Interest and Prices

Foundations of a Theory of Monetary Policy

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Chapter 1

The Return of Monetary Rules

> If it were in our power to regulate completely the price system of the future, the ideal position ... would undoubtedly be one in which, without interfering with the inevitable variations in the relative prices of commodities, the general average level of money prices ... would be perfectly invariable and stable.
>
> And why should not such regulation lie within the scope of practical politics? ... Attempts by means of tariffs, state subsidies, export bounties, and the like, to effect a partial modification of the natural order of [relative prices] almost inevitably involve some loss of utility to the community. Such attempts must so far be regarded as opposed to all reason. Absolute prices on the other hand — money prices — are a matter in the last analysis of pure convention, depending on the *choice of a standard of price* which it lies within our own power to make.
>
> — Knut Wicksell, *Interest and Prices,* 1898, p. 4.

The past century has been one of remarkable innovation in the world's monetary systems. At the turn of the twentieth century, it was taken for granted by practical men that the meaning of a monetary unit should be guaranteed by its convertibility into a specific quantity of some precious metal; debates about monetary policy usually concerned the relative advantages of gold and silver standards, or the possibility of a bimetallic standard. But through fits and starts, the world's currencies have come progressively to be more completely subject to "management" by individual central banks. Since the collapse of the Bretton Woods system of fixed exchange rates in the early 1970s, the last pretense of a connection of the world's currencies to any real commodity has been abandoned. We now live instead in a world of pure "fiat" units of account, the value of each of which depends solely upon the policies of the particular central bank with responsibility for it.

This has brought both opportunities and challenges. On the one hand, variations in the purchasing power of money, with their disruptions of the pattern of economic activity, need no longer result from the vagaries of the market for gold or some other precious metal. The recognition that the purchasing power of money need not be dictated by any "natural" market forces, and is instead a proper subject of government regulation, as proposed by the monetary reformer Knut Wicksell a century ago, should in principle make possible greater stability of the standard of value, facilitating contracting and market exchange. At the same time, the responsibilities of the world's central banks are more complex under a fiat system than they were when the banks' tasks were simply to maintain convertibility of their respective national currencies into gold, and it was not immediately apparent how the banks' new freedom should best be used. Indeed, during the first decade of the new regime, the policies of many industrial nations suffered from a tendency toward chronic inflation, lead to calls from some quarters in the 1980s for a return to a commodity standard.

This has not proven to be necessary. Instead, since the 1980s the central banks of the major industrial nations have been largely successful at bringing inflation down to low and fairly stable levels. Nor does this seem to have involved any permanent sacrifice of other objectives; for example, real GDP growth has been if anything higher on average, and certainly more stable, in the period since inflation has been stabilized in the U.S. Somewhat paradoxically, this period of improved macroeconomic stability has coincided with a *reduction*, in certain senses, in the ambition of central banks' efforts at macroeconomic stabilization. Banks around the world have committed themselves more explicitly to relatively straightforward objectives with regard to the control of inflation, and have found when they do so not only that it is easier to control inflation than previous experience might have suggested, but also that price stability creates a sound basis for real economic performance as well.

What appears to be developing, then, at the turn of another century, is a new consensus in favor of a monetary policy that is disciplined by clear rules intended to ensure a stable standard of value, rather than one that is determined on a purely discretionary basis to serve whatever ends may seem most pressing at any given time. Yet the new monetary rules are not so blindly mechanical as the rules of the gold standard, that defined monetary orthodoxy a century ago. They are instead principles of systematic conduct for institutions that are aware of the consequences of their actions and take responsibility for them, and choose their policies with careful attention to what they accomplish. Indeed, under the current approaches to rule-based policymaking, more emphasis is given to explicit commitments regarding desired economic outcomes, such as a target rate of inflation, than to particular technical indicators that the central bank may find it useful to monitor in achieving that outcome.

The present study seeks to provide theoretical foundations for a rule-based approach to monetary policy of this kind. The development of such a theory is an urgent task, for rule-based monetary policy in the spirit that I have described is possible only in the case that the central banks can develop a conscious and articulate account of what they are doing. It is necessary in order for them to know how to systematically act in a way that can serve their objectives, that are now defined in terms of variables that are much farther from being under the banks' direct control. But it is also necessary in order for them to be able to communicate the nature of their systematic commitments to the public, despite the absence of such mechanical constraints as a commitment to exchange currency for some real commodity. As we explain below, the advantages of a sound monetary policy are largely dependent upon the policy's being *understood* and relied upon by the private sector in arranging its affairs.

And there can be little doubt that the past decade has seen a marked increase in the self-consciousness of central banks about the way in which they conduct monetary policy, and in the explicitness of their communication with the public about their actions and the considerations upon which they are based. A particularly important development in this regard has been the adoption of "inflation targeting" as an approach to the conduct of monetary policy by many of the world's central banks in the 1990s.[1] As we discuss in more detail below, this approach (best exemplified by the practices of such innovators as the Bank of England, the Bank of Canada, the Reserve Bank of New Zealand, and the Swedish Riksbank) is characterized not only by public commitment to an explicit target, but also by a commitment to explain the central bank's policy actions in terms of a systematic decision-making framework that is aimed at achieving this target. This has led to not only to greatly increased communication with the public about the central bank's interpretation of current conditions and the outlook for the future, notably through the publication of detailed *Inflation Reports;* it has also involved fairly explicit discussion of the approach that they follow in deliberating about policy actions, and in some cases even publication of the model or models used in producing the forecasts that play a central role in these deliberations. As a consequence, these banks in particular have found themselves in need of a clear theory of how they can best achieve their objectives, and have played an important role in stimulating reflection on this problem.

It is true that the conceptual frameworks proposed by central banks to deal with their perceived need for a more systematic approach to policy were, until quite recently, largely developed without much guidance from the academic literature on monetary economics. Indeed, the central questions of practical interest for the conduct of policy — how should central banks decide about the appropriate level of overnight interest rates? how should monetary policy respond to the various types of unexpected disturbances that occur? — had in recent decades ceased to be considered suitable topics for academic study. Reasons for this included the trenchant critique of traditional methods of econometric policy evaluation by Lucas (1976); the critique of the use of conventional methods of optimal control in the conduct of economic policy by Kydland and Prescott (1977); and the develop of a new generation of quantitative models of business fluctuations ("real business cycle theory") with more rigorous microeconomic foundations, but which implied no relevance of monetary policy for economic welfare.

[1] See, *e.g.*, Bernanke *et al.* (19xx) for a thorough discussion of this development.

Nonetheless, recent developments, to be discussed in detail in this volume, have considerably changed this picture. The present study will seek to show that it is possible to use the tools of modern macroeconomic theory — intertemporal equilibrium modeling, taking full account of the endogeneity of private-sector expectations — to analyze optimal interest-rate setting in a way that takes seriously the concerns of central bankers, while simultaneously taking account of the "New Classical" critique of traditional policy evaluation exercises. In this way, the basic elements are presented of a theory that can provide a basis for the kind of systematic approach to the conduct of monetary policy which many central banks are currently seeking to develop. In the present chapter, we review some of the key features of this theory, as preparation for the more systematic development that begins in chapter 2.

1 The Importance of Price Stability

A notable feature of the new rule-based approaches to monetary policy has been the increased emphasis given to a particular policy objective: maintaining a low and stable rate of inflation. This is most obvious in the case of the countries with explicit inflation targets. But it also seems to characterize recent policy in the U.S. as well, where the past decade has seen unusual stability of the inflation rate, and where many econometric studies have found evidence of a stronger Fed reaction to inflation variations in recent years. (See further discussion of recent U.S. policy in section 4.1 below.)

Yet the justification of such an emphasis from the standpoint of economic theory may not be obvious. Standard general equilibrium models — and the earliest generation of quantitative equilibrium models of business fluctuations, the "real business cycle" models of the 1980s — imply that the absolute level of prices should be irrelevant for the allocation of resources, which depends only on *relative* prices. Traditional Keynesian macroeconometric models, of course, imply otherwise: variations in the growth rate of wages and prices are found to be associated with substantial variations in economic activity and employment. Yet the existence of such "Phillips curve" relations has typically been held to imply that monetary policy should be used to achieve output or employment goals, rather than giving priority to price stability.

The present study argues instead for a different view of the proper goals of monetary policy. The use of monetary policy to stabilize an appropriately defined price index is in fact an important end to which policy should be directed — at least to a first approximation, it should be the primary aim of monetary policy. But this is *not* — as proponents of inflation targeting sometimes argue — because variations in the rate of inflation have no real effects. Instead, it is exactly because instability of the general level of prices causes substantial real distortions — causing inefficient variation both in aggregate employment and output and in the sectoral composition of economic activity — that price stability is important.

Moreover, the existence of predictable real effects of shifts in monetary policy need not imply that policy should primarily be based on a calculation of its effects on output or employment. For the efficient aggregate level and sectoral composition of real activity is likely to vary over time, as a result of real disturbances of variety of types. The market mechanism performs a difficult computational task — much of the time, fairly accurately — in bringing about a time-varying allocation of resources that responds to these changes in production and consumption opportunities. Because of this, variation over time in employment and output relative to some smooth trend cannot in itself be taken to indicate a failure of proper market functioning. Instead, instability of the general level of prices *is* a good indicator of inefficiency in the real allocation of resources — at least when an appropriate price index is used — because a tendency of prices in general to move in the same direction (either all rising relative to their past values, or all falling) is both a cause and a symptom of systematic imbalances in resource allocation.

This general vision is in many respects an attempt to resurrect a view that was influential among monetary economists prior to the Keynesian revolution. It was perhaps best articulated by the noted Swedish economic theorist Knut Wicksell at the turn of the previous century, along with his followers in the "Stockholm school" of the interwar period (such as Erik Lindahl and Gunnar Myrdahl) and others influenced by Wicksell's work, such as Friedrich Hayek. However, these authors developed their insights without the benefit of either modern general equilibrium theory[2] or macroeconometric modelling techniques, so that it may be doubted whether Wicksellian theory can provide a basis for the kind of quantitative policy analysis in which a modern central bank must engage — and which has become only more essential given current demands for public justification of policy decisions. This book will seek to provide theoretical foundations for the view just sketched that meet modern standards of conceptual rigor, and that are capable of elaboration in a form that can be fit to economic time series.

1.1 Toward a New "Neoclassical Synthesis"

The approach to monetary policy proposed here builds upon advances in the analysis of economic fluctuations, and of the monetary transmission mechanism in particular, over the past few years.[3] The models analyzed in this volume differ in crucial respects from the first two generations of equilibrium business cycle models, namely the "New Classical" models that took Lucas (1972) as their starting point, and the "real business cycle" models pioneered by Kydland and Prescott (1982) and Plosser (1983). Neither of these early illustrations of the possibility of rigorous intertemporal general-equilibrium analysis of short-run fluctuations contained elements that would make them suitable for the analysis of monetary policy. While the Lucas model allows for real effects of unexpected variations in monetary policy (modeled as stochastic variation in the growth rate of the money supply), it implies that any real effects of monetary policy must be purely transitory, and also that monetary disturbances should have *no* real effects to the extent that their effects on aggregate nominal expenditure can be forecast in advance. Yet, as shown chapter 3, VAR evidence on the effects of identified monetary policy shocks is quite inconsistent with these predictions; instead, the effects of monetary policy shocks on aggregate nominal expenditure are forecastable at least 6 months

[2] Of course, Wicksell and his followers were quite familiar with Walrasian general equilibrium theory, and used it as a starting point for their own thought. But at the time, general equilibrium theory meant a *static* model of resource allocation, not obviously applicable to the problems of intertemporal resource allocation with which they were primarily concerned. See, for example, Myrdahl (1931, chap. 2, sec. 4, and chap. 3, sec. 5).

[3] Useful surveys of recent developments include Goodfriend and King (1997) and Gali (2001).

in advance on the basis of federal-funds rate movements, while the (similarly delayed) effects on real activity are substantial and persist for many quarters. Nor is this empirical failure of the model one of minor import for the analysis of monetary policy; the conclusion that only unanticipated monetary policy can have real effects leads fairly directly to the skeptical conclusions of Sargent and Wallace (1975) about the necessary ineffectiveness of any attempt to use monetary policy to stabilize real activity.

The real business cycle (RBC) models of the 1980s offered a very different view of the typical nature of short-run fluctuations in economic activity. But the classic models in this vein similarly imply no scope at all for monetary stabilization policy, because real variables are modeled as evolving in complete independence of any nominal variables; monetary policy is thus (at least implicitly) assumed to be of no relevance as far as fluctuations in real activity are concerned. Since neither the empirical evidence from VAR studies nor the practical experience of central bankers supports this view, we should be reluctant to discuss the nature of desirable monetary policy rules using models of this kind.

Chapters 3 and 4 review a more recent literature that has shown, instead, how models with equally rigorous foundations in intertemporal optimizing behavior can be developed that allow a more realistic account of the real effects of monetary disturbances. These models also imply that systematic monetary policy can make a substantial difference for the way that an economy responds to real disturbances of all sorts, and this is actually the prediction of the models that is of greatest importance for our concerns. VAR models typically do *not* imply that a large part of the variance of fluctuations in real activity should be attributed to monetary policy shocks — that is, to the purely random component of central-bank interest-rate policy — and in any event, one does not really need to understand exactly what the effects of such shocks are, since under almost any view it will be desirable to eliminate such shocks (*i.e.*, to render monetary policy predictable) to the extent possible. (Here, we discuss the ability of our models to account for evidence with regard to the effects of such shocks only because this is the aspect of the effects of monetary policy which can be *empirically identified* under relatively weak, and hence more convincing, identifying assumptions.) On the other hand, we are very interested in what a model implies about the way in which alternative systematic monetary policies determine the effects of real disturbances. The question of practical importance in central banking is never "should we create some random noise this month?", but rather "does this month's news justify a change in the level of interest rates?" To think about this, we need to understand the consequences of different types of possible monetary responses to exogenous disturbances.

The key to obtaining less trivial consequences of systematic monetary policy in the models proposed here is the assumption that prices and/or wages are not continually adjusted, but instead remain fixed for at least short periods (a few months, or even a year) at a level that was judged desirable at an earlier time. However, this postulate does not mean accepting the need for mechanical models of wage and price adjustment of the kind that were at the heart of the Keynesian macroeconometric models of the 1960s. Rather than postulating that prices or wages respond mechanically to some measure of market disequilibrium, they are set optimally, *i.e.*, so as to best serve the interests of the parties assumed to set them, according to the information available at the time that they are set. The delays involved before the next time that prices are reconsidered (or perhaps, before a newly chosen price takes effect) are here taken to be an institutional fact, just like the available production technology. But the resulting constraints are taken account of by the decisionmakers who set them; thus the assumed "stickiness" of prices implies that when they are reconsidered, they are set in a *forward-looking* manner, on the basis of expectations regarding future demand and cost conditions, and not simply in response to current conditions. As a result, expectations turn out to be a crucial factor in the equilibrium relation between inflation and real activity (as argued by Phelps and Friedman in the 1960s). Under certain special assumptions, described in chapter 3, the relation is of exactly the form assumed in the "New Classical" literature: deviations of output from its "natural rate" are proportional to the unexpected component of inflation. However, this is not true more generally; other models, that I would judge to be more realistic, also lead to "expectations-augmented Phillips curve" relations of a sort, but not of the precise sort that implies that anticipated monetary policy cannot have real effects.

It is also important to note that our emphasis upon nominal rigidities does not in any way mean ignoring the real factors in business fluctuations stressed by RBC theory. One important achievement of the RBC literature has been to show that the equilibrium level of output can easily be disturbed by real disturbances of many sorts — variations in the rate of technical progress, variations in government purchases, changes in tax rates, or shifts in tastes of various sorts. We shall not want to abstract from the existence of such disturbances in our models; after all, it is only the existence of real disturbances (*i.e.,* disturbances other than those originating from randomness in monetary policy itself) that gives rise to non-trivial questions about monetary policy, and we shall strive to obtain results that remain valid for as broad a class of possible disturbances as possible. Of course, the predicted effects of real disturbances will not necessarily be the same in the models presented here as in RBC theory, which, in its classic form, assumes complete flexibility of both wages and prices. Instead, in our models, the predicted effects of real disturbances will depend on the nature of monetary policy.

Nonetheless, the predicted evolution of real variables under complete wage and price flexibility — the topic studied in RBC theory — represents an important benchmark in the theory developed here. The level of output that *would* occur in an equilibrium with flexible wages and prices, given current real factors (tastes, technology, government purchases) — what we call the "natural rate" of output, following Friedman (1968) — turns out to be a highly useful concept, even if our theory does not imply that this is what the *actual* level of output will be, regardless of monetary policy. It is the gap between actual output and this natural rate, rather than the level of output as such (or output relative to trend), that is related to inflation dynamics in a properly specified Phillips-curve relation, as we show in chapter 3. It is also this concept of the output gap to which interest rates should respond if a "Taylor rule" is to be a successful approach to inflation stabilization, as we discuss in chapter 4; it is this concept of the output gap that monetary policy should aim to stabilize in order to maximize household welfare, as shown in chapter 6; and it is this concept of the output gap to which optimal interest-rate rules and/or optimal inflation targets should respond, as shown in chapter 8. From the point of view of any of these applications, the fact that the natural rate of output may vary at business-cycle frequencies, as argued in the RBC literature, is of tremendous practical importance. As will be seen, we are also quite interested in the consequences of time variation in what Wicksell (1898) called the "natural rate of interest" — the equilibrium real rate of interest in the case of flexible wages and prices, given current real factors.[4] Once again, RBC theory has a great deal to tell us about the kind of factors that should cause the natural rate of interest to vary. Hence RBC theory, when correctly interpreted, constitutes an important building block of the theory to be developed here.

It is for this reason that Goodfriend and King (1997) speak of models of this kind as representing a "new neoclassical synthesis", in the spirit of the synthesis between Keynesian short-run analysis and neoclassical long-run analysis proposed by Hicks and Samuelson. In the modern, more explicitly dynamic version of such a synthesis, the neoclassical theory (*i.e.*, RBC theory) defines not a static "long-run equilibrium" but rather a dynamic path which represents a sort of *virtual* equilibrium for the economy at each point in time — the equilibrium that one *would* have if wages and prices were not in fact sticky. The evolution of the virtual equilibrium matters because the gaps between actual quantities and their virtual equilibrium values are important measures of the incentives for wage and price adjustment, and hence determinants of wage and price dynamics.

At the same time, the stickiness of prices and/or wages implies that short-run output determination can be understood in a manner reminiscent of Keynesian theory. Indeed, our basic analytical framework in this study will have the structure of a simple model consisting of an "IS equation", a monetary policy rule, and an "AS equation". (The monetary policy rule — which we shall often suppose is something similar to a "Taylor rule" — replaces

[4] This is of course the origin of the "natural rate" terminology — Friedman's concept of a "natural rate of unemployment" appealed to an analogy with Wicksell's "natural rate of interest," a concept with which his readers were presumed already to be familiar. Nowadays, many readers will be more familiar with Friedman's concept, and will find the natural rate of interest most easy to understand as an analogy with Friedman's natural rate.

the "LM equation" of Hicksian pedagogy, since for the most part we are not here interested in the consequences of monetary targeting.) Nonetheless, even for purposes of "short-run" analysis, our model will be less static than an old-fashioned Keynesian model; in particular, *expectations* will be crucial elements in our structural relations (*e.g.*, our "intertemporal IS relation"), so that anything that causes a change in expectations should shift them.

The inclusion of significant forward-looking terms in our key structural relations will have substantial consequences for our analysis of the character of optimal policy, just as Lucas (1976) argued, even if the consequences are not necessarily the ones suggested in the "New Classical" literature. For example, estimated "IS equations" in traditional macroeconometric models often indicate an effect of *lagged* rather than current interest rates on aggregate demand, since the coefficients on lagged rates are found to be more significant than those on a current interest rate, in the case of a regression seeking to explain aggregate real expenditure in terms of observable variables. In the optimization-based model estimated by Rotemberg and Woodford (1997), instead, the observed delay in the effects of an interest-rate innovation on real GDP is explained by an assumption that the interest-sensitive component of private spending is predetermined, though chosen in a forward-looking way. Thus current aggregate demand is assumed to depend on *past expectations* of *current and future* interest rates, rather than past interest rates.

Econometrically, the two hypotheses are not easily distinguished, given the substantial serial correlation of observed interest rates; yet the second hypothesis, I would argue, has a much simpler logic in terms of the optimal timing of expenditure, once one grants the hypothesis of predetermination of spending decisions (just as with pricing decisions). And the specification assumed matters greatly for one's conclusions about the conduct of policy. If expenditure is really affected by lagged interest rates only, it becomes important for the central bank to adjust interest rates in response to its *forecast* of how it would like to affect aggregate demand at a later date; "pre-emptive" actions will be essential. If instead only past expectations of current and future interest rates matter, then *unforecastable* interest rate movements will not affect demand, so that immediate responses to news will serve no purpose; it will instead be important for interest rates to continue to respond to the outlook that *had* been perceived in the past, even if more recent news has substantially modified the bank's forecasts. This *inertial* character of optimal interest-rate policy is discussed further in chapters 7 and 8.

1.2 Microeconomic Foundations and Policy Analysis

The development of a model of the monetary transmission mechanism with clear foundations in individual optimization is important, in our view, for two reasons. One is that it allows us to evaluate alternative monetary policies in a way that avoids the flaw in policy evaluation exercises using traditional Keynesian macroeconometric models stressed by Lucas (1976). Another is that the outcomes resulting from alternative policies can be evaluated in terms of the preferences of private individuals that are reflected in the structural relations of one's model.

Lucas (1976) argued that traditional policy evaluation exercises using macroeconometric models were flawed by a failure to recognize that the relations typically estimated — a "consumption equation," a "price equation," and so on — were actually (at least under the hypothesis of optimizing behavior by households and firms) reduced-form rather than truly structural relations. In particular, in the estimated equations, expectations regarding future conditions (future income in the case of consumers, future costs and future demand in the case of price-setters) were proxied for by current and lagged observable state variables; but the correlation of expectations with those observables ought to be expected to change in the case of a change in the government's policy rule, as contemplated in the policy evaluation exercise.

This problem can be addressed by making use of structural relations that explicitly represent the dependence of economic decisions upon expectations regarding future endogenous variables. The present study illustrates how this can be done, deriving the structural relations that are to be used in the calculation of optimal policy rules from the first-order conditions (Euler equations) that characterize optimal private-sector behavior. These conditions explicitly involve private-sector expectations about the future evolution of endogenous variables, and often they only *implicitly* define private-sector behavior, rather than giving a "consumption equation" or "price equation" in closed form. Our preference for this form of structural relations is precisely that they are ones that should remain invariant (insofar as our theory is correct) under changes in policy that change the stochastic laws of motion of the endogenous variables.

Of course, the mere fact that the structural relations derived here follow from explicit optimization problems for households and firms is no guarantee that they are correctly specified; the (fairly simple) optimization problems that we consider here may or may not be empirically realistic. (Indeed, insofar as we illustrate the principles of our approach in the context of very simple examples, one can be certain that they are not very precise representations of reality.) But this is not an objection to the method that we advocate here; it simply means that there is no substitute for careful empirical research to flesh out the details of a quantitatively realistic account of the monetary transmission mechanism. While the present study does include some discussion of the extent to which the simple models presented here are consistent with empirical evidence, in order to motivate the introduction of certain model elements, no attempt is made here to set out a model that is sufficiently realistic to be used for actual policy analysis in a central bank. Nonetheless, the basic elements of an optimizing model of the monetary transmission mechanism, developed in chapters 3 and 4 of this book, are ones that we believe are representative of crucial elements of a realistic model; and indeed, the illustrative models discussed here have many elements in common with rational-expectations models of the monetary transmission mechanism that are already being used for quantitative policy evaluation at a number of central banks.

A second advantage of proceeding from explicit microeconomic foundations is that in this case, the welfare of private agents — as indicated by the utility functions that underly the structural relations of one's model of the transmission mechanism — provides a natural objective in terms of which alternative policies should be evaluated. In taking this approach, the present study seeks to treat questions of monetary policy in a way that is already standard in other branches of public economics, such as the analysis of optimal tax policy. Nonetheless, the approach has not been common in the literature on monetary policy evaluation, which instead typically evaluates alternative policies in terms of *ad hoc* stabilization objectives for various macroeconomic indicators.

Until recently, welfare-theoretic analyses of monetary policy have been associated exclusively with the problem of reducing the transactions frictions (sometimes called "shoe-leather costs") that account for the use of money in purchases.[5] This is because this was for a long time the only sort of inefficiency present in general-equilibrium monetary models, which typically assumed perfectly flexible wages and prices and perfect competition. Here we show how welfare analysis of monetary policy is also possible in settings that incorporate nominal rigidities. Allowing for these additional frictions — crucial to understanding the real effects of alternative monetary policies — provides a welfare-theoretic justification for additional policy goals.

As shown in chapter 6, taking account of delays in the adjustment of wages and prices provides a clear justification for an approach to monetary policy that aims at price stability. It might seem more obvious that allowing for real effects of monetary policy provides a justification for concern with output stabilization. The stickiness of prices explains why actual output may differ from the "natural rate", and so justifies a concern for the stabilization of the "output gap", *i.e.,* the discrepancy between the actual and natural levels of output. But price stickiness also justifies a concern with price stability. For when prices are not constantly adjusted, instability of the general level of prices creates discrepancies between relative prices owing to the absence of perfect synchronization in the adjustment of the prices of different goods. These relative-price distortions lead in turn to an inefficient sectoral allocation of resources, even when the *aggregate* level of output is correct.

Moreover, our theory implies not only that price stability should matter *in addition* to stability of the output gap, but also that, at least under certain circumstances, inflation stabilization eliminates any need for further concern with the level of real activity. This is

[5]For reviews of that traditional literature, see Woodford (1990) and Chari and Kehoe (1999).

because, at least under the conditions described more precisely in chapter 6, the time-varying efficient level of output is the *same* (up to a constant, which does not affect the basic point) as the level of output that eliminates any incentive for firms on average to either raise or lower their prices. It then follows that there is no conflict between the goal of inflation stabilization and output-gap stabilization, once the welfare-relevant concept of the output gap is properly understood. Furthermore, because of the difficulty involved in measuring the efficient level of economic activity in real time — depending as this does on variations in production costs, consumption needs, and investment opportunities — it may well be more convenient for a central bank to simply concern itself with monitoring the stability of prices.

The development of an explicit welfare analysis of the distortions resulting from inflation variations has advantages beyond the mere provision of a justification for central bankers' current concern with inflation stabilization. For the theory presented here also provides guidance as to *which* price index it is most desirable to stabilize. This is a question of no small practical interest. For example, the stock-market booms and crashes in many industrial nations in the late 1990s led to discussion of whether central banks ought not target an inflation measure that took account of "asset price inflation" as well as goods prices.[6]

The answer provided by the theory developed here is *no.* The prices that monetary policy should aim to stabilize are the ones that are infrequently adjusted, and that consequently can be expected to become misaligned in an environment that requires these prices to move in either direction. Large movements in frequently adjusted prices — and stock prices are among the most flexible of prices — can instead be allowed without raising such concerns, and if allowing them to move makes possible greater stability of the sticky prices, such instability of the flexible prices is desirable.[7] In chapter 6, we show how such a conclusion

[6] For examples of scholarly attention to the question, see Goodhart (20xx) and Cecchetti (2002).

[7] The basic point was already evident to authors of the Stockholm school: "If one desires the greatest possible diminution of the business cycle, ... then one must try to stabilize an index of those prices which are sticky in themselves... Stability of the level of the sticky prices permits a certain freedom for all other price levels, including capital values.... It is evident that [the price of capital goods] is the last price that one should try to stabilize in a capitalist society.... The same is naturally true for all indices of flexible commodity prices" (Myrdal, 1931, pp. 192-193).

can be justified from the point of view of welfare economics. We further show how to develop a quantitative measure of the deadweight loss resulting from stabilization of alternative price indices, so that more subtle distinctions between the relative stickiness of different prices can be dealt with.

In addition to implying that an appropriate inflation target ought not involve asset prices, our theory suggests that not all goods prices are equally relevant. Instead, central banks should target a measure of "core" inflation that places greater weight on those prices that are stickier. Furthermore, insofar as wages are also sticky, a desirable inflation target should take account of wage inflation as well as goods prices. The empirical results discussed in chapter 3 suggest that wages and prices are sticky to a similar extent, suggesting (as we show in chapter 8) that a desirable inflation target should put roughly equal weight on wage and price inflation.

2 The Importance of Policy Commitment

Thus far, we have summarized a theoretical justification for the concern of the inflation-targeting central banks with price stability. But why should it follow that there is a need for public commitment to a target inflation rate, let alone for commitment to a systematic procedure for determining appropriate instrument settings? Why is it not enough to appoint central bankers with a sound understanding of the way the economy works, and then grant them complete discretion to pursue the public interest in the way that they judge best? Should it not follow from our analysis that this would result in price stability, to the extent that this is possible given the instruments available to the central bank and the information available at the time that policy decisions must be made?

We shall argue instead that there is good reason for a central bank to commit itself to a systematic approach to policy, that not only provides an explicit framework for decision-making within the bank, but that is also used to explain the bank's decisions to the public. There are two important advantages of commitment to an appropriately chosen *policy rule* of this kind. One is that the effectiveness of monetary policy depends as much on the public's expectations about future policy as upon the bank's actual actions. Hence a bank must not only manage to make the right decision as often as possible; it is also important that its actions be *predictable*.

The second, and subtler, reason is that even if the public has no difficulty in correctly perceiving the pattern in the central bank's actions — as assumed under the hypothesis of rational expectations — if a bank acts at each date under the assumption that it cannot commit itself to any future behavior (and is not bound by any past commitments), it will choose a systematic pattern of behavior that is suboptimal. We take up each of these arguments in turn.

2.1 Central Banking as Management of Expectations

The first advantage of commitment to a policy rule is that it facilitates public understanding of policy. It is important for the public to understand the central bank's actions, to the greatest extent possible, not only for reasons of democratic legitimacy — though this is an excellent reason itself, given that central bankers are granted substantial autonomy in the execution of their task — but also in order for monetary policy to be most effective.

For successful monetary policy is not so much a matter of effective control of overnight interest rates as it is one of shaping market *expectations* of the way in which interest rates, inflation and income are likely to evolve over the coming year and later. On the one hand, optimizing models imply that private sector behavior should be forward-looking; hence expectations about future market conditions should be important determinants of current behavior. It follows that, insofar as it is possible for the central bank to affect expectations, this should be an important tool of stabilization policy. And given the increasing sophistication of market participants about central banking over the past two decades, it is plausible to suppose that a central bank's commitment to a systematic policy will be factored into private sector forecasts — at least insofar as the bank's actions are observed to match its professed commitments.

Not only do expectations about policy matter, but, at least under current conditions, very little *else* matters. Few central banks of major industrial nations still make much use of credit controls or other attempts to directly regulate the flow of funds through financial markets and institutions. Increases in the sophistication of the financial system have made it more difficult for such controls to be effective, and in any event the goal of improvement of the efficiency of the sectoral allocation of resources stressed above would hardly be served by such controls, which (if successful) inevitably create inefficient distortions in the relative cost of funds to different parts of the economy.

Instead, banks restrict themselves to interventions that seek to control the overnight interest rate in an interbank market for central-bank balances (for example, the federal funds rate in the U.S.). But the current level of overnight interest rates *as such* is of negligible importance for economic decisionmaking; if a change in the overnight rate were thought to imply only a change in the cost of overnight borrowing for that one night, then even a large change (say, a full percentage point increase) would make little difference to anyone's spending decisions. The effectiveness of changes in central-bank targets for overnight rates in affecting spending decisions (and hence ultimately pricing and employment decisions) is wholly dependent upon the impact of such actions upon other financial-market prices, such as longer-term interest rates, equity prices and exchange rates. These are plausibly linked, through arbitrage relations, to the short-term interest rates most directly affected by central-bank actions; but it is the expected future path of short-term rates over coming months and even years that should matter for the determination of these other asset prices, rather than the current level of short-term rates by itself.[8]

Thus the ability of central banks to influence expenditure, and hence pricing, decisions is critically dependent upon their ability to influence market expectations regarding the *future path* of overnight interest rates, and not merely their current level. Better information on

[8]An effect of the same kind is obtained in the basic "neo-Wicksellian" model developed in chapter 4, insofar as the short-run real rate of interest determines not the absolute level of desired private-sector expenditure, but rather the current level relative to the expected future level of expenditure, as a result of an Euler equation for the optimal timing of expenditure. Expected future expenditure, relative to expected expenditure even farther in the future, similarly depends upon expected future short rates, and so on for expectations regarding the still farther future.

the part of market participants about central-bank actions and intentions should increase the degree to which central-bank policy decisions can actually affect these expectations, and so increase the effectiveness of monetary stabilization policy. Insofar as the significance of current developments for future policy are clear to the private sector, markets can to a large extent "do the central bank's work for it," in that the actual changes in overnight rates required to achieve the desired changes in incentives can be much more modest when expected future rates move as well.[9]

An obvious consequence of the importance of managing expectations is that a transparent central-bank decisionmaking process is highly desirable. This has come to be widely accepted by central bankers over the past decade. (See Blinder *et al.*, 2001, for a detailed and authoritative discussion.) But it is sometimes supposed that the most crucial issues are ones such as the frequency of press releases or the promptness and detail with which the minutes of policy deliberations are published. Instead, from the perspective suggested here, what is important is not so much that the central bank's deliberations themselves be public, as that the bank give clear signals about what the public should expect it to do in the future. The public needs to have as clear as possible an understanding of the *rule* that the central bank follows in deciding what it does. Inevitably, the best way to communicate about this will be by offering the public an explanation of the decisions that have already been made; the bank itself would probably not be able to describe how it might act in all conceivable circumstances, most of which will never arise.

Some good practical examples of communication with the public about the central bank's policy commitments are provided by the *Inflation Reports* of the leading inflation-targeting banks. These reports do not pretend to give a blow-by-blow account of the deliberations by

[9]There is evidence that this is already happening, as a result both of greater sophistication on the part of financial markets and greater transparency on the part of central banks, the two developing in a sort of symbiosis with one another. Blinder *et al.* (2001, p. 8) argue that in the period from early 1996 through the middle of 1999, one could observe the U.S. bond market moving in response to macroeconomic developments that helped to stabilize the economy, despite relatively little change in the level of the federal funds rate, and suggest that this reflected an improvement in the bond market's ability to forecast Fed actions before they occur. Statistical evidence of increased forecastability of Fed policy by the markets is provided by Lange *et al.* (2001), who show that the ability of Treasury bill yields to predict changes in the federal funds rate some months in advance has increased since the late 1980s.

which the central bank reached the position that it has determined to announce; but they do explain the *analysis* that justifies the position that has been reached. This analysis provides information about the bank's systematic approach to policy by illustrating its application to the concrete circumstances that have arisen since the last report; and it provides information about how conditions are likely to develop in the future through explicit discussion of the bank's own projections. Because the analysis is made public, it can be expected to shape future deliberations; the bank knows that it should be expected to explain why views expressed in the past are not later being followed. Thus a commitment to transparency of this sort helps to make policy more fully rule-based, as well as increasing the public's understanding of the rule.

It is perhaps worth clarifying further what we intend by "rule-based" policy. We do not mean that a bank should commit itself to an explicit state-contingent plan for the entire foreseeable future, specifying what it would do under every circumstance that might possibly arise. That would obviously be impractical, even under complete unanimity about the correct model of the economy and the objectives of policy, simply because of the vast number of possible futures. But it is not necessary, in order to obtain the benefits of commitment to a systematic policy. It suffices that a central bank commit itself to a systematic way of determining an appropriate response to future developments, without having to list all of the implications of the rule for possible future developments.[10]

Nor is it necessary to imagine that commitment to a systematic rule means that once a rule is adopted it must be followed forever, regardless of subsequent improvements in understanding of the effects of monetary policy on the economy, including experience with the consequences of implementing the rule. If the private sector is forward-looking, and it is possible for the central bank to make the private sector aware of its policy commitments, then there *are* important advantages of commitment to a policy other than discretionary

[10]We show in chapter 8 how policy rules can be designed that can be specified without any reference to particular economic disturbances, but that nonetheless imply an optimal equilibrium response to additive disturbances of an arbitrary type. The targeting rules advocated by Svensson (2001) are examples of rules of this kind.

optimization — *i.e.*, simply doing what seems best at each point in time, with no commitment regarding what may be done later. This is because there are advantages to having the private sector be able to anticipate *delayed* responses to a disturbance, that may not be optimal *ex post* if one re-optimizes taking the private sector's past reaction as given. But one can create the desired anticipations of subsequent behavior — and justify them — without committing to follow a fixed rule in the future no matter what may happen in the meantime.

It suffices that the private sector have no ground to forecast that the bank's behavior will be *systematically* different from the rule that it pretends to follow. This will be the case if the bank is committed to choosing a rule of conduct that is justifiable on certain *principles*, given its model of the economy. (An example of the sort of principles that I have in mind is given in chapter 8.) The bank can then properly be expected to continue to follow its current rule, as long as its understanding of the economy does not change; and as long as there is no *predictable* direction in which its future model of the economy should be different from its current one, private-sector expectations should not be different from those in the case of an indefinite commitment to the current rule. Yet changing to a better rule will remain possible in the case of improved knowledge (which is inevitable); and insofar as the change is justified both in terms of established principles and in terms of a change in the bank's model of the economy that can itself be defended, this need not impair the credibility of the bank's professed commitments.

It follows that rule-based policymaking will necessarily mean a decision process in which an explicit *model* of the economy (albeit one augmented by judgmental elements) plays a central role, both in the deliberations of the policy committee and in explanation of those deliberations to the public. This too has been a prominent feature of recent innovations in the conduct of monetary by the inflation-targeting central banks. While there is undoubtedly much room for improvement both in current models and current approaches to the use of models in policy deliberations, one can only expect the importance of models to policy deliberations to increase in a world of increasingly sophisticated financial markets.

2.2 Pitfalls of Conventional Optimal Control

But it is not enough that a central bank have sound objectives (reflecting a correct analysis of social welfare), that it make policy in a systematic way, using a correct model of the economy and a staff that is well-trained in numerical optimization, and that all this be explained thoroughly to the public. A bank that approaches its problem as one of optimization under *discretion* — deciding afresh on the best action in each decision cycle, with no commitment regarding future actions except that they will be the ones that seem best in whatever circumstances may arise — may obtain a substantially worse outcome, from the point of view of its own objectives, than one that commits itself to follow a properly chosen policy *rule*. As Kydland and Prescott (1977) first showed, this can occur even when the central bank as a correct quantitative model of the policy tradeoffs that it faces at each point in time, and the private sector has correct expectations about the way that policy will be conducted.

At first thought, discretionary optimization might seem exactly what one would want an enlightened central bank to do. All sorts of unexpected events constantly occur that affect the determination of inflation and real activity, and it is not hard to see that, in general, the optimal level of interest rates at any point in time should depend on precisely what has occurred. It is plainly easiest, as a practical matter, to arrange for such complex state-dependence of policy by having the instrument setting at a given point in time be determined only after the unexpected shocks have already been observed. Furthermore, it might seem that the dynamic programming approach to the solution of intertemporal optimization problems provides justification for an approach in which a planning problem is reduced to a series of independent choices at each of a succession of decision dates.

But standard dynamic programming methods are valid only for the optimal control of a system that evolves mechanically in response to the current action of the controller, as in the kind of industrial problems of typical interest in engineering control theory. The problem of monetary stabilization policy is of a different sort, in that the consequences of the central bank's actions depend not only upon the sequence of instrument settings up until the present time, but also upon private-sector expectations regarding future policy. In such a case, sequential (discretionary) optimization leads to a sub-optimal outcome because at each decision point, prior expectations are taken as *given*, rather than as something that can be affected by policy. Nonetheless, the predictable character of the central bank's decisions, taken from this point of view, do determine the (endogenous) expectations of the private sector at earlier dates, under the hypothesis of rational expectations; a commitment to behave differently, that is made credible to the private sector, could shape those expectations in a different way, and because expectations matter for the determination of the variables that the central bank cares about, in general outcomes can be improved through shrewd use of this opportunity.

The best-known example of a distortion created by discretionary optimization is the "inflation bias" analyzed by Kydland and Prescott (1977) and Barro and Gordon (1983). In the presence of a short-run "Phillips curve" tradeoff between inflation and real activity (given inflation expectations), and a target level of real activity higher than the one associated with an optimal inflation rate (in the case of inflation expectations also consistent with that optimal rate), these authors showed that discretionary optimization leads to a rate of inflation that is inefficiently high on average, owing to neglect of the way that pursuit of such a policy raises inflation expectations (causing an adverse shift of the short-run Phillips curve). A variety of solutions to the problem of inflation bias have been proposed. One influential idea is that this bias can be eliminated by assigning the central bank targets for inflation and output that differ from those reflected in the true social welfare function (*i.e.,* the central-bank objective assumed by Kydland and Prescott or Barro and Gordon), without otherwise constraining the central bank's discretion in the selection of policies to achieve its objective. This is one of the primary reasons for the popularity of "inflation targeting", which involves commitment of a central bank to the pursuit of an assigned target rather than being left to simply act as seems best for society at any point in time, while leaving the bank a great deal of flexibility as to the way in which the assigned goal is to be pursued.

However, the distortions resulting from discretionary optimization go beyond simple bias in the *average* levels of inflation or other endogenous variables; this approach to the conduct of policy generally results in suboptimal responses to shocks as well, as shown in chapter 7. For example, various types of real disturbances can create temporary fluctuations in what Wicksell called the "natural rate of interest", meaning (as shown in chapter 4) that the level of nominal interest rates required to stabilize both inflation and the output gap varies over time. However, the amplitude of the adjustment of short-term interest rates can be more moderate — and still have the desired size of effect on spending and hence on both output and inflation — if it is made more *persistent,* so that when interest rates are increased, they will not be expected to quickly return to their normal level, even if the real disturbance that originally justified the adjustment has dissipated. Because aggregate demand depends upon expected future short rates as well as current short rates, a more persistent increase of smaller amplitude can have an equal affect on spending. If one also cares about reducing the volatility of short-term interest rates, a more inertial interest-rate policy of this kind will be preferable; that is, the anticipation that the central bank will follow such a policy leads to a preferable rational-expectations equilibrium. But a central bank that optimizes under discretion has no incentive to continue to maintain interest rates high once the initial shock has dissipated; at this point, prior demand has already responded to whatever interest-rate expectations were held then, and the bank has no reason to take into account any effect upon demand at an earlier date in setting its current interest-rate target.

This distortion in the dynamic response of interest-rate policy to disturbances cannot be cured by any adjustment of the targets that the bank is directed to aim for regardless of what disturbances may occur; instead, policy must be made *history-dependent, i.e.,* dependent upon past conditions even when they are no longer relevant to the determination of the current and future evolution of the variables that the bank cares about. Indeed, in general no *purely forward-looking* decision procedure — one that makes the bank's action at each decision point a function solely of the set of possible paths for its target variables from that time onward — can bring about optimal equilibrium responses to disturbances. Discretionary optimization is an example of such a procedure, and it continues to be when the bank's objective is modified, if the modification does not introduce any history-dependence. But other popular proposals are often purely forward-looking as well. Thus the classic "Taylor rule" (Taylor, 1993) prescribes setting an interest-rate operating target at each decision point as a function of current estimates of inflation and the output gap only (see below), and Taylor (1999) expresses skepticism about the desirability of partial-adjustment dynamics of the kind that characterize most estimated central-bank reaction functions. Popular descriptions of inflation-forecast targeting are typically purely forward-looking as well; the interest-rate setting at each decision point is to be determined purely as a function of the forecast from that date forward for inflation (and possibly other target variables). Thus the intuition that optimal policy *should* be purely forward-looking seems to be fairly commonplace; but when the private sector is forward-looking, any purely forward-looking criterion for policy is almost invariably sub-optimal.

Obtaining a more desirable pattern of responses to random disturbances therefore requires commitment to a systematic policy rule, and not just a (one-time) adjustment of the bank's targets. The primary task of this study is to provide principles that can be used in the design of such rules. By saying that a *policy rule* is necessary, we mean to draw a distinction with two other conceptions of optimal policy. One is discretionary optimization, as just discussed; specifying a rule means a more detailed description of the way in which a decision is to be reached than is involved in a simple commitment to a particular objective. But we also mean to distinguish the approach advocated here from the usual understanding of what an *optimal commitment* involves.

In the literature that contrasts policy commitment with discretionary policymaking, following Kydland and Prescott, "commitment" is generally taken to mean a specification, once and for all, of the state-contingent action to be taken at each subsequent date. An optimal commitment is then a choice of such a state-contingent plan so as to maximize the ex-ante expected value of the policymaker's objective, as evaluated at the initial date t_0 at which the commitment is chosen. This leads to a description of optimal policy in terms of a specification of the instrument setting as a function of the history of exogenous shocks since date t_0.

But the solution to such an optimization problem is not an appealing policy recommendation in practice. For it is generally not *time consistent* — solving the same optimization problem at a later date t_1, to determine the optimal commitment from *that* date onward, will not result in a state-contingent plan from date t_1 onward that continues the plan judged to be optimal at date t_0. This is because the commitment chosen at date t_0 will take account of the consequences of the commitments made for dates t_1 and later for expectations between dates t_0 and t_1, while at date t_1 these expectations will be taken as historical facts that cannot be changed by the policy chosen from then on. (This is just the reason why discretionary optimization does not lead to the same policy as an optimal commitment.) Hence this policy proposal cannot be regarded as proposing a decision procedure that can be used at each date to determine the best action at that date; instead, a state-contingent plan must be determined once and for all, for the rest of time, and thereafter simply implemented, whether it continues to appear desirable or not.

Such a proposal is not a practical one, for two reasons. First, enumeration in advance of all of the possible subsequent histories of shocks will not be feasible — the kinds of situations that the central bank may face at a given date cannot are too various to possibly be listed in advance. And second, the arbitrariness of continuing to stick to a particular specified policy simply because it looked good at a particular past date — the date t_0 at which one happened to make the commitment — is sufficiently unappealing that one cannot imagine a central bank binding itself to behave in this way, or the private sector believing that it had. Here my argument is not that central bankers are *incapable* of commitment to a systematic rule of conduct, so that they are inevitably discretionary optimizers; it is rather that their commitment must be based upon an understanding of the rational justification of the rule, rather than the mere fact that it happens to been chosen (even by themselves) on a past occasion.

Both problems can be avoided by commitment to a systematic rule for determining their policy action at each decision point, that does not reduce to a once-and-for-all specification of the instrument setting as a function of the history of shocks. In chapter 8, it is shown that one can design rules for setting the central bank's interest-rate operating target that lead to optimal dynamic responses to shocks, without the rule specification having to refer to the various disturbances that may have occurred. The disturbances affect the instrument setting, of course; but they affect it either as a result of having affected endogenous variables, such as inflation and output, to which the instrument setting responds, or as a result of being factored into the central bank's projections of the future evolution of the economy under alternative possible instrument settings. Such a rule can result in optimal equilibrium responses to disturbances of any of a vast number of possible types, so that the potential disturbances need not even be listed in advance in order to describe the rule and evaluate its desirability.

The optimal rules derived in accordance with the principles set out in chapter 8 are also time-invariant in form. This means that the optimal rule that would be derived at date t_0, on the basis of a particular structural model of the monetary transmission mechanism and a particular understanding of the central bank's stabilization objectives, will also be derived at date t_1, assuming that the bank's model and objectives remain the same. A commitment to conduct policy in accordance with a rule that is judged optimal on this criterion is thus *time consistent,* in the sense that reconsideration of the matter at a later date on the basis of the same principle will lead to a decision to continue the same course of action as had been intended earlier.[11]

Because of this, adherence to a policy rule need not be taken to mean adoption of a rule at some initial date, after which the rule is followed blindly, without ever again considering its desirability. Instead, *rule-based policymaking* as the term is intended here means that at each decision point an action is taken which conforms to a policy rule, which rule is itself one that is judged to be optimal (from a "timeless perspective" that is made precise in chapter 8) given the central bank's understanding of the monetary transmission mechanism

[11]Note that "time consistency" in the sense that we use the term here does *not* mean that the policymaker does not believe at any time that it is possible to achieve a higher expected value for its objective by deviating from its intended rule. Time consistency does not require this, because this is not the criterion according to which the central bank's action is judged to be optimal at *any* time, including the initial date t_0.

at the time that the decision is made. The desire to follow a rule (and so to avoid the trap of discretionary optimization) does not mean that the bank must refrain from asking itself whether adherence to the rule is consistent with its stabilization objectives. It simply means that whenever this question is taken up, the bank should consider what an optimal *rule* of conduct would be, rather than asking what an optimal *action* is on the individual occasion, and that it should consider the desirability of alternative rules from an impartial perspective that does not amount to simply finding a rationalization for the action that it would like to take on this particular occasion.[12] A central bank might reconsider this question as often as it likes, without this leading it into the kind of sub-optimal behavior that results from discretionary optimization. And when considering the desirability of a policy rule, it is correct for the bank to consider the effects of its being expected to follow the rule indefinitely, even though it does not contemplate *binding* itself to do so; for as long as its view of the policy problem does not change (which it has no reason to expect), a commitment to rule-based policymaking should guarantee that it will continue to act according to the rule judged to be optimal.

Rule-based policymaking in this sense avoids the sorts of rigidity that are often associated with commitment to a "rule", and that probably account for much of the resistance that central bankers often display toward the concept of a policy rule. A commitment to rule-based policymaking does not preclude taking account of all of the information, from whatever sources, that the central bank may have about current economic conditions, including the recognition that disturbances may have occurred that would not have been thought possible a few months earlier. For a policy rule need not specify the instrument setting as a function of a specified list of exogenous states, and indeed it is argued in chapter 8 that an optimal rule should in general *not* take this form. Nor does it preclude changing the form of the

[12] The distinction between these two perspectives is similar to the distinction that is made in ethical theory between "rule utilitarianism" and "act utilitarianism" (Brandt, 1959; Harsanyi, 1982). Act utilitarianism is the view that the right act on any occasion is the one that will maximize social utility in the situation that the actor is in at that time. Rule utilitarianism instead maintains that a right act is one that conforms to the correct rule for this sort of situation, where a correct rule is one that would maximize social utility if *always* followed in all situations of this type.

policy rule when the bank's view of the monetary transmission changes, as it surely will, owing both to institutional change in economies themselves and to the progress of knowledge in economics. Hence it allows the sort of flexibility that is often associated with the term "discretion", while at the same time eliminating the systematic biases that follow from policy analysis that naively applies dynamic-programming principles.

3 Monetary Policy without Control of a Monetary Aggregate

Thus far we have discussed the desirability of a monetary policy rule without saying much about the precise form of rule that is intended. To be more concrete, the present study considers the design of a rule to be used in determining a central bank's operating target for a short-term nominal interest rate. This target will ordinarily be revised at intervals of perhaps once a month (as at the ECB) or eight times a year (as in the U.S.).[13]

Our focus on the choice of an *interest-rate rule* should not surprise readers familiar with the current practice of central banks. Monetary policy decisionmaking almost everywhere means a decision about the operating target for an overnight interest rate, and the increased transparency about policy in recent years has almost meant greater explicitness about the central bank's interest-rate target and about the way in which its interest-rate decisions are made. In such a context, it is natural that adoption of a policy rule should mean commitment to a specific procedure for deciding what interest-rate target is appropriate.

Nonetheless, theoretical analyses of monetary policy have until recently almost invariably characterized policy in terms of a path for the money supply, and discussions of policy rules in the theoretical literature have mainly considered money-growth rules of one type or another. This curious disjunction between theory and practice predates the enthusiasm of the 1970s for monetary targets. Goodhart (1989) complains of "an unhelpful dichotomy, between the

[13] The question of the optimal frequency of reconsideration of the interest-rate target is one of obvious practical interest. But we shall not take it up in this study, as we consider optimal policy in the context of a discrete-time model of the transmission mechanism with "periods" corresponding to the length of the central bank's decision cycle.

theory and the reality of Central Bank operations" that equally characterized the work of John Maynard Keynes and Milton Friedman.

> When either of these two great economists would discuss practical policy matters concerning the level of short-term interest rates, they had no doubts that these were normally determined by the authorities, and could be changed by them, and were not freely determined in the market.... But when they came to their more theoretical papers, they often reverted to the assumption that the Central Bank sets the nominal money stock, or alternatively fixes the level of the monetary base, [with] the demand and supply of money ... equilibrated in the short run ... by market-led developments in nominal interest rates (pp. 330-331).

The present study instead seeks to revive the earlier approach of Knut Wicksell, and considers the advantages of systematic monetary policies that are described in terms of rules for setting a nominal interest rate. While the implied evolution of the money supply is sometimes discussed, the question is often ignored; some of the time, we shall not bother to specify policy (or our economic model) in sufficient detail to determine the associated path of the money supply, or even to tell if one can be uniquely determined in principle. Some readers may fear as a result that we consider an ill-posed question — that the "policy rules" that we study may not represent sufficiently complete descriptions of policy to allow its consequences to be determined, or may not represent states of affairs that the central bank is able to bring about. Hence some general remarks may be appropriate about why it is possible to conceive of the problem of monetary policy as a problem of interest-rate policy, before turning to examples of the specific types of interest-rate rules that we wish to consider.

3.1 Implementing Interest-Rate Policy

An argument that is sometimes made for specifying monetary policy in terms of a rule for base-money growth rather than an interest-rate rule is that central banks do not actually fix overnight interest rates. Even when banks have an operating target for the overnight rate, they typically seek to implement it through open-market operations in Treasury securities or their equivalent — that is, by adjusting the supply of central-bank liabilities to a level that is expected to cause the market for overnight cash to clear near the target rate. Thus it may be argued that the action that the central bank actually takes each day is an adjustment of the nominal magnitude of the monetary base, so that a complete specification of policy should describe the size of this adjustment each day.

But even when banks implement their interest-rate targets entirely through quantity adjustments, as is largely correct as a description of current U.S. arrangements, this conclusion hardly follows. Central banks like the U.S. Federal Reserve determine their quantity adjustments through a two-step procedure: first the interest-rate target is determined by a monetary policy committee (the Federal Open Market Committee in the U.S.) without consideration of the size of the implied open-market operations, and then the appropriate daily open-market operations required to maintain the funds rate near the target are determined by people closer to the financial markets (mainly the Trading Desk at the New York Fed). The higher-level policy decision about the interest-rate target is the more complicated one, made much less frequently because of the complexity of the deliberations involved,[14] and it is accordingly this decision with which the present study is concerned.

Nor is it the case that a central bank's interest-rate target *must* be implemented through choice of an appropriate supply of central-bank liabilities. A central bank can also influence the interest rate at which banks will lend overnight cash to one another through adjustment of the interest rate paid on overnight balances held at the central bank and/or the interest rate at which the central bank is willing to lend overnight cash to banks that run overdrafts on their clearing accounts at the central bank. These are important policy tools outside the U.S., and in some countries are the primary means through which the central bank implements its interest-rate targets.

As is discussed in more detail in Woodford (2001xx), countries like Canada, Australia

[14]The comparative simplicity of the decision about each day's open-market operation is not so much because each day's demand for Fed balances is highly predictable as because the Fed learns immediately how much it has misjudged market demand each day, and can act the following day in response to the previous day's gap between the actual funds rate and the target rate. Owing to intertemporal substitution in the demand for reserves under U.S. regulations, a credible commitment by the Fed to respond the following day is enough to keep the funds rate from deviating too much from the target most of the time. See Taylor (2000) for further discussion of the Trading Desk's reaction function.

and New Zealand now implement monetary policy through a "channel system". In a system of this kind, the overnight interest rate is kept near the central bank's target rate through the provision of standing facilities by the central bank, with interest rates determined by the central bank's current target interest rate $\bar{\imath}_t$. In addition to supplying a certain aggregate quantity of clearing balances (adjusted through open-market operations), the central bank offers a lending facility, through which it stands ready to supply an arbitrary amount of additional overnight balances at an interest rate determined by a fixed spread over the target rate (*i.e.*, $i_t^l = \bar{\imath}_t + \delta$). In the countries just mentioned, the spread δ is generally equal to 25 basis points, regardless of the level of the target rate. Finally, depository institutions that settle payments through the central bank also have the right to maintain excess clearing balances overnight with the central bank at a deposit rate $i_t^d = \bar{\imath}_t - \delta$, where δ is the same fixed spread.

The lending rate on the one hand and the deposit rate on the other then define a channel within which overnight interest rates should be contained.[15] Because these are both standing facilities (unlike the Fed's discount window in the U.S.), no bank has any reason to pay another bank a higher rate for overnight cash than the rate at which it could borrow from the central bank; similarly, no bank has any reason to lend overnight cash at a rate lower than the rate at which it can deposit with the central bank. The result is that the central bank can control overnight interest rates within a fairly tight range regardless of what the aggregate supply of clearing balances may be; frequent quantity adjustments accordingly become less important.

Woodford (2001xx) describes a simple model of overnight interest-rate determination under such a system. In this model, the daily demand for clearing balances by depository institutions depends only on the location of the interbank market rate relative to the channel

[15] It is arguable that the actual lower bound is somewhat above the deposit rate, because of the convenience and lack of credit risk associated with the deposit facility, and similarly that the actual upper bound is slightly above the lending rate, because of the collateral requirements and possible stigma associated with the lending facility. Nonetheless, market rates are observed to stay within the channel established by these rates (except for occasional slight breaches of the upper bound during the early months of operation of Canada's system — see Figure 1.1), and typically near its center.

established by the two standing facilities, rather than on the absolute level of this interest rate. The interbank market then clears at an interest rate

$$i_t = i_t^d + F\left(-\frac{S_t}{\sigma_t}\right)\,(i_t^l - i_t^d), \tag{3.1}$$

where S_t is the aggregate supply of clearing balances (determined by the central bank's open-market operations), σ_t is a factor measuring the degree of uncertainty about payment flows on a given day, and F is a cumulative distribution function that increases monotonically from 0 (when its argument is $-\infty$) to 1 (as the argument approaches $+\infty$).

As noted, the market overnight rate is necessarily within the channel: $i_t^d \leq i_t \leq i_t^l$. Its exact position within the channel should be a decreasing function of the supply of central-bank balances. The model predicts an equilibrium overnight rate at exactly the target rate (the midpoint of the channel) when the supply of clearing balances is equal to

$$S_t = -F^{-1}(1/2)\,\sigma_t. \tag{3.2}$$

If the probability distribution of unexpected payment flows faced by each institution is roughly symmetric, so that $F(0)$ is near one-half, then the aggregate supply of clearing balances required to maintain the overnight rate near the target rate should not vary much with changes in σ_t. Even if this is not quite true, the adjustments of the supply of clearing balances required by (3.2) are unrelated to changes in the target level of interest rates.

Thus achievement of the central bank's operating target does not require any quantity adjustments through open-market operations in response to deviations of the market rate from the target rate; nor are any changes in the supply of central-bank balances required when the bank wishes to change the level of overnight interest rates. The target level of clearing balances in the system (3.2) need be adjusted only in response to "technical" factors (*e.g.*, changes in the volume of payments on certain days that can be expected to affect the σ_i), but *not* on occasions when it is desired to "tighten" or "loosen" monetary policy. Instead, changes in the level of overnight rates, when desired, are brought about through the shifts in the deposit rate and lending rate that automatically follow from a change in the target rate

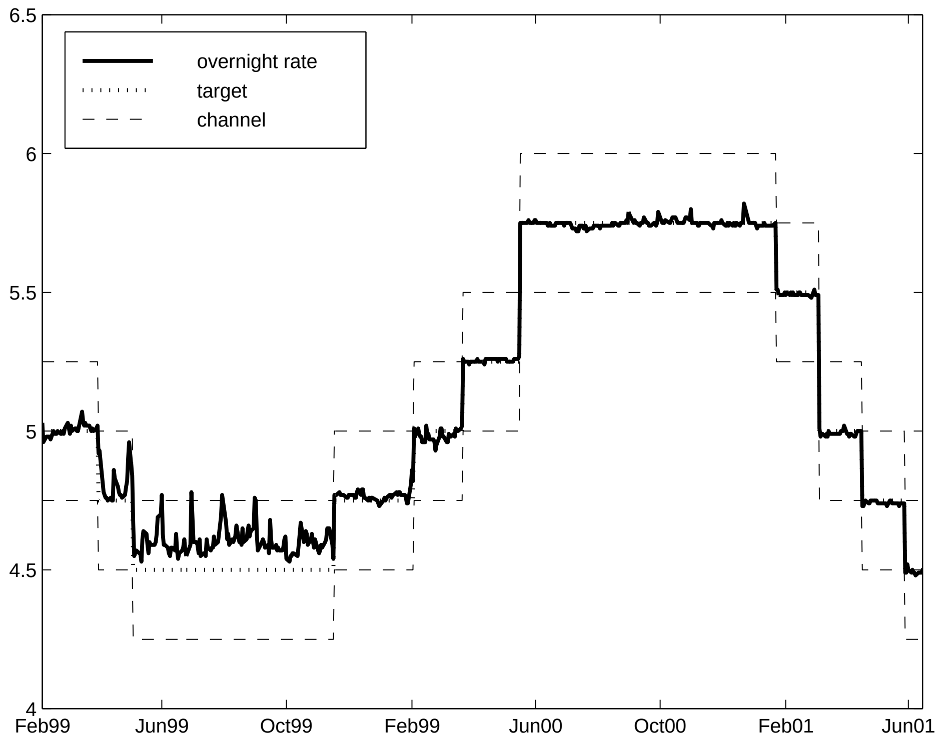

Figure 1.1: The "channel" or operating band and the market overnight rate, since introduction of the LVTS system in Canada. Source: Bank of Canada.

(and constitute the operational meaning of such a change), without any need for quantity adjustments.

This type of system has proven highly effective in Canada, Australia and New Zealand in controlling the level of overnight interest rates. For example, Figure 1.1 plots the overnight rate in Canada since the adoption of the Large-Value Transfer System for payments in February 1999, at which time the standing facilities described above were adopted.[16] One observes that the channel system has been quite effective, at least since early in 2000, at keeping the overnight interest rate not only within the Bank's 50-basis-point "operating band" or channel, but usually within about one basis point of the target rate. Australia and New Zealand similarly now achieve achieve considerably tighter control of overnight interest

[16]A system of the kind described here has been used in Australia since June 1998, and in New Zealand since March 1999.

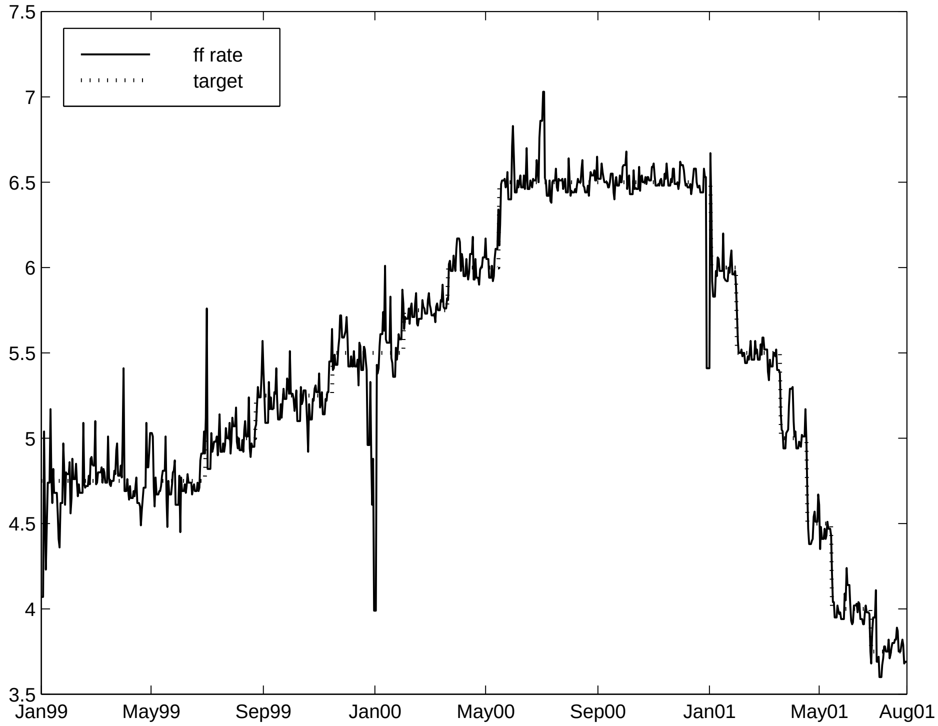

Figure 1.2: The U.S. fed funds rate and the Fed's operating target. Source: Federal Reserve Board.

rates than is achieved under the current operating procedures employed in the U.S.[17] (For purposes of comparison, Figure 1.2 plots the federal funds rate together with the Fed's operating target over the same time period.)

Thus the quantity adjustments of the supply of central-bank balances[18] that are involved in implementation of interest-rate policy are quite different under a channel system as op-

[17]Since March 2000, the standard deviation of $i_t - \bar{\imath}_t$ has been only 1.5 basis points for Australia, 1.1 basis points for Canada, and less than 0.4 basis points for New Zealand, but 13.4 basis points for the U.S. See Woodford (2001xx) for corresponding plots for the other two countries, and for discussion of the differences in the four countries' ability to respond to the "Y2K" panic without loss of control of short-term interest rates.

[18]We refer here to adjustments of the supply of central-bank balances rather than adjustments of the monetary base because in all of the countries under discussion, changes in the public's demand for currency are automatically accommodated by open-market operations that change the monetary base while seeking to insulate the supply of central-bank balances from the effects of such developments. Thus despite the emphasis of the academic literature on monetary-base rules, in practice a quantity-targeting rule that is intended to directly specify the central bank's daily open-market open-market operation would have to specify a target supply of central-bank balances rather than a target value for the monetary base.

posed to the system used in the U.S. In the U.S., policy can be "tightened" *only* by restricting the supply of Fed balances, so that the equilibrium spread between the return available on interbank lending and that available on Fed balances increases; in Canada, instead, there need be no change in supply, as there is no desire to change the spreads $i_t^l - i_t$ or $i_t - i_t^d$. Yet there is no reason to believe that these institutional details have any important consequences for the effects of interest-rate policy on these economies, and hence for the way in which it makes sense for these different central banks to determine their interest-rate operating targets. It follows that our conclusions would be of less universal validity if we were to formulate them in terms of a rule for determining the appropriate size of open-market operations, assuming American institutional arrangements.

Furthermore, for a country with a channel system, it would not be *possible* to formulate our advice in terms of a quantity-targeting rule. On the occasions upon which it is appropriate for the central bank to tighten or loosen policy, this does not imply any change in the appropriate target for the supply of central-bank balances; yet this does not at all mean that the central bank should not act! Because the crucial policy instruments in these countries are in fact the interest rates associated with the two standing facilities, that are in turn directly based on the time-varying interest-rate target, a policy rule for such countries must necessarily be formulated as an interest-rate rule. In fact, this way of specifying monetary policy is equally convenient for a country like the U.S., and is the one that we shall use in this study.

3.2 Monetary Policy in a Cashless Economy

Another case in which a monetary policy prescription would *have* to be specified in terms of an interest-rate rule would be if our advice were to be applicable to a "cashless" economy, by which we mean an economy in which there are *no monetary frictions* whatsoever. In a hypothetical economy of this kind, no central-bank liabilities have any special role to play in the payments system that results in a willingness to hold them despite yielding a lower return than other, equally riskless short-term claims. Consideration of this extreme case is of interest for two reasons.

First, it is possible to imagine that in the coming century the development of electronic payments systems could not only substitute for the use of currency in transactions, but also eliminate any advantage of clearing payments through accounts held at the central bank, as discussed by King (1999). This prospect is highly speculative at present; most current proposals for variants of "electronic money" still depend upon the final settlement of transactions through the central bank, even if payments are made using electronic signals rather than old-fashioned instruments such as paper checks.[19] Yet it is possible that in the future central banks will face the problem of what their role should be in such a world. And the question of how the development of electronic money should be regulated will face them much sooner. If one takes the view that monetary policy can be implemented only by rationing the supply of something that fulfills an essential function in the payments system, it is likely to be judged important to *prevent* the development of alternatives to payments using central-bank money, in order to head off a future in which the central bank is unable to do anything at all on behalf of macroeconomic stabilization — in which it becomes "an army with only a signal corps," in the evocative phrase of Benjamin Friedman (1999).

A second reason why it is useful to consider policy implementation in this hypothetical case is that if we can show that effective interest-rate control is possible even in the complete absence of monetary frictions, it may well simplify our analysis of basic issues in the theory of monetary policy to start from an analysis of the frictionless case, just as a physicist does when analyzing the motion of a pendulum or the trajectory of a cannonball. The appeal of this analytical approach was clear already to Wicksell (1898), who famously began his analysis (though writing at the end of the nineteenth century!) by considering the case of a "pure credit economy", defined as

[19] Charles Freedman (2000), for one, argues that the special role of central banks in providing for final settlement is unlikely ever to be replaced, owing to the unimpeachable solvency of these institutions, as government entities that can create money at will. Some, such as Goodhart (2000), equally doubt that electronic media can ever fully substitute for the use of currency.

> a state of affairs in which money does not actually circulate at all, neither in the form of coin (except perhaps as small change) nor in the form of notes, but where all domestic payments are effected by means of ... bookkeeping transfers (p. 70).

This is the approach that will be taken in the chapters to follow. Our basic model (developed beginning in chapter 2) will be one that abstracts from monetary frictions, in order to direct more attention to more essential aspects of the monetary transmission mechanism, such as the way that spending decisions depend on expected future interest rates as well as current ones, or the way in which fluctuations in nominal expenditure affect real activity. We then pause to consider at various points the modifications of our analysis that are required in order to take account of the monetary frictions that evidently exist, given the observation that non-interest-earning currency continues to be held; it is shown that, as a quantitative matter, these modifications are of relatively minor importance.

In our discussion of interest-rate determination under a present-day channel system, we have supposed that there is a demand for at least a small quantity of central-bank balances for clearing purposes, and these are held despite the existence of a small opportunity cost (25 basis points on average). But once the idea has been accepted that the central bank can vary the overnight interest rate without ever having to vary the size of this return spread, the functioning of the system no longer depends on the existence of a clearing demand. Let us suppose instead that balances held with the central bank cease to be any more useful to commercial banks than any other equally riskless overnight investment. In this case, the demand for central-bank balances will be zero for all interest rates higher than the deposit rate i_t^d. But banks should still be willing to hold arbitrary balances at the central bank as long as the market overnight rate is no higher than the rate paid by the central bank. In this case, it would no longer be possible to induce the overnight cash market to clear at a target rate higher than the rate paid on overnight balances at the central bank; for equation (3.1) reduces to $i_t = i_t^d$ in the case of any positive supply of central-bank balances.

But the central bank could still control the equilibrium overnight rate, by using the deposit rate as its policy instrument.[20] Such a system would differ from current channel systems in that an overnight lending facility would no longer be necessary, so that there would no longer be a "channel".[21] And the rate paid on central-bank balances would no longer be set at a fixed spread δ below the target overnight rate; instead, it would be set at exactly the target rate.

Yet perfect control of overnight rates should still be possible through adjustments of the rate paid on overnight central-bank balances, and changes in the target overnight rate would not have to involve any change in the target supply of central-bank balances, just as is true under current channel systems. Indeed, in this extreme case, any variations that did occur in the supply of central-bank balances would cease to have any effect at all upon the equilibrium overnight rate.

But how can interest-rate variation be achieved without any adjustment at all of the supply of central-bank balances? Informal discussions often treat interest-rate control by the central bank like a species of price control. Certainly, if a government decides to peg the price of some commodity, it may be able to do so, but only by holding stocks of the commodity that are sufficiently large relative to the world market for that commodity, and by standing ready to vary its holdings of the commodity by large amounts as necessary. In the market in question is a large one (more to the point, if either supply or demand in the market is relatively price-elastic) relative to the size of the balance sheet of the government entity seeking to control the price, one doubts that such efforts will be effective. What is different about controlling short-term nominal interest rates?

[20] Grimes (1992) makes a related point, showing that variation of the interest rate paid on central-bank balances would be effective in an environment in which central-bank reserves are no more useful for carrying out transactions than other liquid government securities, so that open-market purchases or sales of such securities are completely ineffective. Hall (1983, 1999) has also proposed this as a method of price-level control in the complete absence of monetary frictions. Hall speaks of control of the interest yield on a government "security", without any need for a central bank at all. But because of the special features that this instrument would need to possess, that are not possessed by privately issued securities — it is a claim only to future delivery of more units of the same instrument, and society's unit of account is defined in terms of this instrument — it seems best to think of it as still taking the same institutional form that it does today, namely, balances in an account with the central bank.

[21] This presumes a world in which no payments are cleared using central-bank balances. Of course, there would be no harm in continuing to offer such a facility as long as the central-bank clearing system were still used for at least some payments.

The difference is that there is no inherent "equilibrium" level of interest rates to which the market would tend in the absence of central-bank intervention, and against which the central bank must therefore exert a significant countervailing force in order to achieve a given operating target.[22] This is because there is no inherent value (in terms of real goods and services) for a fiat unit of account such as the "dollar", except insofar as a particular exchange value results from the monetary policy commitments of the central bank. The basic point was clear to Wicksell (1898, pp. 100-101), who compares relative prices to a pendulum that returns always to the same equilibrium position when perturbed, while the money prices of goods in general are compared to a cylinder resting on a horizontal plane, that can remain equally well in any location on the plane to which it may happen to be moved.[23] Alternative price-level paths are thus equally consistent with market equilibrium in the absence of any intervention that would vary the supply of any real goods or services to the private sector. And associated with these alternative paths for the general level of prices are alternative paths for short-term nominal interest rates.

Of course, this analysis might suggest that while central banks can bring about an arbitrary level of *nominal* interest rates (by creating expectations of the appropriate rate of inflation), they should not be able to significantly affect *real* interest rates, except through trades that are large relative to the economy that they seek to affect. It may also suggest that banks should be able to move nominal rates only by altering inflation expectations; yet banks generally do not feel that they can easily alter expectations of inflation over the near term, so that one might doubt that banks should be able to affect *short-term* nominal rates through such a mechanism.

However, once one recognizes that many prices (and wages) are fairly sticky over short

[22]This does not mean that Wicksell's notion of a "natural" rate of interest determined by real factors is of no relevance to the consideration of the policy options facing a central bank. It is indeed, as argued in chapter 4. But the natural rate of interest is the rate of interest required for *an equilibrium with stable prices;* the central bank nonetheless can arbitrarily choose the level of interest rates (within limits), because it can choose the degree to which prices shall increase or decrease.

[23]This is the ground for his argument — in the quotation from the introduction to his book that begins this chapter — that control of the general level of prices involves no interference with the market mechanism of the kind that is required if some relative price is to be controlled.

time intervals, the arbitrariness of the path of nominal prices (in the sense of their underdetermination by real factors alone) implies that the path of real activity, and the associated path of equilibrium real interest rates, are equally arbitrary. It is equally possible, from a logical standpoint, to imagine allowing the central bank to determine, by arbitrary fiat, the path of aggregate real activity, or the path of real interest rates, or the path of nominal interest rates, as it is to imagine allowing it to determine the path of nominal interest rates.[24] In practice, it is easiest for central banks to exert relatively direct control over overnight nominal interest rates, and so banks generally formulate their short-run objectives (their operating target) in terms of the effect that they seek to bring about in this variable rather than one of the others.

Even recognizing the existence of a very large set of rational expectations equilibria — equally consistent with optimizing private-sector behavior and with market clearing, in the absence of any specification of monetary policy — one might nonetheless suppose, as Fischer Black (1970) once did, that in a fully deregulated system the central bank should have no way of using monetary policy to select among these alternative equilibria. The path of money prices (and similarly nominal interest rates, nominal exchange rates, and so on) would then be determined solely by the self-fulfilling expectations of market participants. Why should the central bank play any special role in determining which of these outcomes should actually occur, if it does not possess any monopoly power as the unique supplier of some crucial service?

The answer is that the unit of account in a purely fiat system is *defined* in terms of the liabilities of the central bank.[25] A financial contract that promises to deliver a certain number

[24]This does not mean, of course, that absolutely any paths for these variables can be achieved through monetary policy; the chosen paths must be consistent with certain constraints implied by the conditions for a rational-expectations equilibrium, for example those presented in chapter 4. But this is true even in the case of the central bank's choice of a path for the price level. Even in a world with fully flexible wages and prices, for example, it would not be possible to bring about a rate of deflation so fast as to imply a negative nominal interest rate.

[25]See Hall (1999) and White (2001) for expressions of similar views. White emphasizes the role of legal tender statutes in defining the meaning of a national currency unit. But such statutes do not represent a restriction upon the means of payment that can be used within a given geographical region — or at any rate, there need be no such restrictions upon private agreements for the point to be valid. What matters is

of U.S. dollars at a specified future date is promising payment in terms of Federal Reserve notes or clearing balances at the Fed (which are treated as freely convertible into one another by Fed). Even in the technological utopia imagined by the enthusiasts of electronic money — where financial market participants are willing to accept as final settlement transfers made over electronic networks in which the central bank is not involved — if debts are contracted in units of a national currency, then clearing balances at the central bank will still define the thing to which these other claims are accepted as equivalent.

This explains why the nominal interest yield on clearing balances at the central bank can determine overnight rates in the market as a whole. The central bank can obviously define the nominal yield on overnight deposits in its clearing accounts as it chooses; it is simply promising to increase the nominal amount credited to a given account, after all. It can also determine this independently of its determination of the quantity of such balances that it supplies. Commercial banks may exchange claims to such deposits among themselves on whatever terms they like. But the market value of a dollar deposit in such an account cannot be anything other than a dollar — *because this defines the meaning of a "dollar"!*

This places the Fed in a different situation than any other issuer of dollar-denominated liabilities.[26] Citibank can determine the number of dollars that one of its jumbo CDs will be worth at maturity, but must then allow the market to determine the current dollar value of such a claim; it cannot determine *both* the quantity that it wishes to issue of such claims and

simply what contracts written in terms of a particular unit of account are taken to mean, and the role of law in stabilizing such meanings is essentially no different than, say, in the case of trademarks.

[26]Costa and De Grauwe (2001) instead argue that "in a cashless society ... the central bank cannot 'force the banks to swallow' the reserves it creates" (p. 11), and speak of the central bank being forced to "liquidate ... assets" in order the redeem the central-bank liabilities that commercial banks are "unwilling to hold" in their portfolios. This neglects the fact that the definition of the U.S. dollar allows the Fed to honor a commitment to pay a certain number of dollars to account-holders the next day by simply crediting them with an account of that size at the Fed — there is no possibility of demanding payment in terms of some other asset valued more highly by the market. Similarly, Costa and De Grauwe argue that "the problem of the central bank in a cashless society is comparable to [that of a] central bank pegging a fixed exchange rate" (footnote 15). But the problem of a bank seeking to maintain an exchange-rate peg is that it promises to deliver a foreign currency in exchange for its liabilities, not liabilities of its own that it freely creates. Costa and De Grauwe say that they imagine a world in which "the unit of account remains a national affair ... and is provided by the state" (p. 1), but seem not to realize that this means defining that unit of account in terms of central-bank liabilities.

the interest yield on them. Yet the Fed can, and does so daily — though at present it chooses to fix the interest yield on Fed balances at zero and only to vary the supply. The Fed's current position as monopoly supplier of an instrument that serves a special function is necessary in order for variations in the quantity supplied to affect the equilibrium spread between this interest rate and other market rates, but *not* in order to allow separate determination of the interest rate on central-bank balances and the quantity of them in existence.

Yes, some may respond, a central bank would still be able to determine the interest rate on overnight deposits at the central bank, and thus the interest rate in the interbank market for such claims, even in a world of completely frictionless financial markets. But would control of *this* interest rate necessarily have consequences for other market rates, the ones that matter for critical intertemporal decisions such as investment spending? The answer is that it must — and all the more so in a world in which financial markets have become highly efficient, so that arbitrage opportunities created by discrepancies among the yields on different market instruments are immediately eliminated. Equally riskless short-term claims issued by the private sector (say, shares in a money-market mutual fund holding very short-term Treasury bills) would not be able to promise a different interest rate than the one available on deposits at the central bank; otherwise, there would excess supply or demand for the private-sector instruments. And determination of the overnight interest rate would also have to imply determination of the equilibrium overnight holding return on longer-lived securities, up to a correction for risk; and so determination of the expected future path of overnight interest rates would essentially determine longer-term interest rates.

The special feature of central banks, then, is simply that they are entities the liabilities of which happen to be used to define the unit of account in a wide range of contracts that other people exchange with one another. There is perhaps no deep, universal reason why this need be so; it is certainly not essential that there be one such entity per national political unit. Nonetheless, the provision of a well-managed unit of account — one in terms of which the equilibrium prices of many goods and services will be relatively stable — clearly facilitates economic life. And given the evident convenience of having a single unit of account be used by most of the parties with whom one wishes to trade, one may well suppose that this function should properly continue to be taken on by the government, even in a world of highly efficient information processing. We here assume a world in which central banks (whether national or supra-national, as in the case of the ECB) continue to fulfill this function, and in which they are interested in managing their fiat currency in the public interest. The present study aims to supply a theory that can help them to do so.

4 Interest-Rate Rules

We have argued that the central problem of the theory of monetary policy is to provide principles that can be used in selecting a desirable rule for setting a central bank's interest-rate operating target. It is perhaps worth saying a bit more at this point about exactly what form of rules we have in mind, and what sort of questions we would like to answer about them. This will provide a more concrete background for the analysis to be developed in the chapters to come.

Probably the earliest example of a prescription for monetary policy in terms of an interest-rate rule is due to Wicksell (1898, 1907). Although writing at a time when the leading industrial nations remained committed to the gold standard, and even most scholars assumed the necessity of a commodity standard of one sort or another, Wicksell foresaw the possibility of a pure fiat standard, and indeed argued that it was essential for the development of "a rational monetary system." He furthermore advocated an interest-rate rule for the management of such a system. His original (1898) statement of the proposed rule was as follows:

> So long as prices remain unaltered the [central] banks' rate of interest[27] is to remain unaltered. If prices rise, the rate of interest is to be raised; and if prices fall, the rate of interest is to be lowered; and the rate of interest is henceforth to be maintained at its new level until a further movement of prices calls for a further change in one direction or the other (p. 189, italicized in original).

Wicksell's proposal can be represented mathematically as a commitment to set the central bank's interest-rate operating target i_t according to a relation of the form[28]

$$i_t = \bar{\imath} + \phi\, p_t, \tag{4.1}$$

where p_t is the log of some general price index (the one that the policy aims to stabilize) and ϕ is a positive response coefficient, or alternatively by a rule of the form

$$\Delta i_t = \phi\, \pi_t \tag{4.2}$$

where $\pi_t \equiv \Delta p_t$ is the inflation rate.[29] We study price-level determination under policy rules of this kind in chapter 2, and argue that such a rule should indeed succeed in stabilizing the price index around a constant level; the principles that determine the equilibrium price level under such a regime are briefly sketched in section xx.

A simple "Wicksellian" rule of this kind has other advantages as well, at least in comparison to other equally simple rules, as discussed in chapter 7. Nonetheless, we shall not confine our attention to rules of this kind. Our primary interest in this study is in the analysis of proposals that are closer in form to the policies currently followed by many central banks. These rules involve an (explicit or implicit) target for the *inflation rate,* rather than for the *price level;* nor are they expressible solely in terms of interest-rate *changes,* so that they are equivalent to a rule that responds to the price level, as in the case of (4.2). The rules typically allow for "base drift" in the price level as a result — even if the inflation rate is kept within a narrow interval at all times, there is no long-run mean reversion in the price level, or even in the price level deflated by some deterministic target path. And as we shall eventually conclude in chapter 8, optimal interest-rate rules are likely to have this property.

[28]Wicksell proposes nothing so specific as a log-linear relation of this kind, of course; he only describes a monotonic relationship. The log-linear specification is useful for the simple calculations of the next section. In chapter 2, we discuss the usefulness of this sort of log-linear approximation of what is necessarily not a globally log-linear rule. The specification (4.1) cannot be maintained for all possible price levels, owing to the requirement that the nominal interest rate be non-negative.

[29]Note that a commitment to set the interest rate according to (4.2) from some date t_0 forward is equivalent to a commitment to set it according to a rule of the form (4.1), where the intercept $\bar{\imath}$ corresponds to $i_{t_0-1} - \phi\, p_{t_0-1}$. Fuhrer and Moore (1995xx) propose a more complicated interpretation of Wicksell's proposal, in which the interest change is instead a function of the price *level.* While their rule is slightly more difficult to analyze, it does not lead to substantially different conclusions about the consequences of commitment to a Wicksellian rule.

4.1 Contemporary Proposals

The best-known example of a proposed rule for setting interest rates is probably the one proposed by John Taylor (1993), both as a rough description of the way that policy had actually been made by the U.S. Federal Reserve under Alan Greenspan's chairmanship, and as a normative prescription (on the basis of stochastic simulations using a number of econometric models). According to the "Taylor rule," as it has come to be known, the Fed's funds-rate operating target i_t is set as a linear function of measures of the current inflation rate and the current gap between real output and potential:

$$i_t = .04 + 1.5\,(\bar{\pi}_t - .02) + 0.5\,(y_t - y_t^p), \tag{4.3}$$

where $\bar{\pi}_t$ is the rate of inflation (the change in the log GDP deflator over the previous four quarters, in Taylor's illustration of the rule's empirical fit), y_t is log output (log real GDP in Taylor's plot), and y_t^p is log "potential" output (log real GDP minus a linear trend, in Taylor's plot). The constants in Taylor's numerical specification indicate an implicit inflation target of two percent per annum, and an estimate of the long-run real federal funds rate of two percent per annum as well, so that a long-run average inflation rate at the target requires a long-run average funds rate of four percent. A slightly simpler rule in the same vein was proposed for the U.K. by Charles Goodhart (1992), according to which "there should be a presumption" that the nominal interest rate would satisfy an equation of the form

$$i_t = .03 + 1.5\,\bar{\pi}_t,$$

and "the Governor should be asked, say twice a year, to account for any divergence from that 'rule' "(p. 324).

The coefficients 1.5 and 0.5 in the Taylor rule are round figures argued to approximately characterize U.S. policy between 1987 and 1992, and that were found to result in desirable outcomes (in terms of inflation and output stability) in simulations.[30] In Taylor's discussions

[30] A similar form of policy rule was advocated, also on the basis of simulation studies, at around the same time by Henderson and McKibbin (1993).

of the rule, he places particular stress upon the importance of responding to inflation above the target rate by raising the nominal interest-rate operating target by *more than the amount* by which inflation exceeds the target; the importance of this "Taylor Principle" is considered in detail in chapters 2 and 4. Taylor (1999xx) argues that the Fed did not adhere to this principle before 1979 (at which time Fed chairman Paul Volcker instituted a radical shift in policy), and this failure may well have been responsible for the greater U.S. macroeconomic instability during the 1960s and 1970s. Taylor illustrates the change in policy by estimating simple Fed reaction functions of the form

$$i_t = \bar{\imath} + \phi_\pi \, (\bar{\pi}_t - \bar{\pi}) + \phi_x \, x_t \tag{4.4}$$

for two different sample periods, using ordinary least squares; his coefficient estimates are shown in Table 1.1. (Here we introduce the notation x_t for the *output gap,* again equated with deviations of log real GDP from trend in Taylor's empirical work.) Nelson (2001) finds that estimates of Taylor-type rules for the U.K. tell a similar story; prior to the adoption of inflation targeting in 1992, U.K. interest rates rose less than one-for-one with increases in inflation (and in the mid-1970s, responded little at all), but since 1992, the long-run inflation response coefficient is estimated to have been nearly 1.3.

Estimates of empirical central-bank reaction functions typically find that a dynamic specification fits the data better, whatever the validity may be of Taylor's (1999xx) preference for a purely contemporaneous specification on normative grounds.[31] For example, Judd and Rudebusch (1998) estimate Fed reaction functions according to which the funds-rate operating target adjusts in response to changes in an implicit desired level of the funds rate $\bar{\imath}_t$ according to partial-adjustment dynamics of the form[32]

$$i_t = (1 - \rho_1) \, \bar{\imath}_t + \rho_1 \, i_{t-1} + \rho_2 \, (i_{t-1} - i_{t-2}). \tag{4.5}$$

[31] This is also true of the estimates for the U.K. reported in Nelson (2001). This is why we refer to Nelson's "long-run inflation response coefficient" in the previous paragraph, rather than to a contemporaneous response coefficient of the kind estimated by Taylor (1999xx).

[32] Here as in all other regressions reported in this section, periods are assumed to be quarters, and quarterly data are used in the estimation.

	ϕ_π	(s.e.)	ϕ_x	(s.e.)	γ	ρ_1	(s.e.)	ρ_2	(s.e)
Taylor (1999)									
1960-79	0.81	(.06)	0.25	(.05)					
1987-97	1.53	(.16)	0.77	(.09)					
Judd-Rudebusch (1998)									
1979-87	1.46	(.26)	1.53	(.80)	1	0.56	(.12)		
1987-97	1.54	(.18)	0.99	(.13)	0	0.72	(.05)	0.43	(.10)
Clarida et al. (2000)									
1960-79	0.83	(.07)	0.27	(.08)		0.68	(.05)		
1979-96	2.15	(.40)	0.93	(.42)		0.79	(.04)		
Orphanides (2001)									
1966-79	1.64	(.38)	0.57	(.12)		0.70	(.07)		
1979-95	1.80	(.48)	0.27	(.30)		0.79	(.11)		

Table 1.1. Alternative estimates of Fed reaction functions.

The desired level of the funds rate in turn depends upon inflation and the output gap in a manner similar to that postulated by Taylor,

$$\bar{\imath}_t = \bar{\imath} + \phi_\pi \, (\bar{\pi}_t - \bar{\pi}) + \phi_x \, (x_t - \gamma x_{t-1}), \tag{4.6}$$

except that the allowance for nonzero γ means that the desired funds rate may respond to the rate of change of the output gap as well as (or instead of) its level. The Judd-Rudebusch estimated coefficients for two different sample periods, corresponding to the Fed chairmanships of Paul Volcker and Alan Greenspan respectively, are also reported in Table 1.1.[33] Taylor's view of the nature of policy in the Greenspan period is largely confirmed, with the exception that Judd and Rudebusch estimate partial-adjustment dynamics implying substantial persistence. They give a similar characterization of policy in the Volcker period,

[33] In their preferred estimates, the value of γ is imposed rather than estimated. The extreme values assumed for the separate periods, however, are suggested by preliminary regressions in which the value of γ is unconstrained.

except that in this period the desired funds rate is found to depend on the rate of change of the output gap, rather than its level.[34]

Many recent discussions of central bank behavior, both positive and normative, argue instead for specifications in which a bank's operating target depends on forecasts. For example, Clarida *et al.* (2000xx) estimate Fed reaction functions of the form[35]

$$\bar{\imath}_t = \bar{\imath} + \phi_\pi \; E[\pi_{t+1} - \bar{\pi}|\Omega_t] + \phi_x \; E[x_t|\Omega_t], \tag{4.7}$$

where Ω_t is the information set assumed to be available to the Fed when setting i_t, and the actual operating target is again related to the desired funds rate $\bar{\imath}_t$ through partial-adjustment dynamics. Like Taylor, these authors find an important increase in the degree to which the Fed's desired level for the funds rate responds to inflation variations since 1979, though in their specification the Fed responds to an inflation *forecast* rather than inflation that has already occurred.[36]

Finally, it should be noted that the view that the Fed has responded more vigorously to inflation variations since 1979 has not gone unchallenged. Attanasio (2001) argues instead that the findings of Taylor and the other authors just cited are distorted by the use of inflation and output-gap data (especially the output-gap estimates) that were not available to the Fed at the time that its interest-rate decisions were made. When he instead estimates Fed reaction functions of the kind assumed by Clarida *et al.* using the forecasts actually produced by Fed staff at the time rather than econometric projections using the data available now, he obtains much more similar estimates for the pre-Volcker and post-Volcker periods, as shown in the table. The inflation-response coefficient ϕ_π is well above one in both periods, according

[34] Judd and Rudebusch also estimate a reaction function for the period (1970-78) corresponding to the chairmanship of Arthur Burns. Like Taylor, in this period they estimate an inflation-response coefficient ϕ_π less than one, though not significantly so in their case.

[35] Clarida *et al.* also estimate variants of the rule in which the forecast horizon is assumed to be more than one quarter in the future. The policies of inflation-targeting central banks have often been represented by rules in which the interest-rate operating target responds to a forecast of inflation as many as 8 quarters in the future. See, *e.g.*, Black *et al.*, (19xx), Batini and Haldane (1999).

[36] Like Taylor, they also suggest that this change has led to greater macroeconomic stability in the later period. They provide a theoretical analysis of why this could have been so, in terms of the vulnerability of an economy to instability due to self-fulfilling expectations in the case of a policy rule of the kind that they estimate for the period 1960-79. Reasons for this are discussed in chapter 4.

to Orphanides' estimates; he instead emphasizes the reduction in the size of ϕ_x as the crucial policy change after 1979, and the key to U.S. macroeconomic stability since the mid-1980s. We shall not seek here to resolve this debate about historical Fed policy, but simply note that much current debate both about the explanation of recent U.S. policy successes and about the reason for past policy failures turns upon claims about the desirability of particular coefficients in Taylor-type rules.

These alternative characterizations suggest a number of questions about the form of a desirable interest-rate rule. One obvious question is whether the variables to which the Fed is described as responding in the Taylor rule and the estimated reaction functions just discussed — some measures of inflation and the output gap — are ones that make sense. Is it desirable for interest rates to be adjusted in response to variations in these variables, and with the signs proposed by Taylor? Is there any ground for thinking it more important to respond to variations in these variables than in others? Is responding to variations in these variables an adequate substitute for attempting to respond to the underlying disturbances that are perceived to be currently affecting the economy?

If it does make sense to respond to these variables, how exactly should they be defined? Which sort of price index is most appropriately used in the inflation measure? Relative to what concept of "potential output" should the "output gap" measure be defined? And how strongly is it desirable to respond to variations in these variables? Is a value of ϕ_π greater than one essential, as argued by Taylor? Is a large value of ϕ_x dangerous, as argued by Orphanides?

We shall also be interested in the most desirable dynamic specification of such an interest-rate rule. Are purely contemporaneous responses, as prescribed by Taylor, preferable? Is there any justification for the more inertial interest-rate dynamics indicated by the estimated reaction functions? If so, how inertial is it desirable for interest-rate policy to be? Is it preferable to respond to forecasts rather than to current or past values of inflation and the output gap? If so, how far in the future should the forecasts look?

Another type of policy rule that has figured prominently both in recent descriptions of actual central-bank behavior and in normative prescriptions is an *inflation-forecast targeting* rule. A classic example is the sort of rule that is often used to explain the current procedures of the Bank of England (*e.g.*, Vickers, 1998). According to the formula, the Bank should be willing to adopt a given operating target i_t for the overnight interest rate at date t if and only if the Bank's forecast of the evolution of inflation over the next two years, conditional upon the interest rate remaining at the level i_t, implies an inflation rate of 2.5 percent per annum (the Bank's current inflation target) two years after date t. This is an example of what Svensson (1999, 2001) calls a "targeting rule" as opposed to an instrument rule. No formula is specified for the central bank's interest-rate operating target; instead, it is to be set at whatever level may turn out to be required in order for the bank's projections to satisfy a certain target criterion. The target criterion need not involve only future inflation; for example, Svensson (1999) advocates a "flexible inflation targeting" rule in which the interest rate is adjusted at date t so as to ensure that

$$E_t\pi_{t+j} + \lambda\, E_t x_{t+k} = \bar{\pi}, \tag{4.8}$$

where $\bar{\pi}$ is the average inflation target, the coefficient $\lambda > 0$ depends on the relative importance of output-gap stabilization, and the horizons j and k are not necessarily the same distance in the future.

We shall also wish to consider the desirable specification of a target criterion in the case of a policy rule of this sort. Again, a basic question is whether it makes sense to define the target criterion in terms of projections for these particular variables, inflation and the output gap, rather than others, such as monetary aggregates. If so, what should determine the relative weight, if any, to be placed on the output-gap forecast? How far in the future should the forecasts in the target criterion look? And is a desirable criterion *purely* forward-looking, as in the case of the two examples just mentioned, or should the inflation target be history-dependent, in addition to (possibly) depending on projected future output gaps?

This study will seek to elaborate a methodology that can be used to give quantitative answers to questions of this sort about optimal policy rules. Of the course, the answers obtained will depend on the details of what one assumes about the nature of the monetary transmission mechanism, and I do not propose to argue for a specific quantitative rule. The aim of the present study is more to suggest a way of approaching the problem than to announce the details of its solution. However, certain model elements recur in many of the models currently used in studies of the effects of monetary policy, both in the academic literature and in central banks; and given the likelihood that a reasonable model will be judged to include these features, we may obtain some tentative conclusions as to the likely form of reasonable policy rules.

4.2 General Criticisms of Interest-Rate Rules

Before taking up specific questions of these kinds about the form of desirable interest-rate rules, it is first necessary to address some more basic issues. Would *any* form of interest-rate rule represent a sensible approach to monetary policy? Proponents of monetary targeting have often argued against interest-rate control *as such* — asserting not that skill is required in the choice of an interest-rate operating target, but that it is a serious mistake to have one at all.

One famous argument, mentioned above, is that of Sargent and Wallace (1975). Sargent and Wallace consider a general class of money-supply rules on the one hand, and a general class of interest-rate rules on the other, and argue that while *any* of the money-supply rules leads to a determinate rational expectations equilibrium (in the context of a particular rational-expectations IS-LM model), *none* of the interest-rate rules do. By *determinacy* of the equilibrium I mean that there is a unique equilibrium satisfying certain bounds, made precise in chapter 2. Sargent and Wallace argue that interest-rate rules lead to *indeterminacy,* meaning that even if one restricts one's attention to bounded solutions to the equilibrium relations (as we shall largely do in this study), there is an extremely large set of equally possible equilibria. These include equilibria in which endogenous variables such as inflation and output respond to random events that are completely unrelated to economic "fundamentals" (*i.e.,* to the exogenous disturbances that affect the structural relations that determine inflation and output), and also equilibria in which "fundamental" disturbances cause fluctuations in equilibrium inflation and output that are arbitrarily large relative to the degree to which the structural relations are perturbed. Thus in such a case, macroeconomic instability can occur due purely to self-fulfilling expectations. This is plainly undesirable, if one's objective is to stabilize inflation and/or output.[37] Hence Sargent and Wallace argue that interest-rate rules can be excluded from consideration as a class; the problem of optimal monetary policy is then properly framed as a question of what the best money-supply rule would be.

However, as McCallum (1981) notes, the Sargent-Wallace indeterminacy result applies, even in the context of their own model, only in the case of interest-rate rules that specify an *exogenous* evolution for the nominal interest rate; this includes the possibility of rules that specify the nominal interest rate as a function of the history of exogenous disturbances, but not rules that make the nominal interest rate a function of *endogenous* variables, such as inflation or output. Yet the Taylor rule, and the other interest-rate rules discussed above, are all rules of the latter sort, so that the Sargent-Wallace result need not apply. The same is shown to be true, in chapters 2 and 4, in the case of the optimizing models of inflation and output determination considered here. Indeed, we find that, at least in the case of the simple model of the monetary transmission mechanism that is most extensively analyzed here, *either* the type of feedback from the general price level to the interest rate (or from changes in the price level to changes in the interest rate) advocated by Wicksell, or the type of feedback from inflation and output to the central bank's interest-rate operating target prescribed by Taylor, would suffice to imply a determinate rational expectations equilibrium. In the case of a level-to-level (or change-to-change) Wicksellian specification, it is only necessary that the sign of the response be the one advocated by Wicksell. In the case of a change-to-level specification like that proposed by Taylor, the "Taylor principle" mentioned above — the requirement that a sustained increase in inflation eventually result in an increase in nominal

[37]The point remains valid if one's objective is, as we shall argue that it should be, to stabilize output relative to its natural rate, rather than output stabilization as such. For the fluctuations in output due purely to self-fulfilling expectations just mentioned will imply fluctuations in the output gap as well.

interest rates that is even larger in percentage points — turns out to be the critical condition that determines whether equilibrium should be determinate or not.[38]

A related criticism of interest-rate targeting also maintains that such a policy is dangerous because it allows instability to be generated by self-fulfilling expectations, but is not based on the possibility of multiple rational-expectations equilibria. Friedman (1968) argues that attempting to control nominal interest rates is dangerous on the basis of Wicksell's (1898, 1907) famous analysis of the "cumulative process". With a nominal interest rate that is fixed at a level below the "natural rate", inflation is generated that increases inflation expectations, which then stimulates demand even further due to the reduction in the real rate, generating even faster inflation, further increasing inflation expectations, and so on without bound.[39] The same process should occur with the opposite sign if the interest rate happens to be set above the natural rate; thus any attempt to fix the nominal interest rate would seem almost inevitably to generate severe instability of the inflation rate. (In Friedman's analysis, there is no indeterminacy of the path of inflation, as inflation expectations are assumed to be a specific function of previously observed inflation.)

As is discussed in chapter 4, this analysis can be formalized in the context of an optimizing model in which inflation and income forecasts are based on extrapolation from past data (*e.g.*, using empirical time-series models). But once again, the classic analysis applies only in the case of a policy that exogenously specifies the path of nominal interest rates. If instead a surge in inflation and output leads to increases in nominal interest rates large enough to raise real rates, then demand should be damped, tending to lower inflation as well — so that there should be no explosive instability of either inflation or output dynamics under adaptive learning. Indeed, the analyses of Bullard and Mitra (2000, 2002) and Preston (2002) find that conformity to the Taylor principle is both a necessary and sufficient condition (at least

[38]Clarida *et al.* (2000xx) argue on this basis that the macroeconomic instability of the 1970s in the U.S. may have been increased by self-fulfilling expectations, given that their estimates (see Table 1.1) imply that the Taylor principle has been satisfied by post-1979 policy but not by previous policy.

[39]This summarizes Friedman's account, rather than Wicksell's original discussion; Wicksell does not discuss endogenous inflation expectations, and so concludes that the price level should rise without bound, rather than the inflation rate. Lindahl (1939) was the first to introduce endogenous inflation expectations into the analysis, and so to conclude that the inflation rate could rise without bound.

within certain simple classes of policy rules) for adaptive learning dynamics to converge to a stationary rational-expectations equilibrium, in which inflation and output fluctuate only in response to "fundamentals".

Thus it is important to realize that these well-known criticisms of interest-rate targeting assume that under such a policy the interest-rate target would remain *fixed,* regardless of the path of inflation. The analyses are quite inapplicable in the case of policy rules such as Wicksell's rule, the Taylor rule or typical inflation-forecast targeting rules, which require that interest rates be raised sharply if inflation is either observed or forecasted to exceed the target rate consistently for a substantial period of time. In fact, in these conventional arguments for monetary targeting, the reason for control of money growth is precisely that this is a policy commitment that ensures that an excessive rate of inflation will lead to interest-rate increases sufficient to curb the growth of nominal expenditure. A fixed target path for the money supply (or more generally, a path that is contingent only upon exogenous state variables, not upon the path of the price level) implies that if the price level grows more rapidly, the private sector will be forced to operate with a lower level of real money balances; this will require interest-rate increases and/or a reduction in real activity sufficient to reduce desired real money balances to the level of the real money supply.

But the same kind of automatic increase in interest rates, curbing expenditure, can be arranged through a simple commitment of the central bank to raise interest rates in response to deviations of the general level of prices from its desired path, as first proposed by Wicksell. And once one recognizes that quantity control is not necessary for such a system to work, it is hard to see why one should wish to encumbered by it. Over the course of the twentieth century, it came to be accepted that no convertibility of national currencies into a real commodity such as gold was necessary in order for central banks to act in a way that controlled the value of their currencies; and once this was accepted, it quickly became evident that nations were better off *not* relying upon such a crude mechanism as a gold standard, which left the value of the national unit of account vulnerable to fluctuations in the market for gold. Similarly, once one accepts that the adjustment of interest rates to head off undesired price-level variation can be managed by central banks without any need for so mechanical a discipline as is provided by a money-growth target, it should be clear that a properly chosen interest-rate rule can be more efficient than monetary targeting, which has the unwanted side effect of making interest rates (and hence the pace of aggregate expenditure) vulnerable to variations in the relation between desired money balances and the volume of transactions.

A more subtle criticism of interest-rate rules as an approach to systematic monetary policy would argue that even if such rules lead to well-defined, well-behaved equilibria, the description of policy in this way may still not be useful to a central bank that wishes to understand, and thus to accurately calibrate, the consequences of its actions. It is often supposed that the key to understanding the effects of monetary policy on inflation must always be the quantity theory of money, according to which the price level is determined by the relation between the nominal money supply on the one hand and the demand for real money balances on the other. It may then be concluded that what matters about *any* monetary policy is the implied path of the money supply, whether this is determined through straightforward monetary targeting or in some more indirect manner.[40] From such a perspective, it might seem that a clearer understanding of the consequences of a central bank's actions would be facilitated by an explicit focus on what evolution of the money supply the bank intends to bring about — that is, by monetary targeting — rather than by talk about interest rate policy that, even if it does imply a specific path for the money supply, does not make the intended path entirely transparent.

The present study aims to show that the basic premise of such a criticism is incorrect. One of the primary goals of Part I of this book is the development of a theoretical framework in which the consequences of alternative interest-rate rules can be analyzed, which does not require that they first be translated into equivalent rules for the evolution of the money supply. Indeed, much of the time we shall analyze the consequences of interest-rate rules without having to solve for the implied path of the money supply, or even having to specify

[40] This is for example the perspective taken in the *Monetary History* of Friedman and Schwartz (1963).

the coefficients of a "money demand" relation. In the case of an economy without monetary frictions — a case that I shall argue is an analytically convenient approximation for many purposes, and that may well represent the future, as discussed above — there will not even be any meaningful money supply or demand to be defined. If instead we take account of the sort of frictions that evidently still exist in an economy like the U.S. at present, then our models will imply an equilibrium path for the money supply along with other endogenous variables. But the factors determining the equilibrium paths of both inflation and output will continue to be nearly the same as in the frictionless economy, so that it does not seem at all natural or useful to try to *explain* the predicted paths of inflation and output as *consequences* of the implied path of the money supply. Instead, it proves to be possible to discuss the determinants of inflation and output in a fairly straightforward way in terms of the coefficients of an interest-rate rule. Thus the characterizations of central-bank policy offered above are found to be quite convenient for analysis of the consequences of one quantitative specification or another. We further show, in chapter 8, that optimal policy can be conveniently represented in terms of specifications of exactly this sort, leading to answers to the very specific questions about interest-rate policy posed in the previous section.

4.3 Neo-Wicksellian Monetary Theory

The non-quantity-theoretic analytical framework developed here develops several important themes from the monetary writings of Knut Wicksell (1898, 1906, 1907). Wicksell argued that even the variations in the price level observed in his own time, under the international gold standard, were not primarily due to variations in the world gold supply, but rather to two other factors — the policies followed by central banks, adjusting the "bank rate" at which they were willing to discount short-term bills, on the one hand, and real disturbances, affecting the "natural rate of interest" on the other. In Wicksell's view, price stability depended on keeping the interest rate controlled by the central bank in line with the "natural rate" determined by real factors (such as the marginal product of capital). Inflation occurred whenever the central banks lowered interest rates, without any decline in the natural rate having occurred to justify this, or whenever the natural rate of interest increased (due, for example, to an increase in the productivity of investment opportunities), without any adjustment of the interest rates controlled by central banks in response. Deflation occurred whenever a disparity was created of the opposite sign.

Whatever the validity of such a non-quantity-theoretic approach for the analysis of price-level determination under the gold standard, Wicksell's approach is a particularly congenial one for thinking about our present circumstances — a world of purely fiat currencies in which central banks adjust their operating targets for nominal interest rates in response to perceived risks of inflation, but pay little if any attention to the evolution of monetary aggregates — to say nothing of the one to which we may be headed, in which monetary frictions become negligible.[41] In such a world, where the concepts of money supply and demand become inapplicable, what is there to determine an equilibrium value for the general level of money prices? One possible answer is the role of *past* prices in determining current equilibrium prices, due either to wage or price stickiness, or to the effect of past prices on expectations regarding future prices (the critical factor in Wicksell's own analysis). Thus once prices have been at a certain level (for whatever arbitrary reason), this historical initial condition ties down their subsequent evolution, though they may subsequently drift arbitrarily far from that level. But probably the most important factor, in general, is *the interest-rate policy of the central bank,* insofar as this responds to the evolution of some price index. A state of affairs in which all wages and prices were 10 percent higher than they presently are would not be equally possible as an equilibrium, if the observation of such a jump in the price level would trigger an increase in interest rates, as called for under either a Wicksellian rule or the Taylor rule.

The way in which the equilibrium price level can be determined by the central bank's interest-rate response to price-level variations, without any reference to the associated fluctuations in any monetary aggregate, can be illustrated very simply. Let us suppose that the equilibrium real rate of interest is determined by real factors (such as time preference

[41] As noted earlier, Wicksell's basic exposition of his theory is for the case of a "pure credit economy."

and the productivity of capital), in complete independence of how nominal quantities may evolve,[42] and let $\{r_t\}$ be an exogenous stochastic process for this real rate. It then follows that the short-term nominal interest rate i_t and the log price level p_t must at all times satisfy the Fisherian relation

$$p_t = E_t p_{t+1} + r_t - i_t, \tag{4.9}$$

assuming rational expectations on the part of the private sector. Because r_t is a certain function of exogenous real factors, rather than the measured real rate of return, this is an *equilibrium* relation — the condition required for equality between aggregate saving and investment — rather than an identity. This "flexible-price IS equation" indicates how the price level that clears the goods market — or equivalently, that equates saving and investment — depends on the expected future price level, real factors affecting saving and investment, and the nominal interest rate controlled by the central bank.

Now suppose that the central bank sets the short-term nominal interest rate according to the Wicksellian rule

$$i_t = \bar{\imath}_t + \phi\, p_t, \tag{4.10}$$

which generalizes (4.1) in allowing for a time-varying intercept, indicating possible shifts over time in monetary policy. Suppose furthermore that $\{\bar{\imath}_t\}$ is another exogenous stochastic process (that is, determined independently of the evolution of prices), that may or may not be correlated with the exogenous fluctuations in the equilibrium real rate of interest. Then substituting (**??**) into (4.9) to eliminate i_t, we obtain a relation of the form

$$p_t = \alpha E_t p_{t+1} + \alpha(r_t - \bar{\imath}_t) \tag{4.11}$$

to determine the equilibrium evolution of the price level, given the exogenous processes $\{r_t, \bar{\imath}_t\}$, where $\alpha \equiv 1/(1+\phi)$ is a coefficient satisfying $0 < \alpha < 1$.

In the case that $\{r_t, \bar{\imath}_t\}$ are bounded processes, equation (4.11) has a unique bounded

[42] In chapter 2, we present assumptions under which this is true in an explicit intertemporal equilibrium model with flexible prices.

solution, obtained by "solving forward," namely

$$p_t = \sum_{j=0}^{\infty} \alpha^{j+1}\, E_t(r_{t+j} - \bar{\imath}_{t+j}). \tag{4.12}$$

Thus the equilibrium price level fluctuates in a bounded interval around the long-run average value

$$\bar{p} \equiv \phi^{-1}\,(\bar{r} - \bar{\imath}),$$

where $\bar{r}, \bar{\imath}$ are the long-run average values of r_t and $\bar{\imath}_t$ respectively. This analysis shows how a policy rule that involves no targets for any monetary aggregate can nonetheless control the long-run price level. It also shows how the determinants of equilibrium inflation can be understood without any reference to the determinants of either the money supply or of money demand — indeed, it does not matter for the analysis just presented whether there is any well-defined demand function for the monetary base.

The account of price-level determination implied by this theory has a strongly Wicksellian flavor. We observe from (4.12) that the equilibrium price level at any date t is increased by either a "loosening" of monetary policy — represented by a reduction of the intercept term $\bar{\imath}_t$ — not justified by any decline in the equilibrium real rate, or by an increase in the equilibrium real rate r_t that is not matched by a tightening of policy. Our forward-looking model also implies that any news that allows the private sector to forecast the future occurrence of either of these things should stimulate inflation immediately.

In the simple model sketched here, there is no distinction of the sort that Wicksell makes between the actual real rate of return and the "natural rate" that would occur in an intertemporal equilibrium with flexible prices. In chapter 4, however, we show how one can usefully introduce such a distinction, in the context of a model with sticky prices. When prices are temporarily sticky, the real rate of return at which borrowing and lending occurs can differ from the natural rate of interest, just as the level of output can differ from its natural rate; and the degree to which both occur depends on the degree of instability of the overall price level, as it is only when the general level of prices is changing that price rigidity creates distortions. Equilibrium condition (4.9) must then be replaced by a more general one, of the form

$$i_t - E_t\pi_{t+1} = r_t^n + \delta(\pi_t, \ldots), \tag{4.13}$$

where r_t^n is the natural rate of interest (here still assumed to depend only on exogenous real factors), and the discrepancy $\delta(\cdot)$ is a function of both current and expected future inflation.[43] The system consisting of conditions (4.10) and (4.13) can again be solved for a unique bounded process for the price level, and the solution is of the form

$$p_t = \sum_{j=0}^{\infty} \psi_j \; E_t(r_{t+j}^n - \bar{\imath}_{t+j})$$

for certain coefficients $\{\psi_j\}$. Thus in the more general case, it is variation in the *natural rate* of interest due to real disturbances of various sorts, to the extent that such variation is not matched by corresponding adjustment of the central bank's reaction function, that causes inflation variation. Just as in Wicksell's theory, real disturbances affecting desired saving and investment are predicted to be important sources of price-level variations; and as in that theory, the implied variation in the natural rate of interest is a useful summary statistic for the way in which a variety of real disturbances should affect the rate of inflation.

In chapters 2 and 4, we show how a similar analysis of equilibrium inflation determination is possible in the case of a rule like the Taylor rule. In this case a positive response of the interest rate to fluctuations in the inflation rate is not sufficient to guarantee a determinate equilibrium (a unique non-explosive equilibrium path for the inflation rate, rather than the price level); it is instead necessary that the response coefficient be greater than one, in accordance with the "Taylor principle" mentioned earlier. But in that case similar results are obtained; equilibrium inflation is a function of current and expected future gaps between the natural rate of interest and the intercept term in the Taylor rule.

We find, then, that it is possible to determine the consequences for inflation dynamics of a given monetary policy rule when it is expressed in terms of an interest-rate rule, without any need to first translate the rule into an implied state-contingent path for the money

[43] In the case of the basic neo-Wicksellian model developed in chapter 4, δ is a function of $\pi_t, E_t\pi_{t+1}$, and $E_t\pi_{t+2}$..

supply. Hence the terms used to describe both actual central-bank policies and simple policy prescriptions in the literature summarized above are not inappropriate ones; we can conveniently analyze the consequences of systematic policies of these types as functions of exactly the coefficients appearing in Table 1.1.

While the usefulness of the neo-Wicksellian framework sketched here is perhaps most evident in the case of an economy without monetary frictions, so that the familiar quantity-theoretic apparatus is plainly inapplicable, it is also equally useful in the case of an economy in which monetary frictions still exist, at least of the modest sort that are indicated by the observed willingness to hold non-interest-earning currency in an advanced economies like that of the U.S. today. In what we have written above, we have not actually relied upon any assumed absence of monetary frictions, except in assuming that the equilibrium real rate of interest (or more generally, the natural rate of interest) is independent of the evolution of nominal variables. But even in the presence of transactions frictions resulting in a demand for base money despite its below-market rate of return, it is unlikely that the natural rate of interest is much affected, as a quantitative matter, by variations in the rate of inflation. (The accuracy of the approximation involved in neglecting such effects is considered numerically in chapters 2 and 4.) Hence the approach proposed here is also appropriate for analysis of the effects of a Taylor rule in an economy like that of the U.S., where changes in the Fed's interest-rate operating target are implemented through adjustments in the supply of (non-interest-earning) Fed balances. The monetary frictions that create a demand for such balances are important for the size of quantity adjustment required to achieve a given change in the funds rate, but of little importance for the effects upon output and inflation of any given change in the path of the funds rate.

Nor are the predictions of the neo-Wicksellian theory really any different from those of a standard quantity-theoretic analysis, despite the apparent dissimilarity of approach. In a quantity-theoretic analysis of inflation determination, central importance is given to the money-demand relation that describes desired real money balances as a function of interest rates and other variables. However, in the case of an interest-rate rule like any of those described above, the money supply varies passively so as to satisfy this relation; hence the relation places no restrictions upon the equilibrium evolution of interest rates and goods prices. Thus in such a system, the equilibrium paths of interest and prices are determined by solving equations (4.9) and (4.10), or equations (4.10) and (4.13) in the case of sticky prices, just as above. Once one knows the equilibrium paths of interest and prices, the money-demand relation can then be used to determined the implied evolution of the money supply as well. But this last relation plays no important role in determining equilibrium inflation under such an analysis. And an insistence upon first solving for the state-contingent path of the money supply implied by the policy rule, and then deriving the equilibrium path of inflation from this along quantity-theoretic lines, would be an unnecessarily roundabout procedure, given that one must first solve for the path of prices and interest rates in order to determine the path of the money supply.

The neo-Wicksellian approach is thus clearly preferable, even granting the existence of a well-defined, econometrically stable money-demand relation, if one wishes to analyze the consequences of interest-rate rules such as the Taylor rule. But, it might be asked, is it clear that *desirable* policy rules should belong to this class, regardless of the current popularity of such prescriptions? Might not a money-growth rule be preferable, in which case a more traditional quantity-theoretic approach would also be necessary in order to explain its effects?

The results of this study suggest that the answer is no. As is shown in chapter 8, it is possible to derive optimal policy rules that indicate how a short-term nominal interest-rate operating target should be set, as a function of the projected evolution of inflation and the output gap, without any reference to the paths of monetary aggregates. It is argued furthermore that this form of prescription has the advantage, relative to other possible characterizations of optimal policy, of being invariant under a larger class of possible exogenous disturbances. For example, in the case of an economy with a well-defined demand for base money, it is possible to compute both the state-contingent evolution of overnight interest rates and the state-contingent evolution of the monetary base in an optimal equilibrium. However, the desired evolution of the monetary base, even when well-defined, will depend upon factors that are of little or no relevance to the desired evolution of interest rates, and this makes it simpler to characterize optimal policy in terms of an interest-rate operating target.

One such factor is the dependence of the optimal path of the monetary base on changes in the transactions technology — for example, available opportunities for substitution among alternative means of payment — that have significant effects on money demand in the presence of a given interest differential between base money and other riskless assets, but have little effect on the relations between interest rates and the incentives for intertemporal substitution of expenditure that determine the desired evolution of interest rates. (In an economy where the financial system is already highly efficient, one expects further innovations to represent movements from one highly efficient system to another, so that the relations between interest rates and the real allocation of resources will remain near that predicted by a model with no financial frictions; but money demand may be greatly affected in percentage terms, as it ceases to be defined in the frictionless limit.) Another is the dependence of the optimal evolution of the monetary base on the details of monetary policy implementation. The desired path of money-market interest rates is largely independent of the rate of interest paid on the monetary base; this instead depends on the intertemporal marginal rates of substitution and marginal products of real investment implied by the desired allocation of resources, and on the desired path for inflation. But the desired path of the monetary base depends greatly on whether it is assumed for institutional reasons that zero interest is paid on base money, or whether instead the interest paid on money varies when other short-term interest rates vary; for the demand for base money depends not on the absolute level of nominal interest rates, but on the *spread* between the interest rate paid on base money and that available on other assets.

Thus even when the desired evolution of the monetary base is well-defined, it is more dependent on special "technical" factors than is the desired evolution of short-term nominal interest rates; this makes a description of optimal monetary policy in terms of a state-contingent money growth rate less convenient. And if, as some forecast, monetary frictions are largely eliminated in the coming century owing to the development of electronic payments media, a description of optimal policy in terms of the desired evolution of a monetary aggregate is likely to become awkward if not altogether impossible. Yet a description of optimal policy in terms of the principles that should regulate the adjustment of an interest-rate operating target should still be possible. Indeed, increasing efficiency of the financial system should only simplify the description of optimal policy in these terms, insofar as the arbitrage relations that connect the overnight interest rate directly targeted by the central bank to other interest rates and asset prices should become simpler and more reliable. Hence the neo-Wicksellian framework proposed here directs attention to precisely those elements of the monetary transmission mechanism that are likely to remain of fundamental importance for the design of effective monetary policies in a world of increasingly efficient financial markets and institutions.

5 Plan of the Book

Part I of this book develops a theoretical framework that can be used to analyze the consequences of alternative monetary policy rules, in a way that takes full account of the consequences of forward-looking private sector behavior. Chapter 2 begins by considering price-level determination when monetary policy is specified by an interest-rate rule, in the case of a model where, for simplicity, prices are completely flexible and the supply of goods is given by an exogenous endowment. This chapter demonstrates the possibility of a coherent theory of price-level determination even in the complete absence of monetary frictions — a special case that is considered repeatedly in what follows, in order to direct attention more closely to the economic relations that are considered to be of more fundamental importance for the characterization of optimal policy. But it also considers price-level determination under an interest-rate rule in a standard optimization-based monetarist framework, allowing a comparison between the consequences of monetary targeting and those of commitment to an interest-rate rule, and an analysis of the extent to which the presence of monetary frictions changes one's conclusions about the effects of an interest-rate rule.

Chapter 3 then introduces endogenous goods supply and nominal price and wage rigidities, so that monetary policy can affect the level of real activity as well as the inflation rate. Considerable attention is given to the microeconomic foundations of the aggregate-supply relations that result from delays in the adjustment of prices or wages, in order to select specifications (from among those that might appear similarly consistent with econometric evidence) with clear behavioral interpretations, that thereby allow one to take account of the "Lucas critique". At the same time, attention is also paid to the need to find a specification of the dynamic relations between real activity and inflation that is consistent with econometric evidence regarding the effects of identified monetary disturbances. A series of modifications of a basic sticky-price model are introduced that can improve the model's fit with estimated responses on various dimensions.

Chapter 4 then integrates the analysis in chapters 2 and 3, considering the effects of interest-rate rules in a framework where monetary policy can affect real activity, and where feedback from measures of real activity to the central bank's interest-rate operating target matter for the predicted effects of such rules. In the neo-Wicksellian framework developed here, inflation dynamics result from the interaction between real disturbances on the one hand and the central bank's interest-rate rule on the other. Wicksell's "natural rate of interest" is shown to play a central role, summarizing the effects of a variety of real disturbances that are relevant for inflation and output-gap determination, in the case of a class of policy rules that may be thought of as generalized Taylor rules. The chapter also includes a first analysis of the consequences of such a framework for the design of desirable policy rules, by considering the conditions under which a Taylor-like rule should be able to stabilize inflation and the output gap.

Chapter 5 completes the theoretical framework by considering the consequences for inflation determination of alternative fiscal policy rules. The analyses of interest-rate rules in chapters 2 and 4 are conducted under a particular assumption about fiscal policy — that it is at least "locally Ricardian" — that is arguably realistic given current fiscal policy commitments in countries like the U.S., but that need not hold, and that may not have held even in the U.S. at all times. Chapter 5 shows that the inflation dynamics implied by a given interest-rate rule may be different in the case of alternative fiscal rules, and uses this analysis to explain the consequences of the bond-price support regime in the U.S. during the 1940s. In fact, the analysis offered later in the book implies that an optimal regime should involve a locally Ricardian fiscal policy, so that the case emphasized in chapters 2 and 4 is argued to be the relevant one for the choice of an optimal monetary policy rule. But it is important to recognize that an optimal policy regime must include the proper sort of fiscal commitment as well, and that a commitment to an "optimal" interest-rate rule need not imply desirable inflation dynamics in the absence of a suitable fiscal commitment.

Part II of the book then considers the optimal conduct of monetary policy in the light of the theoretical framework introduced in the earlier chapters. Chapter 6 begins by considering appropriate stabilization goals for monetary policy. An advantage of the derivation of our model's structural relations from explicit microeconomic foundations in Part I is that it is possible to ask what sort of monetary policy should best serve economic welfare, given the objectives and constraints of the agents whose decisions account for the observed effects of monetary policy. Chapter 6 considers the connection between the obvious measure of economic welfare in such a model — the expected utility of the representative household — and the stabilization of macroeconomic aggregates such as inflation and the output gap.

It is shown that a quadratic approximation to expected utility, which suffices (under certain conditions) for the derivation of a linear approximation to an optimal policy rule, can be expressed in terms of the expected value of squared deviations of certain aggregate variables from target values for those variables; the variables that are relevant, and the details of the quadratic loss function that can be justified on welfare-theoretic grounds, depend on the microeconomic foundations of one's model of the monetary transmission mechanism. In particular, it is shown that different assumptions regarding price and/or wage stickiness imply that price and/or wage inflation should enter the central bank's loss function in different ways. Nonetheless, it is argued that price stability, suitably interpreted — *e.g.*, quite possibly in terms of an index that includes wages as well as the prices of final goods and services — should be an important consideration, though not necessarily the only one, in the selection of a monetary policy rule. Grounds for inclusion of output-gap stabilization and interest-rate stabilization objectives in the loss function as well are considered; but it is argued that in practice, these additional concerns are not likely to justify either an average rate of inflation much greater than zero or substantial variability in the rate of inflation in response to shocks.

Chapter 7 then considers the optimal state-contingent evolution of inflation, output and interest rates in response to real disturbances of various sorts, from the point of view of the sort of loss function argued for in chapter 6 and in the context of a forward-looking model of the monetary transmission mechanism of the kind developed in Part I. An important general issue treated in this chapter is the way in which optimal control techniques must be adapted in the case of control of a forward-looking system. The responses to shocks under an optimal commitment are distinguished from the equilibrium responses under discretionary optimization by the central bank. Particular attention is given to the fact that in general, optimal responses will be *history-dependent* in a way that is inconsistent with any purely forward-looking decision procedure for monetary policy. Alternative approaches to introducing the desired sort of history-dependence into the conduct of policy are surveyed; the one that is emphasized in this study is the possibility of commitment to a *policy rule* that is history-dependent in the desired way.

Finally, chapter 8 considers the problem of choice of a policy rule to implement the desired state-contingent evolution of inflation, output and interest rates, as derived in chapter 7. From the standpoint taken here, this problem of *implementation* of the desired equilibrium is a non-trivial part of the characterization of optimal policy; for the mapping from the history of exogenous disturbances to the desired overnight interest rate at any point in time is not a suitable description of an optimal policy rule, for reasons taken up in this chapter. Instead, it is argued that a more suitable policy prescription should relate the instrument setting to the evolution (observed or projected) of endogenous variables such as inflation and the output gap, as in the proposals mentioned above. A general method is presented for constructing optimal policy rules of this form in the case of a fairly general class of log-linear structural models and quadratic loss functions; the method is then applied to several of the simple optimizing models of the monetary transmission mechanism developed in previous chapters.

It is shown that optimal rules can easily take the form of generalized Taylor rules, or the form of target criteria for a forecast-targeting procedure like that used at the Bank of England. However, even in the case of fairly simple models of the transmission mechanism, the optimal rules are somewhat different from the proposals described above. While there is a fairly clear logic for rules that respond to (and perhaps only to) variations in inflation and in the output gap, the theoretically appropriate measures of inflation and of the output gap may not be the ones used in the characterizations above of current central-bank behavior. And the optimal rules that we obtain are typically different in their dynamic specifications as well. Optimal rules are history-dependent in ways that neither the classic Taylor rule nor familiar descriptions of inflation-forecast targeting are; and while they may well be more forward-looking than the classic Taylor rule, in all of our calibrated examples they are considerably less forward-looking than the procedures currently used at the inflation-targeting central banks.

Conclusions about the precise content of an optimal policy rule, of course, depend on the details of one's model of the transmission mechanism, and we do not attempt here to reach final conclusions in that regard. The answer is likely to be somewhat different for different countries in any event. Our more important goal is to provide a method that individual central banks can use in order to choose sensible systematic policies on the basis of their own research on the nature of the transmission mechanism in their respective economies. It is hoped that the present essay can provide useful guidelines for such an inquiry.

Interest and Prices

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Contents

Chapter 2

Price-Level Determination Under Interest-Rate Rules

While virtually all central banks use a short-term nominal interest rate (typically an overnight rate, such as the federal funds rate in the U.S.) as their instrument, and an extensive empirical literature characterizes actual monetary policy in terms of estimated central bank "reaction functions" for setting such interest rates, the theoretical literature in monetary economics has almost entirely concerned itself with the analysis of policies that are described by alternative (possibly state-contingent) paths for the money supply. The aim of this chapter is to remedy this oversight by presenting a theory of price-level determination under interest-rate rules of the sort that are often taken to describe actual central bank policies.

We shall argue that it is not necessary, in order to understand the consequences of such rules, to first determine their consequences for the evolution of the money supply, and then analyze the equivalent money-supply rule. Instead, it is possible to analyze price-level determination under such rules in terms of an explanatory framework that gives no importance to either the evolution of the money supply or the determinants of money demand. In this neo-Wicksellian framework, the fundamental determinants of the equilibrium price level are instead the real factors that determine the equilibrium real rate of interest, on the one hand, and the systematic relation between interest rates and prices established by the central bank's policy rule, on the other.

We first expound this approach in the context of a purely cashless economy — one in which there are assumed to be no transactions frictions that can be reduced through the use of money balances, and that accordingly provide a reason for holding such balances even when they earn a rate of return that is dominated by that available on other assets. Such a setting — one that is commonly assumed in financial economics and in purely real models of economic fluctuations alike — allows us to display the relations that are of central importance in the neo-Wicksellian theory in their simplest form.

At the same time, neither the usefulness nor the validity of the approach proposed here depends on a claim that monetary frictions do not exist in actual present-day economies. After expounding the theory for the cashless case, we show how the framework can easily be generalized to allow for monetary frictions, modeled in one or another of the ways that are common in monetarist models of inflation determination (by including real balances in the utility function, or assuming a cash-in-advance constraint). We show in this case that equilibrium relations continue to be obtained that are direct generalizations of the ones obtained for the cashless economy, and that need not even imply results that are too different as a quantitative matter, if the monetary frictions are parameterized in an empirically plausible way. Hence the cashless analysis can be viewed as a useful approximation even in the case of an economy where money balances do facilitate transactions to some extent.

In the case of an economy with transactions frictions, one can *also* analyze price-level determination along traditional monetarist lines: one may view the equilibrium price level as being determined by the expected path of the money supply, although the latter quantity is endogenous, in the case of an interest-rate rule such as the Taylor rule, so that money, prices, and interest rates must be simultaneously determined. In the models considered here, this approach would not yield different ultimate conclusions than the neo-Wicksellian analysis, for the system of equilibrium conditions to be solved is actually the same despite the differing direction of approach. Nonetheless, we shall argue that the neo-Wicksellian interpretation of these equilibrium conditions is a particularly fruitful one, not least because it continues to be possible in the limiting case of a cashless economy.

In this chapter, we expound the basic outlines of the neo-Wicksellian theory in the context of a model with flexible prices and an exogenous supply of goods. This allows us to address a number of basic issues in a particularly simple context, and also allows direct comparison of this theory with the standard quantity-theoretic approach, which, when derived from optimizing models, is also most often expounded in a model with flexible prices. A more complete development of the theory is possible only after the introduction of nominal price rigidities in the following chapter.

1 Price-Level Determination in a Cashless Economy

We begin by considering price-level determination in an economy in which both goods markets and financial markets are completely frictionless: markets are perfectly competitive, prices adjust continuously to clear markets, and there exist markets in which state-contingent securities of any kind may be traded. Under the assumption of frictionless financial markets, it is natural to suppose that no "monetary" assets are needed in order to facilitate transactions.

We shall suppose, however, that there exists a monetary *unit of account* in terms of which prices (of both goods and financial assets) are quoted. This unit of account will be defined in terms of a claim to a certain quantity of a liability of the central bank, which may or may not have any physical existence.[1] This liability is not a claim to future payment of anything except future units of the central-bank liability. As argued in section xx of chapter 1, the special situation of the central bank, as issuer of liabilities that promise to pay only additional units of its own liabilities, allows the central bank to fix both the nominal interest yield on its liabilities and the quantity of them in existence.

[1]Under current U.S. arrangements, which are fairly typical, Federal Reserve notes (U.S. currency) and Federal Reserve balances (credits in an account at the Fed, that can be used for clearing purposes and to satisfy reserve requirements) are freely convertible into one another, and a promise to pay "a dollar" may be discharged by transfer to the creditor (or its bank) of either of these types of financial claim, in the amount of one dollar. In a cashless economy of the kind that some envision for the future, currency need no longer exist; in such a world, the "dollar" would be defined by a claim to a one-dollar balance at the Fed. The fact that in such a world there would be no physical dollars (*i.e.*, dollar bills) would not prevent the use of dollar accounts in making payments; after all, even now, the dollar is not a claim to anything *else*, and is accepted in payment only because of the expectation that it can be transferred to someone else in a subsequent transaction.

While we assume that there is no reason why private parties need to hold this particular asset, or receive any benefit from doing so that would not be obtained by holding any other similarly riskless financial claim denominated in terms of the same unit of account, we assume that they choose to hold financial claims on the government along with privately-issued financial claims. The conditions under which the private sector is willing to hold the liabilities of the central bank, along with other government liabilities, are described by arbitrage relations of the kind that are familiar from financial economics. In an equilibrium, where these relations are satisfied, there then exists a well-defined exchange ratio between money and real goods and services.

In a frictionless world of this kind, base money — the monetary liabilities of the central bank — is a perfect substitute for other riskless nominal assets of similarly short maturity, whether these are private obligations or other (non-monetary) government obligations. As a result, variations in the nominal size of the monetary base, due to for example to open-market purchases of other sorts of government obligations by the central bank need have no effect on the prices or interest rates that represent a market equilibrium. Yet this does not mean that in such a world, the central bank has no control over the equilibrium prices of goods in terms of money. As we shall see, the central bank's policy rule is one of the key determinants of the equilibrium price level even in a cashless economy; and it is possible, at least in principle, for the central bank to stabilize the price level around a desired level (or deterministic trend path) through skillful use of the tools at its disposal. But in such a world, the crucial tool available to the central bank will not be open-market operations, but the possibility of adjusting the interest rate paid on central-bank balances.

1.1 An Asset-Pricing Model with Nominal Assets

Consider an economy made up of a large number of identical households. The representative household seeks to maximize the expected value of a discounted sum of period contributions to utility of the form

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t; \xi_t) \right\}. \tag{1.1}$$

Here $0 < \beta < 1$ is a discount factor, and the period contribution to utility u depends upon the level of consumption C_t of the economy's single good. We also allow for exogenous stochastic disturbances ξ_t to the period utility function, which we may think of as representing variation in households' impatience to consume. The presence of this term represents a first simple example of something of considerable importance for our general conception of the problem facing central banks, namely, the existence of real disturbances that should be expected to change the equilibrium real rate of return, and hence the level of nominal interest rates required for price stability. For any given realization of ξ_t, we assume that the period utility function $u(C;\xi_t)$ is concave and strictly increasing in C.

As noted in the introduction, we shall assume *complete financial markets, i.e.,* that available financial assets completely span the relevant uncertainty faced by households about future income, prices, taste shocks, and so on, so that each household faces a single intertemporal budget constraint. Under the assumption of complete markets, a household's *flow* budget constraint each period can be written in the form

$$M_t + B_t \leq W_t + P_tY_t - T_t - P_tC_t. \tag{1.2}$$

Here M_t denotes the household's nominal end-of-period balances in the distinguished financial asset (the monetary base) which represents the economy's unit of account, B_t represents the nominal value (in terms of this unit of account) of the household's end-of-period portfolio of all *other* financial assets (whether privately issued or claims on the government), W_t represents *beginning*-of-period financial wealth (now counting the monetary base along with other assets), Y_t is an exogenous (possibly stochastic) endowment of the single good, P_t is the price of the good in terms of the monetary unit, and T_t represents net (nominal) tax collections by the government. The constraint says that total end-of-period financial assets (money plus bonds) can be worth no more than the value of financial wealth brought into the period, plus non-financial income during the period net of taxes and the value of consumption spending. Interest income is not written explicitly in (1.2), because it is assumed to accrue *between* the discrete dates at which decisions are made; thus W_t already includes the interest earned on bonds held at the end of period $t-1$.[2]

It is important to note that in (1.2), B_t does not refer to the quantity held of some *single* type of bond; as we assume complete markets, households must be able, at least in principle, to hold any of a wide selection of instruments with different state-contingent returns. We need not, however, introduce any notation for the particular types of financial instruments that are traded. (This is one of the conveniences of the assumption of complete markets.) Since any pattern of state-contingent payoffs in the future that a household may desire can be arranged (for the appropriate price), we can write the household's consumption planning and wealth-accumulation problems without any explicit reference to the quantities that it holds of particular assets; and if there are redundant assets, there will not actually be determinate demands for individual assets (our assumption in the case of the monetary base). We distinguish the household's holdings of the monetary base from the rest of its end-of-period portfolio, however, in order to allow us to explicitly discuss the central bank's supply of this asset and the interest paid on it.

In the proposed notation, we may simply represent the household's portfolio choice as a choice of the state-contingent value A_{t+1} of its non-monetary portfolio at the beginning of the next period. Total beginning-of-period wealth in the following period is then given by

$$W_{t+1} = (1+i_t^m)M_t + A_{t+1}, \tag{1.3}$$

where i_t^m is the nominal interest rate paid on money balances held at the end of period t. Note that this implies that W_{t+1}, as a function of the state of the world realized in period $t+1$, is determined by decisions made in period t; thus W_t is a predetermined state variable in (1.2).

At the time of the portfolio decision, A_{t+1} is a random variable, whose value will depend upon the state of the world in period $t+1$. But the household chooses the complete

[2] See equation (1.6) below. Though we often refer to a succession of "periods", as is common in the macroeconomic literature, our models are formally ones in which trading occurs at a sequence of discrete points in time. References to "beginning-of-period" and "end-of-period" portfolios are simply notation to keep track of the effects of trades, not references to different points in time, between which interest may accrue.

specification of this random variable, its value in every possible state. The absence of arbitrage opportunities (a necessary requirement for equilibrium) then requires that there exist a (unique) *stochastic discount factor* (or asset pricing kernel) $Q_{t,t+1}$ with the property that the price in period t of any bond portfolio with random value A_{t+1} in the following period is given by

$$B_t = E_t[Q_{t,t+1}A_{t+1}]. \tag{1.4}$$

(As of date t, $Q_{t,t+1}$ remains a random variable; and E_t refers to the expectation conditional upon the state of the world at date t.) In terms of this discount factor, the riskless *short-term* (one-period) *nominal interest rate* i_t corresponds to the solution to the equation

$$\frac{1}{1+i_t} = E_t[Q_{t,t+1}]. \tag{1.5}$$

Note that if it happens that the representative household chooses to hold a purely riskless portfolio (in nominal terms), so that A_{t+1} is perfectly forecastable at date t, (1.4) states simply that $A_{t+1} = (1+i_t)B_t$. Substituting this into (1.3), and the resulting expression for W_t into (1.2), which holds with equality in equilibrium, we obtain the familiar difference equation

$$M_t + B_t = (1+i_t^m)M_{t-1} + (1+i_{t-1})B_{t-1} + P_tY_t - T_t - P_tC_t \tag{1.6}$$

for the evolution of B_t. This will actually be an equilibrium condition in the case that the government issues only riskless one-period debt; but it is still important, even in that case, to recognize that an individual household's budget constraint allows it the possibility of shifting wealth across states of the world in other ways.[3]

More generally, then, equations (1.2), (1.3) and (1.4) together give a complete description of the household's flow budget constraint. Using (1.3) and (1.4) to eliminate B_t from (1.2),

[3] Condition (1.6) actually represents the correct flow budget constraint if we assume such radically incomplete markets that households can neither borrow nor lend except in terms of the single instrument assumed to be issued by the government. This case of a single traded asset is often considered in the literature on consumption theory (see, *e.g.*, Obstfeld and Rogoff, 1996, sec. 2.3). And in the present context, with identical households, it makes no real difference what we assume about the number of financial markets that are open. However, the characterization of optimal household plans is simplest in the case of complete markets, and the introduction of market valuations for arbitrary random income streams will prove useful, in the next chapter, when we need to consider the optimal pricing decisions of firms.

the constraint can alternatively be written

$$(1 - E_t Q_{t,t+1}(1 + i_t^m))M_t + E_t[Q_{t,t+1}W_{t+1}] \leq W_t + [P_t Y_t - T_t - P_t C_t].$$

Using (1.5), this becomes

$$P_t C_t + \Delta_t M_t + E_t[Q_{t,t+1}W_{t+1}] \leq W_t + [P_t Y_t - T_t]. \tag{1.7}$$

where

$$\Delta_t \equiv \frac{i_t - i_t^m}{1 + i_t}. \tag{1.8}$$

It is clear from this version that the interest-rate differential Δ_t between non-monetary and monetary assets represents the opportunity cost of holding wealth in monetary form. Given its planned state-contingent wealth W_{t+1} at the beginning of the following period, the household can choose any values $C_t, M_t \geq 0$ that satisfy (1.7).[4]

A complete description of the household's budget constraints requires that we also specify a limit on borrowing, to prevent "Ponzi schemes" of the kind that would otherwise be consistent with the infinite sequence of flow budget constraints in an infinite-horizon model. In the spirit of our assumption of perfectly frictionless financial markets, it is natural to suppose that there is no obstacle to borrowing against after-tax endowment income that may be anticipated (even if in only some states of the world) at any future date. The implied constraint is then that the household must hold a net portfolio at the end of period t (possibly including *issuance* of some securities, in order to borrow against future income) such that the wealth W_{t+1} transferred into the next period satisfies the bound

$$W_{t+1} \geq -\sum_{T=t+1}^{\infty} E_{t+1}[Q_{t+1,T}(P_T Y_T - T_T)] \tag{1.9}$$

with certainty, *i.e.*, in *each* state of the world that may be reached in period $t+1$. Here the general stochastic discount factor $Q_{t,T}$ for discounting (nominal) income in period T back to

[4]We assume that money balances must be non-negative because this asset is *defined* as a liability of the central bank, that accordingly cannot be issued by any other parties, even though (under our assumption of complete markets) private securities are issued that are equivalent in terms of their state-contingent payouts. This non-negativity constraint is another reason to single out this asset from the others in writing the household's budget constraints; for we assume no short-sale constraints in the case of any other securities in our model with frictionless financial markets.

an earlier period t is defined by

$$Q_{t,T} \equiv \prod_{s=t+1}^{T} Q_{s-1,s}.$$

(We also use the notation $Q_{t,t} \equiv 1$.) Condition (1.9) then says that a household cannot plan to be indebted in any state in an amount greater than the present value of all subsequent after-tax non-financial income.

The entire infinite sequence of flow budget constraints (1.7) and borrowing limits (1.9) are equivalent to a single *intertemporal* (or lifetime) budget constraint for the household. We note first of all that unless the present value on the right-hand side of (1.9) is well-defined (*i.e.,* the infinite sum converges), the household has no budget constraint: Ponzi schemes are possible, hence unlimited consumption is affordable. Furthermore, if the present value is infinite looking forward from any state of the world, at any date, unbounded consumption is possible not only at that date and in all other states (including along histories under which the state in question never occurs); for with complete markets, it is possible to borrow against that state to finance unbounded consumption in any other state.

We may thus restrict attention to the case in which Ponzi schemes are not possible, because

$$\sum_{T=t}^{\infty} E_t[Q_{t,T}(P_T y_T - T_T)] < \infty \tag{1.10}$$

at all times.[5] The budget constraint is also undefined unless interest rates satisfy the lower bound

$$i_t \geq i_t^m \tag{1.11}$$

at all times. For otherwise, an arbitrage opportunity exists; a household can finance unlimited consumption by shorting riskless one-period bonds (*i.e.,* borrowing at the short riskless rate, assumed to be negative) and using the proceeds partly to hold cash sufficient to repurchase the bonds (repay its debt) a period later and partly to finance additional consumption. Because utility is assumed to be strictly increasing in consumption, such an operation con-

[5]Throughout, it should be understood that when we say that such a relation holds "at all times," this also means in all possible states of the world at each date.

tinues to increase utility no much matter how much it may be engaged in. Hence we may also restrict attention to the case in which (1.11) holds at all times.

We are then able to establish that the infinite sequence of flow budget constraints (1.7) are equivalent to a single intertemporal budget constraint.

PROPOSITION 2.1. Consider positive-valued stochastic processes $\{P_t, Q_{t,T}\}$ satisfying (1.10) and (1.11) at all dates, and let $\{C_t, M_t\}$ be non-negative-valued processes representing a possible consumption and money-accumulation plan for the household. Then there exists a specification of the household's portfolio plan at each date satisfying both the flow budget constraint (1.7) and the borrowing limit (1.9) at each date, if and only if the plans $\{C_t, M_t\}$ satisfy the constraint

$$\sum_{t=0}^{\infty} E_0 Q_{0,t}\,[P_t C_t + \Delta_t M_t] \leq W_0 + \sum_{t=0}^{\infty} E_0 Q_{0,t}[P_t Y_t - T_t]. \tag{1.12}$$

The proof is given in the appendix. Note that the intertemporal budget constraint states simply that the present value of the household's planned consumption over the entire indefinite future, plus the cost to it of its planned money holdings, must not exceed its initial financial wealth plus the present value of its expected after-tax income from sources other than financial assets. One can also show (see the proof in the appendix) that the household's continuation plan, looking forward from any date t (*i.e.*, its plan for dates $T \geq t$, in all of the states that remain possible given the state of the world at date t), must satisfy the corresponding intertemporal budget constraint

$$\sum_{s=t}^{\infty} E_t Q_{t,s}\,[P_s c_s + \Delta_s M_s] \leq W_t + \sum_{s=t}^{\infty} E_t Q_{t,s}[P_s y_s - T_s]. \tag{1.13}$$

The household's optimization problem is then to choose processes C_t, $M_t \geq 0$ for all dates $t \geq 0$, satisfying (1.12) given its initial wealth W_0 and the goods prices and asset prices (indicated by the stochastic discount factors $Q_{t,t+1}$) that it expects to face, so as to maximize (1.1). Given an optimal choice of these processes, an optimal path for W_t may be constructed as in the proof of Proposition 2.1. Given a stochastic process for W_{t+1}, the implied processes for A_{t+1} and for B_t are given by (1.3) and (1.4).

Because this is essentially a standard concave optimization problem subject to a single budget constraint, necessary and sufficient conditions for household optimization are easily given. First of all, (1.10) and (1.11) must hold at all times, since otherwise no optimal plan exists (as more consumption is always possible). Second, since in the cashless economy there is no non-pecuniary benefit to holding money balances, household optimization requires that either

$$M_t = 0 \tag{1.14}$$

or

$$i_t = i_t^m \tag{1.15}$$

at each date and in each possible state (though which condition obtains may differ across dates and across states).

Third, by equating marginal rates of substitution to relative prices, we obtain the first-order conditions

$$\frac{u_c(C_t;\xi_t)}{u_c(C_{t+1};\xi_{t+1})} = \frac{\beta}{Q_{t,t+1}}\frac{P_t}{P_{t+1}}. \tag{1.16}$$

Here U_c is the partial derivative of U with respect to the level of consumption. This condition must hold for each possible state at each date $t \geq 0$, *and* for each possible state that may occur at date $t+1$, given the state that has occurred at date t. ($Q_{t,t+1}$ indicates the value of the discount factor in a particular state at date $t+1$.) Using (1.5), condition (1.16) implies that the short-term nominal interest rate must satisfy

$$1 + i_t = \beta^{-1}\left\{E_t\left[\frac{u_c(C_{t+1};\xi_{t+1})}{u_c(C_t,;\xi_t)}\frac{P_t}{P_{t+1}}\right]\right\}^{-1} \tag{1.17}$$

at each date.

Finally, optimization requires that the household exhaust its intertemporal budget constraint; that is, (1.13) must hold as an equality at each date. Equivalently, the flow budget constraint (1.2) must hold as an equality at each date, and in addition, the household's wealth accumulation must satisfy the *transversality condition*

$$\lim_{T\to\infty} E_t[Q_{t,T} W_T] = 0. \tag{1.18}$$

(Condition (1.13), stated as a strict equality, implies both that (1.2) must hold as a strict equality at each date $T \geq t$, and that (1.18) must hold. Conversely, the latter set of conditions imply that (1.13) holds with strict equality, looking forward from date t.) Finally, given that (1.13) must hold with strict equality, condition (1.10) may equivalently be written

$$E_t\left\{\sum_{T=t}^{\infty} Q_{t,T}\left[P_T C_T + \Delta_T M_T\right]\right\} < \infty. \tag{1.19}$$

We thus obtain a set of conditions — (1.2) as a strict equality; the requirement that either (1.14) or (1.15) hold with equality, in addition to the inequality conditions (1.11) and $M_t \geq 0$; (1.16); (1.18); and (1.19) — that must hold at all times in order for the representative household's actions to be optimal. At the same time, one can show that this set of conditions suffices for optimality as well.

We may now state the complete set of conditions for a rational expectations (or intertemporal) equilibrium in this model. In addition to the conditions just stated for household optimization, markets must clear at all dates. This means that household demands must satisfy

$$C_t = Y_t, \qquad M_t = M_t^s, \qquad A_{t+1} = A_{t+1}^s$$

at all dates. Here M_t^s refers to the supply of base money by the central bank, which we assume to be positive at all dates. A_{t+1}^s refers to the aggregate value at the beginning of period $t+1$ of government bonds in the hands of the public at the end of period t . (In general, it would not suffice for bond-market clearing to require that $B_t = B_t^s$, where B_t^s denotes the market value of government bonds outstanding at the end of period t, as this could allow households to demand a portfolio with different state-contingent payoffs than the aggregate supply of government bonds.) If we specify the supply of government bonds in more primitive terms by specifying the variables $\{B_{t,t+j}^s\}$, where for each date t and each $j \geq 1$, $B_{t,t+j}^s$ denotes the total (nominal) coupons that the government promises to pay at date $t+j$ on bonds that are outstanding at the end of period t, then

$$A^s_{t+1} \equiv \sum_{j=1}^{\infty} E_{t+1}[Q_{t+1,t+j} B^s_{t,t+j}]$$

in each possible state that may be reached at date $t+1$. Finally, note that we abstract here from government purchases of real goods and services (though the model is extended below to allow for them).

Given that $M^s_t > 0$, market-clearing implies that (1.14) cannot hold, and hence that (1.15) must hold at all times. Substituting the market-clearing conditions into conditions (1.16) – (1.17) for household optimization, we obtain equilibrium conditions

$$\frac{u_c(Y_t;\xi_t)}{u_c(Y_{t+1};\xi_{t+1})} = \frac{\beta}{Q_{t,t+1}} \frac{P_t}{P_{t+1}}, \tag{1.20}$$

$$1 + i_t = \beta^{-1} \left\{ E_t \left[\frac{u_c(Y_{t+1};\xi_{t+1})}{u_c(Y_t;\xi_t)} \frac{P_t}{P_{t+1}} \right] \right\}^{-1} \tag{1.21}$$

for each date. Note that the latter relation takes the form of a "Fisher equation" for the nominal interest rate, where the intertemporal marginal rate of substitution of the representative household plays the role of the real-interest factor.

Finally, substituting our solution (1.20) for the stochastic discount factor into (1.18) and (1.19), the latter conditions take the form

$$\lim_{T\to\infty} \beta^T E_t[u_c(Y_t;\xi_T) W^s_T / P_T] = 0, \tag{1.22}$$

$$\sum_{T=t}^{\infty} \beta^T E_t\left[u_c(Y_t;\xi_T) Y_T\right] < \infty. \tag{1.23}$$

(Here we have also used the market-clearing conditions to equate W_T with $W^s_T \equiv (1 + i^m_{T-1}) M^s_{T-1} + A^s_T$, the total supply of nominal claims on the government at the beginning of period T, and (1.15) to substitute for the factor Δ_T in (1.19).) A rational-expectations equilibrium is then a collection of processes that satisfy (1.15), (1.21), (1.22) and (1.23) at all dates $t \geq 0$.

The transversality condition (1.22) can equivalently be written in a possibly more familiar form, in terms of the end-of-period value of total government liabilities, $D_t \equiv M^s_t + B^s_t$.

PROPOSITION 2.2. Let assets be priced by a system of stochastic discount factors that satisfy (1.20), and consider processes $\{P_t, i_t, i_t^m, M_t^s, W_t^s\}$ that satisfy (1.15), (1.21), and (1.23) at all dates, given the exogenous processes $\{Y_t, \xi_t\}$. Then these processes satisfy (1.22) as well if and only if they satisfy

$$\lim_{T\to\infty} \beta^T E_t[u_c(Y_T;\xi_T)D_T/P_T] = 0. \tag{1.24}$$

The proof is given in the appendix. It follows that we can equivalently define equilibrium as follows.

DEFINITION. A *rational-expectations equilibrium* of the cashless economy is a pair of processes $\{P_t, i_t\}$ that satisfy (1.15), (1.21), (1.23), and (1.24) at all dates $t \geq 0$, given the exogenous processes $\{Y_t, \xi_t\}$, and evolution of the variables $\{i_t^m, M_t^s, D_t\}$ consistent with the monetary-fiscal policy regime.

This latter formulation is especially useful in that it allows us to specify fiscal policy in terms of restrictions on the evolution of the total government liabilities, or alternatively, restrictions on the path of the conventional government budget deficit.

Note that we need not include the additional equilibrium condition (1.20) in our definition of rational-expectations equilibrium, if we are interested only in the determination of equilibrium prices and interest rates. (The additional condition must be appended to our system, of course, if we are interested in other equilibrium asset prices.) Nor is there any additional equilibrium condition corresponding to the requirement that (1.2) hold with equality; this condition is necessarily satisfied (when we substitute the market-clearing conditions) as long as the supplies of government liabilities evolve in accordance with the flow *government budget constraint*

$$E_t[Q_{t,t+1}W_{t+1}^s] = W_t^s - T_t - \Delta_t M_t^s. \tag{1.25}$$

We shall assume that the monetary-fiscal policy regime satisfies this constraint at all times. We then have a system of two equalities at each date, (1.15) and (1.21), to determine the two endogenous variables P_t and i_t, together with the bounds (1.23) and (1.24) that the solution must satisfy.

Our notation thus far allows only for fiscal policies consisting of taxes or transfers. But the framework above is easily extended to allow for government purchases of goods and services as well, without any material change being required in the above equilibrium conditions. Let government purchases of the single good in period t be denoted G_t, and suppose that $\{G_t\}$ is an exogenous process, such that $G_t < Y_t$ at all dates.[6] Market clearing then requires that $C_t + G_t = Y_t$ at all dates. Substitution of this relation into the conditions for optimization by the representative household then leads to equilibrium conditions such as

$$1 + i_t = \beta^{-1} \left\{ E_t \left[\frac{u_c(Y_{t+1} - G_{t+1}; \xi_{t+1})}{u_c(Y_t - G_t; \xi_t)} \frac{P_t}{P_{t+1}} \right] \right\}^{-1}, \tag{1.26}$$

generalizing (1.21).

We note that (1.26) is obtained from the previous equation (1.21) by replacing $u_c(Y_t; \xi_t)$ by $u_c(Y_t - G_t; \xi_t)$ each time it occurs. The same is true for the other equilibrium conditions (1.20), (1.23) and (1.24) as well. Alternatively, we obtain the equilibrium conditions for the general case by replacing the "direct" utility function $u(C_t; \xi_t)$ throughout our calculations by the "indirect" utility

$$\tilde{u}(Y_t; \tilde{\xi}_t) \equiv u(Y_t - G_t; \xi_t),) \tag{1.27}$$

indicating the utility flow to the representative household as a function of its "total demand" for resources Y_t, where total demand adds the resources consumed by the government on the household's behalf (its per-capita share of government purchases) to the household's private consumption.[7] In this indirect utility function, $\tilde{\xi}_t$ indicates a vector of disturbances that includes both G_t and the taste shock ξ_t.

[6] One might, of course, also consider fiscal policies under which G_t is endogenously determined, for example as the solution to some welfare-maximization problem of the government's. In the present study, however, we shall assume that government purchases are given exogenously. We allow for endogeneity of the level of net tax collections, as for example in the next section, and this is some importance for our theory of price-level determination. See section xx of chapter 4 for further discussion.

[7] Here we use the same notation Y_t for a choice variable of the household as has previously been used for the exogenous supply of goods. In fact, in the model with endogenous output presented in chapter 4, equilibrium conditions such as (1.27) continue to apply, but with Y_t referring to aggregate demand, and *not* to any exogenously given supply of goods.

The household's problem can then be written as one of choosing the state-contingent evolution of total demand to maximize its expected discounted flow of indirect utility subject to an intertemporal budget constraint of the form (1.12), if C_t in this constraint is taken to refer to total demand, and T_t to the primary government budget surplus (tax collections in excess of government spending).[8] In this case, we can derive exactly the same equilibrium conditions as were obtained earlier, except that the function u is everywhere replaced by $\tilde{u}$. Hence variations in the level of government purchases G_t have exactly the same effect as the taste shock ξ_t; they are simply another source of exogenous variations in the relation $\tilde{u}_c(Y_t; \tilde{\xi}_t)$ between the marginal utility of income to the representative household and aggregate output, and hence of variations in the equilibrium real rate of interest.

1.2 A Wicksellian Policy Regime

We now offer a simple example of a complete specification of monetary and fiscal policy rules for a cashless economy, and consider the determinants of the equilibrium path of the money price of goods under such a regime. Note that, as a consequence of the forward-looking character of households' asset accumulation problems, the determination of equilibrium at any point in time requires that we specify how policy is expected to be conducted into the indefinite future, and in all possible future states. This is one reason for our specification of government policy in terms of systematic *rules* for the determination of both the central bank's actions and the government's budget.

Our specification of monetary policy will be in the spirit of Wicksell's (1898, 1907) proposed rule. As discussed in the previous chapter, this rule can be expressed in terms of a formula for the central bank's interest-rate operating target. We shall also explicitly specify the way in which the central bank adjusts the two instruments at its disposal — the nominal value of the monetary base M_t^s, on the one hand, and the interest rate i_t^m paid on base money

[8]The definition of equilibrium that results from this formulation of the household's problem is equivalent to the standard one, subject to the proviso that the processes $\{G_t, Y_t, \xi_t\}$ are such that the present value of government purchases is finite. Technically, one can imagine an equilibrium in which the present value of per capita output is not finite, though the present value of the resources left for the private sector to consume is finite. But this special case is of little interest and is ignored here.

on the other — in order to achieve its operating target. In a world with monetary frictions (discussed below in section xx), it is possible to use *either* of these instruments to affect the level of short-term nominal interest rates, and as discussed in chapter 1, actual central banks differ in the extent to which they use these two means to implement policy (though almost all central banks formulate policy in terms of an interest-rate operating target). In the cashless economy described above, however, changes in the quantity of base money (for example, through open-market purchases of government securities) have no consequences for the equilibrium determination of interest rates or other variables. (Note that M_t^s does not appear in any of the equilibrium conditions obtained at the end of the previous section.) Hence policy targets must be implemented exclusively through adjustment of the interest paid on base money.

Specifically, we assume a regime under which the interest paid on base money is equal at all times to the central bank's current interest-rate target, determined in response to the bank's assessment of current aggregate conditions. Such a system would resemble the "channel systems" described in chapter 1, under which the interest paid on central-bank balances is equal to the target rate minus a fixed spread; here the spread is assumed to be zero, since in equilibrium, the market interest rate i_t will actually equal i_t^m, and not i_t^m plus any positive spread. It is true that current channel systems pay interest only on central-bank clearing balances, and not on currency. But we can interpret the regime analyzed here to have this property as well; in a cashless world, this would simply mean that currency would not be held in equilibrium (any initially existing currency would be promptly deposited with the central bank in an interest-earning account), so that "base money" M_t^s would correspond to the supply of clearing balances.

Under a Wicksellian rule for the interest-rate target, the interest rate paid on central-bank balances equals

$$i_t^m = \phi(P_t/P_t^*; \nu_t) \tag{1.28}$$

where $P_t^* > 0$ defines a target path for the price level, ν_t is an additional possible exogenous random disturbance to the policy rule (or to its implementation), and $\phi(\cdot; \nu)$ is a non-negative-valued, non-decreasing function for each possible value of the disturbance ν.[9]

Here the function ϕ indicates the rule used by the central bank to sets its operating target, while equation (1.28) indicates the way in which the rule is implemented. The inclusion of a time-varying price-level target P_t^* allows us to treat the case of a rule that seeks to stabilize the price level around a modestly growing trend path — say, a rule that provides for one or two percent inflation per year, perhaps to compensate for bias in the price index that is targeted — rather than necessarily assuming a constant price level target, as Wicksell did. The inclusion of the random disturbance ν_t allows us to consider the effects of random variations in policy, or in its implementation, that we may not wish to model as changes in the target price level itself. This includes the possibility that the central bank may respond to output variations as well as the path of prices, as called for by the Taylor (1993) rule; or that the central bank may respond to perceived variation in the equilibrium real rate of return. (In the present model, both output and the equilibrium real interest rate are purely exogenous; hence systematic responses to these variables can be modeled by the inclusion of an exogenous disturbance term in the policy rule.)

We also need to specify the rule by which the evolution of the monetary base is determined. Here we assume simply that $\{M_t^s\}$ is an exogenous, positive-valued sequence. The logic of the Wicksellian regime requires no variation over time in the supply of base money at all; however, we allow for possible variation over time in the monetary base, in order to analyze the equilibrium consequences of this kind of policy action.

Finally, fiscal policy is specified by a rule for the evolution of the total supply of government liabilities $\{D_t\}$, and by a specification of the composition of government liabilities (debt-management policy) at each point in time. For simplicity, we let $\{D_t\}$ be an exogenous

[9]The function is assumed to be non-negative on the ground that it is not possible for the central bank to drive nominal interest rates to negative levels. We assume that, as under typical current arrangements, the holders of central-bank balances have the right to ask for currency in exchange for such balances at any time, and that it is infeasible to pay negative interest on currency. Hence an attempt to pay negative interest on central-bank balances would lead to zero demand for such balances, and a market overnight interest rate of zero (the rate available on currency), rather than a negative overnight interest rate. The assumed non-negativity of the function requires that $\phi(P/P^*; \nu)$ not be an exactly linear function of $\log(P/P^*)$, though we make use a local log-linear approximation to the function below.

process. One simple example of such a fiscal rule would be a *balanced-budget rule* of the kind analyzed by Schmitt-Grohé and Uribe (2000), where $\Delta D_t = 0$ each period; another would be a policy under which no government bonds are ever issued, so that $D_t = M_t^s$ each period. We also simplify by assuming that all government debt consists entirely of riskless one-period nominal bonds. The variable B_t^s then indicates the supply of such bonds at the end of period t, in terms of their nominal value at the time of issuance (rather than maturity). The implied rule for net tax collections T_t is then given by

$$T_t = (1 + i_{t-1})(D_{t-1} - \Delta_{t-1} M_{t-1}^s) - D_t, \tag{1.29}$$

using the fact that $A_t^s = (1 + i_{t-1})B_{t-1}^s) = (1 + i_{t-1})(D_{t-1} - M_{t-1}^s)$.

A rational-expectations equilibrium is then a set of processes $\{P_t, i_t, i_t^m\}$ that satisfy (1.15), (1.21), (1.23), (1.24), and (1.28) at all dates $t \geq 0$, given the exogenous processes $\{Y_t, \xi_t, M_t^s, D_t\}$.[10] Using (1.15) to eliminate i_t^m in (1.28), we obtain

$$i_t = \phi(P_t/P_t^*; \nu_t), \tag{1.30}$$

as an equilibrium condition linking the paths of interest rates and prices. (Note that this equation directly expresses the interest-rate rule that the central bank implements through its adjustment of the interest rate paid on base money.) We note furthermore that condition (1.23) does not involve any endogenous variables, and thus plays no role in equilibrium determination. We assume processes $\{Y_t, \xi_t\}$ that satisfy this condition; having done so, we can drop (1.23) from our list of requirements for equilibrium. We can thus identify rational-expectations equilibrium with a set of processes $\{P_t, i_t\}$ that satisfy (1.21), (1.24) and (1.30) each period.

We are interested not only in whether a solution to this system of equilibrium conditions *exists,* but in whether these relations suffice to uniquely determine the equilibrium paths of interest rates and prices. The question of the *determinacy* of equilibrium is a preliminary, more basic issue, before we can hope to address the question of what factors affect the

[10] In the case that we allow for government purchases, one should replace u by $\tilde{u}$ in each equation, and ξ_t by $\tilde{\xi}_t$.

equilibrium price level and how they affect it. And there are obvious reasons to worry about determinacy under the kind of regime just described. In the celebrated analysis of Sargent and Wallace (1975), interest-rate rules as such are to be avoided, on the ground that they result in indeterminacy of the equilibrium price level (and hence, in their model, of the equilibrium paths of real variables as well). And it is also often worried that in a cashless economy, there should be nothing to pin down the equilibrium price level, given that there is in such an environment no determinate demand for the monetary base; this is sometimes argued to be an important reason to head off financial innovations that could lead to this kind of world (*e.g.*, Friedman, 2000).

A general analysis of the existence and uniqueness of solutions to our system of equations, for arbitrary monetary and fiscal policy rules, is beyond the scope of this study. We shall instead address here, and in most of this study, a more limited question, namely the *local* determinacy of equilibrium, in the case of policies that involve only small fluctuations over time in the monetary policy rule, and assuming that other exogenous disturbances are similarly small. By local determinacy we mean the question of whether there is a unique equilibrium within a sufficiently small neighborhood of certain paths for the endogenous variables. If so, this equilibrium is at least *locally* unique, and such local uniqueness makes possible a well-defined "comparative statics" analysis of the effects of small disturbances or parameter changes.[11] In fact, we shall analyze the effects of small fluctuations in the price-level target, and other small disturbances, through exactly such a consideration of how the steady-state equilibrium associated with steady trend growth of the price-level target and zero disturbances is perturbed by small stochastic variations in the exogenous variables.

An advantage of restricting our attention to this question is that it can be addressed using purely linear methods; we analyze a log-linear approximation to the structural equations derived above, and characterize the (bounded) log-linear solutions to these equations. This way of characterizing both the important structural relations implied by our model, and the

[11] See section xx of the appendix for further general discussion of the issue of local determinacy and of the methods used in this section to analyze it.

predicted equilibrium evolution of economic time series under alternative policies, is useful not only because of its tractability, but because of the ubiquity of linear time series models in empirical studies. Our approach will thus allow a direct mapping between the predictions of optimizing models of economic behavior and the kinds of structural models and data characterizations already used in quantitative monetary policy analysis. This will facilitate both evaluation of the empirical adequacy of the optimizing models, and productive dialogue between optimization-based and more traditional approaches to policy evaluation.

Furthermore, such an analysis corresponds reasonably closely to the way that the determinacy of rational expectations equilibrium is considered by Sargent and Wallace (1975). That paper assumes a log-linear model, and considers the uniqueness of non-explosive solutions to the log-linear structural equations. Our method will be essentially the same, except that the log-linear structural equations can be justified as a log-linear approximation to exact relations derived from an explicit intertemporal general equilibrium model, and that we shall be more explicit about the class of solutions to be considered. Thus the analysis here suffices to address the particular issue relating to determinacy of equilibrium under interest-rate rules raised by Sargent and Wallace.[12]

We begin by characterizing the steady state near which we shall look for other solutions. Consider an environment in which $Y_t = \bar{Y} > 0$ and $\xi_t = 0$ at all times. Let us assume a policy regime under which the price-level target grows at the constant rate

$$\frac{P^*_{t+1}}{P^*_t} = \bar{\Pi} > \beta$$

at all dates (with some initial $P^*_0 > 0$), $\nu_t = 0$ at all dates, and suppose that the function ϕ satisfies

$$1 + \phi(1; 0) = \frac{\bar{\Pi}}{\beta},$$

so that the policy rule is consistent with the assumed target path for prices. Let us also

[12] In chapter 4, we consider an extension of the model in this chapter which is essentially a rational-expectations IS-LM model, though not identical in structure to the one analyzed by Sargent and Wallace (1975). The determinacy of equilibrium under alternative policy rules is taken up again in that chapter.

assume a fiscal rule under which

$$\frac{D_{t+1}}{D_t} = \gamma_D < \frac{\bar{\Pi}}{\beta}$$

at all dates (given initial government liabilities $D_0 > 0$), and an open-market policy under which $\{M_t^s\}$ is an arbitrary sequence, satisfying

$$0 < M_t^s < D_t \tag{1.31}$$

at all times. Under such a policy regime, we easily observe that the paths

$$P_t = P_t^*, \qquad i_t = \bar{\imath} \equiv \frac{\bar{\Pi} - \beta}{\beta} > 0$$

for all t represent a rational-expectations equilibrium.

Next, let us consider an environment in which there are only small fluctuations in the exogenous variables $Y_t, P_{t+1}^*/P_t^*, \nu_t$, and D_{t+1}/D_t around the constant values specified in the previous paragraph. (Specifically, we suppose that each of these variables remains forever within a bounded interval containing a neighborhood of the steady-state value.) We also continue to assume that the process $\{M_t^s\}$ satisfies the bounds (1.31). We wish to look for rational-expectations equilibria in which the endogenous variables i_t and P_t/P_t^* similarly remain forever within certain neighborhoods of their steady-state values. In this case, condition (1.24) is necessarily satisfied (for any tight enough bounds on the allowable variation in the variables listed above); thus we need only consider the (local) existence and uniqueness of solutions to the system of equations consisting of (1.21) and (1.30).

In the case of tight enough bounds on the variations that we consider in these variables, it suffices that we consider the *bounded* solutions to a system consisting of *log-linear approximations* to conditions (1.21) and (1.30).[13] For the sake of simplicity (and continuity with the assumptions made in more complex examples in the next two chapters), we shall specify that the steady state around which we log-linearize is one in which $\bar{\Pi} = 1$, so that there is zero inflation. (This does not require that we only analyze policies under which there is no

[13] As is further explained in the appendix, this amounts to using the inverse function theorem to demonstrate local uniqueness of the solution to our system of equilibrium conditions, and using the implicit function theorem to give a log-linear local approximation to the solution.

trend growth in the price-level target: it only requires that the target inflation rate is *never very large*.)

A log-linear approximation to (1.21) is then given by[14]

$$\hat{\imath}_t = \hat{r}_t + E_t\pi_{t+1}, \tag{1.32}$$

where

$$\hat{\imath}_t \equiv \log\left(\frac{1+i_t}{1+\bar{\imath}}\right), \qquad \pi_t \equiv \log\left(\frac{P_t}{P_{t-1}}\right).$$

Here the (percentage deviation in the) ex-ante short-term *equilibrium real rate of return* $\hat{r}_t$ is an exogenous process given by

$$\hat{r}_t = \sigma^{-1}[E_t(\hat{Y}_{t+1} - g_{t+1}) - (\hat{Y}_t - g_t)], \tag{1.33}$$

where the constant coefficient

$$\sigma \equiv -\frac{u_c}{u_{cc}\bar{Y}} > 0$$

measures the intertemporal elasticity of substitution in private spending, and the disturbance term

$$g_t \equiv -\frac{u_{c\xi}}{\bar{Y}u_{cc}}\xi_t$$

indicates the percentage increase in output required to keep the marginal utility of income constant, given the change that has occurred in the impatience to consume.

In the case of the model extended to allow for government purchases, equations (1.32) – (1.33) still apply, under the alternative definitions $\sigma \equiv s_C\sigma_C$, where $s_C \equiv \bar{C}/\bar{Y}$ is the steady-state share of private expenditure in total demand and

$$\sigma_C \equiv -\frac{\tilde{u}_c}{\tilde{u}_{cc}\bar{C}} > 0$$

is the intertemporal elasticity of substitution; and $g_t = \hat{G}_t + s_C\bar{C}_t$, where $\hat{G}_t \equiv (G_t - \bar{G})/\bar{Y}$ indicates fluctuations in government purchases, measured in units of steady-state GDP, and

$$\bar{C}_t \equiv -\frac{\tilde{u}_{c\xi}}{\bar{C}\tilde{u}_{cc}}\xi_t$$

[14] Note that the appearance in (1.32) of the inflation rate rather than of the inflation rate relative to steady-state inflation depends on our assumption that the steady-state inflation rate is zero.

indicates the percentage change in private expenditure required to keep marginal utility constant. (In fact, the notation "g_t" is used for the disturbance in (1.33) because variations in government purchases are one of the most obvious reasons for the existence of shifts in this factor.)

A corresponding log-linear approximation to the Wicksellian policy rule (1.30) is given by

$$\hat{\imath}_t = \phi_p \hat{P}_t + \nu_t, \tag{1.34}$$

where $\hat{P}_t \equiv \log(P_t/P_t^*)$, and $\phi_p \geq 0$ represents the elasticity of ϕ with respect to P/P^*, evaluated at the steady-state values of its arguments.[15] To the system of log-linear equations (1.32) and (1.34) we must also adjoin the identity

$$\pi_t = \hat{P}_t - \hat{P}_{t-1} + \pi_t^*, \tag{1.35}$$

where $\pi_t^* \equiv \log(P_t^*/P_{t-1}^*)$ indicates the exogenous fluctuations, if any, in the target inflation rate. We then wish to examine the bounded solutions to the system of log-linear equations (1.32), (1.34), and (1.35).

Using (1.34) and (1.35) to substitute for $\hat{\imath}_t$ and π_{t+1} in the Fisher equation (1.32), we obtain an expectational difference equation in the variable $\hat{P}_t$ alone, given by

$$(1 + \phi_p)\hat{P}_t = E_t \hat{P}_{t+1} + (\hat{r}_t + E_t \pi_{t+1}^* - \nu_t). \tag{1.36}$$

Given a policy rule for which $\phi_p > 0$, as called for by Wicksell, so that $0 < (1 + \phi_p)^{-1} < 1$, this equation can be solved forward (as discussed further in the appendix), to obtain a unique bounded solution

$$\hat{P}_t = \sum_{j=0}^{\infty} (1 + \phi_p)^{-(j+1)} E_t[\hat{r}_{t+j} + \pi_{t+j+1}^* - \nu_{t+j}], \tag{1.37}$$

in the case of any bounded exogenous processes $\{\hat{r}_t, \pi_t^*, \nu_t\}$. Substitution of this solution into (1.34) then yields an associated unique bounded solution for the nominal interest-rate

[15] In writing the disturbance term simply as ν_t, we adopt the normalization under which $\partial\phi/\partial\nu = 1 + \bar{\imath} = \beta^{-1}$, when also evaluated at the steady-state values of the arguments.

dynamics as well, namely

$$\hat{\imath}_t = \sum_{j=0}^{\infty} \phi_p (1+\phi_p)^{-(j+1)} E_t[\hat{r}_{t+j} + \pi^*_{t+j+1} - \nu_{t+j}] + \nu_t, \tag{1.38}$$

while substitution into (1.35) yields a corresponding solution for the equilibrium rate of inflation.

We thus obtain the main result of this section.[16]

Proposition 2.3. Under a Wicksellian policy rule (1.30) with $\phi_p > 0$, the rational-expectations equilibrium paths of prices and interest rates are (locally) determinate; that is, there exist open sets $\mathcal{P}$ and $\mathcal{I}$ such that in the case of any tight enough bounds on the fluctuations in the exogenous processes $\{\hat{r}_t, \pi^*_t, \nu_t\}$, there exists a unique rational-expectations equilibrium in which $P_t/P^*_t \in \mathcal{P}$ and $i_t \in \mathcal{I}$ at all times. Furthermore, equations (1.37) and (1.38) give a log-linear (first-order Taylor series) approximation to that solution, accurate up to a residual of order $\mathcal{O}(||\xi||^2)$, where $||\xi||$ indexes the bounds on the disturbance processes.

See further discussion in the appendix.

Thus there may be a well-defined rational-expectations equilibrium path for the price level, even in a purely cashless economy, and even under a policy rule that is formulated in terms of an interest-rate rule — *i.e.*, a rule for setting a short-term nominal interest rate that is independent of the evolution of any monetary aggregate. It is true that the regime described above assumes an exogenously given path for the monetary base M^s_t. But it should not therefore be assumed that it is the existence of a "money growth target" that is responsible for the existence of a determinate price level; for the equilibrium price level (1.37) is independent of the assumed path of the monetary base.

Indeed, in a cashless economy, a money growth target will *not* succeed in determining

[16] Kerr and King (1996) provide an early discussion of the determinacy of equilibrium under a rule of this kind, in a log-linear framework similar to the one derived here as an approximation to our exact equilibrium conditions. Woodford (1998a) demonstrates determinacy of equilibrium under a Wicksellian regime in the context of a complete intertemporal equilibrium model, along lines similar to those followed here, although the means by which the central bank is assumed to implement its interest-rate operating target is different there.

an equilibrium price level — at least if such a policy is understood to involve a constant (typically zero) rate of interest on the monetary base. In the case that no interest is paid on money, the following result implies that no equilibrium price level is possible at all.

PROPOSITION 2.4. Consider a monetary policy under which the monetary base is bounded below by a positive quantity: $M_t^s \geq \underline{M} > 0$ at all times. (For example, perhaps the monetary base is non-decreasing over time, starting from an initial level $M_0^s > 0$.) Suppose furthermore that government debt is non-negative at all times, so that $D_t \geq M_t^s$. Finally, suppose that $i_t^m = 0$ at all times. Then in the cashless economy described above, there exists no rational-expectations equilibrium path for the price level $\{P_t\}$.

The proof is in the appendix. Results of this kind are often taken (see, *e.g.*, Sargent, 1987, sec. 4.1) to imply that it is not possible to model the determinants of the exchange value of money without introducing some kind of frictions that create a demand for money despite its low rate of return. Hence monetary frictions are thought to be an essential element of any theory of the effects of monetary policy. But we have seen instead that an equilibrium in which money exchanges for goods is possible even in a cashless economy, as long as the rate of interest paid on money is high enough relative to the growth rate of the money supply.

Of course, in actual economies like that of the U.S., interest is *not* paid on the monetary base, so that the policy regime in place would seem to be one to which Proposition 2.4 would apply, were the economy cashless. One might then conclude that monetary frictions are essential to understanding inflation determination in an economy like that of the U.S. But in fact, monetary frictions are essential only to understand one aspect of current U.S. policy — the fact that it is possible for the Fed to implement its operating targets for the federal funds rate without paying interest on Fed balances. As we shall see, the details of how the Fed is able to implement its interest-rate targets are of relatively little importance for the effects of its interest-rate policy on price-level determination; and a cashless model may give a good account of the latter question.

Even when interest is paid on the monetary base at a sufficient rate to allow a monetary equilibrium to exist, if the interest rate on money is *constant,* the equilibrium price level will be indeterminate, as a consequence of Proposition 2.5 in the next section, whether or not the monetary base is kept on a precise deterministic growth path. Determinacy of equilibrium requires a regime under which interest rates respond systematically — in the right way, of course — to variations in the price level. In standard models with monetary frictions, money-growth targeting (with a constant rate of interest on money) is an example of a policy with this property: price-level increases automatically result in increases in short-term nominal interest rates. But in a cashless economy, money-growth targeting has no such consequence for interest rates, and so fails to determine an equilibrium price level. It is actually the presence of a systematic relation between prices and interest rates, of the kind called for Wicksell, that is essential for determinacy of equilibrium, and not control of the money supply as such.

Proposition 2.3 does not simply establish conditions under which a monetary equilibrium is possible: it gives a precise account of the factors that determine the equilibrium price level under such a regime. Equation (1.37) may equivalently be written

$$\log P_t = \sum_{j=0}^{\infty} \varphi_j E_t[\log P^*_{t+j} + \phi_p^{-1}(\hat{r}_{t+j} - \nu_{t+j})], \tag{1.39}$$

where the weights $\varphi_j \equiv \phi_p(1+\phi_p)^{-(j+1)}$ are all positive and sum to one. Thus the equilibrium log price level is equal (up to terms of order $\mathcal{O}(||\xi||^2)$) to a weighted average of the current and expected future log price level targets, plus a deviation term that is itself a weighted average of current and expected future disturbances to the equilibrium real rate of interest and disturbances to the monetary policy rule other than those represented by variations in the "target" price level.[17] The determinants of variations in the equilibrium real interest rate

[17]Note that insofar as we regard the target price level as being implicit in the central bank's reaction function, rather than an explicit target, the distinction between variations in the target price level and variations in ν_t is an arbitrary one; and as we should expect, (1.39) indicates that only the sum $\log P^*_t - (\phi_p^{-1} + 1)\nu_t$ actually matters, in our log-linear approximation, for determination of the equilibrium price level. But for some purposes, it remains useful to distinguish the two components; for example, we allow $\log P^*_t$ to possess a trend, but assume that ν_t is stationary.

are in turn given by (1.33); these depend solely upon exogenous real factors, independent of monetary policy. Note that the associated evolution of the monetary base $\{M_t^s\}$ is *not* among the relevant factors.

The theory of price-level determination under such a regime that we obtain has, in fact, a distinctly Wicksellian flavor. Equation (1.39) indicates that increases in the equilibrium price level result either from exogenous increases in the equilibrium real rate of return (Wicksell's "natural" rate of interest),[18] that are not sufficiently offset by an adjustment of the central bank's operating target, or from a loosening of monetary policy (corresponding either to an increase in P_t^* or a decrease in ν_t), that is not justified by any real disturbance. This emphasis upon the interplay between variations in the equilibrium real rate and the stance of monetary policy (and specifically upon the gap between the current level of the "natural rate" of interest and the interest rate controlled by the central bank) as the source of inflationary or deflationary pressures recalls Wicksell's theory (Wicksell, 1898, 1915).

Of course, our rational-expectations equilibrium version of Wicksellian theory differs in important ways from the original. For example, our solution (1.39) for the equilibrium price level is *forward-looking*, in much the same way as the rational-expectations monetarist analysis presented in section xx below. It is not simply the current equilibrium real rate of return that matters, but a weighted average of current and expected future rates, and it is not simply the current stance of monetary policy that matters, but a weighted average of the current and expected future shifts in the central bank's feedback rule. Perhaps more crucially, our theory does not determine an equilibrium price level as a function of the path (even including expectations about the future path) of the central bank's interest-rate instrument. Rather, the price level depends upon the current and expected future *feedback rules* for determination of the interest rate as a function of the evolution of the price level. (For an exogenously specified interest-rate process leaves the price level indeterminate, as shown above.)

[18] In the simple analysis here, there is no distinction between the actual and "natural" *real* rates of interest. This is introduced in chapter 4, where we find that it continues to be possible to understand price-level determination along similar lines.

Nonetheless, the implications of this theory for the conduct of monetary policy with a view to price stability are reminiscent of Wicksell's prescriptions. First of all, supposing that the target price level P_t^* is constant,[19] and letting $\phi_p > 0$ be fixed, then monetary policy achieves the constant price level $P_t = P^*$ if and only if $\nu_t = \hat{r}_t$ at all times. Failure of policy to track such variations in the "natural rate" with sufficient accuracy was, in Wicksell's account, the primary explanation for price-level instability.[20]

Second, our results imply that for any given degree that the shift factor ν_t fails to track the exogenous variation in $\hat{r}_t$, the price-level instability that results can be reduced by a sharper automatic positive response of the central bank's operating target to price-level increases. The effects upon $\log P_t$ of variations in the gap $\hat{r}_t - \nu_t$ are smaller the larger is ϕ_p, and in fact they can be made arbitrarily small, in principle, by choosing ϕ_p large enough.[21] Thus, a positive automatic response to price-level deviations from target is desirable, not only because it is necessary for determinacy (this would be achieved by even a very small $\phi_p > 0$), but because it reduces the degree to which accurate direct observation of the current equilibrium real rate is necessary in order for price-level variability to be kept at a given level. This too is a theme in Wicksell's discussions of desirable policy.

2 Alternative Interest-Rate Rules

While most central banks organize their monetary-policy deliberations around the choice of an operating target for a short-term nominal interest rate, and pay a great deal of attention

[19]Similar conclusions are reached in the case of a target price level that grows at a constant rate; one must simply add a constant to the value of ν_t required each period to achieve the target.

[20]See, *e.g.*, Wicksell (1915, sec. IV.9). He found evidence for this in the often-remarked tendency of the price level to covary positively with the level of nominal interest rates during the period of the classical gold standard (the so-called "Gibson paradox"). Note that if $\nu_t = 0$, (1.39) implies that $\log P_t$ should vary with exogenous variations in the real rate, while (1.34) then implies that the endogenous variations in the short nominal rate should perfectly coincide with these fluctuations in the price level. A similar conclusion is obtained if ν_t covaries positively, but less than one-for-one, with the variations in r_t, so that the inflationary and deflationary impulses are not completely eliminated.

[21]It is unlikely to be desirable, however, to seek to completely eliminate price-level variation without any need to directly respond to variations in the equilibrium real rate, by choosing a rule with an extremely large elasticity ϕ_p. For in this case, errors in the measurement of the price level by the central bank would have very large effects upon policy, and hence upon the economy, as discussed in Bernanke and Woodford (1997).

to inflation measures in those deliberations, it would be hard to argue that the Wicksellian rule (1.30) represents even a rough description of the current behavior of any central banks. (In particular, central banks clearly accept "base drift" in the price level, rather than seeking to stabilize a price index around an exogenously given target path.) Nor shall we argue that this is actually an optimal rule, though such a rule can be shown to have desirable properties, relative to other rules of equal simplicity (Giannoni, 2000). It is therefore useful to extend our analysis to other types of interest-rate rules. While a complete treatment of the topic would be beyond the scope of this study, we here take up a few additional cases suggested by the discussion of empirical central-bank reaction functions in chapter 1.

2.1 Exogenous Interest-Rate Targets

The simplest sort of interest-rate rule, of course, would be one that involves *no* feedback from any endogenous variables. We might instead suppose that the central bank's interest-rate operating target is given by some exogenous stochastic process $\{\bar{\imath}_t\}$. This need not imply that the sole objective of policy is interest-rate stabilization (or maintenance of "easy money"). A central bank concerned with price stability, and believing in the theoretical model of section 1, might reason that a rational-expectations equilibrium with constant prices is possible only if

$$1 + i_t = 1 + r_t \equiv \beta^{-1} \left\{ E_t \left[\frac{u_c(Y_{t+1}; \xi_{t+1})}{u_c(Y_t; \xi_t)} \right] \right\}^{-1}, \tag{2.40}$$

as a consequence of (1.21). Supposing that it is possible for the central bank to measure the exogenous variation in the right-hand side of this equation in time to use this information in the conduct of policy, would the bank achieve its objective by committing itself to a policy of always adopting the current value of r_t as its operating target? (Such a proposal might appear to be suggested by a Wicksellian analysis of the determinants of inflation, although, as noted in chapter 1 and in the previous section, this is the not the kind of rule actually proposed by Wicksell.)

While such a policy would be *consistent* with the desired rational-expectations equilibrium, it would also be equally consistent with an extremely large class of alternative rational-expectations equilibria, in most of which prices vary randomly. This is true even if, as above, we restrict attention to alternative equilibria that remain forever *near* the reference equilibrium, *i.e.*, the steady state with zero inflation. In fact, this is a consequence of *any* policy commitment that makes the interest-rate operating target purely a function of the economy's exogenous state (*i.e.*, the history of disturbances alone), regardless of how sensibly the exogenous sequence of interest-rate targets may have been chosen.

Proposition 2.5. Let monetary policy be specified by an exogenous sequence of interest-rate targets, assumed to remain forever within a neighborhood of the interest rate $\bar{\imath} > 0$ associated with the zero-inflation steady state; and let these be implemented by setting i_t^m equal to the interest-rate target each period. Let $\{M_t^s, D_t\}$ be exogenous sequences of the kind assumed in Proposition 2.3. Finally, let $\mathcal{P}$ be any neighborhoods of the real number zero. Then for any tight enough bounds on the exogenous processes $\{Y_t, \xi_t\, D_t/D_{t-1}\}$ and on the interest-rate target process, there exists an uncountably infinite set of rational-expectations equilibrium paths for the price level, in each of which the inflation rate satisfies $\pi_t \in \mathcal{P}$ for all t. These include equilibria in which the inflation rate is affected to an arbitrary extent by "fundamental" disturbances (unexpected changes in Y_t or ξ_t), by pure "sunspot" states (exogenous randomness unrelated to the "fundamental" variables), or both.

In this case, any process $\{P_t\}$ that satisfies both (1.21) and (1.24), given the exogenous processes $\{Y_t, \xi_t, i_t^m, M_t^s, D_t\}$, and with the exogenous target i_t^m substituted for i_t in (1.21), represents a rational-expectations equilibrium. For any tight enough bounds on the exogenous processes and on the neighborhood $\mathcal{P}$, (1.24) is necessarily satisfied, so our question reduces to an analysis of the local uniqueness of solutions to (1.21) for a given interest-rate process. As in the previous section, this can be addressed through a consideration of the uniqueness of bounded solutions to the log-linearized equilibrium condition (1.32). This now takes the form

$$\bar{\imath}_t = \hat{r}_t + E_t \pi_{t+1}, \tag{2.41}$$

where $\bar{\imath}_t$ indicates the exogenous fluctuations in the interest-rate target and $\hat{r}_t$ the exogenous fluctuations in r_t owing to random "fundamentals", again given by 1.33). This equation obviously has a unique solution for $E_t\pi_{t+1}$, and if the exogenous terms are bounded, the implied fluctuations in the expected inflation rate will be bounded as well. But this equation does *not* have a unique bounded solution for the stochastic process $\{\pi_t\}$, for absolutely any pattern of bounded fluctuations in the unexpected component of the inflation rate will be consistent with it.

Writing this explicitly, we observe that

$$\pi_t = \bar{\imath}_{t-1} - \hat{r}_{t-1} + \nu_t \tag{2.42}$$

is a bounded solution to (2.41), where $\{\nu_t\}$ represents any mean-zero bounded process that is completely unforecastable a period in advance, *i.e.*, that satisfies

$$E_t\nu_{t+1} = 0$$

at all dates. It then follows from the discussion in section xx of the appendix that we similarly have an uncountably infinite set of bounded solutions to the exact equilibrium condition. (1.21).[22][23]

Here the random variable ν_t may be correlated in an arbitrary way with unforecastable variations in "fundamental" variables such as $\bar{\imath}_t$, $\hat{Y}_t$, and ξ_t; but it may also be completely unrelated to economic fundamentals. Thus even when we restrict attention to nearby solutions, the rational expectations equilibrium price level is quite indeterminate under such a regime. Note furthermore that even though we consider only alternative solutions in which inflation is always within a certain neighborhood of zero, this set of solutions includes alternative

[22]There is actually no need for linearization to establish this, as shown in the appendix. Here we use the method of linear approximation in the analysis of determinacy to illustrate the method, that is used again later in more complicated cases.

[23]It is important to note that this conclusion depends on a particular assumption about the character of fiscal policy. If fiscal policy were not assumed, as it has been here, to be "locally Ricardian," then there might be a locally, or even globally unique equilibrium in the case of an exogenous path for the interest-rate operating target. This occurs, for example, in the analysis of price-level determination under a bond-price support regime in Woodford (2001xx). See further discussion in section xx of chapter 4.

paths for the *price level* that wander arbitrarily far from one another, once sufficient time has passed. This is the basis for the conclusion of Sargent and Wallace (1975) that interest-rate rules are flawed as a general approach to monetary policy, and that policy should instead be formulated in terms of monetary targets.[24]

One might ask, is this sort of price-level indeterminacy really a *problem?* It will be observed in our discussion here that *real* quantities are unaffected by the indeterminacy of the price level, and the same conclusion is true even in the case of the model with monetary frictions considered below in section xx, and even in a more elaborate model, in which output is endogenous and may depend upon the level of real money balances. Thus no variables that actually affect household utility are affected. However, the indeterminacy is plainly undesirable if price stability is a concern, as Sargent and Wallace assume in their analysis of optimal monetary policy. Indeed, since the class of bounded solutions includes solutions in which the unexpected fluctuations in inflation are arbitrarily large, at least *some* of the equilibria consistent with the interest-rate targeting policy are worse (assuming a loss function that penalizes squared deviations of inflation from target, say) than the equilibrium associated with *any* policy that makes equilibrium determinate. Furthermore, similar conclusions are shown below (in chapter 4) to hold in the case of a model with nominal price rigidity, in which case the self-fulfilling expectations also affect real variables, that matter for household utility. Thus, if one evaluates policy rules according to how bad is the *worst* outcome that they might allow, it would be appropriate to assign an absolute priority to the selection of a rule that would guarantee determinacy of equilibrium.

This argument might seem inconsistent with our use above of a purely *local* analysis of determinacy. One response to such a concern would be to refer to the exact analysis in section xx of the appendix, and show that there are indeed solutions involving arbitrarily large unexpected changes in the log price level. But in fact the local analysis is also valid, when correctly interpreted. Let us fix neighborhoods of the steady-state values of π_t, i_t,

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[24] See Walsh (1998, sec. 10.2.1) for an exposition of their analysis in the context of an IS-LM model with rational expectations that is closer to the structure of the model actually used in the original paper.

and so on, that are small enough that the approximation error in the log-linearized relation (1.32) is of an acceptable size, for all paths remaining within these neighborhoods; we shall restrict attention to solutions of this kind. The analysis of the log-linearized equations shows that in the case of the exogenous interest-rate target, there exist solutions in which the inflation rate fluctuates over the *entire* admissible neighborhood, *no matter how small* the fluctuations in the exogenous disturbances may be. Now let us compare such a policy to one that results in a determinate equilibrium (and hence a solution in which π_t and the other endogenous variables are linear functions of the exogenous disturbances, with coefficients that are independent of the assumed shock variances). We observe that by making the exogenous disturbances small enough, we obtain a case in which the inflation variability in at least certain equilibria associated with interest-rate targeting is much greater than in the locally unique equilibrium associated with the other policy. Thus, at least in the case of small enough exogenous disturbances, the conclusion reached from the analysis of the log-linearized equations is correct.

Whether one should only care about the worst possible equilibrium might be doubted, if a particular policy also allows very desirable equilibria, that are better than those associated with any other policies. But in fact, this is unlikely to be a serious problem, once the class of policies that are considered is sufficiently broad; for it is often possible to achieve any desired equilibrium through a policy rule that makes equilibrium determinate, in addition to its being consistent with rules that would make equilibrium indeterminate. There are typically many policy rules consistent with the desired equilibrium; these coincide in what they prescribe should occur in the desired equilibrium, but differ in how policy is specified "off the equilibrium path", and thus may differ as to whether they exclude other nearby equilibria. In such a case, it seems reasonable to accept as a principle of policy design that one should choose one of the rules that makes the desired equilibrium at least locally determinate, if not globally unique. We shall take this perspective in the current study; see chapter 8.

Fortunately, interest-rate rules as such need not imply indeterminacy of equilibrium, as

McCallum (1981) first noted, in the context of the model of Sargent and Wallace. A rule that involves a commitment to feedback from endogenous state variables such as the price level to the level of nominal interest rates can result in a determinate equilibrium, as our analysis of Wicksellian rules in section 1 has shown. We now turn to additional examples that are better descriptions of current policies.

2.2 The "Taylor Principle" and Determinacy

As discussed in chapter 1, the well-known "Taylor rule" (Taylor, 1993) differs from Wicksell's classic proposal in that it directs the central bank to respond to deviations of the *inflation rate* from a target level, without any reference to the absolute level that prices may have reached. Such concern with the inflation rate rather than the level of prices would seem to characterize policy in all advanced nations, at least since the breakdown of the Bretton Woods system in the early 1970's. Thus there is greater relevance for contemporary policy discussions in considering a Taylor rule of the form

$$i_t = \phi(\Pi_t/\Pi_t^*;\ \nu_t) \tag{2.43}$$

for central bank policy, where $\Pi_t \equiv P_t/P_{t-1}$ is the gross inflation rate, Π_t^* is a (possibly time-varying) target rate, ν_t again represents exogenous shifts in this relation, and $\phi(\cdot;\nu)$ is an increasing function for each value of ν. Once again, we shall suppose that in a cashless economy, the central bank's interest-rate operating target is implemented by setting i_t^m each period equal to the right-hand side of (2.43), so that (2.43) holds in equilibrium as a consequence of (1.15). Once again, $\{M_t^s\}$ is allowed to be an arbitrary process, and fiscal policy is specified by an exogenous process $\{D_t\}$.

Again we shall consider equilibria near a zero-inflation steady state. Assuming again that $\phi(1;0) = \beta^{-1} - 1$, such a steady state is an equilibrium in the case that $\Pi_t^* = 1$, $Y_t = \bar{Y} > 0$, and $\nu_t = \xi_t = 0$ at all times. We look for equilibria in which Π_t and i_t fluctuate within neighborhoods of their steady-state values, assuming that the exogenous variables $\{Y_t, \xi_t, \Pi_t^*, \nu_t\}$ all remain forever within neighborhoods of their steady-state values.

A log-linear approximation to (2.43) is given by

$$\hat{\imath}_t = \phi_\pi(\pi_t - \pi_t^*) + \nu_t,$$

or equivalently by

$$\hat{i}_t = \bar{\imath}_t + \phi_\pi \pi_t, \tag{2.44}$$

where now $\pi_t^* \equiv \log \Pi_t^*$, $\phi_\pi > 0$ is the elasticity of ϕ with respect to its first argument, evaluated at the steady state, and $\bar{\imath}_t \equiv \nu_t - \phi_\pi \pi_t^*$ measures the total exogenous shift in the central bank's reaction function. Substitution of this into (1.32) yields an expectational difference equation for the inflation rate,

$$\phi_\pi \pi_t = E_t \pi_{t+1} + (\hat{r}_t - \bar{\imath}_t). \tag{2.45}$$

In the case of a rule that conforms to what we may call the "Taylor principle" — that the central bank should raise its interest-rate instrument *more than one-for-one* with increases in inflation,[25] so that $\phi_\pi > 1$ — we can again solve (2.45) forward, yielding a unique bounded solution of the form

$$\pi_t = \sum_{j=0}^{\infty} \phi_\pi^{-(j+1)} E_t[\hat{r}_{t+j} - \bar{\imath}_{t+j}]. \tag{2.46}$$

We thus obtain the following result.

PROPOSITION 2.6. If monetary policy is characterized by an interest-rate feedback rule of the form (2.43), with $\phi_\pi > 1$, then the rational-expectations equilibrium paths of inflation and the nominal interest rate are (locally) determinate; that is, there exist open sets $\mathcal{P}$ and $\mathcal{I}$ such that in the case of any tight enough bounds on the fluctuations in the exogenous processes $\{\hat{r}_t, \pi_t^*, \nu_t\}$, there exists a unique rational-expectations equilibrium in which $\pi_t \in \mathcal{P}$ and $i_t \in \mathcal{I}$ at all times. Furthermore, equation (2.46) gives a log-linear (first-order Taylor series) approximation to the evolution of inflation in that equilibrium. If instead $0 \leq \phi_\pi < 1$, rational-expectations equilibrium is indeterminate, as in Proposition 2.5.

[25] The rule described in Taylor (1993) obviously conforms to this principle, as it specifies that $\phi_\pi = 1.5$. Discussions of the general desirability of such a principle include Taylor (1995, 1999).

Here the proof follows exactly the same lines as our proofs of Propositions 2.3 and 2.5 above.

When the Taylor Principle is satisfied, we find once again that an interest-rate feedback rule can be compatible with a determinate equilibrium price level.[26] We also observe again that equilibrium inflation is determined by interaction between the real determinants of the equilibrium real rate of return, on the one hand, and the nature of the central bank feedback rule for setting the nominal interest rate, on the other, quite independently of the associated evolution of any monetary aggregate. Moreover, our theory of inflation determination once again has a Wicksellian flavor: increases in the equilibrium real rate that are not offset by sufficient tightening of monetary policy result in inflation, as do loosenings of policy (decreases in $\bar{\imath}_t$) that are not warranted by an exogenous decline in the equilibrium real rate. And once again, it is only current and expected future values of the "gap" variable $\hat{r}_t - \bar{\imath}_t$ that matter for the generation of inflationary or deflationary impulses. Finally, given any stochastic process for the gap variable, the resulting equilibrium inflation variability is smaller the larger is the response elasticity ϕ_π.

The main qualitative difference between this family of rules and the Wicksellian regimes considered earlier is that transitory fluctuations in the gap variable $\hat{r}_t - \bar{\imath}_t$ now give rise to transitory fluctuations in inflation, which however *permanently* shift the absolute level of prices. Thus such a regime almost inevitably results in *price-level drift* (a unit root in the log price level), of the kind that has in fact been observed in all advanced countries in recent decades. This contrasts with the stationarity of the fluctuations in $\hat{P}_t$ under the Wicksellian regime; there, a deterministic trend for P_t^* would suffice to imply trend-stationarity of the equilibrium price level.

Note that this last result holds even in the case of very small positive values of ϕ_p, which correspond to policies that stabilize nominal interest rates to an arbitrarily great extent. Thus price-level drift is not a necessary consequence of policies that achieve a great deal

[26]It should be observed that the price *level*, and not just the inflation rate, is determined in the solution represented by (2.46). This is because the previous period's price level P_{t-1} is given at any date t as a predetermined state variable; unique determination of Π_t then implies a unique equilibrium price level P_t. The "nominal anchor" that allows determination of the absolute price level is thus the dependence of (2.43) on P_{t-1} through its dependence upon Π_t.

of nominal interest-rate smoothing. The result of Goodfriend (1987), according to which a policy that aims to reduce interest-rate variability, among other objectives, results in price-level drift, actually depends upon his specification of the central bank's objectives *other* than interest-rate smoothing. Because the central bank is assumed to care only about the variability of inflation over a short horizon, and not about the size of cumulative changes in the price level, almost any obstacle to complete inflation stabilization (whether due to infeasibility of perfect control, or to the presence of a conflicting objective such as Goodfriend's assumed concern with interest-rate variability) will result in its choosing a path that sacrifices price-level stability for a less variable inflation.[27]

Another important difference between this family of rules and the Wicksellian rules is that here a positive response of the interest-rate operating target to deviations of inflation from its target level does not suffice for determinacy. If we assume instead that $\phi_\pi < 1$, (2.45) has an infinity of bounded solutions, so that equilibrium inflation in this case is indeterminate, just as in the case of pure interest-rate control.[28]

Interestingly, Taylor (1999) finds that U.S. monetary policy during the 1960's and 1970's did not conform to the "Taylor principle", as discussed in chapter 1. At least in the case of a flexible-price model of the kind considered in this chapter, a systematic policy of the kind that Taylor estimates for the period 1960-79 would imply indeterminacy of the equilibrium price level. Clarida, Gali and Gertler (19xx) reach a similar conclusion, on the basis of an estimated forward-looking rule, and propose that the indeterminacy of equilibrium explains the instability of U.S. inflation and real activity during the 1970's.[29]

[27] See, *e.g.*, Van Hoose (1989).

[28] This result was first obtained by Leeper (1991) using similar linearization methods to those employed here. Leeper distinguishes between "active" monetary policies (rules of the form (2.44) with $\phi_\pi > 1$) and "passive" policies (rules with $\phi_\pi < 1$). Our results correspond to Leeper's case in which fiscal policy is "passive", though our specification of fiscal policy in terms of an exogenous path for $\{D_t\}$ does not correspond exactly to any member of his parametric family of fiscal rules. See further discussion in chapter 4, section xx. Kerr and King (1996) also contains an early discussion of the connection between the Taylor principle and determinacy of equilibrium.

[29] Indeterminacy of the equilibrium inflation rate implies indeterminacy of equilibrium real activity as well, in the context of a model with sticky prices of the kind discussed in chapter 4 below. Chari, Christiano and Eichenbaum (1997) also argue that the instability of the 1970's can be attributed to self-fulfilling expectations; but in their analysis, the multiplicity of equilibria results from an absence of commitment on the part of the

Of course, such an interpretation depends both upon an assumption that the interest-rate regressions of these authors correctly identify the character of systematic monetary policy during the period. In fact, an estimated reaction function of this kind could easily be misspecified. For example, consider the equilibrium described by (1.39) in the case of a Wicksellian regime, and suppose that P_t^* grows deterministically at a constant rate. Since in equilibrium $\hat{\imath}_t$ is equal to ϕ_p times $\hat{P}_t$, $\hat{P}_t$ is a stationary series, and π_t is equal to its first-difference (up to a constant), one can show that the (asymptotic) coefficient of a regression of $\hat{\imath}_t$ on π_t (rather than upon the correct variable, $\hat{P}_t$) will equal $\phi_p/2$. This coefficient could easily be much less than one – suggesting violation of the "Taylor principle", and that the price level should be indeterminate – even though in fact, as $\phi_p > 0$, the price level is determinate. Thus Taylor's interpretation of his finding of positive coefficients much less than one on inflation in estimated "Taylor rules" for the classical gold standard period (for example, a coefficient of only 0.02 for the period 1879-91), as indicating an even more extreme version of the kind of unduly passive interest-rate responses seen in the 1960's and 1970's, may well be incorrect.

A less dramatic case of the same problem may bias downwards Taylor's estimate of the inflation response coefficient in the period 1960-79 as well, as Orphanides (20xx) argues.[30] Yet the suggestion that the policy mistakes of the period may have related to a failure to understand the restrictions upon monetary policy required for price-level determinacy is an intriguing one, that suggests that an improved theoretical understanding of this issue could be of considerable practical importance.

Fed, rather than from a commitment to systematic policy of an unfortunate sort.

[30]In Orphanides' analysis, systematic over-estimation of potential output during the 1970s gave policy an inflationary bias, even though the Fed followed a rule similar to the Taylor rule that describes its later behavior. Regressions using corrected estimates of the output gap, rather than the real-time estimates on which policy was actually based then lead to a downward-biased estimate of the inflation-response coefficient, since the period in which policy was looser due to the omitted variable was a period of higher-than-average inflation.

2.3 Inertial Responses to Inflation Variation

The simple class of "Taylor rules" (2.43) allows only the crudest sort of approximation to actual central bank policies, for a number of reasons. One of the more obvious is the allowance for feedback only from the current period's rate of inflation. In practice, monetary policy will never involve feedback from an *instantaneous* rate of inflation (as is sometimes assumed in continuous-time treatments of our problem), because available inflation measures will always be time-averaged over at least a period such as a month. In fact, Taylor's (1993) account of recent U.S. policy assumes that the operating target for the funds rate is a function of inflation over the previous *year*. It is thus desirable to consider rules involving feedback from the rate of inflation averaged over a time longer than one "period". A case in which the analysis remains quite simple (even if it is not realistic as a literal representation of central bank procedures) is that in which the central bank responds to an exponential moving average of past inflation rates of the form

$$\bar{\pi}_t \equiv (1-\delta)\sum_{j=0}^{\infty}\delta^j \pi_{t-j}, \tag{2.47}$$

for some decay factor $0 < \delta < 1$. This case is simple to analyze because the relevant inflation measure evolves according to a simple partial-adjustment formula

$$\bar{\pi}_t = (1-\delta)\pi_t + \delta\bar{\pi}_{t-1}.$$

Let the central bank's log-linear feedback rule be given by

$$\hat{\imath}_t = \bar{\imath}_t + \Phi_\pi \bar{\pi}_t, \tag{2.48}$$

instead of (2.44), where $\bar{\pi}_t$ is defined by (2.47). We then obtain the following result.

Proposition 2.7. Let monetary policy be described by a feedback rule of the form (2.48), at least near the zero-inflation steady state, with $\Phi_\pi \geq 0$. Then equilibrium is determinate if and only if $\Phi_\pi > 1$. When this condition is satisfied, a log-linear approximation to the equilibrium evolution of the smoothed inflation process is given by

$$\bar{\pi}_t = (1-\delta)\sum_{j=0}^{\infty}(\delta + (1-\delta)\Phi_\pi)^{-(j+1)} E_t[\hat{r}_{t+j} - \bar{\imath}_{t+j}]. \tag{2.49}$$

A corresponding approximation to the equilibrium evolution of the single-period inflation rate π_t is then obtained by substituting (2.49) into

$$\pi_t = \frac{\bar{\pi}_t - \delta\bar{\pi}_{t-1}}{1-\delta}. \tag{2.50}$$

The proof is given in the appendix. Note that determinacy is once again obtained if and only if $\Phi_\pi > 1$, as required by the "Taylor principle". Furthermore, in the determinate case, the smoothed inflation rate $\bar{\pi}_t$ bears the same qualitative relation as above to expectations regarding the equilibrium real rate and future monetary policy shifts: it is rises when the natural rate rises without an offsetting tightening of policy, either currently or anticipated to occur in the future.

Estimated central bank reaction functions also typically differ from the simple rule (1.30) in that they incorporate some degree of partial-adjustment dynamics for the interest rate itself: that is, the current setting of the operating target for the interest rate inevitably depends positively upon one or more lags of itself, in addition to measures of current economic conditions such as some measure of inflation. As a simple example, we may consider a rule of the form

$$\hat{\imath}_t = \bar{\imath}_t + \rho(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \phi_\pi \pi_t, \tag{2.51}$$

where again $\phi_\pi \geq 0$, and the coefficient $\rho \geq 0$ measures the degree of intrinsic inertia in the central bank's adjustment of its operating target. Note that when $\rho < 1$, this rule can be represented as a partial-adjustment rule, like those discussed in chapter 1. However, it is also of interest to consider rules with $\rho = 1$ (*i.e.*, rules in which it is the *change* in the operating target that is a function of current inflation, as assumed in some estimated central bank reaction functions, *e.g.*, that of Fuhrer and Moore, 1995b), or even "super-inertial" rules with $\rho > 1$, like those considered by Rotemberg and Woodford (1999).

In the case that $\rho < 1$, the feedback rule (2.51) can actually be equivalently expressed in the form (2.48), simply by solving backwards to eliminate the dependence upon the lagged interest rate. In this alternative representation, the response coefficient Φ_π in (2.48) corresponds to $\phi_\pi/(1-\rho)$ in the new notation, and the decay factor δ in (2.47) corresponds to ρ in the new notation. Thus Proposition 2.7 applies, and equilibrium is determinate if and only if $\Phi_\pi > 1$, again in accordance with the "Taylor principle". (In a case of this kind, the principle must be understood to require that the *eventual* increase in the nominal interest rate as a result of a *sustained* increase in the inflation rate is more than one-for-one.) And once again, an appropriate moving average of inflation is positively related to an average of current and expected future values of the gap between the natural rate of interest and the policy stance measure $\bar{\imath}_t$.

But well-behaved rational expectations equilibria can also exist in the case of rules with $\rho \geq 1$. In fact, we can show the following.

PROPOSITION 2.8. Let monetary policy be described by a feedback rule of the form (2.51), at least near the zero-inflation steady state, with $\phi_\pi, rho \geq 0$. Then equilibrium is determinate if and only if $\phi_\pi > 0$ and

$$\phi_\pi + \rho > 1 \tag{2.52}$$

When these conditions are satisfied, a log-linear approximation to the equilibrium evolution of inflation is given by

$$\pi_t = -\frac{\rho}{\phi_\pi}(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \sum_{j=0}^{\infty}(\phi_\pi + \rho)^{-(j+1)} E_t[\hat{r}_{t+j} - \bar{\imath}_{t+j}], \tag{2.53}$$

where the interest rate evolves according to

$$\hat{\imath}_t = \bar{\imath}_t + \sum_{j=0}^{\infty} \phi_\pi(\phi_\pi + \rho)^{-(j+1)} E_t[\hat{r}_{t+j} - \bar{\imath}_{t+j}].) \tag{2.54}$$

The proof is again in the appendix. When $\rho < 1$, this condition is equivalent to the requirement that $\Phi_\pi > 1$, just discussed. But (2.52) applies more generally. In fact, it shows that if $\rho \geq 1$, *any* positive value for ϕ_π suffices for determinacy. And indeed, if $\rho > 1$, the equilibrium inflation rate is determinate even in the case of moderate negative values of ϕ_π, as is discussed further in the appendix. Determinacy, requires however, that $\phi_\pi \neq 0$, since in the absence of any feedback from inflation, Proposition 2.5 applies.

Equation (2.53) also provides a direct generalization of our earlier solution (2.46) for the equilibrium path of the inflation rate. Again we see that a weighted average of current and expected future "gap" terms determines the current inflation rate, given the lagged interest rate (which affects current inflation in the same way as an exogenous shift in the current stance of monetary policy). The pair of equations (2.53) – (2.54) can be solved iteratively for the entire paths of inflation and the nominal interest rate, given an initial lagged interest rate and the paths of the exogenous disturbances.

For example, suppose that the equilibrium real rate follows an AR(1) process with autoregressive coefficient $0 < \rho_r < 1$, and consider policies of the form (2.51) with $\bar{\imath}_t = 0$ at all times (the central bank reacts only to inflation, with no target changes and no control errors). Then in the solution described by (2.53), the nominal interest rate will be perfectly correlated (positively) with fluctuations in the real rate. But inflation fluctuations are less persistent than the real rate fluctuations, because the effect of an innovation in the real rate at date t upon $E_t\pi_{t+1}$ is $(\rho_r - \rho)$ times its effect upon π_t, whereas the effect upon $E_t\hat{r}_{t+1}$ is ρ_r times its effect upon $\hat{r}_t$. If $\rho = \rho_r$, the inflation fluctuations become purely transitory, and for larger values of ρ, they actually become anti-persistent, *i.e.*, the effect a period after the shock is actually of the *opposite* sign to the initial effect. Figure 2.1 plots impulse responses to a one percent increase in the equilibrium real rate, assuming $\phi_\pi = 0.7, \rho_r = 0.8$, and a variety of values for ρ consistent with (2.52). Panel (a) shows the impulse response of the nominal interest rate for each value of ρ, and panel (b) shows the impulse response of the inflation rate.

This simple example illustrates two desirable features of a more inertial response to inflation variations, relative to the purely contemporaneous specification (2.43). First of all, inertia allows a given degree of reduction of the variance of inflation to be achieved with *less interest-rate variability*. This is an implication of the following result.

Proposition 2.9. Suppose that the equilibrium real rate $\{\hat{r}_t\}$ follows an exogenously given stationary AR(1) process, and let the monetary policy rule be of the form (2.51), with

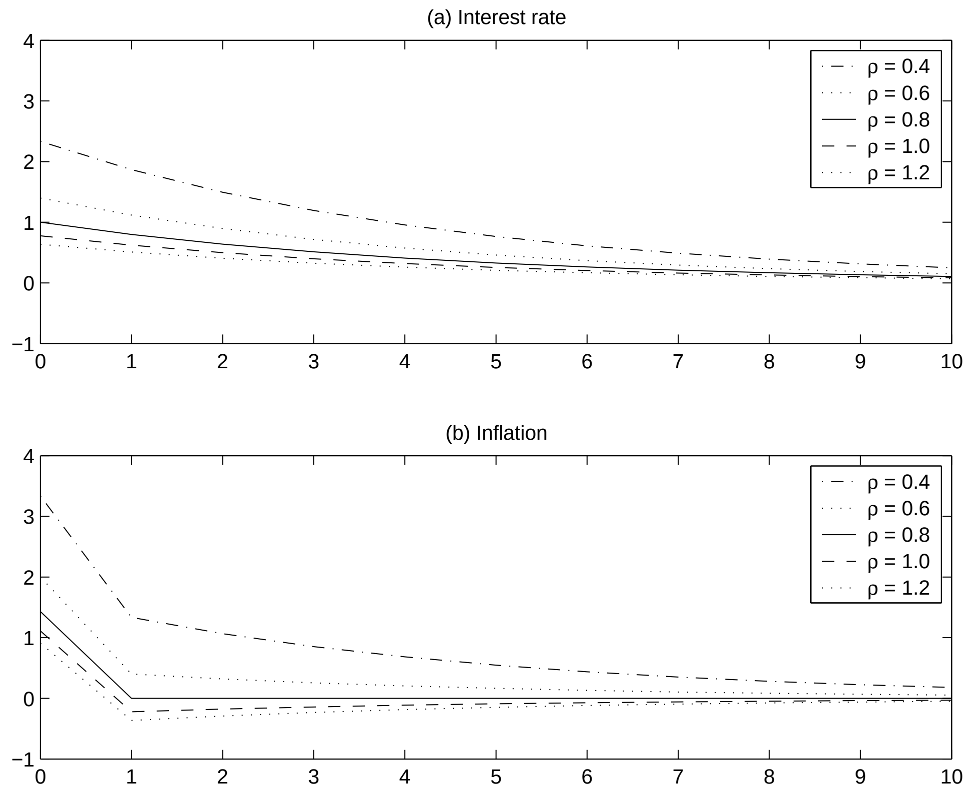

Figure 2.1: Responses to an increase in the natural rate of interest.

$\rho \geq 0, \phi_\pi > 0$ and a constant intercept consistent with the zero-inflation steady state (*i.e.*, $\bar{\imath}_t = 0$). Consider the choice of a policy rule (ρ, ϕ_π) within this class so as to bring about a certain desired unconditional variance of inflation $\mathrm{var}(\pi) > 0$ around the mean inflation rate of zero. For any large enough value of ρ, there exists a ϕ_π satisfying (2.52) such that the unconditional variance of inflation in the stationary rational-expectations equilibrium associated with this rule is of the desired magnitude. Furthermore, the larger is ρ, the *smaller* is the unconditional variance of interest-rate fluctuations $\mathrm{var}(\hat{\imath})$ in this equilibrium.

The proof is in the appendix. We shall argue in chapter 6 that the reduction of interest-rate variability, as well as the reduction of inflation variability, is an appropriate goal of monetary policy. It follows that inertial rules, even *super-inertial* rules (with $\rho > 1$), have an advantage over other members of the class (2.51).

Second, an inertial rule allows a greater degree of stabilization of the *long-run price level,* by making inflation fluctuations less persistent (or even anti-persistent, so that increases in the price level are subsequently offset). We shall argue in chapter 6 that the variability of the rate of inflation over a short horizon is more directly related to welfare losses that monetary policy should seek to minimize than is instability of the long-run price level. Nonetheless, some have argued that stabilization of the long-run price level is also an appropriate goal of policy, for example, in order to facilitate long-term contracting (see, *e.g.,* Hall and Mankiw, 1994).

One possible measure of long-run price-level instability (proposed in Rotemberg and Woodford, 1999) is the variance of innovations in the Beveridge-Nelson (198xx) price-level trend. Let us define the long-run price level as

$$\log P_t^\infty \equiv \lim_{j\to\infty} E_t[\log P_{t+j} - jE(\pi)], \qquad (2.55)$$

where $E(\pi)$ is the unconditional expectation of the rate of inflation.[31] The innovation in this variable at date t is then defined as

$$\log P_t^\infty - E_{t-1}[\log P_t^\infty] = \sum_{j=0}^{\infty}[E_t\pi_{t+j} - E_{t-1}\pi_{t+j}].$$

In the equilibrium just described, this innovation is equal to $(1-\rho)(1-\rho_r)^{-1}(\phi_\pi+\rho-\rho_r)^{-1}$ times the innovation in the equilibrium real rate of return. Thus the variability of such innovations is minimized, and in fact reduced to zero, when $\rho = 1$. In this case, the log price level is actually *trend-stationary,* so that the long-run price level defined in (2.55) is deterministic. For values of ρ near one, the price level still possesses a unit root, but the long-run price level still evolves relatively smoothly.

The fact that a rule with $\rho = 1$ exactly stabilizes the price *level* is less mysterious once one recalls the equivalence between rules of the form (2.51) and rules of the form (2.48). For $0 < \rho < 1$, the rule (2.51) is equivalent to making the interest rate a function of a weighted average of past rates of inflation. As ρ approaches one, the effective weights on past rates

[31] This definition assumes that inflation is a stationary variable, *i.e.,* that $\log P_t$ is difference-stationary. Note that all of the equilibria that we discuss here have this property.

of inflation cease to be discounted relative to more recent rates, so that the rule effectively responds to cumulative inflation over the entire past, which is to say, the price level. In the limit, a rule of this kind thus becomes equivalent to a Wicksellian rule of the form (1.30),[32] which as shown earlier stabilizes the price level.

It may be wondered why rules with $\rho \geq 1$ do not lead to *instrument instability*, *i.e.*, a non-stationary process for the central bank's operating target. In fact, if an arbitrary stationary stochastic process for inflation is fed into (2.51), a non-stationary interest-rate process is almost inevitably implied. However, in a rational expectations equilibrium, the only kind of inflation processes that occur are exactly the kind that *do not* cause the interest rate to be non-stationary. Think again about the Wicksellian regime. For most stationary inflation processes, the price *level* has a unit root, and so a policy that responds to deviations of the price level from a deterministic path should result in a unit root in the interest rate as well, under the reasoning just suggested. The reason this does not happen is that, in equilibrium, a policy of responding to price level deviations makes the price *level* stationary, and not just the rate of inflation. This explains why $\rho = 1$ does not create a problem, but the logic is exactly the same in the case that $\rho > 1$.

Rules with $\rho > 1$ might seem implausible, because commitment to such a rule implies a commitment to continue raising interest rates to higher and higher levels, at an explosive rate, if inflation is ever even temporarily above its target level, as long as it never subsequently falls *below* the target rate. Of course, in equilibrium, these extreme actions need never be taken, as any temporary increase in inflation is followed by subsequent *undershooting* of the target rate, as shown in Figure 2.1(b). But it might be suspected that they fail to be triggered only because of the anticipation that they would be, a "threat" that might properly be considered incredible.[33] In fact, this is not so. Note that in the equilibrium calculated above, equilibrium inflation fluctuates over a bounded interval, as does the nominal interest rate. Hence only the definition of the policy rule (2.51) for inflation rates and lagged interest

[32] Note that when $\rho = 1$, (2.51) is just a first-differenced version of (1.34).

[33] At a conference, Bob Hall compared policies with $\rho > 1$ to the "Doomsday Machine" in *Doctor Strangelove*.

rates within those intervals matters for our conclusion that (2.53) describes a locally unique equilibrium. We could change the specification of the rule in the case of more extreme values of the endogenous variables — in particular, change it to specify that interest rates would never be raised above some finite level, as seems more credible — and still obtain exactly the same conclusions as above. Such a change would at most affect our conclusions about the existence of *other* rational expectations equilibria, that do not remain forever near the steady state, a topic not yet addressed.

A commitment to indefinitely follow a super-inertial instrument rule implies a certain restriction upon the expected evolution of inflation, that must hold in any bounded rational-expectations equilibrium. In the case of a rule of the form (2.51), the restriction takes the following form.

PROPOSITION 2.10. Consider a policy rule of the form (2.51), where $\rho > 1$ and $\{\bar{\imath}_t\}$ is a bounded process, to be adopted beginning at some date t_0. Then any bounded processes $\{\pi_t, \hat{\imath}_t\}$ that satisfy (2.51) for all $t \geq t_0$ must be such that the predicted path of inflation, looking forward from any date $t \geq t_0$, satisfies

$$\sum_{j=0}^{\infty} \rho^{-(j+1)} \phi_\pi \; E_t \pi_{t+j} = -(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}). \qquad (2.56)$$

Conversely, any bounded processes satisfying (2.56) for all $t \geq t_0$ also satisfy (2.51) for all $t \geq t_0$.

The proof is in the appendix. Condition (2.56) is a restriction upon the expected future path of inflation that must be satisfied (in any non-explosive REE) given the commitment to the rule (2.51), that is quite independent of any other assumptions about the conditions required for an equilibrium.

In fact, the policy can alternatively be stated as a commitment to adjust interest rates as necessary in order to ensure that the projected future path of inflation always satisfies the target criterion (2.56). This way of stating the rule is an example of an inflation-forecast *targeting rule,* of the kind discussed by Svensson (1997, 1999, 2001), though it differs from the simpler examples discussed by Svensson in that the target for the weighted average of future inflation forecasts is time-varying, both in response to variation in the lagged level of the nominal interest rate and owing to the exogenous policy shifts represented by the $E_t\bar{\imath}_{t+j}$ terms. We have just shown that a commitment to the instrument rule (2.51) implies that the target criterion (2.56) must be expected to be satisfied at each date. And conversely, a policy that ensures that the target criterion is satisfied at each date must involve an interest-rate path that satisfies (2.51) at each date.

Thus these two superficially different forms of policy commitment are actually *equivalent,* at least as far as the set of bounded rational-expectations equilibria associated with either of them are concerned; and this equivalence is independent of the details of the structural model of the monetary transmission mechanism that is used to predict the consequences of following either rule. Which way of describing the policy commitment should be preferred will depend on which is judged more effective as a way of communicating the policy commitment to the public. In chapter 8, we show that in the context of at least some plausible models of the monetary transmission mechanism, an optimal policy rule can be represented as a more complicated version of a rule of this kind. There we discuss the representation of optimal policy both in terms of a super-inertial instrument rule and in terms of a forecast-targeting rule.

3 Price-Level Determination with Monetary Frictions

We now take up the question of how the equilibrium prices of goods in terms of money are determined in an economy where the monetary liabilities of the central bank *do* facilitate transactions, contrary to our simplifying assumption in the previous two sections. In all actually existing economies, we observe that positive quantities of base money are held by private parties despite the fact that this asset yields a lower return than other very short-term riskless assets; this indicates that there must be advantages to holding money not allowed for in our model above. Our analysis above is only useful in understanding actual economies, then, if we can show that even when such transactions services of money are allowed for, our conclusions about price-level determination are fairly similar to those obtained when abstracting from this complication. Introducing transactions frictions will also allow us to compare our neo-Wicksellian theory of inflation determination with the implications of a traditional quantity-theoretic analysis.

3.1 A Model with Transactions Frictions

We now extend the above analysis to allow for transactions frictions that can be ameliorated through the use of central-bank monetary liabilities. For simplicity, we use a model that has been extensively used in rational-expectations monetarist analyses, the representative-household model of Sidrauski (1967) and Brock (1974, 1975).[34] In this approach, the transactions frictions are not explicitly modeled; instead, the *transactions services* supplied by real money balances are directly represented as an argument of household utility functions.[35]

We again assume a representative-household economy, but now with a household objective of the form

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(C_t, M_t/P_t; \xi_t) \right\}, \tag{3.1}$$

where M_t again measures the household's end-of-period money balances, and P_t is the price of the single good in terms of money in period t. The function u is now an indirect utility function, incorporating the costs of transacting with a given level of money balances; hence the vector of exogenous disturbances ξ_t may now include random variation in the transactions technology, as well as actual preference shocks.[36] For any given realization of ξ_t, we assume that the period utility function $u(c, m; \xi_t)$ satisfies standard neoclassical assumptions: it is

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[34] For other recent expositions of the model, see, *e.g.*, Obstfeld and Rogoff (1996, sec. 8.3) and Walsh (1998, sec. 2.3).

[35] In the alternative analysis presented in section xx of the appendix, the transactions frictions are more represented by a cash-in-advance constraint. On the similarity between models with transactions frictions of that sort and the Sidrauski-Brock model, see also Feenstra (1986), Lucas and Stokey (1987), and Woodford (1998a).

[36] It is important to note that we do not assume that ξ_t is a scalar process; for example, ξ_t might contain two components, one of which represents variation in impatience, and the other of which represents variation in the liquidity services provided by money balances. The use of the single symbol ξ_t to represent both shocks does not imply anything about their assumed correlation; the two components of the vector might, for example, be distributed independently of one another.

concave and strictly increasing in each of the arguments (c, m).[37] We shall also suppose that u implies that both consumption and real balances are (strict) *normal goods; i.e.*, we shall assume that income-expansion paths are upward-sloping in the case of any finite positive relative price for the two "goods".

The household's budget constraints remain of the form assumed in section 1; note that we have already allowed there for a possible return differential between money and other riskless nominal claims, even though in equilibrium no such differential turned out to exist (in the cashless economy). Hence the household's problem is to choose processes $\{C_t, M_t\}$ satisfying (1.12) given its initial wealth W_0, so as to maximize (3.1). Necessary and sufficient conditions for this problem are derived along the same lines as before.

Once again, (1.10) and (1.11) must hold at all times, since otherwise no optimal plan exists. The first-order conditions for optimal choice of the household's money balances now require that $M_t \geq 0$ and

$$\frac{u_m(C_t, M_t/P_t; \xi_t)}{u_c(C_t, M_t/P_t; \xi_t)} \leq \Delta_t$$

at each date, with at least one condition holding with equality at each date; thus at any date at which $M_t > 0$, one must have

$$\frac{u_m(C_t, M_t/P_t; \xi_t)}{u_c(C_t, M_t/P_t; \xi_t)} = \Delta_t. \tag{3.2}$$

(Note that this condition generalizes (1.15) to the case in which utility is increasing in real balances.) Conditions (1.16), (1.17) and the exhaustion of the intertemporal budget constraint are again necessary as before, with only the change that now the marginal utility of consumption must be written $u_c(C_t, m_t; \xi_t)$. And once again this set of conditions can be shown to be both necessary and sufficient for optimality of the household's plan.

[37]The existence of a finite level of money balances at which there is satiation in money, which level will typically be increasing in the level of consumption, creates only minor technical complications. And indeed, many explicit models of transactions frictions, such as cash-in-advance models, imply that such a satiation level should exist. But whether there is satiation does not matter, for present purposes, except for the question of whether it is possible to reduce nominal interest rates all the way to zero. We shall suppose in general that we do not wish to do so, though the reason for this (distortions associated with deflation) will only be introduced after we extend our model to include nominal rigidities.

Substituting the relations implied by market-clearing into the conditions for household optimization, and assuming a policy regime under which $M_t^s > 0$ at all times, we again obtain equilibrium conditions (1.20), (1.21), and (1.22) for each date, together with the conditions

$$\frac{u_m(Y_t, M_t^s/P_t; \xi_t)}{u_c(Y_t, M_t^s/P_t; \xi_t)} = \Delta_t \tag{3.3}$$

and

$$\sum_{T=t}^{\infty} \beta^T E_t \left[u_c(Y_T, M_T^s/P_T; \xi_T) Y_T + u_m(Y_T, M_T^s/P_T; \xi_T) M_T^s/P_T \right] < \infty, \tag{3.4}$$

generalizing (1.15) and (1.23) respectively. These relations, together with a specification of the policy regime, provide a complete description of a rational-expectations equilibrium.

Note that under the assumption that both consumption and real balances are normal goods, u_m/u_c is increasing in consumption and decreasing in real balances. It follows that we can solve (3.3) for equilibrium real balances,[38] obtaining a relation of the form

$$\frac{M_t^s}{P_t} = L(Y_t, \Delta_t; \xi_t). \tag{3.5}$$

Here the *liquidity preference* function L is increasing in Y_t and decreasing in Δ_t, for any value of the disturbance vector ξ_t. Note that equation (3.5) corresponds to the "LM equation" of the Keynesian system, or to the "money market equilibrium" condition of a monetarist model. (In the case that $i_t^m = 0$, as in standard treatments, we can alternatively write the liquidity preference function in terms of y_t and i_t.) From a quantity-theoretic point of view, it is this equilibrium condition that is regarded as determining the price level at each point in time, given the money supply M_t^s at that date.

Finally, we can show once again that condition (1.22) may equivalently be written in the form (1.24).[39] We thus obtain the following generalization of our previous definition.

[38]Technically, monotonicity of u_m/u_c is not quite enough: it implies that there is a well-defined level of equilibrium real money balances for Δ_t in some interval (that may depend on Y_t and ξ_t), but it does not exclude the possibility that no solution exists for interest rates that are either too high or too low. We shall simplify our analysis, when necessary, by assuming boundary conditions on preferences that imply a solution for any $\Delta_t > 0$; specifically, we suppose that u_m becomes unboundedly large as real balances are made small, and arbitrarily small as real balances are made large enough, which may or may not involve satiation at a finite level of real balances. Note that our primary concern here will be with stationary fluctuations around an equilibrium steady state, and for these purposes such boundary conditions are irrelevant.

[39]In the presence of monetary frictions, Proposition 2.2 must be restated as Proposition 2.2′, which is stated and proved in section xx of the appendix.

Definition. A *rational-expectations equilibrium* of the Sidrauski-Brock model is a pair of processes $\{P_t, i_t\}$ that satisfy (1.21), (1.24),[40] (3.4), and (3.5) at all dates $t \geq 0$, given the exogenous processes $\{Y_t, \xi_t\}$, and evolution of the variables $\{i_t^m, M_t^s, D_t\}$ consistent with the monetary-fiscal policy regime.

3.2 Interest-Rate Rules Reconsidered

We now turn to the specification of the monetary-fiscal policy regime. In an economy with monetary frictions, it is no longer necessary for equilibrium that either $i_t = i_t^m$ or $M_t^s = 0$; this increases the range of possible ways in which monetary policy may be specified. The central bank may freely choose (within certain bounds) any two of the variables i_t, i_t^m, and M_t^s, leaving the third to be endogenously determined by the "LM relation" (3.5). In particular, it might choose a target for the monetary base M_t^s while maintaining a fixed rate (zero) for i_t^m, and let market nominal interest rates be endogenously determined, as in many textbook analyses; but it might also choose a short-run operating target for i_t while maintaining a fixed rate (zero) for i_t^m, and let the monetary base be endogenously determined, as under current Fed procedures. The central bank's choice of a rate of an interest rate to pay on central-bank balances now longer implies a particular operating target for short-term market interest rates, and it is now important to distinguish between these two aspects of the monetary policy regime.

In one case, however, the details of the way in which the central bank chooses to implement its interest-rate operating targets will be irrelevant for price-level determination. This is the case, often assumed for pedagogical purposes, in which $u(C, m; \xi)$ is additively separable between the arguments C and m for each possible vector of disturbances ξ.[41] In this familiar case, the marginal utility of consumption is independent of real money balances, just as in the cashless economy, even though now $u_m > 0$ in the case of a low enough level

[40] In writing (1.21) and (1.24) for this model, we must express the marginal utility of consumption as $u_c(Y_t, M_t^s/P_t; \xi_t)$.

[41] See, for example, the presentation of the model by Obstfeld and Rogoff (1996, sec. 8.3) or Walsh (1998, sec. 2.3).

of real balances.

Let monetary policy be specified by an interest-rate rule, such as the Wicksellian rule (1.30), together with an additional equation that specifies either the supply of base money M_t^s or the interest paid on base money i_t^m. If the additional equation specifies the supply of base money, then the interest-rate target is implemented by adjusting i_t^m as necessary in order for the target interest rate i_t to satisfy (3.5), given the supply of base money. (This generalizes the method of policy implementation assumed above for the cashless economy, where (1.15) played the role of equilibrium condition (3.5).) If instead the additional equation specifies the interest paid on money (perhaps specifying $i_t^m = 0$ at all times), then the interest-rate target is implemented by adjusting M_t^s so as to satisfy (3.5). (This is a stylized representation of the current method of policy implementation in countries like the U.S.) We suppose once again that fiscal policy is specified by an exogenous process $\{D_t\}$.

We have shown earlier that in the cashless case, conditions (1.21) and (1.30) alone suffice to determine locally unique rational-expectations equilibrium paths for the variables $\{P_t, i_t\}$, as long as $\phi_p > 0$. The same argument continues to apply here, given that the marginal utility of consumption in (1.21) is once again a function only of the exogenous states (Y_t, ξ_t). Corresponding to these paths will be locally unique paths for $\{M_t^s, i_t^m\}$, obtained by solving (3.5) together with the additional equation that specifies one or the other of these variables each period. Thus the conditions under which the interest-rate rule implies a determinate rational-expectations equilibrium are exactly the same as in a cashless economy, and (1.37) and (1.38) continue to provide a log-linear approximation to the equilibrium evolution of prices and interest rates in response to small enough exogenous disturbances.

The same is true if the interest-rate rule is of a form such as (2.43), (2.48), or (2.51) rather than (1.30). The neo-Wicksellian account of price-level determination developed above continues to apply even in the presence of transactions frictions that allow for a non-negligible interest differential Δ_t in equilibrium. In our model with transactions frictions, the money-demand or "LM" relation (3.5) is a requirement for equilibrium, yet it plays no role in determining the equilibrium evolution of prices under a given interest-rate rule; this relation is relevant only to the question of how the central bank must adjust the instruments under its direct control (M_t^s and i_t^m) so as to *implement* its interest-rate operating targets.

3.3 A Comparison with Money-Growth Targeting

The money-demand relation is, of course, a crucial element in the theory of price-level determination in the case of a money growth rule, *i.e.*, a monetary policy specified in terms of exogenous paths for the monetary base $\{M_t^s\}$ and the interest rate paid on money $\{i_t^m\}$, with the interest rate $\{i_t\}$ left to be determined by the market. Let us consider the case in which M_t^s/M_{t-1}^s and i_t^m remain forever within bounded intervals containing the steady-state values 1 and $0 \leq \bar{\imath}^m < \beta^{-1} - 1$ respectively, and look for rational-expectations equilibria in which $m_t \equiv M_t^s/P_t$ and i_t remain forever near the constant values

$$\bar{m} \equiv L(\bar{Y}, \bar{\Delta}; 0),$$

$$\bar{\imath} = \beta^{-1} - 1 > 0$$

associated with a zero-inflation steady state. (Here $\bar{\Delta} \equiv 1 - \beta(1 + \bar{\imath}^m) > 0$ is the steady-state interest differential.)

Log-linearizing (3.5) around this steady-state, we obtain a relation of the form

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i(\hat{\imath}_t - \hat{\imath}_t^m) + \epsilon_t^m, \tag{3.6}$$

where

$$\hat{m}_t \equiv \log(m_t/\bar{m}), \qquad \hat{\imath}_t^m \equiv \log(1 + i_t^m/1 + \bar{\imath}^m).$$

Here the constant coefficients are

$$\eta_y \equiv \frac{\bar{Y}}{\bar{m}}\frac{\partial L}{\partial y} > 0, \qquad \eta_i \equiv -\frac{1-\bar{\Delta}}{\bar{m}}\frac{\partial L}{\partial \Delta} > 0,$$

with the partial derivatives evaluated at the steady-state values of the arguments of L, and the exogenous disturbance term is

$$\epsilon_t^m \equiv \frac{1}{\bar{m}}\frac{\partial L}{\partial \xi}\xi_t.$$

(The signs asserted above for these coefficients follow from the assumptions regarding preferences stated earlier.) Note that η_y measures the income elasticity of money demand, and η_i the interest semi-elasticity of money demand; numerical values for these coefficients, and for the statistical properties of the disturbance term, can thus be obtained from standard econometric studies of money demand.

We can then study the local determinacy of equilibrium under such a policy by considering the bounded processes $\{\hat{m}_t, \hat{\imath}_t\}$ that satisfy the log-linear equilibrium relations (1.32) and (3.6) at all times. We obtain the following result.

PROPOSITION 2.11. In the context of a Sidrauski-Brock model with additively separable preferences, consider the consequences of a monetary policy specified in terms of exogenous paths $\{M_t^s, i_t^m\}$, together with a fiscal policy specified by an exogenous path $\{D_t\}$. Under such a regime, the rational-expectations equilibrium paths of prices and interest rates are (locally) determinate; that is, there exist open sets $\mathcal{P}$ and $\mathcal{I}$ such that in the case of any tight enough bounds on the fluctuations in the exogenous processes $\{Y_t, \xi_t, M_t^s/M_{t-1}^s, i_t^m, D_t/D_{t-1}\}$, there exists a unique rational-expectations equilibrium in which $P_t/M_t^s \in \mathcal{P}$ and $i_t \in \mathcal{I}$ at all times. Furthermore, a log-linear approximation to the equilibrium path of the price level, accurate up to a residual of order $\mathcal{O}(||\xi||^2)$, takes the form

$$\log P_t = \sum_{j=0}^{\infty} \varphi_j E_t[\log M_{t+j}^s - \eta_i \log(1 + i_t^m) - u_{t+j}] - \log \bar{m}, \tag{3.7}$$

where the weights

$$\varphi_j \equiv \frac{\eta_i^j}{(1+\eta_i)^{j+1}} > 0$$

sum to one, and u_t is a composite exogenous disturbance

$$u_t \equiv \eta_y \hat{Y}_t - \eta_i \hat{r}_t + \epsilon_t^m - \eta_i \log(1 + \bar{\imath}^m).$$

The proof is given in the appendix.

We thus obtain a well-defined rational expectations equilibrium price level under such a policy, for arbitrary bounded fluctuations in the rate of money growth. This is the determinacy result that Sargent and Wallace (1975) stress in their argument for the money supply as the "optimal instrument of monetary policy." However, we have seen that policy rules need not take this form in order to imply a determinate equilibrium path for the price level; interest-rate rules such as those advocated by Wicksell and Taylor also have this property.

Equation (3.7) also provides a simple theory of price-level determination; in the case that one abstracts from disturbances other than the fluctuations in the rate of money growth, it states that the log price level at any point in time is (up to a constant) just a weighted average of current and expected future logs of the money supply.[42] This appealingly simple result may suggest that even in the analysis of other types of possible policy regimes, what matters about any regime is the path of the money supply that it implies.[44] It might then seem natural that alternative strategies for policy should be considered in terms of how one wishes to have the money supply evolve. But as we have seen, a straightforward analysis of the consequences for inflation of alternative policy rules is equally possible without any reference to either the evolution of the money supply or the determinants of money demand.

In fact, our conclusions above about the consequences of alternative interest-rate rules can be viewed as more basic, for Proposition 2.11 is actually a *consequence* of our previous analysis of Wicksellian rules.[45] For solving the money-demand relation (3.5) for the

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[42]This result is most often presented as the implications of a rational-expectations version of the Cagan model of inflation determination.[43] In Cagan's (1956) model, desired log real money balances are a decreasing linear function of the expected rate of growth of the log price level. A (discrete-time) relation of exactly this kind is obtained by substituting (1.32) into (3.6) to eliminate i_t, except that in our case there is also a time-varying intercept for this relation, equal to $\log \bar{m} + u_t + \eta_i \log(1 + i_t^m)$.

[44]The determinacy result in Proposition 2.11 applies only to policies that make the money supply an exogenously specified process, and not to general feedback rules for determination of the money supply. Our derivation of equation (3.7) as an implication of the requirements for rational-expectations equilibrium also goes through more generally, assuming that the infinite sum on the right-hand side is well-defined. But in the case of feedback from endogenous variables (such as the price level) to the money supply, the expressions on the right-hand sides of equations such as (3.7) need not be uniquely defined. That is, there may be a large number of equilibria, in each of which (3.7) is valid, but in each of which the expression on the right-hand side takes a different value, as a result of different evolution of the endogenous determinants of the money supply.

[45]See Taylor (1999) for motivation in terms similar to these of the "Taylor rule". Taylor's discussion, however, elides the distinction between rules such as (1.30), that respond to deviations of the price *level*) from a target path, and rules such as (2.43) that respond to deviations of the *inflation rate* from its target.

equilibrium nominal interest rate, we obtain an equation of the form

$$i_t = \iota(P_t/M_t^s;\ i_t^m, Y_t, \xi_t), \tag{3.8}$$

where ι is an increasing function of its first argument, for any values of i_t^m, Y_t, ξ_t. This is just Keynes' (1936) "liquidity preference theory" of the interest rate; if i_t is graphed as a function of Y_t, suppressing the other arguments, this is the Hicksian "LM curve". When Y_t is exogenous and prices are flexible, as here, it is more useful to think of this as an equilibrium relation between i_t and P_t, as in some presentations of a "flexible-price IS-LM" model.

This equilibrium relation between interest rates and prices, established through the central bank's control of its instruments M_t^s and i_t^m, is just an example of a interest-rate feedback rule of the form (1.30), in which $P_t^* = M_t^s/\bar{m}$, $\nu_t = (i_t^m, Y_t, \xi_t)$,[46] and

$$\phi(p;\nu) = \iota(p/\bar{m};\nu).$$

It then follows that in the log-linear approximation (1.34), $\phi_p = \eta_i^{-1} > 0$ and

$$\nu_t = \hat{\imath}_t^m + \eta_i^{-1}(\eta_y \hat{Y}_t + \epsilon_t^m).$$

Because this is a policy rule under which $\phi_p > 0$, Proposition 2.3 applies, and making the above substitutions for ϕ_p, P_t^*, and ν_t, we find that (3.7) is just the price-level path predicted by our previous result (1.39) for a general rule of the Wicksellian type.

The prior formulation is the more general one, since it applies to Wicksellian rules in which the elasticity of interest-rate response to price-level deviations need not equal exactly η_i^{-1}. Furthermore, it is clear even from (3.7) — which follows from a traditional quantity-theoretic analysis, as shown in the appendix — that the path of the money supply *as such* is not important for price-level determination. What matters is the way in the central bank chooses to adjust the composite variable $\log M_t^s - \eta_i \log(1 + i_t^m)$; it does not matter to what extent this is achieved by varying the money supply as opposed to the interest rate paid on base money (except, of course, that if there is to be trend growth in this variable it must

[46] Here we extend our previous notation in (1.30) by allowing a vector ν_t of exogenous variables to affect the function ϕ.

occur through trend growth in the money supply, owing to the impossibility of reducing i_t^m below zero). The reason for this is simple: it is the shift in this composite variable that indicates the extent to which the central bank's actions shift the equilibrium relation (3.8) between interest rates and prices.

Of course, we could also develop a theory of price-level determination under money-supply feedback rules, in which the path of the monetary base is not specified exogenously, but is instead a specified function of the price level. Wicksellian rules (1.30) could then be equivalently described as rules of this form,

$$M_t^s = M^s(P_t; \P_t^*, i_t^m, Y_t, \xi_t, \nu_t), \tag{3.9}$$

where the function M^s is obtained by substituting (1.30) for i_t in (3.5), and solving for M_t^s as a function of the other variables. But subsuming our theory of price-level determination under interest-rate rules under such a theory of endogenous-money rules is not obviously desirable. First of all, the theory of price-level determination under interest-rate rules expounded above continues to apply in a cashless economy (the form in which we have first expounded it), whereas a theory of endogenous-money rules of the form (3.9) would not be possible in that case.

Even when a well-defined money-demand relation (3.5) exists, it is not obvious that (1.30) and (3.9) are equally useful ways of specifying monetary policy in order to achieve the equilibrium described by (1.39). For (1.30) suffices to specify the aspects of policy that matter for the central bank's stabilization objective (here assumed to be control of the price level), while (3.9) does not — the latter rule must be supplemented by a policy rule for the control of i_t^m in order for the price level to be determined. (Of course, in traditional accounts, it is taken for granted that $i_t^m = 0$.) And even granting a specific, known policy with regard to interest payments on money, (1.30) may be a more useful policy prescription. This is because implementation of (3.9) requires that the central bank take account of the current value of the disturbance ϵ_t^m to the money demand equation in setting its money-supply instrument, while knowledge of this state is irrelevant for purposes of implementation of the interest-rate rule (1.30).

Since the central bank will not be able to estimate this random disturbance with perfect precision in practice, adoption of a monetary-base target necessarily results in interest-rate variations in response to money-demand disturbances that could be avoided through the use of an interest-rate instrument. Interest-rate variations in response to these shocks are not desirable, from the point of view of price-level stabilization, unless they are positively correlated with exogenous variations in the equilibrium real rate of interest. As the primary source of money-demand variation is probably developments in the payments system that have important effects upon money demand without being of particular consequence for equilibrium real rates of return, this seems an implausible line of argument.

The point that money-demand disturbances make an interest-rate operating target more desirable than a money-supply target is of course a central argument in the celebrated analysis of Poole (1970). Poole also argues that if, instead, "IS shocks" are a more important source of instability than "LM shocks," monetary control is superior to interest-rate control. This might make it seem uncertain which concern, in practice, ought to dominate. But let us recall the nature of the argument for a monetary instrument in the case of "IS shocks". It is desirable on stabilization grounds that interest rates rise in response to demand stimulus (in Wicksellian terms, because these disturbances raise the natural rate of interest), but it is assumed that with interest-rate control, interest rates will not rise. Instead, with control of the money supply, such disturbances raise output (or in our flexible-price model, the price level), and as a result increase money demand, leading to an automatic interest-rate increase which aids stabilization. But this argument requires not only that the central bank is not able to respond directly to the disturbance, but that it *cannot condition its interest-rate instrument upon output or the price level* except indirectly, by letting interest rates be affected by an increase in money demand. If we assume instead that the central bank *can* make its instrument a function of current prices and output directly, even though it cannot make it a direct function of exogenous disturbances, then the interest-rate instrument is unambiguously superior.

3.4 Consequences of Non-Separable Utility

The strong irrelevance result obtained in section xx — accordance to which an interest-rate rule such as (1.30) or (2.43) suffices to determine the equilibrium path of prices quite independently of how the central bank implements its interest-rate operating targets through adjustment of the monetary base and/or the interest rate paid on money — depends, of course, on the special assumption of an additively-separable (indirect) utility function $u(C, m; \xi)$. And this assumption is not very realistic, despite its familiarity in textbook treatments; if real balances supply a non-pecuniary yield owing to their usefulness in conducting transactions, it makes sense that the marginal benefit of additional real balances should depend on the volume of purchases that the household makes. (The most plausible assumption, on intuitive grounds, would probably be that $u_{cm} > 0$, so that consumption expenditure and real balances are complements.)

Yet one can justify the neglect of real-balance effects on the marginal utility of income in equations such as (1.21) without assuming either additive separability or a genuinely cashless economy (with the counter-factual implication that money must earn the same rate of return as other riskless assets). In what Woodford (1998a) calls a "cashless limiting economy," the marginal utility of additional real balances becomes quite large as household real balances fall to zero (assuming real purchases of a magnitude near $\bar{Y}$), so that it is possible in equilibrium to have a non-trivial interest-rate differential Δ_t; yet at the same time, the transactions that use money are sufficiently unimportant that variations in the level of real balances sufficient to require a substantial change in the interest-rate differential have only a negligible effect on the marginal utility of real income (or of consumption).

The idea is that in such an economy money is used for transactions of only a very few kinds, though it is essential for those. As a result, positive real balances are demanded even in the case of a substantial interest-rate differential (and hence, a substantial opportunity cost of holding money); but equilibrium real balances are very small relative to national income. Equation (3.3) requires that in such an equilibrium, an increase in real balances *equal in value to a substantial share of aggregate expenditure* would have to increase utility by as much as a substantial percentage increase in consumption; but if equilibrium real balances are tiny in value relative to national income, a substantial *percentage increase* in real balances may still have a negligible effect on utility. As this effect may continue to be negligible for different levels of consumption, the effect on the marginal utility of consumption of a substantial percentage change in real balances may also be negligible. This then justifies neglecting real-balance effects in equations such as (1.21).

Formally, the elasticity of u_m, the marginal utility of additional real money balances, with respect to changes in the level of real expenditure is equal to

$$\upsilon \equiv \frac{\bar{Y} u_{mc}}{u_m},$$

a quantity that we have argued should plausibly have a non-trivial positive value. But what matters for the extent to which variations in the level of real balances affect the Fisher relation (1.21) is the elasticity of u_c, the marginal utility of additional expenditure, with respect to changes in the level of real balances, or

$$\chi \equiv \frac{\bar{m} u_{cm}}{u_c}. \qquad (3.10)$$

We note that $\chi = s_m \upsilon$, where

$$s_m \equiv \frac{\bar{m} u_m}{\bar{Y} u_c} = \bar{\Delta} \frac{\bar{m}}{\bar{Y}}$$

is the flow rate of effective expenditure by households on liquidity services (measured by the interest foregone on the money balances that they hold) expressed as a proportion of national income. It follows that χ and υ must have the same sign. But in a "cashless limiting economy," s_m is infinitesimally small (even though $\bar{\Delta}$ is not), as a result of which χ is infinitesimally small (even though υ) is not.[47] Because χ is negligible, a log-linear approximation to (1.21) again takes the simple form (1.32); yet this is not equivalent to assuming

[47] Woodford (1998xx) displays a parametric family of transactions technologies in which this limiting case is approached as a certain parameter α, measuring the fraction of goods that are purchased using cash, is made arbitrarily small. The paper shows that the equilibrium solution for the state-contingent evolution of prices is continuous in the parameter α, so that the conclusions reached under the assumption that $\chi = 0$, as in a cashless model, provide an accurate approximation to the equilibrium dynamics in the case of any economy with a small enough value of α. At the same time, the interest-rate differential remains bounded away from zero as α is made small; hence the equilibrium of the "cashless limiting economy" is in this respect different from that of a genuinely cashless economy of the kind presented in section xx above.

additive separability, for one need not assume that v is negligible. (The latter quantity matters for the predictions of one's analysis regarding the percentage fluctuations in the money supply that should occur in equilibrium under a particular approach to implementation of the interest-rate rule.)[48]

Thus the case of a "cashless limiting economy" is another in which one may legitimately abstract from any effects of variations in real balances on the equilibrium conditions that determine the evolution of prices under an interest-rate rule. How accurately this limiting case approximates the situation of an actual economy in which central-bank money still provides some valuable services remains, of course, a question for quantitative analysis. We turn, then, to the question of the extent to which our previous results must be modified if we allow for a non-negligible value of the elasticity χ defined in (3.10).

In the general case of the Sidrauski-Brock model, a log-linear approximation to (1.21) takes the form

$$\hat{\imath}_t = \hat{r}_t + E_t \pi_{t+1} - \chi E_t(\hat{m}_{t+1} - \hat{m}_t), \tag{3.11}$$

where $\hat{r}_t$ continues to be defined by (1.33). (Note, however, that $\hat{r}_t$ no longer has the interpretation of being the equilibrium real rate of return; the latter quantity is no longer completely exogenous.) Substituting (3.6) for the equilibrium level of real balances in this equation, we obtain a relation of the form

$$(1 + \eta_i \chi)\hat{\imath}_t = \tilde{r}_t + E_t \pi_{t+1} + \eta_i \chi E_t \hat{\imath}_{t+1} + \eta_i \chi E_t(\hat{\imath}^m_{t+1} - \hat{\imath}^m_t), \tag{3.12}$$

where the composite exogenous disturbance in this relation is given by

$$\tilde{r}_t \equiv \hat{r}_t - \eta_y \chi E_t(\hat{Y}_{t+1} - \hat{Y}_t) - \chi E_t(\epsilon^m_{t+1} - \epsilon^m_t).$$

Then if one specifies the policy rule in a way that allows both i_t and i^m_t to be determined as functions of the path of prices, equation (3.12) alone, together with the policy rule, suffices

[48] The assumption of a "cashless limiting economy" leaves one free to assume elasticities η_y, η_i in the money-demand relation (3.6) in accordance with empirical estimates, whereas the assumption of additive separability would imply a restriction upon these elasticities that is not in accordance with typical estimates, as discussed below.

to determine the equilibrium paths of interest rates and prices, just as in our analyses in sections 1 and 2.

For example, consider a policy regime under which the central bank seeks to maintain a constant interest-rate differential $\bar{\Delta}$ between overnight market rates and the interest paid on the monetary base (as is true of the channel systems described in chapter 1, at least in the case of the interest paid on central-bank balances). In such a case, monetary policy may be specified by an interest-rate rule, such as the Wicksellian rule (1.30), together with the equation

$$\Delta_t = \bar{\Delta} > 0 \tag{3.13}$$

indicating the way in which the interest paid on money varies with the changes in the interest-rate target required by the Wicksellian rule. (We assume that $\bar{\Delta} > 0$ because of the observed preference of central banks with channel systems for maintaining a small positive spread.) Fiscal policy can again be specified by an exogenous sequence $\{D_t\}$.

In order to analyze local equilibrium determination under such a regime, it suffices that we consider the log-linear relations (1.34) and (3.12), together with the equation

$$\hat{\imath}_t = \hat{\imath}^m_t, \tag{3.14}$$

representing the log-linearization of (3.13). Using (3.14) to eliminate $\hat{\imath}^m_t$ from (3.12), we obtain a relation of the form

$$\hat{\imath}_t = \tilde{r}_t + E_t\pi_{t+1}. \tag{3.15}$$

This is of exactly the same form as (1.32), except that the term $\hat{r}_t$ is replaced by $\tilde{r}_t$, a different function of the exogenous disturbances. We observe from (3.15) that $\tilde{r}_t$ can be interpreted as the equilibrium real rate of return in the case that a constant interest-rate differential is maintained. (It is only under the latter stipulation that we can define an equilibrium real rate that is purely exogenous, in the case of non-separable utility.)

We now look for bounded solutions $\{\hat{P}_t, \hat{\imath}_t\}$ to the system of equations (1.34) and (3.15). Because (3.15) is identical to (1.32), except for the replacement of $\hat{r}_t$ by $\tilde{r}_t$, our previous results are directly applicable. Proposition 2.3 then implies that equilibrium is determinate in the case of any rule with $\phi_p > 0$, and a log-linear approximation to the equilibrium price process is given by (1.39), with $\tilde{r}_{t+j}$ replacing $\hat{r}_{t+j}$ in each term.[49] Hence the same theory of price-level determination as derived above continues to apply, with a small modification of our interpretation of the exogenous disturbance term. The same is true for other families of interest-rate rules, such as (2.43) or (2.51).

Suppose instead that policy is implemented in a way that involves a fixed rate of interest on the monetary base (for example, a zero interest rate, as in the U.S. at present). In this case, (3.13) is replaced by the condition $i_t^m = \bar{\imath}$ We can again eliminate $\hat{\imath}_t^m$ from (3.12), and obtain a system of two equations to solve for the equilibrium evolution of prices and interest rates. Similar methods as have been used before can then be employed to derive generalizations of our previous results.

Proposition 2.12. In a Sidrauski-Brock model where utility is not necessarily separable, let monetary policy be specified by a Wicksellian rule (1.30) for the central bank's interest-rate operating target. Suppose that $i_t^m = \bar{\imath}$ at all times, for some $0 \leq \bar{\imath} < \beta^{-1} - 1$; and let fiscal policy again be specified by an exogenous process $\{D_t\}$. Finally, suppose that

$$\chi > -\frac{1}{2\eta_i}. \tag{3.16}$$

Then equilibrium is determinate in the case of any policy rule with $\phi_p > 0$. A log-linear approximation to the locally unique equilibrium price process is given by

$$\log P_t = \phi_p^{-1} \sum_{j=0}^{\infty} \varphi_j E_t \tilde{r}_{t+j} + \sum_{j=0}^{\infty} \tilde{\varphi}_j E_t [\log P_{t+j}^* - \phi_p^{-1} \nu_{t+j}], \tag{3.17}$$

where the weights are given by

$$\varphi_j \equiv \frac{(1 + \eta_i \chi \phi_p)^j \phi_p}{[1 + (1 + \eta_i \chi) \phi_p]^{j+1}}, \tag{3.18}$$

$$\tilde{\varphi}_0 \equiv \frac{(1 + \eta_i \chi) \phi_p}{1 + (1 + \eta_i \chi) \phi_p}, \qquad \tilde{\varphi}_j \equiv \frac{(1 + \eta_i \chi \phi_p)^{j-1} \phi_p}{[1 + (1 + \eta_i \chi) \phi_p]^{j+1}} \quad \text{for } j \geq 1. \tag{3.19}$$

[49] In fact, our previous results for the Wicksellian regime in a cashless economy can be recognized as a special case of the more general result obtained here. For that regime was also one in which (3.13) held at all times, with the value $\bar{\Delta} = 0$; and the composite disturbance $\hat{r}_t$ is just the value of $\tilde{r}_t$ in the case of a cashless economy, since $\chi = 0$ in that case.

The proof is given in the appendix. Here the solution (3.17) generalizes our previous result (1.39), to which it reduces in the case that $\chi = 0$. As long as χ also satisfies $\chi > -(\phi_p \eta_i)^{-1}$, the weights $\{\varphi_j, \tilde{\varphi}_j\}$ are all positive, and again both series of weights sum to one, though for $\chi \neq 0$, the weight $\tilde{\varphi}_j$ no longer exactly equals φ_j.

Thus as long as $\chi \geq 0$ (the case of greatest empirical plausibility), and even for sufficiently small negative values of χ, the same condition as before suffices for determinacy. We also obtain a qualitatively similar theory of price-level determination: again, the log price level is equal to a weighted average of current and future price-level targets, plus a discrepancy that involves current and expected future fluctuations in the (suitably qualified) equilibrium real rate of return $\tilde{r}_t$ and in the reaction-function shift factor ν_t in a manner similar to the previous equation (1.39). The solution (3.17) is also observed to be continuous in χ, so that for any economy in which χ is small enough, the equations derived for the cashless economy provide a close approximation to the equilibrium evolution of prices.

We can similarly extend the results derived earlier for the case of a generalized Taylor rule (2.51).

Proposition 2.13. Let monetary policy instead be specified by an interest-rate rule of the form (2.51), with coefficients $\phi_\pi, \rho \geq 0$, and again suppose that $i_t^m = \bar{\imath}_t^m$ at all times. Finally, suppose again that χ satisfies (3.16).Then equilibrium is determinate if and only if $\phi_\pi > 0$ and (2.52) holds. When these conditions are satisfied, a log-linear approximation to the equilibrium evolution of inflation is given by

$$\pi_t = -\frac{\rho}{\phi_\pi}(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \sum_{j=0}^{\infty} \varphi_j E_t \tilde{r}_{t+j} - \sum_{j=0}^{\infty} \tilde{\varphi}_j E_t \bar{\imath}_{t+j}, \tag{3.20}$$

where the weights are given by

$$\varphi_j \equiv \frac{(1 + \eta_i \chi \phi_\pi)^j}{[(1 + \eta_i \chi)\phi_\pi + \rho]^{j+1}}, \tag{3.21}$$

$$\tilde{\varphi}_0 \equiv \frac{(1 + \eta_i \chi)}{(1 + \eta_i \chi)\phi_\pi + \rho}, \qquad \tilde{\varphi}_j \equiv \frac{[1 + (1 - \rho)\eta_i \chi](1 + \eta_i \chi \phi_\pi)^{j-1}}{[(1 + \eta_i \chi)\phi_\pi + \rho]^{j+1}} \quad \text{for } j \geq 1. \tag{3.22}$$

The proof is in the appendix. Thus the Taylor principle is again necessary and sufficient for determinacy, as long as χ does not take a large negative value. The solution (3.20) generalizes (2.53), to which it reduces when $\chi = 0$. Again the weights $\tilde{\varphi}_j$ are no longer exactly equal to the φ_j weights when $\chi \neq 0$. However, the weights continue all to be positive, if $\rho \leq 1$ and χ is not too negative.

We have found that our qualitative results are largely unaffected by taking account of real-balance effects on the equilibrium real rate of interest. How much are such effects likely to matter quantitatively? The size of the required correction can be numerically calibrated from estimates of money demand. Our simple model implies that the money demand elasticities should be the same at both high and low frequencies, and if we expect the disturbances ϵ_t^m to be unimportant at low frequencies, low-frequency relations among the variables m_t, Y_t, and i_t will be most revealing about these elasticities. As a typical example, Lucas (1999) finds that over the period xxxx, low-frequency variations in real M1 for the U.S. are fairly well fit (up to a constant) by the variations in $\log Y_t - .5 \log i_t$.[50] Linearization of (3.3) in the logs of Y_t, m_t, and i_t indicates that according to our model, $\hat{m}_t$ should be proportional to

$$[\sigma^{-1} + \upsilon]\hat{Y}_t - \beta \log(i_t/\bar{\imath}).$$

(Here it should be recalled that we log-linearize around a steady state with zero inflation, and that we assume zero interest on the monetary base, as is true for the U.S.) Thus $\beta^{-1}[\sigma^{-1} + \upsilon]$ should be the ratio of the income elasticity to the interest elasticity of money demand; according to Lucas' estimates, this ratio is approximately 2.[51]

[50] Our model should really be interpreted as one of demand for the monetary base, but empirical studies more often model the demand for a broader aggregate such as M1. At the low frequencies with which we are here concerned, M1 and the base move roughly in proportion to one another, so we shall use Lucas' estimated elasticities as estimates of the corresponding elasticities of demand for the monetary base.

[51] Here we may observe the way in which the assumption of an additively separable utility function, implying $\upsilon = 0$, would restrict the model's implications regarding money demand in an undesirable way. We would obtain a necessary relation between the degree of interest-sensitivity of private expenditure, on the one hand, and the income-elasticity and interest-elasticity of money demand on the other. A negligible value of χ is instead consistent with arbitrary values of these other elasticities, as long as one assumes a small enough value for s_m.

From this we can infer that

$$\chi = s_m \upsilon = s_m(2\beta - \sigma^{-1}). \tag{3.23}$$

If σ is greater than 0.5, as we shall assume,[52] it follows that $\chi > 0$, as argued above on intuitive grounds. At the same time, no matter how large we assume σ to be, the factor in parentheses in (3.23) cannot exceed 2, so that χ cannot realistically be assigned a value larger than twice the size of s_m. For the U.S., the value of the monetary base is about 25 percent of a quarter's GDP, so that (using the value $\beta = 0.99$ for a quarterly model) s_m is approximately 0.0025. This suggests a value of χ no larger than 0.005. Alternatively, if we use Lucas' estimated coefficients for a semi-logarithmic specification of money demand, namely $\eta_y = 1$ and $\eta_i = 7$ years (or 28 quarters), then the implied ratio of elasticities would be $\eta_y/(1-\beta)\eta_i = 1/0.28 = 3.6$. This would allow χ to be a larger multiple of s_m, but only by a factor of less than two, so that χ should be no larger than 0.01. Since the monetary base is equally small for most industrial economies, a similar conclusion as to the plausible size of real-balance effects would be reached for many economies.

As an illustration of how much allowing for $\chi > 0$ would affect our calculations, consider the solution (3.20) for the case of an inertial Taylor rule. Let us consider a policy rule with $\rho = 0.8$, a fairly typical degree of interest-rate inertia in estimated Fed reaction functions for the U.S. using quarterly data (see Table 1.1), and $\phi_\pi = 0.3$, implying a long-run inflation response coefficient of $\Phi_\pi = 1.5$, Taylor's (1993) value for the Greenspan period. Then the weights φ_j and $\tilde{\varphi}_j$ appearing in the solution (3.20) for various future horizons j are plotted in Figure 2.2. The figure shows the weights both under the assumption that $\chi = 0$ (as in our first approach), and for the positive value $\chi = .02$. The latter is likely to be an over-estimate of the actual size of real-balance effects on the marginal utility of income, but is considered to show that even under the most generous assumptions real-balance effects should not matter greatly. (The value $\eta_i = 28$ quarters is assumed in both cases.)

While the real-balance effect matters for such a calculation, neglecting it would not lead to

[52] See chapter 4 for further discussion of the calibration of this parameter.

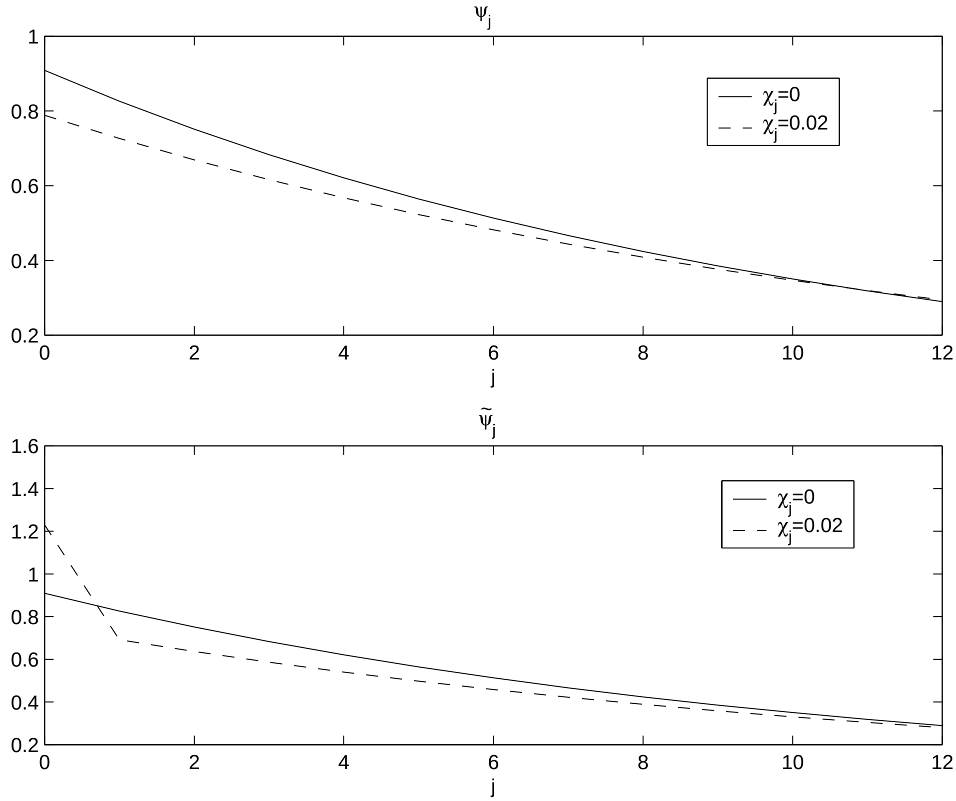

Figure 2.2: The weights φ_j with and without real-balance effects.

extremely misleading conclusions, either. Perhaps the most important qualitative difference is that setting $\chi = 0.02$ results in a slightly larger relative weight on the current period's intercept $\bar{\imath}_t$ as opposed to expected future intercepts. Especially in the (realistic) case that both the equilibrium real rate $\tilde{r}_t$ and the monetary policy disturbance $\bar{\imath}_t$ exhibit substantial positive serial correlation, so that it is only smoothed versions of the coefficients φ_j and $\tilde{\varphi}_j$ that matter in practice, the predictions for inflation under the two assumptions will be quite similar. For example, let us suppose that $\tilde{r}_t$ follows a stationary AR(1) process with serial correlation coefficient $0 < \rho_r < 1$. Then the predicted initial-period jump in the price level in response to a unit positive innovation in the natural rate of interest[53] is given by

$$\Delta \sum_{j=0}^{\infty} \varphi_j \rho_r^j.$$

[53]We understand this to mean a jump of one percentage point *per annum* in the natural rate, meaning that $\tilde{r}_t$ jumps by Δ, where Δ is the length of a period in years.

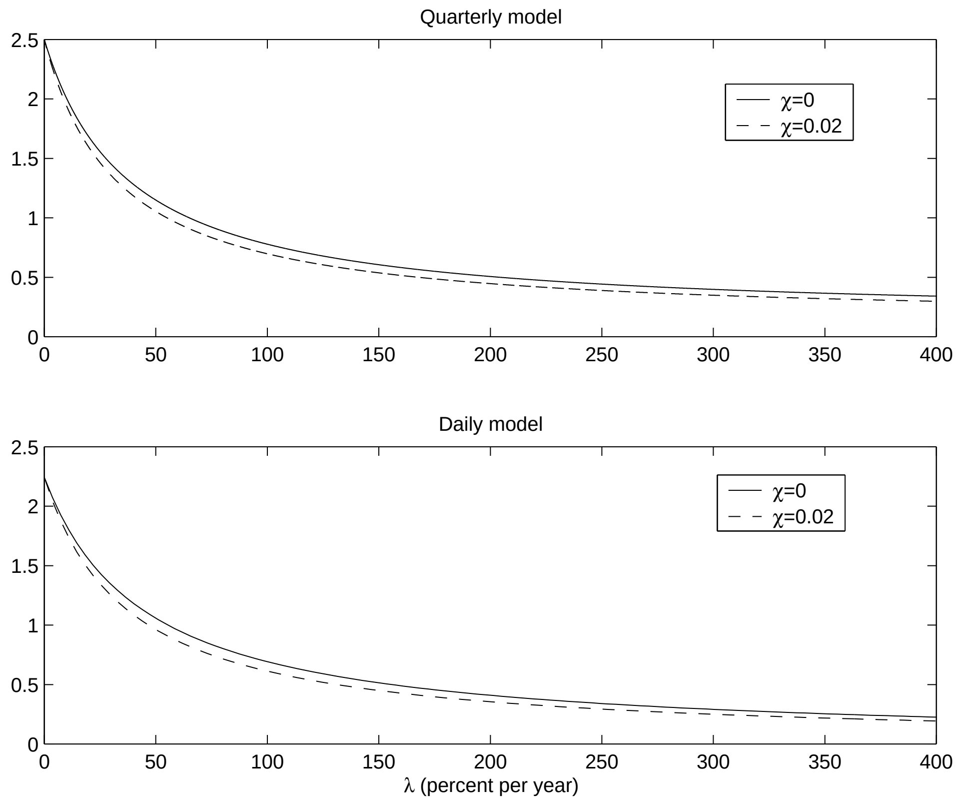

Figure 2.3: Inflation response as a function of shock persistence.

This quantity is plotted, as a function of the degree of persistence of the shocks, in Figure 2.3(a). Here the measure of persistence on the horizontal axis is the rate of decay of the disturbances per unit of calendar time, $\lambda \equiv -(\log \rho_r)/\Delta$, where Δ is the length of a period in years.[54] The results are plotted both under the assumption that $\chi = 0$ and for the upper-bound case $\chi = .02$. We observe that the error involved in neglecting the real-balance effects is quite small.

One case in which real balance effects would matter a great deal, however, is that of a rule (2.43) in which the interest rate depends only upon contemporaneous inflation, in the case that the "periods" are very short. Let us consider the behavior of the solution to our equations as the length of a "period" ($\Delta > 0$ units of calendar time) is made progressively shorter. We shall fix numerical values for the rate of time preference $\delta \equiv (-\log \beta)/\Delta > 0$,

[54]This measure is used, instead of ρ_r itself, in order to allow comparability across models with different period lengths. In panel (a), $\Delta = .25$ years.

the intertemporal elasticity of substitution $\sigma > 0$, steady-state share of expenditure on liquidity services s_m, the interest-rate semi-elasticity of money demand in units of calendar time $\tilde{\eta}_i \equiv \Delta\eta_i > 0$, and the income-elasticity of money demand η_y that are independent of the assumed size of Δ. Then the same reasoning used to derive (3.23) implies that as Δ is made small, the value of χ approaches a well-defined limiting value

$$\bar{\chi} = s_m \left(\frac{\eta_y}{\delta\tilde{\eta}_i} - \sigma^{-1} \right). \tag{3.24}$$

We then obtain the following result.

PROPOSITION 2.14. Consider a sequence of economies with progressively smaller period lengths Δ, calibrated so that $\bar{\chi} \neq 0$. Assume in each case that monetary policy is specified by a contemporaneous Taylor rule (2.43), with a positive inflation-response coefficient $\phi_\pi \neq 1$ that is independent of Δ. Assume also that zero interest is paid on money. Then equilibrium is determinate for all small enough values of Δ if $\phi_\pi > 1$ and $\bar{\chi} > 0$, or if $0 < \phi_\pi < 1$ and $\bar{\chi} < 0$, but not otherwise.

The proof is in the appendix.[55] This result clearly implies that the solution for equilibrium inflation in the short-period limit cannot be a continuous function of χ for values of χ near zero. Thus the value of χ matters in this case, even if it is very small.

However, this failure of continuity in χ in the short-period limit occurs only in the case of a policy rule that makes the interest-rate operating target a purely contemporaneous function of the current period's inflation. As periods are made shorter, the central bank is assumed to respond to a higher-frequency measure of inflation, and in the limit policy is assumed to respond solely to an instantaneous rate of inflation. This is plainly a case

[55] Our result agrees with the continuous-time analysis of Benhabib *et al.* (1998a), who find that the range of values of ϕ_π that result in determinacy depends on the sign of u_{cm}. Note that the result does not contradict Proposition 2.13. That proposition asserts that even in the case of a rule with $\rho = 0$, one has determinacy in the case of $\phi_\pi > 1$ for all values of χ greater than the negative lower bound (3.16). However, in the sequence of economies with progressively shorter period lengths considered in Proposition 2.14, the value of η_i increases as Δ^{-1}. Hence for any sequence of economies in Proposition 2.14 for which $\bar{\chi} < 0$, (3.16) is violated for all small enough values of Δ, and Proposition 2.13 ceases to apply.

of no practical interest. If we assume instead that policy responds to a smoothed inflation measure (2.47), and let the rate of decay (in calendar time) of the exponential weights on past inflation be fixed as we make "periods" shorter, no such problem arises. The same is true in the equivalent case of a policy rule (2.51) with partial adjustment of the interest rate toward a desired level that depends on the current instantaneous rate of inflation, if we hold fixed the rate of adjustment $\psi \equiv -\log \rho/\Delta > 0$ as we make Δ smaller. (We must also assume that ϕ_π is reduced along with Δ, so as to hold fixed the long-run response coefficient $\Phi_\pi \equiv (1-\rho)^{-1}\phi_\pi$.) In this case we obtain the following.

Proposition 2.15. Again consider a sequence of economies with progressively smaller period lengths Δ, and suppose that

$$\bar{\chi} > -\frac{1}{\psi\Phi_\pi\tilde{\eta}_i}. \tag{3.25}$$

Let monetary policy instead be specified by an inertial Taylor rule (2.51), with a long-run inflation-response coefficient $\Phi_\pi \equiv \phi_\pi/(1-\rho)$ and a rate of adjustment $\psi \equiv -\log \rho/\Delta > 0$ that are independent of Δ. Assume again that zero interest is paid on money. Then rational-expectations equilibrium is determinate if and only if $\Phi_p i > 1$, *i.e.,* if and only if the Taylor Principle is satisfied.

The unique bounded solution for the path of nominal interest rates in the determinate case is of the form

$$\hat{\imath}_t = \Lambda\bar{\imath}_t + \Gamma(1-\gamma)\sum_{j=0}^{\infty}\gamma^j E_t\tilde{r}_{t+j} + \tilde{\Gamma}(1-\gamma)\sum_{j=0}^{\infty}\gamma^j E_t\bar{\imath}_{t+j}, \tag{3.26}$$

with the solution for $\{\pi_t\}$ then obtained by inverting (2.51). In this solution, the coefficients $\Lambda, \Gamma, \tilde{\Gamma}$ approach well-defined limiting values as Δ is made arbitrarily small, while the rate of decay of the weights on expected disturbances farther in the future,

$$\xi \equiv -\log\gamma/\Delta > 0,$$

also approaches a well-defined limiting value. Furthermore, these limiting values are all continuous functions of $\bar{\chi}$ for values of $\bar{\chi}$ in the range satisfying (3.25), including values near zero.

Again the proof is in the appendix. Proposition 2.15 implies that in the case of this kind of rule, small non-zero values of χ (of either sign) make no important difference for our conclusions regarding price-level determination, even in the limiting case of arbitrarily short periods.[56] For example, our conclusions above about the small consequences of allowing for a realistic positive value for χ continue to hold in the case of periods shorter than a quarter. As an illustration, Figure 2.3(b) shows the same calculations as in Figure 2.3(a), but for a model in which the "period" is only a day. Thus as long as we assume either a modest degree of time-averaging in the inflation measure to which the central bank responds, or a modest degree of inertia in the central bank's adjustment of its interest-rate operating target in response to inflation variations — both of which are always characteristic of actual central-bank policies — we continue to find that the cashless analysis gives a good approximation to the results obtained under a realistic non-zero value of χ, regardless of the assumed period length.

4 Self-Fulfilling Inflations and Deflations

Thus far we have considered only the problem of *local* determinacy of equilibrium. But it is appropriate also to consider, at least briefly, the question whether rational expectations equilibrium is *globally* unique under one policy rule or another. Certainly we may have greater confidence that a particular policy regime is desirable if a desirable outcome represents not merely a locally unique equilibrium, but the unique rational expectations equilibrium, period. And insofar as regimes may differ in the matter of global uniqueness, even when they are equally consistent with the same desired equilibrium, and equally serve to make it locally determinate, considerations of global uniqueness provide a reasonable further criterion for refining one's policy prescription.

The question of global uniqueness requires that we return to a consideration of the exact, nonlinear equilibrium conditions, as our log-linear approximations can be relied upon to be

[56] Wicksellian policy rules of the form (1.30) can similarly be shown to be well-behaved in the continuous-time limit.

accurate only in the case of equilibria in which the variables remain within a sufficiently small neighborhood of the values at which the log-linearization is done. This makes a complete treatment of the issue rather complex, and beyond the scope of the present study. However, a simple example will serve to illustrate how global multiplicity of equilibrium is possible, despite local determinacy. We shall also give examples of policy regimes that would resolve this problem.

4.1 Global Multiplicity Despite Local Determinacy

Our example illustrates the potential problem of multiplicity of equilibria under a "Taylor rule" as discussed by Schmitt-Grohé and Uribe (1998) and by Benhabib *et al.* (1998b). Consider a deterministic interest-rate feedback rule of the form

$$i_t = \phi(\Pi_t), \tag{4.1}$$

where ϕ is again an increasing continuous function, satisfying $\phi(\Pi) \geq 0$ for all $\Pi > 0$. We suppose once again that this rule incorporates an implicit target inflation rate $\Pi^* > \beta$ satisfying $\phi(\Pi^*) = \beta^{-1}\Pi^* - 1$. The stipulated lower bound – which is of some importance for the present discussion – is necessary because it will be impossible for the central bank to force nominal interest rates to be negative, no matter how much it may increase the monetary base.

Following Benhabib *et al.* (1998b), let fiscal policy now be specified by a rule of the form

$$T_t = \alpha W_t - \frac{i_t}{1+i_t} M_t \tag{4.2}$$

for determination of net tax collections at each date, for some constant $0 < \alpha \leq 1$. Using the flow government budget constraint (1.7), we see that this rule implies that

$$E_t[Q_{t,t+1} W_{t+1}] = (1-\alpha) W_t.$$

Thus this fiscal policy has the "Ricardian" property that the transversality condition (1.18) necessarily holds, regardless of the evolution of the endogenous variables.[57] This means that

[57] Here our terminology follows Benhabib *et al.* (1998a). See chapter 5 for further discussion.

we may omit the transversality condition from our list of requirements for an equilibrium. Note also that under such a fiscal policy, debt management policy is irrelevant for equilibrium determination.

Under such a regime, a rational expectations equilibrium is a pair of processes $\{P_t, i_t\}$ satisfying (4.1) and

$$1 + i_t = \beta^{-1} E_t \left[\frac{\lambda(y_{t+1}, i_{t+1}; \xi_{t+1})}{\lambda(y_t, i_t; \xi_t)} \frac{P_t}{P_{t+1}} \right]^{-1} \tag{4.3}$$

at all dates, together with the bound

$$\sum_{T=t}^{\infty} \beta^T E_t \lambda(y_T, i_T; \xi_T) \left[y_T + \frac{i_T}{1 + i_T} L(y_T, i_T; \xi_T) \right] < \infty. \tag{4.4}$$

Here (4.3) rewrites (1.21) using the function $\lambda_(y, i; \xi)$ that gives the value of U_c as a function of those arguments (by substituting equilibrium real balances for the second argument of U_c), and (4.4) similarly rewrites (1.23), also using (3.2) to substitute for U_m.

The general existence of multiple solutions can be shown by considering the set of perfect foresight equilibria (*i.e.*, deterministic solutions) in the absence of shocks ($y_t = \bar{y}, \xi_t = 0$ for all t). Then, substituting (4.1) into (4.3), we obtain a nonlinear difference equation for the inflation rate,

$$\Pi_{t+1} \lambda(\phi(\Pi_{t+1}))^{-1} = \beta(1 + \phi(\Pi_t))\lambda(\phi(\Pi_t))^{-1}, \tag{4.5}$$

now writing simply $\lambda(i)$ for $\lambda(\bar{y}, i; 0)$. In the cashless limit (or the case of additive separability), this reduces to

$$\Pi_{t+1} = \beta(1 + \phi(\Pi_t)). \tag{4.6}$$

It is clear in this last case that there exists a solution for $\Pi_{t+1} > 0$ in the case of any given $\Pi_t > 0$. Hence starting from any arbitrarily chosen initial inflation rate $\Pi_0 > 0$, we can construct a sequence $\{\Pi_t\}$ that satisfies (4.6) at all dates. Associated with this is a sequence of non-negative interest rates, given by (4.1). As long as these sequences satisfy the bound (4.4), they represent a perfect foresight equilibrium. In the case that desired real money balances $L(\bar{y}, i; 0)$ are bounded above as i approaches zero,[58] because there is satiation at

[58]In fact, it suffices that $iL(\bar{y}, i; 0)$ be bounded. Thus even in the case of the log-log money demand function preferred by Lucas (1999), in which desired real balances decline as $i^{-1/2}$, condition (4.4) is satisfied by all interest-rate sequences.

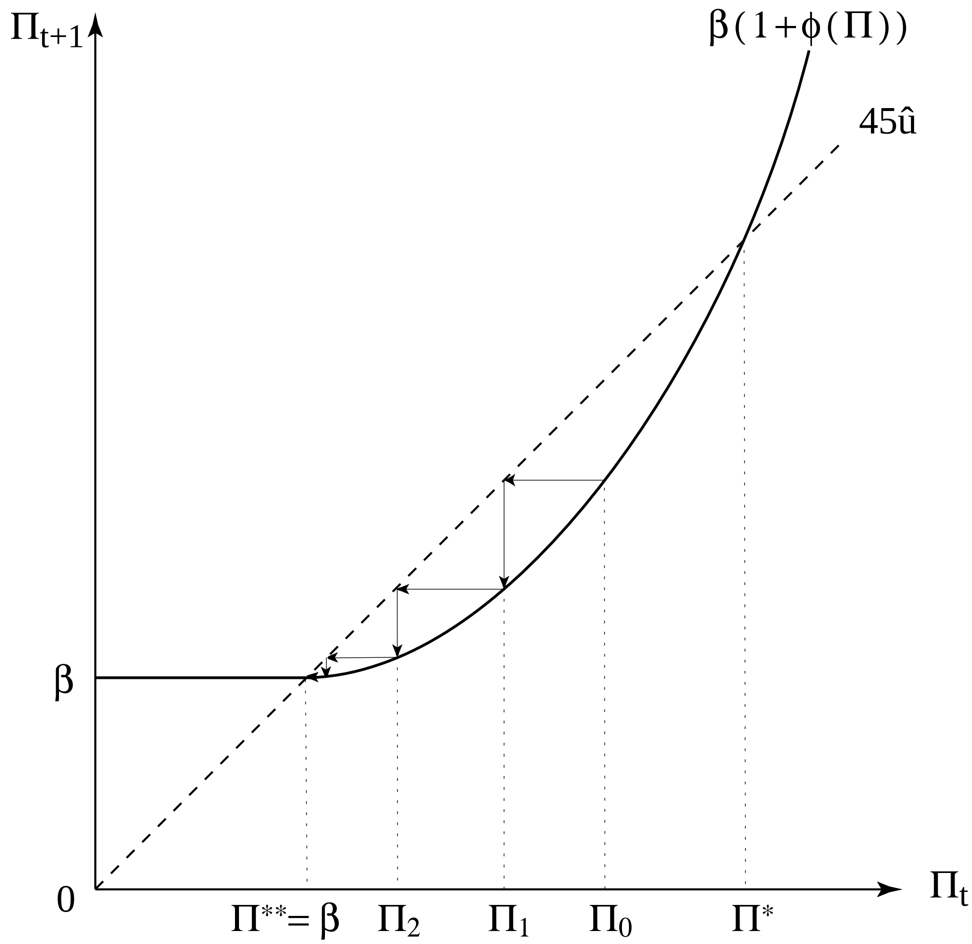

Figure 2.4: A self-fulfilling deflation under a Taylor rule.

a finite level of real money balances, this holds for any sequence $\{i_t\}$. In such a case, it is clear that there exists a continuum of perfect foresight equilibria, one corresponding to each possible initial inflation rate Π_0.

This result obtains even if the rule (4.1) satisfies $\phi_\pi > 1$ near the "target" inflation rate Π^*, so that the "Taylor principle" is satisfied, at least locally. In such a case, one has a large multiplicity of equilibria *globally,* despite local determinacy. This is illustrated in Figure 2.4, where the solid curve plots the locus of pairs (Π_t, Π_{t+1}) that satisfy (4.6). Note that this locus crosses the diagonal at the "target" inflation rate Π^*, indicating that $\Pi_t = \Pi^*$ forever is one solution. The "Taylor principle" implies that the curve cuts the diagonal from below at this point. In such a case, the fact that $\phi(\Pi) \geq 0$ for all Π implies that there also must be another steady state (constant inflation rate satisfying (4.6)), at some lower rate of inflation. At this lower steady state, the curve must cut the diagonal from above; thus, as Benhabib *et al.* stress, the "Taylor principle" *cannot* be globally valid. In the case shown in the figure, the "Taylor principle" is adhered to to the extent possible, which means that the lower steady state corresponds to a zero nominal interest rate. In this case, the lower steady-state inflation rate is $\Pi^{**} = \beta$, corresponding to deflation at the Friedman rate, the rate of time preference of the representative household.

The sequence of inflation rates corresponding to any given initial inflation rate Π_0 may be constructed geometrically as indicated in the figure; for each value of Π_t, one finds the associated value of Π_{t+1} using the curve, then reflects this value down to the horizontal axis using the diagonal, and repeats the construction. In the figure, a value $\Pi_0 < \Pi^*$ is considered. This is consistent with perfect foresight equilibrium only if $\Pi_1 < \Pi_0$, which in turn requires $\Pi_2 < \Pi_1$, and so on. One is able to continue the construction forever, and in the case shown in the figure (where $\phi(\Pi) < \beta^{-1}\Pi - 1$ for all $\Pi > \beta$, while $\phi(\Pi) = 0$ for all $\Pi \leq \beta$), one finds that the inflation rate must decline monotonically over time, approaching the value $\Pi^{**} = \beta$ asymptotically. This indicates the possibility of a *self-fulfilling deflation* under such a regime – inflation that is perpetually lower than the target rate, and eventually, actual deflation, that represents an equilibrium only because even lower inflation is expected in the future. Along such a path, interest rates are constantly being lowered in response to the decline in inflation, but because *expected* future inflation falls at the same time, *real* interest rates are not reduced, and continue to be high enough to restrain demand despite the falling prices.

Such an equilibrium exists for each possible choice of Π_0 in the interval $\beta < \Pi_0 < \Pi^*$. At the same time, for any inflation rate *higher* than the target rate, there exists an equilibrium in which the equilibrium inflation rate *rises* over time, eventually growing unboundedly large. Thus *self-fulfilling inflation* is equally possible under such a regime. Furthermore, because (4.3) need only hold *in expectation* for an inflation process $\{\Pi_t\}$ to constitute a rational expectations equilibrium, there is also an even larger set of equilibria in which the rate of inflation or deflation depends upon "sunspot" variables.[59]

Note that this global multiplicity of solutions does not contradict our previous results with regard to local determinacy. One observes from Figure 2.4 that any deterministic equilibrium other than the one with $\Pi_t = \Pi^*$ forever involves an inflation rate that diverges farther and farther from the target inflation rate as time passes. Thus every other equilibrium eventually leaves a neighborhood of Π^*, even if the initial inflation rate is very close to it. The same can be shown to be true of all of the stochastic equilibria as well,[60] so that the desired equilibrium is indeed locally unique in the sense discussed above. Note also that equilibrium is indeterminate even locally, near the deflationary steady state; for any neighborhood of Π^{**}, there exist a continuum of distinct equilibria in which inflation remains forever within this range. But this too is consistent with our previous results, since the "Taylor principle" is violated near this steady state.

These conclusions are largely unchanged when we take account of real balance effects. As long as $\Pi/\lambda(\phi(\Pi))$ is still a monotonically increasing function of Π, we can solve (4.5) in the same manner as (4.6). In the case that real balances are complementary with private expenditure ($U_{cm} > 0$), as was suggested above to be reasonable, $\lambda(i)$ is a decreasing function, and this condition is necessarily satisfied. And even if $U_{cm} < 0$, the monotonicity condition may still hold – it suffices that λ not be too strongly increasing in i. In particular, as long as $\lambda(i)$ has a finite limiting value for $i = 0$ – which makes sense, as there should be a limit to the value of expenditure, even when it is completely unimpeded by transactions frictions – then the assumptions above about the form of $\phi(\Pi)$ suffice to imply that the curve in Figure 2.4 cuts the diagonal from above at the Friedman rate of deflation. This suffices to imply the existence of a continuum of solutions to (4.5) involving self-fulfilling deflation. These solutions will also satisfy (4.4), and hence represent perfect foresight equilibria, as long as desired real balances are bounded, or indeed, as long $iL(\bar{y}, i; 0)$ has a finite bound for i near zero.

[59] The analysis of this possibility may be conducted along lines like those followed in Woodford (1994a), in the analysis of multiple equilibria under a money growth rule.

[60] Again, see the related analysis in Woodford (1994a).

These results may make it seem that a Taylor rule is not a very reliable way of ensuring a determinate equilibrium price level after all, even if the "Taylor principle" is adhered to except when interest rates become very low (in which case it cannot be). Several responses may be made to this criticism. One is to note that the equilibrium in which inflation is stabilized at the "target" level is nonetheless *locally* unique, which may be enough to allow expectations to coordinate upon that equilibrium rather than one of the others. Here it might seem that the existence of other equilibria with initial inflation rates arbitrarily close to the target rate should make it easy for the economy to "slip" into one of those other equilibria. Indeed, it is often said that in the case of perfect foresight dynamics like those shown in Figure 2.4, the steady state with inflation rate Π^* is "unstable", implying that an economy should be expected almost inevitably to experience either a self-fulfilling inflation or a self-fulfilling deflation under such a regime.

Such reasoning involves a serious misunderstanding of the causal logic of difference equation (4.5). The equation does not indicate how the equilibrium inflation rate in period $t+1$ is determined by the inflation that happens to have occurred in the previous period. If it did, it would be correct to call Π^* an unstable fixed point of the dynamics – even if that point were fortuitously reached, any small perturbation would result in divergence from it. But instead, the equation indicates how the equilibrium inflation rate in period t is determined by *expectations* regarding inflation in the following period. These expectations determine the real interest rate, and hence the incentive for spending, associated with the nominal rate that the central bank sets in response to any given current inflation rate. The equilibria that involve initial inflation rates near (but not equal to) Π^* can only occur as a result of expectations of *future* inflation rates (at least in some states) that are even *farther* from the target inflation rate. Thus the economy can only move to one of these alternative paths if expectations about the future change significantly, something that one may suppose should not easily occur.

Indeed, many analyses of convergence to rational expectations equilibrium as a result of adaptive learning dynamics find that equilibria are stable under the learning dynamics exactly in the case that they are "stable under the backward perfect foresight dynamics," which is exactly the case of the steady state Π^* in Figure 2.4.[61] The key to such results is that any deviation in expected future inflation from the target rate results in an actual inflation rate that is closer to the target rate than is the expected rate. If expectations evolve relatively slowly (as an average of experience over a period of time), then one will persistently observe inflation closer to the target rate than one is expecting, as a result of which expectations eventually adjust toward a value closer to the target rate themselves. But this makes actual inflation even closer to the target rate, and so on, until the process eventually converges to an equilibrium in which both expected and actual inflation equal the target rate forever.

Nonetheless, other types of learning processes, that allow extrapolation of paths diverging from the target steady state, can result in convergence to one of the other equilibria.[62] And even if one regards the target steady state as locally stable, one must worry that a large shock could nonetheless perturb the economy enough that expectations settle upon another equilibrium; thus the problem of self-fulfilling inflations and deflations should probably not be dismissed out of hand. But it is also important to note that this problem is in no way special to the formulation of monetary policy in terms of an interest-rate feedback rule. In particular, exactly the same sort of problems may arise in the case of monetary targeting.

Let us recall the equations that defined equilibrium real balances in the case of a monetary targeting regime, and once again restrict our attention to the case in which there are no exogenous shocks ($\mu_t = \bar{\mu} > \beta, y_t = \bar{y}, \xi_t = 0$ for all t). Solving (3.3) for i_t, and substituting this into the Fisher equation (1.21), we obtain a stochastic difference equation for real balances of the form

$$F(m_t) = \beta/\bar{\mu} E_t G(m_{t+1}), \tag{4.7}$$

[61] For examples of results of this kind, see Grandmont (1985), Grandmont and Laroque (1986), Marcet and Sargent (1989), and Lettau and Van Zandt (1998). Lucas (1986) uses a result of this kind to argue that self-fulfilling inflations should not be expected to occur, in the case of a monetary targeting regime.

[62] See Grandmont and Laroque (19xx) and Lettau and Van Zandt (1998).

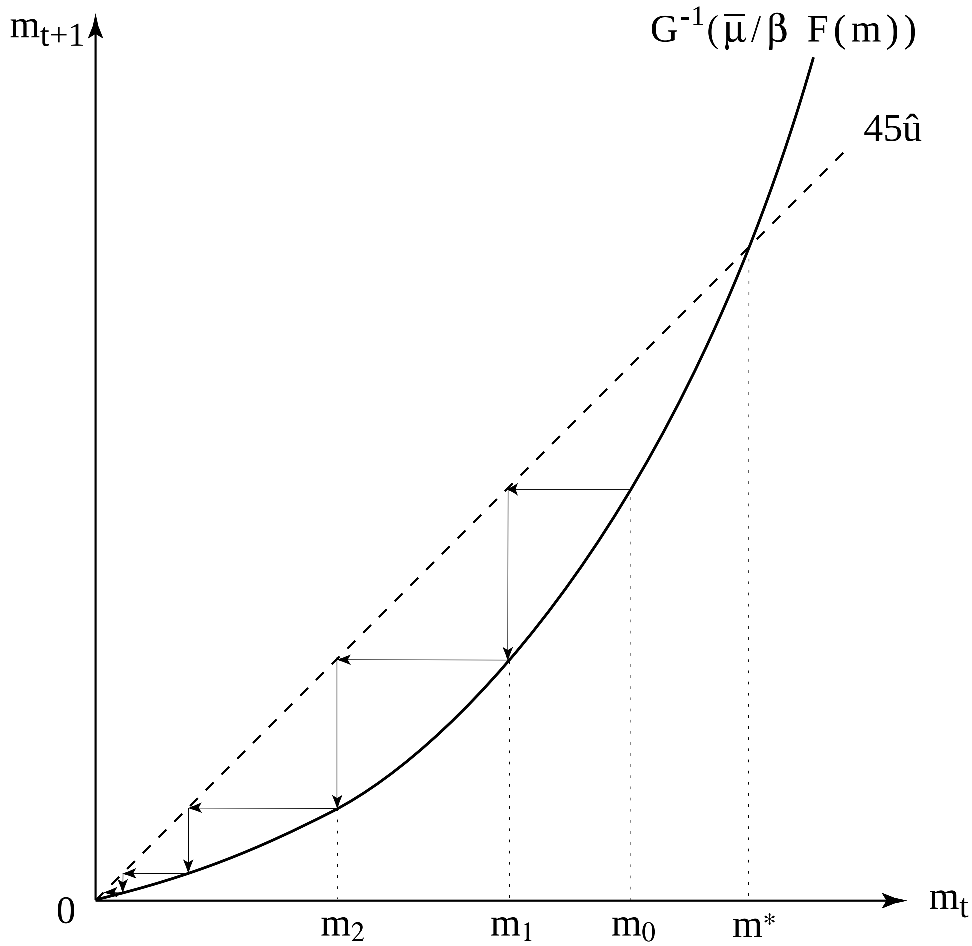

Figure 2.5: A self-fulfilling inflation under monetary targeting.

where

$$F(m) \equiv [U_c(\bar{y}, m; 0) - U_m(\bar{y}, m; 0)]m, \qquad G(m) \equiv U_c(\bar{y}, m; 0)m.$$

This is the exact equation to which (A.19) represents a log-linear approximation, except that we have here suppressed the dependence upon exogenous disturbances.

If we again consider perfect foresight solutions, we look for sequences $\{m_t\}$ that satisfy

$$G(m_{t+1}) = \bar{\mu}/\beta F(m_t). \tag{4.8}$$

In the additively separable case, $G(m)$ is linearly increasing in m, and so can be inverted, while the function $F(m)$ equals m times an increasing function of m. If this latter function

$(U_c - U_m)$ is positive for all $m > 0$ – which is to say, if desired real balances can be made arbitrarily small by making the interest rate high enough – then $F(m)$ is positive, increasing, and convex, and takes the limiting value $F(0) = 0$. In such a case, (4.8) can be uniquely solved for $m_{t+1} > 0$ given any value $m_t > 0$, and the graph of the solution is of the form shown in Figure 2.5. Once again, the steady-state level of real money balances $m^* = L(\bar{y}, (\bar{\mu} - \beta)/\beta; 0)$ is the point at which this graph intersects the diagonal; we note that at this point the curve necessarily cuts the diagonal from below. And once again there is a second steady state, this time at zero real balances.

Just as in the case of the Taylor rule, it will be observed that there is a distinct perfect foresight equilibrium corresponding to each possible value of m_0. If $m_0 < m^*$, real balances must be expected to decrease over time (as in the equilibrium illustrated in the figure), converging to zero asymptotically. This corresponds to a self-fulfilling inflation. For the progressively lower levels of real balances are associated with progressively higher nominal interest rates, which in turn require progressively higher rates of inflation; asymptotically, inflation approaches the rate (that may or may not be finite) that causes complete demonetization of the economy. If instead $m_0 > m^*$, real balances must increase over time, eventually increasing without bound. If desired real balances are finite at all positive interest rates, then such equilibria involve interest rates falling to zero, at least asymptotically, and so eventually involve deflation at a rate approaching the Friedman rate. They thus correspond to the self-fulfilling deflations discovered to be possible in the case of a Taylor rule. Thus, as a general rule, monetary targeting as such avoids neither of these types of potential problem.

In fact, as we shall see, it is possible to choose a policy regime under which equilibrium is globally unique. But here again, there is little difference between the degree to which this is possible when monetary policy takes the form of an interest-rate feedback rule, such as a Taylor rule, and when it instead takes the form of monetary targeting.

4.2 Policies to Prevent a Deflationary Trap

The result that self-fulfilling deflations are possible in the case of monetary targeting may seem surprising, as many papers that consider equilibrium in the case of a constant money supply find that such equilibria are impossible.[63] But the reason for this is that standard analyses do not specify a Ricardian fiscal policy, as we have above. As noted in section 1.2, analyses of monetary targeting typically assume a fiscal policy under which there is zero government debt at all times. Under this specification, the transversality condition (1.24) is no longer redundant; it holds only if the path of real balances satisfies

$$\lim_{T\to\infty} \beta^T E_t[u_c(Y_T, m_T; \xi_T) m_T] = 0. \tag{4.9}$$

As a result, self-fulfilling deflations generally cannot occur in a rational expectations equilibrium, if the money growth rate satisfies $\bar{\mu} \geq 1$, *i.e.*, the money supply is non-decreasing. This is most easily shown in the case that there is satiation in real balances at some finite level. Then for all values of m above this level, $F(m) = G(m)$, and (4.8) requires that

$$m_{t+1} = \bar{\mu}/\beta m_t$$

each period, after real balances exceed the satiation level. At the same time, $U_c(\bar{y}, m_t; 0)$ remains constant, even if preferences are not additively separable. It follows that in the case of any deflationary solution to (4.8),

$$\beta^T U_c(\bar{y}, m_T; 0) m_T = \beta^t \bar{\mu}^{T-t} U_c(\bar{y}, m_t; 0) m_t \geq \beta^t U_c(\bar{y}, m_t; 0) m_t > 0$$

at all dates $T \geq t$, where t is some date at which real balances have already reached the satiation level. Thus the transversality condition is violated in the case of any such path, and it does not represent a perfect foresight equilibrium. A generalization of this argument can be used to exclude stochastic equilibria in which real balances eventually exceed the satiation level as well; as a result one can show that one must have $m_t \leq m^*$ at all times in any rational expectations equilibrium.[64]

[63] See, for example, Brock (1975), Obstfeld and Rogoff (1986), and Woodford (1994a).
[64] See Woodford (1994a) for details.

But this result has nothing to do with the fact that monetary policy is specified in terms of a target path for the money supply. Instead it depends upon the fact that *fiscal* policy is assumed, under such a regime, not to be Ricardian. Government policy implies that not just the money supply, but also the nominal value of government liabilities D_t, grows at the rate $\bar{\mu}$, and it is the latter fact that rules out self-fulfilling deflations. But we could equally well combine this kind of fiscal policy with an interest rate rule, by specifying fiscal policy in terms of a target path for D_t, as in section 2.1 above. We would then obtain exactly the same result.

Let total nominal government liabilities D_t be specified to grow at a constant rate $\bar{\mu} \geq 1$, starting from an initial size $D_0 > 0$, while monetary policy is described by the Taylor rule (4.1). (Note that a balanced-budget policy, in the sense of Schmitt-Grohé and Uribe (1998), is one example of such a fiscal policy.) Let us also suppose, to simplify the analysis, that there exists an inflation rate $\underline{\Pi} > \beta$ such that $\phi(\Pi) = 0$ for all $\Pi \leq \underline{\Pi}$.[65]

Then any deflationary solution to (4.5) involves a zero nominal interest rate in each period after some finite date t, which in turn implies that $\Pi_T = \beta$ for all $T \geq t+1$. It follows that

$$\beta^T \lambda(i_T) D_T / P_T = \beta^t \bar{\mu}^{T-t} \lambda(0) D_t / P_t \geq \beta^t \lambda(0) D_t / P_t > 0$$

at all dates $T > t$, so that the transversality condition (1.24) is violated in the case of any such path.[66] In essence, the specified fiscal policy is too stimulative, in the case of a deflationary path, to be consistent with market clearing; if the private sector consumes only as much as the economy produces, it finds that its real wealth grows explosively, as a result of which it wishes to consume more.[67]

The result just mentioned depends upon interest rates being driven to zero sufficiently quickly in the event that inflation falls. However, even if $\phi(\Pi) > 0$ for all $\Pi > \beta$, as long

[65] One interpretation of this assumption is that the policy rule conforms to the "Taylor principle" except when the zero bound prevents any further decreases in the nominal interest rate.

[66] This is how self-fulfilling deflations are excluded under a Taylor rule in Schmitt-Grohé and Uribe (1998), in the context of a related model.

[67] The effects of fiscal policy upon aggregate demand under a non-Ricardian regime are discussed further in chapter 5.

as $\phi(\Pi) < (\Pi - \beta)/\beta$ for all $\beta < \Pi < \Pi^*$, we can still exclude self-fulfilling deflations in the case of any fiscal policy that ensures a growth rate $\bar{\mu} > 1$ for government liabilities. For this ensures that every deflationary solution involves an inflation rate converging to $\Pi^{**} = \beta$, as shown in Figure 2.4. Hence there must exist a date t such that $1 + \phi(\Pi_T) \geq \bar{\mu}$ for all $T \geq t$. It follows that

$$\beta^T \lambda(i_T) D_T/P_T = \beta^t \lambda(i_t) \prod_{s=t}^{T-1} (1 + i_s)^{-1} \bar{\mu}^{T-t} D_t/P_t \geq \beta^t \lambda(i_t) D_t/P_t > 0,$$

so that again the transversality condition is violated. Finally, even if interest rates level off sooner, so that there exists a lower steady-state inflation rate $\Pi^{**} > \beta$, the same argument works as long as $\bar{\mu} > \beta^{-1}\Pi^{**}$.

Thus, in the case of an appropriate fiscal policy rule, a deflationary trap is not a possible rational expectations equilibrium, even under an interest-rate rule. Furthermore, the type of fiscal commitment that is required to exclude such a possibility is essentially the same in the case of an interest-rate rule as in the case of monetary targeting: fiscal policy must ensure that the nominal value of total government liabilities D_t will not decline, even in the case of sustained deflation.

Indeed, an interest-rate rule has an advantage over the classic formulation of a monetary-targeting regime, in at least one regard. Self-fulfilling deflations may be excluded in the case of an interest-rate rule, even when the target inflation rate Π^* associated with the Taylor rule (or with a Wicksellian regime with a non-constant target path for the price level) implies *deflation* at a rate less than the rate of time preference ($\beta < \Pi^* < 1$). Under the monetary targeting regime (with fiscal policy maintaining zero government debt at all times), instead, the argument given above to exclude self-fulfilling deflations fails when $\bar{\mu} < 1$. In this case, the transversality condition *is* satisfied by all of the solutions to (4.8) that involve inflation rates eventually falling to the level $\Pi_t = \beta$, and so a continuum of perfect foresight equilibria (and similarly a large set of "sunspot" equilibria) exist.[68] Of course, this result could be avoided if we assume a different fiscal policy; it is simply necessary that the nominal value of

[68] See Woodford (1994a) for demonstration of this for a closely related model.

total liabilities be non-decreasing, even as the money supply contracts. Thus Dupor (1999) shows that self-fulfilling deflations can be avoided if the money supply is contracted through open-market operations (that replace the money with corresponding increases in government debt), rather than through the lump-sum tax collections envisioned by Friedman (1969). But this emphasizes that monetary targeting as such is not the key to excluding the possibility of a deflationary trap.

It is sometimes supposed that the conduct of monetary policy through interest-rate control leaves monetary policy impotent in the case of a deflationary trap, because the nominal interest rate instrument cannot be lowered below zero, whereas it is actually still possible to stimulate aggregate demand by increasing the money supply. But monetary control has no such advantage. It is important to remember that *there is no real-balance effect* once the short nominal interest rate falls to zero, even if it is still possible to increase the size of the excess real money balances (*i.e.,* balances in excess of the satiation level) held by the public. This is because higher real balances increase desired spending, for any given expected path of real interest rates, only insofar as they are able to increase the marginal utility of additional expenditure associated with a given level of real expenditure. Once the satiation level of real balances is reached, additional money balances no longer lead to any further relaxation of constraints upon transactional flexibility, and so they cannot stimulate aggregate demand.

Recall that in section 3.3 above, we were able to replace the function $U_c(y_t, m_t; \xi_t)$ describing the marginal utility of additional expenditure by $\lambda(y_t, i_t; \xi_t)$. Thus additional real balances affect the marginal utility of expenditure only insofar as they can reduce the short-term nominal interest rate i_t, which indicates the value of further relaxation of transaction constraints. Once the nominal interest rate cannot be further lowered, an increase in real money balances as such can have no effect upon aggregate demand, through *either* a real-interest-rate effect or a direct real-balance effect. Thus monetary policy is impotent under such circumstances; but this is not a limitation of the use of an interest-rate instrument.

Alternatively, it is sometimes argued that even when increases in the current money supply are ineffective, due to the economy's being in a "liquidity trap" (*i.e.,* money balances already in excess of the satiation level), a commitment to *future* money supply increases can nonetheless stimulate aggregate demand. The idea is that increasing the expected future price level can lower real interest rates even when nominal rates cannot be further reduced (Krugman, 1999). This is right, but there is no special efficacy of a commitment to monetary targets in this regard. Commitment to a high rate of money growth does nothing to exclude a deflationary trap, when coupled with a Ricardian fiscal policy; the deflationary equilibrium simply involves a higher level of excess money balances in that case. Only an anti-deflationary fiscal commitment can solve this problem, and it can do so whether or not there are monetary targets.

On the other hand, the problem of a country like Japan at present may not be so much that it has fallen into a self-fulfilling deflationary trap, despite the existence of an equilibrium with stable prices if only expectations were to coordinate upon it, as that a temporary reduction in the equilibrium real rate of return has made stable prices incompatible with the zero bound on nominal interest rates, as suggested by Krugman. In such a case, the only way to avoid a period of sharp deflation when the disturbance occurs may be a regime that creates expectations of subsequent *inflation*. Let us suppose that the problem of self-fulfilling deflations may be put aside (perhaps because of an appropriate fiscal commitment), and that the economy is expected to converge to a near-steady-state equilibrium after the real disturbance subsides (and the equilibrium real rate returns to its normal positive level). Then whether the path of this sort that is followed involves a sharp initial deflation or not depends upon expected monetary policy after the shock subsides, although, for some expectations regarding future policy, there may be nothing current monetary policy can do to prevent deflation. In such a case, as Krugman argues, the future monetary policy commitment and its credibility are crucial. But the kind of policy commitment that can imply expectations of a suitably high future price level may well be a Wicksellian regime or a Taylor rule, which would determine the expected future price level in the way explained in section 2 above.

4.3 Policies to Prevent an Inflationary Panic

We next consider the opposite sort of instability due to self-fulfilling expectations, the possibility of spontaneous flight from a country's currency, with its loss in value over each time period resulting from expectations of even further declines in value, at an even faster rate, in the near future. Such *self-fulfilling inflations* have generally been considered a more troubling possibility than self-fulfilling deflations in the literature on monetary targeting regimes. However, conditions have been identified under which such equilibria would not exist in the case of a constant money growth rate.

In particular, Obstfeld and Rogoff (1986) show that in the model considered here, if preferences are of the additively separable form considered in section xx above, and in addition

$$\lim_{m \to 0} m u_m(m;0) > 0, \tag{4.10}$$

then no such perfect foresight equilibria exist under monetary targeting. For in this case, the function $F(m)$ is negative for all levels of real balances below a critical level $\underline{m} > 0$, as a result of which the graph of (4.8) cuts the horizontal axis at the point $m_t = \underline{m} > 0$, rather than passing through the origin as shown in Figure 2.5. Thus no equilibrium can ever have $m_t \leq \underline{m}$. But then it follows from (4.8) that no perfect foresight equilibrium can ever have

$$m_t \leq \underline{m}_1 \equiv F^{-1}(\beta/\bar{\mu} G(\underline{m})),$$

as a result of which no perfect foresight equilibrium can ever have

$$m_t \leq \underline{m}_2 \equiv F^{-1}(\beta/\bar{\mu} G(\underline{m}_1)),$$

and so on. One shows in this way that no equilibrium is possible with $m_t < m^*$ at any time.

Condition (4.10) was excluded in our analysis above by the assumption that desired real balances can be made arbitrarily small by sufficiently increasing the interest rate; for it implies that $L(\bar{y}, i; 0)$ is bounded below by $\underline{m}$ for all i. (The existence of such a bound is the key to the above construction, for which additive separability is actually not important.) Obstfeld and Rogoff point out that the conditions required for this to be true are somewhat implausible, though theoretically possible. Observed hyperinflations in several countries have also shown that real balances do indeed fall to a small fraction of their normal level when inflation becomes sufficiently severe, and the money demand functions estimated from such data (as in the classic study by Cagan, 1956) imply that real balances should approach zero in the case of high enough expected inflation. Hence it is not clear that one can rely upon this mechanism to prevent self-fulfilling inflations in an actual economy.

What if monetary policy is instead specified by a Taylor rule of the form (4.1)? A similar argument excluding self-fulfilling inflations would be possible only if the graph of (4.5) becomes vertical at some finite inflation rate $\bar{\Pi}$, so that (4.5) has no solution for Π_{t+1} in the case of $\Pi_t > \bar{\Pi}$. In the additively separable case, this might seem to be impossible, as there could fail to be a solution for Π_{t+1} in (4.6) only if $\phi(\Pi_t)$ is itself not defined. However, it is not clear that a function ϕ that becomes unboundedly large at a finite inflation rate $\bar{\Pi}$ must be excluded as a possible policy. After all, in the case of monetary targeting when (4.10) holds, the policy that excludes self-fulfilling inflations is one of commitment never to supply more than a certain quantity of money, no matter how high this may require interest rates to be driven. This is equivalent to a Wicksellian rule (1.30) in which the function ϕ becomes unboundedly large at a finite level of P_t/P_t^*. If such a policy is considered to be feasible, despite the fact that it commits the central bank to something that is simply impossible in the case of too high a price level, then it is not clear why a Taylor rule that makes ϕ undefined for $\Pi_t \geq \bar{\Pi}$ is not an equally feasible policy. But if we consider a Taylor rule of this kind to represent a credible commitment, then such a strategy to exclude self-fulfilling inflations should actually be superior to monetary targeting. For its applicability would not depend upon the implausible assumption that desired real balances are bounded away from zero.

Furthermore, under some circumstances it is possible to exclude self-fulfilling inflations through commitment to increase interest rates sufficiently sharply at high rates of inflation, even when $\phi(\Pi)$ is well-defined for any finite inflation rate.If $U_{cm} < 0$ at low levels of real balances, the function $\lambda(i)$ is increasing in i, at high levels of i. Suppose that in fact the elasticity of $\lambda(i)$ with respect to $1+i$ is positive, but bounded below one for all high enough i. Then the right-hand side of (4.5) increases with Π_t, for high inflation rates, eventually growing without bound. However, if $\phi(\Pi)$ increases sufficiently rapidly for high values of Π, the function $\Pi/\lambda(\phi(\Pi))$ may be bounded above. In this case, (4.5) has no solution for Π_{t+1} in the case of values of Π_t above some finite bound $\bar{\Pi}$.[69]

But this then allows us to exclude self-fulfilling inflations altogether among the set of perfect foresight equilibria, using an iterative argument like that made above in the case of the lower bound on real balances. Then, assuming a fiscal policy of the kind discussed above, that excludes self-fulfilling deflations, one can show that the equilibrium with inflation forever at the target inflation rate Π^* is the unique perfect foresight equilibrium. In fact, using similar arguments, one can show that it is the unique rational expectations equilibrium, even allowing for stochastic equilibria of arbitrary form. And, at least in the case of sufficiently small random variations in the exogenous variables $\{y_t, \xi_t, \nu_t, \Pi_t^*\}$, the locally unique equilibrium that was approximately characterized in section 2.3 above can similarly be shown to be globally unique.

Even if solutions of these kinds are unavailable, self-fulfilling inflations may be excluded through the addition of policy provisos that apply only in the case of hyperinflation. For example, Obstfeld and Rogoff (1986) propose that the central bank commit itself to peg the value of the monetary unit in terms of some real commodity, by standing ready to exchange the commodity for money, in the event that the real value of the total money supply ever shrinks to a certain very low level. Assuming that this level of real balances is one that would never be reached except in the case of a self-fulfilling inflation, the commitment has no effect except to exclude such paths as possible equilibria. Obstfeld and Rogoff propose this as a solution to the problem of self-fulfilling inflations under a regime that otherwise targets the money supply; but it has no intrinsic connection to monetary targeting, and could equally well be added as a hyperinflation proviso in a regime that otherwise follows a Taylor rule.

[69] This is essentially the way in which self-fulfilling inflations are excluded under a Taylor rule in the analysis of Schmitt-Grohé and Uribe (1998), though their model is not precisely consistent with ours.

1 APPENDIX TO CHAPTER 2

1.1 Proof of Proposition 2.1

PROPOSITION 2.1. Consider positive-valued stochastic processes $\{P_t, Q_{t,T}\}$ satisfying (1.10) and (1.11) at all dates, and let $\{C_t, M_t\}$ be non-negative-valued processes representing a possible consumption and money-accumulation plan for the household. Then there exists a specification of the household's portfolio plan at each date satisfying both the flow budget constraint (1.7) and the borrowing limit (1.9) at each date, if and only if the plans $\{C_t, M_t\}$ satisfy the constraint

$$\sum_{t=0}^{\infty} E_0 Q_{0,t}\left[P_t C_t + \Delta_t M_t\right] \leq W_0 + \sum_{t=0}^{\infty} E_0 Q_{0,t}[P_t Y_t - T_t]. \tag{A.1}$$

PROOF: Substituting s for the time index t in (1.7), taking the present value of both sides of the inequality at an earlier (or no later) date t, and summing over dates s from t through $T-1$, we obtain

$$\sum_{s=t}^{T-1} E_t Q_{t,s}\left[P_s C_s + \Delta_s M_s\right] + E_t[Q_{t,T} W_T] \leq W_t + \sum_{s=t}^{T-1} E_t Q_{t,s}[P_s Y_s - T_s]$$

for any date $T \geq t+1$. Combining this with the bounds (1.9) on the portfolios that may be chosen in the various possible states at date $T-1$ (which are lower bounds upon the values of those portfolios in possible states at date T), one sees that a feasible plan must satisfy

$$\sum_{s=t}^{T-1} E_t Q_{t,s}\left[P_s C_s + \Delta_s M_s\right] \leq W_t + \sum_{s=t}^{\infty} E_t Q_{t,s}[P_s Y_s - T_s].$$

Note that the right-hand side is now independent of the terminal date T. The left-hand side is instead a non-decreasing series in T, given positive goods prices at all dates (necessary for any finite level of consumption to be optimal), interest rates satisfying (1.11), and non-negative levels of consumption and money balances at all times. The right-hand side, which is finite by (1.10), provides an upper bound for this series, which accordingly must converge as T grows. Furthermore, the limiting value of the series must itself satisfy the upper bound. Thus any feasible plan involves a sequence of state-dependent consumption levels and money balances satisfying the intertemporal budget constraint (1.13). In particular, under any feasible plan, the entire infinite streams $\{C_t, M_t\}$ from the initial date $t = 0$ onward must satisfy this constraint for date zero, given the household's initial financial wealth W_0. This establishes the necessity of (A.1).

It remains to show that this constraint is also *sufficient* for processes $\{C_t, M_t\}$ to be attainable. One easily shows that processes that satisfy (A.1) can be achieved, by letting the household's choice of W_t at each date $t \geq 1$ (in each possible state) be given by the value that makes (1.13) hold at that date with equality. Given the hypothesized process for M_t, this then implies a value for A_{t+1} in each possible state, and thus completely specifies the household's portfolio plan at each date $t \geq 0$. The resulting plan obviously satisfies (1.9) at each date, and it is easily verified that it satisfies (1.7), and hence (1.2), at each date as well. Thus the entire sequence of flow budget constraints is equivalent to the single intertemporal constraint (A.1).

1.2 Proof of Proposition 2.2

Proposition 2.2. Let assets be priced by a system of stochastic discount factors that satisfy (1.20), and consider processes $\{P_t, i_t, i_t^m, M_t^s, W_t^s\}$ that satisfy (1.15), (1.21), and (1.23) at all dates, given the exogenous processes $\{Y_t, \xi_t\}$. Then these processes satisfy (1.22) as well if and only if they satisfy

$$\lim_{T\to\infty} \beta^T E_t[u_c(Y_T; \xi_T) D_T / P_T] = 0. \tag{A.2}$$

In this proposition, note that the path of $\{D_t\}$ can be inferred from the processes that are specified using the identity

$$D_t = M_t^s + E_t[Q_{t,t+1}(W_{t+1}^s - (1 + i_t^m) M_t^s)].$$

Proof: Note that (1.15) and (1.21) imply that

$$\beta E_t[u_c(Y_{T+1}; \xi_{T+1})(1 + i_T^m) M_T^s / P_{T+1}] \quad = \quad \beta E_t[u_c(Y_{T+1}; \xi_{T+1})(1 + i_T) M_T^s / P_{T+1}]$$

$$= E_t[u_c(Y_T;\xi_T)M^s_T/P_T].$$

Adding this to the relation

$$\beta E_t[u_c(Y_{T+1};\xi_{T+1})A^s_{T+1}/P_{T+1}] = E_t[u_c(Y_T;\xi_T)B^s_T/P_T]$$

that follows from (1.4) and (1.20), we find that

$$\beta E_t[u_c(Y_{T+1};\xi_{T+1})W^s_{T+1}/P_{T+1}] = E_t[u_c(Y_T;\xi_T)D_T/P_T].$$

It then follows that (1.22) holds if and only if (A.2) does.

In the case of the model with transactions frictions introduced in section xx, a similar proposition continues to hold. A precise statement can be given as follows.

PROPOSITION 2.2′. Let assets be priced by a system of stochastic discount factors that satisfy (1.20), and consider processes $\{P_t, i_t, i^m_t, M^s_t, W^s_t\}$ that satisfy (1.21), (3.3), and (3.4) at all dates, given the exogenous processes $\{Y_t, \xi_t\}$. Then these processes satisfy (1.22) as well if and only if they satisfy

$$\lim_{T\to\infty} \beta^T E_t[u_c(Y_T, M^s_T/P_T;\xi_T)D_T/P_T] = 0. \tag{A.3}$$

In this more general case, (1.21) and (3.3) can be used to show that

$$\begin{aligned}\beta E_t[u_c(Y_{T+1}, M^s_{T+1}/P_{T+1};\xi_{T+1})(1+i^m_T)M^s_T/P_{T+1}] &= E_t[u_c(Y_T, M^s_T/P_T;\xi_T)(1-\Delta_T)M^s_T/P_T] \\ &= E_t[(u_c(Y_T, M^s_T/P_T;\xi_T) - u_m(Y_T, M^s_T/P_T;\xi_T))M^s_T/P\end{aligned}$$

from which it follows as above that

$$\beta E_t[u_c(Y_{T+1}, M^s_{T+1}/P_{T+1};\xi_{T+1})W^s_{T+1}/P_{T+1}] = E_t[u_c(Y_T, M^s_T/P_T;\xi_T)D_T/P_T] - E_t[u_m(Y_T, M^s_T/P_T;\xi_T)M^s_T/P$$

Furthermore, (3.4) implies that

$$\lim_{T\to\infty} \beta^T E_t[u_m(Y_T, M^s_T/P_T;\xi_T)M^s_T/P_T] = 0.$$

Hence (1.22) holds if and only if (A.3) holds.

1.3 Determinacy of Rational-Expectations Equilibrium

[TO BE ADDED][See Woodford (1986)]

1.4 Proof of Proposition 2.3

PROPOSITION 2.3. Under a Wicksellian policy rule (1.30) with $\phi_p > 0$, the rational-expectations equilibrium paths of prices and interest rates are (locally) determinate; that is, there exist open sets $\mathcal{P}$ and $\mathcal{I}$ such that in the case of any tight enough bounds on the fluctuations in the exogenous processes $\{\hat{r}_t, \pi_t^*, \nu_t\}$, there exists a unique rational-expectations equilibrium in which $P_t/P_t^* \in \mathcal{P}$ and $i_t \in \mathcal{I}$ at all times. Furthermore, equations (1.37) and (1.38) give a log-linear (first-order Taylor series) approximation to that solution, accurate up to a residual of order $\mathcal{O}(||\xi||^2)$, where $||\xi||$ indexes the bounds on the disturbance processes.

PROOF: This is a direct application of the implicit function theorem, as discussed in the previous section. As discussed in the text, (1.32) and (1.34) represent log-linear (first-order Taylor-series) approximations to the equilibrium relations (1.21) and (1.30). The existence of a unique bounded solution to the log-linearized relations implies the existence of a locally unique solution to the exact relations as well, in the case of any tight enough bound on the exogenous disturbances, using the inverse function theorem; and that solution to the log-linearized relations provides a first-order Taylor-series approximation to the solution to the exact relations, using the implicit function theorem. If the neighborhoods $\mathcal{P}$ and $\mathcal{I}$ are small enough, any solution to the exact relations restricted to these sets must satisfy the transversality condition (1.24) as well, and so represents a rational-expectations equilibrium.

It thus remains only to demonstrate that the system consisting of (1.32) and (1.34), together with the identity (1.35), has a unique bounded solution when $\phi_p > 0$. As shown in the text, these equations imply (1.36). This is a form of expectational difference equation that occurs repeatedly in this chapter, that may be written in the form

$$z_t = aE_t z_{t+1} + u_t, \tag{A.4}$$

where z_t is an endogenous variable and u_t is an exogenous disturbance process. In the present application, $z_t = \hat{P}_t$, $a = (1+\phi_p)^{-1}$, and

$$u_t = (1+\phi_p)^{-1}(\hat{r}_t + E_t\pi^*_{t+1} - \nu_t).$$

Any expectational difference equation of the form (A.4) has a unique bounded solution $\{z_t\}$ in the case of an arbitrary bounded disturbance process $\{u_t\}$ in the case that $|a| < 1$. Note that (A.4) implies that

$$E_t z_{t+j} = aE_t z_{t+j+1} + E_t u_{t+j}$$

for arbitrary $j \geq 0$. Multiplying this equation by a^j and summing from $j = 0$ through $k-1$, we obtain

$$z_t = a^k E_t z_{t+k} + \sum_{j=0}^{k-1} a^j E_t u_{t+j}. \tag{A.5}$$

Note that this equation must hold for arbitrary k. If $\{z_t\}$ is a bounded process and $|a| < 1$, it follows that

$$\lim_{k\to\infty} a^k E_t z_{t+k} = 0.$$

Then since the left-hand side of (A.5) is independent of k, it follows that the final term on the right must converge in value as k is made unboundedly large, and specifically to the value of the left-hand side. Thus we must have

$$z_t = \sum_{j=0}^{\infty} a^j E_t u_{t+j}. \tag{A.6}$$

(This solution is sometimes said to be obtained by "solving (A.4) forward.")

Equation (A.5) represents not just one possible solution to (A.4), but the unique bounded solution. In the present application, (A.6) yields equation (1.37) in the text. Substitution of this into (1.34) then yields (1.38) as well.

For future reference, it is also useful to consider the case in which $|a| \geq 1$. In this case, the process $\{z_t\}$ recursively defined by

$$z_t = a^{-1}(z_{t-1} - u_{t-1}) + \nu_t \tag{A.7}$$

for all $t \geq 1$, starting from an arbitrary initial condition z_0, represents a solution to (A.4) in the case of any process $\{\nu_t\}$ such that $E_t\nu_{t+1} = 0$ for all t. If $|a| > 1$, (A.7) represents a bounded solution for $\{z_t\}$ in the case of any bounded process $\{\nu_t\}$, assuming that $\{u_t\}$ is bounded as well. Hence there is an extremely large set of bounded solutions $\{z_t\}$ to equation (A.4).

In the case that $|a| = 1$ exactly, not all solutions of the form (A.7) are bounded, even if both $\{u_t\}$ and $\{\nu_t\}$ are bounded processes. Nonetheless, if (A.4) has any bounded solution, it must have an uncountably infinite number of them. Let $\{\bar{z}_t\}$ be one bounded solution. (For example, in the case that

$$v_t \equiv \sum_{j=0}^{\infty} E_t u_{t+j}$$

is well-defined and bounded, as is true for any stationary ARMA process $\{u_t\}$ with bounded innovations, then one bounded solution to (A.4) is given by $\bar{z}_t = v_t$, the solution obtained by solving forward.) Then another bounded solution is recursively defined by

$$z_t = \bar{z}_t + (z_{t-1} - \bar{z}_{t-1} + \nu_t$$

for all $t \geq 1$, starting from an arbitrary initial condition z_0, where $\{\nu_t\}$ is any stochastic process such that $E_t\nu_{t+1} = 0$ for all t, *and* such that $\sum_{t=1}^{T} \nu_t$ remains bounded for arbitrarily large T. This last stipulation can obviously be satisfied by a large number of mean-zero, unforecastable processes; for example, it suffices that ν_t be bounded for each t, and equal to zero with probability one for all t greater than some date T. Hence in this case as well, there are clearly an uncountably infinite number of bounded solutions. Thus the condition that $|a| < 1$ is both necessary and sufficient for the existence of a unique bounded solution to (A.4).

1.5 Proof of Proposition 2.4

Proposition 2.4. Consider a monetary policy under which the monetary base is bounded below by a positive quantity: $M_t^s \geq \underline{M} > 0$ at all times. (For example, perhaps the monetary base is non-decreasing over time, starting from an initial level $M_0^s > 0$.) Suppose furthermore that government debt is non-negative at all times, so that $D_t \geq M_t^s$. Finally, suppose that $i_t^m = 0$ at all times. Then in the cashless economy described above, there exists no rational-expectations equilibrium path for the price level $\{P_t\}$.

PROOF: If $i_t^m = 0$ at all times, (1.15) requires that in any equilibrium, $i_t = 0$ at all times. Then (1.21) requires that

$$\beta E_t[u_c(Y_{t+1};\xi_{t+1}/P_{t+1}] = u_c(Y_t;\xi_t)/P_t$$

at all times, and hence, by iteration, that

$$\beta^T E_t[u_c(Y_T;\xi_T/P_T] = \beta^t u_c(Y_t;\xi_t)/P_t$$

for all $T \geq t$. But this, together with the lower bound $D_t \geq \underline{M} > 0$, implies that

$$\begin{aligned} \beta^T E_t[u_c(Y_T;\xi_T)D_T/P_T] &\geq \underline{M}\beta^T E_t[u_c(Y_T;\xi_T/P_T] \\ &= \underline{M}\beta^t E_t[u_c(Y_t;\xi_t/P_t] > 0, \end{aligned}$$

which contradicts (A.2). Hence no equilibrium is possible.

In fact, equilibrium values could be defined under such a regime for *real* rates of return and asset prices; if we write prices in terms of some real numeraire rather than monetary units, a well-defined equilibrium would exist, but would involve zero exchange value for money. In essence, under the regime described, money is a pure "bubble" — an asset the exchange value of which would have to be sustained purely by the expectation of a future exchange value, and not any dividends ever yielded by the asset — and cannot have an exchange value in a rational-expectations equilibrium, at least not in a representative-household model of the kind assumed here. (In fact, a similar result can be obtained in much more general environments, as shown by Santos and Woodford, 1997.) Instead, under the regime to which Proposition 2.3 applies, interest is paid on money, and — the crucial point — this interest is not simply additional money that remains forever in circulation. Because private-sector nominal claims on the government D_t are assumed to grow at a rate *less* than the rate at which interest is paid on money — recall that $\gamma_D < \bar{\Pi}/\beta = 1 + \phi(1; 0) = 1 + \bar{\imath}^m$ — at least some of the money received as interest payments is eventually redeemed by the government (accepted as payment for taxes), so that money ceases to be a pure "bubble".

1.6 Proof of Proposition 2.5.

PROPOSITION 2.5. Let monetary policy be specified by an exogenous sequence of interest-rate targets, assumed to remain forever within a neighborhood of the interest rate $\bar{\imath} > 0$ associated with the zero-inflation steady state; and let these be implemented by setting i_t^m equal to the interest-rate target each period. Let $\{M_t^s, D_t\}$ be exogenous sequences of the kind assumed in Proposition 2.3. Finally, let $\mathcal{P}$ be any neighborhoods of the real number zero. Then for any tight enough bounds on the exogenous processes $\{Y_t, \xi_t\, D_t/D_{t-1}\}$ and on the interest-rate target process, there exists an uncountably infinite set of rational-expectations equilibrium paths for the price level, in each of which the inflation rate satisfies $\pi_t \in \mathcal{P}$ for all t. These include equilibria in which the inflation rate is affected to an arbitrary extent by "fundamental" disturbances (unexpected changes in Y_t or ξ_t), by pure "sunspot" states (exogenous randomness unrelated to the "fundamental" variables), or both.

As discussed in the text, this can be established using the local method discussed in section xx above. However, in the present case, the equilibrium relations are simply enough to analyze without any resort to linear approximation.

PROOF: Let $\{\nu_t\}$ be any unforecastable mean-zero random variable (or martingale difference) such that $\nu_t < 1$ at all times. Then the inflation process given by

$$\frac{P_t}{P_{t-1}} = \beta \frac{1 + i_{t-1}^m}{u_c(Y_{t-1}; \xi_{t-1})} E_{t-1}[u_c(Y_t; \xi_t)(1 - \nu_t)]\ frac11 - \nu_t$$

satisfies (1.21) at all times. (Note that the solutions (2.42) presented in the text are log-linear approximations to these processes.) In the case of tight enough bounds on both the exogenous variables and the fluctuations in $\{\nu_t\}$, this yields an inflation process such that $\pi_t \in \mathcal{P}$ at all times, and that satisfies (A.2) as well. Hence any such solution represents a rational-expectations equilibrium. The solutions corresponding to different choices of $\{\nu_t\}$ represent distinct equilibria, since in each case the surprise component of inflation is given by

$$\pi_t - E_{t-1}\pi_t = -\log(1-\nu_t).$$

Finally, the variable $\{\nu_t\}$ may be correlated in an arbitrary way with any of the "fundamental" variables, or it may be completely independent of them.

1.7 Proof of Proposition 2.7

PROPOSITION 2.7. Let monetary policy be described by a feedback rule of the form (2.48), at least near the zero-inflation steady state, with $\Phi_\pi \geq 0$. Then equilibrium is determinate if and only if $\Phi_\pi > 1$. When this condition is satisfied, a log-linear approximation to the equilibrium evolution of the smoothed inflation process is given by

$$\bar{\pi}_t = (1-\delta)\sum_{j=0}^{\infty}(\delta + (1-\delta)\Phi_\pi)^{-(j+1)}E_t[\hat{r}_{t+j} - \bar{\imath}_{t+j}]. \tag{A.8}$$

A corresponding approximation to the equilibrium evolution of the single-period inflation rate π_t is then obtained by substituting (A.8) into

$$\pi_t = \frac{\bar{\pi}_t - \delta\bar{\pi}_{t-1}}{1-\delta}. \tag{A.9}$$

PROOF: The solution (A.9) for π_t given the evolution of $\bar{\pi}_t$ is obtained by inverting (2.47). Then substituting (2.48) into (1.32) to eliminate i_t, we obtain

$$\Phi_\pi\bar{\pi}_t = E_t\pi_{t+1} + (\hat{r}_t - \bar{\imath}_t).$$

Substituting (A.9) for π_t in this equation, we obtain an expectational difference equation for the smoothed inflation measure,

$$[\delta + (1-\delta)\Phi_\pi]\bar{\pi}_t = E_t\bar{\pi}_{t+1} + (1-\delta)(\hat{r}_t - \bar{\imath}_t). \tag{A.10}$$

This is again an equation of the form (A.4), allowing us to apply the same method as in the proof of Proposition 2.3. The equation can be solved forward to obtain a unique bounded solution if and only if

$$\delta + (1-\delta)\Phi_\pi > 1,$$

which is to say, if and only if $\Phi_\pi > 1$, as required by the "Taylor principle". When this condition holds, the solution (A.5) is given by (A.8).

1.8 Proof of Proposition 2.8

PROPOSITION 2.8. Let monetary policy be described by a feedback rule of the form (2.51), at least near the zero-inflation steady state, with $\phi_\pi, rho \geq 0$. Then equilibrium is determinate if and only if $\phi_\pi > 0$ and

$$\phi_\pi + \rho > 1 \tag{A.11}$$

When these conditions are satisfied, a log-linear approximation to the equilibrium evolution of inflation is given by (2.53).

PROOF: The proof follows the same lines as in the case of Proposition 2.7. Using (2.51) to eliminate π_{t+1} in (1.32), one obtains an expectational difference equation

$$(\phi_\pi + \rho)\hat{\imath}_t = E_t\hat{\imath}_{t+1} + \phi_\pi \hat{r}_t + \rho\bar{\imath}_t - E_t\bar{\imath}_{t+1},$$

corresponding to (A.10) above, and once again this is of the form (A.4). Applying the same method as in the proof of Proposition 2.3, one finds that there is a unique bounded solution for $\{\hat{\imath}_t\}$ if and only if

$$|\phi_\pi + \rho| > 1 \tag{A.12}$$

is satisfied. Under the sign assumptions made in the statement of the Proposition, condition (A.12) reduces to (A.11). By solving forward, *i.e.*, applying (A.5), one obtains (2.54) as the solution for the interest-rate process.

Corresponding to this solution for the path of the interest rate is a unique solution for $\{\pi_t\}$, obtained by inverting (2.51), if and only if $\phi_\pi \neq 0$. Hence there is a unique bounded solution for $\{\pi_t\}$ if $\phi_\pi > 0$ and (A.11) applies. Using the solution obtained for $\{\hat{\imath}_t\}$, one obtains the solution (2.53) for the inflation process.

If instead $\phi_\pi = 0$, there is a multiplicity of possible solutions for $\{\pi_t\}$, even when the equilibrium path $\{\hat{\imath}_t\}$ is uniquely determined. In fact, Proposition 2.5 again applies in this case. If $\phi_\pi > 0$ but $0 < \phi_\pi + \rho < 1$, there is an uncountably infinite number of solutions for $\{\hat{\imath}_t\}$, as one can show using the method discussed following the proof of Proposition 2.3. To each of these there corresponds a unique associated inflation process, but the set of equilibrium inflation processes is uncountably infinite.

As remarked in the text, determinacy of equilibrium does not require that $\phi_\pi > 0$, though that is the case of primary practical interest. Our analysis above shows that in fact all that is required is that $\phi_\pi \neq 0$ and that (A.12) be satisfied. When $\rho > 1$, the latter condition is satisfied by all non-zero inflation-response coefficients $\phi_\pi > -(\rho - 1)$, which would include moderately negative values. In such a case, (2.53) continues to provide a log-linear approximation to the equilibrium inflation process. The conditions for determinacy would also be satisfied by all $\phi_\pi < -(1 + \rho)$. (Note that this means that determinacy results from sufficiently large negative values of ϕ_π even in the case that $\rho < 1$.) However, the equilibrium obtained in this case depends too crucially upon the assumption of a discrete sequence of dates on which markets are open to be of practical interest. (In the continuous-time limit of the model, no such equilibria are possible. See the discussion below at xxxx.)

1.9 Proof of Proposition 2.9.

Proposition 2.9. Suppose that the equilibrium real rate $\{\hat{r}_t\}$ follows an exogenously given stationary AR(1) process, and let the monetary policy rule be of the form (2.51), with $\rho \geq 0, \phi_\pi > 0$ and a constant intercept consistent with the zero-inflation steady state (*i.e.*, $\bar{\imath}_t = 0$). Consider the choice of a policy rule (ρ, ϕ_π) within this class so as to bring about a certain desired unconditional variance of inflation $\mathrm{var}(\pi) > 0$ around the mean inflation rate of zero. For any large enough value of ρ, there exists a ϕ_π satisfying (A.11) such that the unconditional variance of inflation in the stationary rational-expectations equilibrium associated with this rule is of the desired magnitude. Furthermore, the larger is ρ, the *smaller* is the unconditional variance of interest-rate fluctuations $\mathrm{var}(\hat{\imath})$ in this equilibrium.

PROOF: In the case of an AR(1) process

$$\hat{r}_t = \rho_r \hat{r}_{t-1} + \epsilon_t$$

for the equilibrium real rate, $E_t \hat{r}_{t+j} = \rho_r^j \hat{r}_t$ for all $j \geq 0$. Then for any policy rule (ρ, ϕ_π) that satisfies (A.11) (2.54) implies that

$$\hat{\imath}_t = \frac{\phi_\pi}{\phi_\pi + \rho - \rho_r} \hat{r}_t.$$

Hence $\{\hat{\imath}_t\}$ is also an AR(1) process, with variance and first-order autocovariance

$$\begin{aligned} \mathrm{var}(\hat{\imath}) &= \left(\frac{\phi_\pi}{\phi_\pi + \rho - \rho_r} \right)^2 \mathrm{var}(\hat{r}), \\ \mathrm{cov}(\hat{\imath}_t, \hat{\imath}_{t-1}) &= \rho_r \mathrm{var}(\hat{\imath}). \end{aligned} \tag{A.13}$$

Inverting (2.51) implies that $\pi_t = \phi_\pi^{-1}(\hat{\imath}_t - \hat{\imath}_{t-1})$, from which it follows that

$$\begin{aligned} \mathrm{var}(\pi) &= \phi_\pi^{-2} \mathrm{var}(\hat{\imath}_t - \rho \hat{\imath}_{t-1}) \\ &= \phi_\pi^{-2} [(1 + \rho^2) \mathrm{var}(\hat{\imath}) - 2\rho \mathrm{cov}(\hat{\imath}_t, \hat{\imath}_{t-1})] \\ &= \frac{1 - 2\rho_r \rho + \rho^2}{(\phi_\pi + \rho - \rho_r)^2} \mathrm{var}(\hat{r}). \end{aligned} \tag{A.14}$$

Condition (A.14) can be solved for the required inflation-response coefficient in order to obtain a given degree of variability of inflation, yielding

$$\phi_\pi = \left[(1 - 2\rho_r \rho + \rho^2) frac \mathrm{var}(\hat{r}) \mathrm{var}(\pi) \right]^{1/2} + \rho_r - \rho.$$

(Here we select the positive square root because we know that if a solution exists that satisfies (A.11), it must be such that $\phi_\pi > \rho_r - \rho$.) We note that for all large enough $\rho > 0$, the right-hand side expression must exceed $1 - \rho$, in which case there is indeed a solution satisfying (A.11), as asserted in the Proposition.

This solution can then be substituted for ϕ_π in (A.13), yielding

$$\sigma(\hat{\imath}) = \sigma(\hat{r}) - \frac{\rho - \rho_r}{(1 - 2\rho_r \rho + \rho^2)^{1/2}} \sigma(\pi), \tag{A.15}$$

where $\sigma(x) \equiv (\text{var}(x))^{1/2}$ in the case of any stationary random variable x. The expression

$$\frac{\rho - \rho_r}{(1 - 2\rho_r \rho + \rho^2)^{1/2}} = \frac{\rho - \rho_r}{[(1 - \rho_r^2) + (\rho - \rho_r)^2]^{1/2}}$$

is easily seen to be monotonically increasing in ρ, so that the right-hand side of (A.13) is monotonically decreasing in ρ, for any given value of ϕ_π.

1.10 Proof of Proposition 2.10.

Proposition 2.10. Consider a policy rule of the form (2.51), where $\rho > 1$ and $\{\bar{\imath}_t\}$ is a bounded process, to be adopted beginning at some date t_0. Then any bounded processes $\{\pi_t, \hat{\imath}_t\}$ that satisfy (2.51) for all $t \geq t_0$ must be such that the predicted path of inflation, looking forward from any date $t \geq t_0$, satisfies (2.56). Conversely, any bounded processes satisfying (2.56) for all $t \geq t_0$ also satisfy (2.51) for all $t \geq t_0$.

Proof: Note that (2.51) can equivalently be written

$$\hat{\imath}_{t-1} - \bar{\imath}_{t-1} = -\rho^{-1}\phi_\pi \pi_t + \rho^{-1}\, E_t[\hat{\imath}_t - \bar{\imath}_t].$$

This is yet another stochastic difference equation of the form (A.4), where now $z_t \equiv \hat{\imath}_{t-1} - \bar{\imath}_{t-1}$ happens to be a variable that is predetermined at date t. It follows from the discussion in the proof of Proposition 2.3 that if $|a| < 1$, any bounded solution to (A.4) must satisfy (A.6). In the present case, this result applies if $\rho > 1$, since in this case $0 < \rho^{-1} < 1$, and (2.56) is just the condition corresponding to (A.6).

The converse is established by noting that (A.6) implies that z_t satisfies (A.4). This was implicit in our previous characterization of (A.6) as a "solution" of equation (A.4). In the present application, it does not make sense to call condition (2.56) a "solution" for the the variable $\hat{\imath}_{t-1} - \bar{\imath}_{t-1}$, for this variable is determined at date $t-1$, while the right-hand side of (2.56) depends on information at date t. Instead, (2.56) indicates the way in which the central bank's inflation target at date t varies depending on past conditions.

1.11 Proof of Proposition 2.11.

PROPOSITION 2.11. In the context of a Sidrauski-Brock model with additively separable preferences, consider the consequences of a monetary policy specified in terms of exogenous paths $\{M_t^s, i_t^m\}$, together with a fiscal policy specified by an exogenous path $\{D_t\}$. Under such a regime, the rational-expectations equilibrium paths of prices and interest rates are (locally) determinate; that is, there exist open sets $\mathcal{P}$ and $\mathcal{I}$ such that in the case of any tight enough bounds on the fluctuations in the exogenous processes $\{Y_t, \xi_t, M_t^s/M_{t-1}^s, i_t^m, D_t/D_{t-1}\}$, there exists a unique rational-expectations equilibrium in which $P_t/M_t^s \in \mathcal{P}$ and $i_t \in \mathcal{I}$ at all times. Furthermore, a log-linear approximation to the equilibrium path of the price level, accurate up to a residual of order $\mathcal{O}(||\xi||^2)$, takes the form

$$\log P_t = \sum_{j=0}^{\infty} \varphi_j E_t[\log M_{t+j}^s - \eta_i \log(1 + i_t^m) - u_{t+j}] - \log \bar{m}, \tag{A.16}$$

where the weights

$$\varphi_j \equiv \frac{\eta_i^j}{(1+\eta_i)^{j+1}} > 0$$

sum to one, and u_t is a composite exogenous disturbance

$$u_t \equiv \eta_y \hat{Y}_t - \eta_i \hat{r}_t + \epsilon_t^m - \eta_i \log(1 + \bar{\imath}^m).$$

PROOF: Once again, we may ignore conditions (1.24) and (3.4), as these will be satisfied by any processes $\{P_t/M_t^s, i_t, i_t^m, M_t^s/M_{t-1}^s, D_t/D_{t-1}, Y_t, \xi_t\}$ that satisfy tight enough bounds. It then suffices that we consider the existence of bounded solutions to the system of log-linear relations consisting of (1.32) and (3.6), augmented by the identity

$$\hat{m}_t = \hat{m}_{t-1} + \mu_t - \pi_t, \tag{A.17}$$

where $\mu_t \equiv \log(M_t^s/M_{t-1}^s$ is the exogenous rate of growth in the monetary base. This comprises a system of three expectational difference equations per period to determine the three endogenous variables $\hat{m}_t, \pi_t$, and $\hat{\imath}_t$.

Using (1.32) to eliminate $\hat{\imath}_t$ in (3.6), we obtain a discrete-time rational-expectations version of the Cagan model of inflation determination,

$$\hat{m}_t = -\eta_i E_t \pi_{t+1} + [u_t + \eta_i \log(1 + i_t^m)], \tag{A.18}$$

where u_t is the composite exogenous disturbance defined in the statement of the proposition. Then using (A.17) to substitute for π_{t+1}, (A.18) implies that

$$\hat{m}_t = \alpha E_t \hat{m}_{t+1} + (1-\alpha)[u_t + \eta_i \log(1 + i_t^m) - \eta_i E_t \mu_{t+1}], \tag{A.19}$$

where $\alpha \equiv \eta_i/(1+\eta_i)$.

This is once again an expectational difference equation of the form (A.4). Because $\eta_i > 0$ implies that $0 < \alpha < 1$, (A.19) can be solved forward to obtain a unique bounded solution for $\{\hat{m}_t\}$, given by

$$\hat{m}_t = (1-\alpha) \sum_{j=0}^{\infty} \alpha^j E_t [u_{t+j} + \eta_i \log(1 + i_t^m) - \eta_i \mu_{t+j+1}]. \tag{A.20}$$

Such a unique solution for $\{\hat{m}_t\}$ then implies a unique solution for $\{\hat{\imath}_t\}$, using (3.6), and for $\{\pi_t\}$, using (A.17). It then follows from the discussion in section xx above that there will also be a locally unique solution to the exact equilibrium relations (1.21) and (3.5) in the case of tight enough bounds on the exogenous processes. Furthermore, this solution will satisfy any desired bounds on $\hat{m}_t$ and $\hat{\imath}_t$. (Since $P_t/M_t^s = 1/m_t$, this allows us to ensure that $P_t/M_t^s \in \mathcal{P}$ at all times.) Finally, we observe that (A.19) can be rewritten as (A.16).

1.12 Proof of Proposition 2.12.

PROPOSITION 2.12. In a Sidrauski-Brock model where utility is not necessarily separable, let monetary policy be specified by a Wicksellian rule (1.30) for the central bank's interest-rate operating target. Suppose that $i_t^m = \bar{\imath}$ at all times, for some $0 \leq \bar{\imath} < \beta^{-1} - 1$; and let fiscal policy again be specified by an exogenous process $\{D_t\}$. Finally, suppose that

$$\chi > -\frac{1}{2\eta_i}. \tag{A.21}$$

Then equilibrium is determinate in the case of any policy rule with $\phi_p > 0$. A log-linear approximation to the locally unique equilibrium price process is given by (3.17), where the weights are given by (3.18) – (3.19).

PROOF: Substituting (1.34) for $\hat{\imath}_t$ in (3.12), and setting $\hat{\imath}_t^m = 0$, we obtain the equilibrium relation

$$(1 + (1 + \eta_i\chi)\phi_p)\hat{P}_t = (1 + \eta_i\chi\phi_p)E_t\hat{P}_{t+1} + [E_t\pi^*_{t+1} + (\tilde{r}_t - \nu_t) + \eta_i\chi(E_t\nu_{t+1} - \nu_t)] \quad \text{(A.22)}$$

as a generalization of (1.36). This is once again a stochastic difference equation of the form (A.4). It then follows from our discussion in section xx above that (A.22) can be solved forward to yield a unique bounded solution for $\{\hat{P}_t\}$, if and only if

$$|1 + \eta_i\chi\phi_p| < |1 + \phi_p(1 + \eta_i\chi)|.$$

We observe that as long as (A.21) holds, this determinacy condition is satisfied for all $\phi_p > 0$, just as was concluded in section xx for the case $\chi = 0$. Furthermore, in this case, the unique bounded solution is given by (A.6). Applying this result and rearranging terms, one obtains (3.17).

1.13 Proof of Proposition 2.13.

PROPOSITION 2.13. Let monetary policy instead be specified by an interest-rate rule of the form (2.51), with coefficients $\phi_\pi, \rho \geq 0$, and again suppose that $i_t^m = \bar{\imath}_t^m$ at all times. Finally, suppose that χ satisfies (A.21).Then equilibrium is determinate if and only if $\phi_\pi > 0$ and (A.11) holds. When these conditions are satisfied, a log-linear approximation to the equilibrium evolution of inflation is given by (3.20), where the weights are given by (3.21) – (3.22).

PROOF: Using (2.51) to eliminate π_{t+1} in (3.12), and again setting $\hat{\imath}_t^m = 0$, one obtains a stochastic difference equation for the interest rate of the form

$$[(1 + \eta_i\chi)\phi_\pi + \rho]\hat{\imath}_t = [1 + \eta_i\chi\phi_\pi]E_t\hat{\imath}_{t+1} + \phi_\pi\tilde{r}_t + \rho\bar{\imath}_t - E_t\bar{\imath}_{t+1}. \quad \text{(A.23)}$$

This is again an equation of the form (A.4). It follows that equilibrium is locally determinate if and only if the term in square brackets on the left-hand side (call it γ_0) is larger in absolute value than the term in square brackets on the right-hand side (call it γ_1). Condition (A.21) suffices to guarantee that $\gamma_0 + \gamma_1 > 0$. It then follows that $|\gamma_0| > |\gamma_1|$, so that determinacy obtains, if and only if $\gamma_0 > \gamma_1$. This last inequality is in turn seen to be equivalent to (A.11). In the case that equilibrium is determinate, (A.6) can again be applied, yielding (3.20).

1.14 Proof of Proposition 2.14.

PROPOSITION 2.14. Consider a sequence of economies with progressively smaller period lengths Δ, calibrated so that $\bar{\chi} \neq 0$. Assume in each case that monetary policy is specified by a contemporaneous Taylor rule (2.43), with a positive inflation-response coefficient $\phi_\pi \neq 1$ that is independent of Δ. Assume also that zero interest is paid on money. Then equilibrium is determinate for all small enough values of Δ if $\phi_\pi > 1$ and $\bar{\chi} > 0$, or if $0 < \phi_\pi < 1$ and $\bar{\chi} < 0$, but not otherwise.

PROOF: As discussed in the proof of Proposition 2.13, determinacy obtains if and only if $|\gamma_0| > |\gamma_1|$. In the limit as $\Delta \to 0$, both γ_0 and γ_1 become unboundedly large, while

$$\Delta\gamma_0, \Delta\gamma_1 \to \tilde{\eta}_i \bar{\chi} \phi_\pi. \tag{A.24}$$

Thus both γ_0 and γ_1 have the same sign as $\bar{\chi}$, for all small enough values of Δ. We note furthermore that

$$\gamma_0 - \gamma_1 = \phi_\pi - 1,$$

so that $\gamma_0 > \gamma_1$ if and only if $\phi_\pi > 1$. It then follows that $|\gamma_0| > |\gamma_1|$, and determinacy obtains, for all small enough values of Δ, if and only if either $\phi_\pi > 1$ and $\bar{\chi} > 0$ (so that $\gamma_0 > \gamma_1 > 0$) or $0 < \phi_\pi < 1$ and $\bar{\chi} < 0$ (so that $\gamma_0 < \gamma_1 < 0$).

1.15 Proof of Proposition 2.15.

PROPOSITION 2.15. Again consider a sequence of economies with progressively smaller period lengths Δ, and suppose that

$$\bar{\chi} > -\frac{1}{\psi \Phi_\pi \tilde{\eta}_i}. \tag{A.25}$$

Let monetary policy instead be specified by an inertial Taylor rule (2.51), with a long-run inflation-response coefficient $\Phi_\pi \equiv \phi_\pi/(1-\rho)$ and a rate of adjustment $\psi \equiv -\log \rho/\Delta > 0$ that are independent of Δ. Assume again that zero interest is paid on money. Then rational-expectations equilibrium is determinate if and only if $\Phi_p i > 1$, *i.e.*, if and only if the Taylor Principle is satisfied.

The unique bounded solution for the path of nominal interest rates in the determinate case is of the form

$$\hat{\imath}_t = \Lambda \bar{\imath}_t + \Gamma(1-\gamma)\sum_{j=0}^{\infty} \gamma^j E_t \tilde{r}_{t+j} + \tilde{\Gamma}(1-\gamma)\sum_{j=0}^{\infty} \gamma^j E_t \bar{\imath}_{t+j}, \tag{A.26}$$

with the solution for $\{\pi_t\}$ then obtained by inverting (2.51). In this solution, the coefficients $\Lambda, \Gamma, \tilde{\Gamma}$ approach well-defined limiting values as Δ is made arbitrarily small, while the rate of decay of the weights on expected disturbances farther in the future,

$$\xi \equiv -\log \gamma/\Delta > 0,$$

also approaches a well-defined limiting value. Furthermore, these limiting values are all continuous functions of $\bar{\chi}$ for values of $\bar{\chi}$ in the range satisfying (A.25), including values near zero.

Proof: In this case, because ϕ_π approaches zero at the same rate as Δ (though we again maintain a long-run inflation-response coefficient independent of Δ), γ_0 and γ_1 do not become unboundedly large for small Δ. Instead of (A.24), we obtain

$$\gamma_0, \gamma_1 \to 1 + \tilde{\eta}_i \bar{\chi} \Phi_\pi \psi,$$

and this limiting value is positive given (A.25). Hence $\gamma_0, \gamma_1 > 0$ in the case of any small enough value of Δ, even if $\bar{\chi}$ is (modestly) negative. We furthermore observe that in this case,

$$\gamma_0 - \gamma_1 = (\Phi_p i - 1)(1-\rho).$$

Hence we find once again that $\gamma_0 > \gamma_1$ if and only if $\Phi > 1$. Thus $|\gamma_0| > |\gamma_1|$, and equilibrium is determinate, if and only if $\Phi > 1$.

In the case of determinacy, the equilibrium solution for the nominal interest rate, derived in the course of the proof of Proposition 2.13, is given by

$$\hat{\imath}_t = \bar{\imath}_t + \sum_{j=0}^{\infty} \phi_\pi \varphi_j E_t \tilde{r}_{t+j} - \sum_{j=0}^{\infty} \phi_\pi \tilde{\varphi}_j E_t \bar{\imath}_{t+j},$$

where the coefficients $\{\varphi_j, \tilde{\varphi}_j\}$ are defined in (3.21) – (3.22). This is observed to be of the form (A.26), and allows us to identify the coefficients $\Lambda, \Gamma, \tilde{\Gamma}$, and γ in that representation of the solution.

We then observe that as Δ is made arbitrarily small,

$$\Lambda + \Gamma(1-\gamma) - 1 = \frac{(1+\eta_i \chi)\phi_\pi}{(1+\eta_i \chi)\phi_\pi + \rho} \rightarrow \frac{\tilde{\eta}_i \bar{\chi} \Phi_\pi \psi}{1 + \tilde{\eta}_i \bar{\chi} \Phi_\pi \psi},$$

$$\Gamma = \frac{\phi_\pi}{\phi_\pi + \rho - 1} = \frac{\Phi_\pi}{\Phi_\pi - 1} > 0,$$

$$\tilde{\Gamma} = \frac{1 + (1-\rho)\eta_i \chi}{1 + \eta_i \chi \phi_\pi} \Gamma \rightarrow \frac{1 + \tilde{\eta}_i \bar{\chi} \psi}{1 + \tilde{\eta}_i \bar{\chi} \Phi_\pi \psi} \frac{\Phi_\pi}{\Phi_\pi - 1} > 0,$$

and that ξ has the same limiting value as

$$\frac{1-\gamma}{\Delta} = \frac{1}{\Delta} \frac{\phi_\pi + \rho - 1}{(1+\eta_i \chi)\phi_\pi + \rho} \rightarrow \frac{(\Phi_\pi - 1)\psi}{1 + \tilde{\eta}_i \bar{\chi} \Phi_\pi \psi} > 0.$$

Since $\gamma \rightarrow 1$, we also observe that

$$\Lambda \rightarrow 1 + \frac{\tilde{\eta}_i \bar{\chi} \Phi_\pi \psi}{1 + \tilde{\eta}_i \bar{\chi} \Phi_\pi \psi} > 0.$$

Hence $\Lambda, \Gamma, \tilde{\Gamma}$, and ξ all have well-defined limiting values, and each is a continuous function of $\bar{\chi}$.

Interest and Prices

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Contents

Chapter 3

Optimizing Models with Nominal Rigidities

We turn now to the analysis of models in which monetary policy affects the level of real economic activity, and not just the level of money prices of goods and services. This requires us to go beyond the analysis of endowment economies, as in the previous chapter, and allow instead for endogenous supply decisions. But as is well-known, even when we allow for endogenous supply, monetary policy can have only small effects on the equilibrium allocation of resources in an environment with perfect wage and price flexibility (and identical information on the part of all decision-makers). Hence we also extend our analytical framework to allow for delays in the adjustment of prices and/or wages to changing aggregate conditions. In this way, we allow for non-trivial real effects of monetary policy.

This extension of our framework is also important for a more realistic discussion of central-bank interest-rate policy. In the basic model of chapter 2, the equilibrium real rate of return is completely independent of monetary policy. This means that a central bank can have no effect on nominal interest rates except insofar as it can shift inflation expectations. In our analysis above, we have assumed that it is able to do so, as long as the change in expectations that is called for involves no violation of the postulate of rational expectations on the part of the private sector; but our analysis may give the appearance of assuming precise central-bank control of something upon which banks have little direct influence in reality. The introduction of price stickiness will make our assumption that the central bank can set a short-term nominal interest rate less paradoxical. If private-sector inflation expectations do not change when the central bank seeks to adjust the nominal interest rate, this does not prevent the central bank from achieving its operating target; it simply means that the private sector will perceive the *real* interest rate to have changed, which will affect desired expenditure and hence the degree of utilization of existing productive capacity. (We re-examine the topic of inflation determination under an interest-rate rule in the context of models with nominal rigidities in chapter 4.)

We shall give primary emphasis to models with sticky prices, though we also discuss the consequences of wage stickiness as well. In this we follow most of the literature of the past twenty years, but the choice perhaps deserves brief comment. One reason for emphasizing price stickiness, at least for pedagogical purposes, is simply that models with only sticky prices provide a *simpler* framework for the consideration of basic issues regarding the nature of inflation determination. If we are to talk about the determinants of inflation (which in the context of current policy debate, almost invariably means the rate of increase in goods prices rather than wages), we must model the goods market; and if we are to consider such central issues as the relation between interest rates and expenditure decisions, we need to adjust our nominal interest rate for the expected rate of price inflation as well. On the other hand, it is not equally essential to explicitly consider wage determination. It is possible to model endogenous supply decisions without any reference to a labor market at all, as in the familiar "yeoman farmer" models; and so the simplest models developed here will be of this form, or will be equivalent in their implications to such a model, even if a (completely frictionless) labor market is represented.

It is also often argued that there is more reason to believe that the stickiness of prices matters for the allocation of resources. A well-known criticism of the models of nominal wage rigidity popular in the 1970s was that the mere observation of infrequent changes in individual nominal wages did not in itself prove the existence of a nominal rigidity with any allocative consequences (Barro, 1977; Hall, 1980). Because employment is an ongoing relationship rather than a spot-market transaction, the effective cost to a firm of increased employment of labor inputs at a point in time need not equal the wage paid per hour of work at that time; and under an efficient implicit contract, wages might well be smoother than the effective cost of labor, owing to a preference of workers for a smoother income stream. On the other hand, it is less plausible that the observed rigidity of consumer goods prices should not have allocative consequences, given the absence of a similar kind of ongoing relationship between the suppliers of consumer goods and their customers (Rotemberg, 1987). However, no convincing evidence has ever been offered that the stickiness of nominal wages does *not* result in stickiness of the effective nominal cost of labor inputs; and evidence described below — indicating that the evolution of U.S. inflation can be well explained by the evolution of unit labor costs — suggests that a model of supply costs that treats the reported wage as the true marginal cost of additional hours of labor is not too inaccurate. Thus a more empirically realistic model is likely to involve both wage and price stickiness. We treat models of that kind in section xx below.

We shall give particular attention to the derivation of models with sticky prices and/or wages in which prices and wages are nonetheless set *optimally* on the occasions when they are adjusted. Allowing for optimal price- and wage-setting is important for several reasons. One is that it allows us to highlight the importance of *expectations* for wage and price dynamics. As we shall see in later chapters, forward-looking private sector behavior, in this and other respects, has profound consequences for the optimal conduct of monetary policy. It would thus be a serious mistake to simply assume mechanical wage- and price-adjustment equations (perhaps drawn from the econometric literature), and treat these as structural for purposes of an analysis of optimal monetary policy, as Lucas (1976) so forcefully argued.

Another reason for modeling optimal price- and wage-setting is that we are interested in the welfare evaluation of alternative monetary policies. An especially appealing basis for such evaluation is to ask how alternative possible equilibria compare from the point of view of the private-sector objectives that underlie the behavior assumed in one's model of the effects of alternative policies; but this is only possible insofar as the structural equations of our model of the monetary transmission mechanism are derived from optimizing foundations.

As we shall see in chapter 6, alternative assumptions about the nature of price and wage stickiness imply that alternative stabilization objectives for monetary policy are appropriate.

While we shall give detailed attention to the consequences of assumed delays in the adjustment of prices and/or wages, we shall not attempt here to say anything new about the underlying *reasons* for these delays. Our assumptions about the frequency with which firms adjust their prices, or the time lag that may be involved between the decision about a price and the time that the new price takes effect, are treated as structural features of the environment in which firms sell their products, with the same status as their production functions. How reasonable this is depends on the question that one intends to ask of one's model. The "endogenous growth" has emphasized that, when thinking about the determinants of economies' long-run growth prospects, it is probably a mistake to ignore the endogeneity of production functions — for changes in economic conditions can change the incentives that private parties have to devote resources to research and development, to introduce new products, and so on. On the other hand, for purposes of a comparison of alternative monetary policy rules, it may not be a bad approximation to assume given production functions; the real effects of alternative monetary policies are relatively short-lived, and over this short horizon production possibilities are unlikely to be much affected by the temporary alteration of incentives to innovate that may have occurred.

Similarly, if we wished to analyze the consequences of highly inflationary policies, we would surely *not* want to treat as given the frequency of price adjustment, the degree of indexation of wage contracts, or even the currency in terms of which prices are quoted; we know that practices adjust in all of these respects (and for reasons that are easy to understand) in economies that suffer from sustained high inflation. But our interest in the present study is in the identification of better monetary policies within the class of policies under which inflation is never very great; in fact, we shall make extensive use of approximations that are expected to be accurate only for the analysis of policies of that kind. (It will perhaps not be giving away too much to divulge at this point that, according to our analysis, optimal policy will indeed involve low and stable inflation!) For this purpose, treating the delays involved in price and wage adjustment as structural may not be a bad approximation. The sizes of wage and price increases do clearly vary from year to year, in response to changes in perceived market conditions; practices with regard to the times at which prices or wages are reconsidered, or the units in which they are announced, occur much less often and only in response to more drastic changes in the economic environment.

Finally, we freely grant that the simple models presented here should be viewed only as crude approximations to the actual monetary transmission mechanism. A realistic quantitative model would need to incorporate a large number of complications from which we abstract here, in order to clarify basic concepts. One may wish to add endogenization of the timing of price and wage adjustments to the list of refinements that one would like to incorporate into an eventual, truly accurate model. It is not clear, however, that this particular refinement should be placed too high on the list of refinements when ranked in terms of their likely quantitative importance for the analysis of monetary policy.

Nor is it even clear that any of the models with endogenous timing of price changes that currently exist should be regarded as more realistic than the models presented below, quite apart from the question of complexity. Some feel that models of "state-dependent pricing", such as those of Caplin and Leahy (1991) or Dotsey *et al.* (1999), have "better microfoundations" than do the sorts of models presented here, that assume a given timing for price and wage changes. But this is not obvious. These models assume that firms are constantly re-evaluating the price that they *would* adopt if they were to change their price, and the expected benefits from the change, and then weighing these benefits against the current "menu cost" of a price change to decide whether to actually change their price or not. Yet in reality, the main benefit of infrequent price changes is not lower "menu costs", but reduction of the costs associated with information collection and decision-making (Zbaracki *et al.*, 1999). Obtaining this benefit necessarily means that the timing of the occasions upon which prices are reconsidered will be largely independent of current market conditions; for example, firms often reconsider pricing policy at a particular time of year.

We begin in section 1 with a basic model of monopolistic competition, in which the prices of some goods must be determined a period in advance. This very simple example of price stickiness is useful for introducing a number of basic concepts. It also provides optimizing foundations for a familiar aggregate-supply specification, the "New Classical Phillips curve" used in many well-known analyses of optimal monetary policy, such as those of Kydland and Prescott (1977) and Barro and Gordon (1983). While the literature using this specification has produced a number of insights of more general importance, this relation is quite inadequate as a realistic account of the co-movement of real and nominal variables; it allows, for example, no persistent effects of monetary policy on real activity, and no effects of anticipated policy. These strong conclusions are not general consequences of optimal price-setting, as we show in section 2 through the analysis of a slightly more complex specification, the Calvo (1983) model of staggered price-setting. While still very simple, this model implies an aggregate-supply relation, sometimes called the "New Keynesian Phillips curve", that has proven capable of explaining at least some of the more gross features of inflation dynamics in the U.S. and elsewhere. Section 3 discusses still more complex specifications with increased empirical realism, that introduce delays in the effects of monetary policy changes on inflation. Finally, section 4 discusses models in which nominal wages are sticky as well as prices.

1 A Basic Sticky-Price Model

We begin by displaying the structure of a very basic model, in which monetary policy has real effects as a result of some goods prices being fixed in advance. A number of issues that are easy to analyze in this simple context will turn out also to be relevant to the more realistic models to be developed in later sections.

1.1 Price-Setting and Endogenous Output

In order to be able to model price-setting, we must first extend the representative-household model introduced in the previous chapter in certain respects that are quite distinct from the issue of whether prices are assumed to be *sticky*. In particular, we must allow for endogenous goods supply, rather than simply assuming a given endowment of goods. This requires that we introduce a production technology and at least one variable factor of production (which is labor, in this basic model). We shall be concerned to understand the determinants of the costs of supplying goods, as supply costs are a prime determinant of optimal pricing. We shall furthermore introduce differentiated goods, and monopolistic competition among the suppliers of these goods, as in the "New Keynesian" literature originated by Rotemberg (1982), Mankiw (1985), Svensson (1986) and Blanchard and Kiyotaki (1987), rather than assuming a single good in competitive supply. This last device, which is now quite commonplace, allows individual suppliers a degree of market power, and hence a decision about how to set their prices. It also implies that a supplier that fails to immediately adjust its price in response to a change in demand conditions does not suffer an unboundedly large (percentage) change in its sales, so that it becomes more plausible that prices should not be constantly adjusted.[1]

We thus now assume that the representative household seeks to maximize a discounted sum of utilities of the form

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[u(C_t, M_t/P_t; \xi_t) - \int_0^1 v(h_t(i); \xi_t) di \right] \right\}. \tag{1.1}$$

Here C_t is now an *index* of the household's consumption of each of the individual goods that are supplied, and P_t is a corresponding index of the prices of these goods, while $h_t(i)$ is the quantity of labor of type i supplied. We assume that each of the differentiated goods (indexed by i over the unit interval) uses a specialized labor input in its production (and in this chapter, this will be the *only* variable input); labor of type i is used to produce differentiated good i.

The introduction of differentiated labor inputs is not necessary in order to allow us to

[1] The size of the "menu costs" required to rationalize the failure of suppliers to adjust their prices immediately has been the subject of an extensive literature. We shall not pursue this issue here, though we note that the same sorts of "strategic complementarities" in price-setting that increase the degree of stickiness of the general price level (as discussed below) when price adjustment is asynchronous also tend to reduce the size of the costs of price changes to individual suppliers that are required to rationalize a failure to adjust prices in response to an aggregate demand disturbance. On this latter point, see in particular Ball and Romer (19xx).

analyze monopolistically competitive goods supply. But as we shall see, it is convenient to do so. One reason is that we are able, in this case, to derive a model with factor markets that is equivalent to the frequently-used "yeoman farmer" model, in which households are assumed to directly supply goods. A more important reason is that it turns out that the "strategic complementarity" between different suppliers' pricing decisions is greater when we assume that they do not hire labor from a single homogeneous (competitive) labor market. Because we regard the conclusion obtained in the case of differentiated labor inputs as the more realistic one, we choose this specification as our baseline model. (The case of a single homogeneous labor market is discussed in section 1.4 below.)

The term $v(h_t(i);\xi_t)$ represents the disutility of supplying labor of type i; we assume that for each possible value of ξ, $v(\cdot;\xi)$ is an increasing, convex function.[2] We have written (1.1) as if the representative household simultaneously supplies *all* of the types of labor. However, we might equally well assume that each household specializes in the supply of only *one* type of labor, but that there are an equal number of households supplying each type. In this case, a household that supplies labor of type i seeks to maximize

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t\left[u(C_t, M_t/P_t;\xi_t) - v(h_t(i);\xi_t)\right]\right\}.$$

When not all goods prices are set at the same time, households' wage incomes will be different, depending upon the type of labor they supply. But we may assume that there exist competitive financial markets in which these risks are efficiently shared.

In this case, and if all households start with initial financial assets that give them the same initial intertemporal budget constraints,[3] then since households value consumption streams (and money balances) identically and face the same prices, all households will choose identical consumption and real balances in all states. (Note that while we have allowed for preference

[2] Once again, ξ_t is a vector, so that the use of the same notation for exogenous disturbances to the functions u and v involves no assumption about statistical dependence between the shifts in these two functions.

[3] This means that if any households face different present values of their expected wage incomes as of date zero, they hold initial financial claims W_0 that differ in exactly the way necessary to offset the difference in their expected wages. Note that if this condition has *ever* held, then optimization in the presence of complete financial markets implies that it will hold *forever after*, regardless of which values may be realized for the exogenous disturbances.

shocks ξ_t in (1.1), we assume that these are the same for all households – we here contemplate only *aggregate* shocks.) They will also choose portfolios of financial assets that insure that they continue to have identical intertemporal budget constraints at all subsequent dates.[4] The common intertemporal budget constraint in each state will in turn be exactly that of a household that supplies *all* of the types of labor, and pools the wage income received.

Because each household chooses exactly the same state-contingent consumption plan, the first-order conditions for optimal supply of each type of labor are exactly the same as when a single household type supplies all types of labor so as to maximize (1.1). Thus the conditions that determine equilibrium prices and quantities are the same in the two models. Furthermore, if our welfare criterion in the specialized-labor model is the average level of utility of all households, the level of social welfare associated with a given equilibrium will be measured by the value of (1.1). Thus it makes no difference to our conclusions which version of the model we assume. The fiction that each household directly supplies all types of labor, and so receives its *pro rata* share of the aggregate wage bill of the entire economy, simplifies the exposition in that it allows us to dispense with explicit discussion of the risk-sharing arrangements just referred to.

Following Dixit and Stiglitz (1977), we shall assume that the index C_t is a constant-elasticity-of-substitution aggregator

$$C_t \equiv \left[\int_0^1 c_t(i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}} \tag{1.2}$$

with $\theta > 1$, and that P_t is the corresponding price index

$$P_t \equiv \left[\int_0^1 p_t(i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}. \tag{1.3}$$

Note that (1.3) defines the minimum cost of a unit of the aggregate defined by (1.2), given the individual goods prices $\{p_t(i)\}$; since a household cares only about the number of units

[4] Here our argument relies upon the assumption in (1.1) that the disutility of labor supply is additively separable from the other terms. This implies that even if households expect to work different amounts in particular states, they equalize their marginal utility of income in each state by holding assets that allow them to afford to consume exactly the same amount and hold exactly the same money balances as one another. The more complicated case of non-separable preferences is treated in section xx below.

of this aggregate that it can purchase, deflation by P_t is an appropriate measure of the purchasing power of nominal money balances M_t.

The household's budget constraints are then as in chapter 2, except that the term $p_t c_t$ for nominal consumption expenditure must now be replaced by $\int_0^1 p_t(i)c_t(i)di$, and the term $p_t y_t$ for income from the sale of goods must now be replaced by

$$\int_0^1 w_t(i)h_t(i)di + \int_0^1 \Pi_t(i)di, \tag{1.4}$$

where $w_t(i)$ is the nominal wage of labor of type i in period t, and $\Pi_t(i)$ represents the nominal profits from sales of good i. In writing this last expression, we assume that each household owns an equal share of all of the firms that produce the various goods. Again, given our assumption of complete financial markets, this assumption of distributed ownership is irrelevant. We could also introduce trading in the shares of the firms, without any change in the conditions for a rational expectations equilibrium, except that then equilibrium share prices would also be determined. As these extensions of the framework have no consequences for the equilibrium evolution of goods prices or the quantities of goods supplied, we omit further discussion of them.

As in chapter 2, each household then faces a single intertemporal budget constraint. Optimal (price-taking) household behavior is then described by the conjunction of three sets of requirements. First, the household's consumption spending must be optimally allocated *across differentiated goods* at each point in time, taking as given the overall level of expenditure. Thus the relative expenditures on different goods must be such as to maximize the index (1.2) given the level of total expenditure. As in other applications of the Dixit-Stiglitz model, this requires that purchases of each good i satisfy

$$c_t(i) = C_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta}. \tag{1.5}$$

This rule for distributing expenditure is easily seen to imply that total expenditure will equal $P_t C_t$. Using this substitution, we can write *both* the household's utility and its budget constraints solely in terms of P_t and C_t, without any reference to quantities or prices of the individual goods that are purchased.

Second, taking as given the optimal allocation of consumption expenditure at each date (just described) and the amount of labor supplied (considered below), the household must choose optimal levels of total consumption expenditure at each date, an optimal level of money balances to hold at each date, an optimal amount of financial wealth to accumulate, and an optimal portfolio allocation across the various types of state-contingent bonds that are available. Necessary and sufficient conditions for optimization in this respect are given by *exactly the same* conditions as in chapter 2 – namely, conditions (1.2), (1.12), (1.13), (1.15) and (1.16) of that chapter must again hold at all times, where however P_t now refers to the price index (1.3), c_t is replaced by the index C_t defined in (1.2), and y_t is replaced by Y_t, a similarly defined aggregate of the quantities supplied of the various differentiated goods.[5] This is because both preferences over alternative streams of the consumption aggregate and budget constraints written in terms of affordable paths for the consumption aggregate are exactly the same as in chapter 2.[6]

And finally, the household must choose an optimal quantity of each kind of labor to supply, given the wages that it faces and the value to it of additional income (determined by the consumption-allocation problem just described). The first-order condition for optimal supply of labor of type i at date t is given by

$$\frac{v_h(h_t(i);\xi_t)}{u_c(C_t, m_t;\xi_t)} = \frac{w_t(i)}{P_t}. \tag{1.6}$$

These conditions, together with those listed earlier, comprise a complete set of necessary and sufficient conditions for household optimization.

We turn next to the specification of production possibilities. We assume that each good i has a production function

$$y_t(i) = A_t f(h_t(i)), \tag{1.7}$$

[5] The relative quantities supplied of the various goods must be distributed in the same way as the relative demands implied by (1.5). (When we allow below for government purchases, we assume that the government seeks to maximize a similar aggregate of its purchases, and so distributes its purchases in the same manner.) It then follows that total non-financial income (1.4), which must equal total sales revenues of all of the firms, can be written as P_tY_t.

[6] Here the additive separability of the disutility-of-labor terms in (1.1) is again crucial, allowing us to obtain preferences over paths for the consumption aggregate and real balances that are the same as in our earlier model with no labor supply decision.

where $A_t > 0$ is a time-varying exogenous technology factor, and f is an increasing, concave function. Here labor is represented as the only factor of production (with one specific type of labor being used in the production of each good). We may think of capital as being allocated to each firm in a fixed amount, with capital goods never depreciating, never being produced, and (because they are specific to the firm that uses them) never being reallocated among firms; in this case, the additional argument of the production function may be suppressed. (An extension of the model to allow for endogenous capital accumulation is presented in the next chapter.)

It follows that the variable cost of supplying a quantity $y_t(i)$ of good i is given by

$$w_t(i) f^{-1}(y_t(i)/A_t).$$

Differentiating this, we find that the (nominal) marginal cost of supplying good i is equal to[7]

$$S_t(i) = \frac{w_t(i)}{A_t} \Psi(y_t(i)/A_t),$$

where

$$\Psi(y) \equiv \frac{1}{f'(f^{-1}(y))} \tag{1.8}$$

is an increasing positive function. Substituting the labor supply function (1.6) for the wage, we obtain a relation between the real marginal supply cost and the quantity supplied:

$$s_t(i) \equiv S_t(i)/P_t = s(y_t(i), Y_t; \tilde{\xi}_t),$$

where the *real marginal cost function* is defined by

$$s(y, Y; \tilde{\xi}) = \frac{v_h(f^{-1}(y/A); \xi)}{u_c(Y; \xi) A} \Psi(y/A). \tag{1.9}$$

[7]Here we assume that the producer is a wage-taker, even though we have supposed each good uses a different type of labor with its own market. But our assumption of differentiated labor inputs need not imply that each producer is a monopsonist in its labor market. The only assumption that is important for our results below is that producers that change their prices at different times also hire labor inputs from distinct markets. We might, for example, assume a double continuum of differentiated goods, indexed by (i, j), with an elasticity of substitution of θ between any two goods, as above. We might then assume that all goods with the same index i change their prices at the same time (and so always charge the same price), and are also all produced using the same type of labor (type i labor). The degree of market power of each producer in its product market would then be as assumed here, but the fact that a continuum of producers all bid for type i labor would eliminate any market power in their labor market.

In this last expression, $\tilde{\xi}_t$ represents the complete vector of exogenous disturbances, in which the preference shocks ξ_t have been augmented by the technology factor A_t, we have substituted into the labor supply function the sectoral labor requirement as a function of sectoral output, and we have used the fact that in equilibrium, the index of aggregate consumption C_t must at all times equal the index of output Y_t.[8]

Note also that we have suppressed real balances as an argument of u_c (and hence as an argument of the real marginal cost function) in the denominator. Abstracting from such "real balance effects" can be justified along any of several grounds discussed in chapter 2. It is simplest to suppose that the economy considered in this chapter is a "cashless" one, in which monetary policy is implemented in the way considered in section 1 of chapter 2.[9] As in chapter 2, we shall here assume a cashless economy as our baseline model; the consequences of real balance effects are considered in section 3.2 of chapter 4.

This model of production costs might alternatively be derived from a "yeoman farmer" model, in which households directly supply goods, seeking to maximize

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[u(C_t, M_t/P_t; \xi_t) - \int_0^1 \tilde{v}(y_t(i); \tilde{\xi}_t) di \right] \right\}. \tag{1.10}$$

If we convert the marginal disutility of supply of good i into units of an equivalent quantity of the consumption aggregate, we obtain a "real marginal cost" of good i equal to

$$\frac{\tilde{v}_y(y_t(i); \tilde{\xi}_t)}{u_c(Y_t; \xi_t)}.$$

[8]When we allow for exogenous variation in government purchases, we can still apply this equation, if we understand $u(Y; \xi)$ to mean the function $\tilde{u}(Y_t; \tilde{\xi}_t)$ introduced in section xx of chapter 2, measuring household utility flow as a function of *aggregate demand* rather than consumption expenditure. Under this interpretation, the level of government purchases is just one element of the vector of exogenous disturbances $\tilde{\xi}$ that shifts this relation. When we allow for endogenous variations in investment spending, matters are more complex; in such a case, it is important to remember that it is really $u_c(C; \xi)$ rather than $u_c(Y; \xi)$ that belongs in the denominator of (1.9).

[9]Alternatively, our results apply to a Sidrauski-Brock model in which utility is separable, as discussed in section xx of chapter 2; to a cash-in-advance model of the special type discussed in section xx of the appendix to chapter 2; or to a "cashless limiting economy" of the sort discussed in section xx of chapter 2. They also apply to a much broader class of models with transactions frictions, in the case that monetary policy is implemented through a procedure under which the interest-rate differential Δ_t is held constant, under a suitable reinterpretation of the parameter σ and the disturbance g_t, so that $-\sigma^{-1}(\hat{Y}_t - g_t)$ is the deviation of the log marginal utility of real income from its steady-state level, in the case of a constant interest-rate differential (rather than a constant level of real balances). For discussion of this last case, see section 3.2 of chapter 4, especially footnote xx.

This is in fact identical to (1.9) if the disutility of output supply is given by

$$\tilde{v}(y;\tilde{\xi}) \equiv v(f^{-1}(y/A);\xi).$$

This concept of "real marginal cost" plays exactly the same role in optimal pricing in the yeoman farmer model as does the more conventional concept in the case of supply by firms that purchase inputs, and the results that we obtain below are identical to those that one would obtain from a yeoman farmer model. As noted above, this is one reason for interest in the model with differentiated labor inputs assumed here. However, explicitly modeling the labor market has the advantage of allowing us to derive additional implications of the model. It will also make the extension, below, to a model with sticky wages as well as prices more straightforward.

With our theory of marginal supply costs in place, we now turn to the question of optimal pricing. We shall first examine the case of *perfectly flexible* prices; that is, we shall assume that the supplier of each good chooses a price for it each period, not constrained in any way by the price that has been charged for the good in the past, and with full information about current demand and cost conditions. As usual in a model of monopolistic competition, we assume that each supplier understands that his sales depend upon the price charged for his good, according to the demand function

$$y_t(i) = Y_t \left(\frac{p_t(i)}{P_t}\right)^{-\theta}. \tag{1.11}$$

(The form of the demand curve assumed here follows from (1.5); when all purchases are for private consumption, the index of aggregate demand Y_t corresponds simply to the representative household's choice of the index C_t.) Because good i accounts for only an infinitesimal contribution to households' budgets and their utility from consumption, the supplier of an individual good does not believe that his pricing decision can affect the evolution of either the index of aggregate demand Y_t or the price index P_t; thus $p_t(i)$ is chosen taking the latter two quantities as given. Optimization by the supplier of good i then involves setting a price $p_t(i) = \mu S_t(i)$, where $\mu \equiv \theta/(\theta - 1) > 1$ is the seller's *desired markup,* determined by the usual Lerner formula.

It follows that each supplier will wish to charge a relative price satisfying

$$\frac{p_t(i)}{P_t} = \mu s(y_t(i), Y_t; \tilde{\xi}_t). \tag{1.12}$$

It then follows from (1.11) that the relative supply of good i must satisfy

$$\left(\frac{y_t(i)}{Y_t}\right)^{-1/\theta} = \mu s(y_t(i), Y_t; \tilde{\xi}_t).$$

Because s is increasing in its first argument, this equation must have a unique solution for $y_t(i)$ given Y_t. It follows that in equilibrium, the same quantity must be supplied of each good, and that common quantity must equal Y_t. Equilibrium output must then be given by $Y_t = Y^n(\tilde{\xi}_t)$, where the latter function indicates the solution to the equation

$$s(Y_t^n, Y_t^n; \tilde{\xi}_t) = \mu^{-1}. \tag{1.13}$$

Because s is also increasing in its second argument, this equation as well must have a unique solution for each specification of the exogenous shocks $\tilde{\xi}_t$.

We thus find that in the case of fully flexible prices, equilibrium output is completely *independent of monetary policy*. Given this solution for aggregate output as a function of the exogenous shocks, our model of price-level determination then reduces to exactly the model analyzed in chapter 2 (where an exogenous supply of goods was simply assumed). Thus neither our introduction of endogenous supply nor our assumption of monopolistic competition has any necessary consequences for the effects of monetary policy. But they now make it possible for us to consider other assumptions about pricing behavior, and we shall see that in the case of sticky prices our conclusions *are* different.

The solution to equation (1.13) – which we call the *natural rate of output* following Friedman (1968) – continues to be a useful construct even in the case of sticky prices (though it no longer need equal the *equilibrium*[10] level of output at all times). This is because a log-linear approximation to the real marginal cost function (1.9) is given by

$$\hat{s}_t(i) = \omega \hat{y}_t(i) + \sigma^{-1}\hat{Y}_t - (\omega + \sigma^{-1})\hat{Y}_t^n, \tag{1.14}$$

[10]Some authors would say that it *is* the "equilibrium" level of output, but that the level of output actually observed as a result of sticky prices is instead a "disequilibrium" level of output. We shall avoid this terminology in this study; for us, "equilibrium" always refers to the prediction of our model, whether it involves fully flexible prices or not.

where $\omega > 0$ represents the elasticity of s with respect to its first argument, and $\sigma > 0$ is the intertemporal elasticity of substitution of private expenditure, as in chapter 2. (Here we log-linearize around the steady-state equilibrium in the case of flexible prices and $\tilde{\xi}_t = 0$ at all times.[11] Letting $\bar{Y}$ be the constant level of output in this steady state, we define $\hat{Y}_t$ as in chapter 2, and correspondingly define $\hat{y}_t(i) \equiv \log(y_t(i)/\bar{Y})$, $\hat{Y}_t^n \equiv \log(Y_t^n/\bar{Y})$, and $\hat{s}_t(i) \equiv \log(\mu s_t(i))$.) Thus the natural rate of output provides a useful summary of the way in which disturbances shift the real marginal cost function, whether prices are constantly adjusted or not.

For later purposes it is useful to note that in (1.14), the elasticity ω can be decomposed as

$$\omega = \omega_w + \omega_p, \tag{1.15}$$

where $\omega_w > 0$ is the elasticity of the marginal disutility of work with respect to output increases, and $\omega_p > 0$ is the elasticity of the function Ψ defined in (1.8). Thus ω_w indicates the elasticity of real wage demands with respect to the level of output, holding fixed the marginal utility of income, while ω_p indicates the negative of the elasticity of the marginal product of labor with respect to the level of output.[12]

Of course, our result here that monetary policy is *completely* irrelevant to the determination of real activity is rather special. If, for example, we allow for real balance effects, we shall find that monetary policy can affect equilibrium output even under flexible prices, owing to the effects of expected inflation upon equilibrium real balances. If we furthermore allow for endogenous capital accumulation, we shall find that the natural rate of output depends upon the capital stock, and insofar as real balance effects are able to affect equilibrium capital accumulation, they may have a further effect upon equilibrium output under flexible prices through this channel as well. However, these effects are not plausibly very large in quantitative terms, as studies such as that of Cooley and Hansen (1989) have shown. Thus

[11] The element of $\tilde{\xi}_t$ that measures aggregate technology is here taken to be $a_t \equiv \log A_t$ rather than A_t itself. Thus the steady-state value of the technology factor A_t is normalized as one.

[12] In a model with wage and price-setting, ω_w indicates the degree to which higher economic activity increases workers' desired wages given prices, while ω_p indicates the degree to which higher economic activity increases producers' desired prices given wages.

the conclusion from our simple model remains essentially correct.

1.2 Consequences of Prices Fixed in Advance

We now contrast these results to those that we obtain under a simple form of price stickiness. Let us suppose that all prices $p_t(i)$ must be fixed *a period in advance;* that is, when $p_t(i)$ is chosen, the exogenous disturbances (including possible random variation in monetary policy) realized in periods $t-1$ or earlier are known, but not any of the disturbances that are to be realized only in period t. (Whether the stickiness of prices results because the price that applies in period t has to be announced at an earlier time, or simply because price-setters make their decision on the basis of old information, does not matter for our conclusions.) We shall suppose that the supplier of good i is committed to supply whatever quantity buyers may wish to purchase at the predetermined price $p_t(i)$, and hence to purchase whatever quantity of inputs may turn out to be necessary to fill orders.

When the price $p_t(i)$ is chosen, in period $t-1$, the consequences for sales and profits in period t are not yet known with certainty. Hence we assume that the firm seeks to maximize the *present value* of period t profits, given by

$$E_{t-1}[Q_{t-1,t}\Pi_t(i)], \tag{1.16}$$

where $Q_{t-1,t}$ is the stochastic discount factor introduced in chapter 2, and profits from the sale of good i are given by

$$\Pi_t(i) = p_t(i)y_t(i) - w_t(i)h_t(i).$$

Using (1.7) and (1.11), we can write this objective as a function of the choice variable $p_t(i)$, obtaining

$$E_{t-1}\left\{Q_{t-1,t}[Y_tP_t^\theta p_t(i)^{1-\theta} - w_t(i)f^{-1}(Y_tP_t^\theta p_t(i)^{-\theta}/A_t)]\right\}. \tag{1.17}$$

The supplier of good i then chooses $p_t(i)$ on the basis of information available at date $t-1$ so as to maximize this expression, given the expected state-contingent values of the random variables $Q_{t-1,t}$, Y_t, P_t, $w_t(i)$ and A_t.

The expression inside the square brackets in (1.17) is easily seen to be a concave function of $p_t(i)^{-\theta}$ (which we might alternatively choose as the choice variable), so that the entire expression is similarly concave. It follows that expected profits have a unique maximum at the price that satisfies the first-order condition obtained by differentiating (1.17) with respect to $p_t(i)$. This may be written

$$E_{t-1}\left\{Q_{t-1,t}Y_t P_t^{\theta}[p_t(i) - \mu S_t(i)]\right\} = 0. \tag{1.18}$$

This way of writing the first-order condition indicates that the price is set to equal μ times a weighted average of the values of the nominal marginal cost of supplying good i that are expected to occur in different possible states at date t.

Substituting the demand function (1.11) into the real marginal cost function (1.9), one sees that the marginal cost of supplying good i is a decreasing function of the price $p_t(i)$ (given the values of the variables that the producer cannot affect) in each state. Hence (1.18) has a unique solution for the optimal price $p_t(i)$, that is the same for each good i. In equilibrium, each producer sets an identical price, and this common price will be P_t. Using the result that $p_t(i) = P_t$, and substituting the solution for the stochastic discount factor from equation (xx) of chapter 2, we find that (1.18) requires that

$$E_{t-1}\left\{u_c(Y_t;\xi_t)Y_t \ \ [\mu^{-1} - s(Y_t, Y_t;\tilde{\xi}_t)]\right\} = 0. \tag{1.19}$$

This is a restriction that must be satisfied by the joint distribution of Y_t and the exogenous disturbances $\tilde{\xi}_t$, conditional upon information at date $t-1$; note that it involves no nominal variables. It is a weaker version of our result in the case of flexible prices, that $Y_t = Y_t^n$ at all times. Output equal to the natural rate is equivalent to requiring that $s(Y_t, Y_t;\tilde{\xi}_t) = \mu^{-1}$ at all times; instead, (1.19) requires only that this hold "on average" (where the "average" in question does not involve weights exactly equal to the probability of each state's occurrence).

Together with the stipulation that P_t is predetermined, (1.19) represents an *aggregate supply* relation for this model. We can examine its implications, without needing to specify the rest of the model, if we assume that monetary policy is used to achieve an exogenous target path for nominal GDP, $\mathcal{Y}_t = P_t Y_t$. This sort of aggregate demand specification is very commonly assumed in the literature on sticky-price models, usually by stipulating that nominal GDP is proportional to the money supply, and then that monetary policy is specified by an exogenous target path for the money supply. It is not attractive, for our purposes, to assume either a constant velocity of money or monetary targeting. But we may nonetheless examine an equivalent aggregate demand specification, by assuming that policy is specified in terms of a target path for nominal GDP, that is then achieved by adjusting the interest-rate instrument as necessary. In addition to allowing comparisons with familiar literature, this assumption allows us to examine the consequences of alternative aggregate supply specifications without needing to specify the way in which monetary policy affects aggregate spending. Accordingly, we specify aggregate demand in terms of an exogenous process $\{\mathcal{Y}_t\}$ throughout the present chapter. (The interest-rate adjustments required in order to control aggregate spending are then taken up in chapter 4.)

Substituting $Y_t = \mathcal{Y}_t/P_t$ into (1.19), we observe that the equilibrium price level is given by the solution to the equation

$$E_{t-1}\left\{u_c(\mathcal{Y}_t/P_t;\xi_t)\mathcal{Y}_t[\mu^{-1} - s(\mathcal{Y}_t/P_t, \mathcal{Y}_t/P_t;\tilde{\xi}_t)]\right\} = 0. \tag{1.20}$$

This implies that P_t is a function solely of the joint distribution of $\{\mathcal{Y}_t, \tilde{\xi}_t\}$, conditional upon information at date $t-1$, and that this function is homogeneous of degree one in the distribution of values anticipated for $\mathcal{Y}_t$. Given the value of P_t determined by this *ex ante* distribution, the level of output Y_t is then determined by the *ex post* realization of $\mathcal{Y}_t$. The homogeneity property just referred to implies that Y_t depends only upon the level of $\mathcal{Y}_t$ *relative* to the distribution of levels of nominal spending that were regarded as possible at date $t-1$.

This result can be stated more simply if we make use of a log-linear approximation to the solution (1.20). We log-linearize around the steady-state equilibrium in which $\tilde{\xi}_t = 0$ and $\mathcal{Y}_t/\mathcal{Y}_{t-1} = 1$ at all times, and obtain an expression that approximates the exact solution as long as $\tilde{\xi}_t$ and $\mathcal{Y}_t/\mathcal{Y}_{t-1}$ are always sufficiently *close* to these values. Making use of the log-linear approximation to the real marginal cost function (1.14), we find that the equilibrium price level is approximately given by

$$\log P_t = E_{t-1} \log \mathcal{Y}_t - E_{t-1} \log Y_t^n,$$

from which it follows that

$$\log Y_t = E_{t-1} \log Y_t^n + [\log \mathcal{Y}_t - E_{t-1} \log \mathcal{Y}_t].$$

We then observe that the component of output that can be forecasted a period in advance is always equal to the forecast of the natural rate,

$$E_{t-1} \log Y_t = E_{t-1} \log Y_t^n,$$

and hence is independent of monetary policy. The *unexpected* component of output fluctuations, by contrast, is equal to the unexpected component of nominal GDP (or of nominal GDP growth):

$$\log Y_t - E_{t-1} \log Y_t = \log \mathcal{Y}_t - E_{t-1} \log \mathcal{Y}_t.$$

Thus monetary policy affects real activity in this model only insofar as it causes unexpected variation in nominal spending, and the resulting variations in output must themselves be purely unexpected.

1.3 A "New Classical" Phillips Curve

The above model can be generalized by allowing some prices to be flexible, though others are fixed in advance. This allows us to consider the robustness of our previous conclusions to allowing some prices to be flexible, even in the very short run (as we do in fact observe). It also allows us to derive a "Phillips curve" relation between price movements and output movements, of a kind familiar from the "New Classical" literature of the 1970s.

Suppose now that a fraction $0 < \iota < 1$ of the goods prices are fully flexible – which is to say, set each period on the basis of full information about current demand and cost conditions – while the remaining $1 - \iota$ are set a period in advance, as in the previous subsection. The supplier of each flexible-price good will then set its price each period according to (1.12), while the supplier of each sticky-price good will set its price in advance according to (1.18). The marginal costs of supplying different goods i will differ only insofar as the quantities supplied differ, and these in turn will differ only insofar as the prices of the goods differ. It follows that all flexible-price goods will have a common price p_{1t}, and all sticky-price goods will similarly have a common price p_{2t}. We similarly let y_{1t} denote the common equilibrium output of all flexible-price goods, and y_{2t} the output of sticky-price goods.

Taking a log-linear approximation to the two pricing equations, we obtain

$$\log p_{1t} = \log P_t + \omega \hat{y}_{1t} + \sigma^{-1}\hat{Y}_t - (\omega + \sigma^{-1})\hat{Y}_t^n,$$

$$\log p_{2t} = E_{t-1}[\log P_t + \omega \hat{y}_{2t} + \sigma^{-1}\hat{Y}_t - (\omega + \sigma^{-1})\hat{Y}_t^n],$$

defining $\hat{y}_{it} \equiv \log(y_{it}/\bar{Y})$ for $i = 1, 2$, and again using (1.14) to approximate the real marginal cost function. These approximations apply as long as the fluctuations in p_{it}/P_t, Y_t, and Y_t^n around the values (1, $\bar{Y}$, and $\bar{Y}$ respectively) near which we log-linearize are small enough. Note that up to this log-linear approximation, (log) predetermined prices are set at a value equal to a constant markup over the conditional expectation, at the time that the price is set, of (log) marginal cost. Substituting the demand function (1.11) to eliminate the y_{it} variables, we obtain more simply

$$\log p_{1t} = \log P_t + \zeta(\hat{Y}_t - \hat{Y}_t^n), \tag{1.21}$$

$$\log p_{2t} = E_{t-1}\left[\log P_t + \zeta(\hat{Y}_t - \hat{Y}_t^n)\right],$$

where

$$\zeta \equiv \frac{\omega + \sigma^{-1}}{1 + \omega\theta} > 0. \tag{1.22}$$

These relations imply that up to our log-linear approximation,

$$\log p_{2t} = E_{t-1} \log p_{1t}.$$

But a corresponding log-linear approximation to the aggregate price index (1.3) yields

$$\log P_t = \iota \log p_{1t} + (1 - \iota) \log p_{2t}.$$

It follows that

$$\pi_t - E_{t-1}\pi_t = \log P_t - E_{t-1}\log P_t = \frac{\iota}{1-\iota}(\log p_{1t} - \log P_t).$$

Then using (1.21), we obtain the "New Classical" aggregate supply relation

$$\pi_t = \kappa(\hat{Y}_t - \hat{Y}_t^n) + E_{t-1}\pi_t, \tag{1.23}$$

where

$$\kappa \equiv \frac{\iota}{1-\iota}\zeta.$$

Note that this relation has the form of an "expectations-augmented Phillips curve" of the kind hypothesized by Phelps (1967) and Friedman (1968), in which the specific inflation expectation that is relevant is the expectation at the time at which current predetermined prices were fixed regarding inflation over the interval until the present. This particular form of dependence of aggregate supply upon inflation expectations was stressed in the "New Classical" literature of the 1970s (*e.g.*, Sargent and Wallace, 1975).

This aggregate supply relation implies once again that

$$E_{t-1}\log Y_t = E_{t-1}\log Y_t^n,$$

so that the component of output that can be forecasted a period in advance is still independent of monetary policy. However, unexpected variations in aggregate demand now give rise to inflation variation as well, rather than affecting only output. Again taking the stochastic process for aggregate nominal spending $\mathcal{Y}_t$ as given, (1.23) implies that aggregate output will equal

$$\log Y_t = \log Y_t^n + (1+\kappa)^{-1}(\log \mathcal{Y}_t - E_{t-1}\log \mathcal{Y}_t), \tag{1.24}$$

so that the aggregate price level will equal

$$\log P_t = (E_{t-1}\log \mathcal{Y}_t - \log Y_t^n) + \frac{\kappa}{1+\kappa}(\log \mathcal{Y}_t - E_{t-1}\log \mathcal{Y}_t).$$

How does the degree of price flexibility affect the impact of fluctuations in nominal spending upon real activity? Not surprisingly, a larger number of flexible prices (larger ι)

implies a higher value of κ (steeper short-run Phillips curve), and hence a smaller value of the elasticity of output with respect to unexpected variations in nominal spending in (1.24). In the limit as ι approaches one (all prices flexible), κ becomes unboundedly large, and the output effects of variations in nominal spending approach zero, as found earlier. More interesting is the question whether intermediate values of ι result in effects more like those of the fully-flexible limit or the fully-sticky limit. This turns out to depend upon the size of ζ.

Figure 3.1 plots the elasticity $1/(1+\kappa)$ occurring in (1.24) as a function of ι, for each of several values of ζ. In each case, the function is monotonically decreasing, as one would expect, and takes the same values in the two limiting cases of $\iota = 0$ and $\iota = 1$. However, when $\zeta > 1$, the function is convex (and more so the larger is ζ), while when $\zeta < 1$, the function is concave (and more so the smaller is ζ). Thus if ζ is large, that even *some* goods have flexible prices is enough to prevent variations in nominal spending from having much effect on aggregate output. In such a case, the flexible-price model would provide a reasonable approximation to the evolution of aggregate prices and quantities. But if ζ is small, that even *some* prices are sticky is enough to result in a substantial effect of variations in nominal spending upon output. Indeed, in such a case, the overall price index responds very little to unexpected variations in nominal spending, and the simple model in which all prices were assumed to be predetermined is a reasonably good guide to aggregate outcomes.

There is a simple intuition for the significance of the parameter ζ: it describes the degree of *strategic complementarity* between the price-setting decisions of the suppliers of different goods. Let us consider the following simple game among price-setters: aggregate nominal spending $\mathcal{Y}_t$ is given, and the suppliers of individual goods i are to simultaneously choose the prices $p_t(i)$ for their goods. In what way does a given supplier's optimal price depend upon the level of the prices chosen by the other suppliers? We shall say that the pricing decisions are *strategic complements* if an increase in the prices charged for other goods *increases* the price that it is optimal to charge for one's own good. Correspondingly, one may speak of *strategic substitutes* if an increase in the other prices makes it optimal for one to *reduce* the

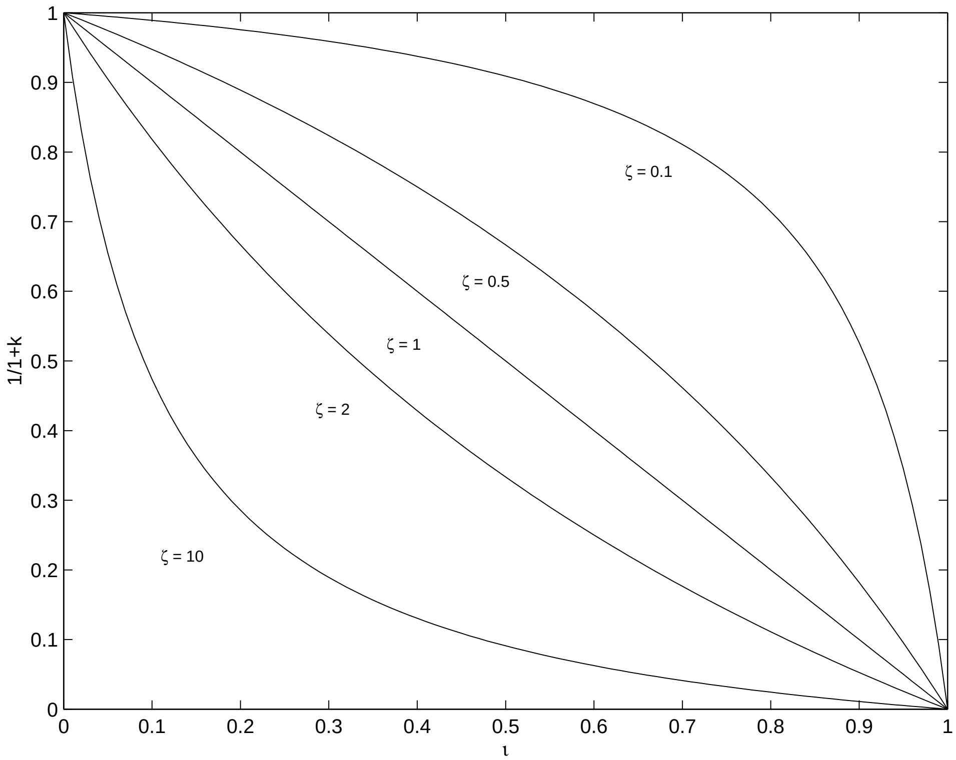

Figure 3.1: Real effects of variation in nominal spending, for alternative degrees of strategic complementarity in price-setting.

price of one's own good.[13]

To determine the answer to this question, we must first consider what one may call the *notional* short-run aggregate supply (SRAS) curve,

$$\frac{p_t^*(i)}{P_t} = \Omega(Y_t; \tilde{\xi}_t),$$

which indicates how each supplier's *desired* relative price varies with the level of aggregate activity. Here the "desired" price $p_t^*(i)$ is the price that the supplier of good i would prefer if the price of good i could be freely set on the basis of demand and cost conditions at date t, independent of any prices charged or announced in the past, and with no consequences for the prices that may be charged at any later dates as well. This is a purely notional supply

[13]See Bulow, Geanakoplos and Klemperer (1985) for the general notion of strategic complementarity, and Haltiwanger and Waldman (1989) for discussion of it in the context of adjustment to aggregate shocks more generally.

curve in that it need not indicate how the prices of any goods are *actually* set, in an economy where prices are sticky; nonetheless, we shall see that the concept is a useful one, even when no prices are perfectly flexible. In the context of our present framework, the notional SRAS curve is given by (1.12). When log-linearized, it takes the form (1.21); thus the coefficient ζ is just the elasticity of the function Ω with respect to Y. Ball and Romer (1990) refer to an economy in which this elasticity is small as being characterized by *real rigidities.*[14]

Best-response curves for individual price-setters can then be derived by substituting the identity $Y_t = \mathcal{Y}_t/P_t$ into the notional SRAS curve, to obtain

$$p_t^*(i) = P_t \Omega(\mathcal{Y}_t/P_t; \tilde{\xi}_t).$$

The degree of strategic complementarity is then indicated by the partial derivative of the right-hand side with respect to P_t. We thus find that strategic complementarity exists[15] (a positive partial derivative) if and only if $\zeta < 1$, while the pricing decisions of the separate firms are strategic substitutes if $\zeta > 1$.

Thus it is the existence or not of strategic complementarity that determines whether the fraction of the suppliers with sticky prices exert a disproportionate effect upon the degree of adjustment of the aggregate price index. The intuition for this is simple: if prices are strategic complements, then the fraction of prices that do not adjust in response to a disturbance to nominal spending lead even the flexible-price suppliers to adjust their prices by less than they otherwise would. If the strategic complementarity is strong enough, there will be little aggregate price adjustment (and so a large output effect) even if the fraction

[14]The analogy is with "nominal rigidity", a situation in which the nominal price charged by a given supplier (rather than its relative price) is insensitive to the quantity that it must supply. Because the nominal SRAS as defined here is independent of any specification of the speed at which prices may be adjusted, the degree to which an economy is characterized by "real rigidities" depends solely upon real factors, the structure of production costs and of demand. The term, though widely used, is somewhat unfortunate, as it suggests that there are costs of adjusting real quantities, whereas in fact it means a situation in which (notional) supply is very *elastic*. We shall thus mainly speak here of "elastic supply" or "strategic complementarity in price-setting" instead.

[15]Note that the question of whether different firms' pricing decisions are strategic complements is only defined relative to a particular pricing game, in which aggregate nominal spending is taken as given, independent of the prices chosen by any of the firms. This is a useful question to consider, however, when analyzing the dynamics of prices and output under an arbitrarily specified stochastic process for nominal spending, as we shall repeatedly do in this chapter.

of flexible-price suppliers is large (though less than one). On the other hand, if prices are strategic substitutes, then the aggregate price level may adjust nearly in proportion to the unexpected change in nominal spending, even when many prices are sticky. For in this case, the fact that some prices do not adjust causes the flexible-price suppliers to compensate for this by adjusting their prices even *more* than they would in a flexible-price equilibrium. In the case of a high degree of strategic substitutability (large ζ), this can result in the aggregate price index adjusting nearly in proportion to the change in nominal spending; but it will not adjust by quite that amount, else there would be no change in real aggregate demand, and so no desire on the part of flexible-price suppliers to change their relative prices.

The possibility of strategic substitutability explains why some authors, such Ohanian *et al.* (1996) and Christiano and Eichenbaum (1996), find in models with sticky prices for only some goods that the partial price stickiness has little effect upon the aggregate effects of monetary shocks. These papers assume parameter values that imply an elasticity ζ greater than one, the case of strategic substitutes. At the same time, the "New Keynesian" literature of the 1980's routinely assumed an elasticity $\zeta < 1$, and hence found that pricing decisions should be strategic complements. This result favored the conclusion that it was plausible to suppose that nominal rigidities mattered a great deal for the character of short-run responses to shocks. It is thus worth considering in further detail possible determinants of the degree of strategic complementarity in pricing, and the plausible size of the elasticity ζ.

1.4 Sources of Strategic Complementarity

In the simple model introduced above, the elasticity of the notional SRAS curve is given by (1.22). Most early "New Keynesian" literature[16] assumes linear utility of consumption. This corresponds in our notation to the limiting case $\sigma^{-1} = 0$, in which case (1.22) reduces to

$$\zeta = \frac{\omega}{1 + \omega\theta}.$$

Given that we must have $\theta > 1$, this expression implies that $\zeta < 1$, so that the pricing decisions of different suppliers are necessarily strategic complements – the conclusion relied

[16]See, for example, the presentation in Blanchard and Fischer (1989, sec. 8.1),

upon in that literature. However, this conclusion is less obviously true if, more realistically, we assume a finite value for the intertemporal elasticity of substitution σ.

Indeed, many of the early authors in the 1990's seeking to incorporate sticky prices into otherwise standard real business cycle models adopted numerical calibrations (derived from the RBC literature) that implied strategic substitutability. One reason for this was the widespread assumption, unlike what we have assumed above, of common economy-wide factor markets, so that the marginal cost of supply would be equal for all goods i at any point in time.[17] In this case the marginal cost of supplying any good is given by

$$S_t = \frac{w_t}{A_t f'(h_t)},$$

where w_t is the wage paid by all producers for the homogeneous labor input, and h_t is the common labor-capital ratio chosen by all producers. Capital is reallocated among firms so as to allow all firms to use the same efficient labor-capital ratio, even if they produce different quantities of their respective goods; this common labor-capital ratio will be given by $h_t = f^{-1}(X_t/A_t)$, where the common output-capital ratio X_t satisfies

$$X_t = \int_0^1 y_t(i)di.$$

Finally, the economy-wide wage w_t satisfies (1.6), where the representative household's labor supply must equal h_t. Marginal supply cost is therefore given by

$$S_t(i) = P_t s(X_t, Y_t; \tilde{\xi}_t),$$

independent of the quantity produced of the individual good i, where s is again the function defined in (1.9).

Using this alternative expression for marginal cost in the derivation of our pricing equations, we again obtain equations such as (1.21), but in which we now define

$$\zeta \equiv \omega + \sigma^{-1} > 0.$$

[17] The consequences of this issue for the degree of strategic complementarity in pricing are stressed by Kimball (1995).

(Note that up to a log-linear approximation, percentage fluctuations in X_t equal percentage fluctuations in Y_t.) In the real business cycle literature, it is standard to assume that $\sigma = 1$. (See, *e.g.*, Cooley and Prescott, 1995.) Under this assumption, $\zeta > 1$, and pricing decisions are strategic substitutes, regardless of the (necessarily positive) value assigned to ω.

On the other hand, even if one were to assume linear utility from consumption (the most extreme possible value for σ), one might easily conclude that pricing decisions should be strategic substitutes, under the assumption of common factor markets. For example, the baseline calibration of Chari *et al.* (2000) implies that $\omega = 1.25$, while $\sigma = 1$. As they assume common factor markets, their parameterization implies a value of $\zeta = 2.25$, and thus a considerable degree of strategic substitutability. But even if they were to assume linear utility of consumption ($\sigma^{-1} = 0$), despite the implausibility of such an extreme value, the assumed degree of diminishing returns to labor in the production function and the assumed degree of increasing marginal disutility of work (which together imply their value of ω) would still imply $\zeta = 1.25$, so that there would still be strategic substitutability. On the other hand, even with their assumed values for ω and σ, if these authors were to assume specific factor markets (as in our analysis above), then (given that they also assign the value $\theta = 10$) they would have found that $\zeta = .17$, using (1.22). This would imply a great degree of strategic complementarity. Thus it is the assumption of common factor markets that makes the most crucial difference for the finding of these and other authors that pricing decisions are strategic substitutes.

The assumption of common factor markets is thus far from innocuous. It is also far from realistic. Of course, it makes sense that high wages in one part of the economy eventually raise wages in the rest, as workers migrate from low-wage to high-wage labor markets. Similarly, it makes sense that high returns to capital in one part of the economy eventually raise the rental rate for capital services throughout the economy, as capital is shifted to higher-return uses (if only through the allocation of new investment). A failure to allow for these factor-price equalization mechanisms makes our model of specific factor markets unrealistic as a model of the long-run effects upon sectoral marginal costs of a *permanent* failure of prices in one sector to adjust.

Nonetheless, the opposite extreme assumption — that all factor prices are *instantaneously* equalized across the suppliers of different goods — is also unrealistic. In the short run, it is not easy for workers to migrate to regions, specialties, or even firms that happen to have temporarily higher labor demand; and it is even less easy for capital goods, once installed, to be reassigned to firms with a temporarily high rate of utilization of their capital stock. This "quasi-fixed" character of factor inputs allows equilibrium factor prices to vary across suppliers for some period following a shock that affects them asymmetrically; and it is mainly these short-run dynamics of factor prices that matter for determining the short-run dynamics of price adjustment in response to shocks, which are what matter for comparing the real allocation of resources under alternative monetary policies.[18]

Factor specificity need not be an all-or-nothing assumption. For example, one might assume (as in Sbordone, 1998) that all suppliers hire the same kind of labor inputs in a single economy-wide labor market, but that each firm has a production function of the form $y_t(i) = A_t f(h_t(i))$, where f is strictly concave. This amounts to assuming that each firm's allocation of capital goods remains fixed, rather than letting capital be reallocated to equalize its rental rate. In this case, real marginal costs are given by

$$\hat{s}_t(i) = \omega_p \hat{y}_t(i) + (\omega_w + \sigma^{-1})\hat{Y}_t - (\omega + \sigma^{-1})\hat{Y}_t^n,$$

where $\omega_p, \omega_w > 0$ are the two components of ω introduced in (1.15). It then follows that

$$\zeta = \frac{\omega + \sigma^{-1}}{1 + \omega_p \theta} > 0. \qquad (1.25)$$

This value of ζ is higher than that implied by (1.22), so that there is less strategic complementarity than in the case of specific labor markets as well (given a finite Frisch elasticity of labor supply, so that $\omega_w > 0$). But it remains a lower value than is obtained under the assumption of common factor markets, *i.e.*, instantaneous equalization of the rental price of

[18] In chapter 4, we show how endogenous reallocation of capital across sectors can be added to the model, so that the rental price of capital must be equalized across sectors in the long run. We nonetheless find that in the presence of adjustment costs for investment, the short-run dynamics of marginal supply costs are not too different from those predicted by our simple model with a fixed allocation of capital to firms.

capital services. For example, Chari *et al.* (2000) assume a Cobb-Douglas production function with a labor share of .67, which implies $\omega_p = .5$. Thus under their baseline calibration, if they were to assume a common labor market but specific capital inputs, they would have obtained $\zeta = .38$, and so a substantial degree of strategic complementarity in price-setting.

There are other reasons as well to believe that the simple model with common factor markets set out above underestimates the likely degree of strategic complementarity in price-setting. One that has received some attention in the literature is the possibility that preferences over differentiated goods need not be of the constant-elasticity (Dixit-Stiglitz) form (1.2). The result is that the elasticity of demand, and hence the desired markup of price over marginal cost, need not be independent of the relative quantities produced of different goods. In particular, Kimball (1995) shows that if the elasticity of demand is lower for the products of suppliers who sell more (because they have relatively low prices), this will increase strategic complementarity, and hence the real effects of variations in nominal spending. For in the event of a decline in nominal spending, price cuts by flexible-price suppliers will lead to an increase in their relative sales (though a reduction in the absolute quantity sold), because the sticky-price suppliers fail to cut their prices. If this increase in relative sales leads to less elastic demand for their products, their desired markups will rise, mitigating the effect upon their desired relative price of the decline in their sales (and hence reduction in their marginal supply cost); this will cause them to cut their prices less, so that the decline in their output is greater.

Following Kimball, let us generalize (1.2) by assuming that the consumption aggregate C_t is implicitly defined by a relation of the form

$$\int_0^1 \psi(c_t(i)/C_t)di = 1, \tag{1.26}$$

where $\psi(x)$ is an increasing, strictly concave function satisfying $\psi(1) = 1$. (Note that this reduces to (1.2) if $\psi(x) = x^{\theta-1/\theta}$.) The demand curve for good i is then implicitly defined by

$$\psi'\left(\frac{y_t(i)}{Y_t}\right) = \psi'(1)\frac{p_t(i)}{P_t}, \tag{1.27}$$

where the price index P_t for aggregate (1.26) is implicitly defined by

$$\int_0^1 \frac{p_t(i)}{P_t}\psi'^{-1}\left(\psi'(1)\frac{p_t(i)}{P_t}\right) di = 1. \tag{1.28}$$

The elasticity of the demand curve (1.27) faced by supplier i is then equal to $\theta(y_t(i)/Y_t)$, where

$$\theta(x) \equiv -\frac{\psi'(x)}{x\psi''(x)}. \tag{1.29}$$

It follows that the desired markup of price over marginal cost for a flexible-price supplier is equal to $\mu(y_t(i)/Y_t)$, where

$$\mu(x) \equiv \frac{\theta(x)}{\theta(x)-1}. \tag{1.30}$$

Note that in general, neither θ nor μ is a constant as assumed before; but we continue to assume a function $\psi(x)$ such that $\theta(x) > 1$ for all x in a neighborhood of 1, so that $\mu(x) > 1$ is well-defined, at least in that neighborhood.

The desired relative price of a flexible-price supplier is then given by

$$\frac{p_t(i)}{P_t} = \mu(y_t(i)/Y_t)s(y_t(i), Y_t; \tilde{\xi}_t), \tag{1.31}$$

generalizing (1.12). Log-linearization of this relation in turn yields

$$\log p_t(i) = \log P_t + \epsilon_\mu(\hat{y}_t(i) - \hat{Y}_t) + s_y(\hat{y}_t(i) - \hat{Y}_t^n) + s_Y(\hat{Y}_t - \hat{Y}_t^n),$$

where ϵ_μ is the elasticity of $\mu(x)$ at the value $x = 1$, and s_y and s_Y are the elasticities of the real marginal cost function with respect to its first two arguments respectively. Using a log-linear approximation to (1.27),

$$\hat{y}_t(i) = \hat{Y}_t - \theta(\log p_t(i) - \log P_t),$$

to eliminate $\hat{y}_t(i)$, and then solving for $\log p_t(i)$, we obtain a log-linear notional SRAS curve, and observe that now

$$\zeta = \frac{s_y + s_Y}{1 + \theta(\epsilon_\mu + s_y)}. \tag{1.32}$$

(Here the coefficient θ refers to $\theta(1)$.) For given values of the other parameters, we observe that a positive ϵ_μ — which means $\theta(x)$ decreasing in x, as discussed above — lowers the value of ζ, thus increasing the degree of strategic complementarity.

In principle, as Kimball shows, ϵ_μ could be an arbitrarily large quantity, and so this factor could result in ζ being arbitrarily small, regardless of the size of the coefficients s_y, s_Y and θ. (Note that $\zeta \to 0$ as $\epsilon_\mu \to \infty$.) Bergin and Feenstra (2000), however, argue that a plausible calibration would assign a value of approximately $\theta\epsilon_\mu = 1$, so that ϵ_μ is positive but not extremely high. If the desired markup depends upon relative demand for different goods as above, one can show that the desired price of supplier i satisfies

$$\log p_t(i) = \frac{1}{1+\theta\epsilon_\mu}\log S_t(i) + \frac{\theta\epsilon_\mu}{1+\theta\epsilon_\mu}\log P_t,$$

regardless of the determinants of nominal marginal cost $S_t(i)$. Bergin and Feenstra note that the literature on exchange-rate pass-through typically finds that a devaluation raises the domestic-currency price of imported goods by only about 0.5 of the percentage of the devaluation. If one interprets this is as measuring the elasticity of $p_t(i)$ with respect to changes in $S_t(i)$ (on the assumption that the devaluation does not affect the foreign-currency marginal cost of supplying the imported goods, or the domestic-currency prices of most goods in the domestic price index), then one may conclude that $\theta\epsilon_\mu$ equals approximately 1.[19]

In the case of homogeneous factor markets ($s_y = 0$, $s_Y = \omega + \sigma^{-1}$), assumed by Bergin and Feenstra, (1.32) reduces to

$$\zeta = \frac{\omega + \sigma^{-1}}{1+\theta\epsilon_\mu}.$$

In this case, their suggested value of ϵ_μ reduces ζ by a factor of 2 relative to what one would obtain under the assumption of Dixit-Stiglitz preferences. This is not enough to radically change one's views about the real effects of monetary policy (as they conclude), but it is nonetheless a non-trivial correction. However, in the case of specific factor markets ($s_y = \omega$, $s_Y = \sigma^{-1}$), the correction matters much less. In this case, (1.32) reduces to

$$\zeta = \frac{\omega + \sigma^{-1}}{1+\theta(\epsilon_\mu + \omega)}.$$

Here setting $\theta\epsilon_\mu$ equal to 1 instead of zero does not make such a great difference, since the term $\theta\omega$ in the denominator is now likely to be much greater than one. For example, in the

[19]Bergin and Feenstra also note that this size of markup elasticity would be implied by the translog specification of preferences that has been popular in econometric studies of demand.

case of the Chari *et al.* values for ω, σ and θ discussed above, allowing for the variation in desired markups only reduces the predicted ζ from 0.17 to 0.16.

Another reason for greater strategic complementarity than is indicated by the baseline analysis above is the economy's input-output structure, stressed by Basu (1995). Our above analysis of marginal supply costs assumes that labor is the only variable factor of production, ignoring the role of intermediate inputs. While this is a familiar assumption in equilibrium business-cycle models (it is, for example, routine in the real business cycle literature), it is far from being literally correct. The production function for "output" as a function of capital and labor inputs that one typically encounters in such models must be interpreted as a functional relation between the *value added* in production (GDP) and *primary* factor inputs, rather than a relation between gross output and all factors of production (including those that are themselves produced). Under certain conditions, and for some purposes, it suffices to model the economy as if this "value-added production function" were the actual production function of individual producers — for example, for purposes of predicting the evolution of real GDP in the context of a flexible-price, perfectly competitive model, as in RBC theory. But as Rotemberg and Woodford (1995) note, it is important not to conflate gross-output and value-added production functions in the case of a model with imperfect competition and prices that do not co-move perfectly with marginal cost.

Rotemberg and Woodford (1995) propose a simple gross-output production function of the form[20]

$$y_t(i) = \min\left[\frac{A_t f(h_t(i))}{1 - s_m}, \frac{m_t(i)}{s_m}\right], \tag{1.33}$$

where $A_t f(h_t(i))$ is the value-added production function (as above), $m_t(i)$ is the quantity of materials inputs used by firm i, and $0 \leq s_m < 1$ is a parameter of the production technology (that can be identified, for purposes of calibration, with the share of materials costs in the

[20] Basu (1995) and Bergin and Feenstra (2000) instead assume that gross output is a Cobb-Douglas function of labor and materials inputs. The simple analysis here, however, which has the advantage of nesting our previous specification as a limiting case, allows us to reach quite similar conclusions about the consequences of the input-output structure for the degree of strategic complementarity in pricing in a highly transparent way. There seems in any event to be no argument other than tractability offered for the Cobb-Douglas specification used in these other papers.

value of gross output). The argument $m_t(i)$ refers to the number of units of the aggregate defined in (1.2) — or more generally in (1.26) — that are purchased for use in producing good i; it is thus a composite of all of the goods produced in the economy, which are all assumed equally to be both final goods and intermediate inputs. If firms allocate their intermediate input purchases across goods j in a cost-minimizing way, the quantity purchased of good j for use in producing good i will equal $m_t(i)(p_t(j)/P_t)^{-\theta}$ (or the corresponding generalization in the case of a non-Dixit-Stiglitz aggregator), and the cost per unit of materials inputs will be the same price index P_t as for consumption purposes. Total demand for firm i's output, given by the sum of final demand and intermediate-input demand from other producers, will still be given by (1.5) — or more generally by (1.27) — where now aggregate demand Y_t is equal to

$$Y_t = C_t + \int_0^1 m_t(i)di. \tag{1.34}$$

It follows from the production function (1.33) that real marginal cost for firm i will equal

$$s_t(i) = (1 - s_m)s_t^{VA}(i) + s_m, \tag{1.35}$$

where $s_t^{VA}(i)$ is the real marginal cost of an additional unit of theoretical value-added (the function of primary input use given by $A_t f(h_t(i))$), and we have used the fact that the price of the materials aggregate is P_t. The marginal cost of supplying value-added is given by the same real marginal cost function derived earlier; for example, in the case of specific factor markets, it is given by

$$s_t^{VA}(i) = s((1 - s_m)y_t(i), Y_t - s_m \int_0^1 y_t(i)di; \tilde{\xi}_t),$$

where $s(y, Y; \tilde{\xi})$ is again defined as in (1.9). The existence of a symmetric steady state then requires that $s_m < \mu^{-1}$, where the latter quantity represents the steady-state level of real marginal cost, given firms' desired markup of $\mu > 1$.[21] Such a steady state involves a constant level of final-goods demand (or real value added) $\bar{Y}$ implicitly defined by[22]

$$s(\bar{Y}, \bar{Y}; 0) = \frac{1 - \mu s_m}{\mu(1 - s_m)}. \tag{1.36}$$

[21] In the case that the desired markup is variable, we continue to use the coefficient μ to refer to the steady-state value, $\mu(1)$.

Regardless of the factor market structure, we find that a log-linear approximation to the real marginal cost function for value added is given by

$$\hat{s}_t^{VA}(i) = s_y \hat{y}_t(i) + s_Y \hat{Y}_t,$$

where the elasticities s_y and s_Y are the same as earlier. Log-linearization of (1.35) then implies that

$$\hat{s}_t(i) = (1 - \mu s_m)(s_y \hat{y}_t(i) + s_Y \hat{Y}_t).$$

It is the appearance of the multiplicative factor $1 - \mu s_m < 1$ here that explains how the economy's input-output structure gives rise to "real rigidity" of the sort that increases the strategic complementarity among pricing decisions. Substitution of the demand curve for $\hat{y}_t(i)$ as before then allows us to solve once again for firm i's desired relative price; we now find that

$$\zeta = \frac{(1 - \mu s_m)(s_y + s_Y)}{1 + \theta[\epsilon_\mu + (1 - \mu s_m) s_y]}, \tag{1.37}$$

generalizing (1.32).

We observe that allowing for a positive materials share lowers ζ, for given values of the other parameters; and once again, this effect could in principle result in an arbitrarily small value for ζ, regardless of the values of the other parameters. (Note that $\zeta \to 0$ as $s_m \to \mu^{-1} < 1$.) However, the share of materials costs in total costs for U.S. manufacturing sectors is typically on the order of 50 or 60 percent; this suggests that a reasonable calibration would be on the order of $\mu s_m = 0.6$.[23] In the case of homogeneous factor markets ($s_y = 0$),

[22]Note that the right-hand side of (1.36) is the the reciprocal of the steady-state "value-added markup" or inefficiency wedge resulting from market power that Rotemberg and Woodford (1995) contrast with the steady-state "gross-output markup" or firm-level markup of price over marginal cost μ. Distinguishing between the two is essential in calibrating an imperfectly competitive model that abstracts from the existence of intermediate inputs.

[23]This would correspond to a materials share of 54 percent, if we assume $\theta = 10$ as do Chari *et al.* Note that Basu (1995) suggests that a share parameter as high as $s_m = 0.9$ could be reasonable, on the ground that the share of intermediate inputs in marginal cost could be substantially higher than their share in average cost. Under such an assumption this correction would be much more important than under the calibration suggested here; for example, the assumption $s_m = 0.9$ combined with an assumption $\theta = 10$ would imply that $\zeta = 0$. But such a calibration is not easy to interpret. If, for example, one proposes that a large fraction of primary input purchases represent fixed or "overhead" costs, this would imply substantial increasing returns (average cost much higher than marginal cost), which would in turn not be consistent with equilibrium in the absence of an implausibly high degree of market power.

Table 3.1: The value of ζ under alternative assumptions.

		$\sigma^{-1} = 1,\ \omega = 1.25$		$\sigma^{-1} = .16,\ \omega = .47$	
$\theta\epsilon_\mu$	μs_m	homo. factor	spec. factor	homo. factor	spec. factor
0	0	2.25	0.17	0.63	0.11
1	0	1.13	0.16	0.32	0.09
0	.6	0.90	0.15	0.25	0.09
1	.6	0.45	0.13	0.13	0.06

this implies that ζ is reduced by a factor of 2.5 (*i.e.*, it is multiplied by 0.4). This is again a significant reduction, and if this effect is combined with the degree of variation in the desired markup argued for above, the value of ζ is only one-fifth the size indicated by the baseline calculation of Chari *et al.* — 0.45 instead of 2.25, which is to say well below 1 (implying strategic complementarity) rather than being well above 1 (implying strategic substitutability). On the other hand, in the case of specific factor markets, this correction matters much less, unless s_m is assumed to be much closer than μ^{-1} than seems realistic. Using the calibration discussed above, assuming $\mu s_m = 0.6$ rather than zero reduces ζ from 0.17 to 0.15 (in the constant-markup case), or from 0.16 to 0.13 (in the variable-markup case). These reductions make no dramatic difference in one's conclusions about the degree of strategic complementarity.

Our conclusions about the effects of these several corrections, in the case of the calibrations just discussed, are summarized in Table 3.1. The left side of the table gives the value of ζ under the assumption that $\omega = 1.25$, $\sigma = 1$, and $\theta = 10$, as assumed by Chari *et al.*, under eight different cases, representing three binary choices. These are the assumption of homogeneous factor markets ($s_y = 0, s_Y = \omega + \sigma^{-1}$) versus specific factor markets ($s_y = \omega$, $s_Y = \sigma^{-1}$); the assumption of a constant desired markup ($\epsilon_\mu = 0$) versus a realistic degree of markup variation ($\theta\epsilon_m u = 1$); and the assumption of no intermediate inputs ($s_m = 0$) versus a realistic intermediate input share in costs ($\mu s_m = 0.6$). In each case, the simple assumption made in the baseline case of Chari *et al.* is the one least favorable to strategic complementarity. Changing any of these assumptions individually reduces ζ substantially relative to their baseline case. However, it is the allowance for factor specificity that matters most, if one accepts that this is the more realistic assumption. For while each of the other two corrections makes a significant difference in the case of homogeneous factor markets, and together they make an even larger difference (as stressed by Bergin and Feenstra), in the case of specific factor markets even their combined effect is not at all dramatic. Furthermore, factor specificity alone reduces the value of ζ much more than the other two factors combined do, if the importance of these latter factors is calibrated in what seems a reasonable way.

The right side of the table offers a similar comparison, when instead lower values are assumed for both σ^{-1} and ω. The values suggested here are those obtained by Rotemberg and Woodford (1997) when a slightly more complicated version of this pricing model is fit to U.S. time series, as discussed further in chapter 4. The higher value of σ (implying much greater interest-sensitivity of private expenditure) is needed in order to account for the observed size of the effects of an identified monetary policy shock on real aggregate demand; the lower value of ω (implying much more elastic labor supply) is needed in order to account for the observed modest declines in real wages that accompany such a large decline in output. Under these alternative assumptions regarding preferences, $\zeta < 1$, implying a modest degree of strategic complementarity, even in the case of homogeneous factor markets, Dixit-Stiglitz preferences, and no intermediate inputs. But once again the degree of real rigidity is increased by modifying any of these last three assumptions. It is interesting to observe in this case that even under the assumption of homogenous factor markets, we obtain a value of $\zeta = .13$, if we make realistic assumptions about the other two sources of real rigidity. Thus a value of ζ in the range between 0.10 and 0.15 does not require implausible assumptions. This is a value that implies substantial strategic complementarity, and as we shall see, enough to explain roughly the observed degree of sluggishness of aggregate price adjustment in response to variations in nominal expenditure, given the observed frequency of price adjustment in economies like that of the U.S.

2 Inflation Dynamics with Staggered Price-Setting

An unsatisfactory feature of the "New Classical" aggregate supply relation derived above is its implication that *only* unanticipated fluctuations in nominal spending have any effect upon real activity, and that equilibrium fluctuations in the output gap must be completely *unforecastable*. These strong predictions were the occasion of a great deal of discussion and criticism during the 1970s and early 1980s (see, *e.g.*, Sheffrin, 1996, chap. 2). They imply that only the *immediate* effects of a monetary policy shock upon nominal expenditure should have any consequences for real activity; delayed effects (effects on nominal spending after the "period" in which the shock occurs) should not affect output at all, only the price level. And such real effects of a monetary policy shock as occur must be purely transitory, *i.e.*, must last no longer than the "period" for which the sticky prices are fixed in advance.

These predictions are quite inconsistent with the effects identified in the "structural VAR" literature. As an example, Figure 3.2 plots the impulse response of nominal GDP to an identified monetary policy shock, according to the structural VAR model of Christiano *et al.* (2001).[24] There is practically no measurable effect of an unexpected interest-rate reduction in quarter zero upon nominal GDP until the second quarter following the monetary policy shock, though there is a strong increase in nominal GDP at that time.[25] But this means that according to the "New Classical" aggregate-supply relation, monetary shocks should have *no* effect upon real activity at all, unless the "period" for which sticky prices are fixed in advance is longer than six months.

Figure 3.3 next shows the estimated impulse response of real GDP to the same shock in

[24] This particular impulse response is not reported in their paper, though the point estimates here are implied by the inflation and real GDP responses that are reported there. I thank Charlie Evans for supplying the data plotted in this and subsequent figures. The impulse responses implied by this particular VAR study are representative of those found in many others; see Christiano *et al.* (1999) for a review.

[25] The fact that there is zero effect upon nominal GDP in quarter 0 is an artifact of the authors' identification scheme, which assumes that any contemporaneous correlation between interest-rate innovations and innovations in either real GDP or inflation is due to feedback from the latter variables to the current interest-rate operating target, as under a Taylor rule. However, the estimated effect in quarter 1 is in no way constrained by the identification scheme; the fact that neither output nor inflation is estimated to be significantly affected in quarter 1 provides some support for the assumption relied upon in the identification scheme.

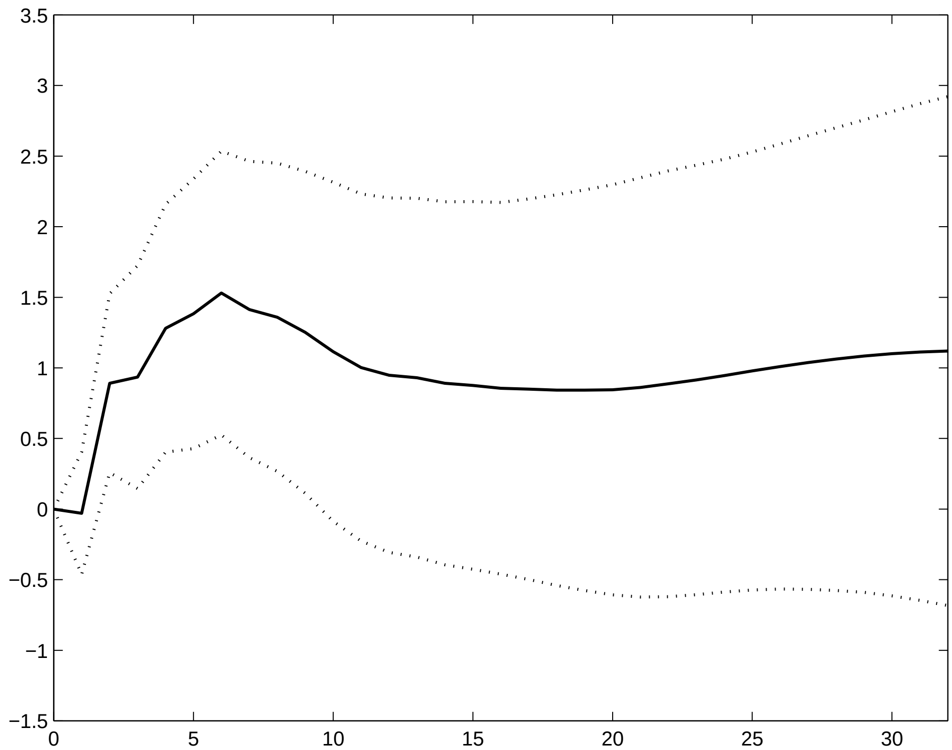

Figure 3.2: Impulse response of nominal GDP to an unanticipated interest-rate reduction in quarter 0. Periods on horizontal axis represent quarters, while the vertical axis measures the effect on log nominal GDP in percentage points. Source: Christiano, Eichenbaum and Evans (2001).

quarter zero. Contrary to the suggestion just mentioned, there is a substantial real effect of the shock. (Note that in both figures, the monetary policy shock is normalized so that the long-run effect on nominal GDP is an increase of one percentage point.) Furthermore, the effect occurs with a substantial delay; there is essentially no effect on output until the second quarter following the policy shock,[26] and the peak output effect occurs only in the sixth quarter following the shock. The estimated effect is still more than two standard errors greater than zero two full years after the shock, and (at least according to the point estimate) the effect is still at more than a third of its maximum level ten quarters after the shock.

[26]It may be wondered why we assert that the shock actually occurs in quarter zero, given that there is no effect on nominal expenditure until quarter two. The occurrence of the policy shock is indicated by the substantial decline in the federal funds rate in quarter zero; see Christiano *et al.* (2001) for a plot of this response.

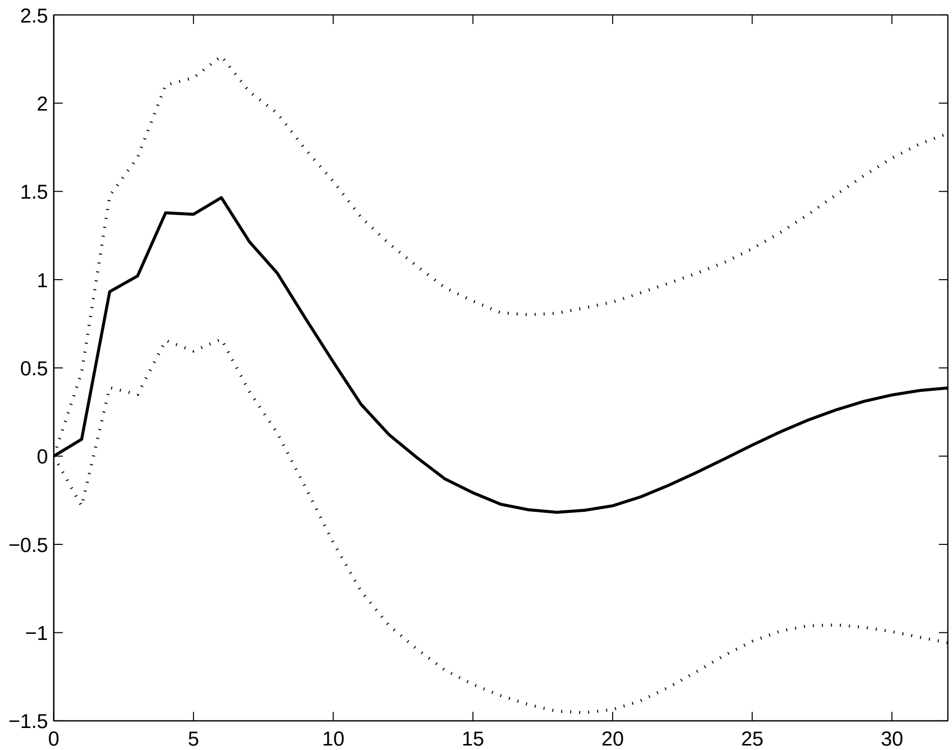

Figure 3.3: Impulse response of real GDP to the same monetary policy shock as in Figure 3.2. The vertical axis measures the effect on log real GDP in percentage points. Source: Christiano, Eichenbaum and Evans (2001).

Such long-lasting effects are inconsistent with the "New Classical" aggregate-supply relation, unless the "period" for which prices are fixed in advance is longer than two years. But survey evidence on price changes (*e.g.*, Blinder *et al.*, 1998) finds that the majority of firms change their prices more frequently than once per year, so that more than half of all prices should have been adjusted at least once within the first two quarters following a shock.

This undesirable feature of the model can be avoided, however — without abandoning the assumption that prices are set optimally (under rational expectations) when they are adjusted, and without assuming counterfactually long intervals between price changes — by assuming that the intervals over which the prices of different goods remain fixed overlap, rather than being perfectly synchronized. As Phelps (1978) and Taylor (1979a, 1980) first pointed out,[27] such staggering of price changes can lead to a sluggish process of adjustment of the overall price level, even when individual prices are adjusted relatively frequently.

This is because, if individual suppliers' pricing decisions are strategic complements, then the fact that some prices are not yet being adjusted will restrain the degree to which prices are changed by those suppliers that do adjust their prices. At a later date, the adjustment of the prices that earlier were sticky will in turn be restrained by the fact that the prices that earlier were adjusted were not changed very much. As a result of this process, the level of prices prior to a shock can continue to have a significant effect on the general level of prices even after most prices have been adjusted at least once since the shock. If the strategic complementarity between pricing decisions is great enough, the adjustment of the general level of prices can be quite slow, even though individual price adjustments are frequent. The consequence will be prolonged effects on real activity of a sustained change in the level of nominal spending.

As we shall see, this theory of pricing can again justify an aggregate supply relation that takes the form of an "expectations-augmented Phillips-curve" relation. However, in this variant case, the kind of inflation expectations that determine the location of the short-run Phillips curve are current expectations regarding future inflation, rather than past expectations regarding current inflation. This might seem a small difference (given the degree of serial correlation in inflation expectations, at least in recent decades), but it is a crucial one; for the aggregate supply relation, together with rational expectations, no longer exclude the possibility of forecastable variations in the output gap. With this modification, the model becomes consistent with the occurrence of prolonged fluctuations in real activity following a monetary policy shock, of at least roughly the kind estimated in the VAR literature.

[27] These first applications of the idea assumed that wages, rather than prices, were fixed for a period of time (owing to wage contracts), and thus concerned the effects of staggered wage negotiations rather than staggered price-setting. But as Blanchard (1983) pointed out, the same principle can be applied to price-setting.

2.1 The Calvo Model of Price-Setting

Here we present a particular example of a model with staggered price-setting, a discrete-time variant of a model proposed by Guillermo Calvo (1983).[28] In this model, a fraction $0 < \alpha < 1$ of goods prices remain unchanged each period, while new prices are chosen for the other $1 - \alpha$ of the goods. For simplicity, the probability that any given price will be adjusted in any given period is assumed to be $1 - \alpha$, independent of the length of time since the price was set, and independent of what the particular good's current price may be.

These last assumptions are plainly unrealistic, but they are very convenient in simplifying the analysis of equilibrium inflation dynamics, as they greatly reduce the size of the state space required to characterize those dynamics. Because each supplier that chooses a new price for its good in period t faces exactly the same decision problem, the optimal price p_t^* is the same for each of them, and so in equilibrium, all prices that are chosen in period t have the common value p_t^*. The remaining fraction α of prices charged in period t are simply a subset of the prices charged in period $t - 1$, with each price appearing in the period t distribution of unchanged prices with the same relative frequency as in the period $t-1$ price distribution. (For this last argument it is crucial that each price has an equal probability of being adjusted in a given period.) Then the Dixit-Stiglitz price index (1.3) in period t satisfies

$$P_t^{1-\theta} \equiv \int_0^1 p_t(i)^{1-\theta} di = (1-\alpha)p_t^{*1-\theta} + \alpha \int_0^1 p_{t-1}(i)^{1-\theta} di,$$

so that

$$P_t = \left[(1-\alpha)p_t^{*1-\theta} + \alpha P_{t-1}^{1-\theta}\right]^{\frac{1}{1-\theta}}. \tag{2.1}$$

It follows that in order to determine the evolution of this price index, we need only know its initial value, and the single new price p_t^* that is chosen each period. The determination of

[28] See also Rotemberg (1987) for discussion of this model, and the similarity of its implications for aggregate dynamics to those of a model with convex costs of price adjustment. The first use of a discrete-time version of Calvo's model of price-setting, in the context of a complete intertemporal equilibrium model of aggregate fluctuations, was in the work of Yun (1996). Other early applications of the same device include Woodford (1996), King and Watson (1996), King and Wolman (1996), and Goodfriend and King (1997). Kimball (1995) also assumes Calvo pricing, albeit in continuous time, in another important early study of an intertemporal equilibrium model with sticky prices.

p_t^*, in turn, depends upon current and expected future demand conditions for the individual good, but (1.11) implies that other prices affect the demand curve for good i only through the value of the price index P_t. Thus we can determine the equilibrium value of the index P_t as a function of its previous period's value, the expected future path of this same index, and current and expected future values of aggregate real variables. There is no need for reference to additional information about past prices.[29]

A supplier that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} [\Pi_T^i(p_t(i))] \right\}, \tag{2.2}$$

by analogy with (1.16), where once again nominal profits each period are given by

$$\Pi_t^i(p) = [Y_t P_t^\theta p^{1-\theta} - w_t(i) f^{-1}(Y_t P_t^\theta p^{-\theta}/A_t). \tag{2.3}$$

Here the factor α^{T-t} multiplying the stochastic discount factor indicates the probability that price $p_t(i)$ will still be charged in period T.[30] The price $p_t(i)$ is chosen on the basis of information available at date t so as to maximize this expression, given the expected state-contingent values of the random variables $Q_{t,T}$, Y_T, P_T, $w_T(i)$ and A_T for all dates $T \geq t$.

Corresponding to (1.18) we obtain in this case the first-order condition

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} Y_T P_T^\theta [p_t(i) - \mu S_T(i)] \right\} = 0. \tag{2.4}$$

Thus there is again a sense in which the price $p_t(i)$ is set equal to μ times a weighted average of the levels of marginal costs expected to prevail in the various future states, and at the

[29] This is only precisely true in the case of our baseline model, with specific factor markets, since in this case the real marginal cost of supplying a given good depends only upon the quantity supplied of that good $y_t(i)$, aggregate output Y_t, and the vector of aggregate shocks $\tilde{\xi}_t$. If we instead assume common factor markets, as in Yun (1996), real marginal cost depends upon an alternative output aggregate (the one that determines aggregate demand for factors), and not solely upon the Dixit-Stiglitz aggregate Y_t (that determines the utility value of output). This means that the equilibrium conditions for the evolution of P_t also involve a second price index, as shown by Yun. However, even in that case, up to a log-linear approximation the equilibrium conditions can be written solely in terms of the index P_t.

[30] Note that (2.2) reduces to (1.16) if the factor α^{T-t} is replaced by one that takes the value 1 if $T = t+1$ and the value zero for all other T.

various future dates, at which the price $p_t(i)$ applies. But once again, (2.4) does not give a closed-form expression for the optimal price, since $S_T(i)$ will depend upon the price of good i.[31] Substituting the demand function (1.11) into the real marginal cost function defined in (1.9), and using this to replace $S_T(i)$ in (2.4), and again substituting the solution for the stochastic discount factor from equation (xx) of chapter 2, we obtain

$$E_t\left\{\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}u_c(Y_T;\xi_T)Y_T P_T^{\theta-1}[p_t^* - \mu P_T s(Y_T P_T^{\theta} p_t^{*-\theta}, Y_T;\tilde{\xi}_T)]\right\} = 0 \tag{2.5}$$

as an equation that implicitly defines the optimal price p_t^*. Note that the monotonicity of s in its first argument implies that the left-hand side of this expression is increasing in p_t^*, so that the optimal price is uniquely defined by this condition. The value of p_t^* then determines the evolution of the price index P_t through (2.1).

Once again, we may usefully approximate the equilibrium dynamics of inflation in the case of small enough disturbances $\tilde{\xi}_t$ by considering a log-linear approximation to these equations. If $\tilde{\xi}_t = 0$ and $Y_t = \bar{Y}$ at all times, equations (2.1) and (2.5) have a solution with zero inflation, in which $P_t = p_t^* = P_{t-1}$ each period. In the case of small enough fluctuations in $\tilde{\xi}_t$ and Y_t around these values, we accordingly look for a solution in which P_t/P_{t-1} and p_t^*/P_t remain always close to 1, though the (log) price level may contain a unit root.[32] The fluctuations in these two stationary variables must approximately satisfy a log-linear approximation to

[31] Authors such as Yun (1996) or Goodfriend and King (1997), who assume common factor markets, are instead able to solve for the optimal price p_t^* in closed form. However, the assumption of common factor markets results in an unrealistic estimate of the likely degree of strategic complementarity in price-setting, as discussed above.

[32] Our algebra at this point is simplified by log-linearizing around a steady state with a zero inflation rate, rather than some other constant inflation rate. The resulting log-linear structural equations thus apply only to the determination of equilibrium in the case of a policy rule that does in fact generate inflation near zero at all times. This does not mean, however, that we can consider only policy rules that make the average inflation rate exactly zero, or that involve a "target" inflation rate of zero. It is only necessary that the average inflation rate be sufficiently small (technically, of order $\mathcal{O}(||\xi||)$, if we want the error in our characterization of the evolution of inflation and other variables to be of order $\mathcal{O}(||\xi||^2)$, where $||\xi||$ is a bound on the size of the disturbances). As we shall argue (see chapter 6) that desirable policies do imply a low average inflation rate, this does not seem an inconvenient restriction, from the standpoint of our goal of characterizing optimal policy. Characterization of the effects of policy in a high-inflation economy might require a more accurate approximation. But the Calvo pricing model itself is implausible as a model of pricing under such circumstances, as many individual prices are likely to be indexed to some broader price index (or to an exchange rate). The model with backward-looking price indexation, discussed in section xx below, would likely be more realistic for such purposes.

(2.1),

$$\pi_t = \frac{1-\alpha}{\alpha}\hat{p}_t^*, \tag{2.6}$$

where $\hat{p}_t^* \equiv \log(p_t^*/P_t)$. They must also approximately satisfy a log-linear approximation to (2.5), given by

$$E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{ \hat{p}_t^* - [\hat{s}_{t,T} + \sum_{\tau=t+1}^{T} \pi_\tau] \right\} = 0, \tag{2.7}$$

where $\hat{s}_{t,T}$ denotes the deviation (from its steady-state value) of the log of real marginal cost in period T for a supplier whose price is p_t^*. (The term in square brackets in (2.7) is thus the deviation of the log of

$$(P_T/P_t)s(Y_T P_T^\theta p_t^{*-\theta}, Y_T; \tilde{\xi}_T)$$

from its steady-state value of μ^{-1}. There are no terms in (2.7) corresponding to stochastic variation in the discount factors in (2.5), because the steady-state value of the term in square brackets in that equation is zero.)

Under our characterization (1.14) of the dependence of marginal supply cost upon a producer's own level of output, the variable $\hat{s}_{t,T}$ can furthermore be approximated by

$$\hat{s}_{t,T} = \hat{s}_T - \omega\theta[\hat{p}_t^* - \sum_{\tau=t+1}^{T} \pi_\tau],$$

where $\hat{s}_T$ denotes the deviation of the log of the *average* level of real marginal cost (real marginal cost for a good with output $y_T(i) = Y_T$) from its steady-state value. Substituting this into (2.7), we can solve for

$$\begin{aligned} \hat{p}_t^* &= (1-\alpha\beta)E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} [(1+\omega\theta)^{-1}\hat{s}_T + \sum_{\tau=t+1}^{T} \pi_\tau] \\ &= \frac{1-\alpha\beta}{1+\omega\theta} \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} E_t \hat{s}_T + \sum_{T=t+1}^{\infty} (\alpha\beta)^{T-t} E_t \pi_T. \end{aligned}$$

Thus the relative price chosen by those suppliers who adjust their prices in period t is a purely forward-looking function of aggregate conditions at that date.

This last expression in turn can be quasi-differenced to yield

$$\hat{p}_t^* = \frac{1-\alpha\beta}{1+\omega\theta}\hat{s}_t + \alpha\beta E_t \pi_{t+1} + \alpha\beta E_t \hat{p}_{t+1}^*.$$

Then using (2.6) to substitute for $\hat{p}_t^*$ on both sides of the above, we obtain a stochastic difference equation for the inflation rate,

$$\pi_t = \xi \hat{s}_t + \beta E_t \pi_{t+1}, \tag{2.8}$$

where

$$\xi \equiv \frac{1-\alpha}{\alpha} \frac{1-\alpha\beta}{1+\omega\theta} > 0.$$

Writing this alternatively as

$$\Delta \log P_t = \xi [\log S_t - \log P_t + \log \mu] + \beta E_t \Delta \log P_{t+1}, \tag{2.9}$$

we see that we have an equation that can be solved for the predicted path of the price index P_t, given the evolution of the average level of nominal marginal cost S_t. Specifically, if $\{\log S_t\}$ is a difference-stationary process, there is a unique solution for the price index process such that $\{\log P_t\}$ is also difference-stationary (*i.e.*, inflation is a stationary process), given by

$$\log P_t = \lambda_1 \log P_{t-1} + (1-\lambda_1)(1-\lambda_2^{-1}) \sum_{j=0}^{\infty} \lambda_2^{-j} [\log \mu + E_t \log S_{t+j}], \tag{2.10}$$

where $0 < \lambda_1 < 1 < \beta^{-1} < \lambda_2$ are the two roots[33] of the characteristic polynomial

$$\beta \lambda^2 - (1 + \beta + \xi)\lambda + 1 = 0. \tag{2.11}$$

This prediction of the Calvo pricing model is in fact independent of any specification of the determinants of average marginal costs; the only feature of the marginal cost function used in deriving (2.8) was the fact that the elasticity of an individual supplier's marginal cost with respect to its own output is equal to $\omega > 0$. Tests of the empirical adequacy of this theory of pricing are therefore appropriately focused upon this prediction, rather than upon the accuracy of the "New Keynesian" aggregate supply relation derived below (which depends as well upon the details of one's theory of supply costs). This is the approach

[33] The roots can be shown to be real and to satisfy the inequalities just stated, as long as $0 < \beta < 1$ and $\xi > 0$.

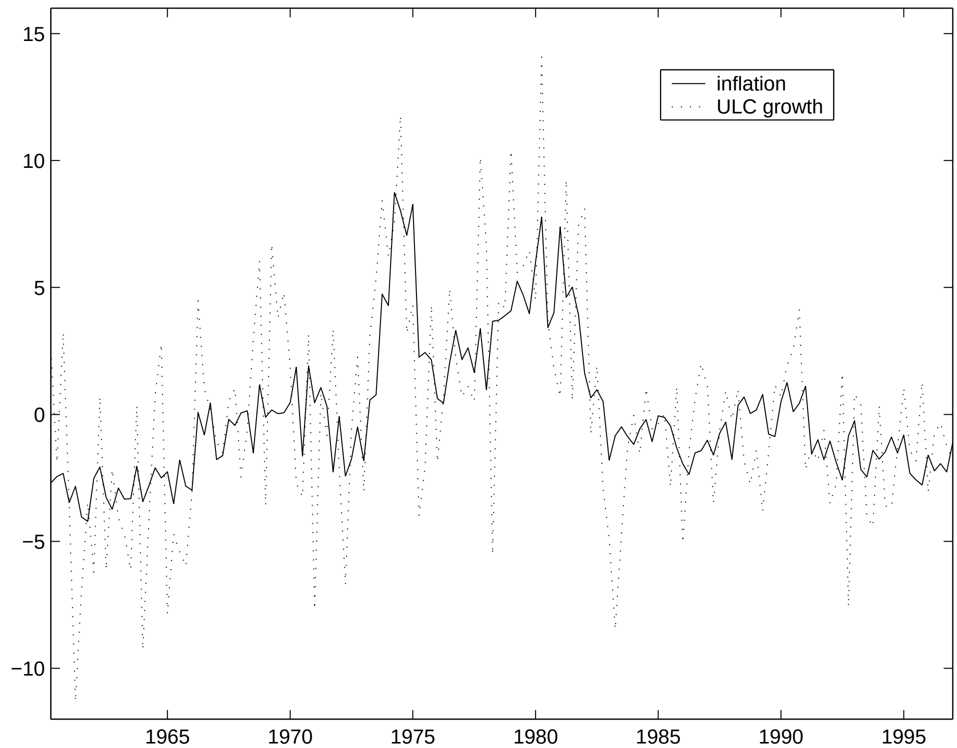

Figure 3.4: U.S. inflation (quarterly change in GDP deflator, in percentage points of equivalent annual rate) compared to growth rate of unit labor cost. Source: Sbordone (2002).

taken by Sbordone (1998, 2002), who uses data on the average level of unit labor cost in the U.S. economy as a measure of nominal marginal cost.[34] She estimates a small atheoretical VAR model with which to forecast the future path of unit labor costs, using quarterly U.S. data over the period 1960-1997. This then allows one to construct an implied series for the forward-looking terms on the right-hand side of (2.10), for any assumed values of β and ξ (which then imply values for λ_1 and λ_2). Starting from an initial condition for the price level (given by its historical value in the initial quarter), one can then simulate (2.10) to obtain a predicted time path for the price level, given the observed path of unit labor costs (and of forecasted future labor costs). One can then compare this prediction to the actual path of

[34]Note that according to the model developed here, average marginal supply cost should be proportional to average unit labor cost under the assumption of a production technology with a constant elasticity of output with respect to the labor input (*e.g.*, the familiar Cobb-Douglas specification). See Rotemberg and Woodford (1999b) for further discussion of this common measure of marginal cost and alternatives.

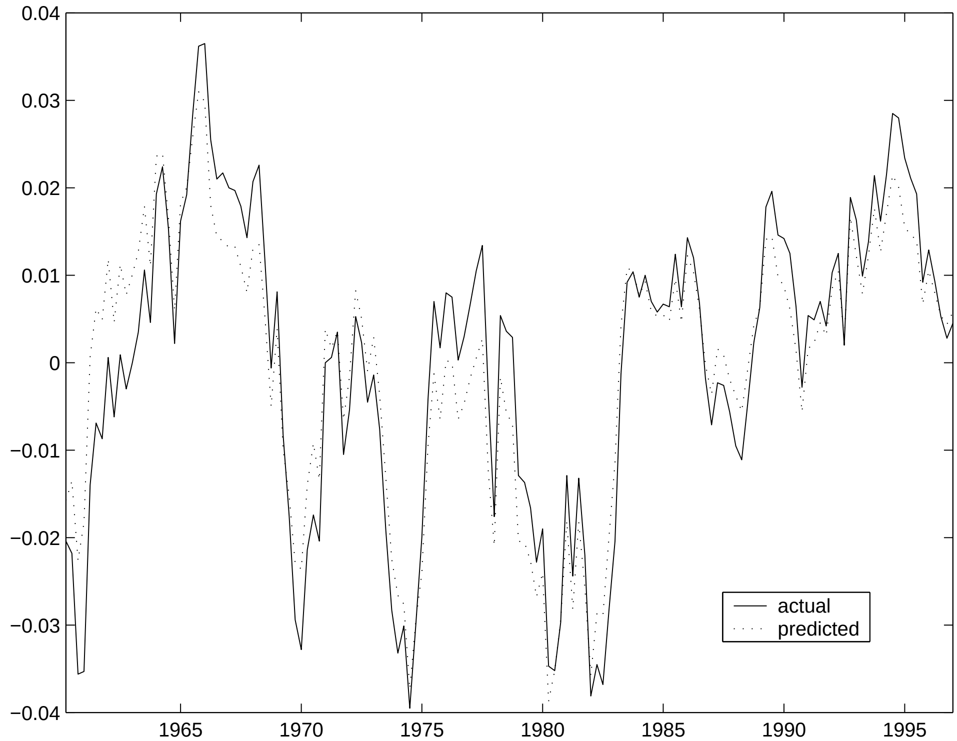

Figure 3.5: Actual path of price/ULC ratio (quarterly U.S. data, reported as log deviation from mean) compared to prediction of the Calvo pricing model. Source: Sbordone (2002).

the aggregate price level over the same period.

Note that in the case of flexible prices (the $\xi \to \infty$ limit of the above model), the predicted price series would simply be given by $\log P_t = \log \mu + \log S_t$, so that predicted inflation would be given by the percentage growth in nominal unit labor costs from one quarter to the next. As shown in Figure 3.4, this would be quite a poor explanation of actual U.S. inflation; in particular, one observes that inflation has been much less volatile than the growth rate of unit labor costs. Alternatively, as shown in Figure 3.5, there has been substantial variation in the ratio of price to unit labor cost from year to year, whereas the model implies that this should be constant. In the case of sticky prices (ξ finite), the model implies less volatile inflation, and in fact, for a certain value of ξ, it predicts a path for the price level quite similar to its actual path. Figure 3.5 compares the actual path of the (demeaned, log) price/ULC ratio to the model's prediction in the case that $\xi = .055$.[35] The fit is quite good;

the mean squared error of the predicted price/ULC series is only 12 percent as large as the variance of the actual price/ULC series. And even this statistic fails to emphasize the extent to which the model succeeds in explaining the timing of quarterly changes in the ratio; the discrepancy between the predicted and actual series is mainly at quite low frequencies. Figure 3.6 similarly compares the actual path of U.S. inflation (measured by growth in the GDP deflator) with the path predicted by the model. The fit is again quite good (which is another way of seeing that the discrepancy in Figure 3.5 is almost entirely at low frequencies).[36]

These results provide persuasive evidence for price stickiness of roughly the sort implied by the Calvo pricing model. The degree to which the model fits U.S. inflation dynamics is perhaps surprising, given that the assumption of a fixed probability of price change for all suppliers has been chosen for analytical convenience rather than out of any belief that it ought to be realistic. Probably this reflects the fact that, given the small value of ξ implied by Sbordone's estimates (and hence the high degree of smoothing of marginal cost in the price dynamics), the details of the distribution of intervals between price changes does not matter much for the evolution of the aggregate price index, but only the average rate at which prices are revised. But the fit of the model *does* clearly depend upon the existence of staggered price changes of the kind assumed by Calvo. The "New Classical" pricing model considered above, for example, would imply that fluctuations in the ratio of price to marginal cost should be unforecastable, so that the predicted P/ULC series would necessarily exhibit no serial correlation. The significant serial correlation of the P/ULC series shown in Figure 3.5, and the closely similar degree of serial correlation of the series predicted by the Calvo

[35] Sbordone selects this value as her estimate, on the ground that it minimizes the mean-squared error of the model's predicted path for $\log P_t$, when β is assigned a value of 1. Note that she actually reports the value of $\xi^{-1} = 18.3$.

[36] Gali and Gertler (1999) obtain similar results for U.S. inflation, using an alternative, instrumental-variables strategy for estimating equation (2.8). Gali and Gertler find that they can statistically reject this baseline pricing model in favor of a generalization in which some price-setters use a backward-looking "rule of thumb", discussed further below in section xx. However, they find that the baseline model already explains historical inflation dynamics quite well, as Figure 3.6 shows; and the validity of the standard errors used to determine that the rejection of the baseline model is statistically significant might also be questioned. Batini *et al.* (2000) similarly find that the Calvo pricing model can explain U.K. inflation dynamics, while Gali *et al.* (2001) obtain similar results for several European countries.

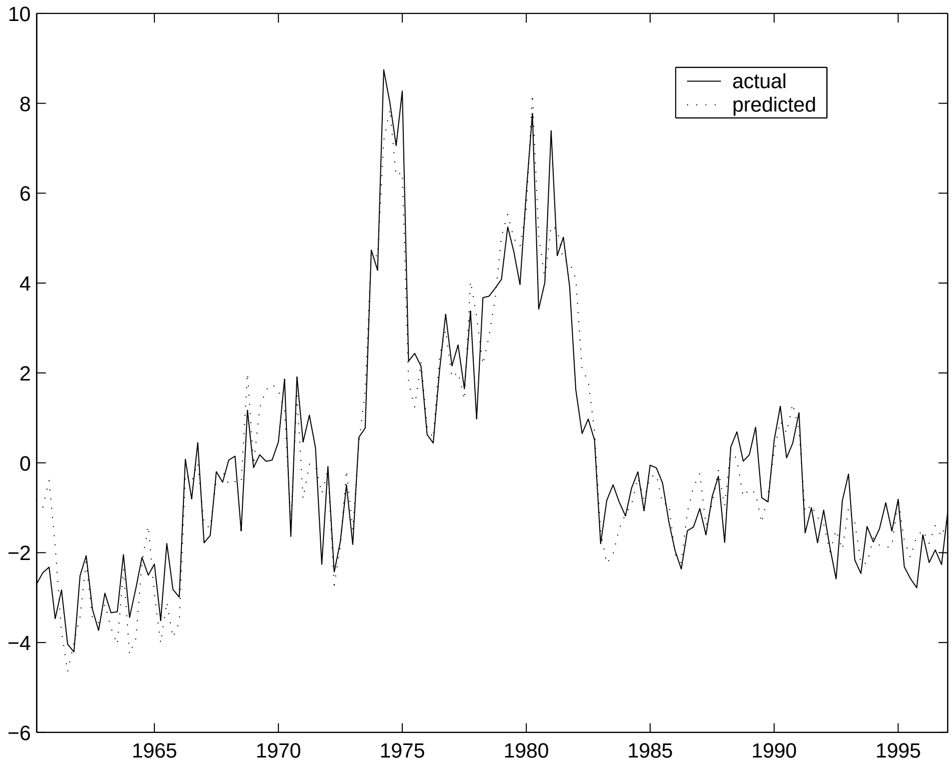

Figure 3.6: Actual path of inflation (quarterly U.S. data) compared to prediction of the Calvo pricing model. Source: Sbordone (2002).

model,[37] suffices to show that the kind of price stickiness allowed in that model cannot similarly explain the U.S. data. Perhaps more surprisingly, Sbordone shows that the U.S. price index cannot be fit nearly as well by any purely backward-looking moving average of unit labor costs, either; the kind of smoothing reflected in the figures depends importantly upon the presence of the forward-looking terms on the right-hand side of (2.10), and not simply upon the partial-adjustment term.[38]

One might, of course, doubt the accuracy of the simple unit-labor-cost measure of marginal cost, for various reasons that are reviewed by Rotemberg and Woodford (1999b).

[37]The autocorrelation functions of both series, with standard errors for the data series, are presented in Sbordone (1998, 2002).

[38]Sbordone shows this formally by separately estimating the weights λ_1 and λ_2^{-1} in (2.10), without imposing the restriction that they correspond to the two roots of (2.11). She finds not only that both coefficients are significantly positive, but that the values that yield the best fit are nearly equal in size, as the Calvo pricing model would imply. (Note that the two roots of (2.11) necessarily satisfy $\lambda_1\lambda_2 = \beta^{-1}$.)

But it seems fortuitous that such a simple model of price-setting can explain actual inflation dynamics so well, if unit labor cost is not in fact a fairly accurate measure of variations in marginal cost, at least at moderate frequencies.[39] In fact, Sbordone experiments with a number of possible corrections to her simple measure of marginal cost, and finds that none of them improves the fit of the pricing model. One might also wonder whether forecasts based on a VAR model with constant coefficients fit to the entire period 1960-97 should correspond very closely to people's expectations, even assuming that their expectations were rational, on the ground that the dynamics of unit labor costs need not have been constant over this period. (One might think, for example, that inflation dynamics have been substantially different since the disinflation of the early 1980s; but this would imply that unit labor cost dynamics should have been different as well.) But, once again, the close fit of the model, and the fact that it fits equally well both before and after the 1979-82 period, suggest that the hypothesis of a common (and at least roughly unbiased) forecasting rule over the entire period is not too inaccurate.

2.2 A "New Keynesian" Phillips Curve

We can use the model of optimal price-setting just derived to obtain an aggregate supply relation — *i.e.*, a structural relation between inflation dynamics and the level of real activity — by adjoining to our above pricing equation a theory of how real marginal costs depend upon the level of real activity. The simple model used earlier in this chapter (recall equation (1.14)) implies that

$$\hat{s}_t = (\omega + \sigma^{-1})(\hat{Y}_t - \hat{Y}_t^n). \tag{2.12}$$

Substituting this into (2.8) then yields an aggregate supply relation of the form

$$\pi_t = \kappa(\hat{Y}_t - \hat{Y}_t^n) + \beta E_t \pi_{t+1}, \tag{2.13}$$

[39] Because of the degree of smoothing involved in (2.10), high-frequency error in the measure of marginal cost will have relatively little effect upon the predicted path for prices. And Figure 3.5 itself indicates at least a small amount of low-frequency specification error, which might be due to low-frequency error in the measure of marginal cost.

where

$$\kappa \equiv (\omega + \sigma^{-1})\xi = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\frac{\omega+\sigma^{-1}}{1+\omega\theta} > 0.$$

Alternatively, the slope coefficient can be written as

$$\kappa = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\zeta, \tag{2.14}$$

where ζ is defined in (1.22). This is what Roberts (1995) calls the "New Keynesian Phillips Curve," as this specification, or one similar to it, is implied by a variety of simple models of optimal price-setting.[40] From (2.14) we observe once again that the short-run Phillips curve is flatter (for any given inflation expectations) the smaller the value of ζ, and thus the greater the degree of strategic complementarity in price-setting. It is also flatter the larger is α, which is to say, the longer the average time interval between price changes.

Our aggregate supply relation again has the form of an expectations-augmented Phillips curve, but now the inflation expectations that shift the curve are current expectations of future inflation, rather than past expectations of the current inflation rate, as in (1.23). The difference turns out to be crucial for the model's ability to allow for forecastable fluctuations in the output gap. It is now possible for non-zero values of $\hat{Y}_t - \hat{Y}_t^n$ to be forecasted at some earlier date $t-j$; this simply requires that $E_{t-j}\pi_t$ not equal $\beta E_{t-j}\pi_{t+1}$.

2.3 Persistent Real Effects of Nominal Disturbances

We return now to the question of the persistence of the real effects of disturbances to nominal spending. Once again, we assume a given stochastic process for aggregate nominal spending $\mathcal{Y}_t$, and consider what processes for P_t and Y_t are then implied by the aggregate supply relation (2.13). As a simple example, suppose that an unexpected disturbance permanently increases $\log \mathcal{Y}$ by a unit amount at date zero, which is then expected to maintain the higher

[40] Notably, the same form of aggregate supply relation, up to our log-linear approximation, is implied by a model with convex costs of price adjustment, as shown by Rotemberg (1987). Note that in Roberts' presentation of the "New Keynesian Phillips Curve," the discount factor β is set equal to one. This simplification may seem appealing, in that it implies a vertical "long-run" inflation-output tradeoff. But correctly accounting for the presence of the discount factor in (2.13) has important consequences for the analysis of optimal policy, as is shown in chapters 6 and 7.

value. Thus

$$E_0 \log \mathcal{Y}_t = 1 \tag{2.15}$$

for all $t \geq 0$, where $\log \mathcal{Y}_t$ is measured as a deviation from the constant level that would have been expected in the absence of the disturbance. Let us consider the expected paths $E_0 \log P_t$ and $E_0 \log Y_t$ that are consistent with (2.13), given the initial condition $\log P_{-1} = 0$, where $\log P_t$ and $\log Y_t$ are also measured as deviations from the levels that would have been maintained but for the shock. We assume that the shock (a purely nominal disturbance) involves no change in the constant expected level for the natural rate of output.

Let $\tilde{p}_t$ denote $E_0 \log P_t - 1$, the extent to which the expected price-level response in period t differs from exact proportionality to the permanent increase in nominal spending. Then we must have $E_0 \log Y_t = -\tilde{p}_t$ for each horizon $t \geq 0$; substituting this into (2.13), we find that the price-level responses must satisfy

$$\tilde{p}_{t-1} - (1 + \beta + \kappa)\tilde{p}_t + \beta\tilde{p}_{t+1} = 0 \tag{2.16}$$

for each $t \geq 0$, starting from the initial condition $\tilde{p}_{-1} = -1$. The unique bounded solution is easily seen to be given by

$$\tilde{p}_t = -\lambda^{t+1}, \tag{2.17}$$

where $0 < \lambda < 1$ is the smaller of the two real roots[41] of the characteristic equation

$$P(\lambda) \equiv \beta\lambda^2 - (1 + \beta + \kappa)\lambda + 1 = 0. \tag{2.18}$$

Since $E_0 \log P_t = 1 + \tilde{p}_t$, we observe that the log price level is expected to rise monotonically, asymptotically reaching a level proportional to the increase in nominal spending. Since $E_0 \log Y_t = -\tilde{p}_t$, we observe that output increases in the period of the disturbance, by an amount less than (though possibly close to) proportional to the increase in nominal spending, then decays monotonically back to its original level. (These impulse responses of the price level and of output are plotted in Figure 3.7, for the illustrative parameter values $\beta = .99$ and $\kappa = .024$.[42])

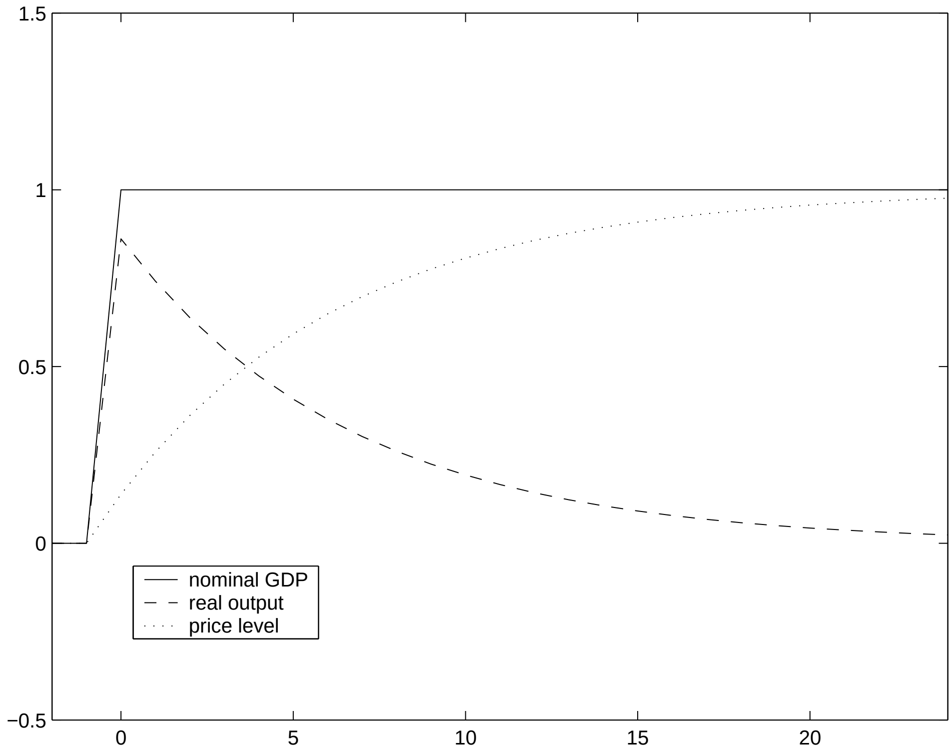

Figure 3.7: Impulse responses to an immediate permanent increase in nominal GDP.

The degree of persistence of the effect upon real activity is thus dependent upon the size of the root λ; the closer λ is to one (its theoretical upper bound), the longer the time it takes for real activity to return to its "potential" level following a nominal disturbance. (Note that λ also determines the size of the initial effect upon real activity in period zero, as well. Thus the same factors that increase the *amplitude* of the effect upon real activity — by making aggregate price adjustment more sluggish — also make the effect more *persistent*.) It is easily seen, in turn, that the smaller root of $P(\lambda)$ varies inversely with the size of κ, with a value that approaches zero in the case of very large κ, and a value that approaches one for κ near zero. Thus a small value of κ (a flat short-run Phillips curve) is required for significant persistence. This in turn could occur either as a result of α being near one (infrequent price

[41]Since $P(0) > 0$, $P(1) < 0$, $P(\beta^{-1}) < 0$, and $P(\lambda) > 0$ for all large enough $\lambda > 0$, it is evident that the equation has two real roots, $0 < \lambda < 1$ and another root greater than β^{-1}.

[42]These values are taken from the estimates of Rotemberg and Woodford (1997), discussed further in chapter 4.

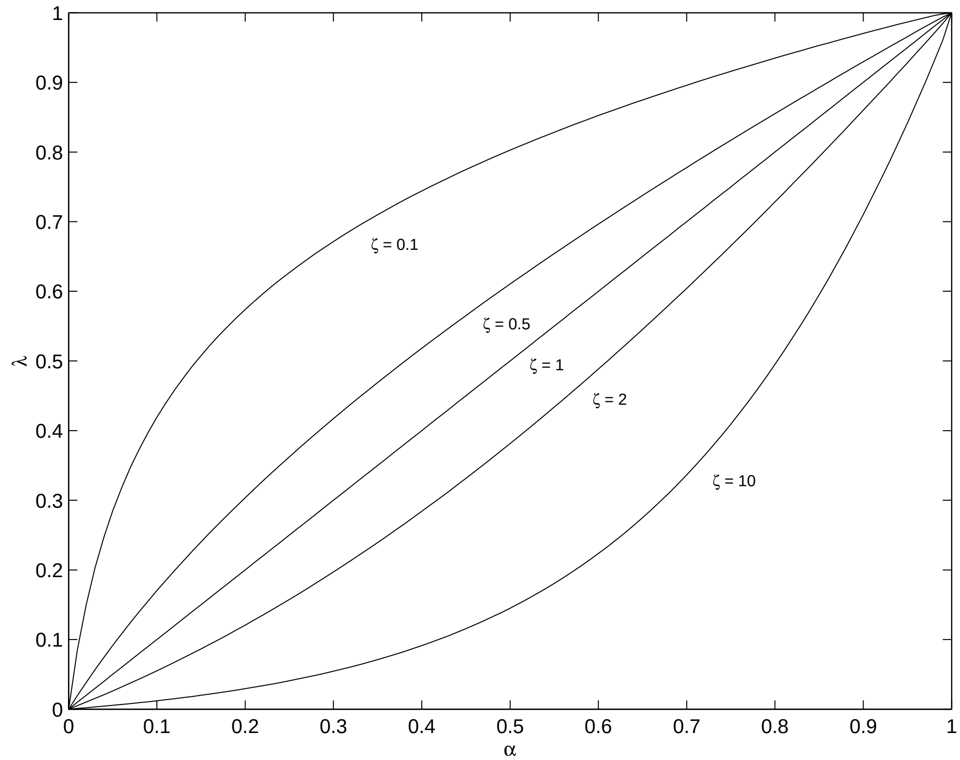

Figure 3.8: Persistence of real effects of an increase in nominal spending as a function of the frequency of price adjustment, for alternative degrees of strategic complementarity in price-setting.

changes) or ζ being small (strong strategic complementarity).

It is perhaps most interesting to observe that the degree of strategic complementarity can have a considerable effect upon the degree of persistence, holding fixed the frequency of price changes. Note that for any value of α, the value of κ can be made arbitrarily small, or arbitrarily large, through assignment of an appropriate value to ζ. The effects of α and ζ upon the degree of persistence are shown quantitatively in Figure 3.8, where the implied value of λ is plotted as a function of these two parameters, assuming the value $\beta = .99$.

One observes that $\lambda > \alpha$ if and only if $\zeta < 1$, *i.e.*, if and only if pricing decisions are strategic complements in the sense discussed above. This can be shown analytically by observing that

$$P(\alpha) = (1 - \alpha)(1 - \alpha\beta)(1 - \zeta),$$

where we have used (2.14) to substitute for κ. Thus $P(\alpha) > 0$, so that $\alpha < \lambda < 1$, if and only if $\zeta < 1$. It is in this case that the rate of adjustment of the aggregate price index is slower than would be expected mechanically as a result of the fact that not all prices have yet had an opportunity to be updated since the occurrence of the shock. (The fraction of prices in period t that have not yet been adjusted even once since the shock is given by α^t.) In this case, Taylor (1980) speaks of the existence of a "contract multiplier" as a result of the staggering of price adjustments.[43] Such a multiplier depends upon the existence of strategic complementarity among different suppliers' pricing decisions, so that the fact that other prices have not fully adjusted makes an individual supplier adjust its own price less. In the case of strategic substitutes ($\zeta > 1$), there is actually even *less* persistence than one would expect for purely mechanical reasons ($\lambda < \alpha$), because the first prices that are adjusted actually over-adjust in reaction to the failure of other prices to adjust in proportion to the increase in nominal spending.

In fact, the connection between the value of ζ and the value of κ indicated by (2.14) continues to hold when additional sources of strategic complementarity are introduced, such as non-CES preferences over differentiated goods, or intermediate inputs. If we generalize our assumptions about demand and production costs in both of these directions, as in section xx above, nominal profits are instead of the form

$$\Pi_t^i(p) = [pY_t d(p/P_t) - w_t(i) f^{-1}((1 - s_m) d(p/P_t) Y_t / A_t) - s_m P_t Y_t d(p/P_t),$$

generalizing (2.3). Here $d(p_t(i)/P_t)$ is the relative demand $c_t(i)/C_t$ implied by (1.27). The corresponding first-order condition for optimal price-setting by a firm that chooses its price at date t is then

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha^{T-t} Q_{t,T} Y_T d(p_t(i)/P_T)[p_t(i) - \mu_T(i) S_T(i)] \right\} = 0,$$

generalizing (2.4). Here the desired markup is given by $\mu_T(i) = \mu(d(p_t(i)/P_T))$, where the function $\mu(x)$ is again defined by (1.30), and marginal cost is now given by (1.35). As before,

[43] In Taylor's original formulation, it is actually the staggered negotiation of wage contracts that gives rise to the "multiplier".

we may log-linearize this to obtain

$$E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \{\hat{p}_{t,T} - [\hat{\mu}_{t,T} + \hat{s}_{t,T}]\} = 0, \tag{2.19}$$

generalizing (2.7). Here we introduce the notation

$$\hat{p}_{t,T} = \hat{p}_t^* - \sum_{\tau=t+1}^{T} \pi_\tau$$

for the (log) relative price at date T of a firm that has set its price at date t, $\hat{y}_{t,T}$ for its (log) output relative to steady state, and

$$\hat{\mu}_{t,T} = \epsilon_\mu(\hat{y}_{t,T} - \hat{Y}_T)$$

for its (log) desired markup relative to steady state.

Noting that

$$\begin{aligned} \hat{p}_{t,T} - [\hat{\mu}_{t,T} + \hat{s}_{t,T}] &= \hat{p}_{t,T} - (\epsilon_\mu + s_y)(\hat{y}_{t,T} - \hat{Y}_T) - (s_y + s_Y)(\hat{Y}_T - \hat{Y}_T^n) \\ &= [1 + \theta(\epsilon_\mu + s_y)]\hat{p}_{t,T} - (s_y + s_Y)(\hat{Y}_T - \hat{Y}_T^n), \end{aligned}$$

we observe that (2.19) can equivalently be written

$$E_t \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left\{\hat{p}_t^* - \zeta(\hat{Y}_t - \hat{Y}_t^n)\right\} = 0, \tag{2.20}$$

where ζ is now given by (1.37). This in turn allows once again to derive an aggregate supply relation of the form (2.13), where κ is again given by (2.14). Thus it continues to be the case that the degree of strategic complementarity is a crucial determinant of the degree of persistence of the real effects of an increase in nominal spending; each of the factors discussed above that increase strategic complementarity increases persistence.

Can staggering of price changes give rise to an empirically realistic degree of persistence, assuming an empirically realistic average interval between price changes? Chari, Kehoe and McGrattan (2000) argue that it cannot. However, their conclusion depends both upon an exaggeration of the size of the "contract multiplier" that would be needed, and an under-estimate of the empirically plausible degree of strategic complementarity. They define the contract multiplier as "the ratio of the half-life of output deviations after a monetary shock with staggered price-setting to the corresponding half-life with synchronized price-setting" (p. 1152). In our analysis above of the response of output under Calvo price-setting to a monetary shock that permanently increases the level of nominal GDP,[44] we have shown that the level of output returns to its steady-state level following a shock as λ^t, where λ is the smaller root of (2.18). Thus the "half-life" with staggered price setting would equal $\log 2/\log\lambda^{-1}$ periods.

By the corresponding half-life "with synchronized price-setting", Chari *et al.* mean the case in which each price that is revised after the shock occurs is immediately adjusted all the way to the new expected long-run price level, so that the fraction of the eventual aggregate price adjustment that has occurred at any time is equal to the fraction of prices that have been revised at least once since the occurrence of the monetary shock. In the case of the Calvo pricing model, the fraction of prices that have not yet been adjusted k periods following a shock is α^{k+1}, so that the "half-life with synchronized price-setting" would equal $\log 2/\log\alpha^{-1}$.[45] The contract multiplier implied by the Calvo pricing model is then equal to $\log\alpha^{-1}/\log\lambda^{-1}$. It follows from our results above that the multiplier is greater than one if and only if $\zeta < 1$, which is to say if and only if there is strategic complementarity in the pricing decisions of different suppliers.[46] In fact, in the continuous-time limit of the Calvo

[44]This is not exactly the form of monetary shock considered by Chari *et al.*, who instead assume an exogenous process for the money supply, and consider its implications within a complete general-equilibrium model of the monetary transmission mechanism. However, in their analysis of a "stripped-down version" of their baseline model, they assume a static money demand function according to which nominal GDP is at all times proportional to the money supply, and also assume a random walk for the (log) money supply; this is thus an experiment of exactly the kind we consider here.

[45]Chari *et al.* instead assume fixed-length price commitments, as a result of which this half life is equal to $N/2$ periods, where is the number of periods that each price remains fixed.

[46]As noted in the previous two footnotes, Chari *et al.* consider a different model of staggered pricing, but obtain this same conclusion; their finding that the multiplier is necessarily less than one in their "stripped-down" model follows from the fact that ζ is necessarily greater than one, for the reasons that we have discussed above. (They call this parameter γ; their equation (34) shows that it must exceed one.) In their analysis, as here, the question of whether a contract multiplier much greater than one is possible amounts largely to a consideration of whether ζ can plausibly be much less than one.

model, one can show that the contract multiplier is equal to

$$\frac{2a^*}{[4a^*(a^*+1)\zeta+1]^{1/2}-1}, \tag{2.21}$$

where a^* is the ratio of the (continuous) rate of price adjustment to the (continuous) rate of time preference, independent of the values of the other parameters.[47] In the case that $a^* >> 1$, this is approximately $\zeta^{-1/2}$, regardless of the size of a^*; this provides an analytical explanation for the finding of Chari *et al.* that the predicted contract multiplier in a model with staggered price-setting depends little upon the assumed length of time between price changes.

Chari *et al.* argue that a very large multiplier would be needed in order for staggered price-setting to account for observed persistence. They fit a univariate ARMA model to detrended real GDP for the postwar U.S., and conclude that the half-life of output fluctuations around trend is approximately 10 quarters. At the same time, they argue that a reasonable length of time to assume that prices are fixed would be only one quarter, implying a half-life with synchronized price-setting of only half a quarter. Thus they argue that a contract multiplier of 20 would be needed; this would be possible only if ζ were quite small (.004 in the case of our continuous-time limit).[48] Instead, they find that their baseline parameter values imply a value of ζ well above one, and so a contract multiplier less than one; and while alternative parameter values can raise the multiplier somewhat, they argue that it cannot plausibly be greater than two. They conclude that one must believe that prices remain fixed for many years in order to account for observed persistence.

However, this way of identifying the persistence of the output effects of monetary shocks assumes that *all* fluctuations of output around a deterministic trend path are due to monetary shocks. There is no reason to assume this; indeed, a central contention of this study is

[47] Here we fix ζ and let α and β decrease with Δ, the period length, so that as $\Delta \to 0$, $\log \beta^{-1}/\Delta$ and $\log \alpha^{-1}/\Delta$ both approach positive constants, ρ and a respectively. Then, expanding the characteristic polynomial (2.18) in powers of Δ, one can show that $\log \lambda^{-1}/\Delta$ approaches a positive constant n as well, which depends upon ζ and the ratio $a^* \equiv a/\rho$. The multiplier (2.21) is then equal to a/n.

[48] This value follows from (2.21) under the assumption that $a^* = 33$, a value suggested by survey evidence on the frequency of price change in the U.S. economy, as discussed below. Note, however, that a similarly small value of ζ is required in the case of any large value for a^*.

that the task of monetary policy is to respond appropriately to a large variety of types of *real* disturbances to which economies are subject. One may attribute great importance to monetary policy without holding that any large fraction of the overall variability of output is due (in the sense of ultimate causation) to the random component of monetary policy; for systematic monetary policy can greatly change the effects of real disturbances. For example, the estimated model of Rotemberg and Woodford (1997), discussed in section xx of chapter 4, attributes only a few percent of the overall variance of real GDP around trend over the sample period to monetary policy shocks; yet the counter-factual simulations reported in that paper show that alternative systematic monetary policies would have implied greatly different paths of real as well as nominal variables.

One needs, then, to identify the real effects of monetary policy shocks *alone* in order to determine the relevant half-life. This may be much shorter than the one estimated by Chari *et al.*, without any implication that monetary policy is unimportant for business fluctuations. For example, the impulse response reported in Figure 3.3 above shows that only three quarters after the peak output response, the level of (log) output has already returned halfway to the level that would have been expected prior to the shock; and two quarters after that, the response has fallen to only 20 percent of its peak level. This suggest a "half-life" of only 2.5 to 3 quarters. The structual VAR of Rotemberg and Woodford (1997), discussed in chapter 4, yields an output response that involves more nearly exponential decay after the quarter in which output is first substantially affected; this response, shown in Figure xx below, exhibits a "half-life" of only about 3 quarters.[49]

Furthermore, survey evidence indicates that many prices remain unchanged for longer than a quarter on average. For example, the survey of Blinder *et al.* (1998) indicates an average time between price changes (for a representative sample of U.S. firms) of 9 months.[50] This

[49] As shown in Figure xx, the peak (and first significant) output contraction occurs two quarters following a contractionary policy shock; log output has returned about half-way to its trend level by the fifth quarter following the shock. Nonetheless, the model of Rotemberg and Woodford is fully consistent with the observed degree of persistence of the deviations of output from trend; compare the predicted and estimated autocorrelation functions reproduced in Figure xx below.

[50] See Rotemberg and Woodford (1997) for discussion of other survey evidence from studies of smaller sectors of the economy.

suggests that the continuous rate of price change should be parameterized as approximately $0.33/quarter$, implying a half-life of 2.08 quarters. Thus the required contract multiplier is only 1.44, within the range that Chari *et al.* find to be possible. In the continuous-time limit of the Calvo pricing model derived above,[51] this requires only that $\zeta = 0.49$, and (as shown in Table 3.1 above) a value this low or even lower is easily consistent with the assumptions about preferences and technology made by Chari *et al.*, once we allow for non-CES preferences and intermediate inputs, or alternatively for at least some degree of factor specificity.[52] A larger value of σ makes this even easier, and as we shall argue in chapter 4, it is most reasonable to calibrate this model with a value of σ much larger than one.

In fact, our discussion above has indicated that a variety of plausible assumptions can justify a value of ζ in the range of 0.10-0.15. This would imply a contract multiplier (in the continuous-time limit of our model) in the range of 2.6 to 3.3, or a "half-life" for the output response (assuming the rate of price adjustment indicated by the survey evidence) between 5.5 and 6.8 quarters. This is a degree of persistence of the effects on real activity of an increase in nominal expenditure considerably greater than the one indicated by the VAR evidence cited by Rotemberg and Woodford, but is comparable to the degree of persistence indicated by some other studies of the effects of identified monetary shocks.[53]

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[51] Here we use (2.21), assuming that $a = 0.33$/quarter and $\rho = 0.01$/quarter, so that $a^* = 33$.

[52] In fact, the model of Rotemberg and Woodford is not able to reproduce their estimated impulse response function for real GDP in response to a monetary policy shock without a considerably smaller value of ζ, approximately 0.13. The difference is obtained because their model is a discrete-time model with "periods" of length three months (which results in a degree of persistence somewhat different than that in the continuous-time limit), because they set $\alpha = 2/3$ (so that exactly one-third of all prices are changed each quarter, rather than the fraction $1 - e^{-a} = 0.28$ implied by integration of the continuous-time model), and because their identified monetary policy shock does not result in an immediate, permanent increase in nominal GDP as assumed above.

[53] The real effects of the identified monetary policy shocks obtained by Rotemberg and Woodford (1997) are less persistent than those obtained in some other VAR studies, mainly using longer sample periods. For example, in the baseline results of Christiano *et al.* (1999), using quarterly data and identifying monetary policy shocks with innovations in the federal funds rate (first column of their Figure 2), the peak contraction in real GDP following an unexpected monetary tightening occurs only five to six quarters after the shock, while output has returned half-way to its original trend path by the eleventh quarter following the shock; this would indicate a "half-life" of five to six quarters. The later results of Christiano *et al.* (2001) indicate a half-life of less than three quarters, however, as noted above.

2.4 Consequences of Persistence in the Growth of Nominal Spending

We have thus far considered only the response of real activity to an unexpected, permanent increase in nominal spending. While this particular thought experiment allows us a clear definition of the degree of persistence, such a perturbation of the expected path of nominal spending has little similarity to the estimated responses of nominal spending to the monetary policy shocks identified in the VAR literature; thus the model's predictions for this case cannot be directly compared to any empirical estimates. Identified monetary policy shocks tend to affect nominal spending only slightly (if at all) in the first few months, with an effect that increases cumulatively over a period of several quarters, eventually bringing expected future nominal spending to a new permanent level.

As a simple case that allows for shocks of this kind, let us assume that the growth rate of nominal spending follows a first-order autoregressive process,

$$\Delta \log \mathcal{Y}_t = \rho \Delta \log \mathcal{Y}_{t-1} + \epsilon_t \tag{2.22}$$

with $0 < \rho < 1$, where ϵ_t is an i.i.d. random variable, assumed for simplicity to have mean zero. This process implies that an innovation ϵ_t increases the conditional expectation $E_t \log \mathcal{Y}_{t+j}$ by an amount $(1-\rho)^{-1}(1-\rho^{j+1})\epsilon_t$ which increases monotonically with j, asymptotically approaching a permanent effect that is $(1-\rho)^{-1} > 1$ times as large as the initial effect.[54] Let us consider the impulse response to a positive innovation $\epsilon_0 = 1-\rho$, which results in a unit increase in the expected long-run level of nominal spending. Again letting $\tilde{p}_t$ denote $E_0 \log P_t - 1$, we find that the price-level responses must satisfy

$$\tilde{p}_{t-1} - (1+\beta+\kappa)\tilde{p}_t + \beta\tilde{p}_{t+1} = \kappa\rho^{t+1}, \tag{2.23}$$

a generalization of (2.16), again starting from initial condition $\tilde{p}_{-1} = -1$. (Because nominal spending does not immediately jump to its expected long-run value, we now have $E_0 \log Y_t =$

[54] This form of stochastic process is used as a rough approximation of actual U.S. time series in Rotemberg (1996).

$[E_0 \log \mathcal{Y}_t - 1] - \tilde{p}_t = -(\tilde{p}_t + \rho^{t+1})$.) The unique bounded solution to (2.23) is given by the solution to the first-order equation,

$$\tilde{p}_t = \lambda \tilde{p}_{t-1} - \frac{\kappa}{\lambda^{-1} - \beta\rho}\rho^{t+1},$$

again starting from the given value for $\tilde{p}_{-1}$, where $0 < \lambda < 1$ is again defined as in (2.17). (Note that in the limiting case $\rho = 0$, this reduces to the same solution (2.17) as before.)

Then using the relations

$$\begin{aligned} E_0 \log P_t &= \tilde{p}_t + 1, \\ E_0 \log Y_t &= -(\tilde{p}_t + \rho^{t+1}), \end{aligned}$$

we can compute the impulse responses of the price level and of aggregate output to an innovation in the nominal spending process. These are plotted in Figure 3.9, for the illustrative parameter values $\beta = .99$, $\kappa = .024$, and $\rho = .4$, along with the impulse response of nominal spending itself.[55] We observe that in this case innovation in nominal spending results in an increase in real activity, that continues to grow larger, reaching its peak only in the second and third quarters after the shock, and then declines monotonically back to its original level. This "hump-shaped" output response is predicted in the case of any large enough value of ρ, specifically, for any value such that $\lambda + \rho > 1$. The responses $\tilde{y}_t \equiv E_0 \log Y_t$ satisfy

$$(1 - \rho L)(1 - \lambda L)\tilde{y}_t = 0$$

for all $t \geq 1$, starting from initial values $\tilde{y}_{-1} = 0$ and

$$\tilde{y}_0 = \frac{(1-\rho)(1-\beta\rho)}{(\lambda^{-1} - \beta\rho)} > 0.$$

A second-order difference equation of this kind has a solution that either decays monotonically back to zero after a peak in the period of the shock, or that first increases to a peak, and then decays monotonically back to zero. Since $\tilde{y}_2 = (\lambda + \rho)\tilde{y}_1$, the second case occurs if and only if $\lambda + \rho > 1$.

[55] The figure also shows the impulse response of the inflation rate, here measured in annual percentage points, so that the inflation rate plotted is defined as $4\,\Delta \log P_t$. The dynamics of the inflation rate are discussed further in section xx below.

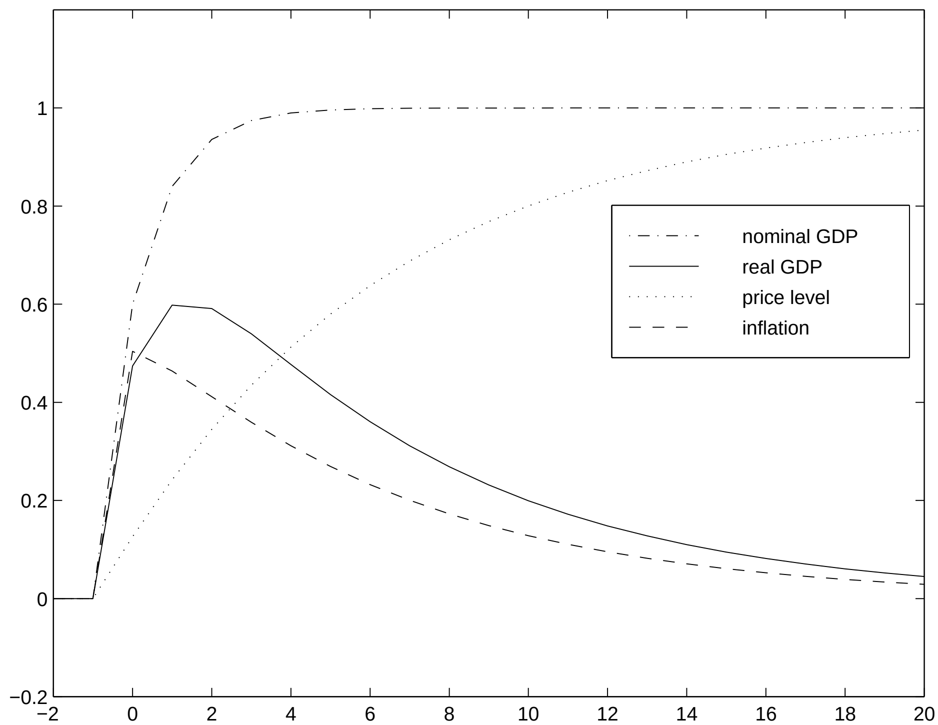

Figure 3.9: Impulse responses to an innovation in nominal GDP, in the case of persistence in nominal GDP growth.

The reason for such an equilibrium response is simple. Even though prices do not change much immediately following the shock, the increase in output is not initially large, because nominal spending has not yet increased much. As nominal spending increases further, real output increases as well. Eventually, prices adjust, and real activity falls back to its original level (or, in a model with trend output growth, back to its original trend).

The "hump-shaped" output response shown in Figure 3.9 is at least roughly of the kind typically estimated using structural VAR methodology (see, *e.g.*, Figure 3.3 above). Indeed, Cochrane (1996) finds that a slightly modified version of the "New Keynesian" aggregate supply relation (2.13) can more closely match the estimated response of output to a monetary policy shock than any of the other simple aggregate supply relations that he considers.[56] A particular advantage of this specification is that it can allow significant output effects of

[56] The form used by Cochrane is actually equation (3.2), with a delay $d = 1$ quarter.

a nominal disturbance even when there is little immediate effect upon nominal spending. This can be seen by considering the above calculation in the case that ρ is close to 1. In such a case, there is little increase in nominal spending in period zero ($log\ \mathcal{Y}_0$ increases only by $1 - \rho$, yet the peak output effect may be an arbitrarily large fraction of 1. (For small enough κ, the value of λ may be made arbitrarily close to 1, so that the adjustment of prices is slow compared to the rate at which expected nominal spending approaches its long-run value.) This contrasts sharply with the prediction of the "New Classical" aggregate supply relation (1.23), which implies that the peak output effect (which must occur in period zero) is bounded above by $1 - \rho$, no matter how flat the short-run Phillips curve may be.

2.5 Consequences of Sectoral Asymmetries

Thus far we have considered only a completely symmetric model, in the sense that all preferences and production relations would remain the same if we interchanged the labels of different goods. In particular, we have assumed that all real disturbances affect supply and demand conditions for all goods in exactly the same way: technical progress lowers the cost of producing each good in exactly the same proportion, and so on. In reality, of course, there are many kinds of disturbances that differentially affect various sectors of the economy. Here we briefly consider an extension of the basic model with staggered pricing that allows for several kinds of asymmetries. One reason for interest in this extension is that in the presence of such asymmetries, it is no longer generally the case that stabilization of an aggregate price index and stabilization of an aggregate output gap are equivalent policies (an important but obviously special implication of the "New Keynesian" Phillips curve (2.13)). Allowing for sectoral asymmetries is also especially important in analyzing monetary policy for an open economy (where, for example, one will want to consider the consequences of shocks that affect the terms of trade), though we do not develop that extension here.[57]

Instead of assuming that the consumption index C_t that enters the utility function of the

[57] In fact, the two-sector model presented here closely resembles the treatment of a two-country monetary union in Benigno (1999).

representative household is defined by the CES index (1.2), let us suppose that it is a CES aggregate of two sub-indices,

$$C_t \equiv \left[(n_1\varphi_{1t})^{\frac{1}{\eta}} C_{1t}^{\frac{\eta-1}{\eta}} + (n_2\varphi_{2t})^{\frac{1}{\eta}} C_{2t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}, \tag{2.24}$$

for some elasticity of substitution $\eta > 0$. The sub-indices are in turn CES aggregates of the quantities purchased of the continuum of differentiated goods in each of the two sectors,

$$C_{jt} \equiv \left[\int_{N_j} c_t(i)^{\frac{\theta-1}{\theta}} di\right]^{\frac{\theta}{\theta-1}},$$

for $j = 1, 2$, where the intervals of goods belonging to the two sectors are respectively $N_1 \equiv [0, n_1]$ and $N_2 \equiv (n_1, 1]$, and once again $\theta > 1$.[58] In the aggregator (2.24), n_j is the number of goods of each type ($n_2 \equiv 1 - n_1$), and the random coefficients φ_{jt} are at all times positive and satisfy the identity $n_1\varphi_{1t} + n_2\varphi_{2t} = 1$. (The variation in the φ_{jt} thus represents a single disturbance each period, a shift in the relative demand for the two sectors' products.)

It follows from this specification of preferences that the minimum cost of obtaining a unit of the sectoral composite good C_{jt} will be given by the sectoral price index

$$P_{jt} \equiv \left[\int_{N_j} p_t(i)^{1-\theta} di\right]^{\frac{1}{1-\theta}}$$

for $j = 1, 2$, and that the minimum cost of obtaining a unit of C_t will correspondingly be given by the overall price index

$$P_t \equiv \left[n_1\varphi_{1t}P_{1t}^{1-\eta} + n_2\varphi_{2t}P_{2t}^{1-\eta}\right]^{\frac{1}{1-\eta}}.$$

The optimal allocation of demand across the various differentiated goods by price-taking consumers will satisfy

$$c_t(i) = C_{jt}\ (p_t(i)/P_{jt})^{-\theta}$$

for each good i in sector j, and the index of sectoral demand will satisfy

$$C_{jt} = n_j\varphi_{jt}\, C_t\ (P_{jt}/P_t)^{-\eta}$$

[58] We need not assume that $\eta > 1$ in order for there to be a well-behaved equilibrium of the two-sector model under monopolistic competition. In fact, the limiting case in which $\eta \to 1$ and the aggregator (2.24) becomes Cobb-Douglas is frequently assumed; see, *e.g.*, Benigno (1999).

for each sector, regardless of total consumption expenditure in a given period. Note that the random factors φ_{jt} appear as multiplicative disturbances in the sectoral demand functions. The aggregators have been normalized so that in the event of a common price p for all goods, the price indices each are equal to that common price, and the demands are equal to $c_t(i) = \varphi_{jt} C_t$ for each good i in sector j.

We furthermore assume a disutility of supplying labor of type i equal to $v(h_t(i);\ \xi_{jt})$ in the case of each good i in sector j; thus we allow for a sector-specific (though not good-specific) disturbance to preferences regarding labor supply. The production function for this good is assumed to be of the form

$$y_t(i) = A_{jt}\ f(h_t(i)),$$

where the function $f(\cdot)$ is common to all goods as before, but we now allow for a sector-specific technology disturbance as well. Assuming that each firm is a wage-taker in a firm-specific labor market as before, nominal profits for firm i in sector j are given by

$$\Pi_t^{ij}(p) = Y_{jt} P_{jt}^{\theta} p^{1-\theta} - w_t(i)\ f^{-1}(A_{jt}^{-1} Y_{jt} P_{jt}^{\theta} p^{-\theta}),$$

generalizing (2.3).

The real marginal cost of supplying any good i in sector j will be given by

$$s^j(y_t(i), Y_t; \tilde{\xi}_t) \equiv \frac{v_h(f^{-1}(y_t(i)/A_{jt}); \xi_{jt})}{u_c(Y_t; \xi_t) A_{jt}}\ \Psi(y_t(i)/A_{jt}),$$

a function of the quantity supplied of the individual good, aggregate output, and sector-specific real disturbances. Because each firm faces a demand curve with constant elasticity θ as before, the desired markup will again be μ. We can then define a natural rate of output Y_{jt}^n for each sector j as the common equilibrium output of each good i in that sector in the case of flexible prices. These levels satisfy

$$\mu\ s^j(Y_{jt}^n, Y_t^n; \tilde{\xi}_t) = \left(\frac{Y_{jt}^n}{n_j \varphi_{jt} Y_t^n} \right)^{-\frac{1}{\eta}},$$

for $j = 1, 2$, where the right-hand side indicates the relative price P_{jt}/P_t required to induce the relative demand Y_{jt}^n / Y_t^n, and the natural rate of aggregate output, Y_t^n, aggregates Y_{1t}^n

and Y_{2t}^n using (2.24). In the case that $\tilde{\xi}_t = 0$ for all t, and in addition $\varphi_{jt} = 1$ for all t and both sectors, then the flexible-price equilibrium involves a common level of output $\bar{Y}$ for all goods, defined in the same way as before. Log-linearizing around this allocation, the real marginal cost function in sector j can be approximated by

$$\hat{s}_t^j(i) = \omega(\hat{y}_t(i) - \hat{Y}_{jt}^n) + \sigma^{-1}(\hat{Y}_t - \hat{Y}_t^n) + \eta(\hat{\varphi}_{jt} + \hat{Y}_t^n - \hat{Y}_{jt}^n), \tag{2.25}$$

generalizing (1.14). (Here $\hat{\varphi}_{jt} \equiv \log \varphi_{jt}$, and the hatted output variables all represent log deviations from $\bar{Y}$.)

Let us now suppose that there is Calvo-style staggered pricing in each of the two sectors, with α_j the fraction of goods prices that remain unchanged each period in sector j. A supplier in sector j that changes its price in period t chooses its new price $p_t(i)$ to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha_j^{T-t} Q_{t,T} [\Pi_T^{ij}(p_t(i))] \right\}, \tag{2.26}$$

generalizing (2.2). This implies a first-order condition that, when log-linearized as before, is of the form

$$E_t \sum_{T=t}^{\infty} (\alpha_j \beta)^{T-t} \left\{ \hat{p}_{jt}^* - [\hat{s}_{t,T}^j - \hat{p}_{jT} + \sum_{\tau=t+1}^{T} \pi_{j\tau}] \right\} = 0, \tag{2.27}$$

where $\hat{p}_{jt}^* \equiv \log(p_{jt}^*/P_{jt})$ is the relative price at date t (relative to other firms in the same sector) of the firms in sector j that have newly revised their prices, $\hat{p}_{jt} \equiv \log(P_{jt}/P_t)$ is the relative price index for sector j (relative to goods prices in the entire economy), $\pi_{jt} \equiv \log(P_{jt}/P_{j,t-1})$ is the rate of inflation in sector j, and $\hat{s}_{t,T}^j$ is the value of $\hat{s}_T^j(i)$ for firms in sector j that last revised their prices at date t. (This generalizes (2.7) above.) Log-linearization of the equation defining the sectoral price index further implies that

$$\pi_{jt} = \frac{1 - \alpha_j}{\alpha_j} \hat{p}_{jt}^*, \tag{2.28}$$

generalizing (2.6).

Equations (2.27) – (2.28) then determine the evolution of the sectoral rate of inflation as a function of expected future inflation in that sector and expectations regarding $\hat{s}_{t,T}^j - \hat{p}_T^j$ at all future dates T. It further follows from (2.25) that

$$\hat{s}^j_{t,T} = \hat{s}^j_T - \omega\theta[\hat{p}^*_{jt} - \sum_{\tau=t+1}^{T} \pi_{j\tau}],$$

where $\hat{s}^j_t$ denotes the deviation of the log of the *average* level of real marginal cost in sector j from its steady-state value. A series of manipulations analogous to those discussed earlier then yields a sectoral inflation equation of the form

$$\pi_{jt} = \xi_j(\hat{s}^j_t - \hat{p}_{jt}) + \beta E_t \pi_{j,t+1}, \tag{2.29}$$

where

$$\xi_j \equiv \frac{1-\alpha_j}{\alpha_j}\frac{1-\alpha_j\beta}{1+\omega\theta} > 0.$$

This generalizes our inflation equation (2.8) for the one-sector model.

Finally, (2.25) implies that

$$\hat{s}^j_t = \omega(\hat{Y}_{jt} - \hat{Y}^n_{jt}) + \sigma^{-1}(\hat{Y}_t - \hat{Y}^n_t) + \eta(\hat{\varphi}_{jt} + \hat{Y}^n_t - \hat{Y}^n_{jt}).$$

Substituting

$$\hat{Y}_{jt} = \hat{\varphi}_{jt} + \hat{Y}_t - \eta\hat{p}_{jt}$$

from the sectoral demand equation, we can alternatively express this as

$$\hat{s}^j_t - \hat{p}_{jt} = (\omega + \sigma^{-1})(\hat{Y}_t - \hat{Y}^n_t) - (1 + \omega\eta)(\hat{p}_{jt} - \hat{p}^n_{jt}),$$

where $\hat{p}^n_{jt}$ is the log relative price index for sector j in a flexible-price equilibrium (a function solely of the exogenous real disturbances). Substituting this into (2.29), we obtain

$$\pi_{jt} = \kappa_j(\hat{Y}_t - \hat{Y}^n_t) + \gamma_j(\hat{p}_{Rt} - \hat{p}^n_{Rt}) + \beta E_t \pi_{j,t+1}, \tag{2.30}$$

generalizing (2.13). Here we have written each of the sectoral relative prices as multiples of the single relative price $\hat{p}_{Rt} \equiv \log(P_{2t}/P_{1t})$ using the identities

$$\hat{p}_{1t} \equiv -n_2\,\hat{p}_{Rt}, \qquad \hat{p}_{2t} \equiv n_1\,\hat{p}_{Rt},$$

$\hat{p}^n_{Rt} = n_1^{-1}\,\hat{p}^n_{2t}$ is the "natural" level of this relative price, and the coefficients are given by

$$\kappa_j \equiv \xi_j\,(\omega + \sigma^{-1}) > 0$$

for $j = 1, 2$, and

$$\gamma_1 \equiv n_2\xi_1\,(1 + \omega\eta) > 0, \qquad \gamma_2 \equiv -n_1\xi_2\,(1 + \omega\eta) < 0.$$

Inflationary pressure in each sector is thus a function of the aggregate level of real activity (relative to its "natural" level) *and* of the sector's relative price index (relative to the "natural" value of this relative price). High aggregate output increases inflationary pressure in both sectors, while a high relative price in one sector reduces inflationary pressure in that sector. Combining equations (2.30) for $j = 1, 2$ with the identity

$$\hat{p}_{Rt} = \hat{p}_{R,t-1} + \pi_{2t} - \pi_{1t}, \tag{2.31}$$

one has a complete system of equations for the evolution of the price indices for both sectors given the evolution of aggregate real activity and the two composite real disturbances $\hat{Y}^n_t$ and $\hat{p}^R_t$. The latter term reflects the various ways in which the real disturbances differentially affect the two sectors, and is given by

$$\hat{p}^n_{Rt} \equiv \frac{1}{\eta}\,[(\hat{\varphi}_{2t} - \hat{\varphi}_{1t}) - (\hat{Y}^n_{2t} - \hat{Y}^n_{1t})].$$

In general, this model implies that inflation in both sectors, and hence aggregate inflation as well, depends on a lagged endogenous variable, $\hat{p}_{R,t-1}$; thus inflation is not so purely forward-looking in this theory as in the fully symmetric (one-sector) case. However, in the case that prices are equally sticky in both sectors ($\alpha_1 = \alpha_2$), $\xi_1 = \xi_2$, and hence

$$n_1\xi_1 + n_2\xi_2 = 0.$$

It follows that if we average the relation (2.30) over the two sectors, weighting each relation by the size of the corresponding sector, we obtain once again equation (2.13) for the evolution of aggregate inflation. Thus we find that the existence of an equilibrium relation of the form (2.13) — implying that there is no incompatibility between stabilization of the overall price index and stabilization of the "output gap" $\hat{Y}_t - \hat{Y}_t^n$ — does not require that all real disturbances affect the demand for and cost of production of each good identically, as assumed earlier. In the case of a model that is otherwise symmetrical (the same degree of price stickiness for all goods, the same form of production function for all goods up to, and so on), a relation of this kind is obtained even in the presence of several types of asymmetric disturbances: disturbances to the relative disutility of supplying different types of labor, disturbances to the relative productivity of labor in different sectors, and shifts in the relative preferences of households for different goods.

On the other hand, if prices are not equally sticky for different types of goods, it ceases to be true that the same policy can simultaneously stabilize the aggregate output gap and the overall inflation rate. The consequences of the resulting tradeoff for optimal stabilization policy are considered in chapter 6.

3 Delayed Effects of Nominal Disturbances on Inflation

We have seen that the assumption of staggered price-setting (as interpreted by Calvo in particular) gives rise to an aggregate-supply relation that fits much better with basic facts about the effects of monetary disturbances than did the simple "New Classical Phillips curve". For this reason, the "New Keynesian Phillips curve" has been employed in many recent discussions of monetary policy that seek to take account of forward-looking private-sector behavior. Nonetheless, even this model has been subject to a good bit of criticism as not fitting too well with econometric evidence regarding the comovements of real and nominal variables.

A central criticism has been that the model implies that inflation should be a more *purely forward-looking* process than it seems to be in reality. Note that (2.13) can be "solved forward" to yield

$$\pi_t = \kappa \sum_{j=0}^{\infty} \beta^j \, E_t[\hat{Y}_{t+j} - \hat{Y}^n_{t+j}]. \tag{3.1}$$

Thus the predicted rate of inflation at any time should depend solely upon the predicted output gaps at that time and later, in a way that is completely independent of either output gaps or inflation in the past. This does not square well with many economists' intuitive view of the inflation process, or with the inflation dynamics implied by the models currently used in most central banks; these models instead assume a substantial degree of *inertia* in the inflation process, so that recent past inflation figures as an important determinant of current inflation.

Clear evidence that the New Keynesian Phillips curve cannot account for the co-movement that is observed between real activity and inflation, or for the persistence of inflation dynamics themselves, is not as easy to obtain as often seems to be assumed. Claims that the equation is grossly at odds with the facts are often based upon the use of one or another conventional "output gap" series as a proxy for $\hat{Y}_t - \hat{Y}^n_t$. For example, (2.13) implies that the series $\pi_{t+1} - \beta^{-1}\pi_t$, which is essentially the rate of acceleration of inflation, should be negatively correlated with $\hat{Y}_t - \hat{Y}^n_t$.[59] Instead, conventional series for the U.S. "output gap" (which subtract one relatively smooth trend or another from a log real GDP series) are generally found to be *positively* correlated with the subsequent acceleration of inflation. But our model gives us good reason to suppose that $\hat{Y}^n_t$ may not be a smooth trend; it should be affected immediately by changes in government purchases or other "autonomous" components of expenditure, and by variations in household impatience to consume or in attitudes toward work, in addition to such slower-moving factors as capital accumulation, technical progress and growth in the labor force. If there are relatively high-frequency variations in $\hat{Y}^n_t$, traditional "gap" measures could easily be *negatively* correlated with $\hat{Y}_t - \hat{Y}^n_t$, owing to policies tending to stabilize output around a smooth trend rather than around the time-varying natural rate (Gali, 1999). The fact that average real unit labor costs — which should

[59] To be precise, it implies that this quasi-acceleration statistic should equal $-\beta^{-1}\kappa\ (\hat{Y}_t - \hat{Y}^n_t)$, plus a forecast error that should be uncorrelated with all period t information, including $\hat{Y}_t - \hat{Y}^n_t$.

correspond to variations in $\hat{Y}_t - \hat{Y}_t^n$ under fairly general assumptions about the nature of the disturbances, as discussed above — are negatively correlated with detrended real GDP suggests that this may well be the case. In fact, real unit labor costs are *negatively* correlated with inflation acceleration in U.S. data, and the New Keynesian Phillips curve accounts quite well for U.S. inflation dynamics when real unit labor costs are used to measure $\hat{Y}_t - \hat{Y}_t^n$, as discussed above.

However, evidence that the simple New Keynesian Phillips curve may indeed be too forward-looking can be found by considering the dynamics of output and inflation in response to identified monetary policy shocks. If the identification of monetary shocks in the VAR studies mentioned earlier is correct, then the estimated impulse responses for real GDP should also correspond to the impulse response for the theoretically correct gap measure, $\hat{Y}_t - \hat{Y}_t^n$, since the path of $\hat{Y}_t^n$ should be unaffected.[60] Hence the estimated impulse responses for real GDP and inflation should satisfy (2.13), or equivalently (3.1).

Typically, they do not. In particular, one generally observes that the main effect of a monetary policy shock on inflation occurs in the quarters *following* those in which the output response is strongest. (For example, Figure 3.10 shows the responses obtained in the study of Christiano *et al.* (2001) discussed earlier.) But this is inconsistent with (3.1), regardless of the assumed parameter values, because this equation states that the inflation response each quarter should be an increasing function of the output responses that are expected in that quarter and *later*. Thus the effect on inflation should *precede* the effect on output, insofar as the latter effect is predictable in advance (as the output impulse response indicates to be the case); and it should *peak earlier* than does the effect on output, since once the peak output effect is reached, the output gaps that can be anticipated from then on are smaller than those that could still be expected a short while earlier. This is not what the VAR studies

[60] This is no longer exactly true once we allow for endogenous capital accumulation, as we do in chapter 4. But even so, the effect upon $\hat{Y}_t^n$ owing to endogenous variation in the capital stock should not be large during the first few quarters. Furthermore, taking account of this effect will only exacerbate the problem sketched here: the true output gap really returns to its previously expected level even *faster* than does real GDP, because increased investment during the early quarters following an interest-rate reduction should raise the natural rate of output. This would only make the tendency of inflation to peak *after* the main effect on the output gap even more dramatic than it appears to be in Figure 3.10.

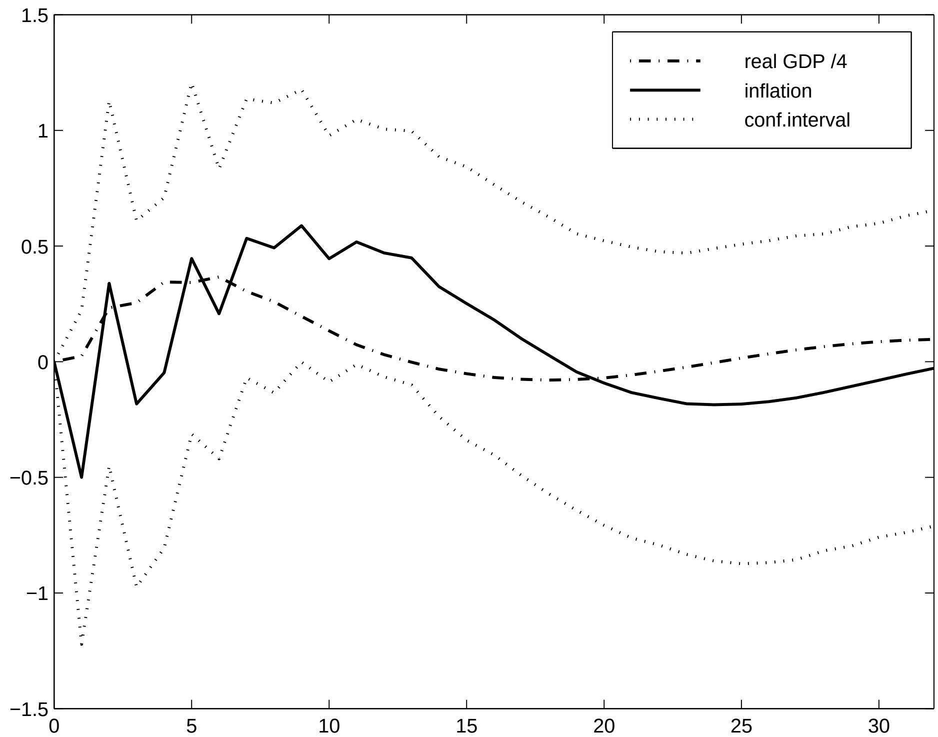

Figure 3.10: Estimated impulse responses of real GDP and inflation to an unexpected interest-rate reduction. Source: Christiano *et al.* (2001).

indicate. Instead, the effects of monetary disturbances on inflation are delayed, and more persistent than would be predicted by our simple model of staggered pricing.

The model of staggered pricing can be extended, however, in a number of ways that make it more realistic in this respect, while continuing to derive inflation dynamics from optimal pricing decisions, subject to certain assumed constraints. Here we give particular attention to two approaches that have been useful in reconciling optimizing models with VAR evidence.

3.1 Staggered Pricing with Delayed Price Changes

One simple way of avoiding the counterfactual prediction that — because of the forward-looking character of inflation in the Calvo model — the rate of inflation should respond *immediately* to a monetary disturbance that is expected to affect nominal expenditure eventually, is to modify the assumption that newly chosen prices take effect immediately. We have assumed in our derivation above that the fraction $1-\alpha$ of prices that change in a given period are chosen optimally, given aggregate conditions in the period in which the new price takes effect. One reason for this assumption was that it allowed us to nest the case of fully flexible prices within our specification (as the limiting case in which $\alpha = 0$). However, the literature on staggered wage- and price-setting has often assumed that new wage contracts and/or price commitments are chosen at a date *before* they first take effect — and as a result, are optimal only conditional upon the information that was available at that earlier date. Allowing for a delay before newly chosen prices take effect will obviously have the consequence that a monetary policy shock will not affect prices until after this delay. It could then also affect output sooner than its effect on inflation.

We can easily modify our presentation above of the discrete-time Calvo model to allow for such delays in the introduction of new prices. Suppose that each of the new prices chosen in period t takes effect only in period $t + d$, for some integer $d \geq 0$. Conditional upon a new price being chosen for a given good in period t, that price applies in periods prior to period $t+d$ with probability zero, in period $t+d$ with probability 1, in period $t+d+1$ with probability α, and more generally in period $t + d + k$ with probability α^k, for any $k \geq 0$. A supplier i that chooses a new price in period t chooses that new price $p_{t+d}(i)$ to maximize

$$E_t \left\{ \sum_{T=t+d}^{\infty} \alpha^{T-t-d} Q_{t,T} [\Pi_T^i(p_{t+d}(i))] \right\},$$

generalizing (2.2), where once again nominal profits each period are given by (2.3). Corresponding to (2.5) we obtain in this case the first-order condition

$$E_t \left\{ \sum_{T=t+d}^{\infty} (\alpha\beta)^{T-t-d} u_c(Y_T;\xi_T) Y_T P_T^{\theta} [p_{t+d}^* - \mu P_T s(Y_T P_T^{\theta} p_{t+d}^{*-\theta}, Y_T; \tilde{\xi}_T)] \right\} = 0$$

to implicitly define the optimal price p_{t+d}^*, that is the same for all suppliers choosing a new price at date t. The value of p_t^* then determines the evolution of the price index P_t through (2.1), just as before.

Through a series of manipulations similar to those presented above, log-linearization of the above first-order condition yields a log-linear aggregate supply relation of the form

$$\pi_t = \kappa E_{t-d}(\hat{Y}_t - \hat{Y}_t^n) + \beta E_{t-d}\pi_{t+1},) \tag{3.2}$$

generalizing (2.13). (All variables and coefficients in this equation have the same definitions as before.) In the case that $d = 1$, this is the form of aggregate supply relation used by Cochrane (1996) and by Bernanke and Woodford (1997). Note that the right-hand side of (3.2) consists entirely of terms that are a function of period $t-d$ information; thus this model implies that inflation π_t is a predetermined variable, depending only upon disturbances in period $t-d$ or earlier. In this case, only fluctuations in $E_{t-d}\hat{Y}_t^n$, the *forecastable component* of the natural rate of output, matter for inflation and output determination.

An alternative interpretation of this model would be not that price changes must actually be determined in advance (say, because advance notice to customers is expected), but rather that when an opportunity to change price arises at date t, the new price (that applies beginning in period t) is chosen on the basis of *old information,* namely, the state of the world as of period $t-d$. This assumption would result in exactly the same optimality criterion for new prices as above, and hence exactly the same aggregate supply relation (3.2). Thus the hypothesis may alternatively be described as one involving *information delays,* as for example in the work of Mankiw and Reis (2001a, 2001b).[61]

Under either interpretation, the model implies that a monetary policy shock in period t has no effect on inflation before period $t+d$. This eliminates an embarrassing feature of the basic "New Keynesian" specification that is especially evident when we recognize that, according to the VAR studies, monetary disturbances affect aggregate nominal expenditure only with a delay (as in Figure 3.2). Suppose that nominal GDP evolves according to a

[61]One advantage of the hypothesis of information delays is that we need not assume the same delay in the case of all types of news. In order to obtain the result that monetary policy shocks do not affect inflation within the first year, for example, it would be necessary to assume a delay of a year in the receipt of information about changes in interest rates; but one might simultaneously assume that other kinds of disturbances are observed by suppliers much more quickly, so that one would not have to assert that *all* price changes are determined entirely by conditions in the previous year and earlier. We do not pursue this extension here, however.

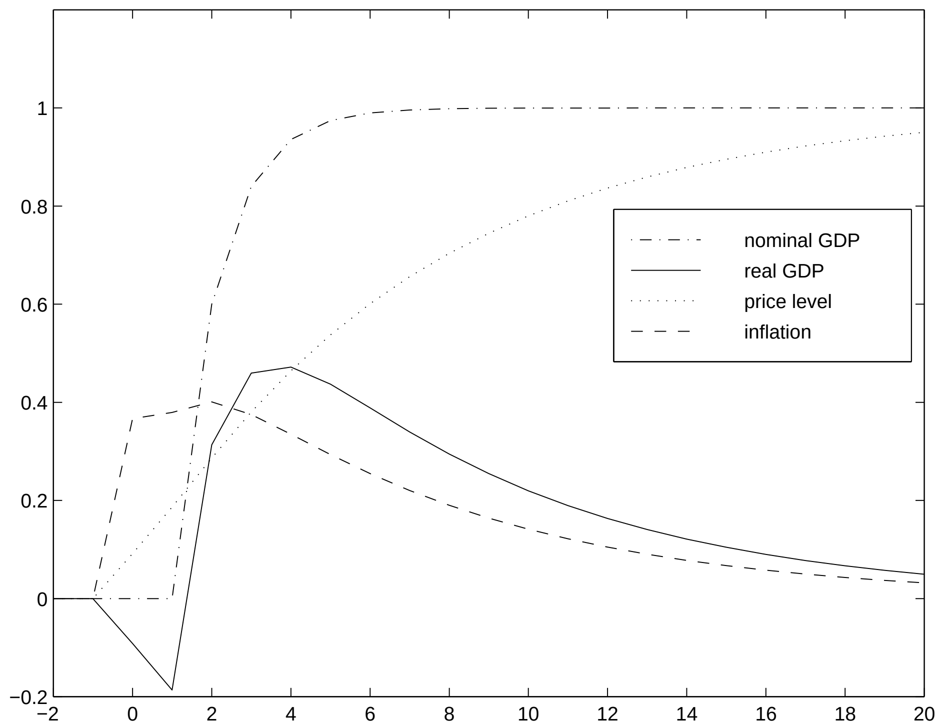

Figure 3.11: Impulse responses to a monetary disturbance with a delayed effect on nominal expenditure ($s = 2$), according to the basic Calvo model.

stochastic process of the form

$$\Delta \log \mathcal{Y}_t = \rho \Delta \log \mathcal{Y}_{t-1} + \epsilon_{t-s}, \tag{3.3}$$

where the integer $s \geq 0$ indicates the lag between the time at which the monetary policy shock occurs and the first time at which it affects nominal GDP. (As before, ϵ_t is assumed to be a mean-zero disturbance realized at date t, completely unforecastable before that date.) We can once again solve for the equilibrium paths of inflation and output given the aggregate supply relation (3.2). We now wish to consider the consequences of alternative assumed delays d in the latter relation.

Figure 3.11 shows the implied impulse responses of inflation and output to a monetary disturbance at date zero according to the standard "New Keynesian" Phillips curve, in the case that the effect of the disturbance on nominal GDP is delayed. Our shock is once again an unexpected loosening of policy that implies an eventual increase in nominal GDP of one percentage point, and the assumed value of ρ is equal to 0.4, as in Figure 3.9; but we now assume that $s = 2$, to match the delay in the effect of an interest-rate reduction on nominal GDP shown in Figure 3.2. (The values assumed for β and κ are the same as in the case of Figure 3.9.)

In the case of the basic "New Keynesian" specification, we obtain the embarrassing prediction that a monetary disturbance that is expected to increase nominal expenditure beginning two periods later should *contract* real activity in the short run. This is because an expectation of higher real activity and/or inflation two periods from now implies that those suppliers who change prices sooner than that (but after learning of the shock) should already raise their prices at a higher than normal rate, in anticipation of high demand and high competitors' prices in the future. Inflation should thus increase immediately in response to the expectation of higher nominal expenditure in the near future; but since nominal expenditure does *not* increase immediately (by hypothesis), this implies a temporary *contraction* of real activity. This is not, of course, at all what estimated output response is like (recall Figure 3.3).

The problem can be solved by assuming a delay $d = 2$ quarters before newly chosen prices take effect (replacing (2.13) by (3.2)). The corresponding impulse responses in this case are shown in Figure 3.12. The predicted responses of all variables are exactly the same as in Figure 3.9, except that the entire impulse response functions are shifted to the right by 2 quarters. Note that the hypothesis of delayed price changes also implies no effect of the monetary shock on output until two quarters later, given the assumption of a delay in the effect of the shock on nominal GDP.[62] The implied response of output in this case is qualitatively fairly similar to what the VAR studies estimate — an effect that is delayed for two quarters, "hump-shaped" thereafter, persistent, and never significantly negative.

This figure still differs from the responses shown in Figure 3.10, though, in that the

[62] Of course, the question remains why an interest-rate reduction should have no effect on nominal GDP until two quarters later. This is a question about the aggregate-demand block of our model, to be deferred until the next chapter.

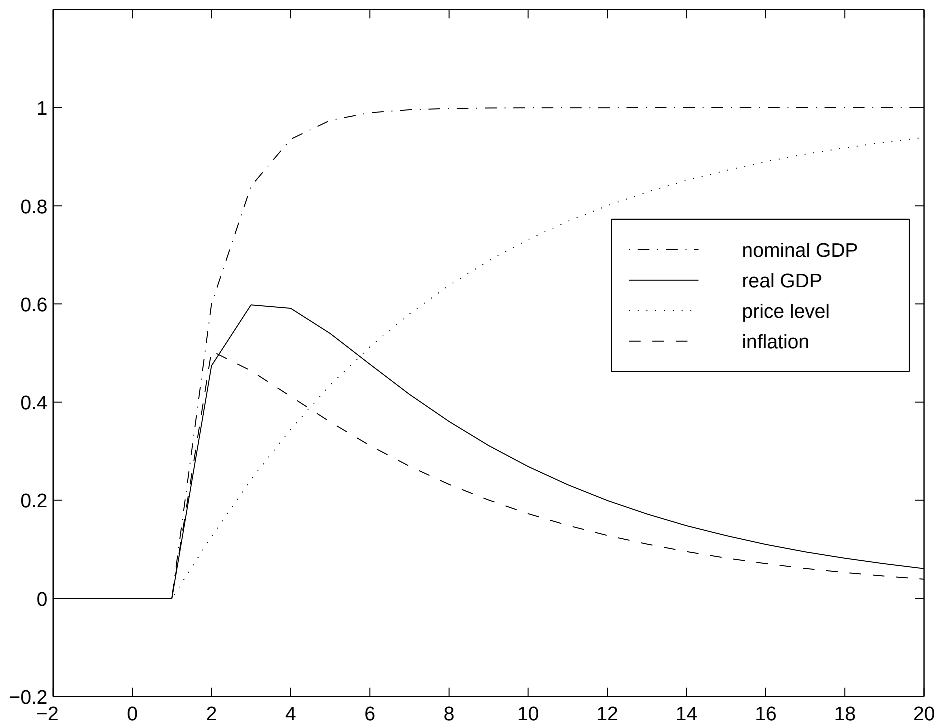

Figure 3.12: Impulse responses to the same monetary disturbance when $d = 2$ quarters in AS relation (3.2).

inflation effect peaks earlier than the output effect. This problem can be ameliorated by assuming an even longer delay d before price changes take effect. (We note that the response shown in Figure 3.10 implies that the price level is no higher than it would have been in the absence of the shock, until six quarters following the shock; so these estimated responses are consistent with d being as long as 6 quarters.) Figure 3.13 shows the corresponding impulse responses in the case that $d = 4$ quarters. We observe now that the effect of the shock on real output peaks before there is any effect on inflation at all (*i.e.*, in the third quarter following the shock).

Even so, the effects on inflation, once the delay d is past, are predicted to appear abruptly; the peak effect on inflation is very clearly in the first quarter in which any effect on prices can occur, with a sharp decline in the inflation effect thereafter. Thus the predicted inflation response still does not exhibit the kind of persistence seen (at least according to the point

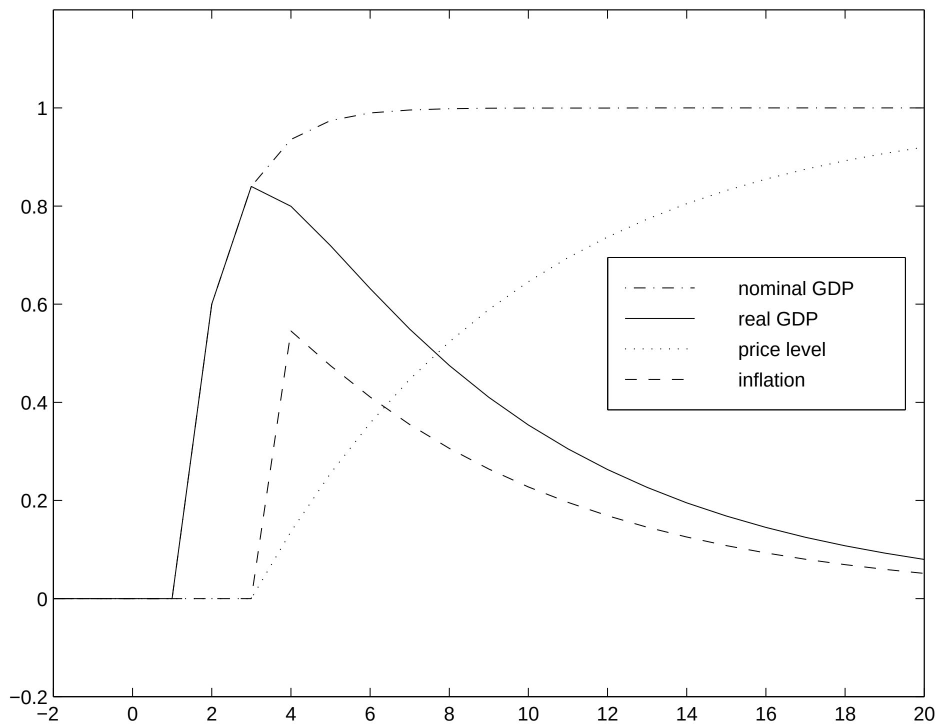

Figure 3.13: Impulse responses to the same monetary disturbance when $d = 4$ quarters.

estimates) in Figure 3.10. A further modification of the Calvo model can help with this.

3.2 Consequences of Indexation to Past Inflation

In the classic Calvo model, it is assumed that prices remain fixed in money terms between those occasions upon which they are re-optimized. While this is a simple way of resolving the question of what to do about prices between revisions, and the assumption conforms with apparent practice at many firms, we have certainly not shown that there is anything optimal about this aspect of the Calvo pricing model; we have instead analyzed optimal pricing policy taking this feature of it as a constraint. One might instead assume that prices are automatically raised in accordance with some mechanical rule between the occasions on which they are reconsidered. If the rule is simple enough, the fact that firms refrain from reconsidering the optimality of their prices for intervals of months at a time will still result in substantial savings on managerial costs, and so it may be plausible that such an interim "rule of thumb" would be used.

One obvious type of more sophisticated interim rule than simply fixing prices in terms of money is one that seeks to correct, in at least a simple way, for increases in the general price index. As an example, Yun (1996) assumes that prices are automatically increased at some rate $\bar{\pi}$ between occasions on which they are reconsidered, where $\bar{\pi}$ is the actual long-run average rate of inflation in the economy. This results in an aggregate supply relation of the form

$$\pi_t = (1-\beta)\bar{\pi} + \kappa\,(\hat{Y}_t - \hat{Y}_t^n) + \beta\; E_t\pi_{t+1},$$

generalizing (2.13). In comparing the expected consequences of monetary policy rules that imply different long-run average rates of inflation, Yun assumes that the parameter $\bar{\pi}$ of firms' pricing policies should change accordingly; this results in a vertical "long-run Phillips curve" relation, unlike the classic Calvo model.

In practice, in economies where inflation has been enough of a problem for indexation of long-term monetary commitments to be worth undertaking, indexation schemes are generally based on a measure of inflation over some relatively short recent time interval, as there is no presumption that inflation can be expected to remain always near some non-zero steady-state value. (Of course, this is related to the fact that there has been little experience of stable commitment to a fixed inflation target in any country, before the past decade!) This suggests that it may be more plausible to assume automatic indexation of price commitments (or wages, as discussed in the next section) to the change in the overall price index over some recent past period. Note, however, that it is not realistic to assume that it should be possible to index individual prices to the *current* price index. Apart from the simultaneity problem that this would create, the assumption that this is possible would not be in the spirit of our assumption that continual monitoring of current conditions in order to maintain a constantly optimal price is too costly to be worthwhile. It is far more plausible, then, to imagine a policy of automatic indexation of one's price (between the occasions on which a full review of the optimality of the price is undertaken) to the change in an overall price index over some *past* time interval.

Christiano *et al.* (2001) and Smets and Wouter (2001) assume partial or full indexation of this kind for both wages and prices, and argue that this extension of the Calvo pricing model improves the empirical fit of their models. Here we examine the consequences of backward-looking indexation for inflation dynamics, continuing for now to assume efficient labor-market contracting. Let us suppose once again that each period a randomly chosen fraction $1-\alpha$ of all prices are reconsidered, and that these are set optimally; but the price of each good i that is *not* reconsidered is adjusted according to the indexation rule

$$\log p_t(i) = \log p_{t-1}(i) + \gamma\,\pi_{t-1}, \tag{3.4}$$

where $0 \le \gamma \le 1$ measures the degree of indexation to the most recently available inflation measure. (Note that even when $\gamma = 1$, as assumed by Christiano *et al.*, nominal rigidities still matter for the effects of aggregate disturbances, because of the one-quarter lag in the indexation.)

This assumption about how prices are adjusted in the interim between re-optimizations affects the way in which prices should be set when they are reconsidered. If we assume, as in the basic Calvo model, that newly-optimized prices take effect immediately, then a new price $p_t(i)$ chosen in period t should be selected to maximize

$$E_t\left\{\sum_{T=t}^{\infty}\alpha^{T-t}Q_{t,T}[\Pi_T^i(p_t(i)\;(P_{T-1}/P_{t-1})^{\gamma})]\right\}.$$

This results in a first-order condition

$$E_t\left\{\sum_{T=t}^{\infty}(\alpha\beta)^{T-t}u_c(Y_T;\xi_T)\;Y_TP_T^{\theta}\;\left(\frac{P_{T-1}}{P_{t-1}}\right)^{\gamma(1-\theta)}\;\left[p_t^* - \mu P_T s(Y_T(p_t^*/P_T)^{-\theta}(P_{T-1}/P_{t-1})^{-\gamma\theta}, Y_T;\tilde{\xi}_T)\right]\right\} = 0$$

to implicitly define the optimal price p_t^*, once again the same for all suppliers choosing a new price at date t. Given the choice of p_t^* each period, the overall price index then evolves according to

$$P_t = \left[(1-\alpha)p_t^{*1-\theta} + \alpha\left(P_{t-1}\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma}\right)^{1-\theta}\right]^{\frac{1}{1-\theta}}, \tag{3.5}$$

generalizing (2.1).

Log-linearization of the first-order condition and of the law of motion (3.5) for the Dixit-Stiglitz price index, together with a series of manipulations analogous to those presented earlier, then yields a log-linear aggregate supply relation of the form

$$\pi_t - \gamma\pi_{t-1} = \kappa\,(\hat{Y}_t - \hat{Y}_t^n) + \beta\, E_t(\pi_{t+1} - \gamma\pi_t),) \tag{3.6}$$

where β, κ, and $\hat{Y}_t^n$ all have the same definitions as in (2.13). The allowance for backward-looking indexation generalizes the New Keynesian Phillips curve in a fairly straightforward way: it is now the quasi-differenced inflation rate, $\pi_t - \gamma\pi_{t-1}$, rather than the inflation rate itself, that is related to the output gap in the way indicated by the previous relation.

In particular, it is still possible to solve (3.6) forward to obtain

$$\pi_t = \gamma\pi_{t-1} + \kappa \sum_{j=0}^{\infty} \beta^j\, E_t[\hat{Y}_{t+j} - \hat{Y}_{t+j}^n],$$

generalizing (3.1). The quasi-differenced inflation rate is still a purely forward-looking function of the expected path of the output gap; but now the inflation rate predicted for periods t and later will depend not only upon the predicted path of the output gap in those periods, but also upon the initial inflation rate π_{t-1}. Thus the extended theory implies *inflation inertia,* to an extent that is greater the larger is the indexation parameter γ.

The difference that is made by a substantial degree of indexation can be illustrated by again considering the predicted impulse responses of inflation and output to a monetary policy shock that results in a persistent increase in the growth rate of nominal GDP. Figure 3.14 shows the predicted impulse responses in the case of aggregate supply relation (3.6), in the case $\gamma = 1$, when (as in Figure 3.9) there is no delay in the effect of the shock on nominal expenditure. We see now that even without the hypothesis of delay before new prices can take effect, it is possible to explain the observed delay in the effect of a monetary disturbance on inflation, relative to its effect on output. We also observe that in this model, the inflation response is "hump-shaped," rather than immediately declining sharply after the quarter of the disturbance, as is predicted by the basic Calvo model (see Figure 3.9).

A number of authors have argued that this kind of modification of the basic Calvo model results in a more realistic specification. Christiano *et al.* (2001) argue that a model with

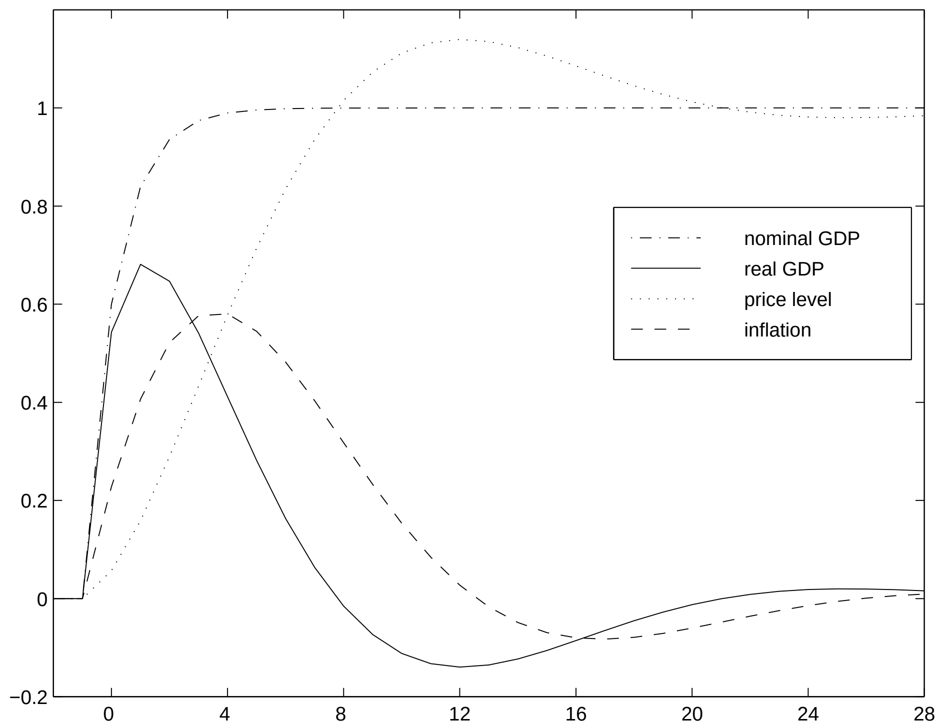

Figure 3.14: Impulse responses to the same monetary disturbance as in Figure 9 ($s = 0$), in the case of backward-looking indexation of prices ($\gamma = 1$).

$\gamma = 1$ better fits their estimated impulse responses than does the standard model with $\gamma = 0$. Smets and Wouter (2001) treat γ as a free parameter (in addition to using a different estimation strategy), and conclude that the best-fitting value of γ is an intermediate value, approximately 0.7. Boivin and Giannoni (2001) also let γ be a free parameter, and estimate (using a strategy based on matching impulse responses, but for a different sample period than that used by Christiano *et al.*) a value of xx.

It should also be noted that when $\gamma = 1$, relation (3.6) is essentially identical to the aggregate supply relation of Fuhrer and Moore (1995a, 1995b), that has been popular in econometric work.[63] Relation (3.6) is also of essentially the same form as the generalization of the Calvo model proposed by Gali and Gertler (1999). Gali and Gertler assume that

[63]In the limiting case $\beta = 1$, the relation is identical to that of Fuhrer and Moore, although the derivation that they offer for their relation is different from the one given here.

prices remain fixed in monetary terms for stochastic intervals of time, as in the Calvo model; but when prices are adjusted, some prices are chosen optimally (as in the Calvo model), while others are adjusted according to a backward-looking rule of thumb that introduces dependence upon lagged inflation. The fraction of suppliers who are backward-looking is treated as a free parameter; variation in this parameter has essentially the same effect as variation in γ in the indexation model. In the limiting case $\beta = 1$, the two models have identical implications; relation (3.6) for any given value of γ can be obtained from the Gali-Gertler model through an appropriate choice of the fraction of backward-looking price-setters. While Gali and Gertler find that the basic Calvo model fits U.S. inflation dynamics fairly well (once real unit labor cost is used as a proxy for the output gap, rather than a traditional output-based measure), their instrumental variables estimates indicate significant rejection of the hypothesis of no backward-looking price-setters. Their point estimates for U.S. data since 1980 indicate a fraction of backward-looking price-setters that would imply (in terms of our notation) a value of γ on the order of 0.6. Gali, Gertler and Lopez-Salido (2000) obtain similar results using European data. Thus there exists a fair amount of consensus — using a variety of empirical proxies for the output gap, and a variety of estimation strategies — that the relation (3.6) better characterizes U.S. inflation dynamics when an indexation parameter between 0.5 and 1.0 is included.

The backward-looking indexation model retains one unfortunate feature of the basic Calvo model, however, even when γ is large. That is that if a monetary disturbance increases nominal GDP only with a lag, as indicated in Figure 3.2, the model predicts that output should initially *contract* in response to such a shock. The reason is that the expectation of a future output increase implies a desire to increase $\pi_t - \gamma\pi_{t-1}$ immediately; so there should be an increase in inflation even before nominal GDP begins to increase.

This problem can be solved, once again, by assuming a delay of d quarters before a newly chosen price takes effect. (In the meantime, a firm's price continues to be adjusted using (3.4).) In this case, the aggregate supply relation instead becomes

$$\pi_t - \gamma\pi_{t-1} = \kappa\, E_{t-d}(\hat{Y}_t - \hat{Y}_t^n) + \beta\, E_{t-d}(\pi_{t+1} - \gamma\pi_t).) \tag{3.7}$$

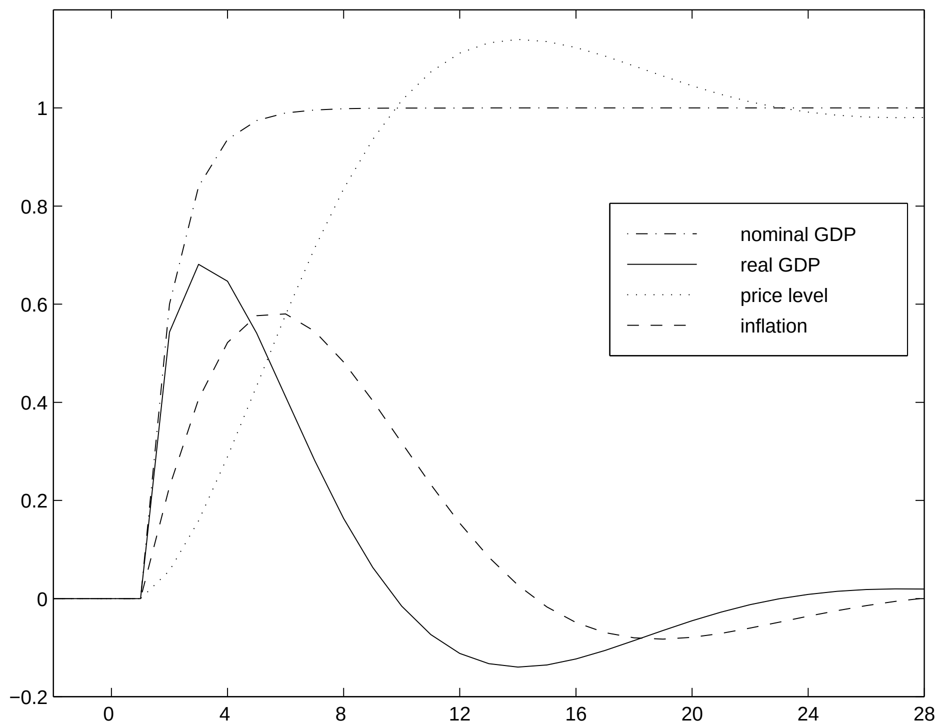

Figure 3.15: Impulse responses to the same monetary disturbance as in Figures 11-13 ($s = 2$), in the case of backward-looking indexation ($\gamma = 1$) and a delay of $d = 2$ quarters in price changes.

Figure 3.15 shows the dynamic responses of output and inflation to an innovation in the process (3.3) for nominal GDP, in the case that $s = 2$ quarters for conformity with Figure 3.2, and in the case of aggregate supply relation (3.7) with $\gamma = 1$ and $d = 2$. In this case there is no effect on output until two quarters following the shock, and then a "hump-shaped" response, as in Figure 3.3. In addition, inflation exhibits a "hump-shaped" response of its own, which peaks later than the output response; in this respect, note that this figure is much more similar to Figure 3.10 than was Figure 3.12 (the case $\gamma = 0$). Figure 3.16 shows the responses to the same disturbance in the case that $d = 4$. This results in a stronger output response and a further delay in the inflation response. This figure results in predicted responses that are most similar to those implied by the VAR estimates shown in Figure 3.10.[64]

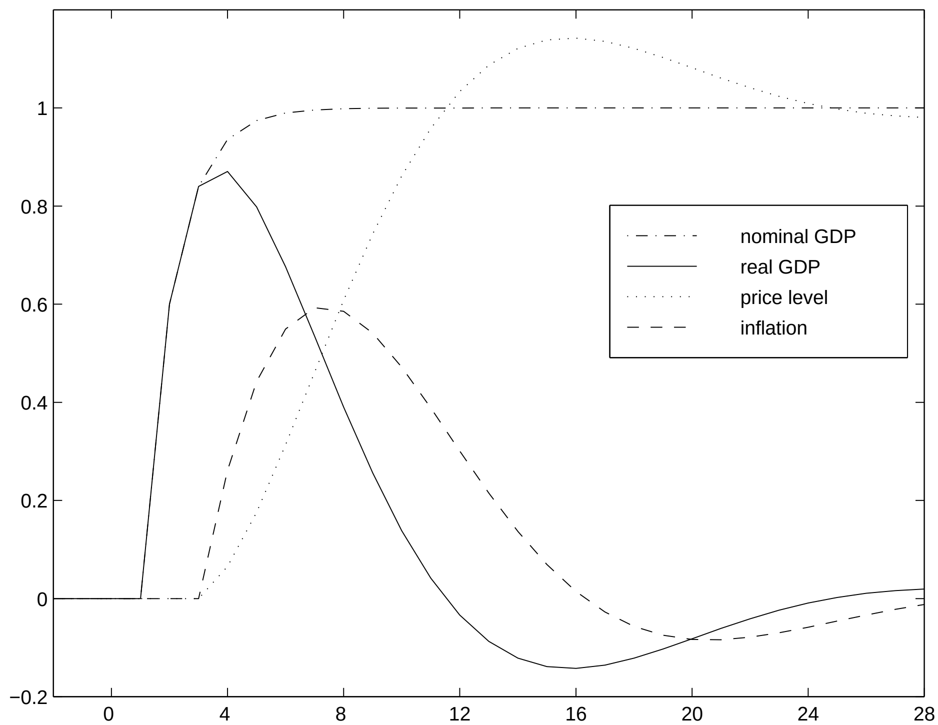

Figure 3.16: Impulse responses to the same monetary disturbance as in Figure 15, in the case that $\gamma = 1$ and $d = 4$ quarters.

4 Consequences of Nominal Wage Stickiness

Thus far the only form of nominal rigidity considered is a delay of one sort or another in the adjustment of the money prices at which goods are offered for sale. In particular, we have not assumed any corresponding stickiness of nominal wages, though this is another familiar explanation for the real effects of monetary policy, emphasized for example in Keynes (1936). We have instead assumed that wages are fully flexible, or equivalently (as far as the derivation of our aggregate supply relations is concerned) that there is efficient contracting between firms and their workers.[65]

[64] We could do better at quantitatively matching the responses shown in Figure 3.10 if we were to treat κ as a free parameter. In the experiments reported here, we have fixed the value of κ at the value estimated by Rotemberg and Woodford (1997). However, their estimates are for a model without backward-looking indexation, in addition to being based on a VAR and a sample period different from those of Christiano *et al.*

Our exclusive emphasis upon price stickiness has allowed our models to take a particularly simple form; in particular, it has been possible, for many purposes, to analyze inflation and output dynamics without any reference to the labor market. Furthermore, as between the two simple hypotheses (*only* sticky prices or *only* sticky wages), the hypothesis of sticky prices is often regarded as more compelling on both theoretical and empirical grounds. We have mentioned in the introduction to this chapter the objection that nominal wages might be constant in the face of disturbances even when the effective cost of marginal hours of labor to firms changes, owing to the existence of implicit contracts between firms and workers. The hypothesis of pure wage stickiness has also often been criticized on account of its implication that real wages should move countercyclically (a criticism of Keynes' model first raised by Dunlop, 19xx, and Tarshis, 19xx). More relevant than overall business-cycle correlations is the finding, in VAR studies such as that of Christiano *et al.* (1997, 2001), of mildly pro-cyclical real wage movements in response to identified monetary policy disturbances. (See, for example, Figure 3.17, which shows the estimated impulse response of the average real wage from the latter study. Here the impulse response of real GDP to the same type of disturbance is also shown for purposes of comparison.) Since these responses ought to be uncorrelated with changes in technology, the failure of the real wage to sharply decline at the time of the increase in real activity is difficult to reconcile with a sticky-wage/flexible-price model.

But even if a model with *only* sticky wages is unappealing, it may be desirable to allow for stickiness of wages as well as prices. Indeed, the study by Christiano *et al.* (1997) criticizes sticky-price (but flexible-wage) models of the monetary transmission mechanism on the ground that they imply *too sharp* a real wage decline in response to a tightening of monetary policy — one so strong that producers' profits ought actually to increase, despite

[65] As far as the relation between price changes and output is concerned, it is enough that goods suppliers face a cost of marginal labor input at each point in time that is equal to the marginal disutility of labor supply expressed in the monetary unit of account. It does not matter whether this occurs because this is the wage that clears a competitive spot market for labor (as assumed explicitly above in our discussion of the relation between unit labor costs and prices), or because an efficient labor contract leads firms to internalize the cost to their workers of requiring additional hours of work.

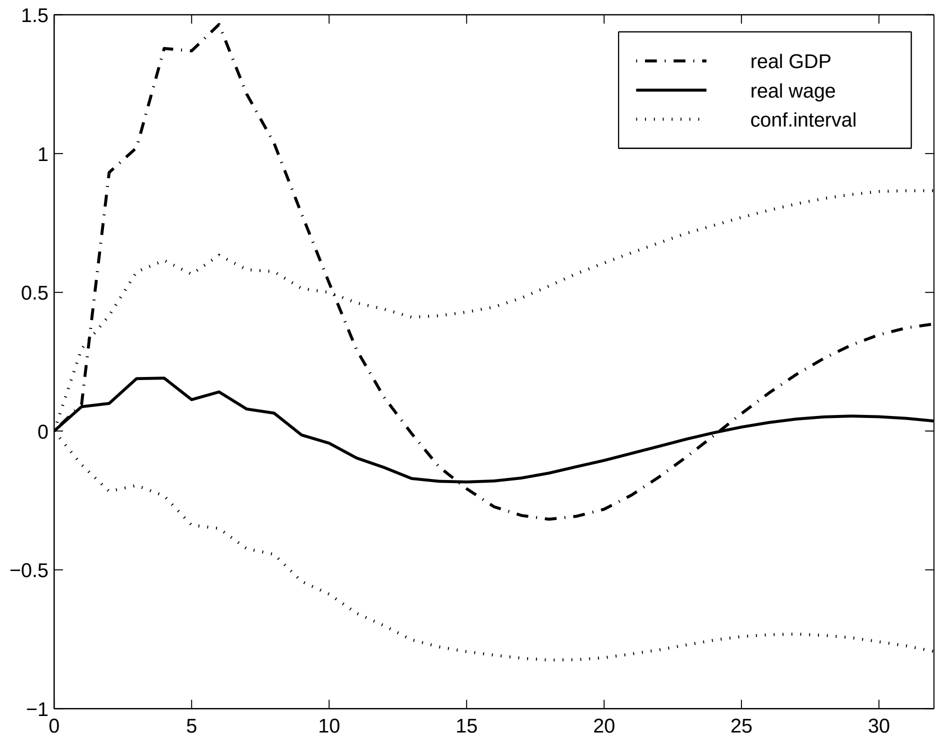

Figure 3.17: Estimated impulse response of the real wage to an unexpected interest-rate reduction. Source: Christiano *et al.* (2001).

their reduced sales (and contrary to fact). This problem can be ameliorated by assuming preference parameters that imply more elastic labor supply, as in the model of Rotemberg and Woodford (1997). But slow adjustment of wages to changes in labor demand may be a more plausible explanation for the relatively modest response of real wages seen in Figure 3.17, and, as we discuss in chapter 6, the choice among these two explanations matters for welfare analysis. Hence we develop here an extension of our baseline model that incorporates wage as well as price stickiness.

4.1 A Model of Staggered Wage-Setting

Here we follow Erceg *et al.* (1999) and model wage-setting in a way that is directly analogous to the model of staggered pricing introduced by Calvo (1983). We introduce wage-setting agents by assuming monopolistic competition among the suppliers of differentiated types of labor, analogous to our treatment above of the goods market. The exposition will be simplest if we follow Erceg *et al.* in assuming a single economy-wide labor market, with the producers of all goods hiring the same kinds of labor and facing the same wages.[66] However, we assume that the labor used to produce each good is a CES aggregate of the continuum of individual types of labor supplied by the representative household, defined by

$$H_t \equiv \left[\int_0^1 h_t(j)^{\frac{\theta_w - 1}{\theta_w}} dj\right]^{\frac{\theta_w}{\theta_w - 1}} \tag{4.1}$$

for some elasticity of substitution $\theta_w > 1$. Here $h_t(j)$ is the labor of type j that is hired; note that the continuum of differentiated types of labor is no longer identified with the continuum of differentiated goods indexed by i, as labor of all types is used in producing each good. It follows that the demand for labor of type j on the part of wage-taking firms will be given by

$$h_t(j) = H_t \left(\frac{w_t(j)}{W_t}\right)^{-\theta_w}, \tag{4.2}$$

where $w_t(j)$ is the (nominal) wage demanded for labor of type j and W_t is a wage index defined analogously with (1.3).

We assume that the wage for each type of labor is set by the monopoly supplier of that type, who then stands ready to supply as many hours of work as turn out to be demanded at that wage. As in our model of monopolistic competition in the goods market, we assume an independent wage-setting decision for each type j, made under the assumption that the choice of that individual wage has no non-negligible effect upon the the wage index W_t or upon the demand H_t for the labor aggregate.[67] We furthermore assume, as in the Calvo

[66]The model presented here differs, however, from that of Erceg *et al.* in that we do not assume that capital can be instantaneously reallocated among firms so as to equalize the return to capital services across firms that change their prices at different times. Instead, as in the baseline model of the previous section, capital is here assumed to be completely immobile. We also adopt less parametric specifications of preferences and technology than in the paper of Erceg *et al.*

[67]Both the assumption of a monopoly supplier of each type of labor and that the supplier of an individual type of labor has no power to affect the wage index W_t indicate that we can no longer assume a continuum of identical households, each supplying the same continuum of types of differentiated labor. Instead, we must assume that households specialize in supplying a particular type of labor, the disutility of which is additively separable from the utility of consumption as before. Risk-sharing among the households that supply types of labor that set their wages at different dates can then result in a common budget constraint for each of the households, which is the same as if each household were to receive its *pro rata* share of the economy's total wage bill, as earlier.

model of staggered pricing, that each of the wages is adjusted with only a probability $1-\alpha_w$ each period, for some $0 < \alpha_w < 1$, which probability is independent of the time since a given wage was last adjusted or the current level of that wage.

It follows that a wage $w_t(j)$ that is adjusted in period t should be chosen to maximize

$$E_t \left\{ \sum_{T=t}^{\infty} (\alpha_w \beta)^{T-t} [\Lambda_T w_t(j) h_T(w_t(j)) - v(h_T(w_t(j)); \xi_T)] \right\}, \tag{4.3}$$

where Λ_T is the representative household's marginal utility of nominal income[68] in period T and the dependence of labor demand $h_T(j)$ upon the wage is given by (4.2). (We have omitted a j in our notation for this last function because the function $h_T(w)$ is the same for all j.) The solution to this problem satisfies the first-order condition

$$E_t \left\{ \sum_{T=t}^{\infty} \alpha_w^{T-t} Q_{t,T} H_T W_T^{\theta_w} [w_t(j) - \mu_w \mathcal{V}(h_T(j), C_T; \xi_T) P_T] \right\} = 0, \tag{4.4}$$

which has a form analogous to (2.4). Here

$$\mathcal{V}(h, C; \xi) \equiv \frac{v_h(h; \xi)}{u_c(C; \xi)}$$

is the marginal rate of substitution between work and consumption for the supplier of a given type of labor, and $\mu_w \equiv \theta_w/(\theta_w - 1) > 1$ is the desired markup of a household's real wage demand over its marginal rate of substitution owing to its monopoly power. If we substitute (4.2) for $h_T(j)$ in (4.4), we obtain a relation that implicitly defines the optimal wage choice w_t^*, which is the same for all wages j that are adjusted at date t. The choice of w_t^* then determines the evolution of the wage index W_t, through a law of motion analogous to (2.1).

Again it is useful to approximate equilibrium wage dynamics in the case of small disturbances using a log-linear approximation, computed under the assumption that W_t/P_t, P_t/P_{t-1} and w_t^*/W_t all remain close to their steady state-values ($\bar{w}, 1$ and 1 respectively) at all times. For the law of motion of the wage index we obtain

$$\pi_t^w \equiv \Delta \log W_t = \frac{1-\alpha_w}{\alpha_w} \hat{w}_t^*, \tag{4.5}$$

[68]Note that although there is technically not a representative household in the present model, in the sense of a household whose trades are equal to its per capita share of all trades in the economy (owing to labor specialization), it remains true that all households have the same consumption budget and choose the same consumption. Hence they have the same marginal utility of income Λ_t at all times, equal to $u_c(C_t; \xi_t)/P_t$.

where $\hat{w}_t^* \equiv \log(w_t^*/W_t)$, analogous to the approximate relation (2.6) for the price index. As an approximation to (4.4) we obtain

$$E_t \sum_{T=t}^{\infty} (\alpha_w \beta)^{T-t} \left\{ \hat{w}_t^* - [\hat{v}_{t,T} - \hat{w}_T + \sum_{\tau=t+1}^{T} \pi_\tau^w] \right\} = 0, \tag{4.6}$$

where $\hat{v}_{t,T}$ denotes the deviation (from its steady-state value) of the log of the marginal rate of substitution between labor and consumption in period T for a type of labor with wage w_t^*, and $\hat{w}_T$ denotes deviation of the log of the real wage W_T/P_T from $\log \bar{w}$, analogous to relation (2.7) for optimal price-setting. Noting furthermore that

$$\hat{v}_{t,T} = \hat{v}_T - \nu\theta_w [\hat{w}_t^* - \sum_{\tau=t+1}^{T} \pi_\tau^w],$$

where $\hat{v}_T$ is the average value of $\hat{v}_{t,T}$ over different types of labor, and

$$\nu \equiv \frac{v_h h(\bar{h};0)}{\bar{h} v_h(\bar{h};0)} > 0,$$

we can solve (4.6) for

$$\hat{w}_t^* = (1-\alpha_w\beta) E_t \sum_{T=t}^{\infty} (\alpha_w\beta)^{T-t} [(1+\nu\theta_w)^{-1}(\hat{v}_T - \hat{w}_T) + \sum_{\tau=t+1}^{T} \pi_\tau^w].$$

This allows us to express the relative wage of types of labor that adjust their wage at date t as a forward-looking function of aggregate conditions.

Manipulations directly analogous to those used in our analysis of staggered pricing allow us to obtain a relation of the form

$$\Delta \log W_t = \xi_w [\hat{v}_t + \log \bar{w} + \log P_t - \log W_t] + \beta E_t [\Delta \log W_{t+1}], \tag{4.7}$$

where

$$\xi_w \equiv \frac{(1-\alpha_w)(1-\alpha_w\beta)}{\alpha_w(1+\nu\theta_w)} > 0.$$

If we continue to assume a production function of the form (1.7) for each good (*i.e.*, a fixed allocation of capital across firms, even though there is a single economy-wide labor market), the corresponding equation for the price dynamics is

$$\Delta \log P_t = \xi_p [\hat{\psi}_t - \log \bar{w} + \log W_t - \log P_t] + \beta E_t [\Delta \log P_{t+1}], \tag{4.8}$$

where $\hat{\psi}_t$ denotes minus the average deviation of the log marginal product of labor from its steady-state value. Here

$$\xi_p \equiv \frac{(1-\alpha_p)(1-\alpha_p\beta)}{\alpha_p(1+\omega_p\theta_p)} > 0,$$

where the parameters previously denoted α and θ are now denoted α_p and θ_p, and ω_p is defined as in (1.15). Note that equation (4.8) is just an alternative way of writing (2.8), given that

$$\hat{s}_t = \log W_t - \log P_t + \hat{\psi}_t.$$

(The only difference is that the expression for ξ_p is different now owing to the assumption of an economy-wide labor market.)

Our assumed forms for the utility and production functions furthermore imply that

$$\begin{aligned} \hat{\psi}_t &= \omega_p \hat{Y}_t - (1+\omega_p)a_t, \\ \hat{v}_t &= (\omega_w + \sigma^{-1})\hat{Y}_t - (\omega + \sigma^{-1})\hat{Y}_t^n + (1+\omega_p)a_t, \end{aligned}$$

where ω_w, ω_p represent the decomposition presented in (1.15) and $a_t \equiv \log A_t$. In deriving the second equation, we use the fact that $\omega_w \equiv \nu\phi$, where

$$\phi \equiv \frac{f(\bar{h})}{\bar{h}f'(\bar{h})} > 1$$

is the elasticity of the representative firm's labor requirement with respect to its level of production. Note that $\log \bar{w} + \hat{v}_t$ represents the equilibrium log real wage in a model with flexible wages, so that the above expressions imply expression (2.12) for the case of flexible wages. Substituting these into (4.7) and (4.8), we obtain a pair of coupled equations for evolution of wages and prices given the path of real activity and the exogenous disturbances. These can be written in the form

$$\Delta \log W_t = \kappa_w(\hat{Y}_t - \hat{Y}_t^n) + \xi_w(\log w_t^n + \log P_t - \log W_t) + \beta E_t[\Delta \log W_{t+1}], \quad (4.9)$$

$$\Delta \log P_t = \kappa_p(\hat{Y}_t - \hat{Y}_t^n) + \xi_p(\log W_t - \log P_t - \log w_t^n) + \beta E_t[\Delta \log P_{t+1}], \quad (4.10)$$

where

$$\kappa_w \equiv \xi_w(\omega_w + \sigma^{-1}) > 0, \qquad \kappa_p \equiv \xi_p\omega_p > 0,$$

and

$$\log w_t^n \equiv \log \bar{w} + (1+\omega_p)a_t - \omega_p \hat{Y}_t^n$$

represents the "natural real wage", *i.e.*, the equilibrium real wage when both wages and prices are fully flexible.

These equations generalize the aggregate supply relation derived earlier for the flexible-wage model in an obvious respect. Note that if we define a particular weighted average of wage and price inflation,

$$\bar{\pi}_t \equiv \frac{\xi_p^{-1}\pi_t + \xi_w^{-1}\pi_{wt}}{\xi_p^{-1} + \xi_w^{-1}}, \tag{4.11}$$

then the corresponding weighted average of (4.9) and (4.10) reduces to

$$\bar{\pi}_t = \kappa(\hat{Y}_t - \hat{Y}_t^n) + \beta E_t \bar{\pi}_{t+1}, \tag{4.12}$$

where

$$\kappa \equiv \frac{\sigma^{-1} + \omega}{\xi_p^{-1} + \xi_w^{-1}} > 0 \tag{4.13}$$

is a coefficient that is smaller the greater the degree of rigidity of *either* wages or prices. In the limit as $\alpha_w \to 0$, $\xi_w \to \infty$, $\bar{\pi}_t$ simply measures price inflation, and (4.12) is again the aggregate supply relation (2.13) obtained earlier, in which (4.13) corresponds once more to (2.14) using the value (1.25) for ζ. More generally, we find that an alternative inflation index is the same kind of purely forward-looking function of the output gap, but this index involves wage inflation with a weight that is greater the greater the relative stickiness of wages. In the limiting case that *only* wages are sticky, (4.12) becomes a "Phillips curve" for wages.

4.2 Sticky Wages and the Real Effects of Nominal Disturbances

We turn now to the implications of wage stickiness for the real effects of fluctuations in nominal expenditure due to a purely monetary disturbance. As is well-known, nominal wage stickiness implies that such disturbances should have temporary effects on real activity, during the time that it takes for wages to adjust, even in the case of fully flexible goods prices. A more subtle question is whether this mechanism should result in real effects that are more or less persistent than those that would result from sticky prices. Some authors (*e.g.*, Andersen, 1998; Huang and Liu, 1998) have argued that sticky wages result in more persistent effects than do sticky prices, and that the assumption of sticky prices (as in our treatment in the earlier part of this chapter) therefore underestimates the likely importance of the real effects of monetary policy.

In order to compare the degree of persistence resulting from wage stickiness with that resulting from price stickiness, it is useful to consider once again the effects of an unexpected permanent increase in nominal GDP, unrelated to any real disturbance (that could affect $\hat{Y}_t^n$ or w_t^n). Let us again assume a shock that results in (2.15) holding for all $t \geq 0$, and consider the expected paths $E_0 \log P_t$, $E_0 \log W_t$ and $E_0 \log Y_t$ that are consistent with the system (4.9) – (4.10), given initial conditions $\log W_{-1} = \log P_{-1} = 0$.

In the case of sticky wages but purely flexible prices, (4.10) reduces to an equilibrium relation between the real wage and the output gap, that can be written in the form

$$\log W_t - \log P_t = \log w_t^n - \frac{\omega_p}{1+\omega_p}(\log \mathcal{Y}_t - \log W_t + \log w_t^n - \log Y_t^n); \tag{4.14}$$

Substituting this into (4.9) yields

$$\Delta \log W_t = \hat{\kappa}[\log \mathcal{Y}_t - \log W_t + \log w_t^n - \log Y_t^n] + \beta E_t[\Delta \log W_{t+1}]$$

for nominal wage dynamics given the evolution of nominal GDP and the real disturbances, where

$$\hat{\kappa} \equiv \frac{\tilde{\kappa}}{1+\omega_p} > 0.$$

This equation has the same form as the equation for *price* dynamics in the model with flexible wages and sticky prices, and so our reasoning in section xx above directly applies. It follows that if we let $\tilde{w}_t$ denote $E_0 \log W_t - 1$, the unique bounded solution is of the form

$$\tilde{w}_t = -\lambda^{t+1},$$

by analogy with (2.17), where $0 < \lambda < 1$ is the smaller root of (2.18) when $\hat{\kappa}$ is substituted for κ. Then noting that (4.14) implies as well that

$$\log Y_t = \log Y_t^n + \frac{1}{1+\omega_p}(\log \mathcal{Y}_t - \log W_t + \log w_t^n - \log Y_t^n),$$

we see that (4.2) implies an output response of the form

$$E_0 \log Y_t = -(1+\omega_p)^{-1}\tilde{w}_t = (1+\omega_p)^{-1}\lambda^{t+1}.$$

The degree of persistence of the output response thus depends upon the root λ, and hence upon the size of $\tilde{\kappa}$, in the same way as in the sticky-price model. The only possibility of a difference between the two models as to the likely degree of persistence would be if it is judged more plausible that $\tilde{\kappa}$ should be small in the case of the sticky-wage model than that κ should be small in the case of the sticky-price model. One possible reason for this, of course, would be if it were observed that wages are in fact adjusted less frequently than prices. But it is not obvious that this is true to any dramatic extent; in an economy like that of the U.S., most wages, like most prices, are adjusted annually if not more often. (And this is in any event not the basis of Andersen's argument, that assumes two-period commitments in each case.)

Let us instead consider the likelihood of there existing a "contract multiplier" in the sense discussed above. We have seen that in the case of the sticky-price model, $\lambda > \alpha_p$, so that output effects decay more slowly than the rate at which prices are revised following the shock, if and only if $\zeta < 1$ (the case of strategic complementarity among pricing decisions), where ζ is again given (for the present setup) by (1.25). We can similarly show that for the sticky-wage model, $\lambda > \alpha_w$ if and only if $\tilde{\zeta} < 1$, where

$$\tilde{\zeta} = \frac{\omega + \sigma^{-1}}{(1+\nu\theta_w)(1+\omega_p)}. \tag{4.15}$$

(Once again, the size of $\tilde{\zeta}$ can be interpreted as a measure of the degree of strategic complementarity among the wage-setting decisions of suppliers of different types of labor.) Our question then reduces to asking whether it is more plausible that $\tilde{\zeta}$ should be less than one in the sticky-wage model than it is that ζ should be less than one in the sticky-price model.

This turns out to be true in the analysis of Andersen (1998), but his simple model omits a number of important considerations. His analysis of the sticky-wage model effectively assumes (in terms of our notation) that $\nu = \sigma^{-1} = 0$ (which also implies that $\omega_w = 0$), and hence concludes that

$$\tilde{\zeta} = \frac{\omega_p}{1 + \omega_p} < 1,$$

implying the existence of a "contract multiplier" regardless of the size of $\omega_p > 0$. However, this result is not as general as it would appear; a contract multiplier less than one is still theoretically possible in the sticky-wage model (if, for example, one assumes $\nu = 0$ but $\sigma < 1$).

Andersen's analysis of the sticky-price model similarly effectively assumes that $\omega_p = \sigma^{-1} = 0$ (which also implies that $\omega_w = \nu$), and hence concludes that $\zeta = \nu$, which may or may not be less than one.[69] But this comparison of the two models is misleading. In fact, as we have seen, the numerators of the expressions for ζ and $\tilde{\zeta}$ are identical, both being equal to $\omega + \sigma^{-1}$; Andersen obtains the value ω_p for the sticky-wage model (which should be small even if ν is large) only because he has assumed $\nu = 0$ in that model. The most important difference between the two cases is instead the presence of the factor $(1 + \nu\theta_w)$ in the denominator of (4.15), whereas the corresponding factor in the denominator of (1.25) is $(1 + \omega_p\theta_p)$ — neither of which factors appear in Andersen's analysis. If one regards ν as being substantially larger than ω_p,[70] and one supposes that there is at least as much substitutability among types of labor as among different goods (so that $\omega_w \geq \omega_p$), then it would indeed follow that $\tilde{\zeta}$ should be significantly smaller than ζ, and wage stickiness would lead to more output persistence than would price stickiness.

This is in fact the basis for the conclusions of Huang and Liu (1998), who also present a comparison of the consequences of wage and price stickiness, but with explicit micro-foundations for the assumed wage- and price-setting equations (and hence consistent assump-

[69] Andersen suggests that the most plausible assumption about preferences would involve $\nu > 1$; smaller values are popular in calibrated quantitative business-cycle models in order to account for the relative acyclicality of real wages, but he proposes that this reflects wage stickiness rather than the nature of preferences regarding labor supply.

[70] It is not obvious that this must be true. For example, Hansen (1995) argues for preferences with $\nu = 0$, as a result of the indivisibility of labor; in his model, variation in hours worked is entirely associated with variation in the number of workers who work a shift of fixed length, rather than variation in the number of hours worked by each member of the labor force, as assumed in the representative-household model used here.

tions about preferences and technologies in the two cases). Their basic model (abstracting from capital accumulation and endogenous velocity of money) is a special case of the one just presented, in which the production function $f(\cdot)$ is assumed to be linear, so that $\omega_p = 0$ and $\omega_w = \nu$, and in which $u(C) = \log C$, so that $\sigma = 1$. They accordingly find that

$$\tilde{\zeta} = \frac{1+\nu}{1+\nu\theta_w} < 1, \qquad \zeta = 1 + \nu > 1,$$

so that there is necessarily strategic complementarity in the sticky-wage model and necessarily none in the sticky-price model.

Yet this conclusion depends upon ignoring a number of reasons discussed earlier for a low value of ζ to be plausible in the model with only sticky prices. In particular, we have shown above that if we assume that the producers of different goods hire labor from distinct labor markets, the value of ζ is instead given by (1.22). In this case, the factor in the denominator is $(1 + \omega\theta_p)$; this is larger than the factor that appears in (1.25), and especially noteworthy is the fact that a large value for ν increases the size of this factor in much the same way as in the case of (4.15). Indeed, under the special parametric assumptions of Huang and Liu, we would find

$$\zeta = \frac{1+\nu}{1+\nu\theta_p},$$

which is just as small as the value that they obtain for $\tilde{\zeta}$, as long as $\theta_p \geq \theta_w$, *i.e.*, as long as price-setters have no more market power than do wage-setters. Thus there is little reason to expect that persistence should be greater in the case of a sticky-wage model than in that of a sticky-price model, once one allows for specific labor markets in the sticky-price model.[71]

It is nonetheless true that wage stickiness will generally increase the size and persistence of the real effects of nominal disturbances, holding fixed the degree of price stickiness. Note that if we subtract (4.10) from (4.9) we obtain

$$\Delta \log w_t = -(\xi_w + \xi_p)(\log w_t - \log w_t^n) + (\kappa_w - \kappa_p)(\log Y_t - \log Y_t^n) + \beta E_t[\Delta \log w_{t+1}], \quad (4.16)$$

[71] Edge (2002) demonstrates this quantitatively, in the case of a complete general-equilibrium model of the monetary transmission mechanism with endogenous capital accumulation.

where $w_t \equiv W_t/P_t$ is an aggregate real wage. In the special case that $\kappa_w = \kappa_p$ (so that wages are sticky to roughly the same extent as are prices), the output term drops out, and the only endogenous variable in (4.16) is the real wage. One easily verifies that this equation has a unique bounded solution for the path of the real wage, given a bounded process for the exogenous disturbance w_t^n; hence in this case, the equilibrium real wage is determined by this equation alone, independent of monetary policy or any other factors that affect only the demand side of the model. Let the solution for w_t be denoted $\bar{w}_t$. Then substitution of this solution into (4.10) gives us an equation for price inflation of the form

$$\pi_t = \kappa(\hat{Y}_t - \hat{Y}_t^n) + E_t\pi_{t+1} + u_t, \tag{4.17}$$

where κ is the common value of κ_w and κ_p, and u_t is an exogenous term defined as

$$u_t \equiv \xi_p(\log \bar{w}_t - \log w_t^n). \tag{4.18}$$

This is again an aggregate supply relation of the same form as (2.13), except that the exogenous intercept is no longer equal to $\hat{Y}_t^n$. It follows that the effect upon output of a monetary disturbance is exactly the same as in our previous analysis, and depends upon the value of κ in the way discussed earlier. (The nature of the exogenous intercept term does not matter for this question, since in any event it is unaffected by a purely monetary disturbance.) We note that in the present case (with both wages and prices sticky), the slope coefficient κ is equal to $\xi_p\omega_p$,, while in the case of flexible wages analyzed earlier, it was equal to $\xi_p(\omega + \sigma^{-1})$. Thus κ is a smaller positive quantity in the present case (assuming that all parameters take the same values, except the parameter α_w determining the degree of wage stickiness), implying both larger and more persistent output effects of a monetary disturbance.

Allowing for wage stickiness can also be important in improving the ability of the model to account for the observed behavior of wages as well as prices. The model with fully flexible wages implies that a monetary contraction should lower wages by more than the decline in prices, for in this limiting case, (4.9) reduces to the equilibrium relation

$$\log W_t - \log P_t = \log w_t^n + (\omega_w + \sigma^{-1})(\hat{Y}_t - \hat{Y}_t^n). \tag{4.19}$$

Thus the predicted decline in real wages must be substantial relative to the decline in output, unless both ω and $\sigma-1$ are quite small in value; in particular, this requires that ν be quite small (as it is necessary that $\omega > \nu$). But if wages are sticky as well as prices, it is possible for the decline in real wages to be small, or even non-existent, even if ν is of substantial magnitude. Indeed, we have just seen that if $\kappa_w = \kappa_p$, there is no effect of a monetary disturbance upon the real wage at all, and this condition can hold regardless of the value of ν. (For any values of the other parameters, it is possible to arrange that $\kappa_w = \kappa_p$ simply by assigning an appropriate value to α_w.) [72]

The model with both wage and price stickiness also allows for more complicated wage dynamics than can be achieved in the flexible-wage model through any choice of the parameters ω and σ, for it ceases to be necessary that the effects of monetary policy on the real wage be any constant multiple of the effects upon real GDP. For arbitrary coefficients $\xi_w, \xi_p > 0$ and bounded processes $\{Y_t, Y_t^n, w_t^n\}$, equation (4.16) has a unique bounded solution for the real wage. This is given by

$$\log w_t = \lambda_1 \log w_{t-1} + \beta^{-1} \sum_{j=0}^{\infty} \lambda_2^{-j-1} E_t z_{t+j}, \tag{4.20}$$

where $0 < \lambda_1 < 1 < \lambda_2$ are the two roots of the characteristic polynomial

$$P(\lambda) \equiv \beta\lambda^2 - (1 + \beta + \xi_w + \xi_p)\lambda + 1 = 0,$$

and the forcing process $\{z_t\}$ is defined by

$$z_t \equiv (\xi_w + \xi_p) \log w_t^n + (\kappa_w - \kappa_p)(\log Y_t - \log Y_t^n).$$

A monetary disturbance affects the path of z_t solely through its effect on Y_t, as the other terms are exogenous. In the case that either wages or prices are perfectly flexible (so that either ξ_w or ξ_p is unboundedly large), both λ_1 and λ_2^{-1} equal zero, and (4.20) implies that $\log w_t$ is a multiple of z_t; but when both are sticky, so that $\xi_w + \xi_p$ is finite, $\log w_t$ is instead proportional to a smoothed version of the forcing process.

[72] If wages are made sufficiently sticky, the model would predict an actual increase in the real wage following a monetary contraction (the prediction of the classic Keynesian model), though evidence such as that presented by Christiano *et al.* (1997) does not indicate that this actually occurs.

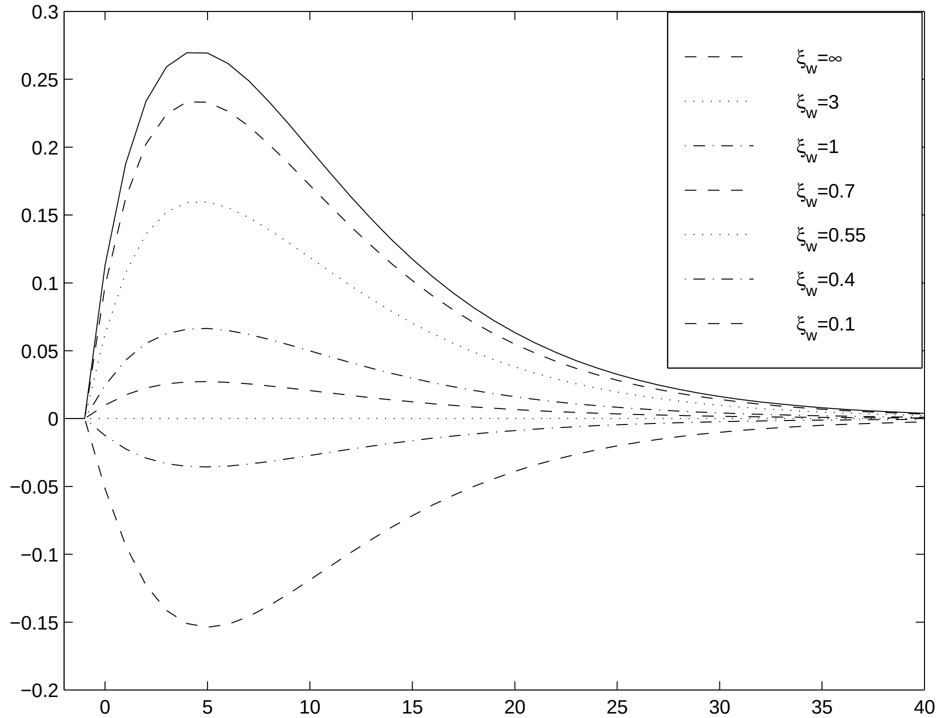

Figure 3.18: Predicted impulse response of the real wage, for alternative degrees of wage stickiness. The solid line indicates the response of real GDP.

The effects of wage stickiness are illustrated in Figure 3.18. Here a given impulse response function for output in response to a monetary disturbance is assumed, and the implied response of the real wage to the same disturbance is then inferred from (4.20). The implied real-wage responses are plotted along with the output response for each of several alternative assumed degrees of nominal wage stickiness, corresponding to different values of ξ_w. Here the assumed response of output is the same as in Figure 3.9, and the values assumed for the parameters $\xi_p, \omega_w, \omega_p$, and σ are again those of Rotemberg and Woodford (1997) (see Table 4.1 of chapter 4). The limiting case $\xi_w = \infty$ (complete wage flexibility) corresponds to the assumptions of Rotemberg and Woodford; the other cases shown each involve some degree of wage stickiness.

As the degree of wage stickiness increases, the size of the real wage increase associated with a given size of effect on output declines. When $\xi_w = 0.55$, κ_w falls to a value equal to that of κ_p, and wages and prices are in a sense "equally sticky"; in this case, there is no longer any real wage response at all to a monetary disturbance that increases output. For even lower values of ξ_w, wages are stickier than prices, and the real wage is predicted to move counter-cyclically.

The estimated real wage response shown in Figure 3.17 suggests that the empirically relevant case is one in which κ_w is slightly larger than κ_p, but not much. Amato and Laubach (1999) reach a similar conclusion when they estimate a model that allows for wage as well as price stickiness, as discussed in section xx of chapter 4. Christiano *et al.* (2001) and Smets and Wouter (2001) similarly find that the impulse responses to identified monetary policy shocks are best fit by a model that incorporates both wage and price stickiness.

Furthermore, allowing for wage as well as price stickiness allows additional ways in which real disturbances may shift the aggregate supply curve, *i.e.*, the short-run relation between inflation and output for given inflation expectations. In the aggregate supply relation (2.13) derived under the assumption of sticky prices but flexible wages, any of a variety of real disturbances can shift this relation — variation in the rate of technical progress, government purchases, and various types of shifts in preferences — but in each case, the aggregate supply relation is shifted exactly to the extent that the disturbance in question changes the natural rate of output (*i.e.*, the equilibrium output level with flexible wages and prices). But in (4.17), this is no longer true, unless the term u_t is zero.

In general this term is not identically zero (*i.e.*, $\bar{w}_t \neq w_t^n$), even when the degree of wage stickiness is exactly that required to make $\kappa_w = \kappa_p$. One notes that $w_t = w_t^n$ is a solution to (4.16) in the case that $\kappa_w = \kappa_p$ only if $\Delta \log w_t^n = \beta E_t \Delta \log w_{t+1}^n$, which in turn is possible, if w_t^n is a stationary process, only if w_t^n is a constant — *i.e.*, if the real disturbances present in the model would never affect the equilibrium real wage in the case of flexible wages and prices. This is true only for extremely special parameter values and/or assumptions about the kinds of real disturbances that can occur. Under any other assumptions, solving (4.16) allows us to determine how various types of real disturbances shift the aggregate supply relation in a way that differs from their effect upon the natural rate of output.[73] Here we do not pursue the topic further, except to note that the existence of a non-zero u_t term implies a tension between the goals of stabilizing inflation and of stabilizing the output gap $\hat{Y}_t - \hat{Y}_t^n$ that does not appear in our baseline sticky-price model. The consequences of this for optimal stabilization policy are taken up in section xxx of chapter 6.

Finally, we note that infrequent reoptimization of wage demands need not mean that wages remain fixed in money terms between the occasions on which they are re-optimized; instead, wages might be indexed to an aggregate price index in the interim, just as in our discussion of goods-price indexation in section xxx. Indeed, indexation schemes of this kind for wages are sometimes a part of multi-year union contracts, so that there is more direct evidence for the idea of indexation in the case of wages than for prices; and in practice, such indexation is always to a lagged price index. Suppose we let $0 \leq \gamma_w \leq 1$ be the indexation rate for wages that are not-reoptimized; that is, if the wage demanded for labor of type j is not reoptimized in period t, it is adjusted according to the indexation rule

$$\log w_t(j) = \log w_{t-1}(j) + \gamma_w \pi_{t-1}.$$

This modifies the relations for optimal wage-setting in a way that is directly analogous to our discussion of optimal price-setting with backward-looking indexation in section xxx.

The result is that the system (4.9)–(4.10) becomes instead

$$\pi_t^w - \gamma_w \pi_{t-1} = \kappa_w(\hat{Y}_t - \hat{Y}_t^n) + \xi_w(\log w_t^n - \log w_t) + \beta E_t[\pi_{t+1}^w - \gamma_w \pi_t], \quad (4.21)$$

$$\pi_t - \gamma_p \pi_{t-1} = \kappa_p(\hat{Y}_t - \hat{Y}_t^n) + \xi_p(\log w_t - \log w_t^n) + \beta E_t[\pi_{t+1} - \gamma_p \pi_t], \quad (4.22)$$

where $\pi_t^w \equiv \Delta \log W_t$ is the rate of wage inflation, $w_t \equiv W_t/P_t$ is the aggregate real wage, γ_p is the rate of indexation of price commitments (called simply γ in section xxx), and the coefficients $\kappa_w, \kappa_p, \xi_w, \xi_p$ are again defined exactly as in (4.9)–(4.10). These equations once

[73] For general parameter values, of course, we are not able to solve for the equilibrium real wage independent of the evolution of real activity, as indicated by (4.20). In the general case there is no exogenous term u_t for which an aggregate supply relation of the simple form (4.17) holds. But our more general point, that when both wages and prices are sticky it is no longer true that inflation stabilization and output-gap stabilization imply one another, continues to hold.

again provide a pair of coupled equations for the evolution of the aggregate wage and price indices, given the path of aggregate output and the real disturbances. Note that in the case that $\gamma_w = \gamma_p = 1$, this represents a simplified version of the aggregate-supply block of the model of Christiano *et al.* (2001), in which we abstract from complications such as endogenous capital accumulation. Smets and Wouters (2001) also estimate a model with an aggregate-supply block of this kind, but treat γ_w and γ_p as free parameters to be estimated. (Their best-fitting values are xxxx.) Both groups of authors find that a model of this type — incorporating staggered wage-setting as well as staggered price-setting, and automatic indexation of both wages and prices to recent past inflation — can account fairly well for the joint dynamics of wages, prices, and real activity.

The existence of the additional terms due to indexation results in inflation inertia much as has already been discussed in the context of the flexible-wage model of section xxx. For example, the inflation dynamics associated with the Fuhrer-Moore aggregate supply relation (3.6) can occur as a result of wage rather than price indexation. In the case of fully flexible (and hence non-indexed) prices, and the special case of a linear production function (so that $\omega_p = 0$), (4.14) reduces to $w_t = w_t^n$, in which case (4.9) implies that

$$\pi_t - \gamma\pi_{t-1} = \kappa(\hat{Y}_t - \hat{Y}_t^n) + \beta E_t[\pi_{t+1} - \gamma\pi_t] + u_t, \tag{4.23}$$

where $\gamma = \gamma_w$, $\kappa = \kappa_w$, and

$$u_t \equiv \beta E_t \Delta \log w_{t+1}^n - \Delta \log w_t^n.$$

Alternatively, similar inflation dynamics can result from a similar degree of indexation of wages and prices. For example, in the special case that $\gamma_w = \gamma_p = \gamma$ and $\kappa_w = \kappa_p = \kappa$, subtracting (4.22) from (4.21) again yields an equation that can be solved for the real wage dynamics independent of monetary policy. Substituting this solution $w_t = \bar{w}_t$ into (4.22) again yields an aggregate supply relation of the form (4.23), but where now u_t is defined as in (4.18).

We thus find that allowing for wage stickiness does not matter all that much, if our goal is simply to construct a positive model of the co-movement of inflation and output, and the way that both can be affected by monetary policy. (Stickiness of wages reduces the slope of the short-run Phillips curve, but a similar degree of flatness could alternatively be obtained by choosing different values for other parameters in a flexible-wage model, without any counterfactual consequences for the dynamics of output and inflation; stickiness of wages creates a new way in which real disturbances can shift the short-run Phillips curve, but similar consequences for the evolution of inflation and the output gap would be obtained by simply postulating an exogenous "cost-push shock" as many authors do.) To this extent, the relative neglect of wage-setting in the early literature on the optimizing models with nominal rigidities can be given a justification.

On the other hand, we shall see that there is an important respect in which it does matter whether one thinks that wages, prices or both are sticky. This has to do with the proper goals of monetary stabilization policy from a welfare-theoretic point of view. If one takes as given an *ad hoc* stabilization goal, defined in terms of the stability of a certain price index and a certain measure of the output gap, then an adequate model for determining how policy can best achieve this goal may well be formulated without having to take a stand on the question of wage stickiness. But if one asks whether it is really appropriate, from the point of view of economic welfare, to define the goal of policy in that way, it turns out to matter after all whether one believes that wages are sticky. Even in the cases discussed above in which monetary policy is unable to affect the evolution of the real wage (so that the effects of purely monetary disturbances on wage and price inflation must be identical), it does not follow that wage and price inflation should be indistinguishable: they will be differentially affected by *real* disturbances. But it then follows that seeking to stabilize wage inflation and seeking to stabilize price inflation will not be equivalent policies. Which is the more appropriate goal? The answer depends on the degree to which wages as opposed to prices are sticky, as we show in chapter 6.

Interest and Prices

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Contents

Chapter 4

A Neo-Wicksellian Framework for the Analysis of Monetary Policy

We are now ready to consider the effects of alternative interest-rate rules for monetary policy, in a setting in which monetary policy has real effects owing to the nominal rigidities discussed in the previous chapter. As in chapter 2, a crucial first issue in the choice of an interest-rate feedback rule is the selection of a rule that results in a determinate equilibrium. We reconsider this issue (previously treated under the case of full price flexibility), and show that the "Taylor principle" — the requirement that interest rates be increased more than one-for-one in response to sustained increases in the inflation rate — continues to be essential for determinacy.

We may then consider the nature of inflation and output determination in the case that a determinate equilibrium exists. Once again, we shall show that the equilibrium evolution of these variables can be understood without reference to the implied path of the money supply, or to the determinants of money demand. When monetary policy is specified in terms of an interest-rate feedback rule — a specification which more directly matches the terms in which monetary policy is discussed within actual central banks — then it is possible to understand the effects of such policies by directly modeling the effects of interest rates upon spending and pricing decisions, without attaching any central importance to the question of how various monetary aggregates may also happen to evolve. This is in fact the approach already taken in many of the econometric models used for policy simulations within central banks.[1] A primary goal of the present exposition will be to show how models with this basic structure — roughly speaking, models that consist of an "IS block," and "aggregate supply (AS) block" and an interest-rate feedback rule — can be derived from explicit optimizing foundations. In this way it is established that a non-monetarist analysis of the effects of monetary policy does not involve any theoretical inconsistency or departure from neoclassical orthodoxy.

Instead, we shall argue that inflation and output determination can be usefully explained in Wicksellian terms — as depending upon the relation between a "natural rate of interest" determined primarily by real factors and the central bank's rule for adjusting the short-term nominal interest rate that serves as its operating target. Increases in output gaps and in inflation result from increases in the natural rate of interest that are not offset by a corresponding tightening of monetary policy (positive shift in the intercept term of the interest-rate feedback rule), or alternatively from loosenings of monetary policy that are not justified by declines in the natural rate of interest.[2]

While this basic approach to inflation determination has already been introduced in chapter 2, it is only in an environment with sticky prices that we are able to introduce the crucial Wicksellian distinction between the actual and the "natural" rate of interest, as the discrepancy between the two arises only as a consequence of failure of prices to adjust sufficiently rapidly. Here we also discuss the underlying real determinants of variation in the natural rate of interest, and discuss the way in which a central bank would respond to such variations in order to maintain stable prices or a stable rate of inflation.

We first expound our neo-Wicksellian analysis in the context of a very simple intertemporal equilibrium model, in which we abstract from endogenous variation in the economy's capital stock. We then extend the model in section 3 to consider the consequences for the

[1] See, *e.g.*, Black *et al.* (1997), Brayton *et al.* (1997), and Coletti *et al.* (1996), for discussions of the models currently used at the U.S. Federal Reserve Board, the Bank of Canada, and the Reserve Bank of New Zealand. A similar approach is also already common in small macro-econometric models used for policy evaluation in the academic literature as well (*e.g.*, Fuhrer and Moore, 1995a), but such models are typically not derived from explicit optimizing foundations.

[2] For Wicksell's views see Wicksell (1898, 1906, 1907). Recent discussions include Humphrey (1992), Fuhrer and Moore (1995b), and Woodford (1998).

monetary transmission mechanism of endogenous capital accumulation. We show that many aspects of the basic model are preserved by this extension. In particular, it is still true that a useful approach to inflation stabilization involves commitment to a "Taylor rule" under which the intercept term varies one-for-one with variation over time in the Wicksellian natural rate of interest. Finally, in section 4 we consider further extensions of the basic framework that incorporate more realistic delays in the effects of monetary policy upon inflation and economic activity.

1 A Basic Model of the Effects of Monetary Policy

We here present a first complete general-equilibrium model of the monetary transmission mechanism, of which we shall make frequent use in the remainder of this essay. (References below to "the baseline model" refer to the model presented in this section.) This model combines the relation between interest-rate targeting by the central bank and intertemporal resource allocation developed in section 1 of chapter 2 with the relation between real activity and inflation developed in section 2 of chapter 3. One should note that the assumptions made in separately deriving these equilibrium relations are in fact mutually consistent, so that our separate partial results can be combined to yield a complete, though highly stylized, model. The resulting framework indicates how interest rates, inflation and real output are jointly determined in a model that abstracts from endogenous variations in the capital stock, and that assumes perfectly flexible wages (or some other mechanism for efficient labor contracting), but monopolistic competition in goods markets, and sticky prices that are adjusted at random intervals in the way assumed by Calvo (1983).[3]

[3]The model expounded here was first presented as a simple example of an optimizing framework for the analysis of alternative monetary policies in Woodford (1994a, 1996). Similar models have been extensively used in the recent literature; see, *e.g.,* Kerr and King (1996), Bernanke and Woodford (1997), Rotemberg and Woodford (1997, 1999a), McCallum and Nelson (1999), and Clarida *et al.* (1999).

1.1 An Intertemporal IS Relation

We recall from chapter 2 that the representative household's optimal intertemporal allocation of consumption spending must satisfy the Euler equation

$$1 + i_t = \beta^{-1} \left\{ E_t \left[\frac{u_c(C_{t+1}; \xi_{t+1})}{u_c(C_t; \xi_t)} \Pi_{t+1}^{-1} \right] \right\}^{-1}, \tag{1.1}$$

where i_t is the riskless one-period nominal interest rate controlled by the central bank. As shown in chapter 2, the simple form (1.1) can be derived either in the case of a utility function that is additively separable between consumption and real money balances, or (our preferred interpretation) in the "cashless limit" discussed in section 3.3 of chapter 2. (The consequences of allowing for non-trivial monetary frictions and a non-separable utility function are taken up in section xx below.) In chapter 2, we neglected labor supply, as we treated the economy's supply of goods as a simple endowment; but even with endogenous labor supply, condition (1.1) is unaffected as long as utility is separable in consumption and leisure, as assumed in chapter 3.[4] Finally, in chapter 2 we assumed a single perishable good, whereas we now assume the existence of a continuum of differentiated goods, in order to allow price-setting by monopolistically competitive producers, as in chapter 3. However, as long as utility depends only upon the Dixit-Stiglitz aggregate of consumption of the various differentiated goods (defined in equation xx of chapter 3), then the Euler equation (1.1) continues to apply, where now C_t refers to the representative household's demand for the consumption aggregate, and $\Pi_t \equiv P_t/P_{t-1}$ refers to the gross rate of increase in the Dixit-Stiglitz price index P_t (defined in equation xx of chapter 3).

Of course, equation (1.1) is not the *only* requirement that must be satisfied for the household's consumption plan to be optimal. Consumption spending must also be optimally allocated each period across the various differentiated goods; this requirement leads to the constant-elasticity demand curve for each of the individual goods (equation xx of chapter

[4] More precisely, we assume that utility is additively separable between a function of consumption and real balances on the one hand, and a function of leisure or hours worked on the other. In the cashless limit, the marginal utility of additional consumption is essentially independent of the level of real money balances, as is assumed in (1.1).

3), assumed in our model of optimal pricing behavior. And (1.1) is not the only first-order necessary condition for optimal intertemporal allocation of aggregate consumption expenditure; in addition, the marginal rate of substitution between consumption at date t and *any possible state* at date $t+1$ must equal the relevant stochastic discount factor (equation xx of chapter 2). However, this much more detailed set of first-order conditions turns out not to be necessary for the derivation of a complete set of equilibrium conditions sufficient to determine interest rates, inflation and output, as long as the central bank's reaction function does not itself involve any asset prices other than the short riskless nominal interest rate.[5] The additional equilibrium conditions are needed only if we wish also to determine the equilibrium values of other asset prices, and we set that question aside here. Finally, optimality also requires exhaustion of the household's intertemporal budget constraint (conditions xx or xx in chapter 2). But, just as in chapter 2, this additional equilibrium condition plays no role in our analysis of equilibrium determination, as long as fiscal policy is "locally Ricardian" (so that the additional equilibrium condition is automatically satisfied in the case of all paths involving small enough deviations of interest rates, inflation and output from certain reference values) and as long as we are concerned solely with local analysis (*i.e.*, with the equilibrium responses to sufficiently small shocks).

In a model where the only source of demand for produced goods is private consumption demand, equilibrium requires that the Dixit-Stiglitz index of aggregate demand Y_t (that figures, for example, in the demand curve for an individual good, given by equation xx of chapter 3) is equal to the representative household's choice of the consumption aggregate C_t. We can thus once again substitute[6] Y_t for C_t in the Euler equation (1.1), and obtain the

[5] If, as is sometimes assumed, the central bank sets its short-rate instrument as a function of an observed long bond rate or a term-structure spread, then a complete model of the consequences of such a rule would have to include a model of the equilibrium term structure of interest rates as well. The statement in the text also assumes that fiscal policy is locally Ricardian (in the sense to be explained in chapter 5), as we shall suppose throughout this chapter, or else that all outstanding government debt consists of riskless one-period nominal bonds, so that again the short nominal rate is the only asset price that matters for equilibrium determination. In the case of a non-Ricardian fiscal policy and long-term government debt, a complete model again must include a model of the term structure, as discussed in section 2.3 of chapter 5.

[6] Recall that we have already made this same substitution in the derivation of our aggregate supply relation in chapter 3. Thus the assumptions made there and those used here are mutually consistent.

equilibrium condition

$$1 + i_t = \beta^{-1} \left\{ E_t \left[\frac{u_c(Y_{t+1}; \xi_{t+1})}{u_c(Y_t; \xi_t)} \Pi_{t+1}^{-1} \right] \right\}^{-1}, \quad (1.2)$$

linking interest rates and the level of real activity. This condition is necessary for equilibrium here, just as in chapter 2, since the derivation there depended in no way upon the assumption that output was given exogenously. However, its interpretation is now somewhat different. When we first encountered this relation (as equation xx in chapter 2), its natural interpretation was as a relation that deternined the equilibrium real rate of return, given the economy's exogenous supply of goods. In the present context it is instead most usefully viewed as the analog, in an intertemporal equilibrium model, of the Hicksian "IS curve".[7] That is, it determines the level of real aggregate demand associated with a given real interest rate, and then since output is demand-determined in the present model, it determines the equilibrium level of output associated with a given real interest rate.

It might seem that this intertemporal "IS relation" depends upon an extremely restrictive conception of the demand for produced goods, namely that all demand is private demand for non-durable consumption goods. However, the model can be understood to allow for government purchases through a simple reinterpretation of the notation, as already noted in chapter 2. The function $u(Y; \xi)$ should be understood to indicate the level of utility from private consumption when aggregate demand is Y, even if that aggregate demand also includes government purchases; as long as government purchases are an exogenous state variable, we can regard them as simply another element of the vector ξ of exogenous random disturbances to the functional relation between utility and Y.

Furthermore, one need not understand the model to assume that investment demand is zero. (This point matters when it comes time to "calibrate" our model for use in quantitative analysis.) A more generous view of our baseline model would be that it abstracts from the effects of variations in private spending (including those classified as investment expenditure

[7] The analogy between this equilibrium relation and the "IS curve" is stressed in particular in Woodford (1994a, 1996), Kerr and King (1996), Bernanke and Woodford (1997), and McCallum and Nelson (1999). An early derivation of an "IS relation" from intertemporal optimization in the same spirit was provided by Koenig (1987, 1993).

in the national income accounts) upon the economy's productive capacity; the theory of marginal supply cost that underlies our model of optimal pricing behavior assumes that the capital stock in each sector of the economy evolves exogenously, so that any variations can be subsumed under variation in the exogenous technology factor A_t. In addition, it assumes that the marginal utility of additional real private expenditure at any point in time is a function solely of the aggregate level of such expenditure, together with exogenous factors — *as if* all forms of private expenditure (including those classified as investment expenditure) were like non-durable consumer purchases. This is not a preposterous theory of "investment" spending, since the existence of convex adjustment costs of the sort assumed in standard neoclassical investment theory *does* imply that the marginal utility of additional investment spending at a given point in time is decreasing in the real quantity of investment spending at that time. However, neoclassical investment theory does imply in general that the marginal utility of additional investment spending *also* depends upon other endogenous factors, such as variations in expected future returns to capital, and our present model must be understood to abstract from variations in these factors. (The implications of a fully-developed neoclassical model of investment demand are presented in section 3 below, and compared to those of the present model.)

This simple model of the effects of real interest rates on aggregate demand is obviously extremely stylized, and it might be wondered why we even bother to derive it from optimizing foundations, if we intend to abstract from so many features of a more realistic equilibrium model. The answer is that the model's simplicity makes it useful as a source of insight into basic issues; yet the consideration of intertemporal optimization introduces some subtleties, even in this simple specification, that we believe are of considerable general importance. The most important advantage of (1.2) over many simple "IS" specifications (including those often assumed in linear rational-expectations models with an IS-LM structure) is that it implies that *expected future* real interest rates, and not just a current short real rate, matter for the determination of aggregate demand.

Note that if we are interested solely in characterizing equilibria involving small fluctuations around a deterministic steady state, it suffices to use a log-linear approximation of (1.2). As in chapter 2, this takes the form

$$\hat{Y}_t = g_t + E_t(\hat{Y}_{t+1} - g_{t+1}) - \sigma(\hat{\imath}_t - E_t\pi_{t+1}), \tag{1.3}$$

where once again the parameter $\sigma > 0$ represents the intertemporal elasticity of substitution in private spending, and g_t is a particular component of the exogenous disturbance ξ_t (which may be interpreted, among other ways, as representing variation in government purchases).[8] Furthermore, in the case of any solutions in which Y_t and g_t are both stationary variables, it follows from (1.3) that

$$\hat{Y}_t = \hat{Y}_\infty + g_t - \sigma \sum_{j=0}^{\infty} E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}), \tag{1.4}$$

using the fact that

$$\lim_{T\to\infty} E_t(\hat{Y}_T - g_T) = \hat{Y}_\infty, \tag{1.5}$$

where $\hat{Y}_\infty$ is the long-run average value of $\hat{Y}_t$ under the policy regime in question.[9]

Thus aggregate demand in this model depends upon *all* expected future short real rates, and not simply upon a current *ex ante* short real rate of return; and unless fluctuations in short rates are both highly unforecastable and highly transitory, expectations of sl future short rates will matter more than the current short rate.[10] The exact way in which

[8]See equations (xx) and (xx) of chapter 2. Here we have written the equilibrium relation somewhat differently. We no longer subsume all sources of variation in the equilibrium real rate of return under a single term $\hat{r}_t$, because output is no longer an exogenous factor. And we now put $\hat{Y}_t$ on the left-hand side, to stress that the equation may now be viewed as determining aggregate demand.

[9]The long-run average value of g_t is assumed to be zero, by definition. The long-run average value of $\log Y_t$ is not necessarily equal to $\log \bar{Y}$, the zero-inflation steady-state level around which we log-linearize. In the case of the "New Keynesian" aggregate supply relation (1.6), a policy that results in a long-run average rate of inflation π_∞ different from zero will also imply a non-zero value for $\hat{Y}_\infty$, namely $(1-\beta)/\kappa$ times π_∞; and while our approximations assume that the inflation rate is *always near* zero, they do not require that inflation be exactly zero on average. If instead we assume the "New Classical" AS relation discussed in chapter 3, or we assume complete indexation of prices to a lagged price index (AS relation (2.23) below, with $\gamma = 1$), then any policy that makes the inflation rate a stationary variable results in $\hat{Y}_\infty = 0$.

[10]One way of interpreting (1.4) is as saying that it is a *long-term* real rate of interest, rather than a short rate, that determines aggregate demand in this model. In fact, the part of the term structure that matters according to (1.4) is the yield on a bond of infinite duration, *i.e.*, the sort of "very long discount" bond discussed by Kazemi (1992) and Fisher and Gilles (2000). These authors show that in an environment of the kind assumed here, the yield on the VLD bond defines the stochastic discount factor that can be used to

expectations of future short rates matter in (1.4) is undoubtedly special, and unlikely to be precisely correct in reality. (We reconsider the question below in the context of a more sophisticated model of investment dynamics.) Nonetheless, the conclusion that expected future short rates matter a great deal is likely to be robust, and this general insight is of considerable importance for the theory of monetary policy, as we shall see. It implies that a central bank's primary impact upon the economy comes about not through the level at which it sets current overnight interest rates, but rather through the way it affects private sector expectations about the likely *future* path of overnight rates. This in turn implies that the credibility of policy commitments must be a paramount concern, that discretionary optimization will almost surely lead to a suboptimal outcome, and that interest-rate smoothing is desirable, among other consequences, as we discuss below in chapters 7 and 8.

1.2 A Complete Model

We may now close our model by combining the above "IS relation" with any of several aggregate supply relations derived in chapter 3. Alternative possible assumptions about the timing of price changes, the information used in price-setting, or the degree of automatic indexation of prices between revisions have no effect upon the derivation above, which simply depends on price-taking behavior by the buyers of goods. As our baseline case we shall assume the "New Keynesian" model of staggered price-setting expounded in section 2 of chapter 3.

In this model, prices are adjusted at random intervals, and remained fixed (in units of the domestic currency) between the dates at which discrete adjustments occur, as proposed by Calvo (1983). The model as expounded in chapter 3 is consistent with the assumptions used in deriving our intertemporal "IS" relation above, in that the marginal utility of income

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price all financial assets; as it happens, it also suffices to determine the optimal level of private expenditure, given the value of the preference shock g_t. The reason is that additive separability of preferences over time allow one to define a "Frisch demand function" for consumption, in which desired consumption at any point in time is a function of the marginal utility of income at that time. The marginal utility of income that enters the Frisch demand function is in turn just the stochastic discount factor that is shown in the asset pricing literature to equal the yield on a VLD bond.

(which matters for optimizing wage demands and hence for marginal supply costs) is assumed to be a decreasing function of the current level of real activity Y_t; this relation is shifted by various exogenous factors, but is independent of all other endogenous variables. (In section 3 below, we consider how both our "IS" and "AS" relations must be modified in order to take account of endogenous capital accumulation.)

As we are here concerned solely with equilibria involving only small fluctuations in inflation and output, it suffices to recall the log-linear approximation to the "AS relation" implied by this model. This is the so-called "New Keynesian Phillips Curve",

$$\pi_t = \kappa(\hat{Y}_t - \hat{Y}_t^n) + \beta E_t \pi_{t+1}, \tag{1.6}$$

where $\kappa > 0$ is a coefficient that depends upon both the frequency of price adjustment and the elasticity of real marginal supply cost with respect to the level of real activity, where $0 < \beta < 1$ is again the discount factor of the representative household, and where $\hat{Y}_t^n$ represents exogenous variation in the "natural rate of output" as a result of any of several types of real disturbances. Let us combine equations (1.3) and (1.6) with an interest-rate rule, such as a "Taylor rule" of the form

$$\hat{\imath}_t = \bar{\imath}_t + \phi_\pi(\pi_t - \bar{\pi}) + \phi_y \hat{Y}_t/4, \tag{1.7}$$

where $\bar{\imath}_t$ is an exogenous (possibly time-varying) intercept, and ϕ_π, ϕ_y and the (implicit) inflation target $\bar{\pi}$ are constant policy coefficients.[11] We then obtain a complete system of equations for determination of the three endogenous processes $\{\hat{\imath}_t, \pi_t, \hat{Y}_t\}$, given the evolution of the exogenous disturbances $\{g_t, \hat{Y}_t^n, \bar{\imath}_t\}$. As long as the only endogenous variables to which the central bank's reaction function responds are inflation and output (as in the specification (1.7)), these three equations suffice for equilibrium determination under such a policy rule. (Dependence upon additional lags of the interest rate instrument, inflation or output, considered below, does not change this conclusion; nor does arbitrary dependence upon exogenous state variables.)

[11] Here we write the coefficient on the output term as $\phi_y/4$ so that ϕ_y corresponds to the output coefficient in a standard "Taylor rule", written in terms of annualized interest and inflation rates. In terms of our notation here, these annualized rates are $4\hat{\imath}_t$ and $4\pi_t$ respectively.

It will often be useful to write our system of equilibrium conditions in terms of the *output gap* $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$. This allows us (at least under our baseline assumptions) to write the AS relation without any residual term; and it will be shown in chapter 6 that (under those same assumptions) it is fluctuations in x_t rather than in $\hat{Y}_t$ that are relevant for welfare. "Taylor rules" are also often specified in terms of a response to variations in the output gap, though a question must raised as to whether the "output gap" measure that would be used in practice corresponds to our theoretical definition here. (We can in any event write our interest-rate rule in terms of the gap, as a purely notational matter, by allowing the intercept to be a function of the natural rate of output.) Our baseline model then consists of the equations

$$x_t = E_t x_{t+1} - \sigma(\hat{\imath}_t - E_t \pi_{t+1} - \hat{r}_t^n), \tag{1.8}$$

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}, \tag{1.9}$$

together with an interest-rate rule such as

$$\hat{\imath}_t = \bar{\imath}_t + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4. \tag{1.10}$$

Note that this equation describes the same family of policy rules as (1.7), but that the exogenous term $\bar{\imath}_t$ is not the same under the two representations of any given rule. We have also here written the "gap" term in our rule as $x_t - \bar{x}$, where $\bar{x} \equiv (1-\beta)\bar{\pi}/\kappa$ is the steady-state value of the output gap consistent with the inflation target $\bar{\pi}$, so that in an equilibrium in which the inflation target is achieved on average, the nominal interest rate $\hat{\imath}_t$ will on average equal $\bar{\imath}_t$.

The intertemporal "IS" relation (1.8) now involves a composite exogenous disturbance term[12]

$$\hat{r}_t^n \equiv \sigma^{-1}[(g_t - \hat{Y}_t^n) - E_t(g_{t+1} - \hat{Y}_{t+1}^n)]. \quad (1.11)$$

This represents deviations of the Wicksellian "natural rate of interest" from the value consistent with a zero-inflation steady state,[13] a concept about which we shall have more to say in the next section. Here it suffices to note that the *only* exogenous disturbance terms in the system consisting of (1.8) – (1.11) are the terms $\hat{r}_t^n$ and $\bar{\imath}_t$. Hence insofar as our policy rule implies a determinate rational expectations equilibrium, it must be one in which fluctuations in both inflation and the output gap are due *solely* to variations in these two factors — variations in the natural rate of interest due to real disturbances, on the one hand, and variations in monetary policy (whether deliberate or accidental) on the other. The exact way in which these factors affect inflation and the output gap is explored further in the next section.

2 Interest-Rate Rules and Price Stability

We turn now to a brief consideration of implications of our baseline framework for the explanation of economic fluctuations and the choice of a monetary policy rule. As in chapter 2, a first question to be addressed concerns the conditions under which an interest-rate rule such as (1.11) implies a determinate rational expectations equilibrium. In the case that equilibrium is determinate, we then inquire as to how equilibrium inflation and real activity are affected by both real disturbances and shifts in monetary policy. Finally, we use this

[12] Note that the framework used by Clarida *et al.* (1999) includes an "IS" relation of exactly this form, but with a disturbance term "g_t" that is described as a "demand shock". This interpretation is somewhat misleading, since it is apparent from (1.11) that any source of transitory variation in the natural rate of output will also affect the natural rate of interest. As a result, the conclusions of Clarida *et al.* about the optimal policy response to "supply shocks" as opposed to "demand shocks" must be interpreted with care. Real disturbances that have a transitory effect upon the natural rate of output are not "supply shocks" in the sense of Clarida *et al.*, because they do not result in any disturbance term in equation (1.6), while they are "demand shocks" in the sense of those authors, because they affect the disturbance term in equation (1.8).

[13] The steady-state value of the natural rate is equal to the value of the nominal interest rate consistent with that same zero-inflation steady state, so (1.11) takes the same form if we interpret $\hat{\imath}_t$ as the (continuously compounded) nominal interest rate itself and $\hat{r}_t^n$ as the (continuously compounded) natural rate of interest.

analysis to consider the design of a monetary policy rule that should maintain stable prices. The question of the extent to which price stability should be the goal of monetary policy is deferred to chapter 6.

2.1 The Natural Rate of Interest

We first consider the relatively simple question of how interest rates must be adjusted in order for monetary policy to be consistent with stable prices. To answer this question, we simply solve our AS and IS relations for the equilibrium paths of output and interest rates, under the assumption of zero inflation at all times. We first observe from the AS relation that $\pi_t = 0$ at all times requires that $x_t = 0$ at all times, *i.e.*, that output equal the natural rate of output at all times. From the derivation of the AS relation in chapter 3, we observe that this conclusion is exact, and not merely a property of the log-linear approximation. For the natural rate of output is exactly the level of output in all sectors for which real marginal cost of supplying each good will equal μ^{-1}, the reciprocal of the desired gross markup. (See equation (xx) of chapter 3.) This latter quantity is equal to marginal revenue for a firm that adjusts its price, in the case that all firms charge identical prices. Thus $Y_t = Y_t^n$ is exactly the condition needed for no firm to wish to charge a price different from the common price charged by all other firms, which is in turn the condition under which firms that adjust their prices will continue to charge the same price as firms that do not, so that there is no inflation.

Substituting these paths for inflation and output into the intertemporal IS relation, we obtain the required path of nominal interest rates. Substituting $\Pi_t = 1$ and $Y_t = Y_t^n$ into (1.2), we see that interest rates must satisfy $i_t = r_t^n$ at all times, where

$$1 + r_t^n \equiv \beta^{-1} \left\{ E_t \left[\frac{u_c(Y_{t+1}^n; \xi_{t+1})}{u_c(Y_t^n; \xi_t)} \right] \right\}^{-1}. \qquad (2.1)$$

That is, the interest rate must at all times equal the Wicksellian *natural rate of interest*, which may be defined as the equilibrium real rate of return in the case of fully flexible prices. Under this definition, we observe a direct correspondence with our previously introduced concept of the natural rate of output;[14] indeed, the natural rate of interest is just the real rate of interest required to keep aggregate demand equal at all times to the natural rate of output.[15] Log-linearizing (2.1), we observe that the exogenous term $\hat{r}_t^n$ in (1.11) corresponds to the percentage deviation of the natural rate of interest from its steady-state value,

$$\hat{r}_t^n \equiv \log\left(\frac{1+r_t^n}{1+\bar{r}^n}\right) = \log(1+r_t^n) + \log\beta.$$

We have thus far referred only to the conditions under which one could obtain complete price stability, in the sense of a constant price level (and hence zero inflation). As we shall see, there is a certain normative interest in this case, as, at least under the assumptions of our baseline model, it would eliminate the distortions resulting from price stickiness. Yet most "inflation targeting" countries instead seek to maintain inflation at a low positive level, and so policies that stabilize inflation at some constant target level $\bar{\pi}$ are also of obvious interest. In fact, in our log-linear approximation, our conclusions are exactly the same in the case, up to certain constant terms. The required path for the output gap will still be a constant (though not zero unless $\bar{\pi} = 0$), and the required path for the nominal interest rate will now be

$$\hat{\imath}_t = \hat{r}_t^n + \bar{\pi}.$$

Though the average values of output and of the nominal interest rate depend upon the target inflation rate, the way in which they should respond to shocks does not (up to a log-linear approximation).

Equation (2.1) (or equally usefully for most purposes, the log-linear version (1.11) provides us with a theory of how various types of real disturbances affect the natural rate of interest, and hence with a theory of how the interest rate controlled by the central bank should respond to those disturbances, in an equilibrium characterized by price stability. To

[14] Of course, Friedman (1968) originally proposed the concept of a "natural rate" of output (or of unemployment) by analogy with Wicksell's concept of a natural rate of interest, a notion that was at that time more familiar!

[15] The concept is thus closely related to Blinder's (1998, chap. 2, sec. 3) notion of the "neutral" rate of interest.

consider the effects of individual disturbances, we need first to recall how various real disturbances affect the natural rate of output. Log-linearization of equation (xx) from chapter 3 implies that

$$\hat{Y}_t^n \equiv \frac{\sigma^{-1} g_t + \omega q_t}{\sigma^{-1} + \omega}, \tag{2.2}$$

where g_t denotes the variation in log output required to maintain a constant marginal utility of real income as in (1.3), q_t correspondingly denotes the variation in log output required to maintain a constant marginal disutility of output supply as in chapter 3, $\sigma > 0$ is the intertemporal elasticity of substitution of private expenditure as in (1.3), and $\omega > 0$ is the elasticity of real marginal cost with respect to a firm's own output as in chapter 3.

These composite disturbance terms can furthermore be expressed in terms of more fundamental disturbances as

$$g_t = \hat{G}_t + s_C \bar{C}_t, \tag{2.3}$$

$$q_t = (1 + \omega^{-1}) a_t + \omega^{-1} \nu \bar{h}_t.$$

Here, as in chapter 2, $\hat{G}_t$ denotes the deviation of government purchases from their steady-state level, measured as a percentage of steady-state output $\bar{Y}$, which shifts the level of private expenditure implied by any given level of aggregate demand $\hat{Y}_t$, and $\bar{C}_t$ denotes the percentage shift in the Frisch (constant marginal utility of income) consumption demand, due to a shift in the utility-of-consumption function. And as in chapter 3, a_t represents variation in the log of the multiplicative technology factor that is common to all sectors, and $\bar{h}_t$ is the percentage shift in the Frisch labor supply, due to a shift in the disutility-of-labor function v. (The exogenous shifts in the Frisch demand schedules are measured at the steady-state values of their arguments.) In addition, $0 < s_C \leq 1$ is the steady-state share of private expenditure in total demand , and $\nu > 0$ is the inverse of the Frisch (or intertemporal) elasticity of labor supply. It then follows from (2.2) that

$$\hat{Y}_t^n = \frac{\sigma^{-1}}{\sigma^{-1} + \omega} (\hat{G}_t + (1 - s_G) \bar{c}_t) + \frac{1}{\sigma^{-1} + \omega} ((1 + \omega) a_t + \nu \bar{h}_t).$$

We observe that *each* of the exogenous disturbances $\hat{G}_t$, $\bar{C}_t$, a_t, and $\bar{h}_t$ increases the natural rate of output, and thus, under a policy aimed at price stability, each of them must be allowed to perturb the equilibrium level of economic activity $\hat{Y}_t$.

Substituting this solution into (1.11), and furthermore assuming (for simplicity) that each of the exogenous disturbances follows an independent first-order autoregressive process, we find that the required interest-rate variations are given by

$$\hat{r}_t^n = (\sigma + \omega^{-1})^{-1}[(1-\rho_G)\hat{G}_t + s_C(1-\rho_c)\bar{C}_t - (1+\omega^{-1})(1-\rho_a)a_t - \omega^{-1}\nu(1-\rho_h)\bar{h}_t], \quad (2.4)$$

where ρ_G, ρ_c, ρ_a, and ρ_h are the coefficients of serial correlation of the four exogenous disturbance processes. Since stationarity requires that $\rho_i < 1$ in each case, we observe that under this assumption, interest rates must increase in response to temporary increases in government purchases or in the impatience of households to consume, and decrease in response to temporary increases in productivity or in the willingness of households to supply labor. In each case, the effects upon the natural rate of interest are larger the more temporary the disturbance (*i.e.,* the less positive the serial correlation).

This prescription may appear quite different from that of Clarida *et al.* (1999), who state (in their "Result 4") that optimal policy involves "adjusting the interest rate to perfectly offset demand shocks," while "perfectly accommodat[ing] shocks to potential output by keeping the nominal interest rate constant". In fact, the variable (their "g_t") here referred to as a "demand shock" corresponds to our natural rate of interest r_t^n.[16] What these authors mean by "perfectly offsetting" movements in this variable is that the central bank's interest-rate instrument should move one-for-one with variations in the natural rate of interest. (Thus "perfectly offsetting" the shocks does not mean that output is insulated from them, but that the *output gap* is.) And what they mean by "perfectly accommodating shocks to potential output" is that, *given* the value of the natural rate of interest, the interest rate should be independent of the natural rate of output. That is, disturbances to the natural rate of output *that do not shift the natural rate of interest* should not affect nominal interest rates. Stated

[16] The variable is evidently thought of as a "demand shock" because it is the disturbance term in the Euler equation (1.8). But because this condition has been written in terms of the *output gap* x_t rather than the level of output $\hat{Y}_t$, the composite disturbance $\hat{r}_t^n$, unlike our variable g_t, cannot properly be regarded as a pure demand shock, if one supposes that transitory disturbances to the natural rate of output occur.

this way, there is no difference between their recommendation and our own.[17] However, it is *not* true, in general, that optimal policy involves no interest-rate response to shocks that affect the natural rate of output, because, as shown by (2.4) such shocks almost always do affect the natural rate of interest to some extent.

It is worth noting that the required interest-rate variations (2.4) in response to the various types of shocks cannot be achieved, in general, through a simple "Taylor rule" under which the nominal interest rate is a function solely of inflation and the deviation of output from trend. In the equilibrium with completely stable prices, inflation does not vary in response to the shocks at all, and so conveys no information about them. Output does vary in response to each of the shocks, but the desired interest-rate response is not proportional to the desired output response across the various types of shocks; indeed, one wants interest rates to vary procyclically in the case of government-purchase or consumption-demand shocks, but countercyclically in response to technology or labor-supply shocks. Thus the central bank will need additional information in order to implement its policy, if complete price stability is its aim.

Analysis of the sources of variation in the natural rate of interest is also important in determining whether complete price stability is necessarily *feasible.* Our analysis above suggests that it should be, insofar as we have been able to solve for paths of output and interest rates that would imply that the IS and AS relations would be satisfied at all times by a zero inflation rate. However, even supposing that the central bank possesses the information required to adjust its interest-rate instrument as required by the above analysis, there is another potential problem, and this is that the natural rate of interest may sometimes be *negative.*[18] If this occurs, then it is not possible for the nominal interest rate to perfectly

[17] Actually, the results referred to in Clarida *et al.* are characterizations of optimizing central bank policy under discretion, which is *not* in general optimal policy, in the sense of the policy that best achieves the central bank's assumed objectives, as we explain in chapter 7. However, in the case that complete stabilization of both inflation and the output gap are possible, doing so corresponds *both* to optimal policy and to the result of discretionary optimization, as we shall see.

[18] Another possible problem is the existence of a non-Ricardian fiscal policy, of a sort that makes a constant price level inconsistent with the condition that households exhaust their intertemporal budget constraints. This potential problem and its implications are taken up in chapter 5.

track the natural rate, owing to the zero lower bound on nominal interest rates. (Recall the discussion of equation (xx) from chapter 2.) Whether the natural rate of interest *is* ever negative is a topic of some debate, though Summers (1991) has suggested that it fluctuates sufficiently in the U.S. for a inflation target several percentage points above zero to be desirable in order to allow more successful stabilization, and Krugman (1998) has argued that it has recently been far below zero in Japan. Here we note simply that our theory allows for variation over time in the natural rate for a variety of reasons, and no reason why it should not sometimes be negative. (Our model does imply a positive *average* level of the natural rate, determined by the rate of time preference of the representative household.) Policy options when the natural rate of interest is temporarily negative are discussed further in section xx below, and in chapters 6 and 7. We argue in the later chapters that in this case it is appropriate not only to choose a non-zero inflation target, but to accept a small amount of inflation variation in order to maintain a lower average rate of inflation despite the constraint on interest-rate policy imposed by the zero bound.

2.2 Conditions for Determinacy of Equilibrium

We have thus far only considered how interest rates would have to vary, in order for there to be an equilibrium with stable prices. Our answer to this question does not yet, in itself, explain what sort of interest-rate rule would be suitable to *bring about* an equilibrium of this kind. In particular, it should *not* be inferred from the above discussion that a suitable policy rule would be simply to set the central bank's interest-rate instrument to equal its estimate of the current natural rate of interest. A policy rule of the form $\hat{\imath}_t = \hat{r}_t^n$ would be *consistent* with the desired equilibrium, but may allow many other, less desirable equilibria as well. Such a rule makes the nominal interest rate a function of purely exogenous state variables, and just as in the flexible-price analysis of chapter 2, *all* such rules imply *indeterminacy* of rational expectations equilibrium. We thus must again take up the question of the determinacy of equilibrium under alternative interest-rate rules, but now in the context of our model with sticky prices and endogenous output variation.

We begin with a formal consideration of interest-rate rules, such as the one just proposed, under which $\{\hat{\imath}_t\}$ is an exogenous process. In this case we wish to solve the system (1.8) – (1.9) for the endogenous variables $\{\pi_t, x_t\}$, given exogenous stationary processes $\{\hat{r}_t^n, \hat{\imath}_t\}$. We observe that this system can be written in the form

$$E_t z_{t+1} = A z_t + a(\hat{r}_t^n - \hat{\imath}_t),$$

where the vector of endogenous variables is

$$z_t \equiv \begin{bmatrix} \pi_t \\ x_t \end{bmatrix},$$

and the matrices of coefficients are

$$A \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ -\beta^{-1}\sigma & 1 + \beta^{-1}\kappa\sigma \end{bmatrix}, \qquad a \equiv \begin{bmatrix} 0 \\ -\sigma \end{bmatrix}.$$

The matrix A has characteristic equation

$$\mathcal{P}(\mu) = \mu^2 - [1 + \beta^{-1}(1 + \kappa\sigma)]\mu + \beta^{-1} = 0.$$

As our parameters satisfy $\kappa, \sigma > 0$ and $0 < \beta < 1$, we observe that $\mathcal{P}(0) > 0$, $\mathcal{P}(1) < 0$, and $\mathcal{P}(\mu) > 0$ again for large enough $\mu > 1$. Hence A has two real eigenvalues, satisfying

$$0 < \mu_1 < 1 < \mu_2.$$

Since neither endogenous state variable is predetermined, the existence of an eigenvalue $|\mu_1| < 1$ implies that rational expectations equilibrium is indeterminate, just as in the flexible-price model of chapter 2 (and in the rational-expectations IS-LM-AS model of Sargent and Wallace (1975)). Here the situation differs from that in chapter 2 in that the alternative stationary solutions include a large number of alternative stochastic processes for output (and also for the expected component of inflation), rather than it being only the *unexpected* component of inflation that fails to be uniquely determined. In the present context it is also clearer that this indeterminacy is undesirable, since in the presence of staggered price-setting, variations in inflation due to self-fulfilling expectations create real distortions (of a kind further characterized in chapter 6).

This result implies that even if the central bank has perfect information about the exogenous fluctuations in the natural rate of interest, a desirable interest-rate rule will also have to involve feedback from endogenous variables such as inflation and/or real activity, if only to ensure determinacy of equilibrium. In fact, if one is seeking to find a rule that implements the equilibrium with completely stable prices (or more generally, a completely stable inflation rate), then neither the variable π_t nor x_t will be useful as a source of *information* about the real disturbances to the economy, for in the desired equilibrium neither variable responds at all to any of the real disturbances.[19] Nonetheless, it may be desirable for the central bank to commit itself to respond to fluctuations in these variables, *in addition* to its response to other sources of information about the real disturbances, in order to render equilibrium determinate.

We illustrate this possibility by considering the determinacy of equilibrium under a Taylor rule of the form (1.10). (Note that it is now necessary to write explicitly the dependence of the interest-rate operating target upon the output gap, since output is here an endogenous variable.) In this case, substitution of (1.10) into (1.8) to eliminate $\hat{\imath}_t$ again yields a system of the form

$$E_t z_{t+1} = A z_t + a(\hat{r}_t^n - \bar{\imath}_t + \bar{\pi}), \tag{2.5}$$

where now

$$z_t \equiv \begin{bmatrix} \pi_t - \bar{\pi} \\ x_t - \bar{x} \end{bmatrix},$$

and

$$A \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma(\phi_\pi - \beta^{-1}) & 1 + \sigma(\phi_x/4 + \beta^{-1}\kappa) \end{bmatrix}, \qquad a \equiv \begin{bmatrix} 0 \\ -\sigma \end{bmatrix}.$$

We observe that

$$\mathrm{tr}A = 1 + \beta^{-1}(1 + \kappa\sigma) + \sigma\phi_x/4, \qquad \det A = \beta^{-1}[1 + \sigma(\phi_x/4 + \kappa\phi_\pi)].$$

[19] This is a common problem for an approach to stabilization policy based upon a commitment to respond solely to deviations of one's target variables from their (constant) target values, discussed in Bernanke and Woodford (1997). It should be noted, however, that if complete stabilization of inflation and the output gap is not desirable — owing, say, to a desire to reduce the degree of interest-rate volatility — then it may be possible to implement an optimal equilibrium through commitment to a rule that responds directly to no variables other than inflation and the output gap, as shown in chapter 8.

Let us furthermore restrict our attention to the case of rules for which $\phi_\pi, \phi_x \geq 0$. Then necessarily $\det A > 1$. We note that a 2×2 matrix with positive determinant has both eigenvalues outside the unit circle (our condition for determinacy) if and only if[20]

$$\det A > 1, \qquad \det A - \mathrm{tr} A > -1, \qquad \det A + \mathrm{tr} A > -1. \tag{2.6}$$

Under our sign restrictions, the first and third of these inequalities necessarily hold, so that both eigenvalues are outside the unit circle if and only if

$$\phi_\pi + \frac{1-\beta}{4\kappa}\phi_x > 1. \tag{2.7}$$

Condition (2.7) for determinacy can be given a simple interpretation. We note that the "New Keynesian Phillips Curve" implies that each percentage point of permanently higher inflation (*i.e.*, quarterly inflation π_t permanently higher by 1/4 of a percent) implies a permanently higher output gap of $(1-\beta)/4\kappa$ percentage points.[21] Hence the left-hand side of (2.7) represents the long-run increase in the nominal interest rate prescribed by (1.10) for each unit permanent increase in the inflation rate. Our condition then corresponds once more to the "Taylor principle": at least in the long run, nominal interest rates should rise by more than the increase in the inflation rate.

We note that contrary to our result in chapter 2, determinacy now depends upon the output response coefficient ϕ_x, and not solely upon the inflation response coefficient ϕ_π; and indeed, a large enough positive value of *either* coefficient suffices to guarantee determinacy. This complicates slightly our interpretation of the Taylor (1999) contrast between pre-Volcker and post-Volcker U.S. monetary policy. Taylor's estimates (discussed above in section 2.3 of chapter 2) imply that $\phi_\pi < 1$ in his pre-Volcker sample; but as they also imply that $\phi_x > 0$ in that period, this does not in itself suffice to imply that equilibrium should have been indeterminate under the earlier policy. Still, plausible numerical values for the parameters of the NKPC imply this, at least if Taylor's point estimates for the policy-rule coefficients

[20] See Proposition 1 of the Appendix.

[21] Thus the "long-run Phillips curve" is not perfectly vertical in this model. We show, however, in chapters 6 and 7 that this does not imply that the optimal long-run inflation rate is positive, even if the optimal output level exceeds the natural rate.

are taken to be correct. For example, if one assumes the parameter values given in Table 1 below (based upon the estimates of Rotemberg and Woodford (1997)), then determinacy would require that the inflation coefficient plus .1 times the output coefficient be greater than one. Taylor's estimates for the period 1960-79 would then imply an interest-rate increase of only .81 + .1(.25) = .84 percentage points per percentage point long-run increase in inflation. Thus just as we concluded in chapter 2, these estimates suggest that equilibrium should have been indeterminate under the pre-Volcker regime, though clearly determinate under the post-Volcker regime.

As discussed in chapter 1, most empirical estimates of Taylor rules incorporate some form of partial adjustment of the short-term interest-rate instrument toward an implicit target that depends upon the current inflation rate and output gap. (We shall also argue in chapter 8 that rules of that kind are desirable on normative grounds.) It is therefore of some interest to consider the effects of interest-rate inertia upon the question of determinacy. For the sake of simplicity we restrict our analysis here to the family of generalized Taylor rules

$$\hat{\imath}_t = \bar{\imath}_t + \rho(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4, \tag{2.8}$$

where we assume that $\rho, \phi_\pi, \phi_x \geq 0$. Substituting (2.8) into (1.8), we again obtain a system of equations that may be written in the form (2.5), but where now where the vector of endogenous variables is

$$z_t \equiv \begin{bmatrix} \pi_t - \bar{\pi} \\ x_t - \bar{x} \\ \hat{\imath}_{t-1} - \bar{\imath}_{t-1} \end{bmatrix},$$

and

$$A \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa & 0 \\ \sigma(\phi_\pi - \beta^{-1}) & 1 + \sigma(\phi_x/4 + \beta^{-1}\kappa) & \sigma\rho \\ \phi_\pi & \phi_x/4 & \rho \end{bmatrix}, \qquad a \equiv \begin{bmatrix} 0 \\ -\sigma \\ 0 \end{bmatrix}.$$

As there is now a predetermined state variable (namely, $\hat{\imath}_{t-1} - \bar{\imath}_{t-1}$), equilibrium is determinate in this case if and only if the 3×3 matrix A has exactly two eigenvalues outside the unit circle.

Necessary and sufficient conditions for determinacy in a system of this form are given by Proposition 2 of the Appendix. We note that in the present case, the characteristic equation of matrix A is of the form

$$\mathcal{P}(\mu) = \mu^3 + A_2\mu^2 + A_1\mu + A_0 = 0,$$

where

$$\begin{aligned} A_0 &= -\beta^{-1}\rho < 0, \\ A_1 &= \rho + \beta^{-1}(1+\rho(1+\kappa\sigma)) + \beta^{-1}\sigma(\kappa\phi_\pi + \phi_x/4) > 0, \\ A_2 &= -\beta^{-1}(1+\kappa\sigma) - 1 - \rho - \sigma\phi_x/4 < 0. \end{aligned}$$

The proposition lists three possible sets of conditions under which there is determinacy. Because of the signs of the coefficients A_i, we see immediately that condition (A.2) is violated and that condition (A.4) must instead hold; thus we can exclude Case I of the proposition. In the present case, the remaining conditions (in addition to (A.4) that, as we have just noted, is necessarily satisfied) required for Case II of the proposition reduce to

$$\phi_\pi + \frac{1-\beta}{4\kappa}\phi_x > 1 - \rho, \tag{2.9}$$

$$\phi_\pi + \frac{1-\rho}{4\kappa}\phi_x + (\beta^{-1} - 1)[\kappa^{-1}\sigma^{-1}(1-\rho)(\beta-\rho) - \rho] > 0. \tag{2.10}$$

The remaining conditions required for Case III[22] are instead (2.9) and

$$\beta^{-1}(1+\kappa\sigma) + \rho + \sigma\phi_x/4 > 2. \tag{2.11}$$

Equilibrium is determinate if and only if the coefficients of the policy rule (2.8) satisfy both (2.9) and at least one of (2.10) and (2.11).

In fact, one can show that under our sign assumptions, (2.9) is necessary and sufficient for determinacy. We prove this by showing that any parameter values that satisfy (2.8) and *not* (2.11) must necessarily satisfy (2.10). We first note that under our sign assumptions,

[22]In the statement of the proposition in the Appendix, another condition listed is (A.6), which is the denial that (2.10) holds. But this is not necessary, for if instead (2.10) holds, determinacy also obtains, as Case II then applies. (The condition is listed in the statement of the proposition simply in order to make the three cases disjoint.) Also condition (A.7) as written in the statement of the proposition allows the coefficient A_2 to be either less than -3 or greater than 3. But as in the present case A_2 is necessarily negative, it is only the possibility that A_2 may be less than -3 that is relevant; this is condition (2.11).

(2.11) can fail to hold only if $\rho < \beta$. (Here we use the fact that $\beta^{-1} + \beta > 2$.) We next observe that the left-hand side of (2.10) is a decreasing function of ρ, for given values of all the other parameters, for all values $\rho < \beta$. Thus the values of ϕ_π required in order for (2.10) not to hold become smaller, the smaller is ρ. On the other hand, the values of ϕ_π consistent with (2.9) become larger, the smaller is ρ. Thus if (2.9) is to be satisfied while (2.10) is not, for any given values of β, κ, σ and ϕ_x, this must occur for the *largest* value of ρ consistent with (2.11) being (weakly) violated. (Note that this last quantity is independent of ϕ_π.)

Furthermore, the left-hand side of (2.10) is an increasing function of ϕ_π. Thus if (2.9) is to be satisfied while (2.10) is not, for any given values of $\beta, \kappa, \sigma, \phi_x$, and ρ, this must occur for the *smallest* value of ϕ_π that is (weakly) consistent with (2.9). (The geometry of these regions is illustrated in Figure 4.1.) It therefore suffices that we consider values of ρ and ϕ_π for which (2.9) and (2.11) hold as equalities, for given values of the other parameters. (This is the point shown by the intersection of the solid line and the dashed line in Figure 4.1.) If (2.10) is not violated in this case, it can never be.

The algebra required to check this is simplest if we solve (2.9) and (2.11) for ϕ_π and ϕ_x as functions of ρ, rather than for ρ and ϕ_π as functions of ϕ_x. We obtain

$$\begin{aligned}\phi_\pi &= (1-\rho) - \frac{1-\beta}{\kappa\sigma}\left[2 - \rho - \frac{1+\kappa\sigma}{\beta}\right],\\ \phi_x &= \frac{4}{\sigma}\left[2 - \rho - \frac{1+\kappa\sigma}{\beta}\right].\end{aligned}$$

Substituting these values into the left-hand side of (2.10), we obtain

$$\frac{1}{\beta\kappa\sigma}(\beta-\rho)^2 > 0,$$

which holds as a strict inequality because $\rho < \beta$. Thus (2.10) holds in this case, and so must hold in any case where (2.9) holds but (2.11) does not. (This is illustrated for particular numerical parameter values in Figure 1.[23]) It follows that condition (2.9) is necessary and sufficient for determinacy.

[23] The values assumed for β, κ, and σ are given in Table 1 below; the value assumed for ϕ_x is .05. This last value has no particular significance, except that the relative locations of the various regions are especially easily seen for a small positive value of this order.

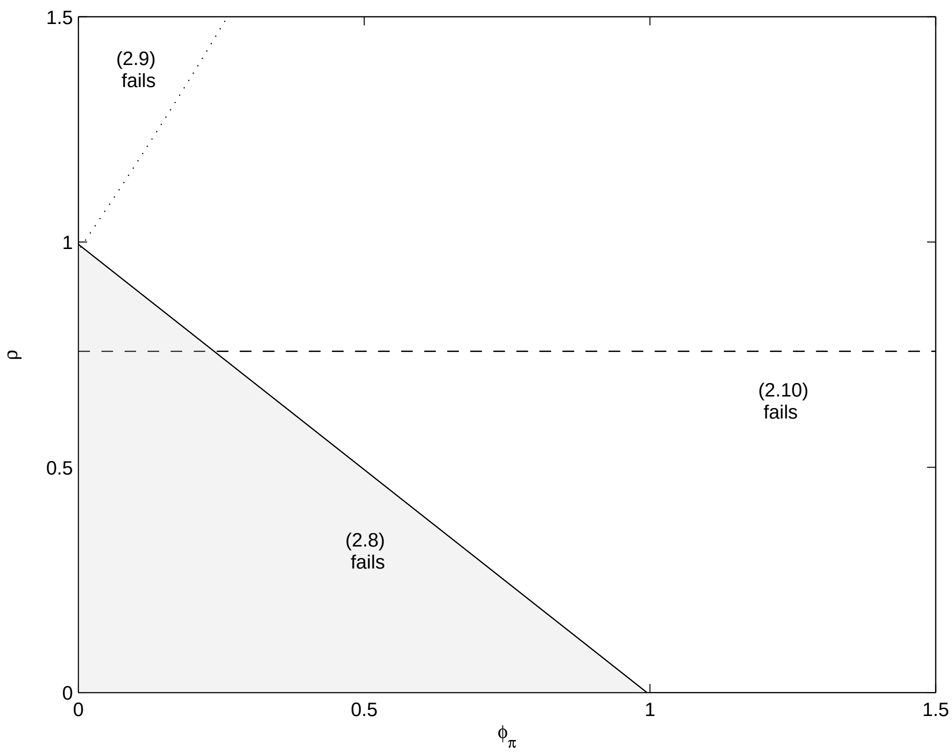

Figure 4.1: Regions in which each of three inequalities fail to hold. Grey region indicates policy rules for which equilibrium is indeterminate; white region indicates determinacy.

Condition (2.9) will be recognized as a generalization of (2.7), and once again it can be interpreted as requiring adherence to the Taylor principle. In the case that $\rho < 1$, the rule (2.8) implies that a sustained increase in inflation of a certain size results in an eventual cumulative increase in the nominal interest rate of $\Phi_\pi \equiv (1-\rho)^{-1}\phi$ times as much; similarly, a sustained increase in the output gap results in an eventual cumulative increase in the interest rate of (1/4 of) $\Phi_x \equiv (1-\rho)^{-1}\phi_x$ times as much.[24] In this case, (2.9) can equivalently be written as

$$\Phi_\pi + \frac{1-\beta}{4\kappa}\Phi_x > 1,$$

which clearly has the same interpretation as (2.7) in the non-inertial case. Furthermore, if $\rho \geq 1$, the eventual cumulative increase in the nominal interest rate is infinite if at least

[24]It may be recalled that the estimated Fed reaction functions described in chapter 1 are described in terms of the values of these long-run response coefficients Φ_π and Φ_x rather than the immediate responses ϕ_π and ϕ_x.

one of ϕ_π or ϕ_x is positive, so that the Taylor principle is necessarily satisfied; but (2.9) is necessarily satisfied in this case as well. Thus (2.9) is equivalent to requiring conformity with the Taylor principle.[25] This result — that the Taylor principle continues to be a crucial condition for determinacy, once understood to refer to *cumulative* responses to a *permanent* inflation increase, even in the case of an inertial interest-rate rule — recalls our finding in chapter 2 in the case of a flexible-price model. The finding that a determinate rational expectations equilibrium necessarily exists for rules with $\rho \geq 1$ ("super-inertial rules") also recalls an earlier result.

Some empirical papers (*e.g.*, Clarida *et al.*, 2000; Bernanke and Boivin, 2000) instead estimate "forward-looking" variants of the Taylor rule, in which interest rates respond to deviations of expected *future* inflation from its target level, instead of responding to the amount that prices have already risen. As a simple example, let us consider the family of rules

$$\hat{\imath}_t = \bar{\imath}_t + \phi_\pi(E_t\pi_{t+1} - \bar{\pi}) + \phi_x(x_t - \bar{x})/4, \tag{2.12}$$

where we again assume that $\phi_\pi, \phi_x > 0$. Substituting (2.12) into (1.8) to eliminate $\hat{\imath}_t$, we again obtain an equation system of the form (2.5), but where now

$$z_t \equiv \begin{bmatrix} \pi_t - \bar{\pi} \\ x_t - \bar{x} \end{bmatrix},$$

[25] In fact, one finds for a wide variety of types of simple interest-rate rules that the Taylor principle is one of the conditions required for determinacy, even if it is not a sufficient condition in itself, as is true here. This should not be too surprising. One observes quite generally — in the case of *any* family of policy rules that involve feedback only from inflation and output, regardless of how many lags of these might be involved — that the boundary between sets of coefficients that satisfy the Taylor principle and those that do not will consist of coefficients for which there is an eigenvalue exactly equal to one. The eigenvalue of one exists for any policy rule with the property that the long-run increase in the nominal interest rate is exactly equal to the long-run increase in the inflation rate, for the associated right eigenvector is one with an element 1 for each current or lagged value of inflation or the interest rate, and an element $(1-\beta)/4\kappa$ for each current or lagged value of the output gap. This is because under the hypothesis about the policy rule, the IS relation, the AS relation and the policy all share the property that the equation continues to be satisfied if inflation, output and interest rates are increased at all dates by the constant factors just mentioned. It follows that a real eigenvalue crosses the unit circle as the sign of the inequality corresponding to the Taylor principle changes. This boundary is therefore one at which the number of unstable eigenvalues increases by one. Often this results in moving from a situation of indeterminacy to determinacy, though we do not seek to establish general conditions for this.

and

$$A \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa \\ \sigma\beta^{-1}(\phi_\pi - 1) & 1 + \sigma(\phi_x/4 - \beta^{-1}\kappa(\phi_\pi - 1)) \end{bmatrix}, \qquad a \equiv \begin{bmatrix} 0 \\ -\sigma \end{bmatrix}.$$

Thus in this case we have

$$\begin{aligned} \mathrm{tr}A &= 1 + \beta^{-1} + \sigma(\phi_x/4 - \beta^{-1}\kappa(\phi_\pi - 1)), \\ \det A &= \beta^{-1}(1 + \sigma\phi_x/4). \end{aligned}$$

Determinacy again obtains if and only if conditions (2.6) are satisfied. In the present case, the first condition takes a slightly different form than before, but it is again necessarily satisfied by all rules satisfying our sign assumptions. The second condition again is equivalent to (2.7). However, the third condition now takes a slightly different form than in the case of conventional (backward-looking) Taylor rules, and is no longer necessarily satisfied by all rules satisfying our sign restrictions. This condition now takes the form

$$\phi_\pi < 1 + \frac{1+\beta}{4\kappa}(\phi_x + 8\sigma^{-1}). \tag{2.13}$$

Equilibrium is then determinate if and only if the coefficients of the policy rule (2.12) satisfy (2.7) and (2.13).

We note that condition (2.7) can once again be interpreted in terms of the Taylor principle. Thus yet again we find that conformity to the Taylor principle is a necessary condition for determinacy. But as in the last case, we here find that it is not sufficient. In particular, condition (2.13) fails to hold for large enough values of ϕ_π, even though the "Taylor principle" is satisfied. Thus adjusting interest rates in response to deviations of expected future inflation from target can give rise to equilibrium fluctuations due purely to self-fulfilling expectations, as shown by Bernanke and Woodford (1997) for a closely related model; this problem does not arise in the case of a strong response to the inflation that has already occurred, no matter how large ϕ_π is made. The region in which equilibrium is determinate for rules of this family is shown in Figure 4.2, again assuming the same structural parameters β, σ, κ as in Figure 4.1. As Clarida *et al.* (2000) also conclude, indeterminacy results from too high a value of ϕ_π only in the case of quite high values relative to empirical estimates

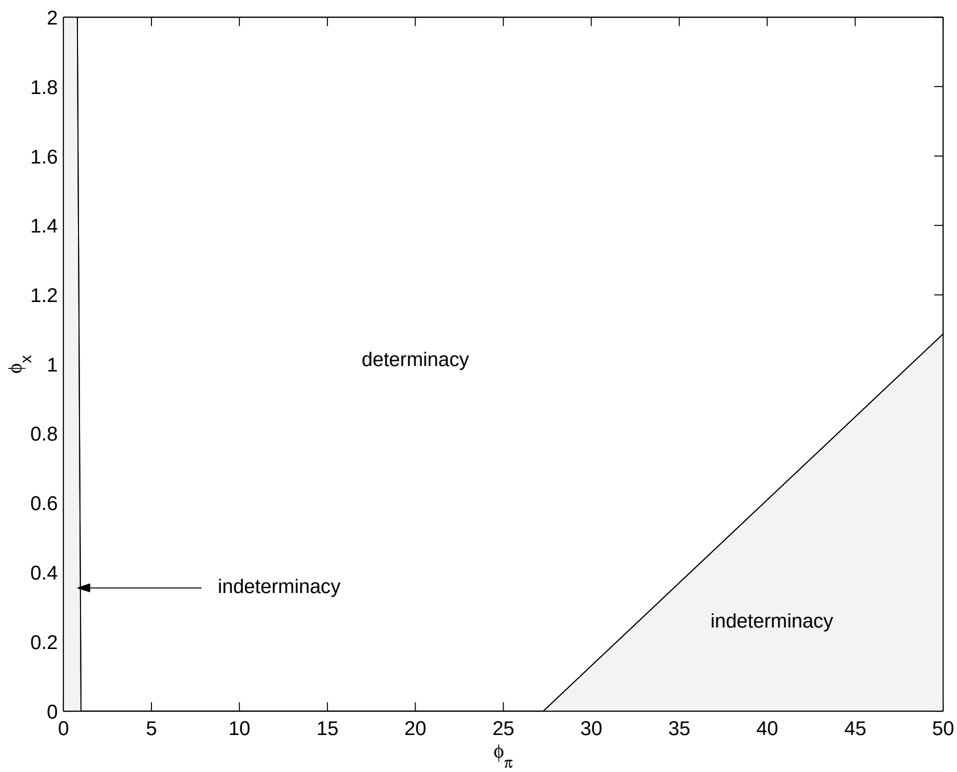

Figure 4.2: Regions of determinacy and indeterminacy for "forward-looking" Taylor rules.

for any central banks. However, this would be a problem if a central bank were to seriously attempt to completely stabilize inflation through extremely tight "targeting" of an inflation forecast, as we discuss below.

The interest-rate rule estimated by Clarida *et al.* (2000) for the Fed is more complicated than rules of the family (2.12), most notably because it allows for interest-rate inertia. Thus in order to interpret their results we need to consider forward-looking rules of the more general family[26]

$$\hat{\imath}_t = \bar{\imath}_t + \rho(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \phi_\pi(E_t\pi_{t+1} - \bar{\pi}) + \phi_x(x_t - \bar{x})/4, \tag{2.14}$$

[26]Note that even this family is still too restricted to include any of the rules actually estimated by Clarida *et al.*, as they also replace the output gap term by a forecast of the future output gap. However, analysis of the family of rules (2.14) does allow insight into how the three key parameters estimated by these authors should be expected to affect the determinacy of equilibrium, and indeed their own numerical examination of the conditions under which equilibrium is determinate considers this family of rules, rather than anything more general. Furthermore, for the reason explained in footnote 25, the Taylor principle has the same importance for the eigenvalues of the equation system if the current output gap is replaced by an expected future output gap; so the most important of our conditions for determinacy is likely to continue to apply.

where we now assume $\rho, \phi_\pi, \phi_x \geq 0$.

This again implies a system of equations that may be written in the form (2.5), but where now

$$A \equiv \begin{bmatrix} \beta^{-1} & -\beta^{-1}\kappa & 0 \\ \sigma\beta^{-1}(\phi_\pi - 1) & 1 + \sigma(\phi_x/4 - \beta^{-1}\kappa(\phi_\pi - 1)) & \sigma\rho \\ \beta^{-1}\phi_\pi & \phi_x/4 - \beta^{-1}\kappa\phi_\pi & \rho \end{bmatrix}, \qquad a \equiv \begin{bmatrix} 0 \\ -\sigma \\ 0 \end{bmatrix}.$$

Once again there is a single predetermined state variable (namely, $\hat{\imath}_{t-1} - \bar{\imath}_{t-1}$), so that equilibrium is determinate if and only if A has exactly two eigenvalues outside the unit circle, and once again necessary and sufficient conditions for this are given by Proposition 2 in the appendix. We observe that conditions (A.1) and (A.2) imply that $\phi_x < 0$, so that Case I is once again impossible under our sign assumptions. It then follows that conditions (A.3) and (A.4) are necessary for determinacy, as these are required by both cases II and III of the proposition. In the present case, (A.3) corresponds once again to (2.9), while (A.4) corresponds to

$$\phi_\pi < 1 + \rho + \frac{1+\beta}{4\kappa}(\phi_x + 8\sigma^{-1}(1+\rho)). \tag{2.15}$$

Note that the latter condition generalizes (2.13).

Once again condition (2.9) corresponds to the Taylor principle, and so we see that once more conformity with that principle is necessary for determinacy. However (as we have already concluded for the case $\rho = 0$), this principle does not *suffice* for determinacy in the case of forward-looking policy rules. In particular, it is necessary that ϕ_π satisfy the upper bound expressed in (2.15). We observe that higher values of ρ relax this constraint, but do not eliminate it. Thus it continues to be true that too large a degree of sensitivity of the interest rate to the inflation forecast results in indeterminacy. This is illustrated in Figure 4.3, where the values of ϕ_π and ρ consistent with a determinate equilibrium are indicated, in the case of rules of the form (2.14) with $\phi_x = 0$, and assuming the same model parameters as in Figure 4.2.[27]

[27] Note that conditions (2.9) and (2.15) are necessary for determinacy, but not sufficient. Determinacy requires in addition that at least one of a further set of three inequalities hold: either (A.5) must hold, or one of the two cases allowed in (A.7) must hold, *i.e.*, either $A_2 > 3$ or $A_2 < -3$. However, for the parameter

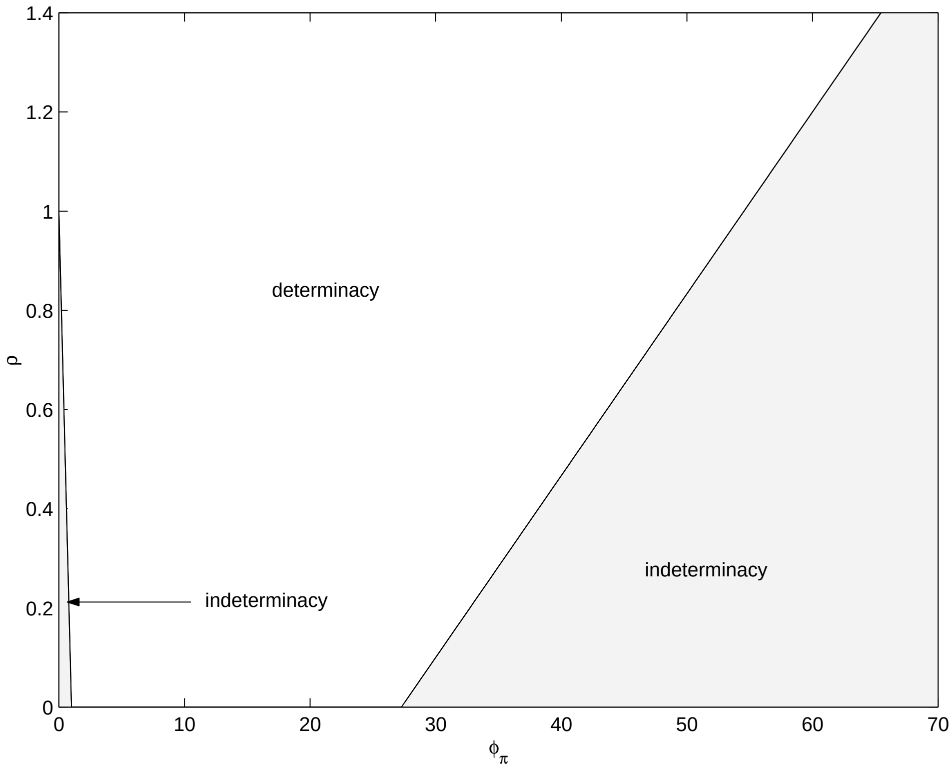

Figure 4.3: Regions of determinacy and indeterminacy for "forward-looking" Taylor rules with interest-rate inertia.

In the case of rules with coefficients in the range that is likely to be of practical interest, however, the other requirements for determinacy are unlikely to be a problem. The requirement for determinacy that is of most practical interest thus remains the Taylor principle. Like Taylor, Clarida *et al.* find that an estimated policy rule for the period 1960-79 involves an insufficient response to inflation to be consistent with determinacy, whereas their estimated rule for the period 1982-96 satisfies the Taylor principle and would imply a determinate equilibrium. For example, their baseline estimates for the earlier period are $\rho = .68$, $\phi_\pi = .27$, $\phi_x = .09$. In the absence of any increase in the output gap, these values imply that a sustained increase in inflation of one percentage point would eventually raise nominal interest rates by only 83 basis points. Thus the Taylor principle is violated unless the

values used in this figure, all policy-rule coefficients that satisfy (2.9) and (2.15) also satisfy at least one of the other three inequalities. Thus these two conditions turn out to be necessary and sufficient for determinacy, at least in this case.

associated increase in the output gap is quite large. Condition (2.9) will be satisfied only if $\kappa < .4(1-\beta)$, which is to say, if $\kappa < .004$; this is much smaller than any plausible value, either from the point of view of the underlying microfoundations of price adjustment or of estimated Phillips curves. On the other hand, their baseline estimates for the later period are $\rho = .79$, $\phi_\pi = .45$, and $\phi_x = .20$. Since in this case $\phi_\pi > 1-\rho$, the Taylor principle is satisfied regardless of the assumed slope of the long-run Phillips curve. On the other hand, because $\phi_\pi < 1$, condition (2.15) is necessarily satisfied as well, and such a policy rule implies a determinate equilibrium.

Another class of possible policy rules that is of at least theoretical interest is that of rules which incorporate a *price level* target — the "Wicksellian" rules considered in chapter 2. In the case of a model with endogenous output, one must also consider the consequences of possible feedback from the level of output, and so the class of rules previously considered is generalized to include all rules of the form

$$\hat{\imath}_t = \bar{\imath}_t + \phi_p(p_t - \bar{p}_t) + \phi_x(x_t - \bar{x})/4, \tag{2.16}$$

where now p_t is the log price level, and $\{\bar{p}_t\}$ is a target path for the log price level, growing deterministically at some rate $\bar{\pi}$ (to which $\bar{x}$ again corresponds). Once again we shall restrict attention to rules with $\phi_p, \phi_x \geq 0$. Giannoni (2000) shows that in the case of rules of this sort, a sufficient condition for determinacy is that $\phi_p > 0$. On the other hand, when $\phi_p = 0$, rules of this sort correspond to rules of the form (1.10) with $\phi_\pi = 0$, and so we already know that determinacy obtains if and only if

$$\phi_x > \frac{4\kappa}{1-\beta}. \tag{2.17}$$

Thus necessary and sufficient conditions are that either $\phi_p > 0$ or ϕ_x exceeds the bound (2.17). These conditions correspond once again to the Taylor principle. For sustained inflation above the rate $\bar{\pi}$ must eventually make $p_t - \bar{p}_t$ arbitrarily large, and then if ϕ_p takes any positive value at all, a rule of the form (2.16) must eventually imply arbitrarily large increases in the level of nominal interest rates. If on the other hand, $\phi_p = 0$, then (2.17) corresponds to the Taylor principle, as discussed earlier. Thus the Taylor principle is yet again determinative. Note also that the weakness of the condition required for determinacy in the case of a rule of the form (2.16) is one of the appealing features of rules of this kind, as discussed by Giannoni.

2.3 Stability under Learning Dynamics

[TO BE ADDED]

2.4 Determinants of Inflation

Having established that interest-rate rules can result in a determinate equilibrium, we turn now to the further characterization of that equilibrium. We are particularly interested in the determinants of equilibrium inflation under the kinds of policies just considered, and the conditions under which fluctuations in the price level can be minimized.

Each of the classes of interest-rate rules considered above has the property that the equilibrium conditions can be written entirely in terms of $\pi_t - \bar{\pi}$, $x_t - \bar{x}$, $\hat{\imath}_t - \bar{\imath}_t$, and $r_t^n - \bar{\imath}_t + \bar{\pi}$. This implies that when equilibrium is determinate, it is possible to solve for the endogenous variables in this list (the first three) as a function of initial conditions and the current and expected future values of the exogenous variable (the last one). In the case of policy rules (1.10) or (2.12), the equation system involves *no* predetermined endogenous variables, so that there are no relevant initial conditions (other than those relating to the path of the exogenous variables). One therefore obtains in these cases a solution of the form

$$\pi_t = \bar{\pi} + \sum_{j=0}^{\infty} \psi_j^{\pi} E_t(\hat{r}_{t+j}^n - \bar{\imath}_{t+j} + \bar{\pi}), \tag{2.18}$$

$$x_t = \bar{x} + \sum_{j=0}^{\infty} \psi_j^{x} E_t(\hat{r}_{t+j}^n - \bar{\imath}_{t+j} + \bar{\pi}), \tag{2.19}$$

$$\hat{\imath}_t = \bar{\imath}_t + \sum_{j=0}^{\infty} \psi_j^{i} E_t(\hat{r}_{t+j}^n - \bar{\imath}_{t+j} + \bar{\pi}). \tag{2.20}$$

In particular, if both eigenvalues of A are outside the unit circle (the condition for determinacy), then A^{-1} is a stable matrix, and one can obtain a unique bounded solution to (2.5) by "solving forward", namely

$$z_t = \sum_{j=0}^{\infty} A^{-j-1} a E_t(\hat{r}^n_{t+j} - \bar{\imath}_{t+j} + \bar{\pi}). \tag{2.21}$$

This allows us to identify the coefficients in equations (2.18) – (2.19). Substitution of the solutions for these variables into the policy rule then allows us to identify the coefficients in (2.20) as well.

In the case of a rule such as (2.8) or (2.14), instead, the lagged nominal interest rate is a predetermined endogenous variable that is relevant for equilibrium determination, because of the way that it enters the policy rule. In cases of this sort, one instead obtains solutions of the form

$$\pi_t = \bar{\pi} + \omega^\pi(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \sum_{j=0}^{\infty} \psi^\pi_j E_t(\hat{r}^n_{t+j} - \bar{\imath}_{t+j} + \bar{\pi}),$$

and similarly for the other endogenous variables.

Thus our model implies that for policy rules of these types, equilibrium inflation depends solely upon the path of the *gap* between the natural rate of interest $\hat{r}^n_t$ and the intercept term $\bar{\imath}_t$ indicating the tightness of central-bank policy. In the case of the inertial interest-rate rules, equilibrium inflation also depends upon a lagged interest rate (specifically, upon $\hat{\imath}_{t-1} - \bar{\imath}_{t-1}$), but in equilibrium this variable will itself be a function of the history of the gaps $\hat{r}^n_{t-j} - \bar{\imath}_{t-j}$. As has already been noted in chapter 2, our theory of inflation determination thus has a distinctively Wicksellian flavor: variations in the rate of inflation depend upon the interaction between the real factors that determine the natural rate of interest on the one hand, and the way in which the central bank adjusts short-term nominal interest rates on the other. Inflation will be stable insofar as the stance of monetary policy is varied to keep up with the exogenous variations in the natural rate of interest that occur as a result of real disturbances, and not varied otherwise; it will be variable insofar as either factor varies other than in perfect tandem with the other. Our analysis here has a more fully Wicksellian character than that presented in chapter 2, because we are now able to distinguish between

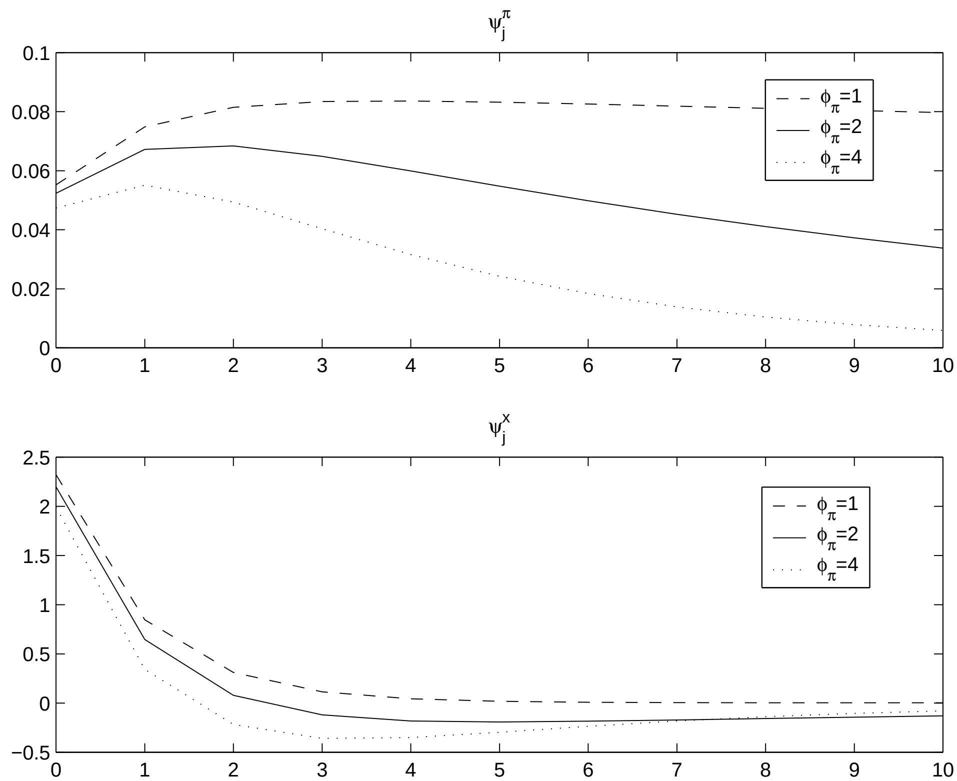

Figure 4.4: Consequences of varying the coefficient ϕ_π in the Taylor rule.

the *natural* rate of interest (which *would* be the equilibrium real rate of return, in the absence of nominal rigidities, and depends purely upon real factors) and the actual real rate of return (which can differ from the natural rate as a result of short-run disequilibrium, and is affected by monetary policy among other factors). We are thus now able to explain economic fluctuations in terms of the development of a gap between the natural and the actual real rate of return, and to discuss the role of monetary policy in helping to minimize such gaps.

Examples of numerical solutions for the coefficients $\{\psi_j^\pi\}$ and $\{\psi_j^x\}$ in the case of the Taylor rules of the form (1.10) are presented in Figures 4.4 and 4.5. Here the numerical values assigned to the structural parameters β, σ, κ are again as in Table 4.1. Here we take as our "baseline" policy rule a Taylor rule with coefficients $\phi_\pi = 2$, $\phi_x = 1$; Figure 4.4 then illustrates the consequences of varying ϕ_π around this baseline value, while Figure 4.5

illustrates the consequences of varying ϕ_x. We observe that for a range of parameter values representing reaction functions similar to actual central-bank policies, the coefficients ψ_j^π and ψ_j^x are positive for all small enough j, which are the coefficients of primary importance in determining the equilibrium responses to typical shocks.[28] Thus we find that higher output gaps and inflation result from increases in the current or expected future natural rate of interest, not offset by a sufficient tightening of monetary policy, or by current or expected future loosening of monetary policy, not justified by a decline in the natural rate of interest. This is essentially a forward-looking variant of the traditional Wicksellian analysis. We also observe that a higher response coefficient on inflation in the Taylor rule results in weaker equilibrium responses of inflation to exogenous disturbances, especially to disturbances expected several quarters in the future; the response of output is also reduced, though less dramatically. A higher response coefficient on the output gap in the Taylor rule instead significantly attenuates the equilibrium response of the output gap to news about the natural rate or monetary policy in the current quarter or the next one, and this also weakens the equilibrium response of inflation.

The consequences of interest-rate inertia in the Taylor rule are shown in Figure 4.6. Here we assume a rule of the form (2.12), with values for ϕ_π and ϕ_x as in the baseline case of Figures 4.4 and 4.5, but with various positive values for ρ. We observe that for given ϕ_π and ϕ_x, a higher value of ρ reduces the equilibrium response of both inflation and output, though the effect is much more dramatic in the case of the inflation response. This should be intuitive, since for given ϕ_π and ϕ_x, a higher ρ implies a larger *eventual* interest-rate response to a sustained increase in inflation or the output gap. When one considers super-inertial rules (*i.e.,* rules with $\rho > 1$), the response of inflation to an increase in the natural rate (not offset by a corresponding tightening of monetary policy) actually becomes *negative.* This is because the output gap still increases, as a result of which interest rates increase; the strong interest-rate inertia then implies an expectation of much higher *future* interest rates

[28]The coefficients for large j would dominate, in computing the effects of news about the natural rate or monetary policy, only if the news were to affect expectations *only* about conditions many quarters in the future.

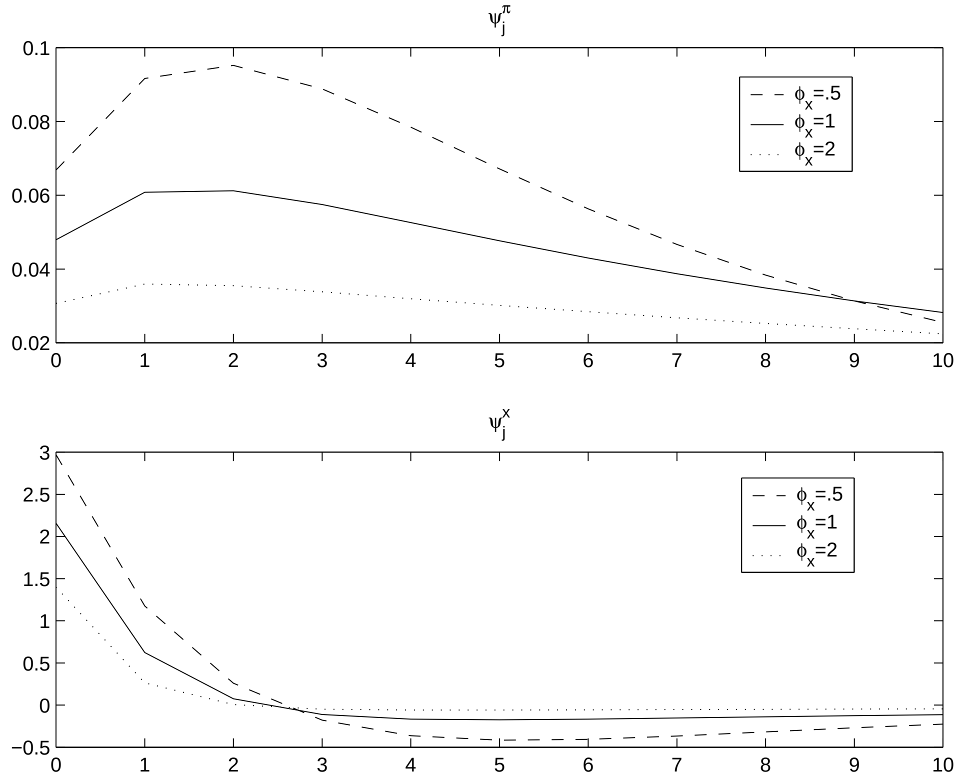

Figure 4.5: Consequences of varying the coefficient ϕ_x in the Taylor rule.

as well, even if the output gap no longer exists. The expectation of future tightening leads to an expectation of *lower* future output gaps, which in turn motivates an immediate reduction in inflation, despite the initially higher output gap. (The reduction in inflation is insufficient to prevent interest rates from rising under the policy rule, as there would otherwise be no increase in expected future interest rates to generate the incentive to disinflation.)

These figures indicate the immediate response of inflation and output to a disturbance that shifts the current and/or expected future values of $\hat{r}_t^n$ or $\bar{\imath}_t$. The figures do not, however, indicate the dynamic response to such disturbances. In the case that $\rho = 0$, inflation and the output gap are both purely forward-looking functions of the current and expected future disturbances, as indicated in (2.18) – (2.19). In this case, the dynamics of the response of inflation and output to a shock are a straightforward consequence of the dynamics of the disturbance itself. (A transitory disturbance must have a purely transitory effect; a more

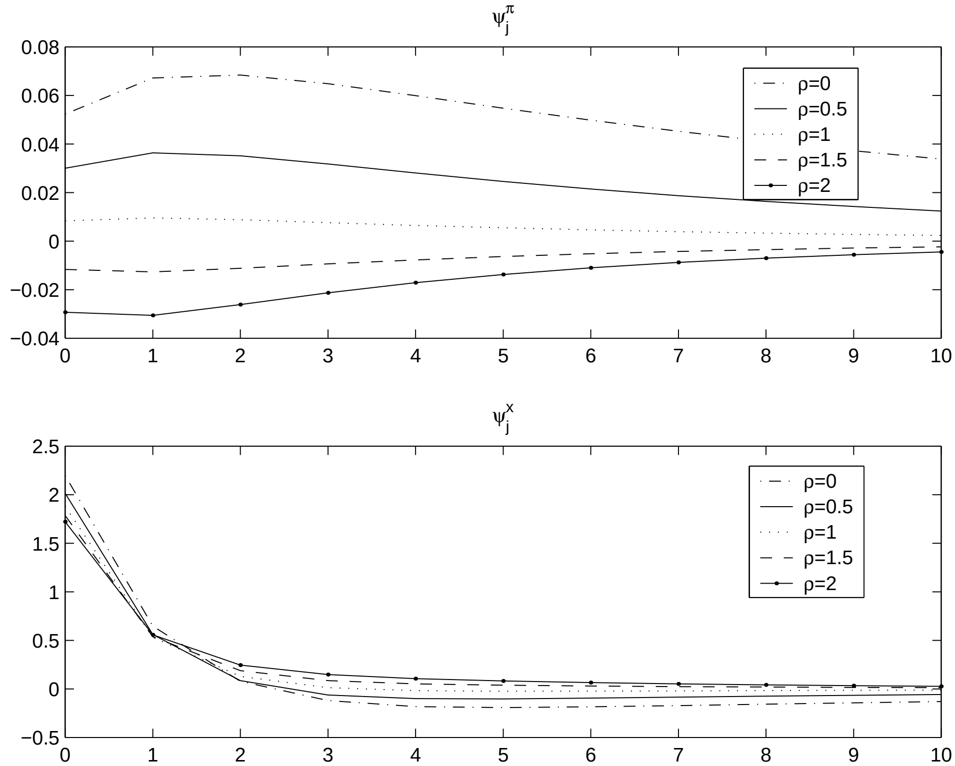

Figure 4.6: Consequences of varying the coefficient ρ in the Taylor rule with interest-rate inertia.

persistent disturbance has a correspondingly more persistent effect, though the effect is also larger, owing the effects of the anticipation of the continued disturbance in the future.) But when the policy rule incorporates feedback from lagged endogenous variables, it is also possible to obtain persistent effects on inflation and output from even a purely transitory disturbance. Since estimated central-bank reaction functions generally incorporate lagged endogenous variables of several sorts, both lagged interest rates and lags of variables such as inflation and the output gap as well (as discussed in chapter 1), it is not implausible to assume such lags in seeking to account for the degree of persistence of the responses of output and inflation to identified monetary policy shocks in historical data. Alternatively, of course, one could simply assume that the monetary policy disturbance $\{\bar{\imath}_t\}$ is serially correlated. This would suffice to allow our model to predict persistent responses to a monetary policy shock, and indeed the two explanations are not even conceptually distinguishable. For example, a policy rule of the form (1.10) where

$$\bar{\imath}_t - \bar{\pi} = \rho(\bar{\imath}_{t-1} - \bar{\pi}) + \epsilon_t$$

and $\{\epsilon_t\}$ is an i.i.d. mean-zero shock is equivalent to a policy rule of the form

$$\hat{\imath}_t = (1-\rho)\bar{\pi} + \rho\hat{\imath}_{t-1}) + \phi_\pi(\pi_t - \rho\pi_{t-1} - (1-\rho)\bar{\pi}) + \phi_x(x_t - \rho x_{t-1} - (1-\rho)\bar{x})/4 + \epsilon_t, \quad (2.22)$$

which now has a serially uncorrelated disturbance term, but feedback from lagged endogenous variables.

As an example of the kind of persistent response to a transitory shock that can result in the case of feedback from lagged endogenous variables, Figure 4.7 presents impulse responses to a monetary policy shock in the case of a policy rule of the form (2.22), where again $\{\epsilon_t\}$ is an i.i.d. mean-zero shock. In the figure, the intertia coefficients are set equal to $\rho = 0.6, 0.7$, or 0.8, while ϕ_π and ϕ_x are chosen to imply the same long-run responses $\Phi_\pi \equiv (1-\rho)^{-1}\phi_\pi = 2$ and $\Phi_x \equiv (1-\rho)^{-1}\phi_x = 1$ in each case. Because these coefficients satisfy (2.7) in each case, equilibrium is determinate. The figure shows the dynamic response to an unexpected monetary tightening (an unexpected increase in ϵ_t that raises the short-term interest rate by one percentage point, for given values of the other arguments of the central-bank reaction function).[29] The baseline case is chosen to be $\rho = 0.7$, because this is approximately the sum of the coefficients on lags of the federal funds rate in the rule estimated by Rotemberg and Woodford (1997), as discussed below, and we wish to provide insight into the theoretical responses obtained in their more complicated model.

One observes that the responses of both output and inflation to such a shock last for many quarters; in the case of the present completely forward-looking model of inflation and output determination, the degree of persistence of all four responses is determined directly by the assumed value of ρ in the policy rule. The amplitude of the equilibrium responses, for any given long-run responses to inflation and output in the policy rule, also depends on the

[29]The responses plotted for the nominal and real interest rate and for inflation are all expressed in percentage points of the equivalent annualized rate, so that "inflation" actually means the variable $4\pi_t$, and so on. The shock increases $4\bar{\imath}_0$ by one percentage point.

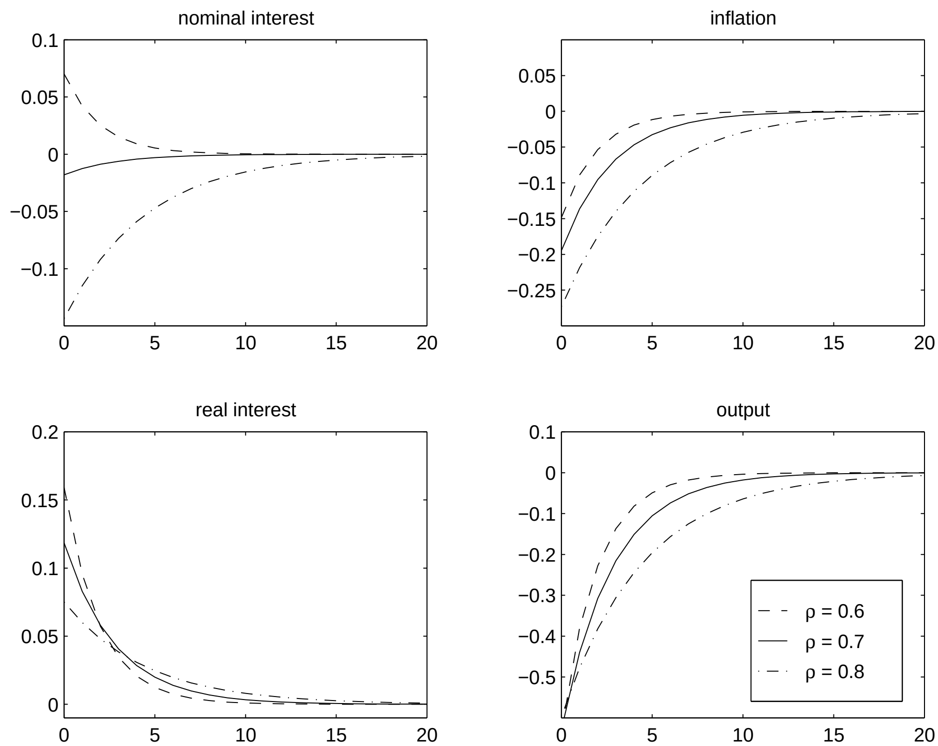

Figure 4.7: Impulse responses to a contractionary monetary policy shock, for alternative degrees of policy inertia.

value of ρ. The initial effect on output is essentially the same regardless of ρ, but the effect is more persistent the larger is ρ; and a more persistent output contraction reduces inflation more (and more persistently). Hence the reduction in inflation is greater, and the effects on both variables are more persistent, for larger values of ρ.

In the case of sufficiently modest values of ρ, a contractionary monetary policy shock is associated with a temporary increase in nominal interest rates; but for $\rho = 0.7$ or larger, the predicted inflation reduction is strong enough that nominal interest rates are actually predicted to *decrease* temporarily. Thus there is predicted to be no "liquidity effect" in the latter cases, a feature that has often been considered an embarrassment for calibrated optimization-based models of the monetary transmission mechanism (see, *e.g.*, Kimball, 1995, or Edge, 2000). Figure 4.7 shows that a "liquidity effect" is possible for some parameter values. However, a more satisfactory resolution of the problem requires that additional delays in the effects of monetary policy be introduced, as discussed in section 4. There we argue that the response shown in Figure 4.7 for the case $\rho = 0.7$ is not too different from the empirically estimated responses for the second quarter following the shock and later. (By that time, nominal interest rates are predicted to return nearly to the level that would have been expected in the absence of the shock, as shown in Figure 4.xx.) It is the estimated responses in the first two quarters that cannot be explained by this simple model; and here the real puzzle is not that nominal interest rates temporarily increase, but rather that output and inflation do not *immediately fall,* as indicated in Figure 4.7. Once that problem is solved, the problem of obtaining a "liquidity effect" is easily solved as well.

Our model can also be used to predict the response of the economy to real disturbances of various sorts, under one or another monetary policy rule. This is important for the explanation of business fluctuations, since it is widely agreed that the greater part of cyclical variation in real activity is ultimately caused by real disturbances rather than by random monetary policy.[30] But it is also important for the choice of a monetary policy rule, for the crucial question for the theory of monetary policy has to do with the choice of the *systematic* component of monetary policy (and not the exogenous random component, which one plainly wishes to eliminate to the extent possible), in the light of the implications of alternative systematic policies for the way that the economy will respond to disturbances that, in their origin, have nothing to do with monetary policy.

We shall not attempt a detailed treatment of the issue here. However, two general lessons from our baseline model are worth pointing out. The first is that, insofar as we are concerned solely with the responses of inflation, the output gap, and nominal interest rates to the real disturbances (and in chapter 6 we shall explain why these are exactly the variables which should be matter from the point of view of social welfare, under the assumptions that underly the present model), and insofar as we restrict attention to policy rules of the general type considered here (and in chapter 8 we shall show that optimal policy can be represented in

[30] This is the implication, for example, of the variance decompositions implied by typical VAR studies. Again see, *e.g.,* Christiano *et al.* (1999).

this way), then the only feature of the real disturbances that matters is *their effect upon the path of the natural rate of interest.* The second is that the responses of inflation and the output gap to a disturbance to the natural rate of interest *are exactly the same* as their responses to a monetary policy shock (disturbance to the $\bar{\imath}_t$ term in the policy rule) that has the same serial correlation properties and the opposite sign. Both conclusions follow from the fact that, in the case of the classes of policy rules considered above, equilibrium inflation and the output gap are functions solely of the path of the "gap" $\hat{r}_t^n - \bar{\imath}_t$. Thus Figure 4.7, for example, also indicates the response of inflation and the output gap to an unexpected reduction in the natural rate of interest, if the natural rate follows a first-order autoregressive process with a coefficient of ρ, and the monetary policy rule is of the form (1.10).

Thus far we have considered inflation and output-gap determination only in the case of a purely forward-looking model of inflation determination, namely, the basic Calvo pricing model introduced in chapter 3. But as discussed in section xx of that chapter, there is a fair amount of evidence suggesting that a model that allows for some degree of inflation inertia can better explain observed inflation dynamics. To what extent does allowance for inflation inertia require us to modify the neo-Wicksellian account just developed?

In fact, inflation inertia of the kind assumed by Christiano *et al.* (2001) makes only a small difference for our qualitative results, though the exact specification matters, of course, for quantitative purposes. Let the aggregate supply relation (1.9) be replaced by

$$\pi_t - \gamma\pi_{t-1} = \kappa x_t + \beta E_t[\pi_{t+1} - \gamma\pi_t], \tag{2.23}$$

where $0 \leq \gamma \leq 1$ indicates the degree of indexation of individual prices to a lagged price index, as in section xx of chapter 3. For simplicity let us again consider a policy rule of the form (1.10). Our complete system of equations for the determination of the equilibrium paths of inflation, output and the nominal interest rate then consists of equations (1.8), (1.10), and (2.23). It is then easily seen that in the case of a policy rule that implies a determinate equilibrium, this equilibrium is described by laws of motion of the form

$$\pi_t = \bar{\pi} + \omega_\pi(\pi_{t-1} - \bar{\pi}) + \sum_{j=0}^{\infty} \psi_j^\pi E_t(\hat{r}_{t+j}^n + \bar{\pi} - \bar{\imath}_{t+j}), \tag{2.24}$$

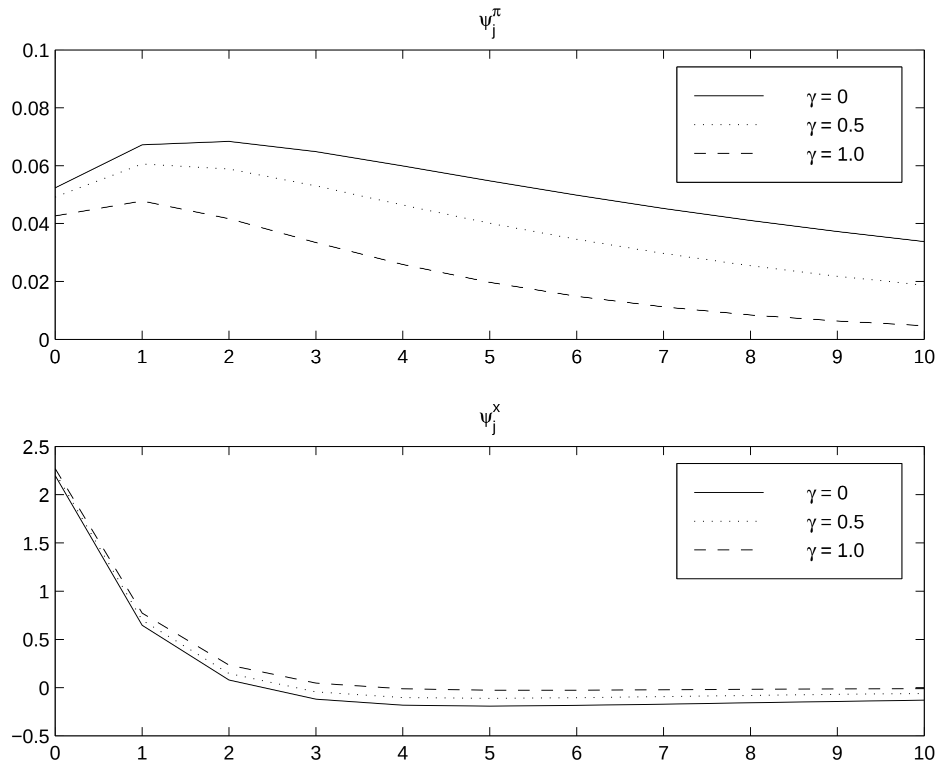

Figure 4.8: Inflation and output-gap responses under a contemporaneous Taylor rule, for alternative degrees of inflation inertia.

$$x_t = \bar{x} + \omega_x(\pi_{t-1} - \bar{\pi}) + \sum_{j=0}^{\infty} \psi_j^x E_t(\hat{r}_{t+j}^n + \bar{\pi} - \bar{\imath}_{t+j}), \tag{2.25}$$

$$\hat{\imath}_t = \bar{\imath}_t + \omega_i(\pi_{t-1} - \bar{\pi}) + \sum_{j=0}^{\infty} \psi_j^i E_t(\hat{r}_{t+j}^n + \bar{\pi} - \bar{\imath}_{t+j}). \tag{2.26}$$

Figure 4.8 plots the numerical values of the coefficients ψ_j^π and ψ_j^x as a function of the horizon j, for three alternative values of γ. Here the assumed values of β, σ and κ are again those given in Table 4.1, while the coefficients assumed in the policy rule are $\phi_\pi = 2, \phi_x = 1$. (Thus the case $\gamma = 0$ in this figure corresponds once again to the baseline cases of Figures 4.4 and 4.5, and to the $\rho = 0$ case of Figure 4.6.) We observe once again that the qualitative impact of news at date t on inflation and the output gap is the same as discussed earlier: an increase in the expected natural rate of interest (now or in the near future) increases both inflation and the output gap, while a tightening of monetary policy lowers both. The main difference made by a positive value of γ in this regard is that the inflation rate is more sensitive to expectations regarding the natural rate and the policy-rule intercept many quarters in the future.

The other difference in the response of inflation and the output gap to these two types of disturbances results from the presence of the $\pi_{t-1} - \bar{\pi}$ terms in each of equations (2.24) – (2.25). When $\gamma = 0$, these terms are zero, but as γ increases, ω_π takes an increasingly larger positive value, while ω_x takes an increasingly larger negative value.[31] In this model, for any given expectations regarding current and future natural rates of interest and monetary policy, the fact of a higher rate of inflation in the past acts as an adverse "supply shock," increasing current inflation while lowering the current output gap. This results in an additional mechanism for the propagation of the effects of fluctuations in the natural rate and/or in the monetary policy rule applied by the central bank. Nonetheless, it continues to be true that the natural rate of interest is a sufficient statistic for the effects of all real disturbances on the evolution of inflation and the output gap; that it is only the gap between the natural rate and the Taylor-rule intercept at each date that matters in this regard; and that at least the immediate effects of disturbances are have the same sign as in our earlier analysis.

2.5 Policy Rules for Inflation Stabilization

We now briefly consider the implications of the model of inflation determination just developed for the design of a monetary policy that would succeed at stabilizing the general level of prices, or more generally at stabilizing the rate of inflation around some target rate.[32] The theory of inflation implied by solutions such as (2.18) yields a simple prescription for a policy under which (if the private sector regards the central bank's policy commitment as

[31]For the parameter values used in Figure 4.8, one obtains $\omega_\pi = 0.42$, $\omega_x = -1.95$ in the case $\gamma = 0.5$, and $\omega_\pi = 0.73$, $\omega_x = -3.17$ in the case $\gamma = 1$.

[32]Here we assume without further discussion a "locally Ricardian" fiscal regime, in the sense discussed in chapter 5; the role of fiscal policy in the design of a regime conducive to price stability is taken up further in that chapter. We also take it for granted that price stability is the goal of monetary policy, without seeking to justify such an objective; the welfare-theoretic justification for such a goal is treated in chapter 6.

perfectly credible) inflation should never deviate from the target rate $\bar{\pi}$. The central bank should commit itself to a policy rule belonging to one of the families discussed in the previous section (or a generalization of these), with the further stipulations that (i) the time-varying intercept term $\bar{\imath}_t$ should track the exogenous variation in the natural rate of interest, so that $\bar{\imath}_t = \hat{r}_t^n + \bar{\pi}$ at all times, and (ii) response coefficients such as ϕ_π, ϕ_x and ρ should be chosen so as to imply a determinate rational expectations equilibrium. The latter proviso implies that the rule should respect the Taylor principle (at least in the case of each of the simple families considered above), but it may also place further restrictions upon the response coefficients as well.

As shown in the previous section, as long as the policy rule involves feedback only from current, lagged, or expected future values of the variables $\pi_t - \bar{\pi}$, $x_t - \bar{x}$ and $\hat{\imath}_t - \bar{\imath}_t$, then *if* equilibrium is determinate, the solution will make each of the variables just listed a function solely of current and expected future values of the "gap" terms $\hat{r}_t^n - \bar{\imath}_t + \bar{\pi}$, plus lagged values of the endogenous variables that enter the central bank's feedback rule and/or the aggregate supply relation.[33] This means that if $\bar{\imath}_t = \hat{r}_t^n + \bar{\pi}$ at all times, there will be no equilibrium fluctuations in the endogenous variables, except those due to non-zero initial values of those variables; the latter variation will be purely deterministic, and will approach zero as time passes, assuming a credible permanent commitment to the policy. Thus such a policy should succeed in principle in complete stabilization of both inflation and the output gap (as here defined).[34] Note that this conclusion holds equally in the case that we assume an aggregate supply relation of the form (2.23), incorporating inflation inertia.

Such an approach to policy would make the natural rate of interest a key concept in monetary policy-making. Furthermore, unlike some discussions (see, *e.g.*, Monetary Policy Committee, 1999) of the "neutral rate of interest", which imply that this should not vary over

[33] A similar conclusion holds if we allow for lagged endogenous terms in the IS relation as well, as discussed in section xx below.

[34] That it is possible to completely stabilize both inflation and the output gap depends, of course, upon the absence in our baseline model of inefficient supply disturbances of the kind discussed in section xx of chapter 3. The nature of the trade-off between inflation stabilization and output-gap stabilization goals that arises in the presence of such disturbances is considered in chapters 6 through 8.

time, our theoretical analysis implies that the natural rate of interest should vary in response to any of a wide range of types of real disturbances, and keeping track of its current value would be an important (and far from trivial) task of central-bank staff under such a regime. Of course, insofar as the policy rule involves feedback from the output gap, as allowed for in our discussion above, implementation of such a rule would also require a central bank to track variations in the current natural rate of output (a topic that already receives a great deal of discussion in most central banks). However, our analysis above implies that this concept is less essential to the central bank's task. For given an ability to adjust $\bar{\imath}_t$ perfectly in response to variations in the natural rate of interest, it would not matter what the responses to endogenous variables are, as long as the response coefficients are large enough to imply determinacy. As we have seen, the latter concern does not require any non-zero response to the output gap. Thus it would be possible, in principle, to commit to a rule such as (1.10) with $\bar{\imath}_t = r_t^n + \bar{\pi}$, $\phi_\pi > 1$, and $\phi_x = 0$, and completely stabilize inflation and the output gap; but the implementation of such rule would not require the central bank to be aware of the current natural rate of output.

On the other hand, simply producing an accurate estimate of the current natural rate of interest and adjusting the bank's operating target for an overnight interest rate accordingly is not quite sufficient for inflation stabilization; a commitment to the right sort of response to variations in endogenous variables such as inflation is also necessary, in order to ensure determinacy. It is true that, if the central bank's adjustment of $\bar{\imath}_t$ so as to track variations in the natural rate of interest is exact, the precise *degree* of response to endogenous state variables is irrelevant, as long as it suffices to put one in the range required for determinacy. This is because, in equilibrium, variations in those variables never occur; the commitment to a response to them matters not in order to change the nature of the desired equilibrium, but simply in order to exclude *other* possible equilibria. However, one must recognize that in practice, perfect tracking of the current natural rate of interest will be impossible, as real-time information about the natural rate will inevitably be imprecise.[35] In this case,

[35] The way in which optimal policy is affected by imperfect observability of the current state of the economy

for any given degree of irreducible variation in the gap process $\hat{r}_t^n - \bar{\imath}_t$ due to limits on the central bank's ability to track variations in the natural rate, the equilibrium fluctuations in inflation implied by a solution such as (2.18) will also depend upon the policy rule's response coefficients.

If one's goal is simply to stabilize inflation (and the output gap, as these goals are equivalent in the present model) to the greatest extent possible, then one would want to choose response coefficients that make the coefficients $\{\psi_j^\pi\}$ and $\{\psi_j^x\}$ as small as possible. In the case of the backward-looking Taylor rules, (1.10) or (2.8), it is possible to make all of these coefficients simultaneously as close to zero as one likes, simply by making the response coefficients ϕ_π and ϕ_x large enough. (In fact, it suffices to make *either* of these coefficients sufficiently large; for an equilibrium with small fluctuations in the output gap must also have small fluctuations in inflation, and vice versa.) Through this approach one can, in principle, make the equilibrium fluctuations in inflation arbitrarily small, even without attempting to track variations in the natural rate of interest at all. This represents an advantage of backward-looking Taylor rules over the forward-looking variants; for in the latter case, making the response coefficient on the inflation forecast too large results in indeterminacy (see Figure 4.2), as is also found by Bernanke and Woodford (1997) in a related model. There is thus a limit on the extent to which inflation fluctuations in response to real disturbances can be reduced using a forward-looking rule, if one restricts attention to rules that result in a determinate equilibrium.[36]

However, complete reliance upon the threat of extreme responses to inflation and/or output-gap variations should they occur, as a means of preventing such variations from ever occurring, is not obviously the most desirable approach. For under such a regime, there is the obvious danger that random noise in the particular measure to which the central bank responds might require violent adjustments of interest rates, that in turn create havoc in

is considered formally in chapter 8.

[36]In the case of rules in the family (2.12), it is still possible to reduce inflation fluctuations to an arbitrary extent without loss of determinacy by making the output-gap coefficient ϕ_x large. But this term in the rule is the same as in a backward-looking (standard) Taylor rule, and a rule with the main weight on that term is essentially equivalent to a backward-looking rule.

the economy. If we suppose that the inflation measure to which the central bank responds represents the average of the prices set by optimizing suppliers, plus an exogenous measurement error term, then a policy rule with a very large coefficient ϕ_π would cause no great trouble *if* price-setters were able to observe the quarter t measurement error disturbance *before* choosing new prices for quarter t. In that case, the equilibrium rate of actual inflation would adjust to (nearly perfectly) offset the measurement error, so that the inflation measure used by the central bank would be nearly perfectly stabilized. This would result in some variation in actual inflation, but this would be limited by the size of the measurement error, and fluctuations in interest rates would be minimal. However, in the more realistic case that quarter t prices are chosen *prior* to private-sector observation of the measurement error in the government statistics, then it would not be possible for private-sector pricing decisions to offset the measurement error, and large interest-rate variations would occur in equilibrium. This could make too extreme a version of such a strategy highly undesirable. Thus if one wishes to stabilize inflation to the greatest extent possible, it is fairly certain that attempting to vary policy in response to estimates of the natural rate of interest will improve policy.

On the other hand, as we show in chapters 6 and 7, there are good reasons why a central bank may not wish to fully stabilize inflation — for example, because the amount of interest-rate variation required would be undesirable. In this case, a constrained-optimal equilibrium will still involve non-trivial fluctuations in inflation and the output gap in response to real disturbances, and as a result, feedback from these endogenous variables may substitute for response to more direct measures of the real disturbances. It then may be possible, as we show in chapter 8, to find an optimal policy rule in which there is no time-varying term $\bar{\imath}_t$ at all.

3 Money and Aggregate Demand

Thus far, our analysis of the effects of monetary policy has made no reference to the evolution of the money supply, which may seem to some a surprising omission. Here we discuss the way in which the analysis can be extended to include analysis of the evolution of the money supply under alternative policy rules, should that be desired, in the case that there exist monetary frictions sufficient for a well-defined money demand function to exist. This extension of the model also makes it possible to consider monetary targeting rules or rules that respond to variations in money growth among the candidate policy proposals. We also consider the extent to which allowance for "real balance effects" would modify the conclusions reached above about inflation determination under interest-rate rules.

3.1 An Optimizing IS-LM Model

We shall now again assume the existence of transactions frictions of the kind considered in chapter 2, that can be represented by the inclusion of liquidity services from real money balances in the household utility function. Once again, this results in a first-order condition for the representative household's optimal demand for money balances of the form

$$\frac{U_m(C_t, m_t; \xi_t)}{U_c(C_t, m_t; \xi_t)} = \frac{i_t - i_t^m}{1 + i_t}, \tag{3.1}$$

where m_t denotes the household's end-of-period real money balances, and i_t^m denotes the interest (if any) paid on such balances. As with our discussion of the intertemporal IS relation above, we note that the first-order conditions describing the optimal behavior of a price-taking household are the same in the case of the sticky-price model considered in this chapter as in the flexible-price model of chapter 2. (The only difference here is that C_t now refers to an index of consumption of a large number of differentiated goods, rather than the single consumer good of chapter 2.) Imposing the requirements that the demands of the representative household equal the economy's aggregate supply of both goods and financial assets, we obtain the equilibrium condition

$$\frac{M_t^s}{P_t} = L(Y_t, \Delta_t; \xi_t), \tag{3.2}$$

where $\Delta_t \equiv (i_t - i_t^m)/(1 + i_t)$ is the interest differential between non-monetary and monetary assets and the money demand function $L(y, \Delta; \xi)$ has the same properties as in chapter 2. When log-linearized around the steady-state equilibrium with zero inflation, we again obtain a relation of the form

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i(\hat{\imath}_t - \hat{\imath}_t^m) + \epsilon_t^m, \tag{3.3}$$

where $\eta_y, \eta_i > 0$ and ϵ_t^m is an exogenous disturbance process.

The other equilibrium conditions used in the analysis thus far continue to apply in the presence of monetary frictions. The only difference between a cashless economy and one in which central-bank money facilitates transactions is that in the latter case, the marginal utility of additional real expenditure by the representative household is given by $U_c(C_t, m_t; \xi_t)$, the value of which will in general depend on the level of real money balances in addition to the level of real expenditure. However, in either the case of preferences additively separable between consumption and real balances or the "cashless limit" discussed in section xx of chapter 2, we may neglect the effect of variations in real money balances on the value of U_c, and the intertemporal IS relation and aggregate supply relation take exactly the form assumed above. The complete system of equilibrium conditions for determination of the nominal interest rate, the price level, and output (in the case of a money-growth rule) or the price level, output and the money supply (in the case of an interest-rate rule) is then given by the IS and AS relations considered above, together with (3.2) or (3.3). This system of equations has essentially the structure of an IS-LM-AS system of the kind familiar from undergraduate textbooks, though here the IS and AS relations are not purely static ones. We note also that the "LM relation" (3.2), considered as an equilibrium relation between i_t and Y_t, is shifted by variations in either M_t^s or i_t^m, which appear as two separate instruments through which monetary policy may be implemented, in addition to potentially being shifted by the exogenous disturbances ξ_t.

In the case of an interest-rate rule of one of the forms considered above, the equilibrium paths of inflation and output are determined by the IS and AS relations as above, but the LM relation now allows us to solve for the implied path of the money supply as well, under an assumption about the path of the interest rate on money (such as the conventional specification $\hat{\imath}^m_t = 0$).[37] In the case of a "cashless limiting economy" — perhaps the most attractive justification for our neglect of real balance effects in the analysis thus far — the absolute size of equilibrium real money balances is assumed to be negligible. However, this does not mean that there may not be non-negligible (and well-defined) fluctuations in equilibrium real balances *relative* to their (extremely small) steady-state level, indicated by the variable $\hat{m}_t$ in (3.2). Similarly, there may be non-negligible, well-defined fluctuations in the rate of growth of the nominal money supply, even in the absence of a well-defined absolute level of the equilibrium money supply.

For example, one may consider how money growth must respond to various kinds of real disturbances, under a policy that succeeds in stabilizing inflation and the output gap. To answer this question, we simply substitute the required interest-rate variations, discussed in section xx above, into equation (3.3) to determine the implied path of the money supply. We find that, in general, the money supply should be allowed to vary in response to all five of the types of exogenous disturbances considered in section xx above, so that a constant money growth rate is certainly not the best way to stabilize inflation .

Nor does the path of the money supply required for price stability necessarily involve "leaning against the wind". For example, *procyclical* variations in the money supply are required in response to temporary fluctuations in productivity, as argued by Ireland (1996);[38] for an increase in a_t raises $\hat{Y}^n_t$ while lowering $\hat{r}^n_t$, thus warranting an increase in $\log M^s_t$ at the same time as an increase in $\hat{Y}_t$. The same is true of temporary labor-supply shocks, and while

[37] It is worth noting, however, that in the absence of a constraint on the evolution of i^m_t, the required path of the money supply is not uniquely determined; for a given interest-rate policy may be implemented using *either* quantity variations or variations in the interest paid on money, or some combination of the two.

[38] Aiyagari and Braun (1998) reach a similar conclusion, in the case of their model with sticky prices, though they assume convex costs of price changes, following Rotemberg (1995), rather than predetermined prices as in the models considered here. These authors also reach a similar conclusion with regard to government-purchase shocks, in the case of their numerical calibration of their model.

the result depends upon parameter values, it is also true of government-purchase shocks and consumption-demand shocks, at least if these are sufficiently persistent. Furthermore, in the case of technology or labor-supply shocks, it is actually desirable for the money supply to be *more* procyclical than would be the case if interest rates were held unchanged; for one actually wants nominal interest rates to *decline* in response to a positive shock. In the case of the other two shocks, this is not true, but it is still possible that holding the nominal interest rate fixed is closer to the optimal response that holding the money supply fixed; in particular, this is necessarily true if the shocks are sufficiently persistent, as in that case the natural rate of interest is affected very little.

Thus variations in the rate of money growth should not, in general, be a very accurate indicator of whether interest rates have been allowed to adjust to the extent that they ought to in response to disturbances. Monetary targeting amounts to an automatic mechanism for bringing about procyclical interest rate variation (and interest-rate increases in response to price-level increases as well); but the required change in the level of interest rates for stabilization — which depends upon the change in the natural rate of interest — might not even have the same *sign* as this. Monetary targeting also causes interest rates to vary in response to money-demand disturbances, even though these should have negligible effects upon the natural rate of interest, so that it is not desirable for interest rates to respond to them. One should therefore expect to be able to do better by adopting an interest-rate rule that (i) achieves a desired degree of response of interest rates to price-level and/or output gap increases through explicit feedback from measures of these variables, as called for in the "Taylor rule," and (ii) incorporates a direct response to changes in the central bank's estimate of the current natural rate of interest.

Our conclusions with regard to the usefulness of monetary targeting contrast with the classic analysis of Poole (1970), according to which monetary targeting should be desirable as long as money-demand disturbances are of less importance than real disturbances to aggregate demand ("IS shocks"). Our government-purchase or consumption-demand shocks presumably correspond to what Poole intends by "IS shocks"; yet even in the case of these shocks, we have argued that some degree of accommodation (allowing the money supply to vary in order to reduce the interest-rate response) is often desirable. And if the shocks are sufficiently persistent, the optimal degree of accommodation of the "IS shift" may be nearly 100 percent. The difference, of course, is that Poole assumes that output stabilization should be the goal of policy, whereas here we assume a goal of stabilizing the output *gap* instead,[39] by which we mean output relative to a natural rate that is affected by "IS shocks" among other real disturbances. If we consider the possibility of technology or labor-supply shocks, neglected by Poole altogether, our results are even more different, and even more strongly support a presumption in favor of accommodation.

But the most important limitation of Poole's analysis is that it assumes that monetary targeting is the only alternative to a complete interest-rate peg. Yet the kind of policy that we have proposed above (or that has been advocated by Wicksell or by Taylor) is not one that fixes nominal interest rates in the face of changing macroeconomic conditions. Once one recognizes that an interest-rate rule may specify interest-rate adjustments in response to changes in the price level and/or in output, then Poole's IS-LM analysis provides no ground whatsoever for belief that it is desirable to respond to changes in monetary aggregates *as well.*

One might, of course, make a case for responding to changes in the growth rate of monetary aggregates if such statistics provide more up-to-date information about prices and/or output than is otherwise available. This seems unlikely; at best, one might on these grounds justify paying attention to monetary aggregates along with a large number of other indicators.[40] And even in this case, it would be most useful to understand the central bank's policy commitment in terms of its response to its estimates of inflation and the output gap, rather than in terms of a commitment to respond to — let alone to stabilize — particular indicators.

[39]This objective is justified in terms of social welfare in chapter 6.

[40]The optimal use of indicator variables given that inflation and the output gap are not perfectly observed in real time is treated in chapter 8.

3.2 Real-Balance Effects

Thus far we have considered only models in which monetary policy affects aggregate demand through the effects of real interest rates on the desired timing of private expenditure. While this is surely the most important single channel through which monetary policy matters, it is sometimes argued that this channel alone is not the only one way in which monetary policy matters, and that under at least some circumstances the neglect of other mechanisms may lead to significantly misleading conclusions. One proposal of this kind with an especially venerable history, is the argument often found in monetarist writing that the level of real money balances held by the private sector should directly influence aggregate demand, for reasons that are independent of the reduction in equilibrium interest rates that will ordinarily accompany a higher real money supply in order to induce the private sector to choose to hold the higher money balances. Pigou (1943) argued for the existence of such a "real balance effect", and proposed that its existence was especially important in guaranteeing the effectiveness of monetary policy even under the circumstances of a "liquidity trap" of the kind posited by Keynes (1936). We take up the case of a liquidity trap below; but first we consider the nature of real balance effects under more ordinary circumstances.

It is first important to note that we have *not* omitted any effect of real money balances upon aggregate demand that results from a simple (Hicksian) wealth effect, as Pigou supposed. The Euler equation (1.1) for the optimal timing of private expenditure, which (when converted to the form (1.2) by imposing market-clearing) has proven to be the crucial equilibrium condition of our Wicksellian framework, does not in any way contradict the contribution of financial wealth to the intertemporal budget constraint of the representative household (discussed in detail in chapter 2). Furthermore, solving our log-linearized Euler equation forward to obtain a relation of the form (1.4) between the level of aggregate expenditure at a given date and a very-long-term real rate (or equivalently, a distributed lead of expected future short real rates) involves no neglect of any wealth effects. It is true that, for an individual household, optimal consumption demand is a function of the household's current financial wealth in addition to its future (after-tax) income expectations and its expectations regarding the available rate of return on its savings. But still, optimal current expenditure *relative* to expected future expenditure depends only on the expected rates of return on assets between now and the future date; for greater financial wealth should imply greater expected future spending in the same proportion as it increases desired current spending. And for households in aggregate, expected spending in the long-run future must be tied down by the exogenous evolution (in the kind of model considered here) of the natural rate of output. Given this anchor for the representative household's expected long-run level of expenditure, the sequence of expected future real interest rates does suffice to determine what optimal expenditure must be in the present.

Nonetheless, there is a type of real-balance effect from which the above analysis has abstracted. Thus far, we have assumed the special class of utility functions described by equation (xx) of chapter 2, that are additively separable between consumption and real money balances. This familiar assumption makes the algebra simpler in many places, but it can hardly be defended as realistic. If utility is obtained from holding money, this must be because money balances facilitate transactions, and it is hardly sensible that the benefits of such balances should be independent of the real volume of transactions that a household actually undertakes. In particular, as already argued in chapter 2, it is plausible that the marginal benefit of additional real balances should be higher when real transactions are greater, so that the cross-partial derivative $U_{cm} > 0$.

We accordingly here consider how our above results extend to the case of a more general preference specification. We shall argue that the simple case treated above remains a good guide to intuition, and indeed that it may be justified as an approximation without any appeal to the plausibility of additive separability. But it is also useful to be able to see how real balance effects require our previous calculations to be modified, if we wish to be more precise.

Let us recall from chapter 2 that in the general case, the equilibrium condition derived from the Euler equation for the optimal timing of private expenditure takes the form

$$1 + i_t = \beta^{-1} \left\{ E_t \left[\frac{U_c(Y_{t+1}, M^s_{t+1}/P_{t+1}; \xi_{t+1})}{U_c(Y_t, M^s_t/P_t; \xi_t)} \Pi^{-1}_{t+1} \right] \right\}^{-1}. \qquad (3.4)$$

(See equation (xx) of chapter 2.) This reduces to equation (1.2) above only under the assumption of additively separable preferences (or the absence of non-negligible transactions frictions). In the general case, a log-linear approximation to this relation takes the form

$$\hat{\imath}_t = \sigma^{-1}[E_t(\hat{Y}_{t+1} - g_{t+1}) - (\hat{Y}_t - g_t)] - \chi(E_t \hat{m}_{t+1} - \hat{m}_t) + E_t \pi_{t+1}, \qquad (3.5)$$

generalizing (1.3), where

$$\chi \equiv \frac{\bar{m} U_{cm}}{U_c}$$

with partial derivatives evaluated at the steady state. Here g_t is again the exogenous factor defined in (2.3), where the shift in the marginal utility of consumption $\bar{C}_t$ is now evaluated at the steady-state values of both C_t and m_t. Note that (3.5) is the same as equation (xx) of chapter 2, except that now we make the dependence on real output explicit, as output is no longer exogenous.

Solving (3.5) forward again yields an "IS equation" of the form

$$\hat{Y}_t = \hat{Y}_\infty + g_t + \chi\sigma\hat{m}_t - \sigma \sum_{j=0}^{\infty} E_t[\hat{\imath}_{t+j} - \pi_{t+j+1}]. \qquad (3.6)$$

If $\chi > 0$, then there is indeed a "real-balance effect" upon aggregate demand. This results not from a wealth effect, but from the fact that even controlling for the path of real interest rates, times when real balances are high are particularly convenient times to spend, due to the way in which money balances facilitate transactions. We shall also see that, contrary to Pigou's suggestion, this effect does not imply that increases in the money supply can increase demand even in a liquidity trap; but this question is deferred until section xx below.

Let us first consider instead the consequences of the existence of a real-balance effect for inflation determination under an interest-rate rule, in the case of only small fluctuations around the zero-inflation steady state (so that the zero lower bound on nominal interest rates is never approached). As in section xx of chapter 2, it is convenient to use (3.3) to eliminate real balances from the intertemporal IS equation (3.5), yielding

$$(1+\eta_i\chi)\hat{\imath}_t = (\sigma^{-1}-\eta_y\chi)(E_t\hat{Y}_{t+1}-Y_t)+E_t\pi_{t+1}+\eta_i\chi E_t\hat{\imath}_{t+1}-\sigma^{-1}(E_tg_{t+1}-g_t)-\chi(E_t\epsilon^m_{t+1}-\epsilon^m_t)-\eta_i\chi(E_t\hat{\imath}^m_{t+1}-\hat{\imath}^m_t). \quad (3.7)$$

This gives us an equilibrium relation written solely in terms of the evolution of interest rates, inflation and output, like (1.3).

Let us then define the natural rate of output Y_t^n as the equilibrium level of output at each point in time that would obtain under flexible prices, given a monetary policy that maintains a constant interest-rate spread Δ_t between non-monetary and monetary riskless short-term assets. This last stipulation in our definition is now necessary, since when utility is non-separable, equilibrium output under flexible prices is no longer independent of monetary policy.[41] Let us similarly define the natural rate of interest r_t^n as the equilibrium real rate of interest under the same hypothetical circumstances. Then we observe from (3.7), that must hold equally whether prices are flexible or sticky, that

$$\hat{r}_t^n = (\sigma^{-1}-\eta_y\chi)(E_t\hat{Y}^n_{t+1}-Y^n_t)-\sigma^{-1}(E_tg_{t+1}-g_t)-\chi(E_t\epsilon_{m,t+1}-\epsilon_{mt}). \quad (3.8)$$

Note that this reduces to our previous definition (1.11) in the case that $\chi = 0$. Using this to substitute for terms on the right-hand side of (3.7), and rearranging terms, we obtain

$$(1-\sigma\eta_y\chi)x_t = (1-\sigma\eta_y\chi)E_tx_{t+1}-\sigma(\hat{\imath}_t-E_t\pi_{t+1}-\hat{r}^n_t)+\sigma\eta_i\chi[E_t(\hat{\imath}_{t+1}-\hat{\imath}^m_{t+1})-(\hat{\imath}_t-\hat{\imath}^m_t)], \quad (3.9)$$

generalizing (1.8), where again $x_t \equiv \hat{Y}_t - \hat{Y}^n_t$. Thus we once more obtain an intertemporal IS relation in terms of the output gap x_t in which the only exogenous disturbance term is the shift in the natural rate of interest $\hat{r}^n_t$.[42]

Allowing for non-trivial monetary frictions and non-separable utility between consumption and real balances also requires modification of our derivation of the aggregate supply

[41] One might, of course, choose other definitions of the natural rate, such as the flexible-price level of output in the case of a monetary policy that results in a stable price level. The choice made here turns out to be convenient in simplifying the welfare analysis of chapter 6.

[42] Note that in the case of flexible prices and an exogenous goods supply, so that necessarily $x_t = 0$ at all times, equation (3.9) reduces to equation (xx) of chapter 2, with $\hat{r}^n_t$ corresponding to the composite exogenous disturbance $\hat{r}_t$ of that model.

relation (1.6). (Throughout chapter 3, we assumed either negligible monetary frictions or separable utility.) In the case of non-separable utility, the marginal utility of income depends upon real money balances as well as real expenditure, so that the household labor supply equation becomes

$$w_t(i) = \frac{v_h(h_t(i); \xi_t}{U_c(C_t, m_t; \xi_t)}.$$

It follows that the real marginal cost of supplying good i, when log-linearized, is given by

$$\hat{s}_t(i) = \omega(\hat{y}_t(i) - q_t) + \sigma^{-1}(\hat{Y}_t - g_t) - \chi \hat{m}_t,$$

where the exogenous disturbances q_t and g_t continue to be defined as before. Substituting (3.3) for real money balances $\hat{m}_t$, and recalling the definition just proposed for the natural rate of output, we can write average real marginal cost in the form

$$\hat{s}_t = \epsilon_{mc}(\hat{Y}_t - \hat{Y}_t^n) + \eta_i \chi (\hat{\imath}_t - \hat{\imath}_t^m), \tag{3.10}$$

where the elasticity of average marginal cost with respect to aggregate output is now equal to

$$\epsilon_{mc} = \omega + \sigma^{-1} - \eta_i \chi, \tag{3.11}$$

and the natural rate of output is now given by

$$\hat{Y}_t^n = \frac{\omega q_t + \sigma^{-1} g_t + \chi \epsilon_t^m}{\epsilon_{mc}}, \tag{3.12}$$

generalizing (2.2).

Expression (3.10) can alternatively be written in the form

$$\hat{s}_t = \epsilon_{mc}\, [x_t + \varphi(\hat{\imath}_t - \hat{\imath}_t^m)],$$

where

$$\varphi \equiv \frac{\eta_i \chi}{\epsilon_{mc}}.$$

The case that would be seem to be of greatest empirical relevance is that in which χ satisfies the bounds

$$0 \leq \chi < \eta_y^{-1}(\omega + \sigma^{-1}),$$

in which case both ϵ_{mc} and φ are positive coefficients.[43] A calculation exactly parallel to that given in chapter 3 then allows us to derive an aggregate supply relation of the form

$$\pi_t = \kappa[x_t + \varphi(\hat{\imath}_t - \hat{\imath}^m_t)] + \beta E_t \pi_{t+1}, \tag{3.13}$$

generalizing (1.6). Here

$$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \frac{\epsilon_{mc}}{1+\omega\theta}$$

as before, but ϵ_{mc} is now given by (3.11). We find that in the non-separable case, inflationary pressure depends not only on the output gap, but also on the interest differential between non-monetary and monetary assets. The greater this return differential is, the greater the cost of holding the money balances needed to facilitate transactions; hence the greater is the marginal cost of supplying a given level of output, and ultimately the incentive to raise prices.

How much of a difference is made by the inclusion of the additional terms in (3.9) and (3.13)? If we assume an interest-rate rule of one of the kinds considered in section xx above, and add the additional stipulation that the interest rate i^m_t paid on the monetary base is adjusted along with all changes in the central bank's interest-rate operating target, so as to maintain a constant spread Δ_t, then our previous conclusions are *entirely unchanged,* under an appropriate calibration of the numerical coefficients. The reason is that in the case that $\hat{\imath}^m_t = \hat{\imath}_t$ at all times, equations (3.9) and (3.13) reduce to *precisely* our previous equations (1.8) and (1.6) respectively, given the new definition (3.8) of the natural rate of interest, with one exception: in our "IS relation" (3.4), the coefficient indicating the interest-sensitivity of real expenditure is no longer σ, as in (1.8), but rather $(\sigma^{-1} - \eta_y \chi)^{-1}$.[44] As long as $\chi < (\eta_y \sigma)^{-1}$ — the empirically realistic case, under the calibration proposed below[45] — then there is no

[43] As discussed in section xx below, Rotemberg and Woodford (1997) argue that a reasonable value for ϵ_{mc} is about .63 for the U.S. economy. We have argued in chapter 2 that a realistic value for $\chi\eta_y$ for the U.S. would be about .01, so this would imply a value for $\sigma^{-1} + \omega$ of about .64. Even if these values are inaccurate, $\sigma^{-1} + \omega$ is likely to be much larger than $\chi\eta_y$.

[44] In our analysis of price-level determination under this kind of policy in chapter 2, where we assumed flexible prices, this difference did not matter, as no fluctuations in the output gap x_t could occur in equilibrium. In the case of price stickiness, the additional qualification is needed.

[45] In a more realistic extension of our model, where we distinguish among different categories of private

qualitative change in our conclusions; we would obtain identical numerical results to those presented in our earlier figures under a suitably different calibration of the value of σ.

Inflation and the output *gap* (the latter, rather than detrended output, being the welfare-relevant quantity, as we shall argue in chapter 6) thus evolve according to exactly the formulas derived earlier, once we take into account the modification of the effects of real disturbances on the natural rate of interest. (Computation of the implied path of detrended output requires that one also take into account the modified expression for the effects of real disturbances on the natural rate of output.) Thus our previous results apply not only to fully (or nearly) "cashless" economies, and to economies in preferences are additively separable between consumption and real balances, but also to economies in which central-bank interest-rate targets are implemented through adjustments of the interest paid on the monetary base, rather than through adjustments of the supply of base money.[46]

If, however, interest-rate adjustments are implemented through variation in the supply of base money, holding fixed the rate of interest paid on the monetary base — as under current U.S. monetary arrangements, for example — then the additional terms in both the IS relation and the AS relation matter, assuming non-trivial monetary frictions and non-separable preferences. Nonetheless, for many purposes the additional terms do not seem likely to have too large an effect on our quantitative conclusions. We can illustrate this by considering a numerical calibration of our model based on U.S. data.

As argued in section xx of chapter 2, evidence on long-run U.S. money demand suggests money-demand elasticities on the order of $\eta_y = 1, \eta_i = 28$ quarters, and a value of $\chi = 0.02$

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expenditure with differing degrees of interest-sensitivity, this would be even more clearly true. For one might well calibrate a lower degree of interest-elasticity (larger σ^{-1}) for those expenditures that are complementary with real money balances (the ones for which $\chi > 0$).

[46]Another case in which the previous results apply would be a hybrid of two of those just mentioned. Suppose that utility each period is given by $U(C_t, m_t^{cb}; \xi_t) + V(m_t^{cu}; \xi_t)$, where m_t^{cb} indicates the real value of the clearing balances held at the central bank to facilitate payments and m_t^{cu} indicates the real value of currency in circulation. (Here the liquidity services provided by clearing balances are assumed to depend on the volume of aggregate transactions in the economy, while the usefulness of currency is assumed to be largely unaffected by variations in aggregate real expenditure.) Then it suffices for the validity of our previous results that the interest rate i_t^{cb} paid on clearing balances be adjusted so as to maintain a constant spread between this and the central bank's interest-rate operating target, as is true under the channel systems mentioned in chapter 1; it does not matter whether there is any interest paid on currency as well.

(or lower) for the elasticity of the marginal utility of expenditure with respect to additional real money balances. Suppose that the other coefficients of our structural equations are based on the parameter values used by Rotemberg and Woodford (1997), shown in Table 4.1 below, and discussed further there. Since Rotemberg and Woodford infer the value of ω from their estimate of ϵ_{mc}, rather than the reverse, allowing for $\chi \neq 0$ does not imply any change in our calibrated value of ϵ_{mc}. Thus the value that we shall assume for φ is (.02)(28)/(.63) = 0.89 quarters.[47] The values assumed for the parameters β, σ, and κ are those reported in Table 4.1.

Using similar methods as above, we can show that in the case that a determinate equilibrium exists,[48] an interest-rate rule of the form (1.10)[49] together with the stipulation that $\hat{\imath}^m_t = 0$ implies equilibrium paths for the endogenous variables of the form

$$\pi_t = \bar{\pi} + \sum_{j=0}^{\infty} \psi^{\pi}_j (E_t \hat{r}^n_{t+j} + \bar{\pi}) - \sum_{j=0}^{\infty} \tilde{\psi}^{\pi}_j E_t \bar{\imath}_{t+j}, \tag{3.14}$$

$$x_t = \bar{x} + \sum_{j=0}^{\infty} \psi^{x}_j (E_t \hat{r}^n_{t+j} + \bar{\pi}) - \sum_{j=0}^{\infty} \tilde{\psi}^{x}_j E_t \bar{\imath}_{t+j}, \tag{3.15}$$

$$\hat{\imath}_t = \bar{\imath}_t + \sum_{j=0}^{\infty} \psi^{i}_j (E_t \hat{r}^n_{t+j} + \bar{\pi}) - \sum_{j=0}^{\infty} \tilde{\psi}^{i}_j E_t \bar{\imath}_{t+j}. \tag{3.16}$$

In the case that $\chi \neq 0$, the coefficients $\tilde{\psi}^{\pi}_j$ and so on are no longer exactly equal to the coefficients ψ^{π}_j and so on. The extent to which allowance for real-balance effects affects the quantitative size of these coefficients is shown in Figures 4.10 and 4.11, for the calibrated parameter values stated in the previous paragraph.

The figures show the results in the case of a rule with $\phi_\pi = 2, \phi_x = 1$, corresponding to the baseline case in the previous Figures 4.4 and 4.5; Figure 4.10 shows the coefficients ψ^{π}_j and ψ^{x}_j, while Figure 4.11 shows the coefficients $\tilde{\psi}^{\pi}_j$ and $\tilde{\psi}^{x}_j$, for various future horizons j. In

[47] This value multiplies a quarterly interest rate; if instead the interest rate is expressed as an annualized rate, φ should equal 0.22 years.

[48] The conditions for determinacy of equilibrium can also be generalized using the same methods as above. One again obtains a set of inequalities that the coefficients of the interest-rate rule must satisfy, and these expressions are continuous functions of χ, so that the set of interest-rate rules that imply a determinate equilibrium remain nearly the same when we assume a small positive value for χ.

[49] Once again, we assume an output-gap target $\bar{x}$ consistent with the inflation target $\bar{\pi}$. In the case that $\chi \neq 0$, this requires that $\bar{x} = (1-\beta)\bar{\pi}/\kappa + \varphi\bar{\pi}$.

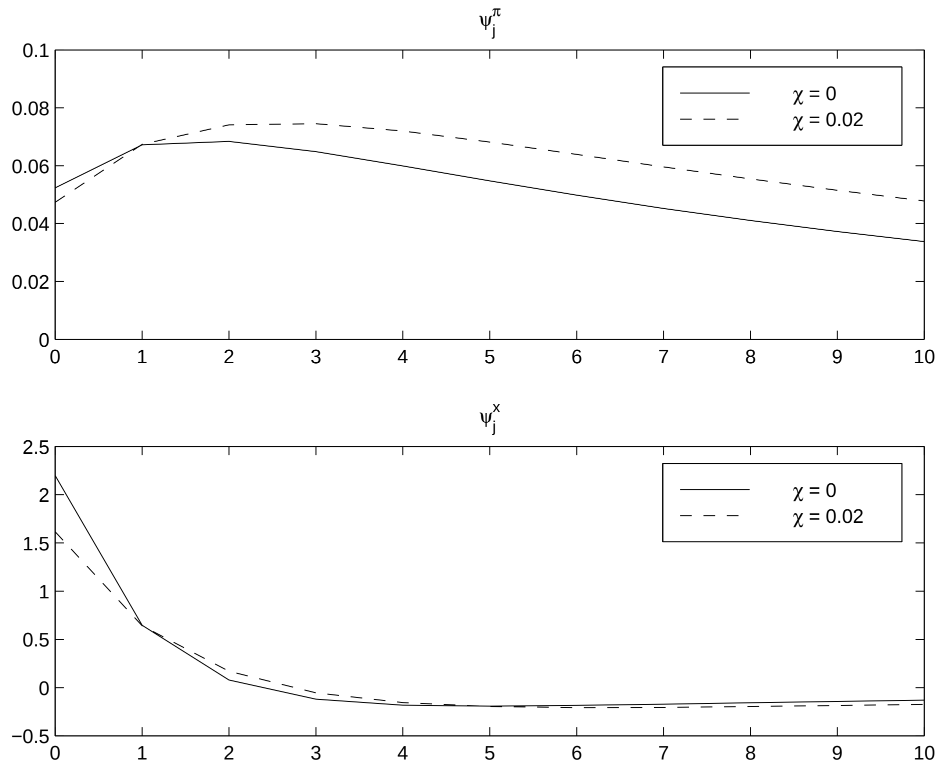

Figure 4.10: Effects of anticipated natural-rate fluctuations under a simple Taylor rule, allowing for real-balance effects.

each panel, the coefficients are computed both for the parameter values stated above (the dashed lines), and for the same values of β, σ, and κ, but under the assumption that χ and φ are equal to zero (as in the earlier figures). We observe that the predicted effects of both types of disturbances on inflation and the output gap are not much affected by allowing for real-balance effects. The main difference is a greater inflationary impact of increases in the natural rate of interest that are foreseen several quarters in advance, as a result of the contribution of the resulting increase in the interest differential to the marginal cost of supply. But even this will matter for predicted inflation dynamics only to the extent that fluctuations in the natural rate of interest are predictable several quarters in advance.

In the case of an interest-rate rule of the form (2.8), again combined with the stipulation

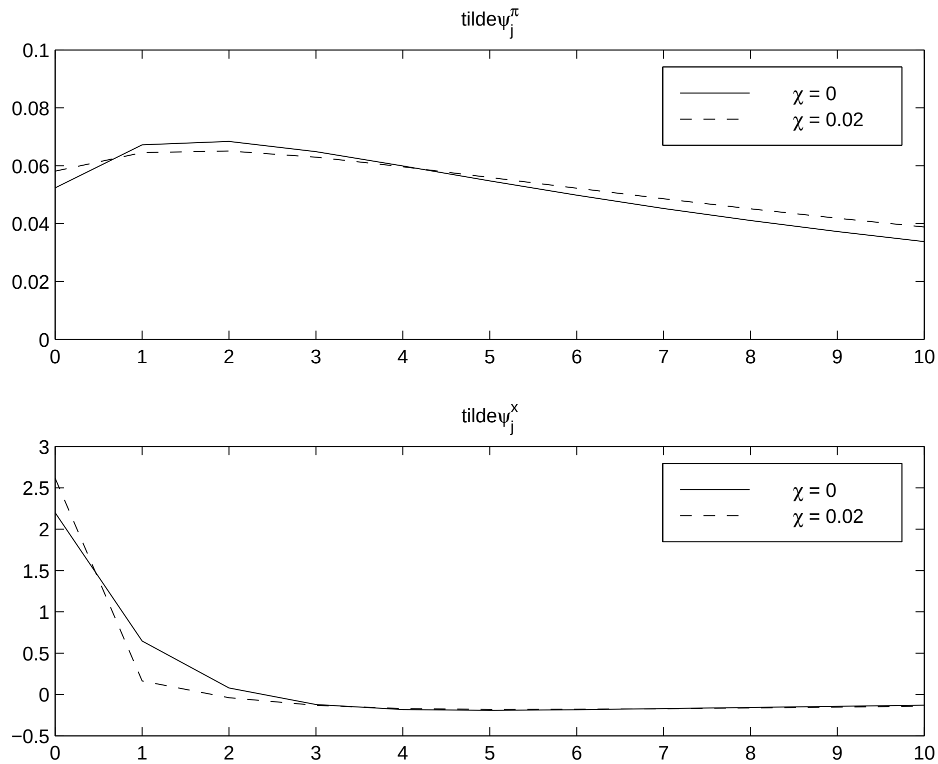

Figure 4.11: Effects of anticipated policy shifts under a simple Taylor rule, allowing for real-balance effects.

that $\hat{\imath}_t^m = 0$, we correspondingly obtain solutions of the form

$$\pi_t = \bar{\pi} + \omega^\pi(\hat{\imath}_{t-1} - \bar{\imath}_{t-1}) + \sum_{j=0}^{\infty} \psi_j^\pi E_t(\hat{r}_{t+j}^n + \bar{\pi}) - \sum_{j=0}^{\infty} \tilde{\psi}_j^\pi E_t \bar{\imath}_{t+j},$$

and similarly for the other endogenous variables. Once again, the coefficients $\tilde{\psi}_j^\pi$ and so on are no longer exactly equal to the coefficients ψ_j^π when $\chi \neq 0$. The quantitative significance of the allowance for real-balance effects is shown in Figures 12 and 13, using the same format as in Figures 10 and 11. Here the assumed coefficients of the policy rule are $\phi_\pi = 0.6, \phi_x = 0.3$, and $\rho = 0.7$, as in the baseline case of Figure 4.6. Our conclusions in this case are essentially the same.

Another way of considering the consequences of real-balance effects for our previous conclusions is to compute the predicted impulse responses of inflation, the output gap and the nominal interest rate to a monetary policy shock using our modified structural equations.

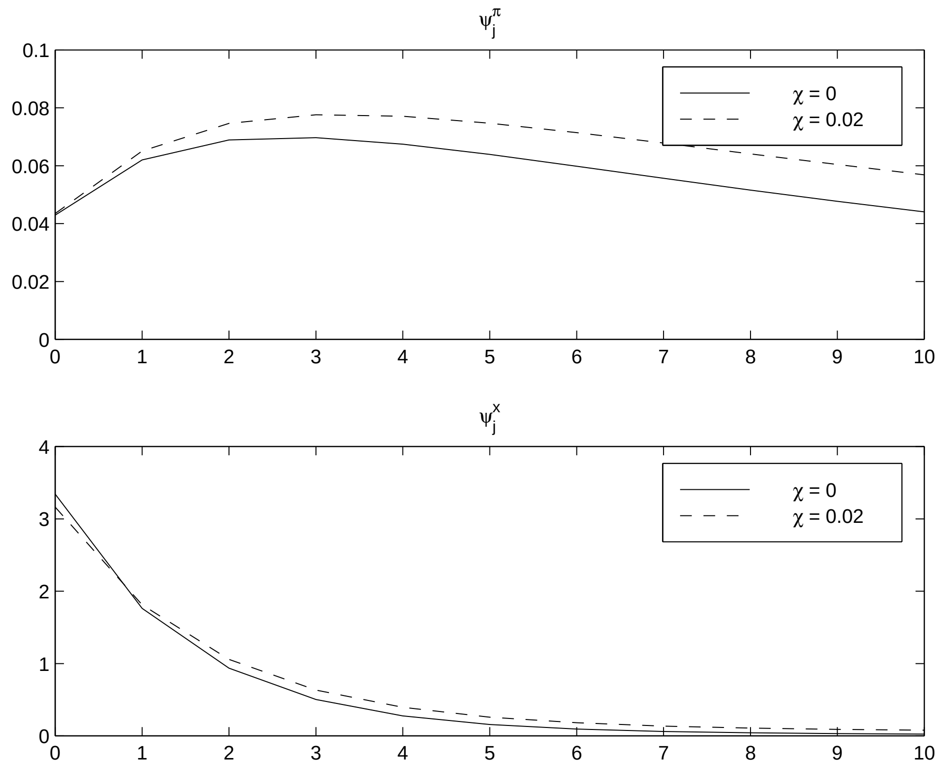

Figure 4.12: Effects of anticipated natural-rate fluctuations under an inertial Taylor rule, allowing for real-balance effects.

Let us again consider a policy rule of the form (2.22), with coefficients $\phi_\pi = 0.6, \phi_x = 0.3$, and $\rho = 0.7$, as in the baseline case of Figure 4.7. The impulse responses to an unexpected monetary tightening, both in the case that $\chi = 0$ (as in Figure 4.7) and in the case that $\chi = 0.02$, are shown in Figure 4.14 below. (The responses shown in that figure are actually for a variant model in which there is no effect on inflation or output during the quarter of the policy shock; but the responses shown for quarter 1 and later are identical to those implied by the model with structural equations (3.9) and (3.13). Note that the responses indicated by the solid lines in Figure 4.14 are identical, for periods 1 and later, to those shown by the solid lines in Figure 4.7.) We observe that the responses of none of the variables are very different when real-balance effects are considered, relative to the overall scale of variation in the variables in question.[50]

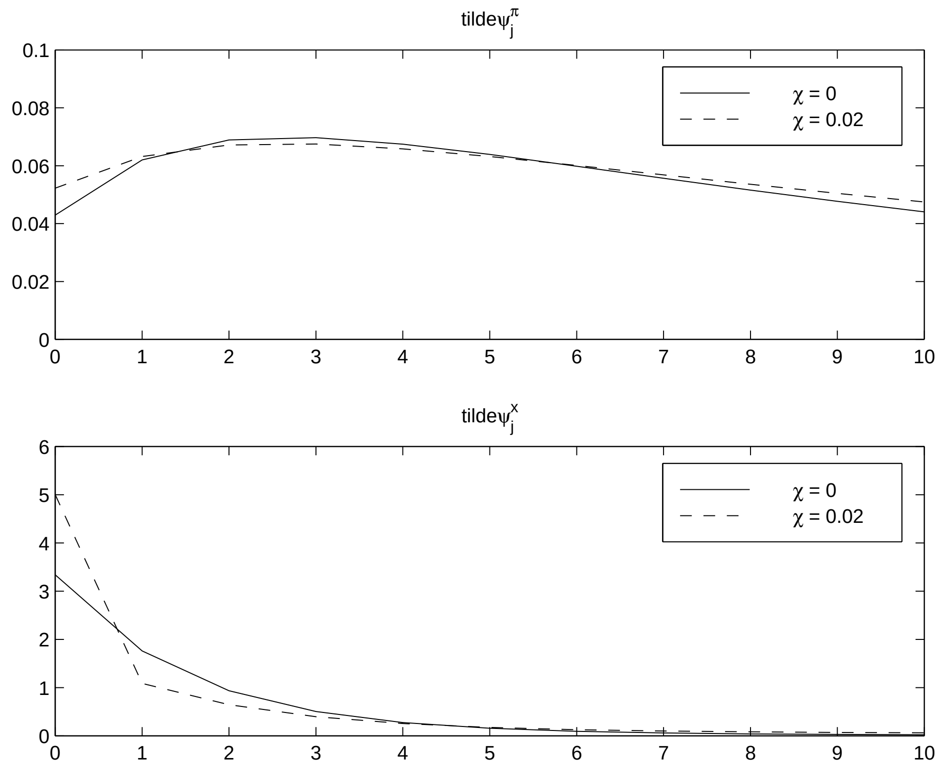

Figure 4.13: Effects of anticipated policy shifts under an inertial Taylor rule, allowing for real-balance effects.

Thus far we have only considered how large real-balance effects might be expected to be, according to our theory, when the model parameters are calibrated on the basis of the observed demand for money in the U.S. One might also wonder whether there is much evidence for real-balance effects in IS or AS relations from econometric estimates of those relations. To answer this question, Ireland (2000) presents maximum-likelihood estimates of a structural model consisting of equations (3.3), (3.9) and (3.13), together with a monetary policy rule, estimated using quarterly U.S. data on inflation, output, nominal interest rates

[50]In the case of the nominal interest rate, allowance for real-balance effects would predict a substantially greater decline, *relative* to the size of the decline predicted in the baseline case (which is very small), if we did not assume a one-quarter delay in the effects of monetary policy shocks on spending and pricing decisions. However, this is of little importance, since empirical estimates (see Figure xx below) indicate that most of the variation in nominal interest rates due to identified monetary policy shocks represents responses that occur prior to any effect on real expenditure — like the response in period zero of Figure xx, rather than the responses in periods one and later. A substantial proportionate effect on the response predicted in those later periods still makes very little difference to the overall predicted variation in nominal interest rates.

and money growth. His estimated value for $\chi\sigma$, the coefficient indicating the size of the effect of variations in real money balances on aggregate demand, is -.02, with a standard error of nearly .04; it is thus not significantly different from zero. Nor is this result obtained only because the effect is not precisely estimated; the 95 percent confidence interval implied by Ireland's results would allow the coefficient to have as large a positive value as .05 or as negative value as $-.09$, but it would not allow values as large as the coefficient of .13 assumed in the case shown in Figures xx-xx. Thus Ireland's estimates would imply real-balance effects even more modest than those exhibited in those figures.

3.3 Monetary Policy in a "Liquidity Trap"

[TO BE ADDED]

4 Delayed Effects of Monetary Policy

The simple optimizing model of the effects of interest-rate changes on aggregate demand presented above differs in a number of respects from the kind of specification that is common in macroeconometric models. One is the degree to which the model is *forward-looking:* (1.4) implies that changes in expected future interest rates should affect current aggregate expenditure as much as do changes in current (short-term) interest rates. However, in practice, expected future interest rates (at least, insofar as these expectations are based on atheoretical time series models) co-move very closely with current short-term interest rates, as shown by Fuhrer and Moore (1995b). Thus the coefficients multiplying current short rates in the expenditure equations of econometric models might simply be proxies for a long distributed lead of expected future short rates. While the two specifications may allow a similar fit to historical data, the forward-looking model has a much clearer rationale in terms of optimizing behavior (for any of a variety of reasons, as noted in chapter 1); and taking account of the forward-looking character of aggregate demand has important consequences for our analysis of optimal policy in Part II of this study.

Another difference between the optimizing model and common econometric specifications, however, gives us more reason to doubt the empirical realism of the theoretical model. This is the fact that our model implies that aggregate expenditure responds to current as opposed to *lagged* interest rates. Hence our model predicts that a monetary policy disturbance should immediately effect real expenditure, as shown for example in Figure xx above (where, in fact, the maximum effect on real activity occurs in the quarter of the shock). Instead, the conventional wisdom in central banks is that monetary policy can have little immediate effect on either real activity or inflation, and VAR studies typically confirm this view. For example, the estimated impulse response of real GDP shown in Figure xx of chapter 3 shows no non-negligible effect until two quarters following the quarter of the monetary policy shock. This would be consistent with an equation for aggregate expenditure involving only interest rates two or more quarters in the past, but not with the theory that we have derived from consideration of the optimal timing of expenditure.

In this section, we briefly discuss ways in which our basic model can be extended to allow for a more realistic delay in the predicted effects of a change in monetary policy. As we shall see, optimization-based theories incorporating such delays continue to imply that interest-rate *expectations* should matter as much as do actual interest rates. And they continue to imply that the key to inflation stabilization should be a policy under which the interest rate controlled by the central bank tracks variations in the natural rate of interest as accurately as is possible. Thus central insights derived from our basic analysis above continue to apply in more realistic settings.

4.1 Consequences of Predetermined Expenditure

One simple way in which one can explain the observed delay in the effect of monetary policy shocks on aggregate expenditure is to assume that expenditure decisions are, to some important extent, made in advance, just as some or all prices are determined in advance in the pricing models introduced in chapter 3. Alternatively, we might assume that many expenditure decisions are based on old information. We shall first present a model in which expenditure decisions are predetermined, *i.e.,* in which aggregate real expenditure in period t must be decided upon in period $t - d$. Later we discuss a closely related model in which expenditure decisions are not based on completely up-to-date information about financial market conditions. In either case, the level of expenditure in period t must be a function only of period $t - d$ information about interest rates, and so can depend on monetary policy shocks in period $t - d$ and earlier, but not on any more recent shocks.

How plausible are such assumptions? In the strict forms in which they have just been stated, both assumptions are obviously too extreme to be entirely realistic; nonetheless, they capture in a simple way a feature of many actual expenditure decisions. Many of the most interest-sensitive components of expenditure, such as investment spending, are in fact predetermined to an important extent, owing both to the existence of planning lags and to the fact that individual projects require expenditure over a period of time ("time to build", in the terminology of Kydland and Prescott, 1982), which expenditure will in most cases be worth continuing once the project has been started. (See, *e.g.,* Edge, 2000, for discussion of both types of delay and their likely quantitative magnitudes.) Once again, while we here model all interest-sensitive expenditure as if it were household consumption (abstracting from any effects of this expenditure on productive capacity), our model should really be interpreted as a model of the timing of private expenditure more generally, and the relevant delays are the ones that apply to the most interest-sensitive components of such expenditure. And even in the case of household consumption, Gabaix and Laibson (2002) have argued that it makes sense to model households as changing their planned consumption levels only intermittently; they show that this hypothesis can help to reconcile the behavior of aggregate consumption with the behavior of asset prices. The simple model of predetermined expenditure decisions proposed here is in the spirit of such a model, though it involves a cruder hypothesis. (The consequences of intermittent adjustment of the kind proposed by Gabaix and Laibson are discussed in section xx below.)

As a simple example, let us suppose that the state-contingent consumption plan from date t onward is chosen to maximize

$$E_{t-d}\left\{\sum_{s=t}^{\infty}\beta^{s-t}u(C_s;\xi_s)\right\}$$

subject to an intertemporal budget constraint of the form

$$\sum_{s=t}^{\infty}E_{t-d}Q_{t-d,s}P_sC_s \leq E_{t-d}Q_{t-d,t}W_t + \sum_{s=t}^{\infty}E_{t-d}Q_{t-d,s}[P_sY_s - T_s], \tag{4.1}$$

where $\{Q_{t,s}\}$ is the same system of stochastic discount factors as in chapter 2. Here d is the assumed length of the delay (in periods) between expenditure decisions and the time that the expenditure actually occurs (or alternatively, the delay in the receipt of new information about aggregate conditions). The intertemporal budget constraint (4.1) is required to hold only in present value discounting back to the state of the economy at date $t-d$, because we assume as before (sequentially) complete financial markets, allowing a household to insure itself at date $t-d$ against the realization of state at date t in which the present value of its subsequent after-tax income is unusually low. The present value $E_{t-d}Q_{t-d,t}W_t$ of period t initial wealth is given as an initial condition, as it follows from period $t-d$ wealth, the consumption path already chosen for periods $t-d$ through $t-1$, after-tax income expectations for those same periods, and financial-market prices.

A necessary condition for an interior solution to this optimization problem is that

$$\beta^d E_{t-d}u_c(C_t;\xi_t) = \Lambda_{t-d}\ E_{t-d}[Q_{t-d,t}P_t] \tag{4.2}$$

at each date $t-d \geq 0$ and each possible state at that date, where $\Lambda_{t-d} > 0$ is the Lagrange multiplier on the household's budget constraint (4.1) looking forward from that date, indicating the shadow value (in terms of period $t-d$ utility) of additional nominal income at date $t-d$. Another necessary condition, given the existence of complete financial markets, is that

$$\Lambda_t Q_{t,s} = \beta^{s-t}\Lambda_s, \tag{4.3}$$

for any two dates $0 \leq t < s$ and any possible state at date s. Note further that using (4.3), we can equivalently write (4.2) as

$$E_{t-d}u_c(C_t;\xi_t) = E_{t-d}[\Lambda_t P_t]. \tag{4.4}$$

One can furthermore show that a system of Lagrange multipliers $\{\Lambda_t\}$ and a consumption plan $\{C_t\}$ such that (i) for each date, C_t depends only on the state of the world at $t-d$; (ii) conditions (4.3) – (4.4) are satisfied in each possible state at each date; and (iii) the intertemporal budget constraint (4.1) holds with equality, looking forward from the initial date; suffice to characterize an optimal plan.[51]

As in section 1, it is useful to give an approximate characterization of the optimal timing of private expenditure in the case of small disturbances by log-linearizing these equilibrium conditions around the deterministic steady-state consumption plan that is optimal in the case of no real disturbances and a monetary policy consistent with zero inflation and a nominal interest rate equal to the rate of time preference. Substituting $Y_t - G_t$ for C_t in (4.4) and log-linearizing, we obtain

$$\hat{Y}_t = g_t - \sigma E_{t-d}\hat{\lambda}_t, \tag{4.5}$$

where $\hat{\lambda}_t \equiv \log(\Lambda_t P_t / u_c(\bar{C};0))$. Here the composite disturbance term is defined as

$$g_t = \hat{G}_t + s_C E_{t-d}\bar{C}_t,$$

generalizing (2.3); note that this need not be predetermined at date $t-d$, if government purchases are not determined as far in advance as is interest-sensitive private expenditure.[52] The disturbances $\hat{G}_t, \bar{C}_t$ and the coefficient $\sigma > 0$ are defined as before.

Equation (4.3) implies that

$$1 + i_t = \beta^{-1}\Lambda_t[E_t\Lambda_{t+1}]^{-1},$$

which when log-linearized becomes

$$i_t = \hat{\lambda}_t + E_t[\pi_{t+1} - \hat{\lambda}_{t+1}].$$

[51] Here it is assumed that the household faces prices such that the right-hand side of (4.1) is well-defined and finite; otherwise, no optimal plan is possible, as before. Thus this condition is also a requirement for equilibrium, as in chapter 2. Similarly, if we assume that households can choose to hold a non-negative quantity of non-interest-bearing currency if they wish, then it is also a requirement for the existence of an optimal plan that the household face a non-negative nominal interest at all times.

[52] The assumption that g_t is not public knowledge at $t-d$ does not require that the government has better information about macroeconomic conditions or a shorter planning horizon than the private sector. This simply indicates exogenous random variation in government purchases that cannot be forecasted d periods in advance on the basis of public information.

This can be solved forward to yield

$$\hat{\lambda}_t = \hat{\lambda}_\infty + \sum_{j=0}^{\infty} E_t(i_{t+j} - \pi_{t+j+1}),$$

which when substituted into (4.5) yields

$$\hat{Y}_t = \hat{Y}_\infty + g_t - \sigma \sum_{j=0}^{\infty} E_{t-d}(\hat{\imath}_{t+j} - \pi_{t+j+1}), \qquad (4.6)$$

generalizing (1.4). Alternatively, our intertemporal "IS relation" can be written in differenced form as

$$\hat{Y}_t = g_t + E_{t-d}(\hat{Y}_{t+1} - g_{t+1}) - \sigma E_{t-d}(i_t - \pi_{t+1}). \qquad (4.7)$$

This is essentially the same form of IS relation as in the basic model, except that now it is *past expectations* of current and future real interest rates that matter, rather than current interest rates or current expectations. Such a specification implies that an unexpected change in monetary policy in period t can have no effect on aggregate real expenditure before period $t + d$. Yet the delayed effect of a monetary policy shock does not occur because it is actual past interest rates, rather than current or expected future rates, that matter for current expenditure; only past expectations regarding real rates of return *from now on* are relevant to the desired substitution between current and future expenditure, even when the decision itself has been made at a past date.

The difference between this specification and an IS relation according to which current real expenditure depends on lagged (but not current) interest rates has quite important implications for the conduct of monetary policy. If expenditure depends on actual lagged interest rates, it would follow that interest rates should be adjusted now to offset disturbances that are expected to affect the output gap in the future, even if these disturbances have no immediate effect; for once the disturbances have their effect, it may be too late for an interest-rate change to have any countervailing effect on expenditure. Thus policy should be forward-looking, and respond immediately to news that affects the economic outlook some quarters in the future. If instead, as in this optimizing model, expenditure depends on *past expectations* of current and future rates, it follows that interest-rate policy affects expenditure only to the extent that it is *forecastable* in advance. There would therefore be no advantage (from the point of view of output-gap stabilization) in responding at all to news except after it has been known to the public for d periods. Instead, it would be important to base current interest rates on past conditions (possibly including past perceptions of the outlook for the future), in order to bring about forecastable interest-rate variations that could be used to offset the effects of predictable disturbances.

Under this alternative specification of our forward-looking model of the determinants of real expenditure, the effects of an (anticipated future) monetary disturbance are similar to those analyzed earlier, only delayed in time. Suppose, for example, that there is a lag of the same length before new price decisions take effect, so that the aggregate-supply relation is of the form

$$\pi_t = \kappa E_{t-d} x_t + \beta E_{t-d} \pi_{t+1},) \tag{4.8}$$

where once again $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$, and $\hat{Y}_t^n$ is defined as before. (Recall equation (xx) of chapter 3.) The component of the output gap that is forecastable d periods in advance furthermore satisfies

$$E_{t-d} x_t = E_{t-d} x_{t+1} - \sigma E_{t-d}(\hat{\imath}_t - \pi_{t+1} - \hat{r}_t^n), \tag{4.9}$$

as a consequence of (4.7). This relation together with

$$x_t = E_{t-d} x_t + (g_t - \hat{Y}_t^n) - E_{t-d}(g_t - \hat{Y}_t^n) \tag{4.10}$$

is in fact equivalent to (4.7).

Finally, let monetary policy be specified by a rule of the form (2.22), as in Figure 4.7. Then for the same parameter values as are assumed in the baseline case ($\rho = 0.7$) of that figure, but with a delay of $d = 1$ quarter, the predicted impulse responses to a contractionary monetary policy shock are those shown by the solid lines in Figure 4.14. Note that for quarters 1 and later following the shock, the responses are identical to those shown in Figure 4.7. The only difference is that there are no effects on output or inflation in the quarter of the shock itself; as a consequence, the initial increase in the nominal interest rate is much larger, as the increase in $\bar{\imath}_t$ is not offset by the reaction to any immediate declines in output

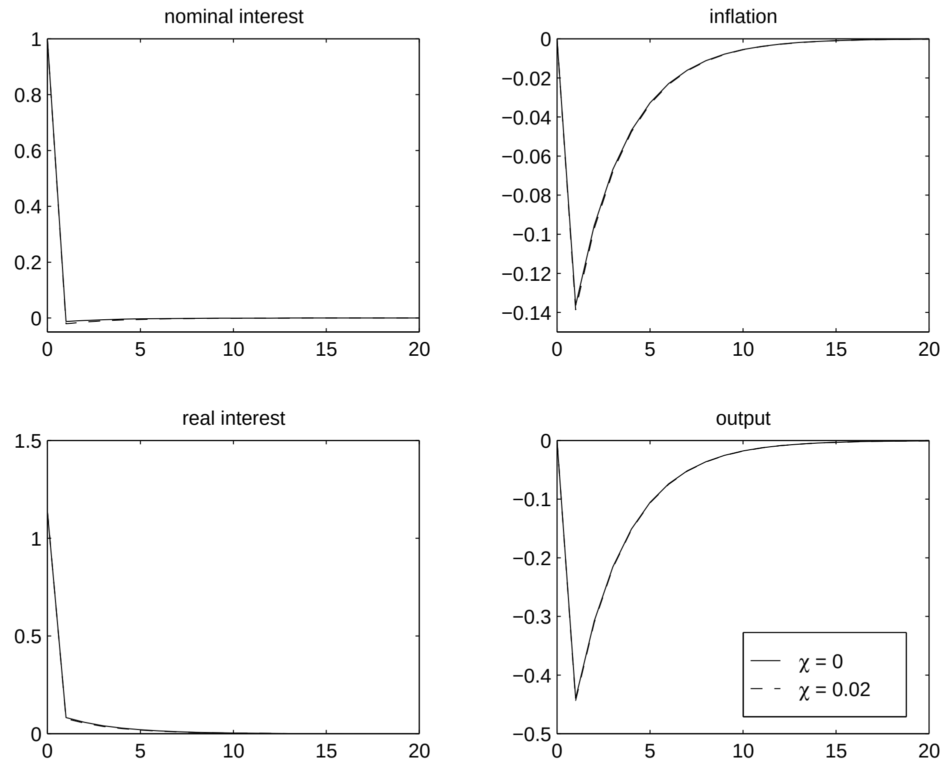

Figure 4.14: Impulse responses to a contractionary monetary policy shock, when both expenditure and prices are predetermined for one quarter.

or inflation (as occurred in Figure 4.7). This modification of the model greatly increases its realism, not only because the effects on output and inflation are delayed, but because of the prediction of a transitory "liquidity effect" of a monetary tightening. Note that the predicted increase in nominal interest rates is weaker and more persistent (or even absent, as in the cases $\rho = 0.7$ or 0.8) in Figure 4.7. The responses shown in Figure 4.14 are much more similar to empirical estimates, discussed further in the next section.[53]

Under this extension of our basic model, the key to inflation and output-gap stabilization continues to be the adjustment of interest rates so as to track the variations in the natural rate of interest due to exogenous real disturbances. For example, suppose that monetary

[53]The figure also shows the predicted impulse responses in the case that we allow for real-balance effects, parameterizing monetary frictions in the way discussed in section xx above. These are not substantially different from those predicted in the cashless case. Hence we continue to abstract from real-balance effects in the remainder of this section and in our discussion of empirical models.

policy is described by a Taylor rule of the form

$$\hat{\imath}_t = \bar{\imath}_t + E_{t-d}[\phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})], \tag{4.11}$$

where $\bar{x}$ is consistent with $\bar{\pi}$ as assumed earlier, and $\bar{\imath}_t$ is again an exogenous intercept term.[54] The evolution of the forecastable components of inflation, the output gap, and the nominal interest rate is then determined by the system of equations consisting of (4.8), (4.9) and the equation obtained by taking the conditional expectation of both sides of (4.11) at date $t-d$. This system of equations has a structure that is precisely analogous to the system consisting of (1.8), (1.9) and (1.10) in the case of the basic model. It implies a determinate rational-expectations equilibrium in exactly the same case as with the previous system, *i.e.,* if and only if the coefficients satisfy the Taylor Principle.

In this case, the unique bounded solution is given by

$$\pi_t = \bar{\pi} + \sum_{j=0}^{\infty} \psi_j^\pi E_{t-d}(\hat{r}^n_{t+j} - \bar{\imath}_{t+j} + \bar{\pi}), \tag{4.12}$$

$$x_t = \bar{x} + \sum_{j=0}^{\infty} \psi_j^x E_{t-d}(\hat{r}^n_{t+j} - \bar{\imath}_{t+j} + \bar{\pi}) + (g_t - \hat{Y}^n_t) - E_{t-d}(g_t - \hat{Y}^n_t), \tag{4.13}$$

$$\hat{\imath}_t = \bar{\imath}_t + \sum_{j=0}^{\infty} \psi_j^i E_{t-d}(\hat{r}^n_{t+j} - \bar{\imath}_{t+j} + \bar{\pi}), \tag{4.14}$$

where the coefficients $\{\psi_j^y\}$ for $y = \pi, x, i$ are exactly the same as before. Here the solutions for the forecastable components of all three variables are given by the forecasts d periods in advance of the solutions previously presented in (2.18) — (2.20). The variables π_t and $\hat{\imath}_t - \bar{\imath}_t$ have no unforecastable components, while the unforecastable component of x_t is given by (4.10).

This solution implies, once again, that inflation and the output gap are both stabilized, to the greatest extent possible, by commitment to a Taylor rule in which the intercept term tracks variation in the natural rate of interest. The only difference is that now it is only necessary to track the fluctuations in the natural rate that can be forecasted d periods in

[54] The assumption that the endogenous terms involve only the components of inflation and the output gap that are forecastable at $t-d$ is purely notational. For it follows from (4.8) that π_t is entirely forecastable at $t-d$, while it follows from (4.10) that the unforecastable component of x_t is purely exogenous.

advance. A rule of the form (4.11) with $\bar{\imath}_t = E_{t-d}\hat{r}_t^n$ and coefficients ϕ_π, ϕ_x satisfying (2.7) implies a determinate equilibrium in which

$$\pi_t = \bar{\pi}, \qquad E_{t-d}x_t = \bar{x}$$

at all times. This obviously stabilizes inflation to the greatest extent possible. Since the forecastable and unforecastable components of the output gap are necessarily uncorrelated,

$$\begin{aligned}\text{var}\{x_t\} &= \text{var}\{E_{t-d}x_t\} + \text{var}\{x_t - E_{t-d}x_t\} \\ &= \text{var}\{E_{t-d}x_t\} + \text{var}\{(g_t - \hat{Y}_t^n) - E_{t-d}(g_t - \hat{Y}_t^n)\},\end{aligned}$$

as a consequence of (4.10). Given that the second term is independent of monetary policy, the most that can be done to stabilize the output gap is to reduce the variance of the forecastable component to zero; the proposed policy achieves this.

Certain complications arise when we seek to combine this alternative model of the effects of interest rates on aggregate demand with aggregate-supply relations of the sort derived in chapter 3. The derivations in chapter 3 assume that period t expenditure is chosen optimally on the basis of period t information, so that (4.5) holds with $d = 0$. The place at which we have used this assumption in chapter 3 is in replacing the marginal utility of income by a function of current consumption, in our account of optimizing labor supply (and hence of the marginal cost of supplying goods). More generally, the (log-linearized) real marginal cost function for the model with flexible wages is given by

$$\hat{s}_t(i) = \omega(\hat{y}_t(i) - \hat{Y}_t^n) + \sigma^{-1}(\hat{Y}_t - \hat{Y}_t^n) - \mu_t, \tag{4.15}$$

where

$$\mu_t \equiv \hat{\lambda}_t - \sigma^{-1}(g_t - \hat{Y}_t) \tag{4.16}$$

is the discrepancy between the (log) marginal utility of real income and the (log) marginal utility of consumption. When period t expenditure is chosen optimally at date t, $\mu_t = 0$, and the aggregate supply relations derived in chapter 3 are correct. But when interest-sensitive private expenditure must be chosen in advance, (4.5) implies only that $E_{t-d}\mu_t = 0$, whereas μ_t need not equal zero.

Now suppose that even though the aggregate index of demand (*i.e.*, the demand for the composite good) is determined d periods in advance, the way that this demand is allocated across the various differentiated goods is *not* committed in advance. It follows that a supplier who considers a price change that will take effect in less than d periods will still calculate the optimal price taking into account the effect on demand for its good from the first period in which the price change takes effect. Then in the case of random intervals between price changes of the kind assumed by Calvo, and a delay of s periods before a newly chosen price takes effect, the aggregate-supply relation should be of the form

$$\pi_t = \kappa E_{t-s} x_t - \frac{\kappa}{\epsilon_{mc}} E_{t-s}\mu_t + \beta E_{t-s}\pi_{t+1}.) \tag{4.17}$$

where $\epsilon_{mc} > 0$ is the elasticity of average real marginal cost with respect to the level of aggregate output.

Only in the case that $s \geq d$ could the $E_{t-s}\mu_t$ term be neglected (as in (4.8) above). If $s < d$, one would instead need to use the form (4.17), together with the relation

$$\mu_t = \sum_{j=0}^{d-1} E_t[\hat{\imath}_{t+j} - \pi_{t+j+1}] + \sigma^{-1}[(\hat{Y}_t - g_t) - E_t(\hat{Y}_{t+d} - g_{t+d})] \tag{4.18}$$

relating μ_t to observables using (4.5).[55] The presence of the $E_{t-s}\mu_t$ term indicates a moderating effect on expected supply costs in period t, and hence on inflationary pressure, of an expectation at $t-s$ of real rates of return between periods t and $t+d-s$ that are higher than those that were anticipated at the time that expenditure was planned for periods t through $t+d-s-1$. Unexpectedly high real rates of return increase the value of income in period t, and so lower average wage demands, even if they occur as a result of shocks that (because unanticipated) are unable to affect aggregate demand.

There appears, however, to be little evidence of an effect of interest rates on supply costs of the kind implied by the $E_{t-s}\mu_t$ term in (4.17). If anything, unexpected interest-rate

[55]See the empirical model of Rotemberg and Woodford (1997) for an illustration.

increases probably *increase* supply costs in the short run, as found by Barth and Ramey (2000) and Christiano *et al.* (2001). It is thus important to note that this effect appears in (4.17), even under the assumption that $s < d$, only in the case that the delay in the effect of interest rates on expenditure is derived from a planning delay rather than in delay in obtaining up-to-date information about financial conditions.

Under an alternative interpretation of (4.6), period t spending decisions are made at date t, but on the basis of an estimate of the household's marginal utility of real income that reflects information available at date $t-d$ rather than the complete information available to financial market participants at date t. Let us suppose that each household has an agent that optimally manages its investments. This agent, with full information about financial market conditions (as well as about the household's tastes and labor income prospects) produces an estimate of the marginal utility of additional wealth; individual household members who go into the goods markets then purchase individual goods to the point at which the marginal utility from an additional dollar of spending on a given good equals the current estimate of the marginal utility of wealth. But suppose there is a time delay in the transmission of the financial advisor's estimate, so that in period t the household's spending decisions are based on the financial advisor's period $t-d$ estimate of what the household's marginal utility of real wealth in period t would be.

Suppose further that the household members simply use this estimate, rather than updating it on the basis of what they should be able to infer about unexpected changes in financial conditions from the prices that they observe.[56] The level of expenditure C_t is then determined at date t to satisfy the first-order condition

$$u_c(C_t; \xi_t) = E_{t-d}[\Lambda_t P_t], \tag{4.19}$$

where the right-hand side is the signal transmitted by the financial advisor at date $t-d$. This has identical implications to (4.4), except that taste shocks ξ_t that are not forecastable at date $t-d$ can still affect private expenditure in this version.[57] Equating C_t with $Y_t - G_t$

[56]There is admittedly an element of bounded rationality in this assumption, that is not required if we assume that the household has committed itself in advance to a particular level of real expenditure.

and log-linearizing, we again obtain (4.5) except that in this case g_t is again defined as in (2.3). We then again obtain an IS relation of the form (4.6).

Under this alternative interpretation, however, each household's labor supply decisions should *also* be affected by its imperfect information about current financial-market conditions. Thus real wages, and hence real marginal costs, should depend on $E_{t-d}\hat{\lambda}_t$ rather than upon the true value of $\hat{\lambda}_t$, as this would be evaluated by the household's financial advisor. But (4.5) implies that $E_{t-d}\hat{\lambda}_t$ can be written as $-\sigma^{-1}(\hat{Y}_t - g_t)$, even if $\hat{\lambda}_t$ cannot. Thus we obtain (4.15) without the μ_t term, and correspondingly (4.17) without the $E_{t-s}\mu_t$ term. That is, we obtain exactly the form of AS relation derived in chapter 3 for the case $d = 0$.

4.2 Small Quantitative Models of the Effects of U.S. Monetary Policy

A model only slightly more complex than those just described is used by Rotemberg and Woodford (1997, 1999a) as a basis for quantitative analysis of alternative interest-rate rules for the U.S. economy. Rotemberg and Woodford assume an intertemporal IS equation of the form (4.7) with a delay of $d = 2$ quarters, and interpret the delay as due to predeterminedness of interest-sensitive private expenditure. Their AS equation is instead a more complex version of (4.8), in which the delay required before revised prices take effect is not the same for all goods. Instead, it is assumed that for a fraction φ of all goods, a new price that is chosen in period t (or at any rate, on the basis of public information in period t) applies to purchases beginning in period $t+1$, while for the remaining goods, a new price chosen in period t takes effect only beginning in period $t+2$. In the case of both types of goods, it is assumed (as in the Calvo model) that a fraction $1-\alpha$ of all goods prices are revised each period, with the price of each good having the same probability of being revised in any given period.

In this case, the aggregate supply relation (4.17) generalizes to

$$\pi_t = \frac{1}{1+\psi}\left\{\kappa E_{t-1}x_t - \frac{\kappa}{\epsilon_{mc}}E_{t-1}\mu_t + \beta E_{t-1}\pi_{t+1}\right\} + \frac{\psi}{1+\psi}\left\{\kappa E_{t-2}x_t + \beta E_{t-2}\pi_{t+1}\right\},) \quad (4.20)$$

[57] An appealing feature of this alternative derivation is that it makes it clear why expenditure at date t on individual goods should still depend on the prices of the individual goods, as we have assumed, rather than only on the forecast of their prices at date $t-d$.

where $\psi \equiv (1-\varphi)/\varphi\alpha > 0$, while (4.18) implies that

$$E_{t-1}\mu_t = E_{t-1}[\hat{\imath}_t - \pi_{t+1}] + \sigma^{-1}[(\hat{Y}_t - g_t) - (\hat{Y}_{t+1} - g_{t+1})]. \tag{4.21}$$

(Note that when $d = 2$, (4.7) implies that the value of the second term in square brackets here is known at date $t-1$, so that we may omit the conditional expectation operator for that term.) The log-linearized structural equations of the Rotemberg-Woodford model then consist of the intertemporal IS relation (4.7) with $d = 2$, the aggregate supply relation obtained by substituting (4.21) into (4.20), a specification of the exogenous disturbance processes $\{g_t, \hat{Y}_t^n\}$, and a monetary policy rule specifying $\hat{\imath}_t$ as function of its own history, current and lagged values of inflation and output, and a serially uncorrelated[58] exogenous monetary policy shock. (Once the processes $\{g_t, \hat{Y}_t^n\}$ have been specified, the evolution of the natural rate of interest $\hat{r}_t^n$ that appears in (4.7) is given by (1.11).)

The monetary policy rule, the laws of motion for the exogenous disturbances, and certain parameters of the structural equations are also specified so as to allow the model to fit as well as possible the joint evolution of short-term nominal interest rates, inflation and output in the U.S. economy. Rotemberg and Woodford characterize the co-movements of these latter three variables by estimating an unrestricted VAR model for the federal funds rate, the rate of growth of the GDP deflator, and the linearly detrended log of real GDP, using quarterly data for the sample period 1980:1-1995:2. These particular measures of the interest rate, inflation and output are used following Taylor (1993), as it is desired that one equation of the VAR should represent an estimate of the Fed's reaction function. The sample period begins at the beginning of 1980 because of the general recognition that an important change in the way that monetary policy was conducted in the U.S. occurred around this time. (Recall the discussion in chapter 1 of the alternative interest-rate rules estimated for different sample periods. An even shorter sample period might be preferred on the same ground — say, a post-1987 sample as in Taylor, 1993, 1999b — but this would allow even less precise estimates

[58] Note that if a sufficient number of lags of the endogenous variables are included, this specification is equivalent to one in which the monetary policy disturbance is allowed to be an arbitrary autoregressive process.

of the impulse responses to a monetary policy shock.)

In the VAR model, $\hat{\imath}_t$, π_{t+1}, and $\hat{Y}_{t+1}$ are regressed on three lags of each of these variables, with the coefficients otherwise unrestricted. (Additional lags are not included as they were not found to be significant.) The particular lags that are included in the case of each variable are chosen in this way because the model implies that $\hat{\imath}_t$, π_{t+1} and $\hat{Y}_{t+1}$ are all part of the same information sets: these variables are known by period t (in particular, before the period $t+1$ interest-rate decision is made), because both inflation and output are predetermined,[59] but not yet known in period $t-1$ (*i.e.*, before the period t interest-rate decision is made). Because the model implies that the period t interest-rate decision cannot affect the determination of either period t output or period t inflation, an OLS regression of $\hat{\imath}_t$ on the lags of all three variables (which include π_t and $\hat{Y}_t$) should identify the coefficients of the monetary policy rule, and the residual of this equation should identify the sequence of monetary policy shocks.[60] The two VAR residuals orthogonal to this one are instead interpreted as the two innovations in the joint exogenous process for the disturbances $\{g_t, \hat{Y}_t^n\}$.

With this identification of the historical monetary policy shocks, the just-identified VAR model can be used to estimate the impulse responses of all three variables to an monetary policy shock. Figure 4.15 plots the estimated responses, together with the associated (+/- 2 s.e.) confidence intervals, in the case of a one-standard-error innovation in the federal funds rate, *i.e.*, an unexpected monetary tightening. By construction, the funds rate increases, while there is no effect on either output or inflation in the quarter of the shock. However, the results also indicate (as in Figure xx of chapter 3) that there is no noticeable effect on output in the following quarter, either, though output sharply declines in the second quarter following the shock; this is the reason for the inclusion of a delay $d = 2$ quarters in the determination of private expenditure in the Rotemberg-Woodford model. The contraction of output relative to trend persists for several quarters, though Rotemberg and Woodford find

[59] Note that it is assumed that the composite exogenous disturbance g_t is known at date $t-1$, *i.e.*, prior to the determination of the period t interest rate. Since government purchases are in fact typically budgeted in advance, this is not implausible.

[60] The assumptions used to identify the monetary policy shock here are common in the structural VAR literature on this question; see, *e.g.*, Christiano *et al.* (1999).

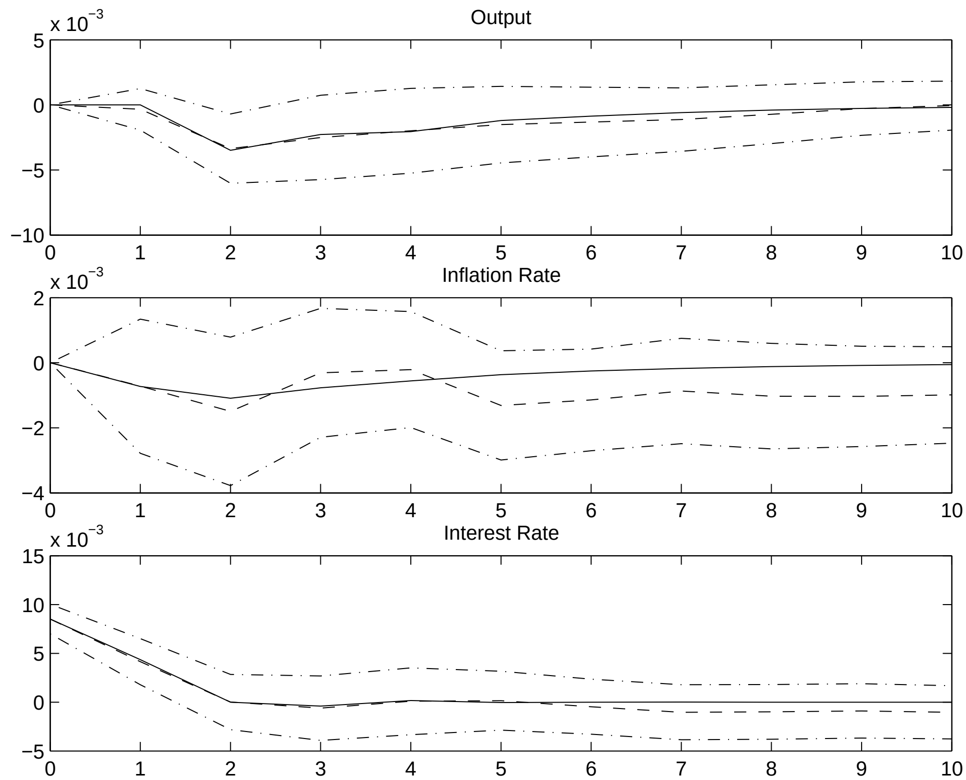

Figure 4.15: Predicted [*solid line*] and estimated [*dashed line*] impulse responses to a monetary policy shock. Dash-dotted lines indicate bounds of the confidence intervals for the estimated responses. Source: Rotemberg and Woodford (1997).

a peak output effect in the second quarter following the monetary shock, which is sooner than what is indicated by most studies using a longer sample.[61] (Compare Figure xx of chapter 3.) The effects of a monetary policy shock on inflation are less well-estimated, but the point estimates indicate lower inflation for many subsequent quarters as a result of the policy tightening.

One equation of the VAR can also be interpreted as indicating the Fed's reaction function over this period. The estimated coefficients on the three lags of the funds rate itself are all

[61] Boivin and Giannoni (2001) compare the results that would be obtained using a similar method to analyze the response to monetary policy shocks in a pre-1980 sample. They argue that much of the difference can be accounted for by the change in their estimated monetary policy rule between the pre-1980 and post-1980 samples, though they also find that a better fit to the pre-1980 responses is possible in the case of a model that incorporates additional grounds for persistence in the effects of interest-rate changes on private-sector expenditure, as discussed below.

positive, and they sum to 0.69; this implies substantial inertia in the Fed's interest-rate policy, as in the estimated rules discussed in chapter 1. The sums of the coefficients on current and lagged values of inflation and detrended output are also both positive, and imply long-run interest-rate responses to sustained increases in these two variables of $\Phi_\pi = 2.13$ and $\Phi_y = 0.47$ respectively. Thus except for the additional dynamics implied by the inclusion of lags of all three variables, the estimated reaction function is similar to Taylor's (1993) characterization of Fed policy under Greenspan's chairmanship.

Rotemberg and Woodford then estimate certain parameters of their model so as to fit the estimated responses to a monetary policy shock as closely as possible, given that monetary policy is described by the estimated interest-rate rule. It can be shown that the model's predicted responses depend only on the parameters β, σ, κ, and a certain function of ω and ψ, in addition to the coefficients of the monetary policy rule.[62] There are thus at most four free parameters that can be chosen (within certain *a priori* bounds) so as to improve the model's fit. Furthermore, because the model implies that $\beta^{-1} - 1$ should equal the long-run average real rate of interest, Rotemberg and Woodford calibrate the discount factor $\beta = 0.99$ on this ground, rather than using information about the responses to shocks to estimate this parameter. They then estimate the values of the other three parameters that minimize the distance between the predicted impulse responses and those implied by the unrestricted VAR estimates. The predicted impulse responses in the case of these parameter values are shown by the solid lines in Figure 4.15.

Note that the model can account quite well for both the size and persistence of the estimated response of real GDP. The predicted output response is in fact essentially the same as the one shown in Figure 4.7 for the inertial policy rule with $\rho = 0.7$, with the magnitude of the policy shock appropriately rescaled[63] and the response delayed for two quarters. For the IS and AS relations of the Rotemberg-Woodford model are identical to those of the

[62] Parameter identification, along with other aspects of the estimation strategy, are discussed in detail in the Appendix to the 1998 working paper version of their paper.

[63] The size of the effective shift in the intercept term of the policy rule in the second period following the shock is found by substituting the predicted responses in the previous two periods for the lagged interest-rate and inflation terms in the rule.

baseline model if one takes expectations conditional upon information two quarters earlier. It follows that the predicted responses of all variables two or more quarters following the shock are the same as in our earlier analysis, if the initial conditions that apply two quarters later are appropriately adjusted; the result follows for the same reason as in the case of the one-quarter delay shown in Figure 4.14. Thus the predicted output response would be exactly the same as in Figures 4.7 or 4.14, but with a two-quarter delay, if the policy rule were of the simpler form assumed in those figures.

The model also accounts well for the estimated response of the funds rate itself to a monetary policy shock: the theoretical model predicts, as the VAR indicates, that the funds rate returns essentially to the level that would have been predicted in the absence of any shock within two quarters following the shock. (This occurs for the same reason that in Figure 4.7 there is very little response of the nominal interest rate to the policy shock, and that in Figure 4.14 the interest rate returns to its normal level after one quarter.) Note that this fact does *not* mean, as is sometimes supposed, that an "interest-rate channel" cannot account for the timing of the observed real effects of monetary policy disturbances, which instead occur only *beginning* two quarters after the shock. The reason is that it is the real rate of interest, and not the nominal rate, that primarily matters for aggregate demand; and if a monetary tightening is expected to lower inflation for several quarters (as predicted both by the theoretical responses and the VAR), this implies a higher real rate of interest that persists for several quarters, even if the nominal rate has returned to its normal level. The predicted and estimated inflation responses do not match as well as in the case of the other two variables, but it should be noted that the estimated responses are highly imprecise.

The parameter values required for this degree of fit are consistent with the *a priori* restrictions implied by the model's microeconomic foundations. The values estimated by Rotemberg and Woodford are shown in Table 4.1. The table also includes their "calibrated" values for several model parameters that cannot be identified from the impulse responses; these are included so as to provide further insight into the economic significance (and plausibility) of the estimated parameters, and also because some of the additional parameters

Table 4.1: Parameter values in the quarterly model of Rotemberg and Woodford (1997).

α	0.66
β	0.99
φ	0.63
ψ	0.88
σ^{-1}	0.16
ν	0.11
ϕ^{-1}	0.75
ω_w	0.14
ω_p	0.33
ω	0.47
ϵ_{mc}	0.63
$(\theta-1)^{-1}$	0.15
ζ	0.14
κ	.024

matter for the welfare evaluation of alternative policies.

Like most of the equilibrium business-cycle literature, they assume a Cobb-Douglas aggregate production function, $f(h) = h^{\lambda}$, and calibrate the elasticity λ to be 0.75 on the basis of the observed labor share in national income.[64] This implies that the component of the elasticity of real marginal cost with respect to output that is due to the diminishing marginal product of labor should equal $\omega_p = 0.33$. They furthermore propose, on the basis of other structural VAR studies that estimate real wage responses, that the elasticity of the real wage with respect to output changes not associated with any change in production possibilities is approximately of the magnitude 0.3. This would imply an overall elasticity of average real marginal cost with respect to aggregate output of $\epsilon_{mc} = 0.63$.

Since in their cashless model, $\epsilon_{mc} = \omega + \sigma^{-1}$, the estimated value of σ^{-1} implies that $\omega = \omega_w + \omega_p = 0.47$, and hence that $\omega_w = 0.14$. Since $\omega_w = \nu\phi$, where ν measures the curvature of the disutility of labor function and ϕ is the inverse of the labor elasticity, the

[64]Because the assumed value of θ implies a ratio of price to marginal cost of 1.15, a labor elasticity of 0.75 implies that one should observe a labor share of $0.75/1.15 = 0.65$, which is about what is observed on average for the U.S.

implied value of ν is $0.75(0.14) = 0.11$. This is a very low degree of curvature, but still a positive one, and so the implied preferences for the representative household satisfy the standard concavity restrictions. While the implication of highly elastic responses of voluntary labor supply to real wage variations may be judged implausible, the problem is a familiar one for equilibrium business-cycle models (such as standard RBC models) that incorporate a wage-taking representative-household model of labor supply.[65]

Given this value for ω, it is then possible to estimate a value for ψ, namely 0.88. This value is also positive, as required by the model. A variety of values of α and φ would be consistent with this value for ψ. Rotemberg and Woodford calibrate $\alpha = 0.66$ on the basis of survey evidence on the typical frequency of price changes in the U.S. economy, such as that of Blinder *et al.* (1998, Table 4.1). (This value of α implies a mean time between price changes of 3 quarters.) The estimate of ψ then implies the value $\varphi = 0.63$, which is between zero and one as required by the theory.

Finally, given these values for α and β, the estimated value of κ is consistent with the theoretical prediction that[66]

$$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\,\zeta$$

in the case that ζ (the measure of "real rigidity" from a variety of sources, discussed in chapter 3) is equal to 0.14. Under the Rotemberg-Woodford assumptions regarding the structure of production costs and demand, the degree of real rigidity should be given by

$$\zeta = \frac{\epsilon_{mc}}{1+\omega\theta},$$

as shown in chapter 3. Hence the required value of ζ is consistent with the values obtained for ϵ_{mc} and ω in the case that the elasticity of substitution among alternative differentiated goods is equal to $\theta = 7.88$. This value is greater than one, as required by the theoretical model, and

[65] Rotemberg and Woodford point out that the assumption of extremely elastic labor supply is not necessary to make their estimated low value of κ consistent with the theoretical restrictions of the model, but only to make the model consistent with the observation of an only modestly procyclical real wage response to monetary policy shocks.

[66] See equation (xx) of chapter 3. This corresponds to the case of specific factor markets, CES preferences over differentiated goods, and no intermediate inputs, as in the baseline model developed in that chapter.

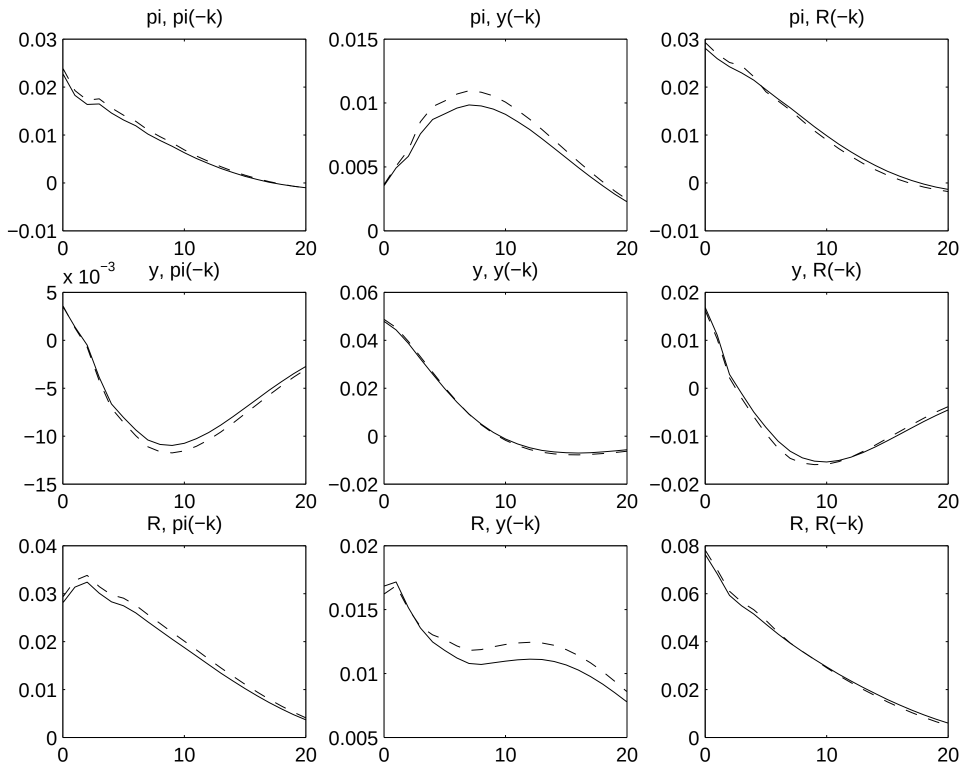

Figure 4.16: Predicted [*solid line*] and estimated [*dashed line*] second moments of quarterly U.S. data, 1980:1-1995:4. Source: Rotemberg and Woodford (1997).

is also a fairly plausible magnitude; it is neither so small as to imply an implausible degree of market power for the typical producer in the U.S. economy (it implies an average markup of prices over marginal cost of less than 15 percent), nor so large as to make it implausible that suppliers should leave their prices unchanged for a period of nine months on average.

It is thus possible to account for the estimated impulse responses to an identified monetary policy shock fairly accurately, assuming parameter values that are not only theoretically possible, but that are also consistent with a variety of other observations about the U.S. economy. Once the parameters of the structural equations have been assigned numerical values in this way, it is then possible to specify the joint stochastic process for the two composite real disturbances $\{g_t, \hat{Y}_t^n\}$ so as to match other features of the estimated joint distribution of the three time series characterized by the Rotemberg-Woodford VAR. Figure 4.16 shows the extent to which this is possible. The figure plots the estimated autocovariance and cross-covariance functions for the three variables (the ones implied by the estimated VAR model), and together with these the predictions of the theoretical model, given the Rotemberg-Woodford specification of the exogenous disturbance processes. These match quite closely. For example, it is worth noting that the model has no difficulty accounting for the observed degree of persistence of either the fluctuations in inflation or in the deviations of output from trend over this sample period. Nor does it have difficulty accounting for such often-remarked features of the data as the negative correlation between output and lagged interest rates and the positive correlation between output and interest-rate leads.

Of course, these last successes of the theoretical model are mainly a result of assuming real disturbances with the proper statistical properties; Rotemberg and Woodford impose no *a priori* restrictions upon the joint law of motion for the two composite disturbances, except that they be stationary processes that evolve independently of the monetary policy shocks. Because the structural VAR model implies that only a small amount of the variability of any of the three variables is ultimately caused by the identified monetary policy disturbances, the restriction that the assumed real disturbances be independent of the identified monetary policy shocks constrains only slightly their ability to choose disturbance processes that imply the desired second moments for the data series. If one were instead to start with tightly parameterized *a priori* assumptions about the laws of motion of the disturbance processes, as is common both in the literature on maximum-likelihood estimation of structural macroeconometric models and in the literature on calibrated equilibrium business-cycle models, one might instead find that the model would fit the properties of the time series less well.[67] Yet the theory developed here gives one no reason to assume particular kinds of "simple" laws of motion for the real disturbances. Indeed, it implies that each of the real disturbances is actually a composite of many different sorts of underlying real disturbances, and that many kinds of real disturbances should affect both g_t and $\hat{Y}_t^n$, albeit with different dynamics. Thus

[67]For criticism of the model on this ground, see Fuhrer (1997), who assumes that the real disturbances should be serially uncorrelated. McGrattan (1999) also finds considerably worse performance when the real disturbance processes are assumed to consist solely of a production-function residual and variation in real government purchases. But the theoretical model allows for preference disturbances as well, which may play an important role in accounting for the historical time series.

there is no reason to expect the two processes to have simple serial correlation properties, or to be uncorrelated with one another.

Because the approach to the design of optimal monetary policy rules emphasized in chapter 8 below makes the form of an optimal policy rule independent of the statistical properties of the real disturbances, I do not here further discuss the details of the disturbance processes estimated by Rotemberg and Woodford. However, it is perhaps of some interest to note the implications of their estimates for the question of the variability of the natural rate of interest. In their IS relation (4.7), it is only the forecastable component $E_t\hat{r}^n_{t+2}$ that matters, owing to the assumption that interest-sensitive private expenditure is predetermined two quarters in advance. Thus it seems most appropriate to ask about the variability of this exogenous term that is implied by the residuals of the estimated IS relation. As reported in Woodford (1999a), this (annualized) series has a standard deviation of 3.72 percentage points. Since the estimated long-run average funds rate and inflation rate imply a mean natural rate of interest of only 2.99 percentage points, this implies that even a one-standard deviation decline in the natural rate involves a natural rate well below zero.[68]

Thus these estimates imply that the natural rate of interest should be negative fairly often. This means that a policy under which the funds rate would always equal $E_{t-2}\hat{r}^n_t + \bar{\pi}$ — as would be necessary, according to the Rotemberg-Woodford model, in order to completely stabilize both $E_{t-2}\pi_t$ and $E_{t-2}x_t$ — would be consistent with the zero lower bound for the funds rate only if the inflation target $\bar{\pi}$ is well above zero. Thus the zero bound creates a tension between inflation stabilization and the pursuit of a low average rate of inflation, as is discussed further in section xx of chapter 6.[69]

[68] Using a quite different approach, Laubach and Williams (2001) estimate a natural-rate series for the U.S. that exhibits substantial variability, though not quite so much. When they assume that the component of the natural rate of interest due to factors other than variation in the economy's long-run growth rate is a stationary AR(2) process, they estimate the standard deviation of the natural rate to be 1.98 percentage points. Their estimate of low-frequency variation implies that the natural rate was as low as only 10 basis points in late 1994. Since their method seeks only to isolate a low-frequency component of natural-rate variations, while theory indicates that higher-frequency variations are likely as well, it is quite plausible that the overall variability of the natural rate should be greater than that estimated by Laubach and Williams, even assuming that they correctly identify low-frequency variations.

[69] The tradeoff between these two objectives is demonstrated quantitatively in the context of the estimated model by Figure 5 in Rotemberg and Woodford (1997).

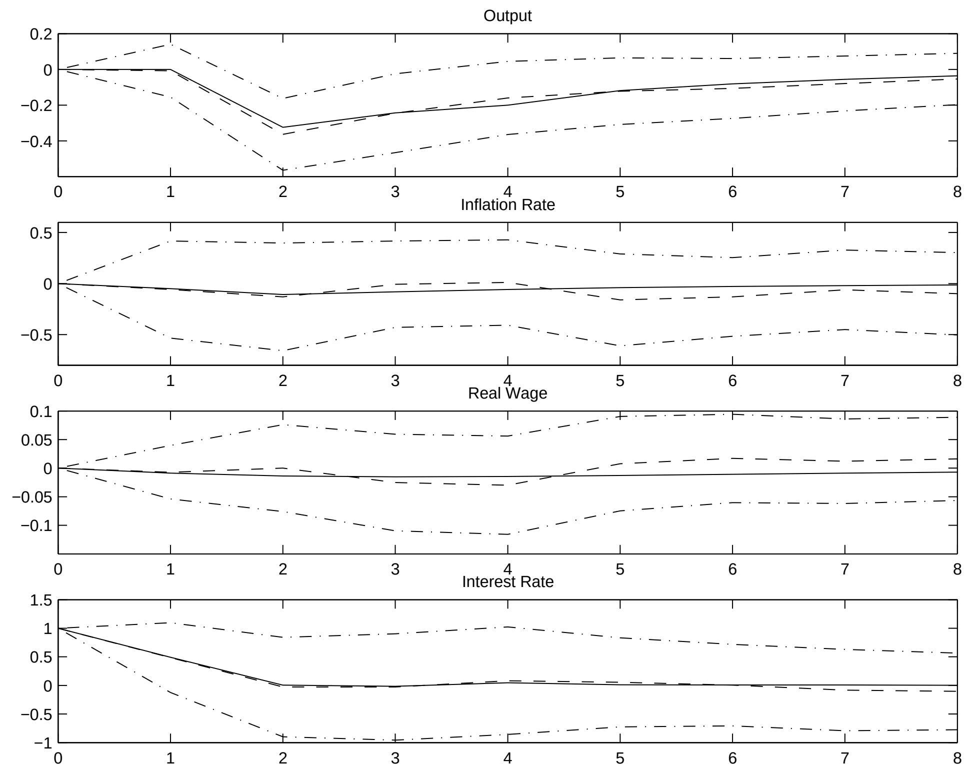

Figure 4.17: Predicted [*solid line*] and estimated [*dashed line*] impulse responses to a monetary policy shock. Source: Amato and Laubach (2001).

Amato and Laubach (2002) extend the analysis of Rotemberg and Woodford by adding a real wage series to the VAR, and estimating the impulse response to wages as well prices to a monetary policy shock. Their estimated impulse responses to an identified monetary policy shock are shown in Figure 4.17. They find that the real wage response is never significantly different from zero, just as in the estimates of Christiano *et al.* (2001) reported in Figure xx of chapter 3. As a result, the estimated responses (taking account of the real wage response along with the others, which are changed very little by the inclusion of the additional series in the VAR) are no longer consistent with the simple Rotemberg-Woodford model, with wage-taking households, for any assumed preference parameters. Amato and Laubach extend the model to allow wages as well as prices to be sticky, along the lines of the models discussed in section xx of chapter 3. The aggregate-supply block of their model consists of a pair of wage and price inflation equations that generalize equations (xx) – (xx) of chapter 3 to incorporate

Table 4.2: Parameter values in the quarterly model of Amato and Laubach (2002).

α_w, α_p	0.66
β	0.99
φ_w, φ_p	0.56
σ^{-1}	0.26
ω_w	0.27
ω_p	0.33
$(\theta_w - 1)^{-1}$	0.13
$(\theta_p - 1)^{-1}$	0.19
ξ_w	.066
ξ_p	.058
κ_w	.035
κ_p	.019

predetermined wage and price changes like those in the Rotemberg-Woodford model.

The numerical parameter values in the Amato-Laubach model are given for comparison purposes in Table 4.2. Here the meaning of the parameters is the same as in the presentation of the model of Erceg *et al.* (2000) in chapter 3, except for the introduction of the parameters φ_w, φ_p (assumed for simplicity to take a common value) that indicate the fraction of newly revised wages and prices respectively that take effect after only one quarter. The values of α_p, β, and ω_p are calibrated as in Rotemberg and Woodford, and $\alpha^w = \alpha_p$ and $\gamma_w = \gamma_p$ are assumed to reduce the number of free parameters. Given the calibrated value for ω_p, the ratio κ_p/ξ_p is also fixed. The free parameters σ, ξ_w, κ_w, κ_p and the common value of ψ in both inflation equations are then estimated to minimize a measure of the distance between the theoretical and estimated impulse responses. These estimates then imply the remaining parameter values listed in the table.

The Amato-Laubach model parameter values imply predicted responses of output, inflation and the nominal interest rate quite similar to those of the Rotemberg-Woodford model, but in addition the model now implies a smaller real wage response, and one that is more gradual and more persistent than the output response. (The responses predicted by their model, given the estimated parameter values, are shown in Figure 4.17.) The model yields these predictions as a result of substantial wage as well as price stickiness, indicated by the small estimated value for κ_w. While κ_w is estimated to be larger than κ_p, so that the real wage is predicted to move slightly procyclically (see Figure xx of chapter 3), the two coefficients are similar in magnitude, and the predicted real wage response is quite small. Note that despite the fact that wages are estimated to be relatively sticky, the estimated model still implies very low curvature of the disutility of labor supply. Even though ω_w can no longer be directly inferred from the relative magnitudes of the output and real-wage responses, it can be inferred from the relative magnitudes of κ_w and ξ_w; the fact that Amato and Laubach estimate that ξ_w is large relative to the size of κ_w can be reconciled with the underlying microeconomic foundations of the model only if the marginal disutility of working does not rise much with increases in employment.

Another weakness of the simple Rotemberg-Woodford model is its failure to predict an inflation response as persistent as the one indicated by the VAR. This response is estimated quite imprecisely by their VAR; but as discussed in chapter 3, a large number of other studies also conclude that inflation responses are both more delayed and more persistent than predicted by their model. A simple way to increase the realism of this aspect of their model is to assume that prices are indexed to past inflation between the occasions on which they are re-optimized, as discussed in section xx of chapter 3.[70] Let us assume flexible wages and staggered price-setting, with individual prices partially indexed to a lagged price index, and again let the parameter $0 \leq \gamma \leq 1$ indicate the degree of indexation between the occasions on which prices are revised. Suppose that there is a delay of $s = 1$ quarter before price revisions take effect, but once again a delay of $d = 2$ quarters before interest-rate disturbances can affect real expenditure. Finally, suppose that this latter delay is interpreted as an information delay rather than actual precommitment of real expenditure, so that the μ_t term disappears from (4.15) as discussed at the end of the previous section.[71] With the

[70] Boivin and Giannoni (2001) also estimate a variant of the Rotemberg-Woodford model that allows for inflation inertia of a similar sort to that discussed here, but derived from the hypothesis of backward-looking "rule-of-thumb" price-setters proposed by Gali and Gertler (1999). Their estimated fraction of backward-looking price-setters for the post-1980 sample period implies a degree of inflation inertia that is only very slightly greater than that implied by the model discussed in the text in the case that $\gamma = 1$.

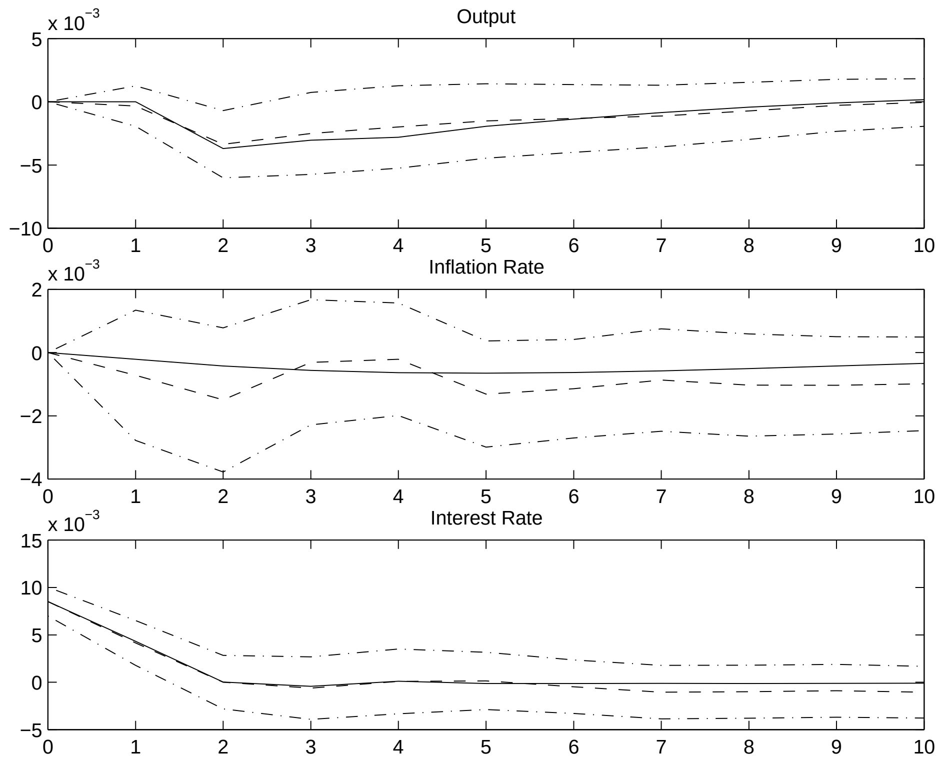

Figure 4.18: Predicted [*solid line*] and estimated [*dashed line*] impulse responses to a monetary policy shock, in the case of a model with inflation inertia. Estimated responses are again taken from Rotemberg and Woodford (1997).

addition of the partial indexation, and specializing to the case just described, the aggregate supply relation (4.17) becomes

$$\pi_t - \gamma\pi_{t-1} = \kappa\, E_{t-1}x_{t+1} + \left(\frac{1-\alpha}{\alpha} + \alpha\beta\right)\, E_{t-1}(\pi_{t+1} - \gamma\pi_t). \tag{4.22}$$

In this variant model, it is no longer necessary to assume that some goods price changes must be determined two quarters in advance (as in the Rotemberg-Woodford model) in order to account for the fact that the maximum decline in inflation does not occur in the quarter

[71] If one instead adopts the alternative interpretation, as in Rotemberg and Woodford, Amato and Laubach, and Boivin and Giannoni, one finds that the model best fits the estimated impulse responses when the coefficient on $E_{t-1}\mu_t$ in the AS relation is made as small as possible. This indicates the superiority of the interpretation proposed here. Note that the problem does not appear in Rotemberg and Woodford (1997) because the $E_{t-1}\mu_t$ term affects only the predicted response of inflation in the first quarter following the quarter of the shock, and in their model that particular response can be matched by assuming a sufficiently large value for $\psi > 0$. Here instead the assumption affects the degree of fit because we impose $\psi = 0$.

Table 4.3: Parameter values in an estimated quarterly model with inflation inertia.

α	0.83
β	0.99
γ	1.00
ψ	0
σ^{-1}	0.22
ζ	0.15
κ	.0052

following the monetary policy shock. As shown in Figure 4.18, the model proposed here predicts a further decline in inflation in the second quarter even when $\psi = 0$; hence we impose the assumption that $\psi = 0$ for the sake of simplicity.

Except for the replacement of (4.20) by (4.22), we continue to assume the same model as in Rotemberg and Woodford. The model parameters σ, κ, and γ are estimated so as to match the same three impulse responses as in Figure 4.8, assuming $\beta = 0.99$ as before, and imposing the constraint that $0 \leq \gamma \leq 1$. The estimated parameter values are shown in Table 4.3.[72] We find the best fit when $\gamma = 1$, implying full indexation to lagged inflation, as assumed by Christiano *et al.* (2001).[73] The resulting aggregate supply relation is fairly similar to the one proposed by Fuhrer and Moore (1995a, 1995b), and used in other small econometric models of the U.S. monetary transmission mechanism, such as Orphanides *et al.* (1997).[74]

[72]These parameters are those that minimize the simple sum of squared deviations between the predicted and estimated responses of the three variables in the first ten quarters following the shock, *i.e.*, the statistic shown in the last column of Table 4.4.

[73]This implies roughly equal weights on lagged inflation and expected future inflation in the aggregate supply relation (4.22), which is similar to what Boivin and Giannoni (2001) find when the estimate a model with a similar, but not quite identical, specification. In addition to introducing inflation inertia through "rule-of-thumb" price-setting of the kind hypothesized by Gali and Gertler (1999), the model of Boivin and Giannoni differs from this one in assuming that only aggregate purchases are predetermined two quarters in advance, and not the purchases of individual goods.

[74]Indeed, the three-equation model, consisting of (4.7), (4.22), and the estimated federal funds-rate reaction function, is quite similar in form to the three-equation model of Fuhrer and Moore (1995b). The main difference is in the way that delays in the effects of monetary policy are introduced here; Fuhrer and Moore assume an AS relation that involves no delay, and an IS relation in which output responds to a long-term real interest rate one quarter earlier, as opposed to the expectation two quarters earlier of the present quarter's long-term real rate (as in Rotemberg and Woodford and here). Other differences in the Fuhrer and

Table 4.4: Root-mean-square deviations between predicted and estimated responses for each of three variables, averaged over quarters 1 through 10 following the monetary policy shock. (Each reported value should be multiplied by 10^{-3}.) The final column indicates the sum of squared deviations over the ten quarters, summing over the three variables. (These values should be multiplied by 10^{-6}.)

Model	rms(y)	rms(π)	rms(i)	SSD
$\gamma = 0$	0.30	0.73	1.02	10.49
$\gamma = 1$	0.40	0.59	0.58	8.48

Because this specification predicts a more inertial response of inflation (and hence of the real rate of interest), the value of σ required to account for the observed output decline following a monetary tightening is now somewhat smaller. In addition, because the effects of low real marginal costs on inflation cumulate over time under the $\gamma = 1$ specification, the size of κ required to account for the observed inflation decline is now much smaller than that indicated in Table 4.1. If we continue to assume that $\alpha = 0.66$, as in the model of Rotemberg and Woodford, this would imply $\zeta = 0.03$, a value that is perhaps too small to be plausible (see the discussion in chapter 3). However, survey evidence on the frequency of price changes no longer requires that α be this small, as one might interpret some of the observed price changes as automatic price changes as a result of indexation, rather than reconsiderations of pricing policy. If, for example, we assume that re-optimizations occur once every six quarters on average (rather than every three quarters, as assumed by Rotemberg and Woodford), then $\alpha = 0.83$. In this case, the estimated value of κ would correspond to $\zeta = 0.15$, as shown in the table. This requires only real rigidities of a magnitude that can be reconciled with plausible assumptions about preferences and technology, as shown in chapter 3. If one is willing to assume an even larger value of α, less real rigidity would be required; for example, if $\alpha = 0.9$, one would only need $\zeta = 0.43$.

The degree of fit to the estimated impulse responses obtained under these parameter values is shown in Figure 4.18. Table 4.4 gives the root-mean-square deviation between the

Moore specification include the inclusion of lagged output terms in the IS relation and the assumption of an interest-rate rule involving only first differences of the funds rate.

predicted and estimated responses in the case of each of the three panels of the figure, and compares these with the corresponding measures for Figure 4.15. Despite the imposition of the restriction that $\psi = 0$, the model that incorporates inflation inertia better fits the estimated responses of both inflation and the nominal interest rate, though at the price of a somewhat worse fit to the estimated output response. (Assuming that $\gamma = 1$ increases the predicted persistence of the effect of the shock on inflation, which better fits the estimated response, but also reduces the predicted persistence of the effect on output, despite the lower assumed value of κ. Thus there is some tension between the assumptions required to account for the persistence of inflation responses and those required to account for the persistence of output responses.) The overall sum of squared deviations is reduced by 19 percent.

While the model with inflation inertia better fits the point estimates for the response of inflation and interest rates to a monetary policy shock implied by the VAR of Rotemberg and Woodford (1997), the imprecision of those estimates makes it hard to say, on the basis of this evidence alone, that a large value of γ is clearly more realistic. However, as discussed in chapter 3, a number of other studies — both single-equation estimates and complete estimated models — have also found that similar aggregate supply relations fit U.S. data better than do specifications that do not incorporate inflation inertia. Hence in part II of this study we consider in some detail the consequences for optimal policy of indexation to lagged inflation of this kind.

4.3 Additional Sources of Delay

[TO BE ADDED]

5 Monetary Policy and Investment Dynamics

One of the more obvious omissions in the baseline model developed above is the absence of any effect of variations in private spending upon the economy's productive capacity, and hence upon supply costs in subsequent periods. This means that we have treated all private expenditure as if it were non-durable consumption expenditure; and while this has kept our analysis of the effects of interest rates on aggregate demand quite simple, one may doubt the accuracy of the conclusions obtained, given the obvious importance of variations in investment spending both in business fluctuations generally and in the the transmission mechanism for monetary policy in particular. We have suggested that the baseline model ought not be interpreted as one in which investment spending is literally constant; in particular, we have argued that the parameter σ in that model ought not be "calibrated" on the basis of studies of intertemporal substitution of consumer expenditure, but should instead be taken to refer to the degree of intertemporal substitutability of overall private expenditure, largely as a result of intertemporal substitution in investment spending. In this section we develop an extended version of the model in which investment spending is explicitly modeled, to see to what extent such a more detailed model has properties different than those of the baseline model, when the latter is calibrated to reflect an elasticity of intertemporal substitution of overall spending that is several times as large as the elasticity of non-durable consumption.

5.1 Investment Demand with Sticky Prices

Our first task is to develop a model of optimizing investment demand by suppliers with sticky prices, and that are demand-constrained as a result. We begin by modifying our production function to include an explicit representation of the effects of variation in the capital stock. The production function for good i is assumed to be of the form

$$y_t(i) = k_t(i) f(A_t h_t(i)/k_t(i)), \tag{5.1}$$

where f is an increasing, concave function as before, with f(0) = 0. Note that in the case that $k_t(i)$ is a constant, this reduces to the form of production function assumed in chapter 3, except that the technology factor A_t is now assumed to multiply the labor input rather than the entire production function ("labor-augmenting" technical progress). We now change the specification of the technology factor so that a one permanent increase in A_t will still result in a one percent long-run increase in equilibrium output, now that the eventual increase in the capital stock is taken into account. (In our model with endogenous capital accumulation, the capital used per unit of labor will eventually increase in proportion to the increase in A_t, in order to maintain a constant long-run relation between the marginal product of capital and the rate of time preference of households.)

We shall assume that each monopolistic supplier makes an independent investment decision each period; there is a separate capital stock $k_t(i)$ for each good, that can be used only in the production of good i, rather than a single capital stock that can "rented" for use in any sector, at a single economy-wide "rental rate" for capital services. The latter assumption is instead remarkably common in the literature on intertemporal general-equilibrium models with sticky prices; important early examples of models of that kind include Hairault and Portier (1993), Kimball (1995), Yun (1996), King and Watson (1996), King and Wolman (1996), and Chari *et al.* (2000), among others. Simplicity probably accounts for this (together with the bad example set by early examples of intertemporal general-equilibrium models with imperfect competition, such as Rotemberg and Woodford, 1992). But the assumption of a single economy-wide rental market for capital is plainly unrealistic, and its consequences are far from trivial in the present context. For it would imply that differences in the demand for goods that have set their prices at different times should result in instantaneous reallocation of the economy's capital stock from lower-demand to higher-demand sectors, and this in turn has an important effect upon the degree to which marginal cost of supply should vary with the demand for a given good. We have shown in chapter 3 that the assumption of economy-wide factor markets greatly reduces the predicted degree of strategic complementarity of the pricing decisions of different suppliers, and in so doing increases the speed of adjustment of the overall level of prices to varying demand conditions. Hence we here assume instead that while all sectors purchase investment goods from the same suppliers (*i.e.*, that the investment goods used by the different sectors are perfect substitutes for their *producers*), these goods cease to be substitutable once they have been purchased for use in production in a particular sector. Capital can be reallocated from low-demand to high-demand sectors only over time, through reduced new investment in the former sectors and increased new investment in the latter; and the speed with which this occurs is limited by the assumption of adjustment costs. The resulting model is more realistic, and also represents a more direct generalization of the constant-capital model developed in chapter 3. (That model implicitly assumed a constant quantity of capital $k_t(i)$ available for the production of each individual good, rather than a constant aggregate capital stock that would be efficiently reallocated each period among sectors. We show below that the constant-capital model can be recovered as a limiting case of the present model, in the limit of very high adjustment costs for investment.)

We assume convex adjustment costs for investment by each firm, of the usual kind assumed in neoclassical investment theory. Increasing the capital stock to the level $k_{t+1}(i)$ in period $t+1$ requires investment spending in the amount $I_t(i) = I(k_{t+1}(i)/k_t(i))k_t(i)$ in period t. Here $I_t(i)$ represents purchases by firm i of the composite good, defined as the usual Dixit-Stiglitz aggregate over purchases of each of the continuum of goods (with the same constant elasticity of substitution $\theta > 1$ as for consumption purchases). In this way, the allocation of investment expenditure across the various goods is in exactly the same proportion as consumption expenditure, resulting in a demand curve for each producer that is again of the form

$$y_t(i) = Y_t \left(\frac{p_t(i)}{P_t} \right)^{-\theta}, \qquad (5.2)$$

but where now aggregate demand is given by $Y_t = C_t + I_t + G_t$, in which expression I_t denotes the integral of $I_t(i)$ over the various firms i. We assume as usual that the function $I(\cdot)$ is increasing and convex; the convexity implies the existence of costs of adjustment. We further assume that near a zero growth rate of the capital stock, this function satisfies $I(1) = \delta$, $I'(1) = 1$, and $I''(1) = \epsilon_\psi$, where $0 < \delta < 1$ and $\epsilon_\psi > 0$ are parameters. This implies that in the steady state to which the economy converges in the absence of shocks (which involves a constant capital stock, as we abstract from trend growth), the steady rate of investment spending required to maintain the capital stock is equal to δ times the steady-state capital stock (so that δ can be interpreted as the rate of depreciation). It also implies that near the steady state, a marginal unit of investment spending increases the capital stock by an equal amount (as there are locally no adjustment costs). Finally, in our log-linear approximation to the equilibrium dynamics, ϵ_ψ will be the parameter that indexes the degree of adjustment costs.

Profit-maximization by firm i then implies that the capital stock for period $t+1$ will be chosen in period t to satisfy the first-order condition

$$\begin{aligned} I'(k_{t+1}(i)/k_t(i)) \quad &= \quad E_t Q_{t,t+1}\Pi_{t+1}\left\{\rho_{t+1}(i) \ + \right. \\ &\left. (k_{t+2}(i)/k_{t+1}(i)) I'(k_{t+2}(i)/k_{t+1}(i)) - I(k_{t+2}(i)/k_{t+1}(i))\right\}, \end{aligned}$$

where $\rho_{t+1}(i)$ is the (real) shadow value of a marginal unit of additional capital for use by firm i in period $t+1$ production, and $Q_{t,t+1}\Pi_{t+1}$ is the stochastic discount factor for evaluating real income streams received in period $t+1$. Expressing the real stochastic discount factor as $\beta\lambda_{t+1}/\lambda_t$, where λ_t is the representative household's marginal utility of real income in period t, and then log-linearizing this condition around the steady-state values of all state variables, we obtain

$$\begin{aligned} \hat{\lambda}_t + \epsilon_\psi(\hat{k}_{t+1}(i) - \hat{k}_t(i)) \quad &= \quad E_t\hat{\lambda}_{t+1} \ + \\ &[1-\beta(1-\delta)]E_t\hat{\rho}_{t+1}(i) + \beta\epsilon_\psi E_t(\hat{k}_{t+2}(i) - \hat{k}_{t+1}(i)), \end{aligned} \tag{5.3}$$

where $\hat{\lambda}_t \equiv \log(\lambda_t/\bar{\lambda})$, $\hat{k}_t(i) \equiv \log(k_t(i)/\bar{K})$, $\hat{\rho}_t(i) \equiv \log(\rho_t(i)/\bar{\rho})$, and variables with bars denote steady-state values.

Note that $\rho_{t+1}(i)$ would correspond to the real "rental price" for capital services if a market existed for such services, though we do not assume one. It is *not* possible in the present model to equate this quantity with the marginal product, or even the marginal revenue product of capital (using the demand curve (5.2) to compute marginal revenue). For suppliers are demand-constrained in their sales, given the prices that they have posted; it is not possible to increase sales by moving down the demand curve. Thus the shadow value of additional capital must instead be computed as the reduction in labor costs through substitution of capital inputs for labor, while still supplying the quantity of output that happens to be demanded. We thus obtain

$$\rho_t(i) = w_t(i)\left(\frac{f(\tilde{h}_t(i)) - \tilde{h}_t(i)f'(\tilde{h}_t(i))}{A_t f'(\tilde{h}_t(i))}\right),$$

where $w_t(i)$ is the real wage for labor of the kind hired by firm i and $\tilde{h}_t(i) \equiv A_t h_t(i)/k_t(i)$ is firm i's effective labor-capital input ratio.[75] We can alternatively express this in terms of the output-capital ratio for firm i (in order to derive an "accelerator" model of investment demand), by substituting (5.1) to obtain

$$\rho_t(i) = \frac{w_t(i)}{A_t} f^{-1}(y_t(i)/k_t(i))[\phi(y_t(i)/k_t(i)) - 1], \tag{5.4}$$

where $\phi(y/k)$ is the reciprocal of the elasticity of the function f, evaluated at the argument $f^{-1}(y/k)$.

We the recall from chapter 3 the first-order condition for optimizing labor supply, which we may write in the form

$$w_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t; \xi_t)}{\lambda_t}, \tag{5.5}$$

again writing labor demand in terms of the demand for good i. Substituting this into (5.4) and log-linearizing, we obtain

$$\hat{\rho}_t(i) = \left(\nu\phi + \frac{\phi}{\phi - 1}\omega_p\right)(\hat{y}_t(i) - \hat{k}_t(i)) + \nu\hat{k}_t(i) - \hat{\lambda}_t - \omega q_t, \tag{5.6}$$

where $\phi > 1$ is the steady-state value of $\phi(y/k)$, *i.e.*, the reciprocal of the elasticity of the production function with respect to the labor input, $\omega_p > 0$ is the negative of the elasticity of $f'(f^{-1}(y/k))$ with respect to y/k, and $\nu > 0$ is once again the elasticity of the marginal disutility of labor with respect to labor supply. (The composite exogenous disturbance q_t is defined as in equation (2.2).) Substituting this into (5.3), we then have an equation to solve for the dynamics of firm i's capital stock, given the evolution of demand $\hat{y}_t(i)$ for its product, the marginal utility of income $\hat{\lambda}_t$, and the exogenous disturbance q_t.

[75] Note that in the case of a flexible-price model, the ratio of $w_t(i)$ to the denominator would always equal marginal revenue, and so this expression would equal the marginal revenue product of capital, though it would be a relatively cumbersome way of writing it.

As the coefficients of these equations are the same for each firm, an equation of the same form holds for the dynamics of the aggregate capital stock (in our log-linear approximation). Our equilibrium condition for the dynamics of the capital stock is thus of the form

$$\begin{aligned} \hat{\lambda}_t + \epsilon_\psi(\hat{K}_{t+1} - \hat{K}_t) &= \beta(1-\delta)E_t\hat{\lambda}_{t+1} + \\ &[1-\beta(1-\delta)][\rho_y E_t\hat{Y}_{t+1} - \rho_k\hat{K}_{t+1} - \omega q_t] + \beta\epsilon_\psi E_t(\hat{K}_{t+2} - \hat{K}_{t+1}), \end{aligned} \quad (5.7)$$

where the elasticities of the marginal valuation of capital are given by

$$\rho_y \quad \equiv \quad \nu\phi + \frac{\phi}{\phi - 1}\omega_p \quad > \quad \rho_k \quad \equiv \quad \rho_y - \nu \quad > \quad 0.$$

The implied dynamics of investment spending are then given by

$$\hat{I}_t = k[\hat{K}_{t+1} - (1-\delta)\hat{K}_t], \quad (5.8)$$

where $\hat{I}_t$ is defined as the percentage deviation of investment from its steady-state level, as a share of steady-state output, and $k \equiv \bar{K}/\bar{Y}$ is the steady-state capital-output ratio.

We have here derived investment dynamics as a function of the evolution of the marginal utility of real income of the representative household. This is in turn related to aggregate spending through the relation $\lambda_t = u_c(Y_t - I_t - G_t; \xi_t)$, which we may log-linearize as

$$\hat{\lambda}_t = -\sigma^{-1}(\hat{Y}_t - \hat{I}_t - g_t), \quad (5.9)$$

where the composite disturbance g_t once again reflects the effects both of government purchases and of shifts in private impatience to consume.[76] Finally, recalling the relation between the marginal utility of income process and the stochastic discount factor that prices bonds, the nominal interest rate must satisfy

$$1 + i_t = \{\beta E_t[\lambda_{t+1}/(\lambda_t\Pi_{t+1})]\}^{-1},$$

which we may log-linearize as

$$\hat{\imath}_t \quad = \quad E_t\pi_{t+1} + \hat{\lambda}_t - E_t\hat{\lambda}_{t+1}. \quad (5.10)$$

[76] Note that the parameter σ in this equation is no longer the intertemporal elasticity of substitution in consumption, but rather $\bar{C}/\bar{Y}$ times that elasticity. In a model with investment, these quantities are not exactly the same, even in the absence of government purchases.

The system of equations (5.7) – (5.10) then comprise the "IS block" of our model. These jointly suffice to determine the paths of the variables $\{\hat{Y}_t, \hat{I}_t, \hat{K}_t, \lambda_t\}$, given an initial capital stock and the evolution of short-term real interest rates $\{\hat{\imath}_t - E_t\pi_{t+1}\}$. The nature of the effects of real interest-rate expectations on these variables is discussed further in section 3.3 below.

5.2 Optimal Price-Setting with Endogenous Capital

We turn next to the implications of an endogenous capital stock for the price-setting decisions of firms. The capital stock affects a firm's marginal cost, of course; but more subtly, a firm considering how its future profits will be affected by the price it sets must also consider how its capital stock will evolve over the time that its price remains fixed.

We begin with the consequences for the relation between marginal cost and output. Real marginal cost can be expressed as the ratio of the real wage to the marginal product of labor. Again writing the factor input ratio as a function of the capital/output ratio, and using (5.5) for the real wage, we obtain

$$s_t(i) = \frac{v_h(f^{-1}(y_t(i)/k_t(i))k_t(i)/A_t;\xi_t)}{\lambda_t A_t f'(f^{-1}(y_t(i)/k_t(i)))}$$

for the real marginal cost of supplying good i. This can be log-linearized to yield

$$\hat{s}_t(i) = \omega(\hat{y}_t(i) - \hat{k}_t(i)) + \nu\hat{k}_t(i) - \hat{\lambda}_t - [\nu\bar{h}_t + (1+\nu)a_t], \tag{5.11}$$

where once again $\omega \equiv \omega_w + \omega_p \equiv \nu\phi + \omega_p > 0$ is the elasticity of marginal cost with respect to a firm's own output, and

$$q_t \equiv \omega^{-1}[\nu\bar{h}_t + (1+\nu)a_t]$$

is the percentage change in output required to maintain a constant marginal disutility of output supply, in the case that the firm's capital remains at its steady-state level.[77]

[77] That is, q_t measures the output change that would be required to maintain a fixed marginal disutility of supply given possible fluctuations in preferences and technology, but not taking account of the effect of possible fluctuations in the firm's capital stock; thus q_t is again an exogenous disturbance term. Note that

Letting $\hat{s}_t$ without the index i denote the average level of real marginal cost in the economy as a whole, we note that (5.11) implies that

$$\hat{s}_t(i) = \hat{s}_t + \omega(\hat{y}_t(i) - \hat{Y}_t) - (\omega - \nu)(\hat{k}_t(i) - \hat{K}_t).$$

Then using (5.2) to substitute for the relative output of firm i, we obtain

$$\hat{s}_t(i) = \hat{s}_t - (\omega - \nu)\tilde{k}_t(i) - \omega\theta\hat{p}_t(i), \tag{5.12}$$

where $\hat{p}_t(i) \equiv \log(p_t(i)/P_t)$ is the firm's relative price, and $\tilde{k}_t(i) \equiv \hat{k}_t(i) - \hat{K}_t$ is its relative capital stock.

As in chapter 3, the Calvo price-setting framework implies that if firm i resets its price in period t, it chooses a price to satisfy the (log-linear approximate) first-order condition

$$\sum_{k=0}^{\infty}(\alpha\beta)^k E_t[\hat{p}_{t+k}(i) - \hat{s}_{t+k}(i)] = 0.$$

Substituting (5.12) for $s_{t+k}(i)$ in this expression, we obtain

$$\sum_{k=0}^{\infty}(\alpha\beta)^k E_t[(1+\omega\theta)\hat{p}_{t+k}(i) - \hat{s}_{t+k} + (\omega - \nu)\tilde{k}_{t+k}(i)] = 0. \tag{5.13}$$

We can as before express the entire sequence of values $\{\hat{p}_{t+k}(i)\}$ as a linear function of the relative price $\hat{p}_t^*$ chosen at date t and aggregate variables (namely, the overall rate of price inflation over various future horizons). However, we cannot yet solve for the optimal choice of $\hat{p}_t^*$, because (5.13) also involves the relative capital stock of firm i at a sequence of future dates, and this depends upon the investment policy of the firm.

We must therefore use the investment theory of the previous section to model the evolution of firm i's relative capital stock. Equation (5.7) implies that

$$\epsilon_\psi(\tilde{k}_{t+1}(i) - \tilde{k}_t(i)) = [1 - \beta(1-\delta)][\rho_y E_t(\hat{y}_{t+1}(i) - \hat{Y}_t) - \rho_k\tilde{k}_{t+1}(i)] + \beta\epsilon_\psi E_t(\tilde{k}_{t+2}(i) - \tilde{k}_{t+1}(i)).$$

the expression given here for q_t in terms of the underlying disturbances differs from that in section 2.1 above, because of our differing specification here of how the technology factor A_t shifts the production function. Nonetheless, this definition of q_t is directly analogous to that used in the case of the constant-capital model; it is actually our use of the notation a_t that is different here.

Again using the demand curve to express relative output as a function of the firm's relative price, this can be written as

$$E_t[Q(L)\tilde{k}_{t+2}(i)] = \Xi E_t \hat{p}_{t+1}(i), \qquad (5.14)$$

where the lag polynomial is

$$Q(L) \equiv \beta - [1 + \beta + (1 - \beta(1-\delta))\rho_k \epsilon_\psi^{-1}]L + L^2,$$

and

$$\Xi \equiv (1 - \beta(1-\delta))\rho_y \theta \epsilon_\psi^{-1} > 0.$$

One can easily show[78] that the lag polynomial can be factored as

$$Q(L) = \beta(1 - \mu_1 L)(1 - \mu_2 L),$$

where the two roots satisfy $0 < \mu_1 < 1 < \beta^{-1} < \mu_2$. We also note that

$$\Xi = -\theta \frac{\rho_y}{\rho_k} Q(1) = \beta\theta \frac{\rho_y}{\rho_k} \mu_2 (1 - \mu_1)(1 - \mu_2^{-1}).$$

It then follows that in the case of any bounded process $\{\hat{p}_t(i)\}$, (5.14) has a unique bounded solution for the evolution of $\{\tilde{k}_t(i)\}$, given an initial capital stock for the firm. This solution is given by

$$\tilde{k}_{t+1}(i) = \mu_1 \tilde{k}_t(i) - z_t(i), \qquad (5.15)$$

which we may integrate forward starting from an initial condition $\tilde{k}_t(i)$; here we define

$$z_t(i) \equiv \beta^{-1} \Xi \sum_{j=1}^{\infty} \mu_2^{-j} E_t \hat{p}_{t+j}(i).$$

Condition (5.13) requires that we evaluate the infinite sum $\sum_{k=0}^{\infty} (\alpha\beta)^k E_t \tilde{k}_{t+k}(i)$. We note that (5.15) implies that

$$E_t \tilde{k}_{t+k+1}(i) = \mu_1 E_t \tilde{k}_{t+k}(i) - E_t z_{t+k}(i)$$

[78]The properties asserted follow directly from the observations that $Q(0) = \beta > 0$, $Q(\beta) < 0$, $Q(1) < 0$, and that $Q(z) > 0$ for all large enough $z > 0$. These conditions imply that $Q(z)$ has two real roots, one between 0 and β and another that is greater than 1.

for all $k \geq 0$. Integrating this law of motion we then find that

$$E_t\tilde{k}_{t+k}(i) = \mu_1^k \tilde{k}_t(i) - \sum_{j=0}^{k-1} \mu_1^{k-1-j} E_t z_{t+j}(i),$$

from which it follows that

$$\sum_{k=0}^{\infty}(\alpha\beta)^k E_t\tilde{k}_{t+k}(i) = \frac{1}{1-\alpha\beta\mu_1}\tilde{k}_t(i) - \frac{\alpha\beta}{1-\alpha\beta\mu_1}\sum_{j=0}^{\infty}(\alpha\beta)^j E_t z_{t+j}(i). \tag{5.16}$$

The final term in this last relation can furthermore be expressed in terms of expected relative prices, yielding

$$\sum_{j=0}^{\infty}(\alpha\beta)^j E_t z_{t+j}(i) = \frac{\Xi}{\beta(1-\alpha\beta\mu_2)}\left[\sum_{j=1}^{\infty}\mu_2^{-j}E_t\hat{p}_{t+j}(i) - \sum_{j=1}^{\infty}(\alpha\beta)^j E_t\hat{p}_{t+j}(i)\right]. \tag{5.17}$$

Now substituting (5.16) – (5.17) for the sum of expected relative capital stocks in (5.13), we obtain a relation that involves only the initial relative capital stock $\tilde{k}_t(i)$. This relation can furthermore be simplified if we average it over all of the firms i that choose new prices at date t. Because the Calvo model assumes that all firms are equally likely to choose new prices at date t, the average value of $\tilde{k}_t(i)$ is zero (even though the average value of $E_t\tilde{k}_{t+k}(i)$ need not be zero for horizons $k > 0$). The average value of $E_t\hat{p}_{t+k}(i)$ can also be expressed as

$$\hat{p}_t^* - \sum_{j=1}^{k} E_t\pi_{t+j},$$

where $\hat{p}_t^*$ denotes the average relative price (average value of $\log p_t(i)/P_t$) for the firms that choose new prices at date t. With these substitutions, (5.13) yields an equation for $\hat{p}_t^*$ of the form

$$(a-b)\hat{p}_t^* = \sum_{k=0}^{\infty}(\alpha\beta)^k E_t\hat{s}_{t+k} + a\sum_{k=1}^{\infty}(\alpha\beta)^k E_t\pi_{t+k} - b\sum_{k=1}^{\infty}\mu_2^{-k}E_t\pi_{t+k}, \tag{5.18}$$

where

$$a \equiv \frac{1+\omega\theta}{1-\alpha\beta} + (\omega-\nu)\frac{\alpha}{1-\alpha\beta}\frac{\Xi}{(1-\alpha\beta\mu_1)(1-\alpha\beta\mu_2)} > 0,$$

$$b \equiv (\omega-\nu)\frac{\alpha}{1-\mu_2^{-1}}\frac{\Xi}{(1-\alpha\beta\mu_1)(1-\alpha\beta\mu_2)} > 0.$$

This allows us to solve for the average relative price chosen at date t by optimizing price-setters, as a function of information at that date about the future evolution of average real marginal costs and the overall rate of price inflation.

As in chapter 3, it is useful to quasi-difference this pricing relation in order to obtain an aggregate supply relation. Equation (5.18) implies that

$$(a-b)E_t[(1-\alpha\beta L^{-1})(1-\mu_2^{-1}L^{-1})\hat{p}_t^*] =$$

$$E_t[(1-\mu_2^{-1}L^{-1})\hat{s}_t] + a\alpha\beta E_t[(1-\mu_2^{-1}L^{-1})\pi_{t+1}] - b\mu_2^{-1}E_t[(1-\alpha\beta L^{-1})\pi_{t+1}]. \quad (5.19)$$

We then recall that in the Calvo pricing model the overall rate of price inflation will be given by

$$\pi_t = \frac{1-\alpha}{\alpha}\hat{p}_t^*.$$

Using this to substitute for $\hat{p}_t^*$ in (5.19), we obtain an inflation equation of the form

$$\pi_t = \xi_0\hat{s}_t - \xi_1 E_t\hat{s}_{t+1} + \psi_1 E_t\pi_{t+1} - \psi_2 E_t\pi_{t+2}, \quad (5.20)$$

where

$$\xi_0 \equiv \frac{1-\alpha}{\alpha}\frac{1}{a-b}, \qquad \xi_1 \equiv \mu_2^{-1}\xi_0,$$

$$\psi_1 \equiv \frac{a(\beta+\mu_2^{-1}) - b(\alpha\beta+\alpha^{-1}\mu_2^{-1})}{a-b}, \qquad \psi_2 \equiv \beta\mu_2^{-1}.$$

Once again, this allows us to solve for equilibrium inflation as a function of the current and expected future average level of real marginal costs across sectors. (The sign of this relationship is investigated numerically below.)

It remains to connect the expected evolution of real marginal costs, in turn, with expectations regarding real activity. Averaging (5.11) over firms i, and substituting (5.9) to eliminate $\hat{\lambda}_t$, we obtain

$$\hat{s}_t = (\omega+\sigma^{-1})\hat{Y}_t - \sigma^{-1}\hat{I}_t - (\omega-\nu)\hat{K}_t - [\sigma^{-1}g_t + \omega q_t]. \quad (5.21)$$

Once again, real marginal costs are increasing in the current level of real activity; but now this relation is affected not merely by exogenous disturbances to tastes and technology, but also by fluctuations in the aggregate capital stock, and by the share of current aggregate demand that is investment as opposed to consumption demand. Equations (5.20) – (5.21) constitute the "aggregate supply block" of our extended model. They jointly replace the aggregate supply relation of our baseline model, and serve to determine equilibrium inflation dynamics as a function of the expected evolution of aggregate real expenditure, the aggregate capital stock, and aggregate investment spending.

5.3 Comparison with the Baseline Model

Our complete extended model then consists of the system of equations (5.7) – (5.10) and (5.20) – (5.21), together with an interest-rate feedback rule such as (1.7) specifying monetary policy. We have a system of seven expectational difference equations per period to determine the equilibrium paths of seven endogenous variables, namely the variables $\{\pi_t, \hat{\imath}_t, \hat{Y}_t, \hat{K}_t, \hat{I}_t, \hat{s}_t, \hat{\lambda}_t\}$, given the paths of three composite exogenous disturbances $\{g_t, q_t, \bar{\imath}_t\}$. It is useful to comment upon the extent to which the structure of the extended (variable-capital) model remains similar, though not identical, to that of the baseline (constant-capital) model.

We have already noted that the equations of the extended model consist of an "IS block"(which allows us to solve for the paths of real output and of the capital stock, given the expected path of real interest rates and the initial capital stock), an "AS block" (which allows us to solve for the path of inflation given the paths of real output and of the capital stock), and a monetary policy rule (which implies a path for nominal interest rates given the paths of inflation and output). In this overall structure it is similar to the baseline model, except that the model involves an additional endogenous variable, the capital stock, which is determined by the "IS block" and taken as an input to the "AS block", along with the level of real activity.[79] It also continues to be the case that real disturbances affect the determination of inflation and output only through their effects upon the two composite disturbances

[79] The structure of the model is thus similar to rational-expectations IS-LM models such as that of Sargent and Wallace (1975), which allows for an endogenous capital stock.

g_t and q_t. Previously, we had emphasized instead the disturbances g_t and $\hat{Y}_t^n$, but these contained the same information as a specification of g_t and q_t. (The appropriate definition of the natural rate of output in the context of the extended model is deferred to the next subsection.) In the case of inflation determination alone (and determination of the output *gap*) we were previously able to further reduce these to a single composite disturbance, $\hat{r}_t^n$. This is no longer possible in the case of the extended model, although, as we discuss in the next subsection, it is still possible to explain inflation determination in terms of the gap between an actual and a "natural" real rate of interest; the problem is that with endogenous variation in the capital stock, the natural rate of interest is no longer a purely exogenous state variable.

We note also that the extended model's "AS block" continues to be nearly as forward-looking as that of the baseline model. The inflation equation (5.20) can once again be "solved forward"[80] to yield a solution of the form

$$\pi_t = \sum_{j=0}^{\infty} \Psi_j E_t \hat{s}_{t+j}, \tag{5.22}$$

where the $\{\Psi_j\}$ are constant coefficients. In the case of the baseline model, the coefficients of this solution are necessarily all positive, and decay exponentially: $\Psi_j = \xi\beta^j$, for some $\xi > 0$.[81] In the extended model, the coefficients are not necessarily all positive. Nonetheless, numerical analysis suggests that for empirically realistic parameter values, one has $\Psi_j > 0$ for all small enough values of j.

This is illustrated in Figure 4.19 in the case of parameter values chosen in the following way. The values used for parameters $\alpha, \beta, \phi, \nu, \omega_p, \omega$, and θ are those given in Table 4.1, drawn from the work of Rotemberg and Woodford (1997). The value used for σ is not the same as in that study, instead, since as discussed earlier, the parameter σ of the baseline model (and similarly of the model of Rotemberg and Woodford) should not be interpreted

[80]The existence of a unique bounded solution of this form depends as usual upon the roots of a characteristic equation satisfying certain conditions, that we do not here examine further. We note however that in the numerical work presented here, we find that the relevant condition is satisfied in the case of what we judge to be empirically realistic parameter values.

[81]This follows from "solving forward" the corresponding inflation equation (xx) of chapter 3.

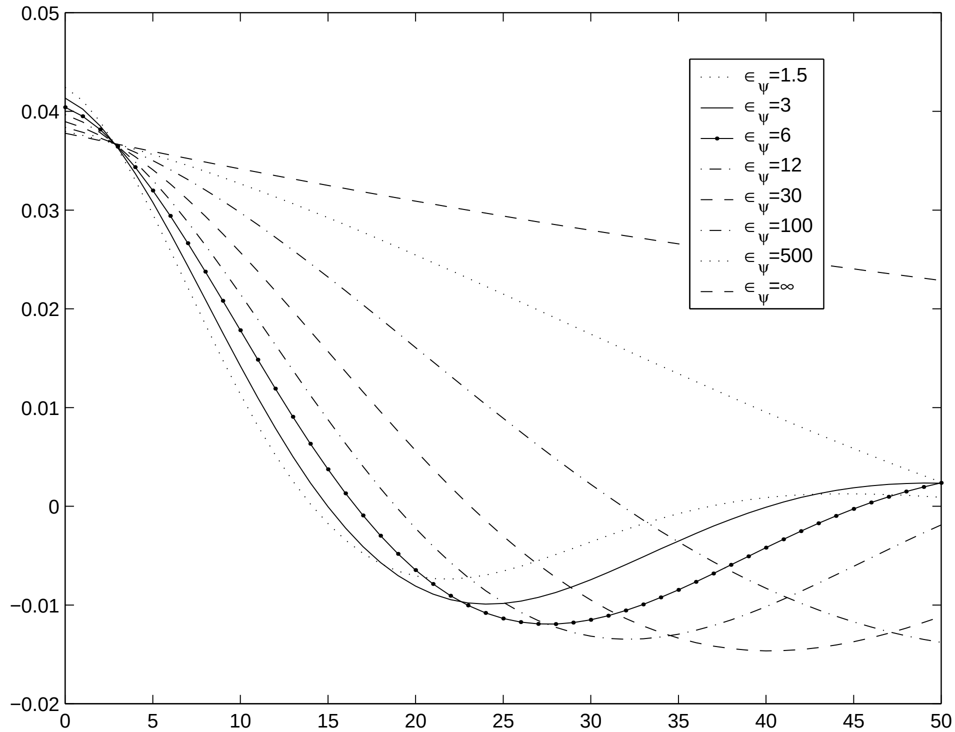

Figure 4.19: The coefficients Ψ_j in inflation equation (5.22), for alternative sizes of investment adjustment costs.

as the intertemporal elasticity of substitution of non-durable consumption expenditure — it instead indicates the substitutability of private expenditure as a whole. In the extended model, instead, σ does refer solely to the substitutability of consumption; so we now calibrate this parameter to equal 1, which is roughly the degree of substitutability typically assumed in the real business cycle literature (see, *e.g.*, Kydland and Prescott, 1982; or King, Plosser and Rebelo, 1988).[82]

We must also assign values to two new parameters, δ and ϵ_ψ, relating to the dynamics of the capital stock. We note that our model implies that the steady-state capital-output ratio $k \equiv \bar{K}/\bar{Y}$ must satisfy

$$\beta^{-1} = \frac{\theta - 1}{\theta} \frac{\phi - 1}{\phi} \frac{1}{k} + (1 - \delta).$$

[82]Strictly speaking, our calibration here is not identical to the standard RBC choice. For as noted above, our σ is actually the consumption share in output times the intertemporal elasticity of substitution of consumption, rather than the elasticity itself; thus a value of 0.7 would be closer to the standard RBC assumption. But we have no ground for choosing a precise value, and so choose 1 as a round number.

Given the values just assumed for β, θ and ϕ in our quarterly model, it follows that the model will predict an average capital-output ratio of 10 quarters (roughly correct for the U.S.) if and only if we assume a quarterly depreciation rate of $\delta = .012$ (about five percent per year). Finally, the figure compares the consequences of a range of different possible positive values for ϵ_ψ. Here the value $\epsilon_\psi = 3$ (indicated by the solid line in the figure) is the one that we regard as most empirically plausible; this results in a degree of responsiveness of overall private expenditure to monetary policy shocks that is similar to that estimated by Rotemberg and Woodford, as we shall see. But we also consider the consequences of smaller and larger values for this crucial new parameter.

In the limit of an extremely large value for ϵ_ψ, the coefficients reduce to those implied by the constant-capital model. One observes from the form given for the polynomial $Q(L)$ in (5.14) that as ϵ_ψ is made large, the two roots approach limiting values $\mu_1 \to 1$, $\mu_2 \to \beta^{-1}$. It then follows that the coefficients in (5.18) approach limiting values $a \to (1+\omega\theta)/(1-\alpha\beta)$, $b \to 0$, and hence that the coefficients in (5.20) approach limiting values

$$\xi_0 \to \xi \equiv \frac{1-\alpha}{\alpha}\frac{1-\alpha\beta}{1+\omega\theta} > 0, \qquad \xi_1 \to \beta\xi > 0,$$

$$\psi_1 \to 2\beta, \qquad \psi_2 \to \beta^2.$$

Thus in the limit, (5.20) takes the form

$$E_t[(1-\beta L^{-1})\pi_t] = \xi E_t[(1-\beta L^{-1})\hat{s}_t] + \beta E_t[(1-\beta L^{-1})\pi_{t+1}].$$

This relation has the same bounded solutions as the simpler relation

$$\pi_t = \xi\hat{s}_t + \beta E_t\pi_{t+1}$$

derived for the baseline model in chapter 3, and in particular it implies that (5.22) holds with coefficients $\Psi_j = \xi\beta^j$. (These are the coefficients indicated by the upper dashed curve in the figure.)

If ϵ_ψ is finite but still quite large, the coefficients $\{\psi_j\}$ again decay only relatively gradually as j increases, though more rapidly than would be predicted by the baseline model.

If instead ϵ_ψ takes a more moderate value (anything in the range that we could consider empirically plausible), the coefficients decline more sharply with j, and indeed become negative if horizons as long as five or six years in the future are considered. Intuitively, the expectation of a high average level of real marginal cost several years in the future is no longer a motive to increase prices now, if firms can instead plan to build up their capital stocks in the meantime. Nonetheless, higher expected future real marginal costs continue to increase inflation, as long as the expectations relate to horizons three years in the future or less. And if the expectations relate to the coming year (*i.e.,* the next four quarters), then the coefficients are not just positive but of roughly the same magnitude as in the baseline model. And it is these coefficients for low j that mainly matter, given that shocks will typically have a relatively transient effect on average real marginal costs. (With flexible prices, average real marginal costs would never vary at all; even with a realistic degree of price stickiness, price adjustment is rapid enough to make mean-reversion in the level of real marginal costs relatively rapid.)

The remaining relation in the "AS block" of the extended model is the real marginal cost relation (5.21. This relation reduces to the same one as in the baseline model if the $\hat{I}_t$ and $\hat{K}_t$ terms are omitted. The relation between real marginal costs and output is no longer as simple in the extended model, owing to the presence of those additional terms. However, insofar as cyclical variation in investment is highly correlated with cyclical variation in output, and cyclical variation in the capital stock is not too great, the implied cyclical variation in real marginal costs in the extended model is not too different. (In this case, the cyclical variation in $\hat{Y}_t - \hat{I}_t$ is highly correlated with, but smaller in amplitude than, the cyclical variation in $\hat{Y}_t$ itself. One corrects for the difference in amplitude by using a substantially larger value for σ^{-1} when calibrating the constant-capital model.) We illustrate this below when we report the impulse response of real marginal costs to a monetary policy shock, in Figure 4.21. Of course, in the limiting case of large ϵ_ψ, the equilibrium fluctuations in both $\hat{I}_t$ and $\hat{K}_t$ are negligible, and the entire "AS block" reduces to an AS relation like that of the baseline model.

The "IS block" of the extended model also retains broad similarities to that of the baseline model. It is first useful to consider the implied long-run average values for capital, output and investment as a function of the long-run average rate of inflation π_∞ implied by a given monetary policy. Equations (5.8) – (5.9) imply that the long-run average values of the various state variables must satisfy

$$\begin{aligned}\hat{\lambda}_\infty &= \rho_y \hat{Y}_\infty - \rho_k \hat{K}_\infty, \\ \hat{I}_\infty &= \delta k \hat{K}_\infty, \\ \hat{\lambda}_\infty &= -\sigma^{-1}(\hat{Y}_\infty - \hat{I}_\infty).\end{aligned}$$

These relations can be solved for $\hat{Y}_\infty, \hat{I}_\infty$ and $\hat{K}_\infty$ as multiples of $\hat{\lambda}_\infty$; this generalizes the relation between $\hat{Y}_\infty$ and $\hat{\lambda}_\infty$ obtained for the baseline model. Equation (5.11) similarly implies that the long-run average level of real marginal cost must satisfy

$$\hat{s}_\infty = \omega \hat{Y}_\infty - (\omega - \nu)\hat{K}_\infty - \hat{\lambda}_\infty;$$

substituting the above solutions, we obtain $\hat{s}_\infty$ as a multiple of $\hat{\lambda}_\infty$ as well. Finally, (5.20) implies that

$$\pi_\infty = \frac{\xi_0 - \xi_1}{1 - \psi_1 + \psi_2}\, \hat{s}_\infty.$$

Using this together with the previous solution allows us to solve for $\hat{\lambda}_\infty$, and hence for $\hat{Y}_\infty, \hat{I}_\infty$ and $\hat{K}_\infty$ as well, as multiples of π_∞.

We turn next to the characterization of transitory fluctuations around these long-run average values. Using (5.8) – (5.9) to eliminate $\hat{Y}_{t+1}$ from (5.7), we obtain a relation of the form

$$E_t[A(L)\hat{K}_{t+2}] = E_t[B(L)\hat{\lambda}_{t+1}] + z_t, \tag{5.23}$$

where $A(L)$ is a quadratic lag polynomial, $B(L)$ is linear, and z_t is a linear combination of the disturbances g_t and q_t. For empirically realistic parameter values, the polynomial $A(L)$ can be factored as $(1 - \tilde{\mu}_1 L)(1 - \tilde{\mu}_2 L)$, where the two real roots satisfy $0 < \tilde{\mu}_1 < 1 < \tilde{\mu}_2$. It follows that there is a unique bounded solution for $\hat{K}_{t+1}$ as a linear function of $\hat{K}_t$, the expectations $E_t\hat{\lambda}_{t+j}$ for $j \geq 0$, and the expectations $E_t z_{t+j}$ for $j \geq 0$. Then solving (5.10) forward to obtain

$$\hat{\lambda}_t = \hat{\lambda}_\infty + \sum_{j=0}^{\infty} E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}), \tag{5.24}$$

and using this to eliminate the expectations $E_t\hat{\lambda}_{t+j}$, we finally obtain a solution of the form

$$\hat{K}_{t+1} = (1 - \tilde{\mu}_1)\hat{K}_\infty + \tilde{\mu}_1 \hat{K}_t - \sum_{j=0}^{\infty} \tilde{\chi}_j E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}) + e_t^k, \tag{5.25}$$

where the $\{\tilde{\chi}_j\}$ are constant coefficients and e_t^k is an exogenous disturbance term (a linear combination of the $\{E_t z_{t+j}\}$). This can be solved iteratively for the dynamics of the capital stock, starting from an initial capital stock and given the evolution of the exogenous disturbances and of real interest-rate expectations.

Equation (5.24) can also be substituted into (5.9) to yield

$$\hat{Y}_t = (\hat{Y}_\infty - \hat{I}_\infty) + \hat{I}_t + g_t - \sigma \sum_{j=0}^{\infty} E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}),$$

a direct generalization of (1.4), which now however takes account of investment spending. Using (5.8) and (5.25) to substitute for $\hat{I}_t$, this expression takes the form

$$\hat{Y}_t = (\hat{Y}_\infty - \Sigma\hat{K}_\infty) + \Sigma\hat{K}_t - \sum_{j=0}^{\infty} \chi_j E_t(\hat{\imath}_{t+j} - \pi_{t+j+1}) + e_t^y, \tag{5.26}$$

where $\Sigma \equiv k[\tilde{\mu}_1 - (1-\delta)]$, $\{\chi_j\}$ is another set of constant coefficients, and e_t^y is another exogenous disturbance term (a linear combination of g_t and of the $\{E_t z_{t+j}\}$). The joint evolution of output and of the capital stock are then determined by the pair of equations (5.25) – (5.26), starting from an initial capital stock and given the evolution of the exogenous disturbances and of real interest-rate expectations.

Except for the need to jointly model the evolution of output and of the capital stock, this system of equations has implications rather similar to those of the "IS relation" (1.4) of the baseline model. In particular, for typical parameter values, the coefficients $\{\chi_j\}$ in (5.26) are all positive, and even of roughly similar magnitude for all j. For example, these coefficients are plotted in Figure 4.20 for a model that is calibrated in the same way as in

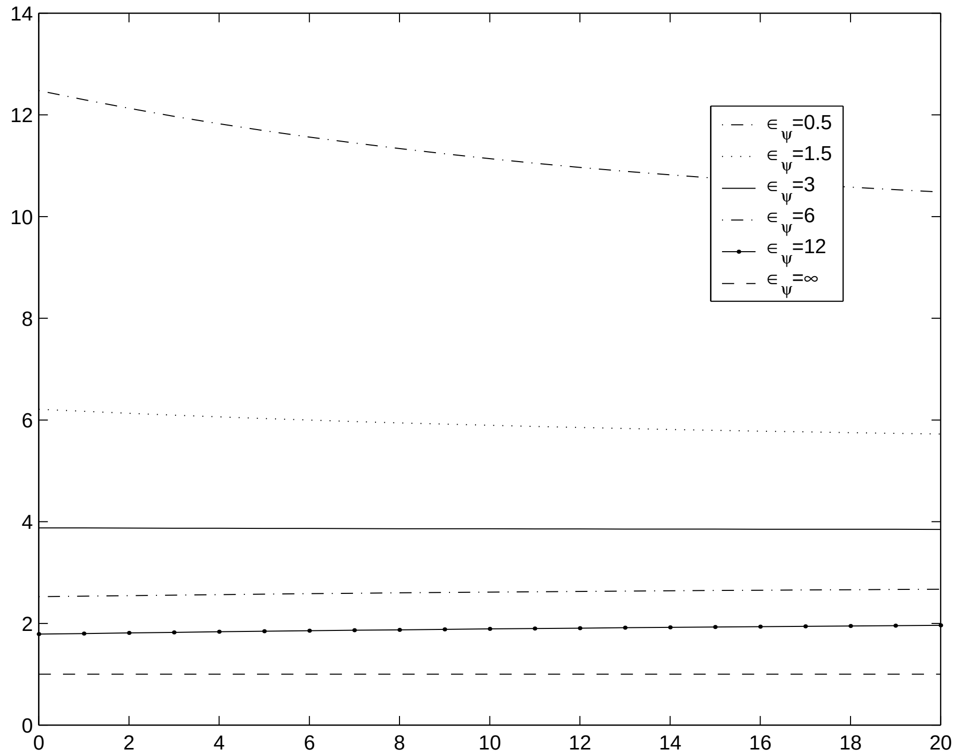

Figure 4.20: The coefficients χ_j in aggregate demand relation (5.26), for alternative sizes of investment adjustment costs.

Figure 4.19, again allowing for a range of different possible values of ϵ_ψ. In the limit of very large ϵ_ψ, the coefficients all approach the constant value σ (here assigned the value 1), as in (1.4). For most lower values of ϵ_ψ, the coefficients are not exactly equal in magnitude, and each coefficient is larger the smaller are the adjustment costs associated with investment spending. However, the coefficients all remain positive, and quite similar in magnitude to one another, especially for values of ϵ_ψ near our baseline value of 3.

We can show analytically that χ_j takes the same value for all j (though a value greater than σ) if it happens that $B(L)$ in (5.23) is of the form $-h(1-\mu_2 L)$, where $h > 0$ and μ_2 is the root greater than one in the factorization of $A(L)$. In this case (5.23) is equivalent to

$$(1-\mu_1 L)\hat{K}_{t+1} = -h\hat{\lambda}_t,$$

and substitution of (5.24) yields a solution for aggregate demand of the form (5.26) with $\chi_j = \sigma + h$ for all $j \geq 0$. For our calibrated parameter values, the root of $B(L)$ coincides with a root of $A(L)$ in this way if and only if ϵ_ψ happens to take a specific value, equal approximately to 3.23. This is in fact not an unrealistic value to assume. Perhaps more interesting, however, is the fact that the coefficients $\{\chi_j\}$ are all reasonably similar in magnitude even when ϵ_ψ is larger or smaller than the critical value.

Thus it continues to be true, as in the baseline model, that changes in interest-rate expectations (due, for example, to a shift in monetary policy) affect aggregate demand through their effect upon a *very long* real rate; the existence of endogenous variation in investment spending simply makes the degree of sensitivity of aggregate demand to the level of the very long real rate greater. For example, we see from the figure that when $\epsilon_\psi = 3$, the degree of interest-sensitivity of aggregate demand is about four times as large as if ϵ_ψ were extremely large; the response to interest-rate changes is thus roughly the same as in a constant-capital model with a value of σ near 4, rather than equal to 1 as assumed here. This justifies our use of a value of σ much larger than 1 when calibrating the baseline model.

However, even if we adjust the value assumed for σ in this way, the predictions of the constant-capital model as to the effects of real interest rate changes are not exactly the same as those of the model with variable capital. This is because lower investment spending as a result of high long real rates of interest soon results in a lower capital stock, and once this occurs aggregate demand is affected through the change in the size of the $\Sigma \hat{K}_t$ term in (5.26). In the case of sufficiently moderate adjustment costs (the empirically realistic case), the value of Σ is negative; for given real interest-rate expectations, a higher existing capital stock depresses investment demand (because returns to existing capital are low).[83] Thus a sustained increase in long real rates of interest will initially depress aggregate demand, in the variable-capital model, by more than it does later on; once the capital stock has fallen this fact helps investment demand to recover, despite the continued high real rates.

[83] For the particular parameter values discussed in the text, our baseline value $\epsilon_\psi = 3$ implies that $\Sigma = -1.246$ in our quarterly model. Note that our model does not require Σ to be negative; one can show that $\Sigma > 0$ (because $\mu_1 > 1 - \delta$) if and only if ϵ_ψ exceeds the critical value $\rho_k(1-\delta)/\delta > 0$. For our calibrated parameter values, this critical value is approximately equal to 114.5, and thus would imply a level of adjustment costs in investment that would be inconsistent with the observed degree of volatility of investment spending.

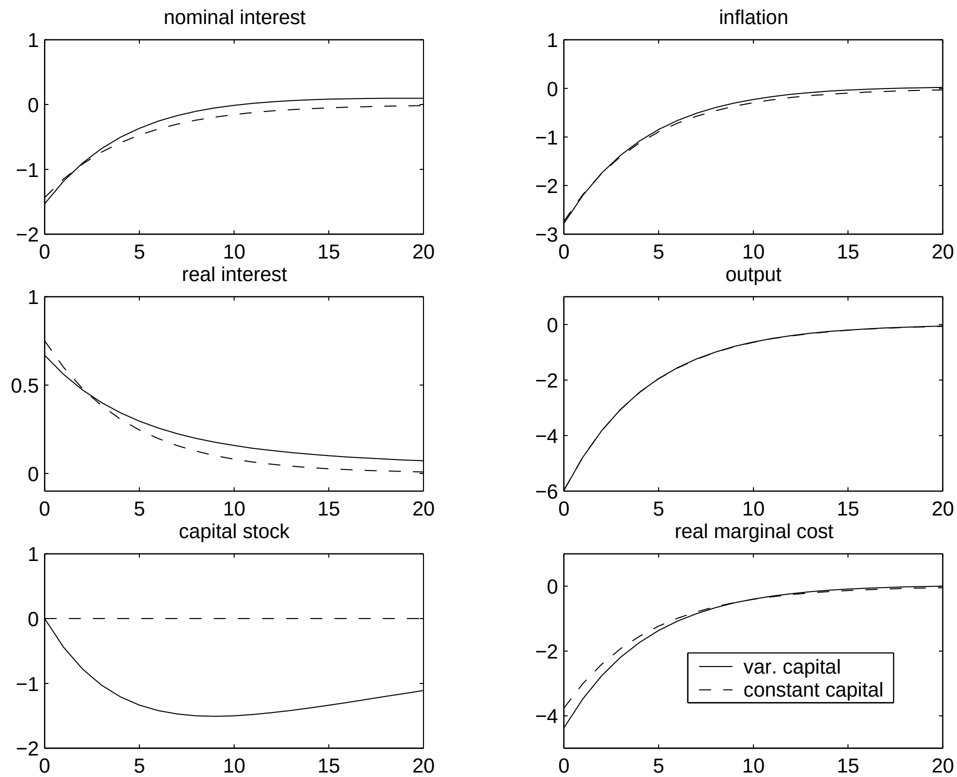

Figure 4.21: Impulse responses to an unexpected monetary tightening: the constant-capital and variable-capital models compared.

The degree to which endogenous variation in the capital stock is likely to matter in practice can be illustrated by considering the predicted effects of a monetary policy shock, in the case of a systematic monetary policy rule again given by (2.22) with coefficients $\phi_\pi = 2, \phi_x = 1$, and $\rho = 0.8$. Impulses responses to an unexpected monetary tightening (again increasing nominal interest rates by one percent per year) are plotted in Figure 4.21, which also reproduces the predictions of the baseline model corresponding to the case $\rho = 0.8$ in Figure 4.7. We observe that when we assume $\epsilon_\psi = 3$ (and all other parameter values as in Figures 4.19 and 4.20), the predicted output response in the extended model is essentially the same as in the baseline model. (It is for this reason that we choose $\epsilon_\psi = 3$ as our baseline calibration of the extended model.)

However, this does not mean that the extended model with $\epsilon_\psi = 3$ and $\sigma = 1$ makes predictions that in all respects identical to those of a constant-capital model with $\sigma = 6.37$.

For example, we see from Figure 4.20 that when $\epsilon_\psi = 3$, the coefficients $\{\chi_j\}$ of the IS relation are approximately constant, but the constant value is a bit less than 4, rather than being greater than 6 as in the baseline model. We could instead arrange for the extended model to predict the same degree of interest-sensitivity of aggregate demand as in the baseline model if we were to assume a value near $\epsilon_\psi = 1.5$. But this would then result in *overprediction* of the output contraction that should result from a monetary policy tightening. The reason has to do with the effects of endogenous variation in the capital stock, abstracted from in the baseline model. The $\Sigma\hat{K}_t$ term in (5.26) contributes a positive stimulus to output (*i.e.*, reduces the size of the output decline) several quarters after the shock, as the low capital stock induces greater investment spending than would otherwise be chosen given the higher-than-average real interest rates.[84] This means that an output response that decays at the same rate as in the constant-capital model (and as in the estimates of Rotemberg and Woodford, 1997, discussed below) requires a *more persistent* increase in real interest rates in the case of the variable-capital model. (One can see from the figure that the simulation does indeed have this property.) On the other hand, because of the very forward-looking character of the model — real interest-rate expectations several years in the future affect aggregate demand to essentially the same extent as the current short-term real rate — a more persistent increase in the short real rate will cause a larger immmediate contraction of aggregate demand, unless the size of the coefficients $\{\chi_j\}$ is reduced. This is achieved in the simulation shown in Figure 4.21 by assuming a large enough value of ϵ_ψ to reduce the interest-rate sensitivity of aggregate demand by a factor of about 40 percent, relative to the baseline model.

The fact that the two alternative models can be calibrated so as to predict very similar output and inflation dynamics in response to the same kind of monetary policy shock indicates that the predictions of the models need not be too different; indeed, their respective

[84]The effect is quite significant. For example, in the simulation shown in Figure 4.21, by the eighth quarter following the shock the positive effect of the $\Sigma\hat{K}_t$ term is 65 percent of the size of the cumulative negative effect of all of the real-interest-rate terms, so that the contraction in aggregate demand is only a bit more than a third of the size it would otherwise have.

abilities to fit empirical evidence as to the response to a particular kind of (historically typical) disturbance might well be quite similar. On the other hand, the mechanisms within the models that produce these predictions are not too closely parallel, owing to the significant effects of endogenous capital accumulation in the extended model. This means that even if the models are calibrated so as to predict similar responses to a particular kind of policy (as in Figure 4.21), it will not follow that the models calibrated in this way would also predict similar responses to all other policies that might be contemplated. Thus while a number of general lessons that may be drawn from the baseline model (*e.g.,* the degree to which aggregate demand should depend upon expected real rates years in the future) are found to be robust to an explicit consideration of endogenous capital accumulation, accurate quantitative conclusions about the nature of optimal monetary policy are likely to require explicit allowance for the dynamics of the capital stock.

5.4 Capital and the Natural Rate of Interest

We now consider the extent to which the concept of the "natural rate of interest", introduced in section 2 in connection with our baseline model, can be extended to a model which allows for endogenous variation in the capital stock. The most important difference in the case of our extended model is that the equilibrium real rate of return under flexible prices is no longer a function solely of current and expected future exogenous disturbances; it depends upon the capital stock as well, which is now an endogenous state variable (and so a function of past monetary policy, among other things, when prices are sticky). Hence if we continue to define the natural rate of interest in this way, it ceases to refer to an exogenous process.

An alternative possibility (pursued in Neiss and Nelson, 2000) would be to define the "natural rate of interest" as what the equilibrium real rate of return would be if prices not only were currently flexible and were expected always to be in the future, but also *had always been in the past* — so that what matters for the computation is not the capital stock that actually exists, but the one that *would* exist under this counterfactual, given the actual history of exogenous real disturbances. Under that definition the natural rate of interest would be exogenous, but at the cost of less connection with equilibrium determination in the actual (sticky-price) economy. It seems odd to define the economy's "natural" level of activity, and correspondingly the associated "natural" level of interest rates, in a way that makes irrelevant the capital stock that actually exists and the effects of this upon the economy's productive capacity. And a clear cost of the alternative definition would be less connection between this concept of "natural" output and the efficient level of output, which clearly depends on the actual capital stock. For this reason, we instead continue to define the "natural rates" of output and interest as those that would result from price flexibility now and in the future, *given* all exogenous and predetermined state variables, including the economy's capital stock. (The connection between this definition and a welfare-based policy objective is taken up in chapter 6.)

Since the equilibrium with flexible prices at any date t depends only on the capital stock at that date[85] and current and expected future exogenous real disturbances, we can write a log-linear approximation to the solution in the form

$$\hat{Y}_t^n = \hat{Y}_t^{ncc} + \eta_y \hat{K}_t,$$

$$\hat{r}_t^n = \hat{r}_t^{ncc} + \eta_r \hat{K}_t,$$

and so on, where the terms $\hat{Y}_t^{ncc}$ and $\hat{r}_t^{ncc}$ refer to exogenous processes (functions solely of the exogenous real disturbances). These "intercept" terms in each expression indicate what the level of real output (or the real interest rate, and so on) *would* be, given current and expected future real disturbances, if prices were flexible *and* the capital stock did not differ from its steady-state level; we shall call this the *constant-capital natural rate* of output (or of interest, and so on). We shall also find it useful to define a "natural rate" of investment $\hat{I}_t^n$ and of the marginal utility of income $\hat{\lambda}_t^n$ in a similar way. We can even define a "natural" capital stock $\hat{K}_{t+1}^n$, as what the capital stock in period $t+1$ would be if it had been chosen in a flexible-price equilibrium in period t, as a function of the actually existing capital stock

[85] This is true up to the log-linear approximation that we use here to characterize equilibrium. More precisely, it would depend on the capital stock in place in each of the firms producing differentiated goods.

$\hat{K}_t$ and the exogenous disturbances at that time; thus we similarly write

$$\hat{K}^n_{t+1} = \hat{K}^{ncc}_{t=1} + \eta_k \hat{K}_t.$$

Finally, we shall use tildes to indicate the "gaps" between the actual and "natural" values of these several variables: $\tilde{Y}_t \equiv \hat{Y}_t - \hat{Y}^n_t$, $\tilde{r}_t \equiv \hat{r}_t - \hat{r}^n_t$, and so on.

Once again, in a flexible-price equilibrium, real marginal cost must at all times be equal to a constant, $(\theta - 1)/\theta$. It then follows from (5.21) that fluctuations in the natural rate of output satisfy

$$\hat{Y}^n_t = \frac{\omega - \nu}{\omega + \sigma^{-1}} \hat{K}_t + \frac{\sigma^{-1}}{\omega + \sigma^{-1}} \hat{I}_t + \frac{\sigma^{-1}}{\omega + \sigma^{-1}} g_t + \frac{\omega}{\omega + \sigma^{-1}} q_t.$$

This relation generalizes equation (2.2) for the baseline model. Note that this does not allow us to solve for the natural rate of output as a function of the capital stock and the real disturbances without also simultaneously solving for the natural rate of investment. However, comparison with (5.21) allows us to derive an expression for real marginal cost in terms of the "gaps",

$$\hat{s}_t = (\omega + \sigma^{-1})\tilde{Y}_t - \sigma^{-1}\tilde{I}_t, \tag{5.27}$$

generalizing equation (xx) of chapter 3.

Because condition (5.7) must also hold in a flexible-price equilibrium, we observe that

$$\hat{\lambda}^n_t + \epsilon_\psi(\hat{K}^n_{t+1} - \hat{K}_t) \quad = \quad \beta(1-\delta)[E_t\hat{\lambda}_{t+1} - \eta_\lambda \tilde{K}_{t+1}] \ +$$

$$[1-\beta(1-\delta)][\rho_y(E_t\hat{Y}^n_{t+1} - \eta_y\tilde{K}_{t+1}) - \rho_k\hat{K}^n_{t+1} - \omega q_t] + \beta\epsilon_\psi[(E_t\hat{K}^n_{t+2} - \eta_k\tilde{K}_{t+1}) - \hat{K}^n_{t+1}].$$

Here we use the fact that in a flexible-price equilibrium, the conditional expectation at t of period $t+1$ output does not correspond to the value of $E_t Y^n_{t+1}$ at date t in the sticky-price equilibrium, for the latter depends upon the value of $\hat{K}_{t+1}$ *in the sticky-price equilibrium,* which is generally not the same as what the period $t+1$ capital stock would be *in a flexible-price equilibrium.* Thus the conditional expectation at t of period $t+1$ output in a flexible-price equilibrium (beginning at t and conditional upon the actual period t capital stock) is actually equal to $E_t\hat{Y}^n_{t+1} - \eta_y\tilde{K}_{t+1}$; and similarly for the expectation at t of other variables determined at $t+1$. Comparing this with (5.7), we see that in the sticky-price equilibrium, the "gap" variables must satisfy

$$\begin{aligned}\tilde{\lambda}_t + \epsilon_\psi \tilde{K}_{t+1} &= \beta(1-\delta)[E_t\tilde{\lambda}_{t+1} + \eta_\lambda \tilde{K}_{t+1}] \;+ \\ &[1-\beta(1-\delta)][\rho_y(E_t\tilde{Y}_{t+1} + \eta_y\tilde{K}_{t+1}) - \rho_k\tilde{K}_{t+1}] + \beta\epsilon_\psi[(E_t\tilde{K}_{t+2} + \eta_k\tilde{K}_{t+1}) - \tilde{K}_{t+1}]. \qquad (5.28)\end{aligned}$$

This equation is similar in form to (5.7), *except* that it is purely forward-looking: it determines the equilibrium size of the gap $\tilde{K}_{t+1}$ without any reference to predetermined state variables such as $\tilde{K}_t$.

Equations (5.8) – (5.10) similarly must hold in a flexible-price equilibrium, implying that the "gaps" must also satisfy equations

$$\tilde{I}_t = k\tilde{K}_{t+1}, \qquad (5.29)$$

$$\tilde{\lambda}_t = -\sigma^{-1}(\tilde{Y}_t - \tilde{I}_t), \qquad (5.30)$$

$$\tilde{r}_t = \tilde{\lambda}_t - (E_t\tilde{\lambda}_{t+1} + \eta_\lambda\tilde{K}_{t+1}). \qquad (5.31)$$

Using equations (5.29) and (5.30) to eliminate $\tilde{\lambda}_t$ and $\tilde{K}_{t+1}$ from (5.28) and (5.31), we are left with a system of two equations that can be written in the form

$$E_t z_{t+1} = A z_t + a\tilde{r}_t, \qquad (5.32)$$

for a certain matrix A and vector a of coefficients, where now

$$z_t \equiv \begin{bmatrix} \tilde{Y}_t \\ \tilde{I}_t \end{bmatrix}.$$

This pair of coupled difference equations generalizes the "gap" version (1.8) of the IS relation of the baseline model.[86]

Let us now close our system by specifying monetary policy in terms of an interest-rate feedback rule of the form

$$\hat{\imath}_t = \bar{\imath}_t + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4, \qquad (5.33)$$

[86]Note, however, that it is typically *not* possible to solve this system "forward" to obtain a solution for the output gap as a function of current and expected future interest-rate gaps, as in equation (2.19) for the baseline model. [ADD MORE]

where we re-introduce the notation $x_t \equiv \tilde{Y}_t$ for the output gap. (Once again, $\bar{x}$ is the steady-state output gap corresponding to the steady-state inflation rate $\bar{\pi}$.) With policy specified by a "Taylor rule" of this kind, the interest-rate gap will be given by

$$\tilde{r}_t = (\bar{\imath}_t - \hat{r}_t^n - \bar{\pi}) - E_t(\pi_{t+1} - \bar{\pi}) + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4. \tag{5.34}$$

Note that in this last relation, the only endogenous variables are "gap" variables *if* we make the further assumption that

$$\bar{\imath}_t = \bar{\imath}_t^{cc} + \eta_r \hat{K}_t, \tag{5.35}$$

where $\bar{\imath}_t^{cc}\}$ is an exogenous process. This implies that in addition to systematic responses to endogenous variation in inflation and in the output gap, the policy rule (5.33) also involves a systematic response to endogenous variation in the capital stock, of a specific sort: the central bank's interest rate operating target is adjusted to exactly the same extent as the natural rate of interest is changed by the variation in the capital stock. This is obviously a special case, but not an entirely implausible one, if we posit a desire to stabilize inflation, and hence to arrange for interest rates to vary one-for-one with variation in the natural rate of interest. For of all the possible sources of variation in the natural rate of interest, variations due to changes in the economy's aggregate capital ought to be the easiest for a central bank to track with some accuracy (owing to the slowness of movements in the capital stock).

A complete system of equilibrium conditions for the determination of the variables $\{\tilde{Y}_t, \tilde{I}_t, \tilde{r}_t, \hat{s}_t, \pi_t\}$ is then given by (5.20), (5.27), (5.32), and (5.34). The system of equations may furthermore be written in the form

$$E_t \hat{z}_{t+1} = \hat{A}\hat{z}_t + \hat{a}(\hat{r}_t^n - \bar{\imath}_t + \bar{\pi}), \tag{5.36}$$

where now

$$\hat{z}_t \equiv \begin{bmatrix} \tilde{Y}_t - \bar{x} \\ \tilde{I}_t - \bar{I} \\ E_t\pi_{t+1} - \bar{\pi} \\ \pi_t - \bar{\pi} \end{bmatrix},$$

$\bar{I}$ is the steady-state value of $\tilde{I}_t$ corresponding to steady inflation at the rate $\bar{\pi}$, and $\hat{A}$ and $\hat{a}$ are again a matrix and vector of coefficients. We obtain this system as follows. The first two rows are obtained by substituting for $\tilde{r}_t$ in (5.32) using (5.34).[87] The third row is obtained by solving (5.20) for $E_t\pi_{t+2}$, and then substituting for $\hat{s}_t$ and $E_t\hat{s}_{t+1}$ using (5.27); one finally substitutes for $E_t\tilde{Y}_{t+1}$ and $E_t\tilde{I}_{t+1}$ using the first two rows of (5.36), just derived. The fourth row is simply an identity.

Because the system (5.36) is purely forward-looking (*i.e.*, there are no predetermined endogenous state variables), a policy rule of the kind defined by (5.33) and (5.35) then results in determinate equilibrium dynamics for inflation and the output gap (among other variables) if and only if the matrix $\hat{A}$ has all four eigenvalues outside the unit circle. When this is true, the system can be "solved forward" in the usual way to obtain a unique bounded solution. The solutions for inflation and the output gap will once again be of the form (2.18) – (2.19), and the implied solution for the nominal interest rate will correspondingly again be of the form (2.20), just as in the baseline model, though the numerical values of the coefficients $\{\psi_j^\pi, \psi_j^x, \psi_j^i\}$ in these expressions will be different. Figure 4.22 plots coefficients $\{\psi_j^\pi, \psi_j^x\}$ for $j = 0$ through 10, in the case of model parameters chosen as in the earlier figures, and a policy rule of the form defined by (5.33) and (5.35), with feedback coefficients $\phi_\pi = 2$, $\phi_x = 1$. Here the solid line indicates the coefficients in the case of the variable-capital model, and the dashed line the coefficients in the case of the baseline model. (The dashed line here corresponds to the baseline case shown in Figures 4 and 5 earlier.)

Once again one finds that, at least in the case of changes in the expected gap $E_t(\hat{r}^n_{t+j} - \bar{\imath}_{t+j})$ only a few quarters in the future, increases in the expected gap increase both inflation and the output gap; and once again, this is true even many quarters in the future in the case of inflation, whereas expected gaps $E_t(\hat{r}^n_{t+j} - \bar{\imath}_{t+j})$ more than a few quarters in the future have little effect upon the output gap. Thus fluctuations in inflation and the output gap can still be explained in essentially the same way as in the constant-capital model. Once again, inflation and positive output gaps result from increases in the natural rate of output that are not fully matched by a tightening of monetary policy, or by loosenings of monetary

[87] Note that all of these equations continue to be valid when we replace variables by the difference of those variables from their steady-state values. We choose to express the equations in this form in (5.36) because the policy rule (5.33) has already been expressed in this form.

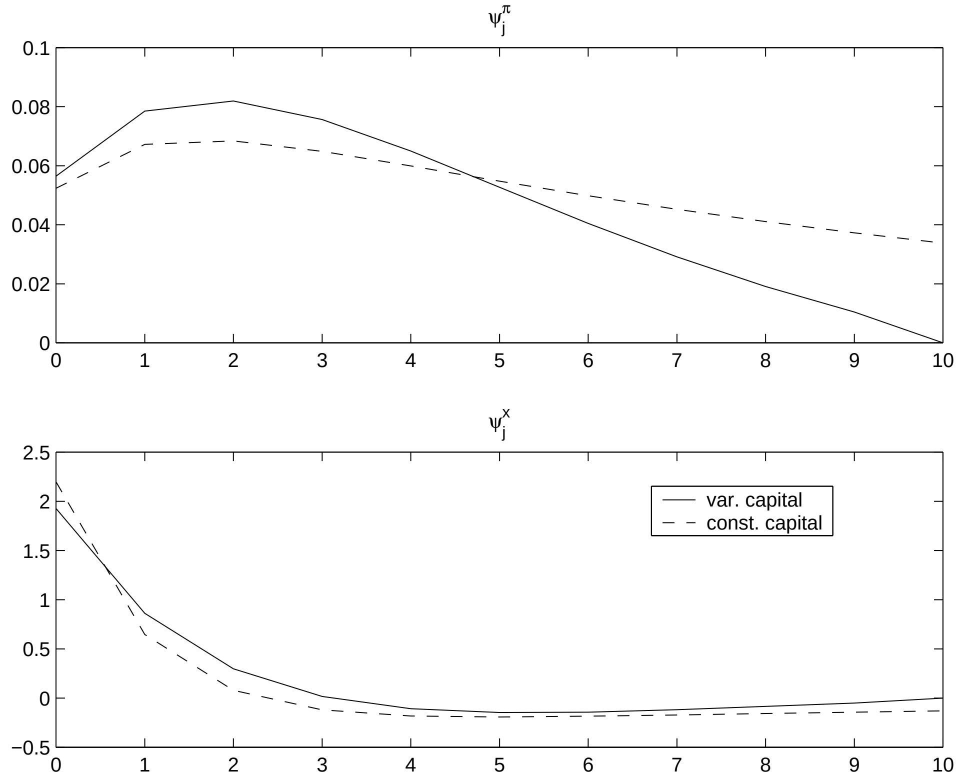

Figure 4.22: Coefficients of the forward-looking solution for inflation and the output gap.

policy not justified by any decline in the natural rate of interest.

An immediate consequence is that once again a possible approach to the goal of inflation stabilization is to commit to a policy rule of the form (5.33) such that (i) the coefficients ϕ_π, ϕ_x are chosen so as to imply a determinate equilibrium, and (ii) the intercept adjustments track variations in the natural rate of interest as well as possible, *i.e.*, the central bank seeks to set $\bar{\imath}_t = \hat{r}_t^n + \bar{\pi}$ at all times. (Note that this is an example of a rule of the form (5.35), with $\bar{\imath}_t^{cc} = \hat{r}_t^{ncc}$.) If it is possible to satisfy this condition with sufficient accuracy, then inflation can in principle be completely stabilized with finite response coefficients. Thus the requirement of tracking variations in the natural rate of interest continues to be as important to the pursuit of price stability as in our analysis of the baseline model.

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Interest and Prices

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Contents

Chapter 6

Inflation Stabilization and Welfare

We turn now to the evaluation of alternative monetary policy rules, using the analytical framework developed in the previous chapters. A first question, of course, is what the goals of monetary policy should be. Most discussions give primary, if not exclusive, attention to two goals in terms of which alternative policies should be evaluated. The first, which has attained particular prominence in recent discussion, is the goal of maintaining a low and stable rate of inflation. It is sometimes argued that this should be the exclusive goal of monetary policy, and various recent developments – the explicit responsibility given to several central banks in the 1990's for achievement of inflation targets, or the exclusive concern with price stability that is specified in the Maastricht treaty as the goal of the European Central Bank – indicate that this view has become influential among policymakers.[1] At the same time, most discussions of actual monetary policy, even now, and even in countries with "inflation targeting" central banks, assume at least some degree of concern with the stabilization of economic activity as well; in the early literature (*e.g.*, Poole, 1970), this was often treated as the primary goal of monetary stabilization policy.

There is thus a fair amount of consensus in the academic literature that a desirable monetary policy is one that achieves a low expected value of a discounted loss function, where the losses each period are a weighted average of terms quadratic in the deviation of inflation from a target rate and in some measure of output relative to potential. But

[1] It may, however, be questioned whether the "inflation targeting" central banks should be understood to care solely about inflation stabilization. See, *e.g.*, Svensson (1999a).

even agreement upon this general form of the objective still allows considerable scope for disagreement about details, that may well matter a great deal for the design of an optimal policy. First of all, obviously, there is the question of the relative weight to be placed upon inflation stabilization and output stabilization. But this is hardly the only ambiguity in the conventional prescription. For instance, which kind of output measure should be stabilized? In particular, should one seek to stabilize output relative to a concept of "potential output" that varies in response to real disturbances that shift the short-run aggregate supply curve, or should one seek to stabilize output relative to a smooth trend?[2]

Similarly, in which sense should price stability be pursued? Should one seek to stabilize deviations of the *price level* from a deterministic target path (as proposed, for example, by Hall and Mankiw, 1994), so that unexpected inflation in excess of one's target rate should subsequently be deliberately counteracted, in order to bring the price level back to its target path? Or should one seek to stabilize deviations of the *inflation rate* from its target level (as assumed, for example, by Svensson, 1997), so that – assuming that the variance of the unforecastable component of inflation cannot be reduced by policy – one should *not* seek to counteract past inflation fluctuations, in order to minimize variation in the forecastable component of inflation? Should greater priority perhaps be given to reducing the variability of *unforecastable* inflation, on the ground that this is what causes unexpected modifications of the real consequences of pre-existing nominal contracts, while forecastable variations in inflation can simply be incorporated into contracts? Or should greater priority be given to stabilization of *forecastable* inflation, on the ground that expected inflation distorts incentives (like an anticipated tax), while unforecastable inflation has no incentive effects (like an unanticipated wealth levy)?

The aim of the present chapter is to show how economic analysis can be brought to bear upon these questions. An important advantage of using a model founded upon private-sector optimization to analyze the consequences of alternative policy rules is that there is a natural

[2]Different answers to this question lead Bean (1983) and West (1986) to reach diametrically opposite conclusions about the case in which nominal GDP targeting would be preferable to money-supply targeting.

welfare criterion in the context of such a model, provided by the preferences of private agents that are displayed in the structural relations that determine the effects of alternative policies. Such a utility-based approach to welfare analysis has long been standard in the theory of public finance. It is not yet too common in analyses of monetary policy, perhaps because it is believed that the main concerns of monetary stabilization policy are assumed away in models with explicit micro-foundations. But we have seen that models founded on individual optimization can be constructed that, thanks to the presence of nominal rigidities, allow for realistic effects of monetary policy upon real variables. Here we shall see those same nominal rigidities provide welfare-economic justification for central bankers' traditional concern for price stability.

Individuals are not assumed, of course, to care directly about prices; their economic welfare depends directly only upon the goods they consume and the amount of effort they expend upon production. But just as taxes can cause deadweight losses because of their effects upon the equilibrium allocation of resources, so can inflation. In a model with nominal rigidities – more specifically, in one in which it is recognized that prices are not adjusted in perfect synchronization with one another (which requires, but is stronger than, the observation that they are not all adjusted continually) – instability of the general price level leads to unnecessary and undesired variation in the relative prices of goods whose prices are adjusted at different times. These relative price distortions result in deadweight losses, just as in the case of distorting taxes. We shall see that this effect can justify not only a loss function that penalizes inflation variations, but indeed – if one assumes parameter values implied by the apparent degree of nominal rigidity in actual economies – a much larger relative weight on inflation variation than upon output variation than is assumed in the loss functions used in many monetary policy evaluation exercises.

Derivation of a utility-based welfare criterion in this way can not only allow us to justify a general concern with price stability, but can furthermore provide exact answers to the questions raised above about the precise formulation of the appropriate loss function. These answers, of course, depend upon the assumptions we make about the structure of the economy; for example, they depend crucially upon the nature of the nominal rigidities that are present. Insofar as the correct structural relations of our model of the economy remain controversial, the proper welfare criterion to use in evaluating policy will remain controversial as well; and our goal here is more to illustrate a method than to reach final conclusions. But insofar as particular parameter values are found to be empirically justified in that they are required in order for our structural equations to fit historical data, they will contain important information about the proper welfare criterion as well.

1 Approximation of Loss Functions and of Optimal Policies

The method that we shall employ in the analysis below derives a quadratic loss function, that represents a quadratic (second-order Taylor series) approximation to the level of expected utility of the representative household in the rational expectations equilibrum associated with a given policy. There are several reasons for our resort to this approximation. One is simply mathematical convenience; with a quadratic approximation to our objective function and linear approximations to our structural equations, we can address the nature of optimal policy within a linear-quadratic optimal control framework that has been extensively studied, and numerical computation of optimal policy is relatively simple. This convenience is especially great when we turn to questions such as the optimal use of indicator variables under circumstances of partial information.

But there are other advantages as well. One is comparability of our results with those of the traditional literature on monetary policy evaluation, which almost always assumes a quadratic loss function of one sort or another. Casting our own results in this familiar form allows us to discuss similarities and differences between our utility-based welfare criterion and those assumed in other studies without letting matters be obscured by superficial differences in functional form that may have relatively little consequence for the results obtained.

And finally, it does not make sense to be concerned with a higher-order approximation to our welfare criterion if we do not plan to characterize the effects of alternative policies with a degree of precision sufficient to allow computation of those higher-order terms. In the first part of this study, we have shown how to derive a log-linear approximation to the equilibrium fluctuations in inflation and output under alternative policies, using a log-linear approximation to the exact structural equations of our model. Using this method, we compute the equilibrium fluctuations in these variables only up to a residual of order $\mathcal{O}(||\xi||^2)$, where $||\xi||$ is a bound on the amplitude of the exogenous disturbances. Given that we do not compute the terms of second order in $||\xi||$ in characterizing equilibrium fluctuations, we cannot expect to compute the terms of third or higher order in $||\xi||$ in evaluating the expected utility of the representative household.[3] Of course, one might also wish to undertake a more accurate approximation of the predicted evolution of the endogenous variables under alternative regimes. However, such a study would introduce a large number of additional free parameters, to which numerical values would have to be assigned for purposes of computation; and there is likely to be little empirical basis for the assignment of such values in most cases, given the degree to which the empirical study of macroeconomic time series makes use of linear models.[4]

However, even a second-order approximation to utility can be computed on the basis of a merely linear approximation to our model structural equation only under special circumstances. We shall assume that these hold in our calculations here, but it is important to be clear about the scope of validity of our results. Let x represent a vector of endogenous variables, and suppose that we wish to evaluate $E[U(x;\xi)]$ under alternative policies, where ξ is a vector of random exogenous disturbances, and $U(\cdot;\xi)$ is a concave function for

[3] There is thus no obvious advantage to the approach sometimes adopted in utility-based welfare analyses, such as Ireland (1997) or Collard *et al.* (1998), of evaluating an exact utility function but using a log-linear approximation to the model's structural equations in order to compute the equilibrium.

[4] A common approach in the quantitative equilibrium business cycle literature, of course, is to assume special functional forms for preferences and technology that allow the higher derivatives of these functions to be inferred from the same small number of parameters as determine the lower-order derivatives, which may then be inferred from first and second moments of the time series alone. This approach often obscures the relation between the properties of the time series and the model parameters that are identified by them, and allows "identifications" that are in fact quite sensitive to the arbitrary functional form assumption. We prefer instead to assume special functional forms as little as possible, but to be clear about the order of approximation that is involved in our calculations.

each possible realization of ξ, and at least twice differentiable. Now suppose that we are able to compute a linear (or log-linear) approximation to the equilibrium responses of the endogenous variables, under a given policy regime, of the form

$$x = x^0 + a'\xi + \mathcal{O}(||\xi||^2), \tag{1.1}$$

where the vectors of coefficients x^{ss} and a may both depend upon policy. (This represents a first-order Taylor series approximation to the exact equilibrium responses $x(\xi)$, assumed to be nonlinear but differentiable, taken around the mean values $\xi = 0$. The conditions under which the solution to the linearized structural equations yield a valid approximation of this kind to the solution to the exact structural equations are discussed in Appendix A.)

Under the assumption that the constant term x^0 in (1.1) is itself of at most order $\mathcal{O}(||\xi||)$, we can take a similar Taylor series expansion of the utility function $U(x)$, and be confident that terms that are of at most order $\mathcal{O}(||x||^3)$ are of at most order $\mathcal{O}(||\xi||^3)$. We then can write

$$U(x;\xi) = \bar{U} + U_x\tilde{x} + U_\xi\xi + \frac{1}{2}\tilde{x}'U_{xx}\tilde{x} + \tilde{x}'U_{x\xi}\xi + \frac{1}{2}\xi'U_{\xi\xi}\xi + \mathcal{O}(||\xi||^3), \tag{1.2}$$

where $\bar{U} \equiv U(\bar{x};0)$, $\tilde{x} \equiv x - \bar{x}$, and all partial derivatives of U are evaluated at $(\bar{x};0)$. We wish to use approximation (1.1) to the equilibrium fluctuations in x to compute the terms of second or lower order in approximation (1.2) to utility. But the term $U_x\tilde{x}$ in (1.2) will generally contain terms of second order that depend upon the neglected second order terms in (1.1). In order to be able to neglect these terms, we must also assume that $U_x(\bar{x};0)$ is at most of order $\mathcal{O}(||\xi||)$. In that case, the neglected terms of order $\mathcal{O}(||\xi||^2)$ contribute only to terms of order $\mathcal{O}(||\xi||^3)$ in $U_x\tilde{x}$.

Assuming this, substitution of (1.1) minus the residual into (1.2) minus the residual yields a correct quadratic approximation to $U(x;\xi)$. Taking the expected value of this expression, and using the fact that we normalize ξ so that $E(\xi) = 0$, we obtain the approximate welfare criterion

$$E[U] = \bar{U} + U_xE[\tilde{x}] + \frac{1}{2}\mathrm{tr}\{U_{xx}\mathrm{var}[x]\} + \mathrm{tr}\{U_{x\xi}\mathrm{cov}(\xi,\tilde{x})\} + \frac{1}{2}\mathrm{tr}\{U_{\xi\xi}\mathrm{var}[\xi]\} + \mathcal{O}(||\xi||^3). \tag{1.3}$$

Here we use the notation $E[z]$ for the expectation of a random vector z, var$[z]$ denotes the variance-covariance matrix, and cov(z_1, z_2) the matrix of covariances between two random vectors z_1, z_2. In expression (1.3) it is understood that the various first and second moments are those that one computes using the linear approximation (1.1).

The validity of this last expression, when the first and second moments are computed using (1.1), thus depends upon two special assumptions. These are that x^0 is only of order $\mathcal{O}(||\xi||)$, and that $U_x(\bar{x}; 0)$ is similarly only of order $\mathcal{O}(||\xi||)$. Technically, we shall suppose that $||\xi||$ is a bound *both* upon the amplitude of the exogenous disturbances, and upon the size of x^0 and $U_x(\bar{x}; 0)$. Our approximation result then refers to a sequence of economies in which $||\xi||$ eventually becomes arbitrarily small; as we progress along this sequence, both the distribution of the disturbances, and certain other parameters of the model that determine x^0 and U_x, are varied so as to respect the changing bound, while keeping the specification of the policy rule the same. What the Taylor theorem guarantees is then that if (1.3)minus the residual yields a higher value for one policy than for another, it will be true for all economies far enough out in this sequence that the first policy yields higher expected utility than the other in the equilibrium of the exact model.

The stipulation regarding x^0 is an assumption about the kind of policy regime which we seek to evaluate, while the stipulation regarding $U_x(\bar{x}; 0)$ is an assumption about the point around which we choose to compute the Taylor expansion in in (1.3). The latter assumption implies that we expand around a state of affairs $\bar{x}$ that is close to being *optimal,* not simply in the sense of being the best we can do using the set of policies under consideration, but in the sense of being near the maximum of $U(x; 0)$ over all possible values of x.[5] Of course, it is really only necessary that U_x be small in directions in which it is possible for the average value of x to differ under alternative policies. Thus it is not necessary for households to

[5]One way to guarantee this would be to stipulate that $\bar{x}$ is in fact the value that maximizes $U(x; 0)$. We do not wish, however, to insist upon this. In some cases, those "first-best" values do not correspond to a possible equilibrium, even in the absence of disturbances. We could linearize the model's structural equations around such values nonetheless, but we prefer to follow convention in linearizing around values that represent a particular equilibrium in the absence of shocks. One advantage of this convention is that our linearized structural equations always have zero constant terms.

be nearly satiated both in consumption and in leisure in order for $\bar{x}$ to be optimal in the necessary sense; it is enough that it not be possible to greatly increase utility by varying both consumption and work effort in a way that is feasible given the economy's production function. But it is somewhat delicate to draw conclusions about the directions in which it is possible for policy to vary the second-order terms in x without actually computing the second-order terms in (1.1), and so we prefer to substitute constraints of this kind into our definition of the objective function $U(x)$, and then require all elements of the vector U_x to be small.

Kim and Kim (1998) provide an example of a problem, relating to the welfare gains from risk-sharing, where this requirement for validity of a welfare calculation based upon the linear approximation (1.1) is not satisfied. They consider the expected utility $E[U(C_i)]$ obtained by a household i in each of two cases. In the first case (autarchy), each household consumes its own random income Y_i, while in the second case (perfect risk-sharing), two households pool their incomes, so that $C_i = (Y_1 + Y_2)/2$ for each. Kim and Kim consider the validity of a log-linear approximation to the relation between consumption and income. In the case of autarchy, the log-linear relation (which is exact) is given by

$$\hat{C}_i = \hat{Y}_i,$$

where as usual hatted variables denote deviations of the logs from the value $\log \bar{Y}$ around which one log-linearizes. In the case of perfect risk-sharing, the log-linear approximation is instead

$$\hat{C}_i = (\hat{Y}_1 + \hat{Y}_2)/2.$$

Substitution of these two log-linear expressions into a quadratic approximation to the utility function,

$$U(C_i) = U(\bar{Y}) + \bar{Y}U'(\bar{Y})\left[\hat{C}_i + \frac{1-\gamma}{2}\hat{C}_i^2\right],$$

where $\gamma > 0$ is the coefficient of relative risk aversion (evaluated at consumption level $\bar{Y}$), does not yield a correct quadratic approximation to utility as a function of $\hat{Y}_1$ and $\hat{Y}_2$. Indeed, if $0 < \gamma < 1$, this approximation implies that expected utility is higher under autarchy.

The problem is that the partial derivative of U with respect to $\log C_i$ (which is equal to $\bar{Y}U'(\bar{Y})$) is non-zero, so that the correct quadratic approximation to expected utility in the case of risk-sharing involves quadratic terms in the Taylor series expansion for $\hat{C}_i$. (The omitted terms raise expected utility in that case, since less variable consumption means a higher expected value for log consumption, by Jensen's inequality.) It does not make sense to assume that this derivative can be made arbitrarily close to zero as we make the bound $||\xi||$ on the amplitude of income variations smaller, either. This could be done only by varying preferences and/or average income as we make $||\xi||$ smaller in such a way that proportional variations in consumption cease to matter much; but this would mean that in the limit, no comparisons between alternative consumption processes would be possible. This difficulty does not arise in the case of our analysis of the welfare gains from macroeconomic stabilization, below, as long as we log-linearize around a level of economic activity (in each sector) that is sufficiently near to being efficient; since an interior optimum does exist in our case, this is possible. But it is important that we check that the derivatives in question are indeed small, under our assumptions, and that the qualifications that this requires to our results be noted.

The assumption that x^0 is small means that the policies considered are all ones with the property that in the absence of shocks, the equilibrium value of x would be near the linearization point $\bar{x}$ — or alternatively, that in equilibrium the *mean* value of x is near $\bar{x}$. Given our assumption about $\bar{x}$, this means that the policies considered are ones under which the equilibrium value x is nearly *optimal*, in the sense discussed above. As we are primarily interested in whether our approximate welfare criterion correctly identifies the optimal policy, the essential requirement is that our model (and the family of available policies) be such that the best available policy can achieve an outcome that is sufficiently close to being fully optimal. Thus the *unavoidable* frictions – the ones that cannot be ameliorated through an appropriate choice of policy – must be small, even if there exist frictions that imply that outcomes under a *bad* policy could be significantly worse.

In the case of our baseline model, the only frictions that prevent equilibrium from being efficient are (i) the market power possessed by suppliers of goods, as a result of monopolistic competition, and (ii) the failure to adjust all goods prices each period. We log-linearize our structural equations around the steady state with zero inflation each period, that represents a possible equilibrium in the absence of real disturbances. In this equilibrium, the failure to adjust prices constantly results in no distortion of the allocation of resources; this allocation is thus nearly optimal as long as the distortion due to market power is sufficiently small. In our calculations below, we assume that it is.

The other assumption required for the validity of the quadratic approximation obtained from our log-linear structural equations is that the policy rules considered be ones under which the equilibrium rate of inflation in the absence of shocks would in fact be near zero – or alternatively, that these policies be ones under which the average rate of inflation is low. This also is assumed; since we conclude that it is optimal for a country with characteristics like those of the U.S. to choose a policy under which the average rate of inflation is slightly, but only slightly, positive, this last assumption is relatively innocuous. However, it is important to realize that under other circumstances – say, an analysis of optimal monetary policy in the presence of a need for significant seignorage revenues – this assumption as well might be inappropriate.

Finally, it is important to note that the conditions required for validity of a quadratic approximation to welfare obtained from log-linear approximations to the structural equations do not relate solely to the structure of the economy; it also matters in which form we choose to express our approximate loss function. Alternative quadratic approximations to U, each equally valid second-order Taylor series expansions (but in terms of different variables), may not yield equally valid approximations to welfare when evaluated using a log-linear solution (1.1) for the model's endogenous variables.

For example, consider a model like that of the next section. Each period's contribution to the utility of the representative household can be approximated by an expression of the form

$$U = a\hat{C} - b\hat{H} + Q_1 + R_1, \tag{1.4}$$

where $\hat{C}, \hat{H}$ denote the percentage deviations in consumption and hours worked respectively, Q_1 is a set of quadratic terms in the log deviations, and the residual R_1 contains terms of third or higher order, or terms that are independent of the policy chosen, that can be neglected. Alternatively, one may eliminate hours using the necessary relation between aggregate consumption and aggregate hours implied by the production function, and obtain a Taylor series expansion of the form

$$U = c\hat{C} + Q_2 + R_2, \tag{1.5}$$

where Q_2, R_2 are other quadratic terms and residual. Under the assumptions just described, the coefficient c is of order $\mathcal{O}(||\xi||)$. It follows that substitution into (1.5) of the solution to our log-linear structural equations yields a valid second-order approximation to utility.

However, the same is not true of substitution of the same solution into (1.4). The structural equations include a production-function relation between $\hat{C}$ and $\hat{H}$, of the form

$$\hat{C} = f\hat{H} + Q_3 + R_3, \tag{1.6}$$

and one of the log-linear structural relations is given by the linear terms in this. In approximation (1.5), $c = a - bf^{-1}$, so that the first-order terms in the two approximations would have the same value. But the terms Q_2 are not equal to Q_1, because of the presence of non-zero quadratic terms Q_3 in (1.6).[6] Hence (1.4) will not yield a correct second-order approximation to welfare, if one substitutes the solutions for $\hat{C}$ and $\hat{H}$ implied by the log-linear structural relations, including the linear part of (1.6). In terms of the criterion set out above, substitution of the log-linear approximate solutions into (1.4) yields an incorrect result because the coefficients a and b are not individually of order $\mathcal{O}(||\xi||)$, even though the linear combination c is. Thus it is not enough that one expand around a near-optimal equilibrium; the expansion must be written in a form that contains no first-order terms that do not involve coefficients of order $\mathcal{O}(||\xi||)$.

[6] Even if the production function is of a constant-elasticity form, the log-linear approximation (1.6) will contain non-zero quadratic terms in the event of variations in government purchases.

2 A Utility-Based Welfare Criterion

We turn now to the computation of a utility-based approximate welfare criterion, of the kind discussed in the previous section, for the case of our baseline model.[7] The natural welfare criterion in our model is the level of expected utility

$$E\left\{\sum_{t=0}^{\infty}\beta^t U_t\right\} \tag{2.1}$$

associated with a given equilibrium, where the period contribution to utility U_t is given by expression (xxx) of chapter 3. Here we shall abstract from the welfare consequences of monetary frictions.[8] Then, substituting the equilibrium condition $C_t = Y_t$,[9] we can express the period utility of the representative household as a function solely of the level of production of each of the goods in period t,

$$U_t = u(Y_t;\xi_t) - \int_0^1 \tilde{v}(y_t(i);\tilde{\xi}_t)di. \tag{2.2}$$

Here

$$Y_t \equiv \left[\int_0^1 y_t(i)^{\frac{\theta-1}{\theta}}di\right]^{\frac{\theta}{\theta-1}} \tag{2.3}$$

is again the index of aggregate demand, where $y_t(i)$ is production (and consumption) in period t of differentiated good i.[10]

The function $\tilde{v}(y;\tilde{\xi})$ indicates the disutility of supplying quantity y. If we assume a "yeoman farmer" model, as in Rotemberg and Woodford (1997, 1999a), this can be interpreted directly as the household's disutility of supplying output. Alternatively, if we assume firms and a labor market, as in section xx of chapter 3, we can define

$$\tilde{v}(y;\tilde{\xi}) \equiv v(f^{-1}(y/A);\xi), \tag{2.4}$$

[7]The derivation follows essentially the lines of that presented in Rotemberg and Woodford (1997, 1999a) in the context of a variant model with additional decision lags.

[8]The welfare criterion derived in this section thus applies in the "cashless limit" introduced in section 3.3 of chapter 2. The additional term that must be added in the case of non-negligible monetary frictions is derived in section 4.1 below.

[9]Recall that we can interpret the model as allowing for government purchases, treating these as a shift term in the utility function $u(Y;\xi)$.

[10]If there is trend growth in productivity and output, the variables Y_t, $y_t(i)$, and A_t (introduced in the next paragraph) should all be interpreted as having been deflated by a common exponential trend factor, to render them stationary.

where $v(h;\xi)$ is the disutility of working h hours in any given production activity, and $Af(h)$ is the output produced using that labor input. Here $\tilde{\xi} \equiv (\xi, a)$ denotes the complete vector of exogenous disturbances, including both the preference shocks ξ[11] and the technology shock $a \equiv \log A$.[12]

Our Taylor series expansions group terms of different powers in the elements of $\tilde{\xi}$, though we shall continue to use the notation $||\xi||$ for the bound on the magnitude of the entire vector of disturbances.

2.1 Output-Gap Stability and Welfare

Under our previous assumptions, u is an increasing, concave function of Y for each possible value of ξ, while $\tilde{v}$ is an increasing, convex function of y for each possible value of $\tilde{\xi}$. Thus (2.2) implies that U_t is a concave function of the entire vector of levels of production of the various goods. Note also that in terms of this notation, the real marginal cost function introduced in chapter 3 is given by

$$s(y, Y; \tilde{\xi}) = \frac{\tilde{v}_y(y;\tilde{\xi})}{u_c(Y;\xi)}. \tag{2.5}$$

It follows that the elasticity of $\tilde{v}_y$ with respect to y is given by $\omega > 0$, the elasticity of real marginal cost with respect to own output, introduced in chapter 3. One may also observe that the elasticity of real marginal cost with respect to aggregate output is given by σ^{-1}, where once again

$$\sigma \equiv -\frac{u_c}{\bar{Y} u_{cc}}$$

measures the intertemporal elasticity of substitution in private expenditure.

The steady-state level of output associated with zero inflation, in the absence of real

[11] Once again, the vector ξ contains many elements, so that the disturbances to the utility of consumption may or may not be correlated with the disturbances to the disutility of working.

[12] We assume a normalization of the productivity measure A such that the unconditional expectation of a is zero. We include a rather than A in our definition of $\tilde{\xi}$ because it is a rather than A that is assumed to be always sufficiently close to zero (in order for our Taylor series approximation to be accurate), and because we wish to approximate the production-function relationship by one that is linear in a rather than linear in A.

disturbances (*i.e.*, when $\tilde{\xi}_t = 0$ at all times) is the quantity $\bar{Y}$ that satisfies

$$s(\bar{Y}, \bar{Y}; 0) = \frac{1-\tau}{\mu} \equiv 1 - \Phi. \tag{2.6}$$

Here τ is the constant proportional tax rate on sales proceeds, and $\mu \equiv \theta/(\theta - 1)$ is the desired markup as a result of suppliers' market power. The parameter Φ then summarizes the overall distortion in the steady-state output level as a result of both taxes and market power. Since the efficient output level $\bar{Y}^e$ for all goods in the absence of shocks satisfies $s(\bar{Y}^e, \bar{Y}^e; 0) = 1$, we observe that $\bar{Y}/\bar{Y}^e$ is a decreasing function of Φ, equal to one when $\Phi = 0$. When Φ is small, we may make use of the log-linear approximation

$$\log(\bar{Y}/\bar{Y}^e) = -(\omega + \sigma^{-1})^{-1}\Phi. \tag{2.7}$$

It is plainly realistic to assume that $\Phi > 0$. However, we shall assume that Φ is small, specifically of order $\mathcal{O}(||\xi||)$, so that (2.7) is accurate up to a residual of order $\mathcal{O}(||\xi||^2)$. This is the assumption of near-efficiency of the steady state level of output with zero inflation that is made in order to allow us to use our log-linear approximations to the model structural equations in welfare comparisons, as explained in the previous section. Note that the introduction of the distorting tax rate τ allows us to contemplate a series of economies in which Φ is made progressively smaller, without this having to involve any change in the size of θ, a parameter that also affects the coefficients of the log-linearized equilibrium conditions.[13]

We now proceed to compute a quadratic Taylor series approximation to (2.2). The first term can be approximated as

$$u(Y_t; \xi_t) = \bar{u} + u_c\tilde{Y}_t + u_\xi\xi_t + \frac{1}{2}u_{cc}\tilde{Y}_t^2 + u_{c\xi}\xi_t\tilde{Y}_t + \frac{1}{2}\xi_t' u_{\xi\xi}\xi_t + \mathcal{O}(||\xi||^3)$$

[13] Rotemberg and Woodford (1997, 1999a) instead assume that τ is of exactly the (negative) size required to offset the distortion due to market power, so that $\Phi = 0$. The intention is to consider optimal monetary stabilization policy as part of a broader analysis of optimal policy, in which another instrument (tax policy) is assigned responsibility for achieving the optimal average level of economic activity, while monetary policy is used to ameliorate the economy's response to shocks. However, it is clear that monetary policy must in practice be chosen in an environment in which such an output subsidy does not, and probably cannot, exist. Furthermore, the fact that the "natural rate" of output is inefficiently low is of importance for certain issues, notably the inflationary bias associated with discretionary policymaking, treated in chapter 7. Hence we here allow for $\Phi > 0$, while still assuming that Φ is of order $\mathcal{O}(||\xi||)$.

$$
\begin{aligned}
&= \bar{u} + \bar{Y}u_c \cdot (\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2) + u_\xi \xi_t + \frac{1}{2}\bar{Y}^2 u_{cc}\hat{Y}_t^2 \\
&\quad + \bar{Y}u_{c\xi}\xi_t\hat{Y}_t + \frac{1}{2}\xi_t' u_{\xi\xi}\xi_t + \mathcal{O}(||\xi||^3) \\
&= \bar{Y}u_c\hat{Y}_t + \frac{1}{2}[\bar{Y}u_c + \bar{Y}^2 u_{cc}]\hat{Y}_t^2 - \bar{Y}^2 u_{cc} g_t \hat{Y}_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{Y}u_c \left\{ \hat{Y}_t + \frac{1}{2}(1-\sigma^{-1})\hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \qquad (2.8)
\end{aligned}
$$

Here the first line represents the usual Taylor expansion, in which $\bar{u} \equiv u(\bar{Y};0)$ and $\tilde{Y}_t \equiv Y_t - \bar{Y}$, and we assume that the fluctuations in $\tilde{Y}_t$ are only of order $\mathcal{O}(||\xi||)$. The second line substitutes for $\tilde{Y}_t$ in terms of $\hat{Y}_t \equiv \log(Y_t/\bar{Y})$, using the Taylor series expansion

$$Y_t/\bar{Y} = 1 + \hat{Y}_t + \frac{1}{2}\hat{Y}_t^2 + \mathcal{O}(||\xi||^3).$$

The third line collects together in the term "t.i.p." all of the terms that are independent of policy, as they involve only constants and exogenous variables, and uses again the notation

$$g_t \equiv -\frac{u_{c\xi}\xi_t}{\bar{Y}u_{cc}}$$

for the percentage variation in output required in order to keep the marginal utility of expenditure u_c at its steady-state level, given the preference shock.[14] The final line collects terms in a useful way; note that the only part of this expression that differs across policies is the expression inside the curly braces.

We may similarly approximate $\tilde{v}(y_t(i);\xi_t)$ by

$$
\begin{aligned}
\tilde{v}(y_t(i);\xi_t) &= \bar{Y}\tilde{v}_y \left\{ \hat{y}_t(i) + \frac{1}{2}(1+\omega)\hat{y}_t(i)^2 - \omega q_t \hat{y}_t(i) \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\
&= \bar{Y}u_c \left\{ (1-\Phi)\hat{y}_t(i) + \frac{1}{2}(1+\omega)\hat{y}_t(i)^2 - \omega q_t \hat{y}_t(i) \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (2.9)
\end{aligned}
$$

where $\hat{y}_t(i) \equiv \log(y_t(i)/\bar{Y})$, ω is the elasticity of real marginal cost with respect to own output discussed above, and

$$q_t \equiv -\frac{\tilde{v}_{y\xi}\xi_t}{\bar{Y}\tilde{v}_{yy}}$$

[14] As in chapters 2 and 4, the exogenous shifts ξ_t in the relation between the marginal utility of expenditure u_c and aggregate demand Y_t should be understood to include variations in the level of government purchases as well as taste shocks. The approximation derived here continues to apply in the presence of government purchases, as long as the function u in (2.8) is understood to refer to the indirect utility function called $\tilde{u}(Y_t;\tilde{\xi}_t)$ in chapter 2.

is the percentage variation in output required to keep the marginal disutility of supply $\tilde{v}_y$ at its steady-state level, given the preference shock. The second line uses (2.5) and (2.6) to replace $\tilde{v}_y$ by $(1-\Phi)u_c$, and the assumption that Φ is of order $\mathcal{O}(||\xi||)$ to simplify. Note that the term premultiplying the expression in curly braces is now the same as in (2.8).

Integrating this expression over the differentiated goods i, we obtain

$$\begin{aligned}\int_0^1 \tilde{v}(y_t(i);\xi_t) &= \bar{Y}u_c\left\{(1-\Phi)E_i\hat{y}_t(i) + \frac{1}{2}(1+\omega)[(E_i\hat{y}_t(i))^2 + \mathrm{var}_i\hat{y}_t(i)] - \omega q_t E_i\hat{y}_t(i)\right\} \\ &\qquad + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= \bar{Y}u_c\left\{(1-\Phi)\hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 - \omega q_t\hat{Y}_t + \frac{1}{2}(\theta^{-1}+\omega)\mathrm{var}_i\hat{y}_t(i)\right\} \\ &\qquad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \tag{2.10}$$

using the notation $E_i\hat{y}_t(i)$ for the mean value of $\hat{y}_t(i)$ across all differentiated goods at date t, and $\mathrm{var}_i\hat{y}_t(i)$ for the corresponding variance. In the second line, we use the Taylor series approximation to (2.3),

$$\hat{Y}_t = E_i\hat{y}_t(i) + \frac{1}{2}(1-\theta^{-1})\mathrm{var}_i\hat{y}_t(i) + \mathcal{O}(||\xi||^3), \tag{2.11}$$

to eliminate $E_i\hat{y}_t(i)$.

Combining (2.8) and (2.10), we finally obtain

$$\begin{aligned} U_t &= \bar{Y}u_c\left\{\Phi\hat{Y}_t - \frac{1}{2}(\sigma^{-1}+\omega)\hat{Y}_t^2 + (\sigma^{-1}g_t + \omega q_t)\hat{Y}_t - \frac{1}{2}(\theta^{-1}+\omega)\mathrm{var}_i\hat{y}_t(i)\right\} \\ &\qquad + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= -\frac{\bar{Y}u_c}{2}\left\{(\sigma^{-1}+\omega)(x_t - x^*)^2 + (\theta^{-1}+\omega)\mathrm{var}_i\hat{y}_t(i)\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \tag{2.12}$$

Here the second line rewrites the expression in terms of the *output gap* $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$, where $\hat{Y}_t^n$ again denotes the (log of the) *natural rate* of output, the equilibrium level of output under complete price flexibility, given by

$$\hat{Y}_t^n \equiv \frac{\sigma^{-1}g_t + \omega q_t}{\sigma^{-1}+\omega}, \tag{2.13}$$

and in terms of the efficient level of the output gap, $x^* \equiv \log(\bar{Y}^e/\bar{Y})$, given by (2.7). (Note that if Φ is positive and of order $\mathcal{O}(||\xi||)$, the same is true of x^*.)

Expression (2.12) represents a quadratic approximation to (2.2), under the assumption that Φ (and hence the inefficiency of the steady-state level of output) is of order $\mathcal{O}(||\xi||)$. It is interesting to observe that the preference and technology shocks $\tilde{\xi}_t$ matter, in this approximation, only through their effects upon a single exogenous state variable, the natural rate of output $\hat{Y}_t^n$. Furthermore, output variability as such does not matter for our utility-based welfare criterion; rather, it is the variability of the *output gap* that matters, and the measure of potential output with respect to which the gap should be measured for purposes of the welfare criterion is the *same* "natural rate" of output that (as shown in chapter 3) determines the short-run relation between output and inflation. Thus we can already offer an answer to one question posed in the introduction to this chapter: it is the output gap, rather than output relative to trend, that one should seek to stabilize, and (if the only distortions in the economy are those associated with monopolistic competition and sticky prices) the relevant output gap is the same one that appears in the short-run aggregate supply curve.

However, (2.12) implies that stabilization of the output gap should not be the sole concern of policy, since the dispersion of output levels across sectors matters as well.[15] In fact, in our baseline framework, there is no reason for equilibrium output to be different for different goods except as a result of relative price distortions that result from sticky prices in an environment where the overall price level is unstable. It is through this channel that price stability turns out to be relevant for welfare, in a way that goes beyond the mere association between inflation and the level of the aggregate output gap.

Specifically, our assumed CES (Dixit-Stiglitz) preferences over differentiated goods imply that each supplier faces a constant-elasticity demand curve of the form

$$\log y_t(i) = \log Y_t - \theta(\log p_t(i) - \log P_t). \tag{2.14}$$

[15]More generally, it is the dispersion of output *gaps* across sectors that matters, along with the *aggregate* output gap. We here assume that the only disturbances $\tilde{\xi}_t$ that affect the natural rate of output have identical effects upon all sectors, so that the dispersion of output gaps across sectors is identical to the dispersion of output levels. In section xx below, we consider the consequences of allowing for shocks with asymmetric effects on different sectors.

It follows from this that

$$\mathrm{var}_i \log y_t(i) = \theta^2 \mathrm{var}_i \log p_t(i),$$

so that (2.12) may equivalently be written

$$U_t = -\frac{\bar{Y} u_c}{2} \left\{ (\sigma^{-1} + \omega)(x_t - x^*)^2 + \theta(1 + \omega\theta)\mathrm{var}_i \log p_t(i) \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \qquad (2.15)$$

Thus we find that, in addition to stabilization of the output gap, it is also appropriate for policy to aim to reduce price dispersion. In our framework, this is achieved by stabilizing the general price level; but the exact way in which fluctuations in the general price level affect price dispersion, and hence welfare, depend upon the details of price-setting.

2.2 Inflation and Relative-Price Distortions

The approximation (2.15) to the utility of the representative household applies to any model with no frictions other than those due to monopolistic competition and sticky prices, regardless of the nature of the delays involved in price-setting. The relation between the price dispersion term and the stability of the general price level depends, instead upon the details of price-setting. Here we do not attempt a general treatment, but illustrate the form of the relation in three simple examples, including our baseline model of staggered price-setting from chapter 3.

As a first example, consider again the case, discussed above in section xx of chapter 3, of an economy in which a fraction $0 < \iota < 1$ of goods prices are fully flexible, while the remaining $1 - \iota$ must be fixed a period in advance. In such an economy, as shown above, the aggregate supply relation takes the familiar "New Classical" form

$$\pi_t = \kappa x_t + E_{t-1}\pi_t, \qquad (2.16)$$

where the slope coefficient is given by

$$\kappa \equiv \frac{\iota}{1-\iota} \frac{\sigma^{-1} + \omega}{1 + \omega\theta} > 0.$$

In this model, in any period all flexible-price goods have the same price, p_t^1, and all sticky-price goods have the same price, p_t^2, which satisfies

$$\log p_t^2 = E_{t-1} \log p_t^1 + \mathcal{O}(||\xi||^2). \qquad (2.17)$$

The overall price index (defined by equation (xx) of chapter 3) furthermore satisfies

$$\log P_t = \iota \log p_t^1 + (1-\iota) \log p_t^2 + \mathcal{O}(||\xi||^2),$$

so that

$$\begin{aligned} \pi_t - E_{t-1}\pi_t &= \iota[\log p_t^1 - E_{t-1} \log p_t^1] + \mathcal{O}(||\xi||^2) \\ &= \iota[\log p_t^1 - \log p_t^2] + \mathcal{O}(||\xi||^2), \end{aligned}$$

using (2.17). It follows that under this assumption about pricing,

$$\begin{aligned} \mathrm{var}_i \log p_t(i) &= \iota(1-\iota)(\log p_t^1 - \log p_t^2)^2 \\ &= \frac{1-\iota}{\iota}(\pi_t - E_{t-1}\pi_t)^2. \end{aligned}$$

As asserted above, equilibrium price dispersion is closely connected with the stability of the general price level; but in this special case, it is only the volatility of the *unexpected component* of inflation that matters.

Substituting this expression into (2.15), we obtain

$$U_t = -\Omega L_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3),$$

where Ω is a positive constant and L_t is a quadratic loss function of the form

$$L_t = (\pi_t - E_{t-1}\pi_t)^2 + \lambda(x_t - x^*)^2, \qquad (2.18)$$

with a relative weight on output gap variability of $\lambda = \kappa/\theta$. We thus obtain precise conclusions regarding both the sense in which aggregate output and inflation variations matter for welfare (it is the output *gap* that matters, and the *unexpected* component of inflation), and the relative weight that should be placed upon the two concerns (the relative weight on output gap variations is proportional to the slope κ of the short-run Phillips curve).

In fact, in the context of this model, there is no tension between the goals represented by the two terms of (2.18). For (2.16) implies that the output gap is itself proportional to the surprise component of inflation. Thus we can simplify (2.18) further, and say that the sole goal of policy should be to minimize the variability of unexpected inflation, or alternatively, that the sole goal should be to stabilize the output gap (when properly measured).[16]

While we obtain a simple result in this case, the model is not a very realistic one, since, as discussed earlier, it is unable to account for the persistence of the observed output effects of monetary disturbances. Let us consider instead, then, the consequences of the kind of staggered pricing assumed in our baseline model, a discrete-time version of the Calvo (1983) pricing model. In this model, a fraction $0 < \alpha < 1$ of all prices remain unchanged each period, with the probability of a price change assumed to be independent of both the length of time since the price was last changed and of the degree to which that good's price is out of line with others. This implies that each period, the distribution of prices $\{p_t(i)\}$ consists of α times the distribution of prices in the previous period, plus an atom of size $(1-\alpha)$ at the price p_t^* that is chosen at date t by all suppliers who choose a new price at that date. As shown in chapter 3, the aggregate supply relation takes in this case the "New Keynesian" form

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1}, \tag{2.19}$$

where now the slope coefficient is given by

$$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \frac{(\sigma^{-1}+\omega)}{1+\omega\theta} > 0. \tag{2.20}$$

Letting

$$\bar{P}_t \equiv E_i \log p_t(i), \qquad \Delta_t \equiv \text{var}_i \log p_t(i),$$

[16]However, if we allow for disturbances to the short-run aggregate supply relation (2.16) that – unlike the preference, technology, or government-purchase shocks considered in chapter 3 – do not shift the efficient level of output to the same extent, then the loss function (2.18) would still be correct, while the output gap that appears in this formula would no longer coincide perfectly with unexpected inflation. In that extension of the model, it would be quite important to know the correct relative weight λ to place on output-gap variations. See section xx below.

we observe from the above recursive characterization of the distribution of prices at date t that

$$\begin{aligned}\bar{P}_t - \bar{P}_{t-1} &= E_i[\log p_t(i) - \bar{P}_{t-1}] \\ &= \alpha E_i[\log p_{t-1}(i) - \bar{P}_{t-1}] + (1-\alpha)(\log p_t^* - \bar{P}_{t-1}) \\ &= (1-\alpha)(\log p_t^* - \bar{P}_{t-1}).\end{aligned}$$

Similar reasoning about the dispersion measure Δ_t yields

$$\begin{aligned}\Delta_t &= \text{var}_i[\log p_t(i) - \bar{P}_{t-1}] \\ &= E_i\{[\log p_t(i) - \bar{P}_{t-1}]^2\} - (E_i \log p_t(i) - \bar{P}_{t-1})^2 \\ &= \alpha E_i\{[\log p_{t-1}(i) - \bar{P}_{t-1}]^2\} + (1-\alpha)(\log p_t^* - \bar{P}_{t-1})^2 - (\bar{P}_t - \bar{P}_{t-1})^2 \\ &= \alpha\Delta_{t-1} + \frac{\alpha}{1-\alpha}(\bar{P}_t - \bar{P}_{t-1})^2.\end{aligned}$$

Finally, substituting the log-linear approximation

$$\bar{P}_t = \log P_t + \mathcal{O}(||\xi||^2),$$

we obtain

$$\Delta_t = \alpha\Delta_{t-1} + \frac{\alpha}{1-\alpha}\pi_t^2 + \mathcal{O}(||\xi||^3) \tag{2.21}$$

as a law of motion for the dispersion of prices. Note that price dispersion is again a function of the degree of instability of the general price level, though now the relation is a dynamic one. Note also that under this assumptions about pricing, both expected and unexpected inflation contribute *equally* to increases in price dispersion.

Integrating forward (2.21) starting from any initial degree of price dispersion Δ_{-1} in the period before the first period for which a new policy is contemplated, the degree of price dispersion in any period $t \geq 0$ under the new policy will be given by

$$\Delta_t = \alpha^{t+1}\Delta_{-1} + \sum_{s=0}^{t} \alpha^{t-s}\left(\frac{\alpha}{1-\alpha}\right)\pi_s^2 + \mathcal{O}(||\xi||^3).$$

Note that the first term will be independent of the policy that is chosen to apply in periods $t \geq 0$. Thus if we take the discounted value of these terms over all periods $t \geq 0$, we obtain

$$\sum_{t=0}^{\infty} \beta^t \Delta_t = \frac{\alpha}{(1-\alpha)(1-\alpha\beta)} \sum_{t=0}^{\infty} \beta^t \pi_t^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3).$$

Substitution of this in turn into (2.15), we find that

$$\sum_{t=0}^{\infty} \beta^t U_t = -\Omega \sum_{t=0}^{\infty} \beta^t L_t + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{2.22}$$

where in this case the normalized quadratic loss function is given by

$$L_t = \pi_t^2 + \lambda(x_t - x^*)^2. \tag{2.23}$$

Here the relative weight on output gap variability is again given by $\lambda = \kappa/\theta$, but now the value of κ referred to is that given in (2.20).[17]

The loss function (2.23) is in fact of a form widely assumed in the literature on monetary policy evaluation (and also in positive models of central bank behavior).[18] Here, however, we are able to present a theoretical justification for the attention to variations in inflation (rather than, say, variations in the price level), as well as for the common assumption that inflation variations are equally costly whether forecastable or not, in terms of the relative-price distortions resulting from price-level instability in the Calvo model of staggered price-setting. We are also able to derive an optimal rate of inflation with respect to which deviations should be measured (namely, zero, as it is in this case that no relative-price distortions result from imperfect synchronization of price changes). And finally, we are again able to derive an optimal relative weight upon output-gap variation as opposed to inflation variation; this depends upon model parameters, but in a way that makes an estimate of the slope of the short-run aggregate supply curve directly informative about the proper size of this weight.[19]

[17] Note that the values of Ω and λ obtained here are slightly different from those that follow from the derivation presented in Rotemberg and Woodford (1999a). The reason is that we are here interested in approximating the expected value of the discounted sum of utilities, conditioning upon the pre-existing degree of price dispersion at date -1, whereas they compute an unconditional expectation. Note that the loss measure that we compute here, for a given policy, will not depend upon the initial price dispersion Δ_{-1}. Nonetheless, it matters whether one conditions upon the value of Δ_{-1} in computing the expected utility. Computing the unconditional expectation, rather than conditioning upon the value of Δ_{-1}, penalizes policies that lead to higher average price dispersion also for the higher average value assumed for Δ_{-1} if one integrates over the unconditional distribution of values for Δ_t associated with a given stationary equilibrium.

[18] See, *e.g.*, Walsh (1998, chap. 8), Clarida *et al.* (1999), or Svensson (1999a).

[19] The size of this weight is of greater interest in the case of this model, since aggregate supply relation (2.19) does not imply that inflation and the output gap should perfectly co-vary under most circumstances. It is true that complete stabilization of one implies complete stabilization of the other, as we discuss further in the next section, and in this sense there is no tension between the two goals if (2.19) holds. But it may not

As we discuss further in section 4.2 below, the estimate of the slope of the short-run aggregate supply curve for the U.S. of Rotemberg and Woodford (1997) implies a value for λ_x on the order of .05, if the output gap is measured in percentage points and inflation is measured as an annualized percentage rate. This value is much lower than the value $\lambda_x = 1$ often assumed in the literature on evaluation of monetary policy rules, on a ground such as "giving equal weight to inflation and output" as stabilization objectives.[20] Our utility-based analysis implies instead that if one assumes the degree of price stickiness that is needed to account for the persistence of the real effects of monetary policy shocks, the distortions associated with inflation are more important than those associated with variation in the aggregate output gap.

As yet another alternative, suppose that prices are indexed to a lagged price index between the occasions on which they are re-optimized, as in the model with "inflation inertia" set out in section xx of chapter 3. In this case, the aggregate supply relation takes the form

$$\pi_t - \gamma\pi_{t-1} = \kappa x_t + \beta E_t[\pi_{t+1} - \gamma\pi_t], \tag{2.24}$$

generalizing (2.19), where γ measures the degree of indexation to the lagged price index, and κ is again defined as in (2.20). Recall that in the periods in which a given price is not re-optimize, it is automatically increased by an amount

$$\log p_t(i) = \log p_{t-1}(i) + \gamma\pi_{t-1}$$

owing to the change in the lagged price index. It follows that (2.21) generalizes to

$$\Delta_t = \alpha\Delta_{t-1} + \frac{\alpha}{1-\alpha}(\pi_t - \gamma\pi_{t-1})^2 + \mathcal{O}(||\xi||^3). \tag{2.25}$$

Price dispersion is increased only when the prices that are re-optimized are increased by an amount different than $\gamma\pi_{t-1}$, the amount by which the prices that are not re-optimized increase. This occurs if and only if the overall rate of inflation π_t differs from $\gamma\pi_{t-1}$.

be possible to achieve complete stabilization, *e.g.*, because of the zero lower bound on nominal interest rates, or informational restrictions upon feasible policies; and in such cases optimal policy will generally depend upon the relative weight placed upon the two goals.

[20]See, *e.g.*, Rudebusch and Svensson (1999) and Williams (1999).

We can integrate (2.25) forward as before, and again obtain an expression of the form (2.22) for discounted utility. But now the normalized quadratic loss function is given by

$$L_t = (\pi_t - \gamma\pi_{t-1})^2 + \lambda(x_t - x^*)^2, \tag{2.26}$$

where $\lambda > 0$ and $x^* \geq 0$ are defined as in (2.23). In the case of full indexation of individual prices to the lagged price index ($\gamma = 1$),[21] this implies that it is the rate of inflation acceleration, $\Delta\pi_t$, rather than the rate of inflation itself, that should be stabilized around zero in order to reduce the distortions associated with price dispersion.[22] Owing to the existence of complete indexation to past inflation, there are no distortions resulting from *constant* inflation, only from *changes* in the rate of inflation. In its implication that steady inflation at any rate causes no harm, this model is like the "New Classical" model (with which it shares the prediction of a vertical "long run Phillips curve"). But in the model with full indexation, inflation *changes* distort the allocation of resources, whether they are predictable in advance or not.

3 The Case for Price Stability

While the loss measures derived above under the various assumptions about the timing of pricing decisions are each different in certain respects, they all share an important common property. This is that the deadweight losses due to relative price distortions can in each case be completely eliminated, in principle, by stabilizing the aggregate price level. The intuition for this result is simple. The aggregate price level is stabilized by creating an environment in which suppliers who choose a new price have no desire at any time to set a price different from the average of existing prices. But if this is so, the average of existing

[21] Note that in this case, the aggregate supply relation is essentially identical to the popular empirical specification proposed by Fuhrer and Moore (1995a, 1995b).

[22] Steinsson (2000) and Amato and Laubach (2001b) similarly find, in the case of an AS relation with inflation inertia owing to the presence of backward-looking "rule-of-thumb" price-setters of the kind proposed by Gali and Gertler (1999), that the utility-based loss function should penalize variations in $\Delta\pi_t$ as well as variations in π_t. The result (2.26) is simpler, however, than the one obtained by these authors, and leads to simpler conclusions regarding the character of optimal policy in the presence of inflation inertia. This is one reason that we emphasize indexation to a lagged price index as a possible source of inflation inertia in this study.

prices never changes, and so the new prices that are chosen at all times are always the same, and eventually all goods prices are equal to that same, constant value. Thus aggregate price stability is a sufficient condition for the absence of price dispersion in our simple framework.

At the same time, in most cases, it is also a necessary condition. This is not true in the pure "New Classical" case, as in that case it is only necessary that there be no *unexpected* changes in the aggregate price level in order for there to be no price dispersion. But this is clearly a highly special case; even if some prices are fully flexible, if the sticky goods prices as in the Calvo model, complete price stability will again be necessary to eliminate distortions. Similarly, even in the model of Mankiw and Reis (2001), which is like the "New Classical" model in assuming that each goods price at any date is set optimally conditional on information available at some prior date (though the dates are different for different goods), complete price stability is necessary to eliminate distortions. This is because in the Mankiw-Reis model, for any horizon k, there are a positive fraction of goods prices that are set more than k periods in advance (or on the basis of information that is more than k periods out of date). This means that relative-price distortions will be created in the case of any disturbance that affects the price level, even if it has no effect upon the price level until k periods later. Since this is true for arbitrary k, distortions are completely eliminated only if exogenous disturbances *never* imply any change in the price level.

Similarly, in the model with staggered pricing and full indexation to a lagged price index, price stability is not necessary for the absence of price dispersion; it is simply necessary that the inflation rate be constant over time. But again this is a highly special case. If the indexation parameter γ takes any value other than one, only zero inflation is consistent with an absence of price dispersion.[23] The same conclusion would be reached if only *some* prices are indexed to the lagged price index.

The argument for the necessity of stability of the general price index for the elimination of

[23]To be precise, an absence of price dispersion will require that prices change at a common rate π_t satisfying the difference equation $\pi_t = \gamma\pi_{t-1}$, given some arbitrary initial rate of inflation. But when $\gamma < 1$, this implies zero inflation every period, at least asymptotically. A *stationary* policy regime that fully eliminates distortions resulting from price dispersion would have to be one with zero inflation at all times.

price dispersion is also independent of the specific details of the way that the timing of price changes is modeled by Calvo (1983). One need not assume that the probability of revision of any given price in a given period is the same for all prices; one might instead assume that prices are revised at fixed time intervals, as in the models of Taylor (1980), Blanchard and Fischer (1989, sec. 8.2), King and Wolman (1999), or Chari *et al.* (2000), among others, or one might endogenize the timing of price revisions as in the model of Dotsey *et al.* (1999). In any of these cases, price dispersion is eliminated only by a policy that completely stabilizes the general price index.

Moreover, price stability is not only the case in which the distortions associated with inefficient output *composition* are eliminated. As we shall see, it is also the route to minimization of the distortions associated with an inefficient *level* of output; and so, in the context of the kind of simple model considered thus far, it is an unambiguously desirable goal for monetary policy. The argument for this is simplest in the case that the equilibrium level of output under flexible prices is optimal, so we take up this case first. But as we shall see, our conclusions require only minor modification even when we allow for the possibility that the natural rate of output is inefficiently low.

3.1 The Case of an Efficient Natural Rate of Output

Here we assume not merely that the inefficiency wedge Φ defined in (2.6) is of order $\mathcal{O}(||\xi||)$, but that it is equal to zero (or at any rate, that it is of order $\mathcal{O}(||\xi||^2)$, so that we may neglect it in our quadratic approximation to expected utility). This implies that $\bar{Y} = \bar{Y}^e$ (or at least that their log difference x^* is of order $\mathcal{O}(||\xi||^2)$), so that the steady-state level of output under flexible prices is efficient (at least to second order). Since we have already verified, above, that percentage *fluctuations* in the natural rate are equal (to second order) to the percentage fluctuations in the efficient level of output, this actually implies that (to second order) the natural rate of output coincides with the efficient level of output at all times.

In this case, we easily obtain a very simple conclusion about the nature of optimal monetary policy. For *each* of the individual terms in our quadratic loss function can be shown to achieve its minimum possible value, zero, if inflation is zero at all times. We have just discussed the fact that this is true of the terms that measure the deadweight loss due to an inefficient composition of output. But in the present case, $x^* = 0$, so that the term in the loss function that involves the aggregate output gap is also minimized (and equal to zero) if and only if $x_t = 0$ at all times. Each of the aggregate supply relations (2.16), (2.19) and (2.24) implies that this will be true in the case of zero inflation at all times.

In the informal argument just given, we have ignored the role of initial conditions that may not be consistent with an equilibrium with zero inflation and zero output gap at all times. If we wish to consider optimal policy from some initial date, however, the initial conditions that happen to exist — not necessarily ones that result from an anticipation of the policy that will be followed from now on — may constrain possible outcomes from now on. Thus it may not be possible to completely stabilize both inflation and the output gap at zero each period from the initial period onward.

In the case of the "New Keynesian" aggregate supply relation (2.19), there is no problem; because this AS relation is purely forward-looking, the set of paths for inflation and the output gap that are feasible from date zero onward are independent of anything that has happened previously. In this case, $\pi_t = 0$ and $x_t = 0$ for all $t \geq 0$ is a possible equilibrium outcome regardless of initial conditions, and so this is plainly the optimal outcome given a loss function (2.23). On the other hand, in the case of the "New Classical" AS relation (2.16), such an outcome represents a possible equilibrium only in the case of initial conditions under which $E_{-1}\pi_0 = 0$. In the case of other initial conditions, it is obvious that an optimal policy will involve $\pi_0 = E_{-1}\pi_0 \neq 0$ and $x_0 = 0$, which reduces the period zero loss (2.18) to its minimum possible value, zero. A loss of zero is also possible in all later periods, through commitment to any path for inflation that is completely predictable a period or longer in advance; in particular, a commitment to $\pi_t = 0$ in all periods $t > 0$ is one way of achieving the minimum possible discounted stream of losses.

Thus a commitment to price stability would again represent an optimal policy, except possibly during a brief transition period. If we seek to choose a time-invariant policy rule that achieves the pattern of behavior that it would be optimal to commit *eventually* to follow — as it is argued in chapters 7 and 8 that we should[24] — then a commitment to price stability would be an example of such a policy. One reason why it might be sensible to commit to follow such a rule from the beginning, even when $E_{-1}\pi_0 \neq 0$, is that a willingness to deliver whatever rate of inflation may have been expected "just this once" may allow the private sector to expect that the same principle would be followed in the future as well, if it ever turned out that $E_{t-1}\pi_t$ differed from zero, in which case there would be no determinate equilibrium level of inflation for the private sector to expect.

In the case of indexation to a lagged price index, resulting in an AS relation (2.24) that implies inflation inertia, matters are still more complex. An equilibrium with $\pi_t = 0$ and $x_t = 0$ for all $t \geq 0$ is possible only in the case of initial conditions under which $\pi_{-1} = 0$. However, it is evident from the form of the AS relation that the set of possible paths for the variables $\pi_t^d \equiv \pi_t - \gamma\pi_{t-1}$ and x_t from period zero onward is independent of the initial conditions, and one also notes that the utility-based loss function (2.26) depends only on the paths of these two variables. It is thus obvious that the optimal policy, from the point of view of minimizing the discounted sum of losses from period zero onward given the economy's prior state, is one under which $\pi_t = \gamma\pi_{t-1}$ and $x_t = 0$ for all $t \geq 0$. Thus the resulting path of inflation will depend on the economy's initial rate of inflation, prior to adoption of the optimal policy; specifically it will satisfy $\pi_t = \gamma^{t+1}\pi_{-1}$.

As long as $\gamma < 1$, optimal policy in this case is again one under which the inflation will *asymptotically* equal zero — and not just on average, but in all possible states of the world. Thus optimal policy will involve a commitment to eventual stabilization of the price level. Furthermore, even in the transition period, the optimal path of inflation will be deterministic, which implies that will be unaffected by any random disturbances that may occur from period zero onward. Thus optimal policy has the immediate consequence that the price level should no longer be affected at all by any random disturbances that may occur,

[24] See the discussion in those chapters of policymaking "from a timeless perspective".

be they disturbances to technology, to preferences, or to the level of government purchases. The existence of inflation inertia only affects the rate at which it is optimal for the central bank to commit to lowering the rate of inflation (assuming an initial positive level) to its long-run target of zero.

It is only in the special case that $\gamma = 1$ (full indexation to the lagged price index) that this result is not obtained. In that case, the optimal commitment given initial conditions is to a constant inflation rate $\pi_t = \pi_{-1}$ for all $t \geq 0$; it is therefore optimal *never* to disinflate, once inflation has been allowed to begin. But such a result plainly depends on the absence, in this simple model, of any distortions associated with steady inflation. If the prices of even a few goods are *not* fully indexed to the lagged price index, zero inflation is required to eliminate relative-price distortions. Similarly, if the adjustment of prices as a result of indexation is not continuous, but occurs only occasionally (though more frequently than re-optimizations of pricing policy), relative-price distortions will again be completely eliminated only in the case of zero inflation. In either case, optimal policy will steadily reduce inflation over time, to a long-run rate of zero, as there is always a motive for some reduction in the inflation rate (as long as the current rate remains positive) but never a motive for increasing it (as opposed to keeping it near its recent level). Thus one finds quite generally that optimal policy will involve a commitment to price stability, at least eventually. And this means that eventually inflation will not only equal zero on average, but zero regardless of the real disturbances that may affect the economy.

This strong conclusion regarding the optimality of complete price stability depends upon various details of our model, as we discuss further in section xx. Nonetheless, it is interesting to remark that it holds despite our having allowed for several different kinds of stochastic disturbances. In particular, our framework allows for exogenous disturbances to technology, to government purchases, to households' impatience to consume, to their willingness to supply labor, or to the transactions technology that determines their demand for money balances. In the face of each of these types of disturbance, it remains optimal, under the circumstances assumed here, for the general level of prices to be held fixed.

The generality of the conclusion results from a simple intuition, stressed by Goodfriend and King (1997). Under the circumstances assumed here, the failure of prices to be continually adjusted is the *only* distortion that prevents rational expectations equilibrium from achieving an optimal allocation of resources. Thus an optimal monetary policy is one that achieves *the same allocation of resources as would occur with flexible prices,* if this is possible. Flexible-price equilibrium models of aggregate fluctuations (*i.e.,* real business cycle models[25]) are then of practical interest, not as descriptions of what aggregate fluctuations should be like *regardless* of the monetary policy regime, but as descriptions of what they would be like under an *optimal* policy regime. Finally, our models of optimal price-setting imply that price stickiness will have no effects upon equilibrium outcomes in the case that monetary policy keeps the general price level completely unchanged over time, since in this case suppliers of goods would not wish to change their prices more frequently even if it were costless for them to do so. Thus complete price stability achieves the optimal allocation of resources.

Verifying that it is in fact possible, in principle, to achieve this first-best allocation through suitable monetary policy requires that we verify that we can solve the equations of our model for the evolution of all variables (including the interest-rate instrument of the central bank) under the assumption that $\pi_t = 0$ at all times. As discussed in chapter 4, this is possible, as long as the natural rate of interest r_t^n is always non-negative; in this case, what is required is that output equal the natural rate of output at all times, and that

[25]Standard real business cycle models (King and Rebelo, 1999) differ from the flexible-price limit of the model assumed here in that product markets are competitive, rather than monopolistically competitive; in that all output is produced using inputs purchased from the same factor markets, so that there is a common level of marginal cost for all firms at any time; and in that the endogenous dynamics of the capital stock in response to shocks is modeled, and indeed emphasized (as the only endogenous propagation mechanism in simple RBC models). However, in the flexible-price limit of our baseline model, all goods prices move together, and similarly the levels of production of each good, so that marginal cost is in fact the same for all firms. If we assume, as in this section, that an output or employment subsidy offsets the distortion due to firms' market power, the flexible-price equilibrium is equivalent to that of a competitive model with a single good. Finally, if we extend the baseline model to take account of capital-accumulation dynamics (which, as we have argued in chapter 4, are not so important for our concerns), then the flexible-price dynamics of our model are fully equivalent to those of a standard RBC model. Note that these models, like our "cashless" model, abstract from real-balance effects upon consumption demand, labor supply, and so on.

the nominal interest rate equal the natural rate of interest at all times. Hence such an equilibrium is possible (under the qualification stated), and thus such an outcome is the one that an optimal policy would aim at.

Further discussion of exactly what kinds of output and interest-rate variations in response to real disturbances this should imply can be found in section xx of chapter 4. Here we recall simply that in the case of none of the types of real disturbances discussed there would it be desirable to use monetary policy to suppress all effects of the real disturbance on aggregate output. Nor can we support, in general, so simple a conclusion as that reached by Ireland (1996), who argues that one should use monetary policy to "insulate aggregate output" against "shocks to demand", while accommodating "shocks to supply". Many readers might assume that "shocks to demand" would include disturbances such as our government-purchase or consumption-demand shocks, but the result just derived here (together with our discussion in chapter 4 of variation in the natural rate of output) implies that it is not optimal to stabilize output in response to these shocks. In fact, in Ireland's theoretical analysis, "shocks to demand" refer solely to money-demand shocks, as this is the only type of exogenous disturbances other than technology shocks that he considers.

3.2 Consequences of an Inefficient Natural Rate of Output

We now consider the extent to which the above conclusions must be modified in the case that (quite realistically) we assume that $\Phi > 0$, so that the equilibrium rate of output under flexible prices would be inefficiently low. The distortions represented by the coefficient Φ, *i.e.*, the market power resulting from monopolistic competition and the constant rate of distorting taxation τ, introduce a wedge between this natural rate of output and the efficient output level. However, this wedge is assumed to be constant over time, so that *percentage changes* in the natural rate still correspond precisely (in our log-linear approximation) to percentage changes in the efficient level of output.[26] Thus, as shown above, the distortions associated with a suboptimal aggregate level of economic activity are still measured a quadratic function

[26] The consequences of time-variation in the size of this wedge are considered below in section xx.

of the output gap, $\lambda(x_t - x^*)^2$, even if now the constant x^* is assumed to be positive. (Note that we assume even in this case that Φ, and hence x^*, is of order $\mathcal{O}(||\xi||)$.)

While this difference matters for the optimal *average* levels of inflation and output – that is, for the deterministic part of our above description of the optimal policy commitment – it has *no* effect (in our log-linear approximation to optimal policy) upon the optimal responses to shocks. We first demonstrate this in the simple context of our "New Classical" model of price-setting. In this case, the normalized quadratic loss function (2.18) can be written

$$L_t = (\pi_t - E_{t-1}\pi_t)^2 - 2\lambda x^* x_t + \lambda x_t^2, \tag{3.1}$$

dropping the term λx^{*2} that is independent of policy. The second term on the right-hand side now indicates a welfare gain from an increase in the expected output gap in any period. However, because x^* is of order $\mathcal{O}(||\xi||)$, a *first-order* approximation to the solution for x_t suffices to give us a *second-order* approximation to this term. Hence we may substitute using the aggregate supply relation (2.16), to obtain

$$L_t = (\pi_t - E_{t-1}\pi_t)^2 - 2\theta^{-1}x^*[\pi_t - E_{t-1}\pi_t] + \lambda x_t^2,$$

recalling that $\lambda = \kappa/\theta$.

Taking the expected discounted value of such terms (and dropping the term $E_{-1}\pi_0$ that is independent of policy), we obtain the utility-based welfare criterion

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t L_t\right\} = -2\theta^{-1}x^*\pi_0 + E_0\left\{\sum_{t=0}^{\infty}\beta^t[(\pi_t - E_{t-1}\pi_t)^2 + \lambda x_t^2].\right\} \tag{3.2}$$

Note that each of the terms proportional to x^* has canceled, except the one indicating a welfare gain from surprise inflation at date zero, the time at which a new policy commitment is adopted. Because it is not possible to commit in advance to an inflation *surprise* at any later date, the corresponding terms for dates $t \geq 1$ do not matter. But this means that allowing for $x^* > 0$ has no effect upon the nature of the optimal policy commitment, except in the initial (transitional) period, when it is possible to take advantage of the fact that private sector expectations of period zero inflation are already given, before the policy is adopted.[27] It is arguable (as we discuss in the next chapter) that it does not make sense to behave differently in this initial period than one commits to behave later, if one wants the commitment to be credible. But regardless of how one manages the transition to the optimal regime, it is optimal to commit to an eventual zero rate of inflation, and to a path for inflation that is unaffected by any stochastic disturbances.[28]

It might be thought that this result depends upon the fact that in the special case in which all prices are changed every period (though some are committed a period in advance), only *unexpected* inflation has an effect upon output. Yet a similar conclusion is obtained in our baseline model, with Calvo price-setting. In this case, we can similarly substitute into (2.23) using the aggregate supply relation (2.19), to obtain

$$L_t = \pi_t^2 - 2\theta^{-1}x^*[\pi_t - \beta E_t\pi_{t+1}] + \lambda x_t^2,$$

noting again that $\lambda = \kappa/\theta$. Taking the expected discounted value, we obtain the utility-based welfare criterion

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t L_t\right\} = -2\theta^{-1}x^*\pi_0 + E_0\left\{\sum_{t=0}^{\infty}\beta^t[\pi_t^2 + \lambda x_t^2]\right\}. \tag{3.3}$$

Once again all of the terms proportional to x^* cancel, except the one indicating welfare gains from a surprise inflation in period zero. Committing in advance to non-zero inflation in any later period does *not* produce any such effect. For the value of the increase in output in any period $t \geq 1$ resulting from higher inflation in period t must be offset by the cost of the *reduction* in output in period $t-1$ as a result of *expectation* of that higher inflation in period t. From the standpoint of the discounted loss criterion (3.2), the costs resulting from the anticipation of the inflation are weighted more strongly (by a factor of $\beta^{-1} > 1$), as they occur earlier in time. On the other hand, the output effect of anticipated inflation, by shifting the short-run aggregate supply curve, is also smaller than the effect of current

[27] Of course, this different prescription in the case of the initial period shows that optimal policy is not *time-consistent* in this case. This issue is taken up in the next chapter.

[28] Of course, in this model, there is no advantage of complete price stability over any other policy that makes inflation completely forecastable a period in advance. But in order to stress the similarity of the results obtained under the alternative aggregate supply specifications, it is worth noting that also in this case there is no *advantage* to any variation in inflation in response to shocks.

inflation, by exactly the factor $\beta < 1$, with the result that the two effects exactly cancel, to first order (which is to say, to second order when multiplied by x^*). Thus once again there is no welfare gain, up to our order of approximation, from a commitment to inflation that can be anticipated in advance. In particular, we find once again that except for transition effects, resulting from the different term in (3.3) for the initial period, it is again optimal to commit to zero inflation, independent of the shocks to the economy.

Nonetheless, the term in (3.3) that is linear in π_0 now affects the optimal commitment for periods later than π_0 as well. That is because of the intertemporal linkage implied by aggregate supply relation (2.19). The welfare gain from inflation at date zero can be obtained with less increase in the period zero output gap (and hence less increase in the λx_0^2 term) if it is accompanied by an increase in expected inflation at date one; and since the welfare loss from such inflation is merely quadratic, it is optimal to commit to some amount of such inflation. Thus the inflation associated with the transition to the optimal regime lasts for more than a single period in this case.

The optimal transition path is characterized in section xx of chapter 7. Here we content ourselves with a few observations about the form of the solution to this problem. First, because the only reason to plan a non-zero inflation rate in period 1 is for the sake of the effect of expected period 1 inflation on the location of the period 0 output-inflation tradeoff, there is no gain from planning on a period 1 inflation rate that is not deterministic. The same is true of planned inflation in all later periods. Thus the optimal commitment from date zero onward involves a deterministic path for inflation; it continues not to be optimal for the inflation rate to respond at all to real disturbances of the various types considered thus far. In addition, as shown in the next chapter, the deterministic path for planned inflation should converge asymptotically to zero, the rate that would be optimal but for the opportunity to achieve an output gain from unexpected inflation in the initial period.

Thus it is optimal (from the point of minimizing discounted losses from date zero onward) to arrange an initial inflation, given that the decision to do so can have no effect upon expectations prior to date zero (if one is not bothered by the non-time-consistency of such a principle of action). The optimal policy involves positive inflation in subsequent periods as well, but there should be a commitment to reduce inflation to its optimal long-run value of zero asymptotically. And the rate at which inflation is committed to decline to zero should be completely unaffected by random disturbances to the economy in the meantime.[29] Thus the assumption that $\Phi > 0$ makes no difference for the conclusions of the previous section with regard to the optimal response to shocks. And if one takes the view (as we shall argue in the next chapter) that one should actually conduct policy as one *would have optimally committed to do* far in the past, thus foregoing the temptation to exploit the private sector's failure to anticipate the new policy, then it is optimal simply to choose $\pi_t = 0$ at all times – *i.e.*, to completely stabilize the price level – just as in the previous section.

It is interesting to note that this result – that the optimal commitment involves a long-run inflation rate of zero, even when the natural rate of output is inefficiently low – does *not* depend upon the existence of a vertical "long-run Phillips curve" tradeoff. For the aggregate supply relation (2.19) in our baseline model implies an upward-sloping relation

$$x^{ss} = (1 - \beta)\kappa^{-1}\pi^{ss}$$

between steady-state inflation $\pi^s s$ and the steady-state output gap x^{ss}. (This is because the expected-inflation term has a coefficient $\beta < 1$, unlike that of the "New Classical" relation (2.16).) It is sometimes supposed that the existence of a long-run Phillips-curve tradeoff, together with an inefficient natural rate, should imply that the Phillips curve should be exploited to some extent, resulting in positive inflation forever, even under commitment. But here that is not true, because the smaller coefficient on the expected-inflation term relative to that on current inflation – which results in the long-run tradeoff – is exactly the size of shift term in the short-run aggregate supply relation that is needed *to precisely eliminate any long-run incentive for non-zero inflation* under an optimal commitment. If one were instead to "simplify" the New Keynesian aggregate supply relation, putting a coefficient of one on expected inflation (as is done in some presentations,[30] presumably in order to conform to

[29] These results agree with those of King and Wolman (1999) in the context of a model with two-period overlapping price commitments in the style of Taylor (1980).

the conventional wisdom regarding the long-run Phillips curve), we would then fail to obtain such a simple result. The optimal long-run inflation rate would actually be found to be *negative*, as the stimulative effects of lower expected inflation would be judged to be worth more than the output cost of lower current inflation – even though there would actually be no long-run output increase as a result of the policy!

And once again, the character of optimal policy in the presence of inflation inertia due to partial indexation to a lagged price index can be determined directly from our results for the Calvo pricing model with no indexation; the optimal time path for π_t in the case of the model without indexation becomes the optimal time path for $\pi_t^d \equiv \pi_t - \gamma\pi_{t-1}$ in the case with indexation. It then follows that once again the optimal path of inflation will be completely deterministic. And while there will be initial inflation (even starting from an initial condition with $\pi_{-1} = 0$) if the central bank allows itself to exploit initial expectations by choosing $\pi_0^d > 0$, optimal policy will involve a commitment to reduce π_t^d asymptotically to zero. In the case of any $\gamma < 1$, this will once again mean a commitment to eventual price stability. And optimal policy has this character despite the fact that the level of output associated with stable prices is inefficiently low, and despite the existence of a positively sloped long-run Phillips curve trade-off, as this is ordinarily defined.

3.3 Caveats

We have seen that, within the class of sticky-price models discussed above, the optimality of a monetary policy that aims at complete price stability is surprisingly robust. Not only does this conclusion not depend upon the fine details of how many prices are set a particular time in advance or left unchanged for a particular length of time, but it remains valid in the case of a considerable range of types of stochastic disturbances, and in the case of an inefficient natural rate of output. Nonetheless, it is likely that some degree of deviation from full price stability is warranted in practice. Some of the more obvious reasons for this are sketched here.

[30] See, *e.g.*, Roberts (1995) or Clarida *et al.* (1999).

First of all, complete price stability may not be *feasible*. We have just argued, in section 3.2, that in our baseline model, it is feasible, because we are able to solve for the required path of the central bank's nominal interest-rate instrument. This is correct, as long as the random disturbances are small enough in amplitude. But if they are larger, such a policy might not be possible, because it might require the nominal interest rate to be *negative* at some times, which, as explained in chapter 2, is not possible under any policy. Specifically, this will occur if it is ever the case that the natural rate of interest is negative. On average, it does not seem that it should be, and thus zero inflation *on average* would seem to be feasible; but it may be temporarily negative as a result of certain kinds of disturbances, and this is enough to make complete price stability infeasible. As a result, a policy will have to be pursued which involves less volatility of the short nominal interest rate in response to shocks, and some amount of price stability will have to be sacrificed for the sake of this.[31] The way in which optimal monetary policy is different in the presence of such a concern is an important concern of chapter 7.

Varying nominal interest rates as much as the natural rate of interest varies may also be desirable as a result of the "shoe-leather costs" involved in economizing on money balances. As argued by Friedman (1969), the size of these distortions is measured by the level of nominal interest rates, and they are eliminated only if nominal interest rates are zero at all times.[32] Taking account of these distortions – from which we have abstracted thus far in our welfare analysis – provides another reason for the equilibrium with complete price stability, even if feasible, not to be fully efficient; for as Friedman argues, a zero nominal interest rate will typically require expected *deflation* at a rate of at least a few percent per year.

One might think that this should make no more difference to our analysis of optimal policy

[31] In general, it will be optimal to back off from complete price stability both by allowing inflation to vary somewhat in response to disturbances, *and* by choosing an average rate of inflation that is somewhat greater than zero, as suggested by Summers (1991), in order to allow more room for interest-rate fluctuations consistent with the zero lower bound. However, the quantitative analysis undertaken below finds that the effect of the interest-rate lower bound on the optimal response of inflation to shocks is more significant than the effect upon the optimal average rate of inflation.

[32] See Woodford (1990) for justification of this relation in a variety of alternative models of the demand for money.

than does the existence of an inefficient natural rate of output due to market power – that it may similarly affect the *deterministic* part of the optimal path for inflation without creating any reason for inflation to vary in response to random shocks. But monetary frictions do not have implications only for the optimal *average* level of nominal interest rates. As with distorting taxes, it is plausible that the deadweight loss is a convex function of the relative-price distortion, so that temporary increases in nominal interest rates are more costly than temporary decreases of the same size are beneficial. In short, monetary frictions provide a further reason for it to be desirable to reduce the *variability* of nominal interest rates, even if one cannot reduce their average level. (At the same time, reducing their average level will require less variable rates, because of the zero floor.) Insofar as these costs are important, they too will justify a departure from complete price stability, in the case of any kinds of real disturbances that cause fluctuations in the natural rate of interest, in order to allow greater stability of nominal interest rates. This tradeoff is treated more explicitly in section xx below.

Even apart from these grounds for concern with interest-rate volatility, it should be recognized that the class of sticky-price models analyzed above are still quite special in certain respects. One of the most obvious is that there are assumed to be no shocks as a result of which the relative prices of any of the goods with sticky prices would vary over time in an *efficient* equilibrium (*i.e.,* the shadow prices that would decentralize the optimal allocation of resources involve no variation in the relative prices of such goods). This is because we have assumed that only goods prices are sticky, that all goods enter the model in a perfectly symmetrical way, and that all random disturbances have perfectly symmetrical effects upon all sectors of the economy. These assumptions are convenient, but plainly an idealization. Yet it should be clear that they are relied upon in our conclusion that stability of the general price level suffices to eliminate the distortions due to price stickiness.

If an efficient allocation of resources requires relative price changes, due to asymmetries in the way that different sticky-price commodities are affected by shocks, this will not be true. We show, however, in section xx below, that even in the presence of asymmetric shocks, it is possible to define a symmetric case in which it is still optimal to completely stabilize the general price level, even though this does not eliminate all of the distortions resulting from price stickiness. But this holds exactly only in a special case, in which different goods are similar, among other respects, in the *degree* of stickiness of their prices. If sectors of the economy differ in their degree of price stickiness (as is surely realistic), then complete stabilization of an aggregate price index will not be optimal. Stabilization of an appropriately defined *asymmetric* price index (that puts more weight on the stickier prices) is a better policy, as argued by Aoki (2001) and Benigno (1999), though even the best policy of this kind need not be fully optimal.

An especially important reason for disturbances to require relative price changes between sticky commodities with sticky prices is that *wages* are probably as sticky as are prices. Real disturbances almost inevitably require real wage adjustments in order for an efficient allocation of resources to be decentralized, and if *both* wages and prices are sticky, it will then not be possible to achieve all of the relative prices associated with efficiency simply by stabilizing the price level – specifically, the real wage will frequently be misaligned, as will be the relative wages of different types of labor if these are not set in perfect synchronization. In such circumstances, complete price stability may not be a good approximation at all to the optimal policy, as Erceg *et al.* (1999) show. As we show in section xx below, stabilization of an appropriately weighted average of prices and wages may still be a good approximation to optimal policy, and fully optimal in some cases. Thus concerns of this kind are not so much reasons not to pursue price stability as they are reasons why care in the choice of the index of prices (including wages) that one seeks to stabilize may be important.

Yet another qualification to our results in this section is that we have assumed a framework in which the flexible-price equilibrium rate of output is efficient, or at most differs from the efficient level by only a (small) constant factor. As we have seen, this assumption is compatible with the existence of a variety of types of economic disturbances, including technology shocks, preference shocks, and variations in government purchases. But it would not hold in the case of other sorts of disturbances, that cause *time variation in the degree* *of inefficiency* of the flexible-price equilibrium. These could include variation in the level of distorting taxes, variation in the degree of market power of firms or workers, or variation in the size of the wage premium that must be paid on efficiency-wage grounds.

In the case of a time-varying gap between the flexible-price equilibrium level of output and the efficient level, complete stabilization of inflation is no longer sufficient for complete stabilization of the welfare-relevant output gap. For while inflation stabilization may imply a level of output at all times equal to the flexible-price equilibrium level, as discussed above, this will no longer minimize the variability of the gap between actual output and the efficient level of output. As a result, complete stabilization of inflation will not generally be optimal. It is not obvious that stabilization of any alternative price index makes sense as a solution to the problem in this case, either, whereas some degree of concern for stabilization of the (appropriately measured) output gap is clearly appropriate, even if it should not wholly displace a concern for inflation stabilization. This is an especially serious challenge to the view that price stability should be the sole goal of monetary policy, if one believes that disturbances of this kind are quantitatively important in practice. Their importance, however, remains a matter of considerable controversy. Furthermore, even if disturbances of this kind are of substantial magnitude, the degree of departure from price stability that can be justified on welfare-theoretic grounds may well be less than is often supposed, as we show in section xx below.

4 Extensions of the Basic Analysis

Here we sketch extensions of our utility-based welfare criterion to incorporate several complications from which we have abstracted in the basic analysis presented in section 2. We shall give particular attention to complications that illustrate some of the reasons just sketched for complete stabilization of the price level not to be optimal.

4.1 Transactions Frictions

In section 2, we have abstracted from the welfare consequences of the transactions frictions that account for the demand for the monetary base. Our results therefore apply to a "cashless" of the kind discussed in chapter 2. Here we consider the way in which they must be modified in order to allow for non-negligible welfare effects of transactions frictions.

As in chapter 2, we may represent the welfare consequences of variations in the degree to which these frictions distort transactions by including real money balances as an additional argument of the utility function of the representative household. This generalization makes real marginal cost, and hence the equilibrium level of output under flexible prices a function of the (endogenous) level of real balances, in addition to the exogenous state of preferences and technology, in the case that the indirect utility function $u(c, m)$ is not additively separable. Substituting the equilibrium level of real balances

$$\hat{m}_t = \eta_y \hat{Y}_t - \eta_i(\hat{\imath}_t - \hat{\imath}_t^m) + \epsilon_t^m \tag{4.1}$$

into the household labor-supply relation, we have shown in chapter 4 that average real marginal cost is given by

$$\hat{s}_t = \epsilon_{mc}[x_t + \varphi(\hat{\imath}_t - \hat{\imath}_t^m)], \tag{4.2}$$

where

$$\epsilon_{mc} \equiv \sigma^{-1} + \omega - \chi\eta_y, \tag{4.3}$$

$$\varphi \equiv \frac{\eta_i \chi}{\epsilon_{mc}}, \tag{4.4}$$

and

$$\hat{Y}_t^n \equiv \frac{\sigma^{-1} g_t + \omega q_t + \chi \epsilon_t^m}{\epsilon_{mc}}. \tag{4.5}$$

In these expressions, we once again use the coefficient $\chi \equiv \bar{m} u_{cm}/u_c$ to measure the degree of complementarity between private expenditure and real balances.

Here we define the natural rate of output $\hat{Y}_t^n$ as the flexible-price equilibrium level of output when the interest rate differential Δ_t is fixed at its steady-state level $\bar{\Delta}$, so that $\hat{Y}_t^n$ is again an exogenous process. This definition also has the advantage that, up to a log-linear approximation, the amount by which the (log) efficient level of output exceeds the (log) natural rate is a constant,[33] in the case that both the steady-state inefficiency wedge Φ and the steady-state interest-rate differential $\bar{\Delta}$ are only of order $\mathcal{O}(||\xi||)$. This constant gap (up to a residual of order $\mathcal{O}(||\xi||^2)$) is given by

$$x^* \equiv \frac{\Phi + s_m \eta_y}{\epsilon_{mc}}, \tag{4.6}$$

where $s_m \equiv \bar{m} u_m / \bar{c} u_c \geq 0$ measures the interest cost of real balances as a fraction of the value of private expenditure. We assume as in chapter 4 that $\epsilon_{mc} > 0$, so that if $\Phi \geq 0$, $x^* > 0$. We observe that the signs of the effects upon the natural rate of output of the various real disturbances discussed in section 3.2 remain the same. Now, however (if $\chi \neq 0$), disturbances to the money demand function, possibly due to shifts in the transactions technology, also affect the natural rate of output.

It follows from (4.2) that in this more general case, the "New Classical" aggregate supply relation takes the form

$$\pi_t = \kappa[x_t + \varphi(\hat{\imath}_t - \hat{\imath}_t^m)] + E_{t-1}\pi_t \tag{4.7}$$

instead of (2.16), where again $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$, and

$$\kappa \equiv \frac{\iota}{1-\iota} \frac{\epsilon_{mc}}{1+\omega\theta} > 0.$$

In the case of Calvo pricing, instead, the aggregate supply relation now takes the form

$$\pi_t = \kappa[x_t + \varphi(\hat{\imath}_t - \hat{\imath}_t^m)] + \beta E_t \pi_{t+1}, \tag{4.8}$$

where again

$$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \frac{\epsilon_{mc}}{1+\omega\theta} > 0.$$

Note that in either case the interest-rate differential appears as a shift factor in the aggregate supply relation, because of its (small) effect on the real marginal cost of supply.

[33] In the present case, the efficient level of output at any point in time is the solution to two equations, stating that real marginal cost is equal to one and that there is satiation in real money balances. Because the second condition implies a zero interest-rate differential regardless of the real disturbances, both the natural rate of output and the efficient level can be defined as output variations in response to real disturbances that maintain real marginal cost constant in the case of a constant interest differential.

We can again approximate the utility from private expenditure using a second-order Taylor series expansion, obtaining

$$\begin{aligned} u(Y_t, m_t; \xi_t) &= \bar{Y} u_c \left\{ \hat{Y}_t + \frac{1}{2}(1-\sigma^{-1})\hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t + s_m \hat{m}_t + \frac{1}{2} s_m (1-\sigma_m^{-1}) \hat{m}_t^2 \right. \\ &+ \left. \chi \hat{m}_t \hat{Y}_t + s_m \sigma^{-1} g_t \hat{m}_t + s_m(\chi + \sigma_m^{-1}) \epsilon_t^m \hat{m}_t \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \end{aligned} \quad (4.9)$$

as a generalization of (2.8). Here we again define the elasticity $\sigma_m \equiv -u_m/\bar{m}u_{mm} > 0$, the exogenous disturbance term g_t is defined as in the cashless model, and

$$\epsilon_t^m \equiv (\chi + \sigma_m^{-1})^{-1} \left[\frac{u_{m\xi}}{u_m} \xi_t - \sigma^{-1} g_t \right]$$

is the exogenous disturbance term in the money-demand relation (4.1).

Once again, we can legitimately substitute into this our log-linear approximate solution to our structural equations only if the coefficients on the linear terms are at most of order $\mathcal{O}(||\xi||)$. This means that we must assume that the economy is sufficiently close to being satiated in money balances. In order to contemplate a series of economies that come as close as we like to this limit, without having to change the specification of preferences or technology (including the transactions technology), it is important to allow for interest payments on the monetary base, which we shall suppose are always close to a steady-state rate of $\bar{\imath}^m$. The steady-state interest differential $\bar{\Delta} \equiv (\bar{\imath} - \bar{\imath}^m)/(1+\bar{\imath})$ as a measure of the degree to which there is not complete satiation in money in the steady state around which we expand, we shall now assume that *both* $\bar{\Delta}$ and Φ are of order $\mathcal{O}(||\xi||)$.

As we consider economies with progressively smaller positive values for $\bar{\Delta}$, obtained by raising the value of $\bar{\imath}^m$, we assume that the steady-state quantities $\bar{Y}, \bar{m}$ approach finite, positive limiting values, and that the partial derivatives of utility also have well-defined limits from this direction. We furthermore assume that the limiting value of u_{mm} is negative (so that σ_m is finite), even though this requires that u_{mm} be discontinuous at the satiation level of real balances.[34] Then in the limit of small $\bar{\Delta}$, we find that s_m is of order $\mathcal{O}(\bar{\Delta})$, as

[34] Note that our assumption that $u_m = 0$ in the case of all levels of real balances in excess of that required for satiation implies that $u_{mm} = 0$ for all values of m *higher* than the limiting value $\bar{m}$, in the case of an income level $\bar{Y}$. This sort of discontinuity typically occurs, for example, in the case of a cash-constraint model of the transactions technology, like that considered in chapter 2, section xx.

is σ_m, though the ratio $s_m\sigma_m^{-1}$ approaches a positive limit as $\bar{\Delta} \to 0$, since

$$\frac{s_m}{\sigma_m} = -\frac{\bar{m}^2 u_{mm}}{\bar{Y} u_c} + \mathcal{O}(\bar{\Delta}).$$

This allows us to drop some of the quadratic terms from (4.9), the coefficients of which are only of order $\mathcal{O}(\bar{\Delta})$.

We note also that in the limit of small $\bar{\Delta}$, the elasticities of the money-demand relation (4.1) reduce to

$$\eta_y = -\frac{\bar{Y} u_{cm}}{\bar{m} u_{mm}} + \mathcal{O}(\bar{\Delta}), \qquad \eta_i = -\frac{u_c}{\bar{m} u_{mm}} + \mathcal{O}(\bar{\Delta}). \tag{4.10}$$

This allows us to substitute $(\eta_i \bar{v})^{-1}$ for $s_m\sigma_m^{-1}$ and $\chi\eta_i\bar{v}$ for η_y in the coefficients of quadratic terms,[35] where $\bar{v} \equiv \bar{Y}/\bar{m}$ is the steady-state "velocity of money". With these substitutions, (4.9) can be written as

$$\begin{aligned} u(Y_t, m_t; \xi_t) &= \bar{Y} u_c \Big\{ \hat{Y}_t + \frac{1}{2}(1-\sigma^{-1})\hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t + s_m \hat{m}_t \\ &+ \chi \hat{m}_t \hat{Y}_t - \frac{1}{2}(\eta_i \bar{v})^{-1}(\hat{m}_t - \epsilon_t^m)^2 \Big\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \tag{4.11}$$

We can then substitute (4.1) for equilibrium real balances $\hat{m}_t$, obtaining

$$\begin{aligned} u(Y_t, m_t; \xi_t) &= \bar{Y} u_c \Big\{ \hat{Y}_t + \frac{1}{2}(1-\sigma^{-1})\hat{Y}_t^2 + \sigma^{-1} g_t \hat{Y}_t + s_m \eta_y \hat{Y}_t - s_m \eta_i (\hat{\imath}_t - \hat{\imath}_t^m) \\ &+ \frac{1}{2}\chi \eta_y \hat{Y}_t^2 + \chi \epsilon_t^m \hat{Y}_t - \frac{1}{2}\bar{v}^{-1}\eta_i(\hat{\imath}_t - \hat{\imath}_t^m)^2 \Big\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \end{aligned} \tag{4.12}$$

Subtracting (2.10) from (4.12), we obtain

$$\begin{aligned} U_t &= \bar{Y} u_c \Big\{ (\Phi + s_m \eta_y)\hat{Y}_t - s_m \eta_i (\hat{\imath}_t - \hat{\imath}_t^m) + [\sigma^{-1} g_t + \omega q_t + \chi \epsilon_t^m]\hat{Y}_t - \frac{1}{2}\bar{v}^{-1}\eta_i(\hat{\imath}_t - \hat{\imath}_t^m)^2 \\ &\quad - \frac{1}{2}\epsilon_{mc}\hat{Y}_t^2 - \frac{1}{2}(\theta^{-1} + \omega)\text{var}_i \hat{y}_t(i) \Big\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \qquad (4.13) \\ &= -\frac{\bar{Y} u_c}{2} \Big\{ \epsilon_{mc}(x_t - x^*)^2 + \bar{v}^{-1}\eta_i(\hat{\imath}_t - \hat{\imath}_t^m + \bar{\Delta})^2 \\ &\quad + (\theta^{-1} + \omega)\text{var}_i \hat{y}_t(i) \Big\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \qquad (4.14) \end{aligned}$$

[35]The advantage of replacing η_y by $\chi\eta_i\bar{v}$ is that it is then clear what form our results take in the familiar special case in which it is assumed that $u_{cm} = 0$; we simply set χ equal to zero in the expressions derived below.

as a generalization of (2.12). Note that in (4.13), the linear terms both have coefficients that are of order $\mathcal{O}(||\xi||)$, as is required for validity of welfare comparisons based on a log-linear solution to our model, as long as both Φ and $\bar{\Delta}$ are of order $\mathcal{O}(||\xi||)$. In (4.14), the optimal level x^* for the output gap is given by (4.6), while the optimal level for the interest-rate differential is zero, since the condition $i_t = i_t^m$, the Friedman (1969) condition for satiation in real money balances, corresponds to $\hat{\imath}_t = \hat{\imath}_t^m - \bar{\Delta} + \mathcal{O}(\bar{\Delta}^2)$.

Finally, substituting for output dispersion as a function of inflation as before, in the case of our baseline (Calvo) model of price-setting, we again obtain an approximate welfare criterion of the form (2.22), where now the normalized loss function is given by

$$L_t = \pi_t^2 + \lambda_x(x_t - x^*)^2 + \lambda_i(\hat{\imath}_t - \hat{\imath}_t^m + \bar{\Delta})^2, \tag{4.15}$$

with weights

$$\lambda_x = \frac{\kappa}{\theta} > 0, \qquad\qquad \lambda_i = \frac{\eta_i}{\bar{v}\epsilon_{mc}}\lambda_x > 0. \tag{4.16}$$

An alternative expression for the weight on the interest-rate term, equivalent under our small-$\bar{\Delta}$ approximation, though applicable only in the case that $\chi \neq 0$, is

$$\lambda_i = \frac{\eta_i}{\eta_y}\varphi\lambda_x. \tag{4.17}$$

Thus taking account of transactions frictions adds an additional term to the loss function, with a positive weight on squared deviations of the interest-rate differential from its optimal size, which is zero.

Note that in the "cashless limit" discussed in chapter 2, $\bar{v}^{-1} \to 0$, so that $\lambda_i \to 0$, and we recover our results above for the cashless model. However, it is important to note that the interest-rate variability term does *not* vanish under the assumption that utility is additively separable between consumption and real balances, so that $u_{cm} = 0$. While this last assumption (which implies that $\chi = 0$) results in the disappearance of real-balance effects from both the aggregate-supply and IS relations of our model of the transmission mechanism, and similarly implies that money-demand disturbances have no effect on the natural rates of output or of interest, it does not imply that $\lambda_i = 0$. Thus it makes a difference whether one assumes that χ is negligible in size because of approximate additive-separability, or instead because equilibrium real balances are small (velocity is large).

One case in which our previous conclusions are largely unaffected is that in which the central bank's interest-rate operating target is implemented through adjustments of the interest paid on the monetary base, so that Δ_t is equal to a fixed spread at all times, regardless of how i_t varies.[36] In this case, the $(\hat{\imath}_t - \hat{\imath}^m_t + \bar{\Delta})^2$ term in (4.15) is a constant, independent of how inflation, output and interest rates vary over time. Optimal policy will then again be one that minimizes a loss function of the form (2.23); the only difference that monetary frictions make would be to the definitions of ϵ_{mc} (and hence of κ and λ_x) and of $\hat{Y}^n_t$. In particular, we will find once again that optimal policy involves complete stabilization of the price level, just as in our analysis of the cashless model in section xx.

If, instead, the interest paid on the monetary base is equal to a constant $\bar{\imath}^m$ at all times (perhaps zero, as in the U.S. at present), then the final term in (4.15) is not irrelevant. In this case, the welfare-theoretic loss function reduces to

$$L_t = \pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (\hat{\imath}_t - i^*)^2, \tag{4.18}$$

where now the optimal nominal interest rate is given by

$$i^* \equiv \log \frac{1 + \bar{\imath}^m}{1 + \bar{\imath}} = -\bar{\Delta} + \mathcal{O}(\bar{\Delta}^2);$$

that is, it is equal to the constant interest rate paid on the monetary base. Note that the final term results in a loss function of the kind assumed by Williams (1999), where the additional term is instead motivated by reference to "aversion to interest-rate variability".

The additional term means that complete stabilization of the price level is no longer optimal, for two reasons. The first is that, as long as $i^m < \bar{\imath} \equiv \beta^{-1} - 1$, the steady-state nominal interest rate that minimizes the last term in (4.15) requires expected deflation, as

[36] As discussed in chapter 1, a number of central banks do already implement policy through channel systems under which the interest rate paid on central bank balances is always equal to the current operating target for the overnight cash rate minus a constant spread. However, even in these countries, no interest is paid on currency, and currency balances continue to constitute most of the monetary base. Thus these countries do not actually represent examples of the ideal system considered here.

argued by Friedman (1969). There is thus now a conflict between the steady-state rate of inflation needed to minimize the first term and that needed to minimize the third. In fact, the long-run inflation rate under an optimal policy commitment, in the absence of stochastic disturbances, is generally intermediate between the two – higher than the Friedman rate (*i.e.*, minus the rate of time preference), but still negative, as shown in chapter 7.

And second, there is now a conflict between the pattern of responses to shocks that minimizes the first term (*i.e.*, no inflation variation at all) and the pattern required to minimize the third term (no interest-rate variation). Insofar as shocks affect the natural rate of interest (and we have shown that many different types of real disturbances all should), nominal interest-rate variations are required to keep inflation stable, and vice versa. In addition, it need not even be true any longer that complete inflation stabilization minimizes the second term – for if $\kappa_i > 0$, the interest-rate variations required to stabilize inflation will result in at least a small amount of output-gap variation as well.

A special case is possible in which no such conflict arises. Suppose that we assume instead the "New Classical" model of pricing, in which all prices are adjusted each period, though some new prices are chosen a period in advance. Let us again suppose that $i_t^m = \bar{\imath}^m$ at all times. In this case, the corresponding normalized loss function is given by

$$L_t = (\pi_t - E_{t-1}\pi_t)^2 + \lambda_x(x_t - x^*)^2 + \lambda_i(\hat{\imath}_t - i^*)^2, \tag{4.19}$$

where the weights are again given by (4.16), but using the definition of κ in (4.7). If we also assume that $\Phi = 0$, then the approximations (4.10) imply that $x^* = -\varphi i^* > 0$, up to a residual of order $\mathcal{O}(||\xi||^2)$. Then the aggregate-supply relation (4.7) can alternatively be written

$$\pi_t = \kappa[(x_t - x^*) + \varphi(\hat{\imath}_t - i^*)] + E_{t-1}\pi_t. \tag{4.20}$$

Then there is no problem with simultaneously minimizing all three terms in (4.19). This simply requires that one set $\hat{\imath}_t = i^*$ each period, and make inflation equal whatever value was forecasted in the previous period, in which case (4.7) implies that $x_t = x^*$ as well. Minimization of the interest-rate variation term has implications only for *expected* inflation, while minimization of the term representing the costs of price dispersion has implications only for *unexpected* inflation, and so, in this special case, there is no conflict between fully achieving both goals at all times.

Even if $\Phi > 0$, we note that

$$x^* + \varphi i^* = \frac{\Phi}{\epsilon_{mc}} > 0. \tag{4.21}$$

Using this to substitute for x^*, the discounted loss measure can be written as

$$\begin{aligned} E_0\left\{\sum_{t=0}^{\infty}\beta^t L_t\right\} &= -2\frac{\Phi}{\theta\epsilon_{mc}}[\pi_0 - \kappa_i\hat{\hat{\imath}}_0] + E_0\left\{\sum_{t=0}^{\infty}\beta^t\left[(\pi_t - E_{t-1}\pi_t)^2\right.\right. \\ &\left.\left. +\lambda_x(x_t+\varphi i^*)^2 + 2\frac{\lambda_x\varphi\Phi}{\epsilon_{mc}}(\hat{\imath}_t - i^*) + \lambda_i(\hat{\imath}_t - i^*)^2\right]\right\}, \end{aligned} \tag{4.22}$$

generalizing (3.2). All terms on the right-hand side except the first one are again minimized by setting $\hat{\imath}_t = i^*$ and $\pi_t = E_{t-1}\pi_t$ each period. (The third term inside the large square brackets is necessarily non-negative each period, because of the equilibrium requirement that $i_t \geq \bar{\imath}^m$, or $\hat{\imath}_t \geq i*$.) The first term indicates that there is an additional welfare gain from unexpected inflation in period zero, because what is decided for this period cannot affect inflation expectations in the previous period; and if one allows oneself to take advantage of that opportunity, the inflation rate in period zero should be chosen to be somewhat higher than had been expected. But thereafter, one will make unexpected inflation equal zero every period, as this is not inconsistent with setting $\hat{\imath}_t = i^*$ in every period. Furthermore, under a policy that is optimal from a timeless perspective, one will simply arrange for zero unexpected inflation and $\hat{\imath}_t = i^*$ each period.

But this case, in which the distortions resulting from price stickiness can be completely eliminated without putting any restriction upon the process that expected inflation may follow, is clearly a very special one. In general, variations in expected inflation as a result of fluctuations in the natural rate of interest (as will be required in order to maintain $\hat{\imath}_t = i^*$ at all times) will result in relative-price distortions. Hence the goal of minimizing the distortions associated with transactions frictions will conflict with that of minimizing the distortions resulting from price stickiness. Before discussing further the nature of this tension between alternative stabilization objectives, we shall argue that a similar concern with nominal interest-rate stabilization can be justified on alternative grounds.

4.2 The Zero Interest-Rate Lower Bound

Even in the case of a cashless economy, incomplete inflation stabilization may be optimal, in order to reduce the variability of nominal interest rates in response to shocks. The reason is the equilibrium requirement that $i_t \geq 0$ at all times.[37] If shocks are sufficiently small, this poses no obstacle to complete inflation stabilization, but if the natural rate of interest is sometimes negative (and by this we mean the natural *short* rate, which is more volatile than the associated natural longer rates), complete stabilization of inflation will be infeasible.

In that case, which seems reasonably likely, it is of some interest to consider the nature of optimal policy subject to the constraint of respecting the zero lower bound. It is reasonably clear that such policy will involve less variation in nominal interest rates than occurs in the natural rate of interest; in particular, market rates will not fall as much as the natural rate does in those states in which it becomes negative. Characterizing the optimal behavior of market rates is a problem beyond the scope of the linear-quadratic optimization methods used here; however, we can consider a related problem that gives some insight into the way in which such a constraint should affect optimal policy. This is to replace the constraint that the nominal interest must be non-negative in every period with a constraint upon its *variability*.

Specifically, Rotemberg and Woodford (1997, 1999a) propose to approximate the effects of the lower bound by imposing instead a requirement that the mean federal funds rate be at least k standard deviations above the theoretical lower bound, where the coefficient k is large enough to imply that violations of the lower bound should be infrequent. The alternative constraint, while inexact, has the advantage that checking it requires only computation of

[37] More generally, the requirement is that $i_t \geq i_t^m$, but here we shall suppose that zero interest is paid on the monetary base, in order to make this constraint as weak as possible. We assume that the payment of negative interest on the monetary base, as proposed by Gesell, Keynes (1936), and more recently, Buiter and xxxx, is technically infeasible.

first and second moments under alternative policy regimes, whereas checking whether the funds rate is predicted to be negative in *any* state would depend upon fine details of the distribution of shocks. In addition, a constraint of this form has the advantage that, assuming linear structural equations and a quadratic loss function, the constrained-optimal policy is a linear rule, just like the unconstrained optimum. Hence our linear methods can still be used to characterize optimal policy.

This can be demonstrated as follows. Note that the constraint can equivalently be expressed as a requirement that the average value of i_t^2 be not more than $K \equiv 1 + k^{-2}$ times the square of the average value of i_t,[38] which latter average must also be non-negative. If we use discounted averages, for conformity with the other terms in our welfare measure, we obtain constraints of the form

$$E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t i_t \right\} \geq 0, \tag{4.23}$$

$$E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t i_t^2 \right\} \leq K \left[E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t i_t \right\} \right]^2. \tag{4.24}$$

Now suppose that we wish to minimize an expected discounted sum of quadratic losses

$$E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t L_t \right\} \tag{4.25}$$

subject to (4.23) – (4.24), and let m_1, m_2 be the discounted average values of i_t and i_t^2 associated with the optimal policy. Then this is also the policy that minimizes (4.25) subject to the two constraints

$$\begin{aligned} E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t i_t \right\} &\geq m_1, \\ E_0 \left\{ (1-\beta) \sum_{t=0}^{\infty} \beta^t i_t^2 \right\} &\leq m_2, \end{aligned}$$

since any policy consistent with both of these also satisfies the weaker constraints (4.23) – (4.24).

[38] By the expression i_t we here actually mean $\log(1 + i_t)$, or $\hat{\imath}_t + \bar{\imath}$.

Then by the Kuhn-Tucker theorem, the policy that minimizes the expected discounted value of (4.25) subject to (4.23) – (4.24) can be shown to also minimize an (unconstrained) loss criterion of the form

$$E_0\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t L_t\right\} - \mu_1 E_0\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t r_t\right\} + \mu_2 E_0\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t r_t^2\right\},$$

where μ_1 and μ_2 are appropriately chosen Lagrange multipliers. (Both multipliers are non-negative, and if the constraint (4.24) binds, $\mu_1 = 2Km_1\mu_2 > 0$.) Finally, the terms in this expression can be rearranged to yield a discounted loss criterion of the form (4.25), but with L_t replaced by

$$\tilde{L}_t \equiv L_t + \tilde{\lambda}_i(\hat{\imath}_t - i^{**})^2, \tag{4.26}$$

where $\tilde{\lambda}_i = \mu_2 \geq 0$ and (if $\mu_2 > 0$)

$$i^{**} = \frac{\mu_1}{2\mu_2} - \bar{\imath} = Km_1 - \bar{\imath}.$$

(There is also a constant term involved in completing the square, but as usual we drop this as it has no effect upon our ranking of alternative policies. Note that we have written the quadratic term in terms of a target value for $\hat{\imath}_t \equiv i_t - \bar{\imath}$ rather than i_t, for consistency with our previous results.)

Thus the optimal policy minimizes the expected discounted value of a quadratic loss function (4.26), subject to the constraints imposed by the structural equations of our model. If the latter are linear, the optimal policy will itself be linear. Note that the effective loss function (4.26) contains a quadratic penalty for interest-rate variations (in the case that constraint (4.24) binds), even if the "direct" social loss function L_t is independent of the path of the interest rate. For example, consider again our baseline model of Calvo pricing, in the "cashless limit". The direct loss function is then given by (2.23), which involves only inflation and the output gap. But if the fluctuations in the natural rate of interest are large enough for (4.24) to bind – *i.e.*, if (4.24) is violated by the solution $\hat{\imath}_t = \hat{r}_t^n$ – then optimal policy actually minimizes a loss function of the form (4.15), exactly as we previously concluded by taking account of transactions frictions. The particular *type* of departure from price stability that is motivated by the need to respect the interest-rate lower bound is exactly the same as the kind that results from taking account of transactions frictions.

The primary qualitative difference between the loss functions motivated in the two ways is that transactions frictions lead to a loss function (4.15) with a "target" interest rate $i^* < 0$ (*i.e.*, lower than the steady-state interest rate $\bar{\imath}$ consistent with zero inflation), while the interest-rate lower bound alone would suggest a "target" interest rate $i^{**} > 0$. For we have shown above that when (4.24) does not bind, optimal policy in the cashless limit involves a deterministic component of inflation that is non-negative (and converging asymptotically to zero); hence average inflation is non-zero. If instead (4.24) does bind, the only reason to choose a different deterministic component for inflation would be in order to relax the constraint, which would involve making average inflation higher (so that the average funds rate can be higher). Thus optimal policy should involve $m_1 > \bar{\imath}$. But since $K > 1$, this implies that $i^{**} > 0$.

If transactions frictions are non-negligible *and* the interest-rate lower bound binds as well, the quadratic interest-rate term in (4.26) is added to the quadratic interest-rate term already present in (4.15). The result is a loss function that again has the same form,

$$\hat{L}_t = \pi_t^2 + \lambda_x (x_t - x^*)^2 + \hat{\lambda}_i (\hat{\imath}_t - \hat{\imath})^2, \tag{4.27}$$

where now $\hat{\lambda}_i = \lambda_i + \mu_2$ is an even larger positive coefficient, while $\hat{\imath}$ is intermediate between i^* and i^{**} (and thus may have either sign). In fact, the value of $\hat{\imath}$, like the value of x^*, matters only for the deterministic component of optimal policy; the optimal responses to shocks depend only upon the weights $\lambda_x, \hat{\lambda}_i$ of the loss function. Thus in this regard both considerations point in the same direction, toward the likely importance of including a quadratic interest-rate term in the loss function. Hence we shall give considerable attention in chapter 7 to the consequences for optimal policy of including such a term.[39]

[39] Note, however, that both considerations justify a concern to reduce the variability of the *level* of interest rates, and *not* a concern with the variability of interest-rate *changes*. The latter sort of "interest-rate smoothing" goal is often assumed to characterize the behavior of actual central banks. As we show in chapter 7, it is possible to justify such the assignment of such a goal to the central bank as part of an optimal *delegation* problem, even if it is not part of the *social* loss function with which are concerned here.

How much are such considerations likely to matter? We investigate this numerically in a calibrated example. The values of the parameters α, β, σ, κ, and ϵ_{mc} are as in Table 4.1, based upon the estimates of Rotemberg and Woodford (1997) discussed in chapter 4.[40] The values for η_y, η_i, and χ given in the table are those implied by estimates of long-run money demand for the U.S., as discussed in chapter 2.[41]

For present purposes, the only aspect of the exogenous disturbances that matters is the implied evolution of the natural rate of interest $\hat{r}_t^n$; for simplicity, we assume that this variable follows a stationary first-order autoregressive process, with a mean of zero, and a standard deviation and serial correlation coefficient as specified in the table. As is explained further in chapter 7, these numerical values are motivated by aspects of the estimated model of Rotemberg and Woodford (1997), though that model involves a more complex specification of the shock processes.

Finally, the structural parameters given in the table imply a value for θ equal to 7.88. Using this, we are able to obtain a theoretical value for λ_x in the utility-based loss function (2.23), which value is also given in the table.[42] We simplify our calculations by assuming that $\Phi = 0$, so that $x^* = 0$.

Using these parameter values, we can explore the tradeoff between minimization of the deadweight losses measured by (2.23) and stabilization of the short-term nominal interest rate. Letting L_t^0 be this loss function (*i.e.*, the loss function abstracting from any costs of interest-rate variability), we can compute its expected discounted value

$$\hat{E}[L^0] \equiv E\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t L_t^0\right\} \tag{4.28}$$

[40] Note that the value of ω reported in Table 4.1 is not exactly the one implied by the values given here, since in Rotemberg and Woodford the value of ω is inferred from the value of ϵ_{mc} assuming that $\chi = 0$.

[41] The value for χ used here is actually slightly higher than that derived in chapter 2, as the value of χ implied by the long-run money demand estimates depends upon the assumed value of σ. However, the values used in both chapters agree to the first three decimal places.

[42] Note that the value given here is equal to $16\kappa/\theta$, rather than κ/θ. This is because we report the loss function weights λ_x, λ_i that are appropriate to use when inflation and interest rates are measured as annualized percentage rates, despite the fact that our model is quarterly. The square of the annualized percentage inflation rate is thus not π_t^2 but $16\pi_t^2$, in terms of the notation used in our earlier theoretical derivations.

Table 6.1: Calibrated parameter values for the quarterly model used for Figure 6.1.

Structural parameters	
α	0.66
β	0.99
σ^{-1}	0.16
κ	.024
ϵ_{mc}	0.63
η_y	1
η_i	28
χ	0.02
Shock process	
$\rho(\hat{r}^n)$	0.35
$\mathrm{sd}(\hat{r}^n)$	3.72
Loss function	
x^*	0
λ_x	.048
λ_i	.077
Interest-Rate Bound	
k	2.26
$\bar{r}$	2.99

in the case of any stochastic processes for inflation and the output gap. Here we use the notation $\hat{E}$ to denote a discounted expectation rather than simply the unconditional expectation of the random variable L_t^0; and the operator E on the right-hand side indicates that we take an unconditional expectation over possible initial states of the exogenous disturbance $\hat{r}_0^n$. We measure the degree of interest-rate variability[43] associated with any possible equilibrium in terms of the statistic

$$V[i] \equiv E\left\{(1-\beta)\sum_{t=0}^{\infty}\beta^t \hat{\imath}_t^2\right\}.$$

We then consider the policies that minimize $\hat{E}[L^0]$ subject to a constraint that $V[i]$ not

[43] The statistic actually measures the average squared deviation of interest rates from the level $\bar{\imath}$ consistent with zero inflation in the absence of disturbances. However, all equilibria on the efficient frontier shown in Figure 6.1 involve zero average values of $\hat{\imath}_t$, since any non-zero steady-state inflation rate would increase the value of *both* $\hat{E}[L^0]$ and $V[i]$, holding constant the way the variables deviate from their steady-state values as a function of the history of disturbances. Among stochastic processes with that property, $V[i]$ is a discounted measure of the series' variability about its mean.

exceed some finite value.

The efficient frontier for these two statistics is shown in Figure 6.1. (The nature of the constrained-efficient policies is discussed in the next chapter.) We observe that it is possible to achieve the theoretical lower bound of zero for $\hat{E}[L^0]$, by completely stabilizing inflation and the output gap as discussed in section 3, only if we are willing to tolerate an interest-rate variability of $V[i] = 13.83$, corresponding to a (discounted) standard deviation of 3.72 percentage points for the federal funds rate. (This is of course just the assumed standard deviation of fluctuations in the natural rate of interest.) Lower interest-rate variability requires that one accept less complete stabilization of inflation and the output gap, indicated by a positive value for $\hat{E}[L^0]$. Complete interest-rate stabilization would require that $\hat{E}[L^0]$ take a value of 2.10, a level of deadweight loss more than twice that associated with steady inflation of one percent per year.[44] Statistics relating to the variability of inflation, the output gap, and the federal funds rate in these two extreme cases are given by the first and last lines of Table 6.2, using measures analogous to $V[i]$ in the case of the other two variables as well. Note that in terms of these measures,

$$\hat{E}[L^0] = V[\pi] + \lambda_x V[x].$$

This frontier indicates how a concern with interest-rate variability, for whatever reason, would affect the degree to which it would be optimal to stabilize inflation and the output gap. We are particularly interested, however, in the degree of attention to this goal that would be justified by either of the two considerations treated above. Taking account of transactions frictions, we would wish to minimize $\hat{E}[L]$, where now L_t is the loss function (4.15) including an interest-rate variability term. The fact that $i^* \neq 0$ in (4.15) affects only the deterministic part of the optimal paths of the endogenous variables, as with our discussion of the case $x^* \neq 0$ in section 3; it has no effect upon the optimal responses to shocks. Thus we can determine the optimal responses to shocks by minimizing $\hat{E}[L]$ with i^* set equal to zero, which amounts to minimizing $\hat{E}[L^0] + \lambda_i V[i]$. This policy corresponds to

[44] A similar efficient frontier in the case of the more complicated model estimated by Rotemberg and Woodford (1997) is shown in Figure 5 of that paper.

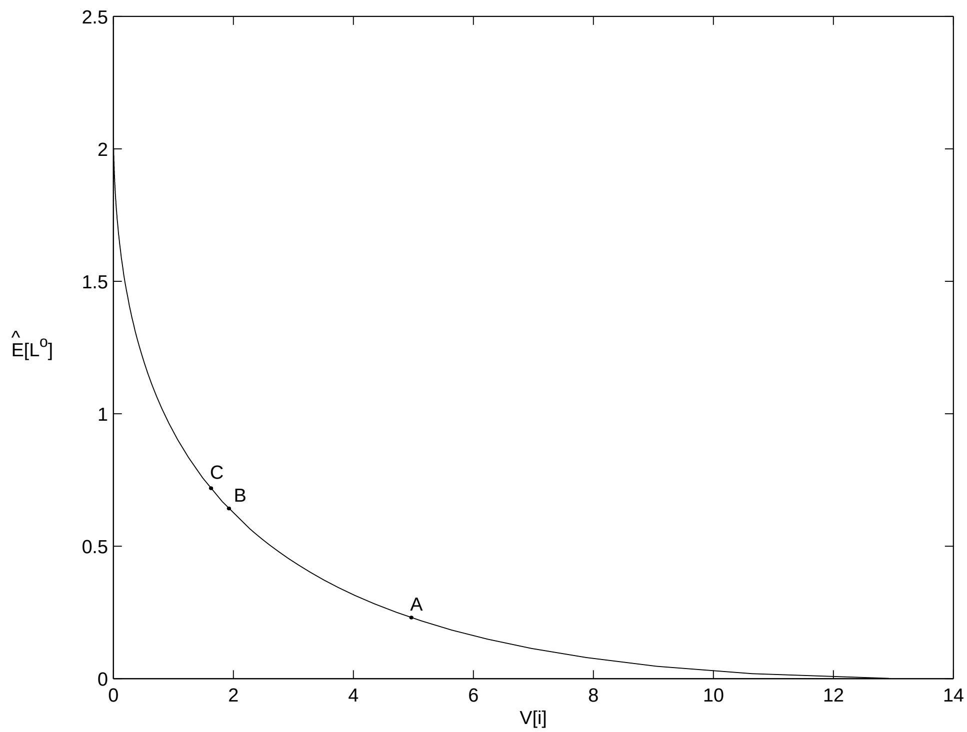

Figure 6.1: The tradeoff between inflation/output-gap stabilization and interest-rate stabilization.

a point on the frontier in Figure 6.1, namely the point at which the slope is equal to $-\lambda_i$, where λ_i is given by (4.17).

Assuming the structural parameters given in Table 6.1, equation (4.4) implies that for the U.S., φ should equal approximately .22 years. Then (4.17) implies that $\lambda_i = .077$, the value also given in the table.[45] This corresponds to point A on the frontier in the figure.[46] Line 2 of Table 6.2 reports the variability of each of the endogenous variables in this equilibrium.

[45]This value, based on an assumption that $\chi = .02$, is probably too large by a factor of two or more, for reasons discussed in chapter 2. Williams (1999), for example, assumes a value (if the weight on the squared inflation term is normalized as one) equal to .02.

[46]In order to show the optimal equilibria under differing assumptions on a single diagram, we use the same structural model in each case – namely, our baseline model, abstracting from real-balance effects, that is also used in the numerical analysis of chapter 7 – simply varying the assumed welfare criterion in each case. This means that in the case of point A, we are actually assuming parameter values in the structural equations ($\chi = 0$) that are not completely consistent with those used to derive the loss function ($\chi = .02$). However, this makes only a small difference to our characterization of optimal policy when transactions frictions are allowed for; the most important effect is upon the loss function, rather than upon the structural equations. Were we to assume additively separable preferences $u(c, m)$, there would be no effect upon the structural equations at all, but the loss function would nonetheless be modified as indicated in (4.15).

Table 6.2: Examples of Policies on the Efficient Frontier.

$\hat{\lambda}_i$	V[π]	V[x]	V[i]	$\hat{E}[L^0]$
0	0	0	13.83	0
.077	.037	4.015	4.961	.231
.236	.130	10.60	1.921	.643
.277	.151	11.75	1.623	.719
∞	.677	29.35	0	2.096

Now suppose instead that we abstract from transactions frictions, but take account of the lower bound on nominal interest rates, or more precisely, that we impose the constraints (4.23) – (4.24). Following Rotemberg and Woodford (1997), we let k equal the ratio of the standard deviation of the funds rate to its mean in the long-run stationary distribution implied by their estimated VAR model of U.S. data. We also assume a value for $\bar{\imath}$, the steady-state real funds rate, equal to the mean real funds rate implied by this same long-run stationary distribution. These values are indicated in Table 6.1.

As shown above, the optimal policy subject to these constraints will minimize $\hat{E}[\tilde{L}]$, where $\tilde{L}_t$ is defined by (4.26) with L_t equal to L_t^0. Once again, the fact that $i^{**} \neq 0$ does not affect the optimal responses to shocks, and so these are as in one of the equilibria on the frontier shown in Figure 6.1. As is explained further in section xx of chapter 7, under our assumed parameter values the implied value of $\tilde{\lambda}_i$ in (4.26), given by the Lagrange multiplier on constraint (4.24), is equal to .236. (See Table 7.1.) The corresponding optimal responses to shocks are those associated with point B on the frontier, which is the point at which the frontier has this steeper slope. The third line of Table 6.2 reports the variability of each of the endogenous variables in this equilibrium. (In the computations reported in the table, the deterministic component of each variable is equal to zero in all periods, so that these statistics refer only to the variations in the variables due to fluctuations in the natural rate of interest.) The higher effective penalty upon interest-rate variability results in less equilibrium variation in nominal interest rates, at the cost of more variation in both inflation and the output gap.

Finally, suppose that in our welfare analysis we take account both of transactions frictions and of the lower bound on nominal interest rates. We then find the policy that minimizes $\hat{E}[\tilde{L}]$ when L_t is given by (4.15) rather than by L_t^0. As noted above, this policy minimizes a criterion of the form $\hat{E}[\hat{L}]$, where $\hat{L}_t$ is given by (4.27). In the case of our assumed parameter values, $\hat{\lambda}_i$ is equal to .277, a larger value than we would obtain taking account of either of these considerations individually. Once again, the optimal responses to shocks correspond to a point on the frontier shown in Figure 6.1, the point labeled C. The variability of the endogenous variables in this equilibrium is indicated on the fourth line of Table 6.2.

It is interesting to note that the value $\hat{\lambda}_i$ is not greatly larger than $\tilde{\lambda}_i$, so that point C is not too much higher than point B on the efficient frontier. This indicates that taking account of transactions frictions does not matter as much for our welfare analysis, once we have already taken account of the interest-rate lower bound, though the reverse is not true. There is a simple intuition for this result. Whether or not we allow for transactions frictions, imposition of constraint (4.24) makes it optimal to choose a policy under which interest-rate variability is roughly of the size that makes (4.24) consistent with an average inflation rate of zero.[47] The level of interest-rate variability that this would involve, the point on the frontier corresponding to it, and the associated weight on interest-rate variations in the effective loss function (which is given by the negative of the slope of the efficient frontier at that point) are all independent of whether the loss function directly penalizes interest-rate variations (because of transactions frictions) or not. For this reason, increasing the weight λ_i on interest-rate variations in the direct loss function L_t has relatively little effect on the weight $\hat{\lambda}_i$ in the effective loss function. The increase in λ_i results in a smaller Lagrange multiplier on constraint (4.24), as there is less desire to vary interest rates even in the absence of the constraint; and so the sum of the two weights increases much less than the

[47] When we take account of only the interest-rate lower bound, it is already optimal to reduce interest-rate variability to a degree consistent with a long-run average inflation rate of only 14 basis points per year. When one takes account of the welfare consequences of transactions frictions as well, it instead becomes optimal to reduce interest-rate variability to an extent consistent with a long-run average inflation rate of negative 11 basis points per year. This does not require a great deal of further reduction in the variability of the short-term interest rate.

increase in λ_i. For this reason, there is not a great loss in accuracy involved in neglecting the welfare consequences of transactions frictions, if the interest-rate lower bound has already been taken account of (as in Rotemberg and Woodford, 1997, 1999a, or Woodford, 1999a). Furthermore, the conclusions reached in those analyses are the same as would have been obtained if transactions frictions were allowed for, but the lower bound were treated as slightly less of a constraint — as for example if one assumed slightly less variability or a slightly higher average value of the natural rate of interest.

4.3 Asymmetric Disturbances

Next we consider the consequences of real disturbances that, unlike those considered in section 2, do not have identical effects upon demand and supply conditions for all goods. Instead, we wish to allow for disturbances of kinds that would affect equilibrium relative prices even in the case that all prices were fully flexible. In the presence of such disturbances, it is generally not possible to arrange for the equilibrium of an economy with sticky prices to reproduce the state-contingent resource allocation of a flexible-price economy simply by choosing a monetary policy that stabilizes an aggregate price index. We shall also allow for asymmetries between sectors in that we shall no longer assume that the frequency of price adjustments must be the same for all goods. (This introduces another reason for different sectors of the economy to be differentially affected by shocks.) What are appropriate stabilization goals in such a case?

The analysis here generalizes the work of Aoki (2001), and is essentially a closed-economy interpretation of the analysis of Benigno (1999).[48] We return again to the model with asymmetric disturbances described in section xx of chapter 3. Our welfare measure is again given by the utility of the representative household, that is once again of the form (2.2), except that now Y_t is a (time-varying) CES aggregate of the sectoral indices of aggregate

[48] In the model of Benigno, there are two countries that specialize in the production of different sets of goods, and real disturbances may have differential effects upon the markets for goods produced in a given country. Here the two sets of goods are instead interpreted as simply two sectors of a single national economy. However, our conclusions with regard to appropriate stabilization objectives directly follow from his analysis of stabilization objectives for a two-country monetary union.

demand Y_{1t} and Y_{2t}, and the disturbances affecting the disutility of output supply (2.4) are now allowed to be different in the case of goods i in different sectors.

A second-order expansion of the equation defining Y_t as a function of the sectoral demand indices is of the form

$$\hat{Y}_t = \sum_j = 1^2 n_j(1+\eta^{-1}\hat{\varphi}_{jt})\hat{Y}_{jt} + \frac{1}{2}n_1 n_2(1-\eta^{-1})(\hat{Y}_{2t}-\hat{Y}_{1t})^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (4.29)$$

where n_j is the fraction of the goods that belong to sector j, η is the elasticity of substitution between the composite products of the two sectors (assumed equal to one by Benigno), and $\hat{\varphi}_{jt}$ is the exogenous disturbance to the sectoral composition of demand (all as in chapter 3). A similar second-order expansion of the index of sectoral demand is of the form

$$\hat{Y}_{jt} = E_i^j \hat{y}_t(i) + \frac{1}{2}(1-\theta^{-1})\text{var}_i^j \hat{y}_t(i) + \mathcal{O}(||\xi||^3), \quad (4.30)$$

for each of the sectors $j = 1, 2$, generalizing (2.11). Here we introduce the notation $E_i^j(\cdot)$ and $\text{var}_i^j(\cdot)$ for the mean and variance of the distribution of values for the different goods i belonging to a given sector j.

Similarly, in the case of any good i in sector j, a second-order expansion of the disutility of output supply can be written in the form

$$\tilde{v}(y_t(i);\xi_t^j) = \bar{Y}u_c\left\{(1-\Phi)\hat{y}_t(i) + \frac{1}{2}(1+\omega)\hat{y}_t(i)^2 - \omega q_{jt}\hat{y}_t(i)\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (4.31)$$

generalizing (2.9), where now

$$q_{jt} \equiv -\frac{\tilde{v}_{y\xi}\xi_t^j}{\bar{Y}\tilde{v}_{yy}}$$

represents the sector-specific variation in the level of output required to maintain a constant marginal disutility of supply. Integrating (4.31) over the goods i belonging to sector j, and using (4.30) to eliminate $E_i^j \hat{y}_t(i)$, we obtain

$$\int_{N_j} \tilde{v}(y_t(i);\xi_t^j)di = n_j\bar{Y}u_c\left\{(1-\Phi)\hat{Y}_{jt} + \frac{1}{2}(1+\omega)\hat{Y}_{jt}^2 - \omega q_{jt}\hat{Y}_{jt} + \frac{1}{2}(\theta^{-1}+\omega)\text{var}_i^j \hat{y}_t(i)\right\}$$

$$+\text{t.i.p.} + \mathcal{O}(||\xi||^3).$$

Then summing over the two sectors and using (4.29) to eliminate $\sum_j n_j \hat{Y}_{jt}$, we obtain

$$\int_0^1 \tilde{v}(y_t(i);\xi_t)di = \bar{Y}u_c\left\{(1-\Phi)\hat{Y}_t + \frac{1}{2}(1+\omega)\hat{Y}_t^2 - \sum_{j=1}^{2} n_j(\omega q_{jt} + \eta^{-1}\hat{\varphi}_{jt})\hat{Y}_{jt}\right.$$
$$\left. +\frac{1}{2}n_1n_2(\eta^{-1}+\omega)(\hat{Y}_{2t}-\hat{Y}_{1t})^2 + \frac{1}{2}(\theta^{-1}+\omega)\sum_{j=1}^{2} n_j \text{var}_i^j \hat{y}_t(i)\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \quad (4.32)$$

generalizing (2.10).

Combining (2.8) and (4.32), we finally obtain

$$\begin{aligned} U_t &= \bar{Y}u_c\left\{\Phi\hat{Y}_t - \frac{1}{2}(\sigma^{-1}+\omega)\hat{Y}_t^2 + \sum_{j=1}^{2}(\sigma^{-1}g_t + \omega q_{jt} + \eta^{-1}\hat{\varphi}_{jt})\hat{Y}_{jt}\right. \\ &\quad \left. -\frac{1}{2}n_1n_2(\eta^{-1}+\omega)(\hat{Y}_{2t}-\hat{Y}_{1t})^2 - \frac{1}{2}(\theta^{-1}+\omega)\sum_{j=1}^{2}\text{var}_i^j \hat{y}_t(i)\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3) \\ &= -\frac{\bar{Y}u_c}{2}\left\{(\sigma^{-1}+\omega)(x_t - x^*)^2 + n_1n_2(\eta^{-1}+\omega)x_{Rt}^2 + (\theta^{-1}+\omega)\sum_{j=1}^{2}\text{var}_i^j \hat{y}_t(i)\right\} \\ &\quad + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \end{aligned} \quad (4.33)$$

generalizing (2.12). Here once again $x_t \equiv \hat{Y}_t - \hat{Y}_t^n$ is the aggregate output gap and we correspondingly define the relative output gap $x_{Rt} \equiv x_{2t} - x_{1t}$, where $x_{jt} \equiv \hat{Y}_{jt} - \hat{Y}_{jt}^n$ is the gap for sector j. In deriving (4.33) from the expression above, we use the fact that the definitions of the aggregate and sectoral natural rates of output in chapter 3 imply that

$$(\sigma^{-1}+\omega)\hat{Y}_t^n = \sigma^{-1}g_t + \sum_{j=1}^{2} n_j q_{jt},$$

$$(\eta^{-1}+\omega)(\hat{Y}_{2t}^n - \hat{Y}_{1t}^n) = \eta^{-1}(\hat{\varphi}_{2t} - \hat{\varphi}_{1t} + +\omega(q_{2t} - q_{1t}).$$

Using the sectoral demand equation to write relative sectoral demand as a function of relative sectoral price, and the demand equation for an individual good to write sectoral output dispersion as a function of sectoral price dispersion this can alternatively be written as

$$U_t = -\frac{\bar{Y}u_c}{2}\left\{(\sigma^{-1}+\omega)(x_t - x^*)^2 + n_1n_2\eta(1+\omega\eta)(\hat{p}_{Rt} - \hat{p}_{Rt}^n)^2\right.$$
$$\left. +\theta(1+\omega\theta)\sum_{j=1}^{2}\text{var}_i^j \log p_t(i)\right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3). \quad (4.34)$$

Here $\hat{p}_{Rt} \equiv \log(P_{2t}/P_{1t})$ is again the (log) sectoral relative price and $\hat{p}_{Rt}^n$ its "natural" value (*i.e.*, the equilibrium relative price under full price flexibility, a function solely of the asymmetric exogenous disturbances).

This quadratic approximation to the utility flow each period is valid regardless of our assumptions about pricing. If we assume Calvo-style staggered price-setting in each sector, then the measure of sectoral price dispersion

$$\Delta_t^j \equiv \mathrm{var}_i^j \log p_t(i)$$

evolves according to the approximate law of motion

$$\Delta_t^j = \alpha_j \Delta_{t-1}^j + \frac{\alpha_j}{1-\alpha_j}\pi_{jt}^2 + \mathcal{O}(||\xi||^3),$$

where π_{jt} is the rate of price inflation in sector j and α_j is the fraction of prices that remain unchanged each period in sector j. Summing over time we then obtain

$$\sum_{t=0}^{\infty} \beta^t \Delta_t^j = \frac{\alpha_j}{(1-\alpha_j)(1-\alpha_j\beta)} \sum_{t=0}^{\infty} \beta^t \pi_{jt}^2 + \text{t.i.p.} + \mathcal{O}(||\xi||^3).$$

Using this substitution for the price dispersion terms in (4.34), we again find that discounted lifetime utility can be approximated by (2.22), where now the period loss function is of the form

$$L_t = \sum_{j=1}^{2} w_j \pi_{jt}^2 + \lambda_x (x_t - x^*)^2 + \lambda_R (\hat{p}_{Rt} - \hat{p}_{Rt}^n)^2, \qquad (4.35)$$

generalizing (2.23). Here the weights (normalized so that $w_1 + w_2 = 1$) are given by

$$w_j \equiv \frac{n_j \kappa}{\kappa_j} > 0, \qquad \lambda_x \equiv \frac{\kappa}{\theta} > 0, \qquad \lambda_R \equiv \frac{n_1 n_2 \eta (1+\omega\eta)}{\sigma^{-1}+\omega}\, \lambda_x > 0,$$

where the coefficients κ_j are defined as in equation (xx) of chapter 3, and

$$\kappa \equiv (n_1 \kappa_1^{-1} + n_2 \kappa_2^{-1})^{-1} > 0$$

is a geometric average of the two.

We now find that deadweight loss depends not only upon the economy-wide average rate of inflation, but on the rate of inflation in each of the sectors individually; the relative weight on inflation variations in sector j is greater the larger the relative size of this sector, and also the smaller the relative value of κ_j, which measures the degree of price stickiness in sector j. (A smaller value of κ_j indicates slower price adjustment in sector j; κ_j is unboundedly large in the limit of perfectly flexible prices in sector j.) The relative weight on aggregate output-gap variations depends as before on the measure κ of the overall degree of price stickiness in the economy. Finally, misalignments of the relative price between the two sectors (relative to what it would be under fully flexible prices) also distort the allocation of resources.[49] The relative weight on this stabilization objective is greater the larger the elasticity of substitution η between the products of the two sectors.

In general, it is not possible to simultaneously satisfy all of these stabilization objectives. In particular, if the natural relative price $\hat{p}^n_{Rt}$ varies over time, it is not possible simultaneously to stabilize inflation in both sectors *and* to eliminate gaps between the relative price and its natural value. Accordingly, in this case we must consider second-best optimal policies. And in general, complete stabilization of the aggregate inflation rate

$$\pi_t \equiv n_1\pi_{1t} + n_2\pi_{2t}$$

is *not* the best available policy. This is most easily seen in the case considered by Aoki, in which prices are fully flexible in one sector, but sticky in the other. If prices are fully flexible in sector j, $\kappa_j^{-1} = 0$, so that $w_j = 0$. In this limiting case, we can also show that the relative-price gap $\hat{p}_{Rt} - \hat{p}^n_{Rt}$ is a constant multiple of the output gap x_t, regardless of policy, so that the same policy completely stabilizes both gap variables; and that complete stabilization of both gaps implies zero inflation in the sticky-price sector.

Hence all three terms in (4.35) with non-zero weights are minimized by the same policy, one which completely stabilizes the price index for the sticky-price sector, and this is clearly the optimal policy in this case. (Aoki interprets this policy as stabilization of an index of "core inflation".) Again the intuition is a simple one: such a policy achieves the same

[49] In Benigno's open-economy application, this corresponds to the real exchange rate, and we obtain a welfare-theoretic justification for a real exchange-rate stabilization objective. However, it should be noted that it is the gap between the real exchange rate and its "natural" level that should be stabilized, rather than the real exchange rate itself.

allocation of resources as would occur under complete price flexibility, since no suppliers in the sticky-price sector have any desire to change their prices more frequently than they already do. But in general, such a policy does not completely stabilize the broader price index, despite the fact that an alternative policy exists that would do so. For if $\hat{p}_{Rt}$ is to track $\hat{p}^n_{Rt}$ while the sticky-price index remains constant, there must be a variable inflation rate in the flexible-price sector.

There is one case in which complete stabilization of π_t continues to be optimal even in the presence of relative-price disturbances; this is the case in which prices are equally sticky in the two sectors (in addition to the sectors being symmetrical in the other ways assumed in chapter 3). When $\alpha_1 = \alpha_2$, so that $\kappa_1 = \kappa_2$, $w_1 = w_2 = 1/2$, and the loss function can alternatively be written

$$L_t = \pi_t^2 + \frac{1}{4}(\hat{p}_{Rt} - \hat{p}_{R,t-1})^2 + \lambda_x(x_t - x^*)^2 + \lambda_R(\hat{p}_{Rt} - \hat{p}^n_{Rt})^2.$$

As shown in chapter 3, in this symmetric case, stabilization of the aggregate inflation rate π_t is equivalent to stabilization of the aggregate output gap x_t, while the relative price $\hat{p}_{Rt}$ evolves in the same way regardless of monetary policy. Thus while it is not possible for any policy to reduce all terms in L_t to zero each period (since $\hat{p}_{Rt} \neq \hat{p}^n_{Rt}$ in general), a policy that completely stabilizes π_t reduces the value of each term to the greatest extent possible, and so is optimal.

More generally, Benigno finds that a policy that completely stabilizes an appropriately weighted average of the sectoral inflation rates,

$$\pi_t^{targ} \equiv \phi\pi_{1t} + (1-\phi)\pi_{2t}, \qquad (4.36)$$

typically provides a reasonably good approximation to optimal policy, if the weight $0 \leq \phi \leq 1$ is properly chosen. (We have just described cases in which each of the values $\phi = 0, 1/2$, or 1 is optimal, suggesting the interest of this general family of rules.) This can be illustrated in a calibrated example.

Suppose that $n_1 = n_2 = 1/2$, $\eta = 1$, and let $\beta, \sigma, \kappa, \omega$ and θ take the values reported in Table 4.1, derived from the study of Rotemberg and Woodford (1997).[50] The implied values

Table 6.3: Calibrated parameter values for the quarterly model used for Figure 6.2.

Additional structural parameters	
n_1, n_2	0.5
η	1
Shock process	
$\rho(\hat{p}_R^n)$	0.8
$\mathrm{sd}(\hat{p}_R^n)$	1.67
Loss function	
x^*	0
λ_x	.048
λ_R	.028

of α_1 and α_2 are then derived from these coefficient values for an arbitrary choice of the relative weight $0 \leq w_2 \leq 1$. (In the case that $w_2 = 0.5$ is chosen, $\alpha_1 = \alpha_2$, and the common value of α is the one reported in Table 4.1.) This allows us to vary the assumed relative stickiness of prices in the two sectors between the two extremes of complete flexibility in sector 2 ($w_2 = 0$) and complete flexibility in sector 1 ($w_2 = 1$), while assuming the same overall degree of price stickiness (as measured by κ). Note that the assumed coefficients λ_x and λ_R in the loss function (4.35) remain the same as we vary w_2; the values implied by the above calibration are indicated in Table 6.3.[51] The tensions between alternative stabilization objectives just discussed exist only insofar as the natural relative price $\hat{p}_{Rt}^n$ is not constant; for purposes of illustration, we assume that this follows an AR(1) process with an autoregressive coefficient of 0.8. The assumed variance of the innovations in this process do not matter for our results (all of our expected losses are proportional to this assumed variance), so it is set equal to 1 without loss of generality.[52]

The solid line in Figure 6.2 plots the minimum attainable value for the expected dis-

[50]Note that the model of Rotemberg and Woodford can be interpreted as a two-sector model in which $\alpha_1 = \alpha_2$. Because no data on relative prices are used in that study, it provides no estimate of η.

[51]As in Table 6.1, the reported weights λ_x and λ_R are sixteen times as large as those implied by the formulas given above, so that they correspond to the relative weights on these terms in the loss function when the inflation rate is measured as an annual rather than a quarterly rate.

[52]Note that an innovation variance of 1 implies a variance of $1/1 - (0.8)^2$ for the disturbance process, or a standard deviation of 1.67.

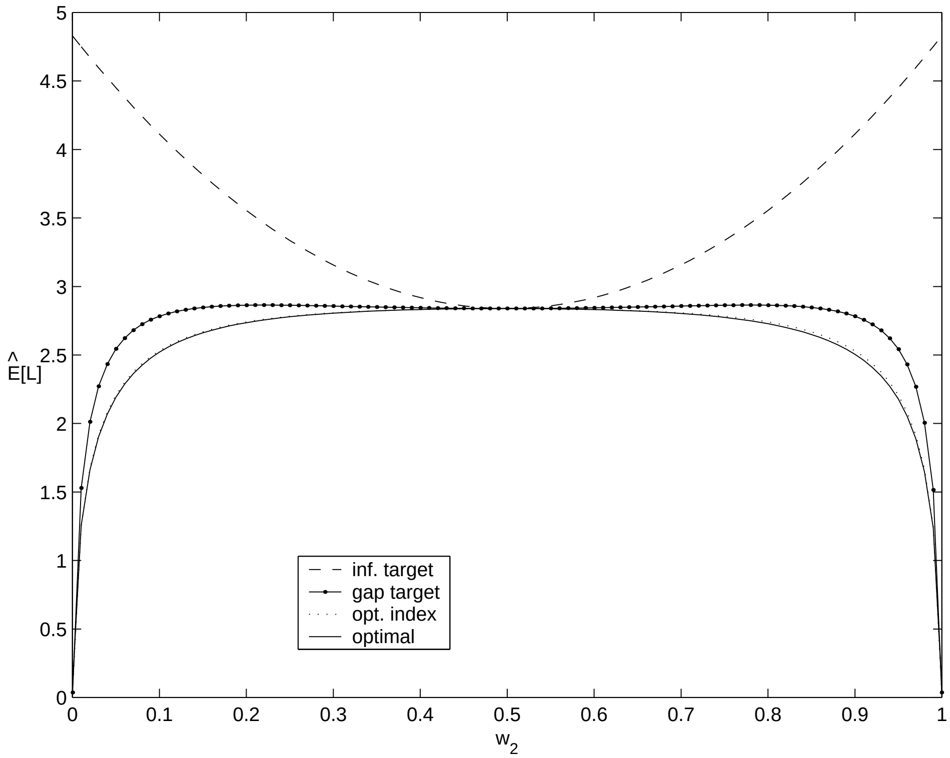

Figure 6.2: Welfare losses under alternative policies with asymmetric disturbances.

counted period loss $\hat{E}[L]$, again defined as in (4.28), for each possible choice of the coefficient w_2 measuring the relative stickiness of sector 2.[53] Only in the case that w_2 takes one of the extreme values (*i.e.*, prices are completely flexible in one sector or the other) is the minimum attainable value zero, for only in this case is it is possible for monetary policy to achieve the allocation of resources associated with complete price flexibility. The expected loss under a policy that strictly targets (completely stabilizes) aggregate inflation is instead shown by the dashed line. The two lines coincide only when $w_2 = 0.5$, the only case in which aggregate-inflation targeting is optimal. Whenever the degrees of price stickiness in the two sectors differ, aggregate-inflation targeting results in greater losses, and the losses associated with this policy are greater the greater the degree of asymmetry, whereas the unavoidable losses are smaller the greater the asymmetry. The expected losses resulting from strict targeting

[53]The precise definition that we assume of constrained-optimal policy in cases like this, as well as the Lagrangian method that we use to characterize it, are explained in chapter 7.

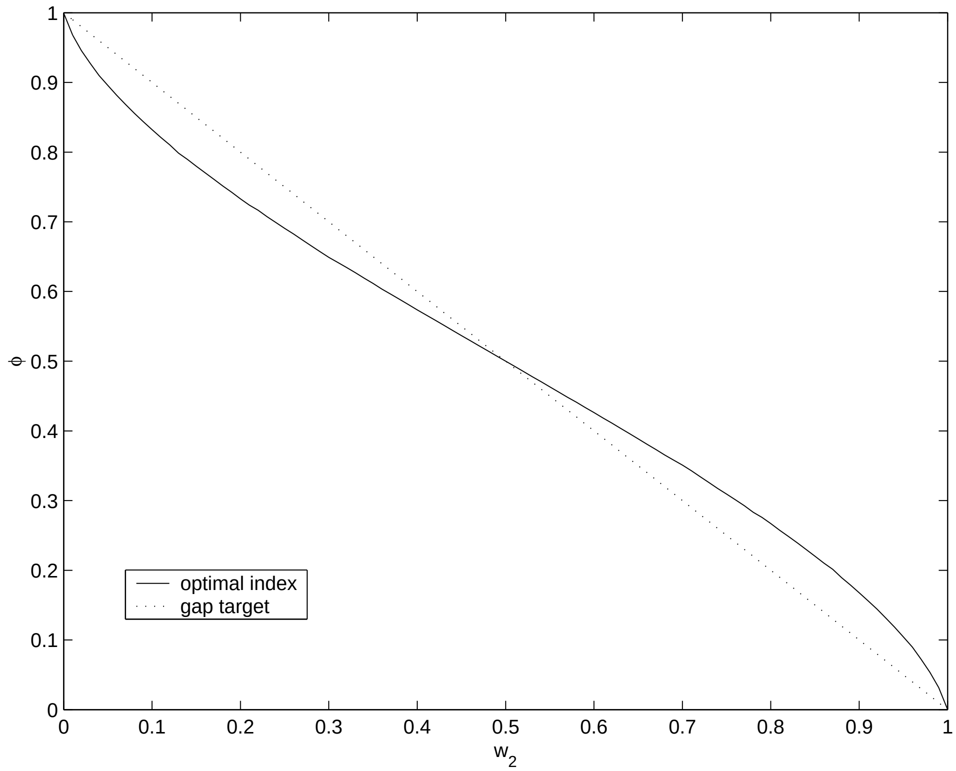

Figure 6.3: The optimal price index to stabilize as a function of the relative stickiness of prices in the two sectors. The dotted line shows the price index that is equivalent to output-gap targeting.

of the weighted index (4.36), where the weight ϕ is optimally chosen for each value of w_2, is instead shown by the dotted line. This coincides with the solid line when $w_2 = 0, 0.5$, or 1 (the three special cases already discussed), but not otherwise; thus these are the only cases in which optimal policy is exactly described by a simple targeting rule of this kind. But even in the other cases, the dotted line is only slightly above the solid line; thus a rule of this kind is a reasonable approximation to optimal policy, if ϕ is properly chosen.

The optimal value of ϕ for each value of w_2 is shown in Figure 6.3. The optimal value are $\phi = 1$ when $w_2 = 0$, $\phi = 0.5$ when $w_2 = 0.5$, and $\phi = 0$ when $w_2 = 1$, for reasons already discussed. More generally, the optimal ϕ is a decreasing function of w_2, as the special cases had already suggested: the near-optimal policy stabilizes an inflation measure that puts more weight on prices in the sector where prices are stickier. As Aoki suggests, this provides theoretical justification for a policy that targets "core inflation" rather than the growth of a broader price index, and offers a theoretical criterion for the construction of such an index. It also explains why it is not appropriate to target an inflation measure that includes "asset price inflation" along with goods price increases, as is sometimes proposed; even if asset prices are also prices, and also can be affected by monetary policy, they are among the prices that are most frequently adjusted in response to new market conditions, and so their movements do not indicate the kind of distortions which we seek to minimize.

The choice of the right rule of the form (4.36) depends, however, upon on an accurate estimate of the relative stickiness of prices in the two sectors. A simple rule that performs relatively well regardless of the value of w_2 is strict targeting of the *output gap:* using monetary policy to ensure that $x_t = 0$ at all times. The expected loss resulting from this rule is shown in Figure 6.3 by the dotted solid line. This policy is only fully optimal in three cases ($w_2 = 0, 0.5$ or 1), and otherwise it is somewhat worse than the best weighted-inflation targeting rule; but it is relatively good over the entire range of possible values of w_2 (unlike the equal-weighted inflation targeting rule, for example), despite involving no coefficients that must be assigned values that vary with the changing value of w_2.

In fact, the output-gap targeting rule is reasonably successful regardless of the value of w_2 because it incorporates the principle of stabilizing more the prices that are most sticky. If one multiplies the inflation equation for sector j (equation (xx) of chapter 3) by w_j and sums over j, one obtains

$$\bar{\pi}_t = \kappa x_t + \beta E_t \bar{\pi}_{t+1}$$

where

$$\bar{\pi}_t \equiv w_1 \pi_{1t} + w_2 \pi_{2t}.$$

From this it follows that output-gap stabilization is equivalent to stabilization of $\bar{\pi}_t$. This is thus a policy of the form (4.36), with $\phi = w_1$. Thus the weights are automatically adjusted to place less weight on the prices in the sector with more flexible prices. This is not done in precisely the optimal way (see the comparison of this function of w_2 with the optimal one in

Figure 6.3), but this simple rule is not too different from the optimal member of the family (4.36) for any value of w_2.

Thus we conclude that even in the case of asymmetric disturbances, stabilization of a price index provides a fairly good recipe for monetary policy, as long as the right price index is chosen. On the other hand, this does not mean that seeking to stabilize the output gap cannot be a sound approach as well, as long as the output gap is properly measured; in fact, in the absence of information about which sector's prices are more sticky, an output-gap target is a more robustly desirable simple policy rule. The choice between the two approaches, then, must turn on which sorts of information the central bank is able to rely upon with more confidence. In practice, banks are likely to be more confident that they can estimate the relative stickiness of different prices with some confidence than that they can accurately track the natural rate of output in real time; for the former question can be studied using past data, while the latter depends upon correctly judging the economy's current state despite the possible occurrence of a vast number of different types of disturbances. For this reason, one may still conclude that an appropriately chosen inflation target represents a sensible approach to policy.

4.4 Sticky Wages and Prices

Similar issues arise if we assume that wages as well as prices are sticky. Once again, some types of real disturbances will modify the "natural" relative price, *i.e.*, the equilibrium real wage under flexible wages and prices, so that no monetary policy can eliminate all of the distortions resulting from wage and price stickiness. Here we analyze welfare-theoretic stabilization goals for the model with sticky wages and prices set out in section xx of chapter 3; our results essentially recapitulate those of Erceg *et al.* (2000).

In this model, all firms hire the same composite labor input; nonetheless, there exist differential demands for the labor supplied by different households j, owing to wage dispersion (as a result of staggered wage adjustment). The demand for each differentiated type of labor is given by

$$h_t(j) = H_t \left(\frac{w_t(j)}{W_t} \right)^{-\theta_w},$$

where $\theta_w > 1$ is the elasticity of substitution among different types of labor on the part of firms. A quadratic expansion of $v(h_t(j); \xi_t)$, integrated over the continuum of different types of labor, yields

$$\int_0^1 v(h_t(j); \xi) dj = \bar{H} v_h \left\{ \hat{H}_t + \frac{1}{2}(1+\nu)\hat{H}_t^2 - \nu \bar{h}_t \hat{H}_t \right.$$
$$\left. + \frac{1}{2}\theta_w(1+\nu\theta_w)\text{var}_j \log w_t(j) \right\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \qquad (4.37)$$

where once again

$$\nu \equiv \frac{\bar{H} v_{hh}}{v_h} > 0.$$

The aggregate demand for the composite labor input H_t is in turn given by

$$H_t = \int_0^1 f^{-1}(y_t(i)/A_t) di, \qquad (4.38)$$

integrating over the demands of each of the firms i. Using a quadratic approximation to an individual firm's labor demand

$$f^{-1}(y_t(i)) = \bar{H} \left\{ 1 + \phi \hat{y}_t(i) + \frac{1}{2}(1+\omega_p)\phi \hat{y}_t(i)^2 \right\} + \mathcal{O}(||\xi||^3),$$

where once again

$$\phi \equiv \frac{\bar{Y}}{\bar{H} f'} > 1, \qquad \omega_p \equiv -\frac{f f''}{(f')^2} > 0,$$

we can expand (4.38) as

$$\begin{aligned}
\hat{H} &= \phi\,(E_i \hat{y}_t(i) - \hat{A}_t) + \frac{1}{2}(1+\omega_p-\phi)\phi\,(E_i \hat{y}_t(i) - \hat{A}_t)^2 \\
&\quad + \frac{1}{2}(1+\omega_p)\phi\ \text{var}_i \hat{y}_t(i) + \mathcal{O}(||\xi||^3) \\
&= \phi\,(\hat{Y}_t - \hat{A}_t) + \frac{1}{2}(1+\omega_p-\phi)\phi\,(\hat{Y}_t - \hat{A}_t)^2 \\
&\quad + \frac{1}{2}(1+\omega_p\theta_p)\theta_p\phi\ \text{var}_i \log p_t(i) + \mathcal{O}(||\xi||^3). \qquad (4.39)
\end{aligned}$$

In the second line we have again used (2.11) to eliminate $E_i \hat{y}_t(i)$ and (2.14) to write $\text{var}_i \hat{y}_t(i)$ as a function of the dispersion of individual goods prices, and adopted the notation θ_p for the elasticity of demand across differentiated goods. Substituting (4.39) for $\hat{H}_t$ in (4.37), and then combining this with (2.8), we obtain the expansion

$$U_t = -\frac{\bar{Y}u_c}{2}\Big\{(\sigma^{-1}+\omega)(x_t - x^*)^2 + \theta_p(1+\omega_p\theta_p)\mathrm{var}_i \log p_t(i)$$
$$+\theta_w\phi^{-1}(1+\nu\theta_w)\mathrm{var}_j \log w_t(j)\Big\} + \text{t.i.p.} + \mathcal{O}(||\xi||^3), \tag{4.40}$$

where x_t and x^* are again defined as in our basic (flexible-wage) model.

This expression for the utility flow each period holds regardless of our assumptions about wage- and price-setting. If, following Erceg *et al.*, we assume Calvo-style staggered adjustment of both wages and prices, we again obtain a representation of the form (2.22), where now the period loss function is of the form

$$L_t = \lambda_p\pi_t^2 + \lambda_w\pi_{wt}^2 + \lambda_x(x_t - x^*)^2. \tag{4.41}$$

Here the weights (normalized so that $\lambda_p + \lambda_w = 1$) are given by

$$\lambda_p = \frac{\theta_p\xi_p^{-1}}{\theta_p\xi_p^{-1} + \theta_w\phi^{-1}\xi_w^{-1}} > 0, \qquad \lambda_w = \frac{\theta_w\phi^{-1}\xi_w^{-1}}{\theta_p\xi_p^{-1} + \theta_w\phi^{-1}\xi_w^{-1}} > 0,$$

$$\lambda_x = \frac{\sigma^{-1}+\omega}{\theta_p\xi_p^{-1} + \theta_w\phi^{-1}\xi_w^{-1}} > 0,$$

where $\xi_w, \xi_p > 0$ are again the elasticities appearing in equations (xx) – (xx) of chapter 3.

Note that the relative weight on output-gap stabilization is again related to the slope of the aggregate supply relation in a similar way as in the basic (flexible-wage) model. In particular, in the case that $\theta_w\phi^{-1} = \theta_p$, the expression for λ_x reduces to κ/θ, where $\kappa > 0$ is the slope of the short-run Phillips-curve relation between a weighted average of price and wage inflation and the output gap (equation (xx) of chapter 3) and $\theta > 1$ is the common value of θ_p and $\theta_w\phi^{-1}$. The weight on inflation stabilization is now, however, divided between a price-inflation stabilization goal and a wage-inflation stabilization goal; the relative weights on each depend on the relative stickiness of wages and prices (as indicated by the relative sizes of ξ_w^{-1} and ξ_p^{-1}). If only prices are sticky, only price inflation matters ($\lambda_w = 0$), and optimal policy involves complete stabilization of prices (as this also completely stabilizes the output gap, in that case). If instead only wages are sticky, only wage inflation matters ($\lambda_p = 0$), and optimal policy involves complete stabilization of wages (which also completely stabilizes the output gap, in this case). In the intermediate case, both goals matter, and complete stabilization is impossible in the presence of fluctuations in the "natural real wage" w_t^n.

An intermediate case in which optimal policy is nonetheless simple to characterize is the special case in which $\theta_w \phi^{-1} = \theta_p$ and $\kappa_w = \kappa_p$, where κ_w, κ_p are the coefficients indicating the effects of output-gap variations on wage and price inflation respectively (see equations (xx) – (xx) of chapter 3). Note that (4.41) can alternatively be written

$$L_t = \bar{\pi}_t^2 + \lambda_p \lambda_w (\log w_t - \log w_{t-1})^2 + \lambda_x (x_t - x^*)^2, \tag{4.42}$$

where

$$\bar{\pi}_t \equiv \lambda_p \pi_t + \lambda_w \pi_{wt}$$

and w_t is the real wage. In the case that $\kappa_w = \kappa_p$, the evolution of the real wage is independent of monetary policy, as shown in chapter 3; hence the middle term in (4.42) is irrelevant. In the case that $\theta_w \phi^{-1} = \theta_p$, the weights λ_p, λ_w define a weighted average of price and wage inflation that is stabilized if and only if the output gap is stabilized. (This follows from equation (xx) of chapter 3.) Hence the two terms in (4.42) that can be affected by monetary policy are both minimized by the same policy, one that completely stabilizes $\bar{\pi}_t$.[54]

This last result suggests that even when both wages and prices are sticky, and L_t cannot be reduced to zero by *any* policy, a policy that stabilizes a weighted inflation measure of the form

$$\pi_t^{targ} \equiv \phi \pi_{1t} + (1 - \phi) \pi_{2t}, \tag{4.43}$$

may be desirable. In fact, except in the special case just described, fully optimal policy cannot be represented by such a simple rule, but numerical experimentation suggests that a targeting rule of this kind can nonetheless be nearly optimal, if the weight ϕ is appropriately chosen.

[54]Even when $x^* = 0$, one can show that this is the optimal policy from the "timeless perspective" that is explained further in chapters 7 and 8. See in particular section xx of chapter 8.

Table 6.4: Calibrated parameter values for the quarterly model used for Figure 6.4.

Additional structural parameters	
θ_p	7.88
$\theta_w \phi^{-1}$	7.88
$\xi_w^{-1} + \xi_p^{-1}$	26.7
κ	.024
Shock process	
$\rho(\hat{w}^n)$	0.8
$\text{sd}(\hat{w}^n)$	1.67
Loss function	
x^*	0
λ_x	.048

This can be illustrated by a calibrated example. We again let the parameters $\beta, \sigma, \omega_w, \omega_p$ and θ_p take the values estimated by Rotemberg and Woodford (1997) and reported in Table 4.1,[55] and in the absence of any direct evidence we assume the same value for $\theta_w \phi^{-1}$ as for θ_p. We wish to let the assumed relative degree of wage as opposed to price stickiness vary between the two extremes of full wage flexibility and full price flexibility on the other; but we assume a given degree of overall nominal rigidity by fixing the value of the slope coefficient κ in the generalized Phillips-curve relation described by equation (xx) of chapter 3. The value assumed for κ is again the one shown in Table 4.1. (Note that in the case of wage flexibility, as assumed by Rotemberg and Woodford, their estimated coefficient κ corresponds to the κ of this model, rather than to κ_p.) This implies a fixed value for the sum $\xi_w^{-1} + \xi_p^{-1}$, shown in Table 6.4,[56] though we wish to vary the relative contributions of the two terms to this sum (between $\xi_w^{-1} = 0$ when wages are fully flexible to $\xi_p^{-1} = 0$ when prices are). We parameterize our assumption about the relative degree of wage and price stickiness by the value of $\lambda_w = \xi_w^{-1}/(\xi_w^{-1} + \xi_p^{-1})$, which we allow to vary over the interval from zero to one. The assumed value of $\xi_w^{-1} + \xi_p^{-1}$ implies a value for λ_x that is independent of the choice

[55] Note that the parameter θ_p is referred to simply as θ in the previous table.

[56] Note that the estimates of Amato and Laubach (2001), reported in Table 4.2, would instead imply a value of 32.4, or a slightly greater overall degree of nominal rigidity than we assume here.

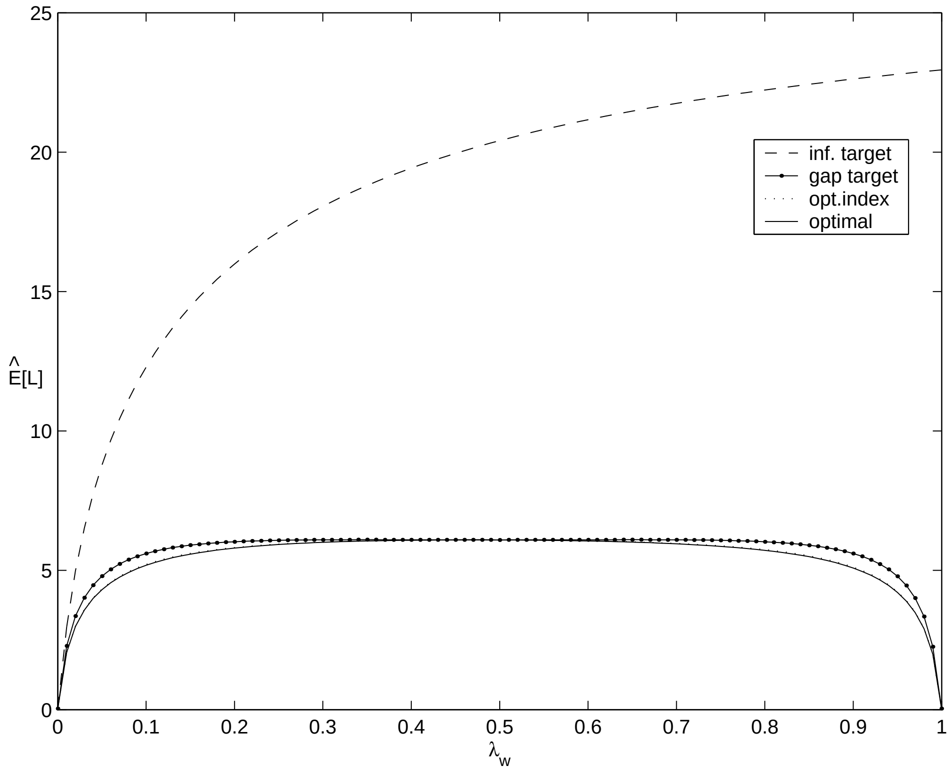

Figure 6.4: Welfare losses under alternative policies with sticky wages and prices.

of λ_w; once again this is the same as in Table 6.1.

The only exogenous disturbance that matters for the optimal state-contingent evolution of wages and prices is the "natural real wage" process $\hat{w}_t^n$. For the sake of illustration, we assume an AR(1) process with the properties listed in Table 6.4. (Note that once again the assumed variance of the disturbance has no effect on our results, other than to scale up the expected value of each term of the loss function equally.)

The solid line in Figure 6.4 then shows the minimum attainable value of $\hat{E}[L]$ associated with these parameter values, for each possible value of λ_w. As in Figure 6.3, distortions can be reduced to zero only in the two extreme cases (here corresponding to full wage flexibility when $\lambda_w = 0$ and full price flexibility when $\lambda_w = 1$). The dashed line instead shows the expected discounted loss associated with a policy of complete stabilization of the rate of price inflation π_t; this policy is optimal only if $\lambda_w = 0$ (completely flexible wages), and grows

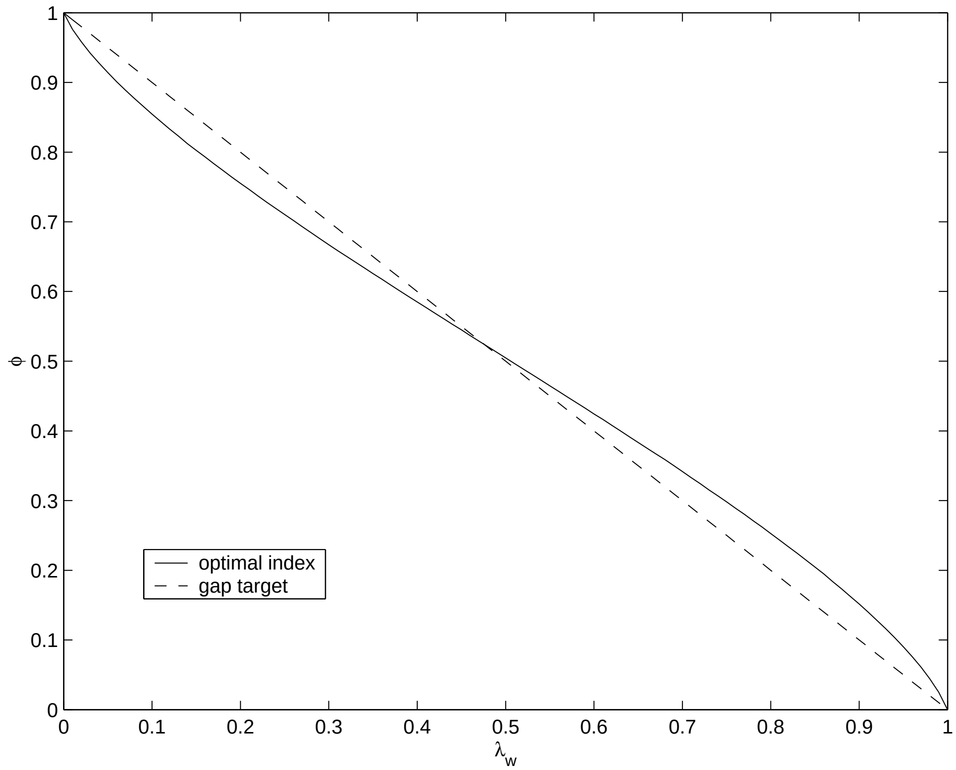

Figure 6.5: The optimal index to stabilize as a function of the relative stickiness of wages and prices. The dotted line shows the generalized inflation-targeting rule that is equivalent to output-gap targeting.

worse the greater the relative stickiness of wages. The expected loss achievable by instead targeting a weighted average of wage and price inflation, when the weight ϕ is optimally chosen, is instead indicated by a dotted line. This only coincides exactly with the solid line (indicating that the optimal policy belongs to this simple family) at three points, when $\lambda_w = 0, 0.48$, or 1. (The intermediate special case is that in which $\lambda_w = (\sigma^{-1}+\omega_w)/(\sigma^{-1}+\omega)$, so that $\kappa_w = \kappa_p$.) However, the dotted line can barely be distinguished from the solid line over the entire interval; hence a well-chosen policy of this form offers a good approximation to optimal policy in all cases. The optimal choice of the weight ϕ is plotted as a function of λ_w in Figure 6.5. Note that it is monotonically decreasing: the optimal weight to put on price as opposed to wage inflation is lower the greater the relative stickiness of wages.

Once again, we also find that output-gap targeting is a fairly robust policy that does not require knowledge of the exact degrees of wage and price stickiness. The expected losses associated with this policy are shown by the dotted solid line in Figure 6.4. Output-gap targeting is only fully optimal in the three special cases just mentioned, and otherwise it is also not as good as the best policy within the generalized inflation-targeting family. In fact, output-gap targeting is equivalent to a generalized inflation-targeting policy, corresponding to an index with $\phi = \lambda_p = 1 - \lambda_w$. But as shown in Figure 6.5, this choice of ϕ is not too different from the optimal one, regardless of the relative stickiness of wages and prices; this is exactly why output-gap targeting is a relatively robust policy prescription. Yet once again, this observation is only of practical importance if direct measures of the output gap in real time are fairly accurate; in practice, the best way to stabilize the output gap may well be to seek to stabilize an appropriate weighted average of wage and price inflation.

4.5 Time-Varying Tax Wedges or Markups

Finally, as discussed above, complete stabilization of inflation ceases to be optimal, even when this implies that aggregate output should perfectly track the equilibrium level of output under flexible wages and prices, if the gap between this latter quantity and the *efficient* level of output is not a constant. In such a case, unlike those considered in the previous two subsections, there are time-varying distortions that do not result from delays in the adjustment of any kind of prices; thus there exists no way of defining the price index to be stabilized that can make price stability an adequate proxy for welfare-maximizing policy. Hence the challenge to price stability as a goal of policy is strongest in this case.

For simplicity, let us assume again a model in which wages are flexible, all prices are equally sticky, and all real disturbances affect the demand for and production costs of all goods symmetrically, as in the models considered in section 2. Let us suppose now, however, that the relative price $p_t(i)/P_t$ at which the supplier of a differentiated good i would be willing to supply a quantity $y_t(i)$ of that good, under conditions of full price flexibility, is given by

$$\tilde{p}(y_t(i), Y_t; \tilde{\xi}_t, \Phi_t) \equiv \frac{1}{1-\Phi_t} \frac{\tilde{v}_y(y_t(i); \tilde{\xi}_t)}{\tilde{u}_c(Y_t; \tilde{\xi}_t)}, \tag{4.44}$$

where Φ_t is an exogenous time-varying composite distortion. Here the term Φ_t includes the distortions resulting from the market power of the supplier of each differentiated good and from the existence of distorting taxes on output, consumption, employment or wage income. We have allowed for these distortions earlier, but assumed them to be constant over time (and represented their composite effect by a *constant* factor $\Phi \geq 0$); we now allow them to vary over time, but assume that their variation is exogenous. The composite distortion Φ_t can also include the effects of a wedge between the representative household's marginal rate of substitution between leisure (less supply of the labor used in producing good i) and consumption and the real wage demanded from producers of good i, as a result of (possibly time-varying) market power in labor supply.

The flexible-price equilibrium level of output for each good Y_t^f is then implicitly defined at each time by the relation

$$\tilde{p}(Y_t^f, Y_t^f; \tilde{\xi}_t, \Phi_t) = 1. \tag{4.45}$$

The time-varying efficient level of output Y_t^e is instead implicitly defined by the relation

$$\frac{\tilde{v}_y(Y_t^e; \tilde{\xi}_t)}{\tilde{u}_c(Y_t^e; \tilde{\xi}_t)} = 1. \tag{4.46}$$

Note that Y_t^e is a function solely of the exogenous disturbances to tastes, technology and government purchases at date t, reflected by the vector $\tilde{\xi}_t$. The flexible-price equilibrium level of output Y_t^f is also a function solely of exogenous disturbances at date t; but the exogenous disturbances that affect it include all of the various disturbances that affect the value of Φ_t. These latter disturbances result in inefficient variations in the flexible-price equilibrium. In particular, log-linearizing both (4.45) and (4.46) and comparing terms, we find that

$$\log Y_t^f = \log Y_t^e - (\omega + \sigma^{-1})^{-1}\Phi_t + \mathcal{O}(||\xi||^2),$$

where $||\xi||$ is now a bound on the size of both disturbances $\tilde{\xi}_t$ and Φ_t.

Which of these ideal levels of output should be defined as the "natural rate"? The definition that conforms most closely to standard usage would define the natural rate of output as the flexible-price equilibrium rate of output *in the case of certain constant values* for the tax wedges and the markups due to market power in price- and/or wage-setting, where the constant values are the long-run average values around which these exogenous series fluctuate. Under this definition, the *average* "output gap" associated with price flexibility — or alternatively, with zero inflation — will again be zero, as above. But price flexibility would no longer imply a zero output gap at all times; for the flexible-price equilibrium level of output varies in response to variations in Φ_t, while our "natural rate" of output (like the efficient level) does not.[57]

The natural rate of output defined in this way also differs from the efficient level of output, although (up to our log-linear approximation) the difference is a constant. For the natural rate Y_t^n is implicitly defined each period by the relation

$$\tilde{p}(Y_t^n, Y_t^n; \tilde{\xi}_t, \bar{\Phi}) = 1, \tag{4.47}$$

where $\bar{\Phi}$ represents the long-run average value of Φ_t. Log-linearization of this indicates that

$$\log Y_t^n = \log Y_t^f + (\omega + \sigma^{-1})^{-1}\hat{\Phi}_t + \mathcal{O}(||\xi||^2), \tag{4.48}$$

where $\hat{\Phi}_t \equiv \Phi_t - \bar{\Phi}_t$, so that

$$\log Y_t^n = \log Y_t^e - (\omega + \sigma^{-1})^{-1}\bar{\Phi} + \mathcal{O}(||\xi||^2).$$

As a result, it continues to be desirable to stabilize the output gap $x_t \equiv \log(Y_t/Y_t^n)$. At the same this definition (rather than defining the output gap relative to Y_t^e directly) has the advantage that in terms of this output gap measure, our aggregate supply relation continues to have a zero constant term.[58]

[57]Note that defining the natural rate in a way that implies that it does not vary in response to variation in the size of the tax wedge is consistent with our definition in the case of monetary frictions in section xx above. There we defined the natural rate as what equilibrium output would be, given current tastes, technology, and government purchases, if wages and prices were flexible *and* the interest-rate differential between non-monetary and monetary assets were at its steady-state level. While time-variation in the distortion associated with an interest-rate differential may cause variation in the flexible-price equilibrium level of output, our definition implies no variation in the natural rate. Here we treat the consequences for aggregate supply of time-variation in the tax wedge in a similar way.

[58]This choice allows us to follow the literature, such as Clarida *et al.* (1999) in representing a non-zero average gap between the level of output consistent with zero inflation and the efficient level by a non-zero target value x^* in the loss function, rather than by a constant in the aggregate-supply relation, while at the same time representing time variation in this gap by a random term in the aggregate-supply relation, rather than a time-varying target for the output gap that appears in the loss function.

Our derivation of the aggregate supply relation under the assumption of Calvo pricing in chapter 3 continues to hold under these assumptions, except that the output gap must be replaced by

$$\hat{Y}_t - \hat{Y}_t^f = \hat{Y}_t - \hat{Y}_t^n + (\omega + \sigma^{-1})^{-1}\hat{\Phi}_t,$$

using (4.48), as it is the gap between actual output and the flexible-price level of output that determines real marginal cost. Hence (2.19) becomes instead

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t, \tag{4.49}$$

where

$$u_t \equiv \frac{\kappa}{\omega + \sigma^{-1}}\hat{\Phi}_t$$

is a composite exogenous disturbance that Clarida *et al.* (1999) refer to as a "cost-push shock".[59]

Because under definition (4.47), the natural rate of output continues to be the same function of preferences, technology and government purchases as before, our derivation above of the normalized utility-based loss function (2.23) still applies, with $\lambda > 0$ and $x^* \geq 0$ defined as before. But the appearance of the random term u_t in the constraint (4.49) implies that it is no longer possible to simultaneously stabilize both inflation and the welfare-relevant output gap.

The tradeoff that exists between these two stabilization goals in the presence of inefficient supply shocks can be illustrated, for a calibrated numerical example, using a figure of a kind popularized by Taylor (1979b). The coefficients β and κ of the AS relation are again calibrated as in Table 6.1; the exogenous disturbance u_t is assumed to be an AR(1) process with serial correlation $\rho(u) = 0.8$ and an innovation variance of one.[60] (Once again,

[59]The terminology is not entirely satisfactory, however, as there is no necessary connection between shocks that affect inflation by increasing costs of production and time variation in the degree of inefficiency of the natural rate of output. Technology shocks, energy price shocks, or variations in real wage demands may all shift the aggregate supply curve in terms of the output relative to trend, without changing the relation between inflation and the output gap as we have defined it.

[60]To be more precise, it is assumed that a one-standard-deviation innovation in the "cost-push shock" raises the annualized inflation rate $4\pi_t$ by one percentage point. In figure 6.6, $V[\pi]$ refers to the variability of the *annualized* inflation rate, and the units on both axes are percentage points squared.

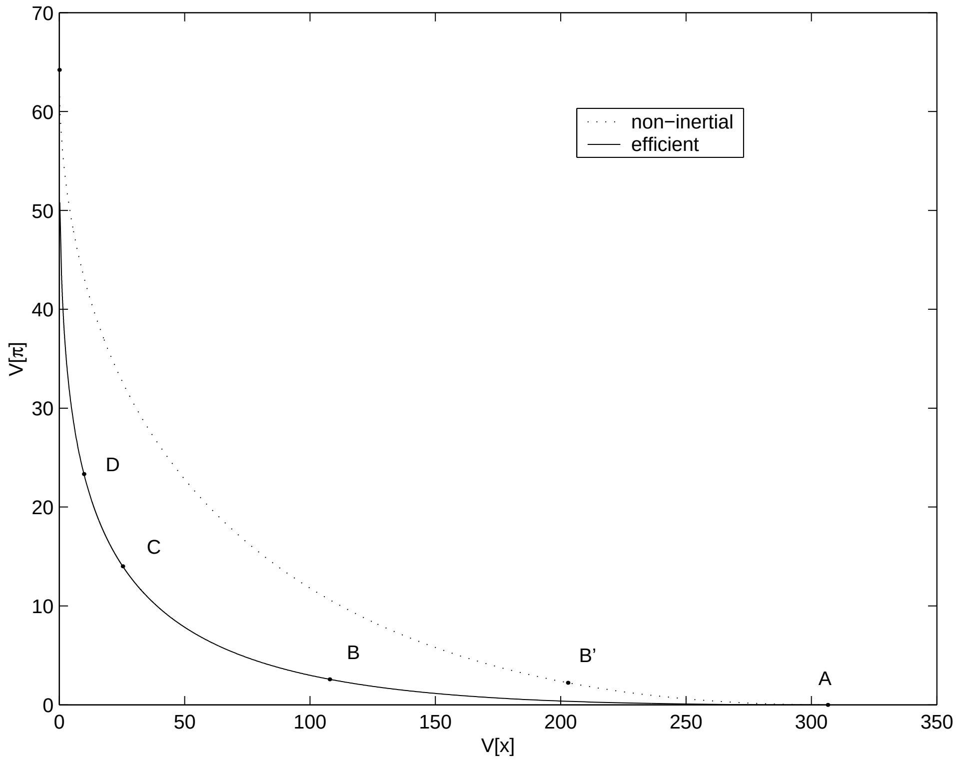

Figure 6.6: The tradeoff between inflation and output-gap stabilization in a model with inefficient supply shocks. The solid line shows the efficient frontier, while the dotted line corresponds to a class of simple rules that are efficient among purely forward-looking policies (see chapter 7).

the assumed variance of the single disturbance is irrelevant for our conclusions, though it determines the scale of the axes in Figure 6.6.) We then compute the *efficient frontier* in $V[\pi] - V[x]$ space, by computing the policy commitment that minimizes $\hat{E}[L]$ for arbitrary values of $\lambda > 0$ in the loss function (2.23). The values of $V[\pi]$ and $V[x]$ associated with each possible value of λ are then plotted as the lower of the two convex curves (*i.e.,* the one closer to the origin) in Figure 6.6.[61] Point A (optimal policy when $\lambda = 0$) indicates how much variability of the output gap must be accepted in order to completely stabilize

[61] The other curve shows the possible combinations of $V[\pi]$ and $V[x]$ that are attainable using a particular family of simple targeting rules, rules that prescribe stabilization of a weighted average of inflation and the output gap. (The equilibria achieved using such rules correspond to the "optimal non-inertial plan" characterized in chapter 7, for alternative assumed relative weights on inflation and output-gap stabilization.) As shown, these rules are generally not on the efficient frontier. This indicates the suboptimality of *purely forward-looking* policy, as is discussed further in the next chapter.

inflation, while point E (optimal policy when λ is unboundedly large) instead indicates how much variability of inflation must be accepted in order to completely stabilize the output gap. Points B, C and D instead represent optimal policies for various finite, positive values of λ.

Since $\lambda > 0$ in the utility-based loss function, complete stabilization of inflation is seen not to be optimal. However, this does not mean that the optimal degree of variability in inflation need be very great. First of all, we observe from the figure that the efficient frontier is quite flat in the lower right region; thus it is possible to substantially reduce the variability of the output gap without too much variability of inflation being required. Instead, on the upper part of the frontier, where efficient policies involve substantial inflation variability, the frontier is quite steep. Thus policies on that part of the frontier would be optimal only if the relative weight λ on output-gap stabilization were quite large. Second, the value of λ that can be justified on welfare-theoretic grounds is likely to be quite small. As shown in Table 4.1, our calibrated parameter values imply a value of $\lambda = 0.05$. In Figure 6.6, the point on the efficient frontier that minimizes this weighted average of the two criteria (and hence that represents optimal policy) is point B. This policy involves quite a modest degree of inflation variability, relative, for example, to the inflation variability that would be required to fully stabilize the output gap.

The value of λ in our utility-based loss function is much smaller than the relative weight on output-gap stabilization that is often assumed in *ad hoc* policy objectives in the literature on monetary policy evaluation. A commonly assumed value would be $\lambda = 1$; this would correspond to point D in the figure. We see that optimal policy involves much more stable inflation than would be chosen under an *ad hoc* criterion of that kind. As another example of a familiar policy recommendation, it has sometimes been proposed that given the existence of "supply shocks", and hence a necessary tension between output stabilization and inflation stabilization, nominal GDP targeting would represent a reasonable balance between the two goals. In our model, in the case that there are no fluctuations in Y_t^e (so that *all* variations in the output level consistent with price stability are inefficient, as arguments for nominal GDP targeting typically assume), nominal GDP targeting is a policy on the efficient frontier, as shown in chapter 7.[62] However, for our calibrated parameter values, nominal GDP targeting would correspond to point C on the frontier; this still involves considerably more variation in inflation than an optimal policy under the utility-based criterion. Thus according to our utility-based analysis, the degree of departure from complete inflation stabilization that can be justified even in the case of inefficient supply shocks of significant magnitude is quite modest.

It should also be noted that the quantitative importance of shocks of this kind is far from clear. The literature on monetary policy evaluation has given considerable emphasis to the tension between the goals of inflation stabilization and output stabilization created by the existence of "supply shocks". It is taken as well established that such shocks are an important factor in practical monetary policy. But the mere existence of "supply shocks," in the sense of real disturbances that shift the short-run Phillips curve, does not imply the existence of *inefficient* supply shocks, the kind of shifts resulting from variation in Φ_t as opposed to the elements of $\tilde{\xi}_t$. As we have seen, the vector ξ_t includes a variety of types of real disturbances that are of clear importance in actual economies, and that can easily cause substantial fluctuations in the flexible-price equilibrium level of output; but in our model, these fluctuations are (to a first-order approximation) also shifts in the *efficient* level of output. While one can also think of possible sources of variation in the flexible-price equilibrium level of output that would clearly not be efficient, such as variations in tax distortions or in market power, such factors have not been clearly established as important sources of short-run fluctuations in economic activity. Thus while it is certainly possible that substantial disturbances of this kind occur, the matter is far from having been established.

We shall nonetheless give considerable attention in the next two chapters to the design of optimal policies in the case that inefficient supply shocks occur. One reason for this is

[62] The policies on the efficient frontier correspond to complete stabilization of $\log P_t + \phi x_t$, for alternative values of $\phi > 0$. The value of λ required to make one of these policies optimal is $16\kappa\phi$. In the case that Y_t^e (and hence Y_t^n) is a constant, so that $x_t = \hat{Y}_t$, nominal GDP targeting represents the efficient policy corresponding to a weight $\lambda = 16\kappa$. In our calibrated example, this corresponds to $\lambda = 0.38$.

that, as explained in chapter 8, we wish to choose policy rules that are robust to alternative beliefs about the nature of the real disturbances to the economy — for example, to alternative beliefs about how frequently particular types of shocks occur. Thus we wish to consider the *possibility* of inefficient supply shocks, and to choose policy rules that would be optimal *whether or not* disturbances of this kind are significant source of macroeconomic instability in a given economy. In order to undertake this challenge, we must consider what optimal policy would seek to achieve in the case that such shocks occur.

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Interest and Prices

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Contents

Chapter 7

Gains From Commitment to a Policy Rule

We now turn to the characterization of optimal policy in the (realistic) case that not all of the stabilization objectives discussed in the previous chapter can simultaneously be achieved, so that it is not possible to fully eliminate all of the distortions that can be affected by monetary policy. It is in this case that the problem of optimal monetary policy becomes non-trivial, in that the character of the optimal equilibrium cannot be read directly from the nature of the loss function derived above. In this chapter we discuss how the constrained-optimal equilibrium pattern of responses to disturbances can be characterized in such a case, and consider in general terms the problem of the choice of a policy rule intended to bring about the desired equilibrium.

The idea that it should be necessary to compromise among several stabilization goals, each desirable in itself but not mutually attainable in an absolute sense, is likely to be intuitive to most practical policymakers. It may be less obvious that it is desirable to attempt to characterize the optimal compromise among these objectives in terms of a *rule* for the systematic conduct of policy. One might think that it should suffice instead to clarify the proper *goals* of policy on the one hand, and to develop a reliable quantitative model of the *effects* of alternative policy actions on the other, and then simply to charge the central bank with the pursuit of the appropriate goals using an appropriate model to inform its decisions. Economists are often willing to suppose that households and firms do a fairly good job of pursuing their objectives without external guidance, even when the optimization problems that they face are quite complex; why not similarly trust that central bankers can be relied upon to behave in something close to an optimal fashion, once their objectives and constraints have been made clear?

But the reason that rules are important in monetary policy is not that central bankers cannot be relied upon to pursue the public interest, or that their highly trained staffs cannot be expected to bring the latest knowledge to bear upon the analysis of the likely effects of alternative policies. It is rather that the central bank's stabilization goals can be most effectively achieved only to the extent that the central bank not only acts appropriately, but is also *understood* by the private sector to predictably act in a certain way. The ability to successfully steer private-sector expectations is favored by a decision procedure that is based on a rule, since in this case the systematic character of the central bank's actions can be most easily made apparent to the public.

Of course, one might imagine that market participants could become accurate predictors of central-bank behavior without any articulation by the central bank of the principles of its behavior, just as they predict the behavior of firms that do not commit themselves to rules of conduct. Yet while expertise of this kind certainly exists, the degree to which central-bank behavior can be confidently predicted simply through extrapolation from past actions is limited; the membership of monetary policy committees changes fairly often (and invites speculation about the consequences for future decisions), and new circumstances are constantly faced that are not too closely analogous to others confronted by the central bank in the recent past. Articulation by the central bank of a rule that guides its decision process (and commitment by the bank to actually follow it!) can greatly improve the predictability of policy by the private sector, even when the rule is less explicit than a mechanical formula that yields an unambiguous prescription on the basis of publicly available data.

Moreover, an optimal pattern of conduct by the central bank will generally *not* correspond to what would result from *discretionary optimization,* as first stressed by Prescott (1977) and Kydland and Prescott (1977, 1980). By discretionary optimization we mean a procedure under which at each time that an action is to be taken, the central bank evaluates the economy's current state and hence its possible future paths from now on, and chooses the optimal current action in the light of this analysis, with no advance commitment about future actions, except that they will similarly be the ones that seem best in whatever state may be reached in the future. This might seem an eminently sensible procedure; it allows the bank to make use of all of its detailed knowledge about current conditions, rather than having to classify the current state as one of the coarsely described possible future states considered at some earlier time. It also avoids the necessity of making an explicit decision about anything other than the action that must currently be taken (although a view about likely future policy will be implicit in the evaluation of alternative possible current actions). And it might seem to involve no loss of efficiency relative to a once-and-for-all optimal plan, insofar as a large literature on dynamic programming and optimal control has stressed the usefulness of recursive methods for the solution of dynamic optimization problems. Under these methods, a dynamic problem is broken into a sequence of individual decision problems, in which the optimal action at each stage is chosen given the state at that time, taking as given the nature of optimizing behavior at stages yet to be reached.

But as Kydland and Prescott showed, these methods are appropriate only for the optimal control of a system that evolves mechanically as a function of its past state, exogenous disturbances, and the current action of the controller. They are not appropriate for the optimal control of a *forward-looking* system, in which people's expectations about future policy are one of the determinants of current outcomes. This is because a dynamic-programming approach considers the optimal action at a given point in time considering only the discounted current and future losses associated with alternative feasible continuation paths, given the system's current state. It thus neglects any effects of the *anticipation* at earlier dates of a different current action than the one that would be judged best by a discretionary optimizer. By credibly committing itself in advance to behave differently, a central bank can steer expectations in a way that furthers its stabilization goals.

It follows, once again, that successful management of expectations is unlikely to be possible except through commitment of the central bank to a fairly explicit rule for the determination of its appropriate action at any point in time. For conscious guidance by a rule is not only an aid to the private sector's understanding of policy; it also makes it more likely that the central bank itself will act correctly. For the temptation to behave in a discretionary fashion must be resisted; optimal policy requires that most of the time the central bank does *not* set interest rates at the level that would be optimal from the point of view of its stabilization objectives, taking as given both past and future policy, and past, present and future private-sector policy *expectations*.[1] This means that good will and a sound understanding of the effects of alternative policy actions and of the economic welfare associated with alternative possible paths for the economy is not enough; there must be a conscious commitment to a criterion for action that will be counterintuitive for a discretionary policymaker, but that actually serves (and can be understood to serve) the bank's goals if pursued in a predictable way.

The importance of creating the right sort of expectations regarding future policy has another important lesson, of a somewhat subtler character, for the way that policy should be conducted. This is that an optimal decision procedure will generally not be *purely forward-looking*, in the sense of allowing the proper current action to be determined solely from an analysis of the set of possible future paths for the economy given its current state. The point seems to be contrary to the intuitions even of many who recognize the importance of commitment to a rule in order to avoid the dangers of discretionary policy. Indeed, many popular current proposals for policy rules are purely forward-looking in character. For example, inflation-forecast targeting as currently practiced at the Bank of England (see, *e.g.,* the description in Vickers, 199xx), or as described in the early analytical literature (*e.g.,* Svensson 1997, 1999; Leitemo, 199xx), selects a current interest-rate target on the basis of the conformity of projections of the economy's future evolution under that policy choice with a criterion that is purely forward-looking (*e.g.*, that RPIX inflation be predicted to equal 2.5

[1] In the thought experiment proposed here, it is assumed that the private sector's current and future expectations about future policy are and will be correct, but that the current action need not be chosen in a way consistent with past expectations regarding this decision.

percent per year at a horizon two years in the future). Similarly, Taylor's classic formulation (Taylor, 1993) of his policy rule prescribes a reaction to current inflation and output-gap estimates that is independent of both past aggregate conditions and past monetary policy.[2] And in discussions elsewhere (*e.g.*, Taylor, 1999a), Taylor stresses the desirability of adjusting the federal funds rate target immediately to changing aggregate conditions, without any of the "partial adjustment" dynamics typically indicated by estimated Fed reaction functions, such as those described in section xx of chapter 1.

The intuition of the proponents of purely forward-looking approaches, presumably, is that it is not desirable for policy to depend on "irrelevant" state variables, that is, on something that affects neither the set of possible future paths for the central bank's target variables nor the proper ranking of alternative paths. Yet such an argument would be incorrect, for the same reason that dynamic programming does not yield a truly optimal policy, as just discussed. Optimal policy must take account of the advantages of the anticipation of the policy at earlier dates; and for this reason it must generally be *history-dependent* rather than purely forward-looking. Past conditions should be taken into account in choosing the current policy setting, because it is desirable that people be able, at the earlier time, to count on the fact that the central bank will subsequently do so. Alternative approaches to the incorporation of history-dependence of an appropriate sort into policy deliberations are discussed in sections xx below.

We begin our discussion of the advantages from policy commitment with a review in section 1 of the most famous disadvantage of discretionary policy, namely the "inflation bias" stressed by Kydland and Prescott (1977) and Barro and Gordon (1983). This issue is one that can be treated in a purely deterministic analysis, and we consider the question first in such a setting. Once we allow in section 2 for random disturbances, the bias in

[2] In the empirical implementation of the rule (Taylor 1993, 1999b), Taylor uses the cumulative increase in the log of the GDP deflator over the previous four quarters as his proposed measure of current inflation; this might appear to imply that Taylor's rule is somewhat history-dependent, unlike the purely contemporaneous "Taylor rules" analyzed in much of the subsequent theoretical literature. However, it appears from Taylor's discussion of the logic of his proposal that the use of a measure of inflation over a year-long period should be understood as a simple attempt to estimate the current value of a target variable that is only imperfectly measured by higher-frequency data, rather than genuine advocacy of history-dependent policy.

the long-run average inflation rate continues to exist, but there is a further problem with discretionary policy as well: suboptimal equilibrium responses to unexpected shocks. Unlike the inflation bias, the distortion of the response to shocks cannot generally be cured by any purely forward-looking policy; it is the emphasis on the inflation bias in many discussions of the disadvantages of discretionary policy that probably accounts for the widespread assumption that the advantages of commitment can be obtained within a decision framework that remains purely forward-looking.

We then consider the problem of implementation of the desired equilibrium responses to disturbances through commitment of the central bank to an appropriate policy rule. A mere characterization of the desired state-contingent evolution of the economy does not suffice as a policy prescription, for a number of reasons outlined in section 3. One of the simplest of these is that a specification of the future evolution of policy (say, a specification of the future path of overnight interest rates, to be concrete) that allows for enough contingencies to represent a desirable policy will be too complex to actually write out in advance.

Instead, there are a variety of ways in which one may specify a *decision procedure* for the central bank that suffices (in the context of an evaluation of current conditions and possible future paths for the economy) to determine a policy action at each date, and such that an understanding that the central bank will follow the procedure determines a rational expectations equilibrium in which the economy's state-contingent evolution is of the desired sort. We compare several alternatives in sections 4 and 5, and argue in section 5 for the particular desirability of a conception of rule-based policymaking under which the central bank seeks in each decision cycle to determine the current interest-rate operating target that is consistent with a projection of the economy's subsequent path that satisfies a "target criterion". Decision procedures of this kind are similar in form to the inflation-forecast targeting currently practiced at the Bank of England and a number of other central banks. But the content of the target criterion is likely to be somewhat different under an optimal rule than under current U.K. practice; in particular, an optimal criterion will be history-dependent, as we illustrate here through a simple example.

1 The Optimal Long-Run Inflation Target

We begin our discussion of methods that can be used to characterize the economy's optimal evolution by considering the optimal rate of inflation in a purely deterministic setting. This problem is non-trivial when it is not possible to simultaneously eliminate all of the distortions that are affected by monetary policy, for one of the reasons discussed in section xx of chapter 6; and the problem provides a simple first setting in which to analyze the distortions resulting from discretionary policy. As it turns out, in the context of the linear-quadratic policy problems that are mainly studied below, the optimal long-run average rate of inflation in the presence of random disturbances continues to be the same one as in the deterministic analysis presented here; this is a consequence of the familiar *certainty-equivalence* principle for such problems. Hence our characterization here of the optimal rate of inflation also applies to the stochastic settings considered in section 2.

1.1 The Inflationary Bias of Discretionary Policy

Let us consider once again the basic neo-Wicksellian model of chapter 4 — a purely cashless economy with exogenous capital, flexible wages (and/or efficient labor contracting), and Calvo-style staggered pricing in goods markets — and suppose that there are no real disturbances ($\tilde{\xi}_t = 0$ for all t). Alternative possible perfect-foresight equilibrium paths for inflation and output must satisfy the "new Keynesian" aggregate-supply relation

$$\pi_t = \kappa x_t + \beta \pi_{t+1} \tag{1.1}$$

for all dates $t \geq 0$, where π_t is the rate of inflation, x_t is the output gap (here equivalent simply to detrended output), and the coefficients satisfy $\kappa > 0, 0 < \beta < 1$. This is of course only a log-linear approximation to the exact relation derived in section xx of chapter 3, valid for the characterization of possible equilibrium paths in which inflation is always near zero. But as we shall see, it is optimal in the present case for inflation to equal exactly zero, so that a comparison of paths in the neighborhood of this particular steady state suffices for a valid characterization of the optimal rate of inflation.

Equilibrium paths must also satisfy another restriction, the intertemporal Euler equation (or IS relation) that relates interest rates to the timing of expenditure. However, in the present case, the optimal paths of inflation and output can be determined without reference to that constraint, which simply determines the path of interest rates associated with any given equilibrium path for inflation and output. We do need to verify that the implied path for nominal interest rates is always non-negative; but we shall see that in the present case, this is true for both the optimal commitment and the equilibrium resulting from discretionary optimization.

We have shown in chapter 6 that a quadratic approximation to the utility of the representative household in this model is a decreasing function of

$$\sum_{t=0}^{\infty} \beta^t [\pi_t^2 + \lambda(x_t - x^*)^2] \tag{1.2}$$

where $\lambda > 0, x^* \geq 0$ are functions of model parameters discussed in that chapter.[3] As discussed in section 1 of chapter 6, a log-linear approximation to the model structural equations suffices for a correct linear approximation to optimal policy only in the case that x^* is small enough. Specifically, since in the present case there are no random disturbances, the solution for the optimal paths for inflation and output obtained by minimizing (1.2) subject to the constraint that (1.1) hold each period will be accurate up to a residual of order $\mathcal{O}(||x^*||^2)$. This suffices, however, for a characterization of the first-order effects of allowing for $x^* > 0$ (*i.e.*, for inefficiency of the natural rate of output). And this, in turn, is enough to allow us to understand the basic character of the inflation bias resulting from discretionary policy, even if we cannot expect our analysis to yield an accurate estimate of the *size* of this bias except when x^* is small.[4]

We therefore consider the choice of monetary policy to minimize (1.2) subject to the sequence of constraints (1.1). Let us first consider the equilibrium outcome under discretionary

[3] It does not matter for our characterization of optimal policy here whether we assume that the loss function parameters are the ones that correctly reflect economic welfare as characterized in chapter 6 or not; our results will be equally applicable if (1.2) is taken to represent an *ad hoc* policy objective of some other kind.

[4] Discuss Benigno-Woodford extension.....

optimization by the central bank. This is fairly simple to characterize, without any need to fully specify the strategic interaction between the central bank and private sector. In the model just recalled, the set of possible equilibrium paths for inflation and output from any period T onward are independent of what inflation, output or interest rates have been in any periods prior to T; nor does the central bank's evaluation of continuation paths from period T onward depend on prior history. Hence in a Markov equilibrium,[5] neither the equilibrium behavior of the private sector nor that of the central bank from period T onward should depend on what has happened earlier. This means that in such an equilibrium, the central bank can (correctly) assume in period t that the action it chooses will have no effect on outcomes in any periods $T \geq t+1$; nor (since the private sector has rational expectations) will it have any effect on the private sector's expectations in period t regarding π_{t+1}.

Let these equilibrium expectations be denoted π^e. (In a Markov equilibrium, inflation expectations are the same at all dates, owing to the time-invariant form of the continuation game.) Then the central bank will perceive itself as being able to choose in period t among inflation-output pairs that satisfy the constraint

$$\pi_t = \kappa x_t + \beta \pi^e. \tag{1.3}$$

As it can no longer affect the contributions to (1.2) from periods prior to t and expects its current decision to have no effect on the contributions from later periods, a discretionary optimizer will choose an action in period t intended to bring about the inflation-output pair that minimizes $\pi_t^2 + \lambda(x_t - x^*)^2$ subject to this constraint. The first-order condition for this static optimization problem is given by

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x^*) = 0. \tag{1.4}$$

Substitution of this into (1.3) implies that the bank will generate inflation satisfying

$$\pi_t = \left(1 + \frac{\kappa^2}{\lambda}\right)^{-1} [\kappa x^* + \beta \pi^e] \tag{1.5}$$

[5]There may, of course, be additional "reputational" equilibria in a policy game of this kind, as analyzed, for example, by Chari *et al.* (199xx) in the context of a sticky-price model with some prices fixed a period in advance. But for present purposes, the demonstration that *one* possible equilibrium under discretion is bad suffices to show that there is a potential gain from commitment to a suitable rule.

in the case of any given expectations π^e.

Rational expectations on the part of the private sector, of course, require that π^e be such that exactly that same inflation rate is generated. Thus the expected rate of inflation under discretionary policy is

$$\pi^e = \frac{\kappa\lambda}{(1-\beta)\lambda + \kappa^2}\, x^* > 0, \tag{1.6}$$

and this will also be the rate of inflation that the central bank chooses to generate each period, given the (correctly) perceived inflation-output tradeoff (1.3). Hence we have obtained the following result.

Proposition 7.1. Consider a cashless economy with flexible wages, Calvo pricing, and no real disturbances. Assume that the initial dispersion of prices $\mathrm{var}\{\log p_{-1}(i)\}$ is small (of order $\mathcal{O}(||\xi||^2)$),[6] and suppose furthermore that real distortions are small ($\Phi = \mathcal{O}(||\xi||)$), so that an approximation to the welfare of the representative household of the form (1.2) is possible, with $x^* > 0$ a small parameter ($x^* = \mathcal{O}(||\xi||)$). Then, at least among inflation paths in which inflation remains forever near enough to zero, there is a unique Markov equilibrium with discretionary optimization by the central bank. In this equilibrium, inflation is constant at the value given by the right-hand side of (1.6), up to an error that is only of order $\mathcal{O}(||\xi||^2)$.

But this outcome is not in fact the best possible rational-expectations equilibrium. Consider instead the problem of choosing bounded deterministic paths for inflation and output to minimize (1.2), subject to the constraint that the sequences must satisfy (1.1) each period. We can write a Lagrangian for this problem of the form

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2}[\pi_t^2 + \lambda(x_t - x^*)^2] + \varphi_t[\pi_t - \kappa x_t - \beta\pi_{t+1}] \right\},$$

where φ_t is a Lagrange multiplier associated with the period t aggregate-supply relation. Differentiation of the Lagrangian with respect to each of its arguments yields a pair of first-

[6]One should recall that this assumption was used in chapter 6 in establishing that welfare is decreasing in (1.2), neglecting terms independent of policy and a residual of only third order. In the equilibrium described in this proposition, the condition continues to hold forever if it holds for the initial price distribution.

order conditions

$$\pi_t + \varphi_t - \varphi_{t-1} = 0, \tag{1.7}$$

$$\lambda(x_t - x^*) - \kappa\varphi_t = 0, \tag{1.8}$$

for each $t \geq 0$, where in (1.7) for $t = 0$ we substitute the value

$$\varphi_{-1} = 0, \tag{1.9}$$

as there is in fact no constraint associated with fulfillment of a period -1 aggregate-supply relation.

Using (1.7) and (1.8) to substitute for π_t and x_t respectively in (1.1), we obtain a difference equation for the evolution of the multipliers,

$$\beta\varphi_{t+1} - \left(1 + \beta + \frac{\kappa^2}{\lambda}\right)\varphi_t + \varphi_{t-1} = \kappa x^*, \tag{1.10}$$

that must hold for all $t \geq 0$, along with the initial condition (1.9). The characteristic equation

$$\beta\mu^2 - \left(1 + \beta + \frac{\kappa^2}{\lambda}\right)\mu + 1 = 0 \tag{1.11}$$

has two real roots

$$0 < \mu_1 < 1 < \mu_2,$$

as a result of which (1.10) has a unique non-explosive solution consistent with the initial condition (1.9), given by

$$\varphi_t = -\frac{\lambda}{\kappa}x^*(1 - \mu_1^{t+1}) \tag{1.12}$$

for all $t \geq 0$. This solution, which is the only one satisfying the relevant transversality condition, represents the optimal perfect-foresight path from standpoint of period zero. Substituting this solution for the multipliers into (1.7), we find that the path of inflation under the optimal commitment is given by

$$\pi_t = (1 - \mu_1)\frac{\lambda}{\kappa}x^*\mu_1^t \tag{1.13}$$

for all $t \geq 0$.

This result indicates that discretionary optimization leads to excessive inflation. Indeed, under the optimal commitment, inflation should asymptotically approach zero, despite the assumption that $x^* > 0$. As has been mentioned in chapter 6, this results because the aggregate-supply relation (1.1) implies that in any perfect-foresight equilibrium, the objective (1.2) must equal

$$-2\frac{x^*}{\theta}\pi_0 + \sum_{t=0}^{\infty} \beta^t[\pi_t^2 + \lambda x_t^2]$$

plus a positive constant. All terms in the above expression except the initial one are minimized by choosing $\pi_t = 0$ each period, regardless of the value of π^*; the presence of the initial term implies an advantage from an initial positive rate of inflation, but because the additional term applies only to inflation in the initial period, it remains optimal to commit to an inflation rate that is eventually zero.

Indeed, it is not obviously desirable to choose a positive inflation rate even initially. It is true that (1.2) is minimized by choosing the inflation path (1.13) rather than zero inflation from the initial date onward. This welfare gain, however, is obtained as a result of the fact that the inflation rate that is chosen initially has no consequences for expectations prior to date zero (that are taken as given at the time of the policy deliberations). This exploitation of the fact that initially existing expectations need not be fulfilled, however, is unattractive. It implies that the optimal policy determined on the above grounds (optimality from the standpoint of date zero) is not *time consistent:* if the same reasoning is used at any later date $t > 0$ to determine the optimal policy commitment from *that* date onward, the policy chosen will not be a continuation of the policy selected at date zero. (This can be seen from the fact that the inflation rate π_t varies with the date t in (1.13), even though the constraints defining the possible inflation paths from any date onward, and the social valuation of alternative inflation paths looking forward from any date, are the same at all times.) This failure of time consistency means that adherence to the policy must be due solely to a willingness to conform to a commitment entered into at an earlier date. In practice, it is difficult to imagine that a central bank would ever regard itself as being committed to a specific sequence of actions chosen at an earlier date, simply because they seemed desirable at the earlier time[7] — whereas it is easier to imagine a bank being committed to a systematic *decision procedure* in the light of which its current actions are always to be justified. Furthermore, the above analysis assumes that it is possible to commmit to an arbitrary time path for inflation, and have this be expected by the private sector; it is assumed to be possible to choose inflation "just this time" while committing never to create inflation in the future. But there is reason to fear that the public should observe the central bank's method of reasoning, rather than its announced future actions, and conclude instead that in the present it should always wish to create inflation "just this time".

A similar problem of time consistency is familiar in the context of fiscal policy, and in that context it is common to conclude that if, say, one wishes the public to be able to confidently expect that capital will not be expropriated in the future, then one should adopt the rule of refusing to expropriate already existing capital, even though the latter action should not have the same kind of effects on investment incentives (since the investment decisions in question have already occurred). Similarly, if one wishes for the public to believe that a non-inflationary policy will eventually be pursued, and there is no difference between the current situation and the one that is anticipated in the future (except that one currently has an opportunity to create inflation without its having been expected), then it makes sense that the central bank should be willing to choose a non-inflationary policy as well. Rather than doing one thing now but promising to behave differently in the future, one should follow a time-invariant policy that is of the kind that one would always wish *to have been expected* to follow. Woodford (1999) calls such a policy "optimal from a timeless perspective."[8]

More specifically, we shall say that a time-invariant policy is optimal from a timeless perspective if the equilibrium evolution from any date t_0 onward (at which date one may consider the justification of the policy) is optimal subject to the constraint that the economy's

[7]Such an understanding of the meaning of policy commitment raises the question posed by Svensson (1999xx): "What is special about date zero?"

[8]For additional discussions of the selection of policy rules from a timeless perspective, see McCallum and Nelson (199xx), Giannoni and Woodford (2002xx), Svensson and Woodford (2002xx), and section xx below.

initial evolution be the one associated with the policy in question. (The presence of such a constraint on the initial outcomes that may be contemplated is a way of committing oneself to forswear the temptation to exploit the already given past expectations regarding those initial outcomes.) In our present context, a time-invariant policy will imply a constant inflation rate $\bar{\pi}$. A constant inflation target $\bar{\pi}$ is optimal from a timeless perspective if the problem of maximizing (1.2) subject to the constraint that the bounded sequences $\{\pi_t, x_t\}$ satisfy (1.1) for each $t \geq 0$, and the additional constraint that $\pi_0 = \bar{\pi}$, has a solution in which $\pi_t = \bar{\pi}$ for all t.[9] The first-order conditions for this latter problem are again given by (1.7) and (1.8) for each $t \geq 0$, but now the initial condition (1.9) is replaced by the requirement that $\pi_0 = \bar{\pi}$. One easily sees that this system of equations has a solution in which $\pi_t = \bar{\pi}$ for all t if and only if $\bar{\pi} = 0$. Hence this is the uniquely optimal inflation target from a timeless perspective.

Proposition 7.2. Consider a cashless economy with flexible wages, Calvo pricing, and no real disturbances, and suppose that the initial dispersion of prices is small (of order $\mathcal{O}(||\xi||^2)$). Then a monetary policy under which inflation is zero for all t is optimal from a timeless perspective, and is the unique policy with this property among all policies under which inflation remains always in a certain interval around zero.

Note that for this proposition, unlike Proposition 7.1, it is not necessary for x^* to be small, since the optimal path is found to be near the zero-inflation steady state regardless of the value of x^*.

Under this analysis of the character of an optimal policy, there is a *constant* inflation bias associated with discretionary policy, which is clearly positive when $x^* = 0$. This is illustrated in Figure 7.1, which plots the time paths of inflation under discretionary policy, under the date-zero-optimal commitment, and under the policy that is optimal from a timeless perspective.[10] Inflation is lowest under the last of these policies, since only in this last case

[9] See section xx below for a definition of optimal policy from a timeless perspective in a more general context.

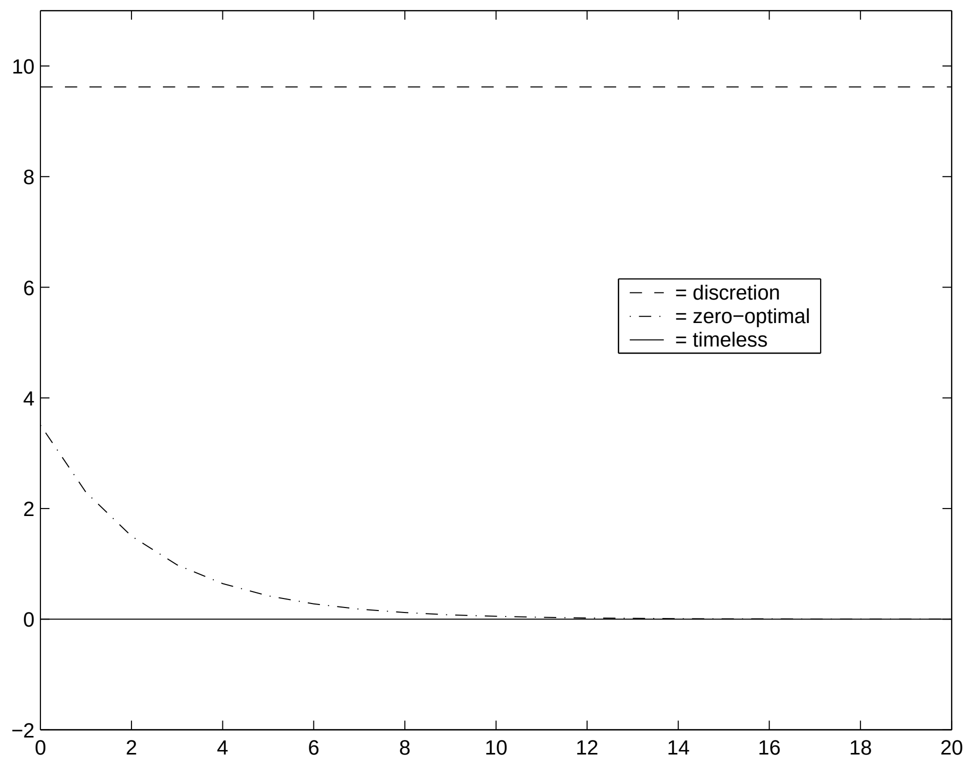

Figure 7.1: Optimal policy from a timeless perspective compared to the result of discretionary optimization, and to the policy commitment that is optimal from the standpoint of period zero.

does the central bank refrain from the temptation to create inflation that cannot affect prior expectations, both in the short run and in the long run.

1.2 Extensions of the Basic Analysis

A striking conclusion of the previous section is that the optimal inflation rate is exactly zero in our baseline model with Calvo pricing, regardless of parameter values, including those that determine the size of the gap x^* between the optimal level of output and that consistent with zero inflation. However, this analysis does neglect various other factors that may bear on the choice of a long-run average inflation target. One is the neglect of monetary frictions of

[10] In this numerical example, the values of β κ are again those given in Table 6.1, λ is assigned the value given there for λ_x, and $x^* = 0.2$. This last value follows from equation (xx) in chapter 6 for the optimal output gap, under the assumptions that the elasticity of substitution among alternative differentiated goods (and hence the elasticity of demand faced by each firm) is equal to 7.88, the value obtained by Rotemberg and Woodford (1997) (see Table 4.1), and that there are no distorting taxes.

the sort that led Friedman (1969) to argue for the optimality of deflation at the rate of time preference.

As shown in the previous chapter, in the presence of non-negligible transactions frictions, the welfare-theoretic loss function takes the form

$$\sum_{t=0}^{\infty} \beta^t [\pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i_t^m)^2], \quad (1.14)$$

where $\lambda_x > 0$ is the coefficient called λ in (1.2), $\lambda_i > 0$ as well, i_t is the short-term nominal interest yield on non-monetary assets, and i_t^m is the interest rate paid on the monetary base.[11] In the event that monetary policy is implemented by varying i_t^m so as to maintain a constant interest differential regardless of the desired level of nominal interest rates — and that this differential is taken to be an institutional datum rather than an aspect of monetary policy — then the additional term in (1.14) relative to (1.2) makes no difference, as it is simply a constant independent of the chosen target path for inflation. In this case, the conclusions of the previous section continue to apply: the optimal policy from a timeless perspective will involve a zero inflation rate at all times. But if, instead, the interest paid on the monetary base is an institutional datum, and $\bar{\imath}^m$ is lower than the rate consistent with zero inflation (for example, if there is zero interest paid on money, as assumed by Friedman), then the rate of inflation that would otherwise be optimal may not be because of its consequences for nominal interest rates and hence the size of the last term in (1.14).

In the case that real balances enter the utility function in an additively separable way, transactions frictions do not affect the structural equations that determine equilibrium inflation and output, as discussed in chapter 4. (While additively separability is not realistic, as discussed earlier, the quantitative magnitude of the real-balance effects that are neglected by such an assumption is likely to be small.) Feasible perfect-foresight equilibrium paths

[11] In the present chapter and the next one, i_t refers to the *continuously compounded* rate of interest on non-monetary assets, $\log(1 + i_t)$ in the notation of the previous chapters, and similarly for i_t^m and the natural rate of interest r_t^n, appearing in equation (xx) below. Hence the interest differential $i_t - i_t^m$ is equal to $\hat{\imath}_t - \hat{\imath}_t^m + \bar{\Delta}$ in the notation of chapter 6, up to a term of order $\mathcal{O}(||\bar{\Delta}||^2)$ that can be neglected in our quadratic approximation to the welfare-theoretic loss function. The change in notation should not create confusion, now that we need no longer discuss the exact nonlinear equations that are log-linearized in deriving the linear structural relations assumed in this chapter and the next.

for inflation, output and nominal interest rates then must satisfy (1.1), together with the corresponding deterministic IS equation

$$x_t = x_{t+1} - \sigma(i_t - \pi_{t+1} - \bar{r}),) \tag{1.15}$$

for periods $t \geq 0$, where $\bar{r} > 0$ represents the constant natural rate of interest. A stationary policy commitment resulting in a constant inflation rate $\bar{\pi}$, output gap $\bar{x}$, and nominal interest rate $\bar{\imath}$ will be optimal from a timeless perspective if the bounded sequences $\{\pi_t, x_t, \imath_t\}$ that minimize (1.14) subject to the constraints that (1.1) and (1.15) hold for each $t \geq 0$, and that

$$\pi_0 = \bar{\pi}, \qquad x_0 = \bar{x},$$

are given by $\pi_t = \bar{\pi}, x_t = \bar{x}, i_t = \bar{\imath}$ for all t.

The Lagrangian for this generalization of our previous problem is of the form

$$\begin{aligned} \mathcal{L} \;\; = \;\; & \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} [\pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^m)^2] + \varphi_{1t}[x_t - x_{t+1} + \sigma(i_t - \pi_{t+1} - \bar{r})] \right. \\ & \left. + \varphi_{2t}[\pi_t - \kappa x_t - \beta \pi_{t+1}] \right\} - \beta^{-1} \varphi_{1,-1}[x_0 + \sigma \pi_0] - \varphi_{2,-1} \pi_0, \end{aligned}$$

where there are now Lagrange multipliers $\varphi_{1t}, \varphi_{2t}$ corresponding to constraints (1.15) and (1.1) respectively, and multipliers $\beta^{-1}\varphi_{1,-1}, \varphi_{2,-1}$ corresponding to the constraints on the values of x_0 and π_0 respectively. (The notation chosen for these last multipliers is selected in order to give the first-order conditions below a time-invariant form.)

The first-order conditions are then

$$\pi_t - \beta^{-1} \sigma \varphi_{1,t-1} + \varphi_{2t} - \varphi_{2,t-1} = 0, \tag{1.16}$$

$$\lambda_x (x_t - x^*) + \varphi_{1t} - \beta^{-1} \varphi_{1,t-1} - \kappa \varphi_{2t} = 0, \tag{1.17}$$

$$\lambda_i (i_t - i^m) + \sigma \varphi_{1t} = 0 \tag{1.18}$$

for each period $t \geq 0$. Conditions (1.17) – (1.18) have a solution with the output gap and interest rate constant over time only if both Lagrange multipliers are also constant over time;

but substituting constant values for the Lagrange multipliers in (1.16) and (1.18), one finds that these relations can be simultaneously satisfied only if

$$\lambda_i(\bar{\imath} - i^m) = -\beta\bar{\pi}.$$

At the same time, equation (1.15) is satisfied by constant values if and only if $\bar{\imath} = \bar{r} + \bar{\pi}$. These two relations are jointly satisfied if and only if the inflation target is equal to

$$\bar{\pi} = -\frac{\lambda_i}{\lambda_i + \beta}\,(\bar{r} - i^m). \tag{1.19}$$

We thus obtain the following result.

Proposition 7.3. Consider an economy with flexible wages, Calvo pricing, and no real disturbances, with transactions frictions such that the quadratic approximation to welfare be given by (1.14), while the log-linear approximate structural relations are of the form (1.1) and (1.15). Suppose that the only feasible monetary policies are ones involving a constant interest rate i^m on the monetary base, where $i^m = \bar{r} - \mathcal{O}(||\xi||)$. Then the policy that is optimal from a timeless perspective involves a constant inflation rate equal to the right-hand side of (1.19).

Note that in the case that $\lambda_i > 0$ and $i^m < \bar{r}$, the optimal inflation target will be negative. In the limiting case of flexible prices, κ, and hence λ_i, becomes unboundedly large, and the optimal rate of deflation is $-(\bar{r} - i^m)$, the rate required to make the real return on money as high as $\bar{r}$, as argued by Friedman (1969). When prices are sticky, and κ is finite, the optimal rate of deflation is more moderate, and the optimal policy is one under which $\bar{\imath} > i^m$; but some deflation is still optimal, in order to reduce the distortions resulting from transactions frictions. In the "cashless limit", in which $\bar{m}$ is negligible, λ_i is much smaller than β, and the optimal rate of inflation is essentially zero, as found in the previous section.

In the case that real-balance effects are non-negligible in the aggregate-supply and IS equations, the same method can be employed as above, but (1.19) takes the more general form[12]

$$\bar{\pi} = -\frac{\lambda_i(\bar{r} - i^m) + \varphi\lambda_x x^*}{\lambda_i + \varphi^2\lambda_x + \beta - (1-\beta)\varphi(\theta^{-1} + \omega)}, \tag{1.20}$$

where the coefficient φ indexes the size of the real-balance effects, as in chapter 4. If we use the calibrated parameter values from Table 6.1, except that x^* is computed from equation (xx: xstaristar) of chapter 6 under the assumption that $\Phi/\epsilon_{mc} = 0.2$ as assumed in the previous section, and we assume that $i^m = 0$, this equation implies an optimal inflation rate of -0.4 percent per year.[13] This involves mild deflation, but much less deflation than would be suggested by the arguments of Friedman (1969) or Lucas (2000), according to which the optimal rate of inflation (under our parameter values and those of Lucas) would be -3 percent per year. And we should recall that these parameter values have been chosen to exaggerate the size of real-balance effects; under the more plausible assumption that $\chi = 0.01$, we would obtain an optimal rate of deflation of only 0.2 percent per year.

It is perhaps surprising to observe that if $\varphi > 0$ (the empirically realistic case, as argued in chapter 2), and if the denominator of (1.20) is positive (as must be true if real-balance effects are not too strong), a higher value of x^* actually *lowers* the optimal inflation target. It is commonly supposed that a higher value of x^* should justify higher inflation, in a model (like the present one) that incorporates an upward-sloping "long-run Phillips curve" relation between inflation and output. Yet this is not so. As shown in chapter 6, in this model, commitment to a higher inflation rate in any period $t > t_0$ (*i.e.*, any period for which the anticipation effects must be considered) does not change the discounted sum of output-gap terms implied by the aggregate-supply relation; for the increase in output in period t is exactly offset by a reduction in output in period $t-1$. The only effects of policy that do

[12]Equation (1.20) applies in the case that this inflation rate implies a nominal interest rate no lower than i^m. In the case that this bound is not satisfied, the analysis must be modified to take account of the constraint on policy resulting from the zero lower bound on nominal interest rates, assumed not to bind in this derivation.

[13]In this calculation, the values of $\lambda_x, \lambda_i, \bar{r}$ and β are taken directly from Table 6.1. The value used for φ is 0.22 years, the value implicit in the Table 6.1 parameter values, as discussed there; similarly, we assume the value $\theta = 7.88$ as in the derivation of the values given in the table. The values for $\epsilon_{mc}, \sigma^{-1}, \chi$, and η_y given in the table imply a value $\omega = 0.48$. In using equation (xx) of chapter 6 to derive x^*, we recall that in the notation of that chapter $i^* = -(\bar{r} - i^m)$. Finally, we note that in the denominator of (1.20), if we use an annual measure for φ, it is necessary to multiply $(1-\beta)$ by 4 to obtain an annual discount rate as well.

not cancel in this way are the real-balance effects on aggregate supply, when these exist. If $\varphi > 0$, a lower nominal interest rate causes a favorable shift in the aggregate-supply relation each period, allowing a higher value for the discounted sum of output gaps. When $x^* > 0$, this channel creates a reason to prefer a lower nominal interest rate than the one consistent with zero inflation, and hence to prefer deflation for reasons independent of Friedman's.

Thus far we have discussed only reasons why the optimal inflation target might be even *lower* than zero. Other considerations may instead justify a positive long-run average inflation rate. These include a desire the prevent the zero lower bound on nominal interest rates from being so tight a constraint on cyclical variation in real rates for stabilization purposes (Summers, 1991), or a desire to make a social norm that prohibits nominal wage declines less of a constraint upon the degree to which real wages can decline when necessary for an efficient allocation of resources (Akerlof *et al.*, 199xx). These latter considerations matter only in the case of random disturbances, and cannot be addressed without considering the optimal responses to such disturbances. It is worth noting, however, that they cast some doubt on the desirability of deliberately aiming for deflation, as opposed to zero inflation or even a very modest positive rate of inflation, of the sort typically aimed at by current inflation targeting central banks.[14]

Our analyses thus far have also argued for the optimality of a policy that would bring about a constant inflation rate in the absence of stochastic disturbances. Yet a number of countries that have actually adopted inflation targets in the past decade have thought it desirable to lower the target inflation rate from an (undesirably high) initial rate of inflation only gradually, over several years, rather than announcing an intention to immediately jump to the inflation rate that is regarded as optimal over the long run. The argument made for the desirability of gradualism of this sort generally involves a belief that there is substantial inertia in the inflationary process, though the models used above do not allow for this. We can consider the effect of inflation inertia on the optimal time path for inflation, still within a

[14]The consequences of the zero lower bound have been considered in section xx of chapter 6, and are considered further in section xx of this chapter. Both analyses display conditions under which a policy would be chosen that involves a positive average rate of inflation.

purely deterministic context, by allowing for indexation of individual prices to a lagged price index, as considered in section xx of chapter 3. In this case, the aggregate-supply relation (1.1) generalizes to

$$\pi_t - \gamma\pi_{t-1} = \kappa x_t + \beta(\pi_{t+1} - \gamma\pi_t), \tag{1.21}$$

where $0 \leq \gamma \leq 1$ indicates the degree of indexation, if we once again abstract from real-balance effects. Hence the possible perfect-foresight paths for inflation and output from any date t_0 onward depend upon the pre-existing rate of inflation $t_0 - 1$.

As shown in chapter 6, the welfare-theoretic loss function (1.2) for a cashless economy generalizes to

$$\sum_{t=0}^{\infty} \beta^t[(\pi_t - \gamma\pi_{t-1})^2 + \lambda(x_t - x^*)^2] \tag{1.22}$$

in the presence of inflation inertia. Because both the constraints (1.21) and the loss function (1.22) are of the same form as before, but with $\pi_t - \gamma\pi_{t-1}$ replacing π_t in the previous equations, the same calculations as before may be directly used to characterize optimal policy (as well as the consequences of discretionary optimization), except that our previous solutions for the path of π_t now apply instead to the path of $\pi_t - \gamma\pi_{t-1}$.

Figure 7.2 shows the implied equilibrium paths for inflation under discretion, under an unconstrained optimal commitment from the standpoint of period zero, and under a policy that is optimal from a timeless perspective, when all parameters are the same as those assumed in Figure 7.1, except that $\gamma = 0.5$, and (since the initial inflation rate now matters) an inflation rate of 10 percent per year prior to period zero is assumed. In all three cases, the fact that the economy starts from a condition of fairly high inflation makes the inflation rate in the early periods higher than it would otherwise be; in fact, one can show that in each of the three cases, the equilibrium value of π_t is given by $\pi_{-1}\gamma^{t+1}$ plus a term that is independent of the initial condition. However, despite this effect of inertial inflation, the comparisons among the three paths remain of the kind discussed earlier: discretion leads to an inflation rate that is permanently higher than the inflation that would occur under an optimal policy, and unconstrained (time-inconsistent) optimization at date zero leads to the choice of a higher inflation rate in the early periods, reflecting an attempt to exploit the

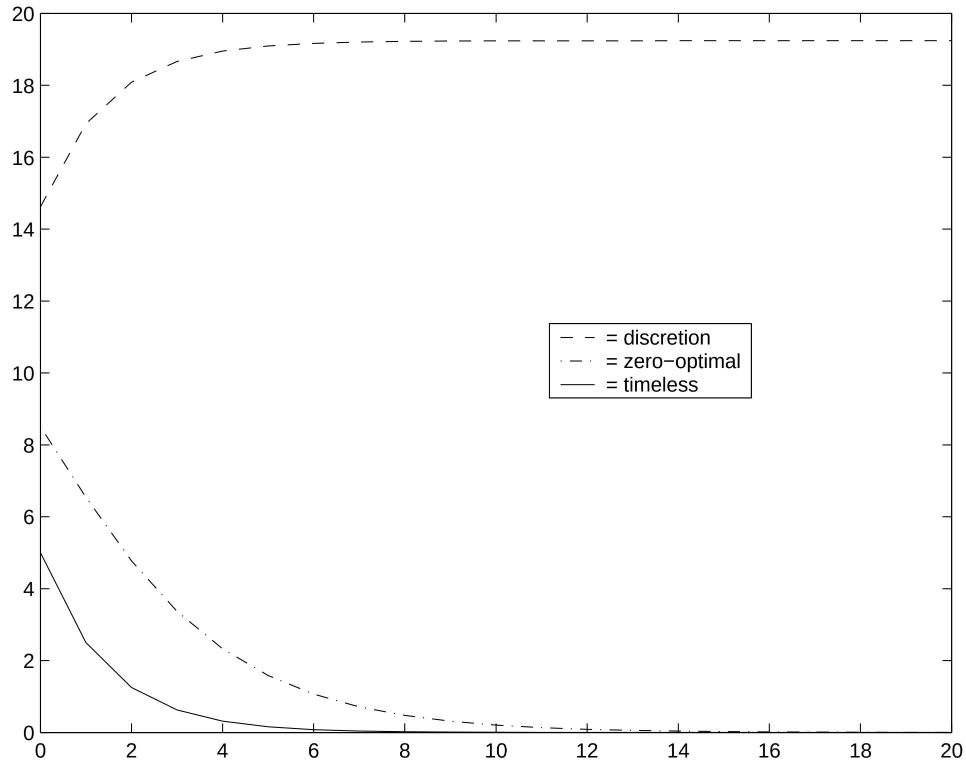

Figure 7.2: Timelessly optimal policy, date-zero-optimal policy, and discretionary policy, in the case of inflation inertia [$\gamma = 0.5$].

initially given expectations of inflation, though it leads to the same inflation rate as under a timelessly optimal policy in the long run.

The proof of Proposition 7.2 above can be directly adapted to provide the following characterization of optimal policy in this case.

PROPOSITION 7.4. Consider an economy of the kind assumed in Proposition 7.2, except that prices are (partially) indexed to a lagged aggregate price index, to a degree measured by $0 \leq \gamma \leq 1$, and suppose that the initial inflation rate satisfies $\pi_{-1} = \mathcal{O}(||\xi||)$. Then the policy that is optimal from a timeless perspective involves a path of inflation along which

$$\pi_t = \gamma \pi_{t-1} \tag{1.23}$$

at each date $t \geq 0$, up to a residual of order $\mathcal{O}(||\xi||)$.

This path has the property that the policy that minimizes discounted losses from any date t_0 onward, given the inflation history up through period $t_0 - 1$ and subject to the constraint that (1.23) hold in period t_0, is one in which (1.23) holds for all $t \geq t_0$. Note that (1.23) can easily be solved to show that $\pi_t = \pi_{-1}\gamma^{t+1}$ for any $t \geq 0$. Once again we find (in the case that $\gamma < 1$) that the optimal long-run inflation rate is zero; but it is now optimal to approach this inflation rate only over a period of time, if the economy starts (for whatever reason) from an initial situation in which inflation has been higher.

In the case of full indexation ($\gamma = 1$), this result implies that it is not optimal *ever* to reduce inflation below whatever its initial level happens to be. Such a result should not be acceptable as having too much practical relevance. It comes about because, in our simple model, all prices are adjusted to reflect past inflation in a perfectly synchronized way, except when they are reoptimized. When $\gamma = 1$, a policy that keeps inflation steady will result in all prices being increased by exactly the same amount each period, so that no price dispersion is created, regardless of the rate of inflation. But in practice, even in an economy with substantial indexation (as one generally observes in economies with chronic high inflation), inflation results in price dispersion, and higher inflation is generally associated with greater dispersion. If our model were extended to allow for this — for example, by supposing that prices are not continuously adjusted in response to variations in the aggregate price index between the occasions on which they are re-optimized, but that these mechanical price adjustments also occur only at certain intervals — then price dispersion would be minimized only with zero inflation, though we would again find that *changes* in the rare of inflation would increase price dispersion and hence lower welfare. We do not attempt to develop an extended indexation model of this kind here. But it is fairly obvious that we can in this way, or in various others, justify the assumption that there are at least some distortions associated with a high rate of inflation, regardless of how steady it may be; and once we introduce even a small amount of such a distortion, it will again be optimal to eventually disinflate, regardless of the degree of inflation inertia.

For example, suppose that we allow again for transactions frictions and suppose that the interest paid on the monetary base is constant. For simplicity, let us consider only the case in which $\chi = 0$ though $\lambda_i > 0$. In this case, the structural equation (1.15) must be combined with (1.21) as constraints on possible perfect-foresight paths, and the welfare-theoretic loss function is of the form

$$\sum_{t=0}^{\infty} \beta^t [(\pi_t - \gamma\pi_{t-1})^2 + \lambda(x_t - x^*)^2 + \lambda_i(i_t - i^m)^2].$$

The first-order condition (1.16) generalizes to

$$(\pi_t - \gamma\pi_{t-1}) - \beta\gamma(\pi_{t+1} - \gamma\pi_t) - \beta^{-1}\sigma\varphi_{1,t-1} + \varphi_{2t} - \varphi_{2,t-1} = 0,$$

while (1.17) – (1.18) continue to apply as written above. The same analysis as above can again be used to derive a unique long-run steady-state rate of inflation that is consistent with these first order conditions, given by

$$\bar{\pi} = -\frac{\lambda_i}{\lambda_i + \beta(1-\gamma)(1-\beta\gamma)}(\bar{r} - i^m) < 0,$$

generalizing (1.19).

While in the case that $\lambda_i = 0$ (the cashless case just considered), this equation yields a determinate outcome only when $\gamma \neq 1$, in the case that $\lambda_i > 0$, there is a uniquely optimal long-run inflation rate even when $\gamma = 1$, namely

$$\bar{\pi} = -(\bar{r} - i^m),$$

the rate of deflation called for by Friedman (1969). It is worth noting that the degree to which it is eventually desirable to disinflate is independent of how large λ_i may be — it is simply necessary to allow for *some* frictions of this kind, that increase with the rate of expected inflation. In fact, if we assume full indexation of prices to a lagged price index, this indexation actually *increases* the degree to which it is optimal to eventually lower the rate of inflation; for it is no longer true that steady deflation creates relative-price distortions owing to the failure of prices to be re-optimized at perfectly synchronized times. This last result, however, is again an example of an extreme conclusion that results from too simple a model of indexation. In reality, even steady deflation is likely to create distortions, for reasons identical to those mentioned in the case of steady inflation; and as a result, the optimal long-run rate of inflation is likely not to be so low as the Friedman rate.

2 Optimal Responses to Disturbances

The inflationary bias resulting from discretionary policy has been much discussed. However, emphasis on the problem of inflation bias has often led to a supposition that the problems resulting from discretion can be cured through a simple adjustment of the targets and/or the relative weights on alternative stabilization objectives assigned to the central bank, while allowing the central bank's decisionmaking framework to be otherwise one of unfettered discretion. For example, our baseline analysis in section 1.1 above implied an inflation bias equal to

$$\frac{\kappa\lambda}{(1-\beta)\lambda+\kappa^2}\, x^*,$$

when the central bank seeks to minimize the true social loss function, in which we have argued that $\lambda > 0, x^* > 0$. But if the central bank instead seeks to minimize a loss function of the form (1.2) with some other coefficients, the preceding analysis still applies; in particular, the equilibrium inflation rate will be zero each period, even under discretion, as long as either $\lambda = 0$, $x^* = 0$, or both. Thus one might suppose that the problem can be solved by appointing a central banker with appropriate preferences, or by charging the central bank with the task of minimizing a particular loss function that differs from true social welfare. The choice of a loss function with $\lambda = 0$ corresponds to Rogoff's (1985) proposal that a "conservative central banker" be chosen, while King (1997) and Blinder (1998) propose that the central bank should have an objective under which $x^* = 0$, *i.e.*, an output target consistent with its inflation target.

But while it is fairly simple to eliminate the bias in the average rate of inflation resulting from discretionary policy through either of these means, this does not suffice to yield an optimal policy framework, for in general the equilibrium responses to shocks that result will be sub-optimal, even if the long-run average values of the various state variables are the optimal ones. It is perhaps obvious that the Rogoff proposal to alter the relative weight on the two stabilization objectives will often result in incorrect responses to disturbances; indeed, Rogoff discusses this as an important qualification to his proposal (so that the optimal degree of "conservativeness" is argued to be less than absolute, in the presence of certain kinds of disturbances). But in the context of a linear-quadratic framework of the kind that we use here to approximate the central bank's problem, an adjustment of the target values of one or more variables affects the average equilibrium values of the endogenous variables resulting from central-bank optimization without having any effect on the equilibrium responses to shocks; hence one might think that the proposal of King and Blinder should not result in any distortion of stabilization policy.[15] But in general, the equilibrium resulting from discretionary optimization is sub-optimal, not only in the long-run average values of variables such as inflation, but *also* in the equilibrium responses of these variables to random shocks, for reasons discussed in the introduction; and an adjustment of the target values assigned to the central bank does nothing to cure this problem.

In this section, we contrast the optimal responses to shocks to those resulting from discretionary optimization in two simple examples, and then discuss the structure of the problem more generally. We show not only that the optimal responses to shocks are generally different from those resulting from discretionary optimization, but that they generally require that equilibrium be history-dependent in a way that cannot result from any purely forward-looking decision procedure for monetary policy. This implies, among other things, that an approach to the implementation of optimal policy that charges the central bank with the minimization of a loss function under discretion, which loss function involves the same target variables as the true social loss function — only with different target values and a different matrix of weights in the quadratic form over deviations from those target values — will

[15] Indeed, King (1997) shows, in the context of a simple model in which an aggregate-supply relation of the "New Classical" form discussed in chapter 3 is combined with an assumed stabilization objective of the form (1.2), that under his proposed modification of the central bank's loss function, the equilibrium resulting from discretionary optimization is optimal. However, this result depends on extremely special features of the example that he considers.

generally be inadequate, as such a decision procedure will be purely forward-looking.

2.1 Cost-Push Shocks

Probably the simplest example of this general problem arises in the case of "cost-push shocks" that shift the aggregate-supply relation when written in terms of the welfare-relevant inflation and output-gap measures.[16] In the case of such disturbances, complete stabilization of both inflation and the output gap is impossible; thus we need not assume any concern with interest-rate stabilization, or other stabilization objectives, in order to conclude that there is an essential tension among the stabilization objectives of monetary policy. In such a case, discretionary policy does not generally lead to optimal responses to shocks.[17]

We can see this by considering the minimization of a social loss function given by the expected value of (1.2), if the aggregate-supply relation each period is given by

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t, \tag{2.1}$$

where u_t is an exogenous cost-push shock. Let us first consider the case of discretionary optimization by the central bank. Let s_t be the exogenous state at date t that contains all information available at that time about current and future cost-push disturbance terms. We observe that the set of possible equilibrium evolutions of inflation and the output gap from period t onward depend only on s_t, and in particular are independent of the past values of all endogenous variables. It follows that in a Markov equilibrium, π_t and x_t should be functions only of s_t. Hence the central bank in period t believes that its policy action in that

[16] Several interpretations of such disturbances are possible, as discussed in sections xx of chapter 6.

[17] If complete stabilization of each of the target variables is simultaneously possible, then there is no difference in the responses to shocks under discretion and under an optimal commitment. A discretionary optimizer finds it optimal to completely stabilize each of the target variables in any given period, given the assumption that they will be stabilized in the future; hence complete stabilization is an equilibrium under discretionary policy, and is also optimal. This is obvious in the case that the target values of the various state variables are mutually consistent with equilibrium in the absence of shocks, so that it is possible for the loss function to equal zero at all times. But even when this is not so — when, for example, x^* is too high to be consistent with zero inflation on average — the same result is true, because the equilibrium responses to shocks, both under discretion and under an optimal commitment, are independent of the assumed target values, which matter only for the average values of the variables in equilibrium.

period can have no effect on terms in the loss function for periods $T \geq t+1$, and also that its action will have no effect on the private sector's expectations $E_t\pi_{t+1}$.

Then the central bank will perceive itself as being able to choose in period t among inflation-output pairs that satisfy the constraint (2.1), for a *given* value of $E_t\pi_{t+1}$ (that depends only on s_t), and it will choose an action in period t intended to bring about the inflation-output pair that minimizes $\pi_t^2 + \lambda(x_t - x^*)^2$ subject to this constraint. The first-order condition for this static optimization problem is again given by (1.4). Substitution of this into (2.1) implies that the bank will generate inflation satisfying

$$\pi_t = \left(1 + \frac{\kappa^2}{\lambda}\right)^{-1} [\kappa x^* + u_t + \beta E_t \pi_{t+1}], \tag{2.2}$$

generalizing (1.5). The Markov solution to this equation is then an inflation process

$$\pi_t = \frac{\kappa\lambda}{(1-\beta)\lambda + \kappa^2}\, x^* + \sum_{j=0}^{\infty} \beta^j \left(\frac{\lambda}{\lambda + \kappa^2}\right)^{j+1} E_t u_{t+j}. \tag{2.3}$$

Proposition 7.5. Consider an economy of the same kind as in Proposition 7.1, except that the aggregate-supply relation (2.1) is perturbed by an exogenous disturbance process $\{u_t\}$ satisfying a uniform bound $||\xi||$. Then there is a neighborhood of zero in which there is a unique Markov equilibrium inflation process under discretionary optimization by the central bank, for any small enough $||\xi||$. In this equilibrium, inflation evolves according to (2.3).

The corresponding Markov solution for output can be obtained by substituting this solution for inflation into (2.1). Note that the long-run average rate of inflation in this equilibrium, given by the constant term in (2.3), is the same as in the deterministic analysis. The equilibrium responses of inflation and output are purely forward-looking; if u_t is a Markov process, with the value of u_t being revealed only in period t, then equilibrium π_t and x_t depend only on the current disturbance u_t, and the effects of a disturbance on the paths of inflation and output will be only as persistent as the disturbance itself.

Now consider instead the nature of an optimal policy commitment.[18] We first consider the state-contingent evolution from some period t_0 onward that minimizes the expected discounted sum of losses from period t_0 onward, conditioning upon the state of the world in period t_0, and subject to the constraint that this evolution represent a possible rational-expectations equilibrium, *i.e.*, that it satisfy (2.1) for all periods $t \geq t_0$. The Lagrangian associated with this problem is of the form

$$\mathcal{L}_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ \frac{1}{2} [\pi_t^2 + \lambda_x (x_t - x^*)^2] + \varphi_t [\pi_t - \kappa x_t - \beta \pi_{t+1}] \right\}. \qquad (2.4)$$

Once again, if there is no welfare loss resulting from nominal interest-rate variation, we may omit the constraint terms corresponding to the IS relation, as these constraints never bind. And in writing the constraint term associated with the period t AS relation, it does not matter that we substitute π_{t+1} for $E_t \pi_{t+1}$; for it is only the conditional expectation of the term at date t_0 that matters in (2.4), and the law of iterated expectations implies that

$$E_{t_0}[\varphi_t E_t \pi_{t+1}] = E_{t_0}[E_t(\varphi_t \pi_{t+1})] = E_{t_0}[\varphi_t \pi_{t+1}]$$

for any $t \geq t_0$.

Differentiating (2.4) with respect to the levels of inflation and output each period, we obtain a pair of first-order conditions of exactly the form (1.7) – (1.8) for each period $t \geq t_0$ (and for each possible state of the world at that date), together with the initial condition

$$\varphi_{t_0 - 1} = 0. \qquad (2.5)$$

Using (1.7) and (1.8) to substitute for π_t and x_t respectively in (2.1), we again obtain a difference equation for the evolution of the multipliers,

$$\beta E_t \varphi_{t+1} - \left(1 + \beta + \frac{\kappa^2}{\lambda}\right) \varphi_t + \varphi_{t-1} = \kappa x^* + u_t, \qquad (2.6)$$

which is a stochastic generalization of (1.10). Once again, the characteristic equation (1.11) has two real roots $0 < \mu_1 < 1 < \mu_2$, as a result of which (2.6) has a unique bounded solution

[18] Early treatments of this problem in the context of the present model with cost-push shocks include Clarida *et al.*, (1999), Woodford (1999xx), and Vestin (2000). Svensson and Woodford (2002xx) discuss a similar problem, but with a one-period delay in the effects of policy on both inflation and output.

for $\{\varphi_t\}$, given by

$$\varphi_t = \mu_1\varphi_{t-1} - (1-\mu_1)\frac{\lambda}{\kappa}x^* - \beta^{-1}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_t u_{t+j}, \tag{2.7}$$

in the case of any bounded disturbance process $\{u_t\}$. Note that this equation can be solved recursively for the evolution of $\{\varphi_t\}$ starting from any initial condition for φ_{-1}. Substituting this solution for the path of the Lagrange multipliers into (1.7) – (1.8), one obtains unique bounded solutions for the paths of inflation and the output gap.

These bounded solutions necessarily satisfy the relevant transversality condition; hence the solution obtained starting from the initial condition (2.5) represents the state-contingent evolution of inflation and output under the t_0-optimal commitment. (Note that the inflation solution thus obtained is a stochastic generalization of (1.13).) However, this solution contains a deterministic component that depends on the time that has elapsed since the date t_0 at which the plan was chosen, and so is not time-consistent. Once again, we prefer a time-invariant policy that is optimal from a timeless perspective, meaning that continuation of the policy from any date t_0 onward leads to an equilibrium from that date onward that minimizes the expected discounted sum of losses, subject to a constraint of the form

$$\pi_{t_0} = \bar{\pi}_{t_0}, \tag{2.8}$$

where the value of $\bar{\pi}_{t_0}$ may depend on the state of the world at date t_0.

Minimization of expected discounted losses subject to the constraint (2.8) leads to a Lagrangian of the same form (2.4), except with the addition of a term representing the additional constraint. This in turn leads to exactly the same system of first-order conditions, except that initial condition (2.8) replaces (2.5). Hence there is a unique bounded solution for the state-contingent evolution of inflation and output from date t_0 onward, in the case of any given specification of the initial condition. This solution will again be of the form (2.7), for some choice of the initial value ϕ_{t_0-1}; the proper choice of this initial multiplier depends on the constraint value $\bar{\pi}_{t_0}$.

In order for such a policy to be time-consistent, we need to select the constraint value $\bar{\pi}_{t_0}$ as a function of exogenous and predetermined variables at date t_0, according to some time-invariant rule that is satisfied by the constrained-optimal state-contingent inflation path (*i.e.*, the solution to the optimization problem with constraint (2.8)) at all dates $t > t_0$. One example of such a specification would be

$$\begin{aligned} \bar{\pi}_{t_0} &= \beta^{-1} \sum_{j=0}^{\infty} \mu_2^{-j-1} E_{t_0} u_{t_0+j} \\ &- \beta^{-1}(1-\mu_1) \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mu_1^{k-1} \mu_2^{-j-1} E_{t_0-k} u_{t_0+j-k}, \end{aligned} \tag{2.9}$$

as a consequence of the following result.

PROPOSITION 7.6. The state-contingent evolution of inflation $\{\pi_t\}$ from some date t_0 onward that minimizes the expected value of (1.2), taking as given the economy's evolution prior to date t_0 and subject to the constraint (2.8), where $\bar{\pi}_{t_0}$ is given by (2.9), is given by

$$\begin{aligned} \pi_t &= \beta^{-1} \sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j} \\ &- \beta^{-1}(1-\mu_1) \sum_{k=1}^{\infty} \sum_{j=0}^{\infty} \mu_1^{k-1} \mu_2^{-j-1} E_{t-k} u_{t+j-k} \end{aligned} \tag{2.10}$$

for all $t \geq t_0$.

The proof is given in the appendix. Note that in the constrained-optimal evolution, inflation in every period after t_0 is chosen to be the same time-invariant function of past disturbances as it has been constrained to be in period t_0. This makes the constraint self-consistent in the desired sense.

One can easily show that (2.9) represents the only specification of $\bar{\pi}_{t_0}$ as a function of the history of exogenous disturbances (only) that is self-consistent in this sense. For regardless of the specification of $\bar{\pi}_{t_0}$, the constrained-optimal evolution from t_0 onward must satisfy (2.7), for some choice of φ_{t_0-1}. Solving for π_t as a function of φ_{t_0-1} and the history of disturbances between dates t_0 and t, and then taking the limit as $t_0 \to -\infty$ for some fixed date t, one obtains (2.10), regardless of the assumed value of ϕ_{t_0-1}.[19] Hence regardless of

[19]The asymptotic independence of the initial condition φ_{t_0-1} results from the fact that $|\mu_1| < 1$.

the specification of $\bar{\pi}_{t_0}$, the dependence of inflation upon the history of disturbances must eventually be of the form (2.10). The only self-consistent specification of $\bar{\pi}_{t_0}$ as a function of the history of disturbances must then be of that same form.

This does not mean, however, that in order for a policy to be optimal from a timeless perspective, it must result in the state-contingent evolution of inflation indicated by (2.10). For one can also find self-consistent specifications of the initial inflation constraint that involve predetermined endogenous variables. As a simple example, the specification

$$\bar{\pi}_{t_0} = (1-\mu_1)\frac{\lambda}{\kappa}x_{t_0-1} + \beta^{-1}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_{t_0}u_{t_0+j} \tag{2.11}$$

also results in a self-consistent constraint.

PROPOSITION 7.7. Consider the same optimization problem as in Proposition 7.6, but with $\bar{\pi}_{t_0}$ given by (2.11). Then in the constrained-optimal state-contingent evolution of inflation and output, the inflation rate satisfies

$$\pi_t = (1-\mu_1)\frac{\lambda}{\kappa}x_{t-1} + \beta^{-1}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_t u_{t+j} \tag{2.12}$$

at each date $t \geq t_0$.

Again, the proof is in the appendix. Hence a time-invariant policy rule that results in a determinate equilibrium in which the inflation rate always satisfies (2.12) will be optimal from a timeless perspective.

Yet another self-consistent constraint would be

$$\bar{\pi}_{t_0} = -(1-\mu_1)(p_{t_0-1} - \bar{p}) + \beta^{-1}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_{t_0}u_{t_0+j}, \tag{2.13}$$

where $p_t \equiv \log P_t$, and $\bar{p}$ is an arbitrary constant, that has the interpretation of a target price level. (It is the long-run average log price level in the equilibrium that is optimal subject to this constraint, as discussed below.) In this case the initial constraint is perhaps most intuitively expressed as a constraint on the initial log price level, $p_{t_0} = \bar{p}_{t_0}$, where

$$\bar{p}_{t_0} = (1-\mu_1)\bar{p} + \mu_1 p_{t_0-1} + \beta^{-1}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_{t_0}u_{t_0+j}. \tag{2.14}$$

For any choice of $\bar{p}$, this is a self-consistent constraint, owing to the following result.

PROPOSITION 7.8. Consider the same optimization problem as in Proposition 7.6, but with $\bar{\pi}_{t_0}$ given by (2.13), or equivalently with a constraint on the initial price level given by (2.14). Then in the constrained-optimal state-contingent evolution of inflation and output, the equilibrium price level each period satisfies

$$\bar{p}_t = (1-\mu_1)\bar{p} + \mu_1 p_{t-1} + \beta^{-1} \sum_{j=0}^{\infty} \mu_2^{-j-1} E_t u_{t+j}. \tag{2.15}$$

Again, the proof is in the appendix. Note that this defines an entire one-parameter family of self-consistent constraints. For one particular choice of $\bar{p}$, namely

$$\bar{p} = p_{t_0-1} + \frac{\lambda}{\kappa} x_{t_0-1},$$

this constraint is identical to (2.11), and the implied state-contingent evolutions are the same. But any other value for $\bar{p}$ would lead to a self-consistent constraint as well.

We see that the state-contingent evolution of inflation under a policy that is optimal from a timeless perspective is not uniquely determined. However, these alternative processes for inflation and output differ only in transitory, deterministic components of the solutions for inflation and the output gap; they agree both as to the long-run average values of both inflation and the output gap, and as to the response of both variables to unexpected shocks in any period from t_0 onward.

PROPOSITION 7.9. Consider again an economy of the kind assumed in Proposition 7.5. Then in the case of any small enough $||\xi||$ and any initial values of predetermined endogenous variables that are close enough to the values associated with the zero-inflation steady state, the long-run average values of inflation and the output gap satisfy

$$\lim_{T\to\infty} E_t \pi_T = 0,$$

$$\lim_{T\to\infty} E_t x_T = 0$$

under the t_0-optimal policy that would be chosen at any date t_0, and the same is true of the equilibrium implemented by any policy that is optimal from a timeless perspective.

Furthermore, let the unexpected change in the forecast of any variable y_{t+m} at date t be denoted

$$I_t[y_{t+m}] \equiv E_t y_{t+m} - E_{t-1} y_{t+m}.$$

Then the effects of unanticipated shocks at any date t on the expected paths of inflation and output are given by

$$\begin{aligned} I_t[\pi_{t+m}] &= \beta^{-1} \sum_{j=0}^{\infty} \mu_2^{-j-1} I_t[u_{t+j}] \\ &- \beta^{-1}(1-\mu_1) \sum_{k=1}^{m} \sum_{j=0}^{\infty} \mu_1^{k-1} \mu_2^{-j-1} I_t[u_{t+m-k+j}], \qquad (2.16) \\ I_t[x_{t+m}] &= -\beta^{-1} \sum_{k=0}^{m} \sum_{j=0}^{\infty} \mu_1^{k} \mu_2^{-j-1} I_t[u_{t+m-k+j}], \qquad (2.17) \end{aligned}$$

for each $m \geq 0$ under a t_0-optimal policy chosen at any date $t_0 \leq t$, and again the same is true of the equilibrium implemented by any policy that is optimal from a timeless perspective. (In each of these characterizations of the paths of inflation and the output gap, the results given are accurate up to an error term of order $\mathcal{O}(||\xi||^2$.)

The proof is in the appendix; essentially, the result follows from the fact that in any of the cases allowed for, the economy's state-contingent evolution must satisfy (2.7).

Comparing Proposition 7.9 with Proposition 7.5, we again find, as in our deterministic analysis in section 1, that there is an inflationary bias to discretionary policy, under the assumption that $x^* > 0$. (The size of the average-inflation bias is exactly the same as determined under the deterministic analysis, as a result of the well-known certainty-equivalence property of both the equilibrium outcome under discretionary optimization and the optimal plan under commitment, in the case of linear-quadratic policy problems of the kind considered here.[20]) However, there is *also* a difference in the responses to shocks under an

[20]For discussions of certainty equivalence in the context of forward-looking models like the ones discussed in this chapter, see, *e.g.*, Backus and Driffill (1986), xxxxxx.

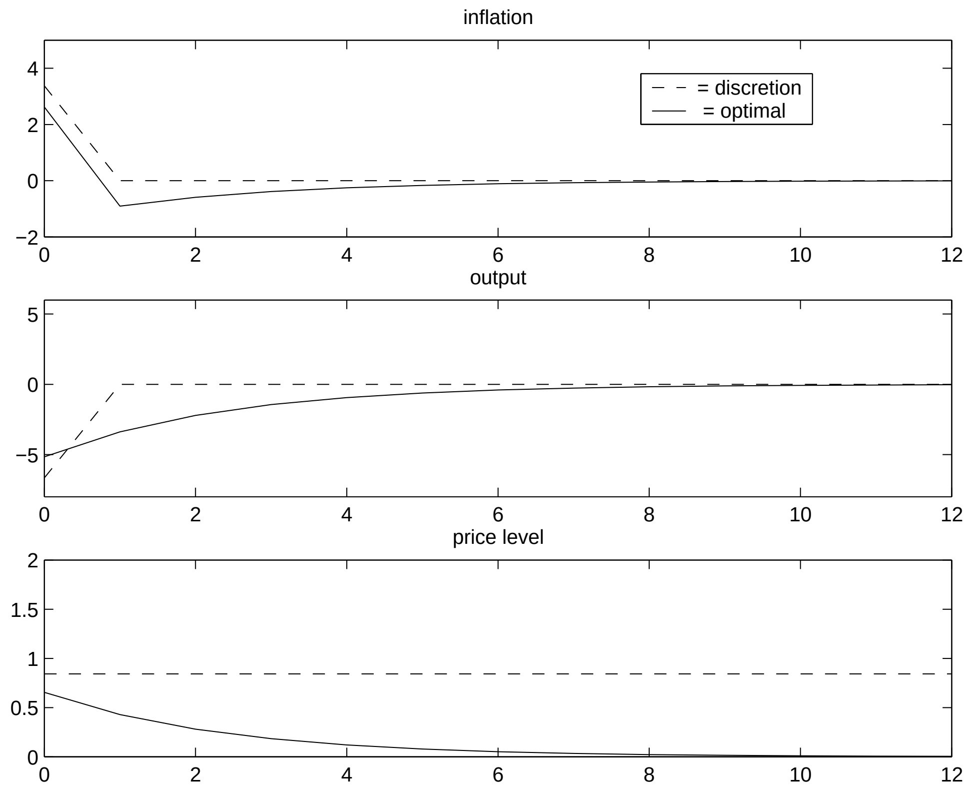

Figure 7.3: Optimal responses to a transitory cost-push shock compared with equilibrium responses under discretionary policy.

optimal policy commitment from those that result from discretionary optimization. And this "stabilization bias" is present even when $x^* = 0$, so that there is no average-inflation bias associated with discretionary policy.[21]

The difference in the equilibrium responses to a "cost-push shock" under discretionary policy and under an optimal commitment are contrasted in Figure 7.3, in the simple case that the cost-push shock is purely transitory, and unforecastable before the period in which it occurs (so that $E_t u_{t+j} = 0$ for all $j \geq 1$). Here the assumed values of β, κ, and λ are again as in Figure 7.1, and the shock in period zero is of size $u_0 = 1$, meaning that a one

[21]Thus the result of King (1997), according to which discretionary policy achieves an optimal state-contingent evolution for the economy as long as the output-gap target x^* in the central bank's loss function is modified so as to be consistent with its inflation target, is special to the specific simple model of aggregate supply that he assumes. Early discussions of "stabilization bias" in the context of other forward-looking models include xxxxx.

percent increase in the general level of prices would be required to prevent any decline in output relative to the natural rate (or more generally, its target level) from occurring, on the assumption that prices will then be stabilized at the new higher level. Once again, the periods represent quarters, and the inflation rate is plotted as an annualized rate, meaning that what is plotted is actually $4\pi_t$.

Under the discretionary policy characterized in Proposition 7.5, the effect of the disturbance on equilibrium inflation and the equilibrium output gap last only as long as the disturbance itself, so that both variables are expected to return to their normal levels by the following quarter. Under an optimal commitment, instead, monetary policy remains tight even after the disturbance has dissipated, so that the output gap returns to zero only much more gradually. As a result of this, while inflation overshoots its long-run target value at the time of the shock, it is held *below* its long-run target value for a time following the shock, so that the unexpected increase in prices is subsequently undone. In fact, as the bottom panel shows, under an optimal commitment, the price level eventually returns to exactly the same path that it would have been expected to follow if the shock had not occurred.

This simple example illustrates a very general feature of optimal policy once one takes account of forward-looking private-sector behavior: optimal policy is almost always *history-dependent* in a way that discretionary policy would not be. Discretionary policy (in a Markov equilibrium) is *purely forward-looking,* in the sense that the action chosen at any date t depends solely upon the set of possible state-contingent paths for the target variables (here, inflation and the output gap) from period t onward, given the economy's state at that time (the values of predetermined or exogenous variables at date t). In the present example, the forward-looking aggregate-supply relation (2.1), which implies that the set of possible equilibrium paths for inflation and output from period t onward are independent of all lagged endogenous variables, together with the assumption that the disturbance u_t is completely unforecastable, imply that the only aspect of the economy's state in period t that affect the set of possible paths for inflation and output from that period onward is the current disturbance u_t. Hence under any purely forward-looking decision procedure for monetary policy, π_t and x_t will depend only on the current disturbance u_t, and the effects of a shock on the paths of these variables will be as transitory as the effect on u_t itself. (The same is true of the associated path of nominal interest rates, if the intertemporal IS equation is of the form (2.23).) Instead, under an optimal policy, both π_t and x_t will depend on disturbances in periods prior to t, in the case of any period $t > t_0$.

Optimal policy is history-dependent (and conventional optimal-control or dynamic-programming methods lead to suboptimal policy) because the *anticipation* by the private sector that future policy will different as a result of conditions at date t — even if those conditions no longer matter for the set of possible paths for the target variables at the later date — can improve stabilization outcomes at date t. Suppose that there is a positive cost-push shock at date t, as illustrated in Figure 7.3. If the transitory disturbance is expected to have no effect on the conduct of policy in later periods (as under any purely forward-looking policy), then the short-run tradeoff between inflation and the output gap in period t is shifted vertically by the amount of the disturbance, u_t, requiring the central bank to choose between an increase in inflation, a negative output gap, or some of each. If instead the central bank is expected to pursue a tighter policy in period $t+1$ and later as a result of the shock in period t, as occurs under the optimal policy depicted in the figure, then the short-run tradeoff between inflation and the output gap is shifted only by the amount of the total change in $u_t + \beta E_t \pi_{t+1}$, which is smaller than the increase in u_t. Hence greater stabilization (less of an increase in inflation, less of a reduction in output, or both) is possible in period t. (The anticipation of tighter policy later restrains price increases in period t, so that less contraction of output is needed in period t to achieve a given degree of moderation of inflationary pressure.) Of course, to achieve this beneficial shift in expectations in period t, it is necessary that the central bank be committed to actually tightening policy later. This will mean less successful stabilization in later periods than would otherwise have been possible; but nonetheless, the discounted sum of expected stabilization losses can be reduced through some use of this tool.

It may not be obvious from Figure 7.3 that commitment to a history-dependent policy can simultaneously improve the stabilization of inflation and of the output gap, since in this numerical example, the overall variability of the output gap is higher under the optimal policy than under discretionary policy. (This need not have been true, but happens to be true for the calibrated parameter values used here, which involve quite a low relative weight λ on output-gap stabilization, for reasons discussed in chapter 6.) The point can be illustrated numerically by computing the inflation/output variance frontier associated with history-dependent as opposed to purely forward-looking policies. These two variance frontiers are shown in Figure 6.6 of the previous chapter, in the case that β and κ are calibrated as in Figure 7.3, and u_t is assumed to be an AR(1) process,

$$u_t = \rho_u u_{t-1} + \epsilon_t^u, \tag{2.18}$$

with $\rho_u = 0.8$. (Here ϵ_t^u is an i.i.d. mean-zero random variable.)

The variance frontier in the case of general linear policies (allowing arbitrary history-dependence) is obtained by computing the optimal responses to disturbances, characterized in Proposition 7.9, for values of $\lambda > 0$ that are allowed to vary over the entire positive real half-line. For each equilibrium in this one-parameter family, we compute the statistics $V[\pi]$ and $V[x]$, where we again use the discounted measure of variability

$$V[y] \equiv (1-\beta)\sum_{t=0}^{\infty}\beta^t \mathrm{var}(y_t)$$

for any random variable $\{y_t\}$. Note that for each value of λ, the date zero-optimal policy minimizes

$$\mathrm{E}\left\{\sum_{t=0}^{\infty}\beta^t[\pi_t^2 + \lambda(x_t - x^*)^2]\right\} \qquad =$$

$$\sum_{t=0}^{\infty}\beta^t\left\{(\mathrm{E}[\pi_t]^2 + \lambda(\mathrm{E}[x_t] - x^*)^2\right\} \quad + \quad (1-\beta)V[\pi] \; + \; (1-\beta)\lambda V[x].$$

Furthermore, the first term on the right-hand side depends only on the deterministic component of policy, while the remaining two terms depend only on the prescribed responses to shocks. As these two aspects of policy can be independently specified, the optimal equilibrium responses to shocks minimize $V[\pi] + \lambda V[x]$. It follows that the responses to shocks characterized in Proposition 7.9 for alternative values of λ describe the policies on the $V[\pi]$-$V[x]$ efficient frontier.

The variance frontier for the more restricted class of purely forward-looking (linear) policies can similarly be obtained by computing the policy from within this class that minimizes $V[\pi] + \lambda V[x]$, for each possible value of $\lambda > 0$. In the present context, this means that we restrict consideration to state-contingent evolutions in which π_t and x_t are linear functions of the current value of u_t,

$$\pi_t = \bar{\pi} + f_\pi u_t, \qquad x_t = \bar{x} + f_x u_t. \tag{2.19}$$

State-contingent evolutions within this family represent possible rational-expectations equilibria if and only if the coefficients satisfy

$$(1-\beta)\bar{\pi} = \kappa\bar{x}, \tag{2.20}$$

$$(1-\beta\rho_u)f_\pi = \kappa f_x + 1, \tag{2.21}$$

as a consequence of (2.1). Finally, in the case of processes of the form (2.19),

$$V[\pi] + \lambda V[x] = [f_\pi^2 + \lambda f_x^2]\mathrm{var}(u).$$

Thus we choose the response coefficients f_π, f_x so as to minimize $f_\pi^2 + \lambda f_x^2$ subject to constraint (2.21). The optimal response coefficients are easily seen to be

$$f_\pi = \frac{1-\beta\rho_u}{\kappa^2\lambda^{-1} + (1-\beta\rho_u)^2} > 0, \qquad f_x = -\frac{\kappa\lambda^{-1}}{\kappa^2\lambda^{-1} + (1-\beta\rho_u)^2} < 0. \tag{2.22}$$

For each member of this one-parameter family of equilibria, the implied values of $V[\pi]$ and $V[x]$ yield a point on the dotted frontier shown in Figure 6.6. (Point B' on this frontier indicates the efficient plan in the case of the value $\lambda = 0.05$ corresponding to the welfare-theoretic loss function.) We observe that this frontier is entirely inside the one computed when policy is allowed to be history-dependent; thus commitment to a history-dependent policy can simultaneously improve the stabilization of both inflation and the output gap.

A striking feature of the impulse responses under an optimal policy displayed in Figure 7.3 is the fact that following a cost-push shock, the unexpected inflation caused by the shock is entirely "undone", so that the price level returns completely to its previously anticipated path. This result is not special to the case of serially uncorrelated, completely unanticipated disturbances assumed in the figure.

PROPOSITION 7.10. Under the same assumptions as in Proposition 7.9, under the t_0−optimal plan, there exists a well-defined long-run expected price level,

$$\lim_{T\to\infty} E_t p_T = p_\infty,$$

that is the same in every period $t \geq t_0$, regardless of the history of disturbances between periods t_0 and t. And the same is true under any policy followed from period t_0 onward that is optimal from a timeless perspective. In the particular case of the latter sort that is optimal subject to the initial constraint (2.13) or (2.14), the long-run expected price level corresponds to the constant $\bar{p}$ in the constraint.

This is also shown in the appendix. Note that the long-run price level p_∞ generally depends on initial conditions (including the pre-existing price level p_{t_0-1}) at the time that the optimal policy is adopted; it is furthermore different (for given initial conditions) under different policies within the set to which the proposition applies. (In the case of initial constraints of the form (2.13) or (2.14), p_∞ is independent of initial conditions at the time that the policy is adopted, as it is in fact determined solely by the constant $\bar{p}$ in the initial constraint; but any long-run price level is possible, under a suitable choice of $\bar{p}$.) Nonetheless, under any policy of the kind specified in the proposition, the intitial conditions plus the form of optimal policy adopted determine a long-run price level, which is not subsequently affected by the realization of any random disturbances. Hence equilibrium fluctuations in the price level are stationary.

This feature of optimal policy may seem counter-intuitive. Indeed, it is often argued that if one wishes to stabilize inflation, and does not care about the absolute level of prices, then surprise deviations from the long-run average inflation rate should *not* have any effect on the inflation rate that policy aims for subsequently: one should "let bygones be bygones,"

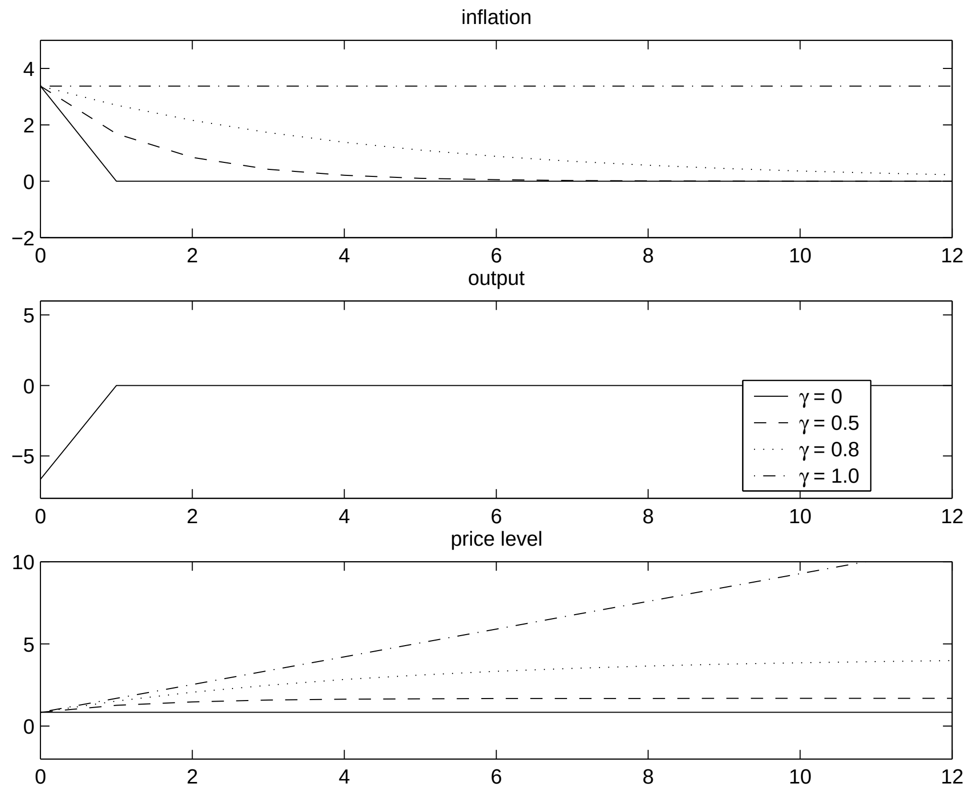

Figure 7.4: Equilibrium responses to a transitory cost-push shock under discretionary policy, for varying degrees of inflation inertia.

even though this means allowing the price level to drift to a permanently different level. The claim is that "undoing" past deviations from the target rate of inflation simply creates additional, unnecessary variability in inflation. And this would be correct if the commitment to subsequently "undo" target misses had no effect on the probability distribution of the unexpected deviations from the inflation target. However, if price-setters are forward-looking, the anticipation that a current increase in the general price level will predictably be "undone" soon gives suppliers a reason not to increase their own prices currently as much as they otherwise would, and so leads to smaller equilibrium deviations from the inflation target in the first place. Hence such a policy can reduce equilibrium inflation variability, and not simply the range over which the absolute price level varies.

Nonetheless, the result that optimal policy involves a stationary (or even trend-stationary) price level depends on fairly special assumptions. For example, in the event of partial index-

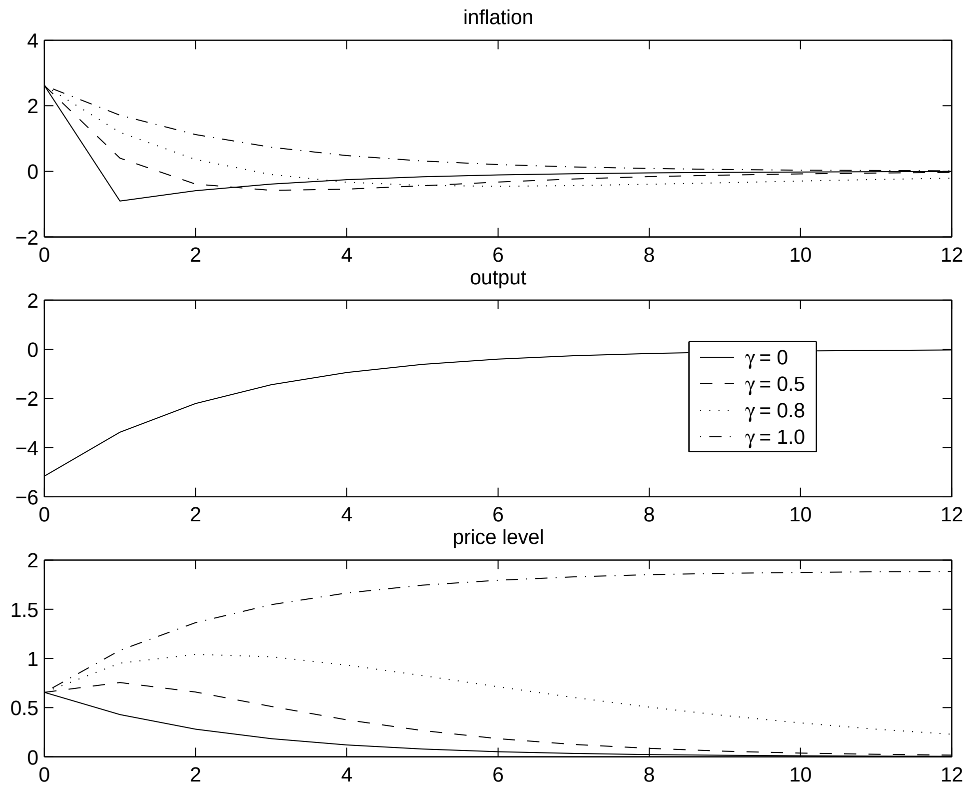

Figure 7.5: Responses to a transitory cost-push shock under optimal policy, again for varying degrees of inflation inertia.

ation of prices to a lagged price index, it ceases to be optimal to completely restore the price level to the path that it would have been expected to follow in the absence of a shock, even though it continues to be optimal for positive inflation surprises to be predictably followed by periods in which inflation is temporarily below its long-run average value. This is illustrated in Figures 7.4 and 7.5, which display the impulse responses to the same kind of shock as in Figure 7.3, but for economies with a range of values for the indexation parameter γ. (The case $\gamma = 0$ corresponds to the basic Calvo pricing model, the case already shown in Figure 7.3.) In Figure 7.4, the responses are shown in the case that the central bank optimizes under discretion, seeking to minimize the expected value of the loss function (1.22). In Figure 7.5, the corresponding responses are shown in the case of an optimal policy commitment.

In the case of discretionary policy, we see that a positive cost-push shock results not only in a burst of inflation during the period of the disturbance, but additional above-normal price increases in subsequent periods, as the rate of inflation only gradually returns to its normal level. As before, the perturbation of the target variable $\pi_t - \gamma\pi_{t-1}$ is as transitory as the disturbance u_t; but the inflation inertia resulting from the automatic indexation creates additional inflation for several more quarters. In the limiting case of full indexation ($\gamma = 1$), inflation remains permanently higher as a result of the transitory cost-push shock.

Instead, under optimal policy, the initial unexpected increase in prices is eventually undone, as long as $\gamma < 1$; and this once again means that inflation eventually undershoots its long-run level for a time. However, for any large enough value of γ, inflation remains greater than its long-run level for a time even after the disturbance has ceased, and only later undershoots its long-run level; the larger is γ, the longer this period of above-average inflation persists. In the limiting case that $\gamma = 1$, the undershooting never occurs; inflation is simply gradually brought back to the long-run target level.[22] In this last case, a temporary disturbance causes a permanent change in the price level, even under optimal policy. (Under optimal policy, the price is an integrated process of order 1, while under discretion it is an integrated process of order 2, since there is a unit root in the *inflation rate*.)

Even if there is not full indexation to a lagged price index, the result that the price level is stationary under optimal policy is relatively fragile, given that welfare does not depend at all on the range of variation in the absolute level of prices. Under many small perturbations of the precise model considered here, optimal policy will involve a price-level process with a unit root; this is true, for example, if there is even a small weight on interest-rate stabilization in the loss function, as discussed in the next section, or if the zero lower bound on nominal interest rates ever binds, as discussed in section xx. The more robust conclusion about optimal policy is not that it is important for the price level to actually be stationary (or trend-stationary); it is rather that it is desirable for an inflationary disturbance to be followed

[22] The reason for this is easily seen. Optimal policy in the case that $\gamma > 0$ is the same as under the characterization in Proposition 7.9, except that π_t must be replaced by the quasi-differenced inflation rate $\pi_t - \gamma\pi_{t-1}$ in each expression. In the case that $\gamma = 1$, the optimal evolution of the inflation rate π_t is the same as the optimal evolution of p_t when $\gamma = 0$. Thus the impulse response of inflation (for $\gamma = 1$) in panel 1 of Figure 7.5 is the same as the impulse response of the price level (under optimal policy) in panel 3 of Figure 7.3. The scales are different because the inflation rate plotted is an annualized rate, $4\pi_t$ rather than π_t.

by a period of tight monetary policy which keeps output below the natural rate for a time, the anticipation of which helps to restrain price increases as a result of the disturbance.

2.2 Fluctuations in the Natural Rate of Interest

We turn now to the optimal response to real disturbances that cause temporary fluctuations in the natural rate of interest r_t^n. In the case that inflation stabilization and output-gap stabilization are the only objectives of monetary policy (as in the case of loss function (1.2)), and the zero interest-rate bound never binds (as assumed thus far), fluctuations in the natural rate of interest do not prevent the central bank from completely stabilizing both inflation and the output gap; it only effects the kind of nominal interest-rate variations that are required in order to achieve this. In such a case, optimal policy continues to involve zero inflation and a zero output gap at all times, just as in the deterministic analysis in section 1.1. In this special case, there is no difference between discretionary policy and an optimal policy commitment, as regards the equilibrium responses to disturbances of this kind: in either case, inflation and the output gap will not respond at all to the disturbance, while the nominal interest-rate operating target will perfectly track the current value of the natural rate of interest.[23]

However, our conclusion is different if the central bank is also concerned to minimize the degree of variability of nominal interest rates, for either of the reasons discussed in section xx of chapter 6, or perhaps for other reasons as well. In this case, as discussed in the previous chapter, it is possible to reduce interest-rate variation at the price of increased variability of inflation and the output gap, and optimal policy will do this to some extent. And once simultaneous satisfaction of all of the stabilization objectives ceases to be possible, it is almost inevitably the case that optimal policy will no longer coincide with discretionary policy, and indeed that optimal policy will no longer be purely forward-looking.

In particular, there are important advantages to a more *inertial* adjustment of interest

[23]These conclusions are easily obtained from the analysis of discretionary policy and optimal commitment in the previous section, since it did not matter in that analysis what we assumed about fluctuations in the natural rate of interest.

rates than would occur under a purely forward-looking policy. As discussed in chapter 4, in an optimizing model, aggregate demand should depend on the expected path of (short-term) real interest rates far into the future, and not simply on their current level. Hence a commit to maintain real rates at a moderately high level for a longer period of time can be as effective a way of preventing a surge in aggregate demand due to real disturbances as a sharper increase in real rates that is expected to be only temporary; but the former policy has the advantage of requiring less variable interest rates. Of course, as in the previous section, conditioning policy on past disturbances rather than current conditions causes distortions; but the gains from anticipation of such behavior can make it nonetheless worthwhile to engage in such behavior to an extent. It can therefore be desirable for the central bank to raise interest rates only gradually in response to an increase in the natural rate of interest, and similarly to lower them again only gradually once the disturbance has passed — even though the type of loss function proposed in chapter 6, section xx, penalizes deviations of the *level* of nominal interest rates from the optimal level, and not large *rates of change* in the interest rate. This is another example of the history-dependence of optimal policy.

Woodford (1999xx) shows that inertial interest-rate adjustment is optimal in the case of the basic neo-Wicksellian model presented in chapter 4. If interest-rate variations matter for welfare, the aggregate-supply relation (2.1) must be augmented by an intertemporal IS relation

$$x_t = E_t x_{t+1} - \sigma[i_t - E_t \pi_{t+1} - r_t^n], \tag{2.23}$$

generalizing (1.15), where now $\{r_t^n\}$ is an exogenous disturbance process. We wish to choose a policy to minimize a social loss function of the form

$$E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [\pi_t^2 + \lambda_x (x_t - x^*)^2 + \lambda_i (i_t - i^*)^2]. \tag{2.24}$$

Here $i^* = i^m$, as in (1.14), if the interest-rate stabilization objective appears solely as a result of transactions frictions; but we may wish to assign a higher value to i^* (and to λ_i) in order to reflect the need to avoid negative nominal interest rates.

Once again, the model is one in which the possible rational-expectations equilibrium paths of inflation, the output gap, and the nominal interest from any date t onward depend only on the real disturbances (r_t^n, u_t) at date t, and upon information at date t about the subsequent evolution of those disturbances. Hence if these disturbances are Markovian, so that their current values contain all available information about their likely future evolution, then any purely forward-looking policy will make all three of the target variables vary solely in response to the current disturbances, and the effects of any disturbance on any of these variables will be only as persistent as the disturbance itself.

The optimal responses to such disturbances can instead be characterized using the same Lagrangian method as in the previous section. We again obtain the first-order conditions (1.16) – (1.18) for each $t \geq t_0$.[24] In order to compute the t_0-optimal commitment, these conditions are solved under the initial conditions

$$\varphi_{1,t_0-1} = \varphi_{2,t_0-1} = 0. \tag{2.25}$$

A policy that is optimal from a timeless perspective is instead only required to minimize (2.24) subject to constraints of the form

$$\pi_{t_0} = \bar{\pi}_{t_0}, \qquad x_{t_0} = \bar{x}_{t_0}. \tag{2.26}$$

The equilibrium associated with such a policy will solve the same system of first-order conditions, but with different initial values for the Lagrange multipliers φ_{1,t_0-1}, φ_{2,t_0-1} than those given in (2.25). The following result, due to Giannoni and Woodford (2002b), guarantees that these conditions determine a unique bounded solution.

Proposition 7.11. For any parameter values $0 < \beta < 1$, $\kappa, \sigma, \lambda_x, \lambda_i > 0$, any bounded processes for the exogenous disturbances $\{r_t^n, u_t\}$, and any specification of the initial lagged Lagrange multipliers $\varphi_{1,t_0}, \varphi_{2,t_0}$, the system of equations consisting of (1.16) – (1.18), (2.1) and (2.23) has a unique bounded solution for the paths of the variables $\{\pi_t, x_t, i_t, \varphi_{1t}, \varphi_{2t}\}$ for periods $t \geq t_0$.

[24] In the general case, the coefficient i^m in (1.18) must be replaced by i^*.

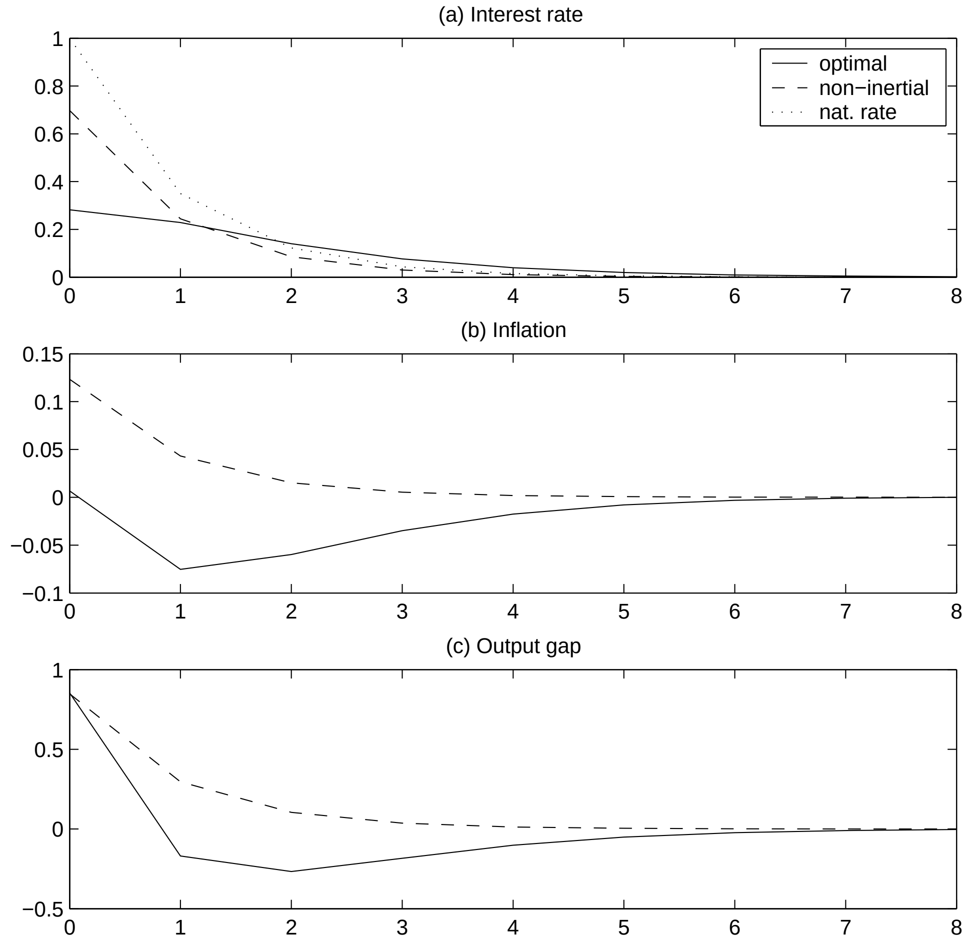

Figure 7.6: Optimal responses to a disturbance to the natural rate of interest.

This is proved in the appendix. It follows that the t_0-optimal commitment involves bounded fluctuations in response to the disturbances, characterized by the unique bounded solution to these equations. And once again, the responses under a timelessly optimal policy are of the same kind.

The optimal responses to a disturbance to the natural rate of interest characterized by these equations are history-dependent, as a result of the presence of the lagged Lagrange multipliers in the first-order conditions. (For example, even in the case that $\{r_t^n\}$ is Markovian, the period t endogenous variables will not be independent of the history of natural-rate disturbances prior to period t.) These optimal responses are illustrated by the solid lines in

Figure 7.6 for the case of an AR(1) process for the natural rate of interest,

$$r_t^n = (1 - \rho_r)\bar{r} + \rho_r r_{t-1}^n + \epsilon_t^r, \tag{2.27}$$

with an autoregressive coefficient $\rho_r = 0.35$. (The numerical values for $\beta, \sigma, \kappa, \lambda_x$, and λ_i used in this example are also again taken from Table 6.1.) The impulse response of the natural rate of interest itself is shown by the dotted line in the first panel of the figure. The equilibrium responses of inflation, output, and the nominal interest rate under the best possible purely forward-looking policy (the "optimal non-inertial plan," characterized in the next section) are also shown for purposes of comparison.

Under an optimal policy, the responses of these variables are not simple multiples of the current deviation of the natural rate of interest from its long-run average value. The nominal interest rate is raised much more gradually in response to an increase in the natural rate than under the optimal purely forward-looking policy, but the higher interest rates are maintained longer than the time for which the disturbance itself persists. This more persistent change in the level of interest rates restrains the initial increase in the output gap to the same extent as under the forward-looking policy, despite the more modest increase in interest rates; and it returns the gap to zero (and even undershoots) quickly, as people come to foresee real rates in the near future that will be even higher than the natural rate. Because the increase in the output gap is much more transitory (and is even soon reversed), the immediate increase in inflation is much smaller under the optimal policy. Yet this improved stabilization of inflation and output requires less volatility of nominal interest rates as well.

The way in which nominal interest rates evolve in an optimal equilibrium can be roughly described as partial adjustment toward a "desired" level that is a function of current and expected future levels of the natural rate; hence interest-rate inertia of the kind suggested by the estimated Fed reaction functions discussed in chapter 1 is actually a feature of optimal policy, rather than an indication of failure to react quickly enough to changing conditions. The nature of the partial-adjustment dynamics can be seen especially clearly in a limiting case of this model, in which $\kappa = 0$, so that the inflation rate never varies (or at any rate evolves exogenously), regardless of the evolution of the output gap.

In this case, our problem reduces to one of choosing state-contingent paths for $\{x_t, i_t\}$ to minimize the discounted sum of losses resulting from output-gap and interest-rate variation, subject to the constraint that (2.23) be satisfied each period (given the exogenous path of inflation). The first-order conditions characterizing optimal policy are then simply (1.17) – (1.18), with the terms involving φ_{2t} omitted. Using the latter of these equations to eliminate the Lagrange multiplier φ_{1t} from the former, we obtain[25]

$$\sigma\lambda_x(x_t - x^*) - \lambda_i[(i_t - i^*) - \beta^{-1}(i_{t-1} - i^*)] = 0.$$

Using this condition, in turn, to eliminate x_t from (2.23), the latter relation becomes[26]

$$i_t - \beta^{-1}i_{t-1} = E_t[i_{t+1} - \beta^{-1}i_t] - \lambda_i^{-1}\lambda_x\sigma^2(i_t - r_t^n). \tag{2.28}$$

This equation in turn has a unique bounded solution of the form

$$i_t = \mu_1 i_{t-1} + (1 - \mu_1)\bar{\imath}_t, \tag{2.29}$$

in the case of any bounded process for the disturbance $\{r_t^n\}$, where

$$\bar{\imath}_t = (1 - \mu_2)\sum_{j=0}^{\infty} \mu_2^{-j} E_t r_{t+j}^n,) \tag{2.30}$$

and $0 < \mu_1 < 1 < \mu_2$ are the two roots of the characteristic equation

$$\mu^2 - \left(1 + \beta^{-1} + \frac{\lambda_x\sigma^2}{\lambda_i}\right)\mu + \beta^{-1} = 0.$$

Equation (2.29) takes the form of partial-adjustment dynamics toward the time-varying desired level of short-term interest rates $\bar{\imath}_t$, with the root μ_1 determining the speed of adjustment. The desired level $\bar{\imath}_t$ is a weighted average of current and expected future natural

[25] In eliminating the lagged multiplier using the same relation, we are in effect choosing a particular time-invariant way of selecting the initial lagged multiplier φ_{1,t_0-1}, using a relation between this multiplier and i_{t_0-1} that *would* have been true in the case of any optimal commitment chosen at a date *earlier* than t_0. Hence the solution obtained here represents at least one of the possible equilibria from date t_0 onward that are optimal from a timeless perspective. Since the equilibrium responses to natural-rate disturbances from date t_0 onward are the same in all such equilibria, the method used here gives the uniquely appropriate characterization of the optimal dynamic response to such a disturbance.

[26] Here we assume that the limiting case with $\kappa = 0$ is one in which prices are so sticky that inflation is zero at all times. If instead inflation varies in response to exogenous "cost-push shocks", then (2.28) again holds, but with the exogenous forcing process r_t^n replaced by the exogenous variation in $r_t^n + E_t\pi_{t+1}$.

rates of interest, with the root μ_2 determining how far in the future the expected values of the natural rate are averaged. In the case that the natural rate process is of the Markovian form (2.27), $\bar{\imath}_t$ is an increasing linear function of r_t^n (an average of r_t^n and the long-run average natural rate $\bar{r}$), with a slope less than one (though closer to one the more persistent are the fluctuations in the natural rate). The optimal inertia coefficient μ_1 in these partial-adjustment dynamics is an increasing function of $\lambda_i/(\lambda_x\sigma^2)$; if interest-rate stabilization is sufficiently important relative to output-gap stabilization, the optimal rate of adjustment of the central bank's operating target in response to a shift in the current natural rate of interest may be quite slow.

While this simple characterization of optimal interest-rate dynamics is exactly true only in the limiting case in which $\kappa = 0$, the optimal interest-rate dynamics shown in Figure 7.6 are fairly similar to those described by (2.29), as is discussed further in Woodford (1999xx). For while $\kappa > 0$ in a realistic model, the slope of the aggregate-supply relation that is estimated econometrically is often fairly small; thus the limiting case just discussed provides considerable insight into the character of optimal interest-rate dynamics.

3 Optimal Simple Policy Rules

We have thus far discussed the character of the state-contingent evolution that one would ideally wish to arrange, and seen that this requires commitment to systematic behavior of a kind that would not be chosen by a discretionary optimizer. We turn now to the question of the type of *policy rule* to which a central bank should commit itself, in order to reap the benefits of policy commitment that we have discussed.

Much of the recent literature has addressed this question by asking which rule would be best within some parametric family of relatively simple rules — for example, the family of contemporaneous Taylor rules

$$i_t = \bar{\imath} + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x(x_t - \bar{x})/4. \tag{3.1}$$

Often it is supposed that commitment to a relatively simple rule of this sort represents the only feasible form of commitment, perhaps because of difficulties of explaining to the public the nature of the central bank's commitment in some more complex case. Here we consider the optimal choice of a policy rule from within such a restricted class, from the point of view of the sort of forward-looking models of the effects of policy commitments presented above. We begin by discussing the optimality criterion that is appropriate in such an exercise.

It might seem that the obvious way in which to choose an optimal rule from within a family such as (3.1) is to compute the rational-expectations equilibrium associated with any given rule in the family, and evaluate a loss function such as (1.14) given the state-contingent evolution of the target variables in that equilibrium. The optimal rule within the family would then be the rule leading to the lowest expected loss. This is the definition of an "optimal simple rule" given, for example, in Currie and Levine (1991).

However, this criterion has undesirable features in the context of a forward-looking model of the effects of policy. As in our discussion of the t_0-optimal commitment in the previous two sections, one is evaluating alternative policies under a criterion which will favor policies that exploit the fact that initial expectations are already given at the time that policy is chosen; and in general, this will lead to a time-inconsistent policy choice, even when policies are restricted to a simple family. The criterion favors policies that create inflation in the initial periods following the policy choice, while committing to negligible average inflation farther in the future. If a rule from within family (3.1) is chosen at a time when negative output gaps are anticipated in the near term (while the long-run average output gap is expected to be zero), a rule will be preferred under which a negative output gap justifies a loosening of policy, exactly because this is expected to create the desired initial inflation without implying long-run inflation. But if the question of the optimal simple rule is reconsidered at a later date, at which a positive output gap is anticipated in the near term, a rule will be preferred under which it is a *positive* output gap that justifies a loosening of policy. The choice of an optimal simple rule on these grounds is time-inconsistent (as Currie and Levine note), for essentially the same reason as in our earlier discussion of the unconstrained policy problem. We wish to propose instead a criterion which will result in a time-consistent selection.

Another disadvantage of this criterion is that even when applied to a class of policies flexible enough to include a policy that is optimal from a timeless perspective, that policy may well *not* be judged optimal within the restricted class of policies.[27] This can be seen from a reconsideration of the deterministic problem treated in section 1.1 above. There the optimal policy from a timeless perspective was found to be one that resulted in zero inflation each period. One might also consider the optimal policy within the restricted class of policies that keep inflation constant at some rate $\bar{\pi}$ for all time. If one simply evaluates (1.2) for the paths of inflation and output associated with any such policy, one obtains a value

$$(1-\beta)^{-1}\left[\bar{\pi}^2 + \lambda\left(\frac{1-\beta}{\kappa}\bar{\pi} - x^*\right)^2\right],$$

which is not minimized at $\bar{\pi} = 0$, in the case that $x^* > 0$. One instead would prefer a somewhat positive inflation rate under this criterion, in order to take advantage of the gains from unanticipated inflation in period t_0, even though the extent to which this is chosen is limited by the restriction that one must choose the same inflation rate for all later periods as well. In order to obtain a criterion that will result in choice of the policy that is optimal from a timeless perspective when the class of simple policies is flexible enough to contain it, it is necessary to evaluate the outcomes associated with alternative rules from a perspective that penalizes a rule for taking advantage of pre-existing expectations at the time of policy choice.

We accordingly propose to evaluate policy rules according to a criterion of the following sort. A quadratic loss criterion such as (1.2), evaluated conditional on the economy's state at date t_0, can be expressed as the sum of two components, $L^{det} + L^{stab}$, where L^{det} depends only on the deterministic component of the equilibrium paths of the target variables, and L^{stab} depends only on the equilibrium responses to unexpected shocks in periods after t_0.

[27] For example, Jensen and McCallum (2002) find that the targeting rule (5.1) below, that brings about the equilibrium characterized by Proposition 7.7 and hence is optimal from a timeless perspective, is not optimal (in the sense that they consider) within a family of simple linear rules that includes rule (5.1). This is because they do not rank alternative policies according to the stabilization loss measure L^{stab} proposed below. For further discussion of the Jensen-McCallum calculations, see xxxxxx.

For example, in the case of (1.2), the deterministic component is given by

$$L^{det} = \sum_{t=t_0}^{\infty} \beta^{t-t_0}[(E_{t_0}\pi_t)^2 + \lambda(E_{t_0}x_t - x^*)^2],$$

while the stabilization component is given by

$$L^{stab} = \sum_{t=t_0}^{\infty} \beta^{t-t_0}[\mathrm{var}_{t_0}(\pi_t) + \lambda \mathrm{var}_{t_0}(x_t)].$$

Under any policy rule that is optimal from a timeless perspective (or, for that matter, under a $t-$optimal commitment chosen at any date $t \leq t_0$), the equilibrium responses to shocks in periods after t_0 will be exactly those that minimize L^{stab}, among the set of possible dynamic responses to shocks that are consistent with the (linear) structural equations. Hence we wish to evaluate rules within more restricted families of linear rules according to the same criterion: the coefficients that determine the equilibrium responses to shocks — *e.g.*, the coefficients ϕ_π, ϕ_x in the case of the family (3.1) — should be chosen so as to minimize L^{stab}, to the extent that this is possible within the class of simple rules considered. This is a criterion that will allow a choice of these coefficients that is independent of the economy's state at date t_0; for that state may affect the predicted deterministic component of the paths of the target variables implied by given response coefficients, but it has no effect on the predicted variances that enter L^{stab}, given a linear model for the evolution of the disturbances. Thus the proposed criterion leads to a time-consistent choice. And it is a criterion that implies that if the family of rules considered is flexible enough to include one that is optimal from a timeless perspective, the response coefficients in that rule will be judged optimal.

Given this choice of the response coefficients, the coefficients that determine the long-run average values of the target variables with which the rule is consistent (the values $\bar{\imath}, \bar{\pi}, \bar{x}$ in (3.1)) are not then chosen to minimize L^{det}. Instead, in accordance with our discussion of deterministic policy problems in section 1, a rule is chosen that is consistent with the long-run average values that occur under the t_0- optimal commitment. (These are the same values as would occur under an optimal commitment chosen at any other date, or under a policy that is optimal from a timeless perspective.) Unlike the criterion of minimizing

L^{det}, this criterion leads to a choice that is independent of the state at t_0, and hence that is time-consistent. And it leads to a rule that is consistent with the same long-run average values of the target variables as any rule that is optimal from a timeless perspective; so if the family of rules under consideration includes such a rule, it will be found optimal within the restricted family. We turn now to some simple examples of policies that could be judged optimal within a restricted family of simple alternatives, though they are not fully optimal.

3.1 The Optimal Non-Inertial Plan

A restricted class of policies of particular interest is the class of *purely forward-looking* policies, under which policy (and hence equilibrium outcomes) at each date depend only on the set of possible evolutions for the target variables that are possible from that date onward. Basing policy solely on projections of the economy's current and possible future states has a certain intuitive appeal, and the forecast-targeting procedures of central banks often seem to have this character; hence it may be of interest to know how different from fully optimal policy the best possible rules of this kind are.

In order to consider this question, we may begin by considering which state-contingent evolution we should wish to bring about, among all those consistent with *any* purely forward-looking policy, and then subsequently ask which policy (or policies) can be used to implement the desired equilibrium. The set of possible state-contingent evolutions to which we may restrict attention consists of those under which the current endogenous non-predetermined state variables z_t depend only on the vector of exogenous states s_t that contains all information available in period t about the disturbances to the structural equations in period t or later, and on the vector Z_t of exogenous disturbances that matter for determination of the variables z_t. For the set of possible evolutions of the economy from date t onward depends only on the values of s_t and Z_t. It follows that if policy depends only on this set, it will also depend only on those variables, and if the policy rule results in a determinate equilibrium, it must be one in which the equilibrium values z_t also depend only on (s_t, Z_t).

We shall call the optimal state-contingent evolution from within this restricted class the *optimal non-inertial plan,* following Woodford (1999xx). To be precise, this is the plan under which (i) the long-run average values of the variables z_t are those associated with a policy that is optimal from a timeless perspective, and (ii) the fluctuations in response to shocks are those that minimize the stabilization loss L^{stab}, subject to the constraint that z_t depend only on (s_t, Z_t).

As an example, consider again the model consisting of aggregate-supply relation (2.1), and suppose once more that social welfare is measured by the expected value of (1.2). For simplicity, suppose that the disturbance u_t evolves according to (2.18), for some $0 \leq \rho_u < 1$. As discussed above, in this case the set of variables (s_t, Z_t) reduces simply to the current value of u_t, and the only possible state-contingent paths that can be implemented by a purely forward-looking (linear) rule are ones in which π_t and x_t are linear functions of the current value of u_t, as in (2.19). Plans of this form are consistent with the equilibrium relation (2.1) if and only if

$$(1-\beta)\bar{\pi} = \kappa \bar{x}, \tag{3.2}$$

$$(1-\beta\rho_u) f_\pi = \kappa f_x + 1. \tag{3.3}$$

Furthermore, in the case of any plan of this form, the stabilization loss is given by

$$L^{stab} = \frac{\beta}{1-\beta}\frac{1-rho_u^2}{1-\beta\rho_u^2}[f_\pi^2 + \lambda f_x^2]\sigma_u^2, \tag{3.4}$$

where σ_u^2 is the unconditional variance of the disturbance process $\{u_t\}$.

It follows from Proposition 7.2 that the long-run average values of inflation and the output gap associated with a timelessly optimal policy are $\bar{\pi} = \bar{x} = 0$. The optimal values of the coefficients (f_π, f_x) are those that minimize (3.4), or equivalently, that minimize $f_\pi^2 + \lambda f_x^2$, subject to constraint (3.3). The solution to this latter problem is given by

$$f_\pi^{oni} = \frac{1-\beta\rho_u}{\kappa^2\lambda^{-1} + (1-\beta\rho_u)^2}, \qquad f_x^{oni} = -\frac{\kappa\lambda^{-1}}{\kappa^2\lambda^{-1} + (1-\beta\rho_u)^2}. \tag{3.5}$$

Thus we obtain the following.

Proposition 7.12. Consider the baseline (Calvo pricing) model, in which the aggregate-supply relation is of the form (2.1), and abstract from any grounds for a concern with interest-rate stabilization, so that the period loss function is of the form (1.2). Let the cost-push disturbance $\{u_t\}$ evolve according to (2.18), for some $0 \leq \rho_u < 1$. Then the optimal non-inertial plan is a state-contingent evolution of the form (2.19) in which $\bar{\pi} = \bar{x} = 0$, and the coefficients f_π, f_x indicating the response to cost-push shocks are given by (3.5).

Applying (2.3) to the case of an AR(1) disturbance process (2.18), we find that the equilibrium responses under discretion are also of the form (2.19), but with

$$\bar{\pi}^{disc} = \frac{\kappa\lambda}{(1-\beta)\lambda + \kappa^2} x^* > 0$$

and

$$f_\pi^{disc} = \frac{\lambda}{\kappa^2 + (1-\beta\rho_u)\lambda} \geq f_\pi^{oni},$$

where the last inequality is strict if $\rho_u > 0$. Thus discretionary policy is suboptimal, even within the class of purely forward-looking policies, except in the special case that $x^* = 0$ and the cost-push shocks are serially uncorrelated. In general, discretionary policy leads both to too high an average rate of inflation (if $x^* > 0$), and to too large an inflation response to cost-push shocks (if $\rho_u > 0$). It follows from (3.3) that too large a positive inflation response f_π also means too small a negative output-gap response f_x.

A similar analysis is possible in the case that welfare is reduced by variation in the level of nominal interest rates, for reasons such as those discussed in chapter 6, section xx. If the period loss function is of the form (2.24), then both (2.1) and (2.23) are constraints on possible equilibrium paths of the target variables $\{\pi_t, x_t, i_t\}$. If we assume that the disturbances to both structural equations are Markovian — in particular, that $\{u_t\}$ evolves according to (2.18) for some $0 \leq \rho_u < 1$, and $\{r_t^n\}$ similarly evolves according to a law of motion (2.27) for some $0 \leq \rho_r < 1$[28] — then under any purely forward-looking policy that

[28] It is not actually necessary to assume that the two disturbances evolve according to independent $AR(1)$ processes, as posited here, in order for the form (3.6) to be correct; one might more generally assume that both r_t^n and u_t are linear functions of (r_{t-1}^n, u_{t-1}) plus an unforecastable innovation term, where the two innovation terms need not be uncorrelated with each other. Certainly there is no economic reason to assume that these disturbance processes must be independent; in the case that steady-state distortions are large, discussed in chapter 6, section xx, many types of disturbances, such as variations in government purchases, will affect r_t^n and u_t simultaneously. However, the case of two independent disturbance processes is easier to treat, and allows us to conduct thought experiments such as the one considered in Figure 7.6.

results in a determinate equilibrium, each of the target variables must evolve according to equations of the form

$$y_t = \bar{y} + f_y u_t + g_y r_t^n, \tag{3.6}$$

where $(\bar{y}, f_y, g_y)$ are constant coefficients for each of the variables $y = \pi, x, i$. The equilibrium relations (2.1) and (2.23) imply two linear restrictions on the coefficients f_y, and another set of two linear restrictions on the coefficients g_y.

Under the assumption that the two disturbance processes are independent, the stabilization loss can be decomposed into two parts,

$$L^{stab} = L^{stab,r} + L^{stab,u},$$

corresponding to the losses resulting from responses to unexpected shocks of the two types. In the case of a plan of the form (3.6), these terms are equal to

$$L^{stab,r} = \frac{\beta}{1-\beta} \frac{1 - rho_r^2}{1 - \beta\rho_r^2} [g_\pi^2 + \lambda_x g_x^2 + \lambda_i g_i^2] \sigma_r^2, \tag{3.7}$$

$$L^{stab,u} = \frac{\beta}{1-\beta} \frac{1 - rho_u^2}{1 - \beta\rho_u^2} [f_\pi^2 + \lambda_x f_x^2 + \lambda_i f_i^2] \sigma_u^2, \tag{3.8}$$

where σ_r^2, σ_u^2 are the unconditional variances of the disturbance processes $\{r_t^n, u_t\}$ respectively. Note that $L^{stab,r}$ involves only the coefficients g_y, while $L^{stab,u}$ involves only the coefficients f_y.

The optimal non-inertial plan is then the state-contingent evolution of the form (3.6)such that (i) the coefficients $\bar{y}$ are the long-run average values under a timelessly optimal policy (characterized in Proposition 7.3), (ii) the coefficients f_y minimize $L^{stab,u}$ subject to the two constraints implied by (2.1) and (2.23), and (iii) the coefficients g_y minimize $L^{stab,r}$ subject to the corresponding two constraints on these coefficients. The solution to this problem, derived in Giannoni and Woodford (2002xx), is presented in section xx of the appendix.[29] Here we note simply that the optimal response coefficients to either of the real disturbances (for example, the coefficients g_y in the case of the natural-rate disturbance) are independent of

[29]Note that the dashed lines plotted in Figure 7.6 indicate the responses implied by the coefficients g_y in the case of the calibrated parameter values assumed in that figure.

the properties of the other disturbance; indeed, they are the same regardless of whether other (independent) disturbances are assumed to exist or not. The optimal response coefficients to a given disturbance are also independent of the degree of variability of that disturbance, though they do depend (as in Proposition 7.12) on the degree of serial correlation of the disturbance.

3.2 The Optimal Taylor Rule

We now consider the optimal choice of a policy rule from within the simple family of Taylor rules (3.1). Let us suppose that inflation and output determination under such a rule are governed by equilibrium relations (2.1) and (2.23) of the basic neo-Wicksellian model introduced in chapter 4. As shown in that chapter,[30] a rule of this kind (with $\phi_\pi, \phi_x \geq 0$) implies a determinate rational-expectations equilibrium if and only if it conforms to the "Taylor principle," *i.e.,* its coefficients satisfy

$$\phi_\pi + \frac{1-\beta}{4\kappa}\phi_x > 1. \tag{3.9}$$

In the case that the disturbance processes are of the form (2.18) and (2.27), this equilibrium is one in which the equilibrium values of π_t, x_t, and i_t will all depend only on the current disturbances r_t^n and u_t. The state-contingent evolution of the target variables will then be of the form (3.6), for certain coefficients $\bar{y}, f_y, g_y$ given in the appendix.

We are thus interested in choosing the coefficients of the policy rule (3.1), from within the class of rules satisfying (3.9), so as to bring about a state-contingent evolution of the form (3.6) that achieves as low as possible a value for L^{stab}, together with the long-run average values $\bar{y}$ associated with a timelessly optimal policy. For at least a certain range of parameter values, it will be possible to choose the coefficients of the Taylor rule so as to implement the

[30]In chapter 4, we considered a version of the "New Keynesian Phillips curve" in which no disturbance term u_t appeared. But the analysis of determinacy of equilibrium there would be unaffected by the addition of such a term. The solutions given in chapter 4 for the equilibrium evolution of inflation and the output gap must be modified to take account of the cost-push disturbances. However, the equations given in chapter 4 remain applicable under the interpretation that the variable called "x_t" in chapter 4 corresponds to $x_t+\kappa^{-1}u_t$ in the notation used in equation (2.1). In the notation of chapter 4, then, the policy rule (3.1) would be one with a time-varying intercept (equal to $\bar{\imath} - (\phi_x/4\kappa)u_t$) in addition to the linear response to $x_t + \kappa^{-1}u_t$.

best possible state-contingent evolution from within the restricted family (3.6), *i.e.*, so as to implement the optimal non-inertial plan. (Note that the 9 coefficients $\bar{y}, f_y, g_y$ must satisfy 6 equalities in order to be consistent with the equilibrium conditions (2.1) and (2.23). Hence a 3-parameter family of policy rules suffices, in principle, to support any plan within this family; it is thus not fortuitous that, for a non-trivial range of parameter values, a Taylor rule can be found that implements the optimal non-inertial plan.) An example of a case in which is this is so is given by the following result of Giannoni and Woodford (2002xx).

Proposition 7.13. Suppose that the disturbance processes in the basic neo-Wicksellian model are of the form (2.18) and (2.27), with a common degree of serial correlation $\rho_r = \rho_u = \rho$. Then if ρ is in the range such that

$$0 < \left[\frac{(1-\beta\rho)(1-\rho)}{\kappa\sigma} - \rho\right]\lambda_i < \frac{(1-\beta)(1-\beta\rho)}{\kappa^2}\lambda_x + 1, \tag{3.10}$$

the optimal non-inertial plan is consistent with a Taylor rule (3.1) with coefficients $\phi_\pi, \phi_x > 0$ that also satisfy (3.9). Hence commitment to this rule implies a determinate equilibrium, and implements the optimal non-inertial plan. It follows that this is the optimal Taylor rule.

The coefficients of this optimal rule are given by

$$\bar{\pi} = \frac{\lambda_i}{\lambda_i + \beta}(i^* - \bar{r}), \qquad \bar{x} = \frac{1-\beta}{\kappa}\bar{\pi}, \qquad \bar{\imath} = \bar{r} + \bar{\pi}, \tag{3.11}$$

$$\phi_\pi = \left[\frac{(1-\beta\rho)(1-\rho)}{\kappa\sigma} - \rho\right]^{-1}\lambda_i^{-1}, \qquad \frac{\phi_x}{4} = (1-\beta\rho)\frac{\lambda_x}{\kappa}\phi_\pi. \tag{3.12}$$

The proof is in the appendix. It is also shown there that the inequalities (3.10) are necessarily satisfied for ρ in some non-empty interval $\underline{\rho} < \rho < \bar{\rho}$, where $0 < \bar{\rho} < 1$, so that this interval also contains some positive values of ρ (but not values of ρ that are too close to one).[31] Hence in an at least some cases, there exists a Taylor rule (with positive feedback coefficients satisfying the Taylor principle) that represents optimal policy, at least among the

[31] In the case of the calibrated parameter values given in Table 6.1 for $\beta, \kappa, \sigma, \lambda_x$, and λ_i, this interval corresponds to $0.17 < \rho < 0.68$.

class of purely forward-looking policies. Nor is this result dependent upon the assumption that $\rho_r = \rho_u$, made in Proposition 7.13 solely for the sake of algebraic simplification. As shown in the appendix, there exists a pair of feedback coefficients (ϕ_π, ϕ_x) consistent with the optimal non-inertial plan for almost all possible parameter values, even when $\rho_r \neq \rho_u$. (Certain functions of the model parameters must be non-zero in order for a system of linear equations to have a solution; but this excludes only certain extremely special parameter values.) One must then check whether the required feedback coefficients satisfy the inequality (3.9), so that the rule implies a determinate equilibrium.[32] Proposition 7.13 shows that some parameter values exist for which the implied feedback coefficients (3.12) satisfy this inequality. Then, since the required feedback coefficients are continuous functions of the model parameters (except at the degenerate parameter values where no solution exists), the inequality is also satisfied for all model parameters (including values of ρ_r and ρ_u) sufficiently close to these ones. For any values of the other parameters, there will thus be an open set of non-negative values for (ρ_r, ρ_u) for which the results announced in Proposition 7.13 obtain, though the algebraic expressions for the optimal feedback coefficients are considerably more complex in the general case.[33]

Optimality of the Taylor rule within the class of purely forward-looking policies implies in particular that there is no gain from adopting a more forward-looking interest-rate feedback rule, at least in the case of this simple model. For example, suppose that we consider rules of the form

$$i_t = \bar{\imath} + \phi_\pi(E_t\pi_{t+k} - \bar{\pi}) + \phi_x(E_t x_{t+k} - \bar{x})/4, \tag{3.13}$$

for some forecast horizon $k > 0$. Such a rule is again purely forward-looking, and so can at

[32]This condition is necessary and sufficient for determinacy, as shown in chapter 4, only under the restriction that $\phi_\pi, \phi_x \geq 0$. Thus we must also verify that these two additional inequalities are satisfied, in order to ensure that the conclusions of Proposition 7.13 hold. Alternatively, we could allow for rules with negative feedback coefficients, as long as they imply a determinate equilibrium; but in that case, there are again additional inequalities that must be checked. The presence of these additional inequalities does not affect the validity of the argument made in the text; the statement "the inequality is also satisfied" should simply be modified to refer to "inequalities."

[33]Proposition xx of Giannoni and Woodford (2002xx) gives an example of an open set of values for (ρ_r, ρ_u) for which this is true, though the conditions established in that result are again only sufficient conditions, and not necessary for existence of an optimal Taylor rule.

best implement the optimal non-inertial plan. However, in a case such as that treated in Proposition 7.13, the Taylor rule already implements this plan, so that a consideration of rules with a forecast horizon $k > 0$ yields no possible improvement. In fact, forward-looking rules may be an inferior approach even to implementation of the optimal non-inertial plan. It is true that it should be equally possible to find a rule of the form (3.13) that is consistent with that state-contingent evolution. For example, in the case treated in Proposition 7.13, there exists a rule of this form that is consistent with the optimal non-inertial plan, for any $k \geq 0$; it is the rule with coefficients $\bar{y}$ as in (3.11) and coefficients ϕ_y given by

$$\phi_y = \rho^{-k}\bar{\phi}_y, \tag{3.14}$$

for $y = \pi, x$, where $\bar{\phi}_y$ refers to the coefficients given in (3.12). However, this alternative policy rule, while equally consistent with the optimal non-inertial plan when $k > 0$, may not also imply determinacy of equilibrium. Indeed, for large enough k, it necessarily does not, as established by Giannoni and Woodford (2002xx).

Proposition 7.14. Consider an economy satisfying the assumptions of Proposition 7.13. Then for all forecast horizons k longer than some critical value, the rule of the form (3.13) that is consistent with the optimal non-inertial plan implies indeterminacy of rational-expectations equilibrium.

The proof is in the Appendix.[34] Thus if the forecast horizon k is sufficiently long, it is not possible to implement the optimal non-inertial plan using a rule of the form (3.13).[35] It follows that, at least when model parameters satisfy (or are close enough to satisfying) the conditions of Proposition 7.13, the best rule in this forward-looking family is not as desirable as the best purely contemporaneous Taylor rule.

[34] Batini and Pearlman (2002) establish a related result for a general family of interest-rate feedback rules in which the current nominal interest rate operating target is a linear function of an inflation forecast and a lagged nominal interest rate.

[35] For example, in the case of the calibrated parameter values given in Table 6.1, the rule (3.13) with coefficients (3.14) implies indeterminacy for all $k \geq 1$.

Yet even in this case, while the Taylor rule is optimal among the class of purely forward-looking policies, it does not follow that one cannot do better; for optimal policy is history-dependent, as shown in section 2.2. We show in chapter 8 that a generalized Taylor rule, that includes dependence of the right kind on lagged variables (in particular, lags of the nominal interest rate), can instead implement a fully optimal equilibrium (understood to mean one that is optimal from a timeless perspective). And even the optimality of the simple Taylor rule among purely forward-looking policies depends on fairly strong restrictions; for example, it will not be true if the disturbances are not (at least jointly, if not individually) Markovian. Moreover, even when one is willing to assume Markovian disturbances, the coefficients of the optimal Taylor rule depend quite critically on the degree of serial correlation of the disturbances, as indicated by (3.12). As shown in chapter 8, it is instead possible to choose a generalized Taylor rule with the property that the *same* feedback coefficients are optimal regardless of the serial correlation properties of the disturbances. Hence the more complex rule is better not only in terms of the expected losses in the case of a particular specification of the disturbance processes, but is also more robust.

4 The Optimal State-Contingent Instrument Path as a Policy Rule

We turn now to the problem of the choosing a policy commitment that would be fully optimal — that would lead not only to the optimal long-run average values of the target variables characterized in section 1, but also to the optimal responses to disturbances characterized in section 2. It might be thought that our characterization of the economy's optimal state-contingent evolution in section 2 has already given as complete a characterization of fully optimal policy as may be desired. For we have shown how to compute the optimal state-contingent paths of the various endogenous variables, including the optimal state-contingent path for the central bank's nominal interest-rate instrument. And it might be supposed that a solution for the optimal state-contingent instrument path — a formula that would tell what the nominal interest-rate operating target should be at each date, as a function of the history of disturbances up to that time — is itself a good example (perhaps even the canonical example) of a fully optimal policy rule. That is, one might propose that commitment to a fully optimal policy should mean a commitment by the central bank to choose its operating target in each decision cycle according to this formula.

We shall argue that this is not a desirable way of deriving an optimal policy rule; but it is first useful to illustrate what such an approach would mean. Let us consider again the model with cost-push shocks and no penalty for interest-rate variations treated in section 2.1. We recall from Proposition 7.6 that equation (2.10) describes the state-contingent evolution of inflation under one kind of policy that would be optimal from a timeless perspective.[36] If we substitute a specific stochastic process for the disturbances $\{u_t\}$, such as (2.18), into this equation we obtain the solution

$$\pi_t = \frac{1}{\beta(\mu_2 - \rho_u)}\{u_t - (1-\mu_1)\sum_{k=1}^{\infty}\mu_1^{k-1}u_{t-k}\} \tag{4.1}$$

for inflation as a function of the history of disturbances up through the current date. We may substitute this in turn into (2.1) to obtain a similar solution for x_t, and substitute both of these solutions into (2.23) to obtain the following solution for the path of the nominal interest rate.

Proposition 7.15. Consider again the policy problem treated in Proposition 7.6, and suppose that the exogenous cost-push disturbances evolve according to (2.18). Then in the timelessly optimal equilibrium characterized in Proposition 7.6, the state-contingent evolution of the nominal interest rate is given by

$$i_t = r_t^n + \left(1 - \frac{\kappa}{\lambda\sigma}\right)\left\{\frac{\mu_1 + \rho_u - 1}{\beta(\mu_2 - \rho_u)}u_t - \frac{1-\mu_1}{\beta(\mu_2-\rho_u)}\sum_{k=0}^{\infty}\mu_1^k u_{t-k}\right\}. \tag{4.2}$$

[36]While this is only one of several possible specifications of the initial inflation constraint $\bar{\pi}_{t_0}$ that is self-consistent in the sense discussed in section 2.1, one can also show that it is the only specification of the initial constraint that leads to an expression (4.2) that is time-invariant, *i.e.*, independent of the date t_0 at which the policy rule is chosen. For example, the alternative specification (2.11) would lead instead to a state-contingent path for the nominal interest rate in periods $t \geq t_0$ that would depend on the value of x_{t_0-1}, and on the number of periods $t - t_0$ that had elapsed since the date at which the commitment was chosen. Because we wish to choose a time-invariant policy rule, in order to address the problem of time consistency discussed above, we accordingly assume that the specification of interest for present purposes is the one discussed in Proposition 7.6.

We might then take equation (4.2) to specify an optimal rule for setting the central bank's interest-rate operating target; indeed, some may suppose that this kind of description of optimal policy, specifying the optimal instrument setting in each possible state of the world (identified by the history of exogenous disturbances), should represent the canonical specification of a policy rule. But this approach to the specification of optimal policy has serious disadvantages. One is that a commitment to the rule by the central bank, even if fully credible and correctly understood by the private sector, need not ensure that the desired (optimal) equilibrium evolution of inflation and output is realized.

For a commitment to set interest rates according to (4.2), regardless of how inflation and output may evolve, is an example of a policy that would specify an exogenous nominal interest-rate path. It then follows from Proposition 4.xx that under the basic model (considered here), rational expectations equilibrium is indeterminate under such a policy. The optimal inflation path (4.2) will be *one* possible equilibrium path for inflation under this policy, but there will also be an uncountably infinite number of other non-explosive equilibrium paths for inflation, includes ones in which inflation (and hence output) responds to cost-push shocks in an entirely different way, and also ones in which inflation and output are affected by pure "sunspot" states. The fact that such a rule does not exclude these other, quite undesirable, equilibria makes this an unattractive approach to the implementation of optimal policy.

We have similarly seen in chapter 4 that commitment to a rule of this kind, that specifies an exogenous path for the nominal interest rate, implies that the minimum-state-variable equilibrium (which in the present case corresponds to the optimal inflation evolution (4.1)) will not be learnable through least-squares regression techniques. Hence from this point of view as well, a rule such as (4.2) represents an undesirable approach to the implementation of optimal policy.

Addressing both of these problems requires that the central bank's policy commitment be specified in a different way, so that the implied interest-rate path will depend on the observed (or projected) paths of inflation and output, and not simply on the bank's evaluation of the history of exogenous disturbances. The Taylor rule is an example of a policy rule with this latter property, and we have seen in chapter 4 that a rule of this kind results both in a determinate equilibrium and in one that is stable under least-squares learning dynamics. However, in the present context, we have seen that a simple (contemporaneous) Taylor rule does not result in an equilibrium that is optimal, as it fails to bring about the history-dependence required for an optimal equilibrium. The question that remains to be addressed, then, is whether we can find a rule that introduces feedback from inflation and/or output to the central bank's interest-rate target of the kind needed to ensure determinacy and learnability, and that at the same time involves the sort of history-dependence needed to implement an optimal equilibrium. We show in the next section how this is possible.

Equation (4.2) is also unappealing as a policy rule, however, for a quite independent reason. The specific formula (4.2) has been shown to be consistent with an optimal state-contingent evolution of inflation and output only under a single very specific assumption about the statistical properties of the cost-push shocks — that they evolve according to a process of the form (2.18) with serial correlation coefficient ρ_u, with the innovation ϵ_t^u completely unforecastable before period t. Were we to assume any other stochastic process for the cost-push shock — something more complex than an AR(1) process, or even an AR(1) process with a different degree of persistence, or even an AR(1) process with the same degree of persistence but with innovations revealed some number of periods in advance — then the solution for the optimal interest rate as a function of the history of the shocks would be given by a different formula.

Yet there is little practical interest in a characterization of optimal policy such as (4.2), which is valid only under the assumption that the real disturbances are of one specific type, no matter how that type is chosen. Suppose that we have estimated the coefficients of the aggregate supply relation (2.1), and that we have an accurate historical series for both inflation and the output gap, and thus can construct a historical series for the disturbance term u_t in this equation. We might then propose to estimate ρ_u using the historical disturbance series, and could then compute the numerical coefficients of a rule of the form (4.2). But a central bank would be highly unlikely to be willing to commit itself to follow the rule even in that case.

For central bankers always have a great deal of highly specific information about the kind of disturbances that have just occurred, which are always somewhat different than those that have been faced at other times. Hence even if it is understood that "typically" cost-push disturbances have had a coefficient of serial correlation of 0.7, there will often be grounds to suppose that the particular shock that has just occurred is likely to be either more persistent or less persistent than a "typical" disturbance. And it is unlikely that central bankers will be willing to commit themselves to stick rigidly to a rule that is believed to lead to outcomes that would be optimal in the case of "typical" disturbances, even in the case that they are aware of the economy's instead being subjected to "atypical" disturbances. In order for an a proposed policy rule to be of practical interest, it must instead be believed that the rule is compatible with optimal (or at least fairly good) outcomes in the case of any of the extremely large number of possible types of disturbances that might be faced on different occasions.

Of course, the sort of analysis that we have used to derive (4.2) can be extended to deal with the case in which there are many different types of real disturbances that may shift the aggregate-supply relation. That is, we may suppose that the residual in equation (2.1) is actually of the form

$$u_t = \sum_j \psi_{jk} \epsilon^j_{t-k}, \tag{4.3}$$

where the $\{\epsilon^j_t\}$ are large set of different types of shocks that may occur in period t, that affect the aggregate-supply relation in that period or later to varying degrees, with varying degrees of persistence, and in ways that are forecastable in advance to varying extents. Given a specification of the dynamic effects on the AS relation of a given type of shock ϵ^j_t, we can compute the response of the nominal interest rate to this particular type of shock in an optimal equilibrium; and we can do this in principle for each of the types of shocks indexed by j, and thus obtain an optimal state-contingent path for the nominal interest rate, where the state in period t is now specified by the histories of realizations of each of the different shocks. But in this case, the formula corresponding to (4.2) will contain separate terms, with different numerical coefficients, for each of the possible types of shocks. Such a description of optimal policy will thus become completely unwieldy in the case of any attempt to capture even in very coarse terms the sorts of differing situations that central banks actually confront at different times.

Giannoni and Woodford (2002a) instead show that if the central bank's policy commitment is described in terms of a relation among endogenous variables that the bank is committed to bring about — rather than in terms of a mapping from exogenous states to the instrument setting, as in (4.2) — it is possible, in a large class of policy problems, to find a rule that is *robustly optimal*, in the sense that the same rule (with given numerical coefficients) continues to be optimal regardless of the assumed statistical properties of the (additive) disturbance terms such as u_t. Indeed, the rule is optimal even if the disturbance terms in the model structural equations are actually composites of an extremely large (not necessarily finite) number of different types of real disturbances, as in (4.3). We illustrate how this is possible in the next section, and discuss the kinds of policy rules to which this approach leads one in greater detail in the next chapter. A rule of this kind represents a policy commitment that a central bank could reasonably make, despite its awareness that it will constantly be receiving quite fine-grained information about current conditions. For a belief that the rule represents a good criterion for judging whether policy is on track does not require the central bank to believe that all shocks are alike, or even that all of the possible types of disturbances to which it may have to respond can all be listed in advance.

5 Commitment to an Optimal Targeting Rule

We now consider an alternative approach to the specification of a policy rule that can implement an optimal equilibrium, and show that this approach can avoid the problems just discussed with a specification of optimal policy in terms of a state-contingent instrument path. The alternative is what Svensson (1999, 20xx) calls a *targeting rule*. Under such a rule, the central bank is committed to adjust its instrument as necessary in order to ensure that a certain *target criterion* is satisfied at all points in time, or more precisely (as this is all that is possible in practice), so that the criterion is *projected* to be satisfied, according to the central bank's forecast of the economy's evolution. The target criterion specifies a condition that the projected evolution of the bank's *target variables* — such as inflation, the output gap, and possibly interest rates as well — must be projected to satisfy if policy is to be regarded as "on track." A simple example would be the criterion that RPIX inflation two years in the future be expected to equal 2.5 per cent per annum; this is the criterion used to explain the policy decisions of the Bank of England under current procedures (Vickers, 1998).

A rule of this kind represents a "higher-level" description of policy than an explicit specification of the instrument setting in each possible state of the world, such as (4.2). The instrument setting that is implied by such a rule at any point in time can only be determined through the use of a quantitative model of the effects of monetary policy on the economy. In each decision cycle, the central bank must use its model (and, of course, the judgment of policymakers) to determine what interest-rate operating target will result in projections that satisfy the target criterion. But a targeting rule is not different, in this respect, from commitment to a rule like the Taylor rule discussed in section 3.2. For in our basic neo-Wicksellian model (the model for which the Taylor rule was shown to constitute an optimal purely forward-looking policy), both inflation and the output gap in period t depend on period t interest rates; hence the policy rule (3.1) does not indicate what the level of interest rates in period t should be, without a calculation of what π_t and x_t are projected to equal in the case of one level of interest rates or another. Our discussion above (as in chapter 4) of the consequences of commitment to such a rule assumed that implementation of an "implicit instrument rule" of this kind is possible.[37]

While such incompleteness of the specification of prescribed central-bank behavior has some disadvantages — for example, it makes it more difficult for the private sector to be

[37]McCallum (19xx) has instead criticized the Taylor rule for not being an "operational" policy proposal, and proposed that one ought instead to consider only candidate rules that explicitly specify the interest-rate operating target as a function of data available to the central bank at the time that the numerical value of this target must be chosen. We do not accept this stricture here, because of our interest in the design of robustly optimal rules.

certain that the central bank is precisely following the announced policy rule — it has the important advantage of making possible a commitment to a rule that is optimal under a much broader range of circumstances, as discussed by Svensson and Woodford (1999) and Giannoni and Woodford (2002a). This can be usefully illustrated through a further consideration of optimal policy in the case treated in sections 2.1 and 4.

5.1 Robustly Optimal Target Criteria

Let us recall the characterization of timelessly optimal policy given in section 2.1. For example, in the case to which Proposition 7.7 applies, the state-contingent evolution of inflation given by equation (2.12) is obtained by solving a system of equations consisting of the first-order conditions (1.7) – (1.8) together with the structural equation (2.1), under the initial condition (2.11). Note furthermore that conditions (1.7) – (1.8) must be satisfied in the equilibrium associated with a timelessly optimal policy for all periods $t \geq t_0$ (for some value of the initial Lagrange multiplier φ_{t_0-1}), regardless of the form of the initial constraint (2.8).

If we use equation (1.8) to substitute for the Lagrange multiplier φ_t in (1.7), we obtain the relation

$$\pi_t + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0 \tag{5.1}$$

that must hold for each $t \geq t_0 + 1$ under any timelessly optimal policy. We cannot show in the same way that the relation must hold at date $t = t_0$, for equation (1.8) need not hold for dates $t < t_0$, and so cannot be used to eliminate φ_{t_0-1}. Nonetheless, the fact that (5.1) must be satisfied for all dates $t \geq t_0 + 1$ under any timelessly optimal policy (and indeed, under a t_0-optimal commitment as well), makes a commitment to ensure that relation (5.1) holds at all times a reasonable candidate for a timelessly optimal policy rule. The following result shows that this guess is correct.

Proposition 7.16. Consider again the problem of choosing monetary policy from date t_0 onward so as to minimize the expected value of (1.2), where the joint evolution of inflation and output must satisfy (2.1) for each date $t \geq t_0$. Let $\{u_t\}$ be a bounded exogenous disturbance process, the statistical character of which is otherwise unspecified. Then if the central bank commits itself to a policy that ensures that (5.1) will be satisfied at each date $t \geq t_0$, there are unique bounded rational-expectations equilibrium processes $\{\pi_t, x_t\}$ for dates $t \geq t_0$ consistent with this policy rule. Furthermore, the equilibrium determined by this policy commitment is the same as the one characterized in Proposition 7.7. Thus the proposed policy rule is optimal from a timeless perspective.

This result is proved in the appendix. Note that the result that the system of equations consisting of (2.1) and (5.1) for each $t \geq t_0$ has a determinate rational-expectations solution is important for two reasons. First, the *existence* of a solution is important, in order for the proposed targeting rule to be *feasible*; Proposition 7.16 implies that there are in fact equilibrium paths for inflation and output that would satisfy the target criterion at all times. (We discuss the instrument settings required to implement such a rule in the next section.) Second, the *uniqueness* of the solution implies that this policy rule, unlike the proposed rule discussed in section 4, is not only *consistent* with an optimal equilibrium, but is furthermore consistent with *no other* equilibria of a less desirable character. Hence a commitment to the targeting rule can be said to *implement* the desired equilibrium, in a way that a commitment to the associated state-contingent interest-rate path does not.

The determinacy result announced in the proposition is established directly in the appendix, through a consideration of the equation system consisting of (2.1) and (5.1). However, the result has a simple intuition; it is a consequence of the existence of a unique bounded solution to the system of equations consisting of (1.7) – (1.8) together with (2.1), that characterize the t_0-optimal plan. For equation (5.1) is equivalent to conditions (1.7) – (1.8) plus the stipulation of an initial Lagrange multiplier[38]

$$\varphi_{t_0-1} = \frac{\lambda}{\kappa}(x_{t_0-1} - x^*). \tag{5.2}$$

[38]Note that (5.2) is just what the multiplier would have to have been equal to if the first-order condition (1.8) also held at date $t = t_0 - 1$, as it would in the case of an optimal commitment chosen at date $t_0 - 1$ or earlier.

However, the initial condition (5.2) is irrelevant for the question whether a determinate solution exists; if a determinate solution exists in the case of the initial condition (2.5), then it also should exist in the case of the initial condition (5.2). Hence a commitment to achievement of the target criterion (5.1) implies a determinate rational-expectations equilibrium.

We thus have an example of a policy rule that results in a determinate equilibrium that is optimal from a timeless perspective. Moreover, the proposed rule is also *robustly* optimal in the sense of Giannoni and Woodford (2002a). For Proposition 7.16 relies on no assumptions about the nature of the exogenous disturbance process $\{u_t\}$, except that it is bounded (as is generally necessary in order for bounded equilibrium paths for inflation and the output gap to be possible under *any* monetary policy, and hence for our approximate characterizations of the equilibrium conditions and of welfare to be valid) and that its effect on the aggregate-supply relation (2.1) is additive (in a log-linear approximation). The first-order conditions from which the target criterion (5.1) is derived are independent of any assumptions about the statistical properties of the disturbances, and so the optimal policy rule obtained in this way is optimal regardless of their character. Hence a commitment to bring about the optimality condition (5.1) is equally sensible regardless of the particular type of shocks that the central bank may believe to have most recently disturbed the economy.

The robustly optimal rule is an example of a flexible inflation targeting rule, in the sense discussed by Svensson (1999, 20xx).[39] The central bank commits itself to adjust the level of nominal interest rates so that the projected inflation rate is consistent at all times with the target criterion. However, the acceptable inflation rate depends on the projected path of the output gap. An inflation rate higher than the long-run target rate (here, zero) is acceptable if the output gap is projected to decline, and a lower inflation rate could be achieved only by reducing the output gap even more sharply, making the left-hand side of (5.1) negative; similarly, an inflation rate lower than the long-run target rate should be sought if even this rate of inflation requires a growing output gap, so that any higher current inflation rate

[39] To be more precise, the type of rule that we consider here corresponds to what Svensson calls a "specific" targeting rule.

would be possible only with a positive value for the left-hand side of (5.1).

The optimal target criterion (5.1) differs, however, from a common conception of "flexible inflation targeting" in that it is the projected *rate of change* of the output gap, rather than the absolute level of the output gap, that should determine the acceptable deviation from the long-run inflation target. This might seem paradoxical, in that it is the absolute level of the output gap, rather than its rate of change, that one wishes to stabilize. But this is simply a reflection, once again, of the fact that optimal policy is not purely forward-looking. The target criterion is history-dependent in the sense that acceptable projections (π_t, x_t) depend on the value of the lagged output gap x_{t-1}, even though the lagged output gap is irrelevant both to the determination of current and future inflation and output gap and to the welfare evaluation of alternative possible paths for those variables from the present time onward. But this sort of history-dependence is exactly what is necessary in order for the targeting rule to bring about the kind of dynamic responses to a cost-push shock shown in Figure 7.3. Once an adverse cost-push shock has caused a negative output gap, the history-dependent target criterion requires that the output gap be restored only gradually to its long-run level, and that inflation be kept below its long-run level during the period in which the output gap is catching up to its long-run level. This kind of dynamic response to a transitory cost-push shock implies that price-level increases due to the cost-push shock will subsequently be undone, and as discussed above, the anticipation that this will be the case restrains price increases at the time of the shock, reducing the extent to which either inflation or a negative output gap are necessary at that time.

The optimal target criterion (5.1) also differs from the kind of target criteria typically used by inflation-targeting central banks to justify their policy settings — such as the criterion used by the Bank of England, mentioned above — in that it specifies an acceptable near-term inflation rate (that is allowed to vary, under stated conditions) rather than a medium-term inflation objective (that should remain always the same, despite short-term inflation variability). It is important to note that this criterion does incorporate a long-run inflation target (namely, zero); because the change in the output gap from one quarter to the next must be zero on average, ensuring that (5.1) holds each quarter will require a zero inflation rate on average. Nonetheless, the rule requires the central bank to justify its instrument setting in each decision cycle by reference to whether it is projected to result in an acceptable near-term inflation rate, and not simply by reference to whether policy continues to be consistent with a projection that inflation should eventually approach the long-term target value.

A flexible (but specific!) target criterion of this kind provides a clearer guide to short-run policy decisions — that is, to the only kind of decisions that a central bank is actually called upon to make — than does a mere specification of the long-run inflation target. After all, the interest-rate decision made at any point in time is of little import for the expected long-run inflation rate, which should depend entirely on how policy is expected to be conducted in the future. A commitment to return inflation to its long-run target rate by a specified (not too distant) horizon may have less trivial implications for current policy, but a horizon that is short enough for such a commitment to determine current policy is likely to result in too rigid a criterion for such a commitment to be desirable. Hence the desirability of a target criterion that specifies the conditions under which near-term deviations from the long-run target are justifiable, rather than merely specifying the long-run target.

Actual inflation-targeting central banks have probably avoided the articulation of a flexible nearer-term target criterion of this kind out of skepticism about whether it is possible to specify in advance all of the conditions under which a given degree of temporary departure from the long-run inflation target should be justifiable. But we have seen that it is possible to derive a robustly optimal target criterion, that correctly determines whether a given degree of departure of projected near-term inflation from the long-run target rate is consistent with the optimal state-contingent inflation path, *regardless* of the size and nature of the disturbances that have most recently affected the economy. If the soundness of such a criterion is accepted, then it ought to be possible for a central bank to commit itself to the conduct of policy in accordance with a nearer-term target criterion of this kind.

While (5.1) represents an example of a robustly optimal target criterion, it is not the only possible criterion with that property. Note that satisfaction of (5.1) each period implies that the quantity $p_t + (\lambda/\kappa)x_t$ never changes (since the left-hand side of (5.1) is just the first difference of this quantity). Hence in any equilibrium that is optimal from a timeless perspective, there exists some value of $\bar{p}$ such that

$$p_t + \frac{\lambda}{\kappa}x_t = \bar{p} \tag{5.3}$$

at all times. This suggests an alternative policy rule, namely, that the central bank commit to ensure that (5.3) holds each period. This too can be shown to be a robustly optimal targeting rule.

PROPOSITION 7.17. Under the same assumptions as in Proposition 7.16, suppose that the central bank commits itself to a policy that ensures that (5.3) will be satisfied at each date $t \geq t_0$. Then there are unique bounded rational-expectations equilibrium processes $\{\pi_t, x_t\}$ for dates $t \geq t_0$ consistent with this policy rule. Furthermore, the equilibrium determined by this policy commitment is the same as the one characterized in Proposition 7.8. Thus the proposed policy rule is optimal from a timeless perspective.

The proof is in the appendix. This type of rule corresponds to the "flexible price-level target" advocated by Hall (1984).[40] Note that our analysis provides a theoretical ground for choosing a particular coefficient on the output gap in such a rule. As shown in chapter 6, in the welfare-theoretic loss function, $\lambda = \kappa/\theta$, where $\theta > 1$ is the elasticity of substitution among alternative goods, and the elasticity of demand faced by each of the monopolistically competitive suppliers. It then follows that the optimal flexible price-level targeting rule stabilizes the value of $p_t + \theta^{-1}x_t$. Since a reasonable calibration of θ must be much larger than one (a valid on the order of 10 is most commonly assumed, in order for the model not to imply an implausible degree of market power), this implies that the weight on the output

[40]The observation that timelessly optimal policies bring about equilibria consistent with a rule of this kind explains our comment, in section xx of chapter 6, that the efficient frontier in Figure 6.xx corresponds to flexible price-level targets with alternative weights on the output gap. The efficient frontier is constructed by computing the optimal state-contingent evolution of inflation and of the output gap in the case of alternative values of λ ranging between 0 and $+\infty$. For each value of λ, the optimal policy is a member of the family (5.3), but with a different weight on the output gap in each case.

gap should be only a small fraction of the weight on the price level.

The targeting rule (5.3) is closely related to (5.1); indeed, committing to (5.1) from some date t_0 onward is equivalent to committing to a rule of the form (5.3) with a particular choice of the price-level target, namely

$$\bar{p} = p_{t_0-1} + \frac{\lambda}{\kappa} x_{t_0-1}.$$

However, the choice of $\bar{p}$ is arbitrary, if one wishes only to ensure that the rule chosen is optimal from a timeless perspective. Different rules in the family lead to the same long-run average inflation rate (though different long-run average price levels), the same long-run average output gap, and the same equilibrium responses to shocks. The associated equilibria differ only in a transitory, deterministic component, as to which we can make no choice from a timeless perspective.

5.2 Implementation of a Targeting Rule

The results in the previous section (Propositions 7.16 and 7.17) describe the state-contingent evolution of inflation and output that should result, in a rational-expectations equilibrium, if the central bank succeeds in ensuring that the target criterion is satisfied at all times. But can a central bank actually ensure this, and hence bring about such an equilibrium? It is appropriate to discuss further what sort of adjustment of its interest-rate instrument this would involve. This means describing the conduct of policy in accordance with such a rule in terms of the associated *reaction function* for the nominal interest rate that is the policy instrument. We can then consider whether such a reaction function determines a unique (or at least a unique non-explosive) rational-expectations equilibrium, and whether this equilibrium should be learnable, along the lines of our analysis of the determinacy and learnability of equilibrium under simple interest-rate rules in chapter 4.

Let us first consider the implementation of the targeting rule (5.1). The policy rule specifies that i_t should be set in period t in such a way that the central bank projects values of π_t and x_t consistent with equation (5.1). The interest-rate decision that this implies depends on the way in which the central bank constructs the projections for current inflation and output conditional on alternative interest-rate decisions.

We shall suppose that the central bank's model of the effects of alternative policies is the correct one, *i.e.*, that it consists of equations (2.1) and (2.23), together with a correct understanding of the laws of motion of the exogenous disturbance processes. But this in itself does not answer the question of what the central bank's projection conditional on its interest-rate decision should be, for according to the equations of the structural model, current inflation and output, given the current nominal interest rate, depend on expectations regarding the economy's future evolution. One way of specifying these expectations would be for the central bank to assume that the private sector expects the economy to evolve in the future (*i.e.*, in period $t+1$ and later) according to the rational-expectations equilibrium described in Proposition 7.16. That is, it assumes that the private sector expects it to succeed in enforcing the target criterion in all periods from $t+1$ onward, even though, for purposes of constructing the conditional projection, the central bank contemplates the consequences of deviation from the interest rate consistent with the policy rule in period t.

This means that the central bank expects the private sector to expect that in period $t+1$, inflation and the output gap will be given by

$$\begin{aligned} \pi_{t+1} &= (1-\mu_1)\frac{\lambda}{\kappa}x_t + \frac{1}{\beta}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_{t+1}u_{t+j+1}, \\ x_{t+1} &= mu_1 x_t + \frac{\kappa}{\beta\lambda}\sum_{j=0}^{\infty}\mu_2^{-j-1}E_{t+1}u_{t+j+1}. \end{aligned}$$

Taking the expectations of these two expressions conditional upon period t information, one obtains solutions for $E_t\pi_{t+1}$ and $E_t x_{t+1}$ as linear functions of x_t and terms of the form $E_t u_{t+j}$. Substituting these solutions into the structural relations (2.1) and (2.23), one can then solve those two relations for π_t and x_t as linear functions of i_t, r_t^n, and terms of the form $E_t u_{t+j}$. The bank's projection of the current-period value of its target $\pi_t + (\lambda/\kappa)x_t$, conditional on its current instrument choice i_t and given its information about the exogenous disturbances, would then equal

$$\frac{-\sigma}{\lambda^{-1}\kappa - \sigma}\frac{1}{\mu_1(1-\mu_1)}\,(i_t - r_t^n) + u_t - \frac{1}{\beta(1-\mu_1)}\frac{\kappa^2+\lambda}{\lambda}\sum_{j=1}^{\infty}\mu_2^{-j}E_t u_{t+j}.$$

If the central bank equates this expression to $(\lambda/\kappa)x_{t-1}$, as required in order for the projection to satisfy (5.1), it obtains a relation that can be solved for i_t, yielding

$$i_t = r_t^n + \mu_1\left(1 - \frac{\lambda\sigma}{\kappa}\right)\left\{\frac{(1-\mu_1)\kappa}{\lambda\sigma}u_t - \frac{\kappa}{\beta\lambda\sigma}\left(1+\frac{\kappa^2}{\lambda}\right)\sum_{j=1}^{\infty}\mu_2^{-j}E_t u_{t+j} - \frac{1-\mu_1}{\sigma}x_{t-1}\right\}. \quad (5.4)$$

This is what Evans and Honkapohja (2002) call the "fundamentals-based reaction function" for implementation of the target criterion (5.1);[41] it gives a formula for the central bank's operating target purely in terms of exogenous and predetermined variables.

However, while this relation is consistent with the desired state-contingent evolution of the interest rate and other variables, a commitment to set interest rates in this way does not necessarily imply a determinate equilibrium; Evans and Honkapohja show that for many parameter values, it does not.[42] For while the right-hand side of (5.4) does not depend solely on exogenous variables, all dependence on either current or expected future endogenous variables has been eliminated, by substituting the values that these variables are expected to take in the desired equilibrium. But a central bank that commits itself to act *as if* the desired equilibrium is being realized regardless of whether this is observed to be the case does not act sufficiently decisively to ensure that this equilibrium is realized, rather than some other one which is less desirable.

An alternative approach, recommended by Evans and Honkapohja, is for the central bank not to substitute out for what the expectations $E_t\pi_{t+1}$ and $E_t x_{t+1}$ *ought* to be, given the economy's current state and the laws of motion that obtain in the desired equilibrium, but

[41] They define this reaction function, like (xxx) below, only for the case of disturbance processes of the special forms (2.18) and (2.27); but the logic of their derivation is the one given here. Note that the determinacy of equilibrium when the central bank commits itself to a reaction function of this form does not depend on the statistical properties of the exogenous disturbance processes; only the coefficient with which the lagged endogenous variable x_{t-1} enters matters for that.

[42] Svensson and Woodford (1999) reach a similar conclusion in the case of a closely related reaction function in the case of a model in which the endogenous components of both inflation and output are predetermined a period in advance.

rather to condition its policy decision on what it *actually observes* current private-sector expectations to be. Under this approach, the central bank produces its projections for current-period inflation and output by solving the structural equations (2.1) and (2.23) for π_t and x_t as functions of i_t, period t expectations, and the exogenous disturbances. In this case, the bank's projection for $\pi_t + (\lambda/\kappa)x_t$ conditional on its current instrument choice is given by

$$-\sigma\frac{\kappa^2+\lambda}{\kappa}\,(i_t - r_t^n) + u_t + \left[\beta + \sigma\left(\frac{\kappa^2+\lambda}{\kappa}\right)\right]\,E_t\pi_{t+1} + \frac{\kappa^2+\lambda}{\kappa}\,E_t x_{t+1}.$$

Equating this to $(\lambda/\kappa)x_{t-1}$ and solving for i_t, one obtains the alternative reaction function

$$i_t = r_t^n + \frac{\kappa}{\sigma(\kappa^2+\lambda)}\,u_t + \left[1 + \frac{\beta\kappa}{\sigma(\kappa^2+\lambda)}\right]\,E_t\pi_{t+1} + \frac{1}{\sigma}\,E_t x_{t+1} - \frac{\lambda}{\sigma(\kappa^2+\lambda)}\,x_{t-1}. \qquad (5.5)$$

Evans and Honkapohja call this an "expectations-based reaction function" intended to implement (5.1).

If the central bank can commit itself to set interest rates in accordance with this reaction function at all times, then rational-expectations equilibrium is necessarily determinate, due to the following result.

PROPOSITION 7.18. Consider an economy in which inflation and output are determined by structural relations of the form (2.1) and (2.23), where the exogenous disturbances $\{u_t, r_t^n\}$ are bounded processes but otherwise unrestricted, and suppose that the central bank sets its nominal interest-rate instrument in accordance with (5.5) in each period $t \geq t_0$. Then there is a determinate rational-expectations equilibrium evolution for inflation, output and the nominal interest rate in periods $t \geq t_0$, and the state-contingent paths of inflation and output are the ones characterized in Proposition 7.7. Hence such a policy is optimal from a timeless perspective.

The proof of this result is simple. Equation (5.5), together with (2.1) and (2.23), implies that (5.1) must hold in each period $t \geq t_0$. (This just reverses the steps in the derivation of (5.5) sketched above.) But the system consisting of equations (2.1) and (5.1) has a unique bounded solution for inflation and output, from Proposition 7.16. Equation (2.23) can then be solved for the associated bounded solution for the path of the nominal interest rate. Furthermore, we know from Proposition 7.16 that the equilibrium determined in this way is the one characterized in Proposition 7.7.

We thus see that a commitment to achieving the flexible inflation target (5.1) is actually an equivalent policy to one that results from commitment to an interest-rate rule of the form (5.5).

[MORE TO BE ADDED]