i v}\left(h_{k}\right)+B_{i} \Psi_{k i U}\left(h_{k}\right)+D_{i} \Psi_{k i D}\left(h_{k}\right)\right] . \text { (6.36a) }
$$

Since only antenna 2 is driven, $A_{i}=0, i \neq 2$ so that

$$
\begin{equation*}
U_{k}=\frac{-j \zeta_{0}}{4 \pi}\left\{A_{2} \Psi_{k 2 V}\left(h_{k}\right)+\sum_{i=1}^{N}\left[B_{i} \Psi_{k i v}\left(h_{k}\right)+D_{i} \Psi_{k i D}\left(h_{k}\right)\right]\right\} . \tag{6.36b}
\end{equation*}
$$

The substitution of (6.36b) in (6.31a) gives for these equations

$$
\begin{align*}
A_{2}\left[\Psi_{k 2 V}\left(h_{k}\right)\right. & \left.-\left(1-\delta_{k 2}\right) \Psi_{k 2 d V}^{f} \cos \beta_{0} h_{k}\right]+\sum_{i=i}^{N} B_{i}\left[\Psi_{k i U}\left(h_{k}\right)\right. \\
& \left.-\Psi_{k i d U}^{f} \cos \beta_{0} h_{k}\right]+\sum_{i=1}^{N} D_{i}\left[\Psi_{k i D}\left(h_{k}\right)-\Psi_{k i d D}^{f} \cos \beta_{0} h_{k}\right]=0 \tag{6.37}
\end{align*}
$$

with $k=1,2, \ldots N$. These equations can be simplified formally by the introduction of the notation

$$
\begin{gather*}
\Phi_{k 2 V}=\Psi_{k 2 V}\left(h_{k}\right)-\left(1-\delta_{k 2}\right) \Psi_{k 2 d V} \cos \beta_{0} h_{k}  \tag{6.38}\\
\Phi_{k i U}=\Psi_{k i U}\left(h_{k}\right)-\Psi_{k i d}^{f} \cos \beta_{0} h_{k}  \tag{6.39a}\\
\Phi_{k i D}=\Psi_{k i D}\left(h_{k}\right)-\Psi_{k i d D} \cos \beta_{0} h_{k} \tag{6.39b}
\end{gather*}
$$

With this notation, (6.37) together with (6.31b) gives the following set of $2 N$ equations for determining the $2 N$ coefficients $B_{i}$ and $D_{i}$ in terms of $A_{2}$ :

$$
\begin{array}{ll}
\sum_{i=1}^{N}\left[\Phi_{k i U} B_{i}+\Phi_{k i D} D_{i}\right]=-\Phi_{k 2 V} A_{2} ; & k=1,2, \ldots N \\
\sum_{i=1}^{N}\left[\Psi_{k i d U}^{h} B_{i}+\Psi_{k i d D}^{h} D_{i}\right]=-\Psi_{k 2 d V}^{h} A_{2} ; & k=1,2, \ldots N \tag{6.41}
\end{array}
$$

These equations may be expressed in matrix form after the introduction of the following notation:

$$
\begin{align*}
& {\left[\Phi_{U}\right]=\left[\begin{array}{lll}
\Phi_{11 U} & \Phi_{12 U} \ldots \Phi_{1 N U} \\
\vdots & & \\
\Phi_{N 1 U} & \ldots & \Phi_{N N U}
\end{array}\right]}  \tag{6.42a}\\
& {\left[\Phi_{D}\right]=\left[\begin{array}{lll}
\Phi_{11 D} & \Phi_{12 D} \ldots \Phi_{1 N D} \\
\vdots & & \\
\Phi_{N 1 D} & \cdots & \Phi_{N N D}
\end{array}\right]} \tag{6.42b}
\end{align*}
$$

$$
\begin{align*}
& {\left[\Psi_{d U}^{h}\right]=\left[\begin{array}{llll}
\Psi_{11 d U}^{h} & \Psi_{12 d U}^{h} & \cdots & \Psi_{1 N d U}^{h} \\
\vdots & & \\
\Psi_{N 1 d U}^{h} & \cdots & & \Psi_{N N d U}^{h}
\end{array}\right]}  \tag{6.43a}\\
& {\left[\Psi_{d D}^{h}\right]=\left[\begin{array}{lll}
\Psi_{11 d D}^{h} & \Psi_{12 d D}^{h} \cdots \Psi_{1 N d D}^{h} \\
\vdots & & \\
\Psi_{N 1 d D}^{h} & \cdots & \Psi_{N N d D}^{h}
\end{array}\right]}  \tag{6.43b}\\
& \left\{\Phi_{2 V}\right\}=\left\{\begin{array}{l}
\Phi_{12 V} \\
\Phi_{22 V} \\
\vdots \\
\Phi_{N 2 V}
\end{array}\right\}  \tag{6.44}\\
& \left\{\Psi_{2 d V}^{h}\right\}=\left\{\begin{array}{l}
\Psi_{12 d V}^{h} \\
\Psi_{22 d V}^{h} \\
\vdots \\
\Psi_{N 2 d V}^{h}
\end{array}\right\} \\
& \{B\}=\left\{\begin{array}{l}
B_{1} \\
B_{2} \\
\vdots \\
B_{N}
\end{array}\right\} \quad\{D\}=\left\{\begin{array}{c}
D_{1} \\
D_{2} \\
\vdots \\
D_{N}
\end{array}\right\} . \tag{6.45}
\end{align*}
$$

The matrix forms of (6.40) and (6.41) are

$$
\begin{align*}
{\left[\Phi_{U}\right]\{B\}+\left[\Phi_{D}\right]\{D\} } & =-\left\{\Phi_{2 V}\right\} A_{2}  \tag{6.46}\\
{\left[\Psi_{d U}^{h}\right]\{B\}+\left[\Psi_{d D}^{h}\right]\{D\} } & =-\left\{\Psi_{2 d V}^{h}\right\} A_{2} . \tag{6.47}
\end{align*}
$$

The $N$ coefficients $B_{i}$ and the $N$ coefficients $D_{i}$ must be determined from these equations for substitution in the equations (6.23) and (6.24) for the currents in the $N$ elements. The coefficient $A_{2}$, which is a common factor, is obtained from (6.29) in terms of the single driving voltage $V_{02}$.

It remains to evaluate the parameters $\Psi$ that occur in the $\Phi$ 's in (6.46) and explicitly in (6.47).

### 6.4 Evaluation of coefficients for the Yagi-Uda array

The equations (6.46) and (6.47) involve the elements of the $N \times N$ matrices $\left[\Phi_{U}\right],\left[\Phi_{D}\right],\left[\Psi_{d U}^{h}\right]$ and $\left[\Psi_{d D}^{h}\right]$. These, in turn, depend on the parameters $\Psi$ introduced in (6.15)-(6.17) and (6.20)-(6.22) and the parameters $\Psi(h)$ defined in (6.32)-(6.34). Since each integral is approximated by a linear combination of two terms with arbitrary coefficients, these can be evaluated by equating both sides in
(6.15)-(6.17) and (6.20)-(6.22) at two values of $z$. The values chosen are $z=0$, and $z=h_{k} / 2$ in addition to $z=h_{k}$ where both sides must vanish.

Specific formulas for the two values of each of the integrals $W$ defined in (6.15)-(6.17) and (6.20)-(6.22) are as follows:

$$
W_{k i \nu}(0) \equiv A_{i}^{-1} \int_{-h_{i}}^{h_{i}} I_{V i}\left(z_{i}^{\prime}\right) K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime} \doteq \int_{-h_{i}}^{h_{i}} M_{0 z_{i}} K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime}
$$

$$
\begin{align*}
& W_{k i v}\left(\frac{h_{k}}{2}\right) \equiv A_{i}^{-1} \int_{-h_{i}}^{h_{i}} I_{V i}\left(z_{i}^{\prime}\right) K_{k i d}\left(\frac{h_{k}}{2}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{6.48a}\\
& \doteq \int_{-h_{i}}^{h_{i}} M_{0 z_{i}^{\prime}} K_{k i d}\left(\frac{h_{k}}{2}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{6.48b}\\
& W_{k i v}(0) \equiv B_{i}^{-1} \int_{-h_{i}}^{h_{i}} I_{U i}\left(z_{i}^{\prime}\right) K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime} \doteq \int_{-h_{i}}^{h_{i}} F_{0 z^{\prime} i} K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime}
\end{align*}
$$

$$
\begin{gather*}
W_{k i U}\left(\frac{h_{k}}{2}\right) \equiv B_{i}^{-1} \int_{-h_{i}}^{h_{i}} I_{U i}\left(z_{i}^{\prime}\right) K_{k i d}\left(\frac{h_{k}}{2}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{6.49a}\\
\doteq \int_{-h_{i}}^{h_{i}} F_{0 z^{\prime} i} K_{k i d}\left(\frac{h_{k}}{2}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{6.49b}\\
W_{k i D}(0) \equiv D_{i}^{-1} \int_{-h_{i}}^{h_{i}} I_{D i}\left(z_{i}^{\prime}\right) K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime} \doteq \int_{-h_{i}}^{h_{i}} H_{0 z^{\prime} i} K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime} \tag{6.50a}
\end{gather*}
$$

$$
\begin{align*}
W_{k i D}\left(\frac{h_{k}}{2}\right) & \equiv D_{i}^{-1} \int_{-h_{i}}^{h_{i}} I_{D i}\left(z_{i}^{\prime}\right) K_{k i d}\left(\frac{h_{k}}{2}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \doteq \int_{-h_{i}}^{h_{i}} H_{0 z^{\prime} i} K_{k i d}\left(\frac{h_{k}}{2}, z_{i}^{\prime}\right) d z_{i}^{\prime} . \tag{6.50b}
\end{align*}
$$

In all of the above, $k=1,2,3, \ldots N$. These are a set of complex numbers which give the values of the integrals (6.20)-(6.22) at the two points $z=0$ and $z=h_{k} / 2$. They are readily evaluated numerically by high-speed computer, or they may be expressed in terms of the tabulated generalized sine and cosine integral functions. Once the $W$ 's in ( 6.48 a$)-(6.50 \mathrm{~b}$ ) have been obtained for all values of $i$ and $k$, the coefficients $\Psi$ may be determined from the equations (6.15)-(6.17) and (6.20)-(6.22). At $z=0$ these become:

$$
\Psi_{k k d V}^{m} \sin \beta_{0} h_{k}+\Psi_{k k d V}^{h}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]=W_{k k V}(0) \quad \text { (6.51a) }
$$

$$
\begin{array}{cc}
\Psi_{k i d V}^{f}\left(1-\cos \beta_{0} h_{k}\right)+\Psi_{k i d V}^{h}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]=W_{k i V}(0) & i \neq k \\
& (6.51 \mathrm{~b})  \tag{6.51b}\\
\Psi_{k i d U}^{f}\left(1-\cos \beta_{0} h_{k}\right)+\Psi_{k i d U}^{h}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]=W_{k i U}(0) & (6.51 \mathrm{c}) \\
\Psi_{k i d D}^{f}\left(1-\cos \beta_{0} h_{k}\right)+\Psi_{k i d D}^{h}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]=W_{k i D}(0) . & (6.51 \mathrm{~d})
\end{array}
$$

At $z=h_{k} / 2$, they are

$$
\Psi_{k k d V}^{m} \sin \left(\beta_{0} h_{k} / 2\right)+\Psi_{k k d V}^{h}\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right]=W_{k k V}\left(\frac{h_{k}}{2}\right)
$$

$$
\begin{gather*}
\Psi_{k i d V}^{f}\left[\cos \left(\beta_{0} h_{k} / 2\right)-\cos \beta_{0} h_{k}\right]+\Psi_{k i d V}^{h}\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right]  \tag{6.52a}\\
=W_{k i V}\left(\frac{h_{k}}{2}\right) \quad i \neq k \tag{6.52b}
\end{gather*}
$$

$$
\begin{align*}
\Psi_{k i d U}^{f}\left[\cos \left(\beta_{0} h_{k} / 2\right)-\right. & \left.\cos \beta_{0} h_{k}\right]+\Psi_{k i d U}^{h}\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] \\
& =W_{k i U}\left(\frac{h_{k}}{2}\right) \tag{6.52c}
\end{align*}
$$

$$
\begin{align*}
\Psi_{k i d D}^{f}\left[\cos \left(\beta_{0} h_{k} / 2\right)-\right. & \left.\cos \beta_{0} h_{k}\right]+\Psi_{k i d D}^{h}\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] \\
& =W_{k i D}\left(\frac{h_{k}}{2}\right) \tag{6.52d}
\end{align*}
$$

The solutions of these equations for the $\Psi$ 's are obtained directly. They are

$$
\begin{align*}
\Psi_{k k d V}^{m}= & \Delta_{1}^{-1}\left\{W_{k k V}(0)\left[\cos \left(\frac{\beta_{0} h_{k}}{4}\right)-\cos \left(\frac{\beta_{0} h_{k}}{2}\right)\right]\right. \\
& \left.-W_{k k V}\left(h_{k} / 2\right)\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\}  \tag{6.53}\\
\Psi_{k k d V}^{h}= & \Delta_{1}^{-1}\left\{W_{k k V}\left(h_{k} / 2\right) \sin \beta_{0} h_{k}-W_{k k V}(0) \sin \left(\beta_{0} h_{k} / 2\right)\right\}(6.54) \\
\Psi_{k i d V}^{f}= & \Delta_{2}^{-1}\left\{W_{k i V}(0)\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right. \\
& \left.-W_{k i V}\left(h_{k} / 2\right)\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\} \quad i \neq k  \tag{6.55}\\
\Psi_{k i d V}^{h}= & \Delta_{2}^{-1}\left\{W_{k i v}\left(h_{k} / 2\right)\left[1-\cos \beta_{0} h_{k}\right]\right. \\
& \left.-W_{k i V}(0)\left[\cos \left(\beta_{0} h_{k} / 2\right)-\cos \beta_{0} h_{k}\right]\right\} \quad i \neq k  \tag{6.56}\\
\Psi_{k i d U}^{f}= & \Delta_{2}^{-1}\left\{W_{k i U}(0)\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right. \\
& \left.-W_{k i V}\left(h_{k} / 2\right)\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\}  \tag{6.57}\\
\Psi_{k i d U}^{h}= & \Delta_{2}^{-1}\left\{W_{k i V}\left(h_{k} / 2\right)\left[1-\cos \beta_{0} h_{k}\right]\right. \\
& \left.-W_{k i U}(0)\left[\cos \left(\beta_{0} h_{k} / 2\right)-\cos \left(\beta_{0} h_{k}\right)\right]\right\}  \tag{6.58}\\
\Psi_{k i d D}^{f}= & \Delta_{2}^{-1}\left\{W_{k i D}(0)\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right. \\
& \left.-W_{k i D}\left(h_{k} / 2\right)\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\} \tag{6.59}
\end{align*}
$$

$$
\begin{align*}
\Psi_{k i d D}^{h}= & \Delta_{2}^{-1}\left\{W_{k i D}\left(h_{k} / 2\right)\left[1-\cos \beta_{0} h_{k}\right]\right. \\
& \left.-W_{k i D}(0)\left[\cos \left(\beta_{0} h_{k} / 2\right)-\cos \beta_{0} h_{k}\right]\right\} \tag{6.60}
\end{align*}
$$

where

$$
\begin{align*}
\Delta_{1}= & \sin \beta_{0} h_{k}\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] \\
& -\sin \left(\beta_{0} h_{k} / 2\right)\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right] \tag{6.61}
\end{align*}
$$

and

$$
\begin{align*}
\Delta_{2}= & {\left[1-\cos \beta_{0} h_{k}\right]\left[\cos \left(\beta_{0} h_{k} / 4\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] } \\
& -\left[\cos \left(\beta_{0} h_{k} / 2\right)-\cos \beta_{0} h_{k}\right]\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right] . \tag{6.62}
\end{align*}
$$

All of the $\Psi$ 's have been determined. The $\Psi(h)$ coefficients are given in (6.32)-(6.34). The elements of the $\Phi$ matrices are obtained from (6.38)-(6.39b). This completes the solution for all of the currents in the elements of the Yagi-Uda array.

### 6.5 Arrays with half-wave elements

When an array includes half-wave parasitic elements the formulation in sections 6.3 and 6.4 is directly applicable. Specifically, when $\beta_{0} h_{i}=\pi / 2$ and element $i$ is parasitic, the current (6.24) has the form

$$
\begin{equation*}
I_{z i}\left(z_{i}\right)=B_{i} \cos \beta_{0} z_{i}+D_{i}\left[\cos \left(\beta_{0} z_{i} / 2\right)-\sqrt{2} / 2\right] . \tag{6.63}
\end{equation*}
$$

If the length of the driven element 2 is such that $\beta_{0} h_{2}$ is near or exactly $\pi / 2$ (as in Fig. 6.8), the alternative form for the current given in (2.35) for the isolated antenna is more convenient since it does not yield an indeterminate form at $\beta_{0} h_{2}=\pi / 2$. That is, in the notation of (6.23),

$$
\begin{equation*}
I_{z 2}\left(z_{2}\right)=A_{2}^{\prime} S_{0 z 2}+B_{2}^{\prime} F_{0 z 2}+D_{2} H_{0 z 2} \tag{6.64}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{0 z 2}=\sin \beta_{0}\left|z_{2}\right|-\sin \beta_{0} h_{2} \tag{6.65}
\end{equation*}
$$

and

$$
\begin{align*}
& A_{2}^{\prime}=-A_{2} \cos \beta_{0} h_{2}=-j\left(2 \pi V_{02} / \zeta_{0} \Psi_{22 d V}^{m}\right)  \tag{6.66a}\\
& B_{2}^{\prime}=B_{2}+A_{2} \sin \beta_{0} h_{2}=B_{2}-A_{2}^{\prime} \tan \beta_{0} h_{2} . \tag{6.66b}
\end{align*}
$$

Note that $A_{2}^{\prime}$ and $B_{2}^{\prime}$ are finite when $\beta_{0} h=\pi / 2$. In this case

$$
\begin{equation*}
S_{0 z 2}=\sin \beta_{0}\left|z_{2}\right|-1, \quad B_{2}^{\prime}=B_{2}+A_{2} \tag{6.6}
\end{equation*}
$$

Since (6.64) is not actually a different distribution from the original in (6.23) but merely a rearrangement that is more convenient when $\beta_{0} h_{2}$ is at or near $\pi / 2$, it is not necessary to repeat the formulation in the preceding sections with $S_{0 z 2}$ substituted for $M_{0 z 2}$. A simple rearrangement of the $2 N$ equations in (6.40) and (6.41) is all that is required. This is accomplished by the substitutions (6.66a) and (6.66b) for $A_{2}$ and $B_{2}$. Specifically, let

$$
\begin{equation*}
A_{2}=-A_{2}^{\prime} \sec \beta_{0} h_{2}, \quad B_{2}=B_{2}^{\prime}+A_{2}^{\prime} \tan \beta_{0} h_{2} \tag{6.68}
\end{equation*}
$$

$$
\begin{gather*}
\Phi_{k 2 V}^{\prime}=\left[\Phi_{k 2 V}-\Phi_{k 2 U} \sin \beta_{0} h_{2}\right] \sec \beta_{0} h_{2}  \tag{6.69}\\
\Psi_{k 2 d V}^{\prime \prime}=\left[\Psi_{k 2 d V}^{h}-\Psi_{k 2 d U}^{h} \sin \beta_{0} h_{2}\right] \sec \beta_{0} h_{2} . \tag{6.70}
\end{gather*}
$$

Also let $B_{i}^{\prime}$ stand for $B_{1}, B_{2}^{\prime}, B_{3}, \ldots B_{N}$. With this notation, the equations (6.40) and (6.41) become:

$$
\begin{array}{ll}
\sum_{i=1}^{N}\left[\Phi_{k i U} B_{i}^{\prime}+\Phi_{k i D} D_{i}\right]=\Phi_{k 2 V}^{\prime} A_{2}^{\prime} ; & k=1,2, \ldots N \\
\sum_{i=1}^{N}\left[\Psi_{k i d U}^{h} B_{i}^{\prime}+\Psi_{k i d D}^{h} D_{i}\right]=\Psi_{k 2 d V}^{\prime h} A_{2}^{\prime} ; & k=1,2, \ldots N . \tag{6.72}
\end{array}
$$

In matrix form these are

$$
\begin{gather*}
{\left[\Phi_{U}\right]\left\{B^{\prime}\right\}+\left[\Phi_{D}\right]\{D\}=\left\{\Phi_{2 V}^{\prime}\right\} A_{2}^{\prime}}  \tag{6.73}\\
{\left[\Psi_{d U}^{h}\right]\left\{B^{\prime}\right\}+\left[\Psi_{d D}^{h}\right]\{D\}=\left\{\Psi_{2 d V}^{\prime}\right\} A_{2}^{\prime}} \tag{6.74}
\end{gather*}
$$

where the four square matrices and the column matrix $\{D\}$ are defined in $(6.42 \mathrm{a}, \mathrm{b}),(6.43 \mathrm{a}, \mathrm{b})$ and $(6.45)$. The other column matrices are

$$
\left\{B^{\prime}\right\}=\left\{\begin{array}{c}
B_{1}  \tag{6.75}\\
B_{2}^{\prime} \\
B_{3} \\
\vdots \\
B_{N}
\end{array}\right\}, \quad\left\{\Phi_{2 V}^{\prime}\right\}=\left\{\begin{array}{c}
\Phi_{12 V}^{\prime} \\
\Phi_{22 V}^{\prime} \\
\Phi_{32 V}^{\prime} \\
\vdots \\
\Phi_{N 2 V}^{\prime}
\end{array}\right\}, \quad\left\{\Psi_{2 d V}^{\prime}\right\}=\left\{\begin{array}{l}
\Psi_{12 d V}^{\prime} \\
\Psi_{22 V V}^{\prime} \\
\Psi_{32 d V}^{\prime} \\
\vdots \\
\Psi_{N 2 d V}^{\prime}
\end{array}\right\}
$$

These equations are to be solved for the $2 N$ coefficients $B_{i}^{\prime}$ and $D_{i}$ in terms of $A_{2}^{\prime}=-j\left(2 \pi V_{02} / \zeta_{0} \Psi_{22 d V}^{m}\right)$. The $\Psi$ functions that occur in these equations are defined in the same manner as in sections 6.3 and 6.4. This is illustrated below for $\beta_{0} h=\pi / 2$.

When $\beta_{0} h=\pi / 2, F_{0 z 2}=M_{0 z 2}=\cos \beta_{0} z$. It follows from (6.48a) and (6.49) that $W_{22 v}(0)=W_{22 U}(0), W_{22 V}\left(h_{2} / 2\right)=W_{22 U}\left(h_{2} / 2\right)$. Hence, from (6.53) and (6.57), (6.54) and (6.58), it follows $\Psi_{k 2 v}\left(z_{k}\right)=\Psi_{k 2 U}\left(z_{k}\right)$ and $\Delta_{1}=\Delta_{2}$. This means that $\Psi_{k 2 d V}^{f}=\Psi_{k 2 d U}^{f}$ and $\Psi_{k 2 d V}^{h}=\Psi_{k 2 d U}^{h}$ when $k \neq 2$ and $\Psi_{22 d V}^{m}=\Psi_{22 d U}^{f}, \quad \Psi_{22 d V}^{h}=\Psi_{22 d U}^{h}$ when $k=2$. As a consequence, $\Phi_{k 2 V}^{\prime}$ becomes indeterminate in the form $0 / 0$ when $k \neq 2$. However, the limiting value as $\beta_{0} h_{2} \rightarrow \pi / 2$ is finite. Thus, (6.69) may be expanded as follows. When $k=2$,

$$
\begin{equation*}
\Phi_{22 V}^{\prime}=-S_{a}\left(h_{2}, h_{2}\right)+E_{a}\left(h_{2}, h_{2}\right)+\Psi_{22 d U}^{f} \tag{6.76a}
\end{equation*}
$$

when $k \neq 2$,

$$
\begin{align*}
\Phi_{k 2 V}^{\prime}= & -S_{b_{k 2}}\left(h_{2}, h_{k}\right)+E_{b_{k 2}}\left(h_{2}, h_{k}\right) \\
& +\frac{\cos \beta_{0} h_{k}}{\Delta_{2}}\left\{[ \operatorname { c o s } \frac { 1 } { 4 } \beta _ { 0 } h _ { k } - \operatorname { c o s } \frac { 1 } { 2 } \beta _ { 0 } h _ { k } ] \left[-E_{b_{k 2}}\left(h_{2}, 0\right)\right.\right. \\
& \left.+E_{b_{k 2}}\left(h_{2}, h_{k}\right)+S_{b_{k 2}}\left(h_{2}, 0\right)-S_{b_{k 2}}\left(h_{2}, h_{k}\right)\right] \\
& +\left(1-\cos \frac{1}{2} \beta_{0} h_{k}\right)\left[E_{b_{k 2}}\left(h_{2}, \frac{h_{k}}{2}\right)\right. \\
& \left.\left.-E_{b_{k 2}}\left(h_{2}, h_{k}\right)-S_{b_{k 2}}\left(h_{2}, \frac{h_{k}}{2}\right)+S_{b_{k 2}}\left(h_{2}, h_{k}\right)\right]\right\} \tag{6.76b}
\end{align*}
$$

where $\Delta_{2}$ is defined in (6.62). Similarly, when $k=2$,

$$
\begin{align*}
\Psi_{22 d V}^{\prime h}= & \frac{1-\sqrt{2}}{\Delta_{2}}\left\{\left[C_{a}\left(h_{2}, \frac{h_{2}}{2}\right)-C_{a}\left(h_{2}, h_{2}\right)\right]\left[1-\frac{1}{\sqrt{2}}\right]\right. \\
& \left.-\left[C_{a}\left(h_{2}, 0\right)-C_{a}\left(h_{2}, h_{2}\right)\right]\left[\cos \frac{\pi}{8}-\frac{1}{\sqrt{2}}\right]\right\} \\
& +\frac{1}{\Delta_{2}}\left\{\left[-S_{a}\left(h_{2}, \frac{h_{2}}{2}\right)+S_{a}\left(h_{2}, h_{2}\right)\right.\right. \\
& \left.+E_{a}\left(h_{2}, \frac{h_{2}}{2}\right)-E_{a}\left(h_{2}, h_{2}\right)\right] \\
& \left.+\frac{1}{\sqrt{2}}\left[S_{a}\left(h_{2}, 0\right)-S_{a}\left(h_{2}, h_{2}\right)-E_{a}\left(h_{2}, 0\right)+E_{a}\left(h_{2}, h_{2}\right)\right]\right\} ; \tag{6.76c}
\end{align*}
$$

when $k \neq 2$,

$$
\begin{align*}
\Psi_{k 2 d V}^{\prime h}= & \frac{1}{\Delta_{2}}\left\{[ 1 - \operatorname { c o s } \beta _ { 0 } h _ { k } ] \left[-S_{b_{k 2}}\left(h_{2}, \frac{h_{k}}{2}\right)+S_{b_{k 2}}\left(h_{2}, h_{k}\right)\right.\right. \\
& \left.+E_{b_{k 2}}\left(h_{2}, \frac{h_{k}}{2}\right)-E_{b_{k 2}}\left(h_{2}, h_{k}\right)\right] \\
& +\left[\cos \frac{1}{2} \beta_{0} h_{k}-\cos \beta_{0} h_{k}\right]\left[S_{b_{k 2}}\left(h_{2}, 0\right)-S_{b_{k 2}}\left(h_{2}, h_{k}\right)\right. \\
& \left.\left.-E_{b_{k 2}}\left(h_{2}, 0\right)+E_{b_{k 2}}\left(h_{2}, h_{k}\right)\right]\right\} . \tag{6.76d}
\end{align*}
$$

The coefficients $B_{i}^{\prime}$ and $D_{i}$ obtained for $\beta_{0} h_{2}=\pi / 2$ from (6.73) and (6.74) with (6.76) are to be used in the current distributions

$$
\begin{gather*}
I_{z 2}\left(z_{2}\right)=A_{2}^{\prime} S_{0 z}+B_{2}^{\prime} F_{0 z}+D_{2} H_{0 z}  \tag{6.77}\\
I_{z i}\left(z_{i}\right)=B_{i}^{\prime} F_{0 z}+D_{i} H_{0 z}, \quad i=1,3, \ldots N . \tag{6.78}
\end{gather*}
$$

In the original analysis of arrays with half-wave elements [2] and in its application to arrays of the Yagi type [3], a somewhat different procedure was used. In effect, this treated the alternative form (6.64) of the distribution of current along a driven half-wave element as an independent representation. The entire procedure carried through in sections 6.3 and 6.4 was repeated with the distribution function $M_{0 z 2}$ replaced by $S_{0 z 2}$. This also involved a simple rearrangement of the integral equations (6.8) so that when $k=2$, the right-hand member is $\left(j 4 \pi / \zeta_{0}\right)\left[\frac{1}{2} V_{02} S_{0 z 2}+C_{2} F_{0 z 2}\right]$.

The alternative procedure is basically equivalent to that outlined in section 6.5 but the two are not identical and involve small quantitative differences when applied to a particular array. In particular, the values of $W_{22 V}(0)$ and $W_{22 V}\left(h_{2} / 2\right)$ from $(6.45 \mathrm{a}, \mathrm{b})$ are necessarily somewhat different when, with $\beta_{0} h=\pi / 2, S_{0 z 2}=\sin \beta_{01} z_{21}-1$ is substituted for $M_{0 z 2}=\cos \beta_{0} z_{2}$ in the integrals. It follows that the two values of $\Psi_{22 d V}^{m}$ as defined in (6.53) are also not quite the same when $S_{0 z 2}$ is used instead of $M_{0 z 2}$. These differences are small and either procedure should give satisfactory results, although in the interest of simplicity and consistency the generalization in section 6.3 is to be preferred.

Reference is here made to the alternative procedure primarily because it was used by Morris in an extensive quantitative study of the Yagi-Uda array. The results of his work, described later in this chapter, differ negligibly from those actually given.

### 6.6 The far field of the Yagi-Uda array; gain

The electric field maintained at distant points by the currents in the $N$ elements of the Yagi-Uda array is readily determined. For the currents

$$
\begin{align*}
& I_{z 2}\left(z_{2}\right)=A_{2} \sin \beta_{0}\left(h_{2}-\left|z_{2}\right|\right)+B_{2}\left(\cos \beta_{0} z_{2}-\cos \beta_{0} h_{2}\right) \\
&  \tag{6.79a}\\
& \quad+D_{2}\left(\cos \frac{1}{2} \beta_{0} z_{2}-\cos \frac{1}{2} \beta_{0} h_{2}\right)  \tag{6.79b}\\
& I_{z i}\left(z_{i}\right)=B_{i}\left(\cos \beta_{0} z_{i}-\cos \beta_{0} h_{i}\right)+D_{i}\left(\cos \frac{1}{2} \beta_{0} z_{i}-\cos \frac{1}{2} \beta_{0} h_{i}\right), \\
& i \neq 2
\end{align*}
$$

the electromagnetic field is

$$
\begin{align*}
& E_{\Theta}\left(R_{2}, \Theta, \Phi\right)=\frac{j \zeta_{0}}{2 \pi}\left\{A_{2} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} F_{m}\left(\Theta, \beta_{0} h_{2}\right)\right. \\
& \left.\quad+\sum_{i=1}^{N} \frac{e^{-j \beta_{0} R_{i}}}{R_{i}}\left[B_{i} G_{m}\left(\Theta, \beta_{0} h_{i}\right)+D_{i} D_{m}\left(\Theta, \beta_{0} h_{i}\right)\right]\right\} \tag{6.80}
\end{align*}
$$

where $F_{m}\left(\Theta, \beta_{0} h\right), G_{m}\left(\Theta, \beta_{0} h\right)$ and $D_{m}\left(\Theta, \beta_{0} h\right)$ are defined in (2.46)(2.48) and $R_{i}$ is the distance from the point of calculation to the centre of element $i$. This may be rearranged as follows:

$$
\begin{equation*}
E_{\Theta N}\left(R_{2}, \Theta, \Phi\right)=-\frac{V_{02}}{\Psi} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} f_{V N}(\Theta, \Phi) \tag{6.81a}
\end{equation*}
$$

Since no ambiguity can arise the symbol $\Psi$ without subscripts and superscripts is used for $\Psi_{22 d V}^{m}$ as defined in (6.53). The field factor in (6.81a) for the $N$-element array is given by

$$
\begin{align*}
f_{V N}(\Theta, \Phi)= & \left\{F_{m}\left(\Theta, \beta_{0} h_{2}\right)+\sum_{i=1}^{N} e^{-j \beta_{0}\left(R_{i}-R_{2}\right)}\left[T_{U i} G_{m}\left(\Theta, \beta_{0} h_{i}\right)\right.\right. \\
& \left.\left.+T_{D i} D_{m}\left(\Theta, \beta_{0} h_{i}\right)\right]\right\} \sec \beta_{0} h_{2} \tag{6.81b}
\end{align*}
$$

In obtaining ( $6.81 \mathrm{a}, \mathrm{b}$ ) the far-field approximation, $R_{i} \doteq R_{2}$, in amplitudes has been made. In the spherical coordinates $R_{2}, \Theta, \Phi$, (Fig. 6.9), and $b_{i, i \pm 1}=b$,

$$
\begin{equation*}
R_{i}-R_{2}=-(i-2) b \sin \Theta \cos \Phi \tag{6.81c}
\end{equation*}
$$

The following set of parameters has been introduced:

$$
\begin{equation*}
T_{U i}=B_{i} / A_{2}, \quad T_{D i}=D_{i} / A_{2} \tag{6.82}
\end{equation*}
$$



Fig. 6.9. Coordinates for 4-element array referred to origin at centre of element 2.

$$
b_{21}=b_{23}=b_{34}=b
$$

where $A_{2}=j 2 \pi V_{02} / \zeta_{0} \Psi \cos \beta_{0} h$. The quantity $E_{\Theta}(\Theta, \Phi) / V_{02}$ is the far field per unit voltage driving element 2.

An alternative expression for the field per unit input current to the driven antenna 2, i.e. $E_{\Theta}(\Theta, \Phi) / I_{z 2}(0)$, is readily obtained with the substitution of $V_{02}=I_{z 2}(0) / Y_{2 N}$ where, from (6.23), the input admittance of antenna 2 when driving the $N$-element array is

$$
\begin{align*}
Y_{2 N}=\frac{I_{z 2}(0)}{V_{02}}= & \frac{j 2 \pi}{\zeta_{0} \Psi \cos \beta_{0} h}\left[\sin \beta_{0} h_{2}+T_{U 2}\left(1-\cos \beta_{0} h_{2}\right)\right. \\
& \left.+T_{D 2}\left(1-\cos \frac{1}{2} \beta_{0} h_{2}\right)\right] . \tag{6.83}
\end{align*}
$$

The result is

$$
\begin{equation*}
E_{\Theta N}\left(R_{2}, \Theta, \Phi\right)=\frac{j \zeta_{0} I_{22}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} f_{I N}(\Theta, \Phi) \tag{6.84a}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{I N}(\Theta, \Phi)=\left\{\frac{F_{m}\left(\Theta, \beta_{0} h_{2}\right)+\sum_{i=1}^{N} e^{-j \beta_{0}\left(R_{i}-R_{2}\right.}\left[T_{U i} G_{m}\left(\Theta, \beta_{0} h_{i}\right)+T_{D i} D_{m}\left(\Theta, \beta_{0} h_{i}\right]\right]}{\sin \beta_{0} h_{2}+T_{U 2}\left(1-\cos \beta_{0} h_{2}\right)+T_{D 2}\left(1-\cos \frac{1}{2} \beta_{0} h_{2}\right)}\right\} . \tag{6.84b}
\end{equation*}
$$

If the driven element is near a half wavelength long, the more convenient alternative form of the current is

$$
\begin{align*}
I_{z 2}\left(z_{2}\right)= & A_{2}^{\prime}\left(\sin \beta_{0}\left|z_{2}\right|-\sin \beta_{0} h_{2}\right)+B_{2}^{\prime}\left(\cos \beta_{0} z_{2}-\cos \beta_{0} h_{2}\right) \\
& +D_{2}\left(\cos \frac{1}{2} \beta_{0} z_{2}-\cos \frac{1}{2} \beta_{0} h_{2}\right) \tag{6.85}
\end{align*}
$$

where $A_{2}^{\prime}=-j 2 \pi V_{02} / \zeta_{0} \Psi$. The currents in the parasitic elements are given by (6.79b). With the notation

$$
\begin{equation*}
T_{U i}^{\prime}=B_{i}^{\prime} / A_{2}, \quad B_{i}^{\prime}=B_{1}, B_{2}^{\prime}, B_{3}, \ldots B_{N} \tag{6.86}
\end{equation*}
$$

the formula for the distant field is

$$
\begin{equation*}
E_{\Theta}\left(R_{2}, \Theta, \Phi\right)=\frac{V_{02}}{\Psi} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} f_{V N}^{\prime}(\Theta, \Phi) \tag{6.87a}
\end{equation*}
$$

where

$$
\begin{align*}
f_{V N}^{\prime}(\Theta, \Phi)=H_{m}(\Theta, & \left.\beta_{0} h_{2}\right)+\sum_{i=1}^{N} e^{-j \beta_{0}\left(R_{i}-R_{2}\right)} \\
& \times\left[T_{U i}^{\prime} G_{m}\left(\Theta, \beta_{0} h_{i}\right)+T_{D i} D_{m}\left(\Theta, \beta_{0} h_{i}\right)\right] \tag{6.87b}
\end{align*}
$$

$H_{m}\left(\Theta, \beta_{0} h\right)$ is defined in (2.51) and, specifically for $\beta_{0} h=\pi / 2$, in (2.52a). As before, $G_{m}\left(\Theta, \beta_{0} h\right)$ and $D_{m}\left(\Theta, \beta_{0} h\right)$ are given in (2.47) and (2.48). Special values for $\beta_{0} h=\pi / 2$ are in ( $2.52 \mathrm{~b}, \mathrm{c}$ ). If desired $E_{\Theta}(\Theta, \Phi)$ as given in $(6.87 \mathrm{a}, \mathrm{b})$ may be referred to the current $I_{z 2}(0)$
instead of the voltage $V_{02}$. In this case

$$
Y_{2 N}=\frac{-j 2 \pi}{\zeta_{0} \Psi}\left[\left(1-\sin \beta_{0} h_{2}\right)+T_{U 2}^{\prime}\left(1-\cos \beta_{0} h_{2}\right)+T_{D 2}\left(1-\cos \frac{1}{2} \beta_{0} h_{2}\right)\right]
$$

so that

$$
\begin{equation*}
E_{\Theta N}\left(R_{2}, \Theta, \Phi\right)=\frac{j \zeta_{0} I_{22}(0)}{2 \pi} \frac{e^{-j \beta_{0} \boldsymbol{R}_{2}}}{R_{2}} f_{I N}^{\prime}(\Theta, \Phi) \tag{6.88}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{i N}^{\prime}(\Theta, \Phi)=\frac{H_{m}\left(\Theta, \beta_{0} h_{2}\right)+\sum_{i=1}^{\vee} e^{\left.-j \beta_{0} R_{i}-R_{2}\right)}\left[T_{v i}^{\prime} G_{m}\left(\Theta, \beta_{0} h_{i}\right)+T_{D_{i}} D_{m}\left(\Theta, \beta_{0} h_{i}\right]\right]}{\left(1-\sin \beta_{0} h_{2}\right)+T_{v 2}^{\prime}\left(1-\cos \beta_{0} h_{2}\right)+T_{D 2}\left(1-\cos \frac{1}{2} \beta_{0} h_{2}\right)} \tag{6.89b}
\end{equation*}
$$

The graphical representations of the normalized field factors $\left|f_{N}(\Theta, \Phi)\right| /\left|f_{N}(\pi / 2,0)\right|$ or $\left|f_{N}^{\prime}(\Theta, \Phi)\right| / / f_{N}^{\prime}(\pi / 2,0) \mid$ in appropriate planes are the field patterns. The field pattern in the equatorial (horizontal) plane is given by $\left|f_{N}(\pi / 2, \Phi)\right| / / f_{N}(\pi / 2,0) \mid$ as a function of $\Phi$. Important field patterns in planes perpendicular to the equatorial plane are with $\Phi=0$ and $\pi$. In this case $\left|f_{N}\left(\Theta,\left\{\begin{array}{l}0 \\ 0\end{array}\right\}\right)\right| /\left|f_{N}(\pi / 2,0)\right|$ is shown graphically as a function of $\Theta$. The ratio of the field in the forward direction (i.e. toward the directors, $\Phi=0$ ) to the field in the backward direction (i.e. toward the reflector, $\Phi=\pi$ ) in the equatorial plane $\Theta=\pi / 2$ is known as the front-to-back ratio. It is given by

$$
\begin{equation*}
R_{F B}=\frac{\left|f_{N}\left(\frac{\pi}{2}, 0\right)\right|}{\left|f_{N}\left(\frac{\pi}{2}, \pi\right)\right|} . \tag{6.90a}
\end{equation*}
$$

The front-to-back ratio in decibels is

$$
\begin{equation*}
r_{F B}=20 \log _{10}\left|f_{N}\left(\frac{\pi}{2}, 0\right) / f_{N}\left(\frac{\pi}{2}, \pi\right)\right| . \tag{6.90b}
\end{equation*}
$$

Note that in all of the ratios involving $f_{N}(\Theta, \Phi)$ either $f_{V N}(\Theta, \Phi)$ or $f_{I N}(\Theta, \Phi)$ may be used.

Since the total power radiated by an array is given by the integral over a great sphere of the normal component of the Poynting vector

$$
\begin{equation*}
\left|S_{\boldsymbol{R}}(R, \boldsymbol{\Theta}, \Phi)\right|=\left|E_{\boldsymbol{\Theta}}(R, \boldsymbol{\Theta}, \Phi)\right|^{2} / 2 \zeta_{0} \tag{6.91}
\end{equation*}
$$

the distribution as a function of $\Theta$ and $\Phi$ of $\left|S_{R}(R, \Theta, \Phi)\right|$ is of interest. The total power supplied to the $N$-element array is

$$
\begin{equation*}
P_{2 N}=\frac{1}{2}\left|I_{z 2}(0)\right|^{2} R_{2 N}=\frac{1}{2}|V(0)|^{2} G_{2 N} \tag{6.92}
\end{equation*}
$$

where $R_{2 N}$ and $G_{2 N}$ are, respectively, the driving-point resistance and conductance of the element 2 when driving the $N$-element parasitic array. With (6.84a) and (6.92)

$$
\begin{align*}
\left|S_{R}\left(R_{2}, \Theta, \Phi\right)\right| & =\frac{P_{2 N}}{4 \pi^{2} R_{2}^{2}} \frac{\zeta_{0}}{G_{2 N}}\left|f_{V N}(\Theta, \Phi)\right|^{2}  \tag{6.93a}\\
& =\frac{P_{2 N}}{4 \pi^{2} R_{2}^{2}} \frac{\zeta_{0}}{R_{2 N}}\left|f_{I N}(\Theta, \Phi)\right|^{2} \tag{6.93b}
\end{align*}
$$

A graphical representation of $\left|f_{N}(\Theta, \Phi) / f_{N}(\pi / 2,0)\right|^{2}$ is known as a power pattern. (Note that $R_{2}$ is a distance, $R_{2 N}$ a resistance.)

If ohmic losses in the conductors of the antennas and in the surrounding dielectric medium (air) are neglected, the total power radiated by an array outside a great sphere of radius $R_{2}$ is the same as the total power supplied at the terminals of the driven element 2 . That is

$$
\begin{align*}
P_{2 N} & =\frac{1}{2}\left|V_{02}\right|^{2} G_{2 N}=\frac{1}{2}\left|I_{z 2}(0)\right|^{2} R_{2 N} \\
& =\int_{0}^{2 \pi} \int_{0}^{\pi}\left|S_{R}\left(R_{2}, \Theta, \Phi\right)\right| R_{2}^{2} \sin \Theta d \Theta d \Phi \tag{6.94}
\end{align*}
$$

With (6.93) and (6.94), formulas are obtained for $R_{2 N}$ and $G_{2 N}$ in terms of the far field. They are

$$
\begin{align*}
& R_{2 N}=\frac{\zeta_{0}}{4 \pi^{2}} \int_{0}^{2 \pi} \int_{0}^{\pi}\left|f_{I N}(\Theta, \Phi)\right|^{2} \sin \Theta d \Theta d \Phi  \tag{6.95a}\\
& G_{2 N}=\frac{\zeta_{0}}{4 \pi^{2}} \int_{0}^{2 \pi} \int_{0}^{\pi}\left|f_{V N}(\Theta, \Phi)\right|^{2} \sin \Theta d \Theta d \Phi \tag{6.95b}
\end{align*}
$$

Actually, both $R_{2 N}$ and $G_{2 N}$ are already known from

$$
I_{z 2}(0) / V_{02}=G_{2}+j B_{2}
$$

when the medium in which the array is immersed is lossless.
The absolute directivity of the $N$-element Yagi array is defined in terms of the power radiated by a fictitious omnidirectional antenna that maintains the same field in all directions as the Yagi array does in the one direction of its maximum, viz., $\Theta=\pi / 2$, $\Phi=0$. This power is

$$
\begin{align*}
P_{\text {Nomni }} & =4 \pi R_{2}^{2}\left|S_{R}\left(R_{2}, \frac{\pi}{2}, 0\right)\right| \\
& =P_{2 N} \frac{\zeta_{0}}{\pi R_{2 N}}\left|f_{I N}(\Theta, \Phi)\right|^{2}=P_{2 N} \frac{\zeta_{0}}{\pi G_{2 N}}\left|f_{V N}(\Theta, \Phi)\right|^{2} \tag{6.96}
\end{align*}
$$

The ratio $P_{N o m n i} / P_{2 N}$ is the absolute directivity. Thus
$D_{N}\left(\frac{\pi}{2}, 0\right)=\frac{P_{N o m n \mathrm{i}}}{P_{2 N}}=\frac{\zeta_{0}}{\pi R_{2 N}}\left|f_{I N}\left(\frac{\pi}{2}, 0\right)\right|^{2}=\frac{\zeta_{0}}{\pi G_{2 N}}\left|f_{V N}\left(\frac{\pi}{2}, 0\right)\right|^{2}$.
This formula is often written with $R_{2 N}$ expressed explicitly as given in ( 6.95 a ). The quantity

$$
\begin{equation*}
G_{N}\left(\frac{\pi}{2}, 0\right)=10 \log _{10} D_{N}\left(\frac{\pi}{2}, 0\right) \tag{6.98}
\end{equation*}
$$

is the absolute gain in decibels.
The absolute directivity of the driven element 2 when isolated is

$$
D_{1}\left(\frac{\pi}{2}, 0\right)=\frac{P_{10 \mathrm{mni}}}{P_{21}}=\frac{\zeta_{0}}{\pi R_{21}}\left|f_{I 1}\left(\frac{\pi}{2}, 0\right)\right|^{2}=\frac{\zeta_{0}}{\pi G_{21}}\left|f_{V 1}\left(\frac{\pi}{2}, 0\right)\right|^{2} \cdot(6.99)
$$

The relative directivity at constant power of the array referred to the isolated driven element is

$$
\begin{equation*}
D_{r}(0)=\frac{D_{N}\left(\frac{\pi}{2}, 0\right)}{D_{1}\left(\frac{\pi}{2}, 0\right)}=\frac{R_{21}\left|f_{I N}\left(\frac{\pi}{2}, 0\right)\right|^{2}}{R_{2 N}\left|f_{I 1}\left(\frac{\pi}{2}, 0\right)\right|^{2}}=\frac{G_{21}\left|f_{V N}\left(\frac{\pi}{2}, 0\right)\right|^{2}}{G_{2 N}\left|f_{V 1}\left(\frac{\pi}{2}, 0\right)\right|^{2}} \tag{6.100}
\end{equation*}
$$

The corresponding relative gain in decibels is
$G_{r}(0)=G_{N}\left(\frac{\pi}{2}, 0\right)-G_{1}\left(\frac{\pi}{2}, 0\right)=10\left[\log _{10} D_{N}\left(\frac{\pi}{2}, 0\right)-\log _{10} D_{1}\left(\frac{\pi}{2}, 0\right)\right]$.

The relative directivity ( 6.100 ) is readily expressed in terms of the electric field in (6.84a). Thus

$$
\begin{equation*}
D_{r}(0)=\frac{\left|E_{\Theta N}\left(R_{2}, \frac{\pi}{2}, 0\right)\right|^{2}}{\left|E_{\Theta 1}\left(R_{2}, \frac{\pi}{2}, 0\right)\right|^{2}} \frac{P_{21}}{P_{2 N}} \tag{6.102}
\end{equation*}
$$

The relative directivity at constant power, $P_{21}=P_{2 N}$, is

$$
\begin{equation*}
D_{r}(0)=\frac{\left|E_{\Theta N}\left(R_{2}, \frac{\pi}{2}, 0\right)\right|^{2}}{\left|E_{\Theta 1}\left(R_{2}, \frac{\pi}{2}, 0\right)\right|^{2}} \tag{6.103}
\end{equation*}
$$

This is equivalent to (6.100).
The relative directivity (6.100) or (6.103) is also the relative forward directivity in the direction $\Theta=\pi / 2, \Phi=0$. The relative
directivity at constant power $P_{21}=P_{2 N}$ in the backward direction $\Theta=\pi / 2, \Phi=\pi$ is defined by

$$
\begin{equation*}
D_{r}(\pi)=\frac{\left|E_{\Theta N}\left(R_{2}, \frac{\pi}{2}, \pi\right)\right|^{2}}{\left|E_{\Theta 1}\left(R_{2}, \frac{\pi}{2}, \pi\right)\right|^{2}}=\frac{R_{21}}{R_{2 N}} \frac{\left|f_{I N}\left(\frac{\pi}{2}, \pi\right)\right|^{2}}{\left|f_{I 1}\left(\frac{\pi}{2}, \pi\right)\right|^{2}}=\frac{G_{21}}{G_{2 N}} \frac{\left|f_{V N}\left(\frac{\pi}{2}, \pi\right)\right|^{2}}{\left|f_{V 1}\left(\frac{\pi}{2}, \pi\right)\right|^{2}} . \tag{6.104}
\end{equation*}
$$

The relative backward gain is

$$
\begin{equation*}
G_{r}(\pi)=10 \log _{10} D_{r}(\pi) . \tag{6.105}
\end{equation*}
$$

Since for a single element rotational symmetry with respect to $\Phi$ gives $f_{N}(\pi / 2,0)=f_{N}(\pi / 2, \pi)$, it follows that

$$
\begin{equation*}
\frac{D_{r}(0)}{D_{r}(\pi)}=\frac{\left|f_{N}\left(\frac{\pi}{2}, 0\right)\right|^{2}}{\left|f_{N}\left(\frac{\pi}{2}, \pi\right)\right|^{2}} \tag{6.106}
\end{equation*}
$$

and

$$
\begin{equation*}
r_{F B}=G_{r}(0)-G_{r}(\pi) \tag{6.107}
\end{equation*}
$$

in decibels. Note that $R_{2}$ is a distance, $R_{21}$ and $R_{2 N}$ resistances.

### 6.7 Simple applications of the modified theory; comparison with experiment

The theory of arrays developed in the preceding sections is like that formulated in the earlier chapters in that the complicated simultaneous integral equations for the currents in the elements are replaced by a set of algebraic equations. This is accomplished by approximating the integrals with an appropriate combination of trigonometric functions. In dealing with arrays of driven elements of equal length it was convenient to use different trigonometric functions for different parts of the integrals and to match these to the integrals at the point of maximum current, $z=z_{m}$, and at the ends, $z= \pm \mathrm{h}$. For use with parasitic elements of unequal length this procedure is modified. Each integral is approximated by a sum of trigonometric terms with coefficients matched to the integral at $z=0, \pm h / 2$ and $\pm h$. In order to illustrate the application of the modified theory and at the same time verify its accuracy it is convenient to consider the simplest cases, the isolated antenna and the two-element parasitic array. Since conventional (sinusoidal) theory fails completely when full-wave elements are involved, the examples are selected deliberately to include such elements.

In Fig. 6.10 are the distributions of current along a full-wave isolated antenna as computed from the modified theory, and as measured. They may be compared with the three-term approximation in Fig. 2.4 where the same experimental data are also shown.


Fig. 6.10. Distribution of current on full-wave antenna; $I(z)=I^{\prime \prime}(z)+j I^{\prime}(z) ; a / \lambda=0.007022$,

$$
h / \lambda=0.5 .
$$

The two theoretical representations, while not identical, are nevertheless both very good approximations of the current. The modified theory does not provide quite as good an overall fit, but is somewhat better in specifying the susceptance-as would be expected since all integrals are matched at $z=0$ and not only at the maximum of current. The admittance in the modified theory is $Y_{0}=(0.926+j 1.350) \times 10^{-3} \mathrm{mhos}$; the value obtained previously is $Y_{0}=(0.976+j 0.988) \times 10^{-3}$ mhos. The measured value after correction for end effects is $(1.025+j 1.676) \times 10^{-3}$ mhos. As indicated in conjunction with Fig. 2.6 a lumped susceptance $B_{0}=0.72 \times 10^{-3}$ mhos must be added to the three-term admittance to give $Y_{0}=(0.976+j 1 \cdot 708) \times 10^{-3}$ mhos. A similar lumped correction is also required with the modified theory, but it is smaller, viz. $B_{0}=0.35$ mhos. It is clear that when suitably corrected to give the right susceptance either theory provides a very acceptable approximation of the current in a dipole.

The distributions of current in an array of two full-wave elements in which element 1 is centre driven and element 2 is parasitic are shown in Fig. 6.11 for four values of $b$, the distance between the parallel antennas. The corresponding field patterns in the equatorial plane are in Fig. 6.12. The distributions of current in Fig. 6.11 may


Fig. 6.11. Theoretical currents in full-wave two-element parasitic array. $h / \lambda=0.5$, $a / \lambda=0.007022$.


Fig. 6.12. Horizontal field patterns of full-wave two-element parasitic array. $h / \lambda=0.5$, $a / \lambda=0.007022$.
be compared with measured values in Fig. 6.13. The agreement is seen to be very good. Equally good agreement has been observed for the field patterns.

As an illustration of the computations for the currents in a twoelement array with elements differing greatly in length, the graphs in Fig. 6.14 are provided. The associated horizontal field patterns are in Fig. 6.15. In the case considered, the driven element is a wavelength long, the parasitic element has successively the three lengths $h_{2}=0.2 \lambda, 0.4 \lambda$, and $0.65 \lambda$. Large changes in the distributions of current are seen to occur in the parasitic element as its length is changed while fixed at the specified distance $b=0.2 \lambda$ from the driven element. Note that except for the shortest length, the currents in the parasitic element differ significantly from the sinusoidal. The current in the driven antenna is only slightly affected by the changes in length of the coupled parasitic antenna, the largest changes occur near the driving point so that the admittance is noticeably modified. Specifically, for the values $h_{2} / \lambda=0 \cdot 2,0 \cdot 4,0 \cdot 65$ the admittances are $(0.916+j 1.041) \times 10^{-3},(0.790+j 1 \cdot 480) \times 10^{-3}$, and $(0.805+j 1 \cdot 510) \times 10^{-3}$ mhos.

A typical computer printout for a two-element parasitic array is in Table 6.1. The coefficients of the trigonometric components of the current, the admittance, the impedance, the current distributions, the horizontal and vertical field patterns, the forward gain, the backward gain and the front-to-back ratio are all given.


Fig. 6.13. Measured currents in full-wave array of two elements (Mack). $h / \lambda=0 \cdot 5, a / \lambda=0.007022$.

Table 6.1



Fig. 6.14. Currents in arrays of two elements of different lengths. $b / \lambda=0 \cdot 2 \cdot a / \lambda=0 \cdot 00702, N=2$.


Fig. 6.15. Horizontal field patterns of arrays of two elements of different lengths. $b / \lambda=0 \cdot 2$, $a / \lambda=0.007022, N=2$.

### 6.8 The three-element Yagi-Uda array $\dagger$

The computed distributions of current and the field pattern for a three-element array consisting of a reflector of length $2 h_{1}=0.51 \lambda$, a driven element of length $2 h_{2}=0 \cdot 50 \lambda$ and a single director of length $2 h_{3}=0.45 \lambda$ are shown in Figs. 6.16 and 6.17. For this array the radius of all elements was taken as $a=0.003369 \lambda$. The drivingpoint impedance of element 2 is $Z_{2}=27 \cdot 4+j 1.27$ ohms. The computed values of the phase angle of the current along the reflector are nearly constant ; it decreases from $74^{\circ} \cdot 5$ at $z / h_{1}=0$ to $72^{\circ} .7$ at $z / h_{1}=0.9$. The phase angle of the current along the driven element decreases from $-2^{\circ} .66$ at $z / h_{2}=0$ to $-8^{\circ} .47$ at $z / h_{2}=0.9$. The phase angle of the current along the director is almost exactly constant, changing only from $-154^{\circ} \cdot 3$ at $z / h_{3}=0$ to $-154^{\circ} .0$ at $z / h_{3}=0.9$. It is clear from Fig. 6.16 that the current in the reflector is so small that it actually contributes negligibly to the field.

[^5]

Fig. 6.16. Currents in three-element Yagi-Uda array.
In order to determine whether the particular length $h_{3}$ and spacing $b_{23}$ is the best value to maintain the largest forward gain $G(0)$ or the maximum front-to-back ratio, the quantities $h_{3} / \lambda$ and $b_{23} / \lambda$ can be varied over a suitable range and the associated forward gain or front-to-back ratio computed. A computer printout for the front-to-back ratio is shown in Fig. 6.18. The ordinates are $2 h_{3} / \lambda=2 H / L$, in a range from 0.50 to 0.36 in steps of 0.01 ; the abscissae are $b_{23} / \lambda=B / L$ in the range from 0.02 to 0.30 in steps of 0.02 . The contours are drawn along estimated lines of constant front-to-back ratio ranging from 1 to 19 . It is seen that the maximum value of front-to-back ratio is close to $b_{23} / \lambda=0.12$ with


Fig. 6.17. Field pattern of three-element Yagi-Uda array with element No. 2 driven.

1-Director Yagi antenna with constant director length and spacing front-to-back ratio in dB


Fig. 6.18. Typical printout for three-element Yagi-Uda array.
$2 h_{3} / \lambda=0.44$. Thus, the distributions of current and the field pattern in Figs. 6.16 and 6.17 do not quite correspond to those for maximum front-to-back ratio. A small readjustment in the length of the director from $2 h_{3}=0.45 \lambda$ to $2 h_{3}=0.44 \lambda$ and an increase in its spacing $b_{23}$ from $0.08 \lambda$ to $0.12 \lambda$ produces an increase in front-to-back ratio from $24 \cdot 14$ to $30 \cdot 70$. If the parameters $2 h_{3} / \lambda$ and $b_{23} / \lambda$ were varied in steps smaller than 0.01 and 0.02 , respectively, an even higher ratio might be obtained within the narrow ranges $2 h_{3}=0.44 \pm 0.01, b_{23}=0.12 \pm 0.02$. A more extended set of contours of the front-to-back ratios is shown in Fig. 6.19 in which the computed numbers have been deleted and only the contours of constant $r_{F B}$ are shown. It is clear that a number of successive maxima in front-to-back ratio are obtained as the distance $b_{23}$ between the director and the driven element is increased. These occur substantially at intervals of $\lambda / 2$ with $2 h_{3}$ between $0.44 \lambda$ and $0.46 \lambda$. Similar computer printouts for forward gain, driving-point resistance and reactance are also shown in Fig. 6.19.

### 6.9 The four, eight and ten director Yagi-Uda arrays $\dagger$

The theory developed and illustrated with simple examples in the preceding sections can be applied to analyse the properties of longer Yagi-Uda arrays. For such arrays the quantities of principal interest include the distributions of current along all elements (since these determine the field), the admittance or impedance of the single driven element, the far-field pattern, the forward gain and the front-to-back ratio. For many purposes the determination of conditions that yield a maximum in the forward gain or in the front-to-back ratio is important. The parameters that may be varied are the length $2 h_{i}$ and radius $a_{i}$ of each element $i$, the distances $b_{i j}$ between the elements $i$ and $j$ and their number $N$. Thus, there are a total of $3 N$ parameters.

Because of the large number of possible combinations, an exhaustive study of the Yagi array would be very costly in both time and money even when a high-speed digital computer is available. An investigation of reasonable proportions must be restricted to a choice and range of parameters that is appropriate to a particular purpose.

In general, the purpose of the Yagi-Uda array is to obtain a highly directive field pattern with large values of the forward gain

[^6]

Fig. 6.19. Contour diagrams constructed with computer printouts for a 1-director Yagi array, (a) forward gain, (b) front-to-back ratio, (c) input resistance, (d) input reactance.
and front-to-back ratio. It has been shown implicitly that these desired properties can be achieved with the array pictured in Fig. 6.8. It consists of the following components : (1) A single driven element No. 2 that is a typical half-wave dipole of length $2 h_{2}=0.5 \lambda$ but with a finite radius $a_{2}$ and a distribution of current that is not assumed in advance to be sinusoidal, but remains to be determined. (2) A single reflecting element No. 1 that is slightly longer ( $2 h_{1}=0.51 \lambda$ ) than the driven antenna is placed at a distance $b_{12}=0.25 \lambda$ from it. The field maintained by the currents induced in a parasitic element of this length and relative location tends to reinforce the field maintained by the currents in the driven element in the forward direction (from 1 to 2 ) and to reduce or cancel it in the opposite or backward direction (from 2 to 1). (3) The balance of the array consists of $N-2$ directors that all have the same half-
length $h_{i}=h$ and that are separated by the same distance $b_{i-1, i}=b$ with $3 \leqslant i \leqslant N$. In order to function as directors, the length $h$ of the $N-2$ parasitic elements must satisfy the inequality, $h_{i}<h_{2}=0.5 \lambda$ if the field maintained by the currents in them is to reinforce in the forward direction the field maintained by the currents in the driven element and in the reflector. If it is required that all antennas have the same radius, $a_{i}=a$, the $3 N$ parameters have been reduced to three; $h, b$ and $N$.

Contour diagrams constructed from computer printouts of the forward gain, the front-to-back ratio, the input resistance and the input reactance are shown in Figs. 6.20 and 6.21 for an array with four identical directors. The parameters are $2 h / \lambda$ and $b / \lambda$ where $h=h_{3}=h_{4}=h_{5}=h_{6}$ and $b=b_{23}=b_{34}=b_{45}=b_{56}$. From these, combinations of $h$ and $b$ may be selected for which the forward gain or the front-to-back ratio is a maximum. For example, the following pairs of values are obtained from Fig. 6.20 to give a maximum front-to-back ratio: $2 h / \lambda=0 \cdot 413,0 \cdot 420,0 \cdot 426,0 \cdot 424$; $b / \lambda=0.033,0.139,0.248,0.360$. These four sets all give a maximum front-to-back ratio, but the field patterns are quite different. These are shown in Fig. 6.22 together with corresponding patterns for similarly optimized one- and two-director arrays. From these it is seen that the field patterns for the most closely spaced condition for maximum front-to-back ratio are practically identical regardless of the number of directors. This is due to the fact that the directors are all so close to the driven element that no minor lobes are possible. As the distance between directors is increased, but limited to values that yield maxima in the front-to-back ratio, minor lobes appear and the beam width is reduced. The currents at the centres of the elements for the arrays that maintain the field patterns in Fig. 6.22 are represented in the form of phasor diagrams in Figs. $6.23 a, b$. The magnitude and angle of $I_{2}(0)$ in each element is shown. Note that for the very closely spaced 4 -director array with $b / \lambda=0.033$, the currents in the directors are almost equal and in phase and much smaller than the current in the driven element. On the other hand, for the largest spacing shown $b / \lambda=0.36$, the currents in the directors are comparable in magnitude with the current in the driven element and their phase differences are close to the progressive phase difference $360^{\circ} b / \lambda=130^{\circ}$ of a wave travelling with the velocity of light from element to element.


Fig. 6.20. Forward gain (a) and front-to-back ratio (b) for a 4-director Yagi array.


Fig. 6.21. Input resistance ( $a$ ) and reactance ( $b$ ) for a 4-director Yagi array.


Fig. 6.22. Horizontal field patterns for Yagi arrays with maxima in front-to-back ratio.
Composite diagrams showing the forward gain, the front-to-back ratio, the input resistance and the input reactance as functions of $b / \lambda$ for $1,2,4$ and 8 director arrays with $h / \lambda=0.43, a / \lambda=0.00337$ are shown in Figs. 6.24a, $b$ and 6.25a,b. From these the major quantities of interest are readily obtained.

A computer printout of a 10 -director Yagi-Uda array $\dagger$ with $2 h / \lambda=0.4$ and $b / \lambda=0.3$ is given in the accompanying Table 6.2.
$\dagger$ The numerical evaluation for the 10 -director Yagi array was done by V. W. H. Chang.


| Element | $\|I(0)\|$ |
| :---: | ---: |
| 1 | 4.35 |
| 2 | 27.34 |
| 3 | 12.55 |
| 4 | 16.68 |


| Element | $\|I(0)\|$ |
| :---: | ---: |
| 1 | 4.55 |
| 2 | 13.32 |
| 3 | 14.17 |
| 4 | 11.05 |

Fig. 6.23a. Phasor diagrams for Yagi arrays with maxima in front-to-back ratio; 1 and 2 directors.


Fig. 6.23b. Like Fig. 6.23a but for 4 directors.

Table 6.2. Computer printout for 8-director Yagi-Uda array

| No. of elements | $=10$ |
| ---: | :--- |
| Half-length of driving antenna | $=0.2500000 \mathrm{E}-00$ |
| Half-length of parasitic antenna | $=0.2000000 \mathrm{E}-00$ |
| Half-length of reflector antenna | $=0.2550000 \mathrm{E}-00$ |
| Radius | $=0.3369000 \mathrm{E}-02$ |
| Spacing between reflector and driving antennas | $=0.2500000 \mathrm{E}-00$ |
| Spacing between parasitic antennas | $=0.3000000 \mathrm{E}-00$ |

Coefficients for current distributions

| Element No. | AR | AI | BR |
| :---: | :---: | :---: | ---: |
| 1 | 0 | 0 | $-0 \cdot 261108 \mathrm{E}-03$ |
| 2 | $0 \cdot 260791 \mathrm{E}-04$ | $-0 \cdot 128598 \mathrm{E}-02$ | $0 \cdot 603188 \mathrm{E}-03$ |
| 3 | 0 | 0 | $0 \cdot 373823 \mathrm{E}-02$ |
| 4 | 0 | 0 | $-0 \cdot 247170 \mathrm{E}-02$ |
| 5 | 0 | 0 | $-0 \cdot 117657 \mathrm{E}-02$ |
| 6 | 0 | 0 | $0 \cdot 302159 \mathrm{E}-02$ |
| 7 | 0 | 0 | $-0 \cdot 130879 \mathrm{E}-02$ |
| 8 | 0 | 0 | $-0 \cdot 176549 \mathrm{E}-02$ |
| 9 | 0 | 0 | $0 \cdot 264461 \mathrm{E}-02$ |
| 10 | 0 | 0 | $-0 \cdot 391583 \mathrm{E}-03$ |

Element No.
1
2
3
4
5
6
7
8
9
10
BI
$0.443744 \mathrm{E}-03$
$0.204197 \mathrm{E}-01$
$-0.956042 \mathrm{E}-03$
$-0.290843 \mathrm{E}-02$
$0.282252 \mathrm{E}-02$
$0.397791 \mathrm{E}-03$
$-0.284188 \mathrm{E}-02$
$0.176994 \mathrm{E}-02$
$0.139167 \mathrm{E}-02$
$-0.278379 \mathrm{E}-02$

DR
0.626141E-02
$0 \cdot 200499 \mathrm{E}-01$
$-0.339069 \mathrm{E}-01$
$0 \cdot 207297 \mathrm{E}-01$
$0 \cdot 102848 \mathrm{E}-01$
$-0.256008 \mathrm{E}-01$
$0 \cdot 109612 \mathrm{E}-01$
$0 \cdot 149901 \mathrm{E}-01$
$-0.223427 \mathrm{E}-01$
$0.390606 \mathrm{E}-02$

DI
$0.121838 \mathrm{E}-01$
$-0.917549 \mathrm{E}-01$
$0.757068 \mathrm{E}-02$
$0.252446 \mathrm{E}-01$
$-0.238580 \mathrm{E}-01$
$-0.357119 \mathrm{E}-02$
$0.241588 \mathrm{E}-01$
$-0 \cdot 149562 \mathrm{E}-01$
$-0.119466 \mathrm{E}-01$
$0 \cdot 234452 \mathrm{E}-01$

## Table 6.2. Computer printout for 8-director Yagi-Uda array-cont.

Current distributions and input admittances
Element No. 1

| $Z / H$ | Real |
| :--- | :---: |
| 0. | $0.163470 \mathrm{E}-02$ |
| 0.1 | $0.161797 \mathrm{E}-02$ |
| 0.2 | $0.156780 \mathrm{E}-02$ |
| 0.3 | $0.148433 \mathrm{E}-02$ |
| 0.4 | $0.136779 \mathrm{E}-02$ |
| 0.5 | $0.121849 \mathrm{E}-02$ |
| 0.6 | $0.103685 \mathrm{E}-02$ |
| 0.7 | $0.823442 \mathrm{E}-03$ |
| 0.8 | $0.578922 \mathrm{E}-03$ |
| 0.9 | $0.304115 \mathrm{E}-03$ |

Imaginary
$0.416262 \mathrm{E}-02$
$0.411786 \mathrm{E}-02$
$0.398398 \mathrm{E}-02$
$0.376216 \mathrm{E}-02$
$0.345436 \mathrm{E}-02$
$0.306328 \mathrm{E}-02$
$0.259232 \mathrm{E}-02$
$0.204556 \mathrm{E}-02$
$0.142768 \mathrm{E}-02$
$0.743922 \mathrm{E}-03$

| $\quad$ Magnitude | Argument |
| :---: | :---: |
| $0.447210 \mathrm{E}-02$ | $68 \cdot 4651$ |
| $0.442432 \mathrm{E}-02$ | $68 \cdot 4550$ |
| $0.428136 \mathrm{E}-02$ | $68 \cdot 4247$ |
| $0.404439 \mathrm{E}-02$ | $68 \cdot 3743$ |
| $0.371530 \mathrm{E}-02$ | $68 \cdot 3042$ |
| $0.329672 \mathrm{E}-02$ | $68 \cdot 2146$ |
| $0.279199 \mathrm{E}-02$ | $68 \cdot 1060$ |
| $0.220508 \mathrm{E}-02$ | 67.9789 |
| $0.154059 \mathrm{E}-02$ | 67.8340 |
| $0.803683 \mathrm{E}-03$ | 67.6719 |

Element No. 2
Imaginary
$-0.516874 \mathrm{E}-02$
$-0.533846 \mathrm{E}-02$
$-0.543588 \mathrm{E}-02$
$-0.544297 \mathrm{E}-02$
$-0.533362 \mathrm{E}-02$
$-0.507442 \mathrm{E}-02$
$-0.462572 \mathrm{E}-02$
$-0.394290 \mathrm{E}-02$
$-0.297779 \mathrm{E}-02$
$-0.168029 \mathrm{E}-02$

| $\quad$ Magnitude | Argument |
| :---: | :---: |
| $0.826517 \mathrm{E}-02$ | -38.6555 |
| $0.832227 \mathrm{E}-02$ | -39.8463 |
| $0.823147 \mathrm{E}-02$ | -41.2718 |
| $0.798446 \mathrm{E}-02$ | -42.9171 |
| $0.756752 \mathrm{E}-02$ | -44.7520 |
| $0.696107 \mathrm{E}-02$ | -46.7358 |
| $0.613943 \mathrm{E}-02$ | -48.8223 |
| $0.507107 \mathrm{E}-02$ | -50.9645 |
| $0.371928 \mathrm{E}-02$ | -53.1176 |
| $0.204333 \mathrm{E}-02$ | -55.2423 |

Element No. 3

| $Z / H$ | Real |
| :--- | :---: |
| $0 \cdot$ | $-0.389258 \mathrm{E}-02$ |
| 0.1 | $-0.385515 \mathrm{E}-02$ |
| 0.2 | $-0.374266 \mathrm{E}-02$ |
| 0.3 | $-0.355451 \mathrm{E}-02$ |
| 0.4 | $-0.328973 \mathrm{E}-02$ |
| 0.5 | $-0.294700 \mathrm{E}-02$ |
| 0.6 | $-0.252470 \mathrm{E}-02$ |
| 0.7 | $-0.202096 \mathrm{E}-02$ |
| 0.8 | $-0.143372 \mathrm{E}-02$ |
| 0.9 | $-0.760799 \mathrm{E}-03$ |


| Imaginary | Magnitude | Argument |
| :---: | :---: | :---: |
| $0.785263 \mathrm{E}-03$ | $0.397100 \mathrm{E}-02$ | 168.6103 |
| $0.777863 \mathrm{E}-03$ | $0.393285 \mathrm{E}-02$ | $168 \cdot 6082$ |
| $0.755602 \mathrm{E}-03$ | $0.381817 \mathrm{E}-02$ | 168.6018 |
| $0.718302 \mathrm{E}-03$ | $0.362636 \mathrm{E}-02$ | 168.5912 |
| $0.665672 \mathrm{E}-03$ | $0.335640 \mathrm{E}-02$ | 168.5765 |
| $0.597315 \mathrm{E}-03$ | $0.300693 \mathrm{E}-02$ | 168.5580 |
| $0.512741 \mathrm{E}-03$ | $0.257624 \mathrm{E}-02$ | 168.5358 |
| $0.41377 \mathrm{E}-03$ | $0.206240 \mathrm{E}-02$ | 168.5102 |
| $0.292589 \mathrm{E}-03$ | $0.146327 \mathrm{E}-02$ | 168.4815 |
| $0.155698 \mathrm{E}-03$ | $0.776567 \mathrm{E}-03$ | 168.4500 |

Table 6.2. Computer printout for 8-director Yagi-Uda array-cont.

| Element No. 4 |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $Z / H$ | Real | Imaginary | Magnitude | Argument |
| 0. | 0.225112E-02 | 0.281162E-02 | $0 \cdot 360177 \mathrm{E}-02$ | 51.2469 |
| $0 \cdot 1$ | $0 \cdot 222970 \mathrm{E}-02$ | $0 \cdot 278474 \mathrm{E}-02$ | $0 \cdot 356740 \mathrm{E}-02$ | $51 \cdot 2456$ |
| $0 \cdot 2$ | 0.216531E-02 | $0.270393 \mathrm{E}-02$ | $0 \cdot 346408 \mathrm{E}-02$ | $51 \cdot 2416$ |
| $0 \cdot 3$ | 0.205751E-02 | 0.256871E-02 | $0 \cdot 329114 \mathrm{E}-02$ | 51.2350 |
| 0.4 | 0.190559E-02 | $0 \cdot 237827 \mathrm{E}-02$ | 0.304753E-02 | 51.2259 |
| 0.5 | 0.170859E-02 | 0.213152E-02 | $0 \cdot 273178 \mathrm{E}-02$ | 51.2144 |
| 0.6 | $0 \cdot 146531 \mathrm{E}-02$ | 0-182713E-02 | 0.234212E-02 | 51.2006 |
| 0.7 | $0 \cdot 117439 \mathrm{E}-02$ | $0 \cdot 146354 \mathrm{E}-02$ | 0.187647E-02 | 51.1848 |
| 0.8 | $0 \cdot 834289 \mathrm{E}-03$ | $0 \cdot 103904 \mathrm{E}-02$ | $0 \cdot 133254 \mathrm{E}-02$ | 51.1671 |
| 0.9 | $0 \cdot 443383 \mathrm{E}-03$ | 0.551817E-03 | $0.707878 \mathrm{E}-03$ | 51.1476 |

Element No. 5

| $Z / H$ | Real | Imaginary |
| :--- | :---: | :---: |
| 0. | $0.115124 \mathrm{E}-02$ | $-0.260615 \mathrm{E}-02$ |
| 0.1 | $0.114022 \mathrm{E}-02$ | $-0.258133 \mathrm{E}-02$ |
| 0.2 | $0.110710 \mathrm{E}-02$ | $-0.250670 \mathrm{E}-02$ |
| 0.3 | $0.105169 \mathrm{E}-02$ | $-0.238177 \mathrm{E}-02$ |
| 0.4 | $0.973653 \mathrm{E}-03$ | $-0.220574 \mathrm{E}-02$ |
| 0.5 | $0.872566 \mathrm{E}-03$ | $-0.197751 \mathrm{E}-02$ |
| 0.6 | $0.747887 \mathrm{E}-03$ | $-0.169575 \mathrm{E}-02$ |
| 0.7 | $0.598995 \mathrm{E}-03$ | $-0.135890 \mathrm{E}-02$ |
| 0.8 | $0.425205 \mathrm{E}-03$ | $-0.965221 \mathrm{E}-03$ |
| 0.9 | $0.225787 \mathrm{E}-03$ | $-0.512883 \mathrm{E}-03$ |


| Magnitude | Argument |
| :---: | :---: |
| $0 \cdot 284910 \mathrm{E}-02$ | $-66 \cdot 0760$ |
| $0 \cdot 282195 \mathrm{E}-02$ | $-66 \cdot 0770$ |
| $0.274030 \mathrm{E}-02$ | $-66 \cdot 0799$ |
| $0 \cdot 260363 \mathrm{E}-02$ | $-66 \cdot 0847$ |
| $0 \cdot 241108 \mathrm{E}-02$ | $-66 \cdot 0913$ |
| $0 \cdot 216147 \mathrm{E}-02$ | $-66 \cdot 0997$ |
| $0 \cdot 185335 \mathrm{E}-02$ | $-66 \cdot 1096$ |
| $0.148506 \mathrm{E}-02$ | $-66 \cdot 1212$ |
| $0 \cdot 105473 \mathrm{E}-02$ | $-66 \cdot 1340$ |
| $0 \cdot 560383 \mathrm{E}-03$ | $-66 \cdot 1481$ |

Element No. 6

| $Z / H$ | Real | Imaginary |
| :--- | :---: | :---: |
| 0 | $-0.280144 \mathrm{E}-02$ | $-0.407170 \mathrm{E}-03$ |
| 0.1 | $-0.277475 \mathrm{E}-02$ | $-0.403259 \mathrm{E}-03$ |
| 0.2 | $-0.269450 \mathrm{E}-02$ | $-0.391507 \mathrm{E}-03$ |
| 0.3 | $-0.256017 \mathrm{E}-02$ | $-0.371848 \mathrm{E}-03$ |
| 0.4 | $-0.237090 \mathrm{E}-02$ | $-0.344178 \mathrm{E}-03$ |
| 0.5 | $-0.212552 \mathrm{E}-02$ | $-0.308354 \mathrm{E}-03$ |
| 0.6 | $-0.182261 \mathrm{E}-02$ | $-0.264202 \mathrm{E}-03$ |
| 0.7 | $-0.146050 \mathrm{E}-02$ | $-0.211518 \mathrm{E}-03$ |
| 0.8 | $-0.103734 \mathrm{E}-02$ | $-0.150081 \mathrm{E}-03$ |
| 0.9 | $-0.551178 \mathrm{E}-03$ | $-0.796550 \mathrm{E}-04$ |


| Magnitude | Argument |
| :---: | :---: |
| $0.283087 \mathrm{E}-02$ | -171.7418 |
| $0.280390 \mathrm{E}-02$ | -171.7424 |
| $0.272279 \mathrm{E}-02$ | -171.7442 |
| $0.258703 \mathrm{E}-02$ | -171.7473 |
| $0.239575 \mathrm{E}-02$ | -171.7516 |
| $0.214777 \mathrm{E}-02$ | -171.7569 |
| $0.184166 \mathrm{E}-02$ | -171.7633 |
| $0.147573 \mathrm{E}-02$ | -171.7707 |
| $0.104814 \mathrm{E}-02$ | -171.7790 |
| $0.556904 \mathrm{E}-03$ | -171.7880 |

Table 6.2. Computer printout for 8-director Yagi-Uda array-cont.
Element No. 7

| $Z / H$ | Real |
| :--- | :---: |
| 0 | $0.118905 \mathrm{E}-02$ |
| 0.1 | $0.117774 \mathrm{E}-02$ |
| 0.2 | $0.114373 \mathrm{E}-02$ |
| 0.3 | $0.108680 \mathrm{E}-02$ |
| 0.4 | $0.100657 \mathrm{E}-02$ |
| 0.5 | $0.902524 \mathrm{E}-03$ |
| 0.6 | $0.774037 \mathrm{E}-03$ |
| 0.7 | $0.620375 \mathrm{E}-03$ |
| 0.8 | $0.440727 \mathrm{E}-03$ |
| 0.9 | $0.234232 \mathrm{E}-03$ |

Imaginary
$0.265022 \mathrm{E}-02$
$0.262496 \mathrm{E}-02$
$0.254901 \mathrm{E}-02$
$0.242187 \mathrm{E}-02$
$0.224275 \mathrm{E}-02$
$0.201056 \mathrm{E}-02$
$0.172395 \mathrm{E}-02$
$0.138136 \mathrm{E}-02$
$0.981069 \mathrm{E}-03$
$0.521244 \mathrm{E}-03$

| Magnitude | Argument |
| :---: | :---: |
| $0 \cdot 290474 \mathrm{E}-02$ | 65.7455 |
| $0 \cdot 287706 \mathrm{E}-02$ | 65.7450 |
| $0 \cdot 279384 \mathrm{E}-02$ | 65.7437 |
| $0 \cdot 265454 \mathrm{E}-02$ | 65.7414 |
| $0 \cdot 245827 \mathrm{E}-02$ | 65.7383 |
| $0 \cdot 220384 \mathrm{E}-02$ | 65.7344 |
| $0 \cdot 188974 \mathrm{E}-02$ | 65.7297 |
| $0 \cdot 151427 \mathrm{E}-02$ | 65.7243 |
| $0 \cdot 107552 \mathrm{E}-02$ | 65.7182 |
| $0 \cdot 571454 \mathrm{E}-03$ | 65.7116 |

Element No. 8

| $Z / H$ | Real | Imaginary |
| :--- | :---: | :---: |
| $0 \cdot$ | $0.164294 \mathrm{E}-02$ | $-0.163338 \mathrm{E}-02$ |
| $0 \cdot 1$ | $0.162728 \mathrm{E}-02$ | $-0.161782 \mathrm{E}-02$ |
| 0.2 | $0.158020 \mathrm{E}-02$ | $-0.157105 \mathrm{E}-02$ |
| 0.3 | $0.150140 \mathrm{E}-02$ | $-0.149275 \mathrm{E}-02$ |
| 0.4 | $0.139038 \mathrm{E}-02$ | $-0.138243 \mathrm{E}-02$ |
| 0.5 | $0.124645 \mathrm{E}-02$ | $-0.123940 \mathrm{E}-02$ |
| 0.6 | $0.106878 \mathrm{E}-02$ | $-0.106281 \mathrm{E}-02$ |
| 0.7 | $0.856408 \mathrm{E}-03$ | $-0.851691 \mathrm{E}-03$ |
| 0.8 | $0.608252 \mathrm{E}-03$ | $-0.604957 \mathrm{E}-03$ |
| 0.9 | $0.323173 \mathrm{E}-03$ | $-0.321454 \mathrm{E}-03$ |


| Magnitude | Argument |
| :---: | :---: |
| $0.231672 \mathrm{E}-02$ | -44.7710 |
| $0.229464 \mathrm{E}-02$ | -44.7712 |
| $0.222828 \mathrm{E}-02$ | -44.7718 |
| $0.211720 \mathrm{E}-02$ | -44.7727 |
| $0.196068 \mathrm{E}-02$ | -44.7741 |
| $0.175777 \mathrm{E}-02$ | -44.7757 |
| $0.150727 \mathrm{E}-02$ | -44.7777 |
| $0.120781 \mathrm{E}-02$ | -44.7800 |
| $0.857871 \mathrm{E}-03$ | -44.7826 |
| $0.455822 \mathrm{E}-03$ | -44.7854 |

Element No. 9

| $Z / H$ | Real | Imaginary |
| :--- | :---: | :---: |
| 0. | $-0.243969 \mathrm{E}-02$ | $-0.131999 \mathrm{E}-02$ |
| 0.1 | $-0.241646 \mathrm{E}-02$ | $-0.130739 \mathrm{E}-02$ |
| 0.2 | $-0.234660 \mathrm{E}-02$ | $-0.126951 \mathrm{E}-02$ |
| 0.3 | $-0.222965 \mathrm{E}-02$ | $-0.120611 \mathrm{E}-02$ |
| 0.4 | $-0.206487 \mathrm{E}-02$ | $-0.111680 \mathrm{E}-02$ |
| 0.5 | $-0.185124 \mathrm{E}-02$ | $-0.100106 \mathrm{E}-02$ |
| 0.6 | $-0.158748 \mathrm{E}-02$ | $-0.858237 \mathrm{E}-03$ |
| 0.7 | $-0.127214 \mathrm{E}-02$ | $-0.687575 \mathrm{E}-03$ |
| 0.8 | $-0.903608 \mathrm{E}-03$ | $-0.488242 \mathrm{E}-03$ |
| 0.9 | $-0.480150 \mathrm{E}-03$ | $-0.259353 \mathrm{E}-03$ |


| Magnitude | Argument |
| :---: | ---: |
| $0.277389 \mathrm{E}-02$ | -151.6237 |
| $0.274746 \mathrm{E}-02$ | -151.6242 |
| $0.266799 \mathrm{E}-02$ | -151.6258 |
| $0.253497 \mathrm{E}-02$ | -151.6285 |
| $0.234754 \mathrm{E}-02$ | -151.6321 |
| $0.210457 \mathrm{E}-02$ | -151.6367 |
| $0.180462 \mathrm{E}-02$ | -151.6422 |
| $0.144607 \mathrm{E}-02$ | -151.6486 |
| $0.102708 \mathrm{E}-02$ | -151.6557 |
| $0.545718 \mathrm{E}-03$ | -151.6634 |

Element No. 10

| $Z / H$ | Real | Imaginary |
| :--- | :---: | :---: |
| $0 \cdot$ | $0.475414 \mathrm{E}-03$ | $0.255409 \mathrm{E}-02$ |
| 0.1 | $0.470794 \mathrm{E}-03$ | $0.252977 \mathrm{E}-02$ |
| 0.2 | $0.456916 \mathrm{E}-03$ | $0.245667 \mathrm{E}-02$ |
| 0.3 | $0.433725 \mathrm{E}-03$ | $0.233429 \mathrm{E}-02$ |
| 0.4 | $0.401134 \mathrm{E}-03$ | $0.216185 \mathrm{E}-02$ |
| 0.5 | $0.359024 \mathrm{E}-03$ | $0.193825 \mathrm{E}-02$ |
| 0.6 | $0.307248 \mathrm{E}-03$ | $0.166218 \mathrm{E}-02$ |
| 0.7 | $0.245641 \mathrm{E}-03$ | $0.133207 \mathrm{E}-02$ |
| 0.8 | $0.174023 \mathrm{E}-03$ | $0.946231 \mathrm{E}-03$ |
| 0.9 | $0.922045 \mathrm{E}-04$ | $0.502831 \mathrm{E}-03$ |


| Magnitude | Argument |
| :---: | :---: |
| 0.259796E-02 | 79.3463 |
| $0 \cdot 257321 \mathrm{E}-02$ | 79.3483 |
| $0 \cdot 249880 \mathrm{E}-02$ | 79.3545 |
| $0 \cdot 237425 \mathrm{E}-02$ | 79.3647 |
| $0 \cdot 219875 \mathrm{E}-02$ | 79.3787 |
| 0.197122E-02 | 79.3965 |
| $0 \cdot 169033 \mathrm{E}-02$ | 79.4177 |
| 0.135453E-02 | 79.4421 |
| $0.962100 \mathrm{E}-03$ | 79.4695 |
| 0.511215E-03 | 79.4994 |

Table 6.2. Computer printout for 8-director Yagi-Uda array-cont.

| Real | Imaginary | Magnitude | Argument |
| :---: | :---: | :---: | ---: |
| Input admittance $=$ | $-0.516874 \mathrm{E}-02$ | $0.826517 \mathrm{E}-02$ | -38.6555 |
| $0.644960 \mathrm{E}-02$ |  |  | 38.6555 |

Horizontal field pattern

| Phi | E | E DB |
| :---: | :---: | :---: |
| 0. | 1.000000 | -0. |
| 5.00 | 0.986511 | -0.1180 |
| 10.00 | 0.944099 | -0.4996 |
| 15.00 | 0.867684 | -1.2328 |
| 20.00 | 0.751469 | -2.4818 |
| 25.00 | 0.593565 | -4.5306 |
| 30.00 | 0.404522 | -7.8611 |
| 35.00 | 0.232860 | -12.6581 |
| 40.00 | 0.225985 | -12.9184 |
| 45.00 | 0.344538 | -9.2553 |
| 50.00 | 0.415399 | -7.6307 |
| 55.00 | 0.390921 | -8.1582 |
| 60.00 | 0.298334 | -10.5060 |
| 65.00 | 0.247244 | -12.1375 |
| 70.00 | 0.299673 | -10.4671 |
| 75.00 | 0.334813 | -9.5039 |
| 80.00 | 0.292586 | -10.6749 |
| 85.00 | 0.225493 | -12.9373 |
| 90.00 | 0.229328 | -12.7909 |
| 95.00 | 0.260556 | -11.6820 |
| 100.00 | 0.241345 | -12.3473 |
| 105.00 | 0.183770 | -14.7145 |
| 110.00 | 0.156452 | -16.1124 |
| 115.00 | 0.176931 | -15.0439 |
| 120.00 | 0.187247 | -14.5517 |
| 125.00 | 0.166602 | -15.5664 |
| 130.00 | 0.130073 | -17.7163 |
| 135.00 | 0.107503 | -19.3716 |
| 140.00 | 0.118454 | -18.5290 |
| 145.00 | 0.146355 | -16.6919 |
| 150.00 | 0.171587 | -15.3103 |
| 155.00 | 0.187308 | -14.5489 |
| 160.00 | 0.193479 | -14.2673 |
| 165.00 | 0.192924 | -14.2923 |
| 170.00 | 0.189266 | -14.4586 |
| 175.00 | 0.185684 | -14.6245 |
| 180.00 | 0.184254 | -14.6917 |
|  |  |  |

$F$ gain $=11.5646 \mathrm{DB}$

$$
\mathrm{B} \text { gain }=-3.1270 \mathrm{DB}
$$

$\mathrm{FTBR}=14.6917 \mathrm{DB}$


Fig. 6.24. Forward gain (a), and front-to-back ratio (b), for a Yagi array with directors of constant length, radius and spacing ( $0.43 \lambda, 0.00337 \lambda$ and $b$, respectively).

The impedance of the driven element when isolated is $Z_{0}=88.94$ $+j 39 \cdot 11$ ohms. Graphs of the currents in all of the elements are shown in Fig. 6.26. The phase angle along each parasitic element is essentially constant. It is represented in Fig. 6.27 as a function of the distance of the element from the driven antenna No. 2. The curve drawn through the points has no physical significance; it serves merely to interrelate the discrete points and thus reveal how nearly constant the phase change from director to director actually is. The electrical separation of adjacent directors is $108^{\circ}$, the average phase difference of the currents is $115^{\circ} 6$. The horizontal field pattern maintained by the currents in the ten-element array is shown in Fig. 6.28.


Fig. 6.25. Input resistance ( $a$ ), and reactance ( $b$ ), for a Yagi array with directors of constant length, radius and spacing $(0.43 \lambda, 0.00337 \lambda$ and $b$, respectively).

### 6.10 Receiving arrays

The study of arrays of cylindrical antennas in all of the earlier sections of the book has been directed specifically to the problem of transmission, which involves the determination of distributions of current, driving-point admittances and field patterns. Arrays of antennas are also used to secure desired directional properties for receivers.
In a transmitting array a single element may be driven, as in parasitic arrays of the Yagi-Uda type, or all the elements may be active as in the broadside or endfire arrays. In these latter the driving voltage is usually supplied from a single power oscillator by

Fig. 6.26. Currents in 10-element Yagi array, element 2 driven; $h_{1}=0.255 \lambda, h_{2}=0.25 \lambda$, $h_{3}=\ldots=h_{10}=0.20 \lambda ; b_{12}=0.25 \lambda, b_{23}=\ldots=b_{9.10}=0.3 \lambda ; a_{1}=a_{2} \ldots=a_{10}=0.00337 \lambda ;$
$\Omega=10$ for $h=0.25 \lambda$

way of a suitable network of transmission lines, transformers and phase shifters. The design of such a feeding system of transmission lines is beyond the scope of this book. However, most transmitting arrays with their associated networks have a single pair of terminals across which the driving voltage is maintained. Since this pair of terminals is directly obvious in the parasitic arrays which have only a single driven element, attention in the following discussion is focused specifically on arrays of this type. Note that all references to the terminals of the driven element in a parasitic array apply equally to the single pair of input terminals of the transmission-line network that drives any other array.

Consider a receiving array of antennas in the incident plane-wave field of a distant transmitter. For convenience let the array be that shown in Fig. 6.8 with a load impedance $Z_{L}$ instead of the generator connected across the terminals of antenna 2. In order to determine all of the properties of this system including, for example, the distributions of current in the elements and the reradiated or scattered field, it is necessary to formulate the coupled integral equations from the boundary condition that requires the tangential


Fig. 6.28. Field of 10 -element Yagi array.
component of the total electric field to vanish on the perfectly conducting surface of each element. Fortunately, if interest is restricted to the transmission of information from a distant transmitter to the load $Z_{L}$, this elaborate analysis is unnecessary since the current in the load between the given terminals can be determined by the application of the reciprocal theorem $\dagger$ to the identical array when driven by the voltage $V_{0}^{e}$ across the same terminals.

The reciprocal theorem applies to two arbitrarily located pairs of terminals, the one, for example, in an array $A$, the other in a simple dipole $D$. First, let the array be used for transmission, the dipole for reception. A generator with EMF $V_{0}^{e}$ and internal impedance $Z_{g}$ is connected across the terminals of the array; a load $Z_{L}$ is connected across the terminals of the dipole. The centre of the driven element 2 in the array is located at the origin of the spherical coordinates $r, \Theta, \Phi$; the receiving dipole is used to measure the field pattern of the array. For this purpose it is moved along the surface of a great sphere so that its axis is always tangent to the electric field maintained by the transmitter. The current $I_{D}(\Theta, \Phi)$ in $Z_{L}$ at the centre of the dipole varies as the dipole is moved. From (2.78) with

[^7](2.79), it is given by
\[

$$
\begin{equation*}
I_{D}(\Theta, \Phi)=\frac{2 h_{e}\left(\frac{\pi}{2}\right) E_{z}^{\mathrm{inc}}}{Z_{0}+Z_{L}}=\frac{-2 h_{e}\left(\frac{\pi}{2}\right) E_{\Theta}\left(R_{2}, \Theta, \Phi\right)}{Z_{0}+Z_{L}} \tag{6.108}
\end{equation*}
$$

\]

where $2 h_{e}(\pi / 2)$ is the effective length of the dipole when its axis is parallel to the incident electric field and perpendicular to the direction of propagation. Note that when the axis of the receiving dipole is tangent to the surface of the great sphere parallel to $E^{\text {inc }}$, the positive direction of the spherical coordinate $\Theta$ is opposite to the positive direction $z$ along the antenna.

The far-zone electric field maintained by the $N$-element Yagi array driven by a generator at the centre of element No. 2 is given by (6.84a). It is

$$
\begin{equation*}
E_{\Theta}\left(R_{2}, \Theta, \Phi\right)=\frac{j \zeta_{0} I_{z 2}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} f_{I N}(\Theta, \Phi) \tag{6.109}
\end{equation*}
$$

where $R_{2}$ is measured from the centre of element No. 2 and the field factor of the array, $f_{I N}(\Theta, \Phi)$ is given by (6.84b). If the driving-point impedance of the array at the terminals of element No. 2 is $Z_{02}$ and the internal impedance of the generator is $Z_{g}$, it follows that

$$
\begin{equation*}
I_{z 2}(0)=\frac{V_{0}^{e}}{Z_{02}+Z_{\mathrm{g}}} . \tag{6.110}
\end{equation*}
$$

With (6.109) and (6.110), (6.108) becomes

$$
\begin{equation*}
I_{D}(\Theta, \Phi)=\frac{2 h_{e}\left(\frac{\pi}{2}\right)}{Z_{0}+Z_{L}} \cdot \frac{j \zeta_{0} V_{0}^{e}}{Z_{02}+Z_{g}} \frac{e^{-j \beta_{0} R_{2}}}{2 \pi R_{2}} f_{I N}(\Theta, \Phi) . \tag{6.111}
\end{equation*}
$$

Now let the generator with its emf $V_{0}^{e}$ and internal impedance $Z_{g}$ be interchanged with the load $Z_{L}$ so that the dipole is the transmitter, the array the receiver. The dipole is again moved over the surface of the same great sphere; the array remains fixed at the origin of coordinates. The current $I_{A}(\Theta, \Phi)$ in the load $Z_{L}$ in the array varies as the location of the transmitter is changed.

The reciprocal theorem states that if the same voltage $V_{0}^{e}$ is applied successively to both antennas and provided $Z_{\mathrm{g}}=Z_{L}$, then

$$
\begin{equation*}
I_{D}(\Theta, \Phi)=I_{A}(\Theta, \Phi) \tag{6.112}
\end{equation*}
$$

for all values of $\Theta$ and $\Phi$. It follows by a rearrangement of (6.111) and with (6.112) that the current in the load $Z_{L}$ of the Yagi array
when used for reception is given by

$$
\begin{equation*}
I_{A}(\Theta, \Phi)=\frac{2 f_{I N}(\Theta, \Phi)}{\beta_{0}\left(Z_{02}+Z_{L}\right)} \cdot \frac{j \zeta_{0} V_{0}^{e}}{Z_{0}+Z_{g}} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} \beta_{0} h_{e}\left(\frac{\pi}{2}\right) \tag{6.113}
\end{equation*}
$$

provided $Z_{L}=Z_{g}$. Since it has been proved $\dagger$ in general that

$$
\begin{equation*}
\beta_{0} h_{e}\left(\frac{\pi}{2}\right)=f_{I}\left(\frac{\pi}{2}, \beta_{0} h\right) \tag{6.114}
\end{equation*}
$$

where $f_{I}\left(\pi / 2, \beta_{0} h\right)$ is the field factor of the dipole given in (2.54b) and evaluted at $\Theta=\pi / 2$, it follows that (6.113) can be expressed as follows:
where

$$
\begin{gather*}
I_{A}(\Theta, \Phi)=-\frac{2 h_{e N}(\Theta, \Phi) E_{\Theta}^{r}}{Z_{02}+Z_{L}}  \tag{6.115}\\
E_{\Theta}^{r}=\frac{j \zeta_{0} I_{2}(0)}{2 \pi} \frac{e^{-j \beta_{0} R_{2}}}{R_{2}} f_{I}\left(\frac{\pi}{2}, \beta_{0} h\right) \tag{6.116}
\end{gather*}
$$

is the field maintained by the dipole at the centre of element No. 2 of the array and where

$$
\begin{equation*}
2 h_{e N}(\Theta, \Phi)=f_{I N}(\Theta, \Phi) / \beta_{0} \tag{6.117}
\end{equation*}
$$

is by definition the effective length of the Yagi array. It follows that the directional properties of the Yagi (or any other array) are the same for reception as for transmission.

The preceding discussion has been concerned with reciprocity with constant applied voltage. If reciprocity is to be preserved with constant power somewhat different conditions must be fulfilled. This problem is considered elsewhere. $\ddagger$

### 6.11 Driven arrays of elements that differ greatly in length

The procedure outlined in section 6.2 for approximating the integrals in the simultaneous integral equations (6.8) for the currents in a parasitic array of unequal elements is quite adequate when the elements do not differ greatly in length. In the Yagi-Uda array the lengths $2 h_{i}$ of the individual elements $i=1, \ldots N$ always lie in a range that extends from slightly greater than $\lambda / 2$ to approximately $\lambda / 3$. Unfortunately, when elements have lengths that encompass the full range permitted by the present theory, viz., $0 \leqslant \beta_{0} h_{i} \leqslant 5 \pi / 4$, the representations ( 6.20 )-(6.22) for the several integrals are not adequate under certain conditions. In particular

[^8]the two-term approximations on the right in (6.20)-(6.22) do not adequately represent the integrals $W_{k i}\left(z_{k}\right)$ on the left whenever element $k$ is quite long ( $\beta_{0} h_{k} \sim \pi$ ) but element $i$ is short $\left(\beta_{0} h_{i} \leqq \pi / 4\right)$. Extensive computations and measurements by W. M. Cheong [6] have shown that the two-term approximations in (6.20)-(6.22) with the two-point fitting used in (6.53)-(6.60) are especially unsatisfactory for points on the longer element in the range $|z|>h / 2$.

A better representation of all of the integrals (6.15)-(6.17) and (6.20)-(6.22) is obtained when full advantage is taken of the threeterm distribution of current given in (6.6) to approximate the integrals. Specifically, let

$$
\begin{align*}
W_{k i v}\left(z_{k}\right) & \equiv \int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \doteq \Psi_{k i d V}^{m} M_{0 z k}+\Psi_{k i d V}^{f} F_{0 z k}+\Psi_{k i d V}^{h} H_{0 z k}  \tag{6.118}\\
W_{k i U}\left(z_{k}\right) & \equiv \int_{-h_{i}}^{h_{i}} F_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \doteq \Psi_{k i d U}^{m} M_{0 z k}+\Psi_{k i d U}^{\prime} F_{0 z k}+\Psi_{k i d U}^{h} H_{0 z k}  \tag{6.119}\\
W_{k i D}\left(z_{k}\right) & \equiv \int_{-h_{i}}^{h_{i}} H_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \doteq \Psi_{k i d D}^{m} M_{0 z k}+\Psi_{k i d D}^{f} F_{0 z k}+\Psi_{k i d D}^{h} H_{0 z k} \tag{6.120}
\end{align*}
$$

The inclusion of the distribution $M_{0 z}$ in the approximate representation of the integrals $W_{k i U}\left(z_{k}\right)$ and $W_{k i D}\left(z_{k}\right)$ is a new departure. In all previous discussions it has been pointed out that the part of the integral that depends on the real part of the kernel is approximately proportional to the distribution in the integrand when the distance $\beta_{0} b_{k i}<1$ (which usually occurs only when $i=k$ and $b_{k k}=a$ ) and that otherwise the entire integral is proportional to combinations of $F_{0 z}=\cos \beta_{0} z-\cos \beta_{0} h$ and $H_{0 z}=\cos \left(\beta_{0} z / 2\right)-\cos \left(\beta_{0} h / 2\right)$. This means that the distribution $M_{0 z}=\sin \beta_{0}(h-|z|)$ can appear on the right only when $M_{0 z^{\prime}}$ appears in the integrand. These statements are still correct. However, the investigations of Cheong [6] have shown that the current induced in the relatively long antenna ( $h \sim \lambda / 2$ ) by a very short one ( $h<\lambda / 4$ ) is not well represented by combinations of $F_{0 z}$ and $H_{0 z}$ alone. These distributions are excellent when the amplitude and phase of the inducing field are approximately constant along the entire length of an antenna. Clearly, this
is not at all true of the field maintained, for example, along a fullwave antenna by the current in an adjacent quite short element. By including the term in $M_{0 z}$, Cheong has obtained an improved over-all representation of the amplitudes of the currents, especially at points at some distance from the centres of the longer elements. On the other hand, since $M_{0 z}$ has a discontinuous slope at $z=0$ (except when $\left.\beta_{0} h=(2 n+1) \pi / 2\right)$ which the actual induced current cannot have, the slope of an approximate representation that makes use of $M_{0 z}$ is necessarily somewhat in error near $z=0$ even though the amplitude is quite well described. The slope of the current is, of course, proportional to the charge per unit length. Fortunately, an incorrect slope with a discontinuity at $z=0$ does not significantly affect the admittance or the far field. These are determined by the magnitude and phase of the current alone.

Since combinations of $F_{0 z}$ and $H_{0 z}$ are excellent approximations of the two integrals in (6.119) and (6.120) except in the special situations just described, it is to be anticipated that the coefficients $\Psi_{k i d U}^{m}$ and $\Psi_{k i d D}^{m}$ will be small except under those conditions. In any event, the three-term representation of the current for all elements including the $M_{0 z}$ terms in (6.119) and (6.120), can only serve to improve the representation of the amplitudes of the currents at the expense of a small error in their slopes near $z=0$.

In order to determine the complex parameters $\Psi$ in (6.118)(6.120), the approximate expressions on the right are made exactly equal to the integrals at the three points $z_{i}=0, z_{i}=h_{i} / 3$ and $z_{i}=2 h_{i} / 3$ instead of only at the two points $z_{i}=0$ and $z_{i}=h_{i} / 2$ used in section 6.4. That is, three equations are obtained from each of the relations (6.118)-(6.120) in the form :

$$
\begin{align*}
W_{k i V}(0)= & \int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i d}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & \Psi_{k i d V}^{m} \sin \beta_{0} h_{k}+\Psi_{k i d V}^{\prime}\left(1-\cos \beta_{0} h_{k}\right) \\
& +\Psi_{k i d V}^{h}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]  \tag{6.121}\\
W_{k i V}\left(h_{k} / 3\right)= & \int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i d}\left(h_{k} / 3, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & \Psi_{k i d V}^{m} \sin \left(2 \beta_{0} h_{k} / 3\right) \\
& +\Psi_{k i d V}^{\prime}\left[\cos \left(\beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k}\right] \\
& +\Psi_{k i d V}^{h}\left[\cos \left(\beta_{0} h_{k} / 6\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] \tag{6.122}
\end{align*}
$$

$$
\begin{align*}
W_{k i V}\left(2 h_{k} / 3\right)= & \int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i d}\left(2 h_{k} / 3, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & \Psi_{k i d V}^{m} \sin \left(\beta_{0} h_{k} / 3\right) \\
& +\Psi_{k i d V}^{f}\left[\cos \left(2 \beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k}\right] \\
& +\Psi_{k i d V}^{h}\left[\cos \left(\beta_{0} h_{k} / 3\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] . \tag{6.123}
\end{align*}
$$

Each integral when evaluated is a complex number. There are, then, three simultaneous complex algebraic equations to evaluate the three complex parameters $\Psi_{k i d V}^{m}, \Psi_{k i d V}^{f}$, and $\Psi_{k i d V}^{h}$ for each pair of values $i$ and $k$. A similar second set of three equations is obtained with the different complex numbers $W_{k i U}(0), W_{k i U}\left(h_{k} / 3\right)$ and $W_{k i U}\left(2 h_{k} / 3\right)$ on the left. These are obtained from the same integrals when $M_{0 z^{\prime} i}$ is replaced by $F_{0 z^{\prime} i}$. The simultaneous solution of these three equations for each pair of values $i$ and $k$ yields the complex parameters $\Psi_{k i d U}^{m}, \Psi_{k i d U}^{f}$ and $\Psi_{k i d U}^{h}$. A third set of three equations is obtained with the quantities $W_{k i D}(0), W_{k i D}\left(h_{k} / 3\right)$ and $W_{k i D}\left(2 h_{k} / 3\right)$ appearing on the left in (6.121)-(6.123). These quantities are defined by the integrals in (6.121)-(6.123) with $M_{0 z^{\prime} i}$ replaced by $H_{0 z^{\prime} i}$. For each pair of values of $i$ and $k$ this third set of three equations yields $\Psi_{k i d D}^{m}, \Psi_{k i d D}^{f}$ and $\Psi_{k i d D}^{h}$. In this manner all values of the parameters $\Psi_{k i d}$ are determined. They have the following forms for each of the subscripts $V, U$ and $D$ on the $\Psi$ 's and $W$ 's:
$\Psi_{k i}^{m}=\Delta^{-1}\left|\begin{array}{lll}W_{k i}(0) & 1-\cos \beta_{0} h_{k} & 1-\cos \left(\beta_{0} h_{k} / 2\right) \\ W_{k i}\left(h_{k} / 3\right) & \cos \left(\beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k} & \cos \left(\beta_{0} h_{k} / 6\right)-\cos \left(\beta_{0} h_{k} / 2\right) \\ W_{k i}\left(2 h_{k} / 3\right) & \cos \left(2 \beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k} & \cos \left(\beta_{0} h_{k} / 3\right)-\cos \left(\beta_{0} h_{k} / 2\right)\end{array}\right|$

$$
\begin{align*}
& \Psi_{k i}^{f}=\Delta^{-1}\left|\begin{array}{lll}
\sin \beta_{0} h_{k} & W_{k i}(0) & 1-\cos \left(\beta_{0} h_{k} / 2\right) \\
\sin \left(2 \beta_{0} h_{k} / 3\right) & W_{k i}\left(h_{k} / 3\right) & \cos \left(\beta_{0} h_{k} / 6\right)-\cos \left(\beta_{0} h_{k} / 2\right) \\
\sin \left(\beta_{0} h_{k} / 3\right) & W_{k i}\left(2 h_{k} / 3\right) & \cos \left(\beta_{0} h_{k} / 3\right)-\cos \left(\beta_{0} h_{k} / 2\right)
\end{array}\right|  \tag{6.125}\\
& \Psi_{k i}^{k}=\Delta^{-1}\left|\begin{array}{lll}
\sin \beta_{0} h_{k} & 1-\cos \beta_{0} h_{k} & W_{k i}(0) \\
\sin \left(2 \beta_{0} h_{k} / 3\right) & \cos \left(\beta_{0} h_{k}(3)-\cos \beta_{0} h_{k}\right. & W_{k}\left(h_{k} / 3\right) \\
\sin \left(\beta_{0} h_{k} / 3\right) & \cos \left(2 \beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k} & W_{k i}\left(2 h_{k} / 3\right)
\end{array}\right|
\end{align*}
$$

where

$$
\Delta=\left|\begin{array}{lll}
\sin \beta_{0} h_{k} & 1-\cos \beta_{0} h_{k} & 1-\cos \left(\beta_{0} h_{k} / 2\right)  \tag{6.127}\\
\sin \left(2 \beta_{0} h_{k} / 3\right) & \cos \left(\beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k} & \cos \left(\beta_{0} h_{k} / 6\right)-\cos \left(\beta_{0} h_{k} / 2\right) \\
\sin \left(\beta_{0} h_{k} / 3\right) & \cos \left(2 \beta_{0} h_{k} / 3\right)-\cos \beta_{0} h_{k} & \cos \left(\beta_{0} h_{k} / 3\right)-\cos \left(\beta_{0} h_{k} / 2\right)
\end{array}\right|
$$

The $N$ simultaneous integral equations for the currents in the elements are

$$
\begin{align*}
& \sum_{i=1}^{N}\left\{A_{i} \int_{-h_{i}}^{h_{i}} M_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}+B_{i}\right\}_{-h_{i}}^{h_{i}} F_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \left.+D_{i} \int_{-h_{i}}^{h_{i}} H_{0 z^{\prime} i} K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}\right\} \\
& =\frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{k}}\left[\frac{1}{2} V_{0 k} M_{0 z k}+U_{k} F_{0 z k}\right]  \tag{6.128}\\
& \quad k=1,2, \ldots N
\end{align*}
$$

where $U_{k}=\frac{-j \zeta_{0}}{4 \pi} \sum_{i=1}^{N} \int_{-h_{k}}^{h_{k}} I_{z i}\left(z_{i}^{\prime}\right) K_{k i}\left(h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}$

$$
\begin{equation*}
=\frac{-j \zeta_{0}}{4 \pi} \sum_{i=1}^{N}\left[A_{i} \Psi_{k i v}\left(h_{k}\right)+B_{i} \Psi_{k i U}\left(h_{k}\right)+D_{i} \Psi_{k i D}\left(h_{k}\right)\right] \tag{6.129}
\end{equation*}
$$

with $\Psi_{k i v}\left(h_{k}\right), \Psi_{k i v}\left(h_{k}\right)$ and $\Psi_{k i D}\left(h_{k}\right)$ defined in (6.32)-(6.34). If the integrals in (6.128) are replaced by their approximate algebraic equivalents, the following set of algebraic equations for the coefficients $A_{i}, B_{i}$ and $D_{i}$ is obtained:

$$
\begin{align*}
\sum_{i=1}^{N} & \left\{A_{i}\left[\Psi_{k i d V}^{m} M_{0 z k}+\Psi_{k i d V}^{f} F_{0 z k}+\Psi_{k i d V}^{h} H_{0 z k}\right]\right. \\
& +B_{i}\left[\Psi_{k i d U}^{m} M_{0 z k}+\Psi_{k i d U}^{f} F_{0 z k}+\Psi_{k i d U}^{h} H_{0 z k}\right] \\
& \left.+D_{i}\left[\Psi_{k i d D}^{m} M_{0 z k}+\Psi_{k i d D}^{f} F_{0 z k}+\Psi_{k i d D}^{h} H_{0 z k}\right]\right\} \\
= & \frac{j 4 \pi}{\zeta_{0} \cos \beta_{0} h_{k}}\left[\frac{1}{2} V_{0 k} M_{0 z k}+U_{k} F_{0 z k}\right] . \tag{6.130}
\end{align*}
$$

Finally, if (6.129) is substituted for $U_{k}$, the set of equations may be arranged as follows:

$$
\begin{align*}
& M_{0 z k} \sum_{i=1}^{N}\left[\left(A_{i} \Psi_{k i d V}^{m}+B_{i} \Psi_{k i d U}^{m}+D_{i} \Psi_{k i d D}^{m}\right) \cos \beta_{0} h_{k}-\frac{j 2 \pi}{\zeta_{0}} V_{0 k}\right] \\
& \quad+F_{0 z k} \sum_{i=1}^{N}\left[\left(A_{i} \Psi_{k i d V}^{f}+B_{i} \Psi_{k i d U}^{f}+D_{i} \Psi_{k i d D}^{f}\right) \cos \beta_{0} h_{k}\right. \\
& \left.\quad-A_{i} \Psi_{k i V}\left(h_{k}\right)-B_{i} \Psi_{k i U}\left(h_{k}\right)-D_{i} \Psi_{k i D}\left(h_{k}\right)\right] \\
& \quad+H_{0 z k} \sum_{i=1}^{N}\left[A_{i} \Psi_{k i d V}^{h}+B_{i} \Psi_{k i d U}^{h}+D_{i} \Psi_{k i d D}^{h}\right] \cos \beta_{0} h_{k}=0 \tag{6.131}
\end{align*}
$$

with $k=1,2, \ldots N$. These equations are satisfied if the coefficient of each of the three distribution functions is allowed to vanish. The result is a set of $3 N$ simultaneous equations for the $3 N$ unknown
coefficients $A, B$ and $D$. They are:

$$
\begin{gather*}
\sum_{i=1}^{N}\left[A_{i} \Psi_{k i d V}^{m}+B_{i} \Psi_{k i d U}^{m}+D_{i} \Psi_{k i d D}^{m}\right]=\frac{j 2 \pi}{\zeta_{0}} \frac{V_{0 k}}{\cos \beta_{0} h_{k}}  \tag{6.132}\\
\sum_{i=1}^{N}\left[A_{i} \Phi_{k i V}+B_{i} \Phi_{k i U}+D_{i} \Phi_{k i D}\right]=0  \tag{6.133}\\
\sum_{i=1}^{N}\left[A_{i} \Psi_{k i d V}^{h}+B_{i} \Psi_{k i d U}^{h}+D_{i} \Psi_{k i d D}^{h}\right]=0 \tag{6.134}
\end{gather*}
$$

with $k=1,2, \ldots N$. In (6.133) the following notation has been introduced:

$$
\begin{align*}
& \Phi_{k i V} \equiv \Psi_{k i V}\left(h_{k}\right)-\Psi_{k i d V}^{f} \cos \beta_{0} h_{k}  \tag{6.135}\\
& \Phi_{k i U} \equiv \Psi_{k i \nu}\left(h_{k}\right)-\Psi_{k i d U}^{f} \cos \beta_{0} h_{k}  \tag{6.136}\\
& \Phi_{k i D} \equiv \Psi_{k i D}\left(h_{k}\right)-\Psi_{k i d D}^{\prime} \cos \beta_{0} h_{k} . \tag{6.137}
\end{align*}
$$

These equations can be expressed in matrix notation. Let

$$
[\Phi]=\left[\begin{array}{llll}
\Phi_{11} & \Phi_{12} & \ldots & \Phi_{1 N}  \tag{6.138}\\
\vdots & & \\
\Phi_{N 1} & \ldots & & \Phi_{N N}
\end{array}\right]
$$

where the $\Phi_{k i}$ 's are defined in (6.135)-(6.137) for each subscript $V, U$ and $D$. Also let

$$
\left[\Psi^{h}\right]=\left[\begin{array}{llll}
\Psi_{11}^{h} & \Psi_{12}^{h} \ldots & \Psi_{1 N}^{h}  \tag{6.139}\\
\vdots & & & \\
\Psi_{N 1}^{h} & \ldots & & \Psi_{N N}^{h}
\end{array}\right], \quad\left[\Psi^{m}\right]=\left[\begin{array}{lll}
\Psi_{11}^{m} & \Psi_{12}^{m} & \ldots \\
\vdots & & \Psi_{1 N}^{m} \\
\Psi_{N 1}^{m} & & \\
& & \\
\Psi_{N N}^{m}
\end{array}\right]
$$

where the $\Psi_{k i}^{h}$ are obtained from (6.127). The following column matrices are needed:

$$
\begin{align*}
&\{A\}=\left\{\begin{array}{c}
A_{1} \\
A_{2} \\
\vdots \\
A_{N}
\end{array}\right\}, \quad\{B\}=\left\{\begin{array}{c}
B_{1} \\
B_{2} \\
\vdots \\
B_{N}
\end{array}\right\}, \quad\{D\}=\left\{\begin{array}{c}
D_{1} \\
D_{2} \\
\vdots \\
D_{N}
\end{array}\right\}  \tag{6.140}\\
&\left\{\begin{array}{l}
j 2 \pi \\
\zeta_{0} \\
\cos \beta_{0} h
\end{array}\right\}=\frac{V_{0}}{\zeta_{0}}\left\{\begin{array}{l}
V_{01} / \cos \beta_{0} h_{1} \\
V_{02} / \cos \beta_{0} h_{2} \\
\vdots \\
V_{0 N} / \cos \beta_{0} h_{N}
\end{array}\right\} . \tag{6.141}
\end{align*}
$$

With this notation, the equivalent matrix equations for determining the coefficients $A_{i}, B_{i}$ and $D_{i}$ are

$$
\begin{gather*}
{\left[\Psi_{d V}^{m}\right]\{A\}+\left[\Psi_{d U}^{m}\right]\{B\}+\left[\Psi_{d D}^{m}\right]\{D\}=\left\{\frac{j 2 \pi}{\zeta_{0}} \frac{V_{0}}{\cos \beta_{0} h}\right\}}  \tag{6.142a}\\
{\left[\Phi_{V}\right]\{A\}+\left[\Phi_{V}\right]\{B\}+\left[\Phi_{D}\right]\{D\}=0}  \tag{6.142b}\\
{\left[\Psi_{d V}^{h}\right]\{A\}+\left[\Psi_{d U}^{h}\right]\{B\}+\left[\Psi_{d D}^{h}\right]\{D\}=0 .} \tag{6.142c}
\end{gather*}
$$

These equations correspond to (6.29) with (6.46) and (6.47) in the simpler case of the Yagi array with two-term fitting of the integrals.

The solutions of (6.132)-(6.134) or (6.142a, b, c) express each of the coefficients $A_{i}, B_{i}$ and $D_{i}$ as a sum of terms in the $N$ voltages $V_{0 k}, k=1,2, \ldots N$. That is

$$
\begin{align*}
& A_{i}=j \frac{2 \pi}{\zeta_{0}} \sum_{k=1}^{N} \frac{V_{0 k}}{\cos \beta_{0} h_{k}} \alpha_{i k}  \tag{6.143}\\
& B_{i}=j \frac{2 \pi}{\zeta_{0}} \sum_{k=1}^{N} \frac{V_{0 k}}{\cos \beta_{0} h_{k}} \beta_{i k}  \tag{6.144}\\
& D_{i}=j \frac{2 \pi}{\zeta_{0}} \sum_{k=1}^{N} \frac{V_{0 k}}{\cos \beta_{0} h_{k}} \gamma_{i k} \tag{6.145}
\end{align*}
$$

where the $\alpha_{i k}, \beta_{i k}$ and $\gamma_{i k}$ are the appropriate cofactors divided by the determinant of the system.

It follows that with the coefficients $A_{i}, B_{i}$ and $D_{i}$ evaluated, the currents in all elements are available in the form :

$$
\begin{align*}
I_{z i}(z)= & j \frac{2 \pi}{\zeta_{0}} \sum_{k=1}^{N} \frac{V_{0 k}}{\cos \beta_{0} h_{k}}\left\{\alpha_{i k} \sin \beta_{0}\left(h_{k}-|z|\right)+\beta_{i k}\left(\cos \beta_{0} z-\cos \beta_{0} h_{k}\right)\right. \\
& \left.+\gamma_{i k}\left[\cos \left(\beta_{0} z / 2\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\}  \tag{6.146}\\
I_{z i}(0)= & j \frac{2 \pi}{\zeta_{0}} \sum_{k=1}^{N} \frac{V_{0 k}}{\cos \beta_{0} h_{k}}\left\{\alpha_{i k} \sin \beta_{0} h_{k}+\beta_{i k}\left(1-\cos \beta_{0} h_{k}\right)\right. \\
& \left.+\gamma_{i k}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\} \\
= & \sum_{k=1}^{N} V_{0 k} Y_{i k} . \tag{6.147}
\end{align*}
$$

In these relations $i=1,2, \ldots \quad N$, and

$$
\begin{align*}
Y_{i k}= & j \frac{2 \pi}{\zeta_{0} \cos \beta_{0} h_{k}}\left\{\alpha_{i k} \sin \beta_{0} h_{k}+\beta_{i k}\left(1-\cos \beta_{0} h_{k}\right)\right. \\
& \left.+\gamma_{i k}\left[1-\cos \left(\beta_{0} h_{k} / 2\right)\right]\right\} . \tag{6.148}
\end{align*}
$$

The quantities $Y_{i k}$, with $k=i$, are the self-admittances of the $N$ elements in the array; the $Y_{i k}$, with $k \neq i$, are the mutual admittances.

They are readily determined from (6.148). Note that in general the self-admittance of an element when coupled to other antennas is not the same as the self-admittance of the same element when isolated.

In matrix form, the equations for the $N$ driving-point currents are
where
and

$$
\begin{gather*}
\left\{I_{z}(0)\right\}=\left[Y_{A}\right]\left\{V_{0}\right\}  \tag{6.149}\\
\left\{I_{z}(0)\right\}=\left\{\begin{array}{l}
I_{z 1}(0) \\
I_{z 2}(0) \\
\vdots \\
I_{z N}(0)
\end{array}\right\}, \quad\left\{V_{0}\right\}=\left\{\begin{array}{l}
V_{01} \\
V_{02} \\
\vdots \\
V_{0 N}
\end{array}\right\}  \tag{6.150}\\
{\left[Y_{A}\right]=\left\{\begin{array}{lll}
Y_{11} & Y_{12} \ldots Y_{1 N} \\
\vdots & \\
Y_{N 1} \ldots & Y_{N N}
\end{array}\right\} .} \tag{6.151}
\end{gather*}
$$

The solution for the currents in the $N$-elements of the array is thus completed in terms of arbitrary voltages. When these are specified, the complete distributions of current are given in the form (6.146). The driving-point admittances $Y_{0 i}$ and impedances $Z_{0 i}$ are given by

$$
\begin{equation*}
Y_{0 i}=\frac{I_{z i}(0)}{V_{0 i}}=\frac{1}{Z_{0 i}} . \tag{6.152}
\end{equation*}
$$

### 6.12 The $\log$-periodic dipole array

An interesting and important example of a curtain of driven elements that all have different lengths and radii and that are unequally spaced is the so-called log-periodic dipole array illustrated in Fig. 6.29. In spite of the fact that in this array all elements are connected directly to an active transmission line, its operation when suitably designed is closely related to that of the Yagi-Uda antenna in which only one element is driven and all others are parasitic. However, unlike the Yagi antenna, the log-periodic array has important broad band properties. These are best introduced in terms of an array of an infinite number of centre-driven dipoles arranged as shown in Fig. 6.29. Let the half-length of a typical element $i$ be $h_{i}$, let its radius be $a_{i}$. The distance between element $i$ and the next adjacent element to the right is $b_{i, i+1}$ where
$i=1,2,3, \ldots$. The array is constructed so that the following parameters

$$
\begin{equation*}
\frac{h_{i}}{h_{i+1}}=\tau, \quad \frac{h_{i+1}}{b_{i, i+1}}=\sigma, \quad 2 \ln \frac{2 h_{i}}{a_{i}}=\Omega \tag{6.153}
\end{equation*}
$$

are treated as constants independent of $i$. As throughout this book, it is assumed that $h_{i} \gg a_{i}$.


Fig. 6.29. Seven elements of an infinite log-periodic array.
If the dipoles individually approximate perfect conductors, the electrical properties of the array (such as the driving-point admittances of the elements and the field pattern of the array) at an angular frequency $\omega_{0}$ depend only on the electrical dimensions $\beta_{0} h_{i}, \beta_{0} b_{i, i+1}$, and $\beta_{0} a_{i}$ where $\beta_{0}=\omega_{0} / c=2 \pi / \lambda_{0}$ and $c$ is the velocity of light. If the angular frequency is changed to $\omega_{n}=\tau^{n} \omega_{0}$, where $n$ is a positive or negative integer, the original electrical properties are determined by $\tau^{-n} \beta_{n} h_{i}, \tau^{-n} \beta_{n} b_{i, i+1}$ and $\tau^{-n} \beta_{n} a_{i}$ where $\beta_{n}=\omega_{n} / c$. However, there are along the array antennas with half-lengths $h_{i+n}=\tau^{-n} h_{i}$ for which $\left(h_{i+n+1} / b_{i+n, i+n+1}\right)=\sigma$ and $2 \ln \left(2 h_{i+n} / a_{i+n}\right)=\Omega$. Since $\beta_{0} h_{i}=\tau^{-n} \beta_{n} h_{i}=\beta_{n} h_{i+n}$, it follows that all properties of the array at the angular frequency $\omega_{0}$ referred to element $i$ are repeated at the angular frequency $\omega_{n}$ but referred to the element $i+n$. This periodicity of the properties with respect to frequency is linear with respect to the logarithm of the frequency. That is, since $\log \omega_{n}=\log \omega_{0}+n \log \tau$, it is clear that any property shown graphically on a logarithmic frequency scale is periodic with
period $\log \tau$. Accordingly, arrays with this construction are known as log-periodic dipole arrays [7-10]. Such arrays are generally driven from a two-wire line in the manner illustrated in Figs. 6.30a, $b$. The arrangement with reversed connections in Fig. $6.30 b$ is the one required for end-fire operation.


Fig. 6.30. Log-periodic array driven from a two-wire line with (a) direct connexions, (b) reversed connexions.

Actual arrays are, of course, never infinite so that the ideal fre-quency-independent properties of the infinite array are modified by asymmetries near the ends. These may be modified by the use of a terminating impedance $Z_{T}$ as shown in Fig. 6.30 which provides an additional parameter. The value $Z_{T}=Z_{c}$, where $Z_{c}$ is the characteristic impedance of the line, is an obvious choice.

### 6.13 Analysis of the log-periodic dipole array

The theory developed in section 6.11 for arrays of antennas with unequal lengths, spacings and radii can be applied directly to the log-periodic dipole array. It is only necessary to specify the drivingpoint voltages to the elements in order to obtain a complete solution for the distributions of current along the elements and their individual input admittances. The driving-point admittance of the array and the complete field pattern are then readily obtained over any frequency range for which the condition $\beta_{0} h_{i} \leqslant 5 \pi / 4$ is satisfied for all elements. Such quantities as the beam width, the directivity, front-to-back ratio and side-lobe level can, of course, be obtained from the field pattern.

Consider specifically the array shown in Fig. 6.30b. The driving voltage is applied to a transmission-line that is connected successively to all of the elements beginning with the shortest. Between each adjacent pair of elements the connexions are reversed by crossing the conductors of the transmission line in order to achieve the desired phase relations. The analysis of this circuit is conveniently carried out following the method introduced by Carrel [9]. The procedure is simply to determine first the matrix equation for the antenna circuit shown in Fig. 6.31a, then the matrix equation for the transmission-line circuit shown in Fig. 6.31b, and finally the matrix equation for the two circuits in parallel. Note that in Fig. 6.31, a generator is connected across each of the $N$ terminals.

The matrix equation for the antenna circuit in Fig. 6.31 $a$ has already been given in (6.149). The elements of the admittance matrix [ $Y_{A}$ ] are the self- and mutual admittances of the antenna array.

The matrix equation for the transmission-line circuit in Fig. 6.31b is readily derived. Consider a typical section of the line between the terminal pairs $i$ and $i+1$ which are separated by a length of line $b_{i, i+1}$ as shown in Fig. 6.32. The relations between the current and voltage at terminals $i$ and those at terminals $i+1$ are readily obtained. $\dagger$ For temporary convenience let $d_{i}=b_{i, i+1}$; also let $\phi$ be any constant phase-shift introduced between adjacent elements in addition to the value $\beta_{0} d$ which is determined by the length of line between elements $i$ and $i+1$.

$$
\begin{gather*}
V_{i}=V_{i+1} \cos \left(\beta_{0} d_{i}+\phi\right)+j I_{i+1}^{\prime} R_{c} \sin \left(\beta_{0} d_{i}+\phi\right)  \tag{6.154a}\\
I_{i}^{\prime \prime} R_{c}=j V_{i+1} \sin \left(\beta_{0} d_{i}+\phi\right)+I_{i+1}^{\prime} R_{c} \cos \left(\beta_{0} d_{i}+\phi\right) \tag{6.154b}
\end{gather*}
$$

where $R_{c}$ is the characteristic resistance of the lossless line.

[^9]
(c)

Fig. 6.31. Schematic diagram of (a) the antenna circuit, $(b)$ the transmission-line circuit, and (c) the antenna and transmission-line circuits connected in parallel.

These equations can be rearranged in the form

$$
\begin{align*}
I_{i}^{\prime \prime} & =-j G_{c}\left[V_{i} \cot \left(\beta_{0} d_{i}+\phi\right)-V_{i+1} \csc \left(\beta_{0} d_{i}+\phi\right)\right]  \tag{6.155a}\\
I_{i+1}^{\prime} & =-j G_{c}\left[V_{i} \csc \left(\beta_{0} d_{i}+\phi\right)-V_{i+1} \cot \left(\beta_{0} d_{i}+\phi\right)\right] \tag{6.155b}
\end{align*}
$$

where $G_{c}=R_{c}^{-1}$ is the characteristic conductance of the lossless line. It follows that

$$
\begin{align*}
& I_{i+1}^{\prime \prime}=-j G_{c}\left[V_{i+1} \cot \left(\beta_{0} d_{i+1}+\phi\right)-V_{i+2} \csc \left(\beta_{0} d_{i+1}+\phi\right)\right] \\
& I_{i+2}^{\prime}=-j G_{c}\left[V_{i+1} \csc \left(\beta_{0} d_{i+1}+\phi\right)-V_{i+2} \cot \left(\beta_{0} d_{i+1}+\phi\right)\right] . \tag{6.156b}
\end{align*}
$$



Fig. 6.32. Section of transmission line between the terminal pairs $i$ and $i+1$ when the voltages $V_{i}$ and $V_{i+1}$ are maintained.

The total current in the generator at the terminals $i+1$ is

$$
\begin{align*}
I_{i+1}=I_{i+1}^{\prime \prime}-I_{i+1}^{\prime}= & j G_{c}\left\{V_{i} \csc \left(\beta_{0} d_{i}+\phi\right)-V_{i+1}\left[\cot \left(\beta_{0} d_{i}+\phi\right)\right.\right. \\
& \left.\left.+\cot \left(\beta_{0} d_{i+1}+\phi\right)\right]+V_{i+2} \csc \left(\beta_{0} d_{i+1}+\phi\right)\right\} . \tag{6.157}
\end{align*}
$$

In particular, when $\phi=\pi$ as in Fig. 6.31b,

$$
\begin{align*}
I_{i+1}= & -j G_{c}\left\{V_{i} \csc \beta_{0} b_{i, i+1}\right. \\
& \left.+V_{i+1}\left(\cot \beta_{0} b_{i, i+1}+\cot \beta_{0} b_{i+1, i+2}\right)+V_{i+2} \csc \beta_{0} b_{i+1, i+2}\right\} . \tag{6.158}
\end{align*}
$$

Also

$$
\begin{equation*}
I_{1}=I_{1}^{\prime \prime}=-j G_{c}\left[V_{1} \cot \beta_{0} b_{12}+V_{2} \csc \beta_{0} b_{12}\right] \tag{6.159}
\end{equation*}
$$

and

$$
I_{N}=-j G_{c}\left[V_{N-1} \csc \beta_{0} b_{N-1, N}+V_{N}\left(\cot \beta_{0} b_{N-1, N}+j y_{N}\right)\right]
$$

since

$$
\begin{equation*}
I_{N}^{\prime \prime}=V_{N} Y_{N}=V_{N} y_{N} G_{c} \tag{6.160a}
\end{equation*}
$$

where

$$
\begin{equation*}
y_{N}=Y_{N} / G_{c}=\left[\frac{Y_{T}+j G_{\mathrm{c}} \tan \beta_{0} b_{T}}{G_{\mathrm{c}}+j Y_{T} \tan \beta_{0} b_{T}}\right] \tag{6.160b}
\end{equation*}
$$

is the normalized admittance in parallel with element $N . Y_{T}=1 / Z_{T}$ is the admittance terminating the final section of line of length $b_{T}=b_{N, N+1} / 2$.

With (6.158), (6.159) and (6.160), the matrix equation for the transmission line has the form

$$
\begin{gather*}
\{I\}=\left[Y_{L}\right]\{V\}  \tag{6.162}\\
\{I\}=\left\{\begin{array}{c}
I_{1} \\
I_{2} \\
\vdots \\
I_{N}
\end{array}\right\}, \quad\{V\}=\left\{\begin{array}{c}
V_{1} \\
V_{2} \\
\vdots \\
V_{N}
\end{array}\right\} \tag{6.163}
\end{gather*}
$$

and


The final step in the analysis of the array in Fig. $6.30 b$ is to connect the transmission-line circuit in Fig. $6.31 b$ in parallel with the antenna circuit in Fig. 6.31a as shown schematically in Fig. $6.31 c$. The same driving voltages are maintained across the $N$ input terminals. Let the total currents in the generators be represented by $I_{i i}=I_{z i}(0)+I_{i}$ where $I_{z i}(0)$ is the current entering antenna $i$ and $I_{i}$ is the current into the transmission line at terminals $i$. The matrix equation for the total current is

$$
\begin{equation*}
\left\{I_{t}\right\}=\left(\left[Y_{A}\right]+\left[Y_{L}\right]\right)\left\{V_{0}\right\}=[Y]\left\{V_{0}\right\} . \tag{6.165}
\end{equation*}
$$

This gives the $N$ currents supplied by $N$ generators connected across the $N$ sets of terminals in Fig. 6.31c. In the actual circuit in Fig. $6.30 b$, there is only one generator, $V_{01}$, and all of the total currents $I_{t i}$ are zero except $I_{t 1}$. Hence, in (6.165)

$$
\begin{gather*}
\left\{I_{t}\right\}=\left\{\begin{array}{c}
I_{t 1} \\
0 \\
0 \\
\vdots \\
0
\end{array}\right\}, \quad\left\{V_{0}\right\}=\left\{\begin{array}{c}
V_{01} \\
\vdots \\
V_{0 N}
\end{array}\right\}  \tag{6.166}\\
{[Y]=\left[Y_{A}\right]+\left[Y_{L}\right] .} \tag{6.167}
\end{gather*}
$$

The voltages $V_{0 i}$ driving the $N$ elements are, therefore, given by

$$
\begin{equation*}
\left\{V_{0}\right\}=[Y]^{-1}\left\{I_{t}\right\} \tag{6.168}
\end{equation*}
$$

in terms of the total current $I_{11}$. The driving-point admittances of
the $N$ elements can be determined as follows. The substitution of

$$
\begin{equation*}
\left\{V_{0}\right\}=\left[Y_{A}\right]^{-1}\left\{I_{z}(0)\right\} \tag{6.169}
\end{equation*}
$$

in (6.165) yields

$$
\begin{equation*}
\left\{I_{t}\right\}=\left[U+\left[Y_{L}\right]\left[Y_{A}\right]^{-1}\right]\left\{I_{z}(0)\right\} \tag{6.170}
\end{equation*}
$$

where $U$ is the unit matrix. Note that $\left[Z_{A}\right]=\left[Y_{A}\right]^{-1}$ is the impedance matrix of the array. The equation (6.170) can be solved for the driving-point currents of the several elements in terms of the driving-point current in element 1. Thus,

$$
\begin{equation*}
\left\{I_{z}(0)\right\}=\left[U+\left[Y_{L}\right]\left[Y_{A}\right]^{-1}\right]^{-1}\left\{I_{t}\right\} . \tag{6.171}
\end{equation*}
$$

These currents with a common phase and amplitude reference value are convenient for calculating the field pattern and for comparing relative amplitudes. The admittances of the $N$ elements are

$$
\begin{equation*}
Y_{0 i}=G_{0 i}+j B_{0 i}=I_{z i}(0) / V_{0 i}, \quad i=1,2, \ldots N \tag{6.172}
\end{equation*}
$$

where $V_{0 i}$ and $I_{z i}(0)$ are given, respectively, by (6.168) and (6.171). The driving-point admittance of the array at the terminals $i=1$ of the first element is

$$
\begin{equation*}
Y_{1}=G_{1}+j B_{1} \doteq I_{11} / V_{01} . \tag{6.173}
\end{equation*}
$$

### 6.14 Characteristics of a typical log-periodic dipole array $\dagger$

A complete determination of the properties of the log-periodic dipole array involves a systematic study in which the several parameters that characterize its operation are varied progressively over adequately wide ranges. These include the degree of taper of the array ( $\tau=h_{i} / h_{i+1}$ ), the relative spacing of the elements ( $\sigma=$ $\left.h_{i+1} / b_{i, i+1}\right)$, the relative thickness of the elements $\left(\Omega=2 \ln \left(2 h_{i} / a_{i}\right)\right.$, the total number of elements $N$, the normalized admittance ( $y_{T}=Y_{T} R_{c}$ ) terminating the transmission line beyond the $N^{\text {th }}$ element, and the phase shift $\phi$ introduced between successive elements in addition to that specified by the electrical distance $\beta_{0} b_{i, i+1}$ between adjacent elements. Such an investigation could also make use of optimization procedures for the forward gain, front-to-back ratio, band width, and other properties of practical interest in a manner similar to that used earlier in this chapter for the Yagi-Uda array. Use of the formulation of sections 6.11-6.13, which takes full account of the coupling among all elements in determining the different distributions of current and the individual
$\dagger$ This section is based on chapter 9 of [6]. Parts of sections $6.14-6.16$ were first published in Radio Science [12].
driving-point admittances, should lead to results of considerable quantitative accuracy to supplement those of earlier, more approximate investigations [7-10]. Although no such exhaustive numerical study is yet available, a complete analysis of a typical log-periodic dipole array has been made by Cheong [6] with a high-speed computer. The parameters for this array are $\tau=0.93, \sigma=0.70$, $\Omega=11 \cdot 4, N=12, Y_{T} R_{c}=1$, and $\phi=\pi$. The results obtained serve admirably to illustrate both the detailed operation of the logperiodic dipole array and the power of the theory.

Consider first the operation of the array at a frequency $\dagger$ such that an element $k$ near its centre is a half wavelength long. At this frequency the admittances of the twelve elements when individually isolated lie on a curve in the complex admittance plane that is very nearly an arc of a circle that extends on both sides of the axis $B_{0}=0$ as shown in Fig. 6.33. Note that element 7 is nearest to resonance with only a small negative susceptance. The actual admittances $Y_{0 i}=G_{0 i}+j B_{0 i}$ of the same elements when driven as parts of the log-periodic array lie on a curve that departs significantly from the circle for the isolated admittances. $\ddagger$ It is roughly circular for the group of elements from No. 3 to No. 9 , but the circle has a much greater radius than that for the isolated elements. Indeed, it is so great that the conductances of a number of elements (Nos. 2 and 3) are negative. This large difference in the driving-point admittances is due to coupling; it indicates a strong interaction between the currents in this group of elements. Note that element 7 is still very nearly resonant. Since the admittance curve near its ends bends inward and comes quite close to the circle for the isolated elements, it must be concluded that the elements near the ends of the array behave much as if they were individually isolated. This is possible only if their currents are relatively small and contribute little to the properties of the array.

In Fig. 6.34 are shown the magnitudes and relative phase angles $\ddagger$ of the complex voltages $V_{0 i}$ that obtain across the input terminals of the elements in the array. The amplitudes are fairly constant for the shorter capacitive elements but they decrease rapidly as soon as the elements are long enough to pass through resonance and become inductive. The phase of the voltages is seen to shift continuously

[^10]

Fig. 6.33. Admittances of elements in a log-periodic dipole array when individually isolated and when in an array with $\tau=0.93, \sigma=0 \cdot 7, \Omega=11 \cdot 4, Y_{T} R_{c}=1, \phi=\pi$ (operating frequency $f_{14}$ ). . Isolated admittances: $\times$, admittances in array.
from element to element along the line. Corresponding curves for the driving-point currents $I_{z i}(0)$ are also in Fig. 6.34. Note particularly that elements 4,5 and 6 all carry larger currents than element 7 which is nearest resonance. Note also that the phase curve for the current crosses that for the voltage at resonance. The shorter elements have leading (capacitive) currents, the longer elements lagging (inductive) currents. The relative powers $\dagger P_{0 i}$ in each element and in the termination are given in Fig. 6.35. Note that in the elements 2 and 3, which have a negative input conductance, the power is negative. This means that power is transferred from the other elements to Nos. 2 and 3 by radiation coupling and then from these back to the feeder. The small rise in voltage shown in Fig. 6.34

[^11]

Fig. 6.34. Relative amplitudes and phases of driving-point currents and voltages for a $\log$-periodic dipole array: $\tau=0.93, \sigma=0.7, \Omega=114, Y_{T} R_{c}=1, \phi=\pi$. . driving-point voltages, $V_{0 i}: \times$, driving-point currents, $I_{s i}(0)$.


Fig. 6.35 . Relative power in the twelve elements and the termination. (Operating frequency $f_{14}$.)
at elements 3 and 4 may be ascribed to elements 2 and 3 acting as generators and not as loads. It is significant that the maximum power per element is not in the resonant element 7 but in the shorter elements 5 and 6 which also have larger currents. This is a consequence of the very much smaller voltage maintained across the terminals of element 7 as compared with the voltages across the terminals of elements 5 and 6.

The roles played by the several elements in the array may be seen most clearly from their currents. The distributions of current $I_{0 i}(z)$ along all twelve elements are shown in Fig. 6.36a referred to the driving voltage $V_{01}$ at the input terminals of the array. Note these distributions differ greatly from element to element-they are not simple sinusoids. The quantity $I_{0 i}(z) / V_{01}$ is represented in its real and imaginary parts; it provides the relative currents that together maintain the electromagnetic field. It is seen that (as predicted from the admittance curves in Fig. 6.33) the currents in the outer elements $1,2,9,10,11,12$ are extremely small so that their contributions are negligible. Clearly, the distant electromagnetic field is determined essentially by the currents in elements 3 to 8 and of


Fig. 6.36. (a) Normalized currents in the elements of a log-periodic dipole array; $I_{z i}(z) / V_{01}$; $\tau=0.93, \sigma=0.7, \Omega=114, Y_{T} R_{c}=1, \phi=\pi$. (b) Normalized currents in the log-periodic array: $I_{z i}(z) / V_{0 i}=\left[I_{i i}^{\prime \prime}(z)+j I_{z i}^{\prime}(z)\right] / V_{0 i} ; \quad I_{z i}(0) / V_{0 i}=Y_{0 i}=G_{0 i}+j B_{0 i} ; \quad \tau=0.93, \quad \sigma=0 \cdot 7$,

$$
\Omega=11 \cdot 4, Y_{T} R_{c}=1, \phi=\pi
$$

these elements $4,5,6$ and 7 predominate. Note in particular that the currents in the shorter-than-resonant elements 4,5 and 6 actually exceed the current in the practically resonant element 7.

The current distributions are also shown in Fig. $6.36 b$ but each current is now referred to its own driving voltage. Thus, the quantities represented are $I_{z i}(z) / V_{0 i}=\left[I_{z i}^{\prime \prime}(z)+j I_{z i}^{\prime}(z)\right] / V_{0 i}$ where $I_{z i}^{\prime \prime}(z)$ is the component in phase with $V_{0 i}, I_{z i}^{\prime}(z)$ the component in phase quadrature. Note that $I_{z i}(0) / V_{0 i}=Y_{0 i}$ so that $I_{z i}^{\prime \prime}(0) / V_{0 i}=G_{0 i}$ and $I_{z i}^{\prime}(0) / V_{0 i}=B_{0 i}$. The power in antenna $i$ is $P_{0 i}=\left|V_{0 i}\right|^{2} G_{0 i}$, but since the value of $V_{0 i}$ differs greatly from element to element as seen in Fig. 6.34, the relative powers in the several elements are not proportional simply to the real parts of the currents $I_{z i}^{\prime \prime}(0)$ in the terminals. However, the distributions in Fig. $6.36 b$ are instructive since they show the negative real parts for elements 2 and 3 that transfer power to the feeding line. They also show that the imaginary parts of the currents in elements 1 to 6 are capacitive, those in elements 7 to 12 inductive. This means that each of the elements 1 to 6 acts as a director for the elements to its right, whereas each of the elements 7 to 12 acts as a reflector for all elements to its left. Actually, the capacitive components of current in elements 3,4 and 5 exceed the conductive components so that relatively little power is supplied to them from the line, and they behave substantially like parasitic directors. The inductive component of current predominates in elements 8 to 12 and these act in major part like parasitic reflectors. However, since the amplitudes of the currents in elements 9 to 12 are quite small, it is clear that the principal reflector action comes from element 8 . In summary, Figs. $6.36 a, b$ indicate that of the twelve elements numbers $1,2,9,10,11$ and 12 may be ignored since their currents are small; elements $5,6,7$ are supplied most of the power from the feeder and behave primarily like driven antennas in an endfire array; elements 3 and 4 act predominantly like parasitic directors; and element 8 is essentially a parasitic reflector. Thus, the log-periodic antenna is very much like a somewhat generalized Yagi-Uda array when driven at a frequency for which the antenna closest to resonance is not too near the ends and the array is long enough to include relatively inactive elements at each end. A lengthening of the array by the addition of one or two or even a great many more elements at either end or at both ends cannot significantly modify the circuit or field properties of the array at the particular frequency since these are determined by the active group.

The normalized far-field pattern in the equatorial or $H$-plane (variable $\Phi$ with $\Theta=\pi / 2$ ) is shown in Fig. 6.37. Note the smoothness of the pattern and the very small minor lobes. As is to be expected this low minor-lobe level is achieved at the expense of the beam width. A comparison with the field pattern in Fig. 6.28 for a 10 -element Yagi-Uda array shows that the latter has larger minor lobes but a much narrower beam. However, the Yagi-Uda array does not have the important frequency-independent properties of the log-periodic dipole array.


Fig. 6.37. Normalized far field of log-periodic array with currents shown in Fig. 6.36a.

### 6.15 Frequency-independent properties of the log-periodic dipole array

The principle underlying the properties of the log-periodic dipole array when driven at the terminals of the shortest element as shown in Fig. $6.30 b$ and operated as illustrated in the preceding section depends upon the following:(1)A small group of about seven dipoles constitutes the active or radiating part of the array. These may be described approximately as including (a) three strongly driven and radiating elements near resonance, (b) three shorter elements each of which combines the functions of a rather weakly driven antenna and a highly active parasitic director, and (c) one longer antenna that acts both as a weakly driven element and a strong parasitic reflector. (2) All other elements in the array and the terminating
admittance $Y_{T}$ have such small currents and so little power that they may be ignored both as loads on the feeding line and as contributing radiators of the far-zone field. (3) The driving-point admittance of the array at the terminals of the shortest element is approximately equal to the characteristic conductance $G_{c}$ of the transmission line. (4) The currents in the active elements maintain a unilateral endfire field pattern with very small minor lobes.

The effect of a change in frequency is to shift the active group toward the terminated end with longer elements when the frequency is lowered. So long as the frequency range is bounded so that neither the shortest nor the longest element in the array is a part of the active group, there can be no significant change in either the circuit or the field properties. The array must behave substantially as if infinitely long. On the other hand, as the frequency is increased or decreased sufficiently to make the element at either end of the array a member of the active group, all of the properties of the array must begin to change. This change becomes drastic when the frequency is varied so much that none of the $N$ elements is near resonance.

The general behaviour of the twelve-element log-periodic dipole array as a function of frequency has been investigated by Cheong [6] using a discrete set of frequencies $f_{1}, f_{2}, \ldots f_{27}$. These are chosen so that the lowest frequency $f_{1}$ is below the resonant frequency of the longest element No. 12 and the highest frequency $f_{27}$ is above the resonant value for the shortest element No. 1 as shown in the table on page 260. In order to distribute the frequencies according to the log-periodic scheme of lengths and spacings, the ratio factor $\sqrt{0.93}$ was chosen so that $f_{j+2} / f_{j}=0.93$ where $j$ is an integer. This provides an intermediate frequency step $f_{j+1} / f_{j}=\sqrt{0.93}$ to achieve a closer approximation of a continuous spectrum. The properties of the array described in the preceding section and represented in Figs. 6.33-6.37 are obtained specifically at the centre frequency $f_{14}$ in this set for which an element (No. 7) near the middle of the array is most nearly resonant.

Consider first a decrease in frequency from $f_{14}$ to $f_{7}$ so that resonance is moved from approximately element 7 to approximately element 10. The corresponding driving-point admittances are shown in the complex admittance plane in Fig. 6.38 together with the admittances of the elements when these are individually isolated. The admittance circle for the isolated antennas and the

Relation between the relative heights of the elements, $h / \lambda$, and the frequencies $f_{i}$.

| $i$ in $f_{i}$ | $h_{1} / \lambda$ | $h_{12} / \lambda$ | $f_{i}$ when $h_{1}=1 \mathrm{~m}$ |
| :---: | :---: | :---: | :---: |
| 1 | 0.0962 | 0.2138 | 28.86 MHz |
| 2 | 0.0998 | 0.2217 | 29.94 |
| 3 | 0.1035 | 0.2299 | 31.05 |
| 4 | 0.1073 | 0.2384 | 32.19 |
| 5 | 0.1113 | 0.2473 | 33.39 |
| 6 | 0.1154 | 0.2564 | 34.62 |
| 7 | 0.1197 | 0.2659 | 35.91 |
| 8 | 0.1241 | 0.2757 | 37.23 |
| 9 | 0.1287 | 0.2859 | 38.61 |
| 10 | 0.1335 | 0.2966 | 40.05 |
| 11 | 0.1384 | 0.3075 | 41.52 |
| 12 | 0.1435 | 0.3188 | 43.05 |
| 13 | 0.1488 | 0.3306 | 44.64 |
| 14 | 0.1543 | 0.3428 | 46.29 |
| 15 | 0.1600 | 0.3555 | 48.00 |
| 16 | 0.1659 | 0.3686 | 49.77 |
| 17 | 0.1721 | 0.3824 | 51.63 |
| 18 | 0.1785 | 0.3966 | 53.55 |
| 19 | 0.1850 | 0.4110 | 55.50 |
| 20 | 0.1918 | 0.4261 | 57.54 |
| 21 | 0.1989 | 0.4419 | 59.67 |
| 22 | 0.2063 | 0.4583 | 61.89 |
| 23 | 0.2139 | 0.4752 | 64.17 |
| 24 | 0.2218 | 0.4928 | 66.54 |
| 25 | 0.2300 | 0.5110 | 69.00 |
| 26 | 0.2385 | 0.5299 | 71.55 |
| 27 | 0.2473 | 0.5494 | 74.19 |

admittance curve $\dagger$ for the array resemble those in Fig. 6.33 but appear to have been moved in a counter-clockwise direction. The admittances of the short elements from 1 to 6 now form a small spiral around the values for the same elements when isolated. The previous tight little spiral of admittances for the longer elements in Fig. 6.33 is completely unwound and the admittance curve for the array no longer comes near to the circle for the admittances of the isolated elements. It is clear that in Fig. 6.38 elements 6 to 12 instead of 3 to 9 as in Fig. 6.33 form the active group. This is further confirmed in Fig. 6.39 which shows the voltages and currents at the driving points of the elements. The voltage amplitudes are quite constant from elements 1 to 8 , then decrease rapidly. The associated current amplitudes are small for elements 1 to 6 , large for elements 6 to 11 and again small for element 12. Evidently, with reference to

[^12]

Fig. 6.38. Like Fig. 6.33 but for lower frequency with resonance near element 10. (Operating frequency $f_{7}$.) . Isolated admittances; $\times$, admittances in array.

Fig. 6.39 (and Fig. 6.34), the group consisting of director-radiators 6,7 and 8 (instead of $3,4,5$ ), radiators $9,10,11$ (instead of $6,7,8$ ) and reflector-radiator 12 (instead of 9 ) are primarily responsible for the properties of the array. These conclusions may also be reached from a study of the current-distribution curves for $I_{z i}(z) / V_{01}$ in Fig. $6.40 a$ and for $I_{z i}(z) / V_{0 i}$ in Fig. 6.40b. The former show clearly that the amplitudes of the currents in elements 1 through 5 are negligibly small. The latter indicate the following: the capacitive currents dominate in elements 6,7 and 8 , in element 9 the capacitive and conductive currents are practically equal, element 10 is nearly resonant with a very small capacitive current, element 11 has large inductive and conductive components, and in element 12 the inductive current exceeds the conductive component. It may be concluded, therefore, that the decrease in frequency which moved


Fig. 6.39. Like Fig. 6.34 but for lower frequency with resonance near element 10. - Driving-point voltages, $V_{0 i} ; \times$, driving-point currents, $I_{z i}(0)$.
resonance from near element 7 to near element 10 has not significantly changed the properties of the active group and, hence, of the array.
If the frequency is decreased still further to $f_{3}$ at which even element No. 12 is too short to be resonant, the admittance curve is that shown in Fig. 6.41. The counter-clockwise rotation of the curves has been increased beyond that in Fig. 6.38 so that now none of the


Fig. 6.40. (a) Like Fig. 6.36a but for lower frequency with resonance near element 10.
(b) Like Fig. $6.36 b$ but for lower frequency with resonance near element 10.


Fig. 6.41. Like Fig. 6.38 but for lower frequency with resonance beyond element 12. (Operating frequency $f_{3}$.)
elements is either inductive or resonant. The small spiral formed by the admittances of the short elements around the circle of their isolated values has two complete turns. It is to be expected, therefore, that elements 1 through 7 must have negligible currents. The active group in Fig. 6.41 includes dipoles 8 to 12. However, none of these is resonant and there are no inductive reflectors. Moreover, since there must be a significant voltage across the terminals of element No. 12, considerable power must be dissipated in the terminating admittance $Y_{T}$. Under these conditions the properties of the array must differ significantly from those existing for the frequencies determining Figs. 6.33 and 6.38. The frequencyindependent behaviour requires at least two radiating and reflecting elements longer than the one nearest resonance.

If the frequency is increased to $f_{19}$ so that element No. 4 is most nearly resonant, the admittance curve takes the form shown in Fig. 6.42. As compared with Fig. 6.33, the curves have been rotated clockwise with respect to the axis $B_{0}=0$. The admittances of the longer elements Nos. 8 through 12 in the array are all clustered close to one end of the circular arc formed by the admittances of the isolated elements. On the other hand, not even the shortest element No. 1 is near the other end of the circular arc. Since a detailed study (in conjunction with Figs. 6.33 and $6.36 a$ ) of the currents and power in the elements longer than resonance has shown that at most two elements longer than the one nearest


Fig. 6.42. Like Fig. 6.33 but for higher frequency with resonance near element 4. (Operating frequency $f_{19}$.) , Isolated admittances: $\times$, admittances in array.
resonance carry significant currents, it follows that all elements from No. 12 down through No. 7 play no significant role in the array. On the other hand, it is clear from Fig. 6.42 that the admittance of element No. 1 does not produce a curve that bends inward toward the circular arc of isolated admittances, but rather outward away from the arc. This is a consequence of the fact that the region of active elements has been moved too close to the end of the array. It is clear from Fig. 6.36a that the active region includes at least four elements shorter than the one nearest resonance. For the frequency $f_{19}$ leading to Fig. 6.42 there are only three such elements available. This means that the frequency responsible for Fig. 6.42 is already somewhat higher than acceptable for the frequencyindependent properties of the array and that the currents in element

No. 1 must differ from the expected since one of the required director-radiators is missing.

The useful range for a frequency-independent behaviour lies between the frequencies at which elements 5 and 10 (or, in general, $N-2$ ) are resonant. In the scale of discrete frequencies used for the twelve-element array this range is approximately $f_{7} \leqslant f \leqslant f_{17}$. The power in the several elements at the frequencies $f_{3}, f_{7}, f_{14}, f_{19}$ and $f_{23}$ is shown in Figs. 6.35 and 6.43. Note that in these figures only the plotted points are significant. The connecting curves serve only to guide the eye.


Fig. 6.43. Like Fig. 6.35 but for frequencies $f_{3}, f_{7}, f_{19}$, and $f_{23}$.
A detailed study of the operation of the twelve-element array over the full range of frequencies from $f_{1}$ to $f_{27}$ has been made by Cheong [6]. Important results in addition to those already discussed are contained in Figs. 6.44-6.46. They may be summarized as follows:

1. As shown in Fig. 6.44, curve $T$, a large fraction of the total power is dissipated in the terminating admittance $Y_{T}=G_{c}$, in the ranges $f<f_{5}$ and $f>f_{26}$. As a consequence only a small fraction of power appears in the dipoles so that little is radiated. It is also clear from Fig. 6.44 that in the range $f_{5} \leqslant f \leqslant f_{26}$ only a small part of the power is dissipated in the terminating admittance, most of it appears in and is radiated from a relatively small group of active dipoles near resonance.


Fig. 6.44. Relative power in the elements and the termination.
2. In the range $f_{5} \leqslant f \leqslant f_{17}$ elements which have half-lengths $h_{i}$ in the range $0.18 \leqslant h_{i} / \lambda \leqslant 0.255$ form the active group. Resonance occurs with $h_{i} / \lambda \doteq 0 \cdot 216$. Elements which have half-lengths $h_{i}$ less than $0 \cdot 18 \lambda$ or greater than $0 \cdot 255 \lambda$ play an insignificant part in the
operation of the array. On the other hand, outside this range of frequencies the shorter and longer elements cannot be ignored.
3. As shown in Fig. 6.45 the driving-point admittance of the array, $Y_{t}$, is reasonably constant at a value very near the characteristic conductance $G_{c}$ of the transmission line over the range $f_{5} \leqslant f \leqslant f_{17}$. Specifically $Y_{T} \doteq(23.0+j 0.0) \times 10^{-3}$ mhos with $G_{c}=20 \times 10^{-3}$ mhos. Outside this range of frequencies $Y_{T}$ varies widely in both real and imaginary parts.
4. The band of frequencies $f_{5} \leqslant f \leqslant f_{17}$ is characterized by a very stable main lobe in the forward direction, i.e. toward the shorter elements and the driving point, and very small side and back lobes. This is clear from Fig. 6.37 and Fig. 6.46. Figure 6.47 shows that the ratio of the forward field to the largest side or back-lobe level is roughly constant near 15 and that the 3 db forward beam width remains quite stable at about $38^{\circ}$ in the range $f_{5} \leqslant f \leqslant f_{17}$. Outside this band of frequencies large side and back lobes appear.


Fig. 6.45. Input admittance $Y_{1}=G_{1}+j B_{1}$ of the log-periodic array.


Fig. 6.46. Like Fig. 6.37 but for a number of different frequencies.
It is important to note that all of the computed data apply to a particular array with a single set of values of the basic parameters $\tau, \sigma, \Omega, Y_{T}, R_{c}$ and $\phi$. An extensive study by high-speed computer of the effects of changes in these parameters and of optimum designs based on the three-term theory is indicated but is not available at the time of this writing. Additional information is given in Cheong [6], Cheong and King [12], and Carrel [9].

### 6.16 Experimental verification of the theory for arrays of unequal dipoles

In order to verify experimentally the predictions of the general theory developed in section 6.11 for arrays of dipoles with a wide range of lengths and spacings, a series of measurements on the


Fig. 6.47. Ratio of the forward field to the largest side lobe and the 3 db beam width over the frequency range $f_{1}$ to $f_{27}$.
twelve-element log-periodic dipole array would be appropriate. However, arrays of this type are driven from two-wire lines in a manner that makes accurate measurements of current distributions, admittances, voltages and field patterns very difficult-especially over a two-to-one or greater range of frequencies. For this reason a less elaborate array arranged to permit precision measurements was preferred by Cheong [6].
As a first step, an extensive experimental study was made of two-coupled dipoles over wide ranges of lengths and spacings in order to verify the adequency of the three-term representation of the currents. When this had been established, a complete array of five elements was constructed after the log-periodic design with the longest element approximately twice as long as the shortest element. This array consisted of monopoles over a very large ground screen. Each element was the extension of the inner conductor of a coaxial line of which the outer conductor pierced the metal ground screen. In order to provide an equivalent for the reversal of the connexions between adjacent pairs of elements, provision was made to permit the insertion of an arbitrary length of coaxial line in addition to a length equal to the spacing of the elements. Since the added phase shift had to be exactly $\pi$ for each different frequency, it was necessary to readjust the length of the sections of coaxial line between the elements for each frequency.

Careful measurements were made of the driving-point admittance, the currents and voltages in amplitude and phase at the base of each element, and the field pattern over a range of some 17 different frequencies that included resonance for the longest and the shortest elements. The agreement between theory and measurement was remarkable in all details, thus confirming the adequacy of the theory for use not only on the five-element array but on an array of any type that satisfies the requirements of the theory. Details and extensive graphs are in the work of Cheong [6] and Cheong and King [13].

## CHAPTER 7

## PLANAR AND THREE-DIMENSIONAL ARRAYS

The study of dipole arrays in chapters 3 through 6 has proceeded from simpler to more complicated configurations. In chapters 3 and 4 all elements are physically alike and arranged to be parallel with their centres uniformly spaced around a circle so that when driven in suitable phase sequences all elements are geometrically and electrically identical. Chapter 5 is also concerned with parallel elements that are structurally alike, but they lie in a curtain with their centres along a straight line of finite length; consequently the electromagnetic environments of the several elements are not all the same. In chapter 6 the requirement that the elements in a curtain array be equal in length is omitted and consideration is given first to arrays of elements that differ only moderately in length, then to arrays in which not only the lengths but also the radii of the elements and the distances between them vary widely. The lifting of each restriction introduces additional complications in the approximate representation of the currents on the elements by simple trigonometric functions and in the reduction of the integrals in the simultaneous integral equations to sums of such functions with suitably defined complex coefficients.

The final generalization which is carried out in this chapter is the omission of the requirement maintained throughout the book until this point, that all elements be non-staggered. The removal of this condition leads to the discussion of arrays of parallel elements that are arranged in a plane as in Fig. 7.1 and in three dimensions as shown in Fig. 7.2. Note that such arrays include arbitrarily staggered elements and collinear elements which do not occur in the circular and curtain arrays considered in chapters 3 through 6 . When the centres of the elements are displaced from a common plane, the halves of many antennas are in different electrical environments so that an even symmetry with respect to their individual centres no longer obtains for the distributions of current. An important new complication is thus introduced:


Fig. 7.1. Planar array of nine identical elements.


Fig. 7.2. Three-dimensional array of twelve identical elements.
components of current with odd symmetries in addition to those with even symmetries.

### 7.1 Vector potentials and integral equations for the currents

Four typical elements in an array of $N$ parallel dipoles are shown in Fig. 7.3. All antennas have their axes parallel to the $Z$-axis


Fig. 7.3. Typical elements in an array of $N$ parallel antennas.
of a system of rectangular coordinates $X, Y, Z$. The centre of the $k^{\text {th }}$ element is at $X_{k}, Y_{k}, Z_{k}$; its radius is $a_{k}$, its half-length $h_{k}$, it is centre driven by a delta-function generator with EMF $V_{0 k}$. As before, the antennas are assumed to be perfectly conducting and electrically thin so that $\beta_{0} a_{k} \ll 1$ for $k=1,2, \ldots, N$. A local axial coordinate $z_{k}$ has its origin at the centre of element $k$.

The vector potential on the surface of antenna $k$ no longer has the simple form given in (2.3), since the even symmetry conditions $I_{z k}\left(-z_{k}\right)=I_{z k}\left(z_{k}\right)$ for the current and $A_{z k}\left(-z_{k}\right)=A_{z k}\left(z_{k}\right)$ for the vector potential no longer apply. However, the vector potential
can be resolved into two parts, one with even symmetry, the other with odd symmetry. Thus

$$
\begin{equation*}
A_{z k}\left(z_{k}\right)=A_{z k}^{\text {even }}\left(z_{k}\right)+A_{z k}^{\text {odd }}\left(z_{k}\right) \tag{7.1}
\end{equation*}
$$

where, in the range $-h_{k} \leqslant z_{k} \leqslant h_{k}$,

$$
\begin{equation*}
A_{2 k}^{\text {even }}\left(z_{k}\right)=\frac{-j}{c}\left[C_{k 1} \cos \beta_{0} z_{k}+\frac{1}{2} V_{0 k} \sin \beta_{0}\left|z_{k}\right|\right] \tag{7.2}
\end{equation*}
$$

as in (2.3), and

$$
\begin{equation*}
A_{z k}^{\text {odd }}\left(z_{k}\right)=\frac{-j}{c} C_{k 2} \sin \beta_{0} z_{k} . \tag{7.3}
\end{equation*}
$$

The vector potential on the surface of antenna $k$ is also given by the sum of integrals,
where

$$
\begin{gather*}
A_{z k}\left(z_{k}\right)=\sum_{i=1}^{N} \frac{\mu_{0}}{4 \pi} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{7.4}\\
G_{k i}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)=\frac{e^{-j \beta_{0} R_{k i}}}{R_{k i}} \tag{7.5a}
\end{gather*}
$$

with

$$
\begin{equation*}
R_{k i}=\sqrt{\left(d_{k i}+z_{i}^{\prime}-z_{k}\right)^{2}+b_{k i}^{2}} \tag{7.5b}
\end{equation*}
$$

As shown in Fig. 7.3, $d_{k i}=\left|Z_{k}-Z_{i}\right|$ is the axial distance between the plane containing the centres of elements $k$ and $i, d_{k k}=0$; $b_{k i}=\sqrt{\left(X_{k}-X_{i}\right)^{2}+\left(Y_{k}-Y_{i}\right)^{2}}, i \neq k$, is the distance between the centre of element $k$ and the projection of the centre of element $i$ onto the plane $z_{k}=0 ; b_{k k}=a_{k}$. The currents $I_{z i}\left(z_{i}\right)$ in the $N$ elements that generate the vector potential on the surface of antenna $k$ as given in (7.4) include even and odd parts with respect to the centres of the respective elements. That is,

$$
\begin{equation*}
I_{z i}\left(z_{i}\right)=I_{z i}^{\text {even }}\left(z_{i}\right)+I_{z i}^{\text {odd }}\left(z_{i}\right) \tag{7.6}
\end{equation*}
$$

where

$$
I_{z i}^{\text {even }}\left(z_{i}\right)=(1 / 2)\left[I_{z i}\left(z_{i}\right)+I_{z i}\left(-z_{i}\right)\right], I_{z i}^{\text {odd }}\left(z_{i}\right)=(1 / 2)\left[I_{z i}\left(z_{i}\right)-I_{z i}\left(-z_{i}\right)\right] .
$$

In order to separate the even and the odd parts of the vector potential in (7.4), the kernel $G_{k i}\left(z_{k}, z_{i}^{\prime}\right)$ in (7.4) must be separated into its even and odd parts. Thus,

$$
\begin{equation*}
G_{k i}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)=G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)+G_{k i}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) \tag{7.7}
\end{equation*}
$$

where, as is readily shown,

$$
\begin{align*}
G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) & =\frac{1}{2}\left[K_{k i}\left(z_{k}-d_{k i}, z_{i}^{\prime}\right)+K_{k i}\left(z_{k}+d_{k i}, z_{i}^{\prime}\right)\right]  \tag{7.8}\\
G_{k i}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) & =\frac{1}{2}\left[K_{k i}\left(z_{k}-d_{k i}, z_{i}^{\prime}\right)-K_{k i}\left(z_{k}+d_{k i}, z_{i}^{\prime}\right)\right] . \tag{7.9}
\end{align*}
$$

The function $K$ occurring in (7.8) and (7.9) is the kernel previously used for non-staggered arrays, viz.,
$K_{k i}\left(z_{k}, z_{i}^{\prime}\right)=K_{k i R}\left(z_{k}, z_{i}^{\prime}\right)+j K_{k i I}\left(z_{k}, z_{i}^{\prime}\right)=\frac{e^{-j \beta_{0} \sqrt{\left(z_{k}-z_{i}^{\prime}\right)^{2}+b_{k i i}^{2}}}}{\sqrt{\left(z_{k}-z_{i}^{\prime}\right)^{2}+b_{k i}^{2}}}$.
Note that when $d_{k i}=0, \quad G_{k i}^{\text {even }}\left(0, z_{k}, z_{i}^{\prime}\right)=K_{k i}\left(z_{k}, z_{i}^{\prime}\right) \quad$ and $G_{k i}^{\text {odd }}\left(0, z_{k}, z_{i}^{\prime}\right)=0$, as required for the previously analysed nonstaggered array. By means of the obvious relation,

$$
K_{k i}\left(z_{k}, z_{i}^{\prime}\right)=K_{k i}\left(-z_{k},-z_{i}^{\prime}\right),
$$

it is readily shown that, when (7.6) and (7.7) are substituted in (7.4), the parts of the integral that involve the products $I^{\text {even }} G^{\text {even }}$, $I^{\text {odd }} G^{\text {odd }}$ are themselves even in $z$, the parts that contain $I^{\text {even }} G^{\text {odd }}$, $I^{\text {odd }} G^{\text {even }}$ are themselves odd in $z$. It follows that the even part of the vector potential is given by

$$
\begin{align*}
4 \pi \mu_{0}^{-1} A_{z k}^{\text {even }}\left(z_{k}\right)= & \int_{-h_{k}}^{h_{k}} I_{z k}^{\text {even }}\left(z_{k}^{\prime}\right) G_{k k}^{\text {even }}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \\
& +\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}^{\text {even }}\left(z_{i}^{\prime}\right) G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& +\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}^{\text {odd }}\left(z_{i}^{\prime}\right) G_{k i}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & \frac{-j 4 \pi}{\zeta_{0}}\left[C_{k 1} \cos \beta_{0} z_{k}+(1 / 2) V_{0 k} \sin \beta_{0}\left|z_{k}\right|\right] \tag{7.11}
\end{align*}
$$

where $k=1,2, \ldots N ; \zeta_{0} \doteq 120 \pi$ ohms; and $\Sigma^{\prime}$ is the sum with $i=k$ omitted. The odd part of the vector potential is contained in

$$
\begin{align*}
4 \pi \mu_{0}^{-1} A_{z k}^{\text {odd }}\left(z_{k}\right)= & \int_{-h_{k}}^{h_{k}} I_{z k}^{\text {odd }}\left(z_{k}^{\prime}\right) G_{k k}^{\text {even }}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \\
& +\sum_{i=1}^{N} I_{z i}^{\text {even }}\left(z_{i}^{\prime}\right) G_{k i}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& +\sum_{i=1}^{N} I_{z i}^{\text {odd }}\left(z_{i}^{\prime}\right) G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & -\left(j 4 \pi / \zeta_{0}\right) C_{k 2} \sin \beta_{0} z_{k} \tag{7.12}
\end{align*}
$$

where $k=1,2, \ldots N$.

The relations on the right in (7.11) and (7.12) are $2 N$ simultaneous integral equations for the even and odd parts of the currents in the $N$ elements.

### 7.2 Vector potential differences and integral equations

In order to determine approximate distributions of current from the two sets of $N$ simultaneous integral equations in the general manner described in earlier chapters, it is convenient to introduce the vector potential differences. This is quite straight forward for the even part of $A_{z k}\left(z_{k}\right)$. Thus, if $4 \pi \mu_{0}^{-1} A_{z k}\left(h_{k}\right)$ is subtracted from both sides of (7.11), the result is

$$
\begin{align*}
4 \pi \mu_{0}^{-1}\left[A_{z k}^{\text {even }}\left(z_{k}\right)-A_{z k}^{\text {even }}\left(h_{k}\right)\right]= & \int_{-h_{k}}^{h_{k}} I_{z k}^{\text {even }}\left(z_{k}^{\prime}\right) G_{k k d}^{\text {even }}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \\
& +\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}^{\text {even }}\left(z_{i}^{\prime}\right) G_{k i d}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& +\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}^{\text {odd }}\left(z_{i}^{\prime}\right) G_{k i d}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & -\left(j 4 \pi / \zeta_{0}\right)\left[\frac{1}{2} V_{0 k}\left(\sin \beta_{0}\left|z_{k}\right|-\sin \beta_{0} h_{k}\right)\right. \\
& \left.+C_{k 1}\left(\cos \beta_{0} z_{k}-\cos \beta_{0} h_{k}\right)\right] \\
= & \left(j 4 \pi / \zeta_{0} \cos \beta_{0} h_{k}\right)\left[\frac{1}{2} V_{0 k} \sin \beta_{0}\left(h_{k}-\left|z_{k}\right|\right)\right. \\
& \left.+U_{k}\left(\cos \beta_{0} z_{k}-\cos \beta_{0} h_{k}\right)\right] \tag{7.13}
\end{align*}
$$

where $k=1,2, \ldots N$ and

$$
\begin{equation*}
U_{k}=\frac{-j \zeta_{0}}{\mu_{0}} A_{z k}^{\text {even }}\left(h_{k}\right), \quad C_{k 1}=\frac{-U_{k}+(1 / 2) V_{0 k} \sin \beta_{0} h_{k}}{\cos \beta_{0} h_{k}} \tag{7.14}
\end{equation*}
$$

as in the corresponding equation with $d_{k i}=0$ for the curtain array. The difference kernel (with extra subscript $d$ ) is defined by

$$
\begin{equation*}
G_{k i d}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)=G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)-G_{k i}^{\text {even }}\left(d_{k i}, h_{k}, z_{i}^{\prime}\right) . \tag{7.15}
\end{equation*}
$$

It is not possible to form an equation like (7.13) with $A_{z k}^{\text {odd }}\left(z_{k}\right)$ since this is an odd function of $z_{k}$ so that if $A_{z k}^{\text {odd }}\left(z_{k}\right)-A_{z k}^{\text {odd }}\left(h_{k}\right)$ is zero at $z_{k}=h_{k}$, it is $-2 A_{z k}^{\text {odd }}\left(h_{k}\right)$ at $z_{z}=-h_{k}$. A convenient alternative $\dagger$ is to subtract the odd function $\left(z_{k} / h_{k}\right) A_{z k}^{\text {odd }}\left(h_{k}\right)$ which is equal to the vector potential at both $z_{k}=h_{k}$ and $z_{k}=-h_{k}$.

Thus, with (7.12),

$$
\begin{align*}
4 \pi \mu_{0}^{-1}\left[A_{z k}^{\text {odd }}\left(z_{k}\right)-\right. & \left.\left(z_{k} / h_{k}\right) A_{z k}^{\text {odd }}\left(h_{k}\right)\right] \\
= & \int_{-h_{k}}^{h_{k}} I_{z k}^{\text {odd }}\left(z_{k}^{\prime}\right) \mathscr{G}_{k k d}^{\text {even }}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \\
& +\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}^{\text {even }}\left(z_{i}^{\prime}\right) \mathscr{G}_{k i d}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& +\sum_{i=1}^{N} \int_{h_{h_{i}}}^{h_{i}} I_{z i}^{\text {odd }}\left(z_{i}^{\prime}\right) \mathscr{G}_{k i d}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
= & -\left(j 4 \pi C_{k 2} / \zeta_{0}\right)\left[\sin \beta_{0} z_{k}-\left(z_{k} / h_{k}\right) \sin \beta_{0} h_{k}\right] \tag{7.16}
\end{align*}
$$

where $k=1,2, \ldots N$ and the difference kernels are given by

$$
\begin{align*}
\mathscr{G}_{k i d}^{e v e n}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) & =G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)-\left(z_{k} / h_{k}\right) G_{k i}^{\text {even }}\left(d_{k i}, h_{k}, z_{i}^{\prime}\right)  \tag{7.17a}\\
\mathscr{G}_{k i d}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) & =G_{k i}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)-\left(z_{k} / h_{k}\right) G_{k i}^{\text {odd }}\left(d_{k i}, h_{k}, z_{i}^{\prime}\right) . \tag{7.17b}
\end{align*}
$$

For each superscript, the kernel may be expanded into its real and imaginary parts as follows:

$$
\begin{equation*}
\mathscr{G}_{k i d}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)=\mathscr{G}_{k i d R}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right)+j \mathscr{G}_{k i d I}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) . \tag{7.1}
\end{equation*}
$$

The desired alternative set of $2 N$ simultaneous integral equations for the even and odd parts of the currents in the $N$ elements is contained in (7.13) and (7.16).

### 7.3 Approximate distribution of current

It has been shown in earlier chapters that the first integral in (7.13) is well approximated by

$$
\begin{align*}
& \int_{-h_{k}}^{h_{k}} I_{z k}^{\text {even }}\left(z_{k}^{\prime}\right) G_{k k d R}^{\text {even }}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime}=\int_{-h_{k}}^{h_{k}} I_{z k}^{\text {even }}\left(z_{k}^{\prime}\right) K_{k k d R}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \\
& \sim I_{z k}^{\text {even }}\left(z_{k}\right)  \tag{7.19}\\
& \text { and } \\
& \int_{-h_{k}}^{h_{k}} I_{z k}^{\text {even }}\left(z_{k}^{\prime}\right) G_{k k d I}^{\text {even }}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime}=\int_{-h_{k}}^{h_{k}} I_{z k}^{\text {even }}\left(z_{k}^{\prime}\right) K_{k k d I}\left(z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \\
& \sim H_{0 z k}  \tag{7.20}\\
& \text { where } \\
& H_{0 z k}=\cos \left(\beta_{0} z_{k} / 2\right)-\cos \left(\beta_{0} h_{k} / 2\right)
\end{align*}
$$

provided $\beta_{0} h_{k} \leqslant 5 \pi / 4$. By the same procedure it is readily shown that the first integral in (7.16) can be separated into analogous parts for which the following relations are good approximations:

$$
\begin{align*}
& \int_{-h_{k}}^{h_{k}} I_{z k}^{\mathrm{odd}}\left(z_{k}^{\prime}\right) \mathscr{G}_{k k d R}^{\mathrm{even}}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \sim I_{z k}^{\mathrm{odd}}\left(z_{k}\right)  \tag{7.22}\\
& \int_{-h_{k}}^{h_{k}} I_{z k}^{\mathrm{odd}}\left(z_{k}^{\prime}\right) \mathcal{G}_{k k d I}^{\mathrm{even}}\left(0, z_{k}, z_{k}^{\prime}\right) d z_{k}^{\prime} \sim E_{0 z k} \tag{7.23}
\end{align*}
$$

where

$$
\begin{equation*}
E_{0 z k}=\sin \left(\beta_{0} z_{k} / 2\right)-\left(z_{k} / h_{k}\right) \sin \left(\beta_{0} h_{k} / 2\right) . \tag{7.24}
\end{equation*}
$$

As a consequence of (7.19) and (7.22) it follows that the trigonometric functions that occur on the right side of (7.13) and (7.16) must also occur as leading terms in the approximate expressions for the currents, together with (7.21) and (7.24). That is, appropriate approximate formulas for the even and odd currents in antenna $k$ are given below. For the even currents

$$
\begin{equation*}
I_{z k}^{\text {even }}\left(z_{k}\right)=A_{k} M_{0 z k}+B_{k} F_{0 z k}+D_{k} H_{0 z k} \tag{7.25a}
\end{equation*}
$$

or the alternative equivalent form:

$$
\begin{equation*}
I_{z k}^{\text {even }}\left(z_{k}\right)=A_{k}^{\prime} S_{0 z k}+B_{k}^{\prime} F_{0 z k}+D_{k} H_{0 z k} \tag{7.25b}
\end{equation*}
$$

where $A_{k}, A_{k}^{\prime}, B_{k}, B_{k}^{\prime}$ and $D_{k}$ are complex coefficients and

$$
\begin{align*}
M_{0 z k} & =\sin \beta_{0}\left(h_{k}-\left|z_{k}\right|\right)  \tag{7.26}\\
S_{0 z k} & =\sin \beta_{0}\left|z_{k}\right|-\sin \beta_{0} h_{k}  \tag{7.27}\\
F_{0 z k} & =\cos \beta_{0} z_{k}-\cos \beta_{0} h_{k}  \tag{7.28}\\
H_{0 z k} & =\cos \left(\beta_{0} z_{k} / 2\right)-\cos \left(\beta_{0} h_{k} / 2\right) . \tag{7.29}
\end{align*}
$$

For the odd currents

$$
\begin{equation*}
I_{z k}^{\text {odd }}\left(z_{k}\right)=Q_{k} P_{0 z k}+R_{k} E_{0 z k} \tag{7.30}
\end{equation*}
$$

where $Q_{k}$ and $R_{k}$ are complex coefficients and

$$
\begin{equation*}
P_{0 z k}=\sin \beta_{0} z_{k}-\left(z_{k} / h_{k}\right) \sin \beta_{0} h_{k} \tag{7.31}
\end{equation*}
$$

$E_{0 z k}$ is defined in (7.24). The above formulas are for $k=1,2, \ldots N$. The approximate formulas (7.25a, b) and (7.30) are obtained specifically from the first integrals in (7.13) and (7.16). When there are no staggered elements ( $d_{i k}=0$ ), it is known that the induced currents are well represented by a linear combination of $F_{0 z k}$ and $H_{0 z k}$. It may be argued that a similar linear combination must also be an acceptable representation of the even parts of the currents induced in staggered elements. This follows from the theoretical and experimental studies referred to in chapter 6 of currents in non-staggered elements that differ greatly in length. If the current induced on a relatively long element (but with $\beta_{0} h_{k} \leqslant 5 \pi / 4$ ) by an adjacent very short antenna is well represented by ( $7.25 \mathrm{a}, \mathrm{b}$ ), it may be concluded that the same must be true of the current induced in antenna $k$ by other coupled elements which maintain a vector potential with an even part that varies less in amplitude and phase along antenna $k$ than the vector potential generated by the currents in a very short element. Since no measurements were available of the currents induced by coupled staggered
elements, a numerical check was made of the degree in which the assumed current distributions satisfy the integral equation. The results were quite satisfactory.

It may be concluded that the current along any element $k$ in an array of $N$ parallel antennas is approximately

$$
\begin{align*}
I_{z k}\left(z_{k}\right)= & A_{k} \sin \beta_{0}\left(h_{k}-\left|z_{k}\right|\right)+B_{k}\left(\cos \beta_{0} z_{k}-\cos \beta_{0} h_{k}\right)+D_{k}\left[\cos \left(\beta_{0} z_{k} / 2\right)\right. \\
& \left.-\cos \left(\beta_{0} h_{k} / 2\right)\right]+Q_{k}\left[\sin \beta_{0} z_{k}-\left(z_{k} / h_{k}\right) \sin \beta_{0} h_{k}\right] \\
& +R_{k}\left[\sin \left(\beta_{0} z_{k} / 2\right)-\left(z_{k} / h_{k}\right) \sin \left(\beta_{0} h_{k} / 2\right)\right] . \tag{7.32}
\end{align*}
$$

If more convenient, the first two terms may be replaced by those in (7.25b). The first three terms for the even part of the current are the same in form as for arrays of parallel, non-staggered elements. They include the term $\sin \beta_{0}\left(h_{k}-\left|z_{k}\right|\right)$ which represents that part of the current excited directly by the generator voltage $V_{0 k}$. No such term is possible for the odd part of the current in a centredriven dipole. The remaining problem is to determine the coefficients in (7.32).

### 7.4 Evaluation of coefficients

The coefficients in the approximate formula (7.32) for the current in a typical element $k$ in an array of $N$ arbitrarily located parallel elements may be evaluated in various ways. The method outlined here is the one selected by V. W. H. Chang in his study of planar and three-dimensional arrays. He preferred to use the following alternative form for the current:

$$
\begin{align*}
I_{z k}\left(z_{k}\right)= & A_{k}^{\prime}\left(\sin \beta_{0}\left|z_{k}\right|-\sin \beta_{0} h_{k}\right)+B_{k}^{\prime}\left(\cos \beta_{0} z_{k}-\cos \beta_{0} h_{k}\right) \\
& +D_{k}\left[\cos \left(\beta_{0} z_{k} / 2\right)-\cos \left(\beta_{0} h_{k} / 2\right)\right] \\
& +Q_{k}\left[\sin \beta_{0} z_{k}-\left(z_{k} / h_{k}\right) \sin \beta_{0} h_{k}\right] \\
& +R_{k}\left[\sin \left(\beta_{0} z_{k} / 2\right)-\left(z_{k} / h_{k}\right) \sin \left(\beta_{0} h_{k} / 2\right)\right] \tag{7.33}
\end{align*}
$$

where $k=1,2, \ldots N$. Instead of substituting the even and odd parts into the integral equation (7.13) and (7.16) he used the simpler integral equation for the total current obtained when (7.4) is equated to (7.1) with (7.2) and (7.3). That is,

$$
\begin{align*}
\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) \frac{e^{-j \beta_{0} R_{k i}}}{R_{k i}} d z_{i}^{\prime}= & -\left(j 4 \pi / \zeta_{0}\right)\left[C_{k 1} \cos \beta_{0} z_{k}+C_{k 2} \sin \beta_{0} z_{k}\right. \\
& \left.+(1 / 2) V_{0 k} \sin \beta_{0}\left|z_{k}\right|\right] . \tag{7.34}
\end{align*}
$$

The substitution of (7.33) in the integral in (7.34) yields $N$ equations with $7 N$ unknowns, viz. the $5 N$ coefficients in (7.33) and the $2 N$
constants $C_{k 1}$ and $C_{k 2}$ with $k=1,2, \ldots N$. The required $7 N$ equations can be obtained by satisfying (7.34) exactly at seven points along each antenna. The points chosen for $z_{k}$ are $h_{k}, 2 h_{k} / 3, h_{k} / 3,0$, $-h_{k} / 3,-2 h_{k} / 3$, and $-h_{k}$. These correspond to the values used in the evaluation of the coefficients for the array of unequal elements in the last sections of chapter 6, but since the currents are now not even functions of $z_{k}$, the negative values $-h_{k},-2 h_{k} / 3$ and $-h_{k} / 3$ must also be used.

The number of unknowns can be reduced by the elimination of the constants $C_{k 1}$ and $C_{k 2}$. The former is conveniently evaluated at $z_{k}=h_{k}$ where the current vanishes; the latter can be obtained from the equation at $z_{k}=2 h_{k} / 3$. Thus, with the notation

$$
\begin{align*}
& U_{k 1}=\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}  \tag{7.35}\\
& U_{k 2}=\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, 2 h_{k} / 3, z_{i}^{\prime}\right) d z_{i}^{\prime} . \tag{7.36}
\end{align*}
$$

(7.34) evaluated at $z_{k}=h_{k}$ and $2 h_{k} / 3$ yields

$$
\begin{equation*}
\left(j 4 \pi / \zeta_{0}\right) C_{k 1}=\left[U_{k 1} \sin \left(2 \beta_{0} h_{k} / 3\right)-U_{k 2} \sin \beta_{0} h_{k}\right] \csc \left(\beta_{0} h_{k} / 3\right) \tag{7.37}
\end{equation*}
$$

$\left(j 4 \pi / \zeta_{0}\right)\left(C_{k 2}+V_{0 k} / 2\right)=\left[U_{k 2} \cos \beta_{0} h_{k}-U_{k 1} \cos \left(2 \beta_{0} h_{k} / 3\right)\right] \csc \left(\beta_{0} h_{k} / 3\right)$.

Note that in the range $\beta_{0} h_{k} \leqslant 5 \pi / 4$, these expressions remain finite.

With (7.37) and (7.38), $C_{k 1}$ and $C_{k 2}$ can be eliminated from (7.34) to obtain

$$
\begin{align*}
& \sum_{i=1}^{N} \sin \left(\beta_{0} h_{k} / 3\right) \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \quad+\sum_{i=1}^{N} \sin \beta_{0}\left(\frac{2}{3} h_{k}-z_{k}\right) \int_{-h}^{h} I_{z i}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& \quad-\sum_{i=1}^{N} \sin \beta_{0}\left(h_{k}-z_{k}\right) \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, \frac{2}{3} h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& =  \tag{7.39}\\
& \frac{j 4 \pi V_{0 k}}{\zeta_{0}} \sin \left(\beta_{0} h_{k} / 3\right) \sin \beta_{0} z_{k} H\left(-z_{k}\right)
\end{align*}
$$

where $H\left(-z_{k}\right)$ is the Heaviside function defined by $H\left(-z_{k}\right)=0$, $z_{k}>0 ; H\left(-z_{k}\right)=1, z_{k} \leqslant 0$.

The next step is to substitute the current (7.33) in the integrals in (7.39). This leads to quantities of the following form:

$$
\begin{align*}
\xi_{k i j}\left(z_{k}\right)= & \sin \left(\beta_{0} h_{k} / 3\right) \int_{-h_{i}}^{h_{i}} J_{z i}^{j}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& +\sin \beta_{0}\left(\frac{2}{3} h_{k}-z_{k}\right) \int_{-h_{i}}^{h_{i}} J_{i i}^{j}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, h_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
& +\sin \beta_{0}\left(h_{k}-z_{k}\right) \int_{-h_{i}}^{h_{i}} J_{z i}^{j}\left(z_{i}^{\prime}\right) G_{k i}\left(d_{k i}, 2 h_{k} / 3, z_{i}^{\prime}\right) d z_{i}^{\prime} \tag{7.40}
\end{align*}
$$

where $k=1,2, \ldots N, i=1,2, \ldots N$ and $j=1,2, \ldots 5$. The notation

$$
\begin{align*}
J_{i z}^{1}\left(z_{i}^{\prime}\right) & =S_{0 z i}=\sin \beta_{0}\left|z_{i}\right|-\sin \beta_{0} h_{i}  \tag{7.41a}\\
J_{z i}^{2}\left(z_{i}^{\prime}\right) & =F_{0 z i}=\cos \beta_{0} z_{i}-\cos \beta_{0} h_{i}  \tag{7.41b}\\
J_{i z}^{3}\left(z_{i}^{\prime}\right) & =H_{0 z i}=\cos \left(\beta_{0} z_{i} / 2\right)-\cos \left(\beta_{0} h_{i} / 2\right)  \tag{7.41c}\\
J_{z i}^{4}\left(z_{i}^{\prime}\right) & =P_{0 z i}=\sin \beta_{0} z_{i}-\left(z_{i} / h_{i}\right) \sin \beta_{0} h_{i}  \tag{7.41d}\\
J_{z i}^{5}\left(z_{i}^{\prime}\right) & =E_{0 z i}=\sin \left(\beta_{0} z_{i} / 2\right)-\left(z_{i} / h_{i}\right) \sin \left(\beta_{0} h_{i} / 2\right) \tag{7.41e}
\end{align*}
$$

is used. Note that for any specified value of $z_{k}$ in a fixed array, (7.40) defines a set of $N$ complex numbers that can be evaluated by highspeed computer. With (7.40) and (7.41a-e), (7.39) becomes:

$$
\begin{gather*}
\sum_{i=1}^{N}\left[A_{i}^{\prime} \xi_{k i 1}\left(z_{k}\right)+B_{i}^{\prime} \xi_{k i 2}\left(z_{k}\right)+D_{i} \xi_{k i 3}\left(z_{k}\right)+Q_{i} \xi_{k i 4}\left(z_{k}\right)+R_{i} \xi_{k i 5}\left(z_{k}\right)\right] \\
=j\left(4 \pi V_{0 k} / \zeta_{0}\right) \sin \left(\beta_{0} h_{k} / 3\right) \sin \beta_{0} z_{k} H\left(-z_{k}\right) \tag{7.42}
\end{gather*}
$$

with $k=1,2, \ldots N$. Five sets of $N$ equations can be obtained from (7.42) if $z_{k}$ is successively made equal to the five values $h_{k} / 3,0$, $-h_{k} / 3,-2 h_{k} / 3$ and $-h_{k}$. These contain the $M=5 N$ unknown coefficients given by the column matrix

$$
\begin{equation*}
\{A\}=\operatorname{tr}\left(A_{1}, \ldots A_{N}, B_{1}, \ldots B_{N}, D_{1}, \ldots D_{N}, Q_{1}, \ldots, Q_{N}, R_{1}, \ldots R_{N}\right) \tag{7.43}
\end{equation*}
$$

where $t r$ indicates the transpose. Let

$$
[\Phi]=\left[\begin{array}{ll}
\Phi_{11} \ldots & \Phi_{1 M}  \tag{7.44}\\
\vdots & \vdots \\
\Phi_{M 1} \ldots & \Phi_{M M}
\end{array}\right]
$$

where $M=5 N$ and

$$
\begin{equation*}
\Phi_{k+(m-1) N, i+(j-1) N}=\xi_{k i j}\left(z_{k}^{m}\right) \tag{7.45}
\end{equation*}
$$

with $j=1,2, \ldots 5 ; m=1,2, \ldots 5 ; k=1,2, \ldots N ; i=1,2, \ldots N$. The
notation $z_{k}^{1}=h_{k} / 3, z_{k}^{2}=0, z_{k}^{3}=-h_{k} / 3, z_{k}^{4}=-2 h_{k} / 3, z_{k}^{5}=-h_{k}$ is used. Also let the following column matrix of $5 N$ terms be defined:

$$
\begin{equation*}
\{W\}=\operatorname{tr}\left(0 \ldots 0,0 \ldots 0, W_{1} \ldots W_{N}, T_{1} \ldots T_{N}, S_{1} \ldots S_{N}\right) \tag{7.46}
\end{equation*}
$$

where

$$
\begin{align*}
W_{k} & =-\left(j 4 \pi V_{0 k} / \zeta_{0}\right) \sin ^{2}\left(\beta_{0} h_{k} / 3\right)  \tag{7.47a}\\
T_{k} & =-\left(j 4 \pi V_{0 k} / \zeta_{0}\right) \sin \left(\beta_{0} h_{k} / 3\right) \sin \left(2 \beta_{0} h_{k} / 3\right)  \tag{7.47b}\\
S_{k} & =-\left(j 4 \pi V_{0 k} / \zeta_{0}\right) \sin \left(\beta_{0} h_{k} / 3\right) \sin \beta_{0} h_{k} \tag{7.47c}
\end{align*}
$$

with $k=1,2, \ldots N$.
With this matrix notation the $5 N$ equations for the $N$ coefficients of the currents in terms of the $N$ driving voltages $V_{0 k}$ with $k=1,2, \ldots N$ are given by the single matrix equation

$$
\begin{equation*}
[\Phi]\{A\}=\{W\} . \tag{7.48}
\end{equation*}
$$

If (7.48) is solved for the $5 N$ coefficients given by (7.43), the $N$ currents $I_{z k}\left(z_{k}\right), k=1,2, \ldots N$ given in (7.33) are known in terms of the $N$ voltages $V_{0 k}$. The currents at the driving points are then given by the matrix equation
where

$$
\begin{gather*}
\left\{I_{z}(0)\right\}=[Y]\left\{V_{0}\right\}  \tag{7.49a}\\
\left\{I_{z}(0)\right\}=\left\{\begin{array}{l}
I_{z 1}(0) \\
\vdots \\
I_{z N}(0)
\end{array}\right\}, \quad\left\{V_{0}\right\}=\left\{\begin{array}{l}
V_{01} \\
\vdots \\
V_{0 N}
\end{array}\right\} . \tag{7.49b}
\end{gather*}
$$

The square matrix

$$
[Y]=\left[\begin{array}{llll}
Y_{11} & Y_{12} \ldots & Y_{1 N}  \tag{7.49c}\\
\vdots & & & \vdots \\
Y_{N 1} & \ldots \ldots \ldots \ldots . . & Y_{N N}
\end{array}\right]
$$

is the admittance matrix. The terms $Y_{i i}$ are the self-admittances, the terms $Y_{i j}, i \neq j$ are the mutual admittances.

The $N$ driving voltages can be expressed in terms of the currents at the driving points in the form

$$
\begin{equation*}
\left\{V_{0}\right\}=[Z]\left\{I_{z}(0)\right\} \tag{7.49d}
\end{equation*}
$$

where $[Z]=[Y]^{-1}$ is the impedance matrix.
The driving-point admittance of element $k$ is defined by

$$
\begin{equation*}
Y_{0 k}=I_{z k}(0) / V_{0 k} . \tag{7.49e}
\end{equation*}
$$

### 7.5 The field patterns

The radiation field of an array of arbitrarily located parallel elements is the superposition of the fields generated by the
individual elements. The far field of element $i$ in such an array is given by

$$
\begin{equation*}
E_{\Theta i}^{r}=\frac{j \omega \mu_{0}}{4 \pi} \frac{e^{-j \beta_{0} R_{i}}}{R_{i}} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) e^{j \beta_{0} z_{i} \cos \Theta} \sin \Theta d z_{i}^{\prime} \tag{7.50a}
\end{equation*}
$$

where $R_{i}$ is the distance from the centre of the antenna $i$ at $X_{i}, Y_{i}, Z_{i}$ to the point of calculation $P$, and $\Theta$ is the angle between the $Z$-axis and the line $0 P$ from the origin of coordinates near the centre of the array.

If the distribution of current (7.33) is substituted in (7.50a), the field of element $i$ can be expressed in the following integrated form:

$$
\begin{align*}
E_{\Theta i}^{r}= & \frac{j \zeta_{0}}{4 \pi} \frac{e^{-j \beta_{0} R_{i}}}{R_{i}}\left[A_{i}^{\prime} H_{m}\left(\Theta, \beta_{0} h_{i}\right)+B_{i}^{\prime} G_{m}\left(\Theta, \beta_{0} h_{i}\right)\right. \\
& \left.+D_{i} D_{m}\left(\Theta, \beta_{0} h_{i}\right)+Q_{i} Q_{m}\left(\Theta, \beta_{0} h_{i}\right)+R_{i} R_{m}\left(\Theta, \beta_{0} h_{i}\right)\right] \tag{7.50b}
\end{align*}
$$

where the individual field factors are as follows:

$$
\left.\begin{array}{rl}
H_{m}\left(\Theta, \beta_{0} h_{i}\right)= & \frac{\beta_{0} \sin \Theta}{2} \int_{-h_{i}}^{h_{i}}\left(\sin \beta_{0}\left|z_{i}^{\prime}\right|-\sin \beta_{0} h_{i}\right) e^{j \beta_{0} z_{i}^{\prime} \cos \Theta} d z_{i}^{\prime} \\
= & \left\{\cos \Theta-\left[1-\cos \beta_{0} h_{i} \cos \left(\beta_{0} h_{i} \cos \Theta\right)\right]\right\} \sec \Theta \csc \Theta \\
H_{m}\left(0, \beta_{0} h_{i}\right)=H_{m}\left(\pi, \beta_{0} h_{i}\right)=0 \\
H_{m}\left(\frac{\pi}{2}, \beta_{0} h_{i}\right)=1-\cos \beta_{0} h_{i}-\beta_{0} h_{i} \sin \beta_{0} h_{i} \\
G_{m}\left(\Theta, \beta_{0} h_{i}\right)= & \frac{\beta_{0} \sin \Theta}{2} \int_{-h_{i}}^{h_{i}}\left(\cos \beta_{0} z_{i}^{\prime}-\cos \beta_{0} h_{i}\right) e^{j \beta_{0} z_{i}^{\prime} \cos \Theta} d z_{i}^{\prime} \\
=\left[\cos \Theta \sin \beta_{0} h_{i} \cos \left(\beta_{0} h_{i} \cos \Theta\right)\right.
\end{array}\right\} \begin{gathered}
\left.\quad-\cos \beta_{0} h_{i} \sin \left(\beta_{0} h_{i} \cos \Theta\right)\right] \sec \Theta \csc \Theta
\end{gathered}
$$

$$
\begin{align*}
& D_{m}\left(\frac{\pi}{2}, \beta_{0} h_{i}\right)=2 \sin \left(\beta_{0} h_{i} / 2\right)-\beta_{0} h_{i} \cos \left(\beta_{0} h_{i} / 2\right)  \tag{7.53b}\\
& D_{m}\left(\frac{\pi}{3}, \beta_{0} h_{i}\right)=D_{m}\left(\frac{2 \pi}{3}, \beta_{0} h_{i}\right)=\frac{\sqrt{3}}{4}\left(\beta_{0} h_{i}-\sin \beta_{0} h_{i}\right)  \tag{7.53c}\\
& Q_{m}\left(\Theta, \beta_{0} h_{i}\right)=\frac{\beta_{0} \sin \Theta}{2} \int_{-h_{i}}^{h_{i}}\left[\sin \beta_{0} z_{i}^{\prime}-\left(z_{i}^{\prime} / h_{i}\right) \sin \beta_{0} h_{i}\right] e^{j \beta_{0 z_{i}^{\prime}} \cos \Theta} d z_{i}^{\prime} \\
& =\left(j / \beta_{0} h_{i}\right)\left[-\beta_{0} h_{i} \cos ^{2} \Theta \cos \beta_{0} h_{i} \sin \left(\beta_{0} h_{i} \cos \Theta\right)\right. \\
& -\sin ^{2} \Theta \sin \beta_{0} h_{i} \sin \left(\beta_{0} h_{i} \cos \Theta\right) \\
& \left.+\beta_{0} h_{i} \cos \Theta \sin \beta_{0} h_{i} \cos \left(\beta_{0} h_{i} \cos \Theta\right)\right] \\
& \times \csc \Theta \sec ^{2} \Theta \\
& Q_{m}\left(0, \beta_{0} h_{i}\right)=Q_{m}\left(\frac{\pi}{2}, \beta_{0} h_{i}\right)=Q_{m}\left(\pi, \beta_{0} h_{i}\right)=0 \\
& R_{m}\left(\Theta, \beta_{0} h_{i}\right)=\frac{\beta_{0} \sin \Theta}{2} \int_{-h_{i}}^{h_{i}}\left[\sin \left(\beta_{0} z_{i}^{\prime} / 2\right)-\left(z_{i}^{\prime} / h_{i}\right)\right. \\
& \left.\times \sin \left(\beta_{0} h_{i} / 2\right)\right] e^{j \beta_{0} z_{i}^{\prime} \cos \theta} d z_{i}^{\prime} \\
& =\left[\frac{j \sin \Theta}{\beta_{0} h_{i} \cos ^{2} \Theta\left(1-4 \cos ^{2} \Theta\right)}\right]\left\{\sin \left(\beta_{0} h_{i} \cos \Theta\right)\right. \\
& \times\left[-2 \beta_{0} h_{i} \cos ^{2} \Theta \cos \left(\beta_{0} h_{i} / 2\right)-\sin \left(\beta_{0} h_{i} / 2\right)\right. \\
& \left.+4 \sin \left(\beta_{0} h_{i} / 2\right) \cos ^{2} \Theta\right]+\beta_{0} h_{i} \sin \left(\beta_{0} h_{i} / 2\right) \\
& \left.\times \cos \Theta \cos \left(\beta_{0} h_{i} \cos \Theta\right)\right\}  \tag{7.55a}\\
& R_{m}\left(\frac{\pi}{2}, \beta_{0} h_{i}\right)=0  \tag{7.55b}\\
& R_{m}\left(\frac{\pi}{3}, \beta_{0} h_{i}\right)=-R_{m}\left(\frac{2 \pi}{3}, \beta_{0} h_{i}\right)=j \frac{\sqrt{3}}{4}\left[\beta_{0} h_{i}+\sin \beta_{0} h_{i}\right. \\
& \left.-\left(8 / \beta_{0} h_{i}\right) \sin ^{2}\left(\beta_{0} h_{i} / 2\right)\right] . \tag{7.55c}
\end{align*}
$$

The radiation field of $N$ parallel antennas is the sum of the contributions from each element. In the far-field approximation

$$
\begin{equation*}
\frac{e^{-j \beta_{0} R_{i}}}{R_{i}}=\frac{e^{-j \beta_{0} R_{0}}}{R_{0}} e^{j \beta_{0}\left(r_{i} \cdot \hat{\mathbf{R}}_{0}\right)} \tag{7.56}
\end{equation*}
$$

where $\mathbf{r}_{i}$ is the vector drawn from the origin near the centre of the array to the centre of antenna $i$ and $\hat{\mathbf{R}}_{0}$ is the unit vector along the line $0 P$ where $P$ is the point of calculation as shown in Fig. 7.4.


Fig. 7.4. Point $P$ in the far field of an array of parallel elements of which element $i$ at $X_{i}, Y_{i}$, $Z_{i}$ is typical.

With (7.56) the far-field of the array is

$$
\begin{align*}
E_{\Theta}^{r}= & \frac{j \zeta_{0}}{4 \pi} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} \sum_{i=1}^{N} e^{j \beta_{0}\left(\mathbf{r}_{i} . \hat{R}_{0}\right)}\left[A_{i}^{\prime} H_{m}\left(\Theta, \beta_{0} h_{i}\right)+B_{i}^{\prime} G_{m}\left(\Theta, \beta_{0} h_{i}\right)\right. \\
& \left.+D_{i} D_{m}\left(\Theta, \beta_{0} h_{i}\right)+Q_{i} Q_{m}\left(\Theta, \beta_{0} h_{i}\right)+R_{i} R_{m}\left(\Theta, \beta_{0} h_{i}\right)\right] . \tag{7.5}
\end{align*}
$$

If the point $P$ where the field is calculated is located by the spherical coordinates $R_{0}, \Theta, \Phi$ and the centre of element $i$ is at $X_{i}, Y_{i}, Z_{i}$, then

$$
\begin{equation*}
\mathbf{r}_{i} \cdot \hat{\mathbf{R}}_{0}=X_{i} \sin \Theta \cos \Phi+Y_{i} \sin \Theta \sin \Phi+Z_{i} \cos \Theta \tag{7.58}
\end{equation*}
$$

### 7.6 The general two-element array $\dagger$

In the introductory analysis of the two-element array in chapter 2 only parallel non-staggered antennas are considered. As a consequence of the resulting even symmetry for the currents in the elements and the vector potentials, a three- or even two-term representation is adequate. The more general five-term approximation of the currents introduced in this chapter includes the previous three terms to describe the even currents and two additional terms

[^13]to represent the odd currents generated by asymmetrical coupling when the elements are collinear as shown in Fig. 7.5 or staggered as in Fig. 7.6.


Fig. 7.5. Two-element collinear array.


Fig. 7.6. Two-element staggered array.

The general formulas for the currents, driving-point admittances and field patterns derived in the preceding sections are readily specialized for the two-element array. Comparative examples of the admittances of coupled full-wave and half-wave elements when driven symmetrically with $V_{01}=V_{02}=1$ and anti-symmetrically with $V_{01}=-V_{02}=1$ as a function of the distance between centres are given in Tables 7.1a, b, c for antennas with $a / \lambda=0.007022$. These tables give the symmetrical admittance $Y^{s}=G^{s}+j B^{s}$ and the anti-symmetrical admittance $Y^{a}=G^{a}+j B^{a}$. The associated self- and mutual admittances are $Y_{1 s}=\left(Y^{s}+Y^{a}\right) / 2$ and $Y_{12}=-\left(Y^{s}-Y^{a}\right) / 2$.

Table 7.1a applies to the non-staggered antennas considered in chapter 2 ; the variable parameter is $b_{12} / \lambda$, the normalized distance between centres. Table 7.1 b is for the collinear pair with the distance ( $d_{12}-2 h$ ) between the adjacent ends as the parameter ( $d_{12}$ is the distance between centres). The admittances in Table 7.1c are for the staggered pair as the centre of element 2 is moved along a $45^{\circ}$ line so that $b_{12}=d_{12}$. The impedances $Z^{s}=1 / Y^{s}$ and $Z^{a}=1 / Y^{a}$ corresponding to the admittances in Tables 7.1a, b, c are shown graphically in Figs. 7.7 and 7.8, respectively, for the symmetrically and anti-symmetrically driven pairs. In these figures $R_{0}$ and $X_{0}$ are the resistance and reactance for infinite separation. The interaction

Table 7.1a. Symmetrical and anti-symmetrical admittances in millimhos of parallel, non-staggered array of two elements;

$$
a / \lambda=0.007022 .
$$

|  | $h / \lambda=0.50$ |  |  | $h / \lambda=0.25$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{12} / \lambda$ | $Y^{s}$ | $Y^{a}$ |  | $Y^{s}$ | $Y^{a}$ |
| 0.05 | $0.813+j 1.397$ | $0.071+j 0.941$ |  | $4.939-j 0.820$ | $4.831-j 53.897$ |
| 0.10 | $1.028+j 1.668$ | $0.146+j 1.122$ |  | $5.572-j 0.290$ | $4.344-j 24.150$ |
| 0.15 | $1.197+j 1.749$ | $0.230+j 1.286$ |  | $6.258+j 0.112$ | $4.385-j 15.386$ |
| 0.25 | $1.448+j 1.627$ | $0.408+j 1.531$ |  | $7.853+j 0.809$ | $4.758-j 8.655$ |
| 0.50 | $1.079+j 0.932$ | $0.865+j 1.774$ |  | $14.321-j 1.496$ | $6.129-j 3.340$ |
| 0.75 | $0.635+j 1.374$ | $1.244+j 1.570$ |  | $8.960-j 7.101$ | $8.318-j 1.318$ |
| 1.00 | $0.872+j 1.657$ | $1.066+j 1.141$ |  | $7.211-j 3.839$ | $11.577-j 2.517$ |
| 1.25 | $1.157+j 1.537$ | $0.728+j 1.359$ |  | $8.543-j 2.095$ | $9.503-j 5.707$ |
| 1.50 | $1.053+j 1.226$ | $0.877+j 1.602$ | $10.725-j 2.875$ | $7.777-j 3.918$ |  |

Table 7.1b. Symmetrical and anti-symmetrical admittances in millimhos of collinear array of two elements ; $a / \lambda=0.007022$.

| $\underline{d_{12}-2 h}$ | $h / \lambda=0.50$ |  | $h / \lambda=0.25$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $Y^{s}$ | $Y^{a}$ | $Y^{s}$ | $Y^{a}$ |
| $\lambda$ |  |  |  |  |
| 0 | $1.050+j 1.581$ | $0.816+j 1.139$ | $5.549-j 1.953$ | 15.281-j6.078 |
| 0.05 | $1.042+j 1.505$ | $0.808+j 1.338$ | 7.508-j1.868 | 11.013--j6.483 |
| 0.10 | $1.035+j 1.465$ | $0.844+j 1.409$ | 8.444-j1.913 | $9 \cdot 529-j 5 \cdot 821$ |
| 0.15 | $1.026+j 1.437$ | $0.875+j 1.440$ | $9 \cdot 100-j 2 \cdot 104$ | 8.858-j5.208 |
| 0.25 | $0 \cdot 999+j 1 \cdot 403$ | $0.919+j 1.461$ | 9.850-j2.764 | 8.412-j4.288 |
| 0.50 | $0.941+j 1.409$ | $0.966+j 1.446$ | $9.521-j 4.026$ | 8.832-j3.243 |
| 0.75 | $0.945+j 1.435$ | $0.965+j 1.422$ | 8.923-j3.800 | 9.415-j3.404 |
| 1.00 | $0.958+j 1.435$ | $0.950+j 1.422$ | $9.042-j 3.445$ | $9.300-j 3.785$ |
| $1 \cdot 25$ | $0.959+j 1.426$ | $0.950+j 1.431$ | 9.295-j3.518 | $9.045-j 3.699$ |
| $1 \cdot 50$ | $0.953+j 1.425$ | $0.956+j 1.432$ | 9.238-j3.708 | $9.103-j 3.517$ |

Table 7.1c. Symmetrical and anti-symmetrical admittances in millimhos of parallel, staggered array of two elements;

$$
a / \lambda=0.007022 .
$$

|  | $h / \lambda=0.50$ |  |  | $h / \lambda=0.25$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $b_{12}$ |  |  |  |  |
| $\lambda$ | $d_{12}$ |  |  |  |  |
| $\lambda$ | $Y^{s}$ | $Y^{a}$ | $Y^{s}$ | $Y^{a}$ |  |
| 0.05 | $0.825+j 1.318$ | $0.086+j 0.608$ | $4.692-j 0.775$ | $11.923-j 76.274$ |  |
| 0.10 | $1.040+j 1.720$ | $0.786+j 0.801$ | $5.456-j 0.470$ | $8.631-j 29.120$ |  |
| 0.15 | $1.189+j 1.832$ | $0.236+j 0.984$ |  | $6.336-j 0.297$ | $7.399-j 16.489$ |
| 0.25 | $1.363+j 1.681$ | $0.420+j 1.379$ | $8.372-j 0.097$ | $6.669-j 8.084$ |  |
| 0.50 | $1.035+j 1.110$ | $0.965+j 1.673$ | $11.077-j 4.641$ | $7.804-j 2.856$ |  |
| 0.75 | $0.785+j 1.474$ | $1.089+j 1.414$ | $8.235-j 3.925$ | $10.159-j 3.106$ |  |
| 1.00 | $1.008+j 1.489$ | $0.906+j 1.350$ | $9.340-j 2.908$ | $8.903-j 4.302$ |  |
| 1.25 | $0.976+j 1.380$ | $0.939+j 1.484$ | $9.483-j 4.115$ | $8.862-j 3.166$ |  |
| 1.50 | $0.918+j 1.400$ | $0.989+j 1.419$ | $8.729-j 3.561$ | $9.650-j 3.650$ |  |



Fig. 7.7. Resistance and reactance of symmetrically driven array of two parallel half-wave dipoles when non-staggered, staggered with $b_{12}=d_{12}$, and collinear; $a / \lambda=0.007022$, $h / \lambda=0.25, V_{02}=V_{01}$.


Fig. 7.8. Like Fig. 7.7 but for anti-symmetrically driven elements, $\boldsymbol{V}_{02}=-V_{01}$.
between the elements is seen to be greatest for the non-staggered pair, smallest for the collinear arrangement. The self- and mutual impedances are given by $Z_{s 1}=\left(Z^{s}+Z^{a}\right) / 2$ and $Z_{12}=\left(Z^{s}-Z^{a}\right) / 2$; they are listed in Tables 7.2a, b.

The current distribution along the lower element of a collinear pair when the adjacent ends are separated by a distance $d_{12}-2 h=0.1 \lambda$ is shown in Fig. 7.9 for both symmetrically and anti-symmetrically driven full-wave elements. Note that in both cases the currents are asymmetrical with respect to the centre of the element. When the excitation is symmetrical ( $V_{01}=V_{02}$ ), the current in the outer half is greatest; when the excitation is antisymmetrical ( $V_{01}=-V_{02}$ ), the current in the inner half is greatest.

The current distribution for a pair of coupled full-wave antennas in the staggered position with $b_{12} / \lambda=d_{12} / \lambda=0.1$ is shown in Fig. 7.10 for symmetric excitation ( $V_{01}=V_{02}$ ) in broken line. Since the two elements are very close together, the interaction is great. When centre driven with equal and opposite voltages, the two conductors form a slightly displaced two-wire line with a large and only slightly asymmetrical reactive current $I_{z 1}^{\prime}(z)$ that is almost sinusoidal, and a very small in-phase component $I_{z 1}^{\prime \prime}(z)$. Since the current induced in each element by that in the other is essentially

Table 7.2a. Self- and mutual impedances of coupled half-wave dipoles; $h / \lambda=0.25 ; a / \lambda=0.007022$.

| $x$ | Parallel, non-staggered |  | Staggered |  | Collinear |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x=b_{12} / \lambda$ |  | $x=b_{12} / \lambda=d_{12} / \lambda$ |  | $x=\left(d_{12}-2 h\right) / \lambda$ |  |
|  | $Z_{s 1}=R_{s 1}+j X_{s 1}$ | $Z_{12}=R_{12}+j X_{12}$ | $Z_{s 1}=R_{s 1}+j X_{s 1}$ | $Z_{12}=R_{12}+j X_{12}$ | $Z_{s 1}=R_{s 1}+j X_{s 1}$ | $Z_{12}=R_{12}+j X_{12}$ |
| 0 |  |  |  |  | $108 \cdot 4+j 39 \cdot 5$ | $51.9+j 17.0$ |
| 0.05 | $99 \cdot 4+j 25 \cdot 6$ | $97 \cdot 7+j 7 \cdot 15$ | $104 \cdot 7+j 23 \cdot 5$ | $102 \cdot 7+j 10 \cdot 7$ | $96 \cdot 4+j 35 \cdot 4$ | $29 \cdot 0-j 4 \cdot 2$ |
| $0 \cdot 10$ | $93 \cdot 1+j 24 \cdot 7$ | 85.9-j15.4 | $95 \cdot 6+j 23 \cdot 6$ | 86.3-j7.9 | $94 \cdot 5+j 36 \cdot 1$ | 18.1-j10.6 |
| 0.15 | $88 \cdot 4+j 28 \cdot 6$ | 71.3-j31.5 | $90 \cdot 1+j 28.9$ | 67.4-j21.5 | $94 \cdot 1+j 36 \cdot 7$ | 10.2-j12.6 |
| 0.25 | $87 \cdot 4+j 37 \cdot 9$ | 38.6-j50.8 | $90 \cdot 1+j 37 \cdot 5$ | 29.4-j36.1 | $94 \cdot 2+j 37 \cdot 3$ | -0.1-j10.8 |
| 0.50 | $97 \cdot 4+j 37 \cdot 9$ | - $28.4-\mathrm{j} 30.7$ | $94 \cdot 1+j 39 \cdot 1$ | -1.7-j30.9 | $94 \cdot 4+j 37 \cdot 2$ | $-5 \cdot 3+j 0 \cdot 5$ |
| 0.75 | $92 \cdot 9+j 36 \cdot 4$ | $-24 \cdot 4+j 17 \cdot 9$ | $94 \cdot 9+j 36 \cdot 8$ | -18.1-j4.6 | $94 \cdot 4+j 37 \cdot 2$ | $0 \cdot 5+j 3 \cdot 2$ |
| 1.00 | $95 \cdot 3+j 37.7$ | $12 \cdot 8+j 19 \cdot 8$ | $94 \cdot 5+j 37.3$ | $4 \cdot 5+j 9 \cdot 8$ | $94 \cdot 4+j 37 \cdot 2$ | 2.2-j0.4 |
| 1.25 | $93 \cdot 9+j 37 \cdot 8$ | 16.5-j9.7 | $94 \cdot 3+j 37 \cdot 2$ | 3.3-j6.8 | $94 \cdot 4+j 37 \cdot 2$ | -0.3-j1.6 |
| 1.50 | $94 \cdot 8+j 37 \cdot 5$ | $-7.8-j 14.2$ | $94 \cdot 4+j 37 \cdot 1$ | $-5 \cdot 7+j 1 \cdot 4$ | $94 \cdot 4+j 37 \cdot 2$ | $-1 \cdot 2+j 0 \cdot 2$ |

Table 7.2b. Self- and mutual impedances of coupled full-wave dipoles; $h / \lambda=0.5 ; a / \lambda=0.007022$.

| $x$ | Parallel, non-staggered |  | Staggered |  | Collinear |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x=b_{12} / \lambda$ |  | $x=b_{12} / \lambda=d_{12} / \lambda$ |  | $x=\left(d_{12}-2 h\right) / \lambda$ |  |
|  | $Z_{s 1}=R_{s 1}+j X_{s 1}$ | $Z_{12}=R_{12}+j X_{12}$ | $Z_{s 1}=R_{s 1}+j X_{s 1}$ | $Z_{12}=R_{12}+j X_{12}$ | $Z_{s 1}=R_{s 1}+j X_{s 1}$ | $Z_{12}=R_{12}+j X_{12}$ |
| 0 |  |  |  |  | 353.7-j509.5 | $-62 \cdot 1+j 70 \cdot 6$ |
| 0.05 | 195.5-j795.5 | $115 \cdot 6+j 260 \cdot 8$ | 285-1-j1078.8 | $56 \cdot 2+j 533 \cdot 7$ | 320.8-j498.4 | $-9 \cdot 8+j 49 \cdot 2$ |
| 0.10 | 190.9-j655.5 | $76 \cdot 8+j 221 \cdot 0$ | 250.5-j825.2 | $7 \cdot 0+j 399 \cdot 4$ | 317.3-j488.9 | $4 \cdot 4+j 33 \cdot 7$ |
| 0.15 | $200 \cdot 5-j 571 \cdot 3$ | $66 \cdot 0+j 182 \cdot 0$ | 239.9-j672.3 | $9 \cdot 4+j 288 \cdot 3$ | 318.6-j484.1 | $10 \cdot 5+j 23 \cdot 1$ |
| 0.25 | 233-8-j476-4 | $71 \cdot 4+j 133 \cdot 5$ | 246.6-j511.3 | $44 \cdot 4+j 152 \cdot 4$ | $322.6-j 481.8$ | $14.1+j 8.7$ |
| 0.50 | 376.4-j457.1 | 154.2-j116 | 354.2-j465-1 | 95.4-j16.6 | 323.5-j484.5 | $4 \cdot 1-j 6 \cdot 4$ |
| 0.75 | 293.7-j495.5 | -16.4-j104.2 | 311.8-j486.2 | -30.2-j42.2 | 323.3-j483.7 | -3.4-j2.4 |
| 1.00 | $342 \cdot 9-j 470 \cdot 3$ | -94.4-j2.3 | 327.3-j485.7 | -15.6-j25.1 | 323.4-j484.0 | $-1.6+j 2 \cdot 1$ |
| 1.25 | 309.3-j493.5 | $3 \cdot 2+j 78 \cdot 2$ | 323.1-j482.0 | 18.4-j0.8 | 323.3-j483.9 | $1 \cdot 3+j 1 \cdot 1$ |
| 1.50 | 333.0-j474.8 | $70 \cdot 0+j 5 \cdot 4$ | 322.5-j484.5 | -8.1-j10-2 | 323.4-j484.0 | 0.8-j0.9 |



Fig. 7.9. Currents in element No. 1 of a symmetrically and anti-symmetrically driven pair of collinear antennas. $I_{z 1}(z)=I_{z 1}^{\prime \prime}(z)+j I_{z 1}^{\prime}(z), V_{02}= \pm V_{01}= \pm 1, a / \lambda=0.007022 . h / \lambda=0.5$. $d_{12} / \lambda=1 \cdot 1$. (Element 1 is below element 2.)
$180^{\circ}$ out of phase, the coupling reinforces the currents excited by the generator voltages. When centre driven by equal and codirectional generators the distribution of current is extremely asymmetrical. The half of each element that is removed from the other has a very large approximately sinusoidal reactive current $I_{z i}^{\prime}(z)$, whereas the adjacent halves have only a small and oppositely directed reactive component. The in-phase component $I_{z i}^{\prime \prime}(z)$ is much greater than when the antennas are anti-symmetrically driven and the asymmetry is in the opposite directions.


Fig. 7.10. Like Fig. 7.9 but for two staggered elements with $b_{12} / \lambda=d_{12} / \lambda=0 \cdot 1$.
The distributions of current for the symmetrically and antisymmetrically driven staggered pair are sketched approximately to scale in Figs. 7.11a, $b$. Note that the more closely coupled adjacent halves of the elements have the greater current when the excitation is asymmetrical, very much the smaller when the excitation is symmetrical. In the former case the coupling between the elements reinforces the generators, in the latter it opposes them.

It is interesting to note that the distribution along the symmetrically driven pair in Fig. 7.11a resembles that along a sleeve dipole. $\dagger$ This is to be expected since the two elements are very closely coupled.

### 7.7 A simple planar array $\ddagger$

The application of the general theory developed earlier in this chapter to planar arrays is conveniently illustrated with the three by three nine-element array shown in Fig. 7.12. This involves nonstaggered, staggered and collinear elements, so that the effects of the different types of coupling on otherwise identical elements can be studied.

[^14]

Fig. 7.11. Currents in two-element staggered array driven (a) symmetrically with $V_{02}=V_{01}$ and (b) anti-symmetrically with $V_{02}=-V_{01}$. The distributions of current are taken from Fig. 7.10.

Consider first the broadside array in which all elements are driven with equal voltages, that is, $V_{0 i}=1, i=1,2, \ldots 9$. Since conventional theory is unable to treat full-wave elements, let an array of nine elements with $h / \lambda=0.5$ be analysed. Let the lateral distances between elements be $b=0.25$ and the axial distance between adjacent ends be $(d-2 h) / \lambda=0.1$ where $d$ is the distance between centres of adjacent collinear elements. With this symmetric excitation, elements 1,2 and 3 are like elements 7,8 and 9 in the even parts of these currents, but the algebraic sign of the odd parts is reversed. The coefficients of the five trigonometric functions in the current distribution given in (7.33) are listed in Table 7.3 for all of the elements. The associated driving-point admittances and impedances are also listed. Note that these differ significantly. The


Fig. 7.12. Planar array of 9 identical, equally-spaced elements.
four different distributions of current are shown in Fig. 7.13 in the form $I_{z i}\left(z_{i}\right)=I_{z i}^{\prime \prime}\left(z_{i}\right)+j I_{z i}^{\prime}\left(z_{i}\right)$ where $I_{z i}^{\prime \prime}\left(z_{i}\right)$ is in phase with $V_{0 i}$, $I_{z i}^{\prime}\left(z_{i}\right)$ in phase quadrature. Note that the currents for elements 7,8 and 9 are like those for 1,2 and 3 but with $-z_{i}$ substituted for $z_{i}$. Elements 4,5 and 6 have even currents. The contribution by the odd currents in elements 1,2 and 3 is seen to be large.

When the same nine-element array is driven to obtain a unilateral endfire pattern with $V_{01}=V_{04}=V_{07}=1, V_{02}=V_{05}=V_{08}=-j$, $V_{03}=V_{06}=V_{09}=-1$, the coefficients for the trigonometric functions in the current distribution (7.33) are listed in Table 7.4. Note that there are now six different sets of coefficients since elements 1 and 3 and their counterparts are no longer electrically identical. The driving-point admittances and impedances are also given in Table 7.4. They are seen to have a wider range of values than in the broadside array. Note that the resistances of the elements

Table 7.3. Nine-element planar array-broadside

$$
\begin{gathered}
a / \lambda=0 \cdot 007022, \quad h / \lambda=0 \cdot 5, \quad b / \lambda=0 \cdot 25, \quad d / \lambda=1 \cdot 1, \\
V_{0 i}=1, \quad i=1,2, \ldots 9
\end{gathered}
$$

Coefficients of trigonometric functions in milliamperes per volt

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i} \dagger$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1.3 .7 .9 | $0.006-j 3.626$ | $0.763-j 0.739$ | $0.287+j 2.849$ | $0.197+j 0.444$ | $-0.008+j 0.134$ |
| 2.8 | $0.010-j 3.580$ | $1.065-j 0.643$ | $0.197+j 3.164$ | $0.317+j 0.604$ | $0.279+j 0.199$ |
| 4.6 | $0.007-j 3.615$ | $0.832-j 0.590$ | $0.176+j 2.596$ | $0+j 0$ | $0+j 0$ |
| 5 | $0.010-j 3.567$ | $1.176-j 0.443$ | $0.087+j 2.821$ | $0+j 0$ | $0+j 0$ |

$\dagger$ Reverse signs for $i=7,8,9$.
Admittances in millimhos and impedances in ohms

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ |
| :---: | :---: | :---: |
| $1,3,7,9$ | $1 \cdot 759+j 1 \cdot 371$ | $353 \cdot 7-j 275 \cdot 7$ |
| 2,8 | $2 \cdot 328+j 1 \cdot 878$ | $260 \cdot 2-j 209 \cdot 9$ |
| 4,6 | $1 \cdot \cdot 40+j 1 \cdot 416$ | $341 \cdot 3-j 262 \cdot 8$ |
| 5 | $2 \cdot 440+j 1 \cdot 955$ | $249 \cdot 6-j 200 \cdot 0$ |

Table 7.4. Nine-element planar array-endfire $a / \lambda=0.007022, \quad h / \lambda=0.5, \quad b / \lambda=0.25, \quad d / \lambda=1.1$ $V_{01}=V_{04}=V_{07}=1, \quad V_{02}=V_{05}=V_{08}=-j$,

$$
V_{03}=V_{06}=V_{09}=-1
$$

Coefficients of trigonometric functions in milliamperes per volt

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i} \dagger$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1,7 | 0.053-j3.674 | 0.321-j0.426 | 0.391-j2.060 | 0.206+j0.463 | $0.288+j 0.213$ |
| 2,8 | -3.675-j0.006 | -0.282-j0.501 | 2.375-j0.027 | $0 \cdot 500-j 0.151$ | 0.196-j0.168 |
| 3,9 | $0.042+j 3.668$ | -0.724-j0.152 | 0.480-j2.201 | $-0.119-j 0.562$ | $-0.223-j 0.124$ |
| 4 | 0.053-j3.664 | 0.381-j0.299 | $0.321+j 1.845$ | $0+j 0$ | $0+j 0$ |
| 5 | -3.665-j0.005 | -0.129-j0.546 | $2 \cdot 137+j 0.010$ | $0+j 0$ | $0+j 0$ |
| 6 | $0.044+j 3.657$ | -0.735-j0.344 | 0.463-j1.918 | $0+j 0$ | $0+j 0$ |

$\dagger$ Reverse signs for $i=7,8,9$.
Admittances in millimhos and impedances in ohms

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ |
| :---: | :---: | :---: |
| 1,7 | $1.034+j 1.208$ | $409 \cdot 0-j 477.8$ |
| 2,8 | $1.030+j 1.811$ | $237.2-j 417.2$ |
| 3,9 | $0.966+j 2.506$ | $134.0-j 347.4$ |
| 4 | $1.084+j 1.250$ | $396 \cdot 0-j 456.7$ |
| 5 | $1.083+j 1.879$ | $230.2-j 399.5$ |
| 6 | $1.008+j 2.607$ | $129.0-j 333.7$ |



Fig. 7.13. Currents in a planar array of 9 elements in broadside. $I_{z i}\left(z_{i}\right)=I_{z i}^{\prime \prime}\left(z_{i}\right)+j I_{z i}^{\prime}\left(z_{i}\right)$, $V_{0 i}=1 ; a / \lambda=0.007022, h / \lambda=0 \cdot 5, b / \lambda=0.25, d / \lambda=1 \cdot 1$.
in the collinear trio in the backward direction $(1,4,7)$ are much greater than the corresponding resistances in the forward trio $(3,6,9)$. This is characteristic of endfire arrays of full-wave elements. The six different currents are shown in Figs. 7.14 and 7.15. The currents in elements 7,8 and 9 are like those in 1,2 and 3 but with $-z_{i}$ substituted for $z_{i}$. Note that the currents in the rear collinear trio $(1,4,7)$ are greater and contribute more to the far field than the currents in the forward trio of the elements ( $3,6,9$ ). The far-field patterns in the horizontal or $H$-plane and the vertical or $E$-plane are shown in Fig. 7.16 for both the broadside and the endfire arrays. The horizontal pattern of the broadside array is bidirectional with maxima at $\Phi=90^{\circ}$ and $270^{\circ}$, the endfire pattern is unidirectional with a broad maximum in the direction $\Phi=0^{\circ}$. The vertical patterns in the direction $\Phi=0$ are seen to be very sharp as would be expected when three full-wave elements (which correspond to six half-wave elements) are stacked. (Note that the vertical pattern shown for the broadside array is not in the direction of the maximum at $\Phi=90^{\circ}$.)


Fig. 7.14. Like Fig. 7.13 but driven in endfire with $V_{01}=V_{04}=V_{07}=1$, $V_{02}=V_{05}=V_{08}=-j, V_{03}=V_{06}=V_{09}=-1$. Currents in elements 1, 2, 3.


Fig. 7.15. Like Fig. 7.14 but for elements 4,5 and 6 .


Fig. 7.16. Horizontal $\left(\Theta=90^{\circ}\right)$ and vertical $\left(\Phi=0^{\circ}\right)$ patterns of a 9 -element planar array of full-wave antennas; broadside and endfire excitation.

When the length of the elements is a half wavelength instead of a full wavelength, it is usually desirable to assign the driving-point currents $I_{z i}(0)$ instead of the voltages $V_{0 i}$. If the array shown in Fig. 7.12 is constructed of half-wave elements with $h / \lambda=0 \cdot 25$, $b_{x} / \lambda=b_{y} / \lambda=0 \cdot 25,(d-2 h) / \lambda=0 \cdot 1$, and the currents are assigned for a broadside pattern with $I_{z i}(0)=2.5$ milliamperes for $i=1,2, \ldots 9$, the coefficients for the trigonometric functions in the expression (7.33) for the currents in the elements are those given in Table 7.5. The required driving voltages $V_{0 i}$ are also listed together with the associated driving-point admittances and impedances. Note that the voltages differ considerably, as do the impedances. This is due entirely to mutual coupling.

The fact that the driving-point currents are all maintained equal and in phase by a suitable choice of voltages does not mean that the several distributions of current are therefore also equal and in phase.

Table 7.5. Nine-element planar array-broadside

$$
\begin{gathered}
a / \lambda=0.007022, \quad h / \lambda=0.25, \quad b / \lambda=0.25, \quad d / \lambda=0.6, \\
I_{z i}(0)=2.5 \times 10^{-3} ; \quad i=1,2, \ldots 9
\end{gathered}
$$

Coefficients of trigonometric functions in milliamperes

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i}^{\dagger}$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1.3 .7 .9 | $-0.810-j 1 \cdot 304$ | $-4.422-j 2.913$ | $20.869+j 5.492$ | $-0.392+j 0.650$ | $2.679-j 6.099$ |
| 2.8 | $-1.233-j 2.300$ | $-5.529-j 5.054$ | $23.204+j 9.404$ | $-0.202+j 1.214$ | $0.897-j 11.409$ |
| 4,6 | $-1.194-j 1 \cdot 213$ | $-5.759-j 2.193$ | $24.120+j 3.342$ | $0+j 0$ | $0+j 0$ |
| 5 | $-1.188-j 2.419$ | $-7.570-j 4.620$ | $27.976+j 7.513$ | $0+j 0$ | $0+j 0$ |

$\dagger$ Reverse signs for $i=7,8,9$.
Admittance in millimhos, impedances in ohms, EMF's in volts

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $V_{0 i}$ |
| :---: | :---: | :---: | :---: |
| $1,3,7,9$ | $8.203+j 4.310$ | $95.5-j 50.2$ | $0.239-j 0.125$ |
| 2,8 | $4.817+j 2.320$ | $168.6-j 81.1$ | $0.421-j 0.203$ |
| 4,6 | $6.369+j 5.538$ | $89.4-j 77.7$ | $0.223-j 0.194$ |
| 5 | $3.713+j 2.663$ | $177.8-j 127.5$ | $0.445-j 0.319$ |

The very different interactions among the several elements necessarily leads to distributions of current that are quite dissimilar in both amplitude and phase. This is shown graphically in Fig. 7.17 for the real and imaginary parts of the currents. The real parts are seen to be more nearly triangular than cosinusoidal; the imaginary parts are quite large and distributed so differently from the real part that the phase angle is very far from constant. This means that even for half-wave elements the conventional assumption that all currents are cosinusoidally distributed and constant in phase along each element is of questionable validity for determining impedances and minor lobe structures.

The far-field pattern of the nine-element broadside planar array of half-wave elements is shown in Fig. 7.18 in the horizontal plane ( $\Theta=90^{\circ}$ ) and the vertical plane in the direction of the maximum horizontal field ( $\Phi=90^{\circ}$ ).

The general five-term theory is also valid for arrays that include parasitic elements. For example, in the nine-element planar array, the upper and lower rows may be parasitic with only elements 4,5 and 6 driven and all constants the same as for the array described in Table 7.5 except that $V_{01}=V_{02}=V_{03}=V_{07}=V_{08}=V_{09}=0$, $V_{04}=V_{05}=V_{06}=1$. The coefficients of the trigonometric


Fig. 7.17. Normalized currents in the nine elements of a planar array. $I_{z i}\left(z_{i}\right)=I_{z i}^{\prime \prime}\left(z_{i}\right)+j I_{z i}^{\prime}\left(z_{i}\right), I_{z i}(0)=2.5$ milliamperes with $i=1,2, \ldots 9$. $a / \lambda=0.007022, h / \lambda=0.25, b / \lambda=0.25, d / \lambda=0.6$.
functions in the distribution of current (7.33) are as given in Table 7.6. The admittances and impedances for the three driven elements are also tabulated. The distributions of the real and imaginary parts of the current referred to the driving voltage are shown in Fig. 7.19. The currents in the collinear parasitic elements are, of course, much smaller than in the driven elements and their distributions are quite different. Note, however, that the current in the middle element No. 5 has quite a different distribution from that of the other two driven elements Nos. 4 and 6.

### 7.8 A three-dimensional array of twenty-seven elements $\dagger$

As a final example of the application of the five-term theory consider a three-dimensional array consisting of three stacked, threeelement, broadside curtains arranged in endfire as shown in Fig. 7.20. Let the lateral distances between the adjacent identical elements be $b_{x} / \lambda=b_{v} / \lambda=0 \cdot 25$, the axial distance between adjacent ends $(d-2 h) / \lambda=0.1$; also let $a / \lambda=0.007022$. If the antennas are

[^15]Table 7.6. Nine-element planar array, with three elements driven

$$
a / \lambda=0.007022, \quad h / \lambda=0.25, \quad b / \lambda=0.25, \quad d / \lambda=0.6
$$

$$
V_{01}=V_{02}=V_{03}=V_{07}=V_{08}=V_{09}=0 ; \quad V_{04}=V_{05}=V_{06}=1
$$

Coefficients of trigonometric functions in milliamperes per volt

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i} \dagger$ | $R_{i} \dagger$ |
| :---: | ---: | :---: | :---: | :---: | :---: |
| 1.3 .7 .9 | $0.020-j 0.038$ | $-0.554-j 0.197$ | $-1 \cdot 305-j 3 \cdot 260$ | $-2 \cdot 225-j 3 \cdot 406$ | $16 \cdot 467-j 33 \cdot 106$ |
| 2.8 | $0.026-j 0.045$ | $-0.519-j 0.340$ | $-1.735-j 4.588$ | $-2.145-j 2.471$ | $16.336-j 27.315$ |
| 4.6 | $-0.386-j 5.485$ | $-8.106-j 14.193$ | $54.503+j 29.666$ | $0+j 0$ | $0+j 0$ |
| 5 | $-0.309-j 5.731$ | $-7.211-j 18.733$ | $45.860+j 62.562$ | $0+j 0$ | $0+j 0$ |

[^16]Admittances in millimhos and impedances in ohms

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ |
| :---: | :---: | :---: |
| 4.6 | $8.244-j 0.019$ | $121 \cdot 3+j 0 \cdot 3$ |
| 5 | $6.530+j 5 \cdot 322$ | $92 \cdot 0-j 75 \cdot 0$ |



Fig. 7.18. Horizontal $\left(\Theta=90^{\circ}\right)$ and vertical $\left(\Phi=90^{\circ}\right)$ patterns of a planar array of nine elements with currents shown in Fig. 7.17: $a / \lambda=0.007022, h / \lambda=0.25, b / \lambda=0.25, d / \lambda=0.6$; $I_{z i}(0)=2 \cdot 5$ milliamperes, $i=1,2, \ldots 9$.
individually a full wavelength long ( $h / \lambda=0.5$ ), the desired unidirectional endfire pattern is well realized when the driving voltages (which directly excite the large sinusoidal components of the currents) are assigned the following values: $V_{1+3 n}=1, V_{2+3 n}=-j$, $V_{3+3 n}=-1$ with $n=0,1,2, \ldots 8$. The unidirectional beam is to be in the positive $x$ direction. With this choice of parameters the five coefficients for the trigonometric components of the current in (7.33) have been computed and listed in Table 7.7. The associated driving-point admittances and impedances are also given. These are seen to vary widely as a necessary consequence of differences in the induced currents. Since the power in each element is given by $P_{0 i}=\frac{1}{2} V_{0 i}^{2} G_{0 i}$, and $V_{0 i}^{2}=V_{0 i} V_{0 i}^{*}=1$, the relative powers are proportional to the driving-point conductances. It is seen from Table 7.7 that the nine elements in the plane $x=-b_{x}$ (which are

Table 7.7. Twenty-seven element three-dimensional endfire array
Main beam in direction $\Phi=0$
$a / \lambda=0.007022, \quad h / \lambda=0 \cdot 5, \quad b_{x} / \lambda=b_{y} / \lambda=0.25, \quad d / \lambda=1 \cdot 1$
$V_{1+3 n}=1, \quad V_{2+3 n}=-j, \quad V_{3+3 n}=-1 ; \quad n=0,1,2, \ldots 8$
Coefficients of trigonometric functions in millimhos per volt

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i} \dagger$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1,7,19, 25 | 0.062-j3.633 | 0.442-j0.687 | 1.033-j2.590 | $0.157+j 0.529$ | $0 \cdot 102+j 0 \cdot 430$ |
| 2, 8, 20, 26 | -3.639-j0.017 | -0.394-j0.815 | 3.419-j0.333 | $0.593-j 0.120$ | $0 \cdot 225-j 0.015$ |
| 3,9,21, 27 | $0.033+j 3.613$ | $-1.355-j 0.336$ | 1.635-j2.430 | $-0.110-j 0.651$ | $-0.309+j 0.062$ |
| 4,22 | 0.062-j3.623 | 0.522-j0.544 | 0.963-j2.358 | $0+j 0$ | $0+j 0$ |
| 5,23 | $-3.628-j 0.017$ | $-0.189-j 0.869$ | $2 \cdot 314+j 0.382$ | $0+j 0$ | $0+j 0$ |
| 6,24 | $0.037+j 3.599$ | - $1.348-j 0.608$ | 1.601-j2.041 | $0+j 0$ | $0+j 0$ |
| 10,16 | 0.075-j3.590 | $0.744-j 0.361$ | $1.063+j 2.997$ | $0.242+j 0.686$ | $0 \cdot 306+j 0.474$ |
| 11,17 | -0.359-j0.025 | $-0.157-j 0.934$ | $3.637+j 0.656$ | $0.762+j 0.006$ | 0.494-j0.002 |
| 12,18 | $0.037+j 3.559$ | $-1.381-j 0.642$ | 1.915-j2.338 | 0.060-j0.746 | $-0.126-j 0.165$ |
| 13 | 0.074-j3.578 | 0-846-j0.422 | $0.985+j 2.675$ | $0+j 0$ | $0+j 0$ |
| 14 | $-2.358-j 0.023$ | 0.097-j0.945 | 3.310-j0.646 | $0+j 0$ | $0+j 0$ |
| 15 | $-0.041+j 3 \cdot 545$ | $-1.330-j 0.931$ | $1.828-j 1.936$ | $0+j 0$ | $0+j 0$ |

$\dagger$ Reverse signs for $i=7,8,9,16,17,18,25,26,27$.
Admittances in millimihos and impedances in ohms

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1,7,19,25 | $1.917+j 1.217$ | 371.9-j236.1 | 10,16 | $2.552+j 1.735$ | 268.0-j182.2 |
| 2, 8, 20, 26 | $1 \cdot 297+j 2 \cdot 630$ | 150.9-j305.9 | 11,17 | $1 \cdot 212+j 3 \cdot 322$ | 96.9-j265.6 |
| 3,9,21, 27 | $1 \cdot 076+j 3 \cdot 102$ | 99.8-j287.7 | 12,18 | $0 \cdot 847+j 3.623$ | 61.2-j261.7 |
| 4,22 | $2 \cdot 008+j 1.269$ | 355.9-j225.0 | 13 | $2.678+j 1.830$ | 254.6-j174.0 |
| 5,23 | $1 \cdot 357+j 2.761$ | 143.4-j291.8 | 14 | $1 \cdot 245+j 3 \cdot 504$ | 90.0-j253.4 |
| 6,24 | $1.096+j 3.257$ | 92.8-j275.8 | 15 | $0.831+j 3.798$ | 55.0-j251.3 |



Fig. 7.19. Currents in the nine elements of a planar array. $I_{z i}\left(z_{i}\right)=I_{z i}^{\prime \prime}\left(z_{i}\right)+j I_{z i}^{\prime}\left(z_{i}\right)$, $V_{01}=V_{02}=V_{03}=V_{07}=V_{08}=V_{09}=0, \quad V_{04}=V_{05}=V_{06}=V_{0}=1 . \quad a / \lambda=0.007022$, $h / \lambda=0.25, b / \lambda=0.25, d / \lambda=0.6$.
the rear elements if the forward direction along the positive $x$-axis is that of the maximum beam), receive the largest amount of power from the generators $\left(9 \cdot 78 V_{0}^{2}\right)$; the nine elements in the plane $x=0$ the next largest amount ( $5 \cdot 78 V_{0}^{2}$ ); and the nine forward elements in the plane $x=b_{x}$ the smallest amount $\left(4 \cdot 51 V_{0}^{2}\right)$. However, the power is reasonably well divided among the elements. It is greatest in the middle elements $10,13,16$ of the rear plane ( $3 \cdot 89 V_{0}^{2}$ ) where induced currents are relatively small; it is least in the middle elements $12,15,18$ of the forward plane $\left(1 \cdot 26 V_{0}^{2}\right)$ where induced currents are relatively large.

The computed currents in 18 of the elements are shown graphically in Figs. 7.21a, b, c. The currents in elements 7, 8, 9, 16, 17, 18, $25,26,27$ are obtained, respectively, from those in elements $1,2,3$, $10,11,12,19,20,21$ with the substitution of $-z_{i}$ for $z_{i}$. Both the real and imaginary parts of the currents on differently situated but otherwise identical elements are seen to vary widely. Those in the outer tiers of elements with centres in the planes $z= \pm d$ exhibit a large asymmetry owing to the one-sidedness of the coupling.

If the full-wave elements in the array are replaced by half-wave elements $(h / \lambda=0.25)$ with the same axial distance $(d-2 h) / \lambda=0.1$ between adjacent ends and all other conditions, including the


Fig. 7.20. Three-dimensional array of 27 identical. equally-spaced elements.
driving voltages unchanged, the coefficients for the trigonometric components of the currents are computed to have the values listed in Table 7.8. The associated driving-point admittances and impedances are also given in Table 7.8. Note their very wide range.

As with the full-wave elements, the power in the nine elements in the rear plane $\left(x=-b_{x}\right)$ is greatest ( $35 \cdot 56 V_{0}^{2}$ ), in the nine elements in the middle plane next greatest ( $4 \cdot 90 V_{0}^{2}$ ), and in the nine elements in the forward plane least ( $0.94 V_{0}^{2}$ ). The distribution of power is seen to be very uneven. Indeed, the currents induced in the central forward elements $12,15,18$ are now so great that these act as negative resistances or generators. The assigned voltage at the terminals of these elements can be maintained only if loads are connected across their terminals instead of generators. This is also true of the central element 14 in the middle plane. In evaluating the powers in the elements in the three planes, the negative values were

Table 7.8. Twenty-seven element three-dimensional endfire array
Main beam in direction $\Phi=0$
$a / \lambda=0.007022, \quad h / \lambda=0.25, \quad b_{x} / \lambda=b_{y} / \lambda=0.25, \quad d / \lambda=0.6$
$V_{1+3 n}=1, \quad V_{2+3 n}=-j, \quad V_{3+3 n}=-1, \quad n=0,1,2, \ldots 8$
Coefficients of trigonometric functions in milliamperes per volt

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i}{ }^{\dagger}$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1.7.19.25 | $-0.432-j 5.436$ | $-9.874-j 12.025$ | $62 \cdot 110+j 20 \cdot 189$ | $-2.912+j 2.092$ | 24.552-j20.379 |
| 2, 8, 20, 26 | $-5.365+j 0.084$ | $-11.235+j 1.644$ | 12.666-j11.274 | $2 \cdot 203+j 0.779$ | -19.840-j6.877 |
| 3, 9, 21, 27 | $0.033+j 5.467$ | $0.526+j 12.608$ | - $3.857-j 24.434$ | 0.807-j1.716 | $-7.485+j 16.453$ |
| 4.22 | $-0.411-j 5.509$ | $-10.672-j 12.054$ | $61 \cdot 129+j 27 \cdot 424$ | $0+j 0$ | $0+j 0$ |
| 5.23 | $-5.415+j 0.079$ | $-11.015+j 1.880$ | 17.355-j11.122 | $0+j 0$ | $0+j 0$ |
| 6, 24 | $0.039+j 5.511$ | $1.022+j 12.365$ | $-5.463-j 28.184$ | $0+i 0$ | $0+j 0$ |
| 10,16 | $-0.373-j 5.669$ | $-9.514-j 16.649$ | $56.729+j 51.632$ | $-2.990+j 0.788$ | $26.280-j 10.367$ |
| 11,17 | $-5.461-j 0.004$ | $-13 \cdot 163+j 0.023$ | 25.922-j0.547 | $1.665+j 0.330$ | $-15 \cdot 317-j 3.610$ |
| 12,18 | $-0.024+j 5.556$ | $-0.643+j 14.403$ | 3.868-j37.307 | 0.504-j1.226 | $-5.271+j 12.429$ |
| 13 | -0.332-j5.754 | $-10.184-j 17.271$ | $53.761+j 60.828$ | $0+j 0$ | $0+j 0$ |
| 14 | -5.505-j0.014 | $-13.073-j 0.071$ | $30.191+j 1.477$ | $0+j 0$ | $0+j 0$ |
| 15 | $-0.026+j 5.595$ | $-0.359+j 14.166$ | $3.450-j 39.822$ | $0+j 0$ | $0+j 0$ |

[^17]Admittances in millimhos, impedances in ohms

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1.7,19,25 | $8.749-j 0.675$ | $113.6+j 8.8$ | 10.16 | $7 \cdot 475+j 4 \cdot 143$ | $102 \cdot 3-j 56 \cdot 7$ |
| 2, 8, 20, 26 | $1.742-j 2.160$ | $226 \cdot 2+j 280 \cdot 5$ | 11.17 | 0.141-j0.109 | $4432 \cdot 9+j 3430 \cdot 8$ |
| 3,9,21, 27 | $0.637+j 0.015$ | 1570.1-j38.0 | 12,18 | $-0.513+j 2.087$ | -111.1-j451.9 |
| 4,22 | $7.643+j 1.488$ | 126.1-j24.5 | 13 | $5 \cdot 895+j 6 \cdot 299$ | $79.2-j 84.6$ |
| 5, 23 | $1.456-j 0.517$ | $609 \cdot 8+j 216 \cdot 6$ | 14 | $-0.367+j 1.275$ | - 208.7-j724.4 |
| 6.24 | $0.617+j 1.400$ | 263.5-j598.0 | 15 | $-0.678+j 3.093$ | $-67.6-j 308.5$ |



(b)

Current in milliamperes per volt


Fig. 7.21. (a) Currents in elements Nos. 1, 4, 10 and 13 of the 27 -element array shown in Fig. 7.20. $V_{1+3 n}=1, V_{2+3 n}=-j, V_{3+3 n}=-1, n=0,1,2, \ldots 8 ; a / \lambda=0 \cdot 007022, h / \lambda=0 \cdot 5$, $b_{x} / \lambda=b_{y} / \lambda=0 \cdot 25, d / \lambda=1 \cdot 1$. (b) Like (a) but for elements $2,5,11$ and 14. (c) Like $(a)$ but for elements $3,6,12$ and 15 .
subtracted since they represent power dissipated in a load, not radiated power. Note that the powers in elements 11 and 17 are not negative but very small. The entire admittance is very low, the input impedance correspondingly high. It might be supposed that these elements contribute negligibly to the radiation field. But this is not necessarily true. The fact that $I_{z 11}(0)$ is near zero does not mean that $I_{z 11}\left(z_{11}\right)$ is everywhere equally small. It may be quite large.
The currents in the elements of the 27 -element array of half-wave dipoles when the driving voltages are assigned to be $V_{1+3 n}=1$, $V_{2+3 n}=-j, V_{3+3 n}=-1$ with $n=0,1, \ldots 8$ are shown in Fig. 7.22. Note that the currents on the elements in the rear plane (top figure) are greater than those in the middle plane (lower left) and still greater than those in the forward plane (lower right). Specifically, the current in element 11 is very small at $z=0$, but quite comparable with the other currents out along the antenna. It is seen from Fig. 7.22 that even with half-wave elements the conventional assumption that the distributions of current along all elements are identical and cosinusoidal is not well satisfied. Since this assumption also implies that the phase of each current is the same along



Fig. 7.22. Like Figs. 7.21a, b, c but with $h / \lambda=0.25$ and $d / \lambda=0.6$.
the antenna as at the driving point, it is of interest to examine the relative phases referred to a common reference, viz. $V_{01}$. This is done in Fig. 7.23 where the phase angles of the currents along all elements are shown. For the elements in the rear plane where induced currents are not of major significance, the phase angle varies relatively little from $z=0$ to $z= \pm h$, much as in an isolated antenna. On the other hand, when induced currents constitute the major parts of the currents in an element, the phase angle varies very widely-as much as $153^{\circ}$ in the middle element 14. It is clear that when large currents are induced in some elements of an array, as in endfire arrangements which maintain a maximum field along the antennas, an assumed current with constant phase cannot be expected to represent even approximately the actual currents in an array.

Since with half-wave antennas the principal component of the current has its maximum value at $z=0$, the progressive phases in the currents required for a specified field pattern can be approximated more closely when the maxima of the currents, i.e. the values $I_{z i}(0)$ are assigned instead of the voltages. Let the values $I_{1+3 n}(0)=2.5 \times 10^{-3}, \quad I_{2+3 n}(0)=-j 2.5 \times 10^{-3}$, $I_{3+3 n}(0)=-2.5 \times 10^{-3}$ amperes be specified for the same 27 element array. The corresponding coefficients for the currents as evaluated by computer are in Table 7.9 together with the required driving voltages and the associated admittances and impedances for the elements. Note that the voltages range from $V_{02}=-0.006-j 0.236$ to $V_{15}=-1 \cdot 173+j 0.403$. The complete distributions of current are in Figs. 7.24a, b, $c$ in the normalized form : $I_{z i}\left(z_{i}\right) / I_{z i}(0)=I_{z i}^{\prime \prime}\left(z_{i}\right) / I_{z i}(0)+j I_{z i}^{\prime}\left(z_{i}\right) / I_{z i}(0)$.

With the driving-point currents specified, the power in each element is conveniently determined from $P_{0 i}=\frac{1}{2}\left|I_{z i}(0)\right|^{2} R_{0 i}$. It is seen to be proportional to $R_{0 i}$ as given in Table 7.9. The distribution of power to the elements with the driving-point currents assigned is quite different from when the voltages are specified. Note that the nine elements in the rear plane $\left(x=-b_{x}\right)$, which with voltages specified received the greatest power, now receive the least ( $411.8 I_{0}^{2}$ ), the middle plane $(x=0)$ is again intermediate ( $523.6 I_{0}^{2}$ ), and the elements in the forward plane ( $x=b_{x}$ ), which with voltages assigned received the least power, now receive the greatest ( $1484 \cdot 8 I_{0}^{2}$ ). However, the division of power is not as extreme as before and there are no elements that have negative resistances and, therefore, feed power into a load instead of receiving

Table 7.9. Twenty-seven-element three-dimensional endfire array

## Main beam in direction $\Phi=0$

$a / \lambda=0.007022, \quad h / \lambda=0.25, \quad b_{x} / \lambda=b_{y} / \lambda=0.25, \quad d / \lambda=0.6$
$I_{1+3 n}(0)=2 \cdot 5, \quad I_{2+3 n}(0)=-j 2 \cdot 5, \quad I_{3+3 n}(0)=-2 \cdot 5$ milliamperes $, \quad n=0,1,2, \ldots 8$
Coefficients of trigonometric functions in milliamperes

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $\varepsilon_{i} \dagger$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1, 7, 19,25 | -0.981-j1.034 | -4.633-j1.870 | $21 \cdot 004+j 2 \cdot 854$ | $-0.096+j 0.670$ | -0.252-j6.390 |
| 2, 8, 20, 26 | $-1 \cdot 289+j 0 \cdot 157$ | - $2.825-j 2.993$ | 5.244-j18.220 | $0 \cdot 702+j 0 \cdot 802$ | $-6.607-j 6 \cdot 543$ |
| 3,9,21,27 | $1.088+j 3.730$ | $4.704+j 8.473$ | - $20.880-j 16.196$ | $-0.066-j 1.859$ | $1.803+j 17.612$ |
| 4,22 | -1.615-j1.016 | $-6.550-j 1.345$ | $25 \cdot 384+j 1 \cdot 124$ | $0+j 0$ | $0+j 0$ |
| 5,23 | $-1.362+j 0.257$ | $-2.582+j 3.699$ | 4.165-j20.288 | $0+j 0$ | $0+j 0$ |
| 6,24 | $2 \cdot 298+j 4 \cdot 128$ | $8 \cdot 353+j 8 \cdot 561$ | - $29.207-j 15 \cdot 136$ | $0+j 0$ | $0+j 0$ |
| 10,16 | $-1.524-j 1.647$ | $-5.991-j 3 \cdot 109$ | $23 \cdot 784+j 4 \cdot 991$ | $0.194+\boldsymbol{j 1 . 0 4 4}$ | -2.972-j9.951 |
| 11,17 | $-2.062+j 0.445$ | $-4 \cdot 486+j 3 \cdot 812$ | 8.275-j20.031 | $1 \cdot 141+j 0.698$ | - 10.697-j5.579 |
| 12,18 | $0.907+j 5.463$ | $4 \cdot 470+j 12 \cdot 196$ | - $20.701-j 22.986$ | 0.010-j2.896 | $1 \cdot 103+j 26 \cdot 825$ |
| 13 | $-2 \cdot 408-j 1.738$ | -8.521-j2.671 | $29 \cdot 404+j 3 \cdot 185$ | $0+j 0$ | $0+j 0$ |
| 14 | $-2 \cdot 273+j 0 \cdot 712$ | $-4 \cdot 393+j 4 \cdot 988$ | 7.235-j23.137 | $0+j 0$ | $0+j 0$ |
| 15 | $2.341+j 0.639$ | $8 \cdot 894+j 13.302$ | -30.910-j23.585 | $0+j 0$ | $0+j 0$ |

$\dagger$ Reverse signs for $i=7,8,9,16,17,18,25,26,27$.
Admittances on millimhos, impedances in ohms, driving EMF's in volts

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $V_{0 i}$ |
| :---: | :---: | :---: | ---: |
| $1,7,19,25$ | $7 \cdot 810+j 6 \cdot 470$ | $75 \cdot 9-j 62 \cdot 9$ | $0 \cdot 190-j 0 \cdot 157$ |
| $2,8,20,26$ | $10 \cdot 580+j 0 \cdot 277$ | $94 \cdot 4-j 2 \cdot 5$ | $-0 \cdot 006-j 0 \cdot 236$ |
| $3,9,21,27$ | $3 \cdot 430+j 0 \cdot 888$ | $273 \cdot 2-j 70.8$ | $-0.683+j 0 \cdot 177$ |
| 4,22 | $4 \cdot 299+j 6 \cdot 231$ | $75 \cdot 0-j 108 \cdot 7$ | $0 \cdot 186-j 0 \cdot 272$ |
| 5,23 | $9 \cdot 898+j 0 \cdot 910$ | $100 \cdot 2-j 9 \cdot 2$ | $-0.023-j 0 \cdot 250$ |
| 6,24 | $2 \cdot 591+j 1 \cdot 353$ | $303 \cdot 2-j 158 \cdot 3$ | $-0.758+j 0 \cdot 396$ |


| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $V_{0 i}$ |
| :---: | ---: | :---: | ---: |
| 10,16 | $4 \cdot 804+j 4 \cdot 082$ | $120 \cdot 9-j 102 \cdot 7$ | $0 \cdot 302-j 0 \cdot 257$ |
| 11,17 | $6 \cdot 459+j 1 \cdot 007$ | $151 \cdot 1-j 23 \cdot 6$ | $-0 \cdot 059-j 0 \cdot 378$ |
| 12,18 | $2 \cdot 447+j 0 \cdot 351$ | $400 \cdot 4-j 57 \cdot 5$ | $-1 \cdot 001+j 0 \cdot 144$ |
| 13 | $2.896+j \cdot 771$ | $128 \cdot 1-j 166 \cdot 8$ | $0.320-j \cdot 447$ |
| 14 | $5 \cdot 622+j 1 \cdot 427$ | $167 \cdot 1-j 42 \cdot 4$ | $-0 \cdot 106-j 0 \cdot 418$ |
| 15 | $1 \cdot 905+j 0 \cdot 654$ | $469 \cdot 5-j 161 \cdot 2$ | $-1 \cdot 173+j 0 \cdot 403$ |

power from a generator. A comparison of the relative powers in all of the elements is shown schematically in Fig. 7.23 in which boxes are located in the three-dimensional pattern of the array. The upper number in each box is the conductance $G_{0 i}$ when the conditions of Table 7.8 with voltages specified obtain; it is proportional to the power $P_{0 i}$ in each element. The lower number in each box is the resistance $R_{0 i}$ for the same array when the conditions of Table 7.9 are maintained with input currents assigned; it is proportional to the power $P_{0 i}$ in each element. The relative distribution of power is seen to be reversed.


Fig. 7.23. Schematic diagram showing the relative powers supplied to the half-wave ( $h / \lambda=0.25$ ) elements in a 27 -element endfire array. The upper number in each box is $G_{0 i}$ (in millimhos) which is proportional to power supplied when the $V_{0 i}$ are specified. The lower number is $R_{0 i}$ (in ohms) which is proportional to power when the driving-point currents $I_{z i}(0)$ are specified.

The distributions of current in Fig. 7.24a,b,c all have the same value at $z_{i}=0$ and the components in phase with the input current are similarly distributed along the antenna in a rough sense. They range from a flattened cosine to a triangle. However, the quadrature currents are by no means negligible (they are presumed not to exist in conventional array theory). Indeed, they are of major significance in those elements which radiate most of the power. Note in particular the very large quadrature currents in all of the elements in the forward plane $x=b_{x}$ which are shown in Fig. 7.24c.




Fig. 7.24. (a) Currents in elements Nos. 1, 4, 10 and 13 of the 27-element array shown in Fig. 7.20. $I_{1+3 n}(0)=2.5 \mathrm{~mA}, I_{2+3 n}(0)=-j 2.5 \mathrm{~mA}, I_{3+3 n}(0)=-2.5 \mathrm{~mA}$, with $n=0,1,2, \ldots 8$. $a / \lambda=0.007022, h / \lambda=0.25, b_{x} / \lambda=b_{y} / \lambda=0.25, d / \lambda=0.6$. (b) Like Fig. 7.24a but for $i=2$, 5, 11 and 14. (c) Like Fig. 7.24a but for $i=3,6,12$ and 15.
(The currents in elements 9, 18 and 24 are like those in 3, 12 and 21 with $-z_{i}$ substituted for $z_{i}$.) These have distributions quite different from the conventionally assumed cosine curve. Evidently the phases are also as far from constant as those shown in Fig. 7.25 for the same array with assigned voltages.

The purpose of an array is to maintain a useful far field. The computed far-field patterns of the 27 -element endfire array shown in Fig. 7.20 are in Fig. 7.26 for all the cases considered in this section, that is, for $h / \lambda=0.5$ with voltages assigned, $h / \lambda=0.25$ with voltages and currents assigned. The horizontal patterns in the equatorial plane $\Theta=90^{\circ}$ all have the principal maximum in the desired direction, $\Phi=0, \Theta=90^{\circ}$. They also have a minor maximum in the backward direction, $\Phi=180^{\circ}, \Theta=90^{\circ}$. This is smallest with the array of half-wave elements with specified input currents, it is largest with the half-wave elements with voltages specified. The array of full-wave elements with voltages specified has a backward lobe of intermediate height. The vertical patterns for the array of half-wave elements are essentially the same when currents or voltages are specified. The former has a very slightly broader main beam and a correspondingly slightly lower minor lobe level. The array of full-wave elements has the narrowest main


Fig. 7.25. Phases of the currents in Fig. 7.22 all referred to $V_{01}$.
beam in the vertical pattern-the array is, of course, axially twice as long. On the other hand, its minor lobe level is correspondingly somewhat higher. Note that since very good approximations of actual currents on all of the elements are used, there are no nulls as would have been obtained with assumed sinusoidal currents with constant phase along each antenna. The details of the minor lobe structure derived from the five-term approximations of the several currents should have an accuracy comparable to that of the major lobe.

If all of the 27 elements are driven in phase, an approximately circular pattern with some undulations is obtained as would be expected; of interest is the fact that in this case, too, a number of the elements have negative driving-point conductances and resistances. This indicates that the induced currents in these elements predominate so that they act as generators and not as loads when connected to a transmission line. Elements with negative resistances are likely to occur in most arrays with large numbers of rather closely coupled elements.

### 7.9 Electrical beam scanning

The major lobe in the endfire patterns shown in Fig. 7.27 is in the direction $\Theta=90^{\circ}, \Phi=0^{\circ}$. This is readily switched electrically to the direction $\Theta=90^{\circ}, \Phi=90^{\circ}$ simply by interchanging the


Fig. 7.26. Horizontal $\left(\Theta=90^{\circ}\right)$ and vertical ( $\Phi=0^{\circ}$ ) patterns of a three-dimensional endfire array of 27 elements; $a / \lambda=0.007022, b_{x} / \lambda=b_{y} / \lambda=0.25,(d-2 h) / \lambda=0.1 ; V_{1+3 n}=1$, $V_{2+3 n}=-j, V_{3+3 n}=-1$ or $I_{1+3 n}(0)=2.5 \mathrm{~mA}, I_{2+3 n}(0)=-j 2.5 \mathrm{~mA}, I_{3+3 n}(0)=-2.5 \mathrm{~mA}$, with $n=0,1,2, \ldots 8$.
phases of the voltages or currents in the broadside rows (parallel to the $y$-axis in Fig. 7.20) and the endfire rows (parallel to the $x$-axis). For example, the assigned voltages would be $V_{0 i}=1$, $1 \leqslant i \leqslant 9 ; V_{0 i}=-j, 10 \leqslant i \leqslant 18 ; V_{0 i}=-1,19 \leqslant i \leqslant 27$ or, if the driving-point currents are assigned, $I_{z i}(0)=2.5 \mathrm{~mA}, 1 \leqslant i \leqslant 9$; $I_{z i}(0)=-j 2 \cdot 5 \mathrm{~mA}, 10 \leqslant i \leqslant 18 ; I_{z i}(0)=-2 \cdot 5 \mathrm{~mA}, 19 \leqslant i \leqslant 27$.

More generally, the direction of the beam is specified by the farfield formula (7.57) in which the field factor of each individual antenna $i$ is given by the square bracket, and the combination of these into a pattern for the array is determined by the phase factors $\exp \left(j \beta_{0} \mathbf{r}_{i} \cdot \mathbf{R}_{0}\right)$. The contribution to the pattern by each element is


Fig. 7.27. Horizontal $\left(\Theta=90^{\circ}\right)$ and vertical patterns of 27 -element three-dimensional endfire array with beam in the directions $\Phi=0^{\circ}$ and $\Phi=45^{\circ}$. Driving voltages specified as in Table 7.10; $a / \lambda=0.007022, h / \lambda=0.5, b_{x} / \lambda=b_{y} / \lambda=0.25, d / \lambda=1 \cdot 1$.
greatest when the amplitudes $A_{i}^{\prime}, B_{i}^{\prime}, D_{i}^{\prime}, Q_{i}$ and $R_{i}$ all include the common factor $\exp \left(-j \beta_{0} \mathbf{r}_{i} \cdot \hat{\mathbf{R}}_{0}\right)$. When this is true the contributions from all the elements arrive in phase in the direction specified by particular values of $\Theta, \Phi$ in ( $\mathbf{r}_{i} \cdot \hat{\mathbf{R}}_{0}$ ) as given in (7.58). This is, evidently, a necessary condition for a maximum in the field pattern. However, it is not a sufficient condition since the directional properties of the individual elements, as given by the square bracket in (7.57) for element $i$, are also involved. These may differ considerably from element to element owing to differences in the distributions of current so that no simple formula for the direction $\Theta_{m}, \Phi_{m}$ of the main lobe on the field pattern can be written down. In the special case of maxima in the equatorial plane ( $\Theta_{m}=90^{\circ}$ )
the presence of the common phase factor

$$
\exp \left(-j \beta_{0} \mathbf{r}_{i} \cdot \hat{\mathbf{R}}_{0}\right)=\exp \left[-j \beta_{0}\left(X_{i} \cos \Phi_{m}+Y_{i} \sin \Phi_{m}\right)\right]
$$

is sufficient to fix the main beam in the direction $\Phi_{m}$. For example, when $\Phi_{m}=0$, the factor is $\exp \left(-j \beta_{0} X_{i}\right)$. This means that when voltages are assigned, these must have the relative phases $\exp \left(j \beta_{0} b_{x}\right)$, 1 , $\exp \left(-j \beta_{0} b_{x}\right)$, respectively, for the elements in the planes $X_{i}=-b_{x}, X_{i}=0$, and $X_{i}=b_{y}$. When $b_{x}=\lambda / 4$ as in the arrays considered in this chapter, the phases are $\exp [j(\pi / 2)]=j, 1$, and $\exp [-j(\pi / 2)]=-j$; or, if $j$ is removed as a common factor, the relative phases are given by $1,-j,-1$, which are the values used in Table 7.7. When the beam is switched to $\Phi_{m}=90^{\circ}$, the phase factor is $\exp \left(-j \beta_{0} Y_{i}\right)$ so that the voltages in the planes $Y_{i}=-b_{y}=-\lambda / 4, Y_{i}=0, Y_{i}=b_{y}=\lambda / 4$ must have the relative phases $\exp [j(\pi / 2)], 1$, and $\exp [-j(\pi / 2)]$. When driving-point currents instead of voltages are assigned, the coefficients apply to them unchanged.

If the direction of the maximum beam is to be $\Theta_{m}=90^{\circ}$, $\Phi_{m}=45^{\circ}$, the coefficients are given by

$$
\exp \left(-j \beta_{0} \mathbf{r}_{i} \cdot \hat{\mathbf{R}}_{0}\right)=\exp \left[-j \beta_{0}\left(X_{i}+Y_{i}\right) \sqrt{2} / 2\right]
$$

For the three-dimensional array of 27 elements shown in Fig. 7.20, the elements are located at $X_{i}=-b_{x}=-\lambda / 4, X_{i}=0, X_{i}=b_{x}=\lambda / 4$ and $Y_{i}=-b_{y}=-\lambda / 4, Y_{i}=0, Y_{i}=b_{y}=\lambda / 4$. Thus, the following relative phases must be assigned to the driving voltages (or currents if these are specified instead of the voltages):

$$
\begin{aligned}
& X_{i}=Y_{i}=-\frac{\lambda}{4}: \exp (j \pi \sqrt{2} / 2) \\
& X_{i}=-\frac{\lambda}{4}, Y_{i}=0 ; X_{i}=0, Y_{i}=-\frac{\lambda}{4}: \exp (j \pi \sqrt{2} / 4) \\
& X_{i}=\frac{\lambda}{4}, Y_{i}=-\frac{\lambda}{4} ; X_{i}=0, Y_{i}=0 ; X_{i}=-\frac{\lambda}{4}, Y_{i}=\frac{\lambda}{4}: 1 \\
& X_{i}=0, Y_{i}=\frac{\lambda}{4} ; X_{i}=\frac{\lambda}{4}, Y_{i}=0 \quad: \exp (-j \pi \sqrt{2} / 4) \\
& X_{i}=Y_{i}=\frac{\lambda}{4}: \exp (-j \pi \sqrt{2} / 2) .
\end{aligned}
$$

Alternatively, if $\exp (j \pi \sqrt{2} / 2)$ is removed as a common factor,
the five-phase coefficients are, respectively, $1, \exp (-j \pi \sqrt{2} / 4)$, $\exp (-j \pi \sqrt{2} / 2), \exp (-j 3 \pi \sqrt{2} / 4)$ and $\exp (-j \pi \sqrt{2})$. With reference to Fig. 7.20, the required assigned voltages are listed in exponential form near the top of Table 7.10 and in complex numerical form later in the table. If these assigned voltages are used in the computer programme, the coefficients for the currents in the 27 elements of the same array analysed in Table 7.7, but with the beam rotated $45^{\circ}$, are as listed in Table 7.10. Note that the number of different currents is greater than when the main beam is in the direction $\Phi=0$ or $\Phi=90^{\circ}$. The currents in the elements with centres in the plane $Z=d$ are, of course, the same as those in the plane $Z=-d$ with $z_{i}$ replaced by $-z_{i}$. The associated admittances and impedances are also given together with the numerical values of the assigned voltages.

Since the voltage magnitudes are $\left|V_{0 i}\right|=1$, the relative powers to the elements are proportional to the driving-point conductances. In general, these are quite comparable in magnitude and range to those in Table 7.7 but the larger values are shifted to the new elements in the backward direction ( $\Phi=225^{\circ}$ ), the smaller values to the new elements in the forward direction $\left(\Phi=45^{\circ}\right)$.

The horizontal field pattern in the plane $\Theta=90^{\circ}$ and the vertical pattern in the plane $\Phi=45^{\circ}$ are shown in Fig. 7.27 together with the corresponding patterns from Fig. 7.26. It is seen that in the horizontal plane the main lobe has been rotated substantially unchanged through $45^{\circ}$, but that the minor lobe structure is somewhat different. The change in the vertical pattern is so small that it can be distinguished only near the peak of a minor lobe. Evidently, the rather narrow beam of a three-dimensional array of full-wave elements in collinear, broadside, and endfire combinations is readily rotated by appropriate changes in the phases of the driving voltages. A similar rotation of the corresponding array of half-wave elements is readily achieved with precisely the same changes in the phases of the assigned driving-point currents.

### 7.10 Problems with practical arrays

The theory developed in this and the preceding chapters provides a complete, practical tool for the quantitative determination of the properties of very general arrays when the active elements are driven by a concentrated EMF at their centres. In practice, antennas are driven from transmission lines that maintain the desired

Table 7.10. Twenty-seven-element three-dimensional endfire array-Beam direction $\Phi=45^{\circ}$

$$
a / \lambda=0.007022, \quad h / \lambda=0.5, \quad b_{x} / \lambda=b_{y} / \lambda=0.25, \quad d / \lambda=1.1
$$

$$
V_{1+3 n}=1, \quad V_{2+3 n}=V_{10+3 n}=\exp (-j \pi \sqrt{2} / 4), \quad V_{3+3 n}=V_{11+3 n}=V_{19+3 n}=\exp (-j \pi \sqrt{2} / 2)
$$

$$
V_{12+3 n}=V_{20+3 n}=\exp (-j 3 \pi \sqrt{2} / 4), \quad V_{21+3 n}=\exp (-j \pi \sqrt{2}), \quad n=0,1,2
$$

Coefficients of the trigonometric functions

| $i$ | $A_{i}^{\prime}$ | $B_{i}^{\prime}$ | $D_{i}^{\prime}$ | $Q_{i}^{\dagger}$ | $R_{i} \dagger$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1,7 | 0.096-j3.641 | 0.147-j0.558 | $1 \cdot 204+j 1 \cdot 871$ | $0 \cdot 184+j 0 \cdot 572$ | $0.334+j 0.528$ |
| 2, 8, 10, 16 | -3.219-j1.657 | -0.308-.j0.769 | $3 \cdot 264+j 0 \cdot 528$ | $0 \cdot 613+j 0.097$ | $0 \cdot 424+j 0.058$ |
| 3,9,19,25 | - $2.898-\mathbf{j 2} 2.189$ | -0.974-j0.365 | 2.792-j2.055 | 0.300-j0.442 | $0 \cdot 104+j 0.025$ |
| 4 | 0.096-j3.631 | 0.216-j0.427 | $1 \cdot 142+j 1.641$ | $0+j 0$ | $0+j 0$ |
| 5,13 | $-3 \cdot 210-j 1.652$ | -0.108-j0.767 | $2 \cdot 994+j 0.473$ | $0+j 0$ | $0+j 0$ |
| 6,22 | $-2.889+j 2.181$ | -0.878-j0.552 | $2 \cdot 632-j 0.181$ | $0+j 0$ | $0+j 0$ |
| 11,17 | $-2 \cdot 878+j 2 \cdot 160$ | -0.793-j0.646 | 3-415-j1.816 | 0.613-j0.473 | 0.398-j0.321 |
| 12, 18, 20.26 | $0.701+j 3.532$ | $-1.307+j 0.030$ | $1 \cdot 112-j 3.167$ | -0.141-j0.697 | -0.906-j0.147 |
| 12, 14 | $-2.867+j 2.154$ | -0.598-j0.817 | 3.147-j1620 | $0+j 0$ | $0+j 0$ |
| 15,23 | $0.702+j 3.519$ | $-1.346-j 0.236$ | 1.152-j2.800 | $0+j 0$ | $0+j 0$ |
| 21,27 | $3 \cdot 500+j 0 \cdot 888$ | $-1.364+j 0.883$ | -0.930-j2.181 | $-0.695-j 0 \cdot 160$ | $-0.207+j 0.471$ |
| 24 | $3 \cdot 491+j 0 \cdot 879$ | $-2 \cdot 161+j 0 \cdot 730$ | -0.596-j1.941 | $0+j 0$ | $0+j 0$ |

$\dagger$ Reverse signs for $i=7,8,9,16,17,18,25,26,27$.
Admittances in millimhos, impedances in ohms

| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $V_{0 i}$ |
| :---: | ---: | :---: | ---: |
| 1.7 | $1.498+j 0.755$ | $532.3-j 268.2$ | $1.0000+j 0.0000$ |
| $2,8,10,16$ | $2.081+j 1.925$ | $258.9-j 239.6$ | $0.4440+j 0.8960$ |
| $3,9,19,25$ | $1.704+j 2.358$ | $201.3-j 278.6$ | $-0.6057-j 0.7957$ |
| 4 | $1.574+j 0.788$ | $508.1-j 254.3$ | $1.0000+j 0.0000$ |
| 5,13 | $2.186+j 2.019$ | $246.9-j 228.0$ | $0.4440+j 0.8960$ |
| 6.22 | $1.789+j 2.464$ | $192.9-j 265.7$ | $-0.6057-j 0.7957$ |


| $i$ | $Y_{0 i}=G_{0 i}+j B_{0 i}$ | $Z_{0 i}=R_{0 i}+j X_{0 i}$ | $V_{0 i}$ |
| :---: | ---: | ---: | :---: |
| 11,17 | $1.366+j 3.339$ | $105.0-j 256.6$ | $-0.6057-j 0.7957$ |
| $12,18,20,26$ | $0.886+j 3.335$ | $74.4-j 280 \cdot 1$ | $-0.9819+j 0.1894$ |
| 14 | $1.408+j 3.524$ | $97.7-j 244.7$ | $-0.6057-j 0.7957$ |
| 15,23 | $0.893+j 3.505$ | $68.3-j 267.9$ | $-0.9819+j 0.1894$ |
| 21,27 | $0.573+j 3.637$ | $42.3-j 268.3$ | $-0.2662+j 0.9639$ |
| 24 | $0.550+j 3.779$ | $37.4-j 257.8$ | $-0.2662+j 0.9639$ |

voltage across the terminals of the antennas, but also introduce the complications that accompany transmission-line end-effects and the coupling between the antenna and the line. There is also the possibility of unbalanced currents on open-wire lines or on the outside surfaces of coaxial lines. These latter can be excited by asymmetrical conditions at the junctions of antennas and feeding lines, or by the intense near fields in an array whenever transmission lines are not in a neutral plane of these fields. Since such currents induced along transmission lines usually contribute significantly to the radiation field and can, therefore, constitute a non-negligible part of the load, both the circuit and ficld properties of an array can be modified greatly whenever they are excited. Important aspects of the problems relating to end-effects and coupling effects between antennas and transmission lines as well as techniques of measurement are considered in the next chapter. However, questions relating to the maintenance of the required voltages for antennas with positive conductances and loads for those with negative conductances in large arrays are not analysed since they involve the specific geometry of each array. A problem of this sort in which elements with both positive and negative resistances play important roles is treated in detail at the end of chapter 6 where the log-periodic array is analysed. This antenna includes not only radiating elements with specified geometrical properties but also a feeding line with definite electrical characteristics. Since it is in the neutral plane, the problems of unbalanced currents are avoided, but those relating to the transfer of power from the radiating elements to the line and vice versa constitute a major aspect of the analysis.

## CHAPTER 8

## TECHNIQUES AND THEORY OF MEASUREMENTS

The preceding chapters are devoted to the development of a theory to predict the characteristics of arrays of physically real dipoles and monopoles. This chapter is concerned with the experimental determination of these characteristics and with the correlation of measured and theoretical results. In practice, the mathematical intricacies of theoretical formulas can be largely avoided by means of a computer programme to which a user need only supply the parameters of a particular array to obtain radiation patterns, driving-point admittances or impedances, and other characteristics. When programmed in this manner, the computer becomes a simulator which can be substituted for the repeated testing and adjusting commonly required in designing an array. However, the value of such a simulator rests entirely on how well predictions agree with observation when the final model is assembled. Comparisons in the preceding chapters between measured and computed results indicate that the theory is capable of describing an actual experimental model with acceptable accuracy. However, in applying the theoretical results to different experimental systems, account must be taken of certain considerations and precautions if such agreement is to be obtained. It is these considerations which are of primary concern in the present chapter.

The required measuring techniques have been discussed in general in several books $\dagger$; the purpose here is to examine difficulties and procedures which apply particularly to arrays of dipoles and monopoles. Although the discussions are not restricted to any particular range of frequencies, most of the procedures have been used in the $100-1200 \mathrm{MHz}$ range.

### 8.1 Transmission lines with coupled loads

Owing to fundamental differences between theoretical and experimental models, theoretical and measured driving-point

[^18]impedances often do not agree if compared blindly. In the former, the radiating elements are in free space and coupled only to one another. In the latter, the radiating elements are coupled not only to one another but to transmission lines as well. Measurements are made on transmission lines that are not infinite but terminated, and equipped with probes that are rather crude approximations of the assumed ideal. Although these differences are commonly ignored, they can have important effects on the measured results.

Adequate theoretical expressions for predicting the principal characteristics of measured radiation patterns are relatively easy to obtain, for they are the result of an integration. This, in effect, yields a kind of average, so that a precise value of the current at any point on the dipole is not required, except possibly in determining the details of a minor lobe structure. In contrast, theoretical expressions for predicting the measured driving-point impedances are very difficult to obtain since they require precise values of the current near the driving point. When transmission lines are attached to antennas, they may have a negligible influence on the radiation patterns but produce important modifications in the boundary conditions and hence in the fields and currents near the driving point. The analysis of a complete system (consisting of the antennas together with their attached transmission lines) as a boundary-value problem has thus far proved to be too complicated to yield solutions, so that effects due to the presence of the transmission lines must be taken into account separately. An exception is the complete analysis of a coaxial line that drives a monopole through a hole in a highly conducting ground screen for which a formal solution is a vailable [9].

In general, if the termination of a transmission line is to be described as a lumped circuit element and to be characterized in a useful manner as an impedance, it should ideally be independent of the circuit to which it is attached. However, such an ideal impedance can be defined only under very special conditions and these are frequently not adequately satisfied at microwave frequencies. $\dagger$ If they are violated, impedance measurements of a given physical load can be expected to yield different results when this is connected to physically or electrically different transmission lines.

Coupling between the transmission line and load usually extends throughout a small region near the line-load junction. This is called

[^19]the terminal zone, and the coupling effects are called terminalzone effects or end-effects.

Many properties of the terminal zones can be determined from the differential equations for the voltage and current along the transmission line. In the usual method of deriving these equations, the line is divided into identical infinitesimal sections, each section is represented by a lumped capacitance, an inductance, and a resistance, and Kirchhoff's laws of ordinary circuits are assumed to apply to each section. The results obtained by this method strictly apply only to infinite, unloaded lines. They can contain no information about terminal-zone effects or radiation which require a derivation based on a more complete theoretical model. Detailed steps of the more exact derivation are given in chapter 2 of [5] and [6]. Regardless of the particular transmission-line model and termination used in the derivation, the following generalized equations are obtained:

$$
\begin{gather*}
\frac{\partial^{2} V(w)}{\partial w^{2}}-\gamma^{2}(w) V(w)=0  \tag{8.1a}\\
I_{1 L}(w)=\frac{1}{z(w)} \frac{\partial V(w)}{\partial w} . \tag{8.1b}
\end{gather*}
$$

The distance, $w$, is measured along the transmission line from the line-load junction. $V(w)$ is the scalar potential difference or voltage between conductors of the transmission line at $w, I_{1 L}(w)$ is the total current in one of the conductors at $w$, and the line is assumed to be perfectly balanced so that $I_{2 L}(w)=-I_{1 L}(w)$.

In the following definitions, $W(w)$ is the vector potential difference between the conductors at $w$, and a subscript $p$ indicates the component of vector potential parallel to the transmission line. A subscript $L$ denotes that part of a quantity which is determined only from currents and charges in the line, and a subscript $T$ denotes that part which is determined only from currents and charges in the termination or load. The various quantities in $(8.1 \mathrm{a}, \mathrm{b})$ are

$$
\begin{aligned}
\gamma(w) & =\sqrt{z(w) y(w) a_{1}(w) \phi_{1}(w)} ; & & \text { propagation 'constant' (8.2) } \\
a_{1}(w) & =\frac{W_{p L}(w)+W_{p T}(w)}{W_{p L}(w)} ; & & \begin{array}{l}
\text { coefficient of inductive (8.3) } \\
\text { coupling }
\end{array} \\
\phi_{1}(w) & =\frac{V_{L}(w)}{V_{L}(w)+V_{T}(w)} ; & & \begin{array}{l}
\text { coefficient of capacitive (8.4) } \\
\text { coupling }
\end{array}
\end{aligned}
$$

$$
\begin{align*}
z(w) & =r(w)+j \omega l^{e}(w) & & \text { impedance per unit } \\
& \doteq j \omega l^{e}(w) & & \text { length } \\
& =j \omega\left[l_{L}^{e}(w)+l_{T}^{e}(w)\right] & & \\
y(w) & =g(w)+j \omega c(w) & & \text { admittance per unit }  \tag{8.6}\\
& \doteq j \omega c(w) & & \text { length } \\
y^{-1}(w) & =y_{L}^{-1}(w)+y_{T}^{-1}(w) & & \\
\beta^{2} & =\omega^{2} \mu \varepsilon & & \\
Z_{c}(w) & =\sqrt{\frac{z(w)}{y(w)} \doteq \sqrt{\frac{l^{e}(w)}{c(w)}}} & & \text { 'characteristic' impedance }
\end{align*}
$$

$\mu$ and $\varepsilon$ are the permeability and permittivity of the material in which the transmission-line conductors are embedded; $\mu=\mu_{r} \mu_{0}$, $\varepsilon=\varepsilon_{r} \varepsilon_{0}$, where $\mu_{0}, \varepsilon_{0}$ are the values for free space.

Vector and scalar potentials are calculated from the Helmholtz integrals of currents and charges over all conductors. These integrals are defined in chapter 1. A detailed discussion of the transmission-line parameters is in [5] and [6], where it is shown that differences in the potential of equipotential rings located just outside the conductors of the transmission line at a distance $w$ from the line-load junction are given approximately by

$$
\begin{gather*}
W_{p}(w)=W_{p L}(w)+W_{p T}(w) \doteq l^{e}(w) I_{1 L}(w)  \tag{8.9}\\
V(w)=V_{L}(w)+V_{T}(w) \doteq \frac{q_{L}(w)}{c(w)} \tag{8.10}
\end{gather*}
$$

where $q_{L}(w)$ is the charge per unit length along one of the trans-mission-line conductors. Those parts due only to currents and charges in the line are

$$
\begin{gather*}
W_{p L}(w) \doteq l_{L}^{e}(w) I_{1 L}(w)  \tag{8.11}\\
V_{L}(w) \doteq \frac{q_{L}(w)}{c_{L}(w)} \tag{8.12}
\end{gather*}
$$

With these approximations, (8.3) and (8.4) become

$$
\begin{align*}
a_{1}(w) & \doteq l^{e}(w) / l_{L}^{e}(w)  \tag{8.13}\\
\phi_{1}(w) & \doteq c(w) / c_{L}(w) . \tag{8.14}
\end{align*}
$$

When coupling between the line and its termination is expressed in terms of the vector potential, it is inductive; if there is no inductive coupling, $a_{1}(w)=1$. Note that if all conductors of a termination are perpendicular to all conductors of the transmission line, there is no
inductive coupling between them. When coupling between the line and its termination is expressed in terms of the scalar potential, it is capacitive; if there is no capacitive coupling, $\phi_{1}(w)=1$.

Because the form of $\gamma(w)$ differs for each line and termination, (8.1a, b) have no general solution. For most specific lines and loads, they are too complicated to yield useful solutions although digital computer techniques [10] could be used to obtain numerical results. However, these would be of limited value since the computation would have to be repeated for each change in the geometrical arrangement of the line or load. An approximate but more general procedure, which is especially useful for experimental work, is developed in the following discussion.

A detailed examination of the parameters in (8.2)-(8.14) reveals that non-uniformities decrease rapidly with distance from the lineload junction. Along the transmission line, $a_{1}(w)$ and $\phi_{1}(w)$ usually differ negligibly from one and $z(w)$ and $y(w)$ are sensibly constant at distances from the line-load junction that exceed ten times the centre-to-centre spacing between the conductors of a two-wire line, or ten times the difference in radii between outer and inner conductors of a coaxial line. This is a rough measure of the extent of the terminal zone. For most transmission lines it is less than $0 \cdot 1 \lambda$. At greater distances from the line-load junction, all parameters are constant and $(8.1 \mathrm{a}, \mathrm{b})$ reduce to the usual linear form :

$$
\begin{gather*}
\frac{d^{2} V(w)}{d w^{2}}-\gamma^{2} V(w)=0  \tag{8.15a}\\
I_{1 L}(w)=\frac{1}{z_{0}} \frac{d V(w)}{d w}  \tag{8.15b}\\
z(w)=z_{0}=r+j \omega l^{e}  \tag{8.16a}\\
y(w)=y_{0}=g+j \omega c  \tag{8.16b}\\
\gamma^{2}(w)=\gamma^{2}=z_{0} y_{0}  \tag{8.16c}\\
\gamma=\alpha+j \beta . \tag{8.16d}
\end{gather*}
$$

since

Thus, except within a small terminal zone, conventional trans-mission-line theory applies and the usual measuring techniques are valid. Changes that occur in the line parameters over short distances near the line-load junction appear as lumped inductances and capacitances in series and parallel with the actual terminating impedance. When a load impedance is determined in the usual manner from measurements on the uniform part of the line, the
quantity determined is always a combination of the actual load impedance with the inductances and capacitances caused by changes of the line parameters within the terminal zone. Approximate account can be taken of such changes if it is assumed that the uniform line parameters of an infinite unloaded line apply everywhere including the terminal zone, and the differences that occur within the terminal zone between the actual parameters and the assumed ones are represented by a balanced network of equivalent lumped series inductances and shunting capacitances, as shown in Figs. $8.1 a$ or $8.1 b$. The lumped elements are defined as follows :

$$
\begin{align*}
L_{T} & =\int_{0}^{d}\left[l^{e}(w)-l_{0}^{e}\right] d w  \tag{8.17a}\\
C_{T} & =\int_{0}^{d}\left[c(w)-c_{0}\right] d w \tag{8.17b}
\end{align*}
$$



Fig. 8.1. (a) Terminal-zone region and network. (b) Alternative representation for terminalzone network.
where $l^{e}(w)=l_{L}^{e}(w)+l_{T}^{e}(w)$ is the true inductance per unit length, $c^{-1}(w)=c_{L}^{-1}(w)+c_{T}^{-1}(w)$ is the true reciprocal capacitance or elastance per unit length, and $l_{0}^{e}$ and $c_{0}^{-1}$ are the corresponding quantities for an infinite line. Everywhere along an infinite line or outside of the terminal zone of a terminated line, the ratio of the
tangential component of vector potential difference to the current and the ratio of scalar potential difference to charge per unit length are constant and given by

$$
\begin{gather*}
W_{p}(w) / I_{1 L}(w)=l_{0}^{e}  \tag{8.18a}\\
V(w) / q_{1 L}(w)=1 / c_{0} . \tag{8.18b}
\end{gather*}
$$

With (8.13) and (8.14), the integrals for $L_{T}$ and $C_{T}$ are

$$
\begin{align*}
L_{T} & =\int_{0}^{d}\left[l_{L}^{e}(w) a_{1}(w)-l_{0}^{e}\right] d w  \tag{8.19a}\\
C_{T} & =\int_{0}^{d}\left[c_{L}(w) \phi_{1}(w)-c_{0}\right] d w . \tag{8.19b}
\end{align*}
$$

One advantage of the procedure of assuming line parameters to be uniform throughout the terminal zone and representing terminalzone non-uniformities by a network of lumped elements is that useful approximate expressions for $L_{T}$ and $C_{T}$ can frequently be derived from considerations of the static and induction fields. Also, $C_{T}$ and $L_{T}$ can be obtained from measurements. If the load is characterized by its impedance and the terminal-zone network is like Fig. 8.1 $a$, the actual load impedance $Z_{L}$, and the apparent load impedance $Z_{a}$ (or the apparent load admittance $Y_{a}$ ), are related by

$$
\begin{gather*}
Y_{a}=\frac{1}{Z_{a}}=\frac{1}{Z_{L}+j \omega L_{T}}+j \omega C_{T}  \tag{8.20}\\
Z_{L}=\frac{Z_{a}}{1-j \omega C_{T} Z_{a}}-j \omega L_{T}=\frac{1}{Y_{a}-j \omega C_{T}}-j \omega L_{T} . \tag{8.21}
\end{gather*}
$$

If the load is characterized by its admittance and the network of Fig. $8.1 b$ is used,

$$
\begin{gather*}
Z_{a}=Y_{a}^{-1}=\frac{1}{Y_{L}+j \omega C_{T}}+j \omega L_{T}  \tag{8.22}\\
Y_{L}=\frac{1}{Z_{a}-j \omega L_{T}}-j \omega C_{T}=\frac{Y_{a}}{1-j \omega L_{T} Y_{a}}-j \omega C_{T} \tag{8.23}
\end{gather*}
$$

The terminology used here is that of the lower-frequency transmission lines. Problems involving waveguides and some of those involving coaxial lines are conveniently solved in terms of propagating and evanescent modes [11]. The latter decay rapidly with distance from a discontinuity and it is this distance which defines the extent of the terminal zone.

### 8.2 Equivalent lumped elements for terminal-zone networks

Whenever an antenna that is ideally approximated by an independent impedance $Z_{L}$ is attached to a transmission line, the impedance that appears to be loading the transmission line is $Z_{a}$, not $Z_{L}$. The apparent impedance, $Z_{a}$, is a combination of $Z_{L}$ and a terminal-zone network consisting of a series reactance $X_{T}=\omega L_{T}$, and a shunting susceptance, $B_{T}=\omega C_{T}$. This network takes account of changes in the parameters of the line as its end is approached and of coupling between the transmission line and the antenna.
$L_{T}$ and $C_{T}$ can be evaluated theoretically, or determined from measured values of $Z_{a}$ and $Z_{L}$. The former is measured directly, the latter is obtained by repeating the measurement of $Z_{a}$ as the distance between the conductors of the transmission line is decreased successively, and then extrapolating the results to a fictitious 'zero' spacing. The extrapolated value of $Z_{a}$ is $Z_{L}$.

One common use of a terminal-zone network is to transform driving-point impedances which have been calculated from an established theory into those which can be measured on a particular transmission line. For this purpose, a single model of the desired antenna and its attached transmission line can be constructed, $Y_{a}=G_{a}+j B_{a}$ measured, and $Z_{L}=R_{L}+j X_{L}$ computed from the theory. Then, from (8.20),

$$
\begin{align*}
\omega C_{T} & =B_{a} \pm \sqrt{\left(G_{a} / R_{L}\right)-G_{a}^{2}}  \tag{8.24a}\\
\omega L_{T} & =-X_{L} \pm \sqrt{\left(R_{L} / G_{a}\right)-R_{L}^{2}} . \tag{8.24b}
\end{align*}
$$

For some models, $Y_{L}$ may be more convenient to compute than $Z_{L}$. From (8.22),

$$
\begin{gather*}
\omega C_{T}=-B_{L} \pm \sqrt{\left(G_{L} / R_{a}\right)-G_{L}^{2}}  \tag{8.25a}\\
\omega L_{T}=X_{a} \pm \sqrt{\left(R_{a} / G_{L}\right)-R_{a}^{2}} . \tag{8.25b}
\end{gather*}
$$

The resulting values of $\omega L_{T}$ and $\omega C_{T}$ can then be used for all other elements of the same kind in the array, as long as the element spacing is not so small that the terminal zones are directly coupled to one another.

The sign to be used in (8.24) and (8.25) is usually the one which makes the magnitudes of $\omega L_{T}$ and $\omega C_{T}$ smallest; in any case, their correct values are the ones that satisfy the imaginary parts of (8.20) and (8.22). Equations (8.24) and (8.25) may involve small
differences between quite large numbers so that high accuracy is required in $Y_{a}$ or $Z_{a}$. Therefore account must be taken of adapters, bends, or connectors which are between the antenna and the point where the measurements are made.

Theoretical determinations of $L_{T}$ and $C_{T}$ can be based on (8.17) or (8.19). Expressions for the inductances and capacitances in the integrands of (8.17) are themselves integrals of the static or induction fields for the particular load and transmission line that is being analysed. An evaluation of these integrals is readily carried out by computer and numerical methods no more complicated than Simpson's rule. Approximate formulas that are applicable to dipoles as end-loads on two-wire lines and to monopoles as endloads on coaxial lines are summarized in the following paragraphs.

Symmetrical dipole as a load on two-wire lines
This model is shown in Fig. 8.2. Approximate expressions for $L_{T}$ and $C_{T}$ are

$$
\begin{equation*}
L_{T} \doteq \frac{\mu}{2 \pi}(b-a) . \quad-C_{T} / c_{0} b \doteq 1 \cdot 5 / \ln (b / a) \tag{8.26}
\end{equation*}
$$



Fig. 8.2. Network for terminal zone of dipole as end-load of two-wire line.
where $a$ is the radius of the conductors and $b$ the distance between the conductors of the transmission line. These expressions are accurate to within about $20 \%$ for $b / a=3$ and improve in accuracy as $b / a$ increases. Expressions with higher accuracy have been derived. $\dagger$

The inductance given by (8.26) accounts for non-uniformities near the end of the transmission line. Most theoretical models assume the antenna to extend from $z=0$ to $z= \pm h$, whereas in some experimental models a section of the antenna may be missing between $z= \pm \frac{1}{2} b$. Account can be taken of this missing section by $\dagger[6]$, p. 50.
subtracting from the measured input reactance the difference in zero-order input reactance between an antenna of length ( $h-b / 2$ ) and one of length $h$ [12]. That is,

$$
\begin{gather*}
\quad X_{g}=\omega L_{g}=\frac{-\zeta_{0} \psi}{2 \pi}\left[\cot \beta_{0}(h-b / 2)-\cot \beta_{0} h\right]  \tag{8.27}\\
\psi=\left\{\begin{array}{l}
\frac{\left|C_{a}(h, 0) \sin \beta_{0} h-S_{a}(h, 0) \cos \beta_{0} h\right|}{\sin \beta_{0} h}, \quad \beta_{0} h<\pi / 2 \\
\left|C_{a}(h, h-\lambda / 4) \sin \beta_{0} h-S_{a}(h, h-\lambda / 4) \cos \beta_{0} h\right|, \quad \beta_{0} h>\pi / 2
\end{array}\right. \tag{8.28b}
\end{gather*}
$$

The integral functions $C_{a}(h, z)$ and $S_{a}(h, z)$ are defined in $(1.56 \mathrm{a}, \mathrm{b})$; they are tabulated in the literature [13]. The total correction for end-effect is a shunt susceptance $B_{T}=\omega C_{T}$ with $C_{T}$ given by (8.26) and a reactance $X_{E} / 2$ in series with each conductor where

$$
\begin{equation*}
X_{E}=X_{T}+X_{g}=\omega\left(L_{T}+L_{g}\right) \tag{8.29}
\end{equation*}
$$

with $L_{T}$ given by (8.26) and $L_{\mathrm{g}}$ by (8.27). The location of $C_{T}$ in the network shown in Fig. 8.2 is arbitrary. It may be connected across the terminals of the dipole or across the line between $L_{g}$ and $L_{T}$ if more convenient.

## Monopole over a ground plane fed by a coaxial line $\dagger$

In this model the outer conductor of the coaxial line ends at the surface of the ground plane, and the inner conductor continues through it to form the monopole shown in Fig. 8.3a. When the coaxial line and monopole are perpendicular to the ground plane, currents on its surface are not coupled inductively to the antenna or the transmission line so that $L_{T}=0$. Since the current on the inner conductor is continuous at the line-load junction, $L_{g}=0$. Hence, the terminal network consists only of a shunting capacitance that is given in Fig. 8.3b. When considered on an admittance basis, this model is especially simple since the terminal-zone correction applies only to the susceptance

$$
\begin{equation*}
G_{L}+j\left(B_{L}+\omega C_{T}\right)=G_{a}+j B_{a} \tag{8.30}
\end{equation*}
$$

so that $G_{a}=G_{L}$ and $B_{a}$ and $B_{L}$ differ only by an additive constant for any particular combination of a coaxial transmission line and monopole antenna.


Fig. 8.3. (a) Monopole over a ground plane. (b) Experimentally determined capacitive end correction for monopole over a ground screen driven by a coaxial line.

Change in conductor radius or spacing of two-wire lines $\dagger$ or coaxial lines $\ddagger$

Applications frequently occur in which the conductors of the dipole or monopole must have diameters different from those of the attached feeding lines, or the base separation at the driving point must be different from the spacing between the conductors of the feeding lines. End-effects associated with these various changes have been analysed separately in the literature and the results can be applied directly, provided the conductors of the

[^20]dipole are extended to form a short section of two-wire line, or the monopole and the associated hole in the ground plane are extended to form a short section of coaxial line. This section of line should be at least twice as long as the terminal zone to prevent coupling between the different terminal regions.

Consider first a two-wire line (Fig. 8.4a) with constant spacing between the axes of its conductors but with conductors of radius $a_{1}$ for $y \leqslant s$ and $a_{r}$ for $y \geqslant s$. The current in each conductor is continuous and in only one direction near the junction at $y=s$; therefore, $L_{T}=0$ and the terminal-zone network consists only of a shunting susceptance. The value of $B_{T}=\omega C_{T}$ for the two-wire line is one-half of that obtained for a coaxial line (Fig. 8.4b) which has a corresponding change in radius of the inner conductor, as long as the ratios $b / a_{1}$ and $b / a_{r}$ are the same for the two-wire line and the coaxial line. $\dagger$

Consider next a two-wire line (Fig. 8.4c) with conductors of constant radius $a$ but with a distance $b_{l}$ between their axes $y \leqslant s$ and $b_{r}$ for $y \geqslant s$. Short sections of conductors normal to the axis of the transmission line join the two parts at $y=s$. The junction and its terminal-zone network are shown in Fig. 8.4c. Approximate formulas for calculating $L_{T l}, L_{T y}, C_{T l}, C_{T r}$ and $C_{T c}$ are straightforward but quite long. $\ddagger$

Terminal-zone networks for many other combinations and junctions are given in [5, 11 and 14]. A commonly used element is the dipole shown in Fig. 8.5, which is centre-driven from a coaxial line perpendicular to the antenna. This model has not been analysed theoretically. Since it is unbalanced, currents are excited on the outside surface of the coaxial line and this becomes a part of the radiating source. The unbalanced condition can be avoided if a shielded two-wire line is used instead of a coaxial line.

### 8.3 Voltages, currents, and impedances of uniform sections of lines

Whenever an array is driven from a single generator, the various non-parasitic elements are connected to the generator through a network of transmission lines that supply at the several terminals currents and voltages which have the necessary amplitudes and phases to produce the desired radiation pattern. In addition,

[^21]

Fig. 8.4. Terminal-zone networks for changes in conductors of two-wire lines and coaxial lines. Within terminal zone of length $d \doteq \lambda / 10$, line parameters are non-uniform. (a) Change in radius of two-wire line conductors; $(b)$ change in radius of inner conductor of coaxial line; (c) change in spacing of two-wire line.


Fig. 8.5. Dipole driven normal to the axis of a coaxial line.
they must give correct impedance matches for a maximum transfer of power. A detailed consideration of the design of power-dividers, phasing and matching networks is beyond the scope of this book. $\dagger$ Experimental procedures for evaluating an array and for measuring the impedances and admittances of the elements are based on the solutions of the linearized transmission-line equations. A short review of relevant forms of the solution and their properties is given in this section.

Near line-load junctions and the ends of a transmission line the propagation 'constant' $\gamma$ is usually a function of position along the line. A practical procedure for taking account of terminal-zone effects with lumped networks and uniform sections of line has already been discussed. Outside the terminal zones the line is essentially uniform and the simple wave equations with constant coefficients, (8.15a) and (8.15b), apply. Solutions of these equations may have many forms. One of the most useful is

$$
\begin{equation*}
V(w)=A e^{\gamma w}+B e^{-\gamma w} \tag{8.31}
\end{equation*}
$$

where $\gamma=\alpha+j \beta$ and $\alpha$ is the attenuation constant in nepers per unit length, $\beta$ the phase constant in radians per unit length. This solution is fitted to a particular line and load when the terminal conditions at the ends of the line are used to determine $A$ and $B$. Note that these conditions must be specified within the terminal zones at the line-load junction or the line-generator junction. Hence, the apparent terminal impedance must be used in determining

[^22]$A$ and $B$. If the line is terminated at $w=0$ by an apparent impedance $Z_{a}$ with current $I(0)$ and voltage drop $V(0)$, it follows that
\[

$$
\begin{equation*}
V(0)=I(0) Z_{a} . \tag{8.32}
\end{equation*}
$$

\]

With (8.32), (8.31) and (8.15b) $A$ and $B$ are readily evaluated and the following expressions obtained:

$$
\begin{align*}
& V(w)=\frac{I(0)}{2}\left[\left(Z_{a}+Z_{c}\right) e^{\gamma w}+\left(Z_{a}-Z_{c}\right) e^{-\gamma w}\right]  \tag{8.33a}\\
& I(w)=\frac{I(0)}{2 Z_{c}}\left[\left(Z_{a}+Z_{c}\right) e^{\gamma w}-\left(Z_{a}-Z_{c}\right) e^{-\gamma w}\right] \tag{8.33b}
\end{align*}
$$

where $Z_{c}$ is the characteristic impedance of the transmission line.
These solutions suggest that an incident wave, travelling in the $-w$ direction from the generator at $w=s$ toward the load at $w=0$, strikes the load and is partially or completely reflected back toward the generator. The incident and reflected parts are

$$
\begin{array}{ll}
V^{+}(w)=\frac{I(0)}{2}\left(Z_{a}+Z_{c}\right) e^{\gamma w} ; & \text { incident wave } \\
V^{-}(w)=\frac{I(0)}{2}\left(Z_{a}-Z_{c}\right) e^{-\gamma w} ; & \text { reflected wave. } \tag{8.34b}
\end{array}
$$

The relative amplitude and phase of the reflected wave are specified in terms of the apparent reflection coefficient of the load, a quantity defined by

$$
\begin{equation*}
\Gamma_{a}=\frac{V^{-}(0)}{V^{+}(0)}=\frac{I^{-}(0)}{I^{+}(0)}=\frac{Z_{a}-Z_{c}}{Z_{a}+Z_{c}}=\frac{Y_{c}-Y_{a}}{Y_{c}+Y_{a}}=\left|\Gamma_{a}\right| e^{j \psi_{a}} . \tag{8.35}
\end{equation*}
$$

$Y_{c}=1 / Z_{c}$ is the characteristic admittance of the line. It follows that

$$
\begin{align*}
V(w) & =\frac{V(0)}{\left(1+\Gamma_{a}\right)}\left[e^{\gamma w}+\Gamma_{a} e^{-\gamma w}\right]=\frac{I(0) Z_{c}}{\left(1-\Gamma_{a}\right)}\left[e^{\gamma w}+\Gamma_{a} e^{-\gamma w}\right]  \tag{8.36}\\
I(w) & =\frac{I(0)}{\left(1-\Gamma_{a}\right)}\left[e^{\gamma w}-\Gamma_{a} e^{-\gamma w}\right]=\frac{V(0) Y_{c}}{\left(1+\Gamma_{a}\right)}\left[e^{\gamma w}-\Gamma_{a} e^{-\gamma w}\right] . \tag{8.37}
\end{align*}
$$

The superposition of the incident and reflected waves yields an interference pattern called a standing wave along the transmission line. When $Z_{a}=Z_{c}, \Gamma_{a}=0, V^{-}(0)=0$, the line is matched. For pure travelling waves outside the terminal zones the line appears to be infinite in length. When $\left|Z_{a}\right| \ll\left|Z_{c}\right|$, as when the load is a short circuit, $\Gamma_{a} \rightarrow-1$, and the incident wave is reflected with a $180^{\circ}$ shift in phase. The voltage and current distributions are pure
standing waves given by

$$
\begin{equation*}
V(w)=I(0) Z_{c} \sinh \gamma w, \quad I(w)=I(0) \cosh \gamma w . \tag{8.38}
\end{equation*}
$$

When $Z_{a} \gg Z_{c}$, the entire incident wave is again reflected but with no change in phase so that $\Gamma_{a}=1$. The distributions of current and voltage are given by ( 8.38 ) with the sinh and cosh interchanged.

The impedance looking toward the load at any point $w$ is given by

$$
\begin{equation*}
Z(w)=V(w) / I(w)=Z_{c}\left|\frac{1+\Gamma_{a} e^{-2 \gamma w}}{1-\Gamma_{a} e^{-2 \gamma w}}\right| \tag{8.39}
\end{equation*}
$$

and the admittance is $Y(w)=1 / Z(w)$.
Alternative expressions in terms of the hyperbolic functions are

$$
\begin{align*}
V(w) / V(0) & =\cosh \gamma w+\left(Y_{a} / Y_{c}\right) \sinh \gamma w  \tag{8.40a}\\
I(w) / I(0) & =\cosh \gamma w+\left(Z_{a} / Z_{c}\right) \sinh \gamma w  \tag{8.40b}\\
V(w) /\left[I(0) Z_{c}\right] & =\sinh \gamma w+\left(Z_{a} / Z_{c}\right) \cosh \gamma w  \tag{8.41a}\\
I(w) /\left[V(0) Y_{c}\right] & =\sinh \gamma w+\left(Y_{a} / Y_{c}\right) \cosh \gamma w  \tag{8.41b}\\
Z(w) / Z_{c} & =\frac{Z_{a} / Z_{c}+\tanh \gamma w}{1+\left(Z_{a} / Z_{c}\right) \tanh \gamma w} . \tag{8.42}
\end{align*}
$$

The preceding equations for current, voltage and impedance express $V(w)$ and $I(w)$ in terms of $V(0)$ and $I(0)$ at the load. They do not involve the actual driving voltage of the generator. A complete solution is obtained by imposing boundary conditions at both ends of the line $\dagger$ in terms of a generator with apparent internal impedance $Z_{g}$ and voltage $V^{e}$ at $w=s$ or $y=s-w=0$ and a load with an apparent impedance $Z_{a}$ at $w=0$ or $y=s$. Specifically

$$
\begin{array}{ll}
y=0: & V_{0}=V^{e}-I_{0} Z_{g} \\
y=s: & V_{s}=I_{s} Z_{a} .
\end{array}
$$

The elimination of $A$ and $B$ in (8.31) yields

$$
\begin{align*}
& V(y)=\frac{V^{e} Z_{c}}{Z_{c}+Z_{g}} \frac{e^{-\gamma s}+\Gamma_{a} e^{-\gamma(2 s-y)}}{1-\Gamma_{g} \Gamma_{a} e^{-2 \gamma s}}  \tag{8.43a}\\
& I(y)=\frac{V^{e}}{Z_{c}+Z_{g}} \frac{e^{-\gamma y}-\Gamma_{a} e^{-\gamma(2 s-y)}}{1-\Gamma_{g} \Gamma_{a} e^{-2 \gamma s}} \tag{8.43b}
\end{align*}
$$

where $\Gamma_{g}$ is the reflexion coefficient corresponding to $Z_{g}$.
The introduction of functions to describe separately the attenuation and the phase characteristics of the terminations makes it
$\dagger$ [5], p. 75.
possible to express currents and voltages in a completely hyperbolic form. These terminal functions are defined as follows:
or

$$
\begin{align*}
& \rho+j \phi=\operatorname{coth}^{-1} \frac{Z}{Z_{c}}=\tanh ^{-1} \frac{Y}{Y_{c}}  \tag{8.44a}\\
& \rho+j \phi^{\prime}=\operatorname{coth}^{-1} \frac{Y}{Y_{c}}=\tanh ^{-1} \frac{Z}{Z_{c}} . \tag{8.44b}
\end{align*}
$$

The corresponding expressions for the currents and voltages are:

$$
\begin{align*}
& \frac{V(w)}{V^{e}}=\frac{\sinh \left(\rho_{g}+j \phi_{g}\right) \cosh \left[\left(\alpha w+\rho_{a}\right)+j\left(\beta w+\phi_{a}\right)\right]}{\sinh \left[\left(\alpha s+\rho_{g}+\rho_{a}\right)+j\left(\beta s+\phi_{g}+\phi_{a}\right)\right]}  \tag{8.45a}\\
& \frac{I(w)}{V^{e}}=\frac{\sinh \left(\rho_{g}+j \phi_{g}\right) \sinh \left[\left(\alpha w+\rho_{a}\right)+j\left(\beta w+\phi_{a}\right)\right]}{\sinh \left[\left(\alpha s+\rho_{g}+\rho_{a}\right)+j\left(\beta s+\phi_{g}+\phi_{a}\right)\right]} . \tag{8.45b}
\end{align*}
$$

The effects of a termination are now shown to be equivalent to those of a section of transmission line with a total loss specified by $\rho$ and a total phase shift specified by $\phi$. Note that the denominator of ( $8.45 \mathrm{a}, \mathrm{b}$ ) includes the total loss and total phase shift of the line plus its terminations at both ends. Impedance and admittance are given by

$$
\begin{align*}
Z(w) & =\operatorname{coth}\left[\left(\alpha w+\rho_{a}\right)+j\left(\beta w+\phi_{a}\right)\right]  \tag{8.46a}\\
Y(w) & =\tanh \left[\left(\alpha w+\rho_{a}\right)+j\left(\beta w+\phi_{a}\right)\right] . \tag{8.46b}
\end{align*}
$$

The terminal functions and the reflexion coefficient are related as follows:

$$
\begin{align*}
\Gamma & =|\Gamma| e^{j \psi}=e^{-2(\rho+j \phi)} \\
|\Gamma| & =e^{-2 \rho}=\frac{\operatorname{coth} \rho-1}{\operatorname{coth} \rho+1}, \quad \psi=-2 \phi  \tag{8.4}\\
\rho & =\frac{1}{2} \ln 1 / \Gamma \left\lvert\,=\operatorname{coth}^{-1} \frac{1+|\Gamma|}{1-|\Gamma|} .\right. \tag{8.48}
\end{align*}
$$

For most transmission lines that are useful as feeders for an array or for measuring sections, the line losses are very small and can often be neglected. Under these conditions

$$
\gamma \doteq j \beta, \quad Z_{c} \doteq R_{c}=\sqrt{l e} / c
$$

and (8.36) and (8.37) give

$$
\begin{align*}
& V(w) \doteq \frac{V(0) e^{j \beta w}}{1+\Gamma_{a}}\left[1+\left|\Gamma_{a}\right| e^{-j\left(2 \beta w-\psi_{a}\right)}\right]  \tag{8.49a}\\
& I(w) \doteq \frac{I(0) e^{j \beta w}}{1-\Gamma_{a}}\left[1-\left|\Gamma_{a}\right| e^{-j\left(2 \beta w-\psi_{a}\right)}\right] . \tag{8.49b}
\end{align*}
$$

These have the following maxima and minima :

$$
\begin{align*}
& \left|V_{\max }(w)\right|=\left|\frac{V(0)}{1+\Gamma_{a}}\right|\left[1+\left|\Gamma_{a}\right|\right], \quad\left|I_{\min }(w)\right|=\left|\frac{I(0)}{1-\Gamma_{a}}\right|\left[1-\left|\Gamma_{a}\right|\right], \\
& 2 \beta w-\psi_{a}=0,2 \pi, \ldots=2 n \pi  \tag{8.50a}\\
& \left|V_{\min }(w)\right|=\left|\frac{V(0)}{1+\Gamma_{a}}\right|\left[1-\left|\Gamma_{a}\right|\right], \quad\left|I_{\max }(w)\right|=\left|\frac{I(0)}{1-\Gamma_{a}}\right|\left[1+\left|\Gamma_{a}\right|\right], \\
& 2 \beta w-\psi_{a}=\pi, 3 \pi, \ldots=(2 n+1) \pi, n=0,1,2 \ldots . \tag{8.50b}
\end{align*}
$$

The ratio of the maximum-to-minimum of either the current or the voltage on a lossless line is called the standing wave ratio and abbreviated SWR.
SWR $=\left|V_{\max }(w) / V_{\min }(w)\right|=\left|I_{\max }(w) / I_{\min }(w)\right|=\left(1+\left|\Gamma_{a}\right|\right) /\left(1-\left|\Gamma_{a}\right|\right)$

$$
\begin{equation*}
=\frac{\left|Z_{a}+Z_{c}\right|+\left|Z_{a}-Z_{c}\right|}{\left|Z_{a}+Z_{c}\right|-\left|Z_{a}-Z_{c}\right|}=\frac{\left|Y_{c}+Y_{a}\right|+\left|Y_{c}-Y_{a}\right|}{\left|Y_{c}+Y_{a}\right|-\left|Y_{c}-Y_{a}\right|}=\operatorname{coth} \rho . \tag{8.51}
\end{equation*}
$$

The distributions of voltage and current are periodic and repeat every half wavelength; the adjacent maxima and minima of voltage or current are separated by a quarter wavelength; the current maxima occur at voltage minima and vice versa. The impedance and admittance looking toward the load also repeat every half wavelength. At maxima and minima of the current or voltage the impedance and admittance are real with the following values [from (8.39)]:

Voltage maxima or current minima

$$
2 \beta \omega-\psi_{a}=0,2 \pi, 4 \pi, \ldots
$$

$Z(w)=R_{c}\left[\frac{1+\left|\Gamma_{a}\right|}{1-\left|\Gamma_{a}\right|}\right]=R_{c}[$ SWR $] ; \quad Y(w)=G_{c}\left[\frac{1-\left|\Gamma_{a}\right|}{1+\left|\Gamma_{a}\right|}\right]$

$$
\begin{equation*}
=G_{c} /[\mathrm{SWR}] . \tag{8.5}
\end{equation*}
$$

Voltage minima or current maxima

$$
2 \beta w-\psi_{a}=\pi, 3 \pi, 5 \pi, \ldots
$$

$Z(w)=R_{c}\left[\frac{1-\left|\Gamma_{a}\right|}{1+\left|\Gamma_{a}\right|}\right]=R_{c} /[$ SWR $] ; \quad Y(w)=G_{c}\left[\frac{1+\left|\Gamma_{a}\right|}{1-\left|\Gamma_{a}\right|}\right]$

$$
\begin{equation*}
=G_{c}[S W R] . \tag{8.53b}
\end{equation*}
$$

The relative distributions of current and voltage for lossless lines are:

$$
\begin{equation*}
V(w) / V(0)=\cos \beta w+j\left(Y_{a} / G_{c}\right) \sin \beta w \tag{8.54a}
\end{equation*}
$$

$$
\begin{align*}
I(w) / I(0) & =\cos \beta w+j\left(Z_{a} / R_{c}\right) \sin \beta w  \tag{8.54b}\\
V(w) /\left[I(0) R_{c}\right] & =\left(Z_{a} / R_{c}\right) \cos \beta w+j \sin \beta w  \tag{8.55a}\\
I(w) /\left[V(0) G_{c}\right] & =\left(Y_{a} / G_{c}\right) \cos \beta w+j \sin \beta w . \tag{8.55b}
\end{align*}
$$

The corresponding impedance is

$$
\begin{equation*}
Z(w)=R_{c}\left[\frac{\left(Z_{a} / R_{c}\right)+j \tan \beta w}{1+j\left(Z_{a} / R_{c}\right) \tan \beta w}\right] \tag{8.56}
\end{equation*}
$$

The admittance is given by an identical expression with $Z_{a}$ and $R_{c}$ replaced by $Y_{a}$ and $G_{c}$.

The input power to a section of line is $P=V I^{*}=|V|^{2} / Z$ with the asterisk denoting the complex conjugate. At a voltage or current maximum ( $8.53 \mathrm{a}, \mathrm{b}$ ) give

$$
\begin{equation*}
P=\left|V_{\max }\right|^{2} / R_{c}[\mathrm{SWR}]=\left|I_{\max }\right|^{2} R_{c} /[\mathrm{SWR}] . \tag{8.5}
\end{equation*}
$$

Equation (8.57) is sometimes of help in measuring the relative power in the branches of a feeding network.

Useful properties of a quarter-wave section of transmission line follow from (8.54) and (8.55). If $I(0)$ and $V(0)$ are the required driving-point current and voltage and, if $\beta w=\pi / 2$,

$$
V\left(\frac{\pi}{2}\right)=j R_{c} I(0), \quad I\left(\frac{\pi}{2}\right)=j V(0) / R_{c}, \quad Z\left(\frac{\pi}{2}\right)=R_{c}^{2} Y_{a} .
$$

### 8.4 Distribution-curve and resonance-curve measuring techniques

A number of techniques have been developed for measuring the apparent impedance or admittance of a load terminating a transmission line. When the load is a monopole or a dipole and the frequency ranges from a few hundred to a few thousand MHz , a choice is usually made between the distribution-curve or the resonance-curve methods. In the distribution-curve method, the total length of the transmission line and its excitation point remain fixed and a loosely-coupled probe is moved along the line to locate a current or voltage minimum and measure either the SWR or the curve width at twice-minimum power. In the resonance-curve or Chipman method a movable short circuit is used to tune the line with its terminations to resonance by adjusting its length. A small loop probe projecting from the short circuit is used to locate a resonance maximum and measure either the curve width at halfmaximum power or the SWR.

There are variations of these methods which are sometimes convenient. Three fixed probes may be used with the distribution-
curve method $\dagger$ in place of a single movable one, or a pair of directional couplers may be used to sample the incident and reflected wave directly. $\ddagger$ An advantage of the resonance-curve method is that it can be used with a receiving as well as with a transmitting antenna. When the antenna is driven, the line can be excited by a loosely coupled stub probe located on the transmission line near the line-load junction. Also, in this method, the locations of the generator and sampling probe can be interchanged.

Distribution-curve and resonance-curve methods are illustrated schematically in Figs. 8.6 and 8.7. The quantities to be measured are the position of a maximum or minimum and either a SWR or a curve width. With reference to accuracy in the measured quantities, three significant figures in the SWR are usually sufficient.


Fig. 8.6. Distribution-curve method of measurement.
(For example, uncertainties of $\pm 0.02$ in the SWR and $\pm 0.02 \lambda$ in the location of a current minimum of a thin quarter-wave monopole driven over a ground screen at a wavelength of about 44 cm may introduce errors of up to $2 \%$ in measured values of $G_{a}$ and up to $5 \%$ in measured values of $B_{a}$ [15].) This is readily achieved except with very high or very low standing wave ratios. When the actual ratio of maximum-to-minimum is measured, the detecting
$\dagger$ See chapter 4 of [5].
$\ddagger$ See p. 235 of [2].

Coaxial line with sliding short-circuiting plunger


Resonance-curve pattern (current)
Fig. 8.7. Resonance-curve method of measurement.
and display system must be linear or obey a square law over a wide range, or a calibrated attenuator must be available. The curve width is generally more convenient to use if the ratios to be measured exceed ten with the distribution-curve method or five with the resonance-curve method. If $\Delta w$ is the curve width at twice-minimum power for a distribution curve or a half-maximum power for a resonance curve, the SWR and apparent terminal attenuation function, $\rho_{a}$, are given by [5] in its eqs. (19) and (23).

$$
\begin{equation*}
\mathrm{SWR}=\operatorname{coth} \pi \frac{\Delta w}{\lambda_{g}} \doteq \frac{\lambda_{\mathrm{g}}}{\pi \Delta w} \doteq \frac{1}{\rho_{a}} \tag{8.58}
\end{equation*}
$$

where $\lambda_{g}$ is the wavelength measured on the transmission line. The approximate equalities in (8.58) are true when $\Delta w / \lambda_{g} \ll 1$. When $\Delta w$ is small or when the SWR is small, the distribution or resonance curve may require graphing on an enlarged scale in order to obtain sufficient accuracy. In determining the location of a distributioncurve minimum or a resonance-curve maximum, readings on each side of the minimum or maximum at several power levels should be averaged to increase the accuracy.

Much of the tedious work of graphing can be avoided with a recorder. The use of a miniature d.c. motor to drive the probe and a coupled miniature selsyn motor permits the direct acquisition of
linear position information at a convenient scale to replace the usual angular position information in a standard antenna pattern recorder.

For either the distribution-curve or the resonance-curve method the detecting probe must be properly tuned and loosely coupled in order to provide high sensitivity without significantly distorting the standing wave pattern. Any distortion or loading introduced by the probe is most pronounced at a current maximum with a current probe and at a voltage maximum with a voltage probe. When a probe is too tightly coupled to the transmission line, the measured SWR is less than the true one and maxima are shifted from points midway between adjacent minima. A simple test for excessive probe coupling is to measure a moderate SWR with probes of different sizes. If there is no change in the measured SWR, the probes are not introducing significant errors. Another useful test is to measure the location of a maximum, the adjacent minima, and the curve width at half-maximum power with a short circuit as the termination. If the probe is introducing no errors, power variations about the maximum should be like $\cos ^{2} \beta w$ [from (8.54)] and the maximum should fall midway between the minima. This test is particularly severe, for the standing wave is very large and only the probe is absorbing power from the line.

A short circuit at or very near the line-load junction is commonly used as reference in determining electrical distances from the junction to a convenient distribution-curve minimum or resonancecurve maximum. If the feeding line is coaxial, this may simply be a conducting plug which makes good contacts with the inner and outer conductors. If the feeding line is an unshielded two-wire line, the short circuit may be provided by a conducting disk which makes good contact with both conductors and which has a radius of at least five times the centre-to-centre spacing of the conductors [16].

When standing wave measurements are made on open two-wire lines, the principal difficulty that is encountered is keeping the lines balanced so that $I_{2}(w)=-I_{1}(w)$. This requires that perfect symmetry be maintained everywhere in the vicinity of the lines. A small probe placed midway between the conductors and connected to a sensitive detector is usually necessary to monitor the condition of balance. When the lines are perfectly balanced, nothing is received on the monitor probe.

The resonance-curve method requires that the distances $y$ and
$w$, which specify the location of the probes with respect to the ends of the line in Fig. 8.7, remain fixed and only the total length, $s$, be varied. One method already suggested is to use a probe attached to a sliding short circuit for sampling the current along the line.


Fig. 8.8. Trombone line stretcher or phase shifter for precision measurements.
However, a movable short circuit that maintains good electrical contact during its motion is difficult to construct. If the load being investigated has a small loss, the resonance-curve maximum is especially sensitive to erratic contacts. When there is sufficient room between inner and outer conductors, a 'non-contacting' short circuit can be used [17, 18]. An alternative method that is very satisfactory is to hold the short-circuit fixed and adjust the line length with a constant-impedance-trombone line as shown in Fig. 8.8. Additional characteristics of the individual components, the effects of probe errors, errors introduced by irregularities in the slotted line, and methods for correcting some of the errors are discussed in [1, 2, and 8].

The measured SWR and location for a minimum or maximum can be used to determine the apparent load impedance or admittance by means of a Smith chart, rectangular or other type of impedance chart, or by direct calculation from formulas. The graphical techniques are generally useful but they frequently introduce uncertainties as large as those involved in the measurement of SWR's and distances. In order to preserve all of the experimental precision it is usually advantageous to determine the
impedances or admittances from formulas. In fact, if many measurements are to be converted, the use of a small digital computer is faster and cheaper since the required averaging can be done on the computer and it is then unnecessary to graph current or voltage distributions.

A useful property of the terminal functions defined in (8.44) is their comparatively slow variation with respect to $\beta_{0} h$. Hence, graphs of these functions are useful for determining errors or irregularities in the measured data.
Distribution-curve method
Assume that a voltage probe is used to measure the SWR and the location of a voltage minimum. Let $w_{n}$ be the distance from the line-load junction to the $n^{\text {th }}$ voltage minimum; in Fig. 8.6, $n=1$. If transmission line losses are negligible over the section of line used in the measurements, the impedance at a voltage minimum is $R_{c} /[\mathrm{SWR}]$. From (8.56),

$$
\frac{1}{[\mathrm{SWR}]}=\left[\frac{Z_{a} / R_{c}+j \tan \beta w_{n}}{1+j\left(Z_{a} / R_{c}\right) \tan \beta w_{n}}\right] .
$$

When this equation is solved for $Z_{a}=R_{a}+j X_{a}$, the result is
$Z_{a}=R_{a}+j X_{a}=\left\{\begin{array}{c}R_{c}\left[\frac{1-j[\mathrm{SWR}] \tan \beta w_{n}}{[\mathrm{SWR}]-j \tan \beta w_{n}}\right] \quad \text { Voltage probe (8.59a) } \\ \frac{R_{c}}{[\mathrm{SWR}]^{2}+\tan ^{2} \beta w_{n}}\left\{[\mathrm{SWR}]\left(1+\tan ^{2} \beta w_{n}\right)\right. \\ \left.+j\left(1-[\mathrm{SWR}]^{2}\right) \tan \beta w_{n}\right\} .\end{array}\right.$
$\beta=2 \pi / \lambda, n$ is the number of the minimum corresponding to $w_{n}$, $\lambda$ is the wavelength along the transmission line.

If a current probe is used and $w_{n}$ is the distance from the line-load junction to a current minimum,
$Z_{a}=R_{a}+j X_{a}=\left\{\begin{array}{c}R_{c}\left[\frac{[\mathrm{SWR}]-j \tan \beta w_{n}}{1-j[\mathrm{SWR}] \tan \beta w_{n}}\right] \text { Current probe (8.59c) } \\ \frac{R_{c}}{1+[\mathrm{SWR}]^{2} \tan ^{2} \beta w_{n}}\left\{[\mathrm{SWR}]\left(1+\tan ^{2} \beta w_{n}\right)\right. \\ \left.+j\left([\mathrm{SWR}]^{2}-1\right) \tan \beta w_{n}\right\} .\end{array}\right.$
The reflexion coefficient is defined by (8.35) with magnitude and phase given by

$$
\begin{equation*}
\left|\Gamma_{a}\right|=\frac{[\mathrm{SWR}]-1}{[\mathrm{SWR}]+1} \tag{8.60a}
\end{equation*}
$$

$$
\psi_{a}= \begin{cases}2\left[\beta w_{n}-\left(n+\frac{1}{2}\right) \pi\right] & \text { Voltage probe }  \tag{8.60b}\\ 2\left[\beta w_{n}-n \pi\right] & \text { Current probe. }\end{cases}
$$

For high SWR's the curve-width method may be used and the SWR calculated from (8.58). If the admittance is to be determined and a voltage probe is used, ( 8.59 c ) and ( 8.59 d ) apply with $G_{a}+j B_{a}$ substituted for $R_{a}+j X_{a}$ and with $G_{c}$ substituted for $R_{c}$. If a current probe is used for admittance measurements, (8.59a) and ( 8.59 b) apply with the indicated substitutions.

The terminal functions defined by (8.44) can be obtained from (8.47) and (8.48). The relations are

$$
\left.\begin{array}{rl}
\rho_{a} & =\operatorname{coth}^{-1} \frac{1+\left|\Gamma_{a}\right|}{1-\left|\Gamma_{a}\right|}=\operatorname{coth}^{-1} \mathrm{SWR}=\frac{1}{2} \ln \frac{\mathrm{SWR}+1}{\mathrm{SWR}-1} \\
& \doteq \pi \Delta w / \lambda, \quad \Delta w=\text { curve width }
\end{array}\right\} \begin{array}{ll}
\left(n+\frac{1}{2}\right) \pi-\beta w_{n} & \text { Voltage probe } \\
\phi_{a} & =-\psi_{a} / 2=\beta w_{n}  \tag{8.62a}\\
\text { Current probe. }
\end{array}
$$

Real and imaginary parts of impedances or admittances in terms of the terminal functions can be found from (8.44a, b). The results are

$$
\begin{align*}
Z_{a} & =R_{a}+j X_{a}=Z_{c}\left\{\frac{\sinh 2 \rho_{a}}{\cosh 2 \rho_{a}-\cos 2 \phi_{a}}-j \frac{\sin 2 \phi_{a}}{\cosh 2 \rho_{a}-\cos 2 \phi_{a}}\right\}  \tag{8.63a}\\
Y_{a} & =G_{a}+j B_{a}=Y_{c}\left\{\frac{\sinh 2 \rho_{a}}{\cosh 2 \rho_{a}+\cos 2 \phi_{a}}+j \frac{\sin 2 \phi_{a}}{\cosh 2 \rho_{a}+\cos 2 \phi_{a}}\right\} . \tag{8.63b}
\end{align*}
$$

Frequently the total distance $w_{n}$ from the line-load junction to a convenient minimum is difficult to measure accurately. Since on a lossless line the impedance is repeated at intervals of $\lambda / 2$ or $\beta w=\pi$ radians, it is necessary only to determine the location of a minimum with respect to an integral number of half wavelengths from the junction. If the load being investigated is removed and a short circuit is placed at the junction, a voltage null will appear at the junction and along the line at each half wavelength from the junction. Let $w_{v}$ be the distance from the $n^{\text {th }}$ voltage null with the short circuit as a load to the nearest voltage minimum with the antenna as a load. Distances toward the generator are positive, those toward the load are negative. Then,

$$
\begin{equation*}
\beta w_{n}=n \pi \pm \beta w_{v} \tag{8.64}
\end{equation*}
$$

and

$$
\tan \left(n \pi \pm \beta w_{v}\right)=\frac{\tan n \pi \pm \tan \beta w_{v}}{1 \pm \tan n \pi \tan \beta w_{v}}= \pm \tan \beta w_{v}
$$

so that $w_{v}$, the shift in the location of a minimum when the unknown impedance is substituted for the short circuit, can be used directly in (8.59a) and (8.59b) if due regard is given to the sign. Similar results hold for a current minimum and ( 8.59 c ) and ( 8.59 d ). In terms of the minimum shift, the phase of the terminal functions and of the reflexion coefficient is

$$
\phi_{a}=-\psi_{a} / 2= \begin{cases}\pi / 2 \mp \beta w_{v} & \text { Voltage probe }  \tag{8.65a}\\ \mp \beta w_{v} & \text { Current probe. }\end{cases}
$$

The various distances are illustrated in Fig. 8.6.

## Resonance-curve method

Let the solution of (8.45) be written for a balanced generator located at an arbitrary point $y=y_{g}$ along the transmission line instead of at $y=0$. Also let the hyperbolic functions be separated into their real and imaginary parts. $\dagger$ Then the magnitudes of currents and voltages along the line are:

$$
\begin{aligned}
& \sqrt{\sinh ^{2}\left(\alpha y_{g}+\rho_{g}\right)+\sin ^{2}\left(\beta y_{g}+\phi_{g}\right)} \\
& |I|=\frac{V^{e}}{R_{c}} \frac{\times \sqrt{\sinh ^{2}\left(\alpha w+\rho_{a}\right)+\sin ^{2}\left(\beta w+\phi_{a}\right)}}{\sqrt{\sinh ^{2}\left(\alpha s+\rho_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta s+\phi_{g}+\phi_{a}\right)}} \\
& \sqrt{\sinh ^{2}\left(\alpha y_{g}+\rho_{g}\right)+\sin ^{2}\left(\beta y_{g}+\phi_{g}\right)} \\
& |V|=V^{e} \frac{\times \sqrt{\sinh ^{2}\left(\alpha w+\rho_{a}\right)+\cos ^{2}\left(\beta w+\phi_{a}\right)}}{\sqrt{\sinh ^{2}\left(\alpha S+\rho_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta s+\phi_{g}+\phi_{a}\right)}} \\
& \sqrt{\sinh ^{2}\left(\alpha w_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta w_{g}+\phi_{a}\right)} \\
& |I|=\frac{V^{e}}{R_{c}} \frac{\times \sqrt{\sinh ^{2}\left(\alpha y+\rho_{g}\right)+\sin ^{2}\left(\beta y+\phi_{g}\right)}}{\sqrt{\sinh ^{2}\left(\alpha s+\rho_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta s+\phi_{g}+\phi_{a}\right)}} \\
& \left.|V|=V^{e} \frac{\sqrt{\sinh ^{2}\left(\alpha w_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta w_{g}+\phi_{a}\right)}}{\sqrt{\sinh ^{2}\left(\alpha s+\rho_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta s+\phi_{g}+\phi_{a}\right)}}\right\} 0 \leqslant y \leqslant y_{g} \\
& \left.|V|=V^{e} \frac{\sqrt{\sinh ^{2}\left(\alpha w_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta w_{g}+\phi_{a}\right)}}{\sqrt{\sinh ^{2}\left(\alpha s+\rho_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta s+\phi_{g}+\phi_{a}\right)}} \quad\right\} 0 \leqslant y \leqslant y_{g} \\
& y_{g} \leqslant y \leqslant s
\end{aligned}
$$

where $w_{g}=s-y_{g}$ is the distance from the load to the generator.

[^23]The first two equations give current and voltage distributions between the load at $y=s$ and the generator as illustrated in Fig. 8.7. The last two equations give the distributions between the generator and the end of the line at $y=0$. If the distance $y_{\mathrm{g}}$ or $w_{\mathrm{g}}$ between the appropriate ends of the line and the generator, and the distance $w$ or $y$ between the appropriate ends of the line and the probe are held constant, while the total length $s$ is changed, the current and voltage vary in the same manner,

$$
\begin{equation*}
I(w) \sim V(w) \sim\left[\sinh ^{2}\left(\alpha s+\rho_{g}+\rho_{a}\right)+\sin ^{2}\left(\beta s+\phi_{\mathrm{g}}+\phi_{a}\right)\right]^{-1 / 2} . \tag{8.66}
\end{equation*}
$$

The line is resonant when (8.66) has its maximum value. Maxima and minima of (8.66) are defined by
$\left(\beta s+\phi_{g}+\phi_{a}\right)=\left\{\begin{array}{lr}n \pi-\frac{1}{2} \sin ^{-1}\left[\frac{\alpha}{\beta} \sinh 2\left(\alpha s+\rho_{g}+\rho_{a}\right)\right] & \text { Maxima } \\ \left(n+\frac{1}{2}\right) \pi+\frac{1}{2} \sin ^{-1}\left[\frac{\alpha}{\beta} \sinh 2\left(\alpha s+\rho_{g}+\rho_{a}\right)\right] & \text { Minima. }\end{array}\right.$
For lossless lines, $\alpha / \beta=0$, and

$$
\begin{array}{ll}
\left(\beta s_{\max }+\phi_{g}+\phi_{a}\right)=n \pi & \text { Maxima } \\
\left(\beta s_{\min }+\phi_{g}+\phi_{a}\right)=\left(n+\frac{1}{2}\right) \pi & \text { Minima } . \tag{8.67b}
\end{array}
$$

With a short circuit at $w=0, \phi_{a}=\pi / 2$; this value is commonly used as a reference. Let $s_{s}$ be a convenient resonant length with a short circuit as the termination at $w=0$ and let $s_{1}$ be the corresponding resonant length with the unknown impedance at $w=0$. When (8.67a) is written successively for both loads and the one is subtracted from the other, the result is

$$
\begin{equation*}
\phi_{a}=\pi / 2+\beta\left(s_{s}-s_{1}\right) . \tag{8.68}
\end{equation*}
$$

For lines with very small losses, $\alpha s \ll 1$, the ratio of maximum-tominimum in (8.66) is

$$
\begin{equation*}
\mathrm{SWR} \doteq \operatorname{coth}\left(\rho_{\mathrm{g}}+\rho_{a}\right) . \tag{8.69}
\end{equation*}
$$

This equation illustrates an important additional requirement in the resonance-curve method that does not occur in the distributioncurve method. In the distribution-curve method, only the parameters that characterize the generator are involved in the measurements ; in the resonance-curve method, $\rho_{\mathrm{g}}$ must be known or the generator must be lightly coupled so that $\rho_{\mathrm{g}} \ll \rho_{a}$ for the loads under investigation. When these conditions are satisfied, $\rho_{a}$ is given by (8.69), and impedances or admittances can be computed directly from (8.63a) or (8.63b). The magnitude of the reflexion
coefficient can be calculated from (8.60a) and its phase, $\psi_{a}$, from

$$
\begin{equation*}
\psi_{a}=-2 \phi_{a}=\beta\left(s_{1}-s_{s}\right)-\pi . \tag{8.70}
\end{equation*}
$$

With the resonance-curve method, it is frequently more convenient to measure the curve width than the SWR, and sometimes it is simpler to measure the curve widths at power levels other than $1 / 2$. For low-loss transmission lines and symmetrical resonance curves, (8.58) is simply

$$
\rho_{a} \doteq \frac{\pi}{\sqrt{p^{2}-1}} \frac{\Delta s}{\lambda}
$$

where $\Delta s$ is the width of the resonance curve at a level $1 / p$ of the maximum.

### 8.5 The measurement of self- and mutual impedance or admittance

At the driving points of the several elements in an array, currents and voltages are related by the usual coupled circuit equations. Let $V_{k}$ be the driving voltage across the terminals of element $k$ in an array of $N$ elements; let $I_{k}(0)$ be the current in the same terminals. Then, if a Kirchhoff equation is written for each element, the following set is obtained:

$$
\begin{gather*}
V_{1}=I_{1}(0) Z_{11}+I_{2}(0) Z_{12}+\ldots I_{k}(0) Z_{1 k}+\ldots I_{p}(0) Z_{1 p}+\ldots I_{N}(0) Z_{1 N} \\
\vdots  \tag{8.71}\\
V_{k}=I_{1}(0) Z_{k 1}+I_{2}(0) Z_{k 2}+\ldots I_{k}(0) Z_{k k}+\ldots I_{p}(0) Z_{k p}+\ldots I_{N}(0) Z_{k N}(8 \\
\vdots \\
V_{N}=I_{1}(0) Z_{N 1}+I_{2}(0) Z_{N 2}+\ldots I_{k}(0) Z_{N k}+\ldots I_{p}(0) Z_{N p}+\ldots I_{N}(0) Z_{N N} .
\end{gather*}
$$

The coefficient $Z_{k p}, p \neq k$, is the mutual impedance between element $k$ and element $p$. As long as the array is in an isotropic medium such as air, $Z_{k p}=Z_{p k} . Z_{k k}$ is the self-impedance of element $k$. The input or driving-point impedance of element $k$ is

$$
\begin{equation*}
Z_{k i n}=\frac{V_{k}}{I_{k}(0)}=\frac{I_{1}(0)}{I_{k}(0)} Z_{k 1}+\ldots Z_{k k}+\ldots \frac{I_{p}(0)}{I_{k}(0)} Z_{k p}+\ldots \frac{I_{N}(0)}{I_{k}(0)} Z_{k N} . \tag{8.72}
\end{equation*}
$$

If the elements are fed by transmission lines, $Z_{k i n}$ is the apparent load impedance of the transmission line. The driving terminals of an antenna coincide with the line-load junction between it and its feeding transmission line. The self-impedance of an element is the input impedance at the terminals of that element when the driving-point currents of all other elements in the array are zero-
that is, when all other elements are open-circuited at their driving points. The mutual impedance between element $k$ and element $p$ is the open-circuit voltage at the driving point of element $p$ per unit current at the driving terminals of element $k$, with the driving points of all elements but $k$ open-circuited.

The relations that involve the admittances are the duals of (8.71). They are

$$
\left.\begin{array}{c}
I_{1}(0)=V_{1} Y_{11}+V_{2} Y_{12}+\ldots V_{k} Y_{1 k}+\ldots V_{p} Y_{1 p}+\ldots V_{N} Y_{1 N} \\
\vdots \\
I_{k}(0)=V_{1} Y_{k 1}+V_{2} Y_{k 2}+\ldots V_{k} Y_{k k}+\ldots V_{p} Y_{k p}+\ldots V_{N} Y_{k N} \\
\vdots  \tag{8.74}\\
I_{N}(0)=V_{1} Y_{N 1}+V_{2} Y_{N 2}+\ldots V_{k} Y_{N k}+\ldots V_{p} Y_{N p}+\ldots V_{N} Y_{N N}
\end{array}\right\}
$$

The self-admittance of an element is its input admittance when the driving-point voltages of all other elements in the array are zerothat is, when all other elements are short-circuited at their driving points. The mutual admittance between element $k$ and element $p$ is the short-circuit current at the driving point of element $p$ per unit voltage at the terminals of element $k$, when the terminals of all elements but $k$ are short-circuited.

Self- and mutual impedances or admittances depend upon the geometrical configuration of each element, the relative orientation and location of the elements in the array, and the total number of elements. Once the self- and mutual impedances or admittances have been determined for an array, they can be used in equations like (8.72) and (8.74) to predict the driving-point impedances or admittances for any set of driving voltages or currents that may be applied to the array.

In principle, there is no difficulty in determining self- and mutual impedances. If known sets of currents or voltages are maintained at the terminals of the several elements and a sufficient number of input impedances or admittances are measured, (8.72) or (8.74) can be inverted and the self- and mutual impedances evaluated. There are, however, two practical difficulties. The first is that the only set of excitation coefficients useful in measuring self- and mutual impedances is that which can be adjusted with high accuracy
independently of the driving-point impedances. The second difficulty is that so many quantities must be determined. In a linear array of $N$ identical, equally spaced elements there are $N / 2$ different self-impedances and $N^{2} / 4$ different mutual impedances if $N$ is even; $(N+1) / 2$ different self-impedances and $\left(N^{2}-1\right) / 4$ different mutual impedances if $N$ is odd. For $N=100$ there are 2550 different quantities to be determined. Fortunately, many of the mutual impedances are sufficiently small so that they can be neglected and many of the self-impedances are alike within tolerable limits. The measuring procedure should provide a rapid indication of such possible simplifications as well as relatively simple steps for determining the significant self- and mutual impedances.

In the case of dipole or monopole arrays, the laborious measurement of self- and mutual quantities may be avoided by applying the two- or three-term theory as discussed in the preceding chapters. The theory has been shown to give good agreement with measured results in applications to circular and Yagi arrays, and this agreement will generally hold for other arrays.

An important especially simple array consists of identical elements uniformly spaced about the circumference of a circle. Since all elements have the same self-impedance, symmetry reduces the total number of unknowns to $N / 2$ if $N$ is even or $(N+1) / 2$ if $N$ is odd. Self- and mutual admittances can be measured as follows: Let element $k$ be driven, all others short-circuited at their driving points. Measure the apparent input admittance of element $k$. From (8.74)

$$
\begin{equation*}
Y_{k i n 1}=Y_{k k} . \tag{8.75}
\end{equation*}
$$

Next, let elements $k$ and $p$ be driven with the voltages $V_{p}=-V_{k}$; let the terminals of all other elements be short-circuited. Again measure the apparent input admittance of element $k$. Then

$$
\begin{equation*}
Y_{k i n 2}=Y_{k k}-Y_{k p}, \quad Y_{k p}=Y_{k i n 1}-Y_{k i n 2} \tag{8.76}
\end{equation*}
$$

Thus, for the circular array, all mutual admittances can be found by successively driving element $k$ and one other element with equal voltages in opposite phases, and each time measuring the input admittance of only element $k$. The driving voltages $V_{k}=-V_{p}$ produce a null in the electromagnetic field along the perpendicular bisector of a chord joining elements $k$ and $p$. This property can be used to obtain the correct voltages by locating a probe at the centre of the array and adjusting the phase and amplitude of element $p$ or
$k$ until no signal is observed in the probe. A $30-40 \mathrm{db}$ range of receiving sensitivity provides an accurate adjustment of the voltages. The short circuits in the elements other than $p$ and $k$ can be placed at the driving terminals or at the ends of lossless sections of transmission lines which are electrically a half wavelength long. This length is critical and must take account of phase shifts in connectors, terminal zones, etc. For monopoles driven over a ground plane by coaxial lines in the manner shown in Fig. 8.3a, the short circuits can consist of very thin plugs in the ends of the coaxial lines. End-effects are quite simple for this model since the terminal-zone network consists of a shunting capacitance. If the measured apparent input admittances given in (8.75) and (8.76) are $Y_{k a 1}$ and $Y_{k a 2}$, then

$$
\begin{align*}
Y_{k k} & =Y_{k a 1}-j \omega C_{T}  \tag{8.77a}\\
Y_{p p} & =Y_{k a 1}-Y_{k a 2} . \tag{8.77b}
\end{align*}
$$

Note that the end-effect contributes only to the self-susceptance.
The self- and mutual impedances of a circular array can be measured with analogous procedures. To determine the selfimpedance, drive one element, open-circuit all others so that all the driving-point currents are zero, and measure the input impedance of the driven element. To determine the mutual admittances, drive the elements successively in pairs with a receiving probe at the centre of the circle to aid in setting $I_{p}(0)=-I_{k}(0)$, and measure the input impedance of a driven element. Equations (8.75) and (8.76) apply with the corresponding impedances substituted for admittances. On an impedance basis, the experimental model of monopoles driven by coaxial lines offers little simplification in terminal-zone effects, and the actual load or input impedances are obtained from the measured apparent ones with (8.21).

In a linear, planar or more general array, the self-admittances or impedances can be measured by the method used for circular arrays, i.e. by driving the element of interest, loading the other elements with short circuits for admittances or open circuits for impedances, and measuring the input admittance or impedance of the driven element. Difficulties arise in measuring mutual admittances by the method of driving the elements in pairs so that $I_{p}(0)=-I_{k}(0)$ or $V_{p}=-V_{k}$, since, in general, there is no simple null in the field at which a receiving probe can be located to aid in the adjustment of the current or voltage. For example, in a

7-element curtain array of identical equispaced elements, there are 4 self- and 8 mutual impedances which must be determined. Among the 8 mutual impedances only $Z_{17}, Z_{26}$ and $Z_{35}$ correspond to pairs of elements symmetrically placed about the centre of the array. For these elements, a null at the centre ensures that each pair is driven by voltages with equal amplitudes and opposite phases.

The open circuit-short circuit method is a traditional procedure for measuring self- and mutual impedances [3, 6]. The self-impedances are measured as already discussed by driving the element of interest and open circuiting the driving points of all other elements. If element $k$ is the driven element, (8.72) becomes

$$
\begin{equation*}
Z_{k i n 1}=V_{k} / I_{k}(0)=Z_{k k} . \tag{8.78}
\end{equation*}
$$

To determine a given mutual impedance a short circuit is substituted for the open circuit in the appropriate element, and the input impedance of the driven element is again measured. If element $k$ is driven and element $p$ is short-circuited, the applicable pair of equations is

$$
\begin{align*}
V_{k} & =I_{k}(0) Z_{k k}+I_{p}(0) Z_{k p}  \tag{8.79a}\\
0 & =I_{k}(0) Z_{k p}+I_{p}(0) Z_{p p} . \tag{8.79b}
\end{align*}
$$

From (8.79a)

$$
\begin{equation*}
Z_{k i n 2}=\frac{V_{k}}{I_{k}(0)}=Z_{k k}+\frac{I_{p}(0)}{I_{k}(0)} Z_{k p} . \tag{8.79c}
\end{equation*}
$$

The use of (8.79b) to eliminate the current ratio in (8.79c), the subsequent solution for $Z_{k p}$, and the expression of $Z_{k k}$ and $Z_{p p}$ in terms of their measured values, $Z_{\text {kin } 1}$ and $Z_{p i n 1}$, yields

$$
\begin{equation*}
Z_{k p}= \pm \sqrt{Z_{p i n 1}\left(Z_{k i n 1}-Z_{k i n 2}\right)} . \tag{8.80}
\end{equation*}
$$

Mutual admittances may be determined in the same manner by an interchange of open and short circuits and the substitution of the appropriate admittances in (8.80). A satisfactory method of providing the required open circuits is to short-circuit the feeding line at an electrical quarter wavelength from the line-load junction.

An alternative procedure $\dagger$ for determining mutual impedances is based on the measurement of both the relative amplitude and phase of the driving-point currents or voltages. Let element $k$ in an array be driven and all other elements be open-circuited. The complete
set ( 8.71 ) then becomes

$$
\begin{align*}
V_{1} & =I_{k}(0) Z_{1 k} \\
& \vdots \\
V_{k} & =I_{k}(0) Z_{k k}  \tag{8.81}\\
& \vdots \\
V_{p} & =I_{k}(0) Z_{p k} \\
& \vdots \\
V_{N} & =I_{k}(0) Z_{N k} .
\end{align*}
$$

It follows that

$$
\begin{gather*}
Z_{k i n}=\frac{V_{k}}{I_{k}(0)}=Z_{k k}  \tag{8.82a}\\
\frac{V_{p}}{V_{k}}=\frac{Z_{p k}}{Z_{k k}} . \tag{8.82b}
\end{gather*}
$$

The relative amplitudes of the voltages immediately indicate which mutual impedances are large enough to be important, and the relative phases need be measured only for these.

Let the open circuits in the elements other than $k$ be provided by identical sections of transmission line that are terminated in short circuits. It follows from (8.56) that the transmission line may be either $\lambda / 4$ or $3 \lambda / 4$ in length, and (8.54) suggests that the apparent voltages across the loads can be measured by a probe placed at a distance $w=\lambda / 2$ from the line-load junction. Thus, if a section of transmission line is assembled with a short circuit at $w=3 \lambda / 4$ and a probe at $w=\lambda / 2$, the apparent driving-point voltages can be measured by interchanging this measuring section with the other loads. When the short circuit is removed, the measuring section can be incorporated in the line feeding the driven element and used to measure $V_{k}$. Note that the probe must be loosely coupled to the line and the electrical distances carefully adjusted. A coaxial measuring section for use with this procedure is shown in Fig. 8.9.

For the measurement of admittances, one element is driven while the others are short-circuited. From (8.73) it is seen that

$$
\begin{gather*}
Y_{k i n}=I_{k}(0) / V_{k}=Y_{k k}  \tag{8.83a}\\
I_{p}(0) / I_{k}(0)=Y_{p k} / Y_{k k} . \tag{8.83b}
\end{gather*}
$$



Fig. 8.9. Coaxial measuring section. For apparent open-circuited load voltages: $w_{p}=\lambda / 2$, $w_{s}=3 \lambda / 4$. For apparent open-circuit load currents: $w_{p}=\lambda / 4, w_{s}=\lambda / 2$.

If a current probe is used in the measuring section, it must be placed at $w=\lambda / 2$, with the short circuit at $w=\lambda$. However, from ( 8.55 a ) with $\beta w=\pi / 2$, it is seen that

$$
\begin{equation*}
V(\lambda / 4)=j I(0) R_{c} \tag{8.84}
\end{equation*}
$$

so that a voltage probe may be used at $w=\lambda / 4$ and the short circuit placed at $w=\lambda / 2$.

### 8.6 Theory and properties of probes

Successful techniques for sampling fields, currents and charges must be based on the responses of physically real probes, not ideal infinitesimal electric and magnetic doublets. Electric-field or charge probes are usually one-dimensional, short thin dipoles or monopoles that have a simple behaviour without serious errors. The usual magnetic-field or current probes, on the other hand, are small loops that have complicated behaviour because they are twodimensional and can be excited in more than one mode. For electrically small loops only the first two modes are important. Because of the manner in which current is distributed around the loop, they are called the circulating or transmission-line mode and the dipole mode, respectively. In the transmission-line mode, there is a continuous current circulating around the loop; currents on opposite sides are equal but in opposite directions in space. In the dipole mode, currents on opposite sides are equal but in opposite directions around the loop, hence in the same direction in space; there is no net circulating current and the probe resembles a small folded dipole. As is shown below, currents in the transmission-line
mode are related to the amplitude of the magnetic field at the centre of the loop ; currents in the dipole mode are related to the amplitude of the electric field at the centre of the loop. Generally, currents in both modes can maintain a potential difference across a load. Hence, when the objective is to measure magnetic fields, the presence of dipole-mode currents in the loop may introduce an error that must be eliminated or corrected.

## Charge or electric-field probes

To examine the properties of a small charge or electric field probet consider the short thin centre-loaded dipole in a linearly polarized field of $\mathbf{E}^{i}$ volts per metre shown in Fig. 8.10a. In the figure $\mathbf{E}^{i}$ and $\mathbf{E}_{p}^{i}$ are in the plane wave front perpendicular to the propagation vector $\mathbf{k} ; \mathbf{E}_{p}^{i}$ and $\mathbf{k}$ are in the plane containing the axis of the antenna. The equivalent circuit, Fig. 8.10b, consists of a Thévenin generator of voltage $V_{g}\left(Z_{L}=\infty\right)$ in a series combination with the load impedance $Z_{L}$ and the input impedance of the antenna $Z_{0} . V_{g}\left(Z_{L}=\infty\right)$ is the open-circuit voltage at the terminals.


Fig. 8.10. Centre-loaded receiving dipole for electric field probe. (a) idealized with no feeding lines; (b) idealized equivalent circuit ; (c) actual with feeding lines; $(d)$ actual equivalent circuit.
$\dagger$ See [6], [19] and [20]. In particular [6], p. 184 and p. 475.

It is given by

$$
\begin{equation*}
V_{g}\left(Z_{L}=\infty\right)=-2 h_{e}(\Theta) E^{i} \cos \psi \tag{8.85}
\end{equation*}
$$

where $h_{e}(\Theta)$ is the complex effective length of a short dipole and $E^{i} \cos \psi=E_{p}^{i}$ is the projection of $E^{i}$ onto the plane containing the axis of the antenna and the direction of advance of the incident plane wave through the centre of the antenna. The load current is

$$
\begin{equation*}
I_{L}=\frac{V_{g}\left(Z_{L}=\infty\right)}{Z_{0}+Z_{L}}=\frac{-2 h_{e}(\Theta) E^{i} \cos \psi}{Z_{0}+Z_{L}}=S_{c} E_{p}^{i} \tag{8.86}
\end{equation*}
$$

where

$$
S_{c}=\frac{-2 h_{e}(\Theta)}{Z_{0}+Z_{L}}
$$

is a sensitivity constant. As indicated in (8.86), the load current is proportional to the average tangential electric field along the dipole. Directions of the field can be determined by rotating the probe until $I_{L}$ is maximum. If the incident electric field is elliptically polarized, it can be resolved into two linearly polarized components along the major and minor axes of the ellipse and an open-circuit voltage defined for each. The total current is the algebraic sum of the currents due to each generator.

For many applications a short monopole over a conducting surface is an effective electric field probe. Such a probe is easily made by extending the inner conductor of a coaxial line. Equations (8.85)-(8.86) still apply if the appropriate value of $Z_{0}$ is used. For a monopole of length $h$ over a ground plane, the input admittance is twice that of a dipole of the same thickness and length $2 h$. With either dipole or monopole probes, most errors of measurement are introduced because the probe is too long or bent or both.

Computed and measured sensitivities $S_{c}$ of some monopole probes are given in Fig. 8.11. Usually, the probe is loaded by a section of transmission line terminated in a matched detector so that $Z_{L}$ can be calculated from (8.39), (8.42) or (8.46a). The complex electrical effective length of a short dipole is

$$
\begin{equation*}
\beta_{0} h_{e}(\Theta)=\frac{1}{2} \beta_{0} h \sin \Theta \tag{8.87a}
\end{equation*}
$$

The input impedance of a short dipole, $\beta_{0} h \leqslant 1$, with $\Omega=2 \ln (2 h / a)=10$ is [6]

$$
\begin{equation*}
Z_{0}=18 \cdot 3 \beta_{0}^{2} h^{2}\left(1+0 \cdot 086 \beta_{0}^{2} h^{2}\right)-j\left(396 \cdot 0 / \beta_{0} h\right)\left(1-0 \cdot 383 \beta_{0}^{2} h^{2}\right) \tag{8.87b}
\end{equation*}
$$



Fig. 8.11. Relative sensitivity $S=S_{c}$ of monopole probes (Whiteside).

When $\beta_{0} h \leqslant 0.5$ and $a \ll h$, the reactance is quite accurately given by

$$
\begin{equation*}
X_{0} \doteq \frac{-60(\Omega-3 \cdot 39)}{\beta_{0} h} \tag{8.87c}
\end{equation*}
$$

the resistance by

$$
\begin{equation*}
R_{0} \doteq 20 \beta_{0}^{2} h^{2}\left(1+0 \cdot 133 \beta_{0}^{2} h^{2}\right) \tag{8.87d}
\end{equation*}
$$

If terminal-zone effects are significant, account must be taken of them in determining the apparent resistance and reactance.

Current or magnetic-field probes
To illustrate the important features of small loops as probes, $\dagger$ consider an unloaded square loop of side $w$, and perimeter $l$ immersed in a linearly polarized electromagnetic field as shown in Fig. 8.12a. A convenient starting point in the analysis is the integral form of the Maxwell equation, $\nabla \times \mathbf{E}=-j \omega \mathbf{B}$ together with $\mathbf{B}=\nabla \times \mathbf{A}$. That is,

$$
\begin{equation*}
\oint_{s} \mathbf{E} \cdot \mathbf{d s}=j \omega \iint_{S} \hat{\mathbf{n}} \cdot \mathbf{B} d S=j \omega \oint \mathbf{A} \cdot \mathbf{d s} \tag{8.88}
\end{equation*}
$$



Fig. 8.12. Loop probe in linearly polarized field. (a) Coordinates; (b) even magnetic field; (c) odd magnetic field; (d) singly-loaded loop; (e) doubly-loaded loop.
$\dagger$ [19] p. 270, [20], [21], [22] and [23].
where $s$ is measured along the contour of the loop, $S$ is the plane area bounded by the contour, $\hat{n}$ is a unit normal to this area in the righthand screw sense with respect to integration around $s, \mathbf{E}$ is the complex amplitude of the total electric field and $\mathbf{B}$ of the total magnetic field at any point on the surface $S$ of the loop. The radius of the wire, $a$, is assumed small, so that a quasi-one-dimensional analysis is adequate. The analysis is no more complicated for a rectangular than for a square loop, but the latter can be shown to have the optimum shape for minimizing averaging errors in a general incident field. $\mathbf{E}$ on the surface of the wire can be related to the total axial current by $\mathbf{E} . d s=z^{i} I d s$ where $z^{i}$ is the internal impedance per unit length of the wire. In general, it is convenient to treat the total field in two parts, the incident field and the reradiated field maintained by the currents induced in the loop, i.e. $\mathbf{E}=\mathbf{E}^{i}+\mathbf{E}^{r}$. With these relations and (1.8a), (8.88) becomes
$-j \omega \iint_{S} \hat{\mathbf{n}} . \mathbf{B}^{i} d S=\oint_{s} z^{i} I(s) d s+\frac{j \omega \mu_{0}}{4 \pi} \oint_{s} \oint_{s} I\left(s^{\prime}\right) \frac{e^{-j \beta_{0} R}}{R} d \mathbf{s}^{\prime} \cdot \mathbf{d s}$
where $\beta_{0}=2 \pi / \lambda$ and $R$ is the distance from the element ds' at $s^{\prime}$ along the axis of the wire to the element ds on its surface. At this point, the assumption is usually made that the loop is sufficiently small to replace $\mathbf{B}^{i}$ by its value at the centre of the loop and $I(s)$ by a constant $I$. Actually a more careful treatment is often required.

Suppose that $\mathbf{B}^{i}$ can be resolved into an even and an odd part with respect to an axis through the centre of the loop. For example, if the loop lies in the $y z$ plane and $\mathbf{B}^{i}$ is a function of $y$ only and is directed parallel to the $x$-axis as in Fig. 8.12, the even and odd parts of $\mathbf{B}^{i}$ with respect to $y$ are

$$
\begin{array}{ll} 
& \mathbf{B}^{i}=\mathbf{B}_{T}^{i}+\mathbf{B}_{D}^{i} \\
\text { even: } & \mathbf{B}_{T}^{i}(y)=\frac{1}{2}\left[\mathbf{B}^{i}(y)+\mathbf{B}^{i}(-y)\right] \\
\text { odd : } & \mathbf{B}_{D}^{i}(y)=\frac{1}{2}\left[\mathbf{B}^{i}(y)-\mathbf{B}^{i}(-y)\right] \tag{8.92}
\end{array}
$$

with the following symmetry conditions:

$$
\begin{equation*}
\mathbf{B}_{T}^{i}(-y)=\mathbf{B}_{T}^{i}(y) ; \quad \mathbf{B}_{D}^{i}(-y)=-\mathbf{B}_{D}^{i}(y) . \tag{8.93}
\end{equation*}
$$

The subscripts $T$ and $D$ denote the transmission-line and dipole modes of the induced currents.

The electric field is related to the magnetic field by the Maxwell equation

$$
\begin{equation*}
\mathbf{E}^{i}=\frac{j}{\omega \mu_{0} \varepsilon_{0}} \nabla \times \mathbf{B}^{i} \tag{8.94}
\end{equation*}
$$

Since $\mathbf{E}^{i}$ is obtained from the first spatial derivative of $\mathbf{B}^{i}, \mathbf{E}^{i}$ is odd when $\mathbf{B}^{i}$ is even, and vice versa. That is,

$$
\begin{equation*}
\mathbf{E}_{T}^{i}(-y)=-\mathbf{E}_{T}^{i}(y) ; \quad \mathbf{E}_{D}^{i}(-y)=\mathbf{E}_{D}^{i}(y) . \tag{8.95}
\end{equation*}
$$

Note that $\mathbf{B}_{T}^{i} \doteq \hat{\mathbf{x}} B_{0}^{i}$ over the area bounded by the loop when $\beta_{0} w \ll 1$; the current, $I_{T}(s)$, is, then, essentially constant around the loop as indicated in Fig. 8.12b. With the symmetry conditions (8.91) and (8.94), (8.89) becomes

$$
\begin{equation*}
-j \omega B_{0}^{i} S=I_{T}\left\{\oint_{s} z^{i} d s+\frac{j \omega \mu}{4 \pi} \oint_{s} \oint_{s} \frac{e^{-j k R}}{R} \mathbf{d s} \cdot \mathbf{d s}\right\}=I_{T} Z_{0} \tag{8.96}
\end{equation*}
$$

The quantity in braces in (8.96) is the impedance $Z_{0}=Y_{0}^{-1}$ of the loop with constant current. $\dagger$ Therefore

$$
\begin{equation*}
I_{T}=I_{T}(0)=-j \omega B_{0}^{i} Y_{0} S=\lambda S_{B}\left(c B_{0}^{i}\right) \tag{8.97}
\end{equation*}
$$

where the sensitivity constant $S_{B}$ for the unloaded loop has been introduced. It is defined by

$$
\begin{equation*}
S_{B}=-j k S Y_{0} / \lambda \tag{8.98}
\end{equation*}
$$

and depends only on the geometry of the probe. The magnetic field is conveniently multiplied by $c=3 \times 10^{8} \mathrm{~m} / \mathrm{sec}$ to give it the same dimensions as $\mathbf{E}^{i}$. Note that the current $I_{T}$ is directly proportional to and, hence, a measure of the incident magnetic field $B_{0}^{i}=B_{T}^{i}(0)$ at the centre of the loop.
$\mathbf{B}_{D}^{i}(y)$ is odd in $y$ and therefore zero at the centre of the loop; it makes no contribution to the surface integral in (8.89). The associated electric field $\mathbf{E}_{\boldsymbol{D}}^{i}$ in (8.95) has a non-zero value at the centre of the loop and is approximately constant over the space occupied by the small loop. It maintains equal and codirectional currents in the two sides of the loop which are parallel to the $z$-axis and hence parallel to $\mathbf{E}_{\boldsymbol{D}}^{i}$ as shown in Fig. 8.12c. These dipole-mode currents are zero at $s= \pm l / 4(y=0, z= \pm w / 2)$. In so far as they are concerned, the loop could be cut at these points and treated as an array of two bent receiving antennas. The current at the centre of each side is

$$
\begin{equation*}
I_{D}(0)=h_{e D} Y_{D} E_{O}^{i}=\lambda S_{E} E_{O}^{i} \tag{8.99}
\end{equation*}
$$

where $h_{e D}$ is the effective length of each half of the array for the dipole noode, $Y_{D}$ is the input admittance at the centre of each antenna when both are driven with equal and codirectional currents, and $E_{0}^{i}=E_{0}^{i}(0)$ is the electric field at the centre of the loop.

[^24]The electric sensitivity constant $S_{E}$ for the unloaded loop has been introduced in (8.99). It is defined as follows:

$$
\begin{equation*}
S_{E}=h_{e D} Y_{D} / \lambda . \tag{8.100}
\end{equation*}
$$

Note that $I_{D}(0)$ is proportional to and, therefore, a measure of the incident electric field $E_{0}^{i}$ at the centre of the loop.

Transmission-line and dipole-mode currents have the following symmetries with respect to the anti-clockwise direction:

$$
\begin{equation*}
I_{T}\left(s+\frac{l}{2}\right)=I_{T}(s) ; \quad I_{D}\left(s+\frac{l}{2}\right)=-I_{D}(s) \tag{8.101}
\end{equation*}
$$

Hence, $I_{T}(s)$ corresponds to the zero-sequence current $I^{(0)}(s)$, and $I_{D}(s)$ to the first-sequence current $I^{(1)}(s)$. Higher-order sequence currents are assumed to be negligible in a sufficiently small loop. The total currents at points $s$ and $s+l / 2$ in the loop are

$$
\begin{gather*}
I(s)=I_{T}(s)+I_{D}(s)  \tag{8.102}\\
I\left(s+\frac{l}{2}\right)=I_{T}(s)-I_{D}(s) \tag{8.103}
\end{gather*}
$$

A very important fact is now evident. The magnetic field that contributes to the surface integral is related only to loop currents in the transmission-line (zero-sequence) mode and not to those in the dipole (first-sequence) mode. Conversely, if the magnetic field at the centre of the loop is to be determined from the current induced in the loop, measurements must involve currents in the transmission-line mode only.

When a small loop is used as a probe, the quantity of primary concern is the load current. Let a load $Z_{L}$ be located at $s=0$, as shown in Fig. 8.12d. The loaded loop can be analysed by replacing the load by a Thévenin generator of voltage $V=-I_{L}(0) Z_{L}$, where $I_{L}(0)$ is the current in the load. This generator maintains a current $V Y(s)=-I(0) Z_{L} Y(s)$ where $Y(s)$ is the input admittance when the loop is driven at the point $s$. The total current at $s=0$ is then

$$
\begin{equation*}
I_{L}(0)=I_{T}(0)+I_{D}(0)-I_{L}(0) Z_{L} Y(0) \tag{8.104}
\end{equation*}
$$

With (8.97) and (8.99), it follows that

$$
\begin{equation*}
I_{L}(0)=\lambda S_{B}^{(1)} c B_{0}^{i}+\lambda S_{E}^{(1)} E_{0}^{i} \tag{8.105}
\end{equation*}
$$

where the sensitivity constants for the singly-loaded loop are defined as follows:

$$
\begin{equation*}
S_{B}^{(1)}=\frac{Y_{L}}{Y_{L}+Y(0)} S_{B} \tag{8.106}
\end{equation*}
$$

$$
\begin{equation*}
S_{E}^{(1)}=\frac{Y_{L}}{Y_{L}+Y(0)} S_{E} \tag{8.107}
\end{equation*}
$$

The importance of the location of the load with respect to the incident fields is now evident. When the load is located at $s=l / 2$ instead of $s=0$ (a change that is equivalent to rotating the loop through $180^{\circ}$ about the $x$-axis), the input admittance of the loop is still the same but the current is given by (8.103) so that

$$
\begin{equation*}
I_{L}\left(\frac{\lambda}{2}\right)=\lambda S_{B}^{(1)} c B_{0}^{i}-\lambda S_{E}^{(1)} E_{0}^{i} \tag{8.108}
\end{equation*}
$$

This equation and (8.105) are useful for determining the relative importance of $I_{D}$. If the load current remains constant when the probe is rotated $180^{\circ}$ about the $x$-axis, $I_{D}$ is negligible and $I_{L}=I_{T}$. If the load current does not remain constant, readings of amplitude and phase may be taken in both positions. Then

$$
\begin{align*}
& I_{T}=\lambda S_{B}^{(1)} c B_{0}^{i}=\frac{1}{2}\left[I_{L}(0)+I_{L}\left(\frac{l}{2}\right)\right]  \tag{8.109a}\\
& I_{D}=\lambda S_{E}^{(1)} E_{0}^{i}=\frac{1}{2}\left[I_{L}(0)-I_{L}\left(\frac{l}{2}\right)\right] . \tag{8.109b}
\end{align*}
$$

Instead of rotating the probe and taking two readings of amplitude and phase, one may use two loads with a hybrid junction to evaluate the sum and the difference.

In the simple example of a linearly polarized electric field indicated in Fig. 8.12, $I_{D}$ is easily eliminated by a simple rotation of the loop until the side containing the load is perpendicular to the electric field. That is, the load is located at $s= \pm l / 4$ in Fig. 8.12. However, when the electric field is elliptically polarized this expedient is unavailable and a doubly-loaded probe is probably the simplest solution in spite of the increased constructional difficulties.

The analysis of a doubly-loaded loop with identical loads $Z_{L}$ at $s=0$ and $s=l / 2$, as shown in Fig. 8.12e, parallels that of the singly-loaded loop. The two load currents are

$$
\begin{align*}
& I_{L 1}=I(0)=I_{T}(0)+I_{D}(0)-\left[I(0) Y(0)+I\left(\frac{l}{2}\right) Y\left(\frac{l}{2}\right)\right] / Y_{L}  \tag{8.110a}\\
& I_{L 2}=I\left(\frac{l}{2}\right)=I_{T}(0)-I_{D}(0)-\left[I(0) Y\left(\frac{l}{2}\right)+I\left(\frac{l}{2}\right) Y(0)\right] / Y_{L} \tag{8.110b}
\end{align*}
$$

The driving-point admittances $Y(0)$ and $Y(l / 2)$ may be resolved
into the zero- and first-sequence admittances $Y^{(0)}$ and $Y^{(1)}$. These can be introduced as follows:

Let

$$
\begin{gather*}
Y(0)=Y^{(0)}+Y^{(1)} ; \quad Y\left(\frac{l}{2}\right)=Y^{(0)}-Y^{(1)} \\
I_{\Sigma}=I_{L 1}+I_{L 2}=\frac{2 Y_{L}}{Y_{L}+2 Y^{(0)}} I_{T}(0)=\lambda S_{B}^{(2)} c B_{0}^{i}  \tag{8.111a}\\
I_{\Delta}=I_{L 1}-I_{L 2}=\frac{2 Y_{L}}{Y_{L}+2 Y^{(1)}} I_{D}(0)=\lambda S_{E}^{(2)} E_{0}^{i} \tag{8.111b}
\end{gather*}
$$

The sensitivity constants for the doubly-loaded probe are defined as follows:

$$
\begin{align*}
& S_{B}^{(2)}=\frac{2 Y_{L}}{Y_{L}+2 Y^{(0)}} S_{B}  \tag{8.111c}\\
& S_{E}^{(2)}=\frac{2 Y_{L}}{Y_{L}+2 Y^{(1)}} S_{E} \tag{8.111d}
\end{align*}
$$

In actual practice, the hybrid junctions used to perform the summing and differencing operations will have good but not infinite isolation, so that the actual measurable currents are

$$
\begin{gather*}
I_{B}=I_{\Sigma}+\gamma I_{\Delta}=\lambda S_{B}^{(2)} c B_{0}^{i}+\gamma \lambda S_{E}^{(2)} E_{0}^{i}  \tag{8.112a}\\
I_{E}=I_{\Delta}+\gamma^{\prime} I_{\Sigma}=\lambda S_{E}^{(2)} E_{0}^{i}+\gamma^{\prime} \lambda S_{B}^{(2)} c B_{0}^{i} \tag{8.112b}
\end{gather*}
$$

where $\gamma$ and $\gamma^{\prime}$ are the coefficients of cross-coupling between the adding and subtracting circuits. It is assumed that they are small.

In the measurement of the magnetic field (especially near the end of a dipole antenna where the polarization of the electric field is highly elliptical) it is particularly important that those parts of the current in the load that are excited in the dipole mode, viz. $I_{D}$, be negligible. To provide a measure of the ability of a probe and loading system to discriminate against such currents, a system error ratio $\varepsilon^{(n)}$ can be defined as the ratio of the output current due to unit parallel electric field ( $E_{0}^{i}=1$ volt $/$ metre) to the output current due to unit normal magnetic field ( $c B_{0}^{i}=1$ volt/metre) ,

$$
\begin{align*}
& \varepsilon^{(1)}=S_{E}^{(1)} / S_{B}^{(1)}  \tag{8.113a}\\
& \varepsilon^{(2)}=S_{E}^{(2)} / S_{B}^{(2)} \tag{8.113b}
\end{align*}
$$

where the superscript indicates the number of loads in the probe. Note that ( 8.113 b ) applies to the combination of the probe and its summing and differencing circuits. The actual ratio of the two currents depends on the ratio of the fields $E_{0}^{i} / c B_{0}^{i}$ and generally
equals $\varepsilon^{(n)}$ only in a plane-wave field. For a system to be capable of measuring the magnetic field with an error of no more than $10 \%$, it is necessary that $\varepsilon^{(n)} \leqslant-20 \mathrm{~dB}$, where $\varepsilon^{(n)}$ in $\mathrm{dB}=20 \log _{10} \varepsilon^{(n)}$.

So far, the discussion has been concerned with square loops, although circular loops are often more desirable. Actually, a comparable analysis of circular loops follows precisely the steps outlined for the square loop including the definition of sensitivity constants, error ratios, etc; differences between the two shapes arise in the theoretical expressions for evaluating the sensitivity constants.

The final step in the practical analysis of loops as probes is to obtain expressions for the sensitivity constants. Consider first the square loop. $Y_{0}$, required in the definition of $S_{B}$ in (8.98), may be found from (8.96) by an expansion of the exponential in a power series. The result for a loop of side $w$ and wire radius $a$ is [23]

$$
\begin{equation*}
Y_{0}=\frac{-j \pi}{\zeta_{0} \beta_{0} w\left(\Omega-4.32+0.37 \beta_{0}^{2} w^{2}\right)} \tag{8.114}
\end{equation*}
$$

where $\zeta_{0}=\sqrt{\mu_{0} / \varepsilon_{0}} \doteq 120 \pi$ ohms and $\Omega=2 \ln (4 w / a)$. Hence

$$
\begin{equation*}
S_{B}=\frac{-\pi w}{\lambda \zeta_{0}\left(\Omega-4 \cdot 32+14 \cdot 6 w^{2} / \lambda^{2}\right)} \tag{8.115}
\end{equation*}
$$

The unloaded electric sensitivity $S_{E}$ is defined by (8.100) in terms of $h_{e D}$ and $Y_{D}$. The effective length for the dipole mode is found by cutting the loop at $s= \pm l / 4$, treating the two halves as a transmitting array, and applying the Rayleigh-Carson reciprocal theorem. $\dagger$ The result is

$$
\begin{equation*}
h_{e D}=\frac{2}{I_{0}} \int_{0}^{I / 4} I(s) d s \tag{8.116}
\end{equation*}
$$

where $I(s)$ is the transmitting current when the array is driven with codirectional currents, and $I_{0}$ is its value at the driving point. To zero order, this current is

$$
\begin{equation*}
I(z) \approx \frac{j 2 \pi V}{\zeta(\Omega-3 \cdot 17)} \frac{\sin \beta_{0}(w-|z|)}{\cos \beta_{0} w} \tag{8.117}
\end{equation*}
$$

With (8.117), (8.116) becomes

$$
\begin{equation*}
h_{e D}=\frac{\cos \frac{1}{2} \beta_{0} w-\cos \beta_{0} w}{\beta_{0} \sin \frac{1}{2} \beta_{0} w} \tag{8.118}
\end{equation*}
$$

$+[6]$, p. 568.

The input admittance is $\dagger$

$$
\begin{equation*}
Y_{D}=\frac{j 2 \pi \tan \beta_{0} w}{\zeta(\Omega-3 \cdot 17)} \tag{8.119}
\end{equation*}
$$

It follows that for the square loop,

$$
\begin{equation*}
S_{E}=\frac{j}{\zeta_{0}(\Omega-3 \cdot 17)} \frac{\tan \beta_{0} w\left(\cos \frac{1}{2} \beta_{0} w-\cos \beta_{0} w\right)}{\sin \frac{1}{2} \beta_{0} w} \tag{8.120}
\end{equation*}
$$

In order to calculate the sensitivity constants $S_{E}, S_{B}$ for loaded loops, note that $Y^{(1)}=Y_{D} / 2$ and $Y^{(0)} \approx Y_{0}$ so that the values from (8.119) and (8.114) can be directly substituted into the following equations for $S_{B}^{(2)}$ and $S_{E}^{(2)}:(8.106),(8.107)$, (8.111c) and (8.111d).

For a circular loop of diameter $w$ and wire radius $a$,
and

$$
\begin{align*}
& Y_{0}=\frac{-j 4}{\zeta_{0} \beta_{0} w\left(\Omega-3 \cdot 52+0.33 \beta_{0}^{2} w^{2}\right)}  \tag{8.121}\\
& S_{B}=\frac{-\pi w}{\lambda \zeta_{0}\left(\Omega-3.52+13 \cdot 0 w^{2} / \lambda^{2}\right)} \tag{8.122}
\end{align*}
$$

where $\Omega=2 \ln (\pi w / a)$.
The unloaded electric sensitivity is found from the response of the loop to an electric field that is uniform in the plane of the loop and pointing in the $z$-direction. The result is $\ddagger$

$$
\begin{equation*}
I(\phi)=E_{z 0}^{i} \frac{w}{j \zeta_{0} a_{1}} \cos \phi \tag{8.123a}
\end{equation*}
$$

where $\phi$ is the angular coordinate measured from the $y$-axis and $a_{1}$ is an expansion parameter calculated by Storer [26]. For $w \leqslant 0 \cdot 1 \lambda$,
and

$$
\begin{gather*}
a_{1} \doteq-(\Omega-3 \cdot 52)\left(1-\beta_{0}^{2} w^{2} / 4\right) / \pi \beta_{0} w  \tag{8.123b}\\
S_{E}=\frac{j 2 \pi^{2} w^{2} / \lambda^{2}}{\zeta_{0}(\Omega-3 \cdot 52)\left(1-9 \cdot 8 w^{2} / \lambda^{2}\right)} \tag{8.123c}
\end{gather*}
$$

for the circular loop. Zero and first-phase-sequence admittances are again needed for evaluating the sensitivity constant for the loaded loop. They can be found from Storer's results [26]. The expressions are complicated but subject to the conditions $w \leqslant 0.03 \lambda$ and $Y_{L}>10 Y^{(1)}, Y^{(0)}$ and $Y^{(1)}$ are

$$
\begin{align*}
Y^{(0)} & \doteq-j 2 \lambda\left[\pi w \zeta_{0}(\Omega-3 \cdot 52)\right]^{-1}  \tag{8.124a}\\
Y^{(1)} & \doteq j 4 \pi w\left[\lambda \zeta_{0}(\Omega-3 \cdot 52)\right]^{-1} \tag{8.124b}
\end{align*}
$$

Generally (8.114)-(8.124b) provide quite accurate results for loop

$$
\dagger[6], \text { p. } 568 .
$$

$\ddagger[25]$, chapter 10 .
diameters or sides $w \leqslant 0.03 \lambda$ and serve as a useful guide for $w \leqslant 0.1 \lambda$. When $w \leqslant 0.03 \lambda$, most of the expressions can be simplified. Note that $Y^{(0)} \gg Y^{(1)}$. They are summarized in Table 8.1.

Table 8.1. Probe characteristics of electrically small loops
Square loop, side $w . w \leqslant 0: 03 \lambda$

$$
\begin{gathered}
S_{B}^{(1)}=\frac{Y_{L}}{Y_{L}+Y^{(0)}} \frac{-\pi w}{\lambda \zeta_{0}(\Omega-4 \cdot 32)} \quad S_{E}^{(1)}=\frac{Y_{L}}{Y_{L}+Y^{(0)}} \frac{j 3 \pi^{2} w^{2}}{\lambda^{2} \zeta_{0}(\Omega-3 \cdot 17)} \\
\varepsilon^{(1)}=-j 3 \pi \frac{w}{\lambda} \frac{\Omega-4 \cdot 32}{\Omega-3 \cdot 17} \\
Y^{(0)}=-j \lambda\left[2 w \zeta_{0}(\Omega-4 \cdot 32)\right]^{-1} \quad Y^{(1)}=j 2 \pi^{2} w\left[\lambda \zeta_{0}(\Omega-3 \cdot 17)\right]^{-1} \\
\Omega=2 \ln (4 w / a)
\end{gathered}
$$

Circular loop, diameter $w . w \leqslant 0.03 \lambda$

$$
\begin{gathered}
S_{B}^{(1)}=\frac{Y_{L}}{Y_{L}+Y^{(0)}} \frac{-\pi w}{\lambda \zeta_{0}(\Omega-3 \cdot 52)} \quad S_{E}^{(1)}=\frac{Y_{L}}{Y_{L}+Y^{(0)}} \frac{j 2 \pi^{2} w^{2}}{\lambda^{2} \zeta_{0}(\Omega-3 \cdot 52)} \\
\varepsilon^{(1)}=-j 2 \pi \frac{w}{\lambda} \\
Y^{(0)}=-j 2 \lambda\left[\pi w \zeta_{0}(\Omega-3 \cdot 52)\right]^{-1} \quad Y^{(1)}=j 4 \pi w\left[\lambda \zeta_{0}(\Omega-3 \cdot 52)\right]^{-1} \\
\Omega=2 \ln (\pi w / a)
\end{gathered}
$$

Square and circular loops. $w \leqslant 0.03 \lambda$

$$
\begin{gathered}
S_{B}^{(2)}=2 \frac{Y_{L}+Y^{(0)}}{Y_{L}+2 Y^{(0)}} S_{B}^{(1)} \quad S_{E}^{(2)}=2 \frac{Y_{L}+Y^{(0)}}{Y_{L}+2 Y^{(1)}} S_{E}^{(1)} \\
\varepsilon^{(2)}=\frac{Y_{L}+2 Y^{(0)}}{Y_{L}+2 Y^{(1)}} \varepsilon^{(1)}
\end{gathered}
$$

The simplified relations of Table 8.1 reveal that the error ratio $\varepsilon^{(1)}$ of singly-loaded probes is independent of the load and approximately a linear function of the length of the side of square probes and of the diameter for circular probes. The magnetic fields measured with a circular loop will have an error less than $10 \%$ provided $w \leqslant 0.016 \lambda$. At 600 MHz this corresponds to $w \sim 0.5 \mathrm{~cm}$; at 3000 MHz to $w \sim 1 \mathrm{~mm}$. It is obviously advantageous to make such measurements at frequencies below 1000 MHz .

Sensitivities and error ratios as functions of loop size are shown in Fig. 8.13 for typical square loops, in Fig. 8.14 for circular loops. The sensitivities are in dB referred to 1 mho , the error ratios are in dB referred to 1 , and magnitudes are absolute, phases are relative. Important characteristics of the graphs are the relatively slow increase in sensitivity as $w / \lambda$ increases beyond about 0.03 or 0.04 , and the minimum of $\varepsilon^{(2)}$ at about $w / \lambda=0.04$, indicating that this size may be a good compromise for probes. From the curves of

Figs. $8.13 c$ and $8.14 c$, a singly-loaded loop with dimensions as large as $w / \lambda=0.1$ is seen to respond nearly as well to the electric field in the dipole mode as to the normal magnetic field in the circulating mode.

A comparison of the theoretical and experimental results in Figs. 8.13-8.14 shows good agreement and suggests that the theoretical results are adequate guides for the design of probes. Graphs of the limiting loop sizes and wire thicknesses required to keep error ratios below given limits are in Fig. 8.15 for singly-loaded loops, in Fig. 8.16 for doubly-loaded loops. In Fig. 8.16, $\gamma$, the crosscoupling coefficient between the adding and subtracting circuits, is assumed to be 1 ; in an actual system it will be at least -20 dB , reducing the indicated error ratio by this amount. Since the effects of changes in wire thickness and load resistance are similar for circular and square loops, the curves of Figs. 8.15 and 8.16 provide a useful qualitative guide for the former.

### 8.7 Construction and use of field probes

In many applications probes are used either in a free-standing arrangement or in conjunction with an image plane. $\dagger$ In the former, shown in Fig. 8.17, the probe is supported at the end of a long rigid tube which contains the feeder lines to the receiving equipment. The supporting tube is attached by a movable carriage to a track on a pivoting arm that permits the accurate placement and orientation of the probe. A loop is usually mounted with its plane perpendicular to the axis of the supporting tube. The free-standing arrangement is versatile and useful for measuring near-zone fields and surface currents on three-dimensional models. A principal disadvantage is that the supporting tube is always present in the field. Its disturbing effects can be reduced with quarter-wave sleeves and absorbing material. An alternative procedure incorporates a rectifying crystal directly in the probe and makes use of resistive wire to measure the d.c. voltage.

The image-plane arrangement, Fig. 8.18, is well-suited for measuring the surface currents on symmetrical models that can themselves be mounted on an image plane [28]. It has the advantage that all cables and supports are contained within the metal walls of the object under investigation or behind the image plane.

[^25]

Fig. 8.13a. Typical magnetic sensitivity of square loops.


Fig. 8.13b. Typical electric sensitivity of square loops.


Fig. 8.13c. Typical error ratios of square loops.


Fig. 8.14a. Typical magnetic sensitivity of circular loops.


Fig. 8.14b. Typical electric sensitivity of circular loops.


Fig. 8.14c. Typical error ratios of circular loops.


Fig. 8.15. Maximum dimension of singly-loaded square loop for given error ratio $\left|\varepsilon^{(1)}\right|$. Theoretical curves independent of $Z_{L}$ (Whiteside).

The probe is mounted at the end of a tube that serves both to move the probe and as the outer conductor of a coaxial line that connects the probe to its receiving system. The entire assembly is contained within a second slotted tube that serves to guide the probe at a constant height along the slot. In an alternative arrangement that permits the probe to move more easily along curved surfaces, the probe is mounted in a short cylindrical block, and a flexible feed line is used in conjunction with only the outer slotted tube, which can be bent to guide the probe along the surface. If the slot is parallel to the direction of the current, it has no significant effect. If it cuts the lines of flow, it must be covered with conducting tape except in the immediate vicinity of the probe.

Examples of probes that may be used in a free-standing arrangement are shown in Fig. 8.19. A balanced charge probe consists of an electrically short dipole formed by $90^{\circ}$ bends in the conductors of


Fig. 8.16. Maximum dimension of doubly-loaded square loop for given error ratio $\left|\varepsilon^{(2)}\right| \leqslant 1$. Theoretical curves, load impedance $Z_{L}=60 r$ ohms (Whiteside).


Fig. 8.17. Arrangement for free-space probe measurements.


Fig. 8.18. Arrangement for probe measurements on an image plane.
a two-wire transmission line, or by bends in the inner conductors of a shielded-pair line or in a pair of adjacent miniature coaxial lines (Fig. 8.19a). Loop probes (Figs. $8.19 b$ and $8.19 c$ ) are made onehalf from a solid brass rod and one-half from a miniature rigid coaxial line. At the junction, which is the location of the load, a small gap is left in the outer conductor and insulator of the coaxial line. Typical gap widths are $1-2 \mathrm{~mm}$. The load $Z_{L}$ is the impedance seen looking into the coaxial line at the gap. For singly-loaded loops, the gap is located symmetrically with respect to the axis of the supporting tube. Typical dimensions of the coaxial line are: outer diameter 0.032 inch, wall thickness 0.004 inch, and an inner conductor of 34 gauge wire. The characteristic impedance is 50 ohms. With doubly-loaded loops, the vertical supporting tube causes some degradation of the error ratio and the bridged loop shown in Fig. 8.19d may be more satisfactory. The mechanical construction must preserve a high degree of symmetry.

(c)

Fig. 8.19. Probes for free-space system. (a) Balanced charge probes; (b) singly-loaded square loop; (c) doubly-loaded circular loop; (d) bridged loop.

Probes for use with the image-plane technique (Fig. 8.20) have the same basic construction as those just described, except that the charge probes are usually monopoles and the current probes are half-loops. The half-loops may be either singly- or doubly-loaded.

The probes just described are shielded loops. Open-wire loops could be used instead of shielded loops, $\dagger$ but they are less convenient in the elimination of dipole-mode currents. With shielded loops the currents induced on the outside of the shield maintain

[^26]

Fig. 8.20. Probes for image-plane equipment. (a) Charge probe; (b) current probe; (c) probe socket.
a potential difference across the gap which is small and located at the point where dipole-mode currents vanish. If the gap is not at this point a dipole-mode voltage is also developed across the load.

Several simple tests can be used to reveal the sensitivity of loop probes to dipole-mode currents. $\dagger$ If the current in the load circuit of the probe is constant under a $180^{\circ}$ rotation of the probe, dipolemode currents are negligible. If the load current is not constant, readings must be taken with the probe in each position and averaged. With the image-plane technique, the probe cannot be rotated but can be tested in a short-circuited coaxial line. Let the probe current be measured with the probe successively at $w=w_{0}=\lambda / 4$ and $w=w_{1}=\lambda / 2$. Then

$$
\varepsilon_{\mathrm{db}}^{(1)}=20 \log \left|I_{L}\left(w_{0}\right) / I_{L}\left(w_{1}\right)\right| .
$$

When doubly-loaded probes are used with summing and differencing circuits, the output currents must be balanced because neither the probe nor the attached lines and loads are perfectly $\dagger[20]$, chapter VII, p. 5.
symmetrical. For this purpose, variable attenuators and phase shifters (or line stretchers) are used in the feed lines. The differencing circuit may be balanced by placing the probe so that it is symmetrically excited and adjusting the attenuators and phase shifters until the difference current is constant under $180^{\circ}$ rotations of the probe. An alternative procedure is to measure individually the difference currents $I_{\Delta 1}$ and $I_{\Delta 2}$ due to the probe loads 1 and 2 when the probe is placed so that it is symmetrically excited. The probe is rotated $180^{\circ}$ and the new currents $I_{\Delta 1}^{\prime}$ and $I_{\Delta 2}^{\prime}$ are measured, and the attenuators and phase shifters adjusted until

$$
\begin{equation*}
I_{\Delta 2} / I_{\Delta 1}=I_{\Delta 1}^{\prime} / I_{\Delta 2}^{\prime} . \tag{8.125}
\end{equation*}
$$

Similar procedures can be used to balance the summing circuits. If a single hybrid junction is used to provide the outputs for both the sum and difference, the balancing adjustment cannot be optimized for both arms simultaneously, but a satisfactory compromise can usually be found.

The measurement of surface distributions of current and charge on good conductors actually involves the measurement of magnetic and electric fields near the surface. Most of the current in a good conductor at high frequencies is concentrated within a very small distance of the surface, $d_{s}$, called the skin depth and given by $\dagger$

$$
\begin{equation*}
d_{s}=(2 / \omega \mu \sigma)^{1 / 2} \tag{8.126}
\end{equation*}
$$

Such a thin layer of current is well approximated by the surface density $K$ on a perfect conductor and is related to the total magnetic field at the surface by the boundary condition

$$
\begin{equation*}
\hat{\mathbf{n}} \times \mathbf{B}=\hat{\mathbf{t}} B_{t}=-K \mu_{0} . \tag{8.127}
\end{equation*}
$$

Similarly, the surface charge $\eta$ is related to the total electric field by

$$
\begin{equation*}
\hat{\mathbf{n}} \cdot \mathbf{E}=E_{n}=-\eta / \varepsilon_{0} \tag{8.128}
\end{equation*}
$$

where $\hat{\mathbf{n}}$ is an outward unit normal from the surface. On thin cylinders, $\mathbf{K}$ and $\eta$ have no angular variation around the cylinder, so that the total axial current and charge per unit length are $\mathbf{I}(z)=2 \pi a \mathbf{K}(z)$ and $q(z)=2 \pi a \eta(z)$. Except near the ends or edges of conductors distributions of $B_{t}$ and $E_{n}$ are often unchanged at very small fractions of a wavelength from the surface, so that probes placed sufficiently near the surface and moved parallel to it sample fields which are proportional to $\mathbf{K}$ and $\eta$. In the imageplane method, the effective centre of the probe is usually quite

[^27]near the surface; in the free-standing method, it is at least a probe radius away. At distances from a surface that are less than a few probe diameters, a probe is tightly coupled to its image so that its distance from the surface must be kept constant. Since the coupling between a probe and an ideal image does not exist near edges and corners, meaningful measurements cannot be made.

The most difficult fields to measure are linearly polarized magnetic fields associated with elliptically polarized electric fields. Figures 8.21-8.23 show the effects of size and orientation of probes in such fields and illustrate the effective use of singly- and doublyloaded loops to make measurements. The near-zone elliptically polarized electric field of a quarter-wave monopole over an image plane is shown in confocal coordinates in Fig. 8.21a. $\dagger$ In Fig. 8.22a are graphs of measurements made along the coordinate $k_{e}=2$ with a singly-loaded square loop with $w / \lambda=0.013$, oriented in the four positions indicated in Fig. 8.21b. Owing to its small size, this loop was relatively insensitive to dipole-mode fields and its orientation with respect to the electric field was not critical. In contrast, for a loop with $w / \lambda=0 \cdot 1$, the orientation is seen in Fig. 8.23a to be very important. When the probe is oriented in the positions marked 'In' and 'Out' the dipole-mode current excited by the component $E_{\varepsilon}$ maintains a voltage across the load, that excited by $E_{\rho}$ does not. In the positions marked 'Right' and 'Left', the dipole-mode currents due to $E_{\rho}$ maintain a voltage across the load, those due to $E_{\varepsilon}$ do not. Since $E_{\varepsilon}$ is nearly proportional to $B_{\Phi}$, whereas $E_{\rho}$ is not, no significant error is introduced in a relative measurement with the probe in the 'In' and 'Out' positions, a very large error in the 'Right' and 'Left' positions. The doubly-loaded loop with its summing and differencing circuits is seen to provide accurate results regardless of orientation even for sizes as large as $w / \lambda=0 \cdot 1$.

### 8.8 Equipment for measuring amplitude and phase

Measurements of the relative amplitudes of fields, currents, and charges are straightforward when the sampling probes are correctly used. The requirements are a signal source with sufficient power and stability in frequency and amplitude during a measurement, and a receiving system that is accurate and obeys a square law or is linear over the relevant range of power levels. For trans-mission-line measurements, a signal source must be isolated from

[^28]


Right


Left


Out


In
(b)

Fig. 8.21. (a) Elliptically polarized electric field near quarter-wave monopole over an image plane; (b) probe orientations for measurements of Figs. 8.22 and 8.23.
the line by at least 10 dB in order that its output be independent of changes in the apparent load impedances. The isolation can be provided by attenuators, resistive cables, isolators, or loosely coupled probes. Bolometers and barretters obey a square law at powers below their saturation and burn-out points. Crystals such as


Fig. 8.22. Relative magnetic field measured with small probe (square loop) (Whiteside).
the 1 N 21 or 1 N 23 are more sensitive than bolometers by about $15-20 \mathrm{~dB}$ but may have response laws with exponents either greater or less than 2 and several crystals may have to be tested before a satisfactory one is found. The superheterodyne receiving system is most sensitive but is susceptible to interference from stray leakage signals. This can be reduced by shielding the receiver with a metal screening, covering coaxial connectors and other possible sources


Fig. 8.23. Relative magnetic field measured with large probe (square loop) (Whiteside).
of spurious radiation with aluminium foil, and by a careful arrangement of the parts of the circuit. The components required for amplitude measurements with a superheterodyne or amplitudemodulated system are indicated in the block diagrams of Figs. $8.24 a$ and $8.24 b$.

The complete receiving system (consisting of a probe, detector, amplifiers, and a display meter or recorder) can be calibrated by

measuring the distance between half-power points of the standingwave pattern on a transmission line that is terminated with a short circuit. For $Z_{a}=0,(8.54)$ and (8.55) give

$$
\frac{I(w)}{I(0)}=\cos \beta w ; \frac{V(w)}{\left[I(0) R_{c}\right]}=j \sin \beta w=j \cos \beta\left(w+\frac{\lambda}{4}\right) .
$$

Power is proportional to $I^{2}$ or $V^{2}$ and hence to $\cos ^{2} \beta w$ or $\cos ^{2} \beta(w+\lambda / 4)$. In either case, if the power level as indicated on the output meter is $P$ and the exponent of the response law of the system is $n$,

$$
P=P_{\max } \cos ^{n}\left(\frac{2 \pi d}{\lambda}\right)
$$

where $d$ is distance measured from a maximum. Let $\Delta w$ be the total distance between points at which $P / P_{\max }=1 / 2$; then, the response law is

$$
\begin{equation*}
n=\frac{\log \frac{1}{2}}{\log \cos \left(\frac{\pi \Delta w}{\lambda}\right)} \tag{8.129}
\end{equation*}
$$


(b)

Calibrated receiving system

Fig. 8.24. (a) Amplitude-modulated system for amplitude measurements; (b) superheterodyne system for amplitude measurements.

Techniques for measuring phase are commonly based on the interference of two signals from the same transmitter. The principal signal is fed to the system under investigation, sampled by a probe, and returned to the receiver. The reference signal, obtained usually from a directional coupler, is fed through an adjustable attenuator and a precision phase shifter and then combined with the signal from the probe. If the signal from the probe is $e_{1}(t)=U \cos \omega t$ and the reference signal is $e_{2}(t)=R \cos \left(\omega t+\phi_{d}\right)$, where $\phi_{d}$ is the difference in phase or electrical path length between the signals, it follows that

$$
\begin{gather*}
e(t)=e_{1}(t)+e_{2}(t)=\begin{array}{l}
U \cos \omega t+R \cos \left(\omega t+\phi_{d}\right)=[U-R] \cos \omega t, \\
\phi_{d}=(2 n+1) \pi .
\end{array}, ~
\end{gather*}
$$

A minimum is observed when $\phi_{d}$ is an odd integral multiple of $\pi$ and a null when, in addition, the amplitudes are equal. In practice, the difference signal is displayed on a meter and, for each probe position, the phase shifter is adjusted until a minimum is indicated by the meter. As in determining transmission-line minima, readings may be taken at equal power levels on each side of a minimum
and averaged to increase accuracy. A superheterodyne receiving system that is based on these principles and designed for use with doubly-loaded probes is shown in the block diagram of Fig. 8.25a. A comparable arrangement for singly-loaded probes and amplitude modulation is given in Fig. 8.25b. Doubly-loaded probes permit accurate measurements with one mechanical adjustment of the probe, and the superheterodyne system provides high sensitivity that permits precise adjustments of the phase but, as is evident from the figures, at a considerable additional complexity.

An effective precision phase shifter can be made from a highquality constant-impedance line stretcher or alternatively the power along a well-matched slotted line can be sampled with an r.f. probe. In the latter case the phase is a linear function of the position of the probe along the line.

Coaxial hybrid junctions are convenient devices for combining the two signals. In a perfect coaxial hybrid junction, a signal fed into one terminal divides equally between the two opposite terminals with $0^{\circ}$ change in phase at the far terminal and $90^{\circ}$ change in phase at the nearest opposite terminal. No signal appears at the adjacent terminal. If the probe signal and the reference signal are fed into adjacent terminals and the electrical path length of the reference signal is properly adjusted, the two signals will be in phase and add at one of the two opposite terminals, while at the other they will be in opposite phase and subtract. The loads at all four terminals of the hybrid junction must be carefully matched.

When carefully used, the procedure just described is capable of yielding accurate results, but it has the serious inconvenience that the depth of the null in the difference signal depends on the relative amplitudes of the input signals. Since deep nulls are required for high precision, frequent readjustment of an attenuator in the phase reference line is necessary. Unfortunately, attenuators either introduce shifts in phase or have very high insertion losses. Figure 8.26 shows a system that avoids this difficulty and produces a deep null independently of the relative amplitude of the two input signals. $\dagger$

In the arrangement of Fig. 8.26, let the signal from the probe be $e_{1}(t)=U\left(1+m \cos \omega_{m} t\right) \cos \omega t$, that from the phase-reference line $e_{2}(t)=R\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\phi_{d}\right) ; \phi_{d}$ is their phase difference,

(b)

Fig. 8.25. (a) Phase-sensitive superheterodyne receiving equipment for use with doublyloaded probes (Whiteside); (b) equipment for phase measurements with amplitude-modulated systems.


Fig. 8.26. Amplitude-insensitive phase detector (Burton).
$m$ is the modulation factor, and $\omega_{m}$ is the modulation frequency. The hybrid junction combines the signals to give
$e_{3}(t)=\frac{R}{2}\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\phi_{d}+\frac{\pi}{2}\right)+\frac{U}{2}\left(1+m \cos \omega_{m} t\right) \cos \omega t$
$e_{4}(t)=\frac{R}{2}\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\phi_{d}\right)+\frac{U}{2}\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\frac{\pi}{2}\right)$.
After detection by the square-law crystals, the signals can be represented as

$$
\begin{aligned}
e_{5}(t)= & \frac{a_{2}^{\prime}}{4}\left[R\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\phi_{d}+\frac{\pi}{2}\right)\right. \\
& \left.+U\left(1+m \cos \omega_{m} t\right) \cos \omega t\right]^{2} \\
e_{6}(t)= & \frac{a_{2}}{4}\left[R\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\phi_{d}\right)\right. \\
& \left.+U\left(1+m \cos \omega_{m} t\right) \cos \left(\omega t+\frac{\pi}{2}\right)\right]^{2}
\end{aligned}
$$

The coefficients $a_{2}$ and $a_{2}^{\prime}$ can be equated by the balancing potentiometer. With balanced currents, the output of the i.f. transformer circuit is

$$
\begin{equation*}
e_{7}(t)=a_{2} R U_{m} \sin \phi_{d} \cos \omega_{m} t \tag{8.131}
\end{equation*}
$$

When $\phi_{d}$ is an integral multiple of $\pi, e_{7}(t)=0$ for all values of $R$ and $U$ that differ from zero.
The principal errors with this procedure come from improper balancing of the coefficients $a_{2}$ and $a_{2}^{\prime}$, mismatches looking into the crystals, and a standing wave on the phase-reference line. With either system it is important that the phase-reference line have no standing wave. A SWR of 1.1 to 1.5 can introduce phase errors of $2.5^{\circ}$ to $12^{\circ}$.

More sophisticated techniques for measuring phase are available [30]. The procedures outlined above are relatively simple and capable of precise results; they are convenient for general laboratory use. Additional details of the individual components are discussed in [1] and [2].

### 8.9 Measurement of radiation patterns

The radiation pattern of an antenna is one of the most important and frequently measured parameters. In the usual procedure the test antenna is mounted at a distance $R=2 D^{2} / \lambda$ from a transmitting antenna and the received power is recorded as a function of the angular intervals of interest. $D$ is the larger of the maximum dimensions of the test antenna and the transmitting antenna. $R$ is chosen to satisfy the requirements for far-field conditions. Its minimum value implies that the incident field closely approximates a plane wave in the volume occupied by the test antenna during the measurement. Also $D / R \ll 1$. The quantity actually measured is the square of the magnitude of the generalized effective length of the receiving antenna. $\dagger$ As discussed in section 6.10 the reciprocal theorem shows that this is the same as the radiation pattern of the antenna when used for transmitting. Occasionally, measurements of the magnitude and phase of the far-zone fields are required. The equipment for either measurement is substantially that described in section 8.8 with the probe replaced by a receiving antenna.

The gain of an antenna in a given direction may be defined as the ratio of the magnitude of the Poynting vector $|S|=\left|E^{2}\right| / 2 \zeta_{0}$
maintained in that direction by the test antenna to the magnitude of $S$ at the same location due to a fictitious isotropic radiator when both have the same input power. It may be measured by comparing the power received on the test antenna to the power received on a reference antenna when both are in the same transmitted field. The reference antenna is one for which the gain relative to an isotropic radiator can be computed. Wave-guide horns and dipoles near resonance are useful for this purpose. The gain of a sufficiently thin half-wave dipole compared to an isotropic radiator is 1.64 or $2 \cdot 15 \mathrm{~dB}$.

When the radiated fields of antennas or arrays are not very directive, there are two potential sources of error in measuring their radiation patterns and gain. The first arises from the coupling of the array to its mount, the second is due to extraneous reflexions of the transmitted field, notably from the ground. Interference from ground reflexions can be reduced by arranging the transmitting antenna with a null of its pattern along the ground and by mounting it on a sufficiently high tower. It can be eliminated by an imageplane form of testing site in which the antennas are mounted close to a level conducting earth. For grazing incidence, the reflexion coefficient of the forward reflexion is close to -1 and the vertical variation of the field at the test antenna can be represented by

$$
\begin{equation*}
E\left(h_{r}\right)=E_{0} \sin \left(\frac{2 \pi h_{r} h_{t}}{\lambda R}\right) \tag{8.132}
\end{equation*}
$$

where $h_{r}$ and $h_{t}$ are the heights of the test antenna and the transmitting antenna, respectively, and $R$ is the distance between them. The test antenna is located at the lowest maximum given by (8.132). This type of testing facility is particularly convenient for use on a flat roof. $\dagger$

A complete image-plane technique [33] is useful for investigating all of the properties-radiated fields, gain, current distributions, and apparent driving-point impedances or admittances - of small arrays of monopoles. In this arrangement, the test array is mounted on a large circular disk that is located several wavelengths from the nearest edges of a large rectangular ground screen; an auxiliary antenna is mounted near one corner. A monopole with corner reflector makes a satisfactory auxiliary antenna that may be used

[^29]

Fig. 8.27. Combined monopole antenna and measuring line; $(a)$ assembled line; $(b)$ parts detail ; dimensions are in inches.
either for transmitting or receiving. Radiation patterns are measured by rotating the large circular disk which is supported on rollers.

If both current distributions along the monopoles and the driving-point admittances are to be measured, a combined antenna and measuring line similar to that shown in Fig. 8.27 is suitable. This model was designed to investigate monopoles with lengths up to $h / \lambda=3 / 4$ at frequencies near 650 MHz . It was part of the fiveelement circular array used to obtain the measurements discussed in chapter 4 . The antenna is a slotted cylinder containing a coaxial line for the probe as discussed in section 8.7. It extends several wavelengths below the image plane inside an additional tube that forms the outer conductor of a second slotted line for measuring admittances. The assembly is mounted vertically below the circular disk in the image plane. When measurements of current and charge distributions are not required, an antenna may be a solid rod and the outer coaxial line need extend only a wavelength below the image plane. When terminated in a suitable connector and fitted with fixed voltage probes it becomes the measuring section discussed in section 8.5.

A minimum permissible size for a ground screen is difficult to specify. Edge reflexions are less troublesome with rectangular shapes than with square or circular ones [34], and their effects can be reduced by locating the test antenna at unequal distances from all edges. The rectangular screen (dimensions: $16 \lambda \times 32 \lambda$ or $24 \times 38$ feet) used for the measurements discussed in chapter 4 produced no measurable interference in either the radiation patterns or driving-point admittances when the circular disk was located about six wavelengths from the nearest edge.

## APPENDIXI

## Tables of $\Psi_{d R}, T^{(m)}$ or $T^{\prime(m)}$ <br> and

Self- and mutual admittances for single elements and circular arrays

Notation:
Self- and mutual admittances are written in the form

$$
Y_{1(m+1)}=G_{1(m+1)}+j B_{1(m+1)}
$$

with the self-admittance given by the row corresponding to $m=0$, the first mutual $Y_{12}$ by the row corresponding to $m=1$, etc. A factor of $10^{-3}$ has been suppressed in the admittances; hence, tabulated values are in millimhos. The characteristic impedance of free space, $\zeta_{0}$, was taken to be $\zeta_{0}=376.730$ ohms.

Table 1. $\Psi_{d R}, T(h)$ or $T^{\prime}\left(\frac{\lambda}{4}\right)$ and admittances for isolated antenna

$$
\left(N=1, a / \lambda=7.022 \times 10^{-3}\right)
$$

| $\beta_{0} h$ | $\frac{h}{\lambda}$ | $\Psi_{d R}$ | $T(h)$ <br> Real | $\begin{array}{r} \text { or } \quad T^{\prime}\left(\frac{\lambda}{4}\right) \\ \text { Imag. } \end{array}$ | $G_{0}$ | $B_{0}$ | $B_{0}+0.72$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1.200 | $0 \cdot 1910$ | 5-31670 | 0.27602 | $-0.74791$ | 4.12999 | 9.59504 | 10.315 |
| 1.350 | 0.2150 | 5.69058 | -0.37945 | $-1.16933$ | 12.28333 | 9.12585 | 9.846 |
| 1.432 | 02280 | 5-88844 | -0.98304 | -0.87529 | 15.51296 | 2.93684 | 3.657 |
| 1.501 | 0.2390 | 6.05385 | $-1.13164$ | $-0.36538$ | $13 \cdot 56922$ | -2.22898 | $-1.509$ |
| 1.570 | 0.2500 | 6.21771 | 2.65166 | 3.79157 | $10 \cdot 17040$ | -4.43037 | $-3.710$ |
| 1.652 | 0.2630 | 6.37511 | $-0.78390$ | 0.20461 | 7.09591 | -4.77136 | -4.051 |
| 1.796 | 0.2860 | 6.54009 | -0.51411 | $0 \cdot 30471$ | 4.24183 | -3.92426 | -3.204 |
| 1.997 | $0 \cdot 3180$ | 6.61380 | -0.32658 | 0.30590 | 2.63298 | $-2.72756$ | -2.008 |
| 2.355 | 0.3750 | 6.44947 | -0.19988 | 0.26239 | 1.63816 | -1.33809 | -0.618 |
| 2.751 | 0.4380 | 6-05835 | -0.16459 | 0.21601 | 1.23747 | $-0.18730$ | 0.533 |
| $3 \cdot 141$ | 0.5000 | 5.73687 | -0.17204 | $0 \cdot 17559$ | 1.02096 | 1.00032 | 1.720 |
| 3.398 | 0.5410 | 5.66947 | -0.19117 | $0 \cdot 15329$ | 0.91729 | 1.91899 | 2.639 |
| 3.649 | 0.5810 | 5.74850 | -0.22145 | $0 \cdot 14008$ | 0.87181 | 2.99706 | 3.717 |
| 3.800 | 0.6050 | 5.86161 | -0.24791 | 0.14155 | 0.91248 | 3.80520 | 4.525 |
| 3.926 | 0.6250 | 5.98717 | -0.27769 | 0.15443 | 1.03854 | 4.65317 | 5.373 |

Table 2. $\Psi_{d R}, T^{(m)}$ and admittances of circular array $\dagger$ of Nelements
$\left(a / \lambda=7.022 \times 10^{-3}\right)$
$h / \lambda=0.25, \quad \beta_{0} h=\pi / 2, \quad \Omega=8.54$


Table 2 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  | $\frac{d}{\lambda}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\Psi_{d \mathrm{R}}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ |  |  |
| 1 | 6.21771 | 6.32221 | 1.96537 | $5 \cdot 27186$ | -14.27616 | 0.00561 | 5.06306 | 0.1875 | 3 |
| 0 | 6.21771 | 0.33433 | $2 \cdot 29021$ | 6.14320 | 1.78558 | 5.70204 | -6.37056 | 0.2500 | 3 |
| 1 | 6.21771 | 4.89530 | 2.04351 | $5 \cdot 48146$ | -10.44863 | 0.22058 | 4.07807 | 0.2500 | 3 |
| 0 | 6.21771 | -0.08089 | 2.69241 | 7.22203 | 2.89935 | 6.23755 | -4.45146 | 0.3125 | 3 |
| 1 | 6.21771 | 4.02973 | 2.14188 | 5.74531 | -8.12687 | 0.49224 | 3.67541 | 0.3125 | 3 |
| 0 | 6.21771 | -0.62328 | 3.27474 | 8.78406 | 4.35423 | 6.96030 | -2.90297 | 0.3750 | 3 |
| 1 | 6.21771 | 3.43500 | 2.25488 | 6.04843 | -6.53158 | 0.91188 | 3.62860 | 0.3750 | 3 |
| 0 | 6.21771 | -1.33434 | 4.27609 | 11.47006 | $6 \cdot 26158$ | 8.08291 | -1.47282 | 0.4375 | 3 |
| 1 | 6.21771 | $2 \cdot 99078$ | 2.38197 | 6.38933 | -5.34001 | 1.69358 | 3.86720 | 0.4375 | 3 |
| 0 | 6.21771 | -2.06303 | 6.38546 | 17.12817 | 8.21617 | 10.22493 | -0.19230 | 0.5000 | 3 |
| 1 | 6.21771 | 2.63905 | 2.52512 | 6.77331 | -4.39654 | 3.45162 | $4 \cdot 20424$ | 0.5000 | 3 |
| 0 | 6.21771 | -0.52523 | 10.97641 | 29.44281 | 4.09124 | 14.62129 | -1.04959 | 0.5625 | 3 |
| 1 | 6.21771 | 2.34955 | $2 \cdot 68812$ | $7 \cdot 21053$ | -3.62001 | $7 \cdot 41076$ | 2.57042 | 0.5625 | 3 |
| 0 | 6.21771 | 7.51808 | 10.33528 | 27.72306 | -17.48391 | $14 \cdot 38485$ | - 7.80740 | 0.6250 | 3 |
| 1 | 6.21771 | 2.10691 | 2.87646 | 7.71574 | -2.96914 | 6.66911 | -4.83826 | 0.6250 | 3 |
| 0 | 6.21771 | 7.73778 | 4.56984 | 12-25800 | -18.07322 | 9.62469 | -7.64428 | 0.6875 | 3 |
| 1 | 6.21771 | 1.90584 | 3.09727 | $8 \cdot 30803$ | -2.42981 | 1.31666 | -5.21447 | 0.6875 | 3 |
| 0 | 6.21771 | 5.86472 | 2.79150 | 7.48783 | -13.04899 | 8.50211 | -5.69179 | 0.7500 | 3 |
| 1 | 6.21771 | 1.75053 | 3.35869 | 9.00926 | - 2.01319 | -0.50714 | $-3.67860$ | 0.7500 | 3 |
| 0 | 6.21771 | 0.40910 | 1.61526 | 4.33274 | 1.58500 | 4.23392 | - 9.94442 | 0.1875 | 4 |
| 1 | 6.21771 | 4.64950 | 2.06586 | $5 \cdot 54141$ | -9.78933 | 0.70316 | 5.84226 |  |  |
| 2 | 6.21771 | 9.12118 | 0.56670 | 1.52011 | -21.78403 | $-1.30749$ | -0.15509 |  |  |
| 0 | 6.21771 | -0.01844 | 1.89994 | 5.09635 | 2.73183 | 4.80273 | -6.56582 | $0 \cdot 2500$ | 4 |
| 1 | 6.21771 | 3.61790 | 2.21455 | 5.94025 | -7.02217 | 0.71557 | 4.42065 |  |  |
| 2 | 6.21771 | 6.57371 | 0.83287 | 2-23406 | -14.95077 | -1.13752 | 0.45635 |  |  |

Table 2 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m | $\Psi_{d R}$ | $\operatorname{Re} T^{\prime(m)}$ | $\operatorname{Im} T^{(m)}$ |  | $B^{(m)}$ | $G_{1(m+1)}$ | $\boldsymbol{B}_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.21771 | -0.63097 | $2 \cdot 25790$ | 6.05652 | 4.37485 | 5.44649 | -4.28694 | 0.3125 | 4 |
| 1 | 6.21771 | 2.96320 | $2 \cdot 39158$ | 6.41511 | $-5.26603$ | 0.78932 | 3.84135 |  |  |
| 2 | 6.21771 | 5.09732 | 1.08085 | 2.89924 | -10.99055 | -0.96862 | 0.97909 |  |  |
| 0 | 6.21771 | $-1.59127$ | $2 \cdot 81668$ | 7.55539 | 6.95074 | 6.25385 | $-2.36381$ | 0.3750 | 4 |
| 1 | 6.21771 | 2.49201 | 2.60144 | 6.97803 | -4.00212 | 1.01286 | 3.83811 |  |  |
| 2 | 6.21771 | $4 \cdot 13220$ | 1-30629 | 3.50394 | -8.40172 | -0.72418 | 1.63832 |  |  |
| 0 | 6.21771 | -3.34395 | 4.05396 | 10.87422 | 11.65209 | 7.56365 | -0.23979 | 0.4375 | 4 |
| 1 | 621771 | $2 \cdot 12940$ | 2.85620 | 7.66138 | -3.02948 | 1.70415 | 4.55110 |  |  |
| 2 | 6.21771 | $3 \cdot 44273$ | $1 \cdot 51270$ | 4.05763 | -6.55232 | -0.09773 | 2.78968 |  |  |
| 0 | 6.21771 | -6.93658 | $9 \cdot 19894$ | 24.67498 | 21.28884 | 11.57104 | 2.89748 | 0.5000 | 4 |
| 1 | 6.21771 | 1.85172 | $3 \cdot 17452$ | 8.51526 | -2.28462 | 5.02408 | 6.60463 |  |  |
| 2 | 6.21771 | 2.91237 | $1 \cdot 70695$ | 4.57867 | -5.12969 | 3.05578 | 5.18210 |  |  |
| 0 | 6.21771 | 12.59562 | 19.02475 | 51.03145 | -31.10376 | 18.82977 | -9.67092 | 0.5625 | 4 |
| 1 | 6.21771 | 1.67463 | $3 \cdot 57861$ | 9.59917 | - 1.80960 | 11.48554 | $-6.78576$ |  |  |
| 2 | 621771 | 2.47658 | 1.89731 | 5.08929 | $-3.96072$ | $9 \cdot 23060$ | $-7.86132$ |  |  |
| 0 | 6.21771 | 9.49153 | 3.80603 | 10.20918 | $-22.77744$ | 9.42201 | $-7.31992$ | 0.6250 | 4 |
| 1 | 6.21771 | 1.66405 | 4.07540 | 10.93173 | -1.78124 | $1 \cdot 14845$ | -4.95942 |  |  |
| 2 | 6.21771 | 2.09596 | 2.09344 | 5.61538 | $-2.93977$ | $-1.50973$ | -5.53868 |  |  |
| 0 | 6.21771 | $6 \cdot 13225$ | 2.43284 | 6.52578 | -13.76660 | 9.33672 | -5.19927 | 0.6875 | 4 |
| 1 | 6.21771 | 1.93857 | 4.59131 | 12.31559 | $-2.51759$ | 0.08397 | -2.94282 |  |  |
| 2 | 6.21771 | 1.74386 | $2 \cdot 30763$ | 6.18991 | -1.99530 | $-2.97887$ | -2.68168 |  |  |
| 0 | 6.21771 | 5.74651 | 2.36377 | $6 \cdot 34051$ | -12.73189 | 9.43757 | -5.02606 | 0.7000 | 4 |
| 1 | 6.21771 | 2.03672 | 4.67782 | 12.54766 | $-2.78086$ | 0.00652 | -2.73032 |  |  |
| 2 | 6.21771 | 1.67501 | $2 \cdot 35405$ | $6 \cdot 31443$ | -1.81063 | -3.11010 | $-2.24520$ |  |  |
| 0 | 6.21771 | 4.63831 | $2 \cdot 26944$ | 6.08749 | $-9.75930$ | 9.77373 | -4.78684 | 0.7500 | 4 |
| 1 | 6.21771 | 2.54993 | 4.87384 | 13.07345 | -4.15749 | -0.19326 | $-2.17155$ |  |  |
| 2 | 6.21771 | $1 \cdot 40005$ | $2 \cdot 55764$ | 6.86054 | -1.07309 | -3.29972 | $-0.62935$ |  |  |

Table 2 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{\prime(m)}$ | $\operatorname{Im} T^{\prime(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.21771 | 0-21514 | 1.40228 | 3.76143 | $2 \cdot 10529$ | 3.54866 | -10.26034 | 0.1875 | 5 |
| 1 | $6 \cdot 21771$ | 3.38329 | 1.96748 | $5 \cdot 27752$ | -6.39288 | 0.85015 | 6.20351 |  |  |
| 2 | $6 \cdot 21771$ | 8.57189 | 0.63877 | 1.71342 | -20.31061 | $-0.74377$ | -0.02069 |  |  |
| 0 | 6.21771 | -0.33422 | 1.66115 | 4.45583 | $3 \cdot 57888$ | 4.20839 | -6.53350 | 0.2500 | 5 |
| 1 | 6.21771 | 2.57552 | $2 \cdot 16720$ | $5 \cdot 81324$ | -4.22614 | 0.80724 | 4.69057 |  |  |
| 2 | 6.21771 | $6 \cdot 18088$ | 0.92448 | 2.47981 | -13.89704 | -0.68352 | 0.36562 |  |  |
| 0 | 6.21771 | $-1.21851$ | 1.99594 | $5 \cdot 35384$ | 5.95086 | 4.92644 | $-3.95320$ | 0.3125 | 5 |
| 1 | 621771 | 1.99727 | 2.40790 | 6.45887 | $-2.67505$ | 0.83996 | $4 \cdot 15493$ |  |  |
| 2 | 6.21771 | 4.79641 | 1.18564 | $3 \cdot 18031$ | -10.18338 | -0.62626 | 0.79710 |  |  |
| 0 | 6.21771 | -2.92127 | 2.65004 | $7 \cdot 10840$ | 10.51830 | $5 \cdot 86009$ | $-1.54167$ | 0.3750 | 5 |
| 1 | 6.21771 | $1 \cdot 50745$ | $2 \cdot 71553$ | 7.28405 | $-1.36117$ | 1.08846 | 4.44406 |  |  |
| 2 | 6.21771 | 3.89003 | 1.42112 | 3.81196 | $-7.75215$ | $-0.46430$ | 1.58593 |  |  |
| 0 | 6.21771 | -4.15869 | 3.23605 | 8.68029 | 13.83752 | 6.43347 | $-0.37358$ | 0.4000 | 5 |
| 1 | 6.21771 | $1 \cdot 32071$ | 2.86853 | 7.69445 | -0.86027 | $1 \cdot 37683$ | 4.92397 |  |  |
| 2 | 6.21771 | $3 \cdot 60681$ | 1.50951 | 4.04908 | -6.99243 | $-0.25343$ | $2 \cdot 18158$ |  |  |
| 0 | 6.21771 | -7.54050 | 5.70219 | $15 \cdot 29540$ | 22.90879 | 8.19364 | $2 \cdot 13276$ | 0.4375 | 5 |
| 1 | 6.21771 | 1.04301 | 3.14776 | 8.44346 | $-0.11536$ | 2.68116 | 6.51145 |  |  |
| 2 | 6.21771 | 3.23949 | 1.63771 | 4.39293 | -6.00714 | 0.86972 | 3.87657 |  |  |
| 0 | 6.21771 | -9.81216 | 8.71608 | 23.37976 | 29.00223 | 10.01690 | 3.62199 | 0.45313 | 5 |
| 1 | 6.21771 | 0.92770 | 3.28782 | 8.81917 | 0.19394 | $4 \cdot 29908$ | 7.64959 |  |  |
| 2 | 6.21771 | 3.10265 | 1.69000 | 4.53321 | $-5.64009$ | $2 \cdot 38234$ | 5.04053 |  |  |
| 0 | 6.21771 | -3.91456 | 26.67988 | 71.56534 | $13 \cdot 18266$ | $20 \cdot 12878$ | 0.97151 | 0.48438 | 5 |
| 1 | 6.21771 | 0.70033 | 3.62681 | 9.72844 | 0.80383 | 13.95875 | $4 \cdot 34305$ |  |  |
| 2 | 6.21771 | 2.85149 | 1.79350 | 4.81084 | -4.96639 | 11.75953 | 1.76253 |  |  |
| 0 | 6.21771 | 11.71804 | 22.64512 | 60.74263 | $-28.74975$ | 18-24252 | $-7.17352$ | 0.5000 | 5 |
| 1 | 6.21771 | 0.59141 | 3.83461 | 10.28586 | 1.09600 | 11.81836 | -4.10811 |  |  |
| 2 | 6.21771 | 2.73538 | 1.84506 | 4.94912 | -4.65492 | 9.43170 | -6.68000 |  |  |

Table 2 (continued)

| Sequence <br> m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{\prime(m)}$ | Sequence admittance |  | Self- and mutual admittances |  | $\frac{d}{\lambda}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ |  |  |
| 0 | 6.21771 | 13.94954 | 12.57851 | 33.74024 | -34.73548 | $13 \cdot 15613$ | $-8.14286$ | 0.51563 | 5 |
| 1 | 6.21771 | 0.48991 | 4.07569 | 10.93251 | 1.36826 | 6.45297 | $-5 \cdot 36780$ |  |  |
| 2 | 6.21771 | 2.62455 | 1.89671 | 5.08769 | -4.35766 | 3.83909 | $-7.92851$ |  |  |
| 0 | 6.21771 | 8.42858 | 4.54611 | $12 \cdot 19435$ | $-19.92622$ | 10.09068 | -4.65915 | 0.5625 | 5 |
| 1 | 6.21771 | 0.31035 | 5.07731 | 13.61924 | 1.84991 | 2.33913 | $-2.61274$ |  |  |
| 2 | 6.21771 | $2 \cdot 31774$ | 2.05426 | 5.51030 | $-3.53468$ | $-1.28730$ | $-5.02080$ |  |  |
| 0 | 6.21771 | $5 \cdot 44546$ | $3 \cdot 30165$ | 8.85624 | -11.92436 | 11.93591 | -3.68524 | 0.6250 | 5 |
| 1 | 6.21771 | 1.26107 | $7 \cdot 19498$ | 19.29960 | -0.70028 | $2 \cdot 17891$ | -1.64603 |  |  |
| 2 | 6.21771 | 1.95089 | $2 \cdot 27860$ | $6 \cdot 11205$ | $-2.55063$ | -3.71874 | $-2.47353$ |  |  |
| 0 | $6 \cdot 21771$ | 4.22835 | 3.18268 | 8.53713 | -8.65964 | 12.14514 | -6.63446 | 0.6875 | 5 |
| 1 | $6 \cdot 21771$ | 4.95667 | $7 \cdot 19221$ | 19.29217 | $-10.61325$ | 1.89086 | $-2.51209$ |  |  |
| 2 | $6 \cdot 21771$ | 1.61255 | $2 \cdot 53586$ | 6.80212 | $-1.64308$ | -3.69486 | 1.49950 |  |  |
| 0 | $6 \cdot 21771$ | 3.96685 | 3.20896 | 8.60762 | -7.95818 | 10.98986 | $-7.20681$ | 0.7100 | 5 |
| 1 | 6.21771 | $5 \cdot 73836$ | 5.99691 | 16.08594 | $-12.71004$ | 1.41714 | $-2.73297$ |  |  |
| 2 | $6 \cdot 21771$ | 1.49504 | 2.64128 | 7.08489 | -1.32788 | $-2.60826$ | 2.35729 |  |  |
| 0 | 6.21771 | 3.64504 | 3.29276 | 8.83240 | $-7.09499$ | 9.21245 | -6.73060 | 0.7500 | 5 |
| 1 | 6.21771 | 5.65993 | 4.08688 | 10.96254 | $-12.49968$ | 0.64516 | $-2.71185$ |  |  |
| 2 | 6.21771 | 1.29053 | 2.85284 | 7.65238 | -0.77932 | -0.83519 | 2.52965 |  |  |
| 0 | 14.83942 | 0.76543 | 1.60211 | 1.80064 | 0.26364 | $1 \cdot 12981$ | -44.14067 | 0.0625 | 8 |
| 1 | 8.04324 | $5 \cdot 28009$ | 1.57354 | 3.26285 | -8.87507 | 0.87168 | 24.88961 |  |  |
| 2 | 3.73120 | $9 \cdot 62409$ | 0.15395 | 0.68816 | -38.54906 | 0.01520 | $-2.53078$ |  |  |
| 3 | 4.39219 | 22.19073 | $-0.04765$ | $-0.18092$ | -80.46642 | $-0.34588$ | $-0.42171$ |  |  |
| 4 | 2.56902 | 16.03502 . | -0.04660 | $-0.30252$ | -97.60802 | $-0.41117$ | 0.53006 |  |  |
| 0 | 8.48656 | 0.21045 | 1-19329 | 2.34512 | 1.55166 | 1.77822 | $-18.48555$ | $0 \cdot 1250$ | 8 |
| 1 | 7.82203 | 2.78655 | 1.76586 | 3.76519 | -3.80931 | 0.93309 | 10.62944 |  |  |
| 2 | 6.21771 | $7 \cdot 19363$ | 0.75322 | 2.02043 | $-16.61362$ | -0.21774 | $-0.81075$ |  |  |
| 3 | 4.61340 | 10.31149 | $0.04917$ | 0.17777 | -33.66265 | $-0.33526$ | 0.07470 |  |  |
| 4 | 3.94887 | 10.77021 | $-0.01094$ | $-0.04619$ | -41.26493 | $-0.19327$ | 0.25043 |  |  |

Table 2 (continued)

| Sequence <br> m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{\prime(m)}$ | Sequence admittance |  | Self- and mutual admittances |  | $\frac{d}{\hat{\lambda}}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ |  |  |
| 0 | 6.21771 | $-0.30497$ | 1.04952 | 2.81521 | 3.50041 | 2.43903 | $-10.55638$ | 0.1875 | 8 |
| 1 | 6.21771 | 1.51389 | 1.60947 | 4.31719 | $-1.37844$ | 0.97192 | 6.99193 |  |  |
| 2 | 6.21771 | 4.52861 | $1 \cdot 20601$ | 3.23497 | -9.46505 | $-0.44869$ | -0.40912 |  |  |
| 3 | 6.21771 | $8 \cdot 56067$ | 0.28472 | 0.76372 | -20-28053 | -0.28443 | 0.30903 |  |  |
| 4 | 6.21771 | 10.58235 | 0.02433 | 0.06525 | -25.70343 | $-0.10143$ | 0.27310 |  |  |
| 0 | 621771 | $-1.61604$ | 1.26374 | $3 \cdot 38982$ | 7.01720 | $3 \cdot 18540$ | -6.05024 | 0.2500 | 8 |
| 1 | 621771 | 0.86391 | 1.85441 | 4.97421 | 0.36504 | 0.97095 | 5.48956 |  |  |
| 2 | 6.21771 | $3 \cdot 22073$ | 1.58968 | 4.26410 | -5.95683 | -0.59988 | 0.18818 |  |  |
| 3 | 6.21771 | 5.99746 | 0.61091 | 1.63868 | -13.40505 | $-0.20834$ | 0.62110 |  |  |
| 4 | 6.21771 | 7.49628 | $0 \cdot 12651$ | 0.33936 | -17.42544 | $-0.12105$ | 0.46977 |  |  |
| 0 | 6.21771 | -3.07274 | 1.59436 | $4 \cdot 27665$ | 10.92459 | 3.66117 | -4.11294 | $0 \cdot 28125$ | 8 |
| 1 | 6.21771 | 0.51587 | 1.99731 | $5 \cdot 35752$ | $1 \cdot 29862$ | 1.03474 | $5 \cdot 39567$ |  |  |
| 2 | 6.21771 | 2.76714 | 1.76689 | 4.73947 | -4.74013 | $-0.57855$ | 0.72395 |  |  |
| 3 | 6.21771 | $5 \cdot 16270$ | 0.79124 | 2.12240 | $-11.16592$ | -0.10905 | 0.98879 |  |  |
| 4 | 6.21771 | 6.44788 | 0.21396 | 0.57391 | $-14.61324$ | $-0.07879$ | 0.82071 |  |  |
| 0 | 6.21771 | -6.26287 | 3.48135 | 9.33828 | 19.48172 | 4.68497 | -1.79227 | 0.3125 | 8 |
| 1 | 621771 | 0.11521 | $2 \cdot 16787$ | $5 \cdot 81502$ | 2.37333 | 1.62530 | 6.06000 |  |  |
| 2 | 6.21771 | $2 \cdot 38182$ | 1.94322 | $5 \cdot 21243$ | $-3.70656$ | -0.02668 | 1.81987 |  |  |
| 3 | 6.21771 | $4 \cdot 50764$ | 0.97178 | 2.60668 | -9.40879 | 0.49098 | 1.89439 |  |  |
| 4 | 6.21771 | $5 \cdot 59887$ | 0.32552 | 0.87318 | $-12.33586$ | 0.47411 | 1.72546 |  |  |
| 0 | $6 \cdot 21771$ | -3-84144 | 14.56160 | 39.05962 | 12.98655 | 8.83417 | $-1.45440$ | 0.34375 | 8 |
| 1 | $6 \cdot 21771$ | -0.37948 | $2 \cdot 39001$ | 6.41089 | 3.70027 | $5 \cdot 31836$ | 4.99896 |  |  |
| 2 | $6 \cdot 21771$ | 2.03733 | $2 \cdot 12667$ | 5.70452 | $-2.78250$ | $3 \cdot 60935$ | 1.01128 |  |  |
| 3 | $6 \cdot 21771$ | 3.98168 | $1 \cdot 14798$ | 3.07932 | $-7.99797$ | $4 \cdot 14048$ | 0.86301 |  |  |
| 4 | $6 \cdot 21771$ | $4 \cdot 90003$ | 0.45641 | 1.22425 | $-10.46134$ | 4.08906 | 0.69445 |  |  |

Table 2 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{lm} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | $6 \cdot 21771$ | 4.74469 | 8.23353 | 22.08539 | -10.04466 | 7.22937 | -3.18958 | 0.3750 | 8 |
| 1 | $6 \cdot 21771$ | $-1.03911$ | 2.71812 | 7.29101 | 5.46964 | 3.22335 | 2.03307 |  |  |
| 2 | $6 \cdot 21771$ | 1.71403 | 2.32753 | 6.24331 | -1.91530 | 1.40099 | -1.88938 |  |  |
| 3 | 6.21771 | $3 \cdot 54993$ | 1.31820 | 3.53589 | -6.83985 | 1.89571 | -2.31900 |  |  |
| 4 | $6 \cdot 21771$ | $4 \cdot 31831$ | 0.59990 | 1.60915 | -8.90094 | 1.81592 | -2.50447 |  |  |
| 0 | 6-21771 | 3.60511 | $5 \cdot 20813$ | 13.97015 | -6.98788 | 6.92141 | -1.54358 | $0 \cdot 40625$ | 8 |
| 1 | $6 \cdot 21771$ | -1.99948 | 3.29915 | 8.84954 | 8.04573 | $2 \cdot 35624$ | 2.53477 |  |  |
| 2 | 6.21771 | 1.39645 | $2 \cdot 56053$ | 6.86828 | -1.06342 | 0.28046 | $-1.55617$ |  |  |
| 3 | $6 \cdot 21771$ | 3.18782 | 1.48290 | 3.97768 | -5.86853 | 0.63578 | -2.38466 |  |  |
| 4 | 6.21771 | 3.82897 | 0.74937 | 2.01009 | -7.58834 | 0.50779 | -2.63218 |  |  |
| 0 | 6.21771 | $2 \cdot 61065$ | 4.38490 | 11.76194 | -4.32035 | 7.89420 | 0.37131 | 0.4375 | 8 |
| 1 | 6.21771 | -3.51453 | 4.63664 | 12.43719 | 12.10963 | 2.58768 | $3 \cdot 29992$ |  |  |
| 2 | 6.21771 | 1.07151 | 2.84911 | 7.64236 | -0.19182 | -0.13872 | -1.30088 |  |  |
| 3 | 6.21771 | 2.87791 | 1.64397 | $4 \cdot 40975$ | -5.03724 | -0.25045 | -2.76242 |  |  |
| 4 | 6.21771 | 3-41217 | 0.89960 | $2 \cdot 41305$ | -6.47035 | -0.52927 | -3.16489 |  |  |
| 0 | 6.21771 | 0.97143 | 2.43161 | 6.52248 | 0.07664 | 1.43500 | -12.49961 | 0.1700 | 20 |
| 1 | 6.21771 | -0.58939 | 0.96375 | 2.58513 | 4.26334 | 1.04357 | 8.55105 |  |  |
| 2 | $6 \cdot 21771$ | 0.64311 | 1.12504 | 3.01778 | 0.95730 | 0.30332 | -0.63750 |  |  |
| 3 | $6 \cdot 21771$ | 1.72937 | 1.05916 | 2.84107 | -1.95645 | -0.02529 | -0.20626 |  |  |
| 4 | 6.21771 | 3.18240 | 0.70144 | 1.88152 | -5.85400 | 0.06437 | -0.05895 |  |  |
| 5 | $6 \cdot 21771$ | $5 \cdot 16224$ | 0.26348 | 0.70676 | -11.16468 | 0.19925 | -0.12010 |  |  |
| 6 | 6.21771 | 7.38030 | 0.04658 | 0.12494 | -17.11434 | 0.22424 | -0.22323 |  |  |
| 7 | 6.21771 | 9.39761 | -0.00104 | -0.00279 | -22.52551 | $0 \cdot 20842$ | -0.27115 |  |  |
| 8 | $6 \cdot 21771$ | 10.99347 | $-0.00840$ | -0.02253 | -26.80619 | 0.20638 | -0.28878 |  |  |
| 9 | $6 \cdot 21771$ | 12.02388 | -0.01052 | -0.02823 | -29.57014 | $0 \cdot 21217$ | -0.30248 |  |  |
| 10 | 6.21771 | 12.38084 | -0.01120 | -0.03004 | -30.52763 | 0.21460 | -0.30893 |  |  |

Table 2 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.21771 | -1.02714 | 1.71496 | $4 \cdot 60017$ | $5 \cdot 43753$ | 2.85913 | -5.81890 | 0.2500 | 20 |
| 1 | $6 \cdot 21771$ | 1.00569 | 3.01550 | 8.08869 | -0.01526 | 1.73291 | 5.81862 |  |  |
| 2 | 6.21771 | $-1.67840$ | 2.02008 | $5 \cdot 41861$ | 7.18446 | 0.20562 | -0.80449 |  |  |
| 3 | 6.21771 | $0 \cdot 34365$ | 1.59237 | $4 \cdot 27133$ | 1.76059 | -0.01856 | -0.36063 |  |  |
| 4 | 6.21771 | 1.51934 | 1.42374 | 3.81899 | -1.39307 | 0.09549 | -0.41945 |  |  |
| 5 | $6 \cdot 21771$ | 2.71476 | 1.03755 | 2.78309 | -4.59963 | -0.06486 | -0.39799 |  |  |
| 6 | $6 \cdot 21771$ | $4 \cdot 11322$ | 0.53003 | 1.42174 | -8.35082 | -0.20028 | -0.11090 |  |  |
| 7 | 6.21771 | 5.62312 | $0 \cdot 16006$ | 0.42935 | -12.40092 | -0.23897 | 0.22478 |  |  |
| 8 | 6.21771 | 6.89325 | 0.02510 | 0.06733 | - 15.80788 | -0.25558 | 0.52470 |  |  |
| 9 | 6.21771 | 7.69158 | -0.00100 | -0.00270 | -17.94931 | -0.25782 | 0.74258 |  |  |
| 10 | $6 \cdot 21771$ | 7.96098 | -0.00395 | -0.01059 | -18.67192 | -0.25483 | 0-82200 |  |  |
| 0 | $6 \cdot 21771$ | 1.05103 | 3.85805 | 10.34872 | -0.13687 | 6.02583 | -1.62248 | 0.3400 | 20 |
| 1 | 6.21771 | -2.34603 | 3.18249 | 8.53661 | 8.97530 | $2 \cdot 94009$ | 4.10157 |  |  |
| 2 | 6.21771 | 0.65817 | 3.07359 | $8 \cdot 24451$ | 0.91691 | -0.50792 | -0.95743 |  |  |
| 3 | $6 \cdot 21771$ | -0.87100 | 7.04184 | 18.88884 | 5.01871 | -1.01616 | -0.19390 |  |  |
| 4 | 6.21771 | -0.43591 | 2.47257 | 6.63236 | 3.85164 | -0.83124 | 0.06052 |  |  |
| 5 | $6 \cdot 21771$ | 1.11157 | 1.93507 | $5 \cdot 19058$ | -0.29926 | 0.06403 | 0.32155 |  |  |
| 6 | 6.21771 | 2.22382 | 1.45216 | 3.89524 | -3.28274 | 1.02681 | 0.06635 |  |  |
| 7 | $6 \cdot 21771$ | 3.25400 | 0.88492 | 2.37368 | -6.04608 | 115323 | -0.41290 |  |  |
| 8 | 6.21771 | 4.24964 | 0.37783 | 1.01349 | -8.71674 | 0.41960 | -0.78701 |  |  |
| 9 | $6 \cdot 21771$ | 5.02288 | $0 \cdot 10007$ | 0.26843 | -10.79085 | -0.57409 | -0.95892 |  |  |
| 10 | $6 \cdot 21771$ | $5 \cdot 31204$ | 0.02995 | 0.08034 | -11.56648 | $-1.02580$ | -0.99404 |  |  |

Table 3. $\Psi_{d R}, T^{(\boldsymbol{m})}$ and admittances of circular array $\dagger$ of Nelements
$\left(a / \lambda=7.022 \times 10^{-3}\right)$
$h / \lambda=3 / 8, \quad \beta_{0} h=3 \pi / 4, \quad \Omega=9.34$

| Sequence <br> m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | Sequence admittance |  | Self- and mutual admittances |  | $\frac{d}{\lambda}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $\boldsymbol{B}_{1(m+1)}$ |  |  |
| 0 | 6.44947 | -0.34647 | 0.34841 | 2.17514 | $-0.42294$ | 1.32680 | $-1 \cdot 12131$ | $0 \cdot 1875$ | 2 |
| 1 | 6.44947 | $-0.12274$ | 0.07664 | 0.47845 | -1.81968 | 0.84834 | 0.69837 |  |  |
| 0 | 6.44947 | $-0.30504$ | 0.39203 | 2.44750 | -0.68161 | 1.55035 | $-1.05823$ | 0.2500 | 2 |
| 1 | 6.44947 | $-0.18438$ | $0 \cdot 10463$ | 0.65319 | $-1.43485$ | 0.89716 | 0.37662 |  |  |
| 0 | 6.44947 | $-0.24661$ | 0.42127 | 2.63003 | $-1.04640$ | 1.72902 | $-1.11269$ | 0.3125 | 2 |
| 1 | 6.44947 | $-0.22537$ | $0 \cdot 13263$ | 0.82801 | -1.17899 | 0.90101 | 0.06630 |  |  |
| 0 | 6.44947 | $-0.17613$ | 0.42372 | 2.64534 | $-1.48639$ | 1.82429 | $-1.24458$ | $0 \cdot 3750$ | 2 |
| 1 | 6.44947 | $-0.25359$ | 0.16070 | 1.00324 | -1.00277 | 0.82105 | -0.24181 |  |  |
| 0 | 6.44947 | $-0.11086$ | 0.39101 | 2.44112 | $-1.89389$ | 1.81048 | $-1.38879$ | 0.4375 | 2 |
| 1 | 6.44947 | $-0.27267$ | 0.18898 | 1-17984 | -0.88369 | 0.63064 | -0.50510 |  |  |
| 0 | 6.44947 | -0.07291 | 0.33159 | 2.07016 | $-2.13080$ | 1.71430 | -1.47116 | $0 \cdot 5000$ | 2 |
| 1 | 6.44947 | $-0.28423$ | 0.21759 | 1.35844 | $-0.81153$ | 0.35586 | -0.65964 |  |  |
| 0 | 6.44947 | $-0.07068$ | 0.26849 | 1.67624 | $-2.14471$ | 1.60735 | -1.46392 | 0.5625 | 2 |
| 1 | 6.44947 | $-0.28878$ | 0.24643 | 1.53846 | $-0.78313$ | 0.06889 | -0.68079 |  |  |
| 0 | 6.44947 | -0.09430 | 0.22053 | 1.37682 | -1.99728 | 1.54683 | $-1.39861$ | 0.6250 | 2 |
| 1 | 6.44947 | -0.28608 | 0.27500 | 1.71683 | $-0.79995$ | $-0.17001$ | -0.59867 |  |  |
| 0 | 6.44947 | $-0.12872$ | 0.19294 | 1.20452 | $-1.78236$ | 1.54531 | $-1.32429$ | 0.6875 | 2 |
| 1 | 6.44947 | -0.27546 | 0.30211 | 1.88610 | $-0.86622$ | -0.34079 | -0.45807 |  |  |
| 0 | 6.44947 | $-0.16365$ | 0.18248 | $1 \cdot 13923$ | $-1.56431$ | 1.58566 | -1.27515 | 0.7500 | 2 |
| 1 | 6.44947 | -0.25628 | 0.32549 | 2.03209 | -0.98599 | $-0.44643$ | -0.28916 |  |  |
| 0 | 6.44947 | $-0.47138$ | 0.35091 | 2.19075 | 0.35693 | 1.04922 | $-1.09414$ | 0.1875 | 3 |
| 1 | 6.44947 | $-0.12274$ | 0.07664 | 0.47845 | $-1.81968$ | 0.57076 | 0.72553 |  |  |
| $\dagger$ Note that $T^{(m)}=T^{(m)}\left(\frac{3 \lambda}{8}\right)$. |  |  |  |  |  |  |  |  |  |

Table 3 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.44947 | -0.44657 | 0.43938 | 2.74308 | 0.20198 | 1.34982 | -0.88924 | 0.2500 | 3 |
| 1 | 6.44947 | -0.18438 | 0.10463 | 0.65319 | -1.43485 | 0.69663 | 0.54561 |  |  |
| 0 | 6.44947 | -0.38446 | 0.54228 | 3-38553 | -0.18574 | 1.68051 | -0.84791 | 0.3125 | 3 |
| 1 | 6.44947 | -0.22537 | 0.13263 | 0.82801 | -1.17899 | 0.85251 | 0.33108 |  |  |
| 0 | 6.44947 | -0.24901 | 0.63644 | 3.97336 | -1.03141 | 1.99328 | -1.01232 | 0.3750 | 3 |
| 1 | $6 \cdot 44947$ | -0.25359 | 0.16070 | 1.00324 | -1.00277 | 0.99004 | -0.00954 |  |  |
| 0 | 6.44947 | -0.02896 | 0.63299 | 3.95182 | -2.40516 | 2.10384 | $-1.39084$ | 0.4375 | 3 |
| 1 | 6.44947 | -0.27267 | 0.18898 | 1-17984 | -0.88369 | 0.92399 | -0.50716 |  |  |
| 0 | 6.44947 | 0.13984 | 0.45236 | $2 \cdot 82412$ | -3.45905 | 1.84700 | -1.69403 | 0.5000 | 3 |
| 1 | 6.44947 | -0.28423 | 0.21759 | $1 \cdot 35844$ | -0.81153 | 0.48856 | -0.88251 |  |  |
| 0 | 6.44947 | 0.12450 | 0.23956 | 1.49557 | -3.36327 | 1.52417 | -1.64317 | 0.5625 | 3 |
| 1 | 6.44947 | -0.28878 | 0.24643 | 1.53846 | -0.78313 | -0.01430 | -0.86005 |  |  |
| 0 | 6.44947 | 0.02068 | 0.12744 | 0.79560 | -2.71512 | 1.40975 | -1.43834 | 0.6250 | 3 |
| 1 | $6 \cdot 44947$ | -0.28608 | 0.27500 | 1.71683 | -0.79995 | $-0.30708$ | -0.63839 |  |  |
| 0 | 6.44947 | -0.07859 | 0.09350 | 0.58373 | -2.09536 | 1.45198 | -1.27594 | 0.6875 | 3 |
| 1 | 6.44947 | -0.27546 | 0.30211 | 1.88610 | -0.86622 | -0.43412 | -0.40971 |  |  |
| 0 | 6.44947 | -0.15342 | 0.09624 | 0.60083 | -1.62816 | 1.55500 | -1.20004 | 0.7500 | 3 |
| 1 | $6 \cdot 44947$ | -0.25628 | 0.32549 | 2.03209 | -0.98599 | -0.47709 | -0.21406 |  |  |
| 0 | $6 \cdot 44947$ | -0.55329 | 0.35643 | $2 \cdot 22523$ | 0.86829 | 0.92702 | -1.05015 | 0.1875 | 4 |
| 1 | 6.44947 | -0.19581 | 0.11142 | 0.69561 | -1.36349 | 0.53340 | 0.80255 |  |  |
| 2 | 6.44947 | -0.03909 | 0.01468 | 009163 | -2.34192 | 0.23141 | 0.31334 |  |  |
| 0 | 6.44947 | $-0.54954$ | 0.48678 | 3.03902 | 0.84487 | 1.28146 | -0.76256 | 0.2500 | 4 |
| 1 | 6.44947 | -0.24507 | 0.15105 | 0.94301 | -1.05600 | 0.70955 | 0.65699 |  |  |
| 2 | 6.44947 | -0.12860 | 0.03216 | 0.20081 | -1.78309 | 0.33845 | 0.29345 |  |  |
| 0 | 6.44947 | -0.48519 | 0.68375 | 4.26873 | 0.44314 | 1.75057 | -0.67078 | 0.3125 | 4 |
| 1 | $6 \cdot 44947$ | -0.27373 | 0.19101 | 1-19247 | -0.87708 | 0.98003 | 0.45381 |  |  |
| 2 | 6.44947 | -0.19443 | 0.05584 | 0.34860 | $-1 \cdot 37211$ | 0.55810 | $0 \cdot 20630$ |  |  |

Table 3 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.44947 | $-0.19693$ | 0.91255 | 5.69715 | $-1.35650$ | - 2.27841 | -0.99994 | 0.3750 | 4 |
| 1 | 6.44947 | $-0.28731$ | 0.23157 | 1.44574 | -0.79228 | 1.29303 | -0.07445 |  |  |
| 2 | 6.44947 | -0.24463 | 0.08409 | 0.52501 | - 1.05872 | 0.83267 | -0.20767 |  |  |
| 0 | 6.44947 | 0.32272 | 0.70498 | 4.40129 | -4.60077 | 2.13046 | $-1.75241$ | $0 \cdot 4375$ | 4 |
| 1 | 6.44947 | -0.28670 | 0.27216 | 1.69915 | $-0.79610$ | 0.91976 | -0.94602 |  |  |
| 2 | 6.44947 | -0.28340 | $0 \cdot 11569$ | 0.72225 | -0.81668 | 0.43131 | $-0.95631$ |  |  |
| 0 | 6.44947 | 0.29755 | 0.22178 | 1.38458 | -4.44362 | 1.54804 | -1.71731 | 0.5000 | 4 |
| 1 | 6.44947 | -0.27039 | $0 \cdot 30999$ | 1.93530 | -0.89794 | $0 \cdot 11189$ | -0.95347 |  |  |
| 2 | 6.44947 | -0.31334 | $0 \cdot 15009$ | 0.93700 | -0.62975 | $-0.38725$ | -0.81938 |  |  |
| 0 | 6.44947 | 0.07700 | 0.06927 | 0.43245 | $-3.06671$ | 1.45636 | $-1.44191$ | 0.5625 | 4 |
| 1 | 6.44947 | $-0.23702$ | 0.33817 | $2 \cdot 11123$ | $-1.10623$ | $-0.18452$ | -0.64456 |  |  |
| 2 | 6.44947 | -0.33597 | 0.18749 | 1.17052 | $-0.48847$ | -0.65487 | -0.33568 |  |  |
| 0 | 6.44947 | -0.06893 | 0.05805 | 0.36244 | -2.15564 | 1.52608 | -1.33380 | 0.6250 | 4 |
| 1 | 6.44947 | $-0.19080$ | 0.34549 | $2 \cdot 15694$ | -1.39480 | -0.26640 | -0.44142 |  |  |
| 2 | 6.44947 | -0.35175 | $0 \cdot 22874$ | 1.42802 | $-0.38996$ | $-0.63085$ | 0.06100 |  |  |
| 0 | 6.44947 | $-0.15616$ | 0.08162 | 0.50959 | $-1.61108$ | 1.56757 | -1.32067 | 0.6875 | 4 |
| 1 | 6.44947 | -0.14737 | 0.32377 | 2.02131 | $-1.66597$ | -0.30213 | -0.31785 |  |  |
| 2 | 6.44947 | $-0.35980$ | 0.27520 | 1.71810 | -0.33968 | -0.45373 | 0.34529 |  |  |
| 0 | 6.44947 | $-0.16904$ | 0.08762 | 0.54702 | $-1.53067$ | 1.56935 | -1.31981 | 0.7000 | 4 |
| 1 | 6.44947 | $-0.14096$ | 0.31630 | 1.97472 | $-1.70593$ | -0.30848 | -0.29848 |  |  |
| 2 | 6.44947 | $-0.36028$ | 0.28527 | 1.78094 | $-0.33673$ | -0.40537 | 0.38611 |  |  |
| 0 | 6.44947 | $-0.21007$ | 0.11293 | 0.70504 | $-1.27449$ | 1.56698 | -1.30361 | 0.7500 | 4 |
| 1 | 6.44947 | $-0.12720$ | 0.28125 | 1.75587 | -1.79184 | -0.33652 | -0.22955 |  |  |
| 2 | 6.44947 | $-0.35715$ | 0.32854 | 2.05113 | $-0.35628$ | $-0.18889$ | 0.48822 |  |  |
| 0 | 6.44947 | -0.61970 | 0.36751 | $2 \cdot 29438$ | 1.28290 | 0.86350 | -0.99648 | 0.1875 | 5 |
| 1 | 6.44947 | $-0.27081$ | 0.14413 | 0.89982 | -0.89527 | 0.53394 | 0.86995 |  |  |
| 2 | 6.44947 | -0.05584 | 0.01790 | 0.11174 | $-2.23737$ | 0.18150 | 0.26974 |  |  |

Table 3 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $\boldsymbol{B}_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.44947 | -0.64486 | 0.55417 | 3.45972 | 1.43995 | 1.28095 | -0.64140 | 0.2500 | 5 |
| 1 | 6.44947 | -0.31193 | 0.19746 | 1.23277 | -0.63857 | 0.76674 | 0.75430 |  |  |
| 2 | 6.44947 | -0.14433 | 0.03840 | 0.23975 | $-1.68489$ | $0 \cdot 32265$ | 0.28637 |  |  |
| 0 | 6.44947 | -0.54344 | 0.93294 | 5.82443 | 0.80675 | 1.96639 | -0.55634 | 0.3125 | 5 |
| 1 | 6.44947 | -0.33241 | 0.25558 | 1.59564 | $-0.51070$ | $1 \cdot 23005$ | 0.51358 |  |  |
| 2 | 6.44947 | -0.20862 | 0.06537 | 0.40812 | -1.28353 | 0.69897 | 0.16796 |  |  |
| 0 | 6.44947 | 0.32595 | 1.11700 | 6.97358 | -4.62090 | 2.43923 | -1.52249 | 0.3750 | 5 |
| 1 | 6.44947 | -0.33217 | $0 \cdot 32142$ | 2.00669 | $-0.51224$ | 1.44711 | -0.66922 |  |  |
| 2 | 6.44947 | $-0.25667$ | 0.09684 | 0.60459 | $-0.98354$ | 0.82007 | -0.87999 |  |  |
| 0 | 6.44947 | 0.57666 | 0.70317 | 4.39000 | $-6.18612$ | 2.02805 | $-1.81601$ | 0.4000 | 5 |
| 1 | 6.44947 | -0.32429 | 0.35016 | $2 \cdot 18607$ | -0.56143 | 0.92523 | $-1.02005$ |  |  |
| 2 | 6.44947 | -0.27237 | 0.11037 | 0.68904 | $-0.88555$ | 0.25574 | -1.16500 |  |  |
| 0 | 6.44947 | 0.42187 | 0.23114 | 1.44306 | $-5.21980$ | 1.60258 | -1.63129 | 0.4375 | 5 |
| 1 | 6.44947 | -0.30048 | 0.39465 | 2.46383 | $-0.71005$ | 0.32744 | -0.88634 |  |  |
| 2 | 6.44947 | -0.29276 | 0.13152 | 0.82111 | $-0.75828$ | -0.40720 | -0.90791 |  |  |
| 0 | 6.44947 | 0.31785 | 0.14604 | 0.91174 | -4.57038 | 1.56449 | -1.52095 | 0.45313 | 5 |
| 1 | 6.44947 | -0.28518 | 0.41283 | 2.57737 | $-0.80556$ | 0.21681 | -0.78337 |  |  |
| 2 | 6.44947 | $-0.30023$ | 0.14063 | 0-87798 | $-0.71161$ | $-0.54318$ | -0.74135 |  |  |
| 0 | 6.44947 | 0.14751 | 0.07588 | 0.47375 | $-3.50693$ | 1.60478 | $-1.38079$ | 0.48438 | 5 |
| 1 | 6.44947 | -0.24284 | 0.44530 | 2.78003 | $-1.06989$ | $0 \cdot 11638$ | -0.63020 |  |  |
| 2 | 6.44947 | -0.31352 | 0.15938 | 0.99503 | $-0.62863$ | -0.68189 | -0.43286 |  |  |
| 0 | 6.44947 | 0.08327 | 0.06638 | 0.41440 | $-3.10585$ | 1.64758 | $-1.35482$ | 0.5000 | 5 |
| 1 | 6.44947 | -0.21527 | 0.45754 | 2.85645 | $-1.24202$ | 0.09445 | -0.58309 |  |  |
| 2 | 6.44947 | -0.31937 | 0.16903 | 1.05530 | $-0.59209$ | -0.71104 | -0.29243 |  |  |

Table 3 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 6.44947 | 0.03044 | 0.06516 | 0.40683 | $-2.77603$ | 1.69002 | $-1.35474$ | 0.51563 | 5 |
| 1 | 6.44947 | -0.18355 | 0.46529 | 2.90484 | -1.44006 | 0.07902 | -0.55239 |  |  |
| 2 | 6.44947 | -0.32471 | 0.17888 | 1-11679 | -0.55877 | -0.72062 | -0.15826 |  |  |
| 0 | 6.44947 | $-0.07925$ | 0.08411 | 0.52513 | -2.09120 | 1.74275 | -1.45800 | 0.5625 | 5 |
| 1 | 6.44947 | -0.07441 | 0.44607 | 2.78486 | -2.12141 | 0.02551 | -0.52578 |  |  |
| 2 | 6.44947 | -0.33765 | 0.20974 | 1.30944 | -0.47798 | $-0.63432$ | 0.20918 |  |  |
| 0 | 6.44947 | $-0.15781$ | $0 \cdot 12414$ | 0.77502 | -1.60074 | 1.56994 | -1.56058 | 0.6250 | 5 |
| 1 | 6.44947 | 0.01552 | 0.31211 | 1.94857 | -2.68287 | -0.11828 | -0.51643 |  |  |
| 2 | 6.44947 | -0.34722 | $0 \cdot 25448$ | 1.58877 | -0.41822 | -0.27918 | 0.49635 |  |  |
| 0 | 6.44947 | $-0.19469$ | $0 \cdot 16288$ | 1.01690 | -1.37050 | 1.40669 | -1.40661 | 0.6875 | 5 |
| 1 | 6.44947 | -0.02918 | 0.17778 | 1.10991 | $-2.40384$ | $-0.27375$ | -0.43291 |  |  |
| 2 | 6.44947 | -0.34575 | 0.30407 | 1.89837 | $-0.42743$ | 0.07886 | 0.45097 |  |  |
| 0 | 6.44947 | $-0.20126$ | $0 \cdot 17513$ | 1.09336 | -1.32949 | $1 \cdot 40201$ | -1.32741 | 0.7100 | 5 |
| 1 | 6.44947 | -0.06173 | $0 \cdot 15075$ | 0.94115 | -2.20057 | -0.31778 | -0.39124 |  |  |
| 2 | 6.44947 | $-0.34162$ | 0.32311 | 2.01720 | -0.45322 | $0 \cdot 16345$ | 0.39020 |  |  |
| 0 | 6.44947 | -0.20643 | 0.19347 | 1.20785 | -1.29723 | 1.45463 | $-1.20877$ | 0.7500 | 5 |
| 1 | 6.44947 | $-0.12027$ | $0 \cdot 12776$ | 0.79763 | -1.83511 | -0.38310 | -0.31211 |  |  |
| 2 | 6.44947 | -0.32801 | 0.35800 | 2.23501 | $-0.53821$ | 0.25971 | 0.26788 |  |  |
| 0 | 8.34383 | $-0.02515$ | 0.40052 | 1.93278 | -1.87751 | 0.51502 | $-1.13221$ | 0.1500 | 20 |
| 1 | 8.25111 | -0.83304 | 0.41564 | 2.02828 | 2.04386 | 0.43561 | 0.84404 |  |  |
| 2 | 7.98204 | -0.53929 | $0 \cdot 24478$ | 1.23478 | 0.63094 | 0.25643 | 0.19158 |  |  |

Table 3 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 3 | 7.56295 | -0.37900 | 0.12488 | 0.66487 | -0.18747 | 0.09682 | -0.09386 |  |  |
| 4 | 7.03486 | -0.24489 | 0.03860 | 0.22095 | -0.96914 | 0.01625 | $-0.14187$ |  |  |
| 5 | 6.44947 | -0.15155 | 0.00584 | 0.03649 | -1.63983 | -0.00505 | -0.16493 |  |  |
| 6 | 5.86408 | -0.11432 | 0.00036 | 0.00248 | -2.05918 | -0.00889 | -0.19133 |  |  |
| 7 | $5 \cdot 33600$ | -0.11294 | -0.00012 | -0.00089 | -2.27337 | -0.01514 | -0.21710 |  |  |
| 8 | 4.91690 | -0.12607 | -0.00015 | -0.00122 | -2.35962 | -0.02320 | -0.23417 |  |  |
| 9 | 4.64783 | -0.13954 | -0.00015 | -0.00131 | -2.37952 | -0.02871 | -0.24255 |  |  |
| 10 | $4 \cdot 55512$ | -0.14497 | -0.00015 | -0.00133 | -2.37994 | -0.03047 | -0.24494 |  |  |
| 0 | 6.44947 | 0.35836 | 0.88997 | $5 \cdot 55618$ | -4.82325 | 1.53249 | -0.96196 | 0.2800 | 20 |
| 1 | 6.44947 | -0.44839 | 0.45566 | $2 \cdot 84473$ | 0.21335 | 1.00305 | 0.09476 |  |  |
| 2 | 6.44947 | 0.09217 | 0.37118 | $2 \cdot 31731$ | -3.16144 | 0.17140 | -0.59347 |  |  |
| 3 | 6.44947 | $-0.62997$ | 0.63638 | 3.97298 | 1.34698 | -0.10521 | -0.31997 |  |  |
| 4 | 6.44947 | -0.50165 | 0.29433 | $1 \cdot 83756$ | 0.54586 | 0.01334 | -0.02415 |  |  |
| 5 | 6.44947 | $-0.40107$ | 0.15795 | 0.98612 | -0.08206 | 0.18889 | 0.13000 |  |  |
| 6 | 6.44947 | -0.31037 | 0.06941 | 0.43331 | -0.64832 | 0.30382 | 0.11621 |  |  |
| 7 | 6.44947 | -0.22595 | 0.02062 | 0.12870 | -1.17535 | 029188 | -0.07330 |  |  |
| 8 | 6.44947 | -0.16422 | 0.00380 | 0.02375 | -1.56073 | 0.16145 | -0.34748 |  |  |
| 9 | 6.44947 | -0.13077 | 0.00038 | 0.00240 | -1.76954 | 0.01048 | -0.57893 |  |  |
| 10 | 6.44947 | $-0.12053$ | -0.00003 | -0.00016 | -1.83351 | $-0.05450$ | -0.66864 |  |  |

Table 4. $\Psi_{d R}, T^{(m)}$ and admittances of circular array $\dagger$ of $N$ elements $\left(a / \lambda=7.022 \times 10^{-3}\right)$
$h / \lambda=0.5, \quad \beta_{0} h=\pi, \quad \Omega=9.92$


Table 4 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 5.73687 | -0.29727 | 0.35169 | 2.04486 | 1.72846 | 0.96023 | 1.29157 | 0.2500 | 3 |
| 1 | 5.73687 | -0.18456 | 0.07187 | 0.41791 | 1.07312 | 0.54232 | 0.21845 |  |  |
| 0 | 5.73687 | -0.22826 | $0 \cdot 40826$ | 2.37378 | 1.32720 | $1 \cdot 15376$ | 1.24493 | 0.3125 | 3 |
| 1 | 5.73687 | -0.20704 | 0.09352 | 0.54375 | 1.20379 | 0.61001 | 0.04114 |  |  |
| 0 | 5.73687 | -0.10932 | 0.42957 | 2.49770 | 0.63563 | 1.27935 | 1.07430 | 0.3750 | 3 |
| 1 | 5.73687 | -0.22249 | 0.11526 | 0.67017 | 1.29364 | 0.60918 | -0.21934 |  |  |
| 0 | 5.73687 | 0.02165 | 0.35891 | 2.08686 | -0.12586 | 1.22683 | 0.85779 | 0.4375 | 3 |
| 1 | 5.73687 | -0.23212 | 0.13704 | 0.79681 | 1.34962 | 1.43002 | -0.49183 |  |  |
| 0 | 5.73687 | 0.06652 | 0.21555 | $1 \cdot 25327$ | -0.38679 | 1.03309 | 0.78749 | 0.5000 | 3 |
| 1 | 5.73687 | -0.23642 | 0.15874 | 0.92300 | $1 \cdot 37462$ | $0 \cdot 11009$ | -0.58714 |  |  |
| 0 | 5.73687 | 0.01613 | 0.10388 | 0.60400 | -0.09378 | 0.89925 | 0.88124 | 0.5625 | 3 |
| 1 | 5.73687 | -0.23541 | 0.18005 | 1.04688 | $1 \cdot 36875$ | -0.14763 | -0.48751 |  |  |
| 0 | 5.73687 | -0.05914 | 0.05632 | 0.32744 | 0.34386 | 0.88534 | 1.00145 | 0.6250 | 3 |
| 1 | 5.73687 | -0.22878 | 0.20024 | 1.16429 | 1.33024 | -0.27895 | -0.32879 |  |  |
| 0 | 5.73687 | $-0.12272$ | 0.04855 | 0.28230 | 0.71352 | 0.93905 | 1.07582 | 0.6875 | 3 |
| 1 | 5.73687 | -0.21618 | $0 \cdot 21798$ | $1 \cdot 26743$ | 1-25697 | -0.32838 | -0.18115 |  |  |
| 0 | 5.73687 | -0.16955 | 0.05915 | 0.34394 | 0.98581 | 1.01050 | 1.09474 | 0.7500 | 3 |
| 1 | 5.73687 | -0.19765 | 0.23111 | 1.34377 | 1-14920 | $-0.33328$ | $-0.05446$ |  |  |
| 0 | 5.73687 | $-0.38958$ | 0.32298 | 1.87795 | $2 \cdot 26516$ | 0.70749 | $1 \cdot 28318$ | 0.1875 | 4 |
| 1 | 5.73687 | -0.19076 | 0.07711 | 0.44835 | 1.10916 | 0.45566 | 0.40398 |  |  |
| 2 | 5.73687 | -0.11166 | 0.00951 | 0.05532 | 0.64925 | 0.25914 | 0.17403 |  |  |
| 0 | 5.73687 | -0.36068 | $0 \cdot 42208$ | $2 \cdot 45413$ | 2.09712 | 0.95895 | 1.38535 | 0.2500 | 4 |
| 1 | 5.73687 | -0.21789 | $0 \cdot 10779$ | 0.62675 | 1.26688 | 0.58149 | 0.29665 |  |  |
| 2 | 5.73687 | -0.15660 | 0.02205 | $0 \cdot 12818$ | 0.91051 | 0.33220 | 0.11847 |  |  |
| 0 | 5.73687 | -0.24528 | 0.54869 | $3 \cdot 19030$ | 1.42614 | $1 \cdot 25828$ | 1-31478 | 0.3125 | 4 |
| 1 | 5.73687 | -0.23260 | $0 \cdot 13859$ | 0.80581 | $1 \cdot 35240$ | 0.73978 | 0.07449 |  |  |
| 2 | 5.73687 | $-0.19403$ | 0.03976 | 0.23119 | $1 \cdot 12817$ | 0.45247 | -0.03762 |  |  |

Table 4 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| m | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\mathrm{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 5.73687 | 0.03603 | 0.57029 | 3.31589 | -0.20949 | 1.41026 | 0.96170 | 0.3750 | 4 |
| 1 | 5.73687 | -0.23660 | 0.16917 | 0.98359 | $1 \cdot 37571$ | 0.73948 | -0.37859 |  |  |
| 2 | 5.73687 | -0.22442 | 0.06156 | 0.35796 | $1 \cdot 30489$ | 0.42667 | -0.41400 |  |  |
| 0 | 5.73687 | $0 \cdot 21558$ | 0.28252 | 1.64266 | $-1.25348$ | $1 \cdot 11283$ | 0.71545 | 0.4375 | 4 |
| 1 | 5.73687 | $-0.22971$ | 0.19830 | $1 \cdot 15297$ | 1.33565 | 0.28499 | $-0.67437$ |  |  |
| 2 | 5.73687 | -0.24835 | 0.08646 | 0.50271 | 1.44399 | -0.04014 | -0.62020 |  |  |
| 0 | 5.73687 | 0.09549 | 0.06553 | 0.38103 | -0.55524 | 0.90838 | 0.86177 | 0.5000 | 4 |
| 1 | 5.73687 | -0.21098 | 0.22272 | 1.29499 | 1.22672 | $-0.07037$ | -0.52603 |  |  |
| 2 | 5.73687 | -0.26639 | 0.11394 | 0.66250 | $1 \cdot 54890$ | -0.38661 | -0.36495 |  |  |
| 0 | 5.73687 | -0.03885 | 0.01820 | $0 \cdot 10581$ | 0.22588 | 0.92235 | 0.98819 | 0.5625 | 4 |
| 1 | 5.73687 | $-0.18107$ | 0.23615 | 1.37308 | 1.05281 | -0.18291 | $-0.34885$ |  |  |
| 2 | 5.73687 | -0.27884 | 0.14403 | 0.83744 | 1.62126 | -0.45073 | $-0.06462$ |  |  |
| 0 | 5.73687 | -0.12339 | 0.02822 | 0.16409 | 0.71746 | 0.96987 | 1.02010 | 0.6250 | 4 |
| 1 | 5.73687 | -0.14652 | 0.23091 | 1.34259 | 0.85195 | -0.21653 | -0.23539 |  |  |
| 2 | 5.73687 | $-0.28533$ | 0.17718 | 1.03020 | 1.65902 | -0.37272 | 0.16814 |  |  |
| 0 | 5.73687 | -0.17437 | 0.05243 | 0.30486 | 1.01388 | 0.98500 | 1.01961 | 0.6875 | 4 |
| 1 | 5.73687 | -0.12132 | 0.20555 | $1 \cdot 19513$ | 0.70539 | $-0.23500$ | -0.15998 |  |  |
| 2 | 5.73687 | -0.28443 | 0.21410 | 1-24487 | 1.65379 | -0.21013 | $0 \cdot 31422$ |  |  |
| 0 | 5.73687 | -0.18188 | 0.05774 | 0.33572 | 1.05751 | 0.98483 | 1.02087 | 0.7000 | 4 |
| 1 | 5.73687 | -0.11866 | 0.19889 | 1-15643 | 0.68991 | -0.23876 | -0.14716 |  |  |
| 2 | 5.73687 | -0.28312 | 0.22199 | 1.29075 | 1.64615 | -0.17160 | 0.33096 |  |  |
| 0 | 5.73687 | -0.20546 | 0.07919 | 0.46045 | 1.19464 | 0.98512 | 1.03626 | 0.7500 | 4 |
| 1 | 5.73687 | -0.11730 | 0.17165 | 0.99803 | 0.68205 | -0.25588 | -0.09792 |  |  |
| 2 | 5.73687 | -0.27283 | 0.25522 | 1.48397 | 1.58631 | -0.01291 | 0.35421 |  |  |
| 0 | 5.73687 | $-0.44337$ | $0 \cdot 35405$ | 2.05857 | 2.57796 | 0.68373 | 1.33527 | 0.1875 | 5 |
| 1 | 5.73687 | -0.23258 | $0 \cdot 10524$ | 0.61191 | $1 \cdot 35233$ | 0.46530 | 0.45724 |  |  |
| 2 | 5.73687 | -0.11985 | 0.01172 | 0.06813 | 0.69687 | 0.22212 | 0.16411 |  |  |

Table 4 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $B_{1(m+1)}$ | $\frac{d}{\lambda}$ | $N$ |
| 0 | 5.73687 | -0.41387 | 0.51736 | 3.00811 | 2.40640 | 1.00601 | 1.46290 | 0.2500 | 5 |
| 1 | 5.73687 | -0.25661 | 0.14733 | 0.85663 | 1.49204 | 0.65756 | 0.35440 |  |  |
| 2 | 5.73687 | -0.16545 | 0.02654 | $0 \cdot 15434$ | 0.96200 | 0.34349 | 0.11736 |  |  |
| 0 | 5.73687 | -0.14964 | 0.75703 | 4.40167 | 0.87007 | 1.43714 | 1.26009 | 0.3125 | 5 |
| 1 | 5.73687 | -0.26445 | $0 \cdot 19253$ | $1 \cdot 11942$ | 1.53760 | 0.93049 | -0.01700 |  |  |
| 2 | 5.73687 | -0.20253 | 0.04688 | 0.27258 | $1 \cdot 17759$ | 0.55178 | $-0.17801$ |  |  |
| 0 | 5.73687 | 0.35041 | 0.42661 | 2.48049 | $-2.03744$ | 1.22362 | 0.72145 | $0 \cdot 3750$ | 5 |
| 1 | 5.73687 | -0.25376 | 0.24152 | 1.40429 | 1.47545 | 0.53554 | -0.66098 |  |  |
| 2 | 5.73687 | -0.23165 | 0.07129 | 0.41452 | 1.34690 | 0.09290 | $-0.71847$ |  |  |
| 0 | 5.73687 | 0.30621 | 0.20277 | 1.17897 | $-1.78041$ | 1.03469 | 0.76885 | 0.4000 | 5 |
| 1 | 5.73687 | -0.24246 | 0.26157 | 1.52085 | 1.40974 | 0.26962 | $-0.63572$ |  |  |
| 2 | 5.73687 | -0.24123 | 0.08193 | 0.47638 | 1.40260 | -0.19748 | -0.63891 |  |  |
| 0 | 5.73687 | 0.14471 | 0.04886 | 0.28409 | -0.84143 | 0.96034 | 0.92211 | 0.4375 | 5 |
| 1 | 5.73687 | -0.21531 | 0.28980 | 1.68502 | 1.25190 | 0.07942 | -0.49057 |  |  |
| 2 | 5.73687 | -0.25353 | 0.09868 | 0.57378 | 1.47410 | -0.41754 | -0.39120 |  |  |
| 0 | 5.73687 | 0.08545 | 0.02679 | 0.15579 | $-0.49686$ | 0.97479 | 0.96515 | 0.45313 | 5 |
| 1 | 5.73687 | -0.19977 | 0.29980 | 1.74317 | 1-16155 | 0.04731 | $-0.44113$ |  |  |
| 2 | 5.73687 | -0.25794 | 0.10593 | 0.61591 | 1.49975 | -0.45681 | -0.28988 |  |  |
| 0 | 5.73687 | -0.00645 | 0.01543 | 0.08970 | 0.03748 | 1.02753 | 0.99888 | $0 \cdot 48438$ | 5 |
| 1 | 5.73687 | -0.16073 | 0.31321 | 1.82113 | 0.93455 | 0.01560 | $-0.37660$ |  |  |
| 2 | 5.73687 | $-0.26553$ | 0.12088 | 0.70284 | 1.54390 | $-0.48451$ | -0.10409 |  |  |

Table 4 (continued)

| Sequence |  |  |  | Sequence admittance |  | Self- and mutual admittances |  | $\frac{d}{\lambda}$ | $N$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m$ | $\Psi_{d R}$ | $\operatorname{Re} T^{(m)}$ | $\operatorname{Im} T^{(m)}$ | $G^{(m)}$ | $B^{(m)}$ | $G_{1(m+1)}$ | $\boldsymbol{B}_{1(m+1)}$ |  |  |
| 0 | 5.73687 | -0.04103 | 0.01824 | $0 \cdot 10608$ | 0.23859 | 1.05256 | 0.99281 | 0.5000 | 5 |
| 1 | 5.73687 | $-0.13765$ | 0.31486 | 1.83072 | 0.80033 | 0.00556 | -0.35896 |  |  |
| 2 | 5.73687 | -0.26871 | 0.12859 | 0.74764 | 1.56240 | $-0.47880$ | -0.01815 |  |  |
| 0 | 5.73687 | $-0.06970$ | 0.02385 | 0.13868 | 0.40525 | 1.07040 | 0.97532 | 0.51563 | 5 |
| 1 | 5.73687 | -0.11303 | 0.31186 | 1.81328 | 0.65720 | $-0.00487$ | -0.34852 |  |  |
| 2 | 5.73687 | -0.27148 | 0.13645 | 0.79337 | 1.57848 | -0.46099 | 0.06349 |  |  |
| 0 | 5.73687 | -0.12986 | 0.04795 | 0.27882 | 0.75507 | 1.05588 | 0.89736 | 0.5625 | 5 |
| 1 | 5.73687 | $-0.04373$ | 0.26898 | 1.56397 | 0.25424 | $-0.05392$ | -0.33909 |  |  |
| 2 | 5.73687 | -0.27718 | $0 \cdot 16104$ | 0.93633 | 1.61162 | -0.33461 | 0.26795 |  |  |
| 0 | 5.73687 | -0.17222 | 0.08301 | 0.48267 | 1.00134 | 0.92501 | 0.89242 | 0.6250 | 5 |
| 1 | 5.73687 | -0.01972 | 0.15994 | 0.92995 | $0 \cdot 11463$ | $-0.15783$ | $-0.30843$ |  |  |
| 2 | 5.73687 | -0.27789 | 0.19628 | 1.14124 | 1.61574 | $-0.06334$ | 0.36289 |  |  |
| 0 | 5.73687 | $-0.18928$ | $0 \cdot 11386$ | 0.66204 | $1 \cdot 10052$ | 0.87378 | 1.02151 | 0.6875 | 5 |
| 1 | 5.73687 | -0.07632 | 0.08466 | 0.49228 | 0.44377 | $-0.24722$ | -0.22979 |  |  |
| 2 | 5.73687 | -0.26826 | 0.23410 | 1.36115 | $1 \cdot 55976$ | $0 \cdot 14135$ | 0.26930 |  |  |
| 0 | 5.73687 | -0.19114 | 0.12315 | 0.71602 | $1 \cdot 11135$ | 0.89069 | 1.06776 | 0.7100 | 5 |
| 1 | 5.73687 | $-0.10210$ | 0.07336 | 0.42653 | 0.59366 | $-0.27077$ | -0.19626 |  |  |
| 2 | 5.73687 | $-0.26143$ | 0.24804 | 1.44218 | 1.52007 | $0 \cdot 18344$ | 0.21805 |  |  |
| 0 | 5.73687 | -0.19034 | 0.13638 | 0.79297 | 1-10669 | 0.95087 | $1 \cdot 12366$ | 0.7500 | 5 |
| 1 | 5.73687 | -0.14431 | 0.06840 | 0.39769 | 0.83910 | $-0.30452$ | -0.13340 |  |  |
| 2 | 5.73687 | $-0.24366$ | 0.27226 | 1.58301 | 1.41671 | 0.22557 | 0.12492 |  |  |

## APPENDIX II

## Summary of the two-term theory for applications

To facilitate practical applications of the theory apart from its detailed development and verification, principal points of the two-term theory, together with the steps required in its utilization, are summarized in this appendix. Numerical results for arrays containing only a few elements can be computed by hand with the aid of the tables in appendix I or those in the literature [1]. Calculations for larger arrays or for element parameters not included in the tables generally require a computer. For these applications, the theory can be conveniently packaged as a computer programme to which the user need only supply input data cards specifying the parameters of the elements and array to obtain a numerical evaluation of any properties of the array [2]. In this form, the theory can be used without considering either intricate mathematics or complicated programming steps once the initial programming is completed. In a reasonably general programme, the parameters that can be specified as input data are the number $N$ of antennas in the array, their radius $a$, length $2 h$, spacing $d$, and the relative driving voltages $V_{i}$ or currents $I_{i}(0)$.

The theory applies to arrays of thin, identical, parallel, non-staggered, centre-driven, highly conducting dipoles which are uniformly spaced around a circle. If the admittances and currents are multiplied by two, they also apply to arrays of vertical monopoles over a large highlyconducting ground plane. The normalized radiation patterns apply to either dipoles or monopoles, but for the latter the ground plane must extend beyond the radius at which the field is evaluated and measured. Dipole lengths should not exceed $1 \cdot 2 \lambda$ and the distance between adjacent elements should be at least $\lambda / 8$; the complete three-term theory is required for high accuracy when $\beta_{0} d_{i k}<1$. To satisfy completely the mathematical conditions of the theory, the element thickness should satisfy $a \leqslant 0.01 \lambda$. However, useful results can be obtained for elements that are two or three times as thick. Comparisons of theoretical and measured results show that predicted radiation patterns are usually well within $\pm 1 \mathrm{~dB}$ of the measured ones, and that calculated self- and mutual admittances are generally within $5 \%-10 \%$ of the measured values except near resonant lengths or spacings. Near resonance, theoretical values are shifted toward larger values of $h / \lambda$ and $d / \lambda$ by approximately one-half of the element radius. The comparison of self- and mutual admittances assumes that proper account has been taken of end-effects. The theoretical drivingpoint admittances apply only to dipoles with no attached transmission
lines, and the presence of these lines must be taken into account as discussed in sections $2.8,4.3$, and 8.2.

The central point of the two-term theory is that the current along each dipole can be accurately represented in the form $I_{k}(z)=s_{k} \sin \beta_{0}(h-|z|)+$ $c_{k}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)$. Driving-point admittances are then given by $Y_{k}=I_{k}(0) / V_{k}=s_{k} \sin \beta_{0} h+c_{k}\left(1-\cos \beta_{0} h\right)$ and the far-zone fields by eqs. (4.23). The principal computing problem is to evaluate the $s_{k}$ and $c_{k}$ coefficients which are functions of all of the parameters of the array. Because the $s_{k}$ and $c_{k}$ coefficients corresponding to different elements are not simply related, the field expressions cannot be summed in a closed form as with the usual one-term sinusoidal theory. However, in the singleterm sinusoidal theory the current is always represented by the sine term alone and this is accurate only for very thin dipoles with lengths very near $2 h=\lambda / 2$. With the two-term theory, general expressions for the currents become indeterminate at $2 h=\lambda / 2$ and correct expressions are found by taking limits as $\beta_{0} h \rightarrow \pi / 2(2 h \rightarrow \lambda / 2)$. The special expressions applicable to $\beta_{0} h=\pi / 2$ are indicated in the text (e.g. (2.35) on p. 56).

For a general array, the evaluation of $s_{k}$ and $c_{k}$ or their equivalents requires the solution of a system of coupled integral equations as in chapters 5 and 6 . However, the symmetry of a circular array of identical equispaced elements permits a reduction of the system of equations to a single integral equation. This reduction is accomplished by the method of symmetrical components in which the elements are all excited with equal amplitudes but with uniformly progressive phase so that the total phase change around the circle is $2 \pi m$ radians where $m$ is an integer ranging from zero to $N-1$. Each value of $m$ gives one of the $N$ phase sequences which is designated by a superscript $m$. The resulting functions are called sequence functions and they form a set that is characteristic of the elements and their spacing in the array. The associated properties of the elements such as the driving-point admittances for arbitrary driving conditions can be obtained from the sequence functions.

In the engineering design of arrays, the quantities of primary concern are radiation patterns and driving-point admittances or impedances. Regardless of whether the objective is to calculate only radiation patterns and driving-point admittances or to perform a complete analysis of the array, the same basic sequence of steps is required in a programme of calculation that utilizes the two-term theory. These steps are as follows:

1. Evaluation of the $C_{b i}, S_{b i}$, and $E_{b i}$ integrals (right part, eqs. (4.13)).
2. Calculation of the sequence sums (left part, eqs. (4.13)).
3. Calculation of required differences of the sequence sums (eqs. (4.12)).
4. Calculation of $\Psi_{d R}$-from eq. (4.16).
5. Calculation of the remaining $\Psi$ functions-from eqs. (4.7)-(4.11).
6. Calculation of $T^{(m)}$ or $T^{(m)}$-from eqs. (4.5).
7. Calculation of $s^{(m)}$ and $c^{(m)}$ or $s^{\prime(m)}$ and $c^{(m)}$-from eqs. (4.15).

Results obtainable at this point:
a. Sequence currents $I^{(m)}(z)$-from eqs. (4.16).
b. Sequence admittances $Y^{(m)}$-from eqs. (4.17).
c. Self- and mutual admittances $Y_{1 k}$-from eqs. (4.19).
d. Self- and mutual impedances $Z_{1 k}$-from eqs. (4.26) and (4.27).
e. Driving-point admittances $Y_{k}$-from eq. (4.24b).
f. Driving-point impedances $Z_{k}$-from reciprocals of driving-point admittances.
8. Calculation of sequence voltage $V^{(m)}$-from eqs. (4.20a).

Note: If driving-point currents are specified, sequence currents $I^{(m)}(0)$ are calculated from eq. (4.20b) and sequence voltages from eq. (4.17).
9. Calculation of $s_{k}$ and $c_{k}$-from eqs. (4.22).

Final results obtainable:
g. Element currents $I_{k}(z)$-from eqs. (4.21).
h. Driving-point admittances-from eqs. (4.24a).
i. Far-zone fields-from eqs. (4.23).
j. Normalized power radiation pattern-from eq. (4.28).

Steps 1-3 will be discussed in detail. The driving-point admittances can be calculated either as in steps $e$ or $h$. Also, driving-point impedances are simply the reciprocals of corresponding driving-point admittances. If the driving-point voltages or currents are to be specified according to some rule so as to provide a particular distribution such as a Tchebyscheff, provision can be made for a subroutine to calculate them as part of step 8 .

The special case of $\beta_{0} h=\pi / 2(h=\lambda / 4)$ is most easily incorporated in a general programme by providing a subroutine to carry out parts of the calculation for this length. If the programme is designed to examine the input data and use the appropriate subroutine whenever $h=\lambda / 4$ is specified, the range of $h / \lambda$ for which the programme can be used is continuous except for values of $h / \lambda$ that differ from $\frac{1}{4}$ by less than about $0 \cdot 001 \lambda$. At these values some of the functions may be sufficiently close to the indeterminate form $0 / 0$ to produce computer overflow.

The integrals of eqs. (4.13) must be evaluated numerically. Although more efficient methods may be found, Simpson's Rule is adequate. Let the integrand $f(z)$ of an integral $I$ be evaluated at an odd number of points which are uniformly spaced at multiples of the interval $\Delta z^{\prime}$. Then, according to Simpson's Rule,

$$
\begin{align*}
I= & \int_{r}^{s} f\left(z^{\prime}\right) d z^{\prime} \\
\doteq & \frac{\Delta z^{\prime}}{3}\left\{f(r)+f(s)+4\left[f\left(r+\Delta z^{\prime}\right)+f\left(r+3 \Delta z^{\prime}\right)+\ldots f\left(s-\Delta z^{\prime}\right)\right]\right. \\
& \left.+2\left[f\left(r+2 \Delta z^{\prime}\right)+f\left(r+4 \Delta z^{\prime}\right)+\ldots f\left(s-2 \Delta z^{\prime}\right)\right]\right\} \tag{1}
\end{align*}
$$

with an approximate error of integration $\varepsilon$ given by

$$
\begin{equation*}
|\varepsilon| \leqslant\left|\frac{I\left(\Delta z^{\prime}\right)-I\left(2 \Delta z^{\prime}\right)}{10}\right| . \tag{2}
\end{equation*}
$$

As a convenient starting point in their evaluation, the integrals may be written in the following form:

$$
\begin{align*}
E(h, z)= & \operatorname{Re} E(h, z)+j \operatorname{Im} E(h, z) \\
= & \int_{0}^{h / \lambda}\left[\left(\frac{\cos 2 \pi \bar{R}_{1}}{\bar{R}_{1}}+\frac{\cos 2 \pi \bar{R}_{2}}{\bar{R}_{2}}\right)-j\left(\frac{\sin 2 \pi \bar{R}_{1}}{\bar{R}_{1}}+\frac{\sin 2 \pi \bar{R}_{2}}{\bar{R}_{2}}\right)\right] d z_{1}^{\prime}  \tag{3a}\\
C(h, z)= & \operatorname{Re} C(h, z)+j \operatorname{Im} C(h, z) \\
= & \int_{0}^{h / \lambda} \cos 2 \pi z_{1}^{\prime}\left[\left(\frac{\cos 2 \pi \bar{R}_{1}}{\bar{R}_{1}}+\frac{\cos 2 \pi \bar{R}_{2}}{\bar{R}_{2}}\right)\right. \\
& \left.-j\left(\frac{\sin 2 \pi \bar{R}_{1}}{\bar{R}_{1}}+\frac{\sin 2 \pi \bar{R}_{2}}{\bar{R}_{2}}\right)\right] d z_{1}^{\prime}  \tag{3b}\\
S(h, z)= & \int_{0}^{h / \lambda} \sin 2 \pi z_{1}^{\prime}\left[\left(\frac{\cos 2 \pi \bar{R}_{1}}{\bar{R}_{1}}+\frac{\cos 2 \pi \bar{R}_{2}}{\bar{R}_{2}}\right)\right. \\
& \left.-j\left(\frac{\sin 2 \pi \bar{R}_{1}}{\bar{R}_{1}}+\frac{\sin 2 \pi \bar{R}_{2}}{\bar{R}_{2}}\right)\right] d z_{1}^{\prime}  \tag{3c}\\
\bar{R}_{1}= & \sqrt{\left(z / \lambda-z_{1}^{\prime}\right)^{2}+\left(b_{i} / \lambda\right)^{2}} \quad \bar{R}_{2}=\sqrt{\left(z / \lambda+z_{1}^{\prime}\right)^{2}+\left(b_{i} / \lambda\right)^{2}} \tag{3d}
\end{align*}
$$

One set of integrals must be evaluated for each element of the array and the sum of these values in subsequent steps takes account of the intercoupling of the elements. The distance from antenna 1 to the $i^{\text {th }}$ antenna is $b_{i}$ given by

$$
\begin{equation*}
b_{i} / \lambda=\frac{(d / \lambda) \sin (i-1) \pi / N}{\sin \pi / N} \tag{4}
\end{equation*}
$$

Thus, for $i=2, b_{2} / \lambda$ corresponds to the distance between adjacent antennas and equals $d / \lambda$.

When $i=1$, the integrals are to be evaluated at the dipole surface and the value to be used for $b_{i}$ is $b_{1} / \lambda=a / \lambda$, the dipole radius. Very small integration intervals $\Delta z^{\prime}$ are required for these integrals because the element is thin, $a / \lambda \ll 1$, and some of the integrands rise rapidly to a sharp maximum of $\lambda / a$ at $z_{1}^{\prime}=z / \lambda$. These integrals with $b_{1}=a$ correspond to a single isolated dipole; they are functions only of $h / \lambda$ and $a / \lambda$ and remain constant as the other array parameters are changed. Their evaluation comprises one of the longest parts of the calculation and considerable computer time can be saved by evaluating them once and supplying their values as input data for subsequent analyses of different arrays that use dipoles of the same $h / \lambda$ and $a / \lambda$.

As a guide in choosing the interval size $\Delta z^{\prime}$, or the number of points $n_{p}$ at which the integrands are to be evaluated, the error $\varepsilon$ of eq. (2) is less than $10^{-5}$ for any of the ' $b$ ' integrals when $d / \lambda>1 / 8$ with the following choice of $n_{p}: h=\lambda / 4, n_{p}=17 ; h=3 \lambda / 8, n_{p}=25 ; h=\lambda / 2, n_{p}=33$. For the ' $a$ ' integrals, the following choice of $n_{p}$ produces an error of $10^{-5}$ or less in the integrals with $a=0.007 \lambda: h=\lambda / 4, n_{p}=193 ; h=\lambda / 2$, $n_{p}=385$. For $a=0.002 \lambda$ and $h=\lambda / 4, n_{p}=1537$ points are necessary to ensure that the error remains less than $10^{-5}$ for all of the integrals.

Each integral is a function of the three parameters $h / \lambda, z / \lambda$, and $b_{i} / \lambda$. The distances $b_{i} / \lambda$ are obtained from eq. (4) with $d / \lambda$ supplied as part of the input data; the element length $h / \lambda$ and radius $a / \lambda$ must also be specified with the input data. For each $h / \lambda$ and $b_{i} / \lambda$, the two-term theory requires an evaluation of the integrals at the values of $z$ in the following list:

$$
\begin{aligned}
& h>\lambda / 4 \text { : ' } b \text { ' integrals }(i>1) \quad \text { ' } a \text { ' integrals }(i=1) \\
& C(h, h), C(h, 0) \quad C(h, h), C(h, 0) \operatorname{Re} C(h, h-\lambda / 4) \\
& S(h, h), S(h, 0) \quad S(h, h), \operatorname{Im} S(h, 0), \operatorname{Re} S(h, h-\lambda / 4) \\
& E(h, h), E(h, 0) \quad E(h, h), E(h, 0)
\end{aligned}
$$

$h<\lambda / 4$ : Required ' $b$ ' integrals are the same but in ' $a$ ' integrals, $\operatorname{Re} C(h, 0)$ and $\operatorname{Re} S(h, 0)$ are required instead of $\operatorname{Re} C(h, h-\lambda / 4)$ and $\operatorname{Re} S(h, h-\lambda / 4)$.
$\begin{aligned} h=\lambda / 4: & \text { ' } b \text { ' integrals }(i>1) \\ & \text { 'a' integrals }(i=1) \\ & C(h, h) \\ & C(h, h), \operatorname{Re} C(h, 0) \\ & (h, h)\end{aligned} \quad S(h, h), \operatorname{Re} S(h, 0), ~ E(h, h) \quad$.
After the required integrals have been evaluated and stored in an ordered array, the sums to be used in calculating $\Psi$ functions may be computed. Four different kinds of sums are used in eqs. (4.6)-(4.11) and each is designated by a subscript as follows: A subscript $\Sigma$ means the sum is to be extended over all values of $i$; a subscript $\Sigma 1$ means that only the integral corresponding to $i=1$ with $b_{i} / \lambda=b_{1} / \lambda=a / \lambda$ is to be used; a subscript $\Sigma 2$ means that the sum excludes $i=1$ but includes all other values of $i$; and a subscript $d$ indicates that a difference of two sums is to be used as in eq. (4.12).
The general arrangement of a programme that might be used to package the two-term theory for applications is shown schematically in Fig. 1 (pp. 419-21).


Fig. 1. Flow chart for applications of two-term theory.



## APPENDIX III

## Summary of formulas for the curtain array

This appendix contains (1) a summary of formulas for the curtain array, (2) tables of $\Psi_{d R}, \Phi_{u}$ and $\Phi_{v}$ for single elements as functions of $\Omega$ and $h / \lambda$ and tables of $\Phi_{u}$ and $\Phi_{v}$ for off-diagonal elements as functions of ( $k-i$ ) $(b / \lambda)$ and $h / \lambda$ (Table 1), (3) sample computer printouts for two cases are included. They give the currents, impedances and field patterns for curtain arrays both for the case of driving-point currents specified and driving voltages specified.

A summary of the curtain array formulas is given below.

1. Driving-point current, voltages, admittances and impedances:
2. Base currents specified.

$$
\left\{I_{2}(0)\right\} \quad \text { specified }
$$

Then

$$
\left\{V_{0}\right\}=-j 60 \Psi_{d R} \frac{\cos \beta_{0} h}{1-\cos \beta_{0} h}\left[\Phi_{u}\right]\left[\left[\Phi_{v}\right]+\frac{\sin \beta_{0} h}{1-\cos \beta_{0} h}\left[\Phi_{u}\right]\right]^{-1}\left\{I_{z}(0)\right\}
$$

and near $\beta_{0} h=n \pi / 2$,

$$
\begin{equation*}
\left\{V_{0}\right\}=j 60 \Psi_{d R}\left[-\sin \beta_{0} h\left[\Phi_{u}^{\prime}\right]+\left(1-\cos \beta_{0} h\right)\left[\Phi_{r}^{\prime}\right]\right]^{-1}\left[\Phi_{u}^{\prime}\right]\left\{V_{0}\right\} . \tag{A3-2}
\end{equation*}
$$

2. Base voltages specified

$$
\left\{V_{0}\right\} \text { specified }
$$

Then

$$
\left.\begin{array}{c}
\left\{I_{z}(0)\right\}=\frac{j}{60 \Psi_{d R}} \frac{1-\cos \beta_{0} h}{\cos \beta_{0} h}\left[\Phi_{u}\right]^{-1}[
\end{array}\left[\Phi_{v}\right]+\frac{\sin \beta_{0} h}{1-\cos \beta_{0} h}\left[\Phi_{u}\right]\right]\left\{V_{0}\right\}
$$

and near $\beta_{0} h=n \pi / 2, n$ odd
$\left\{I_{z}(0)\right\}=\frac{-j}{60 \Psi_{d R}}\left[\Phi_{u}^{\prime}\right]^{-1}\left[-\sin \beta_{0} h\left[\Phi_{u}^{\prime}\right]+\left(1-\cos \beta_{0} h\right)\left[\Phi_{v}^{\prime}\right]\right]\left\{V_{0}\right\}$.
The individual driving-point impedances $Z_{0 i}$ and admittances $Y_{0 i}, i=1,2,3, \ldots N$ are

$$
\begin{equation*}
Z_{0 i}=V_{0 i} / I_{z i}(0), \quad Y_{0 i}=I_{z i}(0) / V_{0 i} \tag{A3-5}
\end{equation*}
$$

II. Distributions of current on the individual elements:

1. $\quad I_{z i}(z)=\frac{j 2 \pi V_{0 i}}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0}(h-|z|)+T\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]$,

$$
\begin{equation*}
\beta_{0} h \neq \frac{n \pi}{2}, n \text { odd } \tag{A3-6}
\end{equation*}
$$

where

$$
T=-\frac{j\left(Y_{0 i} / A\right)+\sin \beta_{0} h}{1-\cos \beta_{0} h}
$$

and

$$
A=\frac{2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}
$$

2. $\quad I_{z}(z)=\frac{-j 2 \pi}{\zeta_{0} \Psi_{d R} \cos \beta_{0} h}\left[\sin \beta_{0} z-\sin \beta_{0} h+T^{\prime}\left(\cos \beta_{0} z-\cos \beta_{0} h\right)\right]$.

$$
\begin{align*}
& \beta_{0} h \text { near } \frac{n \pi}{2}, n \text { odd }  \tag{A3-7}\\
& T^{\prime}=\frac{j\left(Y_{0 i} / A\right)+\sin \beta_{0} h}{1-\cos \beta_{0} h}
\end{align*}
$$

III. Field patterns

1. Array of $N$ elements, $\beta_{0} h \neq n \pi / 2, n$ odd

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta, \Phi)=\frac{-V_{01}}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{0}}}{R_{0} \cos \beta_{0} h_{i=1}^{N}} \sum_{i} C_{i} e^{\left.-j \beta_{0} b(N-2 i+1) / 2\right] \sin \Theta \cos \Phi} \tag{A3-8}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{i}=\xi_{i}\left[F_{m}\left(\Theta, \beta_{0} h\right)-\frac{\left[j\left(Y_{0 i} / A\right)+\sin \beta_{0} h\right]}{1-\cos \beta_{0} h} G_{m}\left(\Theta, \beta_{0} h\right)\right] . \quad \xi_{i}=\frac{V_{0 i}}{V_{01}} \tag{A3-9}
\end{equation*}
$$

2. Array of $N$ elements, $\beta_{0} h$ near $n \pi / 2$

$$
\begin{equation*}
E_{\Theta}^{r}(\Theta, \Phi)=\frac{V_{01}}{\Psi_{d R}} \frac{e^{-j \beta_{0} R_{0}}}{R_{0}} \sum_{i=1}^{N} C_{i}^{\prime} e^{\left.-j \beta_{0} b(N-2 i+1) / 2\right] \sin \Theta \cos \Phi} \tag{A3-10}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{i}^{\prime}=\xi_{i}\left[H_{m}\left(\Theta, \beta_{0} h\right)+\frac{\left[j\left(Y_{0 i} / A\right)+\sin \beta_{0} h\right]}{1-\cos \beta_{0} h} G_{m}\left(\Theta, \beta_{0} h\right)\right] . \tag{A3-11}
\end{equation*}
$$

Numerical values for the functions $\Psi_{d R}, \Phi_{u}$ and $\Phi_{v}$ needed to solve the matrix equations for the driving-point impedances and currents of curtain arrays are given in Table 1. Note that the off-diagonal matrix elements are dependent only on the spacing between the two elements. The integration programme used to compute these functions is an improved Romberg method. The programme is arranged so as to include the correct contributions from the major singularities.

A sample calculation for the case of a three-element endfire array with quarter-wavelength spacing and half-wavelength elements follows. The following headings appear in the printout:

| N | Number of elements |
| :--- | :--- |
| BETAH | $\beta_{0} h$, electric half-length of the antenna |
| BETAB | $\beta_{0} b$, electrical distance between two adjacent elements |
| IZ $(0)$ | $I_{i z}(0)$, driving-point currents |
| V0 | $V_{0 i}$, base voltages |
| ADMTC | $Y_{0 i}$, admittances |

IMPDC $Z_{0_{i}}$, impedances
PSI $\quad \xi_{i}=V_{0 i} / V_{01}$, ratios of base voltages
C $\quad C_{i}$, source strength coefficients for radiation patterns
$\operatorname{IZ}(\mathrm{Z}) \quad I_{z i}(z)$, element currents where $z=0,1 / 4 \beta, 1 / 2 \beta, 3 / 4 \beta, \ldots 3 / \beta$, $\pi / \beta$ where $i=1,2,3, \ldots N$

In the sample printout, values are read from left to right under each heading. For example, driving voltages are read as follows:

V0

$$
\begin{aligned}
& \operatorname{Re}\left(V_{01}\right) \operatorname{Im}\left(V_{01}\right) \quad \operatorname{Re}\left(V_{02}\right) \operatorname{Im}\left(V_{02}\right) \\
& \operatorname{Re}\left(V_{03}\right) \ldots
\end{aligned}
$$

$\operatorname{IZ}(\mathrm{Z})$, or $I_{z i}(z)$, is arranged so that all 14 values for one particular value of $i$ are in one group. The groups are then read vertically down each of the two columns on the page. For example:

$$
\begin{aligned}
& \operatorname{IZ}(\mathrm{Z})=I_{z i}(z) \\
& \operatorname{Re}\left(I_{z 1}(0)\right) \operatorname{Im}\left(I_{z 1}(0)\right) \operatorname{Re}\left(I_{z 1}(1 / 4 \beta)\right) \operatorname{Im}\left(I_{z 1}(1 / 4 \beta)\right) \\
& \vdots \\
& \operatorname{Re}\left(I_{z 1}(3 / \beta)\right) \operatorname{Im}\left(I_{z 1}(3 / \beta)\right) \operatorname{Re}\left(I_{z 1}(\pi / \beta)\right) \operatorname{Im}\left(I_{z 1}(\pi / \beta)\right) \\
& \operatorname{Re}\left(I_{z 2}(0)\right) \ldots \\
& \vdots \\
& \operatorname{Re}\left(I_{z 3}(0) \ldots\right.
\end{aligned}
$$

Note that only terms for values of $z$ equal to or less than $h$ are used. For example, when $h=\pi / 2 \beta$, the last term used is $3 / 2 \beta$. If $h=\pi / \beta$ all terms are used.

## NOTE

The off-diagonal values of $\Phi_{u}$ and $\Phi_{v}$ on pages 431-49 depend only on the distance $b_{i k}$. The left-hand column, i.e. $(k-i) b / \lambda$, is the distance between elements $k$ and $i$. The two columns under each $\Phi_{k i u}$ and $\Phi_{k i v}$ are the real and imaginary parts.

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with driving-point currents specified

|  | $\mathrm{N}=3 \quad \mathrm{PSIDR}=$ | 5.83400 | BETAH $=3.14159$ | BETAB $=1.57080$ | FM $=2.00000$ | $\mathrm{GM}=3.14159$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| IZ(O) |  |  |  |  |  |  |  |
| 1.00000000 | 0. |  | 0. | -1.00000000 |  |  |  |
| -1.00000000 | 0. |  |  |  |  |  |  |
| PHIV |  |  |  |  |  |  |  |
| 0.67210 | -1.66050 | -0.68130 | - -0.85700 | -0.62960 | 0.40710 | -0.68130 | $-0.85700$ |
| 0.67210 | -1.66050 | -0.68130 | - $\quad 0.85700$ | -0.62960 | 0.40710 | -0.68130 | -0.85700 |
| 0.67210 | -1.66050 |  |  |  |  |  |  |
| PHIU |  |  |  |  |  |  |  |
| -6.68980 | 2.89590 | 1.02170 | 1.52470 | 1.10300 | -0.66350 | 1.02170 | 1.52470 |
| -6.68980 | 2.89590 | 1.02170 | 1.52470 | $1 \cdot 10300$ | -0.66350 | 1.02170 | 1.52470 |
| -6.68980 | 2.89590 |  |  |  |  |  |  |
| vo |  |  |  |  |  |  |  |
| 611.59042358 | - 591.04013824 |  | -590.09282684 | -160.23533821 |  |  |  |
| -61.54054213 | 435.34058380 |  |  |  |  |  |  |
| IMPDNC |  |  |  |  |  |  |  |
| 611.59042358 | -591.04013824 |  | $160-23533821$ | -590.09282684 |  |  |  |
| 61.54054213 | -435.34058380 |  |  |  |  |  |  |
| ADMTNC |  |  |  |  |  |  |  |
| 0.00084547 | 0.00081706 |  | 0.00042856 | 0.00157827 |  |  |  |
| 0.00031835 | 0.00225204 |  |  |  |  |  |  |
| PSI |  |  |  |  |  |  |  |
| 1.00000000 | -0. |  | -0.36798476 | -0.61761774 |  |  |  |
| -0.40773164 | 0.31778590 |  |  |  |  |  |  |

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with driving-point currents specified-contd.

| IZ(Z)YOI |  |  |  |
| :--- | ---: | ---: | ---: |
| 0.00084547 | 0.00081706 | 0.00083233 | 0.00009757 |
| 0.00079372 | -0.00060257 | 0.00073204 | -0.00123987 |
| 0.00065114 | -0.00177466 | 0.00055603 | -0.00217371 |
| 0.00045264 | -0.00241222 | 0.00034738 | -0.00247534 |
| 0.00024682 | -0.00235917 | 0.0001578 | -0.00200990 |
| 0.00008406 | -0.00162847 | 0.00003199 | -0.00105940 |
| 0.00000422 | -0.00039905 | -0. | 0.00000000 |
| 0.00081706 | -0.00084547 | 0.00036784 | -0.00057224 |
| -0.00007885 | -0.00028972 | -0.00049524 | -0.00001546 |
| -0.00085544 | 0.00023346 | -0.00113706 | 0.00044160 |
| -0.00132257 | 0.00059599 | -0.00140045 | 0.00068704 |
| -0.00136586 | 0.00070909 | -0.00122094 | 0.00066077 |
| -0.00097471 | 0.00054508 | -0.00064248 | 0.00036922 |
| -0.00024490 | 0.00014412 | 0.00000000 | -0.00000000 |
| -0.00084547 | -0.00081706 | -0.00060772 | -0.00051618 |
| -0.00035847 | -0.00020861 | -0.00011321 | 0.00008653 |
| 0.0001279 | 0.00035089 | 0.00030550 | 0.00056803 |
| 0.000425949 | 0.00072446 | 0.00054593 | 0.00081044 |
| 0.00057869 | 0.00082064 | 0.00054919 | 0.00075440 |
| 0.00045925 | 0.00061586 | 0.00031449 | 0.00041363 |
| 0.00012388 | 0.00016028 | -0.00000000 | -0.00000000 |

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with driving-point currents specified-contd.

| C |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.00153787 |  | -0.00461196 | $-0.00361550$ | -0.00157552 |  |  |  |
|  | 00126944 | 0.00253471 |  |  |  |  |  |
| E |  |  |  |  |  |  |  |
| 0 | 0.01162040 | 2 | 0.01162091 | 4 | 0.01162239 | 6 | 0.01162473 |
| 8 | 0.01162772 | 10 | 0.01163108 | 12 | 0.01163444 | 14 | 0.01163737 |
| 16 | 0.01163937 | 18 | 0.01163985 | 20 | 0.01163815 | 22 | 0.01163358 |
| 24 | 0.01162538 | 26 | 0.01161272 | 28 | 0.01159474 | 30 | 0.01157056 |
| 32 | 0.01153924 | 34 | 0.01149984 | 36 | 0.01145143 | 38 | 0.01139304 |
| 40 | 0.01132374 | 42 | 0.01124261 | 44 | 0.01114879 | 46 | 0.01104144 |
| 48 | 0.01091982 | 50 | 0.01078323 | 52 | 0.01063106 | 54 | 0.01046284 |
| 56 | 0.01027817 | 58 | 0.01007678 | 60 | 0.00985856 | 62 | 0.00962351 |
| 64 | 0.00937182 | 66 | 0.00910378 | 68 | 0.00881989 | 70 | 0.00852080 |
| 72 | 0.00820733 | 74 | 0.00788044 | 76 | 0.00754129 | 78 | 0.00719119 |
| 80 | 0.00683161 | 82 | 0.00646418 | 84 | 0.00609070 | 86 | 0.00571312 |
| 88 | 0.00533357 | 90 | 0.00495435 | 92 | 0.00457800 | 94 | 0.00420730 |
| 96 | 0.00384534 | 98 | 0.00349566 | 100 | 0.00316235 | 102 | 0.00285022 |
| 104 | 0.00256502 | 106 | 0.00231355 | 108 | 0.00210349 | 110 | 0.00194265 |
| 112 | 0.00183734 | 114 | 0.00179002 | 116 | 0.00179765 | 118 | 0.00185201 |
| 120 | 0.00194210 | 122 | 0.00205668 | 124 | 0.00218605 | 126 | 0.00232258 |
| 128 | 0.00246059 | 130 | 0.00259605 | 132 | 0.00272615 | 134 | 0.00284899 |
| 136 | 0.00296339 | 138 | 0.00306866 | 140 | 0.00316450 | 142 | 0.00325087 |
| 144 | 0.00332797 | 146 | 0.00339614 | 148 | 0.00345585 | 150 | 0.00350764 |
| 152 | 0.00355213 | 154 | 0.00358995 | 156 | 0.00362176 | 158 | 0.00364819 |
| 160 | 0.00366990 | 162 | 0.00368751 | 164 | 0.00370158 | 166 | 0.00371265 |
| 168 | 0.00372122 | 170 | 0.00372771 | 172 | 0.00373250 | 174 | 0.00373590 |
| 176 | 0.00373816 | 178 | 0.00373944 | 180 | 0.00373985 | 182 |  |

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with driving-point currents specified-contd.

| E1 |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 0.00492699 | 2 | 0.00492784 | 4 | 0.00493038 | 6 | 0.00493454 |
| 8 | 0.00494024 | 10 | 0.00494736 | 12 | 0.00495573 | 14 | 0.00496517 |
| 16 | 0.00497545 | 18 | 0.00498629 | 20 | 0.00499744 | 22 | 0.00500855 |
| 24 | 0.00501928 | 26 | 0.00502925 | 28 | 0.00503807 | 30 | 0.00504531 |
| 32 | 0.00505054 | 34 | 0.00505330 | 36 | 0.00505313 | 38 | 0.00504956 |
| 40 | 0.00504210 | 42 | 0.00503029 | 44 | 0.00501366 | 46 | 0.00499177 |
| 48 | 0.00496417 | 50 | 0.00493045 | 52 | 0.00489025 | 54 | 0.00484321 |
| 56 | 0.00478903 | 58 | 0.00472746 | 60 | 0.00465829 | 62 | 0.00458138 |
| 64 | 0.00449665 | 66 | 0.00440407 | 68 | 0.00430369 | 70 | 0.00419565 |
| 72 | 0.00408011 | 74 | 0.00395735 | 76 | 0.00382772 | 78 | 0.00369162 |
| 80 | 0.00354953 | 82 | 0.00340200 | 84 | 0.00324966 | 86 | 0.00309318 |
| 88 | 0.00293329 | 90 | 0.00277083 | 92 | 0.00260662 | 94 | 0.00244159 |
| 96 | 0.00227673 | 98 | 0.00211307 | 100 | 0.00195174 | 102 | 0.00179394 |
| 104 | 0.00164104 | 106 | 0.00149450 | 108 | 0.00135605 | 110 | 0.00122765 |
| 112 | 0.00111156 | 114 | 0.00101037 | 116 | 0.00092682 | 118 | 0.00086350 |
| 120 | 0.00082221 | 122 | 0.00080330 | 124 | 0.00080527 | 126 | 0.00082500 |
| 128 | 0.00085846 | 130 | 0.00090162 | 132 | 0.00095089 | 134 | 0.00100342 |
| 136 | 0.00105704 | 138 | 0.00111014 | 140 | 0.00116163 | 142 | 0.00121072 |
| 144 | 0.00125691 | 146 | 0.00129991 | 148 | 0.00133955 | 150 | 0.00137579 |
| 152 | 0.00140866 | 154 | 0.00143824 | 156 | 0.00146467 | 158 | 0.00148810 |
| 160 | 0.00150872 | 162 | 0.00152669 | 164 | 0.00154220 | 166 | 0.00155541 |
| 168 | 0.00156651 | 170 | 0.00157563 | 172 | 0.00158291 | 174 | 0.00158844 |
| 176 | 0.00159233 | 178 | 0.00159464 | 180 | 0.00159541 | 182 |  |

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with base voltages specified


Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with base voltages specified-contd.

| $\mathrm{IZ}(\mathrm{Z})$ |  |  |  |
| :---: | :---: | :---: | :---: |
| 0.00097904 | 0.00070133 | 0.00096382 | -0.00001635 |
| 0.00091912 | -0.00071122 | 0.00084770 | -0.00134006 |
| 0.00075401 | -0.00186379 | 0.00064388 | -0.00224983 |
| 0.00052415 | -0.00247418 | 0.00040226 | -0.00252289 |
| 0.00028580 | -0.00239295 | 0.00018201 | -0.00209241 |
| 0.00009734 | -0.00163998 | 0.00003705 | -0.00106378 |
| 0.00000490 | -0.00039963 | -0. | 0.00000000 |
| 0.00133570 | -0.00100300 | 0.00000815 | -0.00098742 |
| -0.00011568 | -0.00094161 | -0.00079080 | -0.00086845 |
| -0.00137523 | -0.00077247 | -0.00183263 | -0.00065964 |
| -0.00213456 | -0.00053698 | -0.00226225 | -0.00041211 |
| -0.00220776 | -0.00029280 | -0.00197448 | -0.00018647 |
| -0.00157691 | -0.00009973 | -0.00103977 | -0.00003796 |
| -0.00039646 | -0.00000501 | 0.00000000 | -0. |
| -0.00080815 | -0.00202375 | -0.00079559 | -0.00128551 |
| -0.00075868 | -0.00053025 | -0.00069973 | 0.00019506 |
| -0.00062239 | 0.00084533 | -0.00053149 | 0.00138012 |
| -0.00043266 | 0.00176620 | -0.00033205 | 0.00197954 |
| -0.00023592 | 0.00200690 | -0.00015024 | 0.00184656 |
| -0.00008035 | 0.00159850 | -0.00003058 | 0.00101373 |
| -0.00000404 | 0.00039302 | 0. | -0.00000000 |

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with base voltages specified-contd.

| C | 0.00132806 | -0.004430170.00104618 | -0.00224539 | 0.00077446 |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  |  |  |
|  | 0.00048764 |  |  |  |  |  |  |
| E |  |  |  |  |  |  |  |
| 0 | 0.00772204 | 2 | 0.00772032 | 4 | 0.00771517 | 6 | 0.00770652 |
| 8 | 0.00769426 | 10 | 0.00767826 | 12 | 0.00765837 | 14 | $0 \cdot 00763436$ |
| 16 | 0.00760600 | 18 | 0.00757303 | 20 | 0.00753516 | 22 | 0.00749207 |
| 24 | 0.00744344 | 26 | 0.00738894 | 28 | 0.00732822 | 30 | 0.00726093 |
| 32 | 0.00718673 | 34 | 0.00710532 | 36 | 0.00701638 | 38 | 0.00691964 |
| 40 | 0.00681487 | 42 | 0.00670187 | 44 | 0.00658051 | 46 | 0.00645070 |
| 48 | 0.00631244 | 50 | 0.00616582 | 52 | 0.00601098 | 54 | $0 \cdot 00584818$ |
| 56 | 0.00567778 | 58 | 0.00550026 | 60 | 0.00531621 | 62 | 0.00512636 |
| 64 | 0.00493156 | 66 | 0.00473284 | 68 | 0.00453135 | 70 | 0.00432844 |
| 72 | 0.00412563 | 74 | 0.00392462 | 76 | 0.00372729 | 78 | 0.00353575 |
| 80 | 0.00335229 | 82 | 0.00317939 | 84 | 0.00301960 | 86 | 0.00287558 |
| 88 | 0.00274985 | 90 | 0.00264467 | 92 | 0.00256182 | 94 | 0.00250234 |
| 96 | 0.00246640 | 98 | 0.00245317 | 100 | 0.00246091 | 102 | 0.00248718 |
| 104 | 0.00252901 | 106 | 0.00258327 | 108 | 0.00264687 | 110 | 0.00271690 |
| 112 | 0.00279079 | 114 | 0.00286629 | 116 | 0.00294158 | 118 | 0.00301513 |
| 120 | 0.00308578 | 122 | 0.00315262 | 124 | 0.00321502 | 126 | 0.00327255 |
| 128 | 0.00332496 | 130 | 0.00337216 | 132 | 0.00341418 | 134 | 0.00345115 |
| 136 | 0.00348332 | 138 | 0.00351093 | 140 | 0.00353434 | 142 | 0.00355388 |
| 144 | 0.00356995 | 146 | 0.00358290 | 148 | 0.00359313 | 150 | 0.00360101 |
| 152 | 0.00360689 | 154 | 0.00361107 | 156 | 0.00361389 | 158 | 0.00361560 |
| 160 | 0.00361645 | 162 | 0.00361665 | 164 | 0.00361641 | 166 | 0.00361586 |
| 168 | 0.00361513 | 170 | 0.00361437 | 172 | 0.00361362 | 174 | 0.00361297 |
| 176 | 0.00361248 | 178 | 0.00361217 | 180 | 0.00361206 | 182 |  |

Sample output from FAP programme to compute currents, impedances and field patterns for curtain arrays with base voltages specified-contd.

| E1 |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 2.99999997 | 2 | 2.99999905 | 4 | 2.99998534 | 6 | 2.99992591 |
| 8 | 2.99976629 | 10 | 2.99943051 | 12 | 2.99882182 | 14 | 2.99782327 |
| 16 | 2.99629834 | 18 | 2.99409226 | 20 | 2.99103278 | 22 | $2 \cdot 98693159$ |
| 24 | 2.98158598 | 26 | 2.97478038 | 28 | 2.96628857 | 30 | 2.95587531 |
| 32 | 2.94329914 | 34 | 2.92831475 | 36 | $2 \cdot 91067564$ | 38 | 2.89013717 |
| 40 | 2.86645940 | 42 | 2.83941054 | 44 | 2.80877000 | 46 | 2.77433190 |
| 48 | 2.73590815 | 50 | 2.69333220 | 52 | 2.64646170 | 54 | 2.59518206 |
| 56 | 2.53940907 | 58 | $2 \cdot 47909155$ | 60 | 2.41421363 | 62 | $2 \cdot 34479681$ |
| 64 | $2 \cdot 27090105$ | 66 | 2.19262639 | 68 | $2 \cdot 11011279$ | 70 | 2.02354059 |
| 72 | 1.93312947 | 74 | 1.83913770 | 76 | 1.74185981 | 78 | 1.64162478 |
| 80 | 1.53879254 | 82 | 1.43375127 | 84 | $1 \cdot 32691267$ | 86 | 1.21870857 |
| 88 | $1 \cdot 10958548$ | 90 | 1.00000045 | 92 | 0.89041533 | 94 | 0.78129232 |
| 96 | 0.67308814 | 98 | 0.56624962 | 100 | 0.46120823 | 102 | 0.35837606 |
| 104 | 0.25814093 | 106 | $0 \cdot 16086308$ | 108 | 0.06687122 | 110 | 0.02353985 |
| 112 | $0 \cdot 11011214$ | 114 | 0.19262574 | 116 | 0.27090044 | 118 | 0.34479616 |
| 120 | 0.41421304 | 122 | 0.47909095 | 124 | 0.53940848 | 126 | 0.59518150 |
| 128 | 0.64646117 | 130 | 0.69333168 | 132 | 0.73590770 | 134 | 0.77433146 |
| 136 | 0.80876961 | 138 | 0.83941018 | 140 | 0.86645908 | 142 | 0.89013688 |
| 144 | 0.91067540 | 146 | 0.92831454 | 148 | 0.94329897 | 150 | 0.95587515 |
| 152 | 0.96628844 | 154 | 0.97478033 | 156 | 0.98158590 | 158 | 0.98693153 |
| 160 | 0.99103274 | 162 | 0.99409228 | 164 | 0.99629838 | 166 | 0.99782326 |
| 168 | 0.99882185 | 170 | 0.99943053 | 172 | 0.99976631 | 174 | 0.99992595 |
| 176 | 0.99998536 | 178 | 0.99999908 | 180 | 0.99999999 | 182 |  |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.125$ Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$

| $\Omega$ | $\Psi_{d R}$ | $\Phi_{k k u}$ |  | $\Phi_{k k v}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 7.00 | 4.05418 | 2.64345 | 0.19772 | 0.78608 | -0.38223 |
| 7.50 | 4.51933 | 2.97163 | 0.19776 | 0.79628 | $-0.38230$ |
| 8.00 | 4.99202 | 3.30540 | 0.19778 | 0.80428 | -0.38234 |
| 8.50 | 5.47065 | 3.64353 | 0.19779 | 0.81054 | -0.38237 |
| 9.00 | 5.95394 | 3.98508 | 0.19780 | 0.81544 | $-0.38238$ |
| 9.50 | 6.44089 | 4.32928 | 0.19781 | 0.81926 | -0.38239 |
| 10.00 | 6.93071 | 4.67555 | 0.19781 | 0.82224 | -0.38240 |
| 10.50 | 7.42276 | 5.02343 | 0.19781 | 0.82457 | -0.38240 |
| 11.00 | 7.91657 | 5.37257 | 0.19781 | 0.82638 | -0.38240 |
| 11.50 | 8.41174 | 5.72268 | 0.19781 | 0.82780 | -0.38240 |
| 12.00 | 8.90797 | 6.07356 | 0.19781 | 0.82890 | -0.38240 |
| 12.50 | 9.40504 | 6.42503 | 0.19781 | 0.82976 | -0.38240 |
| 13.00 | 9.90275 | 6.77696 | 0.19781 | 0.83043 | -0.38241 |
| 15.00 | 11.89766 | 8.18757 | 0.19781 | 0.83192 | -0.38241 |

Off diagonal values of $\Phi_{w}$ and $\Phi_{v}$ (see note on page 424)

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 0.250 | 0.09815 | 0.11183 | -0.19198 | $-0.21626$ |
| 0.500 | 0.08361 | $-0.03104$ | -0.16178 | 0.05987 |
| 0.750 | $-0.01455$ | $-0.06033$ | 0.02796 | 0.11660 |
| 1.000 | $-0.04655$ | 0.00834 | 0.08992 | -0.01602 |
| 1.250 | 0.00539 | 0.03774 | -0.01034 | -0.07287 |
| 1.500 | 0.03167 | -0.00376 | -0.06116 | 0.00722 |
| 1.750 | -0.00277 | -0.02727 | 0.00532 | 0.05265 |
| 2.000 | -0.02393 | 0.00213 | 0.04619 | -0.00408 |
| 2.250 | 0.00168 | 0.02131 | $-0.00323$ | -0.04114 |
| 2.500 | 0.01921 | -0.00136 | -0.03708 | 0.00262 |
| 2.750 | $-0.00113$ | -0.01748 | 0.00216 | 0.03374 |
| 3.000 | $-0.01603$ | 0.00095 | 0.03095 | -0.00182 |
| 3.250 | 0.00081 | 0.01481 | -0.00155 | -0.02859 |
| 3.500 | 0.01376 | -0.00070 | -0.02656 | 0.00134 |
| 3.750 | $-0.00061$ | $-0.01285$ | 0.00117 | 0.02480 |
| 4.000 | $-0.01205$ | 0.00053 | 0.02325 | -0.00102 |
| 4.250 | 0.00047 | 0.01134 | -0.00091 | -0.02189 |
| 4.500 | 0.01071 | -0.00042 | -0.02068 | 0.00081 |
| 4.750 | $-0.00038$ | -0.01015 | 0.00073 | 0.01960 |
| 5.000 | $-0.00965$ | 0.00034 | 0.01862 | -0.00066 |
| 5.250 | 0.00031 | 0.00919 | -0.00059 | $-0.01773$ |
| 5.500 | 0.00877 | -0.00028 | -0.01693 | 0.00054 |
| 5.750 | $-0.00026$ | -0.00839 | 0.00050 | 0.01620 |
| 6.000 | -0.00804 | 0.00024 | 0.01552 | -0.00046 |
| 6.250 | 0.00022 | 0.00772 | -0.00042 | -0.01490 |
| 6.500 | 0.00742 | $-0.00020$ | -0.01433 | 0.00039 |
| 6.750 | $-0.00019$ | -0.00715 | 0.00036 | 0.01380 |
| 7.000 | $-0.00690$ | 0.00017 | 0.01331 | -0.00033 |
| 7.250 | 0.00016 | 0.00666 | -0.00031 | -0.01285 |
| 7.500 | 0.00644 | -0.00015 | -0.01242 | 0.00029 |
| 7.750 | $-0.00014$ | -0.00623 | 0.00027 | 0.01202 |
| 8.000 | $-0.00603$ | 0.00013 | 0.01165 | -0.00026 |
| 8.250 | 0.00013 | 0.00585 | -0.00024 | -0.01129 |
| 8.500 | 0.00568 | -0.00012 | -0.01096 | 0.00023 |
| 8.750 | -0.00011 | -0.00552 | 0.00021 | 0.01065 |
| 9.000 | $-0.00536$ | 0.00011 | 0.01035 | -0.00020 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.125$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k r}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 9.250 | 0.00010 | 0.00522 | -0.00019 | -0.01007 |
| 9.500 | 000508 | -0.00009 | -0.00981 | 0.00018 |
| 9.750 | -0.00009 | -0.00495 | 0.00017 | 0.00956 |
| 10.000 | -0.00483 | 0.00009 | 0.00932 | -0.00016 |
| 10.250 | 0.00008 | 0.00471 | -0.00016 | -0.00909 |
| 10.500 | 0.00460 | -0.00008 | -0.00888 | 0.00015 |
| 10.750 | -0.00007 | -0.00449 | 0.00014 | 0.00867 |
| 11.000 | -0.00439 | 0.00007 | 0.00847 | -0.00014 |
| 11.250 | 0.00007 | 0.00429 | -0.00013 | -0.00828 |
| 11.500 | 0.00420 | -0.00006 | -0.00810 | 0.00012 |
| 11.750 | -0.00006 | -0.00411 | 0.00012 | 0.00793 |
| 12.000 | -0.00402 | 0.00006 | 0.00777 | -0.00011 |
| 12.250 | 0.00006 | 0.00394 | -0.00011 | -0.00761 |
| 12.500 | 0.00386 | -0.00005 | -0.00746 | 0.00011 |
| 12.750 | -0.00005 | -0.00379 | 0.00010 | 0.00731 |
| 13.000 | -0.00371 | 0.00005 | 0.00717 | -0.00010 |
| 13.250 | 0.00005 | 0.00364 | -0.00009 | -0.00703 |
| 13.500 | 0.00358 | -0.00005 | -0.00690 | 0.00009 |
| 13.750 | -0.00005 | -0.00351 | 0.00009 | 0.00678 |
| 14.000 | -0.00345 | 0.00004 | 0.00666 | -0.00008 |
| 14.250 | 0.00004 | 0.00339 | -0.00008 | -0.00654 |
| 14.500 | 0.00333 | -0.00004 | -0.00643 | 0.00008 |
| 14.750 | -0.00004 | -0.00327 | 0.00008 | 0.00632 |
| 15.000 | -0.00322 | 0.00004 | 0.00621 | -0.00007 |
| 15.250 | 0.00004 | 0.00317 | -0.00007 | -0.00611 |
| 15.500 | 0.00312 | -0.00004 | -0.00601 | 0.00007 |
| 15.750 | -0.00003 | -0.00307 | 0.00007 | 0.00592 |
| 16.000 | -0.00302 | 0.00003 | 0.00583 | -0.00006 |
| 16.250 | 0.00003 | 0.00297 | -0.00006 | -0.00574 |
| 16.500 | 0.00293 | -0.00003 | -0.00565 | 0.00006 |
| 16.750 | -0.00003 | -0.00288 | 0.00006 | 0.00557 |
| 17.000 | -0.00284 | 0.00003 | 0.00548 | -0.00006 |
| 17.250 | 0.00003 | 0.00280 | -0.00006 | -0.00540 |
| 17.500 | 0.00276 | -0.00003 | -0.00533 | 0.00005 |
| 17.750 | -0.00003 | -0.00272 | 0.00005 | 0.00525 |
| 18.000 | -0.00268 | 0.00003 | 0.00518 | -0.00005 |
| 18.250 | 0.00003 | 0.00265 | -0.00005 | -0.00511 |
| 18.500 | 0.00261 | -0.00003 | -0.00504 | 0.00005 |
| 18.750 | -0.00002 | -0.00258 | 0.00005 | 0.00497 |
| 19.000 | -0.00254 | 0.00002 | 0.00491 | -0.00005 |
| 19.250 | 0.00002 | 0.00251 | -0.00004 | -0.00484 |
| 19.500 | 0.00248 | -0.00002 | -0.00478 | 0.00004 |
| 19.750 | -0.00002 | -0.00245 | 0.00004 | 0.00472 |
| 20.000 | -0.00241 | 0.00002 | 0.00466 | -0.00004 |
| 20.250 | 0.00002 | 0.00239 | -0.00004 | $-0.00460$ |
| 20.500 | 0.00236 | -0.00002 | -0.00455 | 0.00004 |
| 20.750 | -0.00002 | -0.00233 | 0.00004 | 0.00449 |
| 21.000 | -0.00230 | 0.00002 | 0.00444 | -0.00004 |
| 21.250 | 0.00002 | 0.00227 | -0.00004 | $-0.00439$ |
| 21.500 | 0.00225 | -0.00002 | -0.00434 | 0.00004 |
| 21.750 | -0.00002 | -0.00222 | 0.00003 | 0.00429 |
| 22.000 | -0.00220 | 0.00002 | 0.00424 | -0.00003 |
| 22.250 | 0.00002 | 0.00217 | -0.00003 | -0.00419 |
| 22.500 | 0.00215 | -0.00002 | -0.00414 | 0.00003 |
| 22.750 | -0.00002 | -0.00212 | 0.00003 | 0.00410 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.125$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d \mathrm{p}}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 23.000 | $-0.00210$ | 0.00002 | 0.00405 | -0.00003 |
| 23.250 | 0.00002 | 0.00208 | $-0.00003$ | -0.00401 |
| 23.500 | 0.00206 | -0.00002 | -0.00397 | 0.00003 |
| 23.750 | $-0.00002$ | -0.00203 | 0.00003 | 0.00393 |
| 24.000 | -0.00201 | 0.00001 | 0.00388 | -0.00003 |
| 24.250 | 0.00001 | 0.00199 | -0.00003 | -0.00384 |
| 24.500 | 0.00197 | -0.00001 | -0.00381 | 0.00003 |
| 24.750 | $-0.00001$ | -0.00195 | 0.00003 | 0.00377 |
| 25.000 | -0.00193 | 0.00001 | 0.00373 | -0.00003 |
| 25.250 | 0.00001 | 0.00191 | $-0.00003$ | -0.00369 |
| 25.500 | 0.00189 | -0.00001 | -0.00366 | 0.00003 |
| 25.750 | $-0.00001$ | -0.00188 | 0.00002 | 0.00362 |
| 26.000 | -0.00186 | 0.00001 | 0.00359 | -0.00002 |
| 26.250 | 0.00001 | 0.00184 | -0.00002 | -0.00355 |
| 26.500 | 0.00182 | $-0.00001$ | $-0.00352$ | 0.00002 |
| 26.750 | -0.00001 | -0.00181 | 0.00002 | 0.00349 |
| 27.000 | -0.00179 | 0.00001 | 0.00345 | -0.00002 |
| 27.250 | 0.00001 | 0.00177 | -0.00002 | -0.00342 |
| 27.500 | 0.00176 | $-0.00001$ | -0.00339 | 0.00002 |
| 27.750 | -0.00001 | -0.00174 | 0.00002 | 0.00336 |
| 28.000 | -0.00173 | 0.00001 | 0.00333 | -0.00002 |
| 28.250 | 0.00001 | 0.00171 | -0.00002 | -0.00330 |
| 28.500 | 0.00169 | -0.00001 | -0.00327 | 0.00002 |
| 28.750 | $-0.00001$ | -0.00168 | 0.00002 | 0.00324 |
| 29.000 | $-0.00167$ | 0.00001 | 0.00321 | -0.00002 |
| 29.250 | 0.00001 | 0.00165 | $-0.00002$ | -0.00319 |
| 29.500 | 0.00164 | -0.00001 | -0.00316 | 0.00002 |
| 29.750 | -0.00001 | $-0.00162$ | 0.00002 | 0.00313 |
| 30.000 | -0.00161 | 0.00001 | 0.00311 | -0.00002 |
| 30.250 | 0.00001 | 0.00160 | -0.00002 | -0.00308 |
| 30.500 | 0.00158 | -0.00001 | -0.00306 | 0.00002 |
| 30.750 | -0.00001 | -0.00157 | 0.00002 | 0.00303 |
| 31.000 | -0.00156 | 0.00001 | 0.00301 | -0.00002 |
| 31.250 | 0.00001 | 0.00155 | $-0.00002$ | -0.00298 |
| 31.500 | 0.00153 | -0.00001 | -0.00296 | 0.00002 |
| 31.750 | $-0.00001$ | $-0.00152$ | 0.00002 | 0.90294 |
| 32.000 | $-0.00151$ | 0.00001 | 0.00291 | -0.00002 |
| 32.250 | 0.00001 | 0.00150 | -0.00002 | -0.00289 |
| 32.500 | 0.00149 | $-0.00001$ | -0.00287 | 0.00002 |
| 32.750 | $-0.00001$ | $-0.00147$ | 0.00002 | 0.00285 |
| 33.000 | -0.00146 | 0.00001 | 0.00283 | $-0.00002$ |
| 33.250 | 0.00001 | 0.00145 | -0.00001 | -0.00280 |
| 33.500 | 0.00144 | -0.00001 | -0.00278 | 0.00001 |
| 33.750 | -0.00001 | $-0.00143$ | 0.00001 | 0.00276 |
| 34.000 | $-0.00142$ | 0.00001 | 0.00274 | -0.00001 |
| 34.250 | 0.00001 | 0.00141 | -0.00001 | -0.00272 |
| 34.500 | 0.00140 | -0.00001 | -0.00270 | 0.00001 |
| 34.750 | -0.00001 | -0.00139 | 0.00001 | 0.00268 |
| 35.000 | $-0.00138$ | 0.00001 | 0.00266 | -0.00001 |
| 35.250 | 0.00001 | 0.00137 | -0.00001 | -0.00264 |
| 35.500 | 0.00136 | $-0.00001$ | -0.00263 | 0.00001 |
| 35.750 | -0.00001 | -0.00135 | 0.00001 | 0.00261 |
| 36.000 | -0.00134 | 0.00001 | 0.00259 | -0.00001 |
| 36.250 | 0.00001 | 0.00133 | -0.00001 | $-0.00257$ |
| 36.500 | 0.00132 | -0.00001 | -0.00255 | 0.00001 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.125$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 36.750 | -0.00001 | $-0.00131$ | 0.00001 | 0.00254 |
| 37.000 | -0.00131 | 0.00001 | 0.00252 | $-0.00001$ |
| 37.250 | 0.00001 | 0.00130 | -0.00001 | $-0.00250$ |
| 37.500 | 0.00129 | -0.00001 | -0.00249 | 0.00001 |
| 37.750 | -0.00001 | $-0.00128$ | 0.00001 | 0.00247 |
| 38.000 | -0.00127 | 0.00001 | 0.00245 | $-0.00001$ |
| 38.250 | 0.00001 | 0.00126 | -0.00001 | -0.00244 |
| 38.500 | 0.00125 | $-0.00001$ | $-0.00242$ | 0.00001 |
| 38.750 | $-0.00001$ | $-0.00125$ | 0.00001 | 0.00241 |
| 39.000 | -0.00124 | 0.00001 | 0.00239 | -0.00001 |
| 39.250 | 0.00001 | 0.00123 | -0.00001 | -0.00238 |
| 39.500 | 0.00122 | $-0.00001$ | $-0.00236$ | 0.00001 |
| 39.750 | $-0.00001$ | $-0.00122$ | 0.00001 | 0.00235 |
| 40.000 | -0.00121 | 0.00001 | 0.00233 | -0.00001 |
| 40.250 | 0.00001 | 0.00120 | $-0.00001$ | -0.00232 |
| 40.500 | 0.00119 | $-0.00001$ | -0.00230 | 0.00001 |
| 40.750 | -0.00001 | -0.00119 | 0.00001 | 0.00229 |
| 41.000 | -0.00118 | 0.00001 | 0.00227 | $-0.00001$ |
| 41.250 | 0.00001 | 0.00117 | -0.00001 | -0.00226 |
| 41.500 | 0.00116 | -0.00000 | $-0.00225$ | 0.00001 |
| 41.750 | -0.00000 | -0.00116 | 0.00001 | 0.00223 |
| 42.000 | $-0.00115$ | 0.00000 | 0.00222 | $-0.00001$ |
| 42.250 | 0.00000 | 0.00114 | $-0.00001$ | -0.00221 |
| 42.500 | 0.00114 | -0.00000 | -0.00219 | 0.00001 |
| 42.750 | -0.00000 | -0.00113 | 0.00001 | 0.00218 |
| 43.000 | -0.00112 | 0.00000 | 0.00217 | $-0.00001$ |
| 43.250 | 0.00000 | 0.00112 | $-0.00001$ | -0.00216 |
| 43.500 | 0.00111 | $-0.00000$ | -0.00214 | 0.00001 |
| 43.750 | -0.00000 | $-0.00110$ | 0.00001 | 0.00213 |
| 44.000 | -0.00110 | 0.00000 | 0.00212 | $-0.00001$ |
| 44.250 | 0.00000 | 0.00109 | -0.00001 | -0.00211 |
| 44.500 | 0.00109 | -0.00000 | -0.00210 | 0.00001 |
| 44.750 | -0.00000 | -0.00108 | 0.00001 | 0.00208 |
| 45.000 | -0.00107 | 0.00000 | 0.00207 | -0.00001 |
| 45.250 | 0.00000 | 0.00107 | $-0.00001$ | -0.00206 |
| 45.500 | 0.00106 | -0.00000 | -0.00205 | 0.00001 |
| 45.750 | $-0.00000$ | -0.00106 | 0.00001 | 0.00204 |
| 46.000 | -0.00105 | 0.00000 | 0.00203 | -0.00091 |
| 46.250 | 0.00000 | 0.00104 | $-0.00001$ | -0.00202 |
| 46.500 | 0.00104 | $-0.00000$ | $-0.00200$ | 0.00001 |
| 46.750 | -0.00000 | $-0.00103$ | 0.00001 | 0.00199 |
| 47.000 | $-0.00103$ | 0.00000 | 0.00198 | -0.00001 |
| 47.250 | 0.00000 | 0.00102 | -0.00001 | -0.00197 |
| 47.500 | 0.00102 | $-0.00000$ | -0.00196 | 0.00001 |
| 47.750 | -0.00000 | -0.00101 | 0.00001 | 0.00195 |
| 48.000 | -0.00101 | 0.00000 | 0.00194 | -0.00001 |
| 48.250 | 0.00000 | 0.00100 | -0.00001 | -0.00193 |
| 48.500 | 0.00100 | -0.00000 | -0.00192 | 0.00001 |
| 48.750 | $-0.00000$ | -0.00099 | 0.00001 | 0.00191 |
| 49.000 | -0.00099 | 0.00000 | 0.00190 | $-0.00001$ |
| 49.250 | 0.00000 | 0.00098 | -0.00001 | -0.00189 |
| 49.500 | 0.00098 | -0.00000 | -0.00188 | 0.00001 |
| 49.750 | -0.00000 | -0.00097 | 0.00001 | 0.00187 |
| 50.000 | $-0.00097$ | 0.00000 | 0.00186 | $-0.00001$ |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.250$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$

| $\Omega$ | $\Psi_{d R}$ | $\Phi_{k u}$ |  | $\Phi_{k k v}$ |  |
| :--- | ---: | ---: | ---: | ---: | ---: |
| 7.00 | 4.73675 | 0.61497 | -1.21658 | 4.73675 | -0.00000 |
| 7.50 | 5.21607 | 0.63565 | -1.21746 | 5.21607 | -0.00000 |
| 8.00 | 5.69991 | 0.65181 | -1.21800 | 5.69991 | -0.00000 |
| 8.50 | 6.18729 | 0.66443 | -1.21832 | 6.18729 | -0.00000 |
| 9.00 | 6.67744 | 0.67427 | -1.21852 | 6.67744 | -0.00000 |
| 9.50 | 7.16976 | 0.68196 | -1.21864 | 7.16976 | -0.00000 |
| 10.00 | 7.66377 | 0.68794 | -1.21871 | 7.66377 | -0.00000 |
| 10.50 | 8.15911 | 0.69261 | -1.21876 | 8.15911 | -0.00000 |
| 11.00 | 8.65547 | 0.69625 | -1.21878 | 8.65547 | -0.00000 |
| 11.50 | 9.65042 | 0.69908 | -1.21880 | 9.15263 | -0.00000 |
| 12.00 | 10.14870 | 0.70129 | -1.21881 | 9.65042 | -0.00000 |
| 12.50 | 10.64736 | 0.70301 | -1.21882 | 10.14870 | -0.00000 |
| 13.00 | 12.64438 | 0.70435 | -1.21882 | 10.64736 | -0.00000 |
| 15.00 | 0.70734 | -1.21883 | 12.64438 | -0.00000 |  |


| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{k i v}$ |  |
| :---: | ---: | ---: | ---: | ---: |
| 0.250 | -0.47248 | -0.67976 | 0.00000 | -0.00000 |
| 0.500 | -0.49881 | 0.20887 | -0.00000 | -0.00000 |
| 0.750 | 0.11054 | 0.37495 | -0.00000 | 0.00000 |
| 1.000 | 0.29570 | -0.06686 | 0.00000 | -0.00000 |
| 1.250 | -0.04437 | -0.24260 | 0.00000 | -0.00000 |
| 1.500 | -0.20507 | 0.03146 | -0.00000 | 0.00000 |
| 1.750 | 0.02341 | 0.17734 | -0.00000 | 0.00000 |
| 2.000 | 0.15607 | -0.01807 | 0.00000 | -0.00000 |
| 2.250 | -0.01436 | -0.13929 | 0.00000 | -0.00000 |
| 2.500 | -0.12573 | 0.0168 | -0.00000 | -0.00000 |
| 2.750 | 0.00968 | 0.11455 | -0.00000 | 0.00000 |
| 3.000 | 0.10517 | -0.00816 | 0.00000 | -0.00000 |
| 3.250 | -0.00696 | -0.09721 | 0.00000 | -0.00000 |
| 3.500 | -0.09036 | 0.00601 | -0.00000 | -0.0000 |
| 3.750 | 0.00524 | 0.08440 | -0.00000 | 0.00000 |
| 4.000 | 0.07918 | -0.00461 | 0.00000 | -0.00000 |
| 4.250 | -0.00409 | -0.07457 | -0.00000 | -0.00000 |
| 4.500 | -0.07046 | 0.00365 | -0.00000 | -0.00000 |
| 4.750 | 0.00328 | 0.06678 | -0.00000 | 0.00000 |
| 5.000 | 0.06346 | -0.00296 | 0.00000 | 0.00000 |
| 5.250 | -0.00269 | -0.06046 | 0.00000 | -0.00000 |
| 5.500 | -0.05772 | 0.00245 | -0.00000 | -0.00000 |
| 5.750 | 0.00224 | 0.05522 | 0.00000 | 0.00000 |
| 6.000 | 0.05293 | -0.00206 | 0.00000 | -0.00000 |
| 6.250 | -0.00190 | -0.05083 | 0.00000 | -0.00000 |
| 6.500 | -0.04888 | 0.00175 | -0.00000 | -0.00000 |
| 6.750 | 0.00163 | 0.04707 | -0.00000 | 0.00000 |
| 7.000 | 0.04540 | -0.00151 | 0.00000 | 0.00000 |
| 7.250 | -0.00141 | -0.04384 | 0.00000 | -0.00000 |
| 7.500 | -0.04238 | 0.00132 | -0.00000 | -0.0000 |
| 7.750 | 0.00124 | 0.04102 | -0.00000 | 0.00000 |
| 8.000 | 0.03974 | -0.00116 | 0.00000 | -0.00000 |
| 8.250 | -0.00109 | -0.03854 | -0.00000 | -0.00000 |
| 8.500 | -0.03741 | 0.00103 | -0.00000 | 0.00000 |
| 8.750 | 0.00097 | 0.03634 | -0.00000 | 0.00000 |
| 9.000 | 0.03533 | -0.00099 | 0.00000 | 0.00000 |
|  |  |  |  |  |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.250$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 9.250 | -0.00087 | -0.03438 | 0.00000 | -0.00000 |
| 9.500 | -0.03348 | 0.00082 | -0.00000 | -0.00000 |
| 9.750 | 0.00078 | 0.03262 | $-0.00000$ | 0.00000 |
| 10.000 | 0.03181 | -0.00074 | 0.00000 | 0.00000 |
| 10.250 | -0.00071 | -0.03103 | 0.00000 | $-0.00000$ |
| 10.500 | -0.03029 | 0.00067 | -0.00000 | $-0.00000$ |
| 10.750 | 0.00064 | 0.02959 | $-0.00000$ | 0.00000 |
| 11.000 | 0.02892 | -0.00061 | 0.00000 | 0.00000 |
| 11.250 | $-0.00059$ | -0.02828 | 0.00000 | $-0.00000$ |
| 11.500 | -0.02766 | 0.00056 | $-0.00000$ | 0.00000 |
| 11.750 | 0.00054 | 0.02707 | 0.00000 | 0.00000 |
| 12.000 | 0.02651 | -0.00052 | 0.00000 | $-0.00000$ |
| 12.250 | -0.00049 | -0.02597 | $-0.00000$ | $-0.00000$ |
| 12.500 | -0.02545 | 0.00048 | $-0.00000$ | $-0.00000$ |
| 12.750 | 0.00046 | 0.02495 | -0.00000 | 0.00000 |
| 13.000 | 0.02447 | -0.00044 | 0.00000 | 0.00000 |
| 13.250 | $-0.00042$ | -0.02401 | 0.00000 | $-0.00000$ |
| 13.500 | -0.02357 | 0.00041 | $-0.00000$ | $-0.00000$ |
| 13.750 | 0.00039 | 0.02314 | $-0.00000$ | 0.00000 |
| 14.000 | 0.02273 | -0.00038 | 0.00000 | 0.00000 |
| 14.250 | -0.00037 | -0.02233 | 0.00000 | $-0.00000$ |
| 14.500 | -0.02194 | 0.00035 | $-0.00000$ | -0.00000 |
| 14.750 | 0.00034 | 0.02157 | $-0.00000$ | 0.00000 |
| 15.000 | 0.02121 | $-0.00033$ | 0.00000 | 0.00000 |
| 15.250 | -0.00032 | -0.02087 | 0.00000 | $-0.00000$ |
| $15 \cdot 500$ | -0.02053 | 0.00031 | $-0.00000$ | $-0.00000$ |
| 15.750 | 0.00030 | 0.02020 | $-0.00000$ | 0.00000 |
| 16.000 | 0.01989 | -0.00029 | 0.00000 | $-0.00000$ |
| 16.250 | -0.00028 | -0.01958 | $-0.00000$ | $-0.00000$ |
| 16.500 | -0.01929 | 0.00027 | $-0.00000$ | 0.00000 |
| 16.750 | 0.00026 | 0.01900 | 0.00000 | 0.00000 |
| 17.000 | 0.01872 | -0.00026 | 0.00000 | $-0.00000$ |
| 17.250 | -0.00025 | -0.01845 | $-0.00000$ | $-0.00000$ |
| 17.500 | -0.01818 | 0.00024 | -0.00000 | $-0.00000$ |
| 17.750 | 0.00024 | 0.01793 | -0.00000 | 0.00000 |
| 18.000 | 0.01768 | -0.00023 | 0.00000 | 0.00000 |
| 18.250 | -0.00022 | -0.01744 | 0.00000 | $-0.00000$ |
| 18.500 | -0.01720 | 0.00022 | $-0.00000$ | $-0.00000$ |
| 18.750 | 0.00021 | 0.01697 | $-0.00000$ | 0.00000 |
| 19.000 | 0.01675 | -0.00021 | 0.00000 | 0.00000 |
| 19.250 | -0.00020 | -0.01653 | 0.00000 | $-0.00000$ |
| 19.500 | -0.01632 | 0.00020 | $-0.00000$ | $-0.00000$ |
| 19.750 | 0.00019 | 0.01611 | $-0.00000$ | 0.00000 |
| 20.000 | 0.01591 | -0.00019 | 0.00000 | 0.00000 |
| 20.250 | -0.00018 | -0.01572 | 0.00000 | $-0.00000$ |
| 20.500 | -0.01552 | 0.00018 | -0.00000 | $-0.00000$ |
| 20.750 | 0.00017 | 0.01534 | -0.00000 | 0.00000 |
| 21.000 | 0.01515 | -0.00017 | 0.00000 | 0.00000 |
| 21.250 | -0.00016 | -0.01498 | 0.00000 | $-0.00000$ |
| 21.500 | -0.01480 | 0.00016 | $-0.00000$ | $-0.00000$ |
| 21.750 | 0.00016 | 0.01463 | $-0.00000$ | 0.00000 |
| 22.000 | 0.01447 | $-0.00015$ | 0.00000 | 0.00000 |
| 22.250 | -0.00015 | -0.01430 | 0.00000 | $-0.00000$ |
| 22.500 | -0.01414 | 0.00015 | $-0.00000$ | $-0.00000$ |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.250$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 22.750 | 0.00014 | 0.01399 | 0.00000 | 0.00000 |
| 23.000 | 0.01384 | -0.00014 | 0.00000 | -0.00000 |
| 23.250 | -0.00014 | -0.01369 | $-0.00000$ | -0.00000 |
| 23.500 | -0.01354 | 0.00013 | -0.00000 | 0.00000 |
| 23.750 | 0.00013 | 0.01340 | 0.00000 | 0.00000 |
| 24.000 | 0.01326 | -0.00013 | 0.00000 | -0.00000 |
| 24.250 | -0.00013 | -0.01312 | $-0.00000$ | -0.00000 |
| 24.500 | -0.01299 | 0.00012 | $-0.00000$ | 0.00000 |
| 24.750 | 0.00012 | 0.01286 | 0.00000 | 0.00000 |
| 25.000 | 0.01273 | -0.00012 | 0.00000 | 0.00000 |
| 25.250 | -0.00012 | $-0.01260$ | 0.00000 | -0.00000 |
| 25.500 | -0.01248 | 0.00011 | $-0.00000$ | -0.00000 |
| 25.750 | 0.00011 | 0.01236 | -0.00000 | 0.00000 |
| 26.000 | 0.01224 | -0.00011 | 0.00000 | 0.00000 |
| 26.250 | -0.00011 | -0.01212 | 0.00000 | -0.00000 |
| 26.500 | -0.01201 | 0.00011 | $-0.00000$ | -0.00000 |
| 26.750 | 0.00010 | 0.01190 | -0.00000 | 0.00000 |
| 27.000 | 0.01179 | -0.00010 | 0.00000 | 0.00000 |
| 27.250 | -0.00010 | -0.01168 | 0.00000 | -0.00000 |
| 27.500 | -0.01157 | 0.00010 | -0.00000 | -0.00000 |
| 27.750 | 0.00010 | 0.01147 | $-0.00000$ | 0.00000 |
| 28.000 | 0.01137 | -0.00009 | 0.00000 | 0.00000 |
| 28.250 | -0.00009 | -0.01127 | 0.00000 | $-0.00000$ |
| 28.500 | -0.01117 | 0.00009 | -0.00000 | -0.00000 |
| 28.750 | 0.00009 | 0.01107 | -0.00000 | 0.00000 |
| 29.000 | 0.01098 | -0.00009 | 0.00000 | 0.00000 |
| 29.250 | -0.00009 | -0.01088 | 0.00000 | -0.00000 |
| 29.500 | -0.01079 | 0.00009 | -0.00000 | -0.00000 |
| 29.750 | 0.00008 | 0.01070 | -0.00000 | 0.00000 |
| 30.000 | 0.01061 | -0.00008 | 0.00000 | 0.00000 |
| 30.250 | -0.00008 | -0.01052 | 0.00000 | -0.00000 |
| 30.500 | -0.01044 | 0.00008 | -0.00000 | -0.00000 |
| 30.750 | 0.00008 | 0.01035 | $-0.00000$ | 0.00000 |
| 31.000 | 0.01027 | -0.00008 | 0.00000 | 0.00000 |
| 31.250 | -0.00008 | -0.01019 | 0.00000 | -0.00000 |
| 31.500 | -0.01010 | 0.00007 | -0.00000 | -0.00000 |
| 31.750 | 0.00007 | 0.01002 | -0.00000 | 0.00000 |
| 32.000 | 0.00995 | -0.00007 | 0.00000 | -0.00000 |
| 32.250 | -0.00007 | -0.00987 | -0.00000 | -0.00000 |
| 32.500 | -0.00979 | 0.00007 | -0.00000 | 0.00000 |
| 32.750 | 0.00007 | 0.00972 | 0.00000 | 0.00000 |
| 33.000 | 0.00965 | -0.00007 | 0.00000 | $-0.00000$ |
| 33.250 | -0.00007 | -0.00957 | -0.00000 | -0.00000 |
| 33.500 | -0.00950 | 0.00007 | -0.00000 | 0.00000 |
| 33.750 | 0.00007 | 0.00943 | 0.00000 | 0.00000 |
| 34.000 | 0.00936 | -0.00006 | 0.00000 | -0.00000 |
| 34.250 | -0.00006 | -0.00929 | -0.00000 | -0.00000 |
| 34.500 | -0.00923 | 0.00006 | -0.00000 | 0.00000 |
| 34.750 | 0.00006 | 0.00916 | 0.00000 | 0.00000 |
| 35.000 | 0.00909 | -0.00006 | 0.00000 | 0.00000 |
| 35.250 | -0.00006 | -0.00903 | 0.00000 | -0.00000 |
| 35.500 | -0.00897 | 0.00006 | -0.00000 | -0.00000 |
| 35.750 | 0.00006 | 0.00890 | -0.00000 | 0.00000 |
| 36.000 | 0.00884 | -0.00006 | 0.00000 | 0.00000 |
| 36.250 | -0.00006 | -0.00878 | 0.00000 | -0.00000 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.250$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{k i u}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 36.500 | $-0.00872$ | 0.00006 | $-0.00000$ | -0.00000 |
| 36.750 | 0.00006 | 0.00866 | $-0.00000$ | 0.00000 |
| 37.000 | 0.00860 | $-0.00005$ | 0.00000 | 0.00000 |
| 37.250 | $-0.00005$ | $-0.00854$ | 0.00000 | -0.00000 |
| 37.500 | -0.00849 | 0.00005 | -0.00000 | -0.00000 |
| 37.750 | 0.00005 | 0.00843 | $-0.00000$ | 0.00000 |
| 38.000 | 0.00838 | -0.00005 | 0.00000 | 0.00000 |
| 38.250 | $-0.00005$ | $-0.00832$ | 0.00000 | -0.00000 |
| 38.500 | -0.00827 | 0.00005 | $-0.00000$ | $-0.00000$ |
| 38.750 | 0.00005 | 0.00821 | $-0.00000$ | 0.00000 |
| 39.000 | 0.00816 | -0.00005 | 0.00000 | 0.00000 |
| 39.250 | $-0.00005$ | -0.00811 | 0.00000 | -0.00000 |
| 39.500 | -0.00806 | 0.00005 | $-0.00000$ | $-0.00000$ |
| 39.750 | 0.00005 | 0.00801 | -0.00000 | 0.00000 |
| 40.000 | 0.00796 | -0.00005 | 0.00000 | 0.00000 |
| 40.250 | $-0.00005$ | -0.00791 | 0.00000 | -0.00000 |
| 40.500 | $-0.00786$ | 0.00005 | $-0.00000$ | -0.00000 |
| 40.750 | 0.00004 | 0.00781 | $-0.00000$ | 0.00000 |
| 41.000 | 0.00776 | -0.00004 | 0.00000 | 0.00000 |
| 41.250 | -0.00004 | -0.00772 | 0.00000 | -0.00000 |
| $41 \cdot 500$ | -0.00767 | 0.00004 | $-0.00000$ | -0.00000 |
| 41.750 | 0.00004 | 0.00762 | -0.00000 | 0.00000 |
| 42.000 | 0.00758 | -0.00004 | 0.00000 | 0.00000 |
| 42.250 | -0.00004 | $-0.00753$ | 0.00000 | -0.00000 |
| 42.500 | -0.00749 | 0.00004 | -0.00000 | -0.00000 |
| 42.750 | 0.00004 | 0.00745 | -0.00000 | 0.00000 |
| 43.000 | 0.00740 | -0.00004 | 0.00000 | 0.00000 |
| 43.250 | -0.00004 | -0.00736 | 0.00000 | -0.00000 |
| 43.500 | $-0.00732$ | 0.00004 | -0.00000 | -0.00000 |
| 43.750 | 0.00004 | 0.00728 | -0.00000 | 0.00000 |
| 44.000 | 0.00723 | -0.00004 | 0.00000 | 0.00000 |
| 44.250 | -0.00004 | -0.00719 | 0.00000 | -0.00000 |
| 44.500 | -0.00715 | 0.00004 | -0.00000 | -0.00000 |
| 44.750 | 0.00004 | 0.00711 | -0.00000 | 0.00000 |
| 45.000 | 0.00707 | $-0.00004$ | 0.00000 | 0.00000 |
| 45.250 | -0.00004 | -0.00703 | 0.00000 | -0.00000 |
| 45.500 | $-0.00700$ | 0.00004 | -0.00000 | 0.00000 |
| 45.750 | 0.00004 | 0.00696 | 0.00000 | 0.00000 |
| 46.000 | 0.00692 | -0.00004 | 0.00000 | -0.00000 |
| 46.250 | $-0.00003$ | -0.00688 | -0.00000 | -0.00000 |
| 46.500 | $-0.00685$ | 0.00003 | $-0.00000$ | 0.00000 |
| 46.750 | 0.00003 | 0.00681 | 0.00000 | 0.00000 |
| 47.000 | 0.00677 | -0.00003 | 0.00000 | -0.00000 |
| 47.250 | $-0.00003$ | -0.00674 | -0.00000 | -0.00000 |
| 47.500 | -0.00670 | 0.00003 | -0.00000 | 0.00000 |
| 47.750 | 0.00003 | 0.00667 | 0.00000 | 0.00000 |
| 48.000 | 0.00663 | $-0.00003$ | 0.00000 | -0.00000 |
| 48.250 | -0.00003 | -0.00660 | -0.00000 | -0.00000 |
| 48.500 | -0.00656 | 0.00003 | -0.00000 | 0.00000 |
| 48.750 | 0.00003 | 0.00653 | 0.00000 | 0.00000 |
| 49.000 | 0.00650 | -0.00003 | 0.00000 | -0.00000 |
| 49.250 | -0.00003 | -0.00646 | -0.00000 | -0.00000 |
| 49.500 | -0.00643 | 0.00003 | -0.00000 | -0.00000 |
| 49.750 | 0.00003 | 0.00640 | -0.00000 | 0.00000 |
| 50.000 | 0.00637 | -0.00003 | 0.00000 | 0.00000 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.375$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$

| $\boldsymbol{\Omega}$ | $\Psi_{d R}$ | $\Phi_{k k u}$ |  | $\Phi_{k k v}$ |  |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 7.00 | 4.19919 | -3.23123 | 2.57558 | 0.21311 | -1.80626 |
| 7.50 | 4.67051 | -3.60298 | 2.57993 | 0.24365 | -1.80933 |
| 8.00 | 5.14769 | -3.97013 | 2.58257 | 0.26761 | -1.81119 |
| 8.50 | 5.62962 | -4.33386 | 2.58417 | 0.28638 | -1.81232 |
| 9.00 | 6.11538 | -4.69506 | 2.58514 | 0.30105 | -1.81301 |
| 9.50 | 6.60417 | -5.05439 | 2.58573 | 0.31251 | -1.81343 |
| 10.00 | 7.09538 | -5.41231 | 2.58609 | 0.32146 | -1.81368 |
| 10.50 | 7.58849 | -5.76918 | 2.58631 | 0.32844 | -1.81383 |
| 11.00 | 8.08311 | -6.12526 | 2.58644 | 0.33388 | -1.81393 |
| 11.50 | 8.57890 | -6.48075 | 2.58652 | 0.33812 | -1.81398 |
| 12.00 | 9.07561 | -6.83578 | 2.58657 | 0.34143 | -1.81402 |
| 12.50 | 9.57305 | -7.19047 | 2.58660 | 0.34401 | -1.81404 |
| 13.00 | 10.07105 | -7.54490 | 2.58661 | 0.34602 | -1.81405 |
| 15.00 | 12.06658 | -8.96103 | 2.58664 | 0.35049 | -1.81407 |


| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{k i v}$ |  |  |  |
| :--- | ---: | ---: | ---: | ---: | :---: | :---: |
| 0.250 | 0.91495 | 1.41028 | -0.64760 | -0.98392 |  |  |
| 0.500 | 1.02366 | -0.50347 | -0.71398 | 0.36254 |  |  |
| 0.750 | -0.30409 | -0.80374 | 0.22402 | 0.56460 |  |  |
| 1.000 | -0.65434 | 0.19794 | 0.46271 | -0.14798 |  |  |
| 1.250 | 0.13707 | 0.54781 | -0.10339 | -0.38918 |  |  |
| 1.500 | 0.46915 | -0.09974 | -0.33434 | 0.07566 |  |  |
| 1.750 | -0.07549 | -0.40923 | 0.05748 | 0.29226 |  |  |
| 2.000 | -0.36234 | 0.05896 | 0.25917 | -0.04501 |  |  |
| 2.250 | 0.04724 | 0.32477 | -0.03613 | -0.23255 |  |  |
| 2.500 | 0.29408 | -0.03866 | -0.21074 | 0.02960 |  |  |
| 2.750 | -0.03220 | -0.26857 | 0.02468 | 0.19258 |  |  |
| 3.000 | -0.24705 | 0.02721 | 0.17724 | -0.02088 |  |  |
| 3.250 | 0.02329 | 0.22868 | -0.01788 | -0.16412 |  |  |
| 3.500 | 0.21281 | -0.02016 | -0.15278 | 0.01548 |  |  |
| 3.750 | -0.01761 | -0.19898 | 0.01353 | 0.14288 |  |  |
| 4.000 | -0.18682 | 0.01552 | 0.13418 | -0.01193 |  |  |
| 4.250 | 0.01377 | 0.17604 | -0.01059 | -0.12646 |  |  |
| 4.500 | 0.16643 | -0.01231 | -0.11957 | 0.00946 |  |  |
| 4.750 | -0.01106 | -0.15781 | 0.00851 | 0.11339 |  |  |
| 5.000 | -0.15003 | 0.01000 | 0.10782 | -0.00769 |  |  |
| 5.250 | 0.00908 | 0.14298 | -0.00698 | -0.10276 |  |  |
| 5.500 | 0.13656 | -0.00828 | -0.09815 | 0.00637 |  |  |
| 5.750 | -0.00758 | -0.13068 | 0.00583 | 0.09393 |  |  |
| 6.000 | -0.12529 | 0.00697 | 0.09006 | -0.00536 |  |  |
| 6.250 | 0.00642 | 0.12033 | -0.00494 | -0.08650 |  |  |
| 6.500 | 0.11574 | -0.00594 | -0.08320 | 0.00457 |  |  |
| 6.750 | -0.00551 | -0.11148 | 0.00424 | 0.08015 |  |  |
| 7.000 | -0.10753 | 0.00513 | 0.07731 | -0.00395 |  |  |
| 7.250 | 0.00478 | 0.10385 | -0.00368 | -0.07467 |  |  |
| 7.500 | 0.10041 | -0.00447 | -0.07219 | 0.00344 |  |  |
| 7.750 | -0.00419 | -0.09719 | 0.00322 | 0.06988 |  |  |
| 8.000 | -0.09417 | 0.00393 | 0.06771 | -0.00303 |  |  |
| 8.250 | 0.00370 | 0.09133 | -0.00285 | -0.06567 |  |  |
| 8.500 | 0.08866 | -0.00348 | -0.06375 | 0.00268 |  |  |
| 8.750 | -0.00329 | -0.08613 | 0.00253 | 0.06194 |  |  |
| 9.000 | 0.08375 | 0.00311 | 0.06023 | -0.00239 |  |  |
| 9.250 | 0.00294 | 0.08150 | -0.00227 | -0.05861 |  |  |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.375$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 9.500 | 0.07936 | -0.00279 | -0.05707 | 0.00215 |
| 9.750 | $-0.00265$ | $-0.07733$ | 0.00204 | 0.05561 |
| 10.000 | -0.07541 | 0.00252 | 0.05423 | -0.00194 |
| 10.250 | 0.00240 | 0.07358 | $-0.00185$ | -0.05291 |
| 10.500 | 0.07183 | -0.00229 | -0.05166 | 0.00176 |
| 10.750 | -0.00218 | $-0.07016$ | 0.00168 | 0.05046 |
| 11.000 | -0.06857 | 0.00208 | 0.04932 | -0.00161 |
| 11.250 | 0.00199 | 0.06705 | -0.00153 | -0.04822 |
| 11.500 | 0.06560 | -0.00191 | -0.04718 | 0.00147 |
| 11.750 | -0.00183 | -0.06421 | 0.00141 | 0.04618 |
| 12.000 | -0.06287 | 0.00175 | 0.04522 | -0.00135 |
| 12.250 | 0.00168 | 0.06159 | $-0.00130$ | -0.04430 |
| 12.500 | 0.06036 | -0.00161 | -0.04342 | 0.00124 |
| 12.750 | -0.00155 | -0.05918 | 0.00120 | 0.04257 |
| 13.000 | -0.05805 | 0.00149 | 0.04175 | -0.00115 |
| 13.250 | 0.00144 | 0.05695 | -0.00111 | -0.04096 |
| 13.500 | 0.05590 | -0.00138 | -0.04021 | 0.00107 |
| 13.750 | -0.00134 | -0.05489 | 0.00103 | 0.03948 |
| 14.000 | -0.05391 | 0.00129 | 0.03877 | -0.00099 |
| 14.250 | 0.00124 | 0.05297 | -0.00096 | -0.03809 |
| 14.500 | 0.05205 | $-0.00120$ | -0.03744 | 0.00092 |
| 14.750 | -0.00116 | -0.05117 | 0.00089 | 0.03681 |
| 15.000 | -0.05032 | 0.00112 | 0.03619 | -0.00086 |
| 15.250 | 0.00109 | 0.04950 | -0.00084 | -0.03560 |
| 15.500 | 0.04870 | -0.00105 | $-0.03503$ | 0.00081 |
| 15.750 | -0.00102 | -0.04793 | 0.00078 | 0.03447 |
| 16.000 | -0.04718 | 0.00099 | 0.03394 | -0.00076 |
| 16.250 | 0.00096 | 0.04646 | -0.00074 | -0.03341 |
| 16.500 | 0.04575 | $-0.00093$ | -0.03291 | 0.00071 |
| 16.750 | $-0.00090$ | -0.04507 | 0.00069 | 0.03242 |
| 17.000 | -0.04441 | 0.00087 | 0.03194 | -0.00067 |
| 17.250 | 0.00085 | 0.04377 | -0.00065 | -0.03148 |
| 17.500 | 0.04314 | $-0.00082$ | -0.03103 | 0.00064 |
| 17.750 | -0.00080 | -0.04253 | 0.00062 | 0.03059 |
| 18.000 | -0.04194 | 0.00078 | 0.03017 | -0.00060 |
| 18.250 | 0.00076 | 0.04137 | -0.00058 | -0.02976 |
| 18.500 | 0.04081 | -0.00074 | -0.02936 | 0.00057 |
| 18.750 | -0.00072 | -0.04027 | 0.00055 | 0.02896 |
| 19.000 | -0.03974 | 0.00070 | 0.02858 | -0.00054 |
| 19.250 | 0.00068 | 0.03922 | -0.00053 | -0.02821 |
| 19.500 | 0.03872 | -0.00066 | -0.02785 | 0.00051 |
| 19.750 | -0.00065 | -0.03823 | 0.00050 | 0.02750 |
| 20.000 | -0.03775 | 0.00063 | 0.02716 | -0.00049 |
| 20.250 | 0.00062 | 0.03729 | -0.00047 | -0.02682 |
| 20.500 | 0.03683 | -0.00060 | -0.02649 | 0.00046 |
| 20.750 | -0.00059 | -0.03639 | 0.00045 | 0.02618 |
| 21.000 | -0.03596 | 0.00057 | 0.02586 | -0.00044 |
| 21.250 | 0.00056 | 0.03553 | -0.00043 | -0.02556 |
| 21.500 | 0.03512 | -0.00055 | -0.02526 | 0.00042 |
| 21.750 | -0.00053 | -0.03472 | 0.00041 | 0.02497 |
| 22.000 | -0.03432 | 0.00052 | 0.02469 | -0.00040 |
| 22.250 | 0.00051 | 0.03394 | -0.00039 | -0.02441 |
| 22.500 | 0.03356 | -0.00050 | -0.02414 | 0.00038 |
| 22.750 | -0.00049 | -0.03319 | 0.00038 | 0.02388 |
| 23.000 | -0.03283 | 0.00048 | 0.02362 | -0.00037 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.375$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 23.250 | 0.00047 | 0.03248 | -0.00036 | -0.02336 |
| 23.500 | 0.03123 | $-0.00046$ | -0.02311 | 0.00035 |
| 23.750 | $-0.00045$ | -0.03180 | 0.00035 | 0.02287 |
| 24.000 | -0.03147 | 0.00044 | 0.02263 | -0.00034 |
| 24.250 | 0.00043 | 0.03114 | -0.00033 | -0.02240 |
| 24.500 | 0.03082 | -0.00042 | -0.02217 | 0.00032 |
| 24.750 | -0.00041 | -0.03051 | 0.00032 | 0.02195 |
| 25.000 | -0.03021 | 0.00040 | 0.02173 | -0.00031 |
| 25.250 | 0.00040 | 0.02991 | -0.00031 | -0.02151 |
| 25.500 | 0.02962 | -0.00039 | -0.02130 | 0.00030 |
| 25.750 | $-0.00038$ | -0.02933 | 0.00029 | 0.02110 |
| 26.000 | -0.02905 | 0.00037 | 0.02089 | -0.00029 |
| 26.250 | 0.00037 | 0.02877 | -0.00028 | -0.02069 |
| 26.500 | 0.02850 | $-0.00036$ | -0.02050 | 0.00028 |
| 26.750 | -0.00035 | $-0.02823$ | 0.00027 | 0.02031 |
| 27.000 | -0.02797 | 0.00035 | 0.02012 | -0.00027 |
| 27.250 | 0.00034 | 0.02771 | -0.00026 | -0.01994 |
| 27.500 | 0.02746 | $-0.00033$ | -0.01975 | 0.00026 |
| 27.750 | -0.00033 | -0.02722 | 0.00025 | 0.01958 |
| 28.000 | -0.02697 | 0.00032 | 0.01940 | -0.00025 |
| 28.250 | 0.00032 | 0.02673 | -0.00024 | -0.01923 |
| 28.500 | 0.02650 | $-0.00031$ | -0.01906 | 0.00024 |
| 28.750 | -0.00031 | $-0.02627$ | 0.00024 | 0.01890 |
| 29.000 | -0.02604 | 0.00030 | 0.01873 | -0.00023 |
| 29.250 | 0.00030 | 0.02582 | $-0.00023$ | -0.01857 |
| 29.500 | 0.02560 | -0.00029 | -0.01842 | 0.00022 |
| 29.750 | -0.00029 | -0.02539 | 0.00022 | 0.01826 |
| 30.000 | -0.02518 | 0.00028 | 0.01811 | -0.00022 |
| 30.250 | 0.00028 | 0.02497 | $-0.00021$ | -0.01796 |
| 30.500 | 0.02476 | -0.00027 | -0.01781 | 0.00021 |
| 30.750 | -0.00027 | -0.02456 | 0.00021 | 0.01767 |
| 31.000 | -0.02436 | 0.00026 | 0.01753 | -0.00020 |
| 31.250 | 0.00026 | 0.02417 | -0.00020 | -0.01738 |
| 31.500 | 0.02398 | $-0.00025$ | -0.01725 | 0.00020 |
| 31.750 | -0.00025 | -0.02379 | 0.00019 | 0.01711 |
| 32.000 | -0.02360 | 0.00025 | 0.01698 | -0.00019 |
| 32.250 | 0.00024 | 0.02342 | -0.00019 | -0.01685 |
| 32.500 | 0.02324 | $-0.00024$ | -0.01672 | 0.00018 |
| 32.750 | $-0.00024$ | $-0.02306$ | 0.00018 | 0.01659 |
| 33.000 | -0.02289 | 0.00023 | 0.01646 | -0.00018 |
| 33.250 | 0.00023 | 0.02272 | -0.00018 | -0.01634 |
| 33.500 | 0.02255 | -0.00023 | -0.01622 | 0.00017 |
| 33.750 | -0.00022 | $-0.02238$ | 0.00017 | 0.01610 |
| 34.000 | -0.02221 | 0.00022 | 0.01598 | -0.00017 |
| 34.250 | 0.00022 | 0.02205 | -0.00017 | -0.01586 |
| 34.500 | 0.02189 | $-0.00021$ | -0.01575 | 0.00016 |
| 34.750 | $-0.00021$ | $-0.02174$ | 0.00016 | 0.01563 |
| 35.000 | -0.02158 | 0.00021 | 0.01552 | -0.00016 |
| 35.250 | 0.00020 | 0.02143 | -0.00016 | -0.01541 |
| 35.500 | 0.02128 | $-0.00020$ | -0.01530 | 0.00015 |
| 35.750 | -0.00020 | $-0.02113$ | 0.00015 | 0.01520 |
| 36.000 | -0.02098 | 0.00019 | 0.01509 | -0.00015 |
| 36.250 | 0.00019 | 0.02084 | $-0.00015$ | -0.01499 |
| 36.500 | 0.02069 | $-0.00019$ | -0.01489 | 0.00015 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.375$ Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{k i v}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 36.750 | -0.00019 | $-0.02055$ | 0.00014 | 0.01478 |
| 37.000 | -0.02041 | 0.00018 | 0.01468 | -0.00014 |
| 37.250 | 0.00018 | 0.02028 | -0.00014 | -0.01459 |
| 37.500 | 0.02014 | $-0.00018$ | -0.01449 | 0.00014 |
| 37.750 | $-0.00018$ | -0.02001 | 0.00014 | 0.01439 |
| 38.000 | -0.01988 | 0.00018 | 0.01430 | -0.00013 |
| 38.250 | 0.00017 | 0.01975 | $-0.00013$ | $-0.01420$ |
| 38.500 | 0.01962 | -0.00017 | -0.01411 | 0.00013 |
| 38.750 | -0.00017 | $-0.01949$ | 0.00013 | 0.01402 |
| 39.000 | -0.01937 | 0.00017 | 0.01393 | $-0.00013$ |
| 39.250 | 0.00016 | 0.01924 | -0.00013 | -0.01384 |
| 39.500 | 0.01912 | -0.00016 | -0.01375 | 0.00012 |
| 39.750 | $-0.00016$ | $-0.01900$ | 0.00012 | 0.01367 |
| 40.000 | -0.01888 | 0.00016 | 0.01358 | $-0.00012$ |
| 40.250 | 0.00016 | 0.01877 | -0.00012 | -0.01350 |
| 40.500 | 0.01865 | -0.00015 | -0.01342 | 0.00012 |
| 40.750 | $-0.00015$ | $-0.01854$ | 0.00012 | 0.01333 |
| 41.000 | -0.01842 | 0.00015 | 0.01325 | -0.00012 |
| 41.250 | 0.00015 | 0.01831 | $-0.00011$ | -0.01317 |
| 41.500 | 0.01820 | -0.00015 | -0.01309 | 0.00011 |
| 41.750 | -0.00014 | $-0.01809$ | 0.00011 | 0.01301 |
| 42.000 | $-0.01798$ | 0.00014 | 0.01294 | -0.00011 |
| 42.250 | 0.00014 | 0.01788 | $-0.00011$ | -0.01286 |
| 42.500 | 0.01777 | $-0.00014$ | -0.01278 | 0.00011 |
| 42.750 | -0.00014 | -0.01767 | 0.00011 | 0.01271 |
| 43.000 | -0.01757 | 0.00014 | 0.01264 | -0.00011 |
| 43.250 | 0.00014 | 0.01746 | -0.00010 | -0.01256 |
| 43.500 | 0.01736 | -0.00013 | -0.01249 | 0.00010 |
| 43.750 | $-0.00013$ | -0.01726 | 0.00010 | 0.01242 |
| 44.000 | -0.01717 | 0.00013 | 0.01235 | -0.00010 |
| 44.250 | 0.00013 | 0.01707 | -0.00010 | -0.01228 |
| 44.500 | 0.01697 | -0.00013 | $-0.01221$ | 0.00010 |
| 44.750 | -0.00013 | $-0.01688$ | 0.0069 | 0.01214 |
| 45.000 | $-0.01679$ | 0.00012 | 0.01207 | -0.00010 |
| 45.250 | 0.00012 | 0.01669 | $-0.00010$ | -0.01201 |
| 45.500 | 0.01660 | -0.00012 | -0.01194 | 0.00009 |
| 45.750 | -0.00012 | $-0.01651$ | 0.00009 | 0.01188 |
| 46.000 | -0.01642 | 0.00012 | 0.01181 | -0.00009 |
| 46.250 | 0.00012 | 0.01633 | -0.00009 | -0.01175 |
| 46.500 | 0.01624 | $-0.00012$ | $-0.01168$ | 0.00009 |
| 46.750 | -0.00012 | -0.01616 | 0.00009 | 0.01162 |
| 47.000 | -0.01607 | 0.00011 | 0.01156 | -0.00009 |
| 47.250 | 0.00011 | 0.01599 | -0.00009 | -0.01150 |
| 47.500 | 0.01590 | -0.00011 | $-0.01144$ | 0.00009 |
| 47.750 | -0.00011 | $-0.01582$ | 0.00009 | 0.01138 |
| 48.000 | -0.01574 | 0.00011 | 0.01132 | -0.00008 |
| 48.250 | 0.00011 | 0.01565 | $-0.00008$ | -0.01126 |
| 48.500 | 0.01557 | $-0.00011$ | $-0.01120$ | 0.00008 |
| 48.750 | -0.00011 | $-0.01549$ | 0.00008 | 0.01115 |
| 49.000 | -0.01542 | 0.00011 | 0.01109 | -0.00008 |
| 49.250 | 0.00010 | 0.01534 | -0.00008 | -0.01103 |
| 49.500 | 0.01526 | $-0.00010$ | -0.01098 | 0.00008 |
| 49.750 | $-0.00010$ | -0.01518 | 0.00008 | 0.01092 |
| 50.000 | -0.01511 | 0.00010 | 0.01087 | -0.00008 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.50$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$

| $\Omega$ | $\Psi_{d R}$ | $\Phi_{k u}$ |  | $\Phi_{k k v}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 7.00 | 2.96068 | -3.65469 | 2.87405 | 0.48701 | -1.64559 |
| 7.50 | 3.42057 | -4.16756 | 2.88308 | 0.52709 | -1.65089 |
| 8.00 | 3.88899 | -4.67618 | 2.88857 | 0.55866 | -1.65410 |
| 8.50 | 4.36420 | -5.18191 | 2.89191 | 0.58344 | -1.65605 |
| 9.00 | 4.84477 | -5.68568 | 2.89393 | 0.60287 | -1.65724 |
| 9.50 | 5.32957 | -6.18815 | 2.89516 | 0.61807 | -1.65796 |
| 10.00 | 5.81769 | -6.68975 | 2.89590 | 0.62995 | -1.65839 |
| 10.50 | 6.30841 | -7.19079 | 2.89635 | 0.63923 | -1.65866 |
| 11.00 | 6.80117 | -7.69146 | 2.89662 | 0.64647 | -1.65882 |
| 11.50 | 7.29552 | -8.19189 | 2.89679 | 0.65212 | -1.65891 |
| 12.00 | 7.79112 | -8.69217 | 2.89689 | 0.65653 | -1.65897 |
| 12.50 | 8.28768 | -9.19235 | 2.89695 | 0.65996 | -1.65901 |
| 13.00 | 8.78501 | -9.69246 | 2.89699 | 0.66264 | -1.65903 |
| 15.00 | 10.77905 | -11.69262 | 2.89704 | 0.66860 | -1.65906 |
|  | Off diagonal values of $\Phi_{u}$ and $\Phi_{v}($ see note on page 424$)$ |  |  |  |  |


| $(k-i)(b / \lambda)$ | $\Phi_{k i u}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 0.250 | 1.02173 | 1.52473 | -0.68124 | --0.85698 |
| 0.500 | $1 \cdot 10298$ | -0.66347 | -0.62956 | 0.40704 |
| 0.750 | -0.45462 | -0.90710 | 0.29109 | 0.51517 |
| 1.000 | $-0.76996$ | 0.32113 | 0.44407 | -0.21742 |
| 1.250 | 0.23434 | 0.66433 | -0.16567 | -0.39044 |
| 1.500 | 0.58100 | -0.17648 | $-0.34702$ | 0.12866 |
| 1.750 | $-0.13673$ | -0.51431 | 0.10187 | 0.31107 |
| 2.000 | $-0.46021$ | 0.10856 | 0.28103 | -0.08216 |
| 2.250 | 0.08803 | 0.41571 | -0.06739 | -0.25573 |
| 2.500 | 0.37862 | -0.07267 | -0.23423 | 0.05612 |
| 2.750 | $-0.06093$ | -0.34732 | 0.04737 | 0.21582 |
| 3.000 | -0.32062 | 0.05177 | 0.19993 | -0.04046 |
| 3.250 | 0.04450 | 0.29760 | -0.03492 | -0.18610 |
| 3.500 | 0.27757 | -0.03864 | -0.17397 | 0.03042 |
| 3.750 | -0.03385 | -0.26000 | 0.02673 | 0.16327 |
| 4.000 | -0.24448 | 0.02989 | 0.15376 | -0.02365 |
| 4.250 | 0.02659 | 0.23067 | -0.02107 | -0.14527 |
| 4.500 | 0.21832 | -0.02379 | -0.13764 | 0.01889 |
| 4.750 | -0.02141 | -0.20720 | 0.01702 | 0.13076 |
| 5.000 | -0.19714 | 0.01937 | 0.12451 | $-0.01542$ |
| 5.250 | 0.01761 | 0.18800 | -0.01403 | -0.11882 |
| 5.500 | 0.17966 | -0.01607 | $-0.11362$ | 0.01282 |
| 5.750 | -0.01473 | -0.17202 | 0.01175 | 0.10885 |
| 6.000 | -0.16500 | 0.01354 | 0.10446 | -0.01082 |
| 6.250 | 0.01250 | 0.15852 | -0.00999 | -0.10040 |
| 6.500 | 0.15253 | -0.01157 | -0.09664 | 0.00925 |
| 6.750 | -0.01074 | -0.14697 | 0.00859 | 0.09315 |
| 7.000 | -0.14180 | 0.00999 | 0.08990 | -0.00800 |
| 7.250 | 0.00932 | 0.13698 | -0.00746 | -0.08687 |
| 7.500 | 0.13247 | -0.00872 | -0.08403 | 0.00698 |
| 7.750 | -0.00817 | -0.12825 | 0.00654 | 0.08137 |
| 8.000 | -0.12429 | 0.00767 | 0.07888 | -0.00615 |
| 8.250 | 0.00722 | 0.12056 | -0.00578 | -0.07653 |
| 8.500 | 0.11706 | -0.00680 | -0.07431 | 0.00545 |
| 8.750 | -0.00642 | -0.11374 | 0.00515 | 0.07222 |
| 9.000 | -0.11061 | 0.00607 | 0.07024 | -0.00487 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.50$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$--contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 9.250 | 0.00575 | 0.10765 | -0.00461 | $-0.06837$ |
| 9.500 | 0.10484 | -0.00545 | -0.06659 | 0.00438 |
| 9.750 | -0.00518 | -0.10217 | 0.00416 | 0.06491 |
| 10.000 | -0.09964 | 0.00493 | 0.06330 | $-0.00395$ |
| 10.250 | 0.00469 | 0.09722 | -0.00376 | -0.06177 |
| 10.500 | 0.09492 | -0.00447 | -0.06032 | 0.00359 |
| 10.750 | -0.00427 | -0.09273 | 0.00342 | 0.05893 |
| 11.000 | -0.09063 | 0.00407 | 0.05760 | $-0.00327$ |
| 11.250 | 0.00390 | 0.08863 | -0.00313 | -0.05633 |
| 11.500 | 0.08672 | -0.00373 | -0.05512 | 0.00300 |
| 11.750 | $-0.00357$ | -0.08488 | 0.00287 | 0.05396 |
| 12.000 | $-0.08312$ | 0.00343 | 0.05284 | $-0.00275$ |
| 12.250 | 0.00329 | 0.08143 | -0.00264 | -0.05177 |
| 12.500 | 0.07981 | -0.00316 | -0.05074 | 0.00254 |
| 12.750 | -0.00304 | -0.07826 | 0.00244 | 0.04976 |
| 13.000 | -0.07676 | 0.00292 | 0.04881 | -0.00235 |
| 13.250 | 0.00281 | 0.07531 | -0.00226 | -0.04789 |
| 13.500 | 0.07393 | -0.00271 | -0.04701 | 0.00218 |
| 13.750 | -0.00261 | -0.07259 | 0.00210 | 0.04616 |
| 14.000 | -0.07130 | 0.00252 | 0.04534 | -0.00203 |
| 14.250 | 0.00243 | 0.07005 | -0.00196 | $-0.04455$ |
| 14.500 | 0.06885 | -0.00235 | -0.04379 | 0.00189 |
| 14.750 | -0.00227 | -0.06768 | 0.00183 | 0.04305 |
| 15.000 | -0.06656 | 0.00220 | 0.04233 | -0.00177 |
| 15.250 | 0.00212 | 0.06547 | -0.00171 | -0.04164 |
| 15.500 | 0.06442 | -0.00206 | -0.04097 | 0.00165 |
| 15.750 | -0.00199 | -0.06340 | 0.00160 | 0.04033 |
| 16.000 | -0.06241 | 0.00193 | 0.03970 | -0.00155 |
| 16.250 | 0.00187 | 0.06145 | $-0.00150$ | -0.03909 |
| 16.500 | 0.06052 | -0.00182 | -0.03850 | 0.00146 |
| 16.750 | -0.00175 | -0.05962 | 0.00142 | 0.03793 |
| 17.000 | -0.05875 | 0.00171 | 0.03737 | $-0.00138$ |
| 17.250 | 0.00166 | 0.05790 | -0.00134 | $-0.03683$ |
| 17.500 | 0.05707 | -0.00161 | -0.03631 | 0.00130 |
| 17.750 | $-0.00157$ | -0.05627 | 0.00126 | 0.03580 |
| 18.000 | -0.05549 | 0.00153 | 0.03531 | -0.00123 |
| 18.250 | 0.00148 | 0.05473 | -0.00119 | -0.03482 |
| 18.500 | 0.05400 | -0.00144 | -0.03435 | 0.00116 |
| 18.750 | -0.00141 | -0.05328 | 0.00113 | 0.03390 |
| 19.000 | -0.05258 | 0.00137 | 0.03345 | -0.00110 |
| 19.250 | 0.00133 | 0.05190 | -0.00107 | $-0.03302$ |
| 19.500 | 0.05123 | -0.00130 | -0.03260 | 0.00105 |
| 19.750 | -0.00127 | -0.05059 | 0.00102 | 0.03219 |
| 20.000 | -0.04995 | 0.00124 | 0.03179 | -0.00099 |
| 20.250 | 0.00121 | 0.04934 | -0.00097 | -0.03139 |
| 20.500 | 0.04874 | -0.00118 | -0.03101 | 0.00095 |
| 20.750 | $-0.00115$ | -0.04815 | 0.00092 | 0.03064 |
| 21.000 | -0.04758 | 0.00112 | 0.03028 | -0.00090 |
| 21.250 | 0.00110 | 0.04702 | -0.00088 | -0.02992 |
| 21.500 | 0.04647 | -0.00107 | -0.02957 | 0.00086 |
| 21.750 | -0.00105 | -0.04594 | 0.00084 | 0.02923 |
| 22.000 | -0.04542 | 0.00102 | 0.02890 | -0.00082 |
| 22.250 | 0.00100 | 0.04491 | -0.00080 | -0.02858 |
| 22.500 | 0.04441 | -0.00098 | -0.02826 | 0.00079 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.50$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| ( $k-i$ )(b/ $\lambda$ ) | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 22.750 | -0.00096 | -0.04392 | 0.00077 | 0.02795 |
| 23.000 | -0.04345 | 0.00094 | 0.02765 | -0.00075 |
| 23.250 | 0.00092 | 0.04298 | -0.00074 | -0.02735 |
| 23.500 | 0.04252 | -0.00090 | -0.02706 | 0.00072 |
| 23.750 | -0.00088 | -0.04208 | 0.00071 | 0.02678 |
| 24.000 | -0.04164 | 0.00086 | 0.02650 | -0.00069 |
| 24.250 | 0.00084 | 0.04121 | -0.00068 | -0.02623 |
| 24.500 | 0.04079 | -0.00082 | -0.02596 | 0.00066 |
| 24.750 | -0.00081 | -0.04038 | 0.00065 | 0.02570 |
| 25.000 | -0.03998 | 0.00079 | 0.02544 | -0.00064 |
| 25.250 | 0.00078 | 0.03958 | -0.00062 | -0.02519 |
| 25.500 | 0.03919 | -0.00076 | -0.02494 | 0.00061 |
| 25.750 | -0.00075 | -0.03881 | 0.00060 | 0.02470 |
| 26.000 | -0.03844 | 0.00073 | 0.02446 | -0.00059 |
| 26.250 | 0.00072 | 0.03807 | -0.00058 | -0.02423 |
| 26.500 | 0.03772 | -0.00070 | -0.02400 | 0.00057 |
| 26.750 | -0.00069 | -0.03736 | 0.00056 | 0.02378 |
| 27.000 | -0.03702 | 0.00068 | 0.02356 | -0.00055 |
| 27.250 | 0.00067 | 0.03668 | -0.00054 | -0.02334 |
| 27.500 | 0.03635 | -0.00065 | -0.02313 | 0.00053 |
| 27.750 | -0.00064 | -0.03602 | 0.00052 | 0.02292 |
| 28.000 | -0.03570 | 0.00063 | 0.02272 | -0.00051 |
| 28.250 | 0.00062 | 0.03538 | -0.00050 | -0.02252 |
| 28.500 | 0.03507 | -0.00061 | -0.02232 | 0.00049 |
| 28.750 | -0.00060 | -0.03477 | 0.00048 | 0.02213 |
| 29.000 | -0.03447 | 0.00059 | 0.02194 | -0.00047 |
| 29.250 | 0.00058 | 0.03417 | -0.00047 | -0.02175 |
| 29.500 | 0.03388 | -0.00057 | -0.02157 | 0.00046 |
| 29.750 | -0.00056 | -0.03360 | 0.00045 | 0.02139 |
| 30.000 | -0.03332 | 0.00055 | 0.02121 | -0.00044 |
| 30.250 | 0.00054 | 0.03304 | -0.00044 | -0.02103 |
| 30.500 | 0.03277 | -0.00053 | -0.02086 | 0.00043 |
| 30.750 | -0.00052 | -0.03251 | 0.00042 | 0.02069 |
| 31.000 | -0.03225 | 0.00052 | 0.02052 | -0.00041 |
| 31.250 | 0.00051 | 0.03199 | -0.00041 | -0.02036 |
| 31.500 | 0.03173 | -0.00050 | -0.02020 | 0.00040 |
| 31.750 | -0.00049 | -0.03148 | 0.00040 | 0.02004 |
| 32.000 | -0.03124 | 0.00048 | 0.01988 | -0.00039 |
| 32.250 | 0.00048 | 0.03100 | -0.00038 | -0.01973 |
| 32.500 | 0.03076 | $-0.00047$ | -0.01958 | 0.00038 |
| 32.750 | -0.00046 | -0.03052 | 0.00037 | 0.01943 |
| 33.000 | -0.03029 | 0.00045 | 0.01928 | -0.00037 |
| 33.250 | 0.00045 | 0.03007 | $-0.00036$ | -0.01914 |
| 33.500 | 0.02984 | -0.00044 | -0.01899 | 0.00035 |
| 33.750 | -0.00043 | -0.02962 | 0.00035 | 0.01885 |
| 34.000 | -0.02940 | 0.00043 | 0.01871 | -0.00034 |
| 34.250 | 0.00042 | 0.02919 | -0.00034 | -0.01858 |
| 34.500 | 0.02898 | -0.00042 | $-0.01844$ | 0.00033 |
| 34.750 | -0.00041 | -0.02877 | 0.00033 | 0.01831 |
| 35.000 | -0.02856 | 0.00040 | 0.01818 | -0.00033 |
| 35.250 | 0.00040 | 0.02836 | -0.00032 | -0.01805 |
| 35.500 | 0.02816 | -0.00039 | -0.01792 | 0.00032 |
| 35.750 | -0.00039 | -0.02796 | 0.00031 | 0.01780 |
| 36.000 | -0.02777 | 0.00038 | 0.01768 | -0.00031 |
| 36.250 | 0.00038 | 0.02758 | -0.00030 | -0.01755 |
| 36.500 | 0.02739 | -0.00037 | -0.01743 | 0.00030 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.50$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 36.750 | -0.00037 | $-0.02720$ | 0.00029 | 0.01732 |
| 37.000 | -0.02702 | 0.00036 | 0.01720 | -0.00029 |
| 37.250 | 0.00036 | 0.02684 | $-0.00029$ | -0.01708 |
| 37.500 | 0.02666 | $-0.00035$ | $-0.01697$ | 0.00028 |
| 37.750 | -0.00035 | $-0.02648$ | 0.00028 | 0.01686 |
| 38.000 | -0.02631 | 0.00034 | 0.01675 | $-0.00028$ |
| 38.250 | 0.00034 | 0.02614 | $-0.00027$ | -0.01664 |
| 38.500 | 0.02597 | -0.00033 | $-0.01653$ | 0.00027 |
| 38.750 | $-0.00033$ | $-0.02580$ | 0.00027 | 0.01642 |
| 39.000 | $-0.02563$ | 0.00033 | 0.01632 | -0.00026 |
| 39.250 | 0.00032 | 0.02547 | $-0.00026$ | -0.01621 |
| 39.500 | 0.02531 | $-0.00032$ | $-0.01611$ | 0.00026 |
| 39.750 | $-0.00031$ | $-0.02515$ | 0.00025 | 0.01601 |
| 40.000 | -0.02499 | 0.00031 | 0.01591 | -0.00025 |
| 40.250 | 0.00031 | 0.02484 | $-0.00025$ | -0.01581 |
| 40.500 | 0.02469 | $-0.00030$ | $-0.01571$ | 0.00024 |
| 40.750 | -0.00030 | $-0.02453$ | 0.00024 | 0.01562 |
| 41.000 | -0.02438 | 0.00029 | 0.01552 | -0.00024 |
| 41.250 | 0.00029 | 0.02424 | $-0.00023$ | -0.01543 |
| 41.500 | 0.02409 | $-0.00029$ | $-0.01534$ | 0.00023 |
| 41.750 | -0.00028 | -0.02395 | 000023 | 0.01524 |
| 42.000 | -0.02380 | 0.00028 | 0.01515 | -0.00023 |
| 42.250 | 0.00028 | 0.02366 | $-0.00022$ | -0.01506 |
| 42.500 | 0.02352 | $-0.00027$ | -0.01497 | 0.00022 |
| 42.750 | -0.00027 | -0.02339 | 0.00022 | 0.01489 |
| 43.000 | -0.02325 | 0.00027 | 0.01480 | -0.00022 |
| 43.250 | 0.00026 | 0.02312 | -0.00021 | -0.01472 |
| 43.500 | 0.02298 | $-0.00026$ | -0.01463 | 0.00021 |
| 43.750 | -0.00026 | $-0.02285$ | 0.00021 | 0.01455 |
| 44.000 | -0.02272 | 0.00026 | 0.01446 | -0.00021 |
| 44.250 | 0.00025 | 0.02259 | -0.00020 | -0.01438 |
| 44.500 | 0.02247 | $-0.00025$ | -0.01430 | 0.00020 |
| 44.750 | -0.00025 | $-0.02234$ | 0.00020 | 0.01422 |
| 45.000 | -0.02222 | 0.00024 | 0.01414 | -0.00020 |
| 45.250 | 0.00024 | 0.02210 | -0.00019 | -0.01407 |
| 45.500 | 0.02197 | $-0.00024$ | -0.01399 | 0.00019 |
| 45.750 | -0.00024 | $-0.02185$ | 0.00019 | 0.01391 |
| 46.000 | -0.02174 | 0.00023 | 0.01384 | -0.00019 |
| 46.250 | 0.00023 | 0.02162 | -0.00019 | -0.01376 |
| 46.500 | 0.02150 | $-0.00023$ | -0.01369 | 0.00018 |
| 46.750 | -0.00023 | $-0.02139$ | 0.00018 | 0.01361 |
| 47.000 | -0.02127 | 0.00022 | 0.01354 | -0.00018 |
| 47.250 | 0.00022 | 0.02116 | -0.00018 | -0.01347 |
| 47.500 | 0.02105 | $-0.00022$ | -0.01340 | 0.00018 |
| 47.750 | -0.00022 | -0.02094 | 0.00017 | 0.01333 |
| 48.000 | -0.02083 | 0.00021 | 0.01326 | -0.00017 |
| 48.250 | 0.00021 | 0.02072 | -0.00017 | -0.01319 |
| 48.500 | 0.02062 | $-0.00021$ | -0.01312 | 0.00017 |
| 48.750 | -0.00021 | -0.02051 | 0.00017 | 0.01306 |
| 49.000 | -0.02041 | 0.00021 | 0.01299 | -0.00017 |
| 49.250 | 0.00020 | 0.02030 | -0.00016 | -0.01292 |
| 49.500 | 0.02020 | $-0.00020$ | $-0.01286$ | 0.00016 |
| 49.750 | -0.00020 | $-0.02010$ | 0.00016 | 0.01279 |
| 50.000 | -0.02000 | 0.00020 | 0.01273 | -0.000i6 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.625$ Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$

| $\Omega$ | $\Psi_{d R}$ | $\Phi_{k k u}$ |  | $\Phi_{k k v}$ |  |
| :---: | ---: | ---: | ---: | ---: | ---: |
| 7.00 | 2.81599 | -2.06379 | 1.28068 | 0.83173 | -1.02685 |
| 7.50 | 3.26383 | -2.40814 | 1.28795 | 0.88239 | -1.03206 |
| 8.00 | 3.72310 | -2.75320 | 1.29236 | 0.92221 | -1.03523 |
| 8.50 | 4.19134 | -3.09919 | 1.29505 | 0.95343 | -1.03716 |
| 9.00 | 4.66659 | -3.44620 | 1.29668 | 0.97785 | -1.03833 |
| 9.50 | 5.14730 | -3.79420 | 1.29766 | 0.99694 | -1.03904 |
| 10.00 | 5.63227 | -4.14312 | 1.29826 | 1.01185 | -1.03947 |
| 10.50 | 6.12057 | -4.49286 | 1.29863 | 1.02348 | -1.03973 |
| 11.00 | 6.61145 | -4.84330 | 1.29885 | 1.03255 | -1.03989 |
| 11.50 | 7.10435 | -5.19435 | 1.29898 | 1.03963 | -1.03998 |
| 12.00 | 7.59882 | -5.54589 | 1.29906 | 1.04514 | -1.04004 |
| 12.50 | 8.09452 | -5.89783 | 1.29911 | 1.04944 | -1.04008 |
| 13.00 | 8.59116 | -6.25011 | 1.29914 | 1.05278 | -1.04010 |
| 15.00 | 10.58370 | -7.66140 | 1.29918 | 1.06024 | -1.04013 |

Off diagonal values of $\Phi_{w}$ and $\Phi_{v}$ (see note on page 424)

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | ---: | ---: | ---: | ---: |
| 0.250 | 0.45164 | 0.59842 | -0.56525 | -0.53538 |
| 0.500 | 0.42851 | -0.45552 | -0.40583 | 0.24351 |
| 0.750 | -0.38272 | -0.42507 | 0.17624 | 0.28315 |
| 1.000 | -0.41223 | 0.29489 | 0.23084 | -0.16312 |
| 1.250 | 0.22288 | 0.38447 | -0.15177 | -0.21024 |
| 1.500 | 0.35176 | -0.16988 | -0.19944 | 0.13674 |
| 1.750 | -0.13186 | -0.31989 | 0.12048 | 0.19073 |
| 2.000 | -0.29107 | 0.10445 | 0.18209 | -0.10500 |
| 2.250 | 0.08436 | 0.26579 | -0.09121 | -0.17329 |
| 2.500 | 0.24383 | -0.06935 | -0.16453 | 0.07932 |
| 2.750 | -0.05790 | -0.22480 | 0.06922 | 0.15603 |
| 3.000 | -0.20826 | 0.04901 | 0.14794 | -0.06068 |
| 3.250 | 0.04198 | 0.19382 | -0.05348 | -0.14033 |
| 3.500 | 0.18113 | -0.03635 | -0.13324 | 0.04738 |
| 3.750 | -0.03176 | -0.16993 | 0.04221 | 0.12666 |
| 4.000 | -0.15997 | 0.02798 | 0.12058 | -0.03778 |
| 4.250 | 0.02484 | 0.15108 | -0.03399 | -0.11496 |
| 4.500 | 0.14310 | -0.02219 | -0.10977 | 0.03071 |
| 4.750 | -0.01994 | -0.13589 | 0.02787 | 0.10497 |
| 5.000 | -0.12936 | 0.01801 | 0.10053 | -0.02539 |
| 5.250 | 0.01635 | 0.12342 | -0.02322 | -0.09641 |
| 5.500 | 0.11799 | -0.01491 | -0.09259 | 0.02131 |
| 5.750 | -0.01365 | -0.11300 | 0.01962 | 0.08904 |
| 6.000 | -0.10842 | 0.01255 | 0.08574 | -0.01812 |
| 6.250 | 0.01157 | 0.10419 | -0.01678 | -0.08265 |
| 6.500 | 0.10027 | -0.01070 | -0.07977 | 0.01558 |
| 6.750 | -0.00993 | -0.09663 | 0.01450 | 0.07707 |
| 7.000 | -0.09325 | 0.00923 | 0.07454 | -0.01353 |
| 7.250 | 0.00861 | 0.09009 | -0.01266 | -0.07217 |
| 7.500 | 0.08714 | -0.00805 | -0.06993 | 0.01186 |
| 7.750 | -0.00754 | -0.08437 | 0.01114 | 0.06783 |
| 8.000 | 0.08177 | 0.00707 | 0.06584 | -0.01047 |
| 8.250 | 0.07702 | -0.000627 | -0.00987 | -0.06396 |
| 8.500 | -0.00592 | -0.07485 | 0.008818 | 0.00932 |
| 8.750 |  |  | 0.06050 |  |
|  |  |  |  |  |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.625$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 9.000 | -0.07279 | 0.00559 | 0.05890 | -0.00834 |
| 9.250 | 0.00530 | 0.07085 | -0.00791 | -0.05739 |
| 9.500 | 0.06900 | -0.00502 | -0.05594 | 0.00751 |
| 9.750 | -0.00477 | -0.06725 | 0.00713 | 0.05457 |
| 10.000 | -0.06558 | 0.00453 | 0.05326 | -0.00679 |
| 10.250 | 0.00431 | 0.06400 | -0.00647 | -0.05201 |
| 10.500 | 0.06249 | -0.00411 | -0.05082 | 0.00617 |
| 10.750 | -0.00392 | -0.06105 | 0.00589 | 0.04968 |
| 11.000 | -0.05967 | 0.00375 | 0.04859 | $-0.00563$ |
| 11.250 | 0.00358 | 0.05835 | $-0.00539$ | -0.04754 |
| 11.500 | 0.05709 | -0.00343 | -0.04654 | 0.00516 |
| 11.750 | -0.00328 | -0.05589 | 0.00495 | 0.04558 |
| 12.000 | $-0.05473$ | 0.00315 | 0.04465 | $-0.00475$ |
| 12.250 | 0.00302 | 0.05362 | $-0.00456$ | -0.04377 |
| 12.500 | 0.05255 | $-0.00290$ | -0.04291 | 0.00438 |
| 12.750 | -0.00279 | $-0.05153$ | 0.00421 | 0.04209 |
| 13.000 | -0.05054 | 0.00268 | 0.04130 | -0.00406 |
| 13.250 | 0.00258 | 0.04959 | -0.00391 | -0.04054 |
| 13.500 | 0.04868 | -0.00249 | -0.03981 | 0.00376 |
| 13.750 | -0.00240 | -0.04780 | 0.00363 | 0.03910 |
| 14.000 | -0.04695 | 0.00231 | 0.03842 | -0.00350 |
| 14.250 | 0.00223 | 0.04613 | -0.00338 | -0.03776 |
| 14.500 | 0.04534 | $-0.00216$ | -0.03712 | 0.00327 |
| 14.750 | -0.00209 | -0.04457 | 0.00316 | 0.03650 |
| 15.000 | -0.04383 | 0.00202 | 0.03590 | $-0.00306$ |
| 15.250 | 0.00195 | 0.04312 | -0.00296 | -0.03532 |
| 15.500 | 0.04242 | -0.00189 | -0.03476 | 0.00286 |
| 15.750 | -0.00183 | -0.04175 | 0.00277 | 0.03422 |
| 16.000 | -0.04110 | 0.00177 | 0.03370 | -0.00269 |
| 16.250 | 0.00172 | 0.04047 | $-0.00261$ | -0.03318 |
| 16.500 | 0.03986 | -0.00167 | -0.03269 | 0.00253 |
| 16.750 | $-0.00162$ | -0.03927 | 0.00246 | 0.03221 |
| 17.000 | -0.03869 | 0.00157 | 0.03174 | -0.00238 |
| 17.250 | 0.00153 | 0.03813 | -0.00232 | -0.03129 |
| 17.500 | 0.03759 | -0.00148 | $-0.03085$ | 0.00225 |
| 17.750 | -0.00144 | $-0.03706$ | 0.00219 | 0.03042 |
| 18.000 | -0.03655 | 0.00140 | 0.03000 | $-0.00213$ |
| 18.250 | 0.00136 | 0.03605 | -0.00207 | -0.02959 |
| 18.500 | 0.03556 | -0.00133 | -0.02920 | 0.00202 |
| 18.750 | -0.00129 | -0.03509 | 0.00196 | 0.02881 |
| 19.000 | -0.03463 | 0.00126 | 0.02844 | -0.00191 |
| 19.250 | 0.00122 | 0.03418 | -0.00186 | -0.02807 |
| 19.500 | 0.03375 | -0.00119 | -0.02772 | 0.00182 |
| 19.750 | -0.00116 | -0.03332 | 0.00177 | 0.02737 |
| 20.000 | -0.03290 | 0.00113 | 0.02703 | -0.00173 |
| 20.250 | 0.00111 | 0.03250 | -0.00168 | -0.02670 |
| 20.500 | 0.03210 | -0.00108 | -0.02638 | 0.00164 |
| 20.750 | -0.00105 | -0.0317? | 0.00160 | 0.02606 |
| 21.000 | -0.03134 | 0.00103 | 0.02576 | $-0.00157$ |
| 21.250 | 0.00101 | 0.03097 | -0.00153 | -0.02546 |
| 21.500 | 0.03061 | -0.00098 | -0.02516 | 0.00150 |
| 21.750 | -0.00096 | -0.03026 | 0.00146 | 0.02488 |
| 22.000 | -0.02992 | 0.00094 | 0.02460 | -0.00143 |
| 22.250 | 0.00092 | 0.02958 | -0.00140 | -0.02432 |
| 22.500 | 0.02925 | -0.00090 | -0.02405 | 0.00137 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.625$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(h / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 22.750 | -0.00088 | -0.02893 | 0.00134 | 0.02379 |
| 23.000 | -0.02862 | 0.00086 | 0.02354 | -0.00131 |
| 23.250 | 0.00084 | 0.02831 | -0.00128 | -0.02328 |
| 23.500 | 0.02801 | -0.00082 | -0.02304 | 0.00125 |
| 23.750 | -0.00080 | -0.02772 | 0.00123 | 0.02280 |
| 24.000 | -0.02743 | 0.00079 | 0.02256 | -0.00120 |
| 24.250 | 0.00077 | 0.02715 | -0.00118 | -0.02233 |
| 24.500 | 0.02687 | -0.00076 | -0.02210 | 0.00115 |
| 24.750 | -0.00074 | -0.02660 | 0.00113 | 0.02188 |
| 25.000 | -0.02633 | 0.00073 | 0.02167 | -0.00111 |
| 25.250 | 0.00071 | 0.02607 | -0.00109 | -0.02145 |
| 25.500 | 0.02582 | -0.00070 | $-0.02124$ | 0.00106 |
| 25.750 | -0.00068 | -0.02557 | 0.00104 | 0.02104 |
| 26.000 | $-0.02532$ | 0.00067 | 0.02084 | -0.00102 |
| 26.250 | 0.00066 | 0.02508 | -0.00100 | -0.02064 |
| 26.500 | 0.02484 | -0.00065 | -0.02045 | 0.00099 |
| 26.750 | $-0.00063$ | $-0.02461$ | 0.00097 | 0.02026 |
| 27.000 | -0.02439 | 0.00062 | 0.02007 | -0.00095 |
| 27.250 | 0.00061 | 0.02416 | -0.00093 | -0.01989 |
| 27.500 | 0.02394 | -0.00060 | -0.01971 | 0.00092 |
| 27.750 | -0.00059 | -0.02373 | 0.00090 | 0.01953 |
| 28.000 | -0.02352 | 0.00058 | 0.01936 | -0.00088 |
| 28.250 | 0.00057 | 0.02331 | $-0.00087$ | -0.01919 |
| 28.500 | 0.02310 | -0.00056 | -0.01902 | 0.00085 |
| 28.750 | -0.00055 | -0.02290 | 0.00084 | 0.01885 |
| 29.000 | -0.02271 | 0.00054 | 0.01869 | -0.00082 |
| 29.250 | 0.00053 | 0.02251 | -0.00081 | -0.01853 |
| 29.500 | 0.02232 | -0.00052 | -0.01838 | 0.00080 |
| 29.750 | -0.00051 | -0.02213 | 0.00078 | 0.01822 |
| 30.000 | -0.02195 | 0.00050 | 0.01807 | -0.00077 |
| 30.250 | 0.00050 | 0.02177 | $-0.00076$ | -0.01792 |
| 30.500 | 0.02159 | -0.00049 | -0.01778 | 0.00074 |
| 30.750 | -0.00048 | -0.02141 | 0.00073 | 0.01763 |
| 31.000 | -0.02124 | 0.00047 | 0.01749 | -0.00072 |
| 31.250 | 0.00046 | 0.02107 | $-0.00071$ | -0.01735 |
| 31.500 | 0.02091 | -0.00046 | -0.01722 | 0.00070 |
| 31.750 | -0.00045 | -0.02074 | 0.00069 | 0.01708 |
| 32.000 | -0.02058 | 0.00044 | 0.01695 | -0.00068 |
| 32.250 | 0.00044 | 0.02042 | $-0.00067$ | -0.01682 |
| 32.500 | 0.02026 | -0.00043 | -0.01669 | 0.00066 |
| 32.750 | -0.00042 | -0.02011 | 0.00065 | 0.01656 |
| 33.000 | -0.01996 | 0.00042 | 0.01644 | -0.00064 |
| 33.250 | 0.00041 | 0.01981 | -0.00063 | -0.01631 |
| 33.500 | 0.01966 | -0.00040 | -0.01619 | 0.00062 |
| 33.750 | -0.00040 | -0.01951 | 0.00061 | 0.01607 |
| 34.000 | -0.01937 | 0.00039 | 0.01595 | -0.00060 |
| 34.250 | 0.00039 | 0.01923 | $-0.00059$ | -0.01584 |
| 34.500 | 0.01909 | -0.00038 | -0.01572 | 0.00058 |
| 34.750 | -0.00038 | -0.01895 | 0.00057 | 0.01561 |
| 35.000 | -0.01882 | 0.00037 | 0.01550 | -0.00057 |
| 35.250 | 0.00037 | 0.01868 | $-0.00056$ | -0.01539 |
| 35.500 | 0.01855 | -0.00036 | -0.01528 | 0.00055 |
| 35.750 | -0.00036 | -0.01842 | 0.00054 | 0.01518 |
| 36.000 | -0.01829 | 0.00035 | 0.01507 | -0.00054 |
| 36.250 | 0.00035 | 0.01817 | -0.00053 | -0.01497 |

Table of elements of $\Phi_{u}$ and $\Phi_{v}$ matrices for $h / \lambda=0.625$
Values for single element $\left(\Phi_{k k u}, \Phi_{k k v}\right.$ and $\left.\Psi_{k k d R}=\Psi_{d R}\right)$-contd.

| $(k-i)(b / \lambda)$ | $\Phi_{\text {kiu }}$ |  | $\Phi_{\text {kiv }}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 36.500 | 0.01804 | -0.00034 | -0.01486 | 0.00052 |
| 36.750 | -0.00034 | -0.01792 | 0.00051 | 0.01476 |
| 37.000 | $-0.01780$ | 0.00033 | 0.01466 | $-0.00051$ |
| 37.250 | 0.00033 | 0.01768 | $-0.00050$ | $-0.01457$ |
| 37.500 | 0.01756 | -0.00032 | -0.01447 | 0.00049 |
| 37.750 | -0.00032 | -0.01745 | 0.00049 | 0.01437 |
| 38.000 | -0.01733 | 0.00031 | 0.01428 | -0.00048 |
| 38.250 | 0.00031 | 0.01722 | -0.00047 | -0.01419 |
| 38.500 | 0.01711 | -0.00031 | -0.01409 | 0.00047 |
| 38.750 | $-0.00030$ | -0.01700 | 0.00046 | 0.01400 |
| 39.000 | -0.01689 | 0.00030 | 0.01391 | -0.00046 |
| 39.250 | 0.00029 | 0.01678 | -0.00045 | -0.01383 |
| 39.500 | 0.01667 | -0.00029 | -0.01374 | 0.00044 |
| 39.750 | -0.00029 | -0.01657 | 0.00044 | 0.01365 |
| 40.000 | -0.01647 | 0.00028 | 0.01357 | -0.00043 |
| 40.250 | 0.00028 | 0.01636 | -0.00043 | -0.01348 |
| 40.500 | 0.01626 | -0.00028 | $-0.01340$ | 0.00042 |
| 40.750 | $-0.00027$ | -0.01616 | 0.00042 | 0.01332 |
| 41.000 | -0.01606 | 0.00027 | 0.01324 | $-0.00041$ |
| 41.250 | 0.00027 | 0.01597 | -0.00041 | -0.01316 |
| 41.500 | 0.01587 | -0.00026 | -0.01308 | 0.00040 |
| 41.750 | -0.00026 | -0.01578 | 0.00040 | 0.01300 |
| 42.000 | $-0.01568$ | 0.00026 | 0.01292 | $-0.00039$ |
| 42.250 | 0.00025 | 0.01559 | -0.00039 | $-0.01285$ |
| 42.500 | 0.01550 | -0.00025 | -0.01277 | 0.00038 |
| 42.750 | -0.00025 | -0.01541 | 0.00038 | 0.01270 |
| 43.000 | -0.01532 | 0.00025 | 0.01262 | -0.00038 |
| 43.250 | 0.00024 | 0.01523 | $-0.00037$ | -0.01255 |
| 43.500 | 0.01514 | -0.00024 | -0.01248 | 0.00037 |
| 43.750 | -0.00024 | -0.01505 | 0.00036 | 0.01241 |
| 44.000 | -0.01497 | 0.00023 | 0.01234 | -0.00036 |
| 44.250 | 0.00023 | 0.01488 | -0.00035 | -0.01227 |
| 44.500 | 0.01480 | -0.00023 | $-0.01220$ | 0.00035 |
| 44.750 | $-0.00023$ | -0.01472 | 0.00035 | 0.01213 |
| 45.000 | -0.01464 | 0.00022 | 0.01206 | -0.00034 |
| 45.250 | 0.00022 | 0.01456 | -0.00034 | $-0.01200$ |
| 45.500 | 0.01448 | -0.00022 | $-0.01193$ | 0.00034 |
| 45.750 | -0.00022 | -0.01440 | 0.00033 | 0.01187 |
| 46.000 | -0.01432 | 0.00021 | 0.01180 | $-0.00033$ |
| 46.250 | 0.00021 | 0.01424 | -0.00032 | $-0.01174$ |
| 46.500 | 0.01416 | -0.00021 | -0.01167 | 0.00032 |
| 46.750 | $-0.00021$ | -0.01409 | 0.00032 | 0.01161 |
| 47.000 | $-0.01401$ | 0.00021 | 0.01155 | -0.00031 |
| 47.250 | 0.00020 | 0.01394 | -0.00031 | -0.01149 |
| 47.500 | 0.01387 | -0.00020 | $-0.01143$ | 0.00031 |
| 47.750 | -0.00020 | -0.01379 | 0.00030 | 0.01137 |
| 48.000 | -0.01372 | 0.00020 | 0.01131 | -0.00030 |
| 48.250 | 0.00019 | 0.01365 | -0.00030 | -0.01125 |
| 48.500 | 0.01358 | -0.00019 | -0.01119 | 0.00030 |
| 48.750 | -0.00019 | -0.01351 | 0.00029 | 0.01114 |
| 49.000 | $-0.01344$ | 0.00019 | 0.01108 | -0.00029 |
| 49.250 | 0.00019 | 0.01337 | -0.00029 | -0.01102 |
| 49.500 | 0.01331 | -0.00019 | -0.01097 | 0.00028 |
| 49.750 | -0.00018 | -0.01324 | 0.00028 | 0.01091 |
| 50.000 | $-0.01317$ | 0.00018 | 0.01086 | -0.00028 |

## APPENDIX IV

## Tables of admittance and impedance curtain arrays

This table is abridged from the report 'Tables for Curtain Arrays' by Ronold W. P. King, Barbara H. Sandler and Sheldon S. Sandler, Cruft Laboratory Scientific Report No. 4 (Series 3), Harvard University, May 1964.

The calculations for the individual elements are given for $\Omega=2 \ln 2 h / a=$ 8.6138 for $\beta_{0} h=\pi / 4$ and for $\Omega=10$ for all other electrical lengths. The admittances are given in millimhos and the impedances in ohms. The vertical listings begin with the first element at the top. The unilateral endfire patterns are prescribed to point in the direction away from the first element toward the last element.

## Broadside array (driving-point currents specified)

$$
\beta_{0} h=0.78539
$$

| $\beta_{0} b=1.57080$ |  | $\beta_{0} b=3.14159$ |  |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $0 \cdot 122+j 3 \cdot 250$ | 11.564-j307.244 | $0 \cdot 122+j 3 \cdot 250$ | 11.564-j307.244 |
| $N=4$ |  | $N=4 \quad 1$ |  |
| $0 \cdot 123+j 3.135$ | 12.532-j318.521 | $0 \cdot 100+j 3 \cdot 204$ | 9.690-j311.824 |
| $0.223+j 3.047$ | 23.894-j326.465 | $0 \cdot 078+j 3 \cdot 170$ | 7.786-j315.273 |
| $0.223+j 3.047$ | 23.894-j326.465 | $0 \cdot 078+j 3 \cdot 170$ | 7.786-j315.273 |
| $0 \cdot 123+j 3.135$ | 12.532-j318.521 | $0 \cdot 100+j 3 \cdot 204$ | 9.690-j311.824 |
| $N=10$ |  | $N=10$ |  |
| $0 \cdot 152+j 3 \cdot 154$ | 15.277-j316.276 | $0 \cdot 101+j 3.210$ | 9.786-j311.261 |
| $0 \cdot 202+j 3.084$ | 21.113-j322.911 | $0 \cdot 076+j 3 \cdot 161$ | 7.615-j316.127 |
| $0 \cdot 170+j 3 \cdot 020$ | 18.620-j330.103 | $0.083+j 3 \cdot 185$ | $8.207-j 313.805$ |
| $0 \cdot 151+j 3 \cdot 030$ | 16.446-j329.235 | $0 \cdot 080+j 3 \cdot 172$ | 7.963-j315.015 |
| $0 \cdot 166+j 3 \cdot 078$ | 17.493-j323.927 | $0.081+j 3 \cdot 178$ | 8.058-j314.481 |
| $0 \cdot 166+j 3.078$ | 17.493-j323.927 | $0.081+j 3.178$ | 8.058-j314.481 |
| $0.151+j 3.030$ | 16.446-j329.235 | $0.080+j 3.172$ | 7.963-j315.015 |
| $0 \cdot 170+j 3.020$ | 18.620-j330.103 | $0.083+j 3 \cdot 185$ | $8.207-j 313.805$ |
| $0.202+j 3.084$ | 21.113-j322.911 | $0.076+j 3.161$ | 7.615-j316.127 |
| $0 \cdot 152+j 3.154$ | 15.277-j316.276 | $0 \cdot 101+j 3.210$ | $9.786-j 311.261$ |
| $N=20$ |  | $N=20$ |  |
| $0 \cdot 140+j 3 \cdot 146$ | 14.158-j317.252 | $0 \cdot 101+j 3 \cdot 211$ | $9.804-j 311.079$ |
| $0.210+j 3.071$ | 22.183-j324.119 | $0 \cdot 076+j 3 \cdot 159$ | 7.599-j316.331 |
| $0 \cdot 184+j 3.029$ | 19.973-j328.914 | $0 \cdot 084+j 3 \cdot 187$ | 8.233-j313.549 |
| $0.141+j 3.045$ | 15.134-j327.710 | $0 \cdot 080+j 3 \cdot 169$ | 7.934-j315.316 |
| $0 \cdot 148+j 3.066$ | 15.754-j325-408 | 0.082+j3.182 | 8.106-j314.096 |
| $0 \cdot 180+j 3.057$ | 19.217-j325.963 | $0 \cdot 081+j 3 \cdot 173$ | 7.998-j314.970 |
| $0.177+j 3.046$ | 18.957-j327.166 | $0.082+j 3.179$ | 8.071 - j314.345 |
| $0 \cdot 150+j 3.053$ | 16.056-j326.795 | $0.081+j 3.175$ | $8.024-j 314.780$ |
| $0 \cdot 151+j 3.058$ | 16.158-j326.263 | $0.081+j 3.178$ | $8.052-j 314.506$ |
| $0.176+j 3.052$ | 18.842-j326.573 | $0.081+j 3 \cdot 176$ | $8.039-j 314.639$ |
| $0.176+j 3.052$ | 18.842-j326.573 | $0 \cdot 081+j 3 \cdot 176$ | 8.039-j314.639 |
| $0 \cdot 151+j 3 \cdot 058$ | 16.158-j326.263 | $0 \cdot 081+j 3 \cdot 178$ | $8.052-j 314.506$ |
| $0 \cdot 150+j 3.053$ | 16.056-j326.795 | $0.081+j 3.175$ | $8.024-j 314.780$ |
| $0.177+j 3.046$ | 18.957-j327.166 | $0.082+j 3.179$ | $8.071-j 314.345$ |
| $0.180+j 3.057$ | 19.217-j325.963 | $0.081+j 3.173$ | 7.998-j314.970 |
| $0.148+j 3.066$ | 15.754-j325.408 | $0.082+j 3.182$ | 8.106-j314.096 |
| $0.141+j 3.045$ | 15.134-j327.710 | $0 \cdot 080+j 3 \cdot 169$ | 7.934-j315.316 |
| $0 \cdot 184+j 3.029$ | 19.973-j328.914 | $0 \cdot 084+j 3 \cdot 187$ | 8.233-j313.549 |
| $0.210+j 3.071$ | 22.183-j324.119 | $0.076+j 3.159$ | 7.599-j316.331 |
| $0 \cdot 140+j 3 \cdot 146$ | 14.158-j317.252 | $0 \cdot 101+j 3 \cdot 211$ | $9.804-j 311.079$ |

Broadside array (base voltages specified)

$$
\beta_{0} h=0.78539
$$

| $\beta_{0} b=1.57080$ |  | $\beta_{0} b=3.14159$ |  |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $N=4^{0.122+j 3.250}$ |  | $0 \cdot 122+j 3 \cdot 250$ | 11.564-j307.244 |
|  |  | $N=4$ |  |
| $0 \cdot 118+j 3 \cdot 137$ | 11.937-j318.359 | $0 \cdot 100+j 3 \cdot 204$ | $9.724-j 311.807$ |
| $0.229+j 3.045$ | 24.512-j326.608 | $0 \cdot 078+j 3 \cdot 170$ | 7.753-j315.290 |
| $0.229+j 3.045$ | 24.512-j326.608 | $0 \cdot 078+j 3 \cdot 170$ | $7.753-j 315.290$ |
| $0 \cdot 118+j 3 \cdot 137$ | 11.937-j318.359 | $0 \cdot 100+j 3 \cdot 204$ | $9.724-j 311.807$ |
| $N=10$ |  | $N=10$ |  |
| $0 \cdot 153+j 3.157$ | 15.298-j316.064 | $0 \cdot 102+j 3 \cdot 210$ | $9.848-j 311.209$ |
| $0.204+j 3.087$ | 21.338-j322.502 | $0.075+j 3.161$ | $7.517-j 316.212$ |
| $0 \cdot 168+j 3.014$ | 18.457-j330.739 | $0.084+j 3.185$ | 8.263-j313.749 |
| $0 \cdot 148+j 3.026$ | 16.125-j329.718 | $0.080+j 3.172$ | $7.931-j 315.051$ |
| $0 \cdot 169+j 3 \cdot 083$ | 17.702-j323.419 | $0 \cdot 082+j 3 \cdot 178$ | 8.068-j314.470 |
| $0 \cdot 169+j 3.083$ | 17.702-j323.419 | $0 \cdot 082+j 3 \cdot 178$ | $8 \cdot 068-j 314.470$ |
| $0.148+j 3.026$ | 16.125-j329.718 | $0 \cdot 080+j 3 \cdot 172$ | 7.931-j315.051 |
| $0 \cdot 168+j 3.014$ | 18.457-j330.739 | $0.084+j 3.185$ | 8.263-j313.749 |
| $0.204+j 3.787$ | 21.338-j322.502 | $0 \cdot 075+j 3.161$ | $7.517-j 316.212$ |
| $0 \cdot 153+j 3 \cdot 157$ | 15.298-j316.064 | $0 \cdot 102+j 3 \cdot 210$ | $9.848-j 311.209$ |
| $N=20$ |  | $N=20$ |  |
| $0.137+j 3.147$ | 13.778-j317.153 | $0 \cdot 102+j 3 \cdot 211$ | $9.844-j 311.084$ |
| $0.215+j 3.071$ | 22.654-j324.092 | $0.074+j 3 \cdot 158$ | $7.449-j 316.530$ |
| $0 \cdot 187+j 3.025$ | 20.346-j329.273 | $0.084+j 3.187$ | $8.278-j 313.535$ |
| $0 \cdot 136+j 3.046$ | 14.591-j327.645 | $0.079+j 3.168$ | 7.842-j315.477 |
| $0.146+j 3.069$ | 15.423-j325.119 | $0.082+j 3 \cdot 181$ | $8 \cdot 116-j 314 \cdot 110$ |
| $0 \cdot 184+j 3.056$ | 19.673-j326.009 | $0 \cdot 080+j 3 \cdot 172$ | 7.932-j315.103 |
| $0.180+j 3.044$ | 19.327-j327.375 | $0 \cdot 082+j 3 \cdot 179$ | $8.064-j 314.379$ |
| $0 \cdot 146+j 3 \cdot 054$ | 15.636-j326.695 | $0 \cdot 080+j 3 \cdot 174$ | 7.966-j314.901 |
| $0.148+j 3.059$ | 15.791-j326.090 | $0 \cdot 081+j 3 \cdot 178$ | $8.062-j 314.490$ |
| $0 \cdot 180+j 3 \cdot 050$ | 19.235-j326.688 | $0.081+j 3.176$ | 8.036-j314.645 |
| $0 \cdot 180+j 3 \cdot 050$ | 19.235-j326.688 | $0.081+j 3.176$ | $8.036-j 314.645$ |
| $0.148+j 3.059$ | 15.791-j326.090 | $0.081+j 3.178$ | $8.062-j 314.490$ |
| $0 \cdot 146+j 3 \cdot 054$ | 15.636-j326.695 | $0 \cdot 080+j 3 \cdot 174$ | 7.966-j314.901 |
| $0 \cdot 180+j 3 \cdot 044$ | 19.327-j327.375 | $0.082+j 3.179$ | 8.064-j314.379 |
| $0.184+j 3.056$ | 19.673-j326.009 | $0 \cdot 080+j 3 \cdot 172$ | 7.932-j315.103 |
| $0.146+j 3.069$ | 15.423-j325.119 | $0 \cdot 082+j 3 \cdot 181$ | 8.116-j314.110 |
| $0 \cdot 136+j 3 \cdot 046$ | 14.591-j327.645 | $0.079+j 3.168$ | 7.842-j315.477 |
| $0 \cdot 187+j 3.025$ | 20.346-j329.273 | $0 \cdot 084+j 3 \cdot 187$ | 8.278-j313.535 |
| $0.215+j 3.071$ | 22.654-j324.092 | $0.074+j 3.158$ | $7.449-j 316.530$ |
| $0.137+j 3.147$ | 13.778-j317.153 | $0 \cdot 102+j 3.211$ | 9.844-j311.084 |

Endfire array (driving-point current specified)

$$
\beta_{0} h=0.78539
$$

| $\beta_{0} b=1 \cdot 57080$ |  | $\beta_{0} b=3.14159$ |  |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $0 \cdot 122+j 3 \cdot 250$ | 11•564-j307.244 | $0 \cdot 122+j 3 \cdot 250$ | 11.564-j307.244 |
| $N=4$ |  | $N=4$ |  |
| $0.059+j 3.193$ | 5.778-j313.123 | $0 \cdot 168+j 3 \cdot 367$ | 14.796-j296.304 |
| $0 \cdot 151+j 3 \cdot 309$ | 13.761-j301.623 | $0 \cdot 195+j 3 \cdot 404$ | 16.735-j292.822 |
| $0 \cdot 150+j 3 \cdot 308$ | 13.659-j301.691 | $0 \cdot 195+j 3 \cdot 404$ | 16.735-j292.822 |
| $0.257+j 3.427$ | 21.769-j290.196 | $0 \cdot 168+j 3 \cdot 367$ | 14.796-j296.304 |
| $N=10$ |  | $N=10$ |  |
| $0.063+j 3.203$ | $6 \cdot 120-j 312 \cdot 060$ | $0 \cdot 183+j 3.442$ | 15.431-j289.678 |
| $0.142+j 3.292$ | 13.121-j303.166 | $0 \cdot 217+j 3.498$ | 17.657-j284.743 |
| $0 \cdot 169+j 3 \cdot 333$ | 15.135-j299.277 | $0 \cdot 228+j 3 \cdot 526$ | 18.267-j282.387 |
| $0.192+j 3.399$ | 16.544-j293.286 | $0 \cdot 233+j 3 \cdot 541$ | 18.520-j281.159 |
| $0 \cdot 196+j 3 \cdot 409$ | 16.847-j292.350 | $0.235+j 3.548$ | 18.619-j280.616 |
| $0.215+j 3.469$ | 17.814-j287.133 | $0.235+j 3.548$ | 18.619-j280.616 |
| $0.209+j 3.457$ | 17.416-j288.192 | $0 \cdot 233+j 3 \cdot 541$ | 18.520-j281.159 |
| $0.236+j 3.528$ | 18.875-j282.157 | $0 \cdot 228+j 3 \cdot 526$ | 18.267-j282.387 |
| $0 \cdot 203+j 3.482$ | 16.706-j286.198 | $0 \cdot 217+j 3 \cdot 498$ | 17.657-j284.743 |
| $0 \cdot 309+j 3 \cdot 580$ | 23.931-j277.234 | $0 \cdot 183+j 3 \cdot 442$ | 15.431-j289.678 |
| $N=20$ |  | $N=20$ |  |
| $0.064+j 3.207$ | $6 \cdot 188-j 311.699$ | $0 \cdot 193+j 3 \cdot 500$ | 15.723-j284.807 |
| $0 \cdot 141+j 3.288$ | 13.044-j303.581 | $0.229+j 3.564$ | 17.992-j279.463 |
| $0.170+j 3.338$ | 15.223-j298.777 | $0 \cdot 243+j 3 \cdot 599$ | 18.651-j276.631 |
| $0.189+j 3.392$ | 16.416-j293.886 | $0 \cdot 250+j 3 \cdot 621$ | 18.964-j274.831 |
| $0.199+j 3.418$ | 17.012-j291.597 | $0.255+j 3.637$ | 19.145-j273.590 |
| $0.211+j 3.458$ | 17.564-j288.089 | $0.258+j 3.649$ | 19.258-j272.700 |
| $0.215+j 3.472$ | 17.802-j286.910 | $0.260+j 3.657$ | 19.335-j272.063 |
| $0 \cdot 224+j 3 \cdot 507$ | 18.174-j283.963 | $0 \cdot 261+j 3.663$ | 19.385-j271.619 |
| $0 \cdot 226+j 3 \cdot 513$ | 18.255-j283.463 | $0.262+j 3.667$ | 19.415-j271.339 |
| $0 \cdot 235+j 3 \cdot 547$ | 18.571-j280.729 | $0.263+j 3.668$ | 19.428-j271.203 |
| $0 \cdot 234+j 3 \cdot 546$ | 18.548-j280.770 | $0 \cdot 263+j 3 \cdot 668$ | 19.428-j271.203 |
| $0 \cdot 243+j 3 \cdot 580$ | 18.869-j278.028 | $0.262+j 3.667$ | 19.415-j271.339 |
| $0.240+j 3.573$ | 18.740-j278.613 | $0.261+j 3.663$ | 19.385-j271.619 |
| $0 \cdot 251+j 3.611$ | 19.130-j275.641 | $0 \cdot 260+j 3.657$ | 19.335-j272.063 |
| $0 \cdot 245+j 3.595$ | 18.836-j276.916 | $0 \cdot 258+j 3.649$ | 19.258-j272.700 |
| $0 \cdot 259+j 3 \cdot 640$ | 19.423-j273.359 | $0.255+j 3.637$ | 19.145-j273.590 |
| $0 \cdot 245+j 3 \cdot 610$ | 18.749-j275.767 | $0.250+j 3.621$ | 18.964-j274.831 |
| $0 \cdot 272+j 3.673$ | 20.021-j270.784 | $0.243+j 3.599$ | 18.651-j276.631 |
| $0 \cdot 232+j 3 \cdot 611$ | 17.682-j275.761 | $0 \cdot 229+j 3.564$ | 17.992-j279.463 |
| $0 \cdot 344+j 3.706$ | 24.834-j267.513 | $0 \cdot 193+j 3 \cdot 500$ | 15.723-j284.807 |

## Endfire array (base voltages specified)

$$
\beta_{0} h=0.78539
$$

| $\beta_{0} b=1 \cdot 57080$ |  | Admittance $\beta_{0} b$ | Impedance 59 |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $0 \cdot 122+j 3 \cdot 250$ | $11.564-j 307.244$ | $0.122+j 3.250$ | 11.564-j307.244 |
| $N=4$ |  | $N=4$ |  |
| $0.053+j 3.191$ | 5.215-j313.290 | $0 \cdot 169+j 3.367$ | 14.886-j296.227 |
| $0 \cdot 149+j 3.309$ | 13.563-j301.579 | $0 \cdot 193+j 3 \cdot 403$ | 16.646-j292.899 |
| $0 \cdot 142+j 3 \cdot 306$ | 12.988-j301.907 | $0.193+j 3.403$ | 16.646-j292.899 |
| $0 \cdot 245+j 3.423$ | 20.806-j290.646 | $0.169+j 3.367$ | 14.886-j296.226 |
| $N=10$ |  | $N=10$ |  |
| $0.059+j 3.202$ | 5.753-j312.223 | $0.187+j 3.446$ | 15.675-j289.334 |
| $0 \cdot 140+j 3 \cdot 294$ | 12.839-j303.024 | $0.217+j 3.499$ | 17.654-j284.681 |
| $0 \cdot 161+j 3 \cdot 329$ | 14.500-j299.689 | $0 \cdot 227+j 3.526$ | 18.207-j282.464 |
| $0 \cdot 187+j 3 \cdot 396$ | 16.167-j293.555 | $0 \cdot 232+j 3 \cdot 540$ | 18.439-j281.305 |
| $0 \cdot 189+j 3 \cdot 403$ | 16.230-j292.965 | $0.234+j 3.546$ | 18.531-j280.791 |
| $0 \cdot 210+j 3 \cdot 464$ | 17.472-j287.643 | $0.234+j 3.546$ | 18.531-j280.791 |
| $0 \cdot 201+j 3 \cdot 450$ | 16.837-j288.855 | $0 \cdot 232+j 3 \cdot 540$ | 18.439-j281-305 |
| $0 \cdot 230+j 3 \cdot 520$ | 18.527-j282.918 | $0 \cdot 227+j 3.526$ | 18.207-j282.464 |
| $0 \cdot 198+j 3 \cdot 477$ | 16.294-j286.701 | $0.217+j 3.499$ | 17.654-j284.681 |
| $0.293+j 3.571$ | 22.792-j278.154 | $0.187+j 3.446$ | 15.675-j289.334 |
| $N=20$ |  | $N=20$ |  |
| $0.061+j 3.206$ | 5.895-j311.808 | $0 \cdot 199+j 3 \cdot 508$ | 16.092-j284.146 |
| $0 \cdot 138+j 3 \cdot 289$ | $12.707-j 303.472$ | $0 \cdot 231+j 3 \cdot 567$ | 18.068-j279.167 |
| $0 \cdot 163+j 3 \cdot 335$ | 14.649-j299.156 | $0 \cdot 243+j 3.600$ | 18.651-j276.528 |
| $0 \cdot 184+j 3 \cdot 389$ | 15.985-j294.162 | $0.249+j 3.621$ | 18.933-j274.843 |
| $0 \cdot 192+j 3 \cdot 411$ | 16.458-j292.213 | $0.254+j 3.636$ | 19.096-j273.675 |
| $0 \cdot 205+j 3.453$ | 17.166-j288.556 | $0.257+j 3.647$ | 19.200-j272.836 |
| $0 \cdot 208+j 3 \cdot 464$ | 17.287-j287.644 | $0 \cdot 259+j 3.655$ | 19.270-j272.234 |
| $0 \cdot 219+j 3 \cdot 501$ | 17.813-j284.541 | $0.260+j 3.660$ | 19.316-j271.815 |
| $0 \cdot 219+j 3 \cdot 504$ | 17.772-j284.267 | $0.261+j 3.664$ | 19.343-j271.550 |
| $0 \cdot 229+j 3 \cdot 539$ | 18.242-j281.386 | $0 \cdot 261+j 3.666$ | 19.356-j271.421 |
| $0 \cdot 227+j 3 \cdot 536$ | 18.087-j281.618 | $0.261+j 3.666$ | 19.356-j271.421 |
| $0.238+j 3.572$ | 18.568-j278.749 | $0.261+j 3.664$ | 19.343-j271.550 |
| $0 \cdot 233+j 3 \cdot 563$ | 18.293-j279.483 | $0.260+j 3.660$ | 19.316-j271.815 |
| $0.246+j 3.601$ | 18.853-j276.425 | $0.259+j 3.655$ | 19.270-j272.234 |
| $0 \cdot 237+j 3.584$ | 18.398-j277.783 | $0.257+j 3.647$ | 19.200-j272.836 |
| $0 \cdot 254+j 3 \cdot 629$ | 19.162-j274.232 | $0 \cdot 254+j 3.636$ | 19.096-j273.675 |
| $0 \cdot 239+j 3 \cdot 600$ | 18.329-j276.574 | $0.249+j 3.621$ | 18.933-j274.843 |
| $0.265+j 3.659$ | 19.715-j271.845 | $0 \cdot 243+j 3.600$ | 18.651-j276.528 |
| $0 \cdot 228+j 3 \cdot 605$ | 17.447-j276.327 | $0.231+j 3.567$ | 18.068-j279.167 |
| $0.324+j 3.693$ | 23.582-j268.702 | $0.199+j 3.508$ | 16.092-j284.146 |

## Broadside array (driving-point currents specified)

$$
\beta_{0} h=1.57079
$$

| $\beta_{0} b=1.57080$ |  | Admittance $\beta_{0} b=$ | Impedance 59 |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $10.449-j 3.889$ | $84 \cdot 059+j 31 \cdot 286$ | 10.499-j3.889 | $84 \cdot 059+j 31 \cdot 286$ |
| $N=4$ |  | $N=4$ |  |
| $12.709+j 3.576$ | 72.913-j20.518 | 14.944-j0.824 | $66.715+j 3.678$ |
| $5 \cdot 807+j 3 \cdot 319$ | 129.803-j74.198 | $18.493+j 1.493$ | 53.723-j 4.336 |
| $5 \cdot 807+j 3 \cdot 319$ | 129.803-j74.198 | $18.493+j 1.493$ | 53.723-j 4.336 |
| $12 \cdot 709+j 3 \cdot 576$ | 72.913-j20.518 | 14.944-j0.824 | $66.715+j 3.678$ |
| $N=10$ |  | $N=10$ |  |
| $10 \cdot 113+j 1.428$ | 96.946-j13.689 | $14.720-j 0.850$ | $67.711+j 3.909$ |
| $7.202+j 2.746$ | 121.234-j46.219 | $18.832+j 1.491$ | $52.771-j 4.177$ |
| $5 \cdot 923+j 5 \cdot 391$ | 92.343-j84.042 | $17.703+j 1.442$ | 56.116-j 4.572 |
| $6 \cdot 572+j 6.018$ | $82.765-j 75.783$ | $18.161+j 1.434$ | 54.721-j 4.321 |
| $8 \cdot 176+j 3 \cdot 532$ | $103.068-j 44.525$ | $17.978+j 1.440$ | 55.268-j 4.428 |
| $8 \cdot 176+j 3 \cdot 532$ | 103.068-j44.525 | $17.978+j 1.440$ | 55.268-j 4.428 |
| $6 \cdot 572+j 6.018$ | 82.765-j75.783 | $18.161+j 1.434$ | $54.721-j 4.321$ |
| $5.923+j 5.391$ | 92.343-j84.042 | $17.703+j 1.442$ | 56.116-j 4.572 |
| $7 \cdot 202+j 2.746$ | 121.234-j46.219 | $18.832+j 1.491$ | 52.771-j 4.177 |
| $10 \cdot 113+j 1 \cdot 428$ | 96.946-j13.689 | 14.720-j0.850 | $67.711+j 3.909$ |
| $N=20$ |  | $N=20$ |  |
| $11.322+j 2.266$ | 84.924-j16.998 | $14.668-j 0.855$ | $67.946+j 3.961$ |
| $6 \cdot 558+j 3.070$ | 125.075-j58.542 | $18.882+j 1.487$ | 52.634-j 4.145 |
| $6 \cdot 008+j 4.532$ | 106.088-j80.025 | $17.645+j 1.450$ | 56.294-j 4.625 |
| $7 \cdot 939+j 6 \cdot 197$ | $78.268-j 61.095$ | $18.234+j 1.427$ | 54.509-j 4.267 |
| $8.713+j 4.980$ | 86.512-j49.440 | $17.883+j 1.452$ | 55.553-j 4.509 |
| $6.889+j 3.994$ | 108.645-j62.989 | $18.112+j 1.431$ | 54.870-j 4.336 |
| $6 \cdot 640+j 4 \cdot 415$ | 104.430-j69.432 | $17.957+j 1.447$ | 55.328-j 4.459 |
| $8.040+j 5.486$ | 84.866-j57.908 | $18.060+j 1.436$ | 55.023-j 4.374 |
| $8.225+j 5.215$ | $86.717-j 54.984$ | $17.997+j 1.443$ | 55.210-j 4.427 |
| $6 \cdot 835+j 4 \cdot 270$ | 105.235-j65.740 | $18.027+j 1.440$ | 55.120-j 4.402 |
| $6 \cdot 835+j 4 \cdot 270$ | 105.235-j65.740 | $18.027+j 1.440$ | 55.120-j 4.402 |
| $8 \cdot 225+j 5 \cdot 215$ | 86.717-j54.984 | $17.997+j 1.443$ | 55.210-j 4.427 |
| $8 \cdot 040+j 5.486$ | 84.866-j57.908 | $18.060+j 1.436$ | 55.023-j 4.374 |
| $6 \cdot 640+j 4.415$ | 104.430-j69.432 | $17.957+j 1.447$ | 55.328-j 4.459 |
| $6.889+j 3.994$ | 108.645-j62.989 | $18.112+j 1.431$ | 54.870-j 4.336 |
| $8.713+j 4.980$ | 86.512-j49.440 | $17.883+j 1.452$ | 55.553-j 4.509 |
| $7.939+j 6.197$ | 78.268-j61.095 | $18.234+j 1.427$ | 54.509-j 4.267 |
| $6 \cdot 008+j 4 \cdot 532$ | 106.088-j80.025 | $17.645+j 1.450$ | 56.294-j 4.625 |
| $6 \cdot 558+j 3 \cdot 070$ | 125.075-j58.542 | $18.882+j 1.487$ | 52.634-j 4.145 |
| $11 \cdot 322+j 2 \cdot 266$ | 84.924-j16.998 | $14.668-j 0.855$ | $67.946+j 3.961$ |

## Endfire array (driving-point current specified)

$$
\beta_{0} h=1.57079
$$

| $\beta_{0} b=1.57080$ |  | Admittance $\beta_{0} b=$ | Impedance <br> 4159 |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $10.449-j 3.889$ | $84.059+j 31.286$ | 10.449-j3.889 | $84.059+j 31.286$ |
| $N=4$ |  | $N=4$ |  |
| 21.021-j0.091 | $47.571+j \quad 0.205$ | 5.122-j3.827 | 125.295+j 93.607 |
| 7.128-j4.119 | $105 \cdot 169+j 60.777$ | 4.399-j3.396 | $142.437+j 109.966$ |
| 7.400-j4.432 | $99.455+j 59.563$ | 4.399-j3.396 | $142 \cdot 437+j 109.966$ |
| 3.999-j2.972 | $161.068+j 119.715$ | 5.122-j3.827 | $125 \cdot 295+j 93.607$ |
| $N=10$ |  | $N=10$ |  |
| 18.995-j1.976 | $52.082+j \quad 5.419$ | 3.533-j3.361 | $148 \cdot 595+j 141.338$ |
| 7.821-j4.265 | $98.548+j 53.747$ | 2.986-j2.896 | $172 \cdot 544+j 167 \cdot 362$ |
| $6 \cdot 462-j 4.040$ | $111 \cdot 269+j 69.563$ | 2.754-j2.736 | $182 \cdot 750+j 181 \cdot 551$ |
| 4.552-j3.546 | $136 \cdot 734+j 106 \cdot 504$ | $2.640-j 2.661$ | $187.898+j 189.374$ |
| 4.458-j3.576 | $136.500+j 109.499$ | 2.591-j2.629 | $190 \cdot 138+j 192.923$ |
| 3.461-j3.112 | $159.762+j 143.670$ | 2.591-j2.629 | $190 \cdot 138+j 192.923$ |
| 3.638-j3.296 | $150.965+j 136.780$ | 2.640-j2.661 | $187.898+j 189.374$ |
| 2.863-j2.783 | $179.606+j 174.555$ | $2.754-j 2.736$ | $182 \cdot 750+j 181.551$ |
| 3.232-j3.222 | $155 \cdot 171+j 154.712$ | 2.986-j2.896 | $172 \cdot 544+j 167 \cdot 362$ |
| 2.450-j2.360 | $211 \cdot 705+j 203 \cdot 950$ | 3.533-j3.361 | $148 \cdot 595+j 141 \cdot 338$ |
| $N=20$ |  | $N=20$ |  |
| 18.250-j2.610 | $53.695+j 7.678$ | 2.755-j2.976 | $167.509+j 180.951$ |
| 8.076-j4.268 | $96.793+j 51.152$ | 2.366-j2.566 | $194 \cdot 207+j 210 \cdot 621$ |
| 6.252-j4.000 | $113.486+j 72.618$ | 2.175-j2.404 | $206.910+j 228.769$ |
| 4.693-j3.602 | $134 \cdot 084+j 102 \cdot 908$ | $2.060-j 2.313$ | $214.737+j 241 \cdot 090$ |
| 4.300-j3.500 | $139.895+j 113.871$ | 1.984-j2.254 | $220.090+j 249.939$ |
| 3.591-j3.196 | $155 \cdot 387+j 138 \cdot 288$ | 1.932-j2.213 | $223.909+j 256.432$ |
| 3.467-j3.172 | $156.992+j 143.647$ | 1.895-j2.184 | $226 \cdot 647+j 261 \cdot 167$ |
| 3.019-j2.926 | $170 \cdot 781+j 165.519$ | $1.870-j 2 \cdot 165$ | $228 \cdot 549+j 264 \cdot 491$ |
| 2.993-j2.945 | $169.756+j 167.026$ | 1.855-j2.152 | $229.753+j 266 \cdot 612$ |
| 2.659-j2.729 | $183 \cdot 170+j 187 \cdot 995$ | 1.847-j2.146 | $230.338+j 267.646$ |
| 2.683-j2.778 | $179.903+j 186.225$ | 1.847-j2.146 | $230 \cdot 338+j 267 \cdot 646$ |
| $2.405-j 2.573$ | $193 \cdot 846+j 207 \cdot 450$ | 1.855-j2.152 | $229.753+j 266.612$ |
| 2.465-j2.651 | $188 \cdot 140+j 202 \cdot 324$ | $1 \cdot 870-j 2 \cdot 165$ | $228 \cdot 549+j 264.491$ |
| $2 \cdot 211-j 2.443$ | $203 \cdot 633+j 225 \cdot 004$ | 1.895-j2.184 | $226 \cdot 647+j 261 \cdot 167$ |
| $2 \cdot 305-j 2.556$ | $194 \cdot 603+j 215 \cdot 766$ | 1.932-j2.213 | $223.909+j 256.432$ |
| 2.053-j2.325 | $213.437+j 241.665$ | 1.984-j2.254 | $220.090+j 249.939$ |
| $2 \cdot 190-j 2.497$ | $198.569+j 226 \cdot 323$ | $2 \cdot 060-j 2 \cdot 313$ | $214.737+j 241.090$ |
| 1.912-j2.196 | $225 \cdot 537+j 258 \cdot 968$ | 2.175-j2.404 | $206.910+j 228.769$ |
| 2.119-j2.521 | $195.412+j 232.494$ | 2.366-j2.566 | $194.207+j 210.621$ |
| 1.787-j1.965 | $253 \cdot 380+j 278 \cdot 536$ | 2.755-j2.976 | $167.509+j 180.951$ |

## Broadside array (driving-point currents specified)

$$
\beta_{0} h=2 \cdot 35620
$$

| $\beta_{0} b=1.57080$ |  | $\beta_{0} b=3.14159$ | Impedance $1159$ |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ | 1.416-j1.335 | $373.800+j 352.429$ |
| $N=4$ |  | $N=4$ |  |
| 1.772-j2.166 | $226.222+j 276.570$ | 1.688-j1.860 | $267.545+j 294.791$ |
| $2 \cdot 885+j 0 \cdot 558$ | 334.125-j 64.674 | 1.761-j2.489 | $189 \cdot 461+j 267 \cdot 725$ |
| $2 \cdot 885+j 0 \cdot 558$ | 334.125-j $64 \cdot 674$ | 1.761-j2.489 | $189.461+j 267.725$ |
| 1.772-j2.166 | $226 \cdot 222+j 276 \cdot 570$ | 1.688-j1.860 | $267.545+j 294.791$ |
| $N=10$ |  | $N=10$ |  |
| 2.546-j1.081 | $332 \cdot 848+j 141 \cdot 276$ | 1.668-j1.738 | $287 \cdot 394+j 299.522$ |
| $2.273-j 0.367$ | $428 \cdot 840+j 69 \cdot 202$ | 1.706-j2.798 | $158.797+j 260 \cdot 557$ |
| 4.977-j1.391 | $186.363+j 52.072$ | 1.767-j2.111 | $233 \cdot 130+j 278 \cdot 516$ |
| 5.319-j2.786 | $147.544+j 77.279$ | 1.743-j2.477 | $189.977+j 270.003$ |
| $2.547-j 0.541$ | $375.674+j 79.816$ | 1.766-j2.302 | $209.859+j 273.425$ |
| $2.547-j 0.541$ | $375.674+j 79.816$ | 1.766-j2.302 | $209.859+j 273.425$ |
| 5.319-j2.786 | $147.544+j 77.279$ | 1.743-j2.477 | $189.977+j 270.003$ |
| 4.977-j1.391 | $186.363+j 52.072$ | 1.767-j2.111 | $233 \cdot 130+j 278 \cdot 516$ |
| 2.273-j0.367 | $428 \cdot 840+j 69.202$ | 1.706-j2.798 | $158.797+j 260.557$ |
| 2.546-j1.081 | $332 \cdot 848+j 141 \cdot 276$ | 1.668-j1.738 | $287 \cdot 394+j 299.522$ |
| $N=20$ |  | $N=20$ |  |
| 1.768-j1.569 | $316.473+j 280.864$ | 1.674-j1.677 | $298.066+j 298.738$ |
| $3.253+j 0.018$ | 307.387-j 1.730 | 1.622-j2.887 | $147.942+j 263.273$ |
| $3.757+j 1 \cdot 121$ | 244.412-j 72.928 | 1.797-j2.006 | $247.732+j 276.631$ |
| 2.666-j1.466 | $288.002+j 158.418$ | 1.660-j2.600 | $174 \cdot 426+j 273 \cdot 219$ |
| $2 \cdot 367-j 1 \cdot 387$ | $314 \cdot 522+j 184 \cdot 290$ | 1.814-j2.142 | $230 \cdot 212+j 271.886$ |
| 3.422-j0.000 | $292.212+j \quad 0.020$ | 1.690-j2.481 | $187 \cdot 563+j 275 \cdot 284$ |
| $3 \cdot 590+j 0 \cdot 494$ | $273.339-j 37.627$ | 1.802-j2.227 | $219 \cdot 579+j 271 \cdot 342$ |
| 2.689-j1.268 | $304 \cdot 189+j 143 \cdot 435$ | 1.721-j2.406 | $196.649+j 274.916$ |
| 2.397-j1.327 | $319 \cdot 318+j 176 \cdot 796$ | 1.779-j2.291 | $211 \cdot 463+j 272 \cdot 272$ |
| $3 \cdot 683+j 0 \cdot 277$ | $269.994-j 20.304$ | 1.751-j2.347 | $204 \cdot 192+j 273 \cdot 668$ |
| $3 \cdot 683+j 0 \cdot 277$ | 269.994-j 20.304 | 1.751-j2.347 | $204 \cdot 192+j 273 \cdot 668$ |
| 2.397-j1.327 | $319 \cdot 318+j 176 \cdot 796$ | 1.779-j2.291 | $211.463+j 272.272$ |
| $2 \cdot 689-j 1 \cdot 268$ | $304 \cdot 189+j 143 \cdot 435$ | 1.721-j2.406 | $196 \cdot 649+j 274 \cdot 916$ |
| $3 \cdot 590+j 0 \cdot 494$ | $273.339-j 37.627$ | 1.802-j2.227 | $219 \cdot 579+j 271 \cdot 342$ |
| 3.422-j0.000 | $292.212+j \quad 0.020$ | 1.690-j2.481 | $187 \cdot 563+j 275 \cdot 284$ |
| 2.367-j1.387 | $314 \cdot 522+j 184 \cdot 290$ | 1.814-j2.142 | $230 \cdot 212+j 271 \cdot 886$ |
| 2.666-j1.466 | $288.002+j 158.418$ | 1.660-j2.600 | 174.426+j273.219 |
| $3 \cdot 757+j 1 \cdot 121$ | $244.412-j 72.928$ | 1.797-j2.006 | $247.732+j 276.631$ |
| $3 \cdot 253+j 0.018$ | 307.387-j 1.730 | 1.622-j2.887 | $147.942+j 263.273$ |
| 1.768-j1.569 | $316.473+j 280 \cdot 864$ | $1.674-j 1.677$ | $298.066+j 298.738$ |

## Broadside array (base voltages specified)

$$
\beta_{0} h=2.35620
$$

| Admittance $\beta_{0} b=$ | Impedance $7080$ | Admittance $\beta_{0} b=$ | 159 Impedance |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ | 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ |
| $N=4$ |  | N $2=4$ |  |
| $2 \cdot 378-j 1.235$ | $331 \cdot 264+j 172 \cdot 017$ | 1.667-j1.929 | $256.499+j 296.734$ |
| $3 \cdot 440-j 0.465$ | $285.452+j 38.575$ | 1.796-j2.409 | $198.923+j 266.764$ |
| 3.440-j0.465 | $285 \cdot 452+j 38 \cdot 575$ | 1.796-j2.409 | $198.923+j 266.764$ |
| 2.378-j1.235 | $331 \cdot 264+j 172 \cdot 017$ | 1.667-j1.929 | $256 \cdot 499+j 296.734$ |
| $N=10$ |  | $N=10$ |  |
| $2 \cdot 326-j 1 \cdot 136$ | $347 \cdot 120+j 169 \cdot 481$ | 1.638-j1.901 | $260 \cdot 172+j 301 \cdot 832$ |
| 3.232-j0.566 | $300 \cdot 210+j 52.564$ | 1.845-j2.474 | $193.668+j 259.699$ |
| 3.597-j0.879 | $262.334+j 64.099$ | 1.719-j2.288 | $209.960+j 279.353$ |
| 3.509-j1.021 | $262.762+j 76.455$ | $1.782-j 2.366$ | $203 \cdot 171+j 269 \cdot 662$ |
| 3.341-j0.911 | $278 \cdot 568+j 75.945$ | 1.756-j2.334 | $205.799+j 273.575$ |
| 3.341-j0.911 | $278.568+j 75.945$ | 1.756-j2.334 | $205.799+j 273.575$ |
| 3.509-j1.021 | $262.762+j 76.455$ | 1.782-j2.366 | $203 \cdot 171+j 269 \cdot 662$ |
| 3.597-j0.879 | $262 \cdot 334+j 64.099$ | 1.719-j2.288 | $209 \cdot 960+j 279 \cdot 353$ |
| 3.232-j0.566 | $300 \cdot 210+j 52.564$ | 1.845-j2.474 | $193 \cdot 668+j 259 \cdot 699$ |
| 2.326-j1.136 | $347 \cdot 120+j 169 \cdot 481$ | 1.638-j1.901 | $260 \cdot 172+j 301 \cdot 832$ |
| $N=20$ |  | $N=20$ |  |
| $2.266-j 1.086$ | $358.924+j 172.045$ | $1.630-j 1.884$ | $262 \cdot 611+j 303.586$ |
| 3.186-j0.506 | $306.136+j 48.569$ | 1.850-j2.475 | $193 \cdot 750+j 259.242$ |
| 3.508-j0.761 | $272.249+j 59.083$ | 1.707-j2.269 | $211 \cdot 723+j 281 \cdot 430$ |
| 3.359-j0.926 | $276.676+j 76.317$ | 1.790-j2.370 | $202.871+j 268.669$ |
| 3•197-j0.897 | $289.962+j 81 \cdot 384$ | 1.738-j2.309 | $208.073+j 276.419$ |
| 3•189-j0.676 | $300 \cdot 102+j 63 \cdot 641$ | 1.773-j2.350 | $204 \cdot 593+j 271 \cdot 231$ |
| 3.205-j0.709 | $297 \cdot 486+j 65.803$ | 1.749-j2.322 | $206.964+j 274.743$ |
| 3.184-j0.826 | $294.272+j 76.330$ | 1.765-j2.341 | $205 \cdot 359+j 272 \cdot 381$ |
| 3.206-j0.798 | $293.755+j 73.121$ | 1.755-j2.329 | $206 \cdot 355+j 273 \cdot 839$ |
| 3.267-j0.771 | $289.930+j 68.405$ | 1.760-j2.335 | $205 \cdot 877+j 273 \cdot 139$ |
| 3.267-j0.771 | $289.930+j 68.405$ | 1.760-j2.335 | $205 \cdot 877+j 273 \cdot 139$ |
| 3.206-j0.798 | $293.755+j 73.121$ | 1.755-j2.329 | $206 \cdot 355+j 273 \cdot 839$ |
| 3.184-j0.826 | $294.272+j 76.330$ | 1.765-j2.341 | $205 \cdot 359+j 272 \cdot 381$ |
| 3.205-j0.709 | $297.486+j 65.803$ | 1.749-j2.322 | $206.964+j 274.743$ |
| 3.189-j0.676 | $300 \cdot 102+j 63.641$ | 1.773-j2.350 | $204.593+j 271.231$ |
| 3.197-j0.897 | $289.962+j 81 \cdot 384$ | 1.738-j2.309 | $208.073+j 276.419$ |
| 3.359-j0.926 | $276 \cdot 676+j 76.317$ | 1.790-j2.370 | $202.871+j 268.669$ |
| 3.508-j0.761 | $272 \cdot 249+j 59.083$ | 1.707-j2.269 | $211.723+j 281.430$ |
| 3.186-j0.506 | $306 \cdot 136+j 48.569$ | 1.850-j2.475 | $193.750+j 259.242$ |
| $2.266-j 1.086$ | $358.924+j 172.045$ | 1.630-j1.884 | $262.611+j 303.586$ |

Endfire array (driving-point current specified)

$$
\beta_{0} h=2 \cdot 35620
$$

| $\beta_{0} b=1 \cdot 57080$ |  | $\beta_{0} b=3.14159$ | Impedance $14159$ |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ | 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ |
| $N=4$ |  | $N=4$ |  |
| 0.768-j2.562 | $107 \cdot 348+j 358.072$ | 1.009-j0.493 | $800 \cdot 083+j 391.021$ |
| 1-181-j0.897 | $536.798+j 407.909$ | 0.932-j0.393 | $910 \cdot 638+j 384 \cdot 363$ |
| 0.995-j1.165 | $423 \cdot 857+j 496 \cdot 201$ | 0.932-j0.393 | $910 \cdot 638+j 384.363$ |
| 0.811-j0.485 | $908.772+j 543.270$ | 1.009-j0.493 | $800.083+j 391.021$ |
| $N=10$ |  | $N=10$ |  |
| 0.719-j1.989 | $160 \cdot 692+j 444 \cdot 648$ | 0.816-j0.100 | $1207 \cdot 569+j 147 \cdot 854$ |
| 1.399-j0.929 | $496 \cdot 162+j 329 \cdot 370$ | 0.718-j0.035 | $1390 \cdot 182+j 68.698$ |
| 0.823-j0.999 | $491.292+j 596.033$ | 0.671-j0.010 | $1489 \cdot 227+j 22 \cdot 603$ |
| 0.942-j0.508 | $822 \cdot 512+j 443 \cdot 818$ | 0.646-j0.004 | $1547.295-j 8.699$ |
| $0.751-j 0.657$ | $754 \cdot 637+j 659 \cdot 352$ | 0.635-j0.010 | 1573.637-j 23.788 |
| 0.750-j0.337 | $1108 \cdot 900+j 498 \cdot 327$ | 0.635-j0.010 | $1573.637-j 23.788$ |
| $0.694-j 0.473$ | $984 \cdot 268+j 670 \cdot 369$ | 0.646-j0.004 | 1547.295-j 8.699 |
| 0.633-j0.227 | $1399.044+j 502.042$ | 0.671-j0.010 | $1489 \cdot 227+j 22 \cdot 603$ |
| 0.656-j0.359 | $1173.727+j 642.474$ | 0.718-j0.035 | $1390 \cdot 182+j 68.698$ |
| $0.537-j 0.150$ | $1727 \cdot 200+j 481 \cdot 071$ | 0.816-j0.100 | $1207 \cdot 569+j 147 \cdot 854$ |
| $N=20$ |  | $N=20$ |  |
| 1.317-j1.946 | $238 \cdot 516+j 352 \cdot 369$ | $0 \cdot 757+j 0 \cdot 160$ | 1263.562-j267.384 |
| 2.159-j0.457 | $443 \cdot 325+j 93.749$ | $0.652+j 0.191$ | 1412.576-j413.497 |
| 1.140-j0.949 | $518.065+j 431 \cdot 501$ | $0.595+j 0.206$ | $1500.532-j 519.963$ |
| 1.353-j0.279 | $709 \cdot 164+j 146 \cdot 166$ | $0 \cdot 558+j 0.216$ | 1558.218-j603.151 |
| 1.018-j0.603 | $727 \cdot 155+j 430 \cdot 732$ | $0.532+j 0.223$ | 1597.740-j669.409 |
| 1.079-j0.213 | $892 \cdot 289+j 176 \cdot 135$ | $0.514+j 0.228$ | 1625-266-j721.834 |
| 0.886-j0.440 | $905.806+j 449.935$ | $0.501+j 0.232$ | 1644.340-j762.222 |
| 0.899-j0.164 | $1076 \cdot 254+j 196.561$ | $0.491+j 0.235$ | 1657.144-j791.705 |
| 0.781-j0.332 | $1084 \cdot 904+j 460 \cdot 938$ | $0.485+j 0.236$ | 1665.037-j811.004 |
| $0.754-j 0.102$ | $1302.915+j 175.574$ | $0.483+j 0.237$ | 1668.794-j820.548 |
| 0.719-j0.235 | $1255 \cdot 957+j 410.737$ | $0.483+j 0.237$ | 1668.794-j820.548 |
| 0.652-j0.051 | $1523 \cdot 627+j 118 \cdot 167$ | $0 \cdot 485+j 0.236$ | 1665.037-j811.004 |
| 0.684-j0.151 | $1394.547+j 308.085$ | $0.491+j 0.235$ | 1657.144-j791.705 |
| 0.588-j0.055 | $1686.099+j 156.772$ | $0.501+j 0.232$ | 1644.340-j762.222 |
| 0.603-j0.142 | $1571 \cdot 217+j 368 \cdot 572$ | $0.514+j 0.228$ | 1625.266-j721.834 |
| 0.518-j0.081 | $1884 \cdot 893+j 293 \cdot 189$ | $0.532+j 0.223$ | 1597.740-j669.409 |
| 0.494-j0.164 | $1824 \cdot 240+j 606 \cdot 325$ | $0 \cdot 558+j 0 \cdot 216$ | 1558.218-j603.151 |
| 0.445-j0.052 | $2218 \cdot 386+j 260 \cdot 922$ | $0.595+j 0.206$ | 1500.532-j519.963 |
| 0.462-j0.131 | $2003 \cdot 505+j 567 \cdot 693$ | $0.652+j 0.191$ | 1412.576-j413.497 |
| 0.383-j0.026 | $2597.795+j 175.311$ | $0.757+j 0.160$ | 1263.562-j267.384 |

## Endfire array (base voltages specified)

$$
\beta_{0} h=2.35620
$$

| $\beta_{0} b=1.57080$ |  | Admittance $\beta_{0} b=$ | Impedance $14159$ |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ | 1.416-j1.335 | $373 \cdot 800+j 352 \cdot 429$ |
| $N=4$ |  | $N=4$ |  |
| 1.587-j1.820 | $272 \cdot 125+j \quad 312 \cdot 111$ | 1.004-j0.597 | $735 \cdot 858+j 437 \cdot 483$ |
| 1.187-j0.653 | $646.777+j 355.615$ | 0.931-j0.300 | $972 \cdot 956+j 313 \cdot 400$ |
| $1.038-j 0.426$ | $824.490+j 338.479$ | 0.931-j0.300 | $972.956+j 313.400$ |
| $0.739+j 0.188$ | $1271 \cdot 479-j 323.466$ | 1.004-j0.597 | $735 \cdot 858+j 437 \cdot 483$ |
| $N=10$ |  | $N=10$ |  |
| 1.568-j1.797 | $275.752+j 316.025$ | 0.833-j0.377 | $996 \cdot 584+j 451 \cdot 392$ |
| 1.220-j0.689 | $621 \cdot 279+j 350.921$ | 0.734-j0.054 | $1355 \cdot 209+j 99.818$ |
| 0.979-j0.326 | $919 \cdot 565+j 305 \cdot 944$ | $0.670+j 0.055$ | 1482.051-j121.931 |
| 0.814-j0.133 | $1195 \cdot 813+j 196 \cdot 023$ | $0.637+j 0.101$ | 1529.885-j243.575 |
| 0.736-j0.056 | $1351 \cdot 188+j 103 \cdot 261$ | $0 \cdot 624+j 0 \cdot 120$ | 1545.965-j297.753 |
| $0.658+j 0.033$ | $1516.022-j 76.287$ | $0 \cdot 624+j 0 \cdot 120$ | 1545.965-j297.753 |
| $0.641+j 0.053$ | $1550.594-j 128.146$ | $0 \cdot 637+j 0 \cdot 101$ | 1529.885-j243.575 |
| $0.560+j 0.162$ | $1646 \cdot 217-j 476 \cdot 624$ | $0.670+j 0.055$ | 1482.051-j121.931 |
| $0.605+j 0.072$ | 1631.125-j 193.964 | 0.734-j0.054 | $1355.209+j 99.819$ |
| $0.487+j 0.468$ | 1066.800-j1025.906 | 0.833-j0.377 | $996 \cdot 584+j 451 \cdot 392$ |
| $N=20$ |  | $N=20$ |  |
| 1.881-j1.656 | $299.529+j 263.639$ | 0.742-j0.258 | $1201.877+j 418.102$ |
| $1.445-j 0.580$ | $595 \cdot 950+j 239 \cdot 114$ | $0.650+j 0.057$ | 1526.390-j132.914 |
| $1.149-j 0.242$ | 833.526+j 175.946 | $0.584+j 0.171$ | 1576.159-j462.011 |
| 1.004-j0.058 | $993 \cdot 159+j \quad 57.727$ | $0.545+j 0.229$ | 1560.743-j655.797 |
| $0.837+j 0.001$ | $1194.570-j \quad 1.535$ | $0.520+j 0.264$ | 1530.486-j776.945 |
| $0.631+j 0.003$ | $1583.581-j \quad 6.286$ | $0.503+j 0.287$ | 1500.443-j856.095 |
| $0.607+j 0.054$ | $1634 \cdot 380-j 145 \cdot 843$ | $0.491+j 0.303$ | 1475.286-j908.707 |
| $0.507+j 0.076$ | 1929.067-j 290.680 | $0.484+j 0.313$ | 1456.283-j943.151 |
| $0 \cdot 516+j 0 \cdot 121$ | $1837.440-j 429.232$ | $0.479+j 0.320$ | 1443.647-j964.120 |
| $0.509+j 0.179$ | 1748.511-j 615.388 | $0.477+j 0.323$ | 1437.399-j974.022 |
| $0.491+j 0.195$ | $1757.837-j 698.636$ | $0.477+j 0.323$ | 1437.399-j974.022 |
| $0.489+j 0.250$ | 1621-102-j 827.013 | $0.479+j 0.320$ | 1443.647-j964.120 |
| $0.442+j 0.240$ | 1748.191-j 947.672 | $0 \cdot 484+j 0.313$ | 1456.283-j943.151 |
| $0.361+j 0.248$ | 1880.501-j1295.187 | $0.491+j 0.303$ | 1475.286-j908.707 |
| $0.335+j 0.238$ | 1986.679-j1409.322 | $0.503+j 0.287$ | 1500.443-j856.095 |
| $0 \cdot 174+j 0.210$ | 2335.692-j2818.250 | $0.520+j 0.264$ | 1530.486-j776.945 |
| $0.249+j 0.214$ | 2307.799-j1986.459 | $0.545+j 0.229$ | 1560.743-j655.797 |
| $0 \cdot 234+j 0 \cdot 310$ | 1552.071-j2055.252 | $0.584+j 0.171$ | 1576.159-j462.011 |
| $0 \cdot 282+j 0 \cdot 211$ | 2274.708-j1702.714 | $0.650+j 0.057$ | 1526.390-j132.914 |
| $0.253+j 0.559$ | $671 \cdot 650-j 1484 \cdot 457$ | 0.742-j0.258 | $1201.877+j 418.102$ |

## APPENDIX IV

## Broadside array (driving-point currents specified)

$$
\beta_{0} h=3.14159
$$

| $\beta_{0} b=1.57080$ |  | $\beta_{0} b=3.14159$ |  |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $0.985+j 1.000$ | 499.710-j507.494 | $0.985+j 1.000$ | 499.710-j507.494 |
| $N=4$ |  | $N=4$ |  |
| $1 \cdot 300+j 1 \cdot 158$ | 428.935-j382.052 | $1 \cdot 122+j 0 \cdot 538$ | $724 \cdot 851-j 347 \cdot 681$ |
| $2 \cdot 605+j 0 \cdot 524$ | 368.900-j 74.220 | $1.051+j 0.284$ | 886.543-j239.267 |
| $2.605+j 0.524$ | 368.900-j 74.220 | $1.051+j 0.284$ | 886.543-j239.267 |
| $1 \cdot 300+j 1 \cdot 158$ | 428.935-j382.052 | $1 \cdot 122+j 0 \cdot 538$ | 724.851-j347.681 |
| $N=10$ |  | $N=10$ |  |
| $1 \cdot 417+j 1 \cdot 159$ | 422.724-j345.951 | $1 \cdot 133+j 0 \cdot 556$ | 711.388-j349.311 |
| $2 \cdot 785+j 0.718$ | 336.688-j 86.845 | $1.031+j 0.271$ | 907.602-j238.628 |
| $2.318+j 0.788$ | 386.673-j131.498 | $1 \cdot 101+j 0 \cdot 324$ | 835.998-j246.013 |
| $2.325+j 0.865$ | 377.747-j140.553 | $1.071+j 0.304$ | 864.154-j245.716 |
| $2.438+j 0.876$ | 363.270-j130.614 | $1.082+j 0.311$ | 853.393-j245.529 |
| $2.438+j 0.876$ | 363.270-j130.614 | $1.082+j 0.311$ | 853.393-j245.529 |
| $2.325+j 0.865$ | 377.747-j140.553 | $1 \cdot 071+j 0 \cdot 304$ | 864.154-j245.716 |
| $2.318+j 0.788$ | 386.673-j131.498 | $1 \cdot 101+j 0 \cdot 324$ | 835.998-j246.013 |
| $2 \cdot 785+j 0.718$ | 336.688-j 86.845 | $1 \cdot 031+j 0 \cdot 271$ | 907.602-j238.628 |
| $1 \cdot 417+j 1 \cdot 159$ | 422.724-j345.951 | $1 \cdot 133+j 0 \cdot 556$ | $711 \cdot 388-j 349 \cdot 311$ |
| $N=20$ |  | $N=20$ |  |
| $1 \cdot 410+j 1 \cdot 161$ | 422.735-j348.012 | $1.135+j 0.561$ | $708 \cdot 268-j 349.951$ |
| $2.761+j 0.700$ | 340.320-j 86.216 | $1 \cdot 028+j 0.270$ | 910.264-j239.139 |
| $2.337+j 0.773$ | 385.711 - 1127.522 | $1 \cdot 105+j 0 \cdot 326$ | 832.672-j245.894 |
| $2 \cdot 343+j 0 \cdot 890$ | 372.969-j141.735 | $1.066+j 0.302$ | 868.043-j246.271 |
| $2.405+j 0.899$ | 364.852-j136.406 | $1.088+j 0.315$ | 848.027-j245.190 |
| $2.409+j 0.834$ | 370.623-j128.334 | $1.074+j 0 \cdot 307$ | 860.805-j246.146 |
| $2.381+j 0.833$ | 374-226-j130.931 | $1.084+j 0.312$ | 852.290-j245.320 |
| $2 \cdot 378+j 0 \cdot 878$ | 370.031-j136.654 | $1.077+j 0.309$ | 857.942-j245.875 |
| $2 \cdot 390+j 0 \cdot 879$ | 368.540-j135.585 | $1.081+j 0 \cdot 311$ | 854.451-j245.464 |
| $2 \cdot 392+j 0.840$ | 372.168-j130.632 | $1.079+j 0.310$ | 856.102-j245.671 |
| $2 \cdot 392+j 0.840$ | 372.168-j130.632 | $1.079+j 0.310$ | 856.102-j245.671 |
| $2 \cdot 390+j 0.879$ | 368.540-j135.585 | $1.081+j 0 \cdot 311$ | 854.451-j245.464 |
| $2 \cdot 378+j 0 \cdot 878$ | $370.031-j 136.654$ | $1.077+j 0.309$ | 857.942-j245.875 |
| $2.381+j 0.833$ | 374.226-j130.931 | $1.084+j 0.312$ | 852.290-j245.320 |
| $2.409+j 0.834$ | 370.623-j1 28.334 | $1.074+j 0.307$ | 860.805-j246.146 |
| $2.405+j 0.899$ | 364.852-j136.406 | $1.088+j 0.315$ | 848.027-j245.190 |
| $2.343+j 0.890$ | 372.969-j141.735 | $1 \cdot 066+j 0 \cdot 302$ | 868.043-j246.271 |
| $2.337+j 0.773$ | 385.711-j127.522 | $1 \cdot 105+j 0 \cdot 326$ | 832.672-j245.894 |
| $2 \cdot 761+j 0.700$ | 340.320-j $86 \cdot 216$ | $1.028+j 0.270$ | 910.264-j239.139 |
| $1 \cdot 410+j 1 \cdot 161$ | $422.735-j 348.012$ | $1 \cdot 135+j 0 \cdot 561$ | 708.268-j349.951 |

## Broadside array (base voltages specified)

$$
\beta_{0} h=3.14159
$$

| $\beta_{0} b=1.57080$ |  | $\beta_{0} b=3 \cdot 14159$ | Impedance $1159$ |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $0.985+j 1.000$ | 499.710-j507.494 | $0.985+j 1.000$ | 499.710-j507.494 |
| $N=4$ |  | $N=4$ |  |
| $1 \cdot 568+j 0 \cdot 815$ | 502.274-j260.992 | $1.078+j 0.563$ | 728.585-j380.868 |
| $2 \cdot 530+j 1 \cdot 186$ | 324.034-j151.902 | $1.085+j 0.255$ | 873.469-j205.141 |
| $2 \cdot 530+j 1 \cdot 186$ | 324.034-j151.902 | $1.085+j 0.255$ | 873.469-j205.141 |
| $1 \cdot 568+j 0 \cdot 815$ | 502.274-j260.992 | $1.078+j 0.563$ | 728.585-j380.868 |
| $N=10$ |  | $N=10$ |  |
| $1 \cdot 624+j 0.957$ | 457.034-j269.359 | $1 \cdot 059+j 0.600$ | 714.585-j405.086 |
| $2 \cdot 329+j 1 \cdot 180$ | 341.631-j173.109 | $1 \cdot 109+j 0 \cdot 198$ | 873.705-j156.039 |
| $2 \cdot 562+j 0 \cdot 859$ | 350.885-j117.687 | $1.058+j 0.363$ | 845.679 - j290.370 |
| $2.452+j 0.762$ | 371.908-j115.630 | $1.092+j 0.285$ | 857.310-j223.673 |
| $2 \cdot 310+j 0 \cdot 894$ | 376.473-j145.732 | $1.076+j 0.317$ | 855.336-j252.234 |
| $2 \cdot 310+j 0 \cdot 894$ | 376.473-j145.732 | $1.076+j 0.317$ | 855.336-j252.234 |
| $2.452+j 0.762$ | 371.908-j115.630 | $1.092+j 0.285$ | 857.310-j223.673 |
| $2 \cdot 562+j 0 \cdot 859$ | 350.885-j117.687 | $1.058+j 0.363$ | 845.679-j290.370 |
| $2 \cdot 329+j 1 \cdot 180$ | 341.631-j173.109 | $1 \cdot 109+j 0 \cdot 198$ | 873.705-j156.039 |
| $1.624+j 0.957$ | 457.034-j269.359 | $1.059+j 0.600$ | 714.585-j405.086 |
| $N=20$ |  | $N=20$ |  |
| $1 \cdot 626+j 0.929$ | 463.503-j264.895 | $1.053+j 0.609$ | 711.428-j411.334 |
| $2 \cdot 368+j 1 \cdot 183$ | 337.980-j168.790 | $1 \cdot 115+j 0 \cdot 190$ | 871.458-j148.401 |
| $2 \cdot 558+j 0 \cdot 901$ | 347.775-j122.478 | $1.051+j 0 \cdot 374$ | $844.861-j 300.449$ |
| $2 \cdot 400+j 0 \cdot 756$ | 379.037-j119.387 | $1 \cdot 101+j 0 \cdot 272$ | 856.088-j211.894 |
| $2 \cdot 319+j 0 \cdot 832$ | 382.028-j136.976 | $1.065+j 0.334$ | 855.035-j268.393 |
| $2 \cdot 393+j 0.910$ | 365.151-j138.824 | $1.090+j 0.294$ | 855.212-j230.435 |
| $2.428+j 0.879$ | 364.115-j131.830 | $1.072+j 0.321$ | 855.872-j256.175 |
| $2 \cdot 376+j 0 \cdot 826$ | 375.550-j130.485 | $1.084+j 0.303$ | 855.462-j238.823 |
| $2 \cdot 362+j 0 \cdot 838$ | 376.066-j133.411 | $1.077+j 0.314$ | 855.814-j249.498 |
| $2 \cdot 408+j 0 \cdot 883$ | 366.082-j134.306 | $1.081+j 0.309$ | 855.664-j244.371 |
| $2 \cdot 408+j 0 \cdot 883$ | 366.082-j134.306 | $1.081+j 0.309$ | 855.664-j244.371 |
| $2 \cdot 362+j 0 \cdot 838$ | 376.066-j133.411 | $1.077+j 0.314$ | 855.814-j249.498 |
| $2.376+j 0.826$ | 375.550-j130.485 | $1.084+j 0.303$ | 855.462-j238.823 |
| $2.428+j 0.879$ | 364.115-j131.830 | $1.072+j 0.321$ | 855.872-j256.175 |
| $2 \cdot 393+j 0.910$ | 365.151-j138.824 | $1.090+j 0.294$ | 855.212-j230.435 |
| $2 \cdot 319+j 0 \cdot 832$ | 382.028-j136.976 | $1.065+j 0.334$ | 855.035-j268.393 |
| $2 \cdot 400+j 0.756$ | 379.037-j119.387 | $1 \cdot 101+j 0.272$ | 856.088-j211.894 |
| $2 \cdot 558+j 0 \cdot 901$ | 347.775-j122.478 | $1.051+j 0.374$ | $844.861-j 300 \cdot 449$ |
| $2 \cdot 368+j 1 \cdot 183$ | 337.980-j168.790 | $1 \cdot 115+j 0 \cdot 190$ | $871.458-j 148.401$ |
| $1 \cdot 626+j 0.929$ | 463.503-j264.895 | $1.053+j 0.609$ | 711.428-j411.334 |

Endfire array (driving-point current specified)

$$
\beta_{0} h=3.14159
$$

| $\beta_{0} b=1 \cdot 57080$ |  | $\beta_{0} b=3.14159$ |  |
| :---: | :---: | :---: | :---: |
| 1 |  | $N=1$ |  |
| $0.985+j 1.000$ | 499.710-j507.494 | 0.985 + $\mathbf{j} 1.000$ | 499.710-j507.494 |
| $N=4$ |  | $N=4$ |  |
| $0.823+j 0.798$ | 626.331-j607.101 | $0.866+j 1.595$ | 262.842-j484.190 |
| $0.438+j 1.615$ | 156.481-j576.885 | $0.665+j 1.803$ | 180.000-j488.263 |
| $0.336+j 1.896$ | 90.012-j511.616 | $0.665+j 1.803$ | 180.000- $\mathbf{4 8 8 8} 263$ |
| $0.058+j 2.375$ | 10.328-j420.763 | $0.866+j 1.595$ | 262.842-j484.190 |
| $N=10$ |  | $N=10$ |  |
| $0.842+j 0.819$ | 610.374-j593.897 | 0.822+j1.788 | 212.309-j461.649 |
| $0.453+j 1.600$ | 163.769-j578.767 | $0.585+j 2.028$ | 131.295-j455.109 |
| $0.470+j 1.926$ | 119.479-j489.972 | $0.528+j 2.138$ | 108.895-j440.821 |
| $0.464+j 2.050$ | 104.931-j464.035 | $0.504+j 2.187$ | 100.117-j434.080 |
| $0.439+j 2.119$ | 93.669-j452.559 | $0.495+j 2.207$ | 96.704-j431.406 |
| $0 \cdot 424+j 2 \cdot 191$ | 85.172-j439.884 | $0.495+j 2.207$ | 96.704-j431.406 |
| $0 \cdot 403+j 2 \cdot 220$ | 79.130-j436.040 | $0 \cdot 504+j 2.187$ | 100.117-j434.080 |
| $0 \cdot 402+j 2 \cdot 279$ | 75.038-j425.585 | $0 \cdot 528+j 2.138$ | 108.895-j440.821 |
| $0.363+j 2.280$ | 68.208-j427.764 | $0.585+j 2.028$ | 131.295-j455.109 |
| $0.241+j 2.621$ | 34-754-j378.347 | $0.822+j 1.788$ | 212.309-j461.649 |
| $N=20$ |  | $N=20$ |  |
| $0.842+j 0.819$ | 610.364-j593.637 | $0.787+j 1.896$ | 186.738-j449.810 |
| $0 \cdot 452+j 1.599$ | 163.874-j579.215 | $0.544+$ j2.129 | 112.526-j440.877 |
| $0.470+j 1.928$ | 119.354-j489.435 | $0.481+j 2.246$ | 91.227-j425.781 |
| $0.463+j 2.047$ | 105.074-j464.774 | $0 \cdot 450+j 2.307$ | 81.435-j417.597 |
| $0 \cdot 440+j 2 \cdot 123$ | 93.566-j451.597 | $0.431+j 2.344$ | 75.825-j412.756 |
| $0 \cdot 422+j 2.185$ | 85.189-j441-258 | $0 \cdot 417+j 2.368$ | 72.217-j409.642 |
| $0 \cdot 407+j 2 \cdot 229$ | 79.317-j434.217 | $0 \cdot 409+j 2.384$ | 69.844-j407.524 |
| $0.394+j 2 \cdot 270$ | 74.339--j427.698 | $0 \cdot 402+j 2.395$ | $68.253-j 406 \cdot 119$ |
| $0.384+j 2.298$ | 70.687-j423.376 | $0 \cdot 399+j 2 \cdot 402$ | 67.288-j405.231 |
| $0 \cdot 374+j 2.329$ | 67.160-j418.519 | $0 \cdot 397+j 2 \cdot 405$ | 66.835-j404.819 |
| $0 \cdot 365+j 2 \cdot 349$ | 64.664-j415.738 | $0 \cdot 397+j 2 \cdot 405$ | 66.835-j404.819 |
| $0.357+j 2.376$ | 61.925-j411.645 | $0 \cdot 399+$ j2.402 | 67.288-j405.231 |
| $0 \cdot 350+j 2 \cdot 387$ | 60.187-j410.030 | $0.402+j 2.395$ | 68.253-j406.119 |
| $0.344+j 2.413$ | 57.913-j406.109 | $0.409+j 2.384$ | 69.844-j407.524 |
| $0.337+j 2.417$ | 56.628-j405.766 | $0.417+j 2.368$ | 72.217-j409.642 |
| $0.334+j 2.446$ | 54.847-j401.269 | $0.431+j 2.344$ | 75.825-j412.756 |
| $0 \cdot 324+j 2.441$ | 53.464-j402.653 | $0.450+j 2.307$ | 81.435-j417.597 |
| $0.329+j 2.474$ | 52.816-j397.207 | $0.481+j 2.246$ | 91.227-j425.781 |
| $0 \cdot 304+j 2.461$ | 49.407-j400.177 | $0.544+\mathrm{j} 2.129$ | 112.526-j440.877 |
| $0 \cdot 205+j 2 \cdot 754$ | 26.829-j361.165 | $0.787+j 1.896$ | 186.738-j449.810 |

## Endfire array (base voltages specified)

$$
\beta_{0} h=3.14159
$$

| $\beta_{0} b=1.57080$ |  | $\beta_{0} b=3 \cdot 14159$ | Impedance 4159 |
| :---: | :---: | :---: | :---: |
| $N=1$ |  | $N=1$ |  |
| $0.985+j 1.000$ | 499.710-j507.494 | $0.985+j 1.000$ | $499.710-j 507.494$ |
| $N=4$ |  | $N=4$ |  |
| $1.062+j 0.567$ | 732.748-j391.436 | $0.776+j 1.612$ | $242.440-j 503.755$ |
| $0.912+j 1.492$ | 298.230-j487.943 | $0.761+j 1.800$ | 199.335-j471.228 |
| $0.860+j 1.706$ | 235.711-j467.372 | $0.761+j 1.800$ | 199.335-j471.228 |
| $0 \cdot 680+j 2 \cdot 257$ | 122.374-j406.108 | $0.776+j 1.612$ | 242.440-j503.755 |
| $N=10$ |  | $N=10$ |  |
| $1.015+j 0.656$ | 694.918-j449.322 | $0 \cdot 636+j 1.827$ | 169.953-j488.127 |
| $0.940+j 1.444$ | 316.569-j486.420 | $0.608+j 2.045$ | 133.598-j449.294 |
| $0 \cdot 826+j 1.754$ | 219.728-j466.639 | $0 \cdot 585+j 2 \cdot 135$ | 119.452-j435.688 |
| $0.719+j 1.948$ | 166.737-j451.867 | $0 \cdot 570+j 2 \cdot 180$ | 112.239-j429.418 |
| $0.663+j 2.032$ | 145.060-j444.764 | $0 \cdot 562+j 2 \cdot 199$ | 109.080-j426.883 |
| $0 \cdot 595+j 2.136$ | 120.953-j434.437 | $0 \cdot 562+j 2 \cdot 199$ | 109.080-j426.883 |
| $0 \cdot 577+j 2 \cdot 155$ | 116.025-j433.058 | $0 \cdot 570+j 2 \cdot 180$ | 112.239-j429.418 |
| $0.512+j 2.251$ | 96.063-j422.314 | $0 \cdot 585+j 2.135$ | 119.452-j435.688 |
| $0 \cdot 547+j 2 \cdot 186$ | 107.662-j430.552 | $0.608+j 2.045$ | 133.598-j449.294 |
| $0 \cdot 506+j 2.472$ | 79.476-j388.190 | $0 \cdot 636+j 1 \cdot 827$ | 169.953-j488.127 |
| $N=20$ |  | $N=20$ |  |
| $1.012+j 0.660$ | 693.386-j451.865 | $0 \cdot 556+j 1.946$ | 135.869-j475.068 |
| $0.943+j 1.439$ | 318.711-j486.074 | $0 \cdot 530+j 2 \cdot 155$ | 107.519-j437.577 |
| $0.822+j 1.760$ | 217.821-j466.334 | $0.505+j 2.249$ | 95.051-j423.302 |
| $0.723+j 1.940$ | 168.775-j452.604 | $0 \cdot 484+j 2 \cdot 305$ | 87.325-j415.556 |
| $0.657+j 2.042$ | 142.835-j443.665 | $0 \cdot 468+j 2 \cdot 340$ | 82.177-j410.927 |
| $0 \cdot 602+j 2 \cdot 123$ | 123.550-j436.002 | $0 \cdot 456+j 2 \cdot 363$ | 78.648-j407.944 |
| $0 \cdot 567+j 2 \cdot 172$ | 112.536-j431.044 | $0.447+j 2.379$ | $76.234-j 405.963$ |
| $0.531+j 2.223$ | 101.632-j425.476 | $0 \cdot 441+j 2 \cdot 390$ | 74.611-j404.642 |
| $0 \cdot 512+j 2.251$ | 96.033-j422.381 | $0 \cdot 437+j 2 \cdot 397$ | 73.614-j403.834 |
| $0 \cdot 485+j 2.291$ | 88.376-j417.712 | $0 \cdot 435+j 2.400$ | 73.133-j403.451 |
| $0 \cdot 474+j 2.307$ | 85.487-j415.897 | $0 \cdot 435+j 2 \cdot 400$ | $73 \cdot 133-j 403 \cdot 451$ |
| $0.451+j 2.343$ | 79.227-j411.531 | $0.437+j 2.397$ | 73.614-j403.834 |
| $0.446+j 2.349$ | 78.112-j410.911 | $0 \cdot 441+j 2 \cdot 390$ | 74.611-j404.642 |
| $0.425+j 2.386$ | 72.358-j406.303 | $0 \cdot 447+j 2.379$ | 76.234-j405.963 |
| $0 \cdot 425+j 2.380$ | 72.675-j407.110 | $0 \cdot 456+j 2 \cdot 363$ | $78.648-j 407.944$ |
| $0 \cdot 403+j 2.423$ | 66.816-j401.672 | $0 \cdot 468+j 2 \cdot 340$ | 82.177-j410.927 |
| 0.410+j2.404 | 68.970-j404.249 | $0 \cdot 484+j 2 \cdot 305$ | 87.325-j415.556 |
| $0 \cdot 376+j 2.462$ | 60.644-j396.977 | $0.505+j 2.249$ | 95.051-j423.302 |
| $0.417+j 2.388$ | $70 \cdot 880-j 406.343$ | $0 \cdot 530+j 2.155$ | 107.519-j437.577 |
| $0 \cdot 408+j 2 \cdot 615$ | 58.233-j373.316 | $0 \cdot 556+j 1.946$ | 135.869-j475.068 |

## APPENDIX V

## Programme for the Yagi-Uda array

Equations used in the programme:

$$
\begin{align*}
& W_{k i V}\left(z_{k}\right)= \int_{-h_{i}}^{h_{i}} M_{0 z i}\left(z_{i}^{\prime}\right) K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
&= \sin \beta_{0} h_{i}\left[C_{b_{k i}}\left(h_{i}, z_{k}\right)-C_{b_{k i}}\left(h_{i}, h_{k}\right)\right] \\
&-\cos \beta_{0} h_{i}\left[S_{b_{k i}}\left(h_{i}, z_{k}\right)-S_{b_{k i}}\left(h_{i}, h_{k}\right)\right] r  \tag{A}\\
& \begin{aligned}
& W_{k i v}\left(z_{k}\right)= \int_{-h_{i}}^{h_{i}} F_{0 z i}\left(z_{i}^{\prime}\right) K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
&= C_{b_{k i}}\left(h_{i}, z_{k}\right)-C_{b_{k i}}\left(h_{i}, h_{k}\right)-\cos \beta_{0} h_{i}\left[E_{b_{k i}}\left(h_{i}, z_{k}\right)-E_{b_{k i}}\left(h_{i}, h_{k}\right)\right] \quad \text { (A) } \\
&\text { (cf. } 6.49 \mathrm{a} \text { and } 6.48 \mathrm{~b})
\end{aligned} \\
& \begin{aligned}
& W_{k i D}\left(z_{k}\right)= \int_{-h_{i}}^{h_{i}} H_{0 z i}\left(z_{i}^{\prime}\right) K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime} \\
&= H_{b_{k i}}\left(h_{i}, z_{k}\right)-H_{b_{k i}}\left(h_{i}, h_{k}\right)-\cos \frac{1}{2} \beta_{0} h_{i}\left[E_{b_{k i}}\left(h_{i}, z_{k}\right)-E_{b_{k i}}\left(h_{i}, h_{k}\right)\right] \text { (C) } \\
&\text { (cf. } 6.50 \mathrm{a} \text { and } 6.50 \mathrm{~b})
\end{aligned}
\end{align*}
$$

Numbered equations refer to equations in chapter 6.
A brief description of the Yagi-Array programme:

1. The programme was written in Fortran II language for the IBM 7094.
2. List of symbols used in the programme:
$\mathrm{N} \quad$ number of elements in the Yagi array.
HL column matrix of $N$ elements, each corresponding to the half-length of the antenna, beginning from element 1 to element $N$.
BKI square matrix of order $N$. The $k i^{\text {th }}$ element corresponds to the separation between the $k^{\text {th }}$ and the $i^{\text {th }}$ element.
A1 $A_{2}$ or $A_{2}^{\prime}$ in eq. (6.23) or eq. (6.64).
A column matrix of $2 N$ elements. The matrix element is arranged as $B_{1}, B_{2}, \ldots B_{N}, D_{1}, D_{2}, \ldots D_{N}$ with $B_{2}$ either primed or unprimed [eq. (6.33) or eq. (6.64)] at output.
NF incremental angle of the field pattern desired $=10^{\circ} / \mathrm{NF}$.
NYESFD if $N Y E S F D=0$, omit field pattern evaluation; otherwise, the field pattern is evaluated.
NYESRS if $N Y E S R S=0$, omit evaluation of residuals; otherwise, residuals are evaluated.
3. The programme assumes that

$$
V_{2}=1 \cdot 0+j 0 \cdot 0 \quad \text { and } \quad V_{k}=0 \text { for } k \neq 2
$$

4. The programme assumes that the radii of the antennas are all the same throughout the array. It also assumes that the separation between the two neighbouring director antennas is the same as that between the elements 2 and 3.
5. Input:

The programme requires 4 input cards for each case.
Card 1: Read N
Card 2; Read HD, HP, HR
where HD is the half-length of the driven antenna (element No. 2)
HP is the length of the parasitic director antennas (element No. 3 through No. N) HR is the half-length of the reflector antenna (element No. 1)
Card 3: Read S1, S2, S3
where $S 1$ is the radius of the antenna
S 2 is the spacing between the reflector and the driven antenna
S3 is the spacing between adjacent parasitic director antennas.
HD, HP, HR, S1, S2 and S3 are in terms of wavelength
Card 4; Read NYESFD, NF, NYESRS
6. Brief description of subroutines:
a. Main programme:

Purpose: To control the flow of the programme
b. FI Subroutine:

Purpose: To evaluate eqs. (A) through (C) and (6.76a)-(6.76d), (6.32)-(6.34), (6.15)-(6.17), (6.20)-(6.22), (6.53)-(6.62). (6.38), (6.39).
$E_{b}(h, z), C_{b}(h, z), S_{b}(h, z), H_{b}(h, z)$ are evaluated by Simpson's rule.
c. Yagi subroutine:

Purpose: To establish eqs. (6.40), (6.41) or (6.71), (6.72) and solve these simultaneous equations by Modified Gauss' method. The A1 and A obtained are the coefficients of the current distributions.
d. Curdis subroutine:

Purpose: To evaluate actual current distribution along the antennas. These are evaluated in 10 equal-distance points. The results are printed at output.
e. Field subroutine:

Purpose: To evaluate (6.84b), (6.89b), (6.90a), (6.90b), (6.97) and input admittance and impedance of the driving antenna of the Yagi array. The results are printed at the output. These are input admittance, input impedance of the driving antenna and field pattern, forward gain, backward gain and front to back ratio.
f. Resid subroutine:

Purpose: To check the result of the programme, evaluate the
differences between the left-hand side and the right-hand side of eq. (6.8) when the $I_{z}$ obtained is substituted back. These differences are called residuals. When $h_{k}=\lambda / 4$, eq. (6.8) is changed to the following form in this subroutine:

$$
\begin{gather*}
\sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) K_{k i d}\left(z_{k}, z_{i}^{\prime}\right) d z_{i}^{\prime}=\frac{-j 4 \pi}{\zeta_{0}}\left[\frac{1}{2} V_{0 k}^{e} S_{0 z k}+C_{k} F_{0 z k}\right], \\
k=1,2, \ldots N \tag{D}
\end{gather*}
$$

where

$$
C_{k}=\frac{j \zeta_{0}}{4 \pi} \sum_{i=1}^{N} \int_{-h_{i}}^{h_{i}} I_{z i}\left(z_{i}^{\prime}\right) K_{k i}\left(0, z_{i}^{\prime}\right) d z_{i}^{\prime} .
$$

The values of the right-hand side of eq. (6.8) or eq. (D) [which is $\left.4 \pi \mu_{0}^{-1} A_{z k}\left(z_{k}\right)\right]$ together with the resultant difference and absolute value of ratio of the difference $4 \pi \mu_{0}^{-1} A_{z k}$ are printed.
g. ECSH subroutine:

Purpose : To evaluate necessary integrals for Resid subroutine.
7. Maximum number of elements of the Yagi array is arbitrarily set to be 20. However, the programme does not use up all the storage positions. If one desires to increase the number of elements beyond 20, one has to redefine all the dimension statements in the programme.

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## Preface

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## LIST OF SYMBOLS

A vector potential, 4
$A_{i}$ coefficient, 131, 139, 188, 191, 192
$A_{z k}^{\text {even }}\left(z_{k}\right), A_{z k}^{\text {odd }}\left(z_{k}\right)$ even and odd parts of vector potential, 275
$A(\Theta, \Phi)$ array factor, 21
a radius of antenna, 3
$a_{n}$ Fourier coefficient, 38
$a_{1}(w)$ coefficient of inductive coupling of line and load, 325, 326
B magnetic vector, 3
$\mathbf{B}^{r}$ magnetic vector in radiation zone, 7
$B_{i}, B_{i R}, j B_{i I}$ coefficient and real and imaginary parts, 131, 139
$B_{D}(y)$ magnetic field in dipole mode of loop, 361
$B_{T}(y)$ magnetic field in transmission line mode of loop, 361
$B_{\Phi}$ cylindrical or spherical component of magnetic field, 6
$B_{\Phi}^{r}$ cylindrical or spherical component of magnetic field in radiation zone, 6
$b$ distance between parallel antennas, 32
$b_{i k}$ distance between antennas $i$ and $k, 32$
$C_{T}$ lumped capacitance for terminal zone, 328
$\operatorname{Cin} x$ integral function, 25
$C_{a}(h, z), C_{b}(h, z)$ integral functions, 24,141
$C_{1}, C_{2}$ constants in integral equation, 48,51
c velocity of light, 6
$c^{(m)}, c^{\prime(m)}$ parameters, 102
$c_{k}$ complex amplitude function, 103
$c(w)$ capacitance per unit length of line, 326
$c_{L}(w)$ part of $c(w)$ due to charges in line, 326
$c_{0}$ capacitance per unit length of uniform line, 329

D directivity, 14
$D_{r}(0)$ relative directivity, 206
$D_{m}\left(\Theta, \beta_{0} h\right)$ field factor, 60
$d_{12}$ distance between centres of collinear antennas, 288

E electric vector, 3
$\mathbf{E}^{r}$ electric vector in radiation zone, 6
$E_{r}, E_{\Theta} \quad$ spherical components of electric field, 6 $E_{\rho}, E_{z}$ cylindrical components of electric field, 6
$E_{z}^{\text {inc }}$ incident electric field, 66
$E_{0 z}$ component in five-term current, 278
$E_{a}(h, z), E_{b}(h, z)$ integral functions, 55, 141
$E_{\Theta}^{r} \Theta$ component of electric field in radiation zone, 6
$F_{0 z}$ component in two-term, three-term and five-term current, 127
$F_{0}\left(\Theta, \beta_{0} h\right), F_{m}\left(\Theta, \beta_{0} h\right)$ field functions of antenna with sinusoidal current, 7, 14
$f_{i}$ discrete frequency applied to array, 259
$f\left(\Theta, \beta_{0} h\right), f^{\prime}\left(\Theta, \beta_{0} h\right) \quad$ field characteristics, $59,60,84,85$
$f_{L}\left(\Theta, \beta_{0} h\right), f_{L}^{\prime}\left(\Theta, \beta_{0} h\right) \quad$ field characteristics, 61,62
$G_{c}$ characteristic conductance of line, 248
$G_{r}(0)$ relative gain, 206
$G_{m}\left(\Theta, \beta_{0} h\right)$ field factor, 60
$G_{N}(\pi / 2,0) \quad$ absolute gain, 206
$G_{k i}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) \quad$ kernel, 275
$G_{k i}^{\text {even }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) \quad$ even part of kernel, 275
$G_{k i}^{\text {odd }}\left(d_{k i}, z_{k}, z_{i}^{\prime}\right) \quad$ odd part of kernel, 275
$g\left(\Theta, \beta_{0} h\right), g^{\prime}\left(\Theta, \beta_{0} h\right) \quad$ field functions of array, 84,85
$H_{0 z}$ component in three-term and five-term current, 127
$H_{m}\left(\Theta, \beta_{0} h\right) \quad$ field factor, 60
$h$ half-length of antenna, 3
$h_{e D}$ effective length of each half of loop for dipole mode, 362
$h_{e}(\pi / 2) \quad$ effective length, 68
$h_{e N}(\Theta, \Phi)$ effective length of $N$-element array, 237
$I^{(m)}(z)$ phase sequence current, 78
$I_{1 L}(w)$ current along transmission line, 325
$I_{z}(z)$ total axial current, 5
$I_{z i}^{\text {even }}\left(z_{i}\right), I_{z i}^{\text {odd }}\left(z_{i}\right) \quad$ even and odd parts of current, 275
$I_{D}(\Theta, \Phi)$ current in receiving antenna, 236

J volume of density of current, 3
$J_{z i}^{j}\left(z_{i}\right)$ current functions, 282
$J_{I}(h, z), J_{R}(h, z)$ shorthand notation for imaginary and real parts of integral, 49

K surface density of current, 4
$K\left(z, z^{\prime}\right)$ kernel in integral equation, 49
$K^{(1)}\left(z, z^{\prime}\right), K^{(2)}\left(z, z^{\prime}\right) \quad$ kernels for phase sequences, 72
$K_{d}\left(z, z^{\prime}\right) \quad$ difference kernel, 51
$K_{k i}\left(z, z^{\prime}\right) \quad$ kernels involving antennas $i$ and $k, 99$
$K_{\text {kid }}\left(z, z^{\prime}\right)$ difference kernels involving antennas $i$ and $k, 99$
$K_{R}\left(z, z^{\prime}\right), K_{I}\left(z, z^{\prime}\right) \quad$ real and imaginary parts of $K\left(z, z^{\prime}\right), 49$
$K_{d}^{(1)}\left(z, z^{\prime}\right), K_{d}^{(2)}\left(z, z^{\prime}\right) \quad$ difference kernels for phase sequences, 72
$K_{11}\left(z, z^{\prime}\right), K_{12}\left(z, z^{\prime}\right) \quad$ kernels for coupled antennas, 71
$K_{11 d}\left(z, z^{\prime}\right), K_{12 d}\left(z, z^{\prime}\right)$ difference kernels for coupled antennas, 71 $K_{k i}\left(z_{k}, z_{i}^{\prime}\right)$ kernel, 276
$K_{k i R}\left(z_{k}, z_{i}^{\prime}\right), K_{k i I}\left(z_{k}, z_{i}^{\prime}\right)$ real and imaginary parts of $K_{k i}\left(z_{k}, z_{i}^{\prime}\right)$
$L_{T}$ lumped inductance for terminal zone, 328
$l^{l}(w)$ external inductance per unit length of line, 326
$l_{L}^{e}(w)$ part of $l^{e}(w)$ due to currents in line, 326
$l_{0}^{e}$ external inductance per unit length of uniform line, 328
$M_{0 z}$ component in two-term, three-term and five-term current, 127
$N$ number of elements in array, 17
$n$ distance between elements in fractions of wavelength, 21
n unit normal, 4
$P$ time-average power, 10
$P_{k}$ time-average power transferred across surface of $k$ th antenna, 29
$P_{0 z}$ component in five-term current, 279
$Q$ denominator of $T$ functions, 54
$Q^{(m)}$ denominator of $T^{(m)}$ functions, 78
$Q_{k}$ complex coefficient, 279
$Q_{m}\left(\Theta, \beta_{0} h\right)$ field factor, 285
$q(z)$ charge per unit length, 5
$R$ distance from arbitrary point to field point, 3, 5, 6
$R_{c} \quad$ characteristic resistance of line, 247
$R_{k}$ complex coefficient, 279
$R_{k i}$ distance between elements on antennas $i$ and $k, 99$
$R_{\text {kih }}$ distance to end of antenna $k$ from point on antenna $i, 99$
$R_{m}^{e} \quad$ radiation resistance, 16
$R_{m}\left(\Theta, \beta_{0} h\right)$ field factor, 285
$R_{0}$ distance to origin, 3
$R_{0}$ distance to centre of antenna, 16
$R_{0}$ driving-point resistance, 25
$R_{1 h}, R_{2 h}$ distances to ends of antenna, 16
$R_{11}, R_{12}$ distances between elements of coupled antennas, 71
$R_{11 h}, R_{12 h}$ distances to ends of coupled antennas, 71
$R_{0}, \Theta, \Phi$ spherical coordinates, 286
$r_{F B}$ front-to-back ratio, 207
$r, \boldsymbol{\Theta}, \boldsymbol{\Phi}$ spherical coordinates, 6
$\hat{\mathbf{r}}, \hat{\boldsymbol{\Theta}}, \boldsymbol{\Phi}$ unit vectors in directions of spherical coordinate axes, 6

S Poynting vector, 9
$S_{B}$ sensitivity constant of unloaded loop, 362
$S_{c}$ sensitivity constant of short antenna, 358
$S_{k i}$ Poynting vector on element $k$ due to current in element $i, 29$
$S_{0 z}$ component in two-term, three-term and five-term current, 128
$S_{B}^{(1)}, S_{E}^{(1)}$ sensitivity constants for singly loaded loop, 363, 364
$S_{B}^{(2)}, S_{E}^{(2)}$ sensitivity constants for doubly loaded loop, 365
$S_{a}(h, z), S_{b}(h, z)$ integral functions, 24,141
SWR standing wave ratio, 340
$s^{(m)}, s^{\prime(m)}$ parameters, 102
$s_{k}$ complex amplitude function, 103
$T, T_{D}, T_{U}, T_{D}^{\prime}, T_{U}^{\prime}$ $T_{D}^{(m)}, T_{U}^{(m)}, T_{D}^{\prime(m)}, T_{U}^{\prime(m)}$
$U$ function proportional to vector potential at end of antenna, 51
$U_{k}$ function proportional to vector potential at end of antenna $k, 98$
$u(z)$ normalized current in unloaded receiving antenna, 67, 68
$V^{(0)}, V^{(1)}$ phase-sequence voltages, 72
$V(w)$ scalar potential difference along transmission line, 325
$V_{L}(w), V_{T}(w)$ parts of $V(w)$ due to charges in line and termination, 326
$V_{0}$ voltage at driving point, 23
$v(z)$ normalized current distribution, 80
$W_{k i D}\left(z_{k}\right)$ normalized vector potential difference of $\cos \frac{1}{2} \beta_{0} z-\cos \frac{1}{2} \beta_{0} h, 238$
$W_{k i v}\left(z_{k}\right)$ normalized vector potential difference of $\cos \beta_{0} z-\cos \beta_{0} h, 238$
$W_{k i v}\left(z_{k}\right)$ normalized vector potential difference of $\sin \beta_{0}(h-|z|), 238$
$W_{p L}(w), W_{p T}(w)$ component of vector potential difference due to currents in line and load, 325
$w$ distance from load along transmission line, 325
$w(z)$ normalized current distribution, 80
$X_{i}, Y_{i}, Z_{i}$ Cartesian coordinates for centre of element $i, 286$
$Y^{(0)}, Y^{(1)} \quad$ zero- and first-phase-sequence admittance of loop, 365
$Y_{a}$ apparent admittance of load terminating line, 329
$Y_{k i n}$ driving-point (input) admittance of element $k, 104$
$Y_{1 \text { in }}$ driving-point (input) admittance, 82
$Y_{L} \quad$ admittance loading loop, 363
$Y_{s 1}, Y_{s 2} \quad$ self-admittance, 80
$Y_{T}$ terminating admittance, 249
$Y(0), Y\left(\frac{l}{2}\right)$ driving-point admittance of loop, 365
$Y_{0}^{(m)}$ phase sequence admittance, 78
$Y_{0}$ admittance of circular loop and square loop with constant current, 362, 366
$Y_{1}$ driving-point admittance of array, 251
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$Z_{i k}$ mutual impedance, 30
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$Z_{L}$ load impedance, 68; for loop, 363
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$\alpha$ attenuation constant of line, 327
$\alpha_{i k}$ cofactor divided by determinant, 243
$\beta$ phase constant of line, 327
$\beta_{i k}$ cofactor divided by determinant, 243
$\beta_{0}$ wave number in air, 5
$\Gamma_{a}$ complex reflexion coefficient of line, 337
$\gamma$ propagation constant of uniform line, 327
$\gamma, \gamma^{\prime}$ coefficients of cross coupling for loop, 365
$\gamma_{i k}$ cofactor divided by determinant, 243
$\gamma(w)$ propagation constant of line, 325
$\Delta$ determinant, 240
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$\Delta s \quad$ width of resonance curve, 350
$\delta$ phase angle of current, 18
$\delta_{i k}$ Kronecker delta, 136
$\delta(z)$ Dirac delta function, 2
$\varepsilon^{(1)}, \varepsilon^{(2)}$ error ratios for singly and doubly loaded loops, 365
$\varepsilon_{0}$ fundamental electric constant (permittivity of free space), 3
$\zeta_{0} \quad$ characteristic impedance of free space, 7
$\eta$ surface density of charge, 4
$\Theta$ spherical coordinate, 6
$\lambda_{0}$ wavelength in free space, 5
$\mu_{0}$ fundamental magnetic constant (permeability of free space), 3
$\xi_{i} \quad$ ratio of voltages, 156
$\rho$ volume density of charge, 3
$\rho, \rho_{a}, \rho_{g}$ apparent terminal attenuation function in general, for load, and for generator, 339
$\rho, \Phi, z \quad$ cylindrical coordinates, 6
$\hat{\boldsymbol{\rho}}, \boldsymbol{\Phi}, \hat{\mathbf{z}}$ unit vectors in directions of cylindrical coordinate axes, 6
$\Sigma$ surface, 9
$\Sigma_{k} \quad$ surface of $k$ th antenna, 29
$\sigma$ spacing ratio for log-periodic antenna, 245
length ratio for log-periodic antenna, 245
Ф cylindrical and spherical coordinate, 6
$\Phi_{k i}$ matrix element, 138
$\Phi_{T k i}^{(m)}$ matrix element, 127
$\phi$ phase shift in section of line, 247
$\phi$ scalar potential, 3
$\phi, \phi_{a}, \phi_{g}$ apparent terminal phase function in general, for load and for generator, 339, 348

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[^0]:    $\dagger$ See [1] chapter V and [2] pp. 178-181.

[^1]:    $\dagger[3]$ in chapter 5 .

[^2]:    $\dagger$ Many of these are described or referred to in [1], chapter 2.

[^3]:    $\dagger$ A more detailed analysis of the receiving antenna is in [1], chapter 4.

[^4]:    $\dagger$ [2], chapter 2.

[^5]:    $\dagger$ This section is based on the work of Dr I. L. Morris [3].

[^6]:    $\dagger$ This section is based on the work of Dr I. L. Morris [3].

[^7]:    $\dagger$ See, for example, [4], p. 690 and [5], p. 216.

[^8]:    $\dagger$ [4], pp. 568-570.
    $\ddagger[4]$, p. 694.

[^9]:    $\dagger$ [11], p. 83, equations (6) and (7).

[^10]:    $\dagger$ Designated as $f_{14}$ in a notation described in section 6.15.
    $\ddagger$ Note that only the plotted points are physically meaningful; the continuous curve serves only to guide the eye.

[^11]:    $\dagger$ Note that only the plotted points are physically significant. The continuous curve serves merely to guide the eye.

[^12]:    $\dagger$ Note that only the plotted points are physically significant. The continuous curve serves merely to guide the eye.

[^13]:    $\dagger$ The computations in this section were planned and programmed by V. W. H. Chang.

[^14]:    $\dagger$ See, for example, R. W. P. King, [2], p. 413, Fig. $30.7 e$.
    $\ddagger$ The computations on this section are those of V. W. H. Chang. Parts of sections 7.7-7.10 were first published in Radio Science [3].

[^15]:    $\dagger$ The computations in this section are those of V. W. H. Chang.

[^16]:    $\dagger$ Reverse the signs for $i=7,8,9$.

[^17]:    $\dagger$ Reverse signs for $i=7,8,9,16,17,18,25,26,27$

[^18]:    $\dagger$ See [1] to [8].

[^19]:    $\dagger$ Chapter 2 of [6].

[^20]:    $\dagger[5]$, p. 368 and p. 411.
    $\ddagger[5]$, p. 377 ; [11], p. 380; [14], p. 96.

[^21]:    $\dagger$ Formulas and graphs for determining $B_{T}$ for changes in the radius of both inner and outer conductors of a coaxial line are in [11]. p. 368 and [14], p. 96.
    $\ddagger$ They are given respectively by (8), (13), (29), (38), and (39) of [5], p. 368 and p. 411.

[^22]:    $\dagger$ See [4] and [5].

[^23]:    $\dagger$ See chapter 4 of [5].

[^24]:    $\dagger$ See chapter 6 of [24].

[^25]:    $\dagger$ See [6], p. 127 and [27].

[^26]:    $\dagger$ See [5], p. 209 and [4], p. 231.

[^27]:    $\dagger$ [20], chapter VII, p. 5.

[^28]:    $\dagger$ See chapter 5 in [6].

[^29]:    $\dagger$ Additional discussions on the measurement of radiation patterns and gain may be found in [3]. [31] and [32], chapters 15 and 16.

