Combustion, Explosion, and Shock Waves, Vol. 37, No. 1, pp. 113–118, 2001

Dependence of the Shape of a Detonation Wave Front on the Detonation Wave Velocity upon Detonation of a Cylindrical Charge

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UDC 534.222

Translated from *Fizika Goreniya i Vzryva*, Vol. 37, No. 1, pp. 127–132, January–February, 2001.
Original article submitted March 24, 2000.

The transition of a system of partial differential equations which describe the stationary flow behind the shock-wave front of a detonation complex upon detonation of a cylindrical charge to a system of ordinary differential equations is performed by means of the series expansion in terms of the radial variable. The necessary equations for determination of the derivatives of solutions with respect to the parameters and the initial conditions for them are formulated. Imposing the condition of continuous extendibility of the solutions leads to equations that allow one to determine the shape of a shock-wave front as a function of wave velocity.

The attempt is made to construct a consistent method of determining the shape of the shock-wave front of a detonation complex upon detonation of a cylindrical charge of finite diameter in the case where the detonation velocity deviates insignificantly from the Chapman–Jouguet velocity. This problem can be solved, disregarding the problems connected with the description of the flow near the charge edge. This problem was considered earlier (see, e.g., [1, 2]), but in the solution of it, the authors restricted themselves only to the determination of the curvature of the front, i.e., the first expansion coefficient of the function describing the front shape in terms of the powers of its argument. The distinguishing feature of the proposed approach is the fundamental possibility of determining the shape of the front with no matter how high accuracy.

We write a system of equations that determines the axisymmetric stationary flow in the presence of a chemical reaction characterized by the unique variable λ (fraction of the reacted substance) in the cylindrical coordinate system:

$$v\,\frac{\partial\rho}{\partial z}+u\,\frac{\partial\rho}{\partial r}+\rho\Big(\frac{\partial v}{\partial z}+\frac{\partial u}{\partial r}+\frac{u}{r}\Big)=0,$$

$$v\,\frac{\partial v}{\partial z}+u\,\frac{\partial v}{\partial r}+\frac{1}{\rho}\,\frac{\partial p}{\partial z}=0,$$

$$v\,\frac{\partial u}{\partial z}+u\,\frac{\partial u}{\partial r}+\frac{1}{\rho}\,\frac{\partial p}{\partial r}=0, \qquad (1)$$

$$v\,\frac{\partial p}{\partial z}+u\,\frac{\partial p}{\partial r}-c^2\Big(v\,\frac{\partial\rho}{\partial z}+u\,\frac{\partial\rho}{\partial r}\Big)=-\frac{\partial E/\partial\lambda}{\partial E/\partial p}\,R\equiv K,$$

$$v\,\frac{\partial\lambda}{\partial z}+u\,\frac{\partial\lambda}{\partial r}=R.$$

Here u and v are the radial and axial velocity-vector components, respectively, E is the internal energy per unit mass, c is the velocity of sound with constant λ, and R is the chemical-reaction rate which depends, in the general case, on the pressure p, the density ρ, and the variable λ (for brevity, the arguments of this function are omitted).

We pass to the coordinate system (l, r) determined by the relation $l = z - z_f(r)$, where $z_f(r)$ is a function describing the front surface in cylindrical coordinates. In coordinates (l, r), the curvilinear front is a plane $l = 0$, which simplifies the writing of the initial boundary conditions. In these coordinates, system (1) is reduced to the form

$$(v+ud)\frac{\partial\rho}{\partial l}+u\,\frac{\partial\rho}{\partial r}+\rho\Big(\frac{\partial v}{\partial l}+d\,\frac{\partial u}{\partial l}+\frac{\partial u}{\partial r}+\frac{u}{r}\Big)=0,$$

$$(v+ud)\frac{\partial v}{\partial l}+u\,\frac{\partial v}{\partial r}+\frac{1}{\rho}\,\frac{\partial p}{\partial l}=0,$$

$$(v+ud)\frac{\partial u}{\partial l}+u\,\frac{\partial u}{\partial r}+d\,\frac{1}{\rho}\,\frac{\partial p}{\partial l}+\frac{1}{\rho}\,\frac{\partial p}{\partial r}=0, \qquad (2)$$

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$$(v+ud)\frac{\partial p}{\partial l}+u\,\frac{\partial p}{\partial r}-c^2\Big((v+ud)\frac{\partial \rho}{\partial l}+u\,\frac{\partial \rho}{\partial r}\Big)=K,$$

$$(v+ud)\frac{\partial \lambda}{\partial l}+u\,\frac{\partial \lambda}{\partial r}=R,$$

where $d=-z_f'(r)$.

With the aim at finding the shape of the front at a given velocity of the detonation wave, we expand the function $z_f(r)$ in powers of r:

$$z_f(r)=\sum_{i=1}^{\infty}a_i r^{2i};$$

other functions:

$$p(r,l)=\sum_{i=0}^{\infty}p_i(l)r^{2i},\quad \rho(r,l)=\sum_{i=0}^{\infty}\rho_i(l)r^{2i},$$

$$v(r,l)=\sum_{i=0}^{\infty}v_i(l)r^{2i},\quad \lambda(r,l)=\sum_{i=0}^{\infty}\lambda_i(l)r^{2i},$$

$$u(r,l)=\sum_{i=0}^{\infty}u_i(l)r^{2i+1}.$$

[Only the even powers of r in the expansion of p, ρ, v, and λ and only the odd powers in the expansion of u follows from the requirement for the analyticity of these functions on the ($r=0$) axis. We also note that the functions with zero subscripts are the values of the corresponding quantities on the axis, except for $u_0(l)$ equal to $\partial u/\partial r$ for $r=0$]. Substituting these series into system (2) and grouping the coefficients at the same powers of r, we obtain the system of ordinary differential equations for the functions $p_i(l)$, $\rho_i(l)$, $v_i(l)$, $u_i(l)$, and $\lambda_i(l)$:

$$\begin{gathered}
\sum_{k+s=m}\Big[\Big(v_k-2\sum_{j+n=k}u_j n a_n\Big)\rho_s' \\
+\rho_s\Big(v_k'-2\sum_{j+n=k}u_j' n a_n\Big)+2(m+1)\rho_s u_k\Big]=0,
\\[2ex]
\sum_{i+k+s=m}\rho_i\Big[\Big(v_k-2\sum_{j+n=k}u_j n a_n\Big)v_s' \\
+2su_k v_s\Big]+p_m'=0,
\\[2ex]
\sum_{i+k+s=m}\rho_i\Big[\Big(v_k-2\sum_{j+n=k}u_j n a_n\Big)u_s'+(2s+1)u_k u_s\Big] \\
-2\sum_{k+s=m}(k+1)a_{k+1}p_s'+2(m+1)p_{m+1}=0,
\\[2ex]
\sum_{k+s=m}\Big[\Big(v_k-2\sum_{j+n=k}u_j n a_n\Big)p_s'+2su_k p_s\Big] \\
-\sum_{i+k+s=m}c_i^2\Big[\Big(v_k-2\sum_{j+n=k}u_j n a_n\Big)\rho_s'+2su_k\rho_s\Big]=K_m,
\\[2ex]
\sum_{k+s=m}\Big[\Big(v_k-2\sum_{j+n=k}u_j n a_n\Big)\lambda_s'+2su_k\lambda_s\Big]=R_m, \\
m=0,1,\ldots\,.
\end{gathered}\tag{3}$$

Here the prime denotes the derivative with respect to the variable l; $R_m(l)$, $K_m(l)$, and $c_m^2(l)$ are the series expansion coefficients of the following functions in terms of the powers of r:

$$R(p,\rho,\lambda)=R\Big(\sum_{i=0}^{\infty}p_i(l)r^{2i},\sum_{i=0}^{\infty}\rho_i(l)r^{2i},\sum_{i=0}^{\infty}\lambda_i(l)r^{2i}\Big),$$

$$K(p,\rho,\lambda)=K\Big(\sum_{i=0}^{\infty}p_i(l)r^{2i},\sum_{i=0}^{\infty}\rho_i(l)r^{2i},\sum_{i=0}^{\infty}\lambda_i(l)r^{2i}\Big),$$

$$c^2(p,\rho,\lambda)=c^2\Big(\sum_{i=0}^{\infty}p_i(l)r^{2i},\sum_{i=0}^{\infty}\rho_i(l)r^{2i},\sum_{i=0}^{\infty}\lambda_i(l)r^{2i}\Big).$$

We consider the dependence of the solutions of this system on the coefficient a_k and the detonation-wave velocity D as the parameters. Resolving the equations of system (3) relative to the derivatives, one can note that, generally, the solutions of these equations can be extended only up to the point $v_0=c_0$. We write, for example, the normalized (i.e., resolved relative to the derivatives) equations for $m=0$:

$$v_0'=\frac{-2c_0^2\rho_0u_0+K_0}{\rho_0(c_0^2-v_0^2)},\quad p_0'=\frac{2v_0c_0^2\rho_0u_0-v_0K_0}{c_0^2-v_0^2},$$

$$\rho_0'=\frac{2v_0^2\rho_0u_0-K_0}{v_0(c_0^2-v_0^2)},$$

$$u_0'=\frac{1}{c_0^2-v_0^2}\Big[\frac{2a_1}{\rho_0}\big(2c_0^2\rho_0u_0-K_0\big)-\big(c_0^2-v_0^2\big)\Big(\frac{2p_1}{\rho_0v_0}+\frac{u_0^2}{v_0}\Big)\Big],$$

$$\lambda_0'=\frac{R_0}{v_0}.$$

(The equations are similar for other values of m.) The solutions of these equations are extendible through the point $v_0=c_0$ only if the expression $2c_0^2\rho_0u_0-K_0$ vanishes simultaneously with the expression $v_0^2-c_0^2$. Precisely the condition of continuous extendibility of solutions through this point (i.e., the absence of a singularity) determines the values of a_k (shape of the front) as a function of D.

To find the dependence $a_k(D)$, we do the following. Let the coefficients a_k be known at a certain velocity D ($a_k=0$ for $D=D_{\rm CJ}$, where $D_{\rm CJ}$ is the velocity of the Chapman–Jouguet ideal detonation in an unbounded medium, i.e., in the case of a plane front). We find the derivatives da_k/dD, d^2a_k/dD^2, etc., thus determining the dependence $a_k(D)$. To do this, we note that the derivative

$$\frac{dp_m}{dD} = \frac{\partial p_m}{\partial D} + \sum_k \frac{\partial p_m}{\partial a_k}\frac{da_k}{dD} \tag{4}$$

(here one can use also the density or velocity derivative instead of pressure derivatives) is finite at the point $M_0 \equiv v_0/c_0 = 1$, whereas the derivatives $\partial p_m/\partial D$ and $\partial p_m/\partial a_k$ tend to infinity in approaching this point; this can be seen after normalization of the differential equations for these functions (see below). Equality (4) also allows us to find da_k/dD. We find these derivatives for $a_k = 0$ and $D = D_{\mathrm{CJ}}$. We write

$$\frac{dp_0}{dD} = \frac{\partial p_0}{\partial D} + \frac{\partial p_0}{\partial a_1}\frac{da_1}{dD} + \frac{\partial p_0}{\partial a_2}\frac{da_2}{dD} + \ldots,$$

$$\frac{dp_1}{dD} = \frac{\partial p_1}{\partial D} + \frac{\partial p_1}{\partial a_1}\frac{da_1}{dD} + \frac{\partial p_1}{\partial a_2}\frac{da_2}{dD} + \ldots,$$

etc. As is shown below, $\partial p_m/\partial a_k$ and other derivatives are different from zero only for $m < k$, and $\partial p_m/\partial D$ is different only for $m = 0$. Whence, considering the limit $M_0 \to 1$, we find that $da_k/dD = 0$ for $k > 1$ and

$$\frac{da_1}{dD} = -\lim_{M_0\to 1}\frac{\partial p_0/\partial D}{\partial p_0/\partial a_1}, \tag{5}$$

i.e., in the D-linear approximation, only the first coefficient of the expansion $z_f(r)$ in terms of the powers of r (curvature of the front) depends on the detonation-wave velocity. We consider the next approximation to find d^2a_k/dD^2. By analogy with the previous case, we write the derivatives d^2p_m/dD^2, discarding, with allowance for the aforesaid, the zero terms:

$$\frac{d^2p_0}{dD^2} = \frac{\partial^2 p_0}{\partial D^2} + \frac{\partial^2 p_0}{\partial a_1^2}\Big(\frac{da_1}{dD}\Big)^2 + 2\,\frac{\partial^2 p_0}{\partial a_1\partial D}\frac{da_1}{dD} + \frac{\partial p_0}{\partial a_1}\frac{d^2a_1}{dD^2} + \frac{\partial p_0}{\partial a_2}\frac{d^2a_2}{dD^2} + \frac{\partial p_0}{\partial a_3}\frac{d^2a_3}{dD^2} + \ldots,$$

$$\frac{d^2p_1}{dD^2} = \frac{\partial^2 p_1}{\partial D^2} + \frac{\partial^2 p_1}{\partial a_1^2}\Big(\frac{da_1}{dD}\Big)^2 + 2\,\frac{\partial^2 p_1}{\partial a_1\partial D}\frac{da_1}{dD} + \frac{\partial p_1}{\partial a_2}\frac{d^2a_2}{dD^2} + \frac{\partial p_1}{\partial a_3}\frac{d^2a_3}{dD^2} + \ldots,$$

$$\frac{d^2p_2}{dD^2} = \frac{\partial^2 p_2}{\partial D^2} + \frac{\partial^2 p_2}{\partial a_1^2}\Big(\frac{da_1}{dD}\Big)^2 + 2\,\frac{\partial^2 p_2}{\partial a_1\partial D}\frac{da_1}{dD} + \frac{\partial p_2}{\partial a_3}\frac{d^2a_3}{dD^2} + \ldots,$$

etc. We show below that the derivatives $\partial^2p_m/\partial a_1^2$ and $\partial^2p_m/\partial a_1\partial D$ are zero for $m > 1$, and $\partial^2p_m/\partial D^2$ is zero for $m > 0$; therefore, considering again the limit $M_0 \to 1$, we obtain $d^2a_k/dD^2 = 0$ for $k > 2$ and

$$\frac{d^2a_2}{dD^2} = -\lim_{M_0\to 1}\Big[\Big(\frac{\partial^2 p_1}{\partial a_1^2}\Big(\frac{da_1}{dD}\Big)^2 + 2\,\frac{\partial^2 p_1}{\partial a_1\partial D}\frac{da_1}{dD}\Big)\Big/\frac{\partial p_1}{\partial a_2}\Big], \tag{6}$$

$$\frac{d^2a_1}{dD^2} = -\lim_{M_0\to 1}\Big[\Big(\frac{\partial^2 p_0}{\partial D^2} + \frac{\partial^2 p_0}{\partial a_1^2}\Big(\frac{da_1}{dD}\Big)^2 + 2\,\frac{\partial^2 p_0}{\partial a_1\partial D}\frac{da_1}{dD} + \frac{\partial p_0}{\partial a_2}\frac{d^2a_2}{dD^2}\Big)\Big/\frac{\partial p_0}{\partial a_1}\Big]. \tag{7}$$

Thus, to calculate da_1/dD, it suffices to know how to find the derivatives of the functions p_m with respect to the parameters a_k and D (in this case, the functions $\partial p_0/\partial a_1$ and $\partial p_0/\partial D$), and, in addition, the functions $\partial^2p_0/\partial D^2$, $\partial^2p_0/\partial a_1^2$, $\partial^2p_0/\partial a_1\partial D$, $\partial p_0/\partial a_2$, $\partial p_1/\partial a_2$, $\partial^2p_1/\partial a_1^2$, and $\partial^2p_1/\partial a_1\partial D$ to calculate d^2a_1/dD^2 and d^2a_1/dD^2.

Similarly, one can, in principle, find any derivative of the coefficients a_k with respect to the detonation velocity D, i.e., to determine the shape of the front with required accuracy.

We now consider the procedure of calculation of the above-mentioned derivatives of the functions p_m with respect to the parameters a_k and D entering formulas (5)–(7). To start with, we note that we are interested in the derivatives at a fixed value of M_0 rather than at a fixed value of l; equations for the latter can be derived by varying the equations of system (3) relative to the parameters a_k or D (see, e.g., [3]). The relationship between these derivatives has the following form:

$$\frac{\partial p_m(M_0,\boldsymbol{a},D)}{\partial a_k} = \frac{\partial p_m(l,\boldsymbol{a},D)}{\partial a_k} + \frac{\partial p_m(l,\boldsymbol{a},D)}{\partial l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_k},$$

$$\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_k} = -\frac{\partial M_0(l,\boldsymbol{a},D)}{\partial a_k}\Big/\frac{\partial M_0(l,\boldsymbol{a},D)}{\partial l}.$$

Here $\boldsymbol{a}$ denotes the set of coefficients a_k, where $k = 1,\ldots,\infty$. A similar relationship between the second-order derivatives has the form

$$\frac{\partial^2 p_m(M_0,\boldsymbol{a},D)}{\partial a_i\partial a_j} = \frac{\partial^2 p_m(l,\boldsymbol{a},D)}{\partial a_i\partial a_j} + \frac{\partial^2 p_m(l,\boldsymbol{a},D)}{\partial a_i\partial l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_j} + \frac{\partial^2 p_m(l,\boldsymbol{a},D)}{\partial a_j\partial l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_i} + \frac{\partial^2 p_m(l,\boldsymbol{a},D)}{\partial^2 l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_i}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_j} + \frac{\partial p_k(l,\boldsymbol{a},D)}{\partial l}\,\frac{\partial^2 l(M_0,\boldsymbol{a},D)}{\partial a_i\partial a_j},$$

$$\frac{\partial^2 l(M_0,\boldsymbol{a},D)}{\partial a_i \partial a_j} = -\Big(\frac{\partial^2 M_0(l,\boldsymbol{a},D)}{\partial a_i \partial a_j} + \frac{\partial^2 M_0(l,\boldsymbol{a},D)}{\partial a_i \partial l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_j} + \frac{\partial^2 M_0(l,\boldsymbol{a},D)}{\partial a_j \partial l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_i} + \frac{\partial^2 M_0(l,\boldsymbol{a},D)}{\partial^2 l}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_j}\,\frac{\partial l(M_0,\boldsymbol{a},D)}{\partial a_i}\Big) \Big/ \frac{\partial M_0(l,\boldsymbol{a},D)}{\partial l}.$$

As has already been mentioned, one can find the equations for determination of the derivatives of the functions $p_m(l,\boldsymbol{a},D)$ with respect to the parameters a_k or D by varying the equations of system (3) relative to the parameters a_k or D. We write, for example, the system of equations for determination of $\partial p_m/\partial a_k$ for $a_k = 0$ and $D = D_{\rm CJ}$ [here $p_m(l) \equiv 0$, $\rho_m(l) \equiv 0$, etc., for $m \geqslant 1$ and $u_m(l) \equiv 0$, and the functions with zero subscripts are corresponding exact functions in the plane case, i.e., upon detonation in an unbounded medium]:

$$\begin{gathered}
\frac{\partial \rho'_m}{\partial a_k}\,v_0 + \rho'_0\,\frac{\partial v_m}{\partial a_k} + \frac{\partial \rho_m}{\partial a_k}\,v'_0 + \rho_0\,\frac{\partial v'_m}{\partial a_k} + 2(m+1)\rho_0\,\frac{\partial u_m}{\partial a_k} = 0,\\
\frac{\partial \rho_m}{\partial a_k}\,v_0 v'_0 + \rho_0\,\frac{\partial v_m}{\partial a_k}\,v'_0 + \rho_0 v_0\,\frac{\partial v'_m}{\partial a_k} + \frac{\partial p'_m}{\partial a_k} = 0,\\
\rho_0 v_0\,\frac{\partial u'_m}{\partial a_k} - 2(m+1)\delta_{k,m+1} p'_0 + 2(m+1)\frac{\partial p_{m+1}}{\partial a_k} = 0, \qquad (8)\\
\frac{\partial v_m}{\partial a_k}\,p'_0 + v_0\,\frac{\partial p'_m}{\partial a_k} - \frac{\partial c^2_m}{\partial a_k}\,v_0\rho'_0 - c_0^2\,\frac{\partial v_m}{\partial a_k}\,\rho'_0 - c_0^2 v_0\,\frac{\partial \rho'_m}{\partial a_k} = \frac{\partial K_m}{\partial a_k},\\
\frac{\partial v_m}{\partial a_k}\,\lambda'_0 + v_0\,\frac{\partial \lambda'_m}{\partial a_k} = \frac{\partial R_m}{\partial a_k}.
\end{gathered}$$

Here

$$\begin{gathered}
\frac{\partial K_m}{\partial a_k} = \frac{\partial K}{\partial \rho}\,\frac{\partial \rho_m}{\partial a_k} + \frac{\partial K}{\partial p}\,\frac{\partial p_m}{\partial a_k} + \frac{\partial K}{\partial \lambda}\,\frac{\partial \lambda_m}{\partial a_k},\\
\frac{\partial R_m}{\partial a_k} = \frac{\partial R}{\partial \rho}\,\frac{\partial \rho_m}{\partial a_k} + \frac{\partial R}{\partial p}\,\frac{\partial p_m}{\partial a_k} + \frac{\partial R}{\partial \lambda}\,\frac{\partial \lambda_m}{\partial a_k},\\
\frac{\partial c^2_m}{\partial a_k} = \frac{\partial c^2}{\partial \rho}\,\frac{\partial \rho_m}{\partial a_k} + \frac{\partial c^2}{\partial p}\,\frac{\partial p_m}{\partial a_k} + \frac{\partial c^2}{\partial \lambda}\,\frac{\partial \lambda_m}{\partial a_k}.
\end{gathered}$$

The derivatives with respect to ρ, p, and λ are taken for $\rho = \rho_0(l)$, $p = p_0(l)$, and $\lambda = \lambda_0(l)$, respectively. Considering this system with the initial conditions (see Appendix), one can conclude that $\partial p_m/\partial a_k$ and the other derivatives are different from zero only for $m < k$ (by virtue of the fact that $m \geqslant k$, the initial conditions are zero, and the equations are homogeneous). The equations for $\partial p_m/\partial D$ can be written in the same way; their consideration allows us to draw a conclusion that $\partial p_m/\partial D = 0$ for $m \neq 0$, and $\partial u_m/\partial D = 0$ for any value of m.

We also present the system that determines $\partial^2 p_m/\partial a_1^2$ for $a_k = 0$ and $D = D_{\rm CJ}$ with allowance for the fact that only the derivatives $\partial p_0/\partial a_1$, $\partial \rho_0/\partial a_1$, $\partial v_0/\partial a_1$, $\partial u_0/\partial a_1$, and $\partial \lambda_0/\partial a_1$ are not zero:

$$\begin{gathered}
\frac{\partial^2 \rho'_m}{\partial a_1^2}\,v_0 + \rho'_0\,\frac{\partial^2 v_m}{\partial a_1^2} + \frac{\partial^2 \rho_m}{\partial a_1^2}\,v'_0 + \rho_0\,\frac{\partial^2 v'_m}{\partial a_1^2} + 2(m+1)\rho_0\,\frac{\partial^2 u_m}{\partial a_1^2}\\
+ \delta_{m,0}\Big(2\,\frac{\partial v_0}{\partial a_1}\,\frac{\partial \rho'_0}{\partial a_1} + 2\,\frac{\partial v'_0}{\partial a_1}\,\frac{\partial \rho_0}{\partial a_1} + 4\,\frac{\partial u_0}{\partial a_1}\,\frac{\partial \rho_0}{\partial a_1}\Big) - 4\,\delta_{m,1}\Big(\rho'_0\,\frac{\partial u_0}{\partial a_1} + \rho_0\,\frac{\partial u'_0}{\partial a_1}\Big) = 0,\\
\frac{\partial^2 \rho_m}{\partial a_1^2}\,v_0 v'_0 + \rho_0\,\frac{\partial^2 v_m}{\partial a_1^2}\,v'_0 + \rho_0 v_0\,\frac{\partial^2 v'_m}{\partial a_1^2} + \frac{\partial^2 p'_m}{\partial a_1^2}\\
+ 2\delta_{m,0}\Big(\frac{\partial \rho_0}{\partial a_1}\,\frac{\partial v_0}{\partial a_1}\,v'_0 + \frac{\partial \rho_0}{\partial a_1}\,\frac{\partial v'_0}{\partial a_1}\,v_0 + \rho_0\,\frac{\partial v_0}{\partial a_1}\,\frac{\partial v'_0}{\partial a_1}\Big) - 4\delta_{m,1}\rho_0\,\frac{\partial u_0}{\partial a_1}\,v'_0 = 0,\\
v_0\,\frac{\partial^2 u'_m}{\partial a_1^2} + 2\delta_{m,0}\Big[\frac{\partial \rho_0}{\partial a_1}\,\frac{\partial u'_0}{\partial a_1}\,v_0 + \rho_0\,\frac{\partial v_0}{\partial a_1}\,\frac{\partial u'_0}{\partial a_1} + \rho_0\Big(\frac{\partial u_0}{\partial a_1}\Big)^2\Big] + 2(m+1)\frac{\partial^2 p_{m+1}}{\partial a_1^2} - 4\delta_{m,0}\,\frac{\partial p'_0}{\partial a_1} = 0, \qquad (9)\\
\frac{\partial^2 v_m}{\partial a_1^2}\,p'_0 + v_0\,\frac{\partial^2 p'_m}{\partial a_1^2} - \frac{\partial^2 c^2_m}{\partial a_1^2}\,v_0\rho'_0 - c_0^2\,\frac{\partial^2 v_m}{\partial a_1^2}\,\rho'_0 - c_0^2 v_0\,\frac{\partial^2 \rho'_m}{\partial a_1^2} + 2\delta_{m,0}\Big(\frac{\partial v_0}{\partial a_1}\,\frac{\partial p'_0}{\partial a_1} - c_0^2\,\frac{\partial v_0}{\partial a_1}\,\frac{\partial \rho'_0}{\partial a_1}\Big)\\
- 4\delta_{m,1}\frac{\partial u_0}{\partial a_1}(p'_0 - c_0^2\rho'_0) = \frac{\partial^2 K_m}{\partial a_1^2},\\
\frac{\partial^2 v_m}{\partial a_1^2}\,\lambda'_0 + v_0\,\frac{\partial^2 \lambda'_m}{\partial a_1^2} + 2\delta_{m,0}\,\frac{\partial v_0}{\partial a_1}\,\frac{\partial \lambda'_0}{\partial a_1} - 4\delta_{m,1}\,\frac{\partial u_0}{\partial a_1}\,\lambda'_0 = \frac{\partial^2 R_m}{\partial a_1^2}.
\end{gathered}$$

Here

$$\frac{\partial^2 R_0}{\partial a_1^2} = \frac{\partial^2 R_0}{\partial p^2}\Big(\frac{\partial p_0}{\partial a_1}\Big)^2 + \frac{\partial^2 R_0}{\partial \rho^2}\Big(\frac{\partial \rho_0}{\partial a_1}\Big)^2 + \frac{\partial^2 R_0}{\partial \lambda^2}\Big(\frac{\partial \lambda_0}{\partial a_1}\Big)^2 + 2\frac{\partial^2 R_0}{\partial \rho \partial p}\frac{\partial \rho_0}{\partial a_1}\frac{\partial p_0}{\partial a_1} + 2\frac{\partial^2 R_0}{\partial \rho \partial \lambda}\frac{\partial \rho_0}{\partial a_1}\frac{\partial \lambda_0}{\partial a_1} + 2\frac{\partial^2 R_0}{\partial \lambda \partial p}\frac{\partial \lambda_0}{\partial a_1}\frac{\partial p_0}{\partial a_1} + \frac{\partial R}{\partial p}\frac{\partial^2 p_0}{\partial a_1^2} + \frac{\partial R}{\partial \rho}\frac{\partial^2 \rho_0}{\partial a_1^2} + \frac{\partial R}{\partial \lambda}\frac{\partial^2 \lambda_0}{\partial a_1^2},$$

$$\frac{\partial^2 R_m}{\partial a_1^2} = \frac{\partial R}{\partial p}\frac{\partial^2 p_m}{\partial a_1^2} + \frac{\partial R}{\partial \rho}\frac{\partial^2 \rho_m}{\partial a_1^2} + \frac{\partial R}{\partial \lambda}\frac{\partial^2 \lambda_m}{\partial a_1^2} \quad \text{for } m > 0,$$

$\partial^2 K_m/\partial a_1^2$ and $\partial^2 c_m^2/\partial a_1^2$ have a similar form. One can conclude from the consideration of this system with the boundary conditions that $\partial^2 p_m/\partial a_1^2 = 0$ for $m > 1$. Similarly, we have $\partial^2 p_m/\partial a_1 \partial D = 0$ for $m > 1$ and $\partial^2 p_m/\partial D^2 = 0$ for $m > 0$.

Equations (8) and (9) and the initial conditions formulated for them in the Appendix allow us to find the derivatives of the solutions with respect to the parameters a_k and D, which, in turn, allow us to find the shape of the shock-wave front in the form of an expansion of the coefficients a_k in terms of the velocity D. A similar procedure can be performed to find the derivative a_k with respect to D of any order; however, it is necessary to note that the complexity of calculations rapidly increases with the order of derivatives.

Thus, the problem of determining the shape of a detonation wave front as a function of the detonation wave velocity upon detonation of a charge of finite diameter has been solved. In addition to the fact that this problem is interest in itself, it is a necessary component of the solution of the more general problem of finding a relation between the basic quantities which characterize the detonation of a cylindrical charge: the detonation wave velocity, the radius of a charge, and the shape of a wavefront.

The author thanks N. M. Kuznetsov and V. G. Grudnitskii for valuable comments and discussions.

APPENDIX

We present the values of the density, pressure, and velocity components behind the front of a curved shock wave in the form

$$\rho_f(r, D, \boldsymbol{a}) = \rho_{id}(D\cos\alpha),$$

$$p_f(r, D, \boldsymbol{a}) = p_{id}(D\cos\alpha),$$

$$v_f(r, D, \boldsymbol{a}) = D\cos\varphi\cos\alpha\sqrt{\tan^2\alpha + \rho_b^2/\rho_{id}^2},$$

$$u_f(r, D, \boldsymbol{a}) = D\sin\varphi\cos\alpha\sqrt{\tan^2\alpha + \rho_b^2/\rho_{id}^2},$$

$$\lambda(r, D, \boldsymbol{a}) = 0,$$

$$\varphi = \arctan(\tan\alpha(\rho_{id}/\rho_b)) - \alpha,$$

where α is the angle between the normal to the front at a given point and the direction of detonation-wave propagation, $\tan\alpha = z_f'(r) = \sum\limits_{i=1}^{\infty} 2ia_i r^{2i-1}$, and ρ_{id} and p_{id} are, respectively, the density and the pressure behind the front of a plane shock wave propagating with the velocity $D\cos\alpha$ in the same initial medium, i.e., at the same values of ρ_b and p_b (the density and the pressure in front of the shock wave). The initial conditions for system (3) have the form

$$p_m|_{l=0} = \frac{\partial^{2m} p_f}{(2m)!\partial r^{2m}}\Big|_{r=0},$$

$$\rho_m|_{l=0} = \frac{\partial^{2m} \rho_f}{(2m)!\partial r^{2m}}\Big|_{r=0},$$

$$v_m|_{l=0} = \frac{\partial^{2m} v_f}{(2m)!\partial r^{2m}}\Big|_{r=0},$$

$$u_{m|l=0} = \frac{\partial^{2m+1} u_f}{(2m+1)!\partial r^{2m+1}}\Big|_{r=0}.$$

The initial conditions for system (8) have the following form:

$$\frac{\partial p_m}{\partial a_k}\Big|_{l=0} = \frac{\partial^{2m+1} p_f}{(2m)!\partial a_k \partial r^{2m}}\Big|_{a_i=0, r=0},$$

$$\frac{\partial \rho_m}{\partial a_k}\Big|_{l=0} = \frac{\partial^{2m+1} \rho_f}{(2m)!\partial a_k \partial r^{2m}}\Big|_{a_i=0, r=0},$$

$$\frac{\partial v_m}{\partial a_k}\Big|_{l=0} = \frac{\partial^{2m+1} v_f}{(2m)!\partial a_k \partial r^{2m}}\Big|_{a_i=0, r=0},$$

$$\frac{\partial u_m}{\partial a_k}\Big|_{l=0} = \frac{\partial^{2m+2} u_f}{(2m+1)!\partial a_k \partial r^{2m+1}}\Big|_{a_i=0, r=0}.$$

We modify their form. For example, for the pressure and the radial velocity component, we write

$$\frac{\partial p_m}{\partial a_k}\Big|_{l=0} = \frac{\partial^{2m}}{(2m)!\partial r^{2m}}\frac{\partial p_f}{\partial \tan\alpha}\frac{\partial \tan\alpha}{\partial a_k}\Big|_{a_i=0, r=0},$$

$$\frac{\partial u_m}{\partial a_k}\Big|_{l=0} = \frac{\partial^{2m+1}}{(2m+1)!\partial r^{2m+1}} \times \frac{\partial u_f}{\partial \tan\alpha}\frac{\partial \tan\alpha}{\partial a_k}\Big|_{a_i=0, r=0}.$$

Since p_f is an even function of tan α, and $\alpha = 0$ for $a_i = 0$, we have $(\partial p_m/\partial a_k)|_{l=0} = 0$. For the same reasons, we have $(\partial \rho_m/\partial a_k)|_{l=0} = 0$ and

$(\partial v_m/\partial a_k)|_{l=0} = 0$. The velocity u_f is an odd function of α; therefore,

$$\frac{\partial u_f}{\partial \tan\,\alpha}\,\frac{\partial \tan\,\alpha}{\partial a_k}\Big|_{a_i=0} \sim r^{2k-1},$$

i.e., only $(\partial u_{k-1}/\partial a_k)|_{l=0}$ are different from zero. Similarly, since

$$\frac{\partial^2 p}{\partial \tan\,\alpha^2}\,\frac{\partial \tan\,\alpha}{\partial a_i}\,\frac{\partial \tan\,\alpha}{\partial a_j}\Big|_{a_k=0} \sim r^{2i-1}r^{2j-1},$$

only $(\partial p_{i+j-1}/\partial a_i\partial a_j)|_{l=0}$, $(\partial \rho_{i+j-1}/\partial a_i\partial a_j)|_{l=0}$, and $(\partial v_{i+j-1}/\partial a_i\partial a_j|_{l=0}$ are different from zero. Using similar considerations, one can show that among the derivatives $\partial/\partial D$, only the derivatives $\partial p_0/\partial D$, $\partial \rho_0/\partial D$, $\partial v_0/\partial D$, $\partial^2 p_0/\partial D^2$, $\partial^2 \rho_0/\partial D^2$, $\partial^2 v_0/\partial D^2$,and $\partial^2 u_0/\partial D\partial a_1$ are not zero.

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