Recursive Models of Dynamic Linear Economies

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Lars Hansen
University of Chicago

Thomas J. Sargent
New York University
and
Hoover Institution

 6 September 2005

Contents

Part II: Representations and Properties

Part III: Extensions

Part IV: Economies as Objects

Acknowledgements

Preface

Part I

Components of an economy

Chapter 1
Introduction

1.1. Introduction

This book views many apparently disparate dynamic economic models as examples of a single class of models that can be adapted and specialized to study diverse economic phenomena. The class of models was created by using recent advances in (i) the theory of recursive dynamic competitive economies;[1] (ii) methods for estimating and interpreting vector autoregression;[2] (iii) linear optimal control theory;[3] and (iv) computer languages for rapidly manipulating linear optimal control systems.[4] We combine these elements to build a class of models for which the competitive equilibria are vector autoregressions that can be swiftly computed, represented, and simulated using the methods of linear optimal control theory. We use the computer language MATLAB to implement the computations. This language has a powerful vocabulary and a convenient structure that liberate time and energy from programming, and thereby spur creative application of linear control theory.

Our goal has been to create a class of models that merge recursive economic theory and with dynamic econometrics.

Systems of autoregressions and of mixed autogregressive, moving average processes are a dominant setting for dynamic econometrics. We constructed our economic models by adopting a version of recursive competitive theory in which an outcome of theorizing is a vector autoregression.

We formulated this class of models because practical difficulties of computing and estimating recursive equilibrium models still limit their use as a tool for thinking about applied problems in economic dynamics. Recursive competitive equilibria were themselves developed as a special case of the Arrow-Debreu competitive equilibrium, both to restrict the range of outcomes possible in the

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1 This work is summarized by Harris (1987) and Stokey, Lucas, and Prescott (1989).
2 See Sims (1980), Hansen and Sargent (1980, 1981, 1990).
3 For example, see Kwakernaak and Sivan (1972), and Anderson and Moore (1979).
4 See the MATLAB manual.

Arrow-Debreu setting and to create a framework for studying applied problems in dynamic economies of long duration. Relative to the general Arrow-Debreu setting, the great advantage of the recursive competitive equilibrium formulation is that equilibria can be computed by solving a discounted dynamic programming problem. Further, under particular additional conditions, an equilibrium can be represented as a Markov process in the state variables. When that Markov process has an invariant distribution to which the process converges, there exists a vector autoregressive representation. Thus, the theory of recursive competitive equilibria holds out the promise of making closer contact with econometric theory than did previous formulations of equilibrium theory.

Two computational difficulties have left much of this promise unrealized. The first is Bellman's "curse of dimensionality" which usually makes dynamic programming a costly procedure for systems with even small numbers of state variables. The second problem is that after a dynamic program has been solved and the equilibrium Markov process computed, the vector autoregression implied by the theory has to be computed by applying classic projection formulas to a large number of second moments of the stationary distribution associated with that Markov process. Typically, each of these computational problems can be solved only approximately. Good research along a number of lines is now being directed at evaluating alternative ways of making these approximations.[5]

The need to make these approximations originates in the fact that for general functional forms for objective functions and constraints, even one iteration on the functional equation of Richard Bellman cannot be performed analytically. It so happens that the functional forms economists would most like to use have been of this general class for which Bellman's equation cannot be iterated upon analytically.

Linear control theory studies the most important special class of problems for which iterations on Bellman's equation *can* be performed analytically: problems with a quadratic objective function and a linear transition function. Application of dynamic programming leads to a system of well understood and rapidly solvable equations known as the matrix Riccati equation.

The philosophy of this book is to swallow hard and to accept up front as primitive descriptions of tastes, technology, and information specifications that satisfy the assumptions of linear optimal control theory. This approach

[5] See Marcet (1989) and Judd (1990). Also see Coleman (1990) and Tauchen (1990).

purchases the ability rapidly to compute equilibria together with a form of equilibrium that is automatically in the form of a vector autoregression. A cost of the approach is that it does not accommodate many specifications that we would like to be able to analyze.

The purpose of this book is to display the versatility and tractability of our class of models. Versions of a wide range of models from modern capital theory and asset pricing theory can be represented within our framework. The equilibria of these models can be computed so easily that we hope that the reader will soon be thinking of improvements to our specifications. We provide formulas and software for the reader to experiment.

1.2. Computer Programs

In writing this book, we put ourselves under a restriction that we should supply the reader with a computer program that implements every equilibrium concept and mathematical representation that we describe. The programs are written in MATLAB, and are described throughout the book. When a MATLAB program is referred to in the text, we place it in `typewriter` font. Similarly, all computer code is placed in `typewriter` font.[6] You will get much more out of this book if you use and modify our programs as you read.

1.3. Organization

This book is organized as follows. Chapter 10 describes the first order linear vector stochastic difference equation, and shows how special cases of it are formed by a variety of models of time series processes that have been studied by economists. This difference equation will be used to represent the information flowing to economic agents within our models. It will also be used to represent the equilibrium of the model.

Chapter 3 defines an economic environment in terms of the preferences of a representative agent, the technology for producing goods, stochastic processes

[6] To run our programs, you will need MATLAB's *Control Toolkit* in addition to the basic MATLAB software.

disturbing preferences and the technology, and the information structure of the economy. The stochastic processes fit into the model introduced in chapter 10, while the preferences, technology, and information structure are specified with an eye toward making the competitive equilibrium one that can be computed by the application of linear control theory.

Chapter 4 describes a social planning problem associated with the equilibrium of the model. The problem is formulated in two ways, first as a variational problem using stochastic Lagrange multipliers, and then as a dynamic programming problem. We describe how to compute the solution of the dynamic programming problem using formulas from linear control theory. The solution of the social planning problem is a first order vector stochastic difference equation of the form studied in chapter 10. We also show how to use the value function for the social planning problem to compute the Lagrange multipliers associated with the planning problem. These multipliers are later used in chapter 6 to compute the equilibrium price system.

Chapter 5 describes the price system and the commodity space that support a competitive equilibrium. We use a formulation that lets the values that appear in agents' budget constraints and objective functions be represented as conditional expectations of geometric sums of streams of future "prices" times quantities. Chapter 5 relates these prices to Arrow-Debreu state contingent prices.

Chapter 6 describes a decentralized version of our economy, and defines and computes a competitive equilibrium. Competitive equilibrium quantities solve a social planning problem. The price system can be deduced from the stochastic Lagrange multipliers associated with the social planning problem.

Chapter 7 describes versions of several dynamic models from the literature that fit easily within our class of models.

Chapter 9 describes the links between our theoretical equilibrium and autoregressive representations of time series of observables. We show how to obtain an autoregressive representation for a list of observable variables that are linear functions of the state variables of the model. The autoregressive representation is naturally affiliated with a recursive representation of the likelihood function for the observable variables. In describing how to deduce the autoregressive representation from the parameters determining the equilibrium of the model, and possibly also from parameters of measurement error processes, we are completing a key step needed to permit econometric estimation of the model's free parameters. Chapter 9 also treats two other topics intimately related to econometric implementation of the models; aggregation over time, and the theory of approximation of one model by another.

Chapter 8 describes fast methods to compute equilibria. We describe how *doubling algorithms* can speed the computation of expectations of geometric sums of quadratic forms, and help to solve dynamic programming problems.

Chapter 11 describes alternative ways to represent demand. It identifies an equivalence class of preference specifications that imply the same demand functions, and characterizes a special subset of them as *canonical* household preferences. Canonical representations of preferences are useful for describing economies with heterogeneity among household's preferences.

Chapter 12 describes a version of our economy with the type of heterogeneity among households allowed when preferences aggregate in a sense introduced by Terrance Gorman . In this setting, affine Engle curves of *common slope* prevail and give rise to a representative consumer. This representative consumer is 'easy to find,' and from the point of view of equilibrium computation of prices and *aggregate* quantities, adequately stands in for the household of chapters 3–6. The allocations to individual consumers require additional computations, which this chapter describes.

Chapter 13 uses our model of preferences to represent multiple goods versions of permanent income models along the lines of Robert Hall's (1978). We retain Hall's specification of the 'storage' technology for accumulating physical assets, and also the restriction on the discount factor, depreciation rate, and gross return on capital that delivered to Hall a martingale for the marginal utility of consumption. Adopting Hall's specification of the storage technology imparts a martingale characterization to the model, but it is hidden away in an 'index' whose increments drive the behavior of consumption demands for various goods, which themselves are not martingales. This model forms a convenient laboratory for thinking about the sources in economic theory of 'unit roots' and 'co-integrating vectors.'

Chapter 14 describes a setting in which there is more heterogeneity among households' preferences, causing the conditions for Gorman aggregation to fail. Households' Engle curves are still affine, but dispersion of their slopes arrests Gorman aggregation. There is another sense, originating with Negishi, in which there is a representative household whose preferences represent a complicated kind of average over the preferences of different types of households. We show

how to compute and interpret this preference ordering over economy-wide aggregates. This average preference ordering cannot be computed before one knows the distribution of wealth evaluated at equilibrium prices.

Chapter 15 describes economies with production and consumption externalities and also distortions due to a government's imposing distorting flat rate taxes. Equilibria of these economies has to be computed by a direct attack on Euler equations and budget constraints, rather than via dynamic programming for an artificial social planning problem.

Chapter 16 describes a recursive version of Jacobson's and Whittle's 'risk sensitive' preferences. This preference specification has the features that, although it violates certainty equivalence – so that the conditional covariance of forecast error distributions impinge on equilibrium decision rules – it does so in a way that preserves linear equilibrium laws of motion, and retains calculation of equilibria and asset prices via simple modifications of our standard formulas. These preferences are a version of those studied by Epstein and Zin ( ) and Weil ( ).

Chapter 17 describes how to adapt our setup to include features of the periodic models of seasonality that have been studied by Osborne (1988), Todd (1990), and Ghysels (1993).

Chapter 20 is a manual of the MATLAB programs that we have prepared to implement the calculations described in this book. The design is consistent with other MATLAB manuals.

The notion of duality and the 'factorization identity' from recursive linear optimal control theory are used repeatedly in Chapter 9 (on representing equilibria econometrically), and chapters 11, 12, and 14 (on representing and aggregating preferences). 'Duality' is the observation that recursive filtering problems (Kalman filtering) have the same mathematical structure as recursive formulations of linear optimal control problems (leading to Riccati equations via dynamic programming). That duality applies so often in our settings in effect 'halves' the mathematical apparatus that we require.

Chapter 2
Stochastic Linear Difference Equations

2.1. Introduction

This chapter introduces the first-order vector linear stochastic difference equation, which we use in two important ways. We use it first to represent the information flowing to economic agents, then again to represent equilibria of our models. The first-order linear stochastic difference equation is associated with a tidy theory of prediction and a host of procedures for econometric application. Their ease of analysis has prompted us to adopt economic specifications that cause our equilibria to have representations in terms of a first-order linear stochastic difference equation.

The first order vector stochastic difference equation is *recursive* because it expresses next period's vector of state variables as a linear function of this period's state vector and a vector of new disturbances to the system. These disturbances form a "martingale difference sequence," and are the basic building block out of which the time series are created. Martingale difference sequences are easy to forecast, a fact that delivers convenient recursive formulas for optimal predictions.

2.2. Notation and Basic Assumptions

Let $\{x_t : t = 1, 2, \ldots\}$ be a sequence of n-dimensional random vectors, i.e. an n-dimensional stochastic process. The vector x_t contains variables observed by economic agents at time t. Let $\{w_t : t = 1, 2, \ldots\}$ be a sequence of N-dimensional random vectors. The vectors $\{w_t\}$ will be treated as building blocks for $\{x_t : t = 1, 2, \ldots\}$, in the sense that we shall be able to express x_t as the sum of two terms. The first is a moving average of past w_t's. The second describes the effects of an initial condition. The $\{w_t\}$ process is used to generate a sequence of information sets $\{J_t : t = 0, 1, \ldots\}$. Let J_0 be generated by x_0 and J_t be generated by $x_0, w_1, \ldots, w_t$, which means that J_t consists of the set of all measurable functions of $\{x_0, w_1, \ldots, w_t\}$.[1] The building block process is assumed to be a martingale difference sequence adapted to this sequence of information sets. We explain what this means by advancing the following

DEFINITION 1: The sequence $\{w_t : t = 1, 2, \ldots\}$ is said to be a *martingale difference sequence* adapted to $\{J_t : t = 0, 1, \ldots\}$ if $E(w_{t+1}|J_t) = 0$ for $t = 0, 1, \ldots$.

In addition, we assume that the building block process is *conditionally homoskedastic*, a phrase whose meaning is conveyed by

DEFINITION 2: The sequence $\{w_t : t = 1, 2, \ldots\}$ is said to be *conditionally homoskedastic* if $E(w_{t+1}w'_{t+1} \mid J_t) = I$ for $t = 0, 1, \ldots$.

It is an implication of the law of iterated expectations that $\{w_t : t = 1, 2, \ldots\}$ is a sequence of (unconditional) mean zero, serially uncorrelated random vectors.[2] In addition, the entries of w_t are assumed to be mutually uncorrelated.

The process $\{x_t : t = 1, 2, \ldots\}$ is constructed recursively using an initial random vector x_0 and a time invariant law of motion:

$$x_{t+1} = Ax_t + Cw_{t+1} , \quad \text{for} \quad t = 0, 1, \ldots, \tag{2.2.1}$$

where A is an n by n matrix and C is an n by N matrix.

Representation (2.2.1) will be a workhorse in this book. First, we will use (2.2.1) to model the information upon which economic agents base their decisions. Information will consist of variables that drive shocks to preferences and to technologies. Second, we shall specify the economic problems faced by the agents in our models and the economic process through which agents' decisions

[1] The phrase "J_0 is generated by x_0" means that J_0 can be expressed as a measurable function of x_0.

[2] Where ϕ_1 and ϕ_2 are information sets with $\phi_1 \subset \phi_2$, and x is a random variable, the law of iterated expectations states that

$$E(x \mid \phi_1) = E(E(x \mid \phi_2) \mid \phi_1).$$

Letting ϕ_1 be the information set corresponding to no observations on any random variables, letting $\phi_2 = J_t$, and applying this law to the process $\{w_t\}$, we obtain

$$E\left(w_{t+1}\right) = E\left(E\left(w_{t+1} \mid J_t\right)\right) = E(0) = 0.$$

are coordinated (competitive equilibrium) so that the state of the economy has a representation of the form (2.2.1).

2.3. Prediction Theory

A tractable theory of prediction is associated with (2.2.1). This theory is used extensively both in computing the equilibrium of the model and in representing that equilibrium in the form of (2.2.1).

The optimal forecast of x_{t+1} given current information is

$$E(x_{t+1} \mid J_t) = Ax_t, \tag{2.3.1}$$

and the one-step-ahead forecast error is

$$x_{t+1} - E(x_{t+1} \mid J_t) = Cw_{t+1}. \tag{2.3.2}$$

The covariance matrix of x_{t+1} conditioned on J_t is just CC':

$$E(x_{t+1} - E(x_{t+1} \mid J_t))(x_{t+1} - E(x_{t+1} \mid J_t))' = CC'. \tag{2.3.3}$$

Sometimes we use a nonrecursive expression for x_t as a function of $x_0, w_1, w_2, \ldots, w_t$. Using (2.2.1) repeatedly, we obtain

$$\begin{aligned} x_t &= Ax_{t-1} + Cw_t \\ &= A^2 x_{t-2} + ACw_{t-1} + Cw_t \\ &= \Big[\sum_{\tau=0}^{t-1} A^\tau C w_{t-\tau}\Big] + A^t x_0. \end{aligned} \tag{2.3.4}$$

Representation (2.3.4) is one type of *moving-average* representation. It expresses $\{x_t : t = 1, 2, \ldots\}$ as a linear function of current and past values of the building block process $\{w_t : t = 1, 2, \ldots\}$ and an initial condition x_0. [3]

[3] Slutsky (1937) argued that business cycle fluctuations could be well modelled by moving average processes. Sims (1980) showed that a fruitful way to summarize correlations between time series is to calculate an impulse response function. In chapter 8, we study the relationship between the impulse response functions calculated by Sims (1980) and the impulse response function associated with (2.3.4).

The moving average piece of representation (2.3.4) is often called an *impulse response function.* An impulse response function depicts the response of current and future values of $\{x_t\}$ to an imposition of a random shock w_t. In representation (2.3.4), the impulse response function is given by entries of the vector sequence $\{A^\tau C : \tau = 0, 1, \ldots\}$.[4]

Shift (2.3.4) forward in time:

$$x_{t+j} = \sum_{s=0}^{j-1} A^s C w_{t+j-s} + A^j x_t. \tag{2.3.5}$$

Projecting both sides of (2.3.5) on the information set $\{x_0, w_t, w_{t-1}, \ldots, w_1\}$ gives[5]

$$E_t x_{t+j} = A^j x_t. \tag{2.3.6}$$

where $E_t(\cdot) \equiv E[(\cdot) \mid x_0, w_t, w_{t-1}, \ldots, w_1] = E(\cdot) \mid J_t$, and x_t is in J_t. Equation (2.3.6) gives the optimal j step ahead prediction.

It is useful to obtain the covariance matrix of the j-step ahead prediction error

$$x_{t+j} - E_t x_{t+j} = \sum_{s=0}^{j-1} A^s C w_{t-s+j} \tag{2.3.7}$$

We have

$$\begin{aligned} E(x_{t+j} - E_t x_{t+j})\ (x_{t+j} - E_t x_{t+j})' \\ = \sum_{k=0}^{j-1} A^k C C' A^{k'} \equiv v_j \end{aligned} \tag{2.3.8a}$$

Note that v_j defined in (2.3.8a) can be calculated recursively via

$$\begin{aligned} v_1 &= CC' \\ v_j &= CC' + A v_{j-1} A', \quad j \geq 2. \end{aligned} \tag{2.3.8b}$$

The matrix v_j is the covariance matrix of the errors in forecasting x_{t+j} on the basis of time t information x_t. To decompose these covariances into parts attributable to the individual components of w_t, we let i_τ be an N-dimensional

[4] Given matrices A and C, the impulse response function can be calculated using the MATLAB program `dimpulse.m`.

[5] For an elementary discussion of linear least squares projections, see Sargent (1987b, chapter IX).

column vector of zeroes except in position τ, where there is a one. Define a matrix $\upsilon_{j,\tau}$ by

$$\upsilon_{j,\tau} = \sum_{k=0}^{j-1} A^k C i_\tau i'_\tau C' A'^k. \tag{2.3.8c}$$

Note that $\sum_{\tau=1}^{N} i_\tau i'_\tau = I$, so that from (2.3.8$a$) and (2.3.8$c$) we have

$$\sum_{\tau=1}^{N} \upsilon_{j,\tau} = \upsilon_j.$$

Evidently, the matrices $\{\upsilon_{j,\tau}, \tau = 1, \ldots, N\}$ give an orthogonal decomposition of the covariance matrix of j-step ahead prediction errors into the parts attributable to each of the components $\tau = 1, \ldots, N$.[6]

The "innovation accounting" methods of Sims (1980) are based on (2.3.8). Sims recommends computing the matrices $\upsilon_{j,\tau}$ in (2.3.8) for a sequence $j = 0, 1, 2, \ldots$. This sequence represents the effects of components of the shock process w_t on the covariance of j-step ahead prediction errors for each series in x_t.

2.4. Transforming Variables to Uncouple Dynamics

A convenient analytical device for the analysis of linear system (2.2.1) is to uncouple the dynamics using the distinct eigenvalues of the matrix A. We use the Jordan decomposition of the matrix A:

$$A = TDT^{-1}, \tag{2.4.1}$$

where T is a nonsingular matrix and D is a matrix constructed as follows. Recall that the eigenvalues of A are the zeroes of the polynomial $\det(\zeta I - A)$. This polynomial has n zeroes because A is n by n. Not all of these zeroes are necessarily distinct, however.[7] Suppose that there are $m \leq n$ distinct zeroes

[6] For given matrices A and C, the matrices $\upsilon_{j,\tau}$ and υ_j are calculated by the MATLAB program `evardec.m`.

[7] In the case in which the eigenvalues of A are distinct, D is taken to be the diagonal matrix whose entries are the eigenvalues and T is the matrix of eigenvectors corresponding to those eigenvalues.

of this polynomial, denoted $\delta_1, \delta_2, \ldots, \delta_m$. For each δ_j, we construct a matrix D_j that has the same dimension as the number of zeroes of $\det(\zeta I - A)$ that are equal to δ_j. The diagonal entries of D_j are δ_j and the entries in the single diagonal row above the main diagonal are all either zero or one. The remaining entries of D_j are zero. Then the matrix D is block diagonal with D_j in the j^{th} diagonal block.

Transform the state vector x_t as follows:

$$x_t^* = T^{-1} x_t. \tag{2.4.2}$$

Substituting into (2.2.1), we have that

$$x_{t+1}^* = D x_t^* + T^{-1} C w_{t+1}. \tag{2.4.3}$$

Since D is block diagonal, we can partition x_t^* according to the diagonal blocks of D or, equivalently, according to the distinct eigenvalues of A. In the law of motion (2.4.3), partition j of x_{t+1}^* is linked only to partition j of x_t^*. In this sense, the dynamics of system (2.4.3) are *uncoupled*. To calculate multi-period forecasts and dynamic multipliers, we must raise the matrix A to integer powers (see (2.3.6)). It is straightforward to verify that

$$A^\tau = T(D^\tau)T^{-1}. \tag{2.4.4}$$

Since D is block diagonal, D^τ is also block diagonal, where block j is just $(D_j)^\tau$. The matrix $(D_j)^\tau$ is upper triangular with δ_j^τ on the diagonal, with all entries of the k^{th} upper right diagonal given by

$$(\delta_j)^{\tau-k}\ \tau!/[k!(\tau-k)!] \quad \text{for} \quad 0 \le k \le \tau, \tag{2.4.5}$$

and zeroes elsewhere. Consequently, raising D to an integer power involves raising the eigenvalues to integer powers. Some of the eigenvalues of A may be complex. In this case, it is convenient to use the polar decomposition of the eigenvalues. Write eigenvalue δ_j in polar form as

$$\delta_j = \rho_j \exp(i\theta_j) = \rho_j[\cos(\theta_j) + i\sin(\theta_j)] \tag{2.4.6}$$

where $\rho_j = \mid \delta_j \mid$. Then

$$\delta_j^\tau = (\rho_j)^\tau \exp(i\tau\theta_j) = (\rho_j)^\tau[\cos(\tau\theta_j) + i\sin(\tau\theta_j)]. \tag{2.4.7}$$

We shall often assume that ρ_j is less than or equal to one, which rules out instability in the dynamics. Whenever ρ_j is strictly less than one, the term $(\rho_j)^\tau$ decays to zero as $\tau \to \infty$. When θ_j is different from zero, eigenvalue j induces an oscillatory component with period $(2\pi/ \mid \theta_j \mid)$.

2.5. Examples

Next we consider some examples of processes that can be accommodated by (2.2.1).

2.5.1. Deterministic seasonals

We use (2.2.1) to represent the model $y_t = y_{t-4}$. Let $n = 4, C = 0$, $x_t = (y_t, y_{t-1}, y_{t-2}, y_{t-3})', x_0 = (0\ 0\ 0\ 1)'$,

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.5.1}$$

In this case the A matrix has four distinct eigenvalues and the absolute values of each of these eigenvalues is one. Two eigenvalues are real $(1, -1)$ and two eigenvalues are imaginary $(i, -i)$, and so have period four. The resulting sequence $\{x_t : t = 1, 2, \ldots\}$ oscillates deterministically with period four. It can be used to model deterministic seasonals in quarterly time series.

2.5.2. Indeterministic seasonals

We want to use (2.2.1) to represent the model

$$y_t = \alpha_4 y_{t-4} + w_t, \tag{2.5.2}$$

where w_t is a martingale difference sequence and $\mid \alpha_4 \mid \leq 1$. We define $x_t = [y_t,\ y_{t-1},\ y_{t-2},\ y_{t-3}]'$, $n = 4$,

$$A = \begin{bmatrix} 0 & 0 & 0 & \alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

With these definitions, (2.2.1) represents (2.5.2). This model displays an "indeterministic" seasonal. Realizations of (2.5.2) display recurrent, but aperiodic, seasonal fluctuations.

2.5.3. Univariate autoregressive processes

We can use (2.2.1) to represent the model

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \alpha_4 y_{t-4} + w_t, \tag{2.5.3}$$

where w_t is a martingale difference sequence. We set $n = 4, x_t = [y_t\ y_{t-1}\ y_{t-2}\ y_{t-3}]'$,

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix A has the form of the *companion matrix* to the vector
$[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$.

2.5.4. Vector autoregressions

Reinterpret (2.5.3) as a vector process in which y_t is a $(k \times 1)$ vector, α_j a $(k \times k)$ matrix, and w_t a $k \times 1$ martingale difference sequence. Then (2.5.3) is termed a *vector autoregression.* To map this into (2.2.1), we set $n = k \cdot 4$,

$$A = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, \; C = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where I is the $(k \times k)$ identity matrix.

2.5.5. Polynomial time trends

Let $n = 2, x_0 = [0\ 1]'$, and

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.5.4}$$

Notice that $D = A$ in the Jordan decomposition of A. It follows from (2.4.5) that

$$A^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \tag{2.5.5}$$

Hence $x_t = (t, 1)'$, so that the first component of x_t is a linear time trend and the second component is a constant.

It is also possible to use (2.2.1) to represent polynomial trends of any order. For instance, let $n = 3, C = 0, x_0 = (0, 0, 1)'$, and

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5.6}$$

Again, $A = D$ in the Jordan decomposition of A. It follows from (2.4.5) that

$$A^t = \begin{bmatrix} 1 & t & t(t-1)/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.5.7}$$

Then $x_t' = [t(t-1)/2, t, 1]$, so that x_t contains linear and quadratic time trends.

2.5.6. Martingales with drift

We modify the linear time trend example by making C nonzero. Suppose that N is one and $C' = [1\ 0]$. Since $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A^t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, it follows that

$$A^\tau C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{2.5.8}$$

Substituting into the moving-average representation (2.3.4), we obtain (2.25)

$$x_{1t} = \sum_{\tau=0}^{t-1} w_{t-\tau} + [1\ t] x_0$$

where x_{1t} is the first entry of x_t. The first term on the right-hand side of the preceding equation is a cumulated sum of martingale differences, and is called a *martingale*, while the second term is a translated linear function of time.

2.5.7. Covariance stationary processes

Next we consider specifications of x_0 and A which imply that the first two moments of $\{x_t : t = 1, 2, \ldots\}$ are replicated over time. Let A satisfy

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 1 \end{bmatrix}, \tag{2.5.9}$$

where A_{11} is an $(n-1) \times (n-1)$ matrix with eigenvalues that have moduli strictly less than one and A_{12} is an $(n-1) \times 1$ column vector. In addition, let $C' = [C_1'\ 0]$. We partition $x_t' = [x_{1t}'\ x_{2t}']$ where x_{1t} has $n-1$ entries. It follows from (2.2.1) that

$$x_{1t+1} = A_{11} x_{1t} + A_{12} x_{2t} + C_1 w_{t+1} \tag{2.5.10}$$

$$x_{2t+1} = x_{2t}. \tag{2.5.11}$$

By construction, the second component, x_{2t}, simply replicates itself over time. For convenience, take $x_{20} = 1$ so that $x_{2t} = 1$ for $t = 1, 2, \ldots$.

We can use (2.5.10) to compute the first two moments of x_{1t}. Let $\mu_t = Ex_{1t}$. Taking unconditional expectations on both sides of (2.5.10) gives

$$\mu_{t+1} = A_{11} \mu_t + A_{12}. \tag{2.5.12}$$

We can solve the nonstochastic difference equation (2.5.12) for the stationary value of μ_t. Define μ as the stationary value of μ_t, and substitute μ for μ_t and μ_{t+1} in (2.5.12). Solving for μ gives $\mu = (I - A_{11})^{-1} A_{12}$. Therefore, if

$$Ex_{10} = (I - A_{11})^{-1} A_{12}, \tag{2.5.13}$$

then Ex_{1t} will be constant over time and equal to the value on the right side of (2.5.13). Further, if the eigenvalues of A_{11} are less than unity in modulus, then starting from any initial value of μ_0, μ_t will converge to the stationary value $(I - A_{11})^{-1} A_{12}$.

Next we use (2.5.10) to compute the unconditional covariances of x_t. Subtracting (2.5.12) from (2.5.10) gives

$$(x_{1t+1} - \mu_{t+1}) = A_{11}(x_{1t} - \mu_t) + C_1 w_{t+1} \tag{2.5.14}$$

From (2.5.14) it follows that

$$\begin{aligned}(x_{1t+1} - \mu_{t+1})(x_{1t+1} - \mu_{t+1})' &= A_{11}(x_{1t} - \mu_t)(x_{1t} - \mu_t)' A_{11}' \\ &+ C_1 w_{t+1} w_{t+1}' C_1' + C_1 w_{t+1}(x_{1t} - \mu_t)' A_{11}' + A_{11}(x_{1t} - \mu_t) w_{t+1}' C_1'.\end{aligned}$$

The law of iterated expectations implies that w_{t+1} is orthogonal to $(x_{1t} - \mu_t)$. Therefore, taking expectations on both sides of the above equation gives

$$V_{t+1} = A_{11} V_t A_{11}' + C_1 C_1',$$

where $V_t \equiv E(x_{1t} - \mu_t)(x_{1t} - \mu_t)'$. Evidently, the stationary value V of the covariance matrix V_t must satisfy

$$V = A_{11} V A_{11}' + C_1 C_1'. \tag{2.5.15}$$

It is straightforward to verify that V is a solution of (2.5.15) if and only if

$$V = \sum_{j=0}^{\infty} A_{11}^j C_1 C_1' A_{11}^{j\prime}. \tag{2.5.16}$$

The infinite sum (2.5.16) converges under the condition that the eigenvalues of A_{11} are less in modulus than unity.[8] If the covariance matrix of x_{10} is V and

[8] Equation (2.5.15) is known as the discrete Lyapunov equation. Given the matrices A_{11} and C_1, this equation is solved by the MATLAB program `dlyap.m`.

the mean of x_{10} is $(I-A_{11})^{-1}A_{12}$, then the covariance and mean of x_{1t} remain constant over time. In this case, the process is said to be *covariance stationary.* If the eigenvalues of A_{11} are all less than unity in modulus, then $V_t \to V$ as $t \to \infty$, starting from any initial value V_0.

From (2.3.8) and (2.5.16), notice that if all of the eigenvalues of A_{11} are less than unity in modulus, then $\lim_{j\to\infty} v_j = V$. That is, the covariance matrix of j-step ahead forecast errors converges to the unconditional covariance matrix of x as the horizon j goes to infinity.[9]

The matrix V can be decomposed according to the contributions of each entry of the process $\{w_t\}$. Let ι_τ be an N-dimensional column vector of zeroes except in position τ, where there is a one. Then

$$I = \sum_{\tau=1}^{N} \iota_\tau \iota_\tau'. \tag{2.5.17}$$

Define a matrix $\tilde{V}_\tau$

$$\tilde{V}_\tau \equiv \sum_{j=o}^{\infty} (A_{11})^j C_1 \iota_\tau \iota_\tau' C_1' (A_{11})^{j\prime} \tag{2.5.18}$$

We have, by analogy to (2.5.15) and (2.5.16), that $\tilde{V}_\tau$ satisfies $\tilde{V}_\tau = A_{11}\tilde{V}_\tau A_{11}' + C_1 i_\tau i_\tau' C_1'$. In light of (2.5.17), (2.5.18), and (2.5.16) we have that

$$V = \sum_{\tau=1}^{N} \tilde{V}_\tau. \tag{2.5.19}$$

The matrix $\tilde{V}_\tau$ has the interpretation of being the contribution to V of the τ^{th} component of the process $\{w_t : t = 1, 2, \ldots\}$. Hence, (2.5.19) gives a decomposition of the covariance matrix V into the portions attributable to each of the underlying economic shocks.

Next, consider the autocovariances of $\{x_t : t = 1, 2, \ldots\}$. From the law of iterated expectations, it follows that

$$\begin{aligned} E[(x_{1t+\tau} - \mu)(x_{1t} - \mu)'] &= E\{E[(x_{1t+\tau} - \mu) \mid J_t](x_{1t} - \mu)'\} \\ &= E[A_{11}^\tau (x_{1t} - \mu)(x_{1t} - \mu)'] \\ &= A_{11}^\tau V. \end{aligned} \tag{2.5.20}$$

[9] The doubling algorithm described in chapter 9 can be used to compute the solution of (2.5.15) via iterations that approximate (2.5.16). The algorithm is implemented in the MATLAB programs `doublej.m` and `doublej2.m` .

Notice that this expected cross-product or *autocovariance* does not depend on calendar time but only on the gap τ between the time indices.[10] Independence of means, covariances, and autocovariances from calendar time defines *covariance stationary* processes. For the particular class of processes we are considering, if the covariance matrix does not depend on calendar time, then none of the autocovariance matrices does.

2.5.8. Multivariate ARMA processes

Specification (2.2.1) assumes that x_t contains all the information that is available at time t to forecast x_{t+1}. In many applications, vector time series are modelled as multivariate autoregressive moving-average (ARMA) processes. Let y_t be a vector stochastic process. An ARMA process $\{y_t : t = 1, 2, \ldots\}$ has a representation of the form:

$$\begin{aligned} y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \cdots + \alpha_k y_{t-k} \\ + \gamma_0 w_t + \gamma_1 w_{t-1} + \cdots + \gamma_k w_{t-k}. \end{aligned} \tag{2.5.21}$$

where $E[w_t \mid y_{t-1}, y_{t-2}, \cdots y_{t-k+1}, w_{t-1}, w_{t-2}, \cdots w_{t-k+1}] = 0$. The requirement that the same number of lags of y enter (2.5.21) as the number of lags of w is not restrictive because some coefficients can be set to zero. Hence we can think of k as being the greater of the two lag lengths. A representation such as (2.5.21) can be shown to satisfy (2.2.1). To see this, we define

$$x_t = \begin{bmatrix} y_t \\ \alpha_2 y_{t-1} + \alpha_3 y_{t-2} \cdots + \alpha_k y_{t-k+1} + \gamma_1 w_t + \gamma_2 w_{t-1} \cdots + \gamma_{k-1} w_{t-k+2} + \gamma_k w_{t-k+1} \\ \alpha_3 y_{t-1} \cdots + \alpha_k y_{t-k+2} + \gamma_2 w_t \cdots + \gamma_{k-1} w_{t-k+3} + \gamma_k w_{t-k+2} \\ \vdots \\ \alpha_k y_{t-1} + \gamma_{k-1} w_t + \gamma_k w_{t-1} \\ \gamma_k w_t \end{bmatrix} \tag{2.5.22}$$

[10] Equation (2.5.20) shows that the matrix autocovariogram of x_{1t} (i.e., $\Gamma_\tau \equiv E[(x_{1t+\tau} - \mu)(x_{1t} - \mu)']$ taken as a function of τ) satisfies the nonrandom difference equation $\Gamma_{t+1} = A_{11}\Gamma_t$ with initial condition $\Gamma_0 = V$.

$$C = \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_k \end{bmatrix} \tag{2.5.23}$$

and

$$A = \begin{bmatrix} \alpha_1 & I & \cdots & 0 \\ \alpha_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_k & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.5.24}$$

It is straightforward to verify that the resulting process $\{x_t : t = 1, 2, \ldots\}$ satisfies (2.2.1).

2.5.9. Prediction of a univariate first order ARMA

Consider the special case of (2.5.21)

$$y_t = \alpha_1 y_{t-1} + \gamma_0 w_t + \gamma_1 w_{t-1} \tag{2.5.25}$$

where y_t is a scalar stochastic process and w_t is a scalar white noise. Assume that $\mid \alpha_1 \mid < 1$ and that $\mid \gamma_1/\gamma_0 \mid < 1$. Applying (2.5.22), we define the state x_t as

$$x_t = \begin{bmatrix} y_t \\ \gamma_1 w_t \end{bmatrix}.$$

Applying (2.5.23) and (2.5.24), we have

$$C = \begin{bmatrix} \gamma_0 \\ \gamma_1 \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_1 & 1 \\ 0 & 0 \end{bmatrix}.$$

We can apply (2.3.6) to obtain a formula for the optimal j-step ahead prediction of y_t. Using (2.3.6) in the present example gives

$$E_t \begin{bmatrix} y_{t+j} \\ \gamma_1 w_{t+j} \end{bmatrix} = \begin{bmatrix} \alpha_1^j & \alpha_1^{j-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ \gamma_1 w_t \end{bmatrix}$$

which implies that

$$E_t y_{t+j} = \alpha_1^j y_t + \alpha_1^{j-1} \gamma_1 w_t. \tag{2.5.26}$$

We can use (2.5.26) to derive a famous formula of John F. Muth (1960). Assume that the system (2.5.25) has been operating forever, so that the initial time is infinitely far in the past. Then using the lag operator L, express (2.5.25) as

$$(1-\alpha_1 L)y_t = (\gamma_0 + \gamma_1 L)w_t.$$

Solving for w_t gives

$$w_t = \gamma_0^{-1}\Big(\frac{1-\alpha_1 L}{1+\frac{\gamma_1}{\gamma_0}L}\Big)\,y_t,$$

which expresses w_t as a geometric distributed lag of current and past y_t's. Substituting this expression for w_t into (2.5.26) and rearranging gives

$$E_t y_{t+j} = \alpha_1^{j-1}\Bigg[\frac{\alpha_1 + \frac{\gamma_1}{\gamma_0}}{1+\frac{\gamma_1}{\gamma_0}L}\Bigg]\,y_t.$$

In the limiting case as $\alpha_1 \to 1$ from below, this formula becomes

$$E_t y_{t+j} = \Bigg[\frac{1 + \frac{\gamma_1}{\gamma_0}}{1+\frac{\gamma_1}{\gamma_0}L}\Bigg]\,y_t, \tag{2.5.27}$$

which is independent of the forecast horizon j. In the limiting case of $\alpha_1 = 1$, it is optimal to forecast y_t for any horizon as a geometric distributed lag of past y's. This is Muth's finding that a univariate process whose first difference is a first order moving average is optimally forecast via an "adaptive expectations" scheme (i.e., a geometric distributed lag with the weights adding up to unity).

2.5.10. Growth

In much of our analysis, we assume that the eigenvalues of A have absolute values less than or equal to one. We have seen that such a restriction still allows for polynomial growth. Geometric growth can also be accommodated by suitably scaling the state vector. For instance, suppose that $\{x_t^+ : t = 1, 2, \ldots\}$ satisfies:

$$x_{t+1}^+ = A^+ x_t^+ + C w_{t+1}^+ \tag{2.5.28}$$

where $E(w_{t+1}^+ \mid J_t) = 0$ and $E[w_{t+1}^+ (w_{t+1}^+)' \mid J_t] = (\varepsilon)^t I$. The positive number ε can be bigger than one. The eigenvalues of A^+ are assumed to have absolute values that are less than or equal to $\varepsilon^{\frac{1}{2}}$, an assumption that we make to assure that the matrix A to be defined below has eigenvalues with modulus bounded above by unity. We transform variables as follows:

$$x_t = (\varepsilon)^{-\frac{t}{2}} x_t^+ \tag{2.5.29}$$

$$w_t = (\varepsilon)^{-\frac{t}{2}} w_t^+ . \tag{2.5.30}$$

The transformed process $\{w_t : t = 1, 2, \ldots\}$ is now conditionally homoskedastic as required because $E[w_{t+1}(w_{t+1})' \mid J_t] = I$. Furthermore, the transformed process $\{x_t : t = 1, 2, \ldots\}$ satisfies (2.2.1) with $A = \varepsilon^{-\frac{1}{2}} A^+$. The matrix A now satisfies the restriction that its eigenvalues are bounded in modulus by unity. The original process $\{x_t^+ : t = 1, 2, \ldots\}$ is allowed to grow over time at a rate of up to .5 log (ε).

2.5.11. A rational expectations model

Consider a model in which a variable p_t is related to a variable m_t via

$$p_t = \lambda E_t p_{t+1} + \gamma m_t , \quad 0 < \lambda < 1 \tag{2.5.31}$$

where

$$m_t = G x_t \tag{2.5.32}$$

and x_t is governed by (2.2.1). In (2.5.31), $E_t(\cdot)$ denotes $E(\cdot) \mid J_t$. This is a rational expectations version of Cagan's model of hyperinflation (here p_t is the log of the price level and m_t the log of the money supply) or a version of Le Roy and Porter's and Shiller's model of stock prices (here p_t is the stock price and m_t is the dividend). Recursions on (2.5.31) establish that a solution to (2.5.31) is $p_t = E_t \gamma \sum_{j=0}^{\infty} \lambda^j m_{t+j}$. Using (2.3.6) and (2.5.32) in this equation gives $p_t = \gamma G \sum_{j=0}^{\infty} \lambda^j A^j x_t$, or $p_t = \gamma G (I - \lambda A)^{-1} x_t$. Collecting our results, we have that (p_t, m_t) satisfies

$$\begin{gathered} \begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} \gamma G (I - \lambda A)^{-1} \\ G \end{bmatrix} x_t \\ x_{t+1} = A x_t + C w_{t+1} . \end{gathered} \tag{2.5.33}$$

System (2.5.33) embodies the cross-equation restrictions associated with rational expectations models: note that the same parameters in A, G that pin down the stochastic process for m_t also enter the equation that determines p_t as a function of the state x_t.

It is useful to show how to derive (2.5.33) using the *method of undetermined coefficients*. Returning to (2.5.31), we guess that a solution for p_t is of the form $p_t = Hx_t$, where H is a matrix to be determined. Given this guess and (2.2.1), it follows that $E_t p_{t+1} = HE_t x_{t+1} = HAx_t$. Substituting this and (2.5.32) into (2.5.31) gives $Hx_t = \lambda HAx_t + \gamma Gx_t$, which must hold for all realizations x_t. This implies that $H = \lambda HA + \gamma G$ or $H = \gamma G(I - \lambda A)^{-1}$, which agrees with (2.5.33).

2.6. The Spectral Density Matrix

Let the mean vector of x_t from the stationary distribution of an $\{x_t\}$ process be denoted μ. Define the *autocovariance function* of the $\{x_t\}$ process to be $C_x(\tau) = E[x_t - \mu]\,[x_{t-\tau} - \mu]'$. The *spectral density matrix* of the $\{x_t\}$ process is defined as

$$S_x(\omega) = \sum_{\tau=-\infty}^{\infty} C_x(\tau) e^{-i\omega\tau}. \tag{2.6.1}$$

Consider an $\{x_t\}$ process governed by (2.2.1), in which x_t is partitioned as in equations (2.5.10),(2.5.11), so that x_{2t} is the constant term. Then the spectral density can be represented as

$$S_x(\omega) = (I - A_{11} e^{-i\omega})^{-1} C_1 C_1' \, (I - A_{11}' e^{+i\omega})^{-1}. \tag{2.6.2}$$

From $S_x(\omega)$,[11] the autocovariances can be recovered via the inversion formula

$$C_x(\tau) = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} S_x(\omega) e^{+i\omega\tau}\, d\omega. \tag{2.6.3}$$

These formulas enable us to compute the spectral and cross-spectral statistics for any of the large variety of models that are special cases of (2.2.1).

---

[11] The MATLAB program `spectral.m` can be used to compute a spectral density matrix. The program requires that the position of the constant term, denoted `nnc`, in x_t be specified. The program then forms the appropriate matrices A_{11} and C_1 in equations (2.5.10),(2.5.11), and applies formula (2.6.2).

2.7. Computer Examples

We now use some MATLAB programs to generate examples that fit into the framework of this chapter.

2.7.1. Deterministic seasonal

We can use the program `dlsim.m` to simulate the model of the deterministic seasonal described above. In using `dlsim.m`, we specify four matrices A, C, G, D whose dimensions must be comparable. In particular, we require that A be $n \times n$, that C be $n \times k$, that G be $\ell \times n$, and that D be $\ell \times k$. For the case of an indeterministic seasonal, we want to create the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$G = I, D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

To accomplish this, we use the following MATLAB code

```
a= [0 0 0 1]
A= compn (a)
```

(This sets A equal to the companion matrix of a.)

```
C=  zeros(4,1)
```

(This sets C equal to a 4×1 matrix of zeros.)

```
G = eye (4)
```

(This sets G equal to the 4×4 identify matrix.)

```
D = zeros(4,1)
```

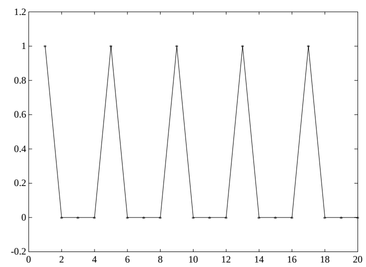

Fig. 2.7.1.a. Deterministic Seasonal.

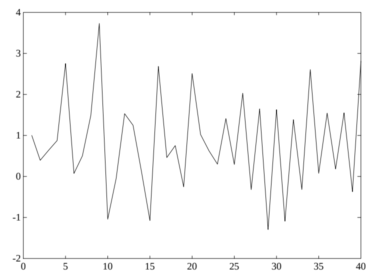

Fig. 2.7.1.b. Indeterministic Seasonal with unit root.

We want to simulate the system

$$x_{t+1} = Ax_t + Cw_{t+1}$$
$$y_t = Gx_t + Dw_{t+1}$$

with an "input" of $w_{t+1} \equiv 0$. We form an input vector w of length 20 by the statement:

$$\texttt{w} = \texttt{zeros}\,(20, 1)$$

We set the initial condition by

$$\texttt{x0} = [\texttt{1 0 0 0}]'$$

To generate the simulation, we set

$$\texttt{y} = \texttt{dlsim}(\texttt{A}, \texttt{C}, \texttt{G}, \texttt{D}, \texttt{w}, \texttt{x0}).$$

This generates the 20×4 matrix y, the i^{th} column of which is the time path taken by the i^{th} state variable (remember that $G = I$ and $D = 0$). We plot the time path of the first component of the state vector in Fig. 2.7.1.a and Fig. 2.7.1.b.

2.7.2. Indeterministic seasonal, unit root

We implement a model of an indeterministic seasonal by altering the preceding example by replacing w with a sequence of i.i.d. normal random variates. We specify that $Ew_{t+1} = 0, Ew_{t+1}^2 = 1$. We accomplish this by the MATLAB phrase

$$\texttt{w} = \texttt{randn}(\texttt{150}, \texttt{1})$$

We have generated a white noise of length 150. We create the simulation by setting

$$\texttt{C} = [\texttt{1 0 0 0}]'$$

$$\texttt{y} = \texttt{dlsim}\,(\texttt{A}, \texttt{C}, \texttt{G}, \texttt{D}, \texttt{w}, \texttt{x0}).$$

We report the first component of y in figure 2. Note the tendency of the system to display explosive oscillations. We invite the reader to calculate the variance of x_{1t} as a function of t.

2.7.3. Indeterministic seasonal, no unit root

We now set

$$\texttt{a}= [\texttt{0 0 0 .7}]$$

$$\texttt{A} = \texttt{compn}(\texttt{a})$$

With all other matrices defined as in the preceding example, we form

$$\texttt{y} = \texttt{dlsim}\,(\texttt{A}, \texttt{C}, \texttt{G}, \texttt{D}, \texttt{w}, \texttt{x0})$$

We plot the component x_{1t} in figure 2.7.2. Notice that the explosive oscillations that were present in Fig. 2.7.1.b are no longer present.

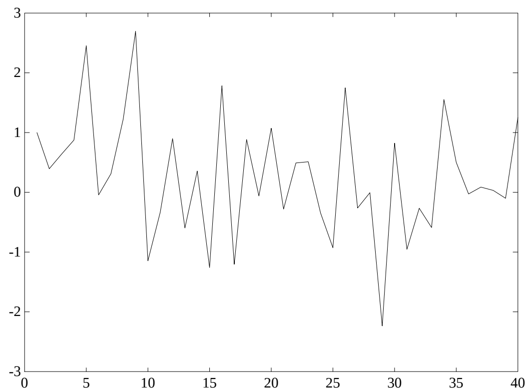

Figure 2.7.2: Indeterministic seasonal with no unit root.

2.7.4. First order autoregression

We want to simulate the first order autoregression

$$x_{1t+1} = .9x_{1t} + w_{t+1},$$

where w_{t+1} is a normally distributed white noise with unit variance. We accomplish this by modifying the MATLAB code of the previous example as follows:

```
x0= [0 0 0 0]'
 a= [.9 0 0 0]
 A= compn(a)
 y= dlsim(A, C, G, D, w, x0)
```

Fig. 2.7.3.a graphs the first component of y, which is the process $\{x_{1t}\}$.

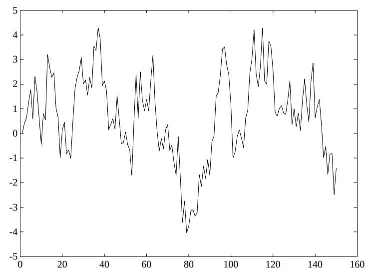

Fig. 2.7.3.a. Simulation of first-order autoregression.

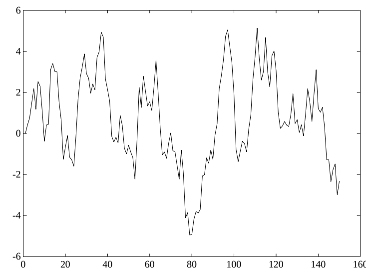

Fig. 2.7.3.b Simulation of second-order autoregression.

2.7.5. Second order autoregression

We simulate the system

$$x_{1t+1} = 1.2x_{1t} - .3x_{1t-1} + w_{t+1}$$

by modifying the code of the preceding example as follows:

```
a= [1.2  -.3  0  0]
A= compn(a)
y= dlsim(A,C,G,D,w,x0)
```

Fig. 2.7.3.b displays the output of $\{x_{1t}\}$.

2.7.6. Growth with homoskedastic noise

We want to simulate the model

$$x_{1t+1} = 1.025 x_{1t} + w_{t+1}$$

where $\{w_{t+1}\}$ continues to be a normal white noise with unit variance. We set the initial condition as $x_{10} = 5$. We modify the MATLAB code of the preceding example as follows:

```
x0= [5 0 0 0]
 a= [1.025 0 0 0 ]
 A= compn(a)
 y= dlsim(A, C, G, D, w, x0)
```

Figure 2.7.4.a displays $\{x_{1t}\}$. Notice the tendency for the randomness to die out, in the sense that the one-step ahead prediction error variance remains unity while the mean level of the process is growing exponentially at rate 1.025 per period.

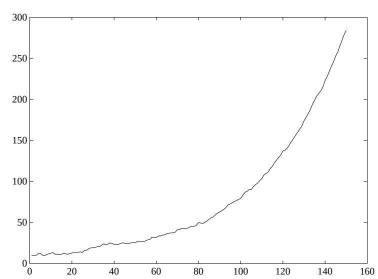

Fig. 2.7.4.a. Growth with homoskedastic noise.

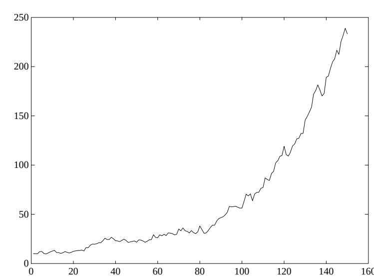

Fig. 2.7.4.b. Growth with heteroskedastic noise.

2.7.7. Growth with heteroskedastic noise

To arrest the tendency of the relative uncertainty to die out in the previous example, we modify it by setting

$$x_{1t+1} = 1.025 x_{1t} + w^*_{t+1}$$

where

$$w^*_t = 1.025^{t/2} w_t$$

and where $\{w_t\}$ continues to be a normal white noise with unit variance. This specification makes the variance of w^*_t equal to $(2.025)^t$.

To simulate this model, we modify the code of the previous example as follows:

```
n= 150
t= [1 : n]'
t= t./2
g= (1.025). ∧ t
wg= w. * g
y= dlsim (A, C, G, D, wg, x0)
```

Figure 2.7.4.b displays $\{x_{1t}\}$. Notice that the randomness now fails to die out.

2.7.8. Second order vector autoregression

We want to simulate the second order vector autoregression

$$\begin{aligned} z_{1t+1} &= .9 z_{1t} + .05 z_{1t-1} + .05 z_{2t} + .01 z_{2t-1} + w_{1t+1} \\ z_{2t+1} &= -.04 z_{1t} - .06 z_{1t-1} + .75 z_{2t} - .1 z_{2t-1} + w_{2t+1} \end{aligned}$$

where $w_t = [w_{1t}, w_{2t}]'$ is a normally distributed vector white noise with identity covariance matrix. To simulate this system, we define

$$x_t = \begin{bmatrix} z_{1t} \\ z_{1t-1} \\ z_{2t} \\ z_{2t-1} \end{bmatrix}$$

We use the MATLAB code

```
A= [.9 .05 .05 .01; 1 0 0 0; -.04 -.06 .75 -.1; 0 0 1 0]
C= [1 0 0 0; 0 0 1 0]
G= zeros (2,4)
G(1,1)= 1
G(2,3)= 1
w= randn (150,2)
D= zeros (2,2)
x0= zeros (4,1)
y= dlsim (A,C,G,D,w,x0)
```

In figure Fig. 2.7.5, we plot the first and third columns of y, which equal $\{z_{1t}, z_{2t}\}$.

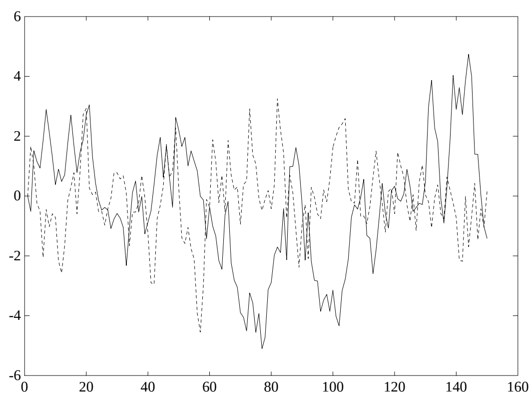

Figure 2.7.5: Simulation of second-order vector autoregression.

We can use the MATLAB command `dimpulse` to compute the impulse response of this system in response to each innovation (w_{1t}, w_{2t}). We employ the following code:

```
i1 = dimpulse(A,C,G,D,1,20)
```

This creates the response over twenty periods of the two variables z_{1t} and z_{2t} to the first innovation w_{1t}. We also use

$$\texttt{i2} = \texttt{dimpulse}(\texttt{A}, \texttt{C}, \texttt{G}, \texttt{D}, 2, 20)$$

This creates the response over twenty periods of the two variables z_{1t} and z_{2t} to the second innovation w_{2t}. We display these impulse response functions in figure 2.7.6.

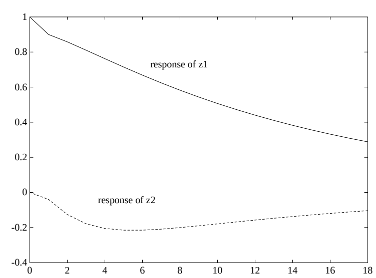

Fig. 2.7.6.a. Response to first innovation.

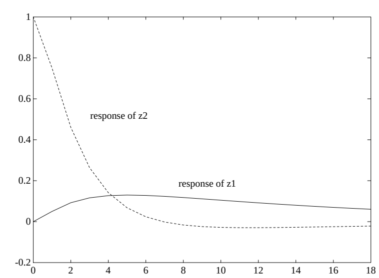

Fig. 2.7.6.b. Response to second innovation.

We can use the MATLAB file `evardec.m` to compute the decomposition of j-step ahead prediction error variances. If we want to compute this decomposition for horizons j extending from 1 to 20, we use the code:

$$[\texttt{tab1}, \texttt{tab2}] = \ \texttt{evardec}(\texttt{A}, \texttt{C}, \texttt{G}, -20, -1, \texttt{eye}(2))$$

The output in tab1 is a $20 \times (1+2)$ table. The first column records the horizon j. For $i = 1, 2$, the $(i+1)^{\text{th}}$ column records the diagonal element of v_j corresponding to the i^{th} variable z_{it}. An orthogonal decomposition of these variances into the parts attributable to w_1 and w_2 is contained in `tab2`. The first column of `tab2` records the horizon j, followed by two columns giving the diagonal element of the matrix $v_{j,1}$ defined by (2.3.8) as the j^{th} row element. Then j is repeated in the fourth column, followed by two columns giving the diagonal element of the matrix $v_{j,2}$.

2.7.9. A rational expectations model

We now indicate how to simulate the model described by equations (2.5.33). We let the variable m_t be generated by

$$m_{t+1} = 1.2m_t - .3m_{t-1} + w_{t+1}.$$

To implement this we set

```
A= compn ([1.2   -.3])
C= [1 0]'
G1= [1 0]
D=  zeros(2,1)
```

We set $\gamma = .5$ and $\lambda = .9$. To implement formula (2.5.33) we set

```
G2 = .5*G1/(eye (2) - .7*A)
```

Then we set

```
G = [G1;G2]
```

To simulate the system we set

```
x0= [1 0]'
w= rand(150, 1)
y= dlsim(A,C,G,D,w,x0)
```

The first column of y is the simulation for m, while the second is the simulation for p. We plot these in figure 2.7.7.a.

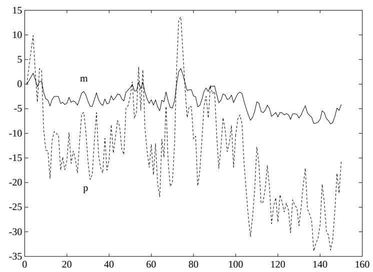

Fig. 2.7.7.a. Simulation of m and p.

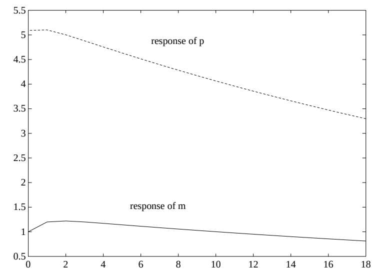

Fig. 2.7.7.b. Impulse Response to innovation in m.

Figure 2.7.7.b gives the response of p_t and m_t to an innovation in money w_1. We compute this by using

$$\texttt{y} = \texttt{dimpulse}(\texttt{A}, \texttt{C}, \texttt{G}, \texttt{D}, 1, 20)$$

To obtain a representation of the solution (2.5.33) in the vector `arma` form

$$d(L) \begin{bmatrix} p_t \\ m_t \end{bmatrix} = \begin{bmatrix} n_1(L) \\ n_2(L) \end{bmatrix} w_t,$$

we use the command

$$[\texttt{n}, \texttt{d}] = \texttt{ss2tf}(\texttt{A}, \texttt{C}, \texttt{G}, \texttt{D}, 1)$$

For our example, we obtain the output

$$\texttt{n} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3.0675 & -.8282 \end{bmatrix}$$

$$\texttt{d} = [\,1 \quad -1.2 \quad .3\,]$$

This output implies that our system has the representation

$$(1 - 1.2L + .3L^2) \begin{bmatrix} m_t \\ p_t \end{bmatrix} = \begin{bmatrix} 1 \\ 3.0675 \quad -.8282L \end{bmatrix} w_t.$$

Notice that the first row of this representation agrees with the process for m_t that we assumed.

2.8. Conclusion

In the following chapter we describe a class of economic structures with prices and quantities that can be represented in terms of a vector linear stochastic difference equation. In particular, the *state* of the economy x_t will be represented by a version of (2.2.1), while a vector y_t containing various prices and quantities will simply be linear functions of the state, i.e., $y_t = Gx_t$. The rest of this book studies how the parameters of the matrices A, C, G can be interpreted as functions of parameters that determine the preferences, technology, and information flows in the economy.

Chapter 3
The Economic Environment

This chapter describes an economic environment with five key components: a sequence of information sets, laws of motion for taste and technology shocks, a technology for producing consumption goods, a technology for producing services from consumer durables and consumption purchases, and a preference ordering over consumption services. A particular economy is selected by specifying a set of matrices A_{22}, C_2, U_b, and U_d that characterize the motion of information sets and of taste and technology shocks; matrices $\Phi_c, \Phi_g, \Phi_i, \Gamma, \Delta_k$, and Θ_k that determine the technology for producing consumption goods; and matrices $\Delta_h, \Theta_h, \Lambda$, and Π and a scalar β that determine the preference ordering over consumption goods. This chapter describes and gives examples of each component of the economic environment.

3.1. Information

Agents have a common information set at each date t. We use a vector martingale difference sequence $\{w_t : t = 1, 2, \ldots\}$ to construct the sequence of information sets $\{J_t : t = 0, 1, \ldots\}$. The initial information set J_0 is generated by a vector $x_0' = (h_{-1}', k_{-1}', z_0')$ of initial conditions, each component of which will be described subsequently. The time t information set J_t is generated by $x_0, w_1, w_2, \ldots, w_t$.

We maintain:

Assumption 1: $E(w_t \mid J_{t-1}) = 0$ and $E(w_t w_t' \mid J_{t-1}) = I$ for $t = 1, 2, \ldots$.

3.2. Taste and Technology Shocks

We use an n_z-dimensional process $\{z_t : t = 0, 1, \ldots\}$ to generate two underlying shocks in our economy. The first shock, denoted b_t, is an n_b-dimensional vector taste shock, and the second shock, denoted d_t, is an n_d-dimensional vector technology or endowment shock. These vectors of shocks are each assumed to be linear functions of the time t *exogenous state vector* z_t:

$$b_t = U_b z_t \text{ and } d_t = U_d z_t \tag{3.2.1}$$

where U_b and U_d are matrices used to select entries of z_t. The law of motion for $\{z_t : t = 0, 1, \ldots\}$ is

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1} \quad \text{for } t = 0, 1, \ldots \tag{3.2.2}$$

where z_0 is a given initial condition. We make the following technical assumption:

Assumption 2: The eigenvalues of the matrix A_{22} have absolute values that are less than or equal to one.

In chapter 2, we showed that (3.2.2) can accommodate a rich variety of time series processes. The matrices U_b and U_d can be chosen to pick off appropriate components of z_t in such a way as to make b_t or d_t follow any of those stochastic processes.

3.3. Technologies

At date t the inputs into production include a scalar household input ℓ_t, an n_k–dimensional vector k_{t-1} of capital stocks available at time t, and the vector d_t of technology shocks. The vector k_{-1} is taken as an initial condition for the economy. The outputs at time t include the time t vector of capital stocks k_t and a composite vector $\bar{o}_t$ that is partitioned into three subvectors, an n_c–dimensional vector of consumption goods c_t, an n_g–dimensional vector of intermediate goods g_t, and an n_i-dimensional vector of investment goods i_t.

The composite output vector $\bar{o}_t$ is constrained by k_{t-1} via the Leontief technology

$$\Phi \bar{o}_t = \Gamma k_{t-1} + d_t. \tag{3.3.1}$$

It is convenient to partition $\Phi = [\Phi_c \ \Phi_g \ \Phi_i]$ conformably with $\bar{o}_t$ so that an alternative representation of (3.3.1) is

$$\Phi_c c_t + \Phi_g g_t + \Phi_i i_t = \Gamma k_{t-1} + d_t. \tag{3.3.2}$$

Entries in the matrix Φ_g can be negative because intermediate goods are used in producing consumption and investment goods. We make the following assumption about Φ:

Assumption 3: $[\Phi_c \ \Phi_g]$ is nonsingular.

This assumption guarantees that the levels of consumption and intermediate goods are determined uniquely by the current period's values of investment and technology shocks and the previous period's capital stock. This assumption can readily be relaxed. Doing so would require that we alter the algorithm to be described in chapter 4 for solving the social planning problem, to accommodate a different definition of the 'control.' In practice, a technology for which assumption 3 is violated can usually be approximated arbitrarily well by another technology for which it is satisfied. We illustrate this below in our descriptions of example technologies 1 and 4.

An alternative specification, which we do not use, would replace the equality in (3.3.1) with a weak inequality, which would allow for the presence of idle capital. For some specifications of (Φ, Γ), it could then turn out to be optimal for there to be idle capital in some time periods. We will eventually describe a Lagrange multiplier on capital that indicates whether idle capital would be preferred to the outcome that we impose by insisting that (3.3.2) hold with equality.

There is an additional constraint to the production of g_t:

$$\mid g_t \mid \leq \ell_t, \tag{3.3.3}$$

where $\mid \cdot \mid$ denotes the norm of a vector. The intermediate goods vector g_t is introduced as a device for modelling symmetric adjustment costs, with the household input ℓ_t being used to measure the magnitude of these costs. In equilibrium, (3.3.3) always holds. For some interesting special cases, g_t does not enter (3.3.1) and hence ℓ_t is zero. In these cases, household inputs into production, such as labor supply, can be modeled as components of c_t.[1]

[1] It is straightforward to extend (3.3.3) to the case in which there are multiple household inputs. Suppose there is a partition g_t^j of g_t corresponding to input ℓ_t^j. Then we would assume: $g_t^j \leq \mid \ell_t^j \mid$ for all j.

Finally, investment goods are used to augment the capital stock for the subsequent time period, with capital possibly depreciating over time:

$$k_t = \Delta_k k_{t-1} + \Theta_k i_t. \tag{3.3.4}$$

We maintain:

ASSUMPTION 4: The absolute values of the eigenvalues of Δ_k are less than or equal to one.

3.4. Examples of Technologies

We provide eight illustrations of the technology (3.3.1), (3.3.3), and (3.3.4).

TECHNOLOGY 1: PURE CONSUMPTION ENDOWMENT

There is a single consumption good that cannot be stored over time. In time period t, there is an endowment d_t of this single good. There is neither a capital stock, nor an intermediate good, nor a rate of investment. Only constraint (3.3.2) is operative, and in this case it simplifies to $c_t = d_t$.

To implement this specification we could set $\Phi_c = 1, \Phi_g = 0, \Phi_i = 0, \Gamma = 0, \Delta_k = 0, \Theta_k = 0$. We can choose A_{22}, C_2, and U_d to make d_t follow any of the variety of stochastic processes described in chapter 1.

However this specification would violate assumption 3 because $[1\ 0]$ is a singular matrix. We can implement this technology by the following specification that does satisfy assumption 3:

$$c_t + i_t = d_{1t}$$

$$g_t = \phi_1 i_t$$

where ϕ_1 is a small positive number. To implement this version, we set $\Delta_k = \Theta_k = 0$ and

$$\Phi_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Phi_i = \begin{bmatrix} 1 \\ \phi_1 \end{bmatrix}, \quad \Phi_g = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad d_t = \begin{bmatrix} d_{1t} \\ 0 \end{bmatrix},$$

Evidently this specification satisfies assumption 3.

We shall eventually use this specification to create a linear-quadratic version of Lucas's (1978) asset pricing model.

TECHNOLOGY 2: SINGLE-PERIOD ADJUSTMENT COSTS

There is a single consumption good, a single intermediate good, and a single investment good. The technology obeys

$$\begin{aligned} c_t &= \gamma k_{t-1} + d_{1t} \quad &, \quad \gamma > 0 \\ \phi_1 i_t &= g_t + d_{2t} \quad &, \quad \phi_1 > 0 \\ \ell_t^2 &= g_t^2 & \\ k_t &= \delta_k k_{t-1} + i_t \quad &, \; 0 < \delta_k < 1 \end{aligned} \tag{3.4.1}$$

where d_{1t} is a random endowment of the consumption good at time t, and d_{2t} is a random disturbance to adjustment costs at time t. Given d_{2t}, investment can be increased or decreased only by adjusting the amount of the intermediate good employed. The larger is the parameter ϕ_1, the higher are adjustment costs. Employment of the intermediate good requires labor input on a one-for-one-basis. Physical capital depreciates over time.

To capture this technology, we specify

$$\Phi_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \Phi_g = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \; \Phi_i = \begin{bmatrix} 0 \\ \phi_1 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} \gamma \\ 0 \end{bmatrix}, \; \Delta_k = \delta_k, \; \Theta_k = 1.$$

We set A_{22}, C_2 and U_d to make $(d_{1t}, d_{2t})' = d_t$ follow one of the stochastic processes described in chapter 2.

This technology embodies a linear quadratic, general equilibrium version of the adjustment-cost technology used in Lucas and Prescott's [1971] model of investment under uncertainty.

TECHNOLOGY 3: MULTI-PERIOD ADJUSTMENT COSTS AND "TIME TO BUILD"

A single consumption good is produced by a single capital good. The capital good can be produced in two ways: a fast and relatively resource-intensive way, and a slow and less resource intensive way. Different amounts of intermediate goods are absorbed in producing investment goods in the fast and the slow ways.

We model this by positing that there are two capital stocks, two investment goods, and four intermediate goods, and that adjustment costs are larger for the faster investment technology.

This technology is represented as

$$c_t = \gamma k_{1t-1} + d_{1t}\ , \qquad \gamma > 0 \tag{3.4.2a}$$
$$k_{1t} = \delta_k k_{1t-1} + k_{2t-1} + i_{1t} \qquad , \qquad 0 < \delta_k < 1 \tag{3.4.2b}$$
$$k_{2t} = i_{2t} \tag{3.4.2c}$$
$$g_{1t} = \phi_1 (i_{1t} + i_{2t}) \qquad , \phi_1 > 0 \tag{3.4.2d}$$
$$g_{2t} = \phi_2 (i_{1t} + k_{2t-1}) \qquad , \phi_2 > 0 \tag{3.4.2e}$$
$$g_{3t} = \phi_3 i_{1t} \qquad , \phi_3 > 0 \tag{3.4.2f}$$
$$g_{4t} = \phi_4 i_{2t} \qquad , \phi_4 > 0 \tag{3.4.2g}$$
$$\ell_t^2 = g_t \cdot g_t \tag{3.4.2h}$$

Equation $(3.4.2a)$ describes how physical capital, k_{1t}, and an endowment shock, d_{1t}, are transformed into the consumption good. Equations $(3.4.2b)$ and $(3.4.2c)$ tell how capital, k_{1t}, can be augmented by "quick investment", i_{1t}, and by "slow investment", i_{2t}. Notice that $(3.4.2b)$ and $(3.4.2c)$ imply that physical capital, k_{1t}, is determined by

$$k_{1t} = \delta_k k_{1t-1} + i_{1t} + i_{2t-1},$$

an equation that exhibits the status of i_{1t} and i_{2t} as 'fast' and 'slow' investment processes, respectively.

Equations $(3.4.2d)$ and $(3.4.2e)$ describe how the intermediate goods, g_{1t} and g_{2t}, are required to produce investment goods. According to $(3.4.2d)$ and $(3.4.2e)$, it is as though two stages of production are required to produce capital, the first stage using intermediate good g_{1t}, and the second stage using intermediate good g_{2t}. According to $(3.4.2d)$ and $(3.4.2e)$, fast investment i_{1t} undergoes *both* stages of production in the same period t, while slow investment i_{2t} undergoes the first stage described by $(3.4.2d)$ in period t and the second stage described by $(3.4.2e)$ in period $(t+1)$.

Equations $(3.4.2f)$ and $(3.4.2g)$ describe some additional inputs of intermediate goods that are specific to the two types of investment processes. We can set $\phi_3 > \phi_4$ to capture the notion that it is more resource intensive to invest quickly. In equation $(3.4.2h)$, '$\cdot$' denotes an inner product.

To map this technology into our setup, we set

$$\Delta_k = \begin{bmatrix} \delta_k & 1 \\ 0 & 0 \end{bmatrix} \ , \ \Theta_k = I$$

$$\Phi_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ , \Phi_g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \ , \Phi_i = \begin{bmatrix} 0 & 0 \\ \phi_1 & \phi_1 \\ \phi_2 & 0 \\ \phi_3 & 0 \\ 0 & \phi_4 \end{bmatrix} \tag{3.4.3}$$

$$\Gamma = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \\ 0 & -\phi_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Recall that the matrices Φ_c, Φ_g, Φ_i multiply the vectors $c_t, [g_{1t}\ g_{2t}\ g_{3t}\ g_{4t}]'$, and $[i_{1t}\ i_{2t}]'$, respectively, while Γ multiplies the vector $[k_{1t-1}, k_{2t-1}]'$. Again, we set U_d, A_{22}, C_2 to make d_{1t} obey one of the processes described in Chapter 2.

This technology captures aspects of those used by Park (1984) and Kydland and Prescott (1982).

Technology 4: Growth

There are a single consumption good, a single investment good, a single capital good, and no intermediate good. Output obeys

$$c_t + i_t = \gamma k_{t-1} + d_t$$

where d_t is a random endowment of output at time t. The motion of capital obeys

$$k_t = \delta_k k_{t-1} + i_t .$$

To represent this technology, we could set $\Phi_c = 1,\ \Phi_i = 1,\ \Phi_g = 0,\ \Gamma = \gamma,\ \Delta_k = \delta_k,\ \Theta_k = 1$.

The reader can verify that this specification of the technology violates assumption 3 ($[\Phi_c\ \ \Phi_g]$ is singular). To analyze such an economy, we could modify some of our calculations to dispense with assumption 3. An alternative way is to approximate the technology with another one that satisfies assumption 3. In particular, assume that

$$c_t + i_t = \gamma k_{t-1} + d_{1t}$$
$$g_t = \phi_1 i_t$$
$$k_t = \delta_k k_{t-1} + i_t$$

where ϕ_1 is a very small positive number and $d_{2t} \equiv 0$. To implement this technology, set

$$\Phi_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \Phi_g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \ \Phi_i = \begin{bmatrix} 1 \\ -\phi_1 \end{bmatrix} \ , \ \Gamma = \begin{bmatrix} \gamma \\ 0 \end{bmatrix},$$

$\Delta_k = \delta_k, \ \Theta_k = 1$.[2]

This technology can be used to create a model of consumption along the lines of Hall (1978) and Flavin (1981), and a linear quadratic version of a model of capital accumulation along the lines of Cass (1965), Koopmans (1965), and Brock and Mirman (1972) . We shall also use later it to represent aspects of a model of economic growth authored by Jones and Manuelli (1988).

Technology 5: Depletable Resource

There is a single consumption good, a single investment good, two intermediate goods and one capital stock. The capital stock is the cumulative stock of the resource that has been extracted. We let investment i_t be the extraction rate, so that

$$k_t = k_{t-1} + i_t. \tag{3.4.4a}$$

All of the amount extracted is consumed, so that

$$c_t = i_t. \tag{3.4.4b}$$

There are two sources of extraction costs. The first, which is coincident with using the first intermediate good g_{1t}, depends on the amount extracted in the current time period

$$g_{1t} = \phi_1 i_t. \tag{3.4.4c}$$

[2] In effect, the modification induces investment to be associated with the use of a small (because $\phi_1 \approx 0$) amount of intermediate goods, which require labor input. The matrix $[\Phi_c \, \Phi_g]$ is now nonsingular, so that assumption 3 is satisfied. When $\phi_1 > 0$, technical conditions are satisfied that are required for the solution of the social planning problem automatically to lie in the space L_0^2 (see Chapters 4 and 5). When ϕ_1 is close to zero, the solution of the social planning problem will closely approximate the solution of the social planning problem for $\phi_1 = 0$, augmented with the restriction that the solution lie in L_0^2.

The second source of extraction costs, captured by the intermediate good g_{2t}, depends on the cumulative amount extracted at period t, which we approximate as $(i_t/2 + k_{t-1})$:[3]

$$g_{2t} = \phi_2(i_t/2 + k_{t-1}). \tag{3.4.4d}$$

To represent this technology, we set

$$\Phi_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \Phi_g = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \Phi_i = \begin{bmatrix} -1 \\ \phi_1 \\ \phi_2/2 \end{bmatrix} \ \Gamma = \begin{bmatrix} 0 \\ 0 \\ -\phi_2 \end{bmatrix},$$

$$\Delta_k = 1, \Theta_k = 1.$$

In this technology, we have included no endowment shock process d_t, so that we can take $U_d = 0, A_{22} = 0, C_2 = 0$. It would be possible to modify the technology in various ways to provide a role for an endowment or technology shock.

Such a technology was used by Hansen, Epple and Roberts [1985] to study alternative arrangements for an exhaustible resource market.

TECHNOLOGY 6: LEARNING BY DOING

There is a single consumption good, a single investment good, a single intermediate good, and a single capital stock. The capital stock is interpreted as the cumulative stock of knowledge, the accumulation of which requires expenditure of current output and the intermediate good. Thus, we set

$$\begin{aligned} c_t + i_t &= \gamma_1 k_{t-1} + d_t \\ k_t &= \delta_k k_{t-1} + (1 - \delta_k) i_t \end{aligned} \tag{3.4.5}$$

Setting $\Theta_k = (1 - \delta_k)$ makes k_t a weighted average of current and past rates of investment. Possession of knowledge (capital) lowers the amount of intermediate goods required to accumulate more knowledge:

$$g_t = \phi i_t - \gamma_2 k_{t-1},$$

where $\phi \geq \gamma_2 > 0$.

[3] We add half the current extraction rate i_t to k_{t-1} to approximate the average amount over the period that has been extracted cumulatively.

To represent this economy, we set

$$\begin{aligned}\Phi_c &= \begin{bmatrix}1\\0\end{bmatrix} \ , \ \Phi_g = \begin{bmatrix}0\\-1\end{bmatrix} \ , \ \Phi_i = \begin{bmatrix}1\\\phi\end{bmatrix}\\ \Gamma &= \begin{bmatrix}\gamma_1\\\gamma_2\end{bmatrix}, \ \Delta_k = \delta_k, \Theta_k = (1-\delta_k).\end{aligned} \tag{3.4.6}$$

Technology 7: Fixed Proportions

There is a single consumption good, a single capital good, and a single "intermediate good" to be interpreted as labor. Labor and capital are required in fixed proportions, apart from the effects of a random "labor-requirements" shock d_{2t}. The technology requires

$$\begin{aligned}c_t + i_t &= \gamma_1 \, k_{t-1} + d_{1t}\\ g_t &= \gamma_2 \, k_{t-1} + d_{2t}\\ g_t^2 &= \ell_t^2\\ k_t &= \delta_k \, k_{t-1} + i_t.\end{aligned}$$

Here g_t represents employment of labor input. The parameter γ_2 determines the nonstochastic part of the capital-labor ratio.

To map this technology into our setup, we set

$$\begin{aligned}\Phi_c &= \begin{bmatrix}1\\0\end{bmatrix}, \ \Phi_i = \begin{bmatrix}1\\0\end{bmatrix}, \ \Phi_g = \begin{bmatrix}0\\1\end{bmatrix},\\ \Gamma &= \begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}, \ \Delta_k = \delta_k, \ \Theta_k = 1.\end{aligned}$$

Technology 8: Interrelated Factor Demand with Costs of Adjustment

To produce output requires physical capital, k_{1t}, and labor, k_{2t}. It is costly to adjust the stock of either factor of production. To adjust capital, the intermediate good g_{1t} must be employed, while to adjust labor, the intermediate good g_{2t} must be employed. To implement this technology, we require $k_{2t} = g_{3t}$, which identifies k_{2t} with the direct input of labor. The technology satisfies

$$\begin{aligned}
c_{1t} + i_t &= [\gamma_1 \, \gamma_2] \begin{bmatrix} k_{1t-1} \\ k_{2t-1} \end{bmatrix} + d_{1t} \\
k_{1t} &= \delta_k k_{1t-1} + i_{1t} \\
k_{2t} &= k_{2t-1} + i_{2t} \\
g_{1t} &= \phi_2 i_{1t} \\
g_{2t} &= \phi_3 i_{2t} \\
g_{3t} &= k_{2t}.
\end{aligned}$$

When $\phi_3 < \phi_2$, it is more costly to adjust capital than labor. To capture this technology, we set

$$\Delta_k = \begin{bmatrix} \delta_k & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \Theta_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Phi_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Phi_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \phi_2 & 0 \\ 0 & \phi_3 \end{bmatrix}, \quad \Phi_g = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; \Gamma = \begin{bmatrix} \gamma_1 & \gamma_2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This technology is a version of one used by Mortensen [1973] and Hansen and Sargent [1981].

3.4.1. Other technologies

Alternative technologies can be constructed that blend features of two or more of those described here. For instance, multiple-period adjustment costs can be incorporated into the growth technology, while learning by doing can be introduced into one of the adjustment cost technologies. Also versions of these single consumption good technologies can be combined to yield technologies for the production of multiple consumption goods.

3.5. Preferences and Household Technologies

We assume a representative household. We postpone until Chapter 12 discussing ways that heterogeneity among consumers can be accommodated within this assumption. We describe preferences in terms of two elements. First we describe a household technology for accumulating a vector of household capital and for using it to produce a vector of consumption services. Then we specify intertemporal preferences for consumption services in different dates and states of the world.

We assume that there is an n_h–dimensional vector of household capital stocks h_{t-1} brought into time t. The vector h_{-1} is taken as an initial condition. The vectors of consumption goods c_t and household capital stocks h_{t-1} are inputs into the household technology at time t. The outputs of this technology are an n_s–dimensional vector of household services s_t and a new vector of stocks of household capital h_t. The relation between inputs and outputs is described by

$$h_t = \Delta_h h_{t-1} + \Theta_h c_t \tag{3.5.1}$$

and

$$s_t = \Lambda h_{t-1} + \Pi c_t. \tag{3.5.2}$$

We maintain the following technical assumption:[4]

Assumption 5: The absolute values of the eigenvalues of Δ_h are less than or equal to one.

Preferences are defined over stochastic processes for household services and household inputs into production. These preferences are separable across components of services, across states of the world, and over time. In particular, preferences are described by the quadratic utility functional:

$$-(\frac{1}{2})E\sum_{t=0}^{\infty}\beta^t\left[(s_t-b_t)\cdot(s_t-b_t)+(\ell_t)^2\right] \mid J_0 \quad , \; 0<\beta<1. \tag{3.5.3}$$

where β is a subjective discount factor.

The household services in this economy play the role of characteristics or attributes in the analyses of Gorman (1980) and Lancaster (1966). We can think

[4] The purpose of this assumption is to assure that under the equilibrium (optimal) decision rule, the state vector for the economy has a transition matrix that is 'stable'.

of consumption c_t at date t as generating a bundle of consumption services in current and future time periods. Thus, the consumption vector c_t generates a vector Πc_t of consumption services at time t and a vector $\Lambda(\Delta_h)^{j-1}\Theta_h c_t$ of consumption services at time $t+j$, for $j \geq 1$. In effect, the household technology puts time and component nonseparabilities into the indirect preference ordering for consumption goods induced by (3.5.3). We do not impose nonnegativity constraints on consumption goods.

3.6. Examples of Household Technology Preference Structures

We describe five examples of household technology-preference structures.

HOUSEHOLD TECHNOLOGY 1: TIME SEPARABILITY

There is a single consumption good which is identical with the single service. There is no household capital. Preferences are described by

$$-\frac{1}{2}E\sum_{t=0}^{\infty}\beta^t\left[(c_t-b_t)^2+\ell_t^2\right] \mid J_0 \quad , \; 0<\beta<1 \tag{3.6.1}$$

where ℓ_t is labor supplied in period t and b_t is a stochastic "bliss point". Notice that when c_t is less than b_t, utility is increasing in consumption. Typically, we would try to specify the parameters of the b_t process and the household and production technologies so that in equilibrium c_t is usually less than b_t.

HOUSEHOLD TECHNOLOGY 2: CONSUMER DURABLES

There are a single consumption good and a single service. A single durable household good obeys

$$h_t = \delta_h h_{t-1} + c_t \quad , \; 0<\delta_h<1.$$

Services at t are related to the stock of durables at the beginning of the period:

$$s_t = \lambda h_{t-1} \; , \; \lambda > 0.$$

Preferences are described by

$$-\frac{1}{2} E \sum_{t=0}^{\infty} \beta^t \left[(\lambda h_{t-1} - b_t)^2 + \ell_t^2 \right] \mid J_0, \tag{3.6.2}$$

where b_t is again a univariate stochastic process that represents a stochastic bliss point. We intend to set parameters so that $(\lambda h_{t-1} - b_t)$ is ordinarily negative, so that utility is rising in consumption services λh_{t-1}.

To implement these preferences, we would set $\Delta_h = \delta_h, \Theta_h = 1, \Lambda = \lambda, \Pi = 0$.

Household Technology 3: Habit Persistence

There is a single consumption good, a single consumption service, and a single household capital stock which is a weighted average of consumption in previous time periods. We want preferences to be

$$-\frac{1}{2} E \sum_{t=0}^{\infty} \beta^t [(c_t - \lambda(1-\delta_h) \sum_{j=0}^{\infty} \delta_h^j c_{t-j-1} - b_t)^2 + \ell_t^2], \\ 0 < \beta < 1 \ , \ 0 < \delta_h < 1 \ , \ \lambda > 0. \tag{3.6.3}$$

Here the bliss point is in effect $b_t + \lambda(1-\delta_h) \sum_{j=0}^{\infty} \delta_h^j c_{t-j-1}$, so that the bliss point shifts in response to a moving average of past consumption. Preferences in this form require an initial condition for the geometric sum $\sum_{j=0}^{\infty} \delta_h^j c_{t-j-1}$, which we specify as an initial condition for the 'stock of household durables,' h_{-1}.

To implement these preferences, let the household capital stock be

$$h_t = \delta_h h_{t-1} + (1-\delta_h) c_t \quad , \ 0 < \delta_h < 1.$$

This implies that

$$h_t = (1-\delta_h) \sum_{j=0}^{t} \delta_h^j c_{t-j} + \delta_h^{t+1} h_{-1}$$

Let consumption services be

$$s_t = -\lambda h_{t-1} + c_t \quad , \ \lambda > 0.$$

We can represent the desired preferences by setting $\Lambda = -\lambda, \ \Pi = 1, \ \Delta_h = \delta_h, \ \Theta_h = 1 - \delta_h$.

The parameter λ governs the strength of habit persistence. When $\lambda = 0$, we recover a version of household technology 1.

Household technology-preferences 3 is a version of the model of habit persistence of Ryder and Heal [1973]. Later we shall use this specification to represent aspects of some ideas of Jones and Manuelli [1988].

Household Technology 4: Adjustment Costs

There is a single consumption good, a single household capital stock equal to consumption, and two consumption services. We want to represent preferences of the form

$$-\frac{1}{2}E\sum_{t=0}^{\infty}\beta^t[(c_t-b_{1t})^2+\lambda^2(c_t-c_{t-1})^2+\ell_t^2] \mid J_0 \qquad (3.6.4)$$
$$0<\beta<1 \quad , \ \lambda>0$$

where b_{1t} is a stochastic bliss process, intended ordinarily to exceed c_t. A consumer with these preferences prefers more c_t to less, but dislikes variability of consumption, as represented by the term $\lambda^2(c_t-c_{t-1})^2$.

To capture such preferences, we set

$$h_t = c_t$$

$$s_t = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} h_{t-1} + \begin{bmatrix} 1 \\ \lambda \end{bmatrix} c_t$$

so that

$$s_{1t} = c_t$$

$$s_{2t} = \lambda(c_t - c_{t-1})$$

We set the first component b_{1t} of b_t to capture the stochastic bliss process, and set the second component identically equal to zero. Thus, we set $\Delta_h = 0, \Theta_h = 1$,

$$\Lambda = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix} \ , \ \Pi = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}.$$

This specification captures a linear-quadratic version of Houthakker and Taylor's (1970) model of adjustment costs or habit persistence.

Household Technology 5: Multiple Consumption Goods

There are two consumption goods and two consumption services. The first consumption service is proportional to the first consumption good, and the second consumption service is a linear combination of the two consumption goods. As in household technology 1, preferences for consumption goods are time separable. There are no durable household goods. We specify

$$\Lambda = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \Pi = \begin{bmatrix} \pi_1 & 0 \\ \pi_2 & \pi_3 \end{bmatrix} \tag{3.6.5}$$

Although preferences for consumption goods are state and date-separable, they are not separable across components. Following Frisch (1932), Heckman and MaCurdy (1980), and Browning, Deaton, and Irish (1985), it is convenient to exploit the separability across time and states and to analyze the implied consumption demands in each state of the world and time period separately. For a given state of the world and time period t, the contribution of c_t to the utility function is

$$-\frac{1}{2}\beta^t(\Pi c_t - b_t)'(\Pi c_t - b_t). \tag{3.6.6}$$

The corresponding marginal utility vector mu_t for consumption is then

$$mu_t = -\beta^t[\Pi'\Pi\, c_t - \Pi'\, b_t]. \tag{3.6.7}$$

Solving (3.6.7) for c_t in terms of mu_t and b_t gives

$$c_t = -(\Pi'\Pi)^{-1}\beta^{-t} mu_t + (\Pi'\Pi)^{-1}\Pi' b_t \tag{3.6.8}$$

Relation (3.6.8) is referred to as the *Frisch demand function* for consumption. We can think of the vector mu_t as playing the role of prices, up to a common factor, for all dates and states. The scale factor is determined by the choice of numeraire.[5]

Notions of substitutes and complements can be defined in terms of these Frisch demand functions. Two goods can be said to be substitutes if the cross-price effect is positive and to be complements if this effect is negative. Hence this classification is determined by the off-diagonal element of $-(\Pi'\Pi)^{-1}$, which

[5] Frisch demand functions are different from Marshallian and Hicks demand functions. In Frisch demand functions, compensation is required to hold the marginal utility of the numeraire good constant.

is equal to $\pi_2\pi_3/\det(\Pi'\Pi)$. If π_2 and π_3 have the same sign, the goods are substitutes. If they have opposite signs, the goods are complements.

This household technology can be modified to incorporate features of the first four household technologies for each of the consumption goods.

3.7. Constraints to Keep the Solutions "Square Summable"

To complete our description of the economic environment, we impose the following additional constraints on the two endogenous state vectors h_t and k_t:

$$E\sum_{t=0}^{\infty}\beta^t h_t \cdot h_t \mid J_0 < \infty \quad \text{and} \quad E\sum_{t=0}^{\infty}\beta^t k_t \cdot k_t \mid J_0 < \infty. \tag{3.7.1}$$

We define the space

$$L_0^2 = [\{y_t\} : y_t \text{ is a random variable in } J_t \text{ and } E\sum_{t=0}^{\infty}\beta^t y_t^2 \mid J_0 < +\infty].$$

We can express (3.7.1) by saying that each component of h_t and each component of k_t belongs to L_0^2.

These restrictions substitute for terminal conditions on the capital stocks. For many specifications of our model, constraints (3.7.1) are redundant because it is optimal for a social planner to stabilize the economy. For such specifications a set of transversality conditions implying (3.7.1) are among the first-order necessary conditions for the planner's problem. For some other specifications, however, the transversality conditions do not imply (3.7.1). For those specifications, we impose (3.7.1) as an additional constraint to give a sensible economic interpretation to the problem.[6] For such specifications, imposing (3.7.1) can be justified informally as a practical way of approximating solutions with nonnegativity constraints on capital stocks.

[6] See the discussion of Hall's model in chapter 4 for an illustration.

3.8. Summary

Information flows in our economy are governed by an exogenous stochastic process z_t that follows

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1},$$

where w_{t+1} is a martingale difference sequence. Preference shocks b_t and technology shocks d_t are linear functions of z_t:

$$\begin{aligned} b_t &= U_b z_t \\ d_t &= U_d z_t \end{aligned}$$

The matrices A_{22}, C_2, U_b, and U_d characterize the laws of motion of b_t and d_t.

There is the following technology for producing consumption goods:

$$\begin{aligned} \Phi_c c_t + \Phi_g g_t + \Phi_i i_t &= \Gamma k_{t-1} + d_t \\ k_t &= \Delta_k k_{t-1} + \Theta_k i_t \\ g_t \cdot g_t &= \ell_t^2 \end{aligned}$$

Here c_t is a vector of consumption goods, g_t a vector of intermediate goods, i_t a vector of investment goods, k_t a vector of physical capital goods, and ℓ_t an amount of labor supplied by the representative household. The matrices $\Phi_c, \Phi_g, \Phi_i, \Gamma, \Delta_k$, and Θ_k determine a particular technology.

Preferences of a representative household are described by

$$\begin{aligned} &-(\frac{1}{2}) E \sum_{t=0}^{\infty} \beta^t [(s_t - b_t) \cdot (s_t - b_t) + \ell_t^2],\ 0 < \beta < 1 \\ &s_t = \Lambda h_{t-1} + \Pi c_t \\ &h_t = \Delta_h h_{t-1} + \Theta_h c_t \end{aligned}$$

where s_t is a vector of consumption services, and h_t is a vector of household capital stocks. A particular set of preferences is specified by naming the matrices $\Lambda, \Pi, \Delta_h, \Theta_h$, and the scalar β.

Having specified the structure of information, technology, and preferences, we must tell how the economy allocates resources in light of what is technically possible and what people want. We do this in the coming chapters.

Chapter 4
Optimal Resource Allocation

We eventually want to use our models to study aspects of competitive equilibria, including time series properties of various spot market prices, asset prices, and rates of return. The first welfare theorem makes competitive equilibrium allocations solve a particular resource allocation problem, which in our setting is a linear-quadratic optimal control problem.

In this chapter, we state the optimal resource allocation problem, and compare two methods for solving it. The first method uses state and date-contingent Lagrange multipliers; the second uses dynamic programming. The first method exposes the direct connection between the Lagrange multipliers and the equilibrium prices in a competitive economy to be analyzed in chapter 6. The second method provides good algorithms for calculating both the law of motion for the optimal quantities and the Lagrange multipliers.

We also describe a set of MATLAB programs that solve the social planning problem and that represent its solution in various ways. We use these programs to solve the social planning problem for six sample economies that are formed by choosing particular examples of the ingredients that were described in chapter 3.

4.1. Planning problem

The social planning problem is to maximize the representative household's utility subject to the resource constraints described in chapter 3. Constraint (3.5) can be substituted directly into the objective function (3.16) to yield

$$-(1/2)E\sum_{t=0}^{\infty}\beta^t[(s_t-b_t)\cdot(s_t-b_t)+g_t\cdot g_t]. \tag{4.1.1}$$

The remaining constraints are all linear:

$$\begin{aligned}\Phi_c c_t+\Phi_g\, g_t+\Phi_i i_t&=\Gamma k_{t-1}+d_t,\\ k_t&=\Delta_k k_{t-1}+\Theta_k i_t,\\ h_t&=\Delta_h h_{t-1}+\Theta_h c_t,\\ s_t&=\Lambda h_{t-1}+\Pi c_t,\end{aligned} \tag{4.1.2}$$

and

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1},\ b_t = U_b z_t \ \text{ and } \ d_t = U_d z_t \tag{4.1.3}$$

for $t = 0, 1, \ldots$ where h_{-1}, k_{-1}, and z_0 are given as initial conditions. The process $\{z_t : t = 0, 1, \ldots)$ is *uncontrollable* in the sense that the social planner cannot influence its evolution. The planner's problem is to choose stochastic processes $\{c_t, s_t, g_t, i_t, k_t, h_t\}_{t=0}^{\infty}$ that maximize (4.1.1) subject to (4.1.2), (4.1.3), and the given initial conditions. All components of the processes chosen by the planner are required to be in the space L_0^2 given by

$$\begin{aligned} L_0^2 = \{y : y_t \ \text{ is in } \ J_t \text{ for } \ t = 0, 1, \ldots, \ \text{and} \\ E \sum_{t=0}^{\infty} \beta^t y_t^2 \mid J_0 < \infty\}. \end{aligned} \tag{4.1.4}$$

Among other things, this requires that the time t decisions depend only on information available at time t.

4.2. Lagrange Multipliers

Our first approach to solving the constrained optimization problem uses Lagrange multipliers. We begin by focussing on the linear constraints given in (4.1.2) and the constraints in (4.1.3) that determine the evolution of the process governing taste and technology shocks. The constraints in (4.1.2) are indexed explicitly by the calendar date t and implicitly by the state of the world (w^t, x_0), where $w^t = (w_1, w_2, \ldots, w_t)$. Associated with these constraints are four vector multiplier processes $\{\mathcal{M}_t^d\}, \{\mathcal{M}_t^k\}, \{\mathcal{M}_t^h\}$, and $\{\mathcal{M}_t^s\}$. Because the constraints are required to hold in all states of the world, the multipliers are stochastic processes, the time t values of which are functions of the state of the world (w^t, x_0). The components of the multiplier processes are in L_0^2.[1]

[1] Chapter 6 discusses the space of stochastic processes in which there exist equilibrium prices that can be used to decentralize the economy. The discussion in chapter 6 also pertains to the Lagrange multipliers of this chapter. The equilibrium prices and the Lagrange multipliers both live in L_0^2.

To calculate the solution to the optimal resource allocation problem, we find the saddle point of the Lagrangian:[2]

$$
\begin{aligned}
\mathcal{L} = -E \sum_{t=0}^{\infty} \beta^t \bigg[& (\frac{1}{2})[(s_t - b_t) \cdot (s_t - b_t) + g_t \cdot g_t] \\
& + \mathcal{M}_t^{d\prime} \cdot (\Phi_c c_t + \Phi_g g_t + \Phi_i i_t - \Gamma k_{t-1} - d_t) \\
& + \mathcal{M}_t^{k\prime} \cdot (k_t - \Delta_k k_{t-1} - \Theta_k i_t) \\
& + \mathcal{M}_t^{h\prime} \cdot (h_t - \Delta_h h_{t-1} - \Theta_h c_t) \\
& + \mathcal{M}_t^{s\prime} \cdot (s_t - \Lambda h_{t-1} - \Pi c_t) \bigg] \mid J_0.
\end{aligned} \tag{4.2.1}
$$

The social planner solves the saddle point problem by choosing contingency plans (stochastic processes) for $\{c_t, g_t, h_t, i_t, k_t, s_t\}$, and for the multipliers $\{\mathcal{M}_t^d\}$, $\{\mathcal{M}_t^k\}$, $\{\mathcal{M}_t^h\}$, and $\{\mathcal{M}_t^s\}$. Each of these objects must be an element of L_0^2.

---

[2] In obtaining the first order conditions for the optimization of (4.5), it is useful to remember the integration operation represented by the conditional expectation operator $E(\cdot \mid J_0)$. Let $f^t(w^t, x_0)$ be the density of (w^t, x_0). Then the representation (4.5) for the Lagrangian is equivalent with

$$
\begin{aligned}
\mathcal{L} = -\sum_{t=0}^{\infty} \beta^t \int \bigg\{ & [(\frac{1}{2})[(s_t - b_t) \cdot (s_t - b_t) + g_t \cdot g_t] \\
& + \mathcal{M}_t^{d\prime}(\Phi_c c_t + \Phi_g g_t + \Phi_i i_t - \Gamma k_{t-1} - d_t) \\
& + \mathcal{M}_t^{k\prime}(k_t - \Delta_k k_{t-1} - \Theta_k i_t) + \mathcal{M}_t^{h\prime}(h_t - \Delta_h h_{t-1} - \Theta_h c_t) \\
& + \mathcal{M}_t^{s\prime}(s_t - \Lambda_h h_{t-1} - \Pi c_t) \bigg\} f^t(w^t, x_0) dw^t.
\end{aligned} \tag{4.2.0$'$}
$$

In this expression, it is understood that each element of $\{s_t,\ b_t, g_t,\ c_t, i_t\ \ k_t, d_t,\ \mathcal{M}_t^d,\ \mathcal{M}_t^k,\ \mathcal{M}_t^h,\ \mathcal{M}_t^k\}$ is to be regarded as a function of (w^t, x_0). The planner is to choose stochastic processes that make each element of $\{c_t, s_t,\ g_t, i_t,\ k_t, h_t,\ \mathcal{M}_t^d,\ \mathcal{M}_t^k,\ \mathcal{M}_t^h,\ \mathcal{M}_t^k\}$ a function of (w^t, x_0), taking as given the initial state vector x_0 and the stochastic processes for b_t and d_t. Expression $(4.2.0')$ emphasizes the fact that each constraint in $(4.1.2)$ applies for each t and each (w^t, x_0), and that a distinct multiplier is attached to each constraint for each (w^t, x_0). In obtaining the first order necessary conditions for an optimum, it is in effect necessary to differentiate $(4.2.0')$ with respect to each choice variable of the planner for each contingency (w^t, x_0). Thus for $(4.2.0)$ the first order condition with respect to $k_t(w^t, x_0)$ is $\beta^t \mathcal{M}_t^k f^t(w^t, x_0) - \beta^{t+1} \int (\Delta_k' \mathcal{M}_{t+1}^k + \Gamma' \mathcal{M}_{t+1}^{d\prime}) f^{t+1}(w^{t+1}, x_0) dw^{t+1} = 0$ or $\beta^t \mathcal{M}_t^k - r\beta^{t+1} \int \left(\Delta_k' \mathcal{M}_{t+1}^k + \Gamma' \mathcal{M}_{t+1}^{d\prime} \right) \frac{f^{t+1}(w^{t+1}, x_0)}{f^t(w^t, x_0)} dw^{t+1} = 0$ or $\beta^t \mathcal{M}_t^k - \beta^t E(\Delta_k' \mathcal{M}_{t+1}^k + \Gamma' \mu_{t+1}^{d\prime}) \mid J_t = 0$. This is the first order condition for k_t displayed in (4.8).

First order necessary conditions can be deduced by computing *Gateaux* or *directional* derivatives around a putative optimum, and by setting them all to zero.[3]

The method of directional derivatives can be illustrated as follows. Let c_t^o be the optimal plan for consumption. Consider a class of admissible perturbations around c_t^o of the form $c_t^o + r\alpha_t$, where r is an arbitrary real number and α_t is an n_c-dimensional random vector in J_t with finite second moments. The vector α_t gives the direction of the derivative. For any direction α_t, we want the optimal setting of r to be zero. We replace c_t^o by the perturbation $c_t^o + r\alpha_t$ in the objective function, differentiate with respect to r, evaluate the result at $r = 0$ and set it equal to zero. This results in

$$-\beta^t E[\alpha_t'(\Phi_c'\mathcal{M}_t^d - \Theta_h'\mathcal{M}_t^h - \Pi'\mathcal{M}_t^s)] = 0, \tag{4.2.2}$$

where we have evaluated the derivative with respect to r at the optimal choice of r, namely $r = 0$. Since α_t can be chosen to be any n_c-dimensional random vector in J_t, (4.2.2) can be satisfied only if $\Phi_c'\mathcal{M}_t^d - \Theta_h'\mathcal{M}_t^h - \Pi'\mathcal{M}_t^s$ is identically zero in every state of nature.[4]

It is useful to illustrate how this method applies to the determination of first order conditions for the terms h_t and k_t, each of which makes two appearances under the sum in (4.2.1), namely as h_t and h_{t-1}, and as k_t and k_{t-1}, respectively. We shall indicate how things work for k_t. Let α_t now be of the same dimension as k_t. For each $t \geq 0$, the terms involving k_t in the sum (4.2.1) are

$$\begin{aligned} &E\{\beta^t \mathcal{M}_t^{k\prime}(k_t^o + r\alpha_t) \\ &\quad - \beta^{t+1}[\mathcal{M}_{t+1}^{k\prime}\Delta_k(k_t^o + r\alpha_t) + \mathcal{M}_{t+1}^{d\prime}\Gamma(k_t^o + r\alpha_t)]\}, \end{aligned}$$

where k_t^o is the optimal capital sequence. Differentiating this expression with respect to r and setting the result to zero for $r = 0$ gives

$$\beta^t E\alpha_t'\{\mathcal{M}_t^k - \beta[\Delta_k'\mathcal{M}_{t+1}^k + \Gamma'\mathcal{M}_{t+1}^d]\} = 0. \tag{4.2.3}$$

---

[3] An alternative approach is to compute *Frechet* derivatives of the Lagrangian (4.2.1) with respect to the stochastic process $\{c_t, g_t, h_t, i_t, k_t, s_t, \mathcal{M}_t^d, \mathcal{M}_t^k, \mathcal{M}_t^h, \mathcal{M}_t^s\}$. These derivatives are taken with respect to entire stochastic processes. To use this approach, we would have to define a sense of differentiation for criterion functions that depend on elements in L_0^2. Such a construction turns out to be straightforward in our context and exploits that fact that the space L_0^2 is a Hilbert space.

[4] See Chapter 6 for further discussion that is pertinent to understanding the use of stochastic Lagrange multipliers.

This equation must hold for all directions α_t that can be chosen as functions of time t information (w^t, x_0). This implies that

$$\mathcal{M}_t^k - E\beta[\Delta_k' \mathcal{M}_{t+1}^k + \Gamma' \mathcal{M}_{t+1}^d] \mid J_t = 0.$$

Applying the law of iterated expectations to the above equation gives

$$E\{\mathcal{M}_t^k - \beta[\Delta_k' \mathcal{M}_{t+1}^k + \Gamma' \mathcal{M}_{t+1}^d]\} = 0,$$

which implies (4.2.3).

In this way, we can compute first-order necessary conditions for all of the processes to be chosen by the social planner. The first-order necessary conditions for maximization with respect to c_t, g_t, h_t, i_t, k_t, and s_t, respectively, are:

$$\begin{aligned} -\Phi_c' \mathcal{M}_t^d + \Theta_h' \mathcal{M}_t^h + \Pi' \mathcal{M}_t^s &= 0, \\ -g_t - \Phi_g' \mathcal{M}_t^d &= 0, \\ -\mathcal{M}_t^h + \beta E(\Delta_h' \mathcal{M}_{t+1}^h + \Lambda' \mathcal{M}_{t+1}^s) \mid J_t &= 0, \\ -\Phi_i' \mathcal{M}_t^d + \Theta_k' \mathcal{M}_t^k &= 0, \\ -\mathcal{M}_t^k + \beta E(\Delta_k' \mathcal{M}_{t+1}^k + \Gamma' \mathcal{M}_{t+1}^d) \mid J_t &= 0, \\ -s_t + b_t - \mathcal{M}_t^s &= 0 \end{aligned} \tag{4.2.4}$$

for $t = 0, 1, \ldots$. In addition, we have the transversality conditions

$$\begin{aligned} \lim_{t\to\infty} \beta^t E[\mathcal{M}_t^{k\prime} k_t] \mid J_0 = 0 \\ \lim_{t\to\infty} \beta^t E[\mathcal{M}_t^{h\prime} h_t] \mid J_0 = 0. \end{aligned} \tag{4.2.5}$$

By way of enforcing (4.2.5), we impose the additional condition that each of the processes $\{c_t, g_t, h_t, i_t, k_t, s_t\}$ belongs to the space L_0^2. This requirement is *stronger* than the transversality conditions, and makes the transversality conditions redundant. In an extended example in an appendix to this chapter, we illustrate the connection between the transversality conditions and the requirement that elements of the solution lie in L_0^2.[5]

The optimal plan can now be computed by solving the stochastic expectational difference equation system formed by augmenting (4.1.2)–(4.1.3) with

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[5] In (4.2.4), we have abused notation a little bit. The Lagrangian is defined in terms of general processes for $\{c_t, g_t, h_t, i_t, k_t, s_t, \mathcal{M}_t^d, \mathcal{M}_t^k, \mathcal{M}_t^h, \mathcal{M}_t^s\}$. In (4.2.4) we have used the same notation for the general processes and the *optimal* choices of these processes.

(4.2.4). This system is to be solved jointly for the process $\{c_t, g_t, h_t, i_t, k_t, s_t, \mathcal{M}_t^d, \mathcal{M}_t^k, \mathcal{M}_t^h, \mathcal{M}_t^s, z_t\}$ subject to the initial conditions for $(h'_{-1}, k'_{-1}, z'_0,)'$ and to the side condition that all individual component processes be in L_0^2. It is feasible to solve this system of difference equations using the invariant subspace methods described in chapter 9. We shall rely on the dynamic programming here.

Before describing dynamic programming, we manipulate some of the first-order conditions given in (4.2.4) to deduce economic interpretations for the Lagrange multipliers. The multipliers have a direct connection to the price system to be used in Chapter 6 to support the optimal resource allocation in a competitive economy.

Solving the sixth equation in (4.2.4) for $\mathcal{M}_t^s$ gives

$$\mathcal{M}_t^s = b_t - s_t. \tag{4.2.6}$$

We can interpret $\mathcal{M}_t^s$ as the marginal utility vector or, equivalently, as the shadow price vector (in terms of utility) for services at date t. Solving the third equation in (4.2.4) forward yields

$$\mathcal{M}_t^h = E[\sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h')^{\tau-1} \Lambda' \mathcal{M}_{t+\tau}^s \mid J_t]. \tag{4.2.7}$$

We interpret $\mathcal{M}_t^h$ as the indirect marginal utility vector for the household capital stock at time t. The infinite discounted sum in (4.2.7) captures the notions that household capital at date t generates services in subsequent time periods, and that the Lagrange multiplier $\mathcal{M}_t^h$ reflects this valuation. The indirect marginal utility vector for consumption at date t is just $\mathcal{M}_t^c \equiv \Theta_h' \mathcal{M}_t^h + \Pi' \mathcal{M}_t^s$ because a vector c_t of consumption goods at time t yields $\Theta_h c_t$ units of household capital and Πc_t units of consumption services at time t.

It is also of interest to deduce marginal valuations or shadow prices (in terms of utility) of investment and productive capital. These can be expressed in terms of the shadow price of consumption and the indirect marginal disutility of intermediate goods g_t. Combining the first two equations in (4.2.4) gives

$$\begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix} \mathcal{M}_t^d = \begin{bmatrix} \Theta_h' \mathcal{M}_t^h + \Pi' \mathcal{M}_t^s \\ -g_t \end{bmatrix}. \tag{4.2.8}$$

Since Assumption 3 is satisfied, the matrix on the left side of (4.2.8) is nonsingular. Solving (4.2.8) for $\mathcal{M}_t^d$, we obtain

$$\mathcal{M}_t^d = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} \Theta_h' \mathcal{M}_t^h + \Pi' \mathcal{M}_t^s \\ -g_t \end{bmatrix}. \tag{4.2.9}$$

The multiplier $\mathcal{M}_t^d$ is used in representing the shadow price of capital in time period t. Solving the fifth equation in (4.2.4) forward gives

$$\mathcal{M}_t^k = E[\sum_{\tau=1}^{\infty} \beta^\tau (\Delta_k')^{\tau-1} \Gamma' \mathcal{M}_{t+\tau}^d \mid J_t]. \tag{4.2.10}$$

We interpret $\mathcal{M}_t^k$ as the shadow price vector for the capital stock.

Representation (4.2.10) can be interpreted as follows. Capital at time t is valued because it is useful for producing output in subsequent time periods. The contribution to value from helping to produce output at time $t+\tau$ is manifested in the term $\Gamma' \mathcal{M}_{t+\tau}^d$. This term is discounted by $\beta^\tau (\Delta_k')^{\tau-1}$, reflecting both the discounting in the consumer's utility function and the depreciation in the capital stock. The vector $\Gamma' \mathcal{M}_t^d$ can be used to ascertain whether there are incentives to hold idle capital at time t. In particular, negative values of this multiplier indicate that a better solution to the social planning problem could be obtained if the equality in resource constraint (2.4) were relaxed to be an inequality.

Finally, the shadow price for new investment is given by $\mathcal{M}_t^i = \Theta_k' \mathcal{M}_t^k$ because a vector i_t of investment goods at time t yields $\Theta_k i_t$ units of capital at time t. In light of the fourth equation in (4.2.4), $\mathcal{M}_t^i$ is also given by $\Phi_i' \mathcal{M}_t^d$, which reflects the resource cost of producing new investment goods. Notice that (4.2.6), (4.2.7), (4.2.9), and (4.2.10) can be used to obtain expressions for the multiplier processes in terms of the endogenous process $\{g_t, s_t\}$ and the exogenous process $\{z_t\}$. The fact that we obtain two equivalent representations for the shadow price of investment is an implicit restriction on the optimal choice of $\{g_t, s_t\}$.

4.3. Dynamic programming

This section briefly describes how the method of dynamic programming can be used to solve the social planning problem.[6] The the nuts and bolts of *linear quadratic* dynamic programming are described in Chapter 9.

Recall that the vector of initial conditions at time zero consists of $x_0' \equiv (h_{-1}', k_{-1}', z_0')$. The social planning problem can be solved in the following way. First, temporarily assume that someone has handed us the solution of the time shifted version of the problem that takes $x_1' \equiv (h_0', k_0', z_1')$ as a given set of initial conditions, and that shifts forward the constraints and objective function one time period. Let $V(x_1)$ be the *optimal value function* that is equal to the objective function of this altered problem evaluated at the initial condition x_1 and the associated optimal plan. Then solve a two-period problem with the objective to maximize:

$$[-.5[(s_0 - b_0) \cdot (s_0 - b_0) + g_0 \cdot g_0] + \beta E V(x_1)] \tag{4.3.1}$$

subject to the linear constraints

$$\begin{aligned} \Phi_c c_0 + \Phi_g g_0 + \Phi_i i_0 &= \Gamma k_{-1} + d_0, \\ k_0 &= \Delta_k k_{-1} + \Theta_k i_0, \\ h_0 &= \Delta_h h_{-1} + \Theta_h c_0, \\ s_0 &= \Lambda h_{-1} + \Pi c_0, \end{aligned} \tag{4.3.2}$$

and

$$z_1 = A_{22} z_0 + C_2 w_1, \ b_0 = U_b z_0 \ \text{ and } \ d_0 = U_d z_0 \tag{4.3.3}$$

The problem is to be solved taking as given the value of the initial state vector x_0. If the function V is concave, the problem can be solved for *policy functions* denoted by the vector valued function $F(x_0)$ that express c_0, g_0, h_0, i_0, k_0, and s_0 as functions of the vector x_0. Then dynamic programming tells us that the optimal values of c_t, g_t, h_t, i_t, k_t and s_t for the original problem are given by $F(x_t)$. So *if* we could somehow discover the function $V(\cdot)$, we would be able to solve the social planning problem simply by solving the two period problem (4.3.1) – (4.3.3).

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[6] See Stokey, Lucas, and Prescott [1989] and Sargent [1987b, chapter 1] for background on dynamic programming and some of its uses in macroeconomics.

We need a way to compute the value function V. Dynamic programming calculates V by exploiting the fact that the objective (4.3.1) evaluated at the optimal policy functions is given by $V(x_0)$. This means that the value function is the same at time zero as it is at time one, and that V solves a fixed-point problem that is formulated as follows. First, solve the two-period optimization problem (4.3.1) – (4.3.3) for a given value function V, then compute the time zero value function $T(V)$. The optimization problem thus induces an *operator* mapping a value function V into a new value function $T(V)$. The *optimal* value function V solves the functional equation $V = T(V)$, known as the *Bellman equation.*

One way to compute V is to iterate on the operator T. Let T^j denote the operator T applied j times. Then the sequence $\{T^j(0) : j = 1, 2, \ldots\}$ of functions converges to V (under some assumptions about our matrices to be described in chapter 9), where 0 is interpreted as a function that is zero over its entire domain. This method works under quite general circumstances.[7]

There is a special structure to the social planning problem. If we let V be a quadratic function of the form $x'Px + \rho$, then $T(V)$ is a quadratic function $x'T_1(P)x+T_2(P,\rho)$. The optimal decision rule depends on P but is independent of the scalar ρ. The optimal value of the matrix P can be calculated by iterating on the T_1 transformation. That is, P can be computed as the limit point of the sequence $\{T_1^j(0) : j = 1, 2, \ldots\}$ where 0 now denotes a matrix with entries that are all zero. However, iteration on the operator T_1 is computationally inefficient. There exists a *doubling algorithm* that speeds up convergence by computing only members of the subsequence $\{T_1^{2^j}(0) : j = 1, 2, \ldots\}$. This and other algorithms are described in Chapter 9.

The time-invariant character of the social planning problem makes the optimal policy functions or *decision rules* time invariant. The time t state vector is $x_t' \equiv (h_{t-1}', k_{t-1}', z_t')$. The time t decision rules depend on x_t. From P, it is straightforward to deduce these rules by solving the two-period problem (4.3.1)

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[7] This method works whenever technical conditions on the social planning problem are satisfied that make it redundant to impose the side conditions (2.24). However, for problems in which those technical conditions aren't satisfied, it is necessary to start the iterations on T from an initial value function of the form $x'W_1x + W_2$ where W_1 is a negative semidefinite matrix with particular eigenvalues less than zero. Pages (BLANK) describe a problem in which it will not work to initiate the iterations on T from an identically zero value function.

– (4.3.3). Since this problem has a quadratic objective function and linear constraints, the contingency plans are all linear in the state vector x_t. We denote these rules $c_t = S_c x_t, g_t = S_g x_t, h_t = S_h x_t, i_t = S_i x_t, k_t = S_k x_t, s_t = S_s x_t$.

Similarly, the law of motion for the state vector is linear:

$$x_{t+1} = A^o x_t + C w_{t+1}$$

where

$$A^o \equiv \begin{bmatrix} A^o_{11} & A^o_{12} \\ 0 & A_{22} \end{bmatrix}, C \equiv \begin{bmatrix} 0 \\ C_2 \end{bmatrix}. \tag{4.3.4}$$

The partitioning of the A^o and C matrices is according to the *endogenous* state vector $(h'_{t-1}, k'_{t-1})'$ and the *exogenous* state vector z_t. The zero restriction on the (2,1) partition of A^o reflects the fact that the exogenous state vector at time $t+1$ does not depend on the endogenous state vector at time t. The zero restriction on the first rows in the partition of C reflects the fact that the endogenous state vector at time $t+1$ is predetermined (i.e., depends only on time t information). The contingency plans for h_t and k_t are embedded in the part of (4.3.4) that determines the endogenous state vector $[h'_t \; k'_t]'$ as a function of x_t. In particular,

$$\begin{bmatrix} S_h \\ S_k \end{bmatrix} = [\, A^o_{11} \quad A^o_{12} \,]. \tag{4.3.5}$$

Notice that the decision rules are recursive in the sense that the time t decision depends on the state vector at time t, which in turn depends on the state vector at time $t-1$. It would be possible to eliminate this dependence via recursive substitutions and to deduce a time-varying representation of the state-contingent decision at time t on current and past values of the noise vector w_t and the initial condition x_0, as in equation (2.5).

Recall that the eigenvalues of A^o determine the growth of the state vector $\{x_t\}$. Since A^o is block triangular, the set of eigenvalues of A^o is the union of the set of eigenvalues of A^o_{11} and the set of eigenvalues of A_{22}. We refer to the first set of eigenvalues as the *endogenous eigenvalues* because A^o_{11} is determined by the solution to the social planning problem. These eigenvalues must have absolute values strictly less than $1/\sqrt{\beta}$ to satisfy the requirement that the elements of $\{x_t\}$ be in L^2_0. We refer to the second set of eigenvalues as the set of *exogenous eigenvalues* because the matrix A_{22} is specified exogenously.

By assumption, the eigenvalues of A_{22} have absolute values that are less than or equal to one.

4.4. Lagrange multipliers as gradients of value function

Associated with the solution of the social planning problem is the quadratic value function $V(x_0) = x_0' P x_0 + \rho$. The function $V(x_0)$ gives the maximal value that the social planner can attain when he starts from initial state x_0.

In this section, we show how the Lagrange multipliers are related to the value function. We attach Lagrange multipliers to each of the constraints, and formulate the Lagrangian associated with iterating once on Bellman's equation. The first-order conditions associated with the saddle point of this Lagrangian restrict the multipliers in terms of the value function. The multipliers become linear functions of the state x_t. Consider the two-period optimization problem that is the time t counterpart to that described by (4.3.1) – (4.3.3).

Form the Lagrangian:

$$\begin{aligned}\mathcal{L} = &-(1/2)[(s_t - b_t)\cdot(s_t - b_t) + g_t \cdot g_t] + \beta E[V(x_{t+1}) \mid J_t] \\ &- \mathcal{M}_t^{d\prime} \cdot (\Phi_c c_t + \Phi_g g_t + \Phi_i i_t - \Gamma k_{t-1} - d_t) \\ &- \mathcal{M}_t^{k\prime} \cdot (k_t - \Delta_k k_{t-1} - \Theta_k i_t) \\ &- \mathcal{M}_t^{h\prime} \cdot (h_t - \Delta_h h_{t-1} - \Theta_h c_t) \\ &- \mathcal{M}_t^{s\prime} \cdot (s_t - \Lambda h_{t-1} - \Pi c_t).\end{aligned} \tag{4.4.1}$$

To obtain the first order conditions for the Lagrangian (4.4.1), recall that $V(x_{t+1}) = x_{t+1}' P x_{t+1} + \rho$. Notice that $\frac{\partial}{\partial h_t} E(x_{t+1}' P x_{t+1} + \rho) \mid I_t = \frac{\partial x_{t+1}}{\partial h_t} \frac{\partial}{\partial x_{t+1}} E(x_{t+1}' P x_{t+1} + \rho) \mid J_t = [I\ 0\ 0] E(2 P x_{t+1}) \mid I_t = 2[I\ 0\ 0] P A^o x_t$. Here the matrix $[I\ 0\ 0]$ satisfies $h_t = [I\ 0\ 0] x_{t+1}$. Similarly $\frac{\partial}{\partial k_t} E(x_{t+1}' P x_{t+1} + \rho) \mid I_t = 2[0\ I\ 0] P A^o x_t$, where $k_t = [0\ I\ 0] x_{t+1}$. Differentiating (4.4.1) with respect to c_t, g_t, h_t, i_t, k_t, and s_t

and using the above expressions for $\frac{\partial}{\partial h_t}$ and $\frac{\partial}{\partial k_t}$ yields

$$\begin{aligned}
-\Phi_c' \mathcal{M}_t^d + \Theta_h' \mathcal{M}_t^h + \Pi' \mathcal{M}_t^s &= 0, \\
- g_t - \Phi_g' \mathcal{M}_t^d &= 0, \\
-\mathcal{M}_t^h + 2\beta [I \ 0 \ 0] P A^o x_t &= 0, \\
- \Phi_i' \mathcal{M}_t^d + \Theta_k' \mathcal{M}_t^k &= 0, \\
-\mathcal{M}_t^k + 2\beta [0 \ I \ 0] P A^o x_t &= 0, \\
- s_t + b_t - \mathcal{M}_t^s &= 0.
\end{aligned} \tag{4.4.2}$$

Solving the third and fifth equations of (4.4.2) for $\mathcal{M}_t^k$ and $\mathcal{M}_t^h$ gives

$$\begin{aligned}
&\mathcal{M}_t^k = M_k x_t \ \text{ and } \ \mathcal{M}_t^h = M_h x_t \ \text{ where} \\
&M_k = 2\beta [0 \ I \ 0] P A^o \\
&M_h = 2\beta [I \ 0 \ 0] P A^o.
\end{aligned} \tag{4.4.3}$$

In comparing (4.4.3) to (4.2.7) and (4.2.10), we see that the derivatives of $E[V(x_{t+1}) \mid J_t]$ with respect to the endogenous state vectors h_t and k_t give expressions in terms of x_t for the conditional expectations of the infinite sums that appear in (4.2.7) and (4.2.10). Solving the sixth equation (4.4.2) for $\mathcal{M}_t^s$ yields

$$\mathcal{M}_t^s = M_s x_t \ \text{ where } \ M_s = (S_b - S_s) \ \text{ and } \ S_b = [0 \ 0 \ U_b]. \tag{4.4.4}$$

Solving the first two equations of (4.4.2) for $\mathcal{M}_t^d$ results in

$$\mathcal{M}_t^d = M_d x_t \ \text{ where } \ M_d = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} \Theta_h' M_h + \Pi' M_s \\ -S_g \end{bmatrix}. \tag{4.4.5}$$

Finally, the shadow price vectors for consumption and investment are given by

$$\mathcal{M}_t^c = M_c x_t \ \text{ where } \ M_c = \Theta_h' M_h + \Pi' M_s \tag{4.4.6}$$

$$\mathcal{M}_t^i = M_i x_t \ \text{ where } \ M_i = \Theta_k' M_k. \tag{4.4.7}$$

Formulas (4.4.3) – (4.4.7) express the Lagrange multipliers for the social planning problem in terms of the optimal value function associated with that problem.

4.5. Planning problem as linear regulator

Our social planning problem can be cast as an *optimal linear regulator problem.* A discounted linear regulator problem has the form:

$$\max_{\{u_t\}} -E\sum_{t=0}^{\infty}\beta^t[x_t'Rx_t + u_t'Qu_t + 2u_t'Wx_t], \quad 0<\beta<1,$$

subject to

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}, \ t \geq 0$$

where $\{w_{t+1}\}$ is a martingale difference sequence adapted to its own history and x_0, x_t is a vector of state variables, and u_t is a vector of control variables; the matrices R, Q, and W are conformable with the objects they multiply. The maximization is subject to the requirement that u_t be chosen to be a function of information known at t, namely $\{x_t, x_{t-1}, \ldots, x_0, u_{t-1}, \ldots, u_0\}$.

To show how our social planning problem maps into the optimal linear regulator problem, we must tell how to choose the objects $[x_t, u_t, w_{t+1}, R, Q, W, A, B, C]$ in the optimal regulator problem. We choose these objects as follows:

$$x_t = \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{bmatrix}, \qquad u_t = i_t$$

and w_t is the martingale difference sequence in (2.2);

$$A = \begin{bmatrix} \Delta_h & \Theta_h U_c[\Phi_c \ \ \Phi_g]^{-1}\Gamma & \Theta_h U_c[\Phi_c \ \ \Phi_g]^{-1}U_d \\ 0 & \Delta_k & 0 \\ 0 & 0 & A_{22} \end{bmatrix}$$

$$B = \begin{bmatrix} -\Theta_h U_c[\Phi_c \ \ \Phi_g]^{-1}\Phi_i \\ \Theta_k \\ 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 \\ 0 \\ C_2 \end{bmatrix}$$

$$\begin{bmatrix} x_t \\ u_t \end{bmatrix}' S \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \begin{bmatrix} x_t \\ u_t \end{bmatrix}' \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}$$

where $S = (G'G + H'H)/2$ and

$$H = [\Lambda \ \vdots \ \Pi U_c[\Phi_c \ \ \Phi_g]^{-1}\Gamma \ \vdots \ \Pi U_c[\Phi_c \ \ \Phi_g]^{-1}U_d - U_b \ \vdots \ -\Pi U_c[\Phi_c \ \ \Phi_g]^{-1}\Phi_i]$$

$$G = U_g[\Phi_c \quad \Phi_g]^{-1}[0 \vdots \Gamma \vdots U_d \vdots -\Phi_i].$$

Here, U_c and U_g are selector matrices to be defined in Appendix A. In Appendix A, we show show constructively that these choices work to map the social planning problem into the linear regulator.

The Bellman equation for the linear regulator is

$$V(x_t) = \max_{u_t}\{-(x_t'Rx_t + u_t'Qu_t + 2u_t'Wx_t) + \beta E_t V(x_{t+1})\} \qquad (4.5.1)$$

where the maximization is subject to

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}.$$

The value function $V(x)$ is quadratic: $V(x_t) = -x_t'Px_t - \rho$, where the matrix P and the scalar ρ satisfy the equations

$$P = R + \beta A'PA - (\beta A'PB + W')(Q + \beta B'PB)^{-1}(\beta B'PA + W) \qquad (4.5.2)$$

$$\rho = \beta(1-\beta)^{-1}\text{trace}(PCC'). \qquad (4.5.3)$$

The solutions of (4.5.1) can be computed by iterating on the T mapping defined above.[8] Chapter 9 describes faster methods of solving these equations.

The optimal control law is given by

$$u_t = -Fx_t \qquad (4.5.4)$$

where

$$F = (Q + \beta B'PB)^{-1}(\beta B'PA + W). \qquad (4.5.5)$$

Substituting (4.5.4) into (4.5.1) gives the optimal closed loop system

$$x_{t+1} = (A - BF)x_t + Cw_{t+1} \qquad (4.5.6)$$

which we represent as

$$x_{t+1} = A^o x_t + Cw_{t+1} \qquad (4.5.7)$$

where $A^o = A - BF$.

[8] The MATLAB program `double.m` implements this algorithm.

We can use the solution of the linear regulator problem to represent the solution of the social planning problem in a useful way. In particular, where

$$
\begin{aligned}
h_t &= S_h x_t & d_t &= S_d x_t \\
k_t &= S_k x_t & c_t &= S_c x_t \\
k_{t-1} &= S_{k1} x_t & g_t &= S_g x_t \\
i_t &= S_i x_t & s_t &= S_s x_t \\
b_t &= S_b x_t
\end{aligned}
$$

we have

$$
\begin{aligned}
\begin{bmatrix} S_h \\ S_k \end{bmatrix} &= [\, A_{11}^o \quad A_{12}^o \,] \\
S_{k1} &= [0 \ I \ 0] \\
S_i &= -F \\
S_d &= [0 \ 0 \ U_d] \\
S_b &= [0 \ 0 \ U_b] \\
S_c &= U_c [\Phi_c \ \Phi_g]^{-1} \{ -\Phi_i S_i + \Gamma S_{k1} + S_d \} \\
S_g &= U_g [\Phi_c \ \Phi_g]^{-1} \{ -\Phi_i S_i + \Gamma S_{k1} + S_d \} \\
S_s &= \Lambda [I \ 0 \ 0] + \Pi S_c
\end{aligned}
$$

Here $\begin{bmatrix} A_{11}^o & A_{12}^o \\ 0 & A_{22} \end{bmatrix} = A - BF$.

We also have a convenient set of formulas for the Lagrange multipliers associated with the social planning problem. Where

$$
\begin{aligned}
\mathcal{M}_t^k &= M_k x_t & \mathcal{M}_t^d &= M_d x_t \\
\mathcal{M}_t^h &= M_h x_t & \mathcal{M}_t^c &= M_c x_t \\
\mathcal{M}_t^s &= M_s x_t & \mathcal{M}_t^i &= M_i x_t
\end{aligned}
$$

we have

$$
\begin{aligned}
M_k &= 2\beta [0 \ I \ 0] P A^o \\
M_h &= 2\beta [I \ 0 \ 0] P A^o \\
M_s &= (S_b - S_s) \\
M_d &= \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} \Theta_h' M_h + \Pi' M_s \\ -S_g \end{bmatrix} \\
M_c &= \Theta_h' M_h + \Pi' M_s \\
M_i &= \Theta_k' M_k .
\end{aligned}
$$

Here the partitions $[0\ I\ 0]$ and $[I\ 0\ 0]$ are conformable with the partition $[h'_{t-1}, k'_{t-1}, z_t]'$ of x_t.

4.6. Solutions for five economies

We now show by example how solutions of social planning problems for our models can be computed by using MATLAB programs. Tables 1 and 2 describe how we have translated the symbols in the model (many of them Greek) into symbols to be manipulated by our programs. The translations are mnemonic, so that it ought to be easy to keep in mind the connections between the expressions in our MATLAB programs and the matrices in our models. We have prepared a battery of programs, to be used in sequence, that compute the objects that define and characterize the solution of the planning problem for a member of our class of models. To use these programs, we first have to feed in the matrices defined in Table 1, using the notation employed in Table 1. We have prepared a number of `.m` files that input these parameters for various particular economies. These files are called `clex*.m`,where the `*` is replaced by a particular integer to denote a particular economy. The economies corresponding to particular `clex*.m` files are listed in the MATLAB manual which we have included as chapter 12 of this book. The `clex*.m` files are MATLAB script files (i.e., they are not functions). To input the paramters of, say, of a version of Hall's economy that we have stored in `clex11.m`, the user just types `clex11`.

Table 1

Correspondence Between Symbols in Model
and Symbols in MATLAB programs

Symbol in Model	Symbol in Computer Program
A_{22}	`a22`
C_2	`c2`
U_b	`ub`
U_d	`ud`
Φ_c	`phic`
Φ_g	`phig`
Φ_i	`phii`
Γ	`gamma`
Δ_k	`deltak`
Θ_k	`thetak`
Δ_h	`deltah`
Θ_h	`thetah`
Λ	`lambda`
Π	`pih`
β	`beta`

Table 2

Correspondence Between Symbols in Solution of the Planning Problem and Symbols in MATLAB programs

Symbol in Model	Symbol in Computer Program
A^o	`ao`
C	`c`
S_c	`sc`
S_g	`sg`
S_s	`ss`
S_k	`sk`
S_i	`si`
S_h	`sh`
S_b	`sb`
S_d	`sd`
M_c	`mc`
M_g	`mg`
M_s	`ms`
M_k	`mk`
M_i	`mi`
M_h	`mh`

The MATLAB programs perform the following tasks:

a. The program `solvea.m` accepts as inputs a collection of matrices that specify a particular economy. It then computes the solution of the planning problem, and for future use creates and stores the matrices listed in table 2.

b. The program `steadst.m` computes the nonstochastic steady state, or equivalently the unconditional mean for the asymptotic stationary distribution, of the state vector, provided that this object is well defined.

c. The program `aarma.m` computes an ARMA representation for the response of a specified list of variables to one of the innovations in the model.

d. The program `aimpulse.m` computes the impulse response function of a specified list of variables to one of the innovations in the model.

e. The program `asimul.m` computes a random or nonrandom simulation of a specified list of variables.

f. The program `asseta.m` computes equilibrium prices for some particular assets to be specified. (The use of this program will be explained in chapter 5.)

The program `solvea.m` makes use of the following two programs in order to solve the social planning problem efficiently.

g. The program `doubleo.m` solves a matrix Riccati equation swiftly via a "doubling algorithm."

h. The program `double2j.m` uses a doubling algorithm to compute variance-like terms that can be represented as particular infinite series of some matrix products.

The user can find out how to use these and all other programs by using the 'help' facility in MATLAB. Thus, to learn how to run the program `solvea.m`, the user just types `help solvea`. In addition, we have included as an appendix to this book a MATLAB manual of all the programs associated with our models.

The purpose of this section is to illustrate how easy it is to use these programs to analyze our models, and how rapidly things can be learned about the structures of our models by representing their solutions in the ways that our programs facilitate. In the spirit of learning by doing, we analyze five related models that can generate a range of behavior for time series of quantities (and also of the equilibrium prices to be studied in chapter 5).

We study a class of models that we form by combining technologies 4 (growth) and 2 (costs of adjustment) with preference specification 3 (habit persistence). By setting the parameters at different particular values, we are able to generate versions of several models that have been studied in the literature. Each of these models is specified by defining preference and technology matrices of the same dimension. To create a new model of this class, we simply reset some parameter values, while leaving the dimensions of the matrices that define the economy unaltered.

Our models are generated by the following specification for preferences, technology, and information.

4.6.1. Preferences

$$-.5E\sum_{t=0}^{\infty}\beta^t[(s_t-b_t)^2+\ell_t^2]|J_0$$
$$s_t=\lambda\, h_{t-1}+\pi c_t$$
$$h_t=\delta_h\, h_{t-1}+\theta_h c_t$$
$$b_t=U_b z_t$$

4.6.2. Technology

$$c_t+i_t=\gamma_1 k_{t-1}+d_{1t}$$
$$k_t=\delta_k\, k_{t-1}+i_t$$
$$g_t=\phi_1 i_t, \qquad \phi_1>0$$
$$\begin{bmatrix} d_{1t} \\ 0 \end{bmatrix}=U_d z_t$$

4.6.3. Information

$$z_{t+1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & 0 \\ 0 & 0 & .5 \end{bmatrix} z_t+\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} w_{t+1}$$
$$U_b=[\,30 \quad 0 \quad 0\,]$$
$$U_d=\begin{bmatrix} 5 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_0=[\,5 \quad 150 \quad 1 \quad 0 \quad 0\,]'$$

Notice that the information process and the initial condition are specified so that the constant is the third state variable. Notice that we have set the b_t process equal to a constant value of 30. There is no random component of the preference shock process. Notice that there is a single nontrivial endowment shock, the second component of d_t having been set to zero via the specification of the matrix U_d. The first component of d_t has been specified to follow a first order autoregressive process with positive mean. The autoregressive parameter for the endowment process has been set at .8. Notice that the third component of the z_t vector is a first order autoregressive process with coefficient .5. However, this component of the z_t vector impinges neither on b_t nor on d_t, given the way that we have specified U_b and U_d. We include the third component of the z_t process in case the reader would like to edit one our files, say, to add a random component to the preference shock b_t.

These specifications of preferences and technology are rich enough to encompass versions of several models that have been popular in the recent macroeconomic literature. The preference specification can accommodate preferences that are quadratic in consumption, as used by Hall [1978]; preferences incorporating habit persistence, as used recently by Becker and Murphy [1988]; and preferences for a durable consumption good, as used by Mankiw [1982]. The technology specification is a version of the one-good 'growth' technology of chapter 2, modified to include costs of adjusting capital.[9] We shall initially set the parameters of the technology to satisfy the necessary condition for consumption to be a random walk in Hall's model, namely, the condition $\beta(\gamma_1 + \delta_k) = 1$. This is also the condition for the 'growth condition' of Jones and Manuelli just to be satisfied. For all of the specifications, we set U_b so that $b_t = 30$ for all t.

By setting the parameter values of this general model to particular values, we can capture the following models.

4.6.4. Brock-Mirman model

Set the *preference* parameters as $\lambda = 0, \pi = 1, \delta_h$ and θ_h arbitrarily. This makes preferences take the form

$$-.5E\sum_{t=0}^{\infty}\beta^t[(c_t - b_t)^2 + \ell_t^2]|J_0.$$

Set the *technology* parameters so that $\gamma_1 > 0, \phi_1 > 0$ but $\phi_1 \approx 0, (\gamma_1+\delta_k)\beta = 1$.

9 The parameters for our first version of Hall's economy are in `clex11.m`; those for our second version of Hall's economy are in `clex12.m`; those for our third version of Hall's economy are in `clex13.m`; those for the Jones-Manuelli model are in `clex10.m`; those for the model with durable consumption goods are in `clex15.m`; and those for Lucas's economy are in `clex14.m`.

4.6.5. A growth economy fueled by habit persistence

Set the technology parameters as in Hall's model, but set the preference parameters to capture preference specification 3 of chapter 2. In particular, set $1 > \delta_h > 0, \theta_h = (1-\delta_h), \pi = 1, \lambda = -1$. This makes preferences assume the form

$$-.5E\sum_{t=0}^{\infty}\beta^t[(c_t - b_t - \lambda(1-\delta_h)\sum_{j=0}^{\infty}\delta_h^j c_{t-j-1})^2 + \ell_t^2]|J_0.$$

4.6.6. Lucas's pure exchange economy

Set preference parameters as in Hall's model, but alter the technology to render capital unproductive, i.e., set $\gamma_1 = 0$.

4.6.7. An economy with a durable consumption good

Set the technology as in Hall's model, but alter preferences to capture the idea that the consumption good is durable. Set $\pi = 0, \lambda > 0, 0 < \delta_h < 1, \theta_h = 1$.

We now illustrate how the solutions of the social planning problem associated with several of these models can be computed and analyzed. Generally, we proceed as follows. First we read in the parameters that represent our economy by way of the matrices listed in Table 1. We have prepared a set of '`.m`' files that read in these matrices for the several economies listed above. Thus, `clex11.m`, `clex12.m`, and `clex13.m` are files that read in matrices corresponding to Hall's model for various different parameter settings. Next, we use `solvea.m` to compute all of the matrices listed in Table 2, which characterize the solution of the planning problem. To compute the vector ARMA representation of any subset of quantities or Lagrange multipliers, we use `aarma.m`. To compute the impulse response functions of any set of quantities and/or Lagrange multipliers to components of $w(t)$, we use the program `aimpulse.m`. Finally, we can use `simul.m` or `asimul.m` to simulate the solution of the model.

4.7. Hall's model

We begin with the version of Hall's model which we solved by hand earlier in this chapter. We begin by setting the parameters in a way that is designed to make consumption follow a random walk. In particular, we set $\phi_1 = .00001, \gamma_1 = .1, \delta_k = .95, \beta = 1/1.05$. Notice that $\beta(\gamma_1 + \delta_k) = 1$. We set the remaining parameters to the values described above.

After reading in the matrices by typing `clex11`, we compute the solution of the planning problem by typing `solvea`. Issuing this command causes the computer to respond as follows:

```
Calculating, please wait
The matrix ao has been calculated for the law of
motion of the entire state vector.  This matrix
satisfies
x(t+1) = ao*x(t) + c*w(t+1).
The endogenous eigenvalues are in the vector endo,
and the exogenous eigenvalues are in the vector exog.
The solution to the model is given by c(t) = sc*x(t),
g(t) = sg*x(t), h(t) = sh*x(t), i(t) = si*x(t)
k(t) = sk*x(t), and s(t) = ss*x(t).
The matrices sc, sg, sh, si, sk, and ss have now been
computed and can be used in other matlab programs.
The matrices sb and sd are constructed so that b(t)
= sb*x(t) and d(t) = sd*x(t) and can be used in other
matlab programs.
The shadow price vectors satisfy Mc(t) = mc*x(t),
Mg(t) = mg*x(t), Mh(t) = mh*x(t), Mi(t) = mi*x(t),
Mk(t) = mk*x(t), Ms(t) = ms*x(t), and Md(t) = md*x(t).
The matrices of these linear combinations can
be used in other matlab programs.
Your equilibrium has been calculated.
You are now ready to experiment with the economy.
```

This is the end of the output that appears on the screen. The solution of the planning problem is stored in the matrices listed in table 2. To inspect these matrices, we just ask MATLAB to show them to us. Thus, issuing the

MATLAB command `ao` results in the output

$$\texttt{ao} = \begin{bmatrix} 0.9000 & 0.0050 & 0.5000 & 0.0200 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.8000 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.8000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.5000 \end{bmatrix}$$

To see the matrix c, we type `c` and elicit the response

$$\texttt{c} = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}$$

Recall that various quantities in the model are determined by premultiplying the state x_t by matrices S_j which are stored by MATLAB in `sj`. For various purposes, it is useful to create a matrix by stacking various `sj`'s on top of one another. For example, we can stack the `s` matrices for consumption, household durables, services, physical investment, and physical capital by issuing the MATLAB command `G=[sc;sh;ss;si;sk]`, which evokes the response

$$\texttt{G} = \begin{bmatrix} 0.0000 & 0.0500 & 5.0000 & 0.2000 & 0.0000 \\ 0.9000 & 0.0050 & 0.5000 & 0.0200 & 0.0000 \\ 0.0000 & 0.0500 & 5.0000 & 0.2000 & 0.0000 \\ 0.0000 & 0.0500 & 0.0000 & 0.8000 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.8000 & 0.0000 \end{bmatrix}$$

The first row of G is S_c, and so on. Similarly, various Lagrange multipliers in the model are determined by premultiplying x_t by the matrices M_j, which are stored by MATLAB in `mj`. We can create a matrix by stacking various `mj`'s by issuing the command `H=[mc;ms;mh;mi;mk]`, which evokes

$$\texttt{H} = \begin{bmatrix} 0.0000 & -0.0500 & 25.0000 & -0.2000 & 0.0000 \\ 0.0000 & -0.0500 & 25.0000 & -0.2000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & -0.0500 & 25.0000 & -0.2000 & 0.0000 \\ 0.0000 & -0.0500 & 25.0000 & -0.2000 & 0.0000 \end{bmatrix}$$

The endogenous and exogenous eigenvalues of A^o or `ao` are stored in `endo` and `exo`, respectively. For the present model, they are given by

$$\texttt{endo} = \begin{bmatrix} .90 \\ 1.0 \end{bmatrix}$$

$$\texttt{exo} = \begin{bmatrix} 1.00 \\ .80 \\ .50 \end{bmatrix}$$

The exogenous eigenvalue of unity corresponds to the constant (unity) in the state vector, while the other two exogenous eigenvalues are also directly inherited from our specification of the A_{22} matrix. The endogenous eigenvalue of .9 is inherited from the depreciation factor of .9 which we set for consumer durables, which is irrelevant in Hall's model because we set $\lambda = 0$. This eigenvalue will become relevant below in specifications in which $\lambda \neq 0$. The eigenvalue of unity reflects the random walk character of consumption in Hall's model. Actually, the second endogenous eigenvalue is not really unity, it is only close to unity. To see this, we switch to a long format in MATLAB by typing `format long` and then we type `endo` to receive the response

$$\texttt{endo} = \begin{bmatrix} 0.90000000000000 \\ 0.99999999999048 \end{bmatrix}$$

The eigenvalue is not exactly unity because of the very small costs of adjusting capital that we have imposed.

The fact that the endogenous eigenvalues of this model are below unity means that it possesses a nonstochastic steady state. To compute the steady state, we set `nnc=3`, which tells the computer that the constant term is the third component of the state vector. Then we type `steadst`, which causes the steady state to be computed and stored in `zs`. To compute the steady state value of consumption, just type `sc*zs`, and so on. For the present model, we obtain

$$\texttt{zs} = \begin{bmatrix} 5.0003 \\ 0.0061 \\ 1.0000 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$

The steady state value of consumption is given by `sc*zs`, which is

$$\texttt{sc} * \texttt{zs} = [\, 5.0003 \,]$$

The steady state value of investment is given by `si*zs`, which is

$$\texttt{si} * \texttt{zs} = [\,0.0003\,]$$

For the present model, these stationary steady state values are of little practical value because of the near unit endogenous eigenvalue. It will take very many periods for the effect of the initial conditions to die out in this model, despite the fact that a steady state for the nonstochastic version of the model does exist.

We can compute an ARMA representation for the impulse response of any quantities or Lagrange multipliers to a given component of the white noise process w_t. We can learn how `aarma.m` works by typing `help aarma`, which delivers the response

```
function[num,den]=aarma(ao,c,sy,ii)
Creates ARMA Representation for linear recursive
equilibrium models.  The equilibrium is
x(t+1) = ao*x(t) + c*w(t+1)
and is created by running SOLVEA. A vector of
observables is given by
y(t) = sy*x(t)
where sy picks off the desired variables.
For example, if we want y=[c',i'], we set
sy=[sc;si].  AARMA creates the representation
den(L)y(t) = num(L)wi(t)
This is an arma representation for the response of
y(t) to the i-th component of w(t).
```

For example, to compute the ARMA representation for the impulse response of c_t, i_t to the first component of w_t, we type `sy=[sc;si]` and `[num,den]=aarma(ao,`

`c,sy,1)` which gives the response

$$\texttt{num} = \begin{bmatrix} 0.0000 & 0.2000 & -0.6400 & 0.7540 & -0.3860 & 0.0720 \\ 0.0000 & 0.8000 & -2.6800 & 3.3040 & -1.7660 & 0.3420 \end{bmatrix}$$

$$\texttt{den} = [\,1.0000 \quad -4.2000 \quad 6.9700 \quad -5.7000 \quad 2.2900 \quad -0.3600\,]$$

This output is to be interpreted as follows. For $i = 0, \ldots, 5$. Define α_i as the element in the $(i+1)$ column of `den`. For $i = 0, \ldots 5$ define ξ_i as the 3×1 matrix that is the $i + 2^{st}$ column of `num`. Define two polynomials in the lag operator L by

$$\alpha(L) = \sum_{i=0}^{5} \alpha_i L^i$$
$$\xi(L) = \sum_{i=0}^{5} \xi_i L^i$$

Let w_{1t} be the first innovation in the system, which drives the endowment process. Then we have the representation

$$\alpha(L) \begin{bmatrix} c_t \\ i_t \\ mc_t \end{bmatrix} = \xi(L) w_{1t}$$

For example, the first row of this representation is

$$(1 - 4.2L + 6.97L^2 - 5.7L^3 + 2.29L^4 - .36L^5) c_t$$
$$= (.2 - .64L + .754L^2 - .386L^3 + .0072L^4) w_{1t}$$

We can also create the impulse response function for a list of variables in response to a particular innovation. We shall compute the impulse response function for the two variables, c, i. To accomplish this, we set `sy` by typing `sy = [sc;si]`. We set `ii` at 1 (we want the response to the first innovation), and specify the number of lags we want to perform the calculation for. We want the impulse response out to forty lags, so we specify `ni=40`. To compute the impulse response, we issue the MATLAB function `aimpulse`, which has the syntax `[z]=aimpulse(ao,c,sy,ii,ni)`, where `sy,ii,ni` have the settings just described.[10] The impulse response function is returned in `z`. In Fig. 4.7.1.a we plot the impulse response functions for this model in response to the first innovation, which is the innovation in the endowment shock. These impulse response functions have shapes that are characteristic of a random walk for consumption and a unit root in capital. For consumption, the impulse response is an open "box" which attains its maximum height immediately. This impulse response is characteristic of a random walk consumption process. For investment, the impulse response has an asymptote.[11]

[10] The MATLAB program `aimpulse.m` takes the inputs we have created from the solution of the social planning problem and feeds them into the MATLAB program `dimpulse.m`, which computes impulse response functions.

[11] In actuality, there is really no asymptote for the impulse response function for either consumption or investment, because the largest eigenvalue is just a little bit less than unity.

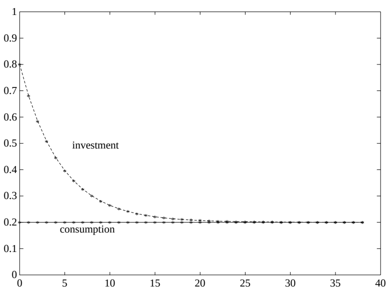

Fig. 4.7.1.a. Impulse response of consumption and investment to an endowment innovation in a version of Hall's model.

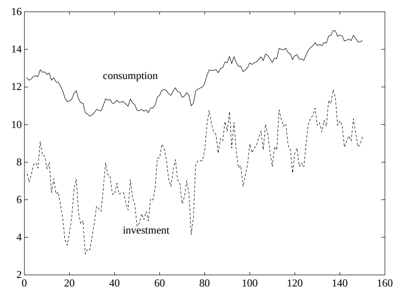

Fig. 4.7.1.b. Simulation of consumption and investment for Hall's model.

We now generate a random simulation of the model for 150 periods. We use the non-interactive program `asimul.m` to generate this simulation. To use this program we must specify an observer matrix `sy` that links the called-for variables to the state. Since we want to simulate the four series c, i, k, and the shadow price of consumption, we set `sy=[sc;si;sk;mc]`. We also have to specify the length of the simulation `t1`, whether we want a random (`k=1`) or nonrandom (`k=2`) simulation, and the initial state vector `x0`. We want a random simulation of length 150 with the initial condition specified above. After setting these parameters, we execute the simulation by commanding `asimul`. We obtain the response:

```
Your simulated vector is in the vector ‘‘y’’.
```

We display aspects of this simulation in Fig. 4.7.1.b.The sample paths of c, k, and the shadow price drift in the fashion that random walks do. For paths that are long enough, a random simulation of this model will eventually

---

In fact, the impulse response functions for both consumption and investment are 'square summable', but it would take a very long realization of them for this behavior to become apparent.

encounter negative values for capital and consumption. The key to rigging samples so that capital and consumption for a long time remain positive with high probability is to select the initial condition for capital large enough and the elements of `c2` small enough.

Figure 4.7.1.b indicates that investment is relatively more variable than consumption, a pattern that is found in aggregate data for a variety of countries. The fact that this version of Hall's model, like the stochastic growth model of Brock and Mirman [1972], so easily delivers this pattern is an important feature that has attracted adherents to this and other versions of 'real business cycle' theories.

4.8. Higher Adjustment Costs

We now turn to a second model which is created by making one modification to the economy we have just studied. The one change we make is to raise the costs associated with adjusting capital. We raise the absolute value of the cost parameter to $\phi_1 = .2$. All other parameters remain as in the previous economy.

We computed the solution of the social planning problem using `solvea.m`. The endogenous eigenvalues were computed to be:

$$\texttt{endo} = \begin{bmatrix} 0.9000 \\ 0.9966 \end{bmatrix}$$

Notice that relative to the previous economy, one endogenous eigenvalue is left unaltered at .9, while the other endogenous eigenvalue has fallen below unity. The endogenous eigenvalue of .9 is inherited from the law of accumulation that we posit for household capital (which in this model is again irrelevant). The drop below unity of the second endogenous eigenvalue is the result of our having increased the costs of adjusting capital. The analysis that we performed on pages BLANK indicates that this is exactly what should occur when adjustment costs increase.

Figure 4.8.1.a reports impulse response functions for the response of c_t and i_t and to an innovation in the endowment process. Notice how these no longer have the tell tale signs of the presence of an endogenous unit eigenvalue. The

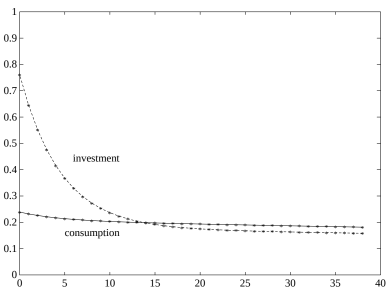

Fig. 4.8.1.a. Impulse response of consumption and investment to an endowment innovation in a version of Hall's model with higher costs of adjusting capital and no random walk in consumption.

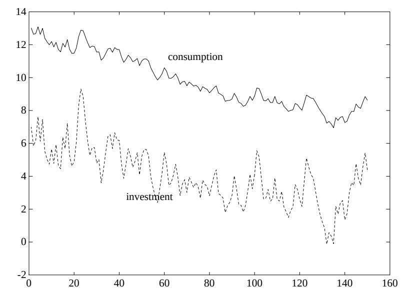

Fig. 4.8.1.b. Simulation of a version of Hall's model with higher costs of adjusting capital and no random walk in consumption.

impulse response for consumption and investment now both appear to be convergent and 'square summable'. Figure 4.8.1.b shows a random simulation beginning from the same value for x_0 used with the earlier version of Hall's model. Notice how consumption, while still smoother than income, has increased high frequency volatility relative to that depicted in figure 4.7.1.a, while the high frequency volatility of investment has decreased. This pattern is a response to the higher costs for adjusting capital. Notice also that there seems to be a downward 'trend' in both consumption and investment. This is a consequence of the decrease in the largest endogenous eigenvalue from being very nearly one in the earlier economy. The present economy has a nonstochastic steady state value for capital of .0000, for consumption of 5.00 (which is the mean of the endowment process), and for investment of .0000, each of which we computed using `steadst.m`. These nonstochastic steady state values correspond to the unconditional means from the asymptotic stationary distribution of our variables. Because the largest endogenous eigenvalue for this economy is .9966 rather than

.9999, the economy is headed toward these mean values much more rapidly than for our previous economy.

4.9. Altered 'growth condition'

We generate our next economy by making two alterations in the preceding economy. First, we raise the adjustment cost parameter from .2 to 1. This will have the effect of further lowering the endogenous eigenvalue that is not .9, and of causing the impulse response functions to dampen faster than they did in the previous economy. Second, we raise the production function parameter from .1 to .15. This will have the effect of raising the optimal stationary value of capital to a positive value for the nonstochastic version of the model. Recall that the optimal stationary value of capital was zero in the previous economy. The nonstochastic steady state values of consumption, investment, and capital are 17.5, 6.25, and 125, respectively, for this economy.

The endogenous eigenvalues are

$$\texttt{endo} = \begin{bmatrix} 0.9000 \\ 0.9524 \end{bmatrix}$$

We also created the impulse response function for c and i, which is reported in figure 4.9.1.a. Notice the much faster rate of damping relative to the impulse responses displayed for the previous economies.

Figure 4.9.1.b displays a random simulation of this economy. Notice that the "transient" behavior displayed by our simulation of the previous economy is not present here. This is a consequence of our having altered the production function parameter value to induce a positive optimal stationary value for the capital stock of 125, and from our having started the simulation at an initial condition of 125 for the capital stock.

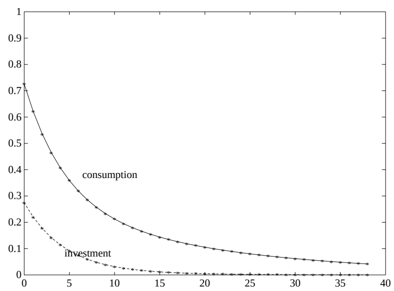

Fig. 4.9.1.a. Impulse response of consumption and investment to an endowment innovation in a version of Hall's model with higher adjustment costs and the 'growth condition' altered.

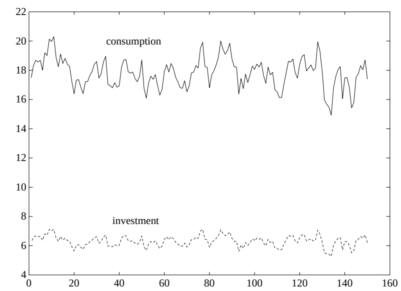

Fig. 4.9.1.a. Simulation of consumption and investment in a version of Hall's model with higher adjustment costs and the 'growth condition' altered.

4.10. A Jones-Manuelli economy

A notable feature of the models for the previous simulations is that consumption, investment, and capital generally failed to *grow*. We now define the matrices and set parameters with a view toward attaining a version of Jones and Manuelli's model of economic growth. We set the parameters of the technology so that Jones and Manuelli's "growth condition" is just satisfied.[12] Our version of Jones and Manuelli's model has the feature that their growth condition is a necessary but not a sufficient condition for growth to occur. Their growth condition makes sustained growth *feasible* in our model. In order for growth to occur, it is also necessary that it be *desirable*, a condition that is determined by the preference parameters λ, δ_h, and θ_h. We set these parameters in order to generate growth.

12 The Jones-Manuelli growth condition on the technology in our notation is $\beta(\gamma+\delta_k) \equiv 1$. This is also a condition that makes the marginal utility of consumption follow a martingale in Hall's model.

In particular, setting λ equal to minus one turns out to generate a preference for growth.[13]

As usual, we compute the equilibrium by using `asolve.m`. For this model, the endogenous eigenvalues are

$$\texttt{endo} = \begin{bmatrix} \texttt{1.0000} + \texttt{0.0000i} \\ \texttt{1.0000} - \texttt{0.0000i} \end{bmatrix}$$

The exogenous eigenvalue of unity is inherited from the law of motion of the unit vector, which is the third state variable. Notice that there are two unit endogenous eigenvalues. With some experimentation, the reader can determine how these two unit endogenous eigenvalues result from specifying the parameters of technology to obey the growth condition, and the parameters of preferences (especially λ) to capture a longing for consumption growth.[14]

Figure 4.10.1.a displays impulse responses of consumption and investment to an innovation in the endowment process. For both consumption and investment, the effect of an innovation actually grows indefinitely over time. This is a product of the second unit endogenous eigenvalue that is inherited from the preference parameter λ.

Figure 4.10.1.b displays a simulation of consumption and investment for this economy. The economy grows. Notice that consumption is much smoother than investment. Notice also that investment typically exceeds consumption. In order to support the 'habit' that fuels growth, the economy has to accumulate physical capital.[15]

We invite the reader to experiment with this economy by altering the settings of some parameter values one at a time relative to the parameter settings that we have made. In particular, we recommend that the following experiments be tried:

---

[13] The parameter values for this economy are stored in `clex10.m`

[14] One unit endogenous eigenvalue stems from setting β, Γ, and Δ_k at the boundary of the Jones-Manuelli growth condition. The other unit endogenous eigenvalue results from setting $\lambda = -1$. The presence of very small positive adjustment costs for capital is what prevents these two endogenous eigenvalues from being exactly unity. The reader can check that they are not exactly unity by using the `format long` command in MATLAB.

[15] It is a feature of models of addiction based on the type of preference specification used here, e.g., Becker and Murphy [1988], that 'addicts' grow wealthier and wealthier over time as they follow a consumption plan that allows for enough accumulation to support their growing addiction.

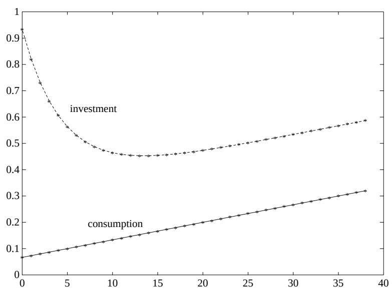

Fig. 4.10.1.a. Impulse response of consumption and investment to an endowment innovation in a Jones-Manuelli economy.

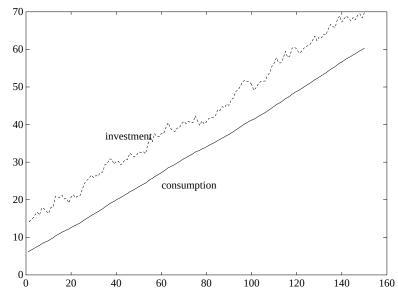

Fig. 4.10.1.b. Simulation of consumption and investment in a Jones-Manuelli economy.

1. Change the value of λ to $-.7$, leaving the other parameters unaltered. Obtain the solution of the planning problem, and inspect the endogenous eigenvalues. Also compute the impulse response function and simulate the model in response to the same initial condition that we used above. Does the economy still grow? Explain.
2. Change the value of β to .94. Recompute the solution of the planning problem. Does the economy grow? Link your explanation to the Jones-Manuelli growth condition.
3. Change the value of $\Gamma(1)$ to .09. Does the economy still grow?

4.11. Durable consumption goods

For our next example economy, we restore the productivity of capital to a value of .1 and raise the level of the parameter measuring adjustment costs for capital to a value of 1. We change the specification of preferences to make the consumption good durable. In particular, we adopt a version of preference specification 2. We implement this by setting λ equal to .1, π equal to zero, and θ_h equal to one. We leave δ_h at the value .9. [16]

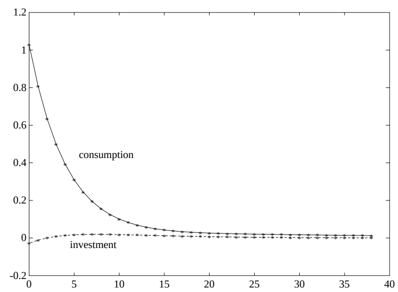

Fig. 4.11.1.a. Impulse response of consumption and investment to an endowment innovation in an economy with a durable consumption good.

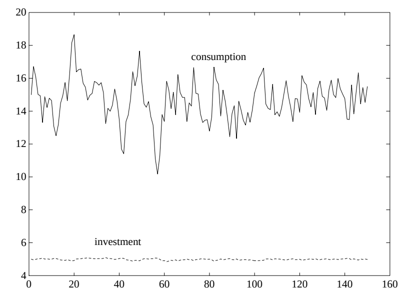

Fig. 4.11.1.b. Simulation of consumption and investment in an economy with a durable consumption good.

Figure 4.11.1.a displays the impulse response functions to an innovation in the endowment process. The impulse response function for consumption and for investment are very different than for our first model. In particular, from the impulse response function, we can see that in choosing consumption, the social planner 'smooths' the endowment shock much less than he does in Hall's original model, in which the planner in effect makes consumption an equal-weight moving average of current and lagged innovations to the endowment process. In the

[16] These parameters settings are created by the file `clex15.m`.

present model, the planner makes consumption a much shorter, more peaked moving average of the endowment process. This shows up in the simulation of consumption and investment, which is reported in figure 4.11.1.b. Notice that now, in contrast to Hall's model, it is investment that is much smoother than consumption. This example thus illustrates how making consumption goods durable tends to undo the strong consumption smoothing result which Hall obtained.

4.12. Summary

In this chapter, we have formulated a planning problem, and described how to compute its solution. We have also described computer programs that solve the planning problem, and that characterize the solution in a variety of ways.

Associated with the solution of the planning problem are a set of Lagrange multipliers, which we have shown how to compute in terms of the derivatives of the value function for the planners dynamic programming problem. In the next two chapters, we shall show how those Lagrange multipliers are related to the price system for a competitive equilibrium. We begin by describing how to represent values.

A. Synthesizing the linear regulator

The social planning problem is to maximize

$$-.5E\sum_{t=0}^{\infty}\beta^t\Big[(s_t-b_t)\cdot(s_t-b_t)+g_t\cdot g_t\Big] \tag{4.A.1}$$

subject to

$$\Phi_c c_t+\Phi_g g_t+\Phi_i i_t=\Gamma k_{t-1}+d_t \tag{4.A.2}$$

$$k_t=\Delta_k k_{t-1}+\Theta_k i_t \tag{4.A.3}$$

$$h_t=\Delta_h h_{t-1}+\Theta_h c_t \tag{4.A.4}$$

$$s_t=\Lambda h_{t-1}+\Pi c_t \tag{4.A.5}$$

$$z_{t+1}=A_{22}z_t+C_2 w_{t+1} \tag{4.A.6}$$

$$\begin{aligned} b_t &= U_b z_t \\ d_t &= U_d z_t \end{aligned} \tag{4.A.7}$$

We define the *state* of the system as $x_t = \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{bmatrix}$ and the *control* as $u_t = i_t$. In defining the control to be i_t, we exploit the assumption that $[\Phi_c \quad \Phi_g]$ is nonsingular.

Solve (4.A.2) for $(c_t \ , \ g_t)$:

$$\begin{bmatrix} c_t \\ g_t \end{bmatrix} = [\Phi_c \, \Phi_g]^{-1} \{\Gamma k_{t-1} + U_d z_t - \Phi_i i_t\}.$$

Let U_c and U_g be selector matrices that pick off the first n_c and the last n_g rows, respectively, of the right side of the above expression, so that the expression can be written

$$\begin{aligned} c_t &= U_c [\Phi_c \, \Phi_g]^{-1} \{\Gamma k_{t-1} + U_d z_t - \Phi_i i_t\} \\ g_t &= U_g [\Phi_c \, \Phi_g]^{-1} \{\Gamma k_{t-1} + U_d z_t - \Phi_i i_t\}. \end{aligned} \tag{4.A.8}$$

Substituting (4.A.8) into (4.A.4) and (4.A.5) gives

$$h_t = \Delta_h h_{t-1} + \Theta_h U_c [\Phi_c \, \Phi_g]^{-1} \{\Gamma k_{t-1} + U_d z_t - \Phi_i i_t\} \tag{4.A.9}$$

$$s_t = \Lambda h_{t-1} + \Pi U_c [\Phi_c \, \Phi_g]^{-1} \{\Gamma k_{t-1} + U_d z_t - \Phi_i i_t\} \tag{4.A.10}$$

Combining (4.A.3), (4.A.9), and (4.A.6) gives the law of motion for the linear regulator

$$\begin{aligned} \begin{pmatrix} h_t \\ k_t \\ z_{t+1} \end{pmatrix} &= \begin{pmatrix} \Delta_h & \Theta_h U_c [\Phi_c \, \Phi_g]^{-1} \Gamma & \Theta_h U_c [\Phi_c \, \Phi_g]^{-1} U_d \\ 0 & \Delta_k & 0 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{pmatrix} \\ &+ \begin{pmatrix} -\Theta_h U_c [\Phi_c \, \Phi_g]^{-1} \Phi_i \\ \Theta_k \\ 0 \end{pmatrix} i_t + \begin{pmatrix} 0 \\ 0 \\ C_2 \end{pmatrix} w_{t+1} \end{aligned} \tag{4.A.11}$$

or

$$x_{t+1} = A x_t + B u_t + C w_{t+1} \tag{4.A.12}$$

where the matrices A, B, and C in (4.A.12) equal the corresponding matrices in (4.A.11).

Now use (4.A.10) to compute $(s_t - b_t) = \Lambda h_{t-1} + \Pi U_c [\Phi_c \, \Phi_g]^{-1} \Gamma k_{t-1} + (\Pi U_c [\Phi_c \, \Phi_g]^{-1} U_d - U_b) z_t - \Pi U_c [\Phi_c \, \Phi_g]^{-1} \Phi_i i_t$. Express this in matrix notation as

$$(s_t - b_t) = [\Lambda \ \vdots \ \Pi U_c [\Phi_c \, \Phi_g]^{-1} \Gamma \ \vdots \ \Pi U_c [\Phi_c \, \Phi_g]^{-1} U_d - U_b \ \vdots \ - \Pi U_c [\Phi_c \, \Phi_g]^{-1} \Phi_i] \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_t \\ i_t \end{bmatrix} \tag{4.A.13}$$

or

$$(s_t - b_t) = [H_s \ \vdots \ H_c] \begin{bmatrix} x_t \\ i_t \end{bmatrix} \tag{4.A.14}$$

where the matrix $[H_s \vdots H_c]$ in $(4.A.14)$ equals the corresponding matrix in $(4.A.13)$.

Next, use $(4.A.8)$ to express g_t as

$$g_t = [0 \vdots U_g[\Phi_c \, \Phi_g]^{-1}\Gamma \vdots U_g[\Phi_c \, \Phi_g]^{-1}U_d \vdots \; - U_g[\Phi_c \, \Phi_g]^{-1}\Phi_i]' \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_t \\ i_t \end{bmatrix} \tag{4.A.15}$$

or

$$g_t = [G_s \vdots G_c] \begin{bmatrix} x_t \\ i_t \end{bmatrix} \tag{4.A.16}$$

where the matrix $[G_s \vdots G_c]$ in $(4.A.16)$ equals the corresponding matrix in $(4.A.15)$.

Define the matrices

$$R = .5(H_s' H_s + G_s' G_s), \; Q = .5(H_c' H_c + G_c' G_c), \; W = .5(H_c' H_s + G_c' G_s). \tag{4.A.17}$$

Then the current period return function for the social planning problem is

$$-(x_t' R x_t + u_t' Q u_t + 2u_t' W x_t). \tag{4.A.18}$$

In view of $(4.A.14)$, $(4.A.16)$, $(4.A.17)$ and $(4.A.18)$, we can represent the objective function in the social planning problem as

$$-E \sum_{t=0}^{\infty} \beta^t (x_t' R x_t + u_t' Q u_t + 2u_t' W x_t), \tag{4.A.19}$$

which is to be maximized over $\{u_t\}_{t=0}^{\infty}$ subject to

$$x_{t+1} = A x_t + B u_t + C w_{t+1}, \quad t \geq 0, \tag{4.A.20}$$

x_0 given. Thus, we have mapped the social planning problem into a discounted optimal linear regulator problem.

B. A Brock-Mirman model

We shall usually use the recursive numerical methods described above to compute a solution of a social planning problem. These computational methods are quick and easy to use. However, to deepen our understanding of the structure of the social planning problem and the role played by various technical assumptions, and also to heighten our appreciation of the ease and power of those recursive numerical methods, it is useful to solve one problem by hand.

We solve a social planning problem for a model with one consumption good and one capital good. We include costs of adjusting the capital stock, but permit them to be zero as a special case. When these costs of adjustment are zero (i.e., when the parameter ϕ in the model is set to zero), the model becomes a linear - quadratic, equilibrium version of Hall's consumption model. To recover Hall's solution of the model when $\phi = 0$, it is necessary to impose a side condition in the form of a version of our restriction (2.24) that forces the capital stock sequence $\{k_t\}$ to belong to L_0^2. The example is a useful laboratory for illustrating the relationships among the presence of costs to control $(\phi > 0)$, the transversality condition, and the side condition that the solution lie in L_0^2. After we work out the answer by hand, we can solve the problem by using the MATLAB program `solvea.m` .

The social planning problem comes from combining versions of our preference specification number 1 and our technology specification number 4: choose a contingency plan for $\{c_t, k_t\}_{t=0}^{\infty}$ to maximize:

$$-E_0 \sum_{t=0}^{\infty} \beta^t [(c_t - b_t)^2 + \ell_t^2] \ , \ 0 < \beta < 1 \tag{4.B.1}$$

subject to

$$c_t + i_t = \gamma k_{t-1} + d_{1t} \ , \gamma > 0 \tag{4.B.2}$$

$$\phi i_t = g_t \ , \ \phi \geq 0 \tag{4.B.3}$$

$$k_t = \delta k_{t-1} + i_t \ , \ 0 < \delta < 1 \tag{4.B.4}$$

$$g_t^2 = \ell_t^2 \tag{4.B.5}$$

$$k_{-1} \ \text{ given} \tag{4.B.6}$$

The stochastic processes b_t and d_{1t} are given by $b_t = U_b z_t$ and $d_{1t} = U_{d1} z_t$, where z_t obeys a version of (1.1). We assume that $\{d_{1t}\}$ and $\{b_t\}$ each belong to L_0^2, and do *not* impose that $\{k_t\}$ belongs to L_0^2.

We begin by forming the Lagrangian

$$\begin{aligned} J = -E_0 \sum_{t=0}^{\infty} \beta^t \{ & \frac{1}{2}[(c_t - b_t)^2 + \ell_t^2] - \lambda_{1t}[\gamma k_{t-1} + d_{1t} - c_t - i_t] \\ & - \lambda_{2t}[g_t - \phi i_t] \\ & - \lambda_{3t}[\delta k_{t-1} + i_t - k_t] - \lambda_{4t}[\frac{1}{2}(\ell_t^2 - g_t^2)]\} \end{aligned} \tag{4.B.7}$$

Here $\{\lambda_{1t}, \lambda_{2t}, \lambda_{3t}, \lambda_{4t}\}_{t=0}^{\infty}$ is a 4-tuple of stochastic Lagrange multipliers. We obtain the first order necessary conditions for a saddle point with respect to $\{c_t, i_t, k_t, \ell_t, g_t \ , \ \lambda_{1t}, \lambda_{2t},$

$\lambda_{3t}\,,\ \lambda_{4t}\}_{t=0}^{\infty}$, and display the transversality condition for capital. First order conditions with respect to c_t, i_t, k_t, l_t, and g_t are:

$$c_t : -(c_t - b_t) - \lambda_{1t} = 0,\ t \geq 0 \tag{4.B.8}$$

$$i_t : -\lambda_{1t} - \phi\lambda_{2t} + \lambda_{3t} = 0,\ t \geq 0 \tag{4.B.9}$$

$$k_t : \gamma\beta E_t\lambda_{1t+1} + \beta\delta E_t\lambda_{3t+1} - \lambda_{3t} = 0,\ t \geq \tag{4.B.10}$$

$$\ell_t : -\ell_t + \lambda_{4t}\ell_t = 0,\ t \geq 0 \tag{4.B.11}$$

$$g_t : \lambda_{2t} - \lambda_{4t}g_t = 0,\ t \geq 0 \tag{4.B.12}$$

In addition, we have the transversality condition

$$\lim_{t\to\infty} E_0\beta^t k_t\lambda_{3t} = 0. \tag{4.B.13}$$

Equation $(4.B.10)$ can be solved forward to yield

$$\lambda_{3t} = \gamma\beta\sum_{j=1}^{\infty}(\delta\beta)^{j-1}E_t\lambda_{1t+j}. \tag{4.B.14}$$

Our strategy is to substitute the above expressions for the multipliers into the first-order condition with respect to k_t to obtain an ‘Euler equation, and to study under what conditions, if any, this equation implies that the marginal utility of consumption is a martingale. Solving the first order conditions for the multipliers, we obtain

$$\lambda_{1t} = b_t - c_t \tag{4.B.15}$$

$$\lambda_{2t} = g_t \tag{4.B.16}$$

$$\lambda_{3t} = \phi g_t + (b_t - c_t) \tag{4.B.17}$$

$$\lambda_{4t} = 1 \tag{4.B.18}$$

Substituting $(4.B.17)$ into $(4.B.10)$ gives the “Euler equation”

$$\begin{aligned} \gamma\beta E_t(b_{t+1} - c_{t+1}) + \beta\delta E_t(\phi g_{t+1} + b_{t+1} - c_{t+1}) \\ = \phi g_t + (b_t - c_t) \end{aligned} \tag{4.B.19}$$

or

$$\begin{aligned} \beta\delta E_t\phi g_{t+1} + \beta(\gamma+\delta)E_t(b_{t+1} - c_{t+1}) \\ = \phi g_t + (b_t - c_t). \end{aligned} \tag{4.B.20}$$

Under the special condition that $\phi = 0$, this equation becomes

$$E_t(b_{t+1} - c_{t+1}) = [\beta(\gamma+\delta)]^{-1}(b_t - c_t), \tag{4.B.21}$$

which states that the shadow price of consumption ($\lambda_{1t} = b_t - c_t$) follows a first-order autoregressive process. Under the further special condition that $\beta(\gamma+\delta) = 1$, the shadow price of consumption follows a martingale.[17] Finally, under the even further special condition that b_t is a martingale, (4.*B*.21) asserts that consumption is a martingale.

The Euler equation (4.*B*.21) is satisfied by the consumption plan

$$c_t = b_t \quad \text{for} \quad t \geq 0. \tag{4.B.22}$$

Solving (4.*B*.2) and (4.*B*.4) for i_t under this plan gives

$$k_t = (\gamma+\delta)k_{t-1} + d_{1t} - b_t. \tag{4.B.23}$$

Note that in the special case that λ_{1t} (and maybe also c_t) is a martingale, $(\gamma+\delta) = 1/\beta$, so that $\{k_t\}$ given by (4.*B*.23) is a "process of exponential order $1/\beta$". This implies that k_t does not belong to L_0^2. Nevertheless, the transversality condition (4.*B*.13) *is* satisfied because $\lambda_{3t} = \phi g_t + (b_t - c_t) = 0$ along this solution, so that

$$\lim_{t\to\infty} \beta^t \lambda_{3t} k_t = 0$$

along this solution.

Thus, when $\phi = 0$, it is optimal to consume bliss consumption always and to adjust the capital stock to support this consumption plan. The difference equation (4.*B*.23) implies that

$$k_t = \xi^t k_0 + \sum_{j=0}^{t-1} \xi^j (d_{1t-j} - b_{t-j})$$

where $\xi \equiv \gamma + \delta$. If $b_t - d_{1t} > \alpha > 0$ for some α for all t, then k_t will eventually become negative and, indeed, will eventually fall below any finite negative number. Such a consumption path is eventually being supported by "borrowing" or by 'negative capital.'

In the interests of attaining an 'Euler equation' for capital, we substitute the following two implications of the constraints into the Euler equation:

$$\begin{aligned} c_t &= (\gamma+\delta)k_{t-1} + d_{1t} - k_t \\ g_t &= \phi k_t - \phi\delta k_{t-1} \end{aligned}$$

After rearrangement, this gives the following Euler equation for capital:

$$\eta E_t\{k_{t+1} - \psi k_t + \beta^{-1} k_{t-1}\} = E_t z_t \tag{4.B.24}$$

where

$$\begin{aligned} \eta &= \beta[\delta\phi^2 + (\gamma+\delta)] \\ \psi &= \frac{\beta\delta^2\phi^2 + \beta(\gamma+\delta)^2 + \phi^2 + 1}{\beta(\delta\phi^2 + (\gamma+\delta))} \\ z_t &= b_t - \beta(\gamma+\delta)b_{t+1} \\ &\quad - d_{1t} + \beta(\gamma+\delta)d_{1t+1} \end{aligned} \tag{4.B.25}$$

---

[17] The condition that $\beta(\gamma+\delta) \equiv 1$ plays the role of a "growth condition" in the model of Jones and Manuelli [1988].

We will solve the Euler equation $(4.B.24)$ using the "certainty equivalence" methods described in Sargent [1987, ch. XIV] and Hansen and Sargent [1980, 1981]. This involves first solving the deterministic version of $(4.B.25)$, and then replacing "feedforward" terms with their expectations conditioned on time t information.

We begin by solving the deterministic version of the Euler equation $(4.B.24)$:

$$\eta\{k_{t+1} - \psi k_t + \beta^{-1} k_{t-1}\} = z_t \tag{4.B.26}$$

Write this as

$$\eta L^{-1}\{1 - \psi L + \beta^{-1} L^2\} k_t = z_t. \tag{4.B.27}$$

We seek a factorization of the polynomial in L:

$$(1 - \psi L + \beta^{-1} L^2) = (1 - \lambda_1 L)(1 - \lambda_2 L) \tag{4.B.28}$$

Evidently

$$\begin{aligned} \psi &= \lambda_1 + \lambda_2 \\ \lambda_1 \lambda_2 &= \beta^{-1}. \end{aligned}$$

Thus we have

$$\lambda_2 = \frac{1}{\lambda_1 \beta} \tag{4.B.29}$$

and

$$\lambda_1 + \frac{1}{\lambda_1 \beta} = \psi. \tag{4.B.30}$$

Equations $(4.B.29)$ and $(4.B.30)$ imply that λ_1 and $\lambda_2 = \frac{1}{\lambda_1 \beta}$ are the intersections of the line of zero slope and height ψ with the curve $\lambda + \frac{1}{\lambda \beta}$ in figure 4.B.1. Since the function $f(\lambda) = \lambda + \frac{1}{\lambda \beta}$ achieves a minimum of $2/\sqrt{\beta}$ at the value $\lambda = 1/\sqrt{\beta}$, it follows that if a solution of $(4.B.30)$ exists, it satisfies, without loss of generality,

$$\begin{aligned} 0 < \lambda_1 &< \frac{1}{\sqrt{\beta}} \\ \lambda_2 &> \frac{1}{\sqrt{\beta}}. \end{aligned}$$

Substituting $(4.B.28)$ into $(4.B.27)$ gives

$$\eta[(1 - \lambda_1 L)(1 - \frac{1}{\lambda_1 \beta} L)] k_{t+1} = z_t. \tag{4.B.31}$$

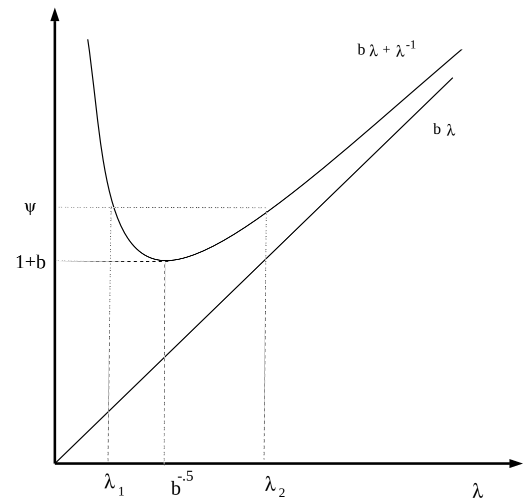

Figure 4.B.1: The function $b\lambda + 1/\lambda$ and its intersections with ψ, which determine the roots λ of the characteristic polynomial $(4.B.28)$.

We start analyzing the solution of $(4.B.31)$ by returning to the special case in which $\phi = 0$. In this case, $(4.B.25)$ implies that

$$\begin{aligned} \psi &= \xi + \frac{1}{\beta\xi}, \quad \xi = \gamma + \delta, \\ \eta &= \beta\xi. \end{aligned} \tag{4.B.32}$$

It then follows immediately from $(4.B.30)$ that we can take

$$\begin{aligned} \lambda_1 &= \frac{1}{\beta\xi} \\ \lambda_2 &= \xi. \end{aligned} \tag{4.B.33}$$

In the special case that the shadow price of consumption is a martingale, $\beta\xi = 1$, so that $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{\beta}$. The Euler equation thus becomes, in the special case that $\phi = 0$,

$$\beta\xi\{(1 - \frac{1}{\beta\xi}L)(1 - \xi L)\}k_{t+1} = z_t.$$

But from the constraints to our problem,

$$(1 - \xi L)k_{t+1} = d_{1t+1} - c_{t+1}$$

Substituting this and the last line of $(4.B.25)$ into the Euler equation gives

$$\beta\xi(1 - \frac{1}{\beta\xi}L)(d_{1t+1} - c_{t+1}) = (\beta\xi - L)(d_{1t+1} - b_{t+1})$$

or

$$(\beta\xi - L)\{(d_{1t+1} - c_{t+1}\} = (\beta\xi - L)(d_{1t+1} - b_{t+1}),$$

an equation that is satisfied by setting $c_t = b_t$ for all t. Thus, our analysis of the Euler equation for capital in the case that $\phi = 0$ reconfirms our earlier derivation that the optimal plan involves setting $c_t = b_t$ and choosing whatever capital path is required to support this.

We begin to study the case when $\phi > 0$ by considering the special case in which ϕ is positive but arbitrarily close to zero. In particular, ϕ can be chosen sufficiently close to zero that in the Euler equation for capital,

$$\eta\{(1 - \lambda_1 L)(1 - \lambda_2 L)\}k_{t+1} = z_t,$$

η is arbitrarily close to $\beta\xi, \lambda_1$ is arbitrarily close $\frac{1}{\beta\xi}$, and λ_2 is arbitrarily close to ξ. This can be verified by using a version of figure 4.B.1.

It is tempting to suppose that since the Euler equation is arbitrarily close to that for the $\phi = 0$ case, the optimal solution for k_t will be close to the solution for k_t found in the $\phi = 0$ case, namely,

$$k_t = \xi^t k_0 + \sum_{j=0}^{t-1} \xi^j (d_{1t-j} - b_{t-j}). \tag{4.B.34}$$

We now show that this supposition is wrong.

Note that when k_t obeys $(4.B.34)$, $i_t = k_t - \delta k_{t-1}$, obeys

$$\begin{aligned} i_t &= \xi^{t-1}(\xi - \delta)k_0 \\ &\quad + d_{1t} - b_t + (\xi - \delta)\sum_{j=0}^{t-2} \xi^j (d_{1t+j-1} - b_{t-j-1}). \end{aligned} \tag{4.B.35}$$

Also, $c_t = b_t \; \forall t$ in this case. When i_t follows $(4.B.35)$, i_t is a process of exponential order ξ. It follows that ϕi_t is also a process of exponential order ξ when $\phi > 0$.

Now since $\ell_t = \phi i_t$ along the optimal path, we have that

$$\sum_{i=0}^{\infty} \beta^t \ell_t^2 = \phi^2 \sum_{t=0}^{\infty} \beta^t i_t^2. \tag{4.B.36}$$

The process i_t^2 is of exponential order ξ^{2t} along the solution $(4.B.35)$. The infinite series $(4.B.36)$ will converge if and only if

$$\beta \cdot \xi^2 < 1, \quad \text{or} \quad \xi < \frac{1}{\sqrt{\beta}}.$$

In the case for which the shadow price of consumption is a martingale, $\xi = 1/\beta > \frac{1}{\sqrt{\beta}}$, so that this condition is violated. In this case, $(4.B.36)$ diverges to $+\infty$.

This means that when investment follows the path $(4.B.35)$, the objective function for the social planning problem diverges to $-\infty$ when $\phi = 0$. Since it is possible to find investment paths that leave the value of the objective function finite, a plan in which the objective function diverges to $-\infty$ cannot be optimal.

Notice the role that the assumption that $\phi > 0$ plays in the above argument.

An alternative argument can be used to show that the path $(4.B.35)$, or one close to it, cannot be optimal when $\phi > 0$ and $\xi > \frac{1}{\sqrt{\beta}}$. This argument involves checking the transversality condition, which is

$$\lim_{t\to\infty} \beta^t k_t \lambda_{3t} = 0.$$

Computing, we have

$$\begin{aligned}
&\lim_{t\to\infty} \beta^t k_t \lambda_{3t} \\
&\quad = \lim_{t\to\infty} \beta^t k_t(\phi g_t + \lambda_{1t}) \\
&\quad = \lim_{t\to\infty} \beta^t k_t[\phi^2(k_t - \delta k_{t-1}) + (b_t - c_t)]. \\
&\quad = \lim_{t\to\infty} \beta^t[\phi^2(k_t^2 - \delta k_t k_{t-1}) + (b_t - c_t)k_t]
\end{aligned}$$

For a solution that involves setting $b_t = c_t$, this becomes

$$\lim_{t\to\infty} \beta^t[\phi^2(k_t^2 - \delta k_t k_{t-1})] = 0 \tag{4.B.37}$$

A necessary and sufficient condition for $(4.B.37)$ to be satisfied is that $\{k_t\}$ be of exponential order less than $\frac{1}{\sqrt{\beta}}$. Along a solution like $(4.B.34)$, this requires that $\xi < \frac{1}{\sqrt{\beta}}$, which is ruled out in the special case that the shadow price of consumption is a martingale. Arguments along these lines can be used to establish generally that when $\phi > 0$, the solutions for i_t and for k_t are required to be of exponential order less than $\frac{1}{\sqrt{\beta}}$.

To solve for the optimal plan when $\phi > 0$, we return to the factored Euler equation $(4.B.31)$:

$$\eta[(1 - \lambda_1 L)(1 - \frac{1}{\lambda_1 \beta} L)]k_{t+1} = z_t \tag{4.81}$$

where $0 < \lambda_1 < 1/\sqrt{\beta}$. Formally, express $(1 - \frac{1}{\lambda_1 \beta} L) = -\frac{1}{\lambda_1 \beta} L(1 - \lambda_1 \beta L^{-1})$. Substitute this into $(4.B.31)$ to get

$$\frac{-\eta}{\lambda_1 \beta}[(1 - \lambda_1 \beta L^{-1})(1 - \lambda_1 L)]k_t = z_t \tag{4.B.38}$$

Operating on both sides of $(4.B.38)$ with the stable (forward) inverse of $(1 - \lambda_1 \beta L^{-1})$ gives

$$(1 - \lambda_1 L)k_t = -\frac{\lambda_1 \beta}{\eta} \frac{1}{(1 - \lambda_1 \beta L^{-1})} z_t \tag{4.B.39}$$

or

$$k_t = \lambda_1 k_{t-1} - \frac{\lambda_1 \beta}{\eta} \sum_{j=0}^{\infty} (\lambda_1 \beta)^j z_{t+j}. \qquad (4.B.40)$$

Since $\lambda_1 < 1/\sqrt{\beta}, \lambda_1 \beta < \sqrt{\beta}$. It follows (in the deterministic case) that the infinite series on the right converges, $\{z_t\}$ being a sequence of exponential order less then $1/\sqrt{\beta}$ (or equivalently, residing in L_0^2).

When $\phi > 0$, equation $(4.B.40)$ gives the unique solution of the Euler equation that satisfies the transversality condition. Because $\lambda_1 < 1/\sqrt{\beta}$, k_t belongs to L_0^2.

4.B.1. Uncertainty

In the case that z_t is a random sequence, the solution when $\phi_1 > 0$ is given by

$$k_t = \lambda_1 k_{t-1} - \frac{\lambda_1 \beta}{\eta} \sum_{j=0}^{\infty} (\lambda_1 \beta)^j E_t z_{t+j} \qquad (4.B.41)$$

That this is the solution can be verified by applying the methods of Sargent [1987, chapter XIV].

Consider applying $(4.B.41)$ in the special case that makes consumption a martingale: $\beta\xi = 1, \eta = \beta\xi = 1, \lambda_1 = 1, b_t = \bar{b}$ for all t. In this case $(4.B.41)$ becomes,

$$k_t - k_{t-1} = -\beta \sum_{j=0}^{\infty} \beta^j E_t (d_{1t+j+1} - d_{1t+j}) \qquad (4.B.42)$$

We can use a summation by parts argument to show that

$$\begin{aligned} E_t \sum_{j=0}^{\infty} \beta^j (d_{1t+j+1} - d_{1t+j}) \\ = (\beta^{-1} - 1) E_t \sum_{j=0}^{\infty} \beta^j d_{1t+j} - \beta^{-1} d_{1t} \end{aligned} \qquad (4.B.43)$$

In particular, note that

$$\begin{aligned} \sum_{j=0}^{\infty} \beta^j (d_{1t+j+1} - d_{1t+j}) \\ = \sum_{j=1}^{\infty} \beta^{j-1} d_{1t+j} - \sum_{j=0}^{\infty} \beta^j d_{1t+j} \\ = (\beta^{-1} - 1) \sum_{j=0}^{\infty} \beta^j d_{1t+j} - \beta^{-1} d_{1t}. \end{aligned}$$

Note that from the constraints

$$c_t = (\gamma + \delta)k_{t-1} - k_t + d_{1t}$$

or

$$c_t = \frac{1}{\beta}k_{t-1} - k_t + d_{1t} \tag{4.B.44}$$

in the special case that $(\gamma + \delta)\beta = 1$, which we are studying. Substituting $(4.B.42)$ and $(4.B.43)$ into $(4.B.44)$ and rearranging gives

$$c_t = \Big(\frac{1}{\beta} - 1\Big)k_{t-1} + (1-\beta)\sum_{j=0}^{\infty}\beta^j E_t d_{1t+j}. \tag{4.B.45}$$

With k_{t-1} interpreted as "assets" and $\{d_{1t}\}$ interpreted as "labor income", representation $(4.B.45)$ matches the representation of the permanent income theory of consumption that is associated with a linear quadratic version of Hall's model.

In this model, $\phi = 0$, so that $(4.B.42)$ and $(4.B.45)$, which emerge from imposing that $\{k_t\}$ reside in L_0^2, are not optimal for the original problem as stated. The solution $(4.B.45)$results from imposing as a side condition on the problem a version of $(4.A.10)$. This side condition is intended to capture the idea that it is not really feasible to drive capital to negative infinity as quickly as the (unrestricted) $\phi = 0$ solution would require.

The solution $(4.B.45)$ is well approximated by the solution of the original problem with $\phi > 0$ but ϕ very close to zero. Instead of imposing the requirement that $\{k_t\}\epsilon L_0^2$ as a sort of "feasibility" condition, setting $\phi > 0$ rigs preferences so that the social planner always *prefers* to make $\{k_t\} \in L_0^2$.

4.B.2. Optimal Stationary States

Temporarily assume that $b_t = \bar{b}$ and $d_{1t} = \bar{d}$ for all t. To solve for the optimal stationary values of c_t and k_t (if they exist), we can use equation $(4.B.20)$ and the following constraints:

$$\phi i_t = g_t \tag{4.52}$$

$$i_t = k_t - \delta k_{t-1} \tag{4.53}$$

$$c_t + i_t = \gamma k_{t-1} + d_{1t} \tag{4.51}$$

Evaluating these at steady state levels $c_t = \bar{c}$ and $k_t = \bar{k}$ for all t gives

$$\bar{c} = (\gamma + \delta - 1)\bar{k} + \bar{d}.$$

Substituting the constraints into the Euler equation $(4.B.20)$ and evaluating at $c_t = \bar{c}$ and $k_t = \bar{k}$ gives

$$\phi^2(\beta\delta - 1)(1-\delta)\bar{k} = [1 - \beta(\gamma + \delta)](\bar{b} - \bar{c})$$

Solving the two preceding equations for $\bar{c}$ and $\bar{k}$ gives

$$\bar{k} = [\phi^2(\beta\delta - 1)(1-\delta) + (1-\beta(\gamma+\delta))(\gamma+\delta-1)]^{-1} \\ (1-\beta(\gamma+\delta)) \cdot (\bar{b} - \bar{d}) \tag{4.B.46}$$

$$\bar{c} = \frac{(\gamma+\delta-1)(1-\beta(\gamma+\delta))}{[\phi^2(\beta\delta-1)(1-\delta) + (1-\beta(\gamma+\delta))(\gamma+\delta-1)]} \\ (\bar{b} - \bar{d}) + \bar{d}. \tag{4.B.47}$$

In the special case that $\phi = 0$, these solutions imply that $\bar{c} = \bar{b}$, so that consumption is at bliss consumption and the steady state value of the multiplier λ_{1t} is zero. When $\phi = 0$, the steady state value of k can be taken to be

$$\bar{k} = \frac{1}{1-(\gamma+\delta)}[\bar{d} - \bar{b}], \tag{4.B.48}$$

a solution that makes sense only when $(\gamma+\delta) < 1$. Note that the constraints imply that capital evolves according to

$$k_t = (\gamma+\delta)k_{t-1} - c_t + d_{1t}.$$

Setting $c_t = \bar{c}$ and $d_{1t} = \bar{d}$ implies

$$k_t = (\gamma+\delta)k_{t-1} - \bar{c} + \bar{d}.$$

The solution of this equation is

$$k_t = (\gamma+\delta)^t k_0 + (\bar{d} - \bar{c}) \sum_{j=0}^{t-1} (\gamma+\delta)^j.$$

This solution converges to the solution $(4.B.48)$ for $\bar{k}$ when $\bar{c} = \bar{b}$ and $(\gamma+\delta) < 1$.

Chapter 5
The Commodity Space

5.1. Valuation

This chapter describes a concept of value that we shall later use to formulate a decentralized version of our model in which the decisions of agents are reconciled in a competitive equilibrium. We describe a commodity space in which both the quantities and prices will reside. The stochastic Lagrange multipliers of chapter 4 are very closely related to the equilibrium prices that we shall compute, and live in the same mathematical space with prices.

The social planning problem studied in chapter 4 produces an outcome in which the process for consumption $\{c_t\}$ is an n-dimensional stochastic process that belongs to L_0^2. To calculate the value $\pi(c)$ of a particular consumption plan $c = \{c_t\}$ from the vantage of time zero, we shall use the representation

$$\pi(c) = E \sum_{t=0}^{\infty} \beta^t p_t^0 \cdot c_t \mid J_0,$$

where p_t^0 belongs to L_0^2. The text of this chapter presents a heuristic justification for so representing the value of $\{c_t\}$. We proceed by reviewing several examples of commodity spaces and valuation functions. The appendix contains a more formal treatment.

5.2. Price systems as linear functionals

We follow Debreu (1954) and express values by using a linear functional π that maps elements of a linear space L into the real line. The space L is taken as the commodity space, elements of which are the vectors of commodities to be evaluated. The functional π assigns values to points in L. It is convenient when the functional π has an inner-product representation, which is a representation in which the value $\pi(c)$ of a commodity point c equals the inner product of c with a point p in another linear space $\tilde{L}$. When such a representation exists, we can write

$$\pi(c) = < c \mid p > \quad \text{for all} \quad c \quad \text{in} \quad L \tag{5.2.1}$$

where $p \varepsilon \tilde{L}$ and $< \cdot \mid \cdot >$ denotes an inner product. In all of the cases that we consider, it turns out that $\tilde{L} = L$, so that c and p reside in the same linear space. Next we consider several examples of a commodity space L, a valuation functional π, and an inner product representation for π.

5.3. A one period model under certainty

Suppose that there is one period and no uncertainty. Let there be n consumption goods. Let c be an $n \times 1$ vector of consumption goods. Let the *commodity space* L be R^n, the n-dimensional Euclidean space. In this case, the *value* of a vector c is given by

$$\pi(c) = < c \mid p > \equiv \sum_{i=1}^{n} c_i p_i$$

where $< \cdot \mid \cdot >$ denotes the inner product, and p is an n-dimensional price vector that belongs to $L = R^n$. Note that both c and p belong to the *same* linear space L.

5.4. One period under uncertainty

Suppose there is again one period, but now there is uncertainty about economic outcomes. Prior to the resolution of uncertainty, the quantity of the i^{th} consumption good is a random variable $c_i(\omega)$, where ω is the state of the world to be realized. Let $c = c(\omega)$ be an n-dimensional random vector whose i^{th} component is $c_i(\omega)$. Let $\text{prob}(\omega)$ be the probability density function of ω.

We want to evaluate a bundle of consumption goods prior to the resolution of uncertainty. Introducing uncertainty serves to increase the dimension of the commodity space, there being a vector $c(\omega)$ for each state of the world $\omega \in \Omega$, where Ω is the set of possible states of the world. When there is an infinite number of states of the world Ω, the commodity space L becomes infinite dimensional. To evaluate a state-contingent bundle of consumption goods prior to the resolution of uncertainty requires a well defined notion of "adding up" or integrating across states of the world.

When the number of states of the world is finite (or countable), it is natural to follow Arrow and Debreu and to define an n-dimensional vector of state-contingent prices $q(\omega)$, where $\Omega = [\omega_1, \omega_1, \ldots, \omega_N]$ is the set of possible states of the world. The value of the random vector c can then be represented as

$$\pi(c) = \sum_{j=1}^{N} c(\omega_j) \cdot q(\omega_j) \equiv < c \mid q > . \tag{5.4.1}$$

Here both c and q are elements of L, the space of n-dimensional random vectors indexed by the state of the world. The i^{th} component of $q(\omega), q_i(\omega)$, is to be interpreted as price of one unit of the i^{th} consumption good contingent on the state of the world being ω.

It is convenient to represent $\pi(c)$ in the alternative form

$$\pi(c) = \sum_{j=1}^{N} c(\omega_j) \cdot p(\omega_j) \text{ prob } (\omega_j), \tag{5.4.2}$$

where $q(\omega_j) = p(\omega_j) \, \text{prob} \, (\omega_j)$. Here c and p are each vectors in L, the space of n-dimensional random vectors. Notice that (5.4.2) implies

$$\pi(c) = Ec \cdot p \equiv < c \mid p > .$$

Representation (5.4.1) is often used in contexts in which there is a finite or countable number of states of the world. We find it easier to use representations that build upon (5.4.2) because we shall be dealing with environments with an uncountable number of states of the world.

5.5. An infinite number of periods and uncertainty

We now come to the main case studied in this book. The n-dimensional vector of consumption goods c_t is indexed both by states of the world and by time. We define an information set J_t as in chapters 2 and 3. Let L be the space of all n-dimensional stochastic processes $\{c_t : t = 0, 1, \ldots\}$ for which c_t is in J_t for all t and for which

$$\sum_{t=0}^{\infty} \beta^t E(c_t \cdot c_t) < \infty. \tag{5.5.1}$$

The constraint that c_t be in J_t is imposed because we want to represent the values only of contingent claims that depend on information available when the contingency is realized. The inequality restriction in (5.5.1) identifies which claims might have finite value.

In addition to integrating over states of the world, we also must sum over points in time. We find it convenient to use the discount factor β in performing this summation. Hence we use the following inner product:

$$< c \mid p > = \sum_{t=0}^{\infty} \beta^t E(c_t \cdot p_t). \tag{5.5.2}$$

In this case, the price system used to represent the valuation functional is an n-dimensional stochastic process $\{p_t : t = 0, 1, \ldots\}$ in L.

5.5.1. Conditioning information

So far we have considered valuation functions that map into the real numbers $I\!R$. This approach suffices for representing competitive equilibrium prices for markets that meet and clear prior to the realization of any information. However, we also want to reopen markets and to study valuations at later points in time, conditioned on information available then.

Consider valuation from the vantage point of time τ. Let valuation be conditioned on the time τ information set J_τ. Let π_τ be a time τ valuation function. We take the domain of π_τ to be the space L_τ consisting of all n-dimensional processes $\{c_{t+\tau} : t = 0,\, 1\, \ldots\}$ where $c_{t+\tau}$ is in J_τ and

$$\sum_{t=0}^{\infty} \beta^t E(c_{t+\tau} \cdot c_{t+\tau}) \mid J_\tau < \infty \tag{5.5.3}$$

with probability one. The range of π_τ is J_τ because valuations reflect the available conditioning information.

There is no longer an inner-product representation for π_τ because the range of π_τ is not the real line. Rather, the range is the space of random variables depending on J_τ. However, we can follow Harrison and Kreps (1979) and Hansen and Richard (1987) by using a conditional inner-product representation:

$$\pi_\tau(c) =< c \mid p >_\tau = \sum_{t=0}^{\infty} \beta^t E(c_{t+\tau} \cdot p_{t+\tau} \mid J_\tau) \tag{5.5.4}$$

where $\{p_{t+\tau} : t = 0, 1, \ldots\}$ is a price process in L. The value assigned by π_τ is a random variable in L^2_τ.

5.6. Lagrange multipliers

While we have focused on representing valuation in a competitive equilibrium, much of our discussion applies to using the method of Lagrange multipliers for solving constrained optimization problems. The vector of Lagrange multipliers for a vector of constraints indexed by states of the world and calendar time can be regarded as a stochastic processes $\{\mathcal{M}_t : t = 0, 1, \ldots\}$ in a space L. The contribution to the Lagrangian is given by a corresponding linear functional μ with an inner product representation

$$\mu(\varepsilon) =< \varepsilon \mid \mathcal{M} >= \sum_{t=0}^{\infty} \beta^t E(\varepsilon_t \cdot \mathcal{M}_t) \tag{5.6.1}$$

where ε_t is the deviation of the constraint at time t.

5.7. Summary

Our purpose in this chapter has been to lay groundwork necessary to decentralize the economy described in chapter 3 into one with a collection of price-taking agents whose decisions are coordinated through markets. The Appendix to this chapter describes the valuation functions that we use in more mathematical detail.

A. Appendix

As was indicated above, we model π as a linear functional on a space L. The space L is assumed to be a *linear* space, by which we mean that for any two members x_1 and x_2 in L and any two real numbers c_1 and c_2 in R, $c_1x_1 + c_2x_2$ are in L. In addition, we suppose that there is an *inner product* $< \cdot \mid \cdot >$ defined on L. This inner product can be used to define a norm $\| x \| =< x \mid x >^{1/2}$ and hence a metric. We take L to be *complete*. This means that all Cauchy sequences in L converge to an element of L. The commodity spaces in all of the examples described in the text are complete linear spaces. The restriction that π be linear requires that $\pi(c_1x_1 + c_2x_2) = c_1\pi(x_1) + c_2\pi(x_2)$. According to the Riesz Representation Theorem, π has an inner product representation whenever π is continuous at zero.

When conditioning information is introduced, it is convenient to work with a space L_J that is linear conditioned on J. For the moment, consider L_J to be a collection of random variables. Products and sums of random variables are also random variables. For L_J to be linear conditioned on J, for any two elements x_1 and x_2 of L_J and any w_1 and w_2 in J, we require that $w_1x_1 + w_2x_2$ is in L_J. Similarly, π_J is conditionally linear if $\pi_J(w_1x_1 + w_2x_2) = w_1\pi_J(x_1) + w_2\pi_J(x_2)$. The rationale for focusing on conditional linearity is that information in J can be used to construct consumption plans or trading strategies. Hansen and Richard (1987) obtained a conditional counterpart to the Riesz Representation Theorem that establishes the existence of a representation $\pi_J(x) = E(x \cdot p \mid J)$ for some p in L_J.

The restriction that L_J be a space of random variables is too limited for our purposes. Instead, we are interested in spaces of n-dimensional stochastic processes. Given an initial probability space $(\Omega, F, \Pr)$ and a sequence $\{F_t : t = 0, 1, \ldots\}$ of subsigma algebras of F, we construct a new probability space $(\Omega^+, F^+, \Pr^+)$ where Ω^+ is the Cartesian product of Ω^+, the nonnegative integers, and the set $\{1, 2, \ldots, n\}$, and where $\Pr^+$ is the product measure of $\Pr$, a measure that assigns $\beta^t(1-\beta)$ to integer t, and $1/n$ to integer j. The sigma algebra F^+ is generated by sets of the form:

$$\{(w, t, j) : w \in f_{t,j}\} \tag{5.A.1}$$

where $\{f_{t,j} : t = 0, 1, \ldots; , j = 1, 2, \ldots, n\}$ is a collection of sets in F such that $f_{t,j}$ is in F_t for all t and j. An n-dimensional stochastic process defined on the original space can be viewed as a random variable on the product space. Thus we can apply the preceding analysis to obtain a conditional inner product representation for $\pi_{\mathcal{T}}$ described in the text.

Chapter 6
A Competitive Economy

6.1. Introduction

This chapter describes a decentralized version of our economy. We assign ownership and decision making to three distinct economic entities, a household and two kinds of firms. We define a *competitive equilibrium.* Versions of the two fundamental theorems of welfare economics are true. We establish these theorems by exhibiting the connection between a competitive equilibrium and a social planning problem. A price system supports the competitive equilibrium, and implies interest rates and prices for derivative assets.

The representative household can be interpreted as a single individual drawn from a population that is homogeneous in all respects. Alternatively, the representative household can be interpreted along lines to be described in chapter 12, as an artificial or "average" household that emerges from aggregating over the preferences and endowments of a collection of households. The representative household owns the technology shock process d_t, and each period sells to firms the current period's realization of the shock process. The household owns the initial stocks h_{-1} of household capital and k_{-1} of productive capital, the latter of which it sells to firms. It sells this initial capital for a value $v_0 \cdot k_{-1}$. The household sells its input ℓ_t to firms. The household uses its resources to purchase consumption goods, which add to its stocks of consumer durables and thereby generate consumption services and utility.

Of the two types of firms, the first type rents capital from firms of type II, rents labor from the household, and buys the current period's realization of the technology shock process d_t from the household. A firm of type I produces new consumption and investment goods, sells the consumption goods to the household, and sells the investment goods to the firms of type II. A firm of type II purchases the initial capital stock k_{-1} and all of the investment goods produced each period, then rents capital to firms of type I.

We use a formulation of a price system which is mathematically convenient, as well as economically interpretable. We let the price system be $[v_0, \{p_t^0, w_t^0, \alpha_t^0, q_t^0,$

$r_t^0\}_{t=0}^\infty]$, where v_0 is a vector that prices the initial capital stock k_{-1}; p_t^0 is an $n_c \times 1$ stochastic process that prices the consumption process c_t; w_t^0 is a scalar stochastic process that prices ℓ_t; α_t^0 is a vector stochastic process that prices the process $\{d_t\}$; q_t^0 is an $n_k \times 1$ vector stochastic process that prices new investment goods; and r_t^0 is an $n_k \times 1$ vector stochastic process of capital rental rates. Each component of $[\{p_t^0, w_t^0, \alpha_t^0, q_t^0, r_t^0\}_{t=0}^\infty]$ resides in the mathematical space L_0^2 defined earlier, namely, $L_0^2 = \Big[\{y_t\}_{t=0}^\infty \colon y_t$ is a random variable in J_t for $t \geq 0$, and $E \sum_{t=0}^\infty \beta^t \ y_t^2 \mid J_0 < +\infty\Big]$. That '$y_t$ is in J_t' means that y_t can be expressed as a measurable function of $J_t = [w^t, x_0]$, where $J_0 = [x_0]$. The square summability requirement, $E \sum_{t=0}^\infty \beta^t y_t^2 \mid J_0 < \infty$, imposes a stochastic version of a requirement that y_t not grow too fast in absolute value.

Stochastic processes for both prices and quantities in our economy must reside in L_0^2. By virtue of a Cauchy-Schwartz inequality, this makes the conditional inner products to be used in the budget constraints and objective functions below well defined and finite in equilibrium.

This chapter formulates and computes a competitive equilibrium. We proceed by first describing the problem for each of our three classes of agents in terms of a Lagrangian. Next we obtain the first order conditions from these Lagrangians. By "matching up" these first-order conditions to the first order conditions found in chapter 3 for the social planning problem, we accomplish two goals. First, we can verify the two fundamental theorems of welfare economics for our economy. Second, we can describe an efficient algorithm for computing the equilibrium price system in terms of the matrices M_k, M_h, M_s, M_d, M_c, and M_i of chapter 3 associated with the multipliers for the social planning problem.

We first describe the problems faced by each of our three types of agents.

6.2. The Problems of Households and Firms

6.2.1. Households

The household chooses stochastic processes for $\{c_t,\, s_t,\, h_t,\, \ell_t\}_{t=0}^{\infty}$, each element of which is in L_0^2, to maximize

$$-\ \frac{1}{2}\ E_0 \sum_{t=0}^{\infty} \beta^t \left[(s_t - b_t) \cdot (s_t - b_t) + \ell_t^2 \right] \tag{6.2.1}$$

subject to

$$E \sum_{t=0}^{\infty} \beta^t\, p_t^0 \cdot c_t \mid J_0 = E \sum_{t=0}^{\infty} \beta^t\, (w_t^0 \ell_t + \alpha_t^0 \cdot d_t) \mid J_0 + v_0 \cdot k_{-1} \tag{6.2.2}$$

$$s_t = \Lambda h_{t-1} + \Pi c_t \tag{6.2.3}$$

$$h_t = \Delta_h h_{t-1} + \Theta_h c_t, \quad h_{-1}, k_{-1} \ \text{ given.} \tag{6.2.4}$$

The household and each firm acts as a price taker. The optimal contingency plan for $(c_t,\, s_t,\, h_t,\, \ell_t)$ must be "realizable" in the sense that time t decisions must be contingent only on information available at time t, i.e., it must reside in L_0^2.

6.2.2. Firms of type I

A firm of type I rents capital and labor, and buys the realization of the endowment process d_t. It uses these inputs to produce consumption goods and investment goods, which it sells.

The firm of type I chooses stochastic processes for $\{c_t, i_t, k_t, \ell_t, g_t, d_t\}$, each element of which is in L_0^2, to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t\, (p_t^0 \cdot c_t + q_t^0 \cdot i_t - r_t^0 \cdot k_{t-1} - w_t^0 \ell_t - \alpha_t^0 \cdot d_t) \tag{6.2.5}$$

subject to

$$\Phi_c\, c_t + \Phi_g\, g_t + \Phi_i\, i_t = \Gamma k_{t-1} + d_t \tag{6.2.6}$$

$$- \ell_t^2 + g_t \cdot g_t = 0 \tag{6.2.7}$$

6.2.3. Firms of type II

A firm of type II is in the business of purchasing investment goods and renting capital to firms of type I. A firm of type II faces as a price taker the vector v_0 and the stochastic processes $\{r_t^0, q_t^0\}$. The firm chooses k_{-1} and stochastic processes for $\{k_t, i_t\}_{t=0}^\infty$ to maximize

$$E \sum_{t=0}^\infty \beta^t (r_t^0 \cdot k_{t-1} - q_t^0 \cdot i_t) \mid J_0 - v_0 \cdot k_{-1} \tag{6.2.8}$$

subject to

$$k_t = \Delta_k k_{t-1} + \Theta_k i_t. \tag{6.2.9}$$

6.3. Competitive Equilibrium

We define a competitive equilibrium for this economy.

DEFINITION: A *competitive equilibrium* is a price system $[v_0, \{p_t^0, w_t^0, \alpha_t^0, q_t^0, r_t^0\}_{t=0}^\infty]$ and an allocation $\{c_t, i_t, k_t, h_t, g_t, d_t\}_{t=0}^\infty$ that satisfy the following conditions:

a. Each component of the price system and the allocation resides in the space L_0^2.
b. Given the price system and given h_{-1}, k_{-1}, the stochastic process $\{c_t, s_t, \ell_t,$
$h_t\}_{t=0}^\infty$ solves the consumer's problem.
c. Given the price system, the stochastic process $\{c_t, i_t, k_t, \ell_t, d_t, g_t\}$ solves the problem of the firm of type I.
d. Given the price system, the vector k_{-1} and the stochastic process $\{k_t, i_t\}_{t=0}^\infty$ solve the problem of the firm of type II.

6.4. Lagrangians

We now formulate each agent's problem as a Lagrangian, and obtain the associated first order conditions.

6.4.1. Households

The household's Lagrangian is

$$
\begin{aligned}
L = - E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ & \big[(s_t - b_t) \cdot (s_t - b_t) + \ell_t^2 \big] / 2 \\
& + \mu_0^w [p_t^0 \cdot c_t - w_t^0 \ell_t - \alpha_t^0 \cdot d_t] \\
& + \mu_t^{s\prime} [s_t - \Lambda h_{t-1} - \Pi c_t] \\
& + \mu_t^{h\prime} [h_t - \Delta_h h_{t-1} - \Theta_h c_t] \Big\} + \mu_0^w \, v_0 \cdot k_{-1}.
\end{aligned}
$$

Here μ_0^w is a scalar and $\{\mu_t^s, \mu_t^h\}$ are sequences of vectors of stochastic Lagrange multipliers. The first order conditions with respect to $s_t,\ \ell_t,\ c_t,$ and h_t, respectively, are:

$$
\begin{aligned}
s_t:& \quad (s_t - b_t) + \mu_t^s = 0, && t \geq 0 \\
\ell_t:& \quad \ell_t - w_t^0 \cdot \mu_0^w = 0, && t \geq 0 \\
c_t:& \quad \mu_0^w \, p_t^0 - \Pi' \mu_t^s - \Theta_h' \ \mu_t^h = 0, && t \geq 0 \\
h_t:& \quad -\beta E_t \ \Lambda' \mu_{t+1}^s - \beta E_t \ \Delta_h' \, \mu_{t+1}^h + \mu_t^h = 0, && t \geq 0
\end{aligned}
$$

Solving these equations, we obtain

$$
\mu_t^s = b_t - s_t, \quad t \geq 0 \tag{6.4.1}
$$

$$
w_t^0 = \ell_t / \mu_0^w, \quad t \geq 0 \tag{6.4.2}
$$

$$
\mu_t^h = E_t \sum_{\tau=1}^{\infty} \beta^\tau \, (\Delta_h')^{\tau-1} \ \Lambda' \, \mu_{t+\tau}^s \quad , \ t \geq 0 \tag{6.4.3}
$$

$$
\mu_0^w \, p_t^0 = \Pi' \, \mu_t^s + \Theta_h' \, \mu_t^h, \quad t \geq 0 \tag{6.4.4}
$$

6.4.2. Firms of type I

The Lagrangian of a type I firm is

$$L_I = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ [p_t^0 \cdot c_t + q_t^0 \cdot i_t - r_t^0 \cdot k_{t-1} - w_t^0 \ell_t - \alpha_t^0 \cdot d_t]$$
$$+ \mathcal{L}_t^{d\prime} [\Gamma k_{t-1} + d_t - \Phi_c c_t - \Phi_g g_t - \Phi_i i_t]$$
$$+ \mathcal{L}_t^{\ell\prime} [(\ell_t^2 - g_t \cdot g_t)/2] \Big\}.$$

Here $\{\mathcal{L}_t^d, \mathcal{L}_t^\ell\}$ is a vector stochastic process of Lagrange multipliers. The first order conditions associated with interior solutions for $c_t, i_t, k_t, \ell_t, d_t$, and g_t, respectively, are

$$c_t: \quad p_t^0 - \Phi_c' \mathcal{L}_t^d = 0, \quad t \geq 0 \tag{6.4.5}$$
$$i_t: \quad q_t^0 - \Phi_i' \mathcal{L}_t^d = 0, \quad t \geq 0 \tag{6.4.6}$$
$$k_t: \quad r_{t+1}^0 - \Gamma' \mathcal{L}_{t+1}^d = 0, \quad t \geq -1 \tag{6.4.7}$$
$$\ell_t: \quad - w_t^0 + \mathcal{L}_t^\ell \ell_t = 0, \quad t \geq 0 \tag{6.4.8}$$
$$d_t: \quad - \alpha_t^0 + \mathcal{L}_t^d = 0, \quad t \geq 0 \tag{6.4.9}$$
$$g_t: \quad - \Phi_g' \mathcal{L}_t^d - g_t \mathcal{L}_t^\ell = 0, \quad t \geq 0 \tag{6.4.10}$$

Solving (6.4.5) and (6.4.10) for $\mathcal{L}_t^d$ gives

$$\mathcal{L}_t^d = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} p_t^0 \\ -g_t \mathcal{L}_t^\ell \end{bmatrix}. \tag{6.4.11}$$

From (6.4.8), the solution for $\mathcal{L}_t^\ell$ satisfies

$$\mathcal{L}_t^\ell = w_t^0 / \ell_t. \tag{6.4.12}$$

Equations (6.4.6), (6.4.7) and (6.4.9) imply

$$q_t^0 = \Phi_i' \ \mathcal{L}_t^d \tag{6.4.13}$$
$$r_t^0 = \Gamma' \ \mathcal{L}_t^d \tag{6.4.14}$$
$$\alpha_t^0 = \mathcal{L}_t^d \tag{6.4.15}$$

6.4.3. Firms of type II

The Lagrangian of a type II firm is

$$L_{II} = E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ (r_t^0 \cdot k_{t-1} - q_t^0 \cdot i_t)$$
$$+ \eta_t' \left(\Delta_k k_{t-1} + \Theta_k i_t - k_t \right) \Big\} - v_0 \cdot k_{-1}$$

where $\{\eta_t\}$ is a sequence of stochastic Lagrange multipliers. The first order conditions for interior solutions with respect to k_t, i_t, and k_{-1}, respectively, are

$$k_t : \quad \beta E_t \, r_{t+1}^0 - \eta_t + \beta E_t \, \Delta_k' \, \eta_{t+1} = 0, \quad t \geq 0 \tag{6.4.16}$$

$$i_t : \quad - q_t^0 + \Theta_k' \, \eta_t = 0, \quad t \geq 0 \tag{6.4.17}$$

$$k_{-1} : \quad r_0^0 + \Delta_k' \eta_0 - v_0 = 0 \tag{6.4.18}$$

Solving (6.4.16) for η_t gives

$$\eta_t = E_t \Big(\sum_{j=1}^{\infty} \beta^j \, \Delta_k'^{(j-1)} \, r_{t+j}^0 \Big), \; t \geq 0 \tag{6.4.19}$$

Equation (6.4.17) implies

$$q_t^0 = \Theta_k' \, \eta_t, \quad t \geq 0 \tag{6.4.20}$$

Equation (6.4.18) implies

$$v_0 = r_0^0 + \Delta_k' \, \eta_0 \tag{6.4.21}$$

6.5. Equilibrium Price System

Our task now is to find stochastic processes of prices, quantities, and Lagrange multipliers that satisfy the first-order conditions for each of our three classes of agents for all time and contingencies. We proceed constructively to link equilibrium prices to the Lagrange multipliers for the planning problem.

Recall the following equations obeyed by the Lagrange multipliers associated with the social planning problem:

$$\mathcal{M}_t^s = b_t - s_t \tag{4.8}$$

$$\mathcal{M}_t^h = E\left[\sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h')^{\tau-1} \Lambda' \mathcal{M}_{t+\tau}^s \mid J_t\right] \tag{4.9}$$

$$\mathcal{M}_t^d = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} \Phi_h' \mathcal{M}_t^h + \Pi' \mathcal{M}_t^s \\ -g_t \end{bmatrix} \tag{4.11}$$

$$\mathcal{M}_t^k = E\left[\sum_{\tau=1}^{\infty} \beta^\tau \, (\Delta')^{\tau-1} \, \Gamma' \, \mathcal{M}_{t+\tau}^d \mid J_t\right] \tag{4.19}$$

In chapter 3, we gave formulas for these multipliers along the optimum of the social planning problem, namely,

$$\mathcal{M}_t^k = M_k x_t \quad \text{and} \quad \mathcal{M}_t^h = M_h x_t \tag{4.21}$$

$$\mathcal{M}_t^s = M_s x_t \tag{4.22}$$

$$\mathcal{M}_t^d = M_d x_t. \tag{4.23}$$

We also defined shadow prices for consumption and investment:

$$\mathcal{M}_t^c = M_c x_t, \; M_c = \Theta_h' M_h + \Pi' M_s \tag{4.24}$$

$$\mathcal{M}_t^i = M_i x_t, \; M_i = \Theta_i' M_k. \tag{4.25}$$

We gave formulas for the matrices M_s, M_k, M_h and M_d in terms of the optimal value function of the social planning problem. The formulas (4.21), (4.22), (4.23), (4.24), (4.25) for the multipliers are evaluated along the solution $x_{t+1} = A^o x_t + Cw_{t+1}$ of the social planning problem.

We can compute the equilibrium price system in terms of the multipliers from the social planning problem. For the time being let μ_0^w be a free parameter. Later we shall indicate how choosing the scalar marginal utility of wealth at time zero, μ_0^w, amounts to specifying a numeraire for our price system. We propose to set

$$p_t^0 = \left[\Pi' \mathcal{M}_t^s + \Theta_h' \mathcal{M}_t^h\right] / \mu_0^w = \mathcal{M}_t^c / \mu_0^w \tag{6.5.1}$$

$$w_t^0 = \mid S_g x_t \mid / \mu_0^w \tag{6.5.2}$$

$$r_t^0 = \Gamma' \mathcal{M}_t^d / \mu_0^w \tag{6.5.3}$$

$$q_t^0 = \Theta_k' \mathcal{M}_t^k / \mu_0^w = \mathcal{M}_t^i / \mu_0^w \tag{6.5.4}$$

$$\alpha_t^0 = \mathcal{M}_t^d / \mu_0^w \tag{6.5.5}$$

$$v_0 = \Gamma' \mathcal{M}_0^d / \mu_0^w + \Delta_k' \mathcal{M}_0^k / \mu_0^w . \tag{6.5.6}$$

We shall verify that with this price system, values can be assigned to the Lagrange multipliers for each of our three classes of agents that cause all of their first-order necessary conditions to be satisfied at these prices and at the quantities associated with the optimum of the social planning problem.

For the household, we set

$$\mu_t^s = \mathcal{M}_t^s \tag{6.5.7}$$

$$\mu_t^h = \mathcal{M}_t^h \tag{6.5.8}$$

With these choices of multipliers, equations (6.4.1), (6.4.2), (6.4.3) and (6.4.4) are evidently satisfied at the proposed equilibrium prices (6.5.1) – (6.5.6) and at the quantities associated with the optimum of the social planning problem.

For the firm of type I, we set

$$\mathcal{L}_t^d = \mathcal{M}_t^d / \mu_0^w \tag{6.5.9}$$

$$\mathcal{L}_t^\ell = 1 / \mu_0^w . \tag{6.5.10}$$

With the settings (6.5.9) for $\mathcal{L}_t^d$, (6.5.10) for $\mathcal{L}_t^\ell$, and the price process (6.5.1), equation (6.4.11) becomes equivalent with (4.11) from the social planning problem. Equation (6.5.3) for r_t^0 implies that the firm's marginal condition (6.4.14) is satisfied along the solution of the social planning problem. Similarly, (6.4.20) implies that (6.4.15) is satisfied. Formula (6.5.4) for q_t^0 together with the fourth equation of (4.8) ($-\Phi_i' \mathcal{M}_t^d + \Theta_k' \mathcal{M}_t^k = 0$) implies that (6.4.13) is satisfied along the solution of the social planning problem. Finally, (6.5.9)–(6.5.10) imply that (6.4.10) is equivalent with the second equation of (4.8) ($-g_t - \Phi_g' \mathcal{M}_t^d = 0$). Thus, with settings (6.5.8), (6.5.9), price system (6.5.1)–(6.5.6) implies that firm I's first order necessary conditions are satisfied along the quantity path implied by the social optimum.

For the firm of type II, we set

$$\eta_t = \mathcal{M}_t^k / \mu_0^w. \tag{6.5.11}$$

With this setting, (6.5.6) and (3.19) imply that (6.4.19) (and thus (6.4.16)) is satisfied. Also, (6.4.20) is evidently satisfied as is (6.4.21). Thus, the first order conditions for firms of type II are all satisfied at price system (6.5.1)–(6.5.6) along the solution of the social planning problem. We are finished.

In summary, the price system (6.5.1)–(6.5.6) supports the allocation associated with the optimum of the social planning problem as a competitive equilibrium. The direction of argument can be reversed to establish that a competitive equilibrium solves the social planning problem. This argument uses a competitive equilibrium allocation and price system to define multiplier processes that satisfy the first order conditions for the social planning problem.[1]

The scalar μ_0^w that appears as a free parameter in (6.5.1)–(6.5.6) is evidently the marginal utility of wealth at time zero. In setting this parameter, we select a numeraire for our price system. For example, the j^{th} consumption good at time zero can be selected as the numeraire by setting

$$\mu_0^w = \bar{e}_j \, \mathcal{M}_t^c = \bar{e}_j \, M_c x_0$$

where $\bar{e}_j$ is a $(1 \times n_c)$ vector consisting of zeros in each location except the jth, where there is a one. For the jth consumption good at time zero to be a valid numeraire, we require that $\bar{e}_j \, M_c x_0$ not equal zero. This is imposed in:

Assumption 5.1: The random variable $\bar{e}_j \, M_c x_0$ selected as numeraire differs from zero with probability one.

[1] Since the solution of the social planning problem is unique, so is the competitive equilibrium.

6.6. Asset Pricing

We can use the main idea behind "arbitrage pricing theory" to motivate asset pricing formulas. Arbitrage pricing theory extracts solely from the weak implication of equilibrium that assets must be priced so that budget sets offer no opportunities for earning sure returns with a zero commitment of resources.

To illustrate this approach, imagine altering the representative household's problem (6.2.1) – (6.2.4) by supplying it with one additional opportunity. The household can go into the securities business on the side by issuing securities that promise to pay off a stream of the $(n_c \times 1)$ vector of consumption goods $\{y_t\}$. We assume that $\{y_t\} \in L_0^2$. Suppose there is a market in such securities and that the price at time 0 of one unit of such security is a_0. If the household sells S of these securities, its revenue at time 0 is Sa_0. To cover itself in all contingencies, the household must purchase state contingent claims to consumption in the amount $\{y_t\}$ for each unit of the security issued. The cost of purchasing these claims to support the sale of S securities is

$$S \cdot E \sum_{t=0}^{\infty} \beta^t \, p_t^0 \cdot y_t \mid J_0.$$

With this opportunity opened up to the household, the following term must be added to the right side of household's budget constraint (6.2.2):

$$S(a_0 - E \sum_{t=0}^{\infty} \beta^t \, p_t^0 \cdot y_t \mid J_0).$$

If $a_0 > E\sum_{t=0}^{\infty} \beta^t \, p_t^0 \cdot y_t \mid J_0$, the household can make the present value of consumption as large as it wants by setting S equal to a suitable positive number, i.e., by *selling* the security whose price is a_0. However, for our economy, it is not *feasible* for the consumer to achieve any such desired present value of consumption. Therefore, in equilibrium we cannot have $a_0 > E\sum_{t=0}^{\infty} \beta^t \, p_t^0 \cdot y_t \mid J_0$. Similarly, we cannot have $a_0 < E\sum_{t=0}^{\infty} \beta^t \, p_t^0 \cdot y_t \mid J_0$, because that would confront the household with the opportunity to make the present value of consumption as large as it wants by *buying* the security at prices a_0, then selling the returns y_t in the market for state contingent claims. Therefore, we must have

$$a_0 = E \sum_{t=0}^{\infty} \beta^t p_t^0 \cdot y_t \mid J_0. \tag{6.6.1}$$

We can use (6.6.1) to derive formulas for various special $\{y_t\}$ processes, and thereby recover versions of Lucas's asset pricing model [1978], and theories of the term structure of interest rates. We derive more explicit formulas for assets with payoffs of the form

$$y_t = U_a \, x_t \tag{6.6.2}$$

where U_a is an $n_c \times n$ matrix. Substituting (6.6.2) and the pricing formula $p_t^0 = M_c x_t / \mu_0^w$ into (6.6.1) gives

$$a_0 = E \sum_{t=0}^{\infty} \beta^t \, x_t' \, Z_a x_t \mid J_0 \tag{6.6.3}$$

where

$$Z_a = U_a' M_c / \mu_0^w . \tag{6.6.4}$$

We shall now show that a_0 can be represented as

$$a_0 = x_0' \, \mu_a \, x_0 + \sigma_a \tag{6.6.5}$$

where

$$\mu_a = \sum_{\tau=0}^{\infty} \beta^\tau \, (A^{o\prime})^\tau \; Z_a \, A^{o\tau} \tag{6.6.6}$$

$$\sigma_a = \frac{\beta}{1-\beta} \text{ trace } Z_a \sum_{\tau=0}^{\infty} \beta^\tau \, (A^o)^\tau \, CC'(A^{o\prime})^\tau . \tag{6.6.7}$$

According to (6.6.5), the asset price a_0 turns out to be the sum of a constant σ_a, which reflects a "risk premium," and a quadratic form in the state vector x_t. To understand why σ_a reflects a risk premium, notice how the parameters in C influence σ_a but do not influence μ_a.

To derive (6.6.5), first express (6.6.3) as[2]

$$a_0 = E \sum_{t=0}^{\infty} \beta^t \text{ trace } [Z_a x_t x_t'] \mid J_0 . \tag{6.6.8}$$

For $t \geq 1$, (1.5) implies that

$$E x_t \, x_t' \mid J_0 = \sum_{\tau=0}^{t-1} (A^o)^\tau \, CC'(A^{o\prime})^\tau + (A^o)^t \, x_0 \, x_0' \, (A^{o\prime})^t . \tag{6.6.9}$$

[2] An alternative way to derive these formulas is described in chapter [ ] (seasonality).

Substituting (6.6.9) into (6.6.8) and rearranging gives

$$\begin{aligned} a_0 = \sum_{t=1}^{\infty} \beta^t \ \text{trace} \ \left[Z_a \sum_{\tau=0}^{t-1} (A^o)^\tau \, CC' \, (A^{o\prime})^\tau \right] \\ + \ \text{trace} \ \ Z_a \sum_{t=0}^{\infty} \beta^t \, (A^o)^t \, x_0 \, x_0' \, (A^{o\prime})^t . \end{aligned} \tag{6.6.10}$$

Exchanging orders of summation in the first term on the right of (6.6.10) gives

$$\begin{aligned} \sum_{t=1}^{\infty} \beta^t \ \text{trace} & \left[Z_a \sum_{\tau=0}^{t-1} (A^o)^\tau \, CC' \, (A^{o\prime})^\tau \right] \\ &= \text{trace} \ Z_a \sum_{\tau=0}^{\infty} \ \sum_{t=\tau+1}^{\infty} \ \beta^t \, (A^o)^\tau \, CC' \, (A^{o\prime})^\tau \\ &= \ \frac{\beta}{1-\beta} \ \text{trace} \ Z_a \sum_{\tau=0}^{\infty} \beta^\tau \, (A^o)^\tau \, CC' \, (A^{o\prime})^\tau \\ &\equiv \ \sigma_a \end{aligned}$$

which establishes (6.6.7).

The second term on the right side of (6.6.10) can be transformed (by repeatedly using the rule $\text{trace}\, AB = \text{trace}\, BA$) to

$$x_0' \ \sum_{t=0}^{\infty} \beta^t \, (A^{o\prime})^t \, Z_a \, (A^o)^t \, x_0 \equiv x_0' \, \mu_a x_0 ,$$

which defines the matrix μ given in (6.6.6). This completes our verification of the asset pricing formulas (6.6.5) – (6.6.7).

To implement (6.6.5) requires the application of numerical methods to calculate the matrices μ_a and σ_a that satisfy (6.6.6) and (6.6.7). An efficient 'doubling algorithm' for calculating these matrices is described in chapter 8.

As an application of (6.6.3) – (6.6.5), let us compute the value of a title to one unit of the stream of the jth endowment shock, $\{d_{jt}\}_{t=0}^{\infty}$. Let $d_{jt} = e_j \, x_t$, where e_j is a selection vector that picks off the appropriate linear combination of x_t. From (6.5.5) we have that the time zero value of the time t shock d_{jt} is

$$d_{jt} \, M^d x_t / \mu_0^w = x_t' \, e_j' \, M^d x_t / \mu_0^w .$$

The value of the entire stream is then given by

$$E\sum_{t=0}^{\infty}\beta^t\, x_t'\, Z_a x_t \mid J_0$$

where $Z_a = e_j'\, M^d/\mu_0^w$. This matches (6.6.3), so that formulas (6.6.5)–(6.6.7) are applicable.

6.7. Term Structure of Interest Rates

The value at time zero of a sure claim to one unit of the first consumption good at time zero is evidently given by

$$R_1^0 = \beta E[\bar{e}_1 \cdot p_1^0] \mid J_0$$

or

$$R_1^0 = \beta\, \bar{e}_1 \cdot M_c\, A^o\, x_0/\mu_0^w. \tag{6.7.1}$$

Here R_1^0 is the reciprocal of the gross one-period sure interest rate at time zero. For longer horizons, we have

$$R_j^0 = \beta^j\, E[\bar{e}_1 \cdot p_j^0] \mid J_0,\ j \geq 1$$

or

$$R_j^0 = \beta^j\, \bar{e}_1 \cdot M_c\, A^{oj}\, x_0/\mu_0^w. \tag{6.7.2}$$

Here R_j^0 is the reciprocal of the gross interest factor for a sure claim on the first consumption good j periods into the future at time zero.

6.8. Re-opening Markets

The competitive equilibrium prices state– and date–contingent commodities that are traded at time zero. After time zero, markets are "closed," with traders simply executing agreements entered into at time zero. As usual in Arrow-Debreu models, markets can be opened in subsequent time periods, but are redundant in the sense that zero trades occur. However, for the purpose of extracting the time series implications of our model, it is useful to compute the prices in such re-opened markets.

Suppose that markets re-open at some time $t \geq 1$, and that the household and firms re-evaluate their contingency plans at new prices. The household now values consumption services from time t forward. Only goods from time t forward enter the valuations appearing in the budget sets and objective functions of each of our agents. We use L_t^2 as the commodity space, defined as

$$L_t^2 = [\{y_s\}_{s=t}^{\infty} : \ y_s \ \text{ is a random variable in } \ J_s \ \text{ for } \ s \geq t$$
$$\text{and } E \sum_{s=t}^{\infty} \beta^{s-t} \, y_s^2 \mid J_t < +\infty]$$

Expectations conditioned on J_t replace those conditioned on J_0 in the intertemporal budget constraint of the household and the cash flow evaluations of the firms. For convenience, we use the jth consumption good at time t as the numeraire. For this choice to deliver a valid numeraire, we invoke

Assumption 5.2: The random variable $\bar{e}_j \, M^c \, x_t$ differs from zero with probability one.

We set the household's marginal utility of time t wealth, μ_t^w, equal to $\bar{e}_j \, M^c \, x_t$ in order to select the time t, j^{th} consumption good as numeraire. With these specifications, we can simply replicate the time zero analysis to obtain equilibrium prices from the vantage point of time t. This yields the following price system:

$$p_s^t = M_c x_s / [\bar{e}_j M_c x_t], \qquad s \geq t \tag{6.8.1}$$

$$w_s^t = \mid S_g x_s \mid / [\bar{e}_j M_c x_t], \quad s \geq t \tag{6.8.2}$$

$$r_s^t = \Gamma' M_d x_s / [\bar{e}_j M_c x_t], \quad s \geq t \tag{6.8.3}$$

$$q_s^t = M_i x_s / [\bar{e}_j \, M_c x_t], \qquad s \geq t \tag{6.8.4}$$

$$\alpha_s^t = M_d x_s / [\bar{e}_j \, M_c x_t], \quad s \geq t \tag{6.8.5}$$

$$v_t = [\Gamma' M_d + \Delta_k' M_k] x_t / \, [\bar{e}_j \, M_c x_t] \tag{6.8.6}$$

Of particular interest are the *spot market* prices implied by (6.8.1) – (6.8.6), namely, $p_t^t, w_t^t, r_t^t, q_t^t, \alpha_t^t$.

6.8.1. Recursive price system

Prescott and Mehra [1980] and Lucas [1982, JME] extensively utilized recursive formulas expressed in terms of one period forward state contingent claims prices. Counterparts to their recursive pricing formulas are easy to express for our framework. In particular, one-period forward claims on consumption are priced by the function

$$p_{t+1}^t = M_c \, x_{t+1} / \bar{e}_j \, M_c x_t.$$

At time t, claims on consumption j-period forward are priced by

$$p_{t+j}^t = M_c x_{t+j} / \bar{e}_j M_c x_t.$$

Evidently, p_{t+j}^t can be built up recursively using the equality

$$\begin{aligned} p_{t+j}^t &= p_{t+j}^{t+1} \; \bar{e}_j \, p_{t+1}^t \\ &= \frac{M_c \, x_{t+j}}{\bar{e}_j \, M_c x_{t+1}} \quad \bar{e}_j \quad \frac{M_c \, x_{t+1}}{\bar{e}_j \, M_c \, x_t}. \end{aligned}$$

This is a version of a recursive pricing formula often used in formulations of recursive competitive equilibria.

6.8.2. Non-Gaussian asset prices

The time t value of a permanent claim to a stream $y_s = U_a x_s, s \geq t$ is given by

$$a_t = (x_t' \, \mu_a \, x_t + \sigma_a)/(\bar{e}_j \, M_c \, x_t) \tag{6.8.7}$$

where μ_a and σ_a satisfy (6.6.6) and (6.6.7) with $Z_a = U_a' M_c$. Notice how (6.8.7) makes the asset price a_t a nonlinear function of the state vector x_t. Suppose, for example, that the w_t process is Gaussian. This implies that the equilibrium x_t process given by is a multivariate normal process. Even so, the asset prices determined by (6.8.7) are not normally distributed, being determined as the ratio of a quadratic form in the Gaussian state vector x_t to a linear form in x_t. Thus, the asset prices generated by this "most Gaussian of economies" are highly nonlinear stochastic processes.

The term structure of interest rates on perfectly safe claims on the first consumption good j periods ahead is characterized by the gross interest factors

$$R_j^t = \beta^j \bar{e}_1 \cdot M_c A^{oj} x_t / [\bar{e}_j M_c x_t], \quad j \geq 1, \; t \geq 0 \tag{6.8.8}$$

which generalizes (6.7.2).

6.9. Summary of Pricing Formulas

For convenience, we now summarize our formulas for the competitive equilibrium price system. They are:

(6.58) $$p_s^t = M_c x_s / [\bar{e}_j \, M_c \, x_t], \; s \geq t$$

(6.59) $$w_s^t = \mid S_g x_s \mid / [\bar{e}_j \, M_c \, x_t], \; s \geq t$$

(6.60) $$r_s^t = \Gamma' \, M_d x_s / [\bar{e}_j \, M_c \, x_t], \; s \geq t$$

(6.61) $$q_s^t = M_i x_s / [\bar{e}_j \, M_c \, x_t], \; s \geq t$$

(6.62) $$\alpha_s^t = M_d x_s / [\bar{e}_j \, M_c \, x_t], \; s \geq t$$

$$\upsilon_t = [\Gamma' M_d + \Delta_k' M_k]\, x_t / \bar{e}_j\, M_c\, x_t \tag{6.63}$$

The asset that entitles the owner to the stream of returns $y_t = U_a x_t$ is priced according to

$$a_t = (x_t'\, \mu_a\, x_t + \sigma_a)/[\bar{e}_j\, M_c\, x_t] \tag{6.64}$$

where

$$\mu_a = \sum_{\tau=0}^{\infty} \beta^\tau\, (A^{o\prime})^\tau\ Z_a\, A^{o\tau} \tag{6.51}$$

$$\sigma_a = \ \frac{\beta}{1-\beta} \text{ trace } Z_a \sum_{\tau=0}^{\infty} \beta^\tau\, (A^o)^\tau CC'\, (A^{o\prime})^\tau \tag{6.52}$$

$$Z_a = U_a'\, M_c$$

The term structure of interest rates is determined by

$$R_j^t = \beta^j\ \bar{e}_1\, M_c\, A^{oj}\, x_t / [\bar{e}_j\, M_c\, x_t], \tag{6.65}$$

which gives the price at t of a sure claim on the first consumption good j periods ahead.

6.10. Asset Pricing Example

We[3] use the simple pure exchange one good model that is contained in `clex14.m` to illustrate our asset pricing formulas. The economy in `clex14.m` is a linear-quadratic version of an economy that Lucas (1978) used to develop an equilibrium theory of asset prices.

The economy is a member of the special class of structures described in chapter 3. The economy is described as follows:

---

[3] Note to Sargent. This section was 0 in /mnt2/linquad on the SUN. This file is hsch5in.tex. The figures are in hsch*.ps. The MATLAB program used to generate this is hschap5.m. A diary of these runs is in hsch5 in the /linquad directory.

6.10.1. Preferences

$$
\begin{aligned}
-.5E \sum_{t=0}^{\infty} \beta^t [(c_t - b_t)^2 + \ell_t^2] | J_0 \\
s_t &= c_t \\
b_t &= U_b z_t
\end{aligned}
$$

6.10.2. Technology

$$
\begin{aligned}
c_t &= d_{1t} \\
k_t &= \delta_k \, k_{t-1} + i_t \\
g_t &= \phi_1 i_t, \qquad \phi_1 > 0 \\
\begin{bmatrix} d_{1t} \\ 0 \end{bmatrix} &= U_d z_t
\end{aligned}
$$

6.10.3. Information

$$
\begin{aligned}
z_{t+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & 0 \\ 0 & 0 & .5 \end{bmatrix} z_t + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} w_{t+1} \\
U_b &= [\, 30 \quad 0 \quad 0 \,] \\
U_d &= \begin{bmatrix} 5 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
x_0 &= [\, 5 \quad 150 \quad 1 \quad 0 \quad 0 \,]'
\end{aligned}
$$

To compute the asset prices in this economy we issue the following MATLAB commands:

```
clex14
solvea
t1 = 100;
nt = t1;
sy=sc;
asimul
pay=sd(1,:)
asseta
```

The program `asseta` constructs a simulation of length nt of the price and rate of return of an asset that yields a stream of returns equal to $pay * x_t$, where the user specifies the matrix pay. Here we specified that $pay = sd(1,:)$, so that we are pricing a perpetual claim on the endowment process d_{1t}, which is the asset that Lucas priced in his 1978 paper. If the user desires to price a vector of assets, he should simply feed in the matrix pay such that $pay * x_t$ is the payout vector of those assets. Let nn be the number of rows of pay, i.e., nn is the number of assets being priced. The program asseta creates a vector y of length nt that equals the vector `[mrs, payoff, asset prices, returns]`, where `mrs` is the one period intertemporal marginal rate of substitution; `payoff` is the payoff on the asset(s), which equals $pay * x_t$; `asset prices` is the series of asset prices; and `ret` is the one period gross realized rate of return on the asset(s). For $j = 1, 2, 5$, the program also creates the reciprocals of the j-period ahead gross rates of return on safe assets, and stores them in the vectors $R1, R2, R5$.

We have computed asset prices for two versions of this economy. The first has the parameter settings listed above, while the second alters the autoregressive parameter in the endowment process to be .4 rather than .8. Figures 6.10.1 through 6.10.3 record the results of one hundred period simulations for each of these two economies. Figure 6.10.1 displays the simulated value of the asset price for the first economy. Figures 6.10.2 displays the gross rates of return on the 'Lucas tree' and on a sure one-period bond. We computed the correlation coefficient between these two returns to be -.49. For this economy, the 'risk premium' term in the price of the Lucas tree, namely σ_a in formula (6.8.7), is calculated to be -12.80. To give an idea of how the term structure of interest rates moves in this economy, Figure 6.10.2.b displays the *net* rates of return on one period and five period sure bonds. (We computed the net rate of return on j-periods bonds by taking the log of the gross rate of return and dividing by j.) Notice the tendency of the term structure to slope upward when rates are low, and to slope downward when rates are high.

Figures 6.10.3.a and 6.10.3.b record rates of return for the 'Lucas tree' and for sure bonds in the economy with the autoregressive parameter for the endowment process equaling .4. Figure 6.10.3.a shows the gross rates of return on the 'Lucas tree' and on a sure one-period bond. The correlation between these two was computed to be -.62. From Figure 6.10.3.b, we see that the tendency for the yield curve to slope upward when rates are low and to slope downward when rates are high has been accentuated relative to our first economy. For the

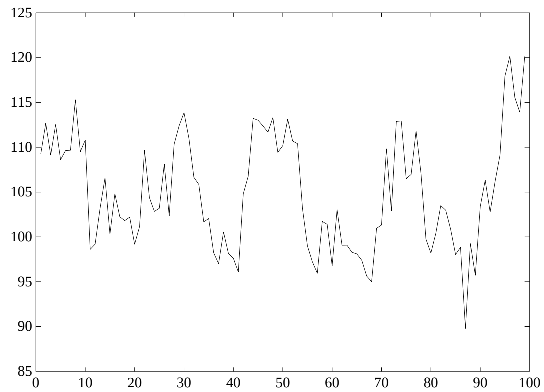

Figure 6.10.1: Price of a 'stock' entitling the owner to a perpetual claim on the dividends of a 'Lucas tree' when the autoregressive parameter for the endowment process is .8.

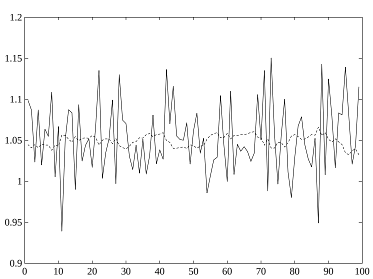

Fig. 6.10.2.a. Realized one period gross rates of return on a Lucas tree (solid line) and on a sure one period bond (dotted line) when the autoregressive parameter for the endowment process is .8.

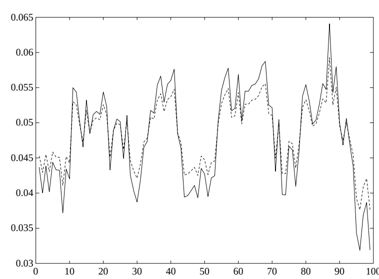

Fig. 6.10.2.b. Net rates of return on a one-period (solid line) and a five period (dotted line) when the autoregressive parameter for the endowment process is .8.

second economy, the 'risk premium' term σ_a in the price of the Lucas tree is calculated to be -5.90.

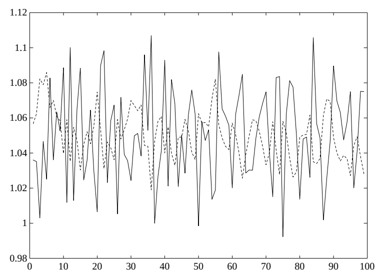

Fig. 6.10.3.a. Realized one period gross rates of return on a Lucas tree (solid line) and on a sure one period bond (dotted line) when the autoregressive parameter for the endowment process is .4.

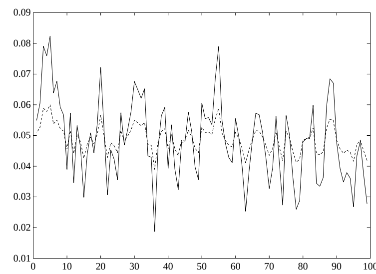

Fig. 6.10.2.b. Net rates of return on a one-period (solid line) and a five period (dotted line) when the autoregressive parameter for the endowment process is .4.

The pure exchange economy in `clex14.m` is one of the simplest to which our asset pricing formulas can be applied. Indeed, for this simple economy, the pricing formulas can be worked out by hand, as the exercises at the end of this chapter indicate. In chapter 4, we shall apply these formulas and our computer programs in much richer contexts in which one can't get very far 'by hand'

6.11. Exercises

1. Consider an economy that consists of technology specification 1 and preference specification 1. The social planning problem is simply to maximize

$$E_0 - \sum_{t=0}^{\infty} \beta^t \, \{\frac{1}{2} \, (c_t - b_t)^2\}$$

subject to

$$c_t = d_t$$

Assume that $b_t = \bar{b} > 0$ for all t and that $d_t = \xi_0 + \xi_1 \, d_{t-1} + \varepsilon_{dt}$, where ε_{dt} is a white noise with mean zero and variance $\sigma_\varepsilon^2, \xi_0 > 0$, and $\mid \xi_1 \mid < 1/\sqrt{\beta}$. The endowment process $\{d_t\}$ is produced by "trees", there being one tree for each (representative) household. The household owns the "tree" at the beginning of time (time $t = 0$).

a. Carefully define a competitive equilibrium for this economy. In your definition, describe a particular decentralization scheme, being careful to tell who owns what and who trades with whom.
b. Calculate the time-zero equilibrium price system.
c. Let υ_0 be the time zero value in terms of the time zero consumption good of a title to the entire stream of dividends $\{d_t\}_{t=0}^{\infty}$. Prove that υ_0 satisfies

$$\upsilon_0 = \Big[E_0 \sum_{t=0}^{\infty} \beta^t \, \bar{b} \, d_t - E_0 \sum_{t=0}^{\infty} \beta^t \, d_t^2\Big] / (b - d_0)$$

d. Compute the gross one period sure rate of interest.
e. Compute the gross two period sure rate of interest.

2. Consider an economy with preference specification 1. The technology is specified as

$$c_t = d_t - G_t$$

where G_t is government purchases. We assume that $G_t = U_G \, z_t$, and $d_t = U_d \, z_t$. Assume that $b_t = \bar{b} > 0$ for all t.

Assume that the government levies state-contingent lump sum taxes τ_t on the household at time t, where

$$\tau_t = \tau_t(w^t, x_0)$$

where $w^t = (w_1, w_2, \ldots, w_t)$. Lump sum taxes τ_t are denominated in units of the time t consumption good.

a. Formulate the government's time zero budget constraint.
b. Define a competitive equilibrium for this economy.
c. Compute a time zero equilibrium price system.
d. Define a formula like the one derived under (c) in problem 1 for the time zero value of a title to the dividends from the tree.
e. Suppose that the lump sum taxes are on *trees*, not on the household. Derive a formula for the value of a tree at time zero.
f. Compute the gross interest rate on sure one period loans.

3. Consider an economy defined by the social planning problem: maximize the utility of the representative household

$$-\left(\frac{1}{2}\right) E \sum_{t=0}^{\infty} \beta^t \left[(c_t - b)^2 + \ell_t^2\right] \mid J_0, \; 0 < \beta < 1, \, b > 0 \tag{1}$$

subject to the technology

$$c_t = d_t + \phi g_t \; , \quad \phi > 0 \tag{2}$$

$$g_t = \ell_t . \tag{3}$$

Here d_t is an exogenous process describing the flow of dividends from a single tree (per representative household). The dividend obeys the stochastic process

$$d_0 \;\text{ given} \;\; , \quad b > d_0 > 0$$

$$d_t = d_0 + w_t, \quad t \geq 1$$

where w_t is an independently and identically distributed random process with

$$E w_t = 0$$

$$E w_t^2 = \sigma_w^2 .$$

In (1), b is a constant, c_t is consumption at t, and ℓ_t is labor supplied at t; E is the mathematical expectation operator, and J_0 is information available at

$t = 0$, namely, d_0. Equation (2) describes how consumption is related to the exogenous level of dividends at t and the amount $\phi g_t = \phi \ell_t$ produced through the application of labor.

a. Solve the social planning problem, finding the optimal strategy for consumption and the labor supply.

Now decentralize the economy as follows. Let households own the stock of one tree initially. Households sell the tree to a representative firm at time zero (before d_0 has been realized). Households sell their labor to the firm each period. The firm buys the tree at the beginning of time zero, hires labor, and sells output to the household.

b. State the maximum problems of the representative household and the representative firm for the decentralized economy.
c. Find a representation for the time zero Arrow-Debreu price system that supports the solution of the social planning problem as a competitive equilibrium. Give formulas for the price of consumption goods and for the wage of labor.
d. Derive a formula for the time zero price of trees in terms of the parameters of preferences, technology, and stochastic process for dividends. (Get as far you can in deriving a closed form).
e. Give a formula for the gross interest rate on sure one period loans.

Chapter 7
Applications

7.1. Introduction

7.2. Partial Equilibrium Interpretation

The models studied in this book can be reinterpreted as partial equilibrium models which employ the notion of a *representative firm*, and which generalize the preference and technology specifications of Lucas and Prescott (1971). The idea is that there is a large number of identical firms that produce the same goods and sell them in competitive markets. Because they are all identical, we carry along only one of these firms, and let it produce the entire output in the industry (which is harmless under constant returns to scale). But we have to be careful in our analysis because this representative firm's decisions play two very different roles: as a stand–in for the 'average' competitive producer, and as producer of the entire industry's output. We make the firm act as a competitor in solving its optimum problem.

Demand is governed by the system (11.3.14), with p_t^0 simply being replaced by p_t, namely,

$$\begin{aligned} c_t &= -\Pi^{-1}\Lambda h_{t-1} + \Pi^{-1}b_t - \Pi^{-1}E_t\{\Pi'^{-1} - \Pi'^{-1}\Theta_h' \\ &\qquad [I - (\Delta_h' - \Lambda'\Pi'^{-1}\Theta_h')\beta L^{-1}]^{-1}\Lambda'\Pi'^{-1}\beta L^{-1}\}p_t \qquad (7.2.1) \\ h_t &= \Delta_h h_{t-1} + \Theta_h c_t \end{aligned}$$

Here c_t is a vector of consumption goods. Through this demand system, the representative firm's output decisions influence the evolution of the market price. However, we want the representative firm to ignore this influence in making its output decisions.

A representative firm takes as given and beyond its control the stochastic process $\{p_t\}_{t=0}^{\infty}$. The firm sells its output c_t in a competitive market each period. Only spot markets convene at each date $t \geq 0$. The firm also faces an exogenous process of cost disturbances d_t.

The firm chooses stochastic processes $\{c_t, g_t, i_t, k_t\}_{t=0}^{\infty}$ to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{p_t \cdot c_t - g_t \cdot g_t/2\}$$

subject to

$$\begin{aligned} \Phi_c c_t + \Phi_i i_t + \Phi_g g_t &= \Gamma k_{t-1} + d_t \\ k_t &= \Delta_k k_{t-1} + \Theta_k i_t \end{aligned} \tag{7.2.2}$$

given k_{-1}. This problem is not well posed until we describe perceived laws of motion for the processes $\{p_t, d_t\}_{t=0}^{\infty}$ that the firm does not control, but which influence its returns. Specifying the law for the exogenous process $\{d_t\}$ is easy, because the representative firm's decisions are assumed *not* to influence it. The situation is different with the price process, because the price *is* influenced by the output decisions of the representative firm. Despite this influence, we want the firm to behave *competitively*, that is, to regard the price process as beyond its control. We want to specify the firm's beliefs about the evolution of the price so that: (a) the firm has 'rational expectations', i.e., its beliefs about the evolution of prices allow it to forecast future prices optimally, given the information that it has at each moment; and (b) the firm acts competitively and treats the price process as given and beyond its control.

We assume that the firm takes as given a law of motion for spot prices and for the information variables that help to predict spot prices. We model this forecasting problem as follows. The firm observes the *state of the market* X_t at t, and believes that the law of motion for the spot price is

$$\begin{aligned} p_t &= m_p X_t \\ X_{t+1} &= a_p X_t + C w_{t+1} \end{aligned} \tag{7.2.3}$$

where $X_t = [h'_{t-1}, K'_{t-1}, z'_t]'$, where K_t is the market-wide capital stock, which the firm takes as given and beyond its control. The firm believes that the cost shock process evolves according to $d_t = S_d X_t$. The *state* for the firm at date t is

$$\tilde{x}_t = [X'_t, k'_{t-1}]'.$$

The firm's problem is a discounted linear regulator problem. Under our assumptions about the technology, the firm's *control* can be taken to be i_t. The solution of the firm's problem is a decision rule for investment of the form

$$i_t = -f_i \tilde{x}_t. \tag{7.2.4}$$

This decision rule and equations (7.2.2) then determine $[c_t, g_t, k_t]$ as linear functions of $\tilde{x}_t$. The matrix f_i in the above equation is a function of all of the matrices describing the firm's constraints, including a_p and m_p. The firm's rule for c_t, implied by (7.2.2) and (7.2.4) can be represented as

$$c_t = f_c \tilde{x}_t. \tag{7.2.5}$$

Equation (7.2.2) implies that the firm's capital evolves according to

$$k_t = \Delta_k k_{t-1} - \Theta_k f_i \tilde{x}_t. \tag{7.2.6}$$

At this point, but not earlier, impose that the 'representative firm is representative' by setting $k_t \equiv K_t$ in (7.2.6), use it to deduce the *actual* law of motion for K_t, and then use this to fill in the rows corresponding to K_t of the actual law of motion for X_t:

$$X_{t+1} = a_a X_t + C w_{t+1}. \tag{7.2.7}$$

To get the rows corresonding to h_t, use (7.2.5) together with the law of motion $h_t = \Delta_h h_{t-1} + \Theta_h c_t$.

To get a formula for the actual law of motion of the price, use (11.1.1) and the actual law of motion (7.2.7) for $x_t = X_t$ to solve for a consumption process. Put the consumption process and preference shock into (11.3.1) and solve for μ_t^s. Then solve (11.3.3) forward for μ_t^h; substitute into (11.3.2) to solve for p_t^0. Set $p_t = p_t^0$, then express the motion of prices as

$$p_t = m_a X_t. \tag{7.2.8}$$

The system (7.2.7), (7.2.8) describes the *actual* law of motion for spot prices that is induced by the firm's optimizing behavior and market clearing when the firm's *perceived* law of motion for the spot prices is (7.2.3). The firm's optimization problem and market clearing thus induce a mapping from a *perceived* law of motion (a_p, m_p) for spot prices to an *actual* law (a_a, m_a).

Definition: A *rational expectations equilibrium* (or a *partial equilibrium*) is a fixed point of the mapping from the perceived law of motion for spot prices to the actual law of motion for spot prices.

An equivalent definition is:

Definition: A *partial equilibrium* is a stochastic process $\{p_t, c_t, i_t, g_t, k_t, K_t, h_t\}_{t=0}^{\infty}$, each element of which belongs to L_0^2, such that:

i. Given $\{p_t\}_{t=0}^{\infty}$, in particular given the law of motion (7.2.3), $\{c_t, i_t, g_t, k_t\}_{t=0}^{\infty}$ solve the firm's problem.

ii. $\{p_t, c_t, h_t\}_{t=0}^{\infty}$ satisfy the demand system (7.2.1).

iii. $\{k_t\}_{t=0}^{\infty} = \{K_t\}_{t=0}^{\infty}$.

This is a version of Lucas and Prescott's (1971) rational expectations competitive equilibrium, which they used to study investment under uncertainty with adjustment costs. The following proposition states the relationship between a partial equilibrium and our earlier notion of competitive equilibrium:

Proposition: Let $\{c_t, s_t, i_t, g_t, k_t, p_t^0, w_t^0, \alpha_t^0, r_t^0, q_t^0\}_{t=0}^{\infty}, v_0$ be a competitive equilibrium. Then $\{p_t^0, c_t, i_t, g_t, k_t, k_t, h_t\}_{t=0}^{\infty}$ is a partial equilibrium.

This proposition can be proved directly by verifying that the proposed partial equilibrium satisfies the first order necessary and sufficient conditions for the firm's problem in the partial equilibrium, and that the proposed $\{p_t, c_t, h_t\}_{t=0}^{\infty}$ process satisfies the demand system (11.3.14).

7.2.1. Partial equilibrium investment under uncertainty

Our partial equilibrium structure includes many examples of linear rational expectations models (e.g., Sargent (1987, chapter XVI), Eichenbaum (1983), and Hansen and Sargent (1991, Two Difficulties). Here is how we can apply these ideas to a version of Lucas and Prescott's (1971) model of investment under uncertainty. There is one good produced with one factor of production (capital) via a linear technology. A representative firm maximizes

$$E \sum_{t=0}^{\infty} \beta^t \{p_t c_t - g_t^2/2\},$$

subject to the technology

$$\begin{aligned} c_t &= \gamma k_{t-1} \\ k_t &= \delta_k k_{t-1} + i_t \\ g_t &= f_1 i_t + f_2 d_t, \end{aligned}$$

where d_t is a cost shifter, $\gamma > 0$, and $f_1 > 0$ is a cost parameter and $f_2 = 1$. Demand is governed by

$$p_t = \alpha_0 - \alpha_1 c_t + u_t,$$

where u_t is a demand shifter with mean zero and α_0, α_1 are positive parameters. Assume that u_t, d_t are uncorrelated first-order autoregressive processes.

Lucas and Prescott computed rational expectations equilibrium quantities by forming a social planning problem with criterion

$$E \sum_{t=0}^{\infty} \beta^t \left\{ \int_0^{c_t} (\alpha_0 - \alpha_1 \nu + u_t) d\,\nu - .5 g_t^2 \right\},$$

where the integral under the demand curve is 'consumer surplus.' Consumer surplus equals

$$(\alpha_0 + u_t) c_t - \frac{\alpha_1}{2} c_t^2.$$

To map this model into our framework, set $\Lambda = 0, \Delta_h = 0, \Theta_h = 0, \Pi^2 = \alpha_1, b_t = \frac{\alpha_0}{\Pi} + \frac{1}{\Pi} u_t$. Notice that with this specification,

$$(s_t - b_t)^2/2 = (\alpha_0 + u_t) c_t - \frac{\alpha_1}{2} c_t^2 + b_t^2/2.$$

The term in b_t^2 can be ignored because it influences no decisions. With this specification, our social planning problem is equivalent with Lucas and Prescott's. After we have computed the equilibrium quantities by solving the social planning problem, we can compute the 'marginal utility price'

$$\begin{aligned} p_t &= \Pi (b_t - s_t) \\ &= \alpha_0 + u_t - \alpha_1 c_t, \end{aligned}$$

where we are using $\alpha_1 = \Pi^2$.

7.3. Introduction

The remainder of chapter provides more examples of models that conform to our framework. Most of these examples were originally stated as partial equilibrium models. The appendix of the chapter describes a scheme for pricing objects that until now were unpriced because they were sheltered from the market by the warmth of the household. We use this decentralization when we want to price some of the household capital stocks.

7.4. A Housing Model

Rosen and Topel (1988) formulated a partial equilibrium model of a housing market consisting of a linear demand curve relating a stock of housing inversely to a rental rate; an equilibrium condition relating the price of houses to the discounted present value of rentals, adjusted for depreciation; and a quadratic cost curve for producing houses.

7.4.1. Demand

We can capture Rosen and Topel's specification by sweeping house rentals into the household sector. See the appendix of this chapter for an account of a decentralization that supports this interpretation. Rosen and Topel expressed the demand side of their model in terms of the two equations

$$R_t = b_t + \alpha h_t$$
$$p_t = E_t \sum_{\tau=0}^{\infty} (\beta \delta_h)^\tau R_{t+\tau}$$

where h_t is the stock of housing at time t, R_t is the rental rate for housing, p_t is the price of new houses, and b_t is a demand shifter; $\alpha < 0$ is a demand parameter, and δ_h is the depreciation factor for houses. We cast this demand specification within our class of models by letting the stock of houses h_t evolve according to

$$h_t = \delta_h h_{t-1} + c_t, \quad \delta_h \in (0,1),$$

where c_t is the rate of production of new houses. Houses produce services s_t according to $s_t = \bar{\lambda} h_t$ or $s_t = \lambda h_{t-1} + \pi c_t$, where $\lambda = \bar{\lambda}\delta_h, \pi = \bar{\lambda}$. We can take

$\bar{\lambda}\rho_t^0 = R_t$ as the rental rate on housing at time t, measured in units of time t consumption (housing).

Demand for housing services is

$$s_t = b_t - \mu_0 \rho_t^0,$$

where the price of new houses p_t is related to ρ_t^0 by $\rho_t^0 = \pi^{-1}[p_t - \beta\delta_h E_t p_{t+1}]$. This equation, which is a special case of equation (11.3.7) from chapter 11, imposes the feature of the present specification that $\delta_h - \lambda\pi^{-1}\theta_h = 0$, is a version of Rosen and Topel's equation (12). It can be solved to yield $p_t = \bar{\lambda}E_t \sum_{\tau=0}^{\infty}(\beta\delta_h)^\tau \rho_t^0$, a version of Rosen and Topel's equation (14). The parameter $\bar{\lambda}$ governs the slope of the demand curve for housing, in terms of the rental rate for housing.

7.4.2. House producers

Rosen and Topel's representative firm maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t [p_t c_t - \Omega(c_t, c_t - c_{t-1}, e_t)],$$

where $\Omega(c_t, c_t - c_{t-1}, e_t)$ is the cost of producing new houses, and $\{e_t\}$ is a cost shifter. The function Ω incorporates costs of adjusting the rate of production of new houses. The firm takes the stochastic process for p_t as given. Costs are given by

$$\Omega(c_t, c_t - c_{t-1}, e_t) = g_t \cdot g_t$$

where

$$\begin{aligned} g_{1t} &= f_1 c_t + f_2 e_t \\ g_{2t} &= f_3(c_t - c_{t-1}), \end{aligned}$$

where e_t is our cost-shifter. To map this into our specification, we use the technology

$$\begin{aligned} f_1 c_t - g_{1t} &= 0 k_{t-1} - f_2 e_t \\ c_t - i_t &= 0 \\ f_3 c_t - g_{2t} &= f_3 k_{t-1} \\ k_t &= 0 k_{t-1} + i_t. \end{aligned}$$

7.5. Cattle Cycles

Rosen, Murphy, and Scheinkman (1994) used a partial equilibrium model to interpret recurrent cycles in U.S. cattle prices. Their model has a static linear demand curve interacting with a 'time-to-grow' structure for raising cattle. Let p_t be the price of freshly slaughtered beef, m_t the feeding cost of preparing an animal for slaughter, $\tilde{h}_t$ the one-period holding cost for a mature animal, $\gamma_1 \tilde{h}_t$ the one-period holding cost for a yearling, and $\gamma_0 \tilde{h}_t$ the one period holding cost for a calf. The costs $\{\tilde{h}_t, m_t\}_{t=0}^{\infty}$ are exogenous stochastic processes, while the stochastic process $\{p_t\}_{t=0}^{\infty}$ is determined by a rational expectations equilibrium. Let $\tilde{x}_t$ be the breeding stock, and $\tilde{y}_t$ be the total stock of animals. The law of motion for stocks is

$$\tilde{x}_t = (1-\delta)\tilde{x}_{t-1} + g\tilde{x}_{t-3} - c_t, \tag{7.5.1}$$

where c_t is a rate of slaughtering. The total head count of cattle is

$$\tilde{y}_t = \tilde{x}_t + g\tilde{x}_{t-1} + g\tilde{x}_{t-2}, \tag{7.5.2}$$

which is the sum of adults, calves, and yearlings, respectively.

A representative farmer maximizes

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \{ p_t c_t - & \tilde{h}_t \tilde{x}_t - (\gamma_0 \tilde{h}_t)(g\tilde{x}_{t-1}) - (\gamma_1 \tilde{h}_t)(g\tilde{x}_{t-2}) - m_t c_t \\ & - \Psi(\tilde{x}_t, \tilde{x}_{t-1}, \tilde{x}_{t-2}, c_t) \}, \end{aligned} \tag{7.5.3}$$

where

$$\Psi = \frac{\psi_1}{2}\tilde{x}_t^2 + \frac{\psi_2}{2}\tilde{x}_{t-1}^2 + \frac{\psi_3}{2}\tilde{x}_{t-2}^2 + \frac{\psi_4}{2}c_t^2. \tag{7.5.4}$$

The maximization is subject to the law of motion (7.5.1), taking as given the stochastic laws of motion for the exogenous random processes and the equilibrium price process, and the initial state $[\tilde{x}_{-1}, \tilde{x}_{-2}, \tilde{x}_{-3}]$. Here $(\psi_j, j = 1,2,3)$ are small positive parameters that represent quadratic costs of carrying stocks, and ψ_4 is a small positive parameter. The costs in (7.5.4) are implicitly taken into account by Rosen, Murphy, and Scheinkman, and motivate their decision to "solve stable roots backwards and unstable roots forwards." To capture Rosen, Murphy, and Scheinkman's solution, we shall set each of the ϕ_j's to a positive but very small number.

Demand is governed by

$$c_t = \alpha_0 - \alpha_1 p_t + \tilde{d}_t, \tag{5}$$

where $\alpha_0 > 0$, $\alpha_1 > 0$, and $\{\tilde{d}_t\}_{t=0}^{\infty}$ is a stochastic process with mean zero representing a demand shifter.

7.5.1. Mapping cattle farms into our framework

We show how to map the model of Rosen, Murphy, and Scheinkman into our general setup.

7.5.2. Preferences

Set $\Lambda = 0, \Delta_h = 0, \Theta_h = 0, \Pi = \alpha_1^{-1}, b_t = \Pi \tilde{d}_t + \Pi \alpha_0$. With these settings, first-order condition (6.13) for the household's problem becomes

$$c_t = \Pi^{-1} b_t - \Pi^{-2} p_t,$$

or

$$c_t = \alpha_0 - \alpha_1 p_t + \tilde{d}_t.$$

7.5.3. Technology

The law of motion for capital is

$$\begin{bmatrix} \tilde{x}_t \\ \tilde{x}_{t-1} \\ \tilde{x}_{t-2} \end{bmatrix} = \begin{bmatrix} (1-\delta) & 0 & g \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{x}_{t-2} \\ \tilde{x}_{t-3} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} i_t,$$

or

$$k_t = \Delta_k k_{t-1} + \Theta_h i_t.$$

Here $i_t = -c_t$.

We use adjustment costs to capture the holding and slaughtering costs. We set

$$g_{1t} = f_1 \tilde{x}_t + f_2 \tilde{h}_t,$$

or

$$g_{1t} = f_1 [(1-\delta)\tilde{x}_{t-2} + g\tilde{x}_{t-3} - c_t] + f_2 \tilde{h}_t.$$

We set

$$g_{2t} = f_3 \tilde{x}_{t-1} + f_r \tilde{h}_t$$
$$g_{3t} = f_5 \tilde{x}_{t-1} + f_6 \tilde{h}_t .$$

Notice that

$$g_{1t}^2 = f_1^2 \tilde{x}_t^2 + f_2 \tilde{h}_t^2 + 2 f_1 f_2 \tilde{x}_t \tilde{h}_t$$
$$g_{2t}^2 = f_3^2 \tilde{x}_{t-1}^2 + f_2 \tilde{h}_t^2 + 2 f_3 f_4 \tilde{x}_{t-1} \tilde{h}_t$$
$$g_{3t}^2 = f_5^2 \tilde{x}_{t-2}^2 + f_6 \tilde{h}_t^2 + 2 f_5 f_6 \tilde{x}_{t-2} \tilde{h}_t .$$

Thus, we set

$$f_1^2 = \frac{\psi_1}{2} \quad f_2^2 = \frac{\psi_2}{2} \quad f_3^2 = \frac{\psi_3}{2}$$
$$2 f_1 f_2 = 1 \quad 2 f_3 f_4 = \gamma_0 g \quad 2 f_5 f_6 = \gamma_1 g$$

To capture the feeding costs we set $g_{4t} = f_7 c_t + f_8 m_t$, and set $f_7^2 = \frac{\psi_4}{2} \quad 2 f_7 f_8 = 1$. Thus, we set

$$\begin{bmatrix} 1 \\ f_1 \\ 0 \\ 0 \\ -f_7 \end{bmatrix} c_t + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} i_t + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_{1t} \\ g_{2t} \\ g_{3t} \\ g_{4t} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 \\ f_1(1-\delta) & 0 & g f_1 \\ f_3 & 0 & 0 \\ 0 & f_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_{t-1} \\ \tilde{x}_{t-2} \\ \tilde{x}_{t-3} \end{bmatrix} + \begin{bmatrix} 0 \\ f_2 \tilde{h}_t \\ f_4 \tilde{h}_t \\ f_6 \tilde{h}_t \\ f_8 m_t \end{bmatrix} .$$

We set $d_t = U_d z_t$ where

$$U_d = \begin{bmatrix} 0 \\ f_2 U_h \\ f_4 U_h \\ f_6 U_h \\ f_8 U_m \end{bmatrix} ,$$

where $[U_h, U_m]$ are selector vectors that pick off $\tilde{h}_t$ and m_t from the exogenous state vector z_t. We specify the information matrices $[A_{22}, C_2]$ to incorporate Rosen, Murphy, and Scheinkman's specification that $[\tilde{h}_t, m_t, \tilde{d}_t]$ consists of three uncorrelated first order autoregressive processes.[1]

[1] This model is estimated by Anderson, Hansen, McGrattan, and Sargent (1996).

7.6. Models of Occupational Choice and Pay

Aloyisius Siow (1984) and Sherwin Rosen (1995) and have used pure 'time-to-build' structures to represent entry cycles into occupations, and also inter-occupational wage movements. It is easiest to incorporate these models into our framework by putting production into the household technology, using the decentralization described in the appendix to generate prices.

7.6.1. A one-occupation model

Rosen [1995] studied a partial equilibrium model determining a stock of 'engineers' N_t; the number of new entrants into engineering school, n_t; and the wage level w_t of engineers. It takes k periods of schooling to become an engineer. The model consists of the following equations: first, a demand curve for engineers

$$w_t = -\alpha_d N_t + \epsilon_{1t}\ ,\ \alpha_d > 0; \tag{7.6.1}$$

second, a time-to-build structure of the education process

$$N_{t+k} = \delta_N N_{t+k-1} + n_t\ ,\ 0 < \delta_N < 1; \tag{7.6.2}$$

third, a definition of the discounted present value of each new engineering student

$$v_t = \beta^k E_t \sum_{j=0}^{\infty} (\beta \delta_N)^j w_{t+k+j}; \tag{7.6.3}$$

and fourth, a supply curve of new students driven by v_t

$$n_t = \alpha_s v_t + \epsilon_{2t}\ ,\ \alpha_s > 0. \tag{7.6.4}$$

Here $\{\epsilon_{1t}, \epsilon_{2t}\}$ are stochastic processes of labor demand and supply shocks. A *partial equilibrium* is a stochastic process $\{w_t, N_t, v_t, n_t\}_{t=0}^{\infty}$ satisfying these four equations, and initial conditions $N_{-1}, n_{-s}, s = 1, \ldots, -k$.

We can represent this model by sweeping the time-to-build structure and the demand for engineers into the household technology, and putting the supply of new engineers into the technology for producing goods. Here is how. We take the household technology to be

$$s_t = [\lambda_1 \ 0 \ \dots \ 0] \begin{bmatrix} h_{1t-1} \\ h_{2t-1} \\ \vdots \\ h_{k+1,t-1} \end{bmatrix} + 0 \cdot c_t$$

$$\begin{bmatrix} h_{1t} \\ h_{2t} \\ \vdots \\ h_{k,t} \\ h_{k+1,t} \end{bmatrix} = \begin{bmatrix} \delta_N & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} h_{1t-1} \\ h_{2t-1} \\ \vdots \\ h_{k,t-1} \\ h_{k+1,t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} c_t$$

$$b_t = \epsilon_{1t}$$

This specification sets Rosen's $N_t = h_{1t-1}, n_t = c_t, h_{s+1,t-1} = n_{t-s}, s = 1, \dots, k$, and uses the home-produced good to capture the demand for labor. Here λ_1 embodies Rosen's demand parameter α_d.[2]

To capture Rosen's supply curve, we use the physical technology

$$c_t = i_t + d_{1t}$$
$$\varphi_1 i_t = g_t$$

where d_{1t} is proportional to Rosen's supply shock ϵ_{2t}, and where the adjustment cost parameter φ_1 varies directly with Rosen's supply curve parameter α_s.

Rosen showed that the equilibrium decision role for new entrants (our c_t) must satisfy the condition

$$n_t = f_1 E_t N_{t+k} + f_2 \epsilon_{1t} + f_3 \epsilon_{2t}$$

where $f_1 < 0$.

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2 In the definition of Λ in the household technology, we would replace the zeros with $\varepsilon > 0$ as a trick to acquire detectability; see chapter 9 and its appendix for the definition and role of detectability.

7.6.2. Skilled and unskilled workers

We can generalize the preceding model to two occupations, called skilled and unskilled, to obtain alternative versions of a model estimated by A. Siow (1984). The model consists of the following elements: first, a demand curve for labor

$$\begin{bmatrix} w_{ut} \\ w_{st} \end{bmatrix} = \alpha_d \begin{bmatrix} N_{ut} \\ N_{st} \end{bmatrix} + \epsilon_{1t};$$

where α_d is a (2×2) matrix of demand parameters and ϵ_{1t} is a vector of demand shifters; second, time-to-train specifications for skilled and unskilled labor, respectively:

$$\begin{aligned} N_{st+k} &= \delta_N N_{st+k-1} + n_{st} \\ N_{ut} &= \delta_N N_{ut-1} + n_{ut}; \end{aligned}$$

where N_{st}, N_{ut} are stocks of the two types of labor, and n_{st}, n_{ut} are entry rates into the two occupations; third, definitions of discounted present values of new entrants to the skilled and unskilled occupations, respectively:

$$\begin{aligned} v_{st} &= E_t \beta^k \sum_{j=0}^{\infty} (\beta\delta_N)^j w_{st+k+j} \\ v_{ut} &= E_t \sum_{j=0}^{\infty} (\beta\delta_N)^j w_{ut+j}, \end{aligned}$$

where w_{ut}, w_{st} are wage rates for the two occupations; and fourth, supply curves for new entrants:

$$\begin{bmatrix} n_{st} \\ n_{ut} \end{bmatrix} = \alpha_s \begin{bmatrix} v_{ut} \\ v_{st} \end{bmatrix} + \epsilon_{2t}. \tag{7.6.5}$$

As an alternative to (7.6.5), Siow simply used the 'equalizing differences' condition

$$v_{ut} = v_{st}. \tag{7.6.6}$$

We capture this model by pushing most of the 'action' into the household sector. Households decide what kind of durable good to accumulate, namely, unskilled labor or skilled labor. Unskilled labor and skilled labor can be combined to produce services, which we specify to generate the demands labor. We let c_{1t}, c_{2t} be rates of entry n_{ut}, n_{st} into unskilled and skilled labor, and constrain them to satisfy $c_{1t} + c_{2t} = i_t + d_{1t}$, the rate of total new entrants. To generate the upward sloping supply curves (7.6.5), we specify that $\phi_1 i_t + \phi_2 c_{2t} = g_t$. The technology is thus

$$\begin{bmatrix} 1 & 1 \\ 0 & -\phi_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{1t} \\ c_{2t} \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -\phi_1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} i_{1t} \\ i_{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} g_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} k_{t-1} + \begin{bmatrix} d_{1t} \\ 0 \\ 0 \end{bmatrix},$$

where d_{1t} is a supply shifter. To get Siow's model, we set $\phi_1 = \phi_2 = 0$, in which case d_{1t} becomes an exogenous supply of new entrants into the labor force.

We specify the law of motion for household capital

$$\begin{bmatrix} h_{1t} \\ h_{2t} \\ h_{3t} \\ \vdots \\ h_{k+1,t} \\ h_{k+2,t} \end{bmatrix} = \begin{bmatrix} \delta_N & 0 & 0 & 0 & \cdots & 0 \\ 0 & \delta_N & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} h_{1t-1} \\ h_{2t-1} \\ h_{3t-1} \\ \vdots \\ h_{k+1,t-1} \\ h_{k+2,t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{1t} \\ c_{2t} \end{bmatrix}.$$

where $h_{1t-1} = N_{ut-1}, h_{2t-1} = N_{st}, h_{j+2,t-1} = n_{s,t-j}, j = 1, \ldots, k$. We generate the demand for labor by specifying services as

$$\begin{bmatrix} s_{1t} \\ s_{2t} \end{bmatrix} = \bar{\Lambda} \begin{bmatrix} h_{1t} \\ h_{2t} \end{bmatrix} = \bar{\Lambda} e \Delta_h h_{t-1} + \bar{\Lambda} e \Theta_h c_t$$

where e is a selector vector that verifies $\begin{bmatrix} h_{1t} \\ h_{2t} \end{bmatrix} = e h_t$ We set the preference shock process $b_t = [\, b_{1t} \quad b_{2t} \quad 0 \quad 0\,]'$ to capture the shifters in the demands for labor.

7.7. A Cash-in-Advance Model

We want to use our framework to mimic a situation in which households are in a cash-in-advance environment in which they face a sequence of budget constraints

$$\tilde{c}_t + \frac{m_t}{p_t} + B_t \leq y_t + R_{gt-1}B_{t-1} + \frac{m_{t-1}}{p_t}$$
$$\tilde{c}_t \leq \frac{m_{t-1}}{p_t}$$

Here R_{gt-1} is the gross rate of return on indexed bonds B_{t-1} held from $t-1$ to t; p_t is the price level at t; $\tilde{c}_t$ is time t consumption;

and m_t is currency held from t to $t+1$. The household's preferences are ordered by $E_0 \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t)$. (We use $\tilde{c}_t$ to denote consumption in order to separate this notation from the c_t of our framework, which is soon to be defined.) Using the cash-in-advance constraint at equality in the budget constraint gives

$$B_t + \frac{\tilde{c}_{t+1}}{R_t} = y_t + B_{t-1}R_{gt-1},$$

where $R_t = \frac{p_t}{p_{t+1}}$ is the gross rate of return on currency between t and $t+1$. The force of the cash-in-advance restriction is that decisions about time t money-holding influence time $t+1$ consumption $\tilde{c}_{t+1}$, but time t consumption is predetermined.

7.7.1. Reinterpreting the household technology

We can specify the household technology to capture key elements of the cash-in-advance specification. We can use a 'back-solving' approach, and let $R = \frac{p_t}{p_{t+1}}$ be a constant rate of return on currency. We shall set $\tilde{c}_t = s_t$ and $c_t = m_t/p_t$, and sweep the cash-in-advance specification into a one-period time delay between a decision to consume (i.e., hold real balances) and when consumption goods are actually enjoyed. Thus, we take the household technology to be

$$s_t = Rc_{t-1},$$

which we accomplish by taking $\Lambda = R, \Pi = 0, \delta_h = 0, \theta_h = 1$. When there is inflation, $R < 1$. When $R > 1$, there is deflation. Preferences are of the usual kind

$$-.5E_0 \sum_{t=0}^{\infty} \beta^t [(s_t - b_t) \cdot (s_t - b_t) + \ell_t^2].$$

With these matchups, the time t 'seignorage' component of government revenue is

$$\begin{aligned}\tilde{r}_t &= \frac{m_t - m_{t-1}}{p_t} \\ &= c_t - s_t.\end{aligned}$$

This means that given R, the present value of seignorage revenues can be computed using the methods in chapter 8. With these specifications, an equilibrium with present value government budget balance can be computed, possibly including the inflation rate parameter R along with some of the τ's over which we search for an equilibrium.

Once an equilibrium is computed, the time series for real balances can be found from

$$\frac{m_t}{p_t} = c_t,$$

and the price level and nominal level of currency can be computed using the assumed R.

7.8. Taxation in a Vintage Capital Model

Owens (1994) has studied the effects of taxation on prices of new and old commercial buildings. His analysis requires keeping track of the age distribution of capital, which we can accomplish by specifying, for example,

$$\begin{bmatrix} k_{1t} \\ k_{2t} \\ k_{3t} \\ k_{4t} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \delta_h & 0 & 0 & 0 \\ 0 & \delta_h & 0 & 0 \\ 0 & 0 & \delta_h & 1 \end{bmatrix} \begin{bmatrix} k_{1t-1} \\ k_{2t-1} \\ k_{3t-1} \\ k_{4t-1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} i_t.$$

Here k_{1t} is new capital, k_{2t} is one year old capital, and so on. We could also include a time-to-build aspect, but have not here. To differentiate among the services produced by capital of different ages, we specify

$$c_t = \Gamma_c k_{t-1},$$

where we make c_t a vector that is comparable in dimension with k_{t-1} and Γ_c is diagonal. 'Office services' are then produced according to

$$s_t = [\, \pi_1 \quad \pi_2 \quad \dots \quad \pi_n \,]\, c_t.$$

We can set the π vector to make new office space more desirable than old office space.

We let the government tax capital of different ages differently, which puts terms like $E_0 \sum \beta^t \tau_k r_t^0 \cdot k_t$ or $E_0 \sum \beta^t \tau_{kk} p_{kt}^0 \cdot k_{t-1}$ in the budget constraints of households and the government, as in the framework of chapter 14. The tax matrices τ_k, τ_{kk} can be chosen to model different kinds of policies for depreciating capital

for tax purposes.

This framework can be used to model the effects on the prices of capital of alternative policies for capital taxation.

A. Decentralizing the Household

It can be useful to decentralize the household sector in order to price household services and stocks. Suppose that the household buys a vector of services from firms of type III at the price of services ρ_t^0. The household sells its initial stocks of both physical and household capital and also its labor and endowment process to firms. The price of the initial stock of household capital is $\tilde{v}_0$. The household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t [(s_t - b_t) \cdot (s_t - b_t) + \ell_t^2\}$$

subject to the budget constraint

$$E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot s_t = E_0 \sum_{t=0}^{\infty} \beta^t (w_t^0 \ell_t + \alpha_t^0 \cdot d_t) + (v_0 \cdot k_{-1} + \tilde{v}_0 \cdot h_{-1}). \qquad (7.A.1)$$

Firms of type III

Firms of type III purchase the consumption vector c_t, rent household capital, and produce and sell household services and additions to the stocks of household capital. Type III firms sell s_t to households at price ρ_t^0 and new household capital $\Theta_h c_t$ to firms of type IV at price p_{ht}^0. Firms of type III rent household capital from firms of type IV at a rental price r_{ht}^0, and maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{\rho_t^0 \cdot s_t + p_{ht}^0 \Theta_h c_t - r_{ht}^0 \cdot h_{t-1} - p_t^0 \cdot c_t\}$$

subject to

$$s_t = \Lambda h_{t-1} + \Pi c_t.$$

Firms of type IV

Firms of type IV purchase new household capital from firms of type III, and rent existing household capital to firms of type III at rental price r^0_{ht}. Firms of type IV maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t \{ r^0_{ht} \cdot h_{t-1} - p^0_{ht} \Theta_h c_t \} - \tilde{v}_0 \cdot h_{-1}$$

subject to

$$h_t = \Delta_h h_{t-1} + \Theta_h c_t.$$

Computing Prices

If we formulate the optimum for a firm of type III, obtain the associated first order necessary conditions and rearrange, we get the following restrictions on prices:

$$\begin{aligned} p^0_t &= \Theta_h' p_{ht} + \Pi' \rho^0_t \\ r^0_{ht} &= \Lambda' \rho^0_t. \end{aligned} \tag{7.A.2}$$

From the first order conditions for a firm of type IV, obtain

$$p^0_{ht} = E_t \beta [\Delta_h' p^0_{ht+1} + r^0_{ht+1}]. \tag{7.A.3}$$

We can use $(7.A.2)$ with $(7.A.3)$ to obtain

$$p^0_t = \Theta_h' E_t [\sum_{j=1}^{\infty} \beta^{(j)} \Delta'^{(j-1)} r^0_{ht+j}] + \Pi' \rho^0_t.$$

This is a generalization of Siow's equilibrium condition $(7.6.6)$. For us $p^0_t = M_c x_t$ is the vector of shadow prices of new entrants into the two types of profession.

We have already shown how to compute the price ρ^0_t. Indeed, this decentralization is a way to set up an explicit market in the 'implicit' services priced by ρ^0_t. The prices of stocks of household capital can be computed from the multipliers on h_{t-1} and c_t in the social planning problem.

Chapter 8
Efficient Computations

8.1. Introduction

This chapter describes fast algorithms for computing the value function and the optimal decision rule of our social planning problem.[1] The decision rule determines the allocation. The value function determines competitive equilibrium prices. The optimal value function and the optimal decision rules can be computed by iterating to convergence on the T operator associated with Bellman's functional equation. These iterations can be accelerated by using ideas from linear optimal control theory. We avail ourselves of these faster methods because we want to analyze high dimensional systems.

This chapter is organized as follows. First, we display a transformation that removes both discounting and cross-products between states and controls. This transformation simplifies the algebra without altering the substance. Next we describe invariant subspace methods for solving an optimal linear regulator problem, which are typically faster than iterating on the Bellman equation. We then describe a closely related method called the *doubling algorithm*, which 'skips steps' in iterating on the Bellman equation. The calculations can be further accelerated by partitioning the state vector to take advantage of the pattern of zeros in A and B. Next we discuss fast methods for computing equilibria for *periodic economies*. We describe the periodic optimal linear regulator problem, and show how to solve it rapidly. We conclude the chapter by describing how our calculations can be adapted to handle Hansen and Sargent's (1995) recursive formulation of Jacobsen's and Whittle's risk-sensitive preferences, which will be used in Chapter **@robust@**.

This chapter focuses mostly on *nonstochastic* optimal linear regulator problems. As indicated in chapter 3, the optimal decision rule for a stochastic optimal

[1] Parts of this chapter use results described in Anderson, Hansen, McGrattan, and Sargent (1995). Also see Kwakernaak and Sivan [1972] for what is mostly a treatment of continuous time systems.

linear regulator problem equals the optimal decision rule for the associated nonstochastic optimal linear regulator problem. Furthermore, from chapter 3, the matrices determining the Lagrange multipliers depend only on the piece of the optimal value function associated with the nonstochastic part of our problem.

Throughout this chapter, we study solutions of our control problem that satisfy the additional condition

$$E \sum_{t=0}^{\infty} \beta^t (|x_t|^2 + |u_t|^2) < \infty, \tag{8.1.1}$$

where x_t is the state and u_t is the control. In an appendix, we describe conditions on the matrices determining returns and the transition law that are sufficient by themselves to imply condition (8.1.1).[2]

8.2. The Optimal Linear Regulator Problem

Consider the following version of the optimal linear regulator problem: choose a contingency plan for $\{u_t\}_{t=0}^{\infty}$ to maximize

$$-E \sum_{t=0}^{\infty} \beta^t [x_t' R x_t + u_t' Q u_t + 2 u_t' W x_t], \ 0 < \beta < 1 \tag{8.2.1}$$

subject to

$$x_{t+1} = A x_t + B u_t + C w_{t+1}, \ t \geq 0, \tag{8.2.2}$$

where x_0 is given. In (8.2.1) – (8.2.2), x_t is an $n \times 1$ vector of state variables, and u_t is a $k \times 1$ vector of control variables. In (8.2.2), we assume that w_{t+1} is a martingale difference sequence with $E w_t w_t' = I$, and that C is a matrix conformable as required to x and w. We also impose condition (8.1.1). We temporarily assume that R and Q are positive definite matrices, although in practice we use weaker assumptions about both matrices.

A standard way to solve this problem is the method of dynamic programming. Let $V(x)$ be the optimal value associated with the program starting from initial state vector $x_0 = x$. Bellman's functional equation is

$$V(x_t) = \max_{u_t} \Big\{ -(x_t' R x_t + u_t' Q u_t + 2 u_t' W x_t) + \beta E_t V(x_{t+1}) \Big\} \tag{8.2.3}$$

[2] For conditions sufficient to imply this condition, see Kwakernaak and Sivan [1972], Anderson and Moore [1979], and Anderson, Hansen, McGrattan, and Sargent (1995).

where the maximization is subject to (8.2.2). One way to solve this functional equation is to iterate on a version of (8.2.3), thereby constructing a sequence $V_j(x_t)$ of successively better approximations to $V(x_t)$. In particular, let

$$V_{j+1}(x_t) = \max_{u_t}\Big\{-(x_t' R x_t + u_t' Q u_t + 2u_t' W x_t) + \beta E_t V_j(x_{t+1})\Big\}, \tag{8.2.4}$$

where again the maximization is subject to (8.2.2). Suppose that we initiate the iterations from $V_0(x) = 0$ (which is the appropriate terminal value function for a one-period problem). Then direct calculations show that successive iterates on (8.2.4) take the quadratic form

$$V_j(x_t) = -x_t' P_j x_t - \rho_j, \tag{8.2.5}$$

where P_j and ρ_j satisfy the equations

$$\begin{aligned} P_{j+1} = R + \beta A' P_j A &- (\beta A' P_j B + W) \\ &\times (Q + \beta B' P_j B)^{-1} (\beta B' P_j A + W') \end{aligned} \tag{8.2.6}$$

$$\rho_{j+1} = \beta \rho_j + \beta \text{ trace } P_j C C'. \tag{8.2.7}$$

Equation (8.2.6) is the *matrix Riccati difference equation.* Notice that it involves only $\{P_j\}$ and is independent of $\{\rho_j\}$. Notice also that C, which multiplies the noises impinging on the system and so determines the variances of innovations to information in the system, affects the $\{\rho_j\}$ sequence but not the $\{P_j\}$ sequence. We can say that $\{P_j\}$ is independent of the system's noise statistics.[3]

Let P and ρ be the limits of (8.2.6) and (8.2.7), respectively. Then the value function $V(x_t)$ that satisfies the Bellman equation (8.2.3) is given by

$$V(x_t) = -x_t' P x_t - \rho,$$

where P and ρ are the limit points of iterations on (8.2.6) and (8.2.7) starting from $P_0 = 0, \rho = 0$.

The decision rule that attains the right side of (8.2.4) is given by

$$u_t = -F_j x_t$$

[3] This fact is what permits us to focus on nonstochastic problems in devising our algorithms.

where

$$F_j = (Q + \beta B' P_j B)^{-1} (\beta B' P_j A + W'). \tag{8.2.8}$$

The optimal decision rule for the original problem is given by $u_t = -F x_t$, where $F = \lim_{j \to \infty} F_j$, or

$$F = (Q + \beta B' P B)^{-1} (\beta B' P A + W'). \tag{8.2.9}$$

According to (8.2.9), the optimum decision rule for u_t is independent of the parameters C, and so of the noise statistics.

The limit point P of iterations on (8.2.6) evidently satisfies

$$\begin{aligned} P = R + \beta A' P A &- (\beta A' P B + W) \\ &\times (Q + \beta B' P B)^{-1} (\beta B' P A + W') \end{aligned} \tag{8.2.10}$$

This equation in P is called the *algebraic matrix Riccati equation*.

One way to solve an optimal linear regulator problem is to iterate directly on (8.2.6) and (8.2.7). Faster algorithms are available. First, we describe a useful transformation that simplifies some of the formulas.

8.3. Transformations to eliminate discounting and cross-products

The following transformation eliminates both discounting and cross-products between states and controls. Define the transformed control v_t and transformed state $\hat{x}_t$ by

$$v_t = \beta^{t/2} (u_t + Q^{-1} W' x_t), \quad \hat{x}_t = \beta^{t/2} x_t. \tag{8.3.1}$$

Notice that

$$v_t' Q v_t = \beta^t \,[\, x_t' \ u_t' \,] \begin{bmatrix} W Q^{-1} W' & W \\ W' & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}.$$

It follows that

$$\beta^t \,[\, x_t' \ u_t' \,] \begin{bmatrix} R & W' \\ W & Q \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} = \hat{x}_t' R^* \hat{x}_t + v_t' Q v_t$$

where $R^* = R - W Q^{-1} W'$. The transition law (8.2.2) can be represented as

$$\hat{x}_{t+1} = A^* \hat{x}_t + B^* v_t + \beta^{\frac{t+1}{2}} C w_{t+1}$$

where $A^* = \beta^{1/2}(A - BQ^{-1}W'), B^* = \beta^{1/2}B$. Therefore, regulator problem (8.2.1) – (8.2.2) is equivalent to the following regulator problem without cross-products between states and controls and without discounting: choose $\{v_t\}$ to maximize

$$-E\sum_{t=0}^{\infty}[\hat{x}_t' R^* \hat{x}_t + v_t' Q v_t] \tag{8.3.2}$$

subject to

$$\hat{x}_{t+1} = A^*\hat{x}_t + B^* v_t + \beta^{\frac{t+1}{2}} C w_{t+1}, \tag{8.3.3}$$

where

$$P = R^* + A^{*\prime} P A^* - A^{*\prime} P B^* (Q + B^{*\prime} P B^*)^{-1} B^{*'} P A^* \tag{8.3.4}$$

$$F^* = (Q + B^{*\prime} P B^*)^{-1} B^{*\prime} P A^*, \tag{8.3.5}$$

it being understood that P is the positive semi-definite solution of (8.3.4).

The optimal closed loop system in terms of transformed variables is

$$\hat{x}_{t+1} = (A^* - B^* F^*)\hat{x}_t + \beta^{\frac{t+1}{2}} C w_{t+1} \tag{8.3.6}$$

Multiplying both sides of this equation by $\beta^{-(\frac{t+1}{2})}$ gives

$$x_{t+1} = \beta^{-\frac{1}{2}}(A^* - B^* F^*) x_t + C w_{t+1}. \tag{8.3.7}$$

8.4. Stability Conditions

We shall typically restrict the undiscounted linear regulator (8.3.2), (8.3.3) defined by the matrices (A^*, B^*, R^*, Q) to satisfy some conditions from control theory designed to render the problem well behaved.

In particular, let $DD' = R^*$, so that D is said to be a factor of R^*. Our conditions are cast in terms of the concepts of *stabilizability* and *detectability* defined in Appendix A. We make

ASSUMPTION A1: The pair (A^*, B^*) is *stabilizable*. The pair (A^*, D) is *detectable*.

Then there obtains:

Stability Theorem: Under assumption A1: (i.) starting from any negative semi-definite matrix P_o, iterations on the matrix Riccati difference equation converge; and (ii.) The eigenvalues of $(A^* - B^* F^*)$ are stable.

In the next section, we describe a class of algorithms that exploit the stabilizing property of the optimal $(A^* - B^* F)$.[4]

8.5. Invariant Subspace Methods

Following Vaughan [1970], a literature has developed fast algorithms for computing the limit point of the matrix Riccati equation (8.2.6), based on an eigenstructure of a matrix associated with the Riccati equation. These methods work with a Lagrangian formulation of the problem and with the linear restrictions that stability condition (8.1.1) imposes on the multipliers and the state vector. These conditions restrict the matrix P that solves the algebraic matrix Riccati equation.

Without loss of generality, we work with the undiscounted deterministic optimal linear regulator problem: choose $\{u_t\}_{t=0}^{\infty}$ to maximize

$$-\sum_{t=0}^{\infty}\{x_t' R x_t + u_t' Q u_t\} \tag{8.5.1}$$

subject to

$$x_{t+1} = A x_t + B u_t. \tag{8.5.2}$$

4 Because the eigenvalues of $(A^* - B^* F^*)$ are less than unity in modulus, it follows that the eigenvalues of $A^o = \beta^{-\frac{1}{2}}(A^* - B^* F^*)$ are less than $\frac{1}{\sqrt{\beta}}$ in modulus.

8.5.1. Px as Lagrange multiplier

It is convenient to write a Lagrangian for the Bellman equation:

$$V(x) = \max\{-(x'Rx + u'Qu + 2\mu'[Ax + Bu - \tilde{x}]) + V(\tilde{x})\},$$

where $\tilde{x}$ is next period's value of the state, μ is a vector of multipliers, and $V(x) = -x'Px$ where the matrix P solves the matrix Riccati equation. The first-order condition for the above Lagrangian with respect to $\tilde{x}$ implies that $\mu = Px$. Thus, as usual, the multipliers are linked to the gradient of the value function.

8.5.2. Invariant subspace methods

Invariant subspace methods compute P indirectly by restricting the initial vector of the multipliers μ to stabilize the solution for x_t, u_t, as required by (8.1.1). For now, we assume that A is invertible. We move to the space of sequences, and let $\{\mu_t\}_{t=0}^{\infty}$ be a sequence of vectors of Lagrange multipliers. Form the Lagrangian

$$J = -\sum_{t=0}^{\infty}\{x_t'Rx_t + u_t'Qu_t + 2\mu_{t+1}'[Ax_t + Bu_t - x_{t+1}]\} - 2\mu_0'(\bar{x}_0 - x_0). \quad (8.5.3)$$

Here $\bar{x}_0$ is the given initial level of x_0. First order necessary conditions for the maximization of J with respect to $\{u_t\}_{t=0}^{\infty}$ and $\{x_t\}_{t=0}^{\infty}$ are

$$u_t: \qquad Qu_t + B'\mu_{t+1} = 0\ , \qquad t \geq 0 \quad (8.5.4)$$

$$x_t: \qquad \mu_t = Rx_t + A'\mu_{t+1}, \qquad t \geq 0. \quad (8.5.5)$$

Solve (8.5.4) for u_t and substitute into (8.5.2) to obtain

$$x_{t+1} = Ax_t - BQ^{-1}B'\mu_{t+1}. \quad (8.5.6)$$

Represent (8.5.5) and (8.5.6) as

$$L\begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix} = N\begin{bmatrix} x_t \\ \mu_t \end{bmatrix}, \quad (8.5.7)$$

where

$$L = \begin{bmatrix} I & BQ^{-1}B' \\ 0 & A' \end{bmatrix}, \quad N = \begin{bmatrix} A & 0 \\ -R & I \end{bmatrix}.$$

Represent (8.5.7) as

$$\begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix} = M_f \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} \tag{8.5.8}$$

or

$$\begin{bmatrix} x_t \\ \mu_t \end{bmatrix} = M_b \begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix} \tag{8.5.9}$$

where

$$M_f = L^{-1}N = \begin{bmatrix} A + BQ^{-1}B'A'^{-1}R & -BQ^{-1}B'A'^{-1} \\ -A'^{-1}R & A'^{-1} \end{bmatrix}, \tag{8.5.10}$$

and

$$M_b = N^{-1}L = \begin{bmatrix} A^{-1} & A^{-1}BQ^{-1}B' \\ RA^{-1} & RA^{-1}BQ^{-1}B' + A' \end{bmatrix}. \tag{8.5.11}$$

Evidently $M_b = M_f^{-1}$. The matrices M_f and M_b each have the property that their eigenvalues occur in reciprocal pairs: if λ_o is an eigenvalue, then so is λ_o^{-1}. We postpone a proof of the 'reciprocal pairs' property of the eigenvalues to the subsequent section on the doubling algorithm, where it will follow simply by verifying that M_b and M_f are examples of *symplectic* matrices.

Because its eigenvalues occur in reciprocal pairs, we can represent the matrix M_f in (8.5.8) via a Schur decomposition

$$M_f = VWV^{-1}, \tag{8.5.12}$$

where V is a nonsingular matrix,

$$W = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix},$$

where W_{11} is a stable matrix, and W_{22} is an unstable matrix. In terms of transformed variables $y_t^* = V^{-1}y_t \equiv V^{-1}\begin{bmatrix} x_t \\ \mu_t \end{bmatrix}$, the system can be written

$$y_{t+1}^* = Wy_t^*. \tag{8.5.13}$$

Let $V^{-1} = \begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix}$, where the partitions conform in size to those of W. The solution of (8.5.13) is

$$y_t^* = \begin{bmatrix} W_{11}^t & \phi_t \\ 0 & W_{22}^t \end{bmatrix} \begin{bmatrix} V^{11}x_0 + V^{12}\mu_0 \\ V^{21}x_0 + V^{22}\mu_0 \end{bmatrix}, \tag{8.5.14}$$

where $\phi_0 = W_{12}, \phi_{j+1} = W_{11}^j W_{12} + \phi_j W_{22}$ for $j \geq 0$. Because W_{22} is an unstable matrix, to guarantee that $\lim_{t\to\infty} y_t^* = 0$, we require that

$$V^{21}x_0 + V^{22}\mu_0 = 0, \tag{8.5.15}$$

which sets an initial condition that replicates itself over time in the sense that recursions on (8.5.14) imply

$$V^{21}x_t + V^{22}\mu_t = 0, \tag{8.5.16}$$

for all $t \geq 0$. Equation (8.5.15) implies

$$\mu_0 = -(V^{22})^{-1}V^{21}x_0.$$

Substituting (8.5.16) into (8.5.13) and using $\begin{bmatrix} x_t \\ \mu_t \end{bmatrix} = Vy_t^*$ gives

$$\begin{aligned} x_{t+1} &= V_{11}W_{11}(V^{11} - V^{12}(V^{22})^{-1}V^{21})x_t \\ \mu_{t+1} &= V_{21}W_{11}(V^{11} - V^{12}(V^{22})^{-1}V^{21})x_t. \end{aligned} \tag{8.5.17}$$

However, as noted above, $\mu_t = Px_t$, where P solves the algebraic Riccati equation (8.3.4). Therefore, (8.5.17) implies that $PV_{11} = V_{21}$ or

$$P = V_{21}V_{11}^{-1} = -(V^{22})^{-1}V^{21}. \tag{8.5.18}$$

Equation (8.5.18) is our formula for P.

8.5.3. Distorted Economies

The invariant subspace method can also be applied to compute solutions of distorted economies whose equilibrium conditions can be arranged into the form (8.5.7). Examples of such economies are described in Chapter 15, where equilibrium conditions of the form (8.5.7) cannot be interpreted as the first order conditions of any linear quadratic control problem. For these economies, the matrix M_f in general fails to have eigenvalues in reciprocal pairs. It may or may not be possible to sort the eigenvalues into equal numbers of stable and unstable ones, which are to become the eigenvalues of W_{11} and W_{22}, respectively. Whether it *is* possible becomes a check for the existence and uniqueness of a stable solution of the model. The condition that there exist a unique solution of (8.5.8) with $|x_t|^2 < \infty$ is that there exists a Schur decomposition (8.5.12) of M_f in which half the eigenvalues of M_f are stable, and the other half are unstable. An excess of stable eigenvalues indicates nonuniqueness; an excess of unstable eigenvalues indicates nonexistence of a stable solution. Where a unique solution exists, it can be computed using formula (8.5.18).[5]

8.5.4. Transition Dynamics

Invariant subspace algorithms can be adapted to solve models in which elements of the matrices determining preferences, technologies, information, and government policies vary deterministically over time, before some date T_1, after which they are constant. The procedure is to use the algorithm (8.5.18) to solve the model for $t \geq T_1$, then to work backwards for earlier dates.

We want to compute an equilibrium in which the L, N matrices are time-varying in a simple deterministic way, say, due to once and for all changes in tax rates at some date $t = T_1 > 0$. Suppose that we want to solve

$$N \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} = L \begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix}, \quad t \geq T_1 \tag{8.5.19}$$

and

$$\tilde{N} \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} = \tilde{L} \begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix}, \quad 0 \leq t < T_1, \tag{8.5.20}$$

where $\tilde{N}, \tilde{L}$ are the 'temporary' versions of the matrices whose 'permanent' values are L, N.

[5] See Blanchard and Kahn (1981) and Whiteman (1983).

For $t \geq T_1$, we use the solution of the 'permanent' system, with

$$V^{21} x_t + V^{22} \mu_t = 0. \tag{8.5.21}$$

In particular, (8.5.21) implies that

$$G_{1T_1} x_{T_1} + G_{2T_1} \mu_{T_1} = 0, \tag{8.5.22}$$

where $G_{1T_1} = V^{21}, G_{2T_1} = V^{22}$.

We know that

$$\begin{bmatrix} x_{T_1} \\ \mu_{T_1} \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix} \begin{bmatrix} V^{11} x_{T_1-1} + V^{12} \mu_{T_1-1} \\ V^{21} x_{T_1-1} + V^{22} \mu_{T_1-1} \end{bmatrix}. \tag{8.5.23}$$

We want to impose restriction (8.5.22) on (8.5.23) and use it to solve for x_{T_1-1} as a linear function of μ_{T_1-1}. A couple of lines of algebra leads to the restriction

$$\begin{aligned} &[G_{1T_1} V_{11} + G_{2T_1} V_{21}][W_{11} V^{11} + W_{12} V^{21}] + [G_{1T_1} V_{12} + G_{2T_1} V_{22}] W_{22} V^{21} x_{T_1-1} \\ +&\Big\{[G_{1T_1} V_{11} + G_{2T_1} V_{21}][W_{11} V^{12} + W_{12} V^{22}] + [G_{1T_1} V_{12} + G_{2T_1} V_{22}] W_{22} V^{22}\Big\} \mu_{T_1-1} \\ &\equiv G_{1,T_1-1} x_{T_1-1} + G_{2,T_1-1} \mu_{T_1-1} = 0. \end{aligned} \tag{8.5.24}$$

This equation can be written as

$$G_{1T_1-1} x_{T_1-1} + G_{2T_1-1} \mu_{T_1-1} = 0, \tag{8.5.25}$$

and it can be solved for μ_{T_1-1} as a linear function of x_{T_1-1}. Equations (8.5.24) and (8.5.25) define (G_{1t}, G_{2t}) as a function of (G_{1t+1}, G_{2t+1}). We use (8.5.25) to 'backdate' the $G_{it}, i = 1, 2$, matrices, and iterate back to $t = 0$.

These calculations will produce time-varying versions of *all* of our equilibrium matrices $A^o, C, S_c, M_c, \ldots$ for $t = 0, 1, \ldots, T_1$ described in chapters 4 and 6.

8.6. The Doubling Algorithm

The algebraic matrix Riccati equation can be solved with a *doubling algorithm.*[6] The algorithm shares with invariant subspace methods the prominent role it assigns to the matrix M_b of equation (8.5.9).

We start with a finite horizon version of our problem for horizon $t = 0, \ldots, \tau - 1$, which leads to a two point boundary problem. We continue to assume that A is nonsingular, iterate on (8.5.8), and impose the boundary condition $\mu_\tau = 0$ to get

$$\hat{M} \begin{bmatrix} x_\tau \\ 0 \end{bmatrix} = \begin{bmatrix} x_0 \\ \mu_0 \end{bmatrix}, \tag{8.6.1}$$

where

$$\hat{M} = M_f^{-\tau} = M_b^\tau. \tag{8.6.2}$$

We want to solve (8.6.2) for μ_0 as a function of x_0, and from this solution deduce a finite-horizon approximation to P. Partitioning $\hat{M}$ conformably with the state-co-state partition, we deduce $\hat{M}_{11} x_\tau = x_0, \quad \hat{M}_{21} x_\tau = \mu_0$. Therefore, we choose $\mu_0 = \hat{M}_{21}(\hat{M}_{11})^{-1} x_0$, and set the matrix

$$P = \hat{M}_{21}(\hat{M}_{11})^{-1}. \tag{8.6.3}$$

The plan is efficiently to compute $\hat{M}$ for large horizon τ, then use (8.6.3) to compute P. We can accelerate the computations by choosing τ to be a power of two and using

$$M_f^{-2^{k+1}} = (M_f^{-2^k}) M_f^{-2^k}. \tag{8.6.4}$$

Thus, for $\tau = 2^j$, the matrix $\hat{M} = M_f^{-\tau}$ can be computed in j iterations instead of 2^j iterations, inspiring the name *doubling* algorithm.

Because M_f^{-1} has unstable eigenvalues, direct iterations on (8.6.4) can be unreliable. Therefore, the *doubling algorithm* transforms iterations on (8.6.4) into other iterations whose important objects converge. These iterations exploit the fact that the matrix M_f is *symplectic* (see Appendix B). The eigenvalues of symplectic matrices come in reciprocal pairs. The product of symplectic matrices is symplectic; for any symplectic matrix S, the matrices $S_{21}(S_{11})^{-1}$ and $(S_{11})^{-1} S_{12}$ are both symmetric; and

$$\begin{aligned} S_{22} &= (S_{11}')^{-1} + S_{21}(S_{11})^{-1} S_{12} \\ &= (S_{11}')^{-1} + S_{21}(S_{11})^{-1} S_{11} (S_{11})^{-1} S_{12}. \end{aligned}$$

[6] This section is based on Anderson, Hansen, McGrattan, and Sargent (1995). For another discussion of the doubling algorithm, see Anderson and Moore [1979, pp. 158–160].

Therefore, a $(2n \times 2n)$ symplectic matrix can be represented in terms of three $(n \times n)$ matrices $\alpha = (S_{11})^{-1}, \beta = (S_{11})^{-1}S_{12}, \gamma = S_{21}(S_{11})^{-1}$, the latter two matrices being symmetric.

These properties of symplectic matrices inspire the following parameterization of $M_f^{-2^k}$

$$M_f^{-2^k} = \begin{bmatrix} \alpha_k^{-1} & \alpha_k^{-1}\beta_k \\ \gamma_k\alpha_k^{-1} & \alpha_k' + \gamma_k\alpha_k^{-1}\beta_k \end{bmatrix}, \tag{8.6.5}$$

where the $n \times n$ matrices $\alpha_k, \beta_k, \gamma_k$ satisfy the recursions

$$\begin{aligned} \alpha_{k+1} &= \alpha_k(I + \beta_k\gamma_k)^{-1}\alpha_k \\ \beta_{k+1} &= \beta_k + \alpha_k(I + \beta_k\gamma_k)^{-1}\beta_k\alpha_k' \\ \gamma_{k+1} &= \gamma_k + \alpha_k'\gamma_k(I + \beta_k\gamma_k)^{-1}\alpha_k. \end{aligned} \tag{8.6.6}$$

To initialize, we use representation (8.5.11) for $M_b = M_f^{-1}$ to induce the settings: $\alpha_0 = A, \gamma_0 = R, \beta_0 = BQ^{-1}B'$.

Anderson, Hansen, McGrattan, and Sargent (1996) describe a version of the doubling algorithm modified to build in a positive definite terminal value matrix P_o. Their scheme initializes iterations on (8.6.6) as follows:

$$\begin{aligned} \alpha_0 &= (I + BQ^{-1}B'P_o)^{-1}A \\ \beta_0 &= (I + BQ^{-1}B'P_o)^{-1}BQ^{-1}B' \\ \gamma_0 &= R - P_o + A'P_o(I + BQ^{-1}B'P_o)^{-1}A. \end{aligned} \tag{8.6.7}$$

The modified algorithm then works as follows:

1. Initialize $\alpha_0, \beta_0, \gamma_0$ according to (8.6.7).
2. Iterate on (8.6.6).
3. Form P as the limit of $\gamma_k + P_o$.

We have assumed that A is nonsingular, but Anderson (1985) argues that the doubling algorithm is applicable also in circumstances where A is singular.[7] Anderson, Hansen, McGrattan, and Sargent (1996) report the results of computations in which the doubling algorithm is among the fastest and most reliable available algorithms for solving several example economies.

7 See Anderson, Hansen, McGrattan, and Sargent (1996) for conditions under which the matrix sequences $\{\alpha_k\}, \{\beta_k\}, \{\gamma_k\}$ converge.

8.7. Partitioning the State Vector

Undiscounted versions of the control problem solved by our social planner assume a form for which it is natural to partition the state vector to take advantage of the pattern of zeros in A and B. This leads to a control problem of the form: choose $\{u_t\}_{t=0}^{\infty}$ to maximize

$$-\sum_{t=0}^{\infty}\{\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}' \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + u_t' Q u_t\} \tag{8.7.1}$$

subject to

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_t, \tag{8.7.2}$$

with $[x_{10}', x_{20}']'$ given.[8]

For this problem, the operator associated with Bellman's equation is

$$T(P) = R + A'PA - A'PB(Q + B'PB)^{-1}B'PA. \tag{8.7.3}$$

Partitioning P and $T(P)$ conformably with the partition $\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}$ makes the $(1,1)$ and $(1,2)$ components of $T(P)$ satisfy

$$T_{11}(P_{11}) = R_{11} + A_{11}' P_{11} A_{11} - A_{11}' P_{11} B_1 (Q + B_1' P_{11} B_1)^{-1} B_1' P_{11} A_{11} \tag{8.7.4}$$

$$\begin{aligned} T_{12}(P_{11}, P_{12}) = {} & R_{12} + A_{11}' P_{11} A_{12} \\ & - A_{11}' P_{11} B_1 (Q + B_1' P_{11} B_1)^{-1} B_1' P_{11} A_{12} \\ & + [A_{11}' - A_{11}' P_{11} B_1 (Q + B_1' P_{11} B_1)^{-1} B_1'] P_{12} A_{22} \end{aligned} \tag{8.7.5}$$

Equation (8.7.4) shows that T_{11} depends on P_{11}, but not on other elements of the partition of P. From (8.7.5), T_{12} depends on P_{11} and P_{12}, but not on P_{22}. Because T maps symmetric matrices into symmetric matrices, the $(2,1)$

[8] System (8.7.1) – (8.7.2) is called a *controllability canonical form* (see Kwakernaak and Sivan [1972]). Two things distinguish a controllability canonical form: (1) the pattern of zeros in the pair $(A, B,)$ and (2) a requirement that (A_{11}, B_1) be a controllable pair (see Appendix A of this chapter). A controllability canonical form adopts a description of the state vector that separates it into a part x_{2t} that cannot be affected by the controls, and a part x_{1t} that can be controlled in the sense that there exists a sequence of controls $\{u_t\}$ that sends x_1 to any arbitrarily specified point within the space in which x_1 lives.

block of T is just the transpose of the $(1,2)$ block. Finally, the $(2,2)$ block of T depends on P_{11}, P_{12}, and P_{22}.

Partition the optimal feedback matrix $F = [F_1 \ F_2]$, where the partition is conformable with that of x_t. Then the optimal control is

$$u_t = [F_1 \ F_2] \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}.$$

Let P_{11}^f be the fixed point of (8.7.4) and let P_{12}^f be the fixed point of $T_{12}(P_{11}^f, P_{12})$. Then F_1 and F_2 are given by

$$F_1 = (Q + B_1' P_{11}^f B_1)^{-1} B_1' P_{11}^f A_{11} \tag{8.7.6}$$

$$F_2 = (Q + B_1' P_{11}^f B_1)^{-1} (B_1' P_{11}^f P_{12} + B_1' P_{12}^f A_{22}) \tag{8.7.7}$$

Equation (8.7.6) shows that F_1 depends only on P_{11}^f, while F_2 depends on P_{11}^f and P_{12}^f, but not on P_{22}^f, the fixed point of T_{22}.

We aim to compute $[F_1, F_2]$ and the multipliers described in chapter 3, which turn out only to depend on P_{11}^f and P_{12}^f. We can compute these objects rapidly by using the structure exposed by (8.7.4) and (8.7.5). First, note that the T_{11} operator identified by (8.7.4) is formally equivalent with the T operator of (8.7.3), except that $(1,1)$ subscripts appear on A and R, and a (1) subscript appears on B. Thus, the T_{11} operator is simply the operator whose iterations define the matrix Riccati difference equation for the small optimal regulator problem determined by the matrixes (A_{11}, B_1, Q, R_{11}). We can compute P_{11}^f by using any of the algorithms described above for this smaller problem. We have chosen to use the doubling algorithm (8.6.6).

Second, given a fixed point P_{11}^f of T_{11}, we apply another sort of doubling algorithm to compute the fixed point of $T_{12}(P_{11}^f, \cdot)$. This mapping has the form

$$T_{12}(P_{11}^f, P_{12}) = D + G' P_{12} H \tag{8.7.8}$$

where $D = R_{12} + A_{11}' P_{11}^f A_{12} - A_{11}' P_{11}^f B_1 (Q + B_1' P_{11}^f B_1)^{-1} B_1' P_{11}^f A_{12}, G = [A_{11} - B_1 (Q + B_1' P_{11}^f B_1)^{-1} B_1' P_{11}^f A_{11}], H = A_{22}$. Notice that $G = A_{11} - B_1 F_1$, where F_1 is computed from (8.7.6). When x_{2t} is set to zero for all t, the law of motion for x_{1t} under the optimal control is thus given by

$$x_{1t+1} = G x_{1t}.$$

For problems for which condition (8.1.1) is either automatically satisfied or else imposed, the eigenvalues of G and H each have absolute values strictly less than unity. That the eigenvalues of G and H are both less than unity assures the existence of a limit point to iterations on (8.7.8). The limit point satisfies the *Sylvester equation*

$$P_{12} = D + G' P_{12} H, \tag{8.7.9}$$

which is to be solved for P_{12}. The limit point of iterations on T_{12} initiated from $P_{12}(0) = 0$ can be represented

$$P_{12}^f = \sum_{j=0}^{\infty} G'^j D H^j, \tag{8.7.10}$$

whose status as a fixed point of $T_{12}(P_{11}^f, \cdot)$ can be verified directly. However, iterations on (8.7.9) would not be an efficient way to compute P_{12}. Instead, we recommend using this doubling algorithm. Compute the following objects recursively:

$$\begin{aligned} G_j &= G_{j-1} G_{j-1} \\ H_j &= H_{j-1} H_{j-1} \\ P_{12,j} &= P_{12,j-1} + G'_{j-1} P_{12,j-1} H_{j-1} \end{aligned} \tag{8.7.11}$$

where we set $P_{12,0} = D, G_0 = G, H_0 = H$. By repeated substitution it can be shown that

$$P_{12,j} = \sum_{i=0}^{2^j-1} G^{i\prime} D H^i. \tag{8.7.12}$$

Each iteration doubles the number of terms in the sum.[9] , [10]

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[9] This algorithm is implemented in the MATLAB program `double2j.m`.

[10] The $(1,2)$ partition of P is simply $P_{12}^{f\prime}$. We could derive an algorithm similar to (8.7.11) to compute P_{22}^f, but we don't need to compute P_{22}^f, which is used to compute neither $[F_1 \; F_2]$ nor the Lagrange multipliers that determine the price system associated with our equilibrium.

8.8. The Periodic Optimal Linear Regulator

In chapter 17, we study a class of models of seasonality whose social planning problems form a periodic optimal linear regulator problem: choose $\{u_t\}_{t=0}^{\infty}$ to maximize

$$\sum_{t=0}^{\infty}\{x'_{s(t)}R_{s(t)}x_t + u'_tQ_{s(t)}u_t\} \tag{8.8.1}$$

subject to

$$x_{t+1} = A_{s(t)}x_t + B_{s(t)}u_t. \tag{8.8.2}$$

Here $s(t)$ is a periodic function that maps the integers into a subset of the integers:

$$s : (\cdots - 1, 0, 1, \cdots) \rightarrow [1, 2, \cdots, p]$$
$$s(t + p) = s(t) \text{ for all } t.$$

In problem (8.8.1) - (8.8.2), the matrices A_s, B_s, Q_s, and R_s that define the linear regulator problem are each periodic with common period p.

Associated with problem (8.8.1) – (8.8.2) is the following version of the matrix Riccati difference equation:

$$\begin{aligned} P_t &= R_{s(t)} + A'_{s(t)}P_{t+1}A_{s(t)} \\ &\quad - A'_{s(t)}P_{t+1}B_{s(t)}(Q_{s(t)} + B'_{s(t)}P_{t+1}B_{s(t)})^{-1}B'_{s(t)}P_{t+1}A_{s(t)}. \end{aligned} \tag{8.8.3}$$

Under conditions that generalize assumption A1, which were discussed by Richard Todd [1983], iterations on (8.8.3) yield p convergent subsequences, whose limit points we denote $P_1, P_2, \ldots, P_p$. The optimal decision rule in period t is

$$u_t = -F_{s(t)}x_t, \tag{8.8.4}$$

where

$$F_{s(t)} = -(Q_{s(t)} + B'_{s(t)}P_{s(t+1)}B_{s(t)})^{-1}B'_{s(t)}P_{s(t+1)}, A_{s(t)}. \tag{8.8.5}$$

Thus, the optimal decision rules themselves have period p.

One way to compute the optimal decision rules is to iterate on (8.8.3) to convergence of the p subsequences, and then to use (8.8.5). Faster algorithms can be obtained by adapting calculations described earlier in this chapter. In the next section, we show how doubling algorithms apply to the periodic linear regulator problem, and also how the ‘controllability canonical form’ can be exploited.

8.9. A Periodic Doubling Algorithm

First-order conditions for the periodic linear regulator can be represented as

$$\begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix} = M_{f,s(t)} \begin{bmatrix} x_t \\ \mu_t \end{bmatrix}, \tag{8.9.1}$$

where $M_{f,s(t)}$ is the periodic counterpart to the matrix M_f defined in (8.5.10). Iterating this equation p times and using the periodic structure of $s(t)$ gives

$$\begin{bmatrix} x_{t+p} \\ \mu_{t+p} \end{bmatrix} = \Gamma_p \begin{bmatrix} x_t \\ \mu_t \end{bmatrix}, \tag{8.9.2}$$

where

$$\Gamma_p \equiv M_{f,p-1} M_{f,p-2} \cdots M_{f,1} M_{f,p}. \tag{8.9.3}$$

The matrix Γ_p is the product of p symplectic matrices, and therefore is symplectic. Equation (8.9.2) at $t = p$ can be represented

$$\Gamma_p^{-1} \begin{bmatrix} x_{2p} \\ \mu_{2p} \end{bmatrix} = \begin{bmatrix} x_p \\ \mu_p \end{bmatrix}, \tag{8.9.4}$$

where

$$\Gamma_p^{-1} = M_{f,p}^{-1} M_{f,1}^{-1} \cdots M_{f,p-1}^{-1}. \tag{8.9.5}$$

Iterating (8.9.4) $\tau - 1 \geq 1$ times and imposing the same boundary condition used in (8.6.1) gives

$$\hat{M} \begin{bmatrix} x_{p\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} x_p \\ \mu_p \end{bmatrix}, \tag{8.9.6}$$

where $\hat{M} = \Gamma_p^{-\tau}$. An argument used earlier implies that the doubling algorithm can be applied to our redefined $\hat{M}$ to compute

$$P_p = \hat{M}_{21} (\hat{M}_{11})^{-1}. \tag{8.9.7}$$

It is straightforward to compute the remaining $p-1$ value functions. Notice that (8.9.4) implies

$$\hat{M} \begin{bmatrix} x_{p\tau} \\ 0 \end{bmatrix} = M_{f,p-1} \begin{bmatrix} x_{p-1} \\ \mu_{p-1} \end{bmatrix},$$

or

$$M_{f,p-1}^{-1} \hat{M} \begin{bmatrix} x_{p\tau} \\ 0 \end{bmatrix} = \begin{bmatrix} x_{p-1} \\ \mu_{p-1} \end{bmatrix}.$$

The same argument used above now implies that

$$\mu_1 = \tilde{M}_{21}(\tilde{M}_{22})^{-1}x_1 \equiv P_1 x_1,$$

where $\tilde{M} = \hat{M}_{p-1} \equiv M_{f,p-1}^{-1}\hat{M}$ is symplectic because it is the product of two symplectic matrices. The product of two symplectic matrices Z_1, Z_2 has representation

$$Z_1 Z_2 = \bar{Z} = \begin{bmatrix} \tilde{\alpha}^{-1} & \tilde{\alpha}^{-1}\tilde{\beta} \\ \tilde{\gamma}\tilde{\alpha}^{-1} & \tilde{\alpha}^{1} + \tilde{\gamma}\tilde{\alpha}^{-1}\tilde{\beta} \end{bmatrix}$$

where

$$\begin{aligned} \tilde{\alpha} &= \alpha_2(I + \beta_1\gamma_2)^{-1}\alpha_1 \\ \tilde{\gamma} &= \gamma_1 + \alpha_1'\gamma_2(I + \beta_1\gamma_2)^{-1}\alpha_1 \\ \tilde{\beta} &= \beta_2 + \alpha_2(I + \beta_1\gamma_2)^{-1}\beta_1\alpha_2'. \end{aligned} \tag{8.9.8}$$

We can use this feature to compute P_{p-1} from the γ term produced by this representation of multiplication.

Iterating this argument leads us to compute $P_{p-2}, \ldots, P_1$ as the corresponding γ matrices in the successive multiplications used to form $\hat{M}_{p-2} = M_{f,p-2}^{-1}\hat{M}_{p-1}, \ldots, \hat{M}_1 = M_{f,1}^{-1}\hat{M}_2$.

Thus, the algorithm works as follows.

1. Initialize $\alpha_0, \beta_0, \gamma_0$ according to (8.6.7).
2. Use the algorithm (8.9.8) for multiplying symplectic matrices to form Γ_p^{-1} defined as in (8.9.5).
3. Iterate on (8.6.6).
4. Form P_p as the limit of $\gamma_k + P_o$.
5. Successively form $\hat{M}_{p-1}, \hat{M}_{p-2}, \ldots, \hat{M}_1$ using (8.9.8), and set the corresponding γ terms to $P_{p-1}, P_{p-2}, \ldots, P_1$.

Having computed $P_1, \ldots, P_p$, we can use (8.8.5) to compute the optimal decision rules. The optimal feedback laws are periodic, so that $u_t = -F_{s(t)}x_t$. The matrices $F_1, \ldots F_p$ are computed from

$$F_j = (Q_j + B_j{}'P_{j+1}B_j)^{-1}B_j{}'P_{j+1}A_j,$$

where it is understood that $P_{p+1} = P_1$.

8.9.1. Partitioning the state vector

We can also apply the partitioning technique to the periodic optimal linear regulator problem in order to accelerate the computations. We partition the state vector into $\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}$ exactly as above. With the appropriate specification of R_s, Q_s, A_s , and B_s , we obtain a periodic version of the $T_{11}(P_{11})$ mapping described in equation (8.7.4). Use our procedures to compute $P_1, P_2, \ldots, P_p$ as described above, then set $P^f_{11,j} = P_j$ for $j = 1, \ldots, p$.

The T_{12} mapping for the periodic model becomes

$$T_{12,k}(P^f_{11,k+1}, P_{12,k+1}) = D_k + {G_k}' P_{12,k+1} H_k \tag{8.9.9}$$

where

$$\begin{aligned} D_k &= R_{12,k} + A'_{11,k} P^f_{11,k+1} A_{12,k} - A'_{11,k} P^f_{11,k+1} B_{1k} \\ &\quad \times (Q_k + B'_{1k} P^f_{11,k+1} B_{1k})^{-1} B'_{1k} P_{11,k+1} A_{12,k} \\ G_k &= [A_{11,k} - B_{1k}(Q_k + B'_{1k} P^f_{11,k+1} B_{1k})^{-1} B'_{1k} P_{11,k+1} A_{11,k} \\ H_k &= A_{22,k} \end{aligned} \tag{8.9.10}$$

In (8.9.9) – (8.9.10), P^f_{11k+1} is the fixed point for $P_{11,k+1}$ corresponding to period $k+1$. Iterations on (8.9.9) will give rise to a sequence consisting of p convergent subsequences, whose limit points we call $P^f_{12,1}, \ldots, P^f_{12,p}$. We desire to compute these limiting matrices.

We begin by creating an operator $\bar{T}_{12,1}$ whose fixed point is $P^f_{12,1}$. We define

$$\bar{T}_{12,1}(P_{12,1}) = \bar{D}_1 + \bar{G}'_1 P_{12,1} \bar{H}_1 \tag{8.9.11}$$

where $\bar{D}_1 = D_1 + G'_1 D_2 H_1 + \cdots + G'_1 G'_2 \cdots G'_{p-1} D_p H_{p-1} H_{p-2} \cdots H_1 \bar{G}'_1 = G'_1 G'_2 \cdots G'_p \bar{H}_1 = H_p H_{p-1} \cdots H_1$. We can compute the fixed point of (8.9.11) by using the standard doubling algorithm that is described in section blank and that is implemented in the MATLAB program `double2j.m`.

Once we have computed $P^f_{12,1}$, we can compute $P^f_{12,j}$ for $j = p, p-1, \ldots, 2$ by using

$$\begin{aligned} P^f_{12,p} &= D_p + G'_p P^f_{12,1} H_1 \\ P^f_{12,j} &= D_j + G'_j P^f_{12,j+1} H_j \quad , \; j = p-1, p-2, \cdots, 2 \end{aligned} \tag{8.9.12}$$

The optimal feedback laws $u_t \equiv -F_{s(t)}, x_t$ can be computed as follows. Let $F_{s(t)} = [F_{1s(t)} \; F_{2s(t)}]$, where the partition of $F_{s(t)}$ matches that of the state vector into $x_1(t), x_2(t)$. Then we have

$$\begin{aligned} F_{1j} &= (Q_j + B'_{1j} P^f_{11,j+1} B_{1j})^{-1} B'_{1j} P^f_{11,j+1} A_{11,j} \\ F_{2j} &= (Q_j + B'_{1j} P^f_{11,j+1} B_{1j})^{-1} (B'_{1j} P^f_{11,j+1} A_{12,j} + B'_{1j} P^f_{12,j+1} A_{22,j}) \\ &\text{for } j = 1, \ldots, p. \end{aligned} \tag{8.9.13}$$

The optimal closed loop system is then

$$x_{t+1} = (A_{s(t)} - B_{s(t)}F_{s(t)})x_t. \tag{8.9.14}$$

8.10. Linear Exponential Quadratic Gaussian Control

In chapter 16, we shall reinterpret some of our economies in terms of risk-sensitive control theory. In this section, we describe how to adapt the preceding computational strategies to handle versions of the 'risk-sensitivity corrections' of Jacobsen (1973, 1977) and Whittle (1990). We use Hansen and Sargent's (1995) method of implementing discounting. The resulting specification preserves the computational ease of the original linear quadratic specification, while relaxing 'certainty equivalence.' Let $V_{t_1}(x_{t_1}) = -x'_{t_1} P_{t_1} x_{t_1} - \eta_{t_1}$. Let $\beta \in (0,1)$ and consider the sequence $\{V_t(x_t)\}_{t=t_0}^{t_1}$ of value functions generated by the following constrained optimization problems:

$$V_t(x_t) = \max_{u_t} \Big\{ -(x'_t R x_t + u'_t Q u_t) + \beta \frac{2}{\sigma} \ \log E_t \exp \ \frac{\sigma}{2} V_{t+1}(x_{t+1}) \Big\} \tag{8.10.1}$$

subject to

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}, \tag{8.10.2}$$

where w_{t+1} is an $(N \times 1)$ martingale difference sequence with Gaussian density

$$f(w_{t+1}) = \frac{1}{(2\pi)^{N/2} |\Sigma|^{1/2}} \ \exp\{- \frac{1}{2} \, w'_{t+1} \, \Sigma^{-1} \, w_{t+1}\}. \tag{8.10.3}$$

Usually, we shall set the covariance matrix $\Sigma = E w_t w'_t = I$. We momentarily retain the more general notation in order to state a useful lemma in greater generality.

In solving this *discounted linear exponential quadratic Gaussian* (LEQG) control problem, we use the following lemma due to Jacobson (1973).

LEMMA (JACOBSON): Let $w_{t+1} \sim \mathcal{N}(0, \Sigma)$ and $x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$. Suppose that the matrix $(\Sigma^{-1} - \sigma C' P_{t+1} C)$ is positive definite. Then

$$E_t \exp\{\frac{\sigma}{2} \, x'_{t+1} \, P_{t+1} \, x_{t+1}\} =$$

$$\int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}|\Sigma|^{1/2}} \exp\{-\frac{1}{2} w_{t+1}' \Sigma^{-1} w_{t+1}\} \exp\{\frac{\sigma}{2} x_{t+1}' P_{t+1} x_{t+1}\} \tag{8.10.4}$$

$$= k \exp\Big\{\frac{\sigma}{2} (Ax_t + Bu_t)' \tilde{P}_{t+1} (Ax_t + Bu_t)\Big\}$$

where

$$\tilde{P}_{t+1} = P_{t+1} + \sigma P_{t+1} C (\Sigma^{-1} - \sigma C' P_{t+1} C)^{-1} C' P_{t+1} \tag{8.10.5}$$

$$k = \sqrt{\frac{\mid (\Sigma^{-1} - \sigma C' P_{t+1} C)^{-1} \mid}{\mid \Sigma \mid}}. \tag{8.10.6}$$

This concludes the statement of the lemma.

Let $V_{t+1}(x_{t+1}) = -x_{t+1}' P_{t+1} x_{t+1} - \eta_{t+1}$, and apply the lemma to evaluate the term inside the braces on the right side of (8.10.1):

$$\begin{aligned} x_t' R x_t + u_t' Q u_t + \beta \frac{2}{\sigma} \log E_t \exp\Big\{\frac{\sigma}{2} [x_{t+1}' P_{t+1} x_t + \eta_{t+1}]\Big\} \\ = x_t' R x_t + u_t' Q u_t + \beta (Ax_t + Bu_t)' \tilde{P}_{t+1} (Ax_t + Bu_t) \\ + \text{ constant} \end{aligned} \tag{8.10.7}$$

where $\tilde{P}_{t+1}$ is given by equation (8.10.5). Maximizing the right hand side of (8.10.7) with respect to u_t gives the linear decision rule $u_t = -F_t x_t$, where F_t is determined by the recursions:

$$\tilde{P}_{t+1} = P_{t+1} + \sigma P_{t+1} C (\Sigma^{-1} - \sigma C' P_{t+1} C)^{-1} C' P_{t+1} \tag{8.10.8}$$

$$F_t = \{Q + \beta B' \tilde{P}_{t+1} B\}^{-1} \beta B' \tilde{P}_{t+1} A \tag{8.10.9}$$

$$\begin{aligned} P_t = R + \beta A' \tilde{P}_{t+1} A \\ - \beta^2 A' \tilde{P}_{t+1} B (Q + \beta B' \tilde{P}_{t+1} B)^{-1} B' \tilde{P}_{t+1} A. \end{aligned} \tag{8.10.10}$$

Notice that in the special case that $\sigma = 0$, these equations are versions of the Riccati difference equation and the associated decision rule. Notice also that when $\sigma \neq 0$, equations (8.10.8), (8.10.9), and (8.10.10) imply that the decision rules F_t depend on the innovation variances of the exogenous processes (note the appearance of C in (8.10.8)).

We can obtain a more compact version of these recursions as follows. Apply the matrix identity $(a - b\,d^{-1}c)^{-1} = a^{-1} + a^{-1}b(d - ca^{-1}b)^{-1}ca^{-1}$ to (8.10.10) using the settings $a^{-1} = \beta \tilde{P}_{t+1}$, $b = -B$, $d = Q$, $c = B'$ to obtain

$$\begin{aligned}\beta \tilde{P}_{t+1} - \beta \tilde{P}_{t+1} B (B'(\beta \tilde{P}_{t+1})\, B + Q)^{-1} B'(\beta \tilde{P}_{t+1}) \\ = (\frac{1}{\beta}\tilde{P}_{t+1} + BQ^{-1}B')^{-1}.\end{aligned}$$

Substituting into the right side of (8.10.10) gives

$$P_t = R + A'(\frac{1}{\beta}\tilde{P}_{t+1} + BQ^{-1}B')^{-1}A\ . \tag{8.10.11}$$

Now apply the same matrix identity to the right side of (8.10.8) to obtain

$$\tilde{P}_{t+1} = (P_{t+1}^{-1} - \sigma C\Sigma C')^{-1}. \tag{8.10.12}$$

Substituting (8.10.12) into (8.10.11) gives the version

$$P_t = R + A'(\beta^{-1}P_{t+1}^{-1} + BQ^{-1}B' - \sigma\beta^{-1}C\Sigma C')^{-1}A. \tag{8.10.13}$$

Collecting results, we have that the solution of the problem can be represented via the recursions (8.10.13), (8.116), (8.10.9). We are interested in problems for which recursions on these equations converge as $t \to -\infty$. In situations in which convergence prevails, we can avail ourselves of a doubling algorithm to accelerate the computations.

8.10.1. Doubling algorithm

It suffices to consider the undiscounted ($\beta = 1$) version of our problem, because we can transform a discounted problem into an undiscounted one. Represent the Riccati equation (8.10.13) in the form (see Appendix C)

$$P_t = R + A'(P_{t+1}^{-1} + J)^{-1}A \tag{8.10.14}$$

where $J = BQ^{-1}B' - \sigma C\Sigma C'$. The doubling algorithm applies with

$$M_f^{-1} = M_b = \begin{bmatrix} A^{-1} & A^{-1}J \\ RA^{-1} & A' + RA^{-1}J \end{bmatrix},$$

and with the settings $\alpha_0 = A,\ \gamma_0 = R,\ \beta_0 = J$. To compute the solution with terminal value matrix P_o, use the initializations $\alpha_0 = (I + JP_o)^{-1}A, \beta_0 = (I + JP_o)^{-1}J, \gamma_0 = -P_o + R + A'P_o(I + JP_o)^{-1}A$. The algorithm then works as follows.

1. Initialize $\alpha_0, \beta_0, \gamma_0$ according to the formulas just given.
2. Iterate on (8.6.6).
3. Form P as the limit of $\gamma_k + P_o$.

A. Concepts of Linear Control Theory

Assume in the deterministic linear regulator (8.5.1)–(8.5.2) that matrix R is positive semi-definite and that Q is positive definite. Sufficient conditions for existence and stability of a solution of the deterministic linear regulator are typically stated in terms defined in the following four definitions.

DEFINITION: The pair (A, B) is *stabilizable* if $y'B = 0$ and $y'A = \lambda y'$ for some complex number λ and some complex vector y implies that $|\lambda| < 1$ or $y = 0$.

DEFINITION: The pair (A, B) is *controllable* if $y'B = 0$ and $y'A = \lambda y'$ for some complex number λ and some complex vector y implies that $y = 0$.

DEFINITION: The pair (A, D) is *detectable* if $D'y = 0$ and $Ay = \lambda y$ for some complex number λ and some complex vector y implies that $|\lambda| < 1$ or $y = 0$.

DEFINITION: The pair (A, D) is *observable* if $D'y = 0$ and $Ay = \lambda y$ for some complex number λ and some complex vector y implies $y = 0$.

Stabilizability and controllability evidently form a pair of concepts, with controllability implying stabilizability, but not *vice versa* (i.e., controllability is a more restrictive assumption. Similarly, detectability and observability form a pair of concepts, with observability implying detectability, but not *vice versa*.

Stabilizability is equivalent with existence of a time-invariant control law that stabilizes the state vector. Controllability implies that there exists a sequence of controls that can attain an arbitrary value for the state vector starting from any initial state vector, within n periods, where n is the dimension of the state. When (A, B) is controllable, the entire state vector is ‘endogenous,’ in the sense of being potentially ‘under control.’

The concepts of detectability and observability are applied to the pair of matrices (A, D), where $DD' = R$ (i.e., D is a factor of R).

Assume (a.) that the pair (A, B) is stabilizable, which implies that it is *feasible* to stabilize the state vector; and that (b.) the pair (A, D) is detectable, which means that it is *desirable* to stabilize the state vector. Together, assumptions (a.) and (b.) imply that the optimal control stabilizes the state vector.

When R is nonsingular, the pair (A, D) is observable, and the value function is strictly concave.

B. Symplectic Matrices

We now define symplectic matrices[11] and state some of their properties.

DEFINITION: A $(2n \times 2n)$ matrix Z is said to be symplectic if $Z'JZ = J$, where

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

The following properties of symplectic matrices follow directly from the definition of a symplectic matrix:

PROPERTY 1: If the matrix Z is symplectic, then so is any positive integer power of Z.

PROPERTY 2: If Z_1 and Z_2 are both $(2n \times 2n)$ symplectic matrices, then their product $Z_1 Z_2$ is also symplectic.

PROPERTY 3: If a symplectic matrix Z is written in partitioned form

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

and if Z_{11}^{-1} exists, then

$$Z_{22} = (Z_{11}')^{-1} + Z_{21} Z_{11}^{-1} Z_{12}$$

PROPERTY 4: The eigenvalues of any symplectic matrix Z occur in reciprocal pairs, *i.e.*, if λ_i is an eigenvalue of a symplectic matrix Z, then so is λ_i^{-1}.

To establish property 4, that from the definition that any symplectic matrix Z satisfies $Z^{-1} = J^{-1} Z' J$. Since Z^{-1} and Z' are thus related by a similarity transformation, they have common eigenvalues. This implies that the eigenvalues of Z must occur in reciprocal pairs.

Property 3 means that if Z_{11}^{-1} exists, then a symplectic matrix Z can be represented in the form

$$Z = \begin{bmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ \gamma\alpha^{-1} & \alpha' + \gamma\alpha^{-1}\beta \end{bmatrix} \tag{8.B.1}$$

[11] See Anderson and Moore [1979, pp. 160–161] and also Anderson, Hansen, McGrattan, and Sargent (1996).

Let Z_j, for $j = 1,2$, be two symplectic matrices, each represented in the form $(8.B.1)$:

$$Z_j = \begin{bmatrix} \alpha_j^{-1} & \alpha_j^{-1}\beta_j \\ \gamma_j\alpha_j^{-1} & \alpha_j' + \gamma_j\alpha_j^{-1}\beta_j \end{bmatrix}. \tag{8.B.2}$$

It can be verified directly that the product $Z_1 Z_2 = \bar{Z}$ has the same form, namely,

$$Z_1 Z_2 = \bar{Z} = \begin{bmatrix} \tilde{\alpha}^{-1} & \tilde{\alpha}^{-1}\tilde{\beta} \\ \tilde{\gamma}\tilde{\alpha}^{-1} & \tilde{\alpha}' + \tilde{\gamma}\tilde{\alpha}^{-1}\tilde{\beta} \end{bmatrix}. \tag{8.B.3}$$

where

$$\begin{aligned} \tilde{\alpha} &= \alpha_2(I + \beta_1\gamma_2)^{-1}\alpha_1 \\ \tilde{\gamma} &= \gamma_1 + \alpha_1'\gamma_2(I + \beta_1\gamma_2)^{-1}\alpha_1 \\ \tilde{\beta} &= \beta_2 + \alpha_2(I + \beta_1\gamma_2)^{-1}\beta_1\alpha_2' \end{aligned} \tag{8.B.4}$$

This algorithm is implemented in our MATLAB program `mult.m`.

C. Alternative forms of Riccati equation

It is useful to display alternative forms of the Riccati equation

$$P = R + A'PA - A'PB(Q + B'PB)^{-1}B'PA. \tag{8.C.1}$$

We first apply the following matrix identity from Noble and Daniel [1977, p. 29]. Assume that d^{-1} and a^{-1} exist. Then $(a - bd^{-1}c)^{-1} = a^{-1} + a^{-1}b[d - ca^{-1}b]^{-1}ca^{-1}$. Apply this identity, setting $a^{-1} = P_{t+1}, b = -B, d = Q, c = B'$ to obtain

$$(P_{t+1}^{-1} + BQ^{-1}B')^{-1} = P_{t+1} - P_{t+1}B(B'P_{t+1}B + Q)^{-1}B'P_{t+1}.$$

Substituting the above identity into $(8.C.1)$ establishes

$$P_t = R + A'(P_{t+1}^{-1} + BQ^{-1}B')^{-1}A. \tag{8.C.2}$$

Now write $(8.C.2)$ as

$$\begin{aligned} P_t &= R + A'P_{t+1}P_{t+1}^{-1}(P_{t+1}^{-1} + BQ^{-1}B')^{-1}A \\ P_t &= R + A'P_{t+1}(P_{t+1}^{-1}P_{t+1} + BQ^{-1}B'P_{t+1})^{-1}A \\ P_t &= R + A'P_{t+1}(I + BQ^{-1}B'P_{t+1})^{-1}A \end{aligned}$$

Assume that A^{-1} exists, and write the preceding equation as

$$P_t = R + A' P_{t+1}(A^{-1} + A^{-1}BQ^{-1}B'P_{t+1})^{-1}$$

$$P_t = A' P_{t+1}(A^{-1} + A^{-1}BQ^{-1}B'P_{t+1})^{-1}$$

$$+ R(A^{-1} + A^{-1}BQ^{-1}B'P_{t+1})(A^{-1} + A^{-1}BQ^{-1}B'P_{t+1})^{-1}$$

This equation can be represented as

$$\begin{aligned} P_t = \{RA^{-1} + [A' + RA^{-1}BQ^{-1}B']P_{t+1}\} \\ \{A^{-1} + A^{-1}BQ^{-1}B'P_{t+1}\}^{-1}. \end{aligned} \tag{8.C.3}$$

Equation (8.C.3) takes the form

$$P_t = \{C + DP_{t+1}\} \times \{E + FP_{t+1}\}^{-1} \tag{8.C.4}$$

where

$$C = RA^{-1}$$

$$D = A' + RA^{-1}BQ^{-1}B'$$

$$E = A^{-1}$$

$$F = A^{-1}BQ^{-1}B',$$

which can be represented as

$$\begin{bmatrix} X_t \\ Y_t \end{bmatrix} = \begin{bmatrix} E & F \\ C & D \end{bmatrix} \begin{bmatrix} X_{t+1} \\ Y_{t+1} \end{bmatrix},$$

where $P_t = Y_t X_t^{-1}$. Notice that

$$\begin{bmatrix} E & F \\ C & D \end{bmatrix} = M_f^{-1} = M_b,$$

and that $\lim_{t \to -\infty} P_t = \lim_{t \to -\infty} Y_t X_t^{-1}$ can be computed as the limit of the γ_j term in the representation of the symplectic matrix $M_f^{-2^j}$.

Part II

Representations and Properties

Chapter 9
Representation and Estimation

This chapter shows how our models restrict moments of observed prices and quantities, and how observations can be used to make inferences about the parameters of our models. Earlier chapters have prepared a state-space representation that expresses states x_t and observables y_t as linear functions of an initial x_0 and histories of martingale difference sequences w_t. The w_t's are shocks to endowments and preferences whose histories are observed by the agents in the economy. The econometrician does not see those shocks, at least directly. Therefore, to prepare a model for estimation we obtain another representation of it that is cast in terms of shocks that can in principle be recovered from an econometrician's observations. By using the Kalman filter we shall obtain what is known as an 'innovations representation'. It is a workhorse. It can be transformed to yield a Wold representation or a vector autoregression for observables.[1] An important approach to estimation, approximation, and aggregation over time is to deduce the restrictions that the models of the economy and of data collection impose on the innovations representation. The Kalman filter does this efficiently, and enables a recursive way of calculating a Gaussian likelihood function.

We describe how to obtain maximum likelihood and generalized method of moments estimates of a model's parameters, using both time domain and frequency domain methods. As by-products of time domain estimation, we deduce autoregressive and Wold representations for observables. As a by-product of frequency domain estimation, we recover a theory of specification error. We also study the effects of aggregation over time, and how to estimate a model specified at a finer time interval than pertains to the available data. These methods must be augmented to incorporate data on asset prices, which are non-linear functions of the state of the economy. The last part of the chapter describes how asset prices, returns, and other nonlinear functions of the state can be used in estimation.

The Kalman filter is intimately connected to the optimal linear regulator (i.e., the linear-quadratic dynamic programming problem). Remarkably, the key

[1] See Sims (1972, 1980), Whittle (1983), and Sargent (1987) for definitions and discussions of the Wold and autoregressive representations.

mathematical formula associated with the Kalman filter is the same matrix Riccati equation that solves the linear regulator. Furthermore, the same key 'factorization identity' occurring with the Kalman filter plays a role in linear-quadratic optimization theory. In chapters 11 and 14, we shall use a 'factorization identity' to provide information about alternative representations of preferences.

9.1. The Kalman Filter

We regard a vector of time t data y_t as error-ridden measures of linear combinations of the state vector x_t. We append a measurement equation to the equilibrium law of motion of the state to attain the following state space system:

$$\begin{aligned} x_{t+1} &= A^o x_t + C w_{t+1} \\ y_t &= G x_t + v_t \end{aligned} \tag{9.1.1}$$

where v_t is a martingale difference sequence of measurement errors that satisfies $E v_t v_t' = R, E w_{t+1} v_s' = 0$ for all $t+1 \geq s$. Here G is a matrix whose rows are composed of entries of the S_j and M_j matrices, computed for example in chapters 4, 6, and 7, that select those components of quantities and prices for which data are available.[2] We assume that x_0 is a random vector with known mean $\hat{x}_0$ and covariance matrix $E(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)' = \Sigma_0$. Using (9.1.1), we have $E y_0 = G \hat{x}_0$.

We adopt the notation $y_0^t = [y_t, y_{t-1}, \ldots, y_0]$, $y^t = y_{-\infty}^t$. For any variable z_t, $t \geq 0$, we let $\hat{z}_t = \hat{E}[z_t | y_0^{t-1}, \hat{x}_0]$, where $\hat{E}(\cdot)$ is the linear least squares projection operator. Also, we occasionally use the notation $\hat{E}_t z_t = \hat{E}[z_t | y_0^t, \hat{x}_0]$. We want recursive formulas for $\hat{y}_t, \hat{x}_t$. We begin by constructing an innovation process $\{a_t\}$ such that $[a_0^t, \hat{x}_0]$ forms an orthogonal basis for the information set $[y_0^t, \hat{x}_0]$. We recursively calculate the projections $\hat{x}_{t+1}$ and $\hat{y}_t$ by regressing on the orthogonal basis $[a_0^t, \hat{x}_0]$.

The orthogonal basis for $[y_0^t, \hat{x}_0]$ is constructed using a Gram-Schmidt process. Begin with the regression equation $y_0 = E y_0 + a_0 = G \hat{x}_0 + a_0$ or

$$a_0 = y_0 - G \hat{x}_0,$$

[2] Later we shall permit serially correlated measurement errors. It is easy to modify the calculations to permit $E w_{t+1} v_t'$ to be nonzero.

where the residual a_0 satisfies the least squares normal equation $Ea_0' = 0$. Evidently, $[y_0, \hat{x}_0]$ and $[a_0, \hat{x}_0]$ span the same linear space. Next, form a_1 as the residual from a regression of y_1 on $[y_0, \hat{x}_0]$, or equivalently, a regression of y_1 on $[a_0, \hat{x}_0]$: $a_1 = y_1 - \hat{E}[y_1 \mid y_0, \hat{x}_0]$ or

$$a_1 = y_1 - \hat{E}[y_1 \mid a_0, \hat{x}_0];$$

a_1 is by construction orthogonal to a_0 and $\hat{x}_0$; i.e., $E(a_1 a_0') = 0, E(a_1) = 0$. Continuing in this way, form $a_t = y_t - \hat{E}[y_t \mid y_0^{t-1}, \hat{x}_0] = y_t - \hat{E}[y_t \mid a_0^{t-1}, \hat{x}_0]$, where $E(a_t a_s') = 0$ for $s = 0, \ldots, t-1$ and $E(a_t) = 0$. We call a_t the *innovation* in y_t.

It is useful to represent a_t as follows. From the second equation of (9.1.1) and from the fact that v_t is orthogonal to y_{t-s} and x_{t-s} for $s \geq 1$, it follows that

$$\hat{y}_t = G\hat{x}_t$$

and that

$$y_t = G\hat{x}_t + G(x_t - \hat{x}_t) + v_t.$$

By subtracting the first equation from the second, we find that the innovation a_t in y_t satisfies

$$a_t \equiv y_t - \hat{y}_t = G(x_t - \hat{x}_t) + v_t. \tag{9.1.2}$$

Calculate the second moment matrix of a_t to be

$$\begin{aligned} E a_t a_t' &= GE(x_t - \hat{x}_t)(x_t - \hat{x}_t)'G' + Ev_t v_t' \\ &= G\Sigma_t G' + R \equiv \Omega_t \end{aligned}$$

where $\Sigma_t \equiv E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$. We shall soon give a recursive formula for Σ_t.

From the first equation in (9.1.1), it follows that

$$\hat{E}_t x_{t+1} = A^o \hat{E}_t x_t = A^o \hat{E}_{t-1} x_t + A^o(\hat{E}_t x_t - \hat{E}_{t-1} x_t), \tag{9.1.3}$$

where again $\hat{E}_t$ denotes projection on $[y_0^t, \hat{x}_0]$. Express the projection $\hat{E}_t x_t = Ex_t + \sum_{j=0}^t \Gamma_j a_j$ where $x_t = \hat{E}_t x_t + \psi_t$, ψ_t is a least squares residual vector, and the regression coefficients Γ_j are determined by the least squares orthogonality conditions $E\psi_t a_s' = 0$ for $s = 0, \ldots, t$. Because $[a_0^t, \hat{x}_0]$ is an orthogonal basis for $[y_0^t, \hat{x}_0]$, these orthogonality conditions imply

$$(Ex_t a_t')(\Omega_t)^{-1} = \Gamma_t, \tag{9.1.4}$$

where $Ea_t a_t' = \Omega_t$. To compute $Ex_t a_t'$, first notice that $\hat{E}_{t-1} x_t = Ex_t + \sum_{j=0}^{t-1} \Gamma_j a_j$. Then $x_t = \hat{E}_{t-1} x_t + \Gamma_t a_t + \psi_t$ can be interpreted in terms of the regression equation

$$(x_t - \hat{E}_{t-1} x_t) = \Gamma_t a_t + \psi_t, \tag{9.1.5}$$

where $\Gamma_t a_t = \hat{E}[(x_t - \hat{E}_{t-1} x_t)|a_t]$. Evidently, $E(x_t - \hat{E}_{t-1} x_t) a_t' = Ex_t a_t'$. Use (9.1.2) to compute $E(x_t - \hat{E}_{t-1} x_t) a_t' = \Sigma_t G'$. It follows that (9.1.4) becomes

$$\Gamma_t = \Sigma_t G' (G \Sigma_t G' + R)^{-1}, \tag{9.1.6}$$

and from (9.1.5) that

$$\hat{E}_t x_t = \hat{E}_{t-1} x_t + \Gamma_t a_t. \tag{9.1.7}$$

Substituting (9.1.7) into (9.1.3) gives $\hat{x}_{t+1} = A^o \hat{x}_t + A^o \Gamma_t (y_t - G \hat{x}_t)$ or

$$\hat{x}_{t+1} = A^o \hat{x}_t + K_t a_t, \tag{9.1.8}$$

where $a_t = y_t - G\hat{x}_t$, and where from (9.1.6) K_t must satisfy

$$K_t = A^o \Sigma_t G' (G \Sigma_t G' + R)^{-1}. \tag{9.1.9}$$

Equation (9.1.9) expresses the 'Kalman gain' K_t in terms of the state covariance matrix $\Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$.

We need an equation for Σ_t. Subtract $\hat{x}_{t+1} = A^o \hat{x}_t + K_t (y_t - G\hat{x}_t)$ from the first equation of (9.1.1) to obtain $x_{t+1} - \hat{x}_{t+1} = (A^o - K_t G)(x_t - \hat{x}_t) + C w_{t+1} - K_t v_t$. Multiply each side of this equation by its own transpose and take expectations to obtain

$$\begin{aligned} \Sigma_{t+1} = {} & (A^o - K_t G) \Sigma_t (A^o - K_t G)' \\ & + CC' + K_t R K_t'. \end{aligned} \tag{9.1.10}$$

Substituting (9.1.9) into (9.1.10) and rearranging gives a matrix Riccati difference equation for Σ_t:

$$\begin{aligned} \Sigma_{t+1} = {} & A^o \Sigma_t A^{o\prime} + CC' \\ & - A^o \Sigma_t G' (G \Sigma_t G' + R)^{-1} G \Sigma_t A^{o\prime}. \end{aligned} \tag{9.1.11}$$

The recursive (9.1.9) and (9.1.11) for Σ_t, K_t determine the Kalman filter. They are to be initialized from a given Σ_0. Later we discuss alternative ways to choose Σ_0.

9.2. Innovations Representation

The Kalman filter lets us associate with representation (9.1.1) an 'innovations representation':

$$\begin{aligned} \hat{x}_{t+1} &= A^o \hat{x}_t + K_t a_t \\ y_t &= G\hat{x}_t + a_t, \end{aligned} \tag{9.2.1}$$

where $Ea_t a_t' \equiv \Omega_t = G\Sigma_t G' + R$. This time varying representation is obtained starting from arbitrary initial conditions $\hat{x}_0, \Sigma_0$. We can rearrange (9.2.1) into the form of a whitening filter

$$\begin{aligned} a_t &= y_t - G\hat{x}_t \\ \hat{x}_{t+1} &= A^o \hat{x}_t + K_t a_t, \end{aligned} \tag{9.2.2}$$

which can be used for recursively constructing a record of innovations $\{a_t\}_{t=0}^T$ from an $\hat{x}_0$ and a record of observations $\{y_t\}_{t=0}^T$. The filter defined by (9.2.2) is called a "whitening filter" because it accepts as "input" the serially correlated process $\{y_t\}$ and produces as "output" the serially uncorrelated (i.e., "white") vector stochastic process $\{a_t\}$. The process $\{a_t\}$ is said to be a *fundamental* white noise for the $\{y_t\}$ process, because it equals the one-step ahead prediction error in a linear least squares projection of y_t on the history of y.[3]

Later, we shall use the whitening filter in several ways. We shall use it to study how the innovations $\{a_t\}$ from a population vector autoregression for $\{y_t\}$ are related to the $\{y_t\}$ process and to the underlying martingale process $\{w_t\}$ of information flowing to agents. We shall also use it to construct a recursive representation of a Gaussian likelihood function for a sample drawn from the $\{y_t\}$ process.

[3] See Sims (1972), Hansen and Sargent (1991, chapter 2), and Sargent (1987, chapter XI) for the role such an error process plays in the construction of Wold's representation theorem.

9.3. Convergence results

For the purpose of obtaining a time-invariant counterpart to (9.2.1), we introduce two assumptions.

ASSUMPTION A1: The pair $(A^{o\prime}, G')$ is stabilizable.

ASSUMPTION A2: The pair $(A^{o\prime}, C)$ is detectable.

See the appendix to chapter 8 for definitions of stabilizability and detectability. Assumptions A1 and A2 are typically met for our applications. Under A1 and A2, two useful results occur. The first is that iterations on the matrix Riccati difference equation (9.1.11) converge as $t \to \infty$, starting from any positive semi-definite initial matrix Σ_0. The limiting matrix $\Sigma_\infty \equiv \lim_{t\to\infty} \Sigma_t$ is the unique positive semi-definite matrix Σ that satisfies the algebraic matrix Riccati equation[4]

$$\begin{aligned} \Sigma = {} & A^o \Sigma A^{o\prime} + CC' \\ & - A^o \Sigma G' (G \Sigma G' + R)^{-1} G \Sigma A^{o\prime}. \end{aligned} \tag{9.3.1}$$

If we initiate the Kalman filter by choosing $\Sigma_0 = \Sigma_\infty$, then from (9.1.11) and (9.1.9), we obtain a time invariant K_t matrix, call it K. Under this circumstance, representation (9.2.1) becomes time invariant. The stationary covariance matrix of the innovations is given by $\Omega = E a_t a_t' = G\Sigma G' + R$, where $\Sigma = \Sigma_\infty = \Sigma_0$.

The second useful result is that Assumptions A1 and A2 imply that $A^o - KG$ is a stable matrix, i.e., its eigenvalues are strictly less than unity in modulus. Later we shall see how the stability of the matrix $A^o - KG$ plays a key role in a convenient formula for the autoregressive representation for the $\{y_t\}$ process.

[4] The limiting form of (9.1.10) is evidently a discrete Lyapunov or Sylvester equation. See chapter 8.

9.3.1. Time-Invariant Innovations Representation

The infinite-horizon time invariant Kalman filter defines a matrix valued function, which we express as

$$\left[K, \Sigma\right] = \texttt{kfilter}\ (A^o, G, V_1, V_2, V_3) \tag{9.3.2}$$

where $V_1 = CC', V_2 = Ev_t v_t', V_3 = Ew_{t+1}v_t', \Sigma = E_{t-1}(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$. For our model, we can use (9.3.2) with the following settings for the matrices V_1, V_2, V_3: $V_1 = CC', V_2 = R, V_3 =$ a matrix of zeros conformable to x and y.[5]

By using the function `kfilter`, we can evidently associate with representation (9.1.1) a time-invariant innovations representation (9.2.1) in which K_t is constant.

9.4. Serially Correlated Measurement Errors

It is useful to adapt the preceding calculations to cover the case in which the measurement errors v_t in (9.1.1) are serially correlated.[6] Modify (9.1.1) to be

$$\begin{aligned} x_{t+1} &= A^o x_t + C w_{t+1} \\ y_t &= G x_t + v_t \\ v_t &= D v_{t-1} + \eta_t \end{aligned} \tag{9.4.1}$$

where D is a matrix whose eigenvalues are strictly below unity in modulus and η_t is a martingale difference sequence that satisfies

$$\begin{aligned} E\eta_t \eta_t' &= R \\ E w_{t+1} \eta_s' &= 0 \quad \text{for all } t \text{ and } s. \end{aligned}$$

In (9.4.1), v_t is a serially correlated measurement error process that is orthogonal to the x_t process.

Define the quasi-differenced process

$$\overline{y}_t \equiv y_{t+1} - D y_t. \tag{9.4.2}$$

5 The function `kfilter` defined in (9.3.2) solves a version of (9.1.9) and (9.1.11) for Σ_∞ and K_∞, a version that has been generalized to permit arbitrary covariance between w_{t+1} and v_t, which is required for several of our applications.

6 The calculations in this section imitate those of Anderson and Moore [1979].

From (9.4.1) and the definition (9.4.2) it follows that

$$\overline{y}_t = (GA^o - DG)x_t + GCw_{t+1} + \eta_{t+1}$$

Thus, $(x_t, \overline{y}_t)$ is governed by the state space system

$$\begin{aligned} x_{t+1} &= A^o x_t + Cw_{t+1} \\ \overline{y}_t &= \overline{G} x_t + GCw_{t+1} + \eta_{t+1} \end{aligned} \tag{9.4.3}$$

where $\overline{G} = GA^o - DG$. This state space system has nonzero covariance between the state noise Cw_{t+1} and the "measurement noise" $(GCw_{t+1} + \eta_{t+1})$. Define the covariance matrices $V_1 = CC', V_2 = GCC'G' + R, V_3 = CC'G'$. By applying the Kalman filter to (9.4.3), we obtain a gain sequence K_t with which to construct the associated innovations representation

$$\begin{aligned} \hat{x}_{t+1} &= A^o \hat{x}_t + K_t u_t \\ \overline{y}_t &= \overline{G} \hat{x}_t + u_t \end{aligned} \tag{9.4.4}$$

where $\hat{x}_t = \hat{E}[x_t \mid \overline{y}_0^{t-1}, \hat{x}_0], u_t = \overline{y}_t - \hat{E}[\overline{y}_t \mid \overline{y}_0^{t-1}, \hat{x}_0], \Omega_1 \equiv Eu_t u_t' = \overline{G}\Sigma_t \overline{G}' + V_2$. Using definition (9.4.2), it follows that $[y_0^{t+1}, \hat{x}_0]$ and $[\overline{y}_0^t, \hat{x}_0]$ span the same space, so that $\hat{x}_t = \hat{E}[x_t \mid y_0^t, \hat{x}_0], u_t = y_{t+1} - \hat{E}[y_{t+1} \mid y_0^t, \hat{x}_0]$. Thus, u_t is the innovation in y_{t+1}.

9.5. Combined System

It is useful to have a formula that gives a state space representation for y_t driven by the innovations to y_t. We obtain this by combining the innovations system (9.4.4) for $\overline{y}_t$ with the system

$$y_{t+1} = Dy_t + \overline{y}_t. \tag{9.5.1}$$

The system (9.5.1) accepts $\{\overline{y}_t\}$ as an "input" and produces $\{y_t\}$ as an "output". The two systems (9.4.4) and (9.5.1) can be combined in a series to give the state space system:

$$\begin{aligned} \begin{bmatrix} \hat{x}_{t+1} \\ y_{t+1} \end{bmatrix} &= \begin{bmatrix} A^o & 0 \\ \overline{G} & D \end{bmatrix} \begin{bmatrix} \hat{x}_t \\ y_t \end{bmatrix} + \begin{bmatrix} K_t \\ I \end{bmatrix} u_t \\ y_t &= [\,0 \quad I\,] \begin{bmatrix} \hat{x}_t \\ y_t \end{bmatrix} + [0] u_t \end{aligned} \tag{9.5.2}$$

The MATLAB program `evardec.m` uses the time-invariant version of (9.5.2), obtained using `kfilter.m`, to obtain a decomposition of the j-step ahead prediction error variance associated with the Wold representation for y_t.[7]

9.6. Recursive Formulation of Likelihood Function

The Kalman filter enables a recursive algorithm for computing a Gaussian likelihood function for a sample of observations $\{y_s\}_{s=0}^T$ on a $(p \times 1)$ vector y_t. We assume that these data are governed by the innovations representation (9.2.1) The likelihood function of $\{y_s\}_{s=0}^T$ is defined as the density $f(y_T, y_{T-1}, \ldots, y_0)$. It is convenient to factor the likelihood function

$$\begin{aligned} f(y_T, y_{T-1}, \ldots, y_0) = f_T(y_T|y_{T-1}, \ldots, y_0) f_{T-1}(y_{T-1}|y_{T-2}, \ldots, y_0) \cdots \\ f_1(y_1|y_0) f_0(y_0). \end{aligned} \tag{9.6.1}$$

The Gaussian likelihood function for an $n \times 1$ random vector y with mean μ and covariance matrix V is

$$\mathcal{N}(\mu, V) = (2\pi)^{-n/2} |V|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(y-\mu)' V^{-1} (y-\mu)\right). \tag{9.6.2}$$

Evidently, from (9.1.1), the distribution $f_0(y_0)$ is $\mathcal{N}(Gx_0, \Omega_0)$, where $\Omega_t = G\Sigma_t G' + R$ and Σ_t is the covariance matrix of x_t around $\hat{x}_t$. Further, $f(y_t|y_{t-1}, \ldots, y_0) = \mathcal{N}(G\hat{x}_t, \Omega_t)$. It is easy to verify that the distribution $g_t(a_t)$ of the innovation a_t is $\mathcal{N}(0, \Omega_t)$ Thus, $f_0(y_0)$ equals $g_0(a_0)$, the distribution of the initial innovation. More generally, from (9.2.1), the conditional density $f_t(y_t|y_{t-1}, \ldots, y_0)$ equals the density $g_t(a_t)$ of a_t. Then the likelihood (9.6.1) can be represented as

$$g_T(a_T) g_{T-1}(a_{T-1}) \ldots g_1(a_1) g_0(a_0). \tag{9.6.3}$$

Expression (9.6.3) implies that the logarithm of the likelihood function for y_0^T is

$$-.5 \sum_{t=0}^{T} \{p \ln(2\pi) + \ln|\Omega_t| + a_t' \Omega_t^{-1} a_t\}. \tag{9.6.4}$$

---

[7] The MATLAB program `series.m` can be used to obtain the time-invariant system (9.5.2) from the two systems (9.4.4) and (9.5.1).

9.6.1. Initialization

Two alternative sets of assumptions are commonly used to initiate the Kalman filter, corresponding to different information about y_0.

(a.) The distribution of the initial y_0 is treated as if it were conditioned on an infinite history of y's. This idea is implemented by specifying that x_0 has mean $\hat{x}_0 = E[x_0|y_{-1}, y_{-2}, \ldots]$, and a covariance matrix $\Sigma_0 = \Sigma_\infty$ coming from the steady state of the Kalman filter. In this case, the time-invariant Kalman filter can be used to construct the Gaussian log likelihood:

$$-.5 \sum_{t=0}^{T} \{p \ln(2\pi) + \ln|\Omega| + a_t' \Omega^{-1} a_t\}, \tag{9.6.5}$$

where $\Omega = G\Sigma_\infty G' + R$, and where the innovations a_t are computed using the steady state Kalman gain K. This procedure amounts to replacing $f_0(y_0)$ in (9.6.1) with $f(y_0|y_{-\infty}^{-1})$.

(b.) The initial value y_0 is drawn from the stationary distribution of y, meaning that it is associated with an x_0 governed by the stationary distribution of x_t, an assumption implemented by initiating the Kalman filter with $\Sigma_0 = \Sigma_x$, where Σ_x is the asymptotic stationary covariance matrix of x.

Assumptions (a) and (b) pertain to how one selects the matrix Σ_0. Under each of assumptions (a) and (b), it is common to set $\hat{x}_0$ equal to the unconditional mean of x, provided that this exists.

9.6.2. Non-existence of a stationary distribution

Approach (b) assumes that the law of motion $x_{t+1} = A^o x_t + C w_{t+1}$ is such that the $\{x_t\}$ process has an asymptotic stationary distribution, and cannot be used without modification in models that violate this assumption. When an asymptotic stationary distribution doesn't exist, one procedure is to assume a 'diffuse' initial distribution over the piece of x_0 that has no stationary distribution. The models described in chapter 13, with their co-integrated equilibrium consumption processes, necessitate such a procedure.

In the appendix, we describe a method for coping with this situation, inspired by ideas of Kohn and Ansley (19XXX). It is most useful for us to describe their idea in the context of models with serially correlated measurement errors, which we treat in the next section.

9.6.3. Serially correlated measurement errors

When we use the state space model with serially correlated measurement errors (9.4.1), some adjustments are called for in the above procedures for forming the log likelihood. These adjustments are occasioned by

the timings in the definitions of $\hat{x}_t, u_t$. In particular, the notation now denotes $\hat{x}_t = E[x_t | y^t]$ and $\Sigma_t = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$. These changes mean that the distribution $g_{t-1}(u_{t-1})$ equals $f_t(y_t | y_{t-1}, \ldots, y_0)$. So corresponding to the factorization (9.6.1) we have

$$g_{T-1}(u_{T-1}) g_{T-2}(u_{T-2}) \ldots g_0(u_0) g_{-1}(u_{-1}). \tag{9.6.6}$$

To deduce the appropriate distribution of y_0, or equivalently, of u_{-1}, notice that the time 0 version of the 'whitener' is

$$\begin{aligned} u_{-1} &= y_0 - D y_{-1} - \overline{G} \hat{x}_{-1} \\ \hat{x}_0 &= A^o \hat{x}_{-1} + K_0 u_{-1}, \end{aligned}$$

where K_0 is the time 0 value for the Kalman gain. It is natural to start the system with $y_{-1} = GEx$ and $\hat{x}_{-1} = Ex$, where Ex is the stationary mean of x_t,[8] and to initiate the Kalman filter from the mean of the stationary distribution of x. So the Gaussian log likelihood function is

$$-.5 \sum_{t=-1}^{T-1} \{ p \ln(2\pi) + \ln |\Omega_t| + u_t' \Omega_t^{-1} u_t \}. \tag{9.6.7}$$

We now indicate how these procedures can be adapted to handle models for which no stationary distribution for x_t exists, following procedures of Kohn and Ansley (BLANK). The idea is to factor the likelihood function as

$$\begin{aligned} f(y_T, y_{T-1}, \ldots, y_0) = f_T(y_T | y_{T-1}, \ldots, y_0) f_{T-1}(y_{T-1} | y_{T-2}, \ldots, y_0) \cdots \\ f_m(y_m | y_{m-1}, \ldots, y_0) f(y_{m-1}, \ldots, y_0). \end{aligned} \tag{9.6.8}$$

[8] Notice that G and not $\overline{G}$ appears in the equation for the unconditional mean.

Kohn and Ansley assign a 'diffuse prior' to that subset of the state vector that does not possess a stationary distribution, and let the remaining piece of x_0 be distributed according to its stationary distribution. This specification embodies an 'improper prior' distribution for $(y_{m-1}, \dots, y_0)$. Under this specification, we use the first m observations of y_t to estimate x_{m-1}, then form $\hat{x}_{m-1}, \Sigma_{m-1}$ from which to initiate the Kalman filter for the system $(9.4.3)$ with serially correlated measurement errors. The Kalman filter is applied to compute the likelihood for the sample $\{y_s\}_{s=m}^T$. In addition, we can adjust $(9.6.8)$ to account for the first m observations. Details are given in the appendix.

9.7. Wold Representation

For the purpose of describing the relationship of the time-invariant innovations representation to the Wold and autoregressive representations, we shall avail ourselves when needed of:

Assumption A3: The eigenvalues of A^o are all less than unity in modulus, except possibly for one associated with a constant.

A Wold representation for a stationary stochastic process y_t is a moving average of the form

$$y_t = Ey + \sum_{j=0}^{\infty} \Gamma_j \epsilon_{t-j},$$

where $\epsilon_t = y_t - \hat{E}[y_t | y^{t-1}]$, and $\sum_{j=0}^{\infty} \text{trace} \Gamma_j \Gamma_j' < +\infty$. (Below, we shall for the most part set the unconditional mean vector Ey to zero, to conserve on notation.) We can attain a Wold representation by manipulating the innovations system in a way that amounts to driving the date for the initial $\hat{x}_0$ arbitrarily far into the past. Thus, the first equation of $(9.4.4)$ can be solved recursively for

$$\hat{x}_{t+1} = \sum_{j=0}^{t} (A^o)^j K u_{t-j} + (A^o)^{t+1} \hat{x}_0.$$

Now assume that $\hat{x}_0$ was itself formed by having observed the history y^{-1}, so that

$$\hat{x}_0 = (I - A^o L)^{-1} K u_{-1} + \mu_x,$$

where μ_x is the unconditional mean of x. Under this specification for $\hat{x}_0$,

$$\hat{x}_{t+1} = (I - A^o L)^{-1} K u_t + \mu_x. \tag{9.7.1}$$

Below, we shall omit the unconditional mean term, by assuming that $\mu_x = 0$.

To get a Wold representation for y_t, substitute (9.4.2) into (9.4.4) to obtain

$$\begin{aligned} \hat{x}_{t+1} &= A^o \hat{x}_t + K u_t \\ y_{t+1} - D y_t &= \overline{G} \hat{x}_t + u_t. \end{aligned} \tag{9.7.2}$$

Then (9.7.2) and (9.7.1) can be used to get a Wold representation for y_t:

$$y_{t+1} = [I - DL]^{-1} [I + \overline{G}(I - A^o L)^{-1} K L] u_t, \tag{9.7.3}$$

where again L is the lag operator. Also, from (9.7.2) a "whitening filter" for obtaining $\{u_t\}$ from $\{y_t\}$ is given by

$$\begin{aligned} u_t &= y_{t+1} - D y_t - \overline{G} \hat{x}_t \\ \hat{x}_{t+1} &= A^o \hat{x}_t + K u_t. \end{aligned} \tag{9.7.4}$$

9.8. Vector Autoregression for $\{y_t\}$

We can use the innovations representation and some results from linear algebra to derive a convenient formula for the one-step-ahead linear least squares forecast of y_t based on the history of the $\{y_t\}$ process. We begin by deriving a version of the *factorization identity*, which asserts equality between two representations of the spectral density matrix of the observables. We will encounter a mathematically equivalent form of this identity in Chapter 11 when we discuss observationally equivalent representations of preferences.

9.8.1. The factorization identity

For the model with serially uncorrelated measurement errors, we have two alternative representations for an observed process $\{y_t\}$, the original state space representation (9.1.1) and the innovations representation (9.2.1). Because they describe the *same* stochastic process $\{y_t\}$, they give two alternative representations of the spectral density matrix of $\{y_t\}$, an outcome that expresses the *factorization identity*.

The original state space representation is

$$\begin{aligned} x_{t+1} &= A^o x_t + C w_{t+1} \\ y_t &= G x_t + v_t, \end{aligned} \tag{9.8.1}$$

where w_{t+1} is a martingale difference sequence of innovations to agents' information sets, and v_t is another martingale difference sequence of measurement errors. We assumed that w_{t+1}, v_t are mutually orthogonal stochastic processes.

The first line of representation (9.8.1) can be written $L^{-1} x_t = (I - A^o L)^{-1} C w_{t+1}$ or $x_t = (L^{-1} - A^o)^{-1} C w_{t+1}$. It follows that the covariance generating function of $\{x_t\}$ satisfies

$$S_x(z) = (zI - A^o)^{-1} C C' (z^{-1} I - (A^o)')^{-1}.$$

Using this expression and the second line of (9.1.1), together with the observation that v_t is orthogonal to the process x_t, shows that the covariance generating function of y_t is given by

$$S_y(z) = G(zI - A^o)^{-1} C C' (z^{-1} I - (A^o)')^{-1} G' + R. \tag{9.8.2}$$

Representation (9.2.1) implies $\hat{x}_t = (L^{-1} - A^o)^{-1} K a_t$, and

$$y_t = [G(L^{-1} - A^o)^{-1} K + I] a_t. \tag{9.8.3}$$

Because a_t is a white noise with covariance matrix $G \Sigma G' + R$, it follows that the covariance generating function of $\{y_t\}$ equals

$$S_y(z) = [G(zI - A^o)^{-1} K + I][G \Sigma G' + R][K' (z^{-1} I - A^{o\prime})^{-1} G' + I]. \tag{9.8.4}$$

Expressions (9.8.2) and (9.8.4) are alternative representations for the covariance generating function $S_y(z)$. Equating them leads to the *factorization identity*:

$$\begin{aligned} &G(zI - A^o)^{-1} C C' (z^{-1} I - A^{o\prime})^{-1} G' + R = \\ &\quad [G(zI - A^o)^{-1} K + I][G \Sigma G' + R][K' (z^{-1} I - A^{o\prime})^{-1} G' + I]. \end{aligned} \tag{9.8.5}$$

The importance of the factorization identity hinges on the fact that, under assumptions A1 and A2, the zeros of the polynomial $\det[G(zI-A^o)^{-1}K+I]$ all lie inside the unit circle, which means that in the representation (9.8.3) for y_t, the polynomial in L on the right hand side has a one-sided inverse in nonnegative powers of L, so that a_t lies in the space spanned by y^t. We establish this result in the next section, then apply it in subsequent ones.

9.8.2. Location of zeros of characteristic polynomial

We utilize two theorems from the algebra of partitioned matrices. Let a, b, c, d be appropriately conformable and invertible matrices. Then

$$(a - bd^{-1}c)^{-1} = a^{-1} + a^{-1}b(d - ca^{-1}b)^{-1}ca^{-1} \tag{9.8.6}$$

and

$$\det(a)\det(d + ca^{-1}b) = \det(d)\det(a + bd^{-1}c). \tag{9.8.7}$$

Apply equality (9.8.6) to $[I + G(zI - A^o)^{-1}K]^{-1}$ with the settings $a = I, b = -G, d = (zI - A^o), c = K$, to get

$$[I + G(zI - A^o)^{-1}K]^{-1} = I - G[zI - (A^o - KG)]^{-1}K. \tag{9.8.8}$$

Apply equality (9.8.7) with the settings $a = I, b = G, d = (zI - A^o), c = K$ to get

$$\det(zI - (A^o - KG)) = \det(zI - A^o)\det(I + G(zI - A^o)^{-1}K),$$

or

$$\det[I + G(zI - A^o)^{-1}K] = \frac{\det(zI - (A^o - KG))}{\det(zI - A^o)}. \tag{9.8.9}$$

It follows from (9.8.9) that the *zeros* of $\det[I + G(zI - A^o)^{-1}K]$ are the eigenvalues of $A^o - KG$, and the *poles* of $\det[I + G(zI - A^o)^{-1}K]$ are the eigenvalues of A^o. Assumptions A1 and A2 guarantee that the eigenvalues of $A^o - KG$ are less than unity in modulus. We have already made assumptions that assure that the eigenvalues of A^o are less than unity in modulus. These conditions on the eigenvalues together with equations (9.8.8) and (9.8.9) permit us to obtain the Wold and autoregressive representations of $\{y_t\}$ in convenient forms.

9.8.3. Wold and autoregressive representations (white measurement errors)

From (9.8.8), we have that

$$[G(I - A^o L)^{-1} K L + I]^{-1} = I - G[I - (A^o - KG)L]^{-1} KL. \qquad (9.8.10)$$

For the model with serially uncorrelated measurement errors, the Wold representation for $\{y_t\}$ is

$$y_t = [G(I - A^o L)^{-1} KL + I] a_t. \qquad (9.8.11)$$

Applying the inverse of the operator on the right of (9.8.11) and using (9.8.10) gives

$$y_t = G[I - (A^o - KG)L]^{-1} K y_{t-1} + a_t, \qquad (9.8.12)$$

which decomposes y_t into an innovation a_t and a one-step ahead linear least squares predictor

$$E[y_t | y^{t-1}] = G[I - (A^o - KG)L]^{-1} K y_{t-1}. \qquad (9.8.13)$$

Equation (9.8.12) is equivalent with

$$y_t = \sum_{j=1}^{\infty} G(A^o - KG)^{j-1} K y_{t-j} + a_t. \qquad (9.8.14)$$

Equation (9.8.14) is a vector autoregressive representation for y_t. Thus, the Kalman filter allows us to move from the original state space representation to a vector autoregression.

9.8.4. Serially correlated measurement errors

With few modifications, the preceding analysis can be adapted to calculate the vector autoregressive representation and the one-step ahead prediction for y_t for the case in which the measurement errors are vector first-order autoregressive processes. We have seen that the Wold representation in this case takes the form

$$y_{t+1} = [I - DL]^{-1}[I + \overline{G}(I - A^o L)^{-1} K L] u_t. \tag{9.8.15}$$

Operating on both sides of (9.8.15) with the inverse of the operator in L on the right side, and using (9.8.10), we obtain

$$[I - DL]\{I - \overline{G}[I - (A^o - K\overline{G})L]^{-1} K L\} y_{t+1} = u_t,$$

or

$$y_{t+1} = \{D + (I - DL)\,\overline{G}[I - (A^o - K\overline{G})L]^{-1} K\}\, y_t + u_t, \tag{9.8.16}$$

where recall that $u_t = y_{t+1} - \hat{E}[y_{t+1}|y^t]$. The above equation can be expressed in the alternative forms

$$\begin{aligned} y_{t+1} = D y_t &+ \overline{G}\sum_{j=1}^{\infty}(A^o - K\overline{G})^{j-1} K y_{t-j+1} \\ &- D\overline{G}\sum_{j=1}^{\infty}(A^o - K\overline{G})^{j-1} K y_{t-j} + u_t, \end{aligned}$$

or

$$\begin{aligned} y_{t+1} =& [D + \overline{G}K] y_t + \sum_{j=1}^{\infty}[\overline{G}(A^o - K\overline{G})^j K \\ & - D\overline{G}(A^o - K\overline{G})^{j-1} K] y_{t-j} + u_t. \end{aligned} \tag{9.8.17}$$

These equations express y_{t+1} as the sum of the one-step ahead linear least squares forecast and the one-step prediction error.[9]

[9] The MATLAB program `varrep.m` uses (9.8.17) to obtain a vector autoregressive representation for an equilibrium set of y_t's, given $[A^o, C, G, D, R]$.

9.9. Innovations in y_{t+1} as Functions of Innovations w_{t+1} and η_{t+1}

By coupling the original state space system with the associated innovations representation, it is possible to express the innovations in the $\{y_t\}$ process as functions of the disturbances $\{w_t\}$ and the measurement errors $\{v_t\}$. Having a method for expressing this connection can be useful when we want to interpret the innovations in $\{y_t\}$ as functions of the shocks impinging on agents' information sets and the measurement errors.

The state space system is

$$\begin{aligned} x_{t+1} &= A^o x_t + C w_{t+1} \\ \overline{y}_t &= \overline{G} x_t + G C w_{t+1} + \eta_{t+1}, \end{aligned} \tag{9.9.1}$$

which corresponds to an innovations representation, which can be expressed as the "whitener"

$$\begin{aligned} \hat{x}_{t+1} &= (A^o - K\overline{G})\hat{x}_t + K\overline{y}_t \\ u_t &= \overline{y}_t - \overline{G}\hat{x}_t. \end{aligned} \tag{9.9.2}$$

Substituting the second equation of (9.4.3) into the first equation of (9.9.2) gives

$$\hat{x}_{t+1} = (A^o - K\overline{G})\hat{x}_t + K\overline{G}x_t + KGCw_{t+1} + K\eta_{t+1}. \tag{9.9.3}$$

Using (9.9.3), systems (9.4.3) and (9.9.2) can be combined to give the consolidated system

$$\begin{bmatrix} x_{t+1} \\ \hat{x}_{t+1} \end{bmatrix} = \begin{bmatrix} A^o & 0 \\ K\overline{G} & A^o - K\overline{G} \end{bmatrix} \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + \begin{bmatrix} Cw_{t+1} \\ KGCw_{t+1} + K\eta_{t+1} \end{bmatrix} \tag{9.9.4}$$

$$u_t = [\overline{G} \ \ -\overline{G}] \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix} + [GCw_{t+1} + \eta_{t+1}]$$

In system (9.9.4), the "inputs" are the innovations to agents' information sets, namely, w_{t+1}, and the innovations to the measurement errors, namely, η_{t+1}. The "output" of the system is the innovation to y_{t+1}, namely $u_t = y_{t+1} - \hat{E}y_{t+1} \mid y^t$. By computing the impulse response function of system (9.9.4), we can study how the innovations u_t depend on current and past values of w_{t+1} and η_{t+1}. Versions of formula (9.9.4) are useful for studying the range of issues considered by Hansen and Sargent [1991, "Two Difficulties"].[10] In the next section, we illustrate one such issue in the context of a permanent income example.

[10] The MATLAB programs `white1.m` and `white2.m` use formula (9.9.4) to compute impulse response functions of u_t with respect to w_t and η_t, respectively.

9.10. Innovations in the y_t's and the w_t's in a Permanent Income Model

This section illustrates some of the preceding ideas in the context of an economic model that implies that the econometrician's information set spans a smaller space than agents' information. The context is a class of models which impose a form of expected present value budget balance. As we shall see, expected present value budget balance is characterized by a condition that implies that the moving average representation associated with the model, which records the response of the system to the w_t's, fails to be invertible. The outcome is that the innovations in the autoregressive representation don't coincide with the w_t's. Representation (9.9.4) can be used to compute a distributed lag expressing the innovations as functions of the lagged w_t's.

We consider the following version of Hall's model in which the endowment process is the sum of two orthogonal autoregressive processes. Preferences, technology, and information are specified as follows:

9.10.1. Preferences

$$-\frac{1}{2}E\sum_{t=0}^{\infty}\beta^t[(c_t-b_t)^2+\ell_t^2] \mid J_0$$

9.10.2. Technology

$$\begin{aligned} c_t+i_t &= \gamma k_{t-1}+d_t \\ \phi_1 i_t &= g_t \\ k_t &= \delta_k k_{t-1}+i_t \\ g_t\cdot g_t &= \ell_t^2 \end{aligned}$$

9.10.3. Information

$$A_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & .9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$U_d = \begin{bmatrix} 5 & 1 & 1 & .8 & .6 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$U_b = [\, 30 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,]$$

We specify that $\gamma = .05, \delta_k = 1, \beta = 1/1.05, \phi_1 = .00001$. Note that $\beta(\delta_k + \gamma) = 1$, which is the condition for consumption to be a random walk in Hall's model. The preference shock is constant at 30, while the endowment process is the sum of a constant (5) plus two orthogonal processes. In particular, we have specified that $d_t = 5 + d_{1t} + d_{2t}$, where

$$\begin{aligned} d_{1t} &= .9 d_{1t-1} + w_{1t} \\ d_{2t} &= \tilde{w}_{2t} + .8\tilde{w}_{2t-1} + .6\tilde{w}_{2t-2} + .4\tilde{w}_{2t-3} \end{aligned}$$

where $(w_{1t}, \tilde{w}_{2t}) = (w_{1t}, 4w_{2t})$. Notice that we have set

$$E \begin{bmatrix} w_{1t} \\ \tilde{w}_{2t} \end{bmatrix} \begin{bmatrix} w_{1t} \\ \tilde{w}_{2t} \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix}.$$

Here d_{1t} is a first order autoregressive process, while d_{2t} is a third order pure moving average process.

We define the household's net of interest deficit as $c_t - d_t$. Hall's model imposes "expected present value budget balance," in the sense that $E \sum_{j=0}^{\infty} \beta^j (c_{t+j} - d_{t+j}) \mid J_t = \beta^{-1} k_{t-1}$ for all t,[11] which implies that the present value of the moving average coefficients in the response of the deficit to innovations in agents' information sets must be zero. That is, let the moving average representation of $(c_t, c_t - d_t)$ in terms of the w_t's be[12]

$$\begin{bmatrix} c_t \\ c_t - d_t \end{bmatrix} = \begin{bmatrix} \sigma_1(L) \\ \sigma_2(L) \end{bmatrix} w_t, \tag{9.10.1}$$

[11] See Sargent [1987] and Hansen, Roberds, and Sargent [1990].

[12] Without loss of generality, the covariance matrix of w_t can be chosen to be the identity matrix.

where $\sigma_1(L)$ and $\sigma_2(L)$ are each (1×2) matrix polynomials, and $\sigma(L) = \sum_{j=0}^{\infty} \sigma_j L^j$. Then Hall's model imposes the restriction

$$\sigma_2(\beta) = [\,0 \quad 0\,]. \tag{9.10.2}$$

The agents in this version of Hall's model observe J_t at t, which includes the history of each component of w_t up to t. This means that agents see histories of both components of the endowment process d_{1t} and d_{2t}. Let us now put ourselves in the shoes of an econometrician who has data on the history of the pair $[\,c_t, d_t\,]$, but not directly on the history of w_t. We imagine the econometrician to form a record of consumption and the deficit $[\,c_t, c_t - d_t\,]$, and to obtain a Wold representation for the process $[\,c_t, c_t - d_t\,]$. Let this representation be denoted[13]

$$\begin{bmatrix} c_t \\ c_t - d_t \end{bmatrix} = \begin{bmatrix} \sigma_1^*(L) \\ \sigma_2^*(L) \end{bmatrix} u_t, \tag{9.10.3}$$

where $\sigma^*(L)$ is one-sided in nonnegative powers of L, and u_t is a serially uncorrelated process with mean zero and $E u_t u_t' = I$; u_t is the innovation of $[\,c_t, c_t - d_t\,]$ relative to the history $[c^{t-1}, c^{t-1} - d^{t-1}]$. In representation (9.10.3), u_t is the object that would appear in the Gaussian log likelihood function, as in formula (9.6.4).

It is natural to ask whether the impulse response functions $\sigma^*(L)$ in the Wold representation (or vector autoregression) (9.10.3) estimated by the econometrician "resemble" the impulse response functions $\sigma(L)$ that depict the response of $(\,c_t, c_t - d_t\,)$ to the innovations to agents' information. A way to attack this question is to ask whether the history of the $\{u_t\}$ process of innovations to the econometrician's information set in (9.10.3) *reveals* the history of the $\{w_t\}$ process impinging on agents' information sets. In the present model, the answer to this question is 'no' precisely because restriction (9.10.2) holds. In particular, (9.10.2) implies that the history of u_t 's in (9.10.3) spans a *smaller* linear space than does the history of w_t 's.

Here is the reason. The u_t 's in (9.10.3) are constructed to lie in the space spanned by the history of the $[\,c_t, c_t - d_t\,]$ process.[14] Technically, this implies

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[13] Without loss of generality, the covariance matrix of u_t can be chosen to be the identity matrix.

[14] Recall the construction underlying Wold's representation theorem, e.g., see Sargent [1987, chapter XI].

that the operator $\sigma^*(L)$ in (9.10.3) is invertible, so that (9.10.3) can be expressed as

$$u_t = \sigma^*(L)^{-1} \begin{bmatrix} c_t \\ c_t - d_t \end{bmatrix},$$

where $\sigma^*(L)^{-1}$ is one-sided in nonnegative powers of L, and where the coefficients in its power series expansion are square summable. Given that $\sigma^*(z)\sigma^*(z^{-1})'$ is of full rank, a necessary condition for $\sigma^*(L)^{-1}$ to exist (i.e., to have a representation as a square-summable polynomial in nonnegative powers of L) is that det $\sigma^*(z)$ have no zeros inside the unit circle.

Condition (9.10.2) then rules out the possibility that $\sigma^*(L)$ is related to $\sigma(L)$ by a relation of the form $\sigma^*(L) = U\sigma(L)$ where U is a nonsingular 2×2 matrix. For (9.10.2) implies that det $\sigma(z)$ has a zero at β, which is *inside* the unit circle. In circumstances in which $[c_t, c_t - d_t]$ is a full rank process,[15] the history of $[c_t, c_t - d_t]$ generates a smaller information set than does the history of the w_t process.

When u_t spans a smaller space than w_t, u_t will typically be a distributed lag of w_t that is not concentrated at zero lag:

$$u_t = \sum_{j=0}^{\infty} \alpha_j \ w_{t-j}. \tag{9.10.4}$$

Thus the econometrician's news u_t potentially responds with a lag to the agents' news w_t. The calculations leading to representation (9.9.4) can be used to compute the vector distributed lag α_j.

To illustrate these ideas in the context of the present version of Hall's model, figures 9.10.1.a and 9.10.1.b display the impulse response functions of $[c_t, c_t - d_t]$ to the two innovations in the endowment process. Consumption displays the characteristic "random walk" response with respect to each innovation. Each endowment innovation leads to a temporary surplus followed by a permanent net-of-interest deficit. The temporary surplus is used to accumulate a stock of capital sufficient to support the permanent net of interest deficit that is to follow it. Restriction (9.10.2) states that the temporary surplus just offsets the permanent deficit in terms of expected present value. For each innovation, we computed the present value of the response of $(c_t - d_t)$ to be zero, as predicted by (9.10.2).

---

15 By a full rank process we mean that $\sigma^*(z)\sigma^*(z^{-1})$.

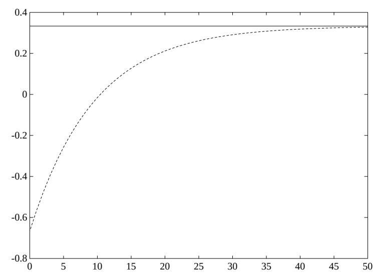

Fig. 9.10.1.a. Impulse response of consumption and deficit to first endowment innovation. The dotted line denotes the deficit, the dark line consumption.

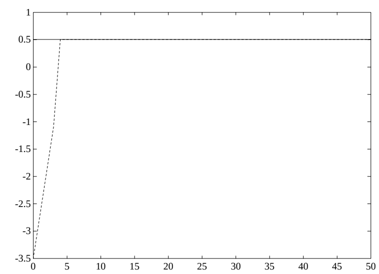

Fig. 9.10.1.b. Impulse response of consumption and deficit to second endowment innovation. The dotted line denotes the deficit, the dark line consumption.

Figures 9.10.2.a and 9.10.2.b report the impulse responses from the Wold representation, which we obtained using the programs `varma.m` and `varma2.m`. The innovation covariance matrix for the u_t's was

$$E\sigma_0^* \sigma_0^{*\prime} = \begin{bmatrix} .3662 & -1.9874 \\ -1.9874 & 12.8509 \end{bmatrix}.$$

Notice that consumption responds only to the first innovation in the Wold representation, and that it responds with an impulse response symptomatic of a random walk. That consumption responds only to the first innovation in the vector autoregression is indicative of the Granger-causality imposed on the $[\,c_t, c_t - d_t\,]$ process by Hall's model: consumption Granger causes $c_t - d_t$, with no reverse causality.

Unlike consumption, the response of the deficit $(c_t - d_t)$ to the innovations in the vector autoregression depicted in figures 9.10.2.a and 9.10.2.b fail to match up qualitatively with the patterns displayed in figures 9.10.1.a and 9.10.1.b. In particular, the present values $(\sigma_2^*(\beta))$ of the response of $c_t - d_t$ to u_t are $(6.0963, 6.6544)$. By construction, $\sigma_2^*(\beta)$ cannot be zero because $\sigma_2^*(L)$ is invertible.

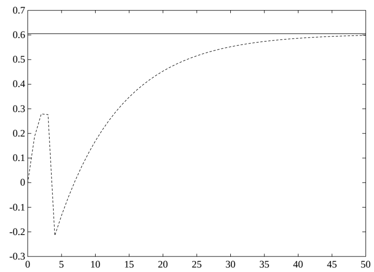

Fig. 9.10.2.a. Impulse response of consumption and deficit to first innovation in Wold representation. The dotted line denotes the deficit, the dark line consumption.

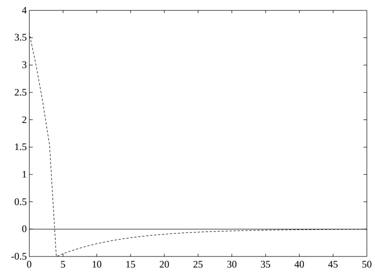

Fig. 9.10.2.b. Impulse response of consumption and deficit to second innovation in Wold representation. The dotted line denotes the deficit, the dark line consumption.

Figures 9.10.3.a and 9.10.3.b display the impulse responses of u_t to w_t, the kind of representation depicted in equation (9.10.4). While the responses of the innovations to consumption are concentrated at lag zero for both components of w_t, the responses of the innovations to $(c_t - d_t)$ are spread over time (especially the response to w_{1t}). Thus, the innovations to $(c_t - d_t)$ as revealed by the vector autoregression depend on what to economic agents is "old news".

Hansen, Roberds, and Sargent [1991] describe how such issues impinge on strategies for econometrically testing present value budget balance. Hansen and Sargent [1991] and Marcet [1991] more generally study the link between innovations in a vector autoregression and the innovations in agents' information sets.

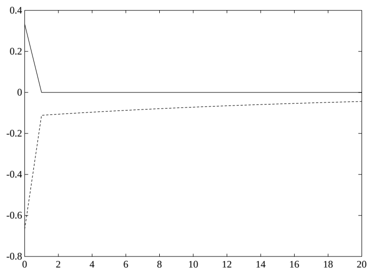

Fig. 9.10.3.a. Impulse response of innovations in Wold representation for consumption and deficit to first endowment innovation. The dotted line denotes the deficit, the dark line consumption.

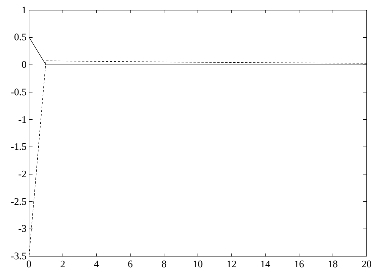

Fig. 9.10.3.b. Impulse response of innovations in Wold representation for consumption and deficit to second endowment innovation. The dotted line denotes the deficit, the dark line consumption.

9.11. Frequency Domain Estimation

We now describe how to estimate the free parameters of the model (9.4.1) using the frequency domain approximation to the likelihood function of Hannan [1970].We assume a model for which y_t is asymptotically stationary. Let the mean vector for the observable $\{y_t\}$ process be denoted μ. The mean vector μ is a function of the parameters of the model. The *spectral density matrix* of the $\{y_t\}$ process is defined as

$$S_y(\omega) = \sum_{\tau=-\infty}^{\infty} C_y(\tau) e^{-i\omega\tau} \tag{9.11.1}$$

where $C_y(\tau) = E[y_t - \mu][y_{t-\tau} - \mu]'$. For the model (9.4.1), the spectral density can be represented as

$$\begin{aligned} S_y(\omega) = & G(I - A^o e^{-i\omega})^{-1} C C' (I - A^{o\prime} e^{+i\omega})^{-1} G' \\ & + (I - D e^{-i\omega})^{-1} R (I - D' e^{+i\omega})^{-1} \end{aligned} \tag{9.11.2a}$$

and the unconditional means can be represented via a function

$$Ey_t \equiv \mu = \mu(A^o, G). \tag{9.11.2b}$$

The autocovariances can be recovered from $S_y(\omega)$ via the inversion formula[16]

$$C_y(\tau) = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} S_y(\omega) e^{+i\omega\tau} d\omega. \tag{9.11.3}$$

Let y_t be a $(p\times 1)$ vector. Suppose that a sample of observations on $\{y_t\}_{t=1}^T$ is available. Define the *Fourier transform* of $\{y_t\}_{t=1}^T$ as

$$y(\omega_j) = \sum_{t=1}^T y_t e^{-i\omega_j t}, \omega_j = \frac{2\pi j}{T}, j = 1,\ldots,T. \tag{9.11.4}$$

The *periodogram* of $\{y_t\}_{t=1}^T$ is defined as

$$J_y(\omega_j) = \frac{1}{T} y(\omega_j)\overline{y}(\omega_j)', \tag{9.11.5}$$

where the overbar denotes complex conjugation.

Following Hannan [1970], the Gaussian log likelihood of $\{y_t\}_{t=1}^T$ as a function of the free parameters determining $A^o, C, D,$ and R can be approximated as

$$\begin{aligned} L^* = &-\left(\frac{1}{2}\right)(T+Tp)\log 2\pi - \sum_{j=1}^{T/2+1} \log\{\det S_y(\omega_j)\} \\ &- \sum_{j=1}^{T/2+1} \text{trace}\big[S_y(\omega_j)^{-1} J_y(\omega_j)\big] \\ &- \frac{T}{2}\text{trace}\Big\{S_y(0)^{-1}\big[T^{-1}\sum_{t=1}^T y_t - \mu\big]\ \big[T^{-1}\sum_{t=1}^T y_t - \mu\big]'\Big\} \end{aligned} \tag{9.11.6}$$

In (9.11.6), p is the dimension of the y_t vector.

The free parameters determining A^o, C, D, and R can be estimated by maximizing the right side of (9.11.6) with respect to them. Notice that the *data* $\{y_t\}_{t=1}^T$ enter the right side of (9.11.6) only through the sample mean

16 The MATLAB programs `spectral.m` and `spectr1.m` can be used to compute a spectral density matrix for one of our models. These programs implement formula (9.11.2).

$T^{-1}\sum_{t=1}^{T} y_t$ and the periodogram $J_y(\omega_j)$, while the *theory* enters through relation (9.11.2) which determines μ and $S_y(\omega_j)$ as functions of the free parameters. Parameter estimation uses any of a variety of hill-climbing algorithms on (9.11.6).[17]

An advantage of frequency domain estimation is that it avoids the need, associated with time domain estimation, to deduce a Wold representation for y_t. Notice that estimation proceeds without factoring the spectral density matrix (9.11.2).

9.12. Approximation Theory

When an economist estimates a misspecified model, how are the probability limits of the parameters that he estimates related to the parameters of a "true" model? This question is not well posed until one states an alternative model relative to which the model at hand is regarded as misspecified. If such an alternative model is on the table, then the question can be answered by adapting the analysis of approximation used by Christopher Sims [1972] and Halbert White [1982]. A modification of (9.11.6) underlies the theory of approximation.

To state a complete theory of approximation, these elements are required: (1) a model that in truth generates the data (to speak of approximation, it is necessary to specify *what* is being approximated); (2) the model being estimated; and (3) the method of parameter estimation. We make the following assumption about these three elements. The true model is a member of the class of models described earlier in this book, with parameters denoted by a vector δ. The true mean vector for the observables is $\nu(\delta)$, and the true spectral density matrix is $\overline{S}_y(\omega, \delta)$, where $\overline{S}_y(\omega, \delta)$ is determined by a version of (9.11.2a) with parameters $\overline{A}^o, \overline{C}, \overline{G}, \overline{R}, \overline{D}$, which depend on the parameter vector δ. The *estimated model* is *another* version of (9.11.2), where the parameters determining the matrices A^o, C, G, R, D, are denoted α, the spectral density matrix is $S_y(\omega, \alpha)$, and the mean vector is $\mu(\alpha)$. The method of estimation is maximum likelihood. It can be shown (see Hansen and Sargent (1993)) that the probability limits of the free

[17] For example, see Bard (1974 XXXX).

parameters α satisfy

$$\begin{aligned}\text{plim}\ \hat{\alpha} = \arg\max_{\alpha}\Big\{ & -\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det S_y(\omega,\alpha)\,d\omega \\ & - \frac{1}{2\pi}\int_{-\pi}^{\pi} \text{trace}\big[S_y(\omega,\alpha)^{-1}\,\overline{S}_y(\omega,\delta)\big]\,d\omega \\ & - [\nu - \mu(\alpha)]\,S_y(0,\alpha)^{-1}[\nu-\mu(\alpha)]'\Big\}\ .\end{aligned} \tag{9.12.1}$$

The right side of (9.12.1) is obtained from (9.11.6) by appropriately taking limits as $T \to \infty$. Roughly speaking, taking limits replaces the periodogram $J_y(\omega_j)$ with the spectral density for the *true* model $\overline{S}_y(\omega_j)$, and replaces the sample mean with the true mean vector.

9.13. Aggregation Over Time

In this section, we describe how to use the Kalman filter to calculate the likelihood for data that are "aggregated over time." We formulate a model that generates observations in state space form and then use the Kalman filter to derive an associated innovations representation from which the Gaussian log likelihood function can be constructed.

Let the original equilibrium model have the state space form

$$\begin{aligned} x_{t+1} &= A^o x_t + C w_{t+1} \\ y_t &= G x_t \end{aligned} \tag{9.13.1}$$

where w_{t+1} is a martingale difference sequence with $E w_{t+1} w_{t+1}' | J_t = I$. We assume that the model is formulated to apply at a finer time interval than that for which data are available. For example, the model (9.13.1) may apply to weekly or monthly data, while only quarterly or annual data may be available to the economist. Furthermore, some of the observed data may be averages over time of the $\{y_t\}$ data generated by (9.13.1), as when "flow" data are generated by averaging over time. (Data on output, consumption, and investment flows are usually generated in this way.) Others of the data may simply be point-in-time "skip sampled" versions of the data. That is, "quarterly" data are formed by sampling every thirteenth observation of the "weekly" data. We want to catalogue the restrictions imposed on these time aggregated data by the model

(9.13.1). We accomplish this by deducing the likelihood function of these data as a function of the free parameters for (9.13.1).

We perform our analysis of aggregation over time in two steps. First, we expand the state space by including enough lagged states to accommodate whatever averaging over time of data is occurring. Let m be the number of dates over which data are potentially to be averaged. Then we form the augmented system

$$\begin{bmatrix} x_{t+1} \\ x_t \\ x_{t-1} \\ \vdots \\ x_{t-m+2} \end{bmatrix} = \begin{bmatrix} A^o & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \begin{bmatrix} x_t \\ x_{t-1} \\ x_{t-2} \\ \vdots \\ x_{t-m+1} \end{bmatrix} + \begin{bmatrix} C \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} w_{t+1}$$

or

$$\overline{x}_{t+1} = \overline{A}\overline{x}_t + \overline{C} w_{t+1} \tag{9.13.2}$$

where

$$\overline{x}_{t+1} = \begin{bmatrix} x_{t+1} \\ x_t \\ x_{t-1} \\ \vdots \\ x_{t-m+2} \end{bmatrix}, \ \overline{A} = \begin{bmatrix} A^o & 0 & \cdots & 0 & 0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \ \overline{C} = \begin{bmatrix} C \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{9.13.3}$$

Once we have formed $\overline{A}$ and $\overline{C}$, it is easy to form the appropriate model for averaged data. For example, suppose that we are interested in forming the model governing three period averages of consumption. We would set m equal to 3 in (9.13.2) and (9.13.3), and could then model averaged consumption via the observer equation $y_t = \overline{G}\overline{x}_t$ where $\overline{G} = [S_c \ S_c \ S_c]$. The MATLAB program `avg.m` obtains the matrices $\overline{A}$ and $\overline{C}$ of (9.13.3) for a given m, thereby accomplishing the first step in our analysis of aggregation over time.

The second step is actually to perform the aggregation over time by skipping observations on a representation of the form (9.13.1) or (9.13.2). Let an equilibrium be represented in the state space form

$$\begin{aligned} x_{t+1} &= Ax_t + Cw_{t+1}, \quad t = 0, 1, 2, \ldots \\ y_t &= Gx_t \end{aligned} \tag{9.13.4}$$

where the first line of (9.13.4) could correspond either to (9.13.2) or to its special case, the first line of (9.13.1). Suppose that data on y_t are available only every $r > 1$ periods, where r is an integer. Then the data are generated by the model

$$\begin{aligned} x_{t+r} &= A_r x_t + w^r_{t+r}, \quad t = 0, r, 2r, \ldots \\ y_t &= G x_t \end{aligned} \tag{9.13.5}$$

where

$$\begin{aligned} A_r &= A^r \\ w^r_{t+r} &= A^{r-1} C w_{t+1} + A^{r-2} C w_{t+2} + \cdots + A C w_{t+r-1} + C w_{t+r} \end{aligned} \tag{9.13.6}$$

Represent (9.13.5),(9.13.6) as the state space system

$$\begin{aligned} x_{s+1} &= A_r x_s + w^r_{s+1} \quad , \ s = 0, 1, 2, \ldots \\ y_s &= G x_s \end{aligned} \tag{9.13.7}$$

where w^r_{s+1} is a martingale difference sequence with contemporaneous covariance matrix

$$\begin{aligned} E w^r_s w^{r\prime}_s &= CC' + ACC'A' + \cdots + A^{r-1} CC' A^{r-1\prime} \\ &\equiv V. \end{aligned} \tag{9.13.8}$$

Now suppose that only error-corrupted observations on the time aggregated $\{y_s\}$ data are available, and that the measurement errors are first-order serially correlated. To capture this assumption, augment (9.13.7) – (9.13.8) to become the state space system

$$\begin{aligned} x_{s+1} &= A_r x_s + w^r_{s+1} \\ y_s &= G x_s + \upsilon_s \\ \upsilon_s &= D \upsilon_{s-1} + \eta_s \end{aligned} \tag{9.13.9}$$

where $E\eta_s \eta_s' = R$ and $E w_{s+1} \eta_s' = 0$ for all t and s.

System (9.13.9) is a version of the state space system (9.4.1). Proceeding as in our analysis of (9.4.1), define $\overline{y}_s \equiv y_{s+1} - D y_s$ and $G_r = (G A_r - DG)$. Then (9.13.9) implies the system

$$\begin{aligned} x_{s+1} &= A_r x_s + w^r_{s+1} \\ \overline{y}_s &= G_r x_s + G w^r_{s+1} + \eta_{s+1}. \end{aligned} \tag{9.13.10}$$

Define the covariance matrices $E w^r_s w^{r'}_s = V \equiv V_1, E(G w^r_{s+1} + \eta_{s+1})(G w^r_{s+1} + \eta_{s+1})' = GVG' + R \equiv V_2, E w^r_{s+1} (G w^r_{s+1} + \eta_{s+1})' = VG' = V_3$. Use the function `kfilter` to obtain $[K, \Sigma] = \texttt{kfilter}(A_r, G_r, V_1, V_2, V_3)$. Then an innovations representation for system (9.13.10) is

$$\begin{aligned} \hat{x}_{s+1} &= A_r \hat{x}_s + K a_s \\ \overline{y}_s &= G_r \hat{x}_s + a_s \end{aligned} \tag{9.13.11}$$

where $\hat{x}_s = E[x_s \mid \overline{y}_0^{s-1}], a_s = \overline{y}_s - E[\overline{y}_s \mid \overline{y}_0^{s-1}], \Omega_1 \equiv E a_s a_s' = G_r \Sigma G_r' + V_2$. Once again, the innovations representation (9.13.11) can be used to form the residuals a_t recursively, and thereby to form the Gaussian log likelihood function.[18]

We illustrate the programs `avg.m` and `aggreg.m` by showing how they can be used to analyze the effects of aggregation over time in the context of our equilibrium version of Hall's model. We want to deduce the univariate Wold representation for consumption data that are constructed by taking a three period moving average, and then "skip sampling" every third period. The following MATLAB code performs these calculations:

```
clex11;                        reads in parameters of Hall's economy
solvea;                        computes the equilibrium
[AA,CC]= avg(a0,C,3);          forms state for three period averaging
G = [sc sc sc];                forms observer for three-period moving average
                                of consumption
R = .0001; D = 0;              sets parameters of measurement error process
[Ar,Cr,aa,bb,cc,dd,V1] =
    aggreg (AA,CC,G,D,R,3)
y = dimpulse(aa,bb,cc,dd,1,20);    forms moving average representation
```

We have set the parameters of Hall's model at the values that make unaveraged consumption follow a random walk. Notice that we set R and D so that only a very small measurement error is present in consumption. The impulse response function for skip-sampled three period moving average consumption reveals the following representation for the skip-sampled moving average data $\overline{c}_t$:

$$\overline{c}_t - \overline{c}_{t-1} = a_t + .2208 a_{t-1}$$

where $a_t = \overline{c}_t - E(\overline{c}_t \mid \overline{c}_{t-1}, \overline{c}_{t-2}, \ldots)$. Thus, the first difference of $\overline{c}_t$ is a first-order moving average process. These calculations recover a version of Holbrook

[18] The MATLAB program `aggreg.m` constructs the innovations representation (9.13.11) from inputs in the form of the state space representation (9.13.4) and the parameters R and D of the measurement error model (9.13.2).

Working's [1960] findings about the effects of skip sampling a moving average of a random walk.

9.14. Simulation Estimators

We have described how to estimate the free parameters of a model using data that are possibly error-ridden linear functions of the state vector x_t. In our models, quantities and (scaled Arrow-Debreu) prices are linear functions of the state, but asset prices and rates of return are non-linear functions of the state. In this section, we describe how observations of non-linear functions of the state can be used in estimation.

The equilibrium transition law for the state vector x_t is given by

$$x_{t+1} = A^o(\theta)x_t + C(\theta)w_{t+1}, \quad E w_t w_t' = I \tag{9.14.1}$$

where the $r \times 1$ vector θ contains the free parameters of preferences, technologies, and information. We partition the data into two parts, $(z_{1t}, t = 0, \ldots T)$ and $(z_{2t}, t = 0, \ldots T)$, where the z_{1t}'s are *linear* functions of the state x_t, and the z_{2t}'s are *nonlinear* functions of the state. Assume that z_{1t} is $k \times 1$ and z_{2t} is $m \times 1$. The data are related to the state x_t and measurement errors v_t as follows:

$$\begin{aligned} z_{1t} &= G(\theta)x_t + v_{1,t} \\ z_{2t} &= f(x_t, v_{2,t}, \theta), \end{aligned}$$

where

$$E \begin{pmatrix} w_{t+1} \\ v_t \end{pmatrix} \begin{pmatrix} w_{t+1} \\ v_t \end{pmatrix}' = \begin{pmatrix} Q(\theta) & W(\theta) \\ W(\theta)' & R(\theta) \end{pmatrix},$$

and where $Q(\theta) = C(\theta)C(\theta)'$.

The Gaussian log likelihood function of $\{z_{1t}\}_{t=0}^T$ is

$$L(\theta) = \sum_{t=0}^T \ell_t = -\frac{1}{2} \sum_{t=0}^T \Big[p \log(2\pi) + \log|\Omega_t| + a_t' \Omega_t^{-1} a_t \Big]$$

where z_t is $p \times 1$ and $a_t = z_{1,t} - E[z_{1,t} | z_{1,t-1}, \ldots, z_{1,0}]$ is the innovation vector from the 'innovations representation' and $\Omega_t = E a_t a_t'$.

Maximizing the log likelihood function with respect to θ is equivalent with a particular Generalized Method of Moments (GMM) procedure using observations on $(z_{1t}, t = 0, \dots T)$. Note that the first-order order conditions for maximizing the log likelihood function are

$$\frac{\partial L}{\partial \theta} = 0.$$

To see how this matches up with GMM, compute the *score vector* $s_t = \frac{\partial \ell_t}{\partial \theta}$ which has elements,

$$\frac{\partial \ell_t}{\partial \theta_i} = -\frac{1}{2}\text{tr}\left\{\left(\Omega_t^{-1}\frac{\partial \Omega_t}{\partial \theta_i}\right)\left(I - \Omega_t^{-1}a_t a_t'\right)\right\} - \left(\frac{\partial a_t}{\partial \theta_i}\right)'\Omega_t^{-1}a_t.$$

Using the notation of Hansen (1982), the GMM estimator of θ minimizes

$$J_T(\theta) = g_T(\theta)'\mathcal{W}_T g_T(\theta) \tag{9.14.2}$$

where

$$g_T(\theta) = \frac{1}{T+1}\sum_{t=0}^{T} s_t(\theta)$$

and $\mathcal{W}_T$ is *any* positive definite $r \times r$ weighting matrix. Notice that $g_T(\theta) = \frac{\partial L}{\partial \theta}$, so that for any positive definite weighting matrix, (9.14.2) is minimized by the minimizer of $L(\theta)$. The irrelevance of the weighting matrix $\mathcal{W}_T$ reflects the property that from the viewpoint of GMM, this is a 'just-identified' system, with as many moment conditions as free parameters.

Suppose that we want to use the observations in z_{1t} and in z_{2t} to estimate θ. We can apply a method described by Ingram and Lee. Given the law of motion in (9.14.1) and a realization from a pseudo-random number generator for $\{w_{j+1}, v_{1j}, v_{2j}\}_{j=0}^{N}$, we can generate a pseudo-random realization of the series $\{z_{1j}, z_{2j}\}_{j=0}^{N}$. Let $q(\cdot)$ be a given function of the data. Use the data and the simulation of the model, respectively, to compute the two moment vectors:

$$H_T(z) = \frac{1}{T+1}\sum_{t=0}^{T} q(z_{1t}, z_{2t})$$

$$H_N(\theta) = \frac{1}{N+1}\sum_{j=0}^{N} q(z_{1j}, z_{2j}; \theta).$$

Define $h_T(\theta)$ as follows

$$\begin{aligned} h_T(\theta) &= \frac{1}{T+1}\sum_{t=0}^{T}\Big[q(z_{1t}, z_{2t}) - \frac{1}{n+1}\sum_{j=0}^{n} q(z_{1j}, z_{2j}; \theta)\Big] \\ &= H_T(z) - H_N(\theta), \end{aligned}$$

where $n+1 = (N+1)/(T+1)$ and $N+1$ is some integer multiple of $T+1$. Then the estimation strategy for obtaining θ is to minimize

$$J_T(\theta) = \begin{bmatrix} \partial L/\partial\theta \\ h_T(\theta) \end{bmatrix}' \mathcal{W}_T \begin{bmatrix} \partial L/\partial\theta \\ h_T(\theta) \end{bmatrix}$$

for some weighting matrix $\mathcal{W}_T$. To estimate $\mathcal{W}_T$, we can use the two-stage procedure in Hansen (1982), which is to start with $\mathcal{W}_T = I$ and then construct the weighting matrix associated with the resulting estimate of θ.

A. Initialization of the Kalman Filter

This appendix describes how Kohn and Ansley's idea for estimating the initial state can be applied in the context of our class of models. Aside from numerical issues, Kohn and Ansley's procedure is equivalent to using *all* of the data $\{y_s\}_{s=0}^{T}$, and initializing the Kalman filter from a partitioned covariance matrix designed to approximate

$$\Sigma_0 = \begin{bmatrix} +\infty I & +\infty \mathbf{1} \\ +\infty Q' & \Sigma_{0,22} \end{bmatrix},$$

where $\Sigma_{0,22}$ is the asymptotic covariance matrix of that piece of the state vector that has an asymptotically stationary distribution, and $\mathbf{1}$ is a matrix of ones. The $+\infty I$ pertains to elements of the state that have no asymptotic stationary distribution. In practice, $+\infty$ is approximated by a large positive scalar. This procedure was used by Harvey and Pierse.[19] This procedure ought to be close to Kohn and Ansley's, though the literature contains examples of cases in which the numerical properties of the '$+\infty \approx$ a big number' approach are poor. For that reason, it is good to have in hand procedures like the one we shall describe.

[19] Another approach has been to use an 'inverse filter' in which the recursions are cast in terms of the inverse of Σ_t.

For convenience, we temporarily work with the state-space system[20]

$$\begin{aligned} x_{t+1} &= Ax_t + Cw^*_{t+1} \\ y_t &= Gx_t + Qw^*_{t+1}, \end{aligned} \tag{9.A.1}$$

where w^*_{t+1} is a martingale difference sequence with identity for its conditional covariance matrix. In the interest of eventually imputing a diffuse prior to the initial values of that part of the state vector that has no stationary distribution, we represent the initial state as

$$x_0 = \phi\eta + \psi + N\nu,$$

where ψ is an $n \times 1$ vector with all zeros except possibly for one value of one, which locates the constant in the state; and ν is normally distributed with mean zero and covariance I, and η is normally distributed with mean zero and covariance kI, where the random vectors ν and η are assumed to be independent. We use $\phi\eta$ to represent the piece of the initial state that has no stationary distribution, and $N\nu$ to represent the piece with a stationary distribution. We attain a diffuse prior on the stationary distribution by driving k to $+\infty$. Our plan is to project x_m on $y_{m-1}, \ldots, y_0$, while driving $k \to +\infty$, and then to initialize the Kalman filter from the resulting estimators of the distribution of x_m.

By iterating on the state equation $(9.A.1)$, we can write:

$$x_m = A^m\phi\eta + A^m\psi + H_m w^m \tag{9.A.2}$$

where $w^{m\prime} = (\,\nu' \quad {w_1}^{*\prime} \quad \ldots \quad w^{*\prime}_m\,)$ and

$$H_m = (\,A^{m-1}N \quad A^{m-2}C \quad \ldots \quad C\,).$$

Now create a vector $Y^{m-1\prime} = (\,y'_0 \quad y'_1 \quad \ldots \quad y'_{m-1}\,)$ that obeys:

$$Y^{m-1} = M_m\eta + \alpha + G_m w^m \tag{9.A.3}$$

[20] It is easy to map (9.4.3), which describes the state-space system wih serially correlated measurement errors, into this form. Define $w^*_{t+1} = \begin{pmatrix} w_{t+1} \\ \eta_{t+1} \end{pmatrix}$ and represent (9.4.3) as

$$\begin{aligned} x_{t+1} &= Ax_t + (C\ O)\ w^*_{t+1} \\ y_t &= \overline{G}x_t + (GC\ I)\ w^*_{t+1}. \end{aligned}$$

where

$$M_m = \begin{pmatrix} G\phi \\ GA\phi \\ \vdots \\ GA^{m-1}\phi \end{pmatrix}, \quad \alpha = \begin{pmatrix} G \\ GA \\ \vdots \\ GA^{m-1} \end{pmatrix}\psi$$

and

$$G_m = \begin{pmatrix} GN & Q & 0 & 0 & \cdots & 0 & 0 \\ GAN & GC & Q & 0 & \cdots & 0 & 0 \\ GA^2N & GAC & GC & Q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ GA^{m-2}N & GA^{m-3}C & GA^{m-4}C & GA^{m-5}C & \cdots & Q & 0 \\ GA^{m-1}N & GA^{m-2}C & GA^{m-3}C & GA^{m-4}C & \cdots & GC & Q \end{pmatrix}$$

Transform equation $(9.A.2)$ as follows. Regress $H_m w^m$ onto $G_m w^m$, and denote the residual as $R_m w^m$, to obtain the representation

$$H_m w^m = H_m^* G_m w^m + R_m w^m, \tag{9.A.4}$$

where $H_m^* = (EH_m w^m w^{m-1\prime} G_m')(EG_m w^m w^{m-1\prime} G_m')^{-1}$ is a matrix of least squares regression coefficients and $R_m = H_m - H_m^* G_m$. Thus $H_m^* = H_m G_m'(G_m G_m{}')^{-1}$. Also, since $G_m w^m = Y^{m-1} - M_m\eta - \alpha$, $(9.A.4)$ implies the representation

$$H_m w^m = H_m^*(Y^{m-1} - M_m\eta - \alpha) + R_m w^m.$$

Rewrite state equation $(9.A.2)$ as:

$$x_m = (A^m\phi - H_m^* M_m)\eta + A^m\psi - H_m^*\alpha + H_m^* Y^{m-1} + R_m w^m. \tag{9.A.5}$$

Next we compute some conditional expectations and covariances. Initially, we use $(9.A.5)$ and the facts that (i) by assumption, w^m is orthogonal to η, and (ii) by construction, $R_m w^m$ is orthogonal to $G_m w^m$, to compute:

$$E(x_m|Y^{m-1},\eta) = (A^m\phi - H_m^* M_m)\eta + A^m\psi - H_m^*\alpha + H_m^* Y^{m-1},$$

and

$$\text{cov}(x_m|Y^{m-1},\eta) = R_m R_m'.$$

To compute the conditional expectation and covariance matrix conditioning only Y^{m-1}, we first compute the projection of η on $Y^{m-1}-\alpha$:

$$\eta = \beta^*(Y^{m-1}-\alpha)+\varepsilon,$$

where ε is a least squares residual. We compute $E\eta(Y^{m-1}-\alpha)$ and the second moment matrix of $Y^{m-1}-\alpha$ and use them in the projection formula:

$$\beta^* = (kM_m')(kM_mM_m'+G_mG_m')^{-1}.$$

Premultiply by $[M_m'(G_mG_m')^{-1}M_m]^{-1}[M_m'(G_mG_m')^{-1}M_m]$ to get $\beta^* = [M_m'(G_m G_m')^{-1}M_m]^{-1}M_m'(G_mG_m')^{-1}[kM_mM_m'(kM_mM_m'+G_mG_m')^{-1}]$. If we drive $k\rightarrow +\infty$, the last term in square brackets approaches the identity matrix, so that we have

$$E(\eta|Y^{m-1}-\alpha) = [M_m'(G_mG_m{}')^{-1}M_m]^{-1}M_m'(G_mG_m{}')^{-1}(Y^{m-1}-\alpha). \quad (9.A.6)$$

Notice that $\varepsilon = \beta^*(M_m\eta+G_mw^m)-\eta = (\beta^*M_n-I)\eta+\beta^*G_mw^m$, and that $(\beta^*M_m-I)=0$. It follows that

$$\text{cov}(\eta|Y^{m-1}-\alpha) = [M_m'(G_mG_m{}')^{-1}M_m]^{-1}. \quad (9.A.7)$$

Using these results and applying the Law of Iterated Expectations to $(9.A.5)$ gives:[21]

$$E(x_m|Y^{m-1}) = (A^{m-1}\phi - H_m^*M_m)[M_m'(G_mG_m')^{-1}M_m]^{-1}$$
$$M_m'(G_mG_m')^{-1}(Y^{m-1}-\alpha)+A^{m-1}\psi - H_m^*\alpha+H_m^*Y^{m-1}, \quad (9.A.8)$$

and

$$\text{cov}(x_m|Y^{m-1}) = R_mR_m' + (A^{m-1}\phi - H_m^*M_m)[M_m'(G_mG_m')^{-1}M_m]^{-1}$$
$$(A^{m-1}\phi - H_m^*M_m)'. \quad (9.A.9)$$

The Kalman filter is to be initialized by using these values of $\hat{x}_m, \Sigma_m$, then applied to compute $(9.6.8)$, using observations $\{y_s\}_{s=m}^T$.

[21] Note that equations $(9.A.6)$ and $(9.A.7)$ result from applying generalized least squares to the system of equations $(9.A.3)$, where η is regarded as a matrix of constants and M_m is a matrix of regressors.

When we apply this procedure with $(9.A.1)$ corresponding to the system $(9.4.3)$, we should interpret Y^{m-1} in the preceding development as $\overline{Y}^{m-1}$ which corresponds to Y^m in the *real* data. In this case, we should interpret $\hat{x}_m, \Sigma_m$ according to definitions of the $(\hat{\cdot})$ variables defined for the system with serially correlated measurement errors.[22]

We can also include a contribution to the likelihood function to account for the initial observations used to form $\hat{x}_m$. Begin with $(9.A.3)$ and let $\Omega = G_m G_m'$, which we take to be nonsingular. Suppose that M_m is dimensioned r by s where $r > s$ so that η is 'overidentified.' Construct two matrices labeled $M^{\perp}$ and M^*, dimensioned $(r-s) \times r$ and $s \times r$, respectively, to satisfy:

$$M^{\perp}\Omega^{-1}M_m = 0$$

$$M^*\Omega^{-1}M^{\perp\prime} = 0,$$

and construct the nonsingular matrix:

$$D = \begin{pmatrix} M^*\Omega^{-1} \\ M^{\perp}\Omega^{-1} \end{pmatrix}.$$

Define:

$$z_1 = M^*\Omega^{-1}Y^m$$

$$z_2 = M^{\perp}\Omega^{-1}Y^m.$$

Notice that conditioned on η, z_1 and z_2 are uncorrelated. Moreover, by construction z_2 does not depend on η.

We deduce the initial likelihood contribution as follows. First note that

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = DY^m.$$

Transforming the z's introduces a Jacobian term:

$$\log \det D,$$

which is the first contribution to the likelihood.

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[22] Notice that with serially correlated measurement errors, $(9.A.8)$, $(9.A.9)$ give the appropriate initial conditions for the Kalman filter, because of the dating conventions that make u_t the innovation to y_{t+1}.

The second and third contributions are the likelihoods of the z's. Conditioned on η, the likelihood can be factored. Only the first term in the factorization depends on η, and is present in the 'exactly identified' case. The quadratic form term converges to zero for this contribution. We deduce the $\log\det$ term by taking the limit as k goes to infinity of:

$$\log\det(kM^*\Omega^{-1}M_m M_m'\Omega^{-1}M^{*\prime} + M^*G_m G_m' M^{*\prime}) =$$

$$n_1 \log k + \log\det(M^*\Omega^{-1}M_m M_m'\Omega^{-1}M^{*\prime} + \frac{1}{k}M^*G_m G_m' M^{*\prime})$$

where n_1 is the dimension of z_1. Taking the limit and neglecting the term $n_1 \log k$, which is the same for all settings of the parameter values and so can be ignored, leaves the term:

$$\log\det(M^*\Omega^{-1}M_m M_m'\Omega^{-1}M^{*\prime}).$$

The z_2 contribution to the likelihood retains both a log det and a quadratic form contribution. Notice that the z_2 term is absent in the 'exactly identified' case.

Chapter 10
Semiparametric Estimation with Limited Information

10.1. Introduction

This chapter describes semiparametric estimation of transmission mechanisms under limited information.

10.2. Underlying Economic Model

Consider the following economic model. The information available to economic agents at time t is denoted J_t. There is an endogenous state vector k_{t-1} which we will think of as a vector of capital stocks. The capital stocks evolve according to the evolution equation:

$$k_t = \Delta k_{t-1} + \Theta i_t \tag{10.2.1}$$

where i_t is a vector of flow variables which we refer to as investment goods. The absolute values of eigenvalues of the matrix Δ are presumed to be strictly less than one. There is also an exogenous state vector z_t with dynamics given by:

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1} \tag{10.2.2}$$

where $\{w_t\}$ is a martingale difference sequence adapted to $\{J_t\}$ with a conditional covariance matrix I. It is known that a first-order linear specification of the dynamics is quite flexible because it can represent multivariate ARMA models of arbitrary orders. The composite state vector at time t is denoted

$$x_t \equiv \begin{pmatrix} k_{t-1} \\ z_t \end{pmatrix}.$$

The recursive solution to the model gives investment i_t as a function of the state vector x_t:

$$i_t = S x_t = S_k k_{t-1} + S_z z_t.$$

Substituting this solution into equations (10.2.1) and (10.2.2), we find that

$$x_{t+1} = Ax_t + Cw_{t+1} \tag{10.2.3}$$

where

$$A \equiv \begin{pmatrix} \Delta - \Theta S_k & \Theta S_z \\ 0 & A_{22} \end{pmatrix}, C \equiv \begin{pmatrix} 0 \\ C_2 \end{pmatrix}.$$

Thus, equation (10.2.3) gives the evolution of the state vector process when the optimal or equilibrium investment rule is imposed.

The state vector z_t may enter into the decision rule or equilibrium investment relation for one of two reasons. Some components of z_t may enter directly into the objective functions of economic agents; and other components may simply be used in forecasting future values of these variables. Let z_t^1 denote a vector of the former components, which we presume are related to z_t via:

$$z_t^1 = H_1 z_t$$

where H_1 is just a matrix that selects elements from z_t. The solution for investment can often be represented as:

$$i_t = S_k k_{t-1} + S_0 z_t^1 + S_f \sum_{j=0}^{\infty} (\Lambda)^j S_1 E(z_{t+j}^1 | J_t) \tag{10.2.4}$$

and hence

$$S_z = S_0 H_1 + S_f (I - \Lambda)^{-1} S_1 H_1$$

(see Hansen and Sargent 1981 and Sargent 1987). One way to obtain a solution of this form is to stack the first order conditions for the endogenous state vector and its corresponding co-state vector into an expectational first-order difference equation driven by the forcing process $\{z_t^1\}$, then to solve that difference equation. A similar structure can be obtained even when the endogenous state vector is not the solution to an optimal resource allocation problem. The estimation method we describe below exploits the feedforward structure of the solution for investment whereby investment depends on current and expected future values of the forcing process $\{z_t^1\}$.

10.3. Econometrician's information and the implied orthogonality conditions

We presume that the econometrician observes a time series of investment $i_t, t = 1, 2, ..., T$. With knowledge of the depreciation matrix Δ, he can construct an approximate capital series, 'approximate' because the initial capital stock k_0 may be unknown. Because the dominant eigenvalue of Δ is strictly less than unity in modulus, the approximation error vanishes as the sample size T gets large. The econometrician also observes some but not all of the vector z_t.

Partition

$$z_t^1 = \begin{pmatrix} y_t \\ u_t \end{pmatrix}$$

where y_t is observed by the econometrician by u_t is not. To have any hope of identifying the parameters in the underlying model, we assume that the exogenous state vector process can be uncoupled in the following way:

$$z_{t+1}^y = A_y z_t^y + C_y w_{t+1}^y$$

and

$$z_{t+1}^u = A_u z_t^u + C_u w_{t+1}^u$$

where

$$z_t = \begin{pmatrix} z_t^y \\ z_t^u \end{pmatrix}, w_t = \begin{pmatrix} w_t^y \\ w_t^u \end{pmatrix}$$

and

$$y_t = H_y z_t^y \quad \text{and} \quad u_t = H_u z_t^u.$$

To guarantee asymptotic stationarity, we restrict the absolute values of the eigenvalues of A_u to be strictly less than one. The $\{u_t\}$ process gives us one interpretation of why investment is not an exact function of variables observed by an econometrician. Therefore, we rewrite the investment relation as:

$$i_t = S_k k_{t-1} + S_{0,y} y_t + S_f \sum_{j=0}^{\infty} (\Lambda)^j S_{1,y} E(y_{t+j} | J_t) + e_t$$

where

$$e_t \equiv S_{0,u} u_t + S_f \sum_{j=0}^{\infty} (\Lambda)^j S_{1,u} E(u_{t+j} | J_t)$$

and

$$S_j = \begin{pmatrix} S_{j,y} & S_{j,u} \end{pmatrix}, j = 0, 1.$$

The process $\{e_t\}$ provides an error term that can be interpreted along the lines of Hansen and Sargent (1980). By construction it is uncorrelated with the process $\{y_t\}$ at all leads and lags. This uncorrelatedness can be directly exploited in estimating the parameters of endogenous dynamics of the model, *e.g.*, the parameters governing the transmission mechanism. In fact, this can be accomplished in ways that permit a robust specification of the dynamics associated with the underlying shock process $\{u_t\}$. In other words, a semiparametric estimation method is possible in this setting.

Another source of omitted information is in the forecasting of future values of the process $\{y_t\}$. For instance, let K_t denote the information set used by the econometrician, constructed in a way so that at least it is no larger than J_t. This gives rise to an additional model "specification" error, say f_t, as emphasized by Shiller (1978) and Hansen and Sargent (1980, 1982). Thus, the investment equation used by an econometrician is given by:

$$i_t = S_k k_{t-1} + S_{0,y} y_t + S_f \sum_{j=0}^{\infty} (\Lambda)^j S_{1,y} E(y_{t+j}|K_t) + e_t + f_t \tag{10.3.1}$$

where

$$f_t \equiv S_f \sum_{j=0}^{\infty} (\Lambda)^j S_{1,y} [E(y_{t+j}|J_t) - E(y_{t+j}|K_t)].$$

By the Law of Iterated Expectations, the error term f_t is uncorrelated with current and past values of $\{y_t\}$, but can be correlated with future values of this process. As a consequence, the orthogonality conditions that are robust to misspecifying the information set are:

$$E[(f_t + e_t) y_{t-j}{}'] = 0 \quad \text{for} \quad j = 0, 1, ... \tag{10.3.2}$$

The presence of the component f_t in the disturbance term is what limits the orthogonality conditions to be one-sided. Future values of y_t may be correlated with the disturbance term in the investment equation.

Since the information set K_t is a misspecified version of J_t, unless one is willing to specify the omitted information precisely, it is most convenient to envision an econometrician modeling the evolution equation for $\{y_t\}$ in a flexible manner. Although information is omitted, so long as $\{y_t\}$ has a state-space representation, we know that it can be represented as a multivariate version of an ARMA model, although the autoregressive and moving-average orders will be unknown to the econometrician. For the sake of simplicity, we assumethat the $\{y_t\}$ process is stationary and has a moving-average representation:

$$y_t = B(L)v_t \tag{10.3.3}$$

where the operator B has a one-sided inverse, and current and past values of v_t also generate the information set K_t. In what follows, it is not necessary to limit B to be a ratio of polynomials, as in ARMA models. More general dynamics can be accommodated. As we will see, this in effect introduces an infinite dimensional nuisance parameter into the moment conditions (10.3.2).

Finally, note that by omitting information relative to that used by economic agents, we cannot expect to deduce impulse response functions that are interpretable in terms of the economic shocks impinging on the decision maker. In other words, the response of investment or capital stock to an innovation in $\{y_t\}$ (i.e., in economic agents' information set) will be contaminated. Nevertheless, we will still be in a position to identify parameters of the endogenous dynamics.

10.4. An Adjustment Cost Example

A linear-quadratic model of adjustment costs has a solution for investment that is of the form given by (3) with a scalar investment and capital stock and a scalar Λ that we will denote by λ. For simplicity, we presume that the obervable forcing process $\{y_t\}$ is also scalar. Write the econometric relation for investment in feedforward form as:

$$\begin{aligned} i_t &= \rho k_{t-1} + \psi_o y_t + \psi_f \sum_{j=0}^{\infty} (\lambda)^j E(y_{t+j}|K_t) + e_t + f_t \\ k_t &= \delta k_{t-1} + i_t. \end{aligned} \tag{10.4.1}$$

The operator B enters into the model solution because of its role in the solution to the prediction problem:

$$y_t^p = E(\sum_{j=0}^{\infty} \lambda^j y_{t+j}|K_t).$$

It is known from Hansen and Sargent (1980) that

$$y_t^p = B^*(L)v_t$$

where

$$B^*(\zeta) = \frac{\zeta B(\zeta) - \lambda B(\lambda)}{\zeta - \lambda}. \tag{10.4.2}$$

Substituting this formula into (10.4.1) and solving for the econometric disturbance term, we obtain:

$$f_t + e_t = i_t - \rho k_{t-1} - \psi_o y_t - \psi_f B^*(L)v_t. \tag{10.4.3}$$

Prior to investigating the estimation of endogenous dynamics as captured by the parameters $\rho, \psi_o, \psi_f, \delta, \lambda$ we will study the impact of estimating B in both parametric and nonparametric settings.

10.5. A Slightly Simpler Estimation Problem

Let Y_t denote a random vector, each entry of which is a linear combination of current and past values of y_t. Suppose the unconditional moment condition used in estimation is:

$$E[(f_t + e_t)Y_t] = 0.$$

A component of these moment conditions that depends on B is:

$$\beta_o \equiv E[Y_t B^*(L)v_t], \tag{10.5.1}$$

and for the moment let us suppose that β_o is the parameter of interest.

10.5.1. Scalar Parameterizations of B

As a preliminary step to studying nonparametric estimators of B, we initially consider very simple scalar parameterizations of B:

$$B_\alpha = B + \alpha F.$$

We suppose that for sufficiently small values of α we can invert the operator B_α. We must explore what happens to the moment condition for small perturbations in α.

Define:

$$\phi_t(\alpha) \equiv B_\alpha(L)^{-1} y_t.$$

Differentiating ϕ_t with respect to α and evaluating we find that

$$B(L)D\phi_t(0) + F(L)v_t = 0,$$

or

$$D\phi_t(0) = -[B(L)]^{-1}F(L)v_t. \tag{10.5.2}$$

Then differentiating the moment relation:

$$\beta(\alpha) = E[z_t^y B_\alpha^*(L)\phi_t(\alpha)]$$

we find that

$$d\beta(0)/d\alpha = E\{Y_t[F^*(L)v_t]\} + E\{Y_t[B^*(L)D\phi_t(0)]\}. \tag{10.5.3}$$

The * notation is used to denote the transformation of an operator given by (10.4.2).

Let α_T denote the maximum likelihood estimator for $\alpha = 0$ for sample size T, and let

$$\beta_T \equiv (1/T)\sum_{t=1}^{T} Y_T B_{\alpha_T}^*(L)\phi_{t,T}(\alpha_T)$$

denote the sample estimator of β_o where the notation $\phi_{t,T}(\alpha_T)$ denotes the time t approximation for v_t using the estimator α_T. Then the sampling error in β_T as an estimator of β_o can be decomposed into two components:

$$\sqrt{T}(\beta_T - \beta_o) \approx (1/\sqrt{T})\sum_{t=1}^{T}[Y_t B^*(L)v_t - \beta_o] + [d\beta(0)/d\alpha]\sqrt{T}\alpha_T. \tag{10.5.4}$$

The first term is the usual central limit approximation for sample moment estimators while the second term accounts for the additional sampling error induced by having to estimate B. It is the second term that we turn our attention to.

The limiting distribution of $\{\alpha_T\}$ is determined by the score of the conditional likelihood of y_t conditioned on the past. This score is given by

$$s_t = -v_t D\phi_t(0) + E[v(t) D\phi_t(0)]. \tag{10.5.5}$$

The first term of the score comes from differentiating the quadratic form in the one-step ahead forecast error of y_t, and the second term from differentiating the log variance term of the time t contribution to the conditional log likelihood. The score variable has mean zero conditioned on K_{t-1} and the resulting score process is a martingale difference sequence. Then

$$\sqrt{T}\alpha_T \approx \frac{\sum_{t=1}^{T} s_t}{\sqrt{T} E({s_t}^2)}. \tag{10.5.6}$$

In light of the fundamental role played by the score variable in deteriming the limiting distribution for the estimator sequence $\{\alpha_T\}$, it will prove to be very useful to represent the derivative $d\beta(0)/d\alpha$ as an expected cross product of some random vector with the score s_t. We now deduce what that random vector is by obtaining an alternative expression for the right-hand side of (10.5.3). Note that

$$\begin{aligned} E\{Y_t[F^*(L)v_t]\} &= E\left[Y_t\left(\left[\frac{LF(L)}{L-\lambda}\right] v_t\right)\right] \\ &= -E\left[Y_t\left(\left[\frac{LB(L)}{L-\lambda}\right] D\phi_t(0)\right)\right] \end{aligned} \tag{10.5.7}$$

where the first equality follows because future values of v_t are orthogonal to Y_t and the second equality follows from formula (10.5.2) for $D\phi_t(0)$. Substituting (10.5.7) into (10.5.3) and using formula (10.4.2) for B^* results in:

$$\begin{aligned} d\beta(0)/d\alpha &= -E\left[Y_t\left(\left[\frac{\lambda B(\lambda)}{L-\lambda}\right] D\phi_t(0)\right)\right] \\ &= -\lambda B(\lambda) E\left[\left(\left[\frac{1}{L^{-1}-\lambda}\right] Y_t\right) D\phi_t(0)\right] \\ &= -\lambda B(\lambda) E\left[\left(\left[\frac{1}{1-\lambda L}\right] Y_{t-1}\right) D\phi_t(0)\right] \end{aligned} \tag{10.5.8}$$

where the second equality follows from the joint stationarity of the composite process $\{[Y_t, D\phi_t(0)]\}$. Formula (10.5.8) is almost what we want, except that we need an expression in terms of s_t instead of $D\phi_t(0)$. This can be obtained by noting that

$$E\left[\left(\left[\frac{1}{1-\lambda L}\right]Y_{t-1}\right)D\phi_t(0)\right] = -E\left[v_t\left(\left[\frac{1}{1-\lambda L}\right]Y_{t-1}\right)s_t\right] \qquad (10.5.9)$$

which can be verified as follows. Compute the expectation on the right-hand side by conditioning first on K_{t-1} and using the two facts that (i) $D\phi_t(0)$ is the sum of a term in v_t and $E[D\phi_t(0)|K_{t-1}]$, and (ii) the third moment of v_t is zero. Then apply the Law of Iterated Expecations again to obtain the left-hand side of (10.5.9). Combining (10.5.8) and (10.5.9), we obtain the desired formula:

$$d\beta(0)/d\alpha = \lambda B(\lambda)E\left[v_t\left(\left[\frac{1}{1-\lambda L}\right]Y_{t-1}\right)s_t\right]. \qquad (10.5.10)$$

Armed with this formula, we can think of the time t contribution of the "correction term" for estimating B as the outcome from running a least squares regression of $\lambda B(\lambda)v_t\left(\left[\frac{1}{1-\lambda L}\right]Y_{t-1}\right)$ onto the score s_t. This interpretation can be seen by substituting (10.5.7) and (10.5.10) into (10.5.4) and interpreting $\frac{d\beta(0)/d\alpha}{E({s_t}^2)}$ as a population regression coefficient. Although we performed this computation for an affine scalar parameterization of B, it can be mimicked for any sufficiently smooth one dimensional parameterization. The correction term will continue be interpretable as a regression score.

10.6. Multidimensional Parameterizations of B

As a further step in studying the impact on β_o of using a nonparametric estimator of B, we now briefly consider what happens when we use parameterizations that have more than one dimension but are still finite dimensional. This turns out to be an easy extension of our previous analysis. Let s_t be the score vector associated with any such nondegenerate parameterization. (By "nondegenerate" we simply mean that the second moment of the score vector is nonsingular, a local identification condition.) The entries of the score vector s_t can be represented as in (14), and the derivative matrix $\partial\beta(0)/\partial\alpha'$ is given by the expected cross product:

$$\partial\beta(0)/\partial\alpha' = \lambda B(\lambda)E\left[v_t\left(\left[\frac{1}{1-\lambda L}\right]Y_{t-1}\right){s_t}'\right].$$

Therefore,

$$\sqrt{T}(\beta_T - \beta_o) \approx (1/\sqrt{T}) \sum_{t=1}^{T} [Y_t B^*(L) v_t - \beta_o] \\ + \lambda B(\lambda) E\left[v_t \left(\left[\frac{1}{1-\lambda L} \right] Y_{t-1} \right) {s_t}' \right] [E(s_t {s_t}')]^{-1} (1/\sqrt{T}) \sum_{t=1}^{T} s_t.$$

Again the correction term for the first stage estimation of B has a regression interpretation: regress $\lambda B(\lambda) v_t \left(\left[\frac{1}{1-\lambda L} \right] Y_{t-1} \right)$ onto the score *vector* s_t.

10.7. Nonparametric Estimation of B

Since the derivative matrix has an expected cross product representation for any finite dimensional parameterization, we can use an insight from Stein (1956) and Levit (1975), developed more fully by Van der Vaart (1991) and Newey (1993), to deduce the asymptotic distribution when B is estimated nonparametrically. We simply ask what happens to the population regression of $\lambda B(\lambda) v_t \left(\left[\frac{1}{1-\lambda L} \right] Y_{t-1} \right)$ onto the linear space of time t scores for all possible parameterizations of B. Since the elements of the regressand can be viewed as scores of hypothetical parameterizations, the resulting limiting distribution for β_o is

$$\sqrt{T}(\beta_T - \beta_o) \approx \sum_{t=1}^{T} \left[Y_t B^*(L) v_t - \beta_o + v_t \left(\left[\frac{1}{1-\lambda L} \right] Y_{t-1} \right) \right]. \qquad (10.7.1)$$

This additive decomposition gives a time series counterpart to the "correction terms" for semiparametric M-estimators derived by Newey (1993), (*e.g.*, see formula (3.10) in Newey).[1]

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[1] One diffference between Newey's derivation and ours is that Newey works with score vectors for the entire process of observables. Given the additive structure of our model we can work with the simpler scores of maximum likelihood estimators of B using only data on $\{y_t\}$.

10.8. Back to the Adjustment Cost Model

Let us now revert to the estimation problem of interest posed using the adjustment cost model. We let β_o denote the parameter vector governing the endogenous dynamics and view $\rho, \psi_o, \psi_f, \delta$ and ρ as functions of the unknown parameter vector β_o. When the capital stock is not directly observable, the generated stock sequence will also depend on β_o through its dependence on the depreciation factor δ. Suppose that we estimate β_o using a GMM estimator that exploits the unconditional moment restriction:

$$E[Y_t(f_t + e_t)] = 0.$$

Then the usual GMM inference works with an additional correction term in which the derivatives of the moment conditions are computed by differentiating with respect to β and evaluating these derivatives at the true value of β_o and B. Let this derivative be denoted d_o, and let a_o denote the limiting matrix that selects the moment conditions to be used in estimation. Then

$$\sqrt{T}(\beta_T - \beta_o) \approx -(a_o d_o)^{-1} a_o (1/\sqrt{T} \sum_{t=1}^{T} \left[Y_t(f_t + e_t) + \lambda B(\lambda) \left(\left[\frac{1}{1 - \lambda L} \right] Y_{t-1} \right) v_t \right].$$

Chapter 11
Representation of Demand

11.1. Introduction

This chapter derives demand schedules from our preference specification

$$\begin{aligned} h_t &= \Delta_h h_{t-1} + \Theta_h c_t \\ s_t &= \Lambda h_{t-1} + \Pi c_t \end{aligned} \tag{11.1.1}$$

with preference shock $b_t = U_b z_t$. An equivalence class of preferences $(\Delta_h, \Theta_h, \Pi, \Lambda, U_b)$ give rise to identical demand schedules. Among such preferences, particular ones that we call *canonical* are easiest to work with.

We apply the concept of canonical representation of preferences to a version of Becker and Murphy's model of rational addiction. The chapter also uses demand curves to to construct partial equilibrium interpretations of our models. This chapter sets the stage for the studies of aggregation in chapters BLANK and BLANK.

11.2. Canonical Representations of Services

We begin with a definition.

DEFINITION: A representation of a household service technology $(\Delta_h, \Theta_h, \Pi, \Lambda, U_b)$ is said to be *canonical* if it satisfies the following two requirements:

i. Π is nonsingular.
ii. The absolute values of the eigenvalues of $(\Delta_h - \Theta_h \Pi^{-1} \Lambda)$ are strictly less than $1/\sqrt{\beta}$.

A canonical household service technology maps any given service process $\{s_t\}$ in L_0^2 into a corresponding consumption process $\{c_t\}$ for which the implied household capital stock process $\{h_t\}$ is also in L_0^2. To verify this, we use the canonical representation to obtain a recursive representation for the consumption process in terms of the service process:

$$\begin{aligned} c_t &= -\Pi^{-1}\Lambda h_{t-1} + \Pi^{-1}s_t \\ h_t &= (\Delta_h - \Theta_h\Pi^{-1}\Lambda)h_{t-1} + \Theta_h\Pi^{-1}s_t \end{aligned} \tag{11.2.1}$$

The restriction on the eigenvalues of the matrix $(\Delta_h - \Theta_h\Pi^{-1}\Lambda)$ keeps the household capital stock $\{h_t\}$ in L_0^2.

11.3. Dynamic Demand Functions for Consumption Goods

We postpone constructing a canonical representation, and proceed immediately to use one to construct a dynamic demand schedule. In Chapter 6 we derived the following first-order conditions for the household's optimization problem:

$$s_t = b_t - \mu_t^s \tag{11.3.1}$$

$$\Pi'\mu_t^s = -\Theta_h'\mu_t^h + \mu_0^w p_t^0 \tag{11.3.2}$$

$$\mu_t^h = \beta E_t(\Lambda'\mu_{t+1}^s + \Delta_h'\mu_{t+1}^h). \tag{11.3.3}$$

As a prelude to computing demand for consumption, we compute the demand for services. Our strategy is to use (11.3.2) and (11.3.3) to solve for the multiplier μ_t^s, and then to substitute this solution into (11.3.1). Shift (11.3.2) forward one time period and solve (11.3.2) for μ_{t+1}^s. Substitute this expression into (11.3.3):

$$\mu_t^h = \beta E_t(-\Lambda'\Pi^{-1\prime}\Theta_h'\mu_{t+1}^h + \mu_0^w\Lambda'\Pi^{-1\prime}p_{t+1}^0 + \Delta_h'\mu_{t+1}^h). \tag{11.3.4}$$

Solve (11.3.4) forward to obtain:

$$\mu_t^h = \mu_0^w E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h' - \Lambda'\Pi^{-1\prime}\Theta_h')^{\tau-1}\Lambda'\Pi^{-1\prime}p_{t+\tau}^0. \tag{11.3.5}$$

Because we are using a canonical household service technology, the infinite sum on the right side of (11.3.5) converges (in L_0^2). Therefore, the service demand can be expressed as

$$s_t = b_t - \mu_0^w \rho_t^0 \tag{11.3.6}$$

where

$$\rho_t^0 \equiv \Pi^{-1\prime}\Big[p_t^0 - \Theta_h' E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h' - \Lambda' \Pi^{-1\prime} \Theta_h')^{\tau-1} \Lambda' \Pi^{-1\prime} p_{t+\tau}^0\Big]. \tag{11.3.7}$$

Equations (11.3.6) and (11.3.7) represent the service demands in terms of expected future prices of the consumption good. The random vector ρ_t^0 is the implicit rental price for services expressed in terms of current and expected future prices of consumption goods. Equation (11.2.1) transforms $\{s_t\}$ in L_0^2 into $\{c_t\}$ in L_0^2.

11.3.1. The multiplier μ_0^w

The service demands given in (11.3.6) depend on the endogenous scalar multiplier μ_0^w. To compute μ_0^w, we partition the household capital and service sequences into two components. One component is a service sequence obtained from the *initial* endowment of household capital. The other component is the service sequence obtained from *market purchases* of consumption goods. The service sequence $\{s_{i,t}\}$ obtained from the initial endowment of household capital evolves according to:

$$\begin{aligned} s_{i,t} &= \Lambda h_{i,t-1} \\ h_{i,t} &= \Delta_h h_{i,t-1} \end{aligned} \tag{11.3.8}$$

where $h_{i,-1} = h_{-1}$. The service sequence $\{s_{m,t}\}$ obtained from purchases of consumption satisfies:

$$s_{m,t} = b_t - s_{i,t} - \mu_0^w \rho_t^0. \tag{11.3.9}$$

We can compute the time zero cost of the sequence $\{s_{m,t}\}$ in one of two equivalent ways. One way is to compute the time zero cost of the consumption sequence $\{c_t\}$ needed to support the service demands using the price sequence

$\{p_t^0\}$. Another way is to use the implicit rental sequence $\{\rho_t^0\}$ directly to compute the time zero costs of $\{s_{m,t}\}$. In the appendix to this chapter, we verify that the two measures of costs agree:

$$E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot s_{m,t} = E_0 \sum_{t=0}^{\infty} \beta^t p_t^0 \cdot c_t. \tag{11.3.10}$$

It is reasonable that, starting from $h_{-1} = 0$, the value of services equals the value of the associated consumption stream.

It follows from (11.3.9) that

$$E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot s_{m,t} = E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot (b_t - s_{i,t}) - \mu_0^w E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot \rho_t^0. \tag{11.3.11}$$

Substitute (11.3.10) and (11.3.11) into the consumer's budget constraint (6.2), and solve for the time zero marginal utility of wealth μ_0^w:

$$\mu_0^w = \frac{E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot (b_t - s_{i,t}) - W_0}{E_0 \sum_{t=0}^{\infty} \beta^t \rho_t^0 \cdot \rho_t^0}, \tag{11.3.12}$$

where W_0 denotes initial period wealth given by

$$W_0 = E_0 \sum_{t=0}^{\infty} \beta^t (w_t^0 \ell_t + \alpha_t^0 \cdot d_t) + v_0 \cdot k_{-1}. \tag{11.3.13}$$

Taken together, (11.3.6), (11.3.7), (11.3.12) and (11.3.13) give the demand functions for consumption services. A recursive representation for the dynamic demand function for consumption goods is obtained by substituting for s_t in (11.2.1).

11.3.2. Dynamic Demand System

Substituting (11.3.6) and (11.3.7) into (11.2.1) gives

$$\begin{aligned} c_t &= -\Pi^{-1}\Lambda h_{t-1} + \Pi^{-1} b_t - \Pi^{-1}\mu_0^w E_t\{\Pi'^{\,-1} - \Pi'^{\,-1}\Theta_h' \\ &\qquad [I - (\Delta_h' - \Lambda'\Pi'^{\,-1}\Theta_h')\beta L^{-1}]^{-1}\Lambda'\Pi'^{-1}\beta L^{-1}\} p_t^0 \qquad (11.3.14) \\ h_t &= \Delta_h h_{t-1} + \Theta_h c_t \end{aligned}$$

Equation system (11.3.14) can be regarded as a system of dynamic demand functions for consumption, which express consumption demand at date t as a function of future scaled Arrow-Debreu prices p_t^0 and, as mediated through the state variable h_{t-1}, past values of consumption.

11.3.3. Foreshadow of Gorman aggregation

In the chapter 12, we shall explore how the dynamic demand schedule for consumption goods opens up the possibility of satisfying Gorman's (1953) conditions for aggregation in a heterogeneous consumer version of the model. The first equation of (11.3.14) amounts to an *Engle curve* for consumption that is linear in individual wealth with a coefficient on μ_o^w (which depends on wealth)[1] that only depends on prices. In a model of consumers who have the same household technologies $(\Delta_h, \Theta_h, \Lambda, \Pi)$ but possibly different preference shock processes, the coefficient on wealth is the same for all consumers. Gorman showed that when Engel curves satisfy this property, there exists a unique community or aggregate preference ordering over aggregate consumption that is independent of the distribution of wealth. This property will be exploited in chapter 12 when we solve for the equilibrium of a multiple consumer version of our economy. The community dynamic demand schedule for a heterogeneous agent economy will be obtained by summing the individual Engel curves.

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[1] Through (11.3.12) the multiplier μ_0^w depends on wealth in an affine relationship.

11.4. Computing Canonical Representations

In deriving a dynamic demand function, we assumed that the representation of the household service technology is canonical. Now we start with a preference shock process $\{b_t\}$ and a specification of $(\Delta_h, \Theta_h, \Lambda, \Pi)$ that is *not* necessarily canonical and show how to find a canonical representation. In the appendix, we establish that for any $(\Delta_h, \Theta_h, \Lambda, \Pi)$, there exists a *canonical* service technology $(\Delta_h, \Theta_h, \hat{\Lambda}, \hat{\Pi})$ and accompanying preference shock process $\{\hat{b}_t\}$ that induces an identical preference ordering over consumption. In the text, we display the mechanics of how to compute the canonical technology and associated preference shock process, assigning the technical details to the appendix.[2] These mechanics are closely related to mathematics of innovations representations.

11.4.1. Heuristics

We study two polynomials in the lag operator L:

$$\sigma(L) = \Pi + \Lambda L[I - \Delta_h L]^{-1}\Theta_h$$
$$\hat{\sigma}(L) = \hat{\Pi} + \hat{\Lambda} L[I - \Delta_h L]^{-1}\Theta_h .$$

As explained in the appendix, when $c_t = 0 \forall t < 0$, applying the operator $\sigma(L)$ to c_t gives s_t, so that $s_t = \sigma(L)c_t$. For two household technologies $[\Delta_h, \Theta_h, \Pi, \Lambda]$ and $[\hat{\Delta}_h, \Theta_h, \hat{\Pi}, \hat{\Lambda}]$ to give rise to the same preference ordering over $\{c_t\}$ it is necessary that

$$\sigma(\beta^{.5}L^{-1})'\sigma(\beta^{.5}L) = \hat{\sigma}(\beta^{.5}L^{-1})'\hat{\sigma}(\beta^{.5}L).$$

If the $[\hat{\Lambda}, \hat{\Pi}]$ technology is to be canonical, it is necessary that $\hat{\sigma}(\beta^{.5}L)$ be *invertible*.

In the appendix, we verify the following version of the *factorization identity*:

$$[\Pi + \beta^{1/2}L^{-1}\Lambda(I - \beta^{1/2}L^{-1}\Delta_h)^{-1}\Theta_h]'[\Pi + \beta^{1/2}L\Lambda(I - \beta^{1/2}L\Delta_h)^{-1}\Theta_h]$$
$$= [\hat{\Pi} + \beta^{1/2}L^{-1}\hat{\Lambda}(I - \beta^{1/2}L^{-1}\Delta_h)^{-1}\Theta_h]'[\hat{\Pi} + \beta^{1/2}L\hat{\Lambda}(I - \beta^{1/2}L\Delta_h)^{-1}\Theta_h],$$

where $[\hat{\Lambda}, \hat{\Lambda}]$ satisfy (10.16), (10.19), and (10.20) below. As part of the factorization identity, it is proved that the $[\hat{\Lambda}, \hat{\Pi}]$ representation satisfies both of the requirements to achieve the status of a *canonical* representation. Thus, to attain a canonical household technology, we have to implement this factorization. We can do this by solving a control problem.

[2] The MATLAB program `canonpr.m` computes a canonical representation.

11.4.2. An auxiliary problem that induces a canonical representation

An artificial optimization problem and the associated optimal linear regulator facilitate computing a canonical representation. Thus, confront a household with the optimization problem: choose $\{c_t\}_{t=0}^{\infty}$ to maximize

$$-.5\sum_{t=0}^{\infty}\beta^t(s_t-b_t)\cdot(s_t-b_t)$$

subject to

$$\begin{aligned} h_t &= \Delta_h h_{t-1}+\Theta_h c_t \\ s_t &= \Lambda h_{t-1}+\Pi c_t. \end{aligned}$$

The recursive solution to this optimization problem contains all of the ingredients for a canonical service technology.

This optimization problem is a version of one a household confronts in a competitive equilibrium, except that we have eliminated the budget constraint. For a canonical technology, the solution to this optimization problem is trivial: choose $\{c_t\}$ so that the implied service sequence matches the preference shock sequence, $s_t = b_t \;\forall\, t$. However, when the service technology is not canonical, it might not be feasible to construct a consumption process that attains that goal, in which case the optimization problem is not trivial.

We simplify the household optimization problem further by initially setting the preference shock process to zero for all $t \geq 0$. In making this simplification, we are exploiting the fact that for the optimal linear regulator problem, the feedback part of the decision rule can be computed independently of the feedforward part, and that the $\{b_t\}$ process influences only the feedforward part. In this optimization problem it is feasible to *stabilize* the state vector $\{h_t\}$ so that it satisfies the square summability requirement. For instance, one can set the consumption process to zero for all $t \geq 0$. So long as it is also *optimal* to stabilize the household capital stock process, it is known that there will be a unique positive semidefinite matrix P satisfying the algebraic Riccati equation:[3]

$$\begin{aligned} P = \Lambda'\Lambda + \beta\Delta_h' P\Delta_h - (\beta\Delta_h' P\Theta_h + \Lambda'\Pi) \\ (\Pi'\Pi+\beta\Theta_h' P\Theta_h)^{-1}(\beta\Theta_h' P\Delta_h+\Pi'\Lambda). \end{aligned} \tag{11.4.1}$$

[3] We require that assumption A1 and the stability theorem of chapter 9 apply to this control problem.

The optimal choice of consumption can be represented as

$$c_t = -(\Pi'\Pi + \beta\Theta_h' P\Theta_h)^{-1}(\beta\Theta_h' P\Delta_h + \Pi'\Lambda)h_{t-1}. \tag{11.4.2}$$

When this optimal rule is implemented, the evolution equation for the household capital stock is

$$h_t = [\Delta_h - \Theta_h(\Pi'\Pi + \beta\Theta_h' P\Theta_h)^{-1}(\beta\Theta_h' P\Delta_h + \Pi'\Lambda)]h_{t-1}, \tag{11.4.3}$$

where the eigenvalues of the matrix multiplying h_{t-1} are strictly less than $1/\sqrt{\beta}$.[4] With this in mind, we choose $\hat{\Pi}$ and $\hat{\Lambda}$ so that

$$\hat{\Pi}^{-1}\hat{\Lambda} = (\Pi'\Pi + \beta\Theta_h' P\Theta_h)^{-1}(\beta\Theta_h' P\Delta_h + \Pi'\Lambda). \tag{11.4.4}$$

For this choice, condition (ii) for a canonical service technology is met.

We still have to construct $\hat{\Pi}$. In the appendix, it is shown as an implication of the factorization identity that we should set $\hat{\Pi}$ to be a factor of the symmetric positive definite matrix $(\Pi'\Pi + \beta\Theta_h' P\Theta_h)$:

$$(\Pi'\Pi + \beta\Theta_h' P\Theta_h) = \hat{\Pi}'\hat{\Pi}. \tag{11.4.5}$$

Any factorization will work so long as $\hat{\Pi}$ is a square matrix. Since $(\Pi'\Pi + \beta\Theta_h' P\Theta_h)$ is nonsingular, $\hat{\Pi}$ satisfies condition (i) for a canonical representation.

In summary, (11.4.1), (11.4.4), and (11.4.5) compute a $(\hat{\Pi}, \hat{\Lambda})$ that corresponds to a canonical representation. The service process $\{\hat{s}_t\}$ for this new household technology satisfies:

$$\hat{s}_t = \hat{\Lambda} h_{t-1} + \hat{\Pi} c_t. \tag{11.4.6}$$

We also need to construct a preference shock process to accompany the canonical service technology. One way to do this is simply to reintroduce the preference shock process $\{b_t\}$ into the auxiliary household optimization problem, and recompute the optimal decision rule for consumption. The decision rule can be represented as:

$$c_t = -(\hat{\Pi})^{-1}\hat{\Lambda} h_{t-1} + (\hat{\Pi})^{-1}\hat{U}_b z_t \tag{11.4.7}$$

[4] This condition on the eigenvalues of the 'closed loop system' follows from the assumption that it is optimal to stabilize the system (i.e., that the system is detectable).

for some matrix $\hat{U}_b$. As discussed in chapter 4, the *feedback* portion of this decision rule $[(\hat{\Pi})^{-1}\hat{\Lambda}]$ is the same as for the problem in which the preference shock process was set to zero. The feedforward part $[(\hat{\Pi})^{-1}\hat{U}_b]$ can be computed using the method described in chapter 4, which permits the optimal decision rule to be calculated efficiently in two steps. Using those methods, the shock process associated with the canonical service technology is

$$\hat{b}_t = \hat{U}_b z_t. \tag{11.4.8}$$

An alternative method for computing $\{\hat{b}_t\}$ is more useful and revealing. As shown in the appendix to this chapter, two preference representations having the same demand functions give rise to the same preference ordering over consumption paths. Therefore, the marginal utilities are also the same across the two preference representations, and in particular across the two specifications of household technologies and preference shock processes. Equality between the indirect marginal utility of consumption and the current and expected future marginal utilities of consumption services and (11.3.1), (11.3.2), and (11.3.3) implies that the two preference shock processes must satisfy:

$$\Pi' b_t + \Theta_h' E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h')^{\tau-1} \Lambda' b_{t+\tau} = \hat{\Pi}' \hat{b}_t + \Theta_h' E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h')^{\tau-1} \hat{\Lambda}' \hat{b}_{t+\tau}. \tag{11.4.9}$$

Let the left side of (11.4.9) be denoted $\tilde{b}_t$ for each t. Since the $(\hat{\Lambda}, \hat{\Pi})$ technology is canonical, it follows that we can solve (11.4.9) for $\hat{b}_t$:

$$\hat{b}_t = \hat{\Pi}^{-1\prime} \tilde{b}_t - \hat{\Pi}^{-1\prime} \Theta_h' E_t \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h' - \hat{\Lambda}' \hat{\Pi}^{-1\prime} \Theta_h')^{\tau-1} \hat{\Lambda}' \hat{\Pi}^{-1\prime} \tilde{b}_{t+\tau}. \tag{11.4.10}$$

Relation (11.4.10) is derived by applying an operator identity to equation (11.4.9).

11.5. Operator Identities

For canonical household technologies a matrix identity is

$$[\Pi + \Lambda(I - \Delta_h L)^{-1}\Theta_h L]^{-1} = \{\Pi^{-1} - \Pi^{-1}\Lambda[I - (\Delta_h - \Theta_h \Pi^{-1}\Lambda)L]^{-1}\Theta_h \Pi^{-1} L\}.$$

The identity shows that for canonical representations of preferences $(\Delta_h, \Theta_h, \Pi, \Lambda, U_b)$, there are two equivalent ways of expressing the mapping between sequences $\{s_t\}$ and sequences $\{c_t\}$. To establish the identity, assume that $h_{-1} = 0$, or equivalently that $c_t = 0 \ \forall\ t < 0$. Note that the second equation of representation (11.2.1) implies

$$h_t = [I - (\Delta_h - \Theta_h \Pi^{-1}\Lambda)L]^{-1}\Theta_h \Pi^{-1} s_t.$$

Lagging this one period and substituting into the first equation of (11.2.1) gives

$$c_t = \{\Pi^{-1} - \Pi^{-1}\Lambda[I - (\Delta_h - \Theta_h \Pi^{-1}\Lambda)L]^{-1}\Theta_h \Pi^{-1} L\} s_t.$$

This equation shows how to obtain sequences $\{c_t\} \in L_0^2$ that are associated with arbitrary sequences $\{s_t\} \in L_0^2$. Now recall that the household technology implies

$$s_t = [\Pi + \Lambda(I - \Delta_h L)^{-1}\Theta_h L] c_t,$$

which expresses $\{s_t\} \in L_0^2$ as a function of $\{c_t\} \in L_0^2$. The assumption that (Λ, Π) is *canonical* implies that the operator $[\Pi + \Lambda(I - \Delta_h L)^{-1}\Theta_h L]$ mapping sequences from L_0^2 into L_0^2 is invertible, which implies the identity.

Here is how to derive the 'dual' or transposed version of the identity, which is the one used to get (11.4.10). Use (11.3.3) to deduce

$$\mu_t^h = (I - \beta\Delta_h' L^{-1})^{-1}\beta\Lambda' L^{-1}\mu_{t+1}^s.$$

Then use (11.3.2) to deduce

$$(\dagger) \qquad \mu_0^w p_t = [\Pi' + \Theta_h'(I - \beta\Delta_h' L^{-1})^{-1}\beta L^{-1}]\mu_t^s.$$

Alternatively, solve (11.3.2) for μ_t^s,

$$\mu_t^s = \Pi'^{-1}(-\Theta_h' \mu_t^h + \mu_0^w p_t^0).$$

Substitute this into (11.3.3) to get

$$(\ddagger) \qquad \mu_t^s = \{\Pi'^{-1} - \Pi'^{-1}\Theta_h'[I - (\Delta_h' - \Lambda'\Pi'^{-1}\Theta_h')\beta L^{-1}]^{-1}\Lambda'\Pi'^{-1}\beta L^{-1}\}\mu_0^w p_t^0.$$

When $(\Delta_h, \Theta_h, \Pi, \Lambda)$ is canonical, the operator on the right side of $(\dagger)$ has an inverse equal to the operator on the right side of $(\ddagger)$:

$$[\Pi' + \Theta_h'(I - \beta\Delta_h' L^{-1})^{-1}\beta L^{-1}]^{-1} = \{\Pi'^{-1} - \Pi'^{-1}\Theta_h'[I - (\Delta_h' - \Lambda'\Pi'^{-1}\Theta_h')\beta L^{-1}]^{-1}\Lambda'\Pi'^{-1}\beta L^{-1}\}.$$

In the appendix to this chapter, we use Fourier transforms to show that the alternative service technology $(\hat{\Lambda}, \hat{\Pi})$ and preference shock process $\{\hat{b}_t\}$ induce the same preference ordering for consumption goods as did the original ones.

11.6. Becker-Murphy Model of Rational Addiction

We illustrate our analysis with a discrete-time version of the habit-persistence model advocated by Becker and Murphy (1988). The household technology is a parametric version of induced preferences for consumption of the form suggested by Pollak (1970), Ryder and Heal (1973), and Stigler and Becker (1977). The household technology has a single consumption good, two services, and a single household capital stock. The household capital measures a *habit* stock constructed to be a geometrically-weighted average of current and past consumptions:

$$h_t = \delta_h h_{t-1} + (1-\delta_h)c_t, \tag{11.6.1}$$

where $0 < \delta_h < 1$. The first service is proportional to consumption, and the second one is a linear combination of consumption and the habit stock:

$$s_t = \begin{bmatrix} \pi_1 & 0 \\ \pi_2 & \pi_3 \end{bmatrix} \begin{bmatrix} c_t \\ h_t \end{bmatrix}. \tag{11.6.2}$$

We normalize π_1 and π_3 to be strictly positive. Imagine for a moment that c_t and h_t are distinct consumptions goods and that there is no intertemporal connection between them. Then recall from our discussion of preferences for multiple consumption goods in Chapter 3, that the Frisch classification of complements is equivalent to requiring π_2 to be negative.

In light of the evolution equation (11.6.1) for the household capital stock, this notion of complementarity is limiting because it ignores the fact that h_t is a weighted average of current and past consumptions. For this reason, we consider a related notion of complementarity referred to by Ryder and Heal (1973) and Becker and Murphy (1988) as *adjacent complementarity.* Substituting (11.6.1) into (11.6.2) we obtain the following service technology:

$$s_t = \Lambda h_{t-1} + \Pi c_t, \tag{11.6.3}$$

where

$$\Lambda = \begin{bmatrix} 0 \\ \pi_3\delta_h \end{bmatrix} \quad \text{and} \quad \Pi = \begin{bmatrix} \pi_1 \\ \pi_2 + \pi_3(1-\delta_h) \end{bmatrix}.$$

Service technology (11.6.3) is clearly not canonical: simply note that two services are constructed from one underlying consumption good, so we cannot construct a consumption sequence to support any hypothetical admissible service sequence.

To capture the notion of *adjacent complementarity*, we consider a canonical representation for household services. The canonical household service technology has a single service and can be expressed as:

$$\hat{s}_t = \hat{\Lambda} h_{t-1} + \hat{\Pi} c_t, \tag{11.6.4}$$

where $\{\hat{s}_t\}$ is a scalar service process and $(\hat{\Lambda}, \hat{\Pi})$ satisfies:

$$\mid \delta_h - (1-\delta_h)\hat{\Lambda}/\hat{\Pi} \mid < 1/\sqrt{\beta}. \tag{11.6.5}$$

We normalize the scalar $\hat{\Pi}$ to be positive so that increases in time t consumption increase the time t canonical service $\hat{s}_t$. When specialized to this parametric model, Ryder and Heal's (1973) notion of *adjacent complementarity* becomes the restriction that $\hat{\Lambda}$ must be negative. In this case, (11.6.5) implies that

$$0 \leq \delta_h - (1-\delta_h)\hat{\Lambda}/\hat{\Pi} \leq 1/\sqrt{\beta}. \tag{11.6.6}$$

As shown by Becker and Murphy (1988), adjacent complementarity ($\hat{\Lambda} \leq 0$) implies that $\pi_2 \leq 0$. The converse is not true, however. To see the relation between $\hat{\Lambda}$ and π_2, multiply both sides of (9.64) by $(1-\beta^{1/2}\zeta^{-1}\delta_h)(1-\beta^{1/2}\zeta\delta_h)$ to obtain:

$$\begin{gathered}
\Pi'\Pi\,(1-\beta^{1/2}\zeta^{-1}\delta_h)(1-\beta^{1/2}\zeta\delta_h) + \beta\Lambda'\Lambda(1-\delta_h)^2 + \\
\beta^{1/2}\zeta^{-1}(1-\delta_h)(1-\beta^{1/2}\zeta\delta_h)\Lambda'\Pi + \beta^{1/2}\zeta(1-\delta_h)(1-\beta^{1/2}\zeta^{-1}\delta_h)\Lambda'\Pi \\
= \hat{\Pi}^2(1-\beta^{1/2}\zeta^{-1}\delta_h)(1-\beta^{1/2}\zeta\delta_h) + \beta\hat{\Lambda}\hat{\Lambda}(1-\delta_h)^2 + \\
\beta^{1/2}\zeta^{-1}(1-\delta_h)(1-\beta^{1/2}\zeta\delta_h)\hat{\Lambda}\hat{\Pi} + \beta^{1/2}\zeta(1-\delta_h)(1-\beta^{1/2}\zeta^{-1}\delta_h)\hat{\Lambda}\hat{\Pi}.
\end{gathered} \tag{11.6.7}$$

This equality holds for all ζ except $\zeta = 0$. Evaluate both sides of (11.6.7) at $\zeta = \beta^{1/2}\delta_h$:

$$\begin{aligned}
&\beta\Lambda'\Lambda(1-\delta_h)^2 + (1-\delta_h)(1-\beta\delta_h^2)\Lambda'\Pi/\delta_h \\
&\quad = \beta\hat{\Lambda}^2(1-\delta_h)^2 + (1-\delta_h)(1-\beta\delta_h^2)\hat{\Lambda}\hat{\Pi}/\delta_h.
\end{aligned} \tag{11.6.8}$$

The right side of (11.6.8) can be expressed as

$$\begin{aligned}
&\beta\hat{\Lambda}^2(1-\delta_h)^2 + (1-\delta_h)(1-\beta\delta_h^2)\hat{\Lambda}\hat{\Pi}/\delta_h \\
&\quad = \beta(1-\delta_h)\hat{\Lambda}\hat{\Pi}\{[(1-\delta_h)\hat{\Lambda}/\hat{\Pi} - \delta_h] + (1/\beta\delta_h)\}.
\end{aligned} \tag{11.6.9}$$

Since $\hat{\Lambda}\hat{\Pi} < 0$ and inequality (11.6.6) is satisfied, it follows that

$$\begin{aligned}\beta(1-\delta_h)\hat{\Lambda}\hat{\Pi}\{[(1-\delta_h)\hat{\Lambda}/\hat{\Pi}-\delta_h]+(1/\beta\delta_h)\} \\ \leq \beta(1-\delta_h)\hat{\Lambda}\hat{\Pi}[-1/\sqrt{\beta}+(1/\beta\delta_h)] \\ \leq 0.\end{aligned} \tag{11.6.10}$$

Combining (11.6.10) and (11.6.8), we have that if $\hat{\Lambda} \leq 0$, then

$$\beta\Lambda'\Lambda(1-\delta_h)^2 + (1-\delta_h)(1-\beta\delta_h^2)\Lambda'\Pi/\delta_h \leq 0. \tag{11.6.11}$$

Inequality (11.6.11) is satisfied only when $\Lambda'\Pi \leq 0$. This in turn requires that $\pi_2 \leq 0$ because

$$\begin{gathered}\Lambda'\Pi = \pi_3\delta_h[\pi_2 + \pi_3(1-\delta_h)], \\ 0 < \delta_h < 1 \quad \text{and} \quad \pi_3 > 0.\end{gathered} \tag{11.6.12}$$

Inequality (11.6.6) permits $\delta_h - (1-\delta_h)\hat{\Pi}/\hat{\Lambda}$ to exceed one. In this case, growth in consumption is required to support most constant service sequences, although this growth will be dominated by $\{\beta^{t/2} : t = 0, 1, \ldots\}$. This is a household technology with an *extreme* form of *addiction* to the consumption good. Note that

$$\delta_h - (1-\delta_h)\hat{\Lambda}/\hat{\Pi} = \delta_h(1 + \hat{\Lambda}/\hat{\Pi}) - \hat{\Lambda}/\hat{\Pi}. \tag{11.6.13}$$

Therefore, instability is implied whenever $-\hat{\Lambda}$ exceeds $\hat{\Pi}$ in the canonical household service technology.

A. Fourier transforms

This appendix applies Fourier transforms to establish some key equalities asserted in the text. We begin with some background on 0.

11.A.1. Primer on transforms

For a two-sided scalar sequence $\{c_j\}_{j=-\infty}^{\infty}$, the $z-$transform is defined as the complex valued function

$$c(z) = \sum_{j=-\infty}^{\infty} c_j z^j,$$

where z is a scalar complex number.[5] The inversion formula asserts

$$c_k = \frac{1}{2\pi i} \int_{\Gamma} c(z) z^{-k-1} dz$$

where Γ is any closed contour around zero in the complex plane, and the integration is complex integration counterclockwise along the path Γ. If we take Γ to be the unit circle and set $z = e^{-i\omega}$, we get the following version of the inversion formula

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} c(e^{-i\omega}) e^{i\omega k} d\omega.$$

We denote transform pairs with the notation

$$\{c_k\} \leftrightarrow c(z).$$

The *convolution* of two sequences $\{y_k\}, \{x_k\}$, is denoted $\{y * x\}$ and is defined as

$$\{y * x\}_{k=-\infty}^{\infty} \equiv \{ \sum_{s=-\infty}^{\infty} y_s x_{k-s} \}_{k=-\infty}^{\infty}.$$

Direct calculations establish the *convolution property*

$$\{y * x\}_{k=-\infty}^{\infty} \leftrightarrow x(z) y(z).$$

We have the linearity property that for any scalars (a, b)

$$a\{x_k\} + b\{y_k\} \leftrightarrow a x(z) + b y(z).$$

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[5] For descriptions of Fourier and z-transforms, see Gabel and Roberts (1973), Liu and Liu (19**). For some of their uses in time series economics see Nerlove, Grether and Carvalho (1979) and Sargent (1979).

11.A.2. Time reversal and Parseval's formula

Let $\tilde{c}_{-k} = c_k$ for all k. Then $\{\tilde{c}_k\}_{k=-\infty}^{\infty}$ has transform

$$\tilde{c}(z) = \sum_{k=-\infty}^{\infty} \tilde{c}_k z^k = \sum_{k=-\infty}^{\infty} c_k z^{-k} = c(z^{-1}).$$

Applying the convolution theorem to $c(z)c(z^{-1})$ gives

$$c(z)c(z^{-1}) \leftrightarrow \{\sum_{s=-\infty}^{\infty} c_s c_{s-k}\}_{k=-\infty}^{\infty}.$$

Applying the inversion formula gives

$$\sum_{s=-\infty}^{\infty} c_s c_{s-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} c(e^{-i\omega})c(e^{i\omega})e^{i\omega k}d\,\omega.$$

If we set $k = 0$, we obtain Parseval's equality:

$$\sum_{s=-\infty}^{\infty} c_s^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |c(e^{-i\omega})|^2 d\,\omega.$$

11.A.3. One sided sequences

There are two types of one-sided sequences (also called 'half-infinite' sequences). A sequence is called a *causal* sequence if $c_k = 0\ \forall k < 0$, and is *anti-causal* if it has zero elements $\forall\ k > 1$. A one-sided causal sequence can be obtained by setting to zero all elements of a two-sided sequence with negative subscripts. Let $\{u_k\}_{k=-\infty}^{\infty}$ be the *step sequence*, which is zero for $k < 0$, and 1 for $k \geq 0$. Evidently $\{u_k c_k\}$ is always a one-sided sequence.

11.A.4. Useful properties

1. z_0 is said to be a *pole of order* $m \geq 1$ of $c(z)$ if $\lim_{z \to z_0}(z - z_0)^m c(z) \neq 0$.

2. $c(z)$ is the transform of a causal sequence if all of its poles lie *outside* the unit circle.

3. $c(z)$ is the transform of an anti-causal sequence if all of its poles lie *inside* the unit circle.

4. If $c(z)$ is either causal or anti-causal, the inversion formula can be implemented by 'long division.'

5. Initial value theorem:

$$\lim_{z \to 0} c(z) = c_0.$$

6. Final value theorem:

$$\lim_{k \to \infty} c_k = \lim_{z \to 1}(1 - z)c(z).$$

11.A.5. One sided transforms

A one-sided transform is defined as

$$c^+(z) = \sum_{k=0}^{\infty} c_k z^k \equiv [c(z)]_+,$$

where $[\quad]_+$ is the 'annihilation operator' that sets to zero all coefficients on negative powers of z. The same inversion formulas hold, with $c^+(z)$ replacing $c(z)$. Notice that $c^+(z) = c(z)$ only if $\{c_k\}$ is causal. We shall adopt the notation

$$\mathscr{F}(c)(z) = c^+(z).$$

For one-sided transforms, we have the *shift theorem*

$$\mathscr{F}(\{c_{t-n}\})(z) = z^n \mathscr{F}(\{c_t\})(z) + \sum_{k=1}^{n} z^{n-k} c_{-k}.$$

11.A.6. Discounting

For the purpose of introducing discounting, we shall work with the alternative transformation defined by

$$\mathscr{T}(\{c_t\}_{t=0}^{\infty})(z) \equiv \mathscr{F}(\{c_t \beta^{t/2}\}_{t=0}^{\infty})(z),$$

so that $\mathscr{T}(y)$ is the ordinary transform of $\{\beta^{t/2} y_t\}$. The inversion formula is then

$$\beta^{t/2} y_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathscr{T}(e^{-i\omega}) e^{i\omega t} d\omega,$$

and the shift theorem is

$$\mathscr{T}(\{c_{t-n}\})(z) = (\beta^{.5} z)^n \mathscr{F}(\{c_t\})(z) + \sum_{k=1}^{n} (z\beta^{.5})^{n-k} c_{-k}.$$

11.A.7. Fourier transforms

Below we shall work with vector versions of the transforms $\mathscr{T}$. Consider a vector sequence $y = \{y_t\}$ satisfying

$$\sum_{t=0}^{\infty} \beta^t y_t \cdot y_t < \infty, \tag{11.A.1}$$

define the transform:

$$\mathscr{T}(y)(\zeta) \equiv \sum_{t=0}^{\infty} \beta^{t/2} y_t \zeta^t. \tag{11.A.2}$$

This transform is at least well-defined for $\mid \zeta \mid < 1$ and can also be defined through an appropriate limiting argument for $\mid \zeta \mid = 1$.[6] For vector sequences $\{y_t\}$ and $\{\hat{y}_t\}$ satisfying (11.A.1), Parseval's formula is

$$(1/2\pi) \int_{-\pi}^{\pi} \mathscr{T}(y)[\exp(i\theta)] \cdot \mathscr{T}(\hat{y})[\exp(-i\theta)] d\theta = \sum_{t=0}^{\infty} \beta^t (y_t \cdot \hat{y}_t). \tag{11.A.3}$$

[6] The boundary of the unit circle can be parameterized by $\zeta = \exp(i\theta)$ for $\theta \in (-\pi, \pi]$. Using this parameterization, the infinite series on the right side of (11.4.6) converges in L^2 where the L^2 space is constructed using Lebesgue measure on $(-\pi, \pi]$.

We use Fourier 0 to represent our dynamic household technologies. It follows from (11.2.1) and the definitions of s_{mt} and s_{it} that

$$\begin{aligned}\Pi\mathscr{T}(c)(\zeta) &= -\beta^{1/2}\zeta\Lambda\mathscr{T}(h_m)(\zeta) + \mathscr{T}(s_m)(\zeta)\\ \mathscr{T}(h_m)(\zeta) &= \beta^{1/2}\zeta(\Delta_h - \Theta_h\Pi^{-1}\Lambda)\mathscr{T}(h_m)(\zeta) + \Theta_h\Pi^{-1}\mathscr{T}(s_m)(\zeta)\end{aligned} \tag{11.A.4}$$

where $h_{m,-1} = 0$. The transforms of the consumption sequence and the market service sequence are related by

$$\mathscr{T}(c)(\zeta) = \mathscr{C}(\zeta)\mathscr{T}(s_m)(\zeta) \tag{11.A.5}$$

where

$$\mathscr{C}(\zeta) \equiv \Pi^{-1}\Big\{I - \beta^{1/2}\zeta\Lambda[I - \beta^{1/2}\zeta(\Delta_h - \Theta_h\Pi^{-1}\Lambda)]^{-1}\Theta_h\Pi^{-1}\Big\}. \tag{11.A.6}$$

The matrix function $\mathscr{C}$ of a complex variable ζ represents the mapping from *desired* consumption services into the consumption goods *required* to support those services.

11.A.8. Verifying Equivalent Valuations

Our derivation of the dynamic demand functions for consumption goods relied on two intermediate results: (a) equivalent 0 of market services and consumption goods asserted in (11.3.10); and (b) for a given specification of preferences and household technology, the existence of a canonical service technology that induces the same preference ordering over consumption streams. To establish these intermediate results we use Fourier transforms.

We now show establish the valuation equivalence asserted in (11.3.10). Applying Parseval's formula (11.A.3), we have that

$$\begin{aligned}\sum_{t=0}^{\infty}\beta^t p_t^0\cdot c_t &= (1/2\pi)\int_{-\pi}^{\pi}\mathscr{T}(p^0)[\exp(i\theta)]\cdot\mathscr{T}(c)[\exp(-i\theta)]d\theta\\ &= (1/2\pi)\int_{-\pi}^{\pi}\mathscr{T}(p^0)[\exp(i\theta)]\cdot\{\mathscr{C}[\exp(-i\theta)]\mathscr{T}(s_m)[\exp(-i\theta)\}]d\theta\\ &= (1/2\pi)\int_{-\pi}^{\pi}\{\mathscr{C}[\exp(-i\theta)]'\mathscr{T}(p^0)[\exp(i\theta)]\}\cdot\mathscr{T}(s_m)[\exp(-i\theta)]d\theta.\end{aligned} \tag{11.A.7}$$

Formula $(11.A.7)$ gives us the following candidate for the transform of the rental sequence for consumption services: $\mathscr{C}(\zeta^{-1})'\mathscr{T}(p^0)(\zeta)$. The rental sequence $\{\tilde{\rho}_t^0\}$ associated with this transform is given by:

$$\begin{aligned}\tilde{\rho}_t^0 &\equiv \Pi^{-1\prime}\Big\{I - \beta L^{-1}\Theta_h'[I - \beta L^{-1}(\Delta_h - \Theta_h \Pi^{-1}\Lambda)']^{-1}\Lambda'\Pi^{-1\prime}\Big\}p_t^0 \\ &= \Pi^{-1\prime}\Big[p_t^0 - \Theta_h' \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h - \Theta_h \Pi^{-1}\Lambda)'^{\tau-1}\Lambda'\Pi^{-1\prime}p_{t+\tau}^0\Big].\end{aligned} \tag{11.A.8}$$

Using this rental sequence, it follows from $(11.A.5)$ that

$$\sum_{t=0}^{\infty} \beta^t p_t^0 \cdot c_t = \sum_{t=0}^{\infty} \beta^t \tilde{\rho}_t^0 \cdot s_{mt}. \tag{11.A.9}$$

Notice that the candidate rental sequence $\{\tilde{\rho}_t^0\}$ violates the information constraints because $\tilde{\rho}_t^0$ will not necessarily be in J_t. From the vantage point of valuation, all that we require is equality of the expectations of the infinite sums in $(11.A.9)$ conditioned on J_0. It follows from the Law of Iterated Expectations that

$$E_0 \tilde{\rho}_t^0 \cdot s_t = E_0 \rho_t^0 \cdot s_t \tag{11.A.10}$$

where

$$\rho_t^0 \equiv E_t \tilde{\rho}_t^0, \tag{11.A.11}$$

since hypothetical service vectors s_t are restricted to be in the information set J_t. Taking expectations of both sides of $(11.A.11)$ conditioned on J_0 and substituting from $(11.A.11)$ establishes the value equivalence given in $(11.3.10)$.

11.A.9. Equivalent representations of preferences

We now turn to task (b), to show that the candidate canonical representation of the service technology implies the same induced preference ordering for consumption. There are two preference representations on the table (Λ, Π), $(\hat{\Lambda}, \hat{\Pi})$, where the objects with hats are canonical. Again we partition the household capital stock and the consumption service process into two components. Similar to $(11.A.4)$ we have that

$$\begin{aligned} \mathscr{T}(s_m)(\zeta) &= \beta^{1/2}\zeta\Lambda\mathscr{T}(h_m)(\zeta) + \Pi\mathscr{T}(c)(\zeta) \\ \mathscr{T}(h_m)(\zeta) &= \beta^{1/2}\zeta\Delta_h\mathscr{T}(h_m)(\zeta) + \Theta_h\mathscr{T}(c)(\zeta). \end{aligned} \tag{11.A.12}$$

Hence

$$\mathscr{T}(s_m)(\zeta) = \mathscr{S}(\zeta)\mathscr{T}(c)(\zeta) \tag{11.A.13}$$

where

$$\mathscr{S}(\zeta) \equiv [\Pi + \beta^{1/2}\zeta\Lambda(I - \beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h]. \tag{11.A.14}$$

The function $\mathscr{S}$ represents the mapping from consumption goods into market supplied consumption services. An analogous argument leads to the formula:

$$\mathscr{T}(\hat{s}_m)(\zeta) = \hat{\mathscr{S}}(\zeta)\mathscr{T}(c)(\zeta) \tag{11.A.15}$$

where

$$\hat{\mathscr{S}}(\zeta) \equiv [\hat{\Pi} + \beta^{1/2}\zeta\hat{\Lambda}(I - \beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h], \tag{11.A.16}$$

where objects with hats, including $\hat{s}_m$, correspond to the canonical representation. It is straightforward to show that

$$\mathscr{T}(s_i)(\zeta) = \Lambda(I - \beta^{1/2}\zeta\Delta_h)^{-1}h_{-1} \tag{11.A.17}$$

and

$$\mathscr{T}(\hat{s}_i)(\zeta) = \hat{\Lambda}(I - \beta^{1/2}\zeta\Delta_h)^{-1}h_{-1}. \tag{11.A.18}$$

The time t contribution to the consumers' utility function can be expressed as:

$$\begin{aligned} -(1/2)\beta^t&\Big[(b_t - s_{i,t} - s_{m,t})\cdot(b_t - s_{i,t} - s_{m,t})\Big] \\ &= -(1/2)\beta^t\Big[s_{m,t}\cdot s_{m,t} + 2s_{m,t}\cdot s_{i,t} - 2s_{m,t}\cdot b_t + \\ &\qquad (b_t - s_{i,t})\cdot(b_t - s_{i,t})\Big]. \end{aligned} \tag{11.A.19}$$

Note that the fourth term is not affected by the consumption choice, and thus can be ignored.

We now study the Fourier representations of the sums:

$$\sum_{t=0}^{\infty} \beta^t s_{m,t} \cdot s_{m,t}, \sum_{t=0}^{\infty} \beta^t s_{m,t} \cdot s_{i,t} \text{ and } \sum_{t=0}^{\infty} \beta^t s_{m,t} \cdot b_t. \tag{11.A.20}$$

11.A.10. First term: factorization identity

The first infinite sum in (11.A.20) can be represented as:

$$\begin{aligned} &\sum_{t=0}^{\infty} \beta^t s_{m,t} \cdot s_{m,t} = \\ &(1/2\pi) \int_{-\pi}^{\pi} \Big\{ \mathscr{T}(c)[\exp(i\theta)] \Big\}' \mathscr{S}[\exp(i\theta)]' \mathscr{S}[\exp(-i\theta)] \\ &\qquad \mathscr{T}(c)[\exp(-i\theta)] d\theta. \end{aligned} \tag{11.A.21}$$

To show that $(\hat{\Pi}, \hat{\Lambda})$ and $\{\hat{b}_t\}$ imply the same induced preferences for consumption goods, we must first establish the factorization:

$$\mathscr{S}(\zeta^{-1})' \mathscr{S}(\zeta) = \hat{\mathscr{S}}(\zeta^{-1})' \hat{\mathscr{S}}(\zeta). \tag{11.A.22}$$

To verify this result, note that

$$\begin{aligned} &[\Pi + \beta^{1/2} \zeta^{-1} \Lambda (I - \beta^{1/2} \zeta^{-1} \Delta_h)^{-1} \Theta_h]' [\Pi + \beta^{1/2} \zeta \Lambda (I - \beta^{1/2} \zeta \Delta_h)^{-1} \Theta_h] \\ &\quad = \Pi' \Pi + \beta \Theta_h' (I - \beta^{1/2} \zeta^{-1} \Delta_h)^{-1\prime} \Lambda' \Lambda (I - \beta^{1/2} \zeta \Delta_h)^{-1} \Theta_h \\ &\quad + \beta^{1/2} \zeta^{-1} \Theta_h' (I - \beta^{1/2} \zeta^{-1} \Delta_h)^{-1\prime} \Lambda' \Pi \\ &\quad + \beta^{1/2} \zeta \Pi' \Lambda (I - \beta^{1/2} \zeta \Delta_h)^{-1} \Theta_h. \end{aligned} \tag{11.A.23}$$

Since P satisfies the algebraic Riccati equation(11.4.1), it follows that

$$\begin{aligned} &\Lambda' \Lambda = P - \beta \Delta_h' P \Delta_h + \hat{\Lambda}' \hat{\Lambda} \\ &= (I - \beta^{1/2} \zeta^{-1} \Delta_h)' P (I - \beta^{1/2} \zeta \Delta_h) + \beta^{1/2} \zeta^{-1} \Delta_h' P (I - \beta^{1/2} \zeta \Delta_h) \\ &\quad + \beta^{1/2} \zeta (I - \beta^{1/2} \zeta^{-1} \Delta_h)' P \Delta_h + \hat{\Lambda}' \hat{\Lambda}. \end{aligned} \tag{11.A.24}$$

Therefore,

$$\begin{aligned}
&\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\Lambda'\Lambda(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h \\
&\quad=\Theta_h'P\Theta_h+\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\Delta_h'P\Theta_h \\
&\quad+\beta^{1/2}\zeta\Theta_h'P\Delta_h(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h \\
&\quad+\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h.
\end{aligned} \tag{11.A.25}$$

Furthermore, it follows from (11.4.4) and (11.4.5) that

$$\begin{aligned}
\hat{\Pi}'\hat{\Lambda}&=\hat{\Pi}'\hat{\Pi}(\hat{\Pi})^{-1}\hat{\Lambda} \\
&=(\beta\Theta_h'P\Theta_h+\Lambda'\Pi).
\end{aligned} \tag{11.A.26}$$

Substituting (11.A.25) and (11.A.26) into (11.A.23) results in

$$\begin{aligned}
&[\Pi+\beta^{1/2}\zeta^{-1}\Lambda(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1}\Theta_h]'[\Pi+\beta^{1/2}\zeta\Lambda(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h] \\
&\quad=\Pi'\Pi+\beta\Theta_h'P\Theta_h+\beta\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h \\
&\quad+\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}(\beta\Delta_h'P\Theta_h+\Lambda'\Pi) \\
&\quad+\beta^{1/2}\zeta(\Pi\Lambda'+\beta\Theta_h'P\Delta_h)(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h \\
&\quad=\hat{\Pi}'\hat{\Pi}+\beta\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h \\
&\quad+\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Pi} \\
&\quad+\beta^{1/2}\zeta\hat{\Lambda}'\hat{\Pi}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h \\
&=[\hat{\Pi}+\beta^{1/2}\zeta^{-1}\hat{\Lambda}(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1}\Theta_h]'[\hat{\Pi}+\beta^{1/2}\zeta\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Theta_h]
\end{aligned} \tag{11.A.27}$$

which proves factorization (11.A.22).

11.A.11. Second term

The second infinite sum in (11.A.20) can be represented as

$$\sum_{t=0}^{\infty} \beta^t s_{m,t} \cdot s_{i,t} = (1/2\pi) \int_{-\pi}^{\pi} \{\mathscr{T}(c)[\exp(i\theta)]\}' \mathscr{S}[\exp(i\theta)]' \Lambda \\ [I - \beta^{1/2} \exp(-i\theta)\Delta_h]^{-1} h_{-1} d\theta. \tag{11.A.28}$$

We will verify that

$$\mathscr{S}(\zeta^{-1})' \Lambda \Delta_h (I - \beta^{1/2} \zeta \Delta_h)^{-1} = \\ \hat{\mathscr{S}}(\zeta^{-1})' \hat{\Lambda} \Delta_h (I - \beta^{1/2} \zeta \Delta_h)^{-1} + \beta^{1/2} \zeta^{-1} \Theta_h' (I - \beta^{1/2} \zeta^{-1} \Delta_h)^{-1\prime} P \Delta_h. \tag{11.A.29}$$

It then follows that

$$(1/2\pi) \int_{-\pi}^{\pi} \{T(c)[\exp(i\theta)]\}' S[\exp(i\theta)]' \Lambda \Delta_h [I - \beta^{1/2} \exp(-i\theta)\Delta_h]^{-1} h_{-1} d\theta \\ = (1/2\pi) \int_{-\pi}^{\pi} \{T(c)[\exp(i\theta)]\}' \hat{S}[\exp(i\theta)]' \hat{\Lambda} \Delta_h [I - \beta^{1/2} \exp(-i\theta)\Delta_h]^{-1} h_{-1} d\theta \tag{11.A.30}$$

because

$$(1/2\pi) \int_{-\pi}^{\pi} \{\mathscr{T}(c)[\exp(i\theta)]\}' \beta^{1/2} \exp(i\theta) \\ \Theta_h' [I - \beta^{1/2} \exp(i\theta)\Delta_h]^{-1\prime} P \Delta_h h_{-1} d\theta = 0. \tag{11.A.31}$$

Relation (11.A.31) holds since $\mathscr{T}(c)(\zeta)' \beta^{1/2} \zeta \Delta_h' (I - \beta^{1/2} \zeta \Delta_h)^{-1\prime}$ has a power series expansion and is zero when $\zeta = 0$ and $P \Delta_h h_{-1}$ can be viewed a constant function with a trivial power series expansion. Relation (11.A.31) then follows from Parseval's formula (11.A.3) where $\beta^{t/2} y_t$ is constructed from the t^{th} coefficient of the power series expansion for the first function and $\beta^{t/2} \tilde{y}_t$ from the t^{th} coefficient of the power series expansion for the second function.

It remains to establish (11.A.29). Note that the left side of (11.A.29) can be expanded as follows:

$$[\Pi + \beta^{1/2} \zeta^{-1} \Lambda (I - \beta^{1/2} \zeta^{-1} \Delta_h)^{-1} \Theta_h]' \Lambda (I - \beta^{1/2} \zeta \Delta_h)^{-1} \Delta_h \\ = \Pi' \Lambda (I - \beta^{1/2} \zeta \Delta_h)^{-1} \Delta_h + \beta^{1/2} \zeta^{-1} \Theta_h' (I - \beta^{1/2} \zeta^{-1} \Delta_h)^{-1\prime} \Lambda' \Lambda \\ (I - \beta^{1/2} \zeta \Delta_h)^{-1} \Delta_h. \tag{11.A.32}$$

It follows from the algebraic Riccati equation (11.4.1) that

$$\begin{aligned}\Lambda'\Lambda &= P(I-\beta^{1/2}\zeta\Delta_h)+\beta^{1/2}\zeta P\Delta_h-\beta\Delta_h' P\Delta_h+\hat{\Lambda}'\hat{\Lambda}\\ &= P(I-\beta^{1/2}\zeta\Delta_h)+\beta^{1/2}\zeta(I-\beta^{1/2}\zeta^{-1}\Delta_h')P\Delta_h+\hat{\Lambda}'\hat{\Lambda},\end{aligned}\tag{11.A.33}$$

and hence

$$\begin{aligned}&\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\Lambda'\Lambda(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h\\ &=\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}P\Delta_h+\beta\Theta_h' P\Delta_h(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h\\ &+\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h\end{aligned}\tag{11.A.34}$$

Substituting (11.A.34) and (11.A.26) into (11.A.32) gives

$$\begin{aligned}&[\Pi+\beta^{1/2}\zeta^{-1}\Lambda(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1}\Theta_h]'\Lambda(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h\\ &\qquad=\Pi'\Lambda(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h+\\ &\qquad\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\Lambda'\Lambda(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h\\ &\qquad=(\Pi'\Lambda+\beta\Theta_h' P\Delta_h)(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h+\\ &\qquad\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}P\Delta_h+\\ &\qquad\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h\\ &\qquad=\hat{\Pi}'\hat{\Lambda}/(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h+\\ &\qquad\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}\hat{\Lambda}'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h+\\ &\qquad\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}P\Delta_h\\ &=[\hat{\Pi}+\beta^{1/2}\zeta^{-1}\hat{\Lambda}(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1}\Theta_h]'\hat{\Lambda}(I-\beta^{1/2}\zeta\Delta_h)^{-1}\Delta_h+\\ &\qquad\beta^{1/2}\zeta^{-1}\Theta_h'(I-\beta^{1/2}\zeta^{-1}\Delta_h)^{-1\prime}P\Delta_h\end{aligned}\tag{11.A.35}$$

which establishes (11.A.29).

11.A.12. Third term

The third sum in (11.*A*.20) can be represented as

$$\sum_{t=0}^{\infty}\beta^t s_{m,t}\cdot b_t = \\ (1/2\pi)\int_{-\pi}^{\pi}\{\mathscr{T}(c)[\exp(i\theta)]\}'\mathscr{S}[\exp(i\theta)]'\mathscr{T}(b)[\exp(-i\theta)]d\theta. \tag{11.A.36}$$

Note that

$$\mathscr{S}(\zeta^{-1})'\mathscr{T}(b)(\zeta) = \hat{\mathscr{S}}(\zeta^{-1})'\hat{\mathscr{S}}(\zeta^{-1})'^{-1}\mathscr{S}(\zeta^{-1})'\mathscr{T}(b)(\zeta). \tag{11.A.37}$$

With this in mind, we define

$$\hat{b}_t = E\{[\hat{\mathscr{S}}(L^{-1})']^{-1}\mathscr{S}(L^{-1})'b_t \mid J_t\}. \tag{11.A.38}$$

Then by reasoning similar to that leading to result (a), we have that

$$\sum_{t=0}^{\infty}\beta^t s_{m,t}\cdot b_t = \sum_{t=0}^{\infty}\beta^t \hat{s}_{m,t}\cdot\hat{b}_t. \tag{11.A.39}$$

Taken together (11.*A*.22), (11.*A*.29) and (11.*A*.37) show that the induced preference ordering for consumption is the same for $(\hat{\Lambda},\hat{\Pi})$ and $\{\hat{b}_t\}$ as it is for the original specification (Λ,Π) and $\{b_t\}$. This establishes result (b).

Chapter 12
Gorman Heterogeneous Households

12.1. Introduction

This chapter and the next describe methods for computing equilibria of versions of our economies in which consumers have heterogeneous preferences and endowments. In each chapter, we adopt simplifications that make it possible for us to cope with the complications introduced by heterogeneity. In the present chapter, we adopt a version of Terrance Gorman's (1953) specification. We describe a class of heterogeneous consumer economies that satisfy Gorman's conditions for aggregation, which lets us compute equilibrium *aggregate* allocations and prices *before* computing allocations to individuals.[1]

In the following chapter, we adopt a more general specification of heterogeneity that causes us to depart from the representative consumer framework of Gorman. In particular, we adapt the idea of Negishi (1960), who described a social welfare function that is maximized, subject to resource and technological constraints, by a competitive equilibrium allocation. For Negishi, that social welfare function is a "linear combination of the individual utility functions of consumers, with the weights in the combination in inverse proportion to the marginal utilities of income." Because Negishi's weights depend on the allocation through the marginal utilities of income, computing a competitive equilibrium via constrained maximization of a Negishi-style welfare function requires solving a fixed point problem in the weights. In the following chapter, we apply such a fixed point approach. The beauty of Gorman's aggregation conditions is that, when they apply, time series aggregates and market prices can be computed without resorting to Negishi's fixed point approach.

In the present chapter, consumers are permitted to differ *only* with respect to their endowments and the process $\{b_t\}$ that disturbs their preferences. We

[1] The discussion in this chapter is patterned after the material in section 3 of Hansen (1987).

assume that all consumers have a common information set that includes observations on past values of the economy wide capital stocks h_{t-1}, k_{t-1}, and the common exogenous state variables in z_t that drive each of the individual preference shock processes and the technology shock process $\{d_t\}$. Preferences of the individual consumers can be aggregated simply by summing both preference shocks and initial endowments across consumers, thereby forming a representative consumer. We can compute all *aggregate* aspects of a competitive equilibrium of the economy with heterogeneous consumers by forming the representative consumer and proceeding as in chapters 3, 4, and 5. We show how to calculate individual allocations by using the demand functions that were described in the previous chapter.

In the next section, we briefly describe Gorman aggregation in a standard static section before adapting it to our purposes.

12.2. A Digression on Gorman Aggregation

Suppose for the moment that there are n consumption goods, taking into account indexation by dates and states, and that consumption of person $j = 1, \ldots, J$ is denoted c^j. Let c^a denote the aggregate amount of consumption to be allocated among consumers. Associated with c^a is an Edgeworth box and a collection of Pareto optimal allocations. From the Pareto optimal allocations, one can construct utility allocation surfaces describing the frontier of alternative feasible utility assignments to individual consumers. Imagine moving from the aggregate vector c^a to some other vector $\tilde{c}^a$ and hence to a new Edgeworth box. If neither the original box nor the new box contain one another, then it is possible that the utility allocation surfaces for the two boxes may *cross*, in which case there exists no ordering of aggregate consumption that is independent of the utility weights assigned to individual consumers.

Before describing a special case in which an aggregate social preference ordering *does* exist, we illustrate a situation in which there *doesn't* exist a social preference ordering that is independent of the aggregate allocation.

Figures 12.2.1 and Fgpareto2f describe efficient allocations in a two person, two good, pure exchange economy with a structure of preferences that *violate* the

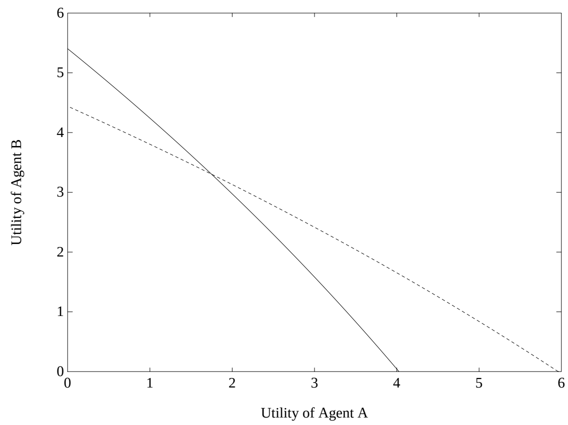

Figure 12.2.1: Utility allocation for Agents A and B for endowment vectors E = (8,3) and E = (3,8).

Gorman aggregation conditions. Agent A has utility function $U^A = X_A^{1/3} Y_A^{2/3}$, while consumer B has utility function given by $U^B = X_B^{2/3} Y_B^{1/3}$, where the aggregate endowment pair is $E = (X_A + X_B, Y_A + Y_B)$. Figure 12.2.1 shows two utility possibility frontiers, one associated with $E = (8, 3)$, a second one associated with $E = (3, 8)$.[2] The fact that the utility possibility frontiers in figure 12.2.1 cross indicates that the two aggregate endowment vectors $(8, 3), (3, 8)$ cannot be ranked in a way that ignores how utility is distributed between consumers A and B.

In the same economy, Figure 12.2.2 shows the Edgeworth boxes and contract curves with the two allocations $E = (8, 3)$ and $E = (3, 8)$.

For a given endowment, the slope of the consumers' indifference curves at the tangencies between indifference curves that determines the contract curve varies as one moves along the contract curve. This means that for a given aggregate endowment, the competitive equilibrium price depends on the allocation between consumers A and B. It follows that for this economy, one cannot expect to determine equilibrium prices independently of the equilibrium allocation.

[2] A utility possibility frontier is the locus of pairs (U_A, U_B) that solve $U_A = \max_{X_A, Y_A} X_A^{1/3} Y_A^{2/3}$ subject to the constraints $X_B^{2/3} Y_B^{1/3} \geq U^B, (X_A + X_B, Y_A + Y_B) = E$.

Gorman (1953) described restrictions on preferences under which it *is* possible to obtain a community preference ordering. Whenever Gorman's conditions are satisfied, there occur substantial simplifications in solving multiple-consumer optimal resource allocation problems: in intertemporal contexts, it becomes possible first to determine the optimal allocation of aggregate resources over time. Then the aggregate consumption can be allocated among consumers by allocating utility levels to each person.

To understand Gorman's restrictions, imagine specifying the preferences of consumer j in one of two equivalent ways: in terms either of a family of indifference curves indexed by the utility level, or in terms of a family of compensated demand functions. Following Gorman (1953), let $\psi_j(p)$ denote the baseline indifference curve for person j parameterized in terms of a price vector (or vector of utility gradients) p. In addition, let $\psi_c(p)$ denote a common indifference curve for all consumers used to measure deviations from the baseline curves. This lets the compensated demand function for person j be represented as

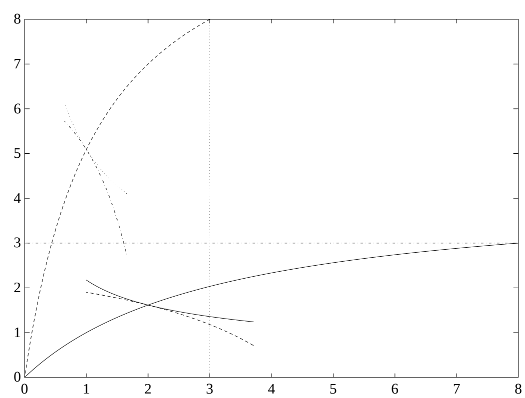

Figure 12.2.2: Overlapping Edgeworth Boxes for endowment vectors E = (8,3) and E = (3,8).

$$c^j = \psi_j(p) + u^j \psi_c(p) \tag{12.2.1}$$

where u^j is a scalar utility index for person j. The baseline functions ψ_j and the common function ψ_c are the derivatives of concave functions that are positively homogeneous of degree 1. Hence these functions are homogeneous of degree zero in prices, assuring that the slopes of indifference curves should depend only on the *ratio* of prices. The baseline indifference curves are either the *highest* or *lowest* indifference curves, corresponding respectively to cases in which the utility indices are restricted to be nonpositive or nonnegative. As noted by Gorman, when preferences are of this form, there is a well defined compensated demand function for a fictitious representative consumer obtained by aggregating (12.2.1):

$$c^a = \psi_a(p) + u^a \psi_c(p) \tag{12.2.2}$$

where

$$u^a = \sum u^j \text{ and } \psi_a = \sum \psi_j. \tag{12.2.3}$$

In this case, optimal resource allocation in a heterogeneous consumer economy simplifies as follows. Preferences (12.2.2), define a *community preference ordering* for aggregate consumption. This preference–ordering can be combined with a specification of the technology for producing consumption goods to determine the optimal allocation of aggregate consumption.

Mapping (12.2.2) can be inverted to obtain a *gradient* vector p that is independent of how utilities are allocated across consumers. Since ψ_c and ψ_a are homogeneous of degree zero, gradients are only determined up to a scalar multiple. Armed with p, we can then allocate utility among J consumers while respecting the adding up constraint given in (12.2.3). The allocation of aggregate

consumption across goods and the associated gradient are determined independently of how the aggregate utility is divided among the individual consumers.

A decentralized version of this analysis proceeds as follows. Let W^j denote the wealth of consumer j and W^a denote aggregate wealth. Then W^j should satisfy:

$$W^j = p \cdot c^j = p \cdot \psi_j(p) + u^j p \cdot \psi_c(p). \tag{12.2.4}$$

Solving (12.2.4) for u^j gives

$$u^j = [W^j - p \cdot \psi_j(p)]/p \cdot \psi_c(p). \tag{12.2.5}$$

Hence the Engel curve for consumer j is given by

$$c^j = \psi_j(p) - p \cdot \psi_j(p)/p \cdot \psi_c(p) + W^j \psi_c(p)/p \cdot \psi_c(p). \tag{12.2.6}$$

Notice that the *coefficient* on W^j is the same for all j since $\psi_c(p)/p \cdot \psi_c(p)$ is only a function of the price vector p in a decentralized economy. The individual allocations can be determined from the Engel curves by substituting for p the gradient vector obtained from the single consumer optimal allocation problem. Individual consumption c^j as given by (12.2.6) depends directly on prices through the functions ψ_j and ψ_c and indirectly through the evaluation of wealth.

For the specifications of preferences adopted in this book, the baseline indifference curves are degenerate because they do not depend on p. A finite-dimensional counterpart to this circumstance occurs when

$$\psi_j(p) = \chi^j, \tag{12.2.7}$$

where χ^j is a vector with the same dimension as c^j. With this specification, the rules for allocating consumption across individuals become linear in aggregate consumption. To see this, observe that an implication of (12.2.2) is

$$\psi_c(p) = (c^a - \chi^a)/u_a. \tag{12.2.8}$$

Substituting (12.2.8) into (12.2.1) gives

$$c^j - \chi^j = (u^j/u^a)(c^a - \chi^a), \tag{12.2.9}$$

so that there is a common scale factor (u^j/u^a) across all goods for person j. Hence the fraction of total utility assigned to consumer j determines his fraction of the vector $(c^a - \chi^a)$.

Here is an example. Suppose that the preferences of consumer j are represented using the utility function:

$$U^j(c^j) = -[(c^j - \chi^j)' V (c^j - \chi^j)]^{1/2}. \tag{12.2.10}$$

The compensated demand schedule is then obtained by solving the first-order conditions:

$$V(c^j - \chi^j)/U^j(c^j) = \mu^j p \tag{12.2.11}$$

$$U^j(c^j) = u^j,$$

where μ^j is a Lagrange multiplier. Substitute the second equation into the first and solve for $c^j - \chi^j$:

$$c^j - \chi^j = u^j \mu^j V^{-1} p. \tag{12.2.12}$$

Substitute the right side of (12.2.12) into the utility function and solve for the multiplier μ^j:

$$\mu^j = 1/(p' V^{-1} p)^{1/2} \tag{12.2.13}$$

Hence the compensated demand function is given by

$$c^j = b^j + u^j V^{-1} p/(p' V^{-1} p)^{1/2}. \tag{12.2.14}$$

In this example,

$$\psi_j(p) = \chi^j \quad \text{and} \quad \psi_c(p) = V^{-1} p/(p' V^{-1} p)^{1/2}. \tag{12.2.15}$$

Notice that to obtain a representation of preferences linear in the utility index requires using a particular monotonic transformation of the utility function. In our example, the quadratic form on the right side of (12.2.10) is raised to the one-half power.

12.3. An Economy with Heterogeneous Consumers

We now specify a multi-consumer version of our dynamic linear economy designed to satisfy counterparts to Gorman's conditions for aggregation. There is a collection of consumers, indexed by $i = 1, 2, \ldots, I$. Consumers differ both in their preferences and in their endowments, but not in their information. Consumer i has preferences that are ordered by

$$-\left(\frac{1}{2}\right) E \sum_{t=0}^{\infty} \beta^t \left[(s_t^i - b_t^i) \cdot (s_t^i - b_t^i) + \ell_t^{i2}\right] \mid J_0 \tag{12.3.1}$$

where $\{s_t^i\}$ is linked to $\{h_t^i\}$ and $\{c_t^i\}$ via

$$s_t^i = \Lambda\, h_{t-1}^i + \Pi\, c_t^i \tag{12.3.2}$$

$$h_t^i = \Delta_h\, h_{t-1}^i + \Theta_h\, c_t^i, \tag{12.3.3}$$

and h_{-1}^i is given. In (12.3.1), (12.3.2), (12.3.3), the i superscript pertains to consumer i. The preference disturbance b_t^i is determined by

$$b_t^i = U_b^i\, z_t \tag{12.3.4}$$

where z_t continues to be governed by (3.2). The $i^{\rm th}$ consumer maximizes (12.3.1) subject to (12.3.2), (12.3.3) and the budget constraint

$$E \sum_{t=0}^{\infty} \beta^t\, p_t^0\, \cdot c_t^i \mid J_0 = E \sum_{t=0}^{\infty} \beta^t \,(w_t^0\, \ell_t^i + \alpha_t^0\, \cdot d_t^i) \mid J_0 + v_0\, \cdot k_{-1}^i, \tag{12.3.5}$$

where k_{-1}^i is given. The $i^{\rm th}$ consumer owns an endowment process d_t^i, governed by the stochastic process $d_t^i = U_d^i\, z_t$.

Each consumer observes the aggregate information J_t at time t, as well as the idiosyncratic capital stocks k_{t-1}^i and h_{t-1}^i. The information set J_t continues to be defined as $J_t = [w^t, x_0]$.

This specification confines heterogeneity among consumers to differences in the preference $\{b_t^i\}$ processes, represented by different selections of U_b^i; differences in the endowment $\{d_t^i\}$ processes, represented by different selections of U_d^i; differences in h_{-1}^i; and differences in k_{-1}^i. The matrices Λ, Π, Δ_h, Θ_h do *not* depend on i. This makes everybody's demand curve have the form of (10.15), with different μ_0^w (reflecting different wealth levels) and different b_t processes.

Prices and the aggregate real variables can be computed by synthesizing a representative consumer and solving a version of the social planning problem that was described in chapter 3. Use the settings $h_{-1} = \sum_i h_{-1}^i$, $k_{-1} = \sum_i k_{-1}^i\, U_b, = \sum_i U_b^i$, and $U_d = \sum_i U_d^i$. This gives us aggregate quantities and prices. We let μ_o^{wa} denote the multiplier on wealth in the budget constraint of the representative (or average) household. To compute individual individual allocations requires more work, to which we now turn.

12.4. Allocations

A direct way to compute individual allocations would be to solve the problem each household faces in the competitive equilibrium at the competitive equilibrium prices. For a *fixed* Lagrange multiplier on the household's budget constraint, the household's problem can be expressed as an optimal linear regulator, with a state vector augmented to reflect the aggregate state variables determining the scaled Arrow-Debreu prices. It is possible to compute the allocation to a particular household by using an iterative scheme to calculate the Lagrange multiplier that assures that the household's budget constraint is satisfied, but this is not the procedure that we recommend. Instead note that the allocation rule for labor is

$$\ell_t^j = (\mu_0^{wj}/\mu_0^{wa})\ell_t^a. \tag{12.4.1}$$

If we substitute this expression for ℓ_t^j into versions of (10.13) and (10.14) for the j th consumer, we get the following version of the household's budget constraint:

$$\mu_0^{jw} E_0 \sum_{t=0}^{\infty} \beta^t \{\rho_t^0 \cdot \rho_t^0 + (w_t^0/\mu_0^{aw})\ell_t^a\} = E_0 \sum_{t=0}^{\infty} \beta^t \{\rho_t^0 \cdot (b_t^j - s_{ti}^j) - \alpha_t^0 \cdot d_t^j\} - v_0 k_{-1}^j.$$

Solve this equation for μ_0^{wj}, using a doubling algorithm. With μ_0^{jw} in hand, we can use the first-order conditions for services and the canonical service technology to solve for the equilibrium allocation to household j. For a canonical service technology, the first-order conditions for consumption services are:

$$s_t^j - b_t^j = \mu_0^{jw} \rho_t^0. \tag{12.4.2}$$

Given ρ_t^0, which we know from the aggregate allocation and (10.8), we can solve (12.4.2) for s_t^j, then plug s_t^j into the 'inverse canonical representation' to solve for c_t^j:

$$\begin{aligned} c_t^j &= -\Pi^{-1}\Lambda h_{t-1}^j + \Pi^{-1} s_t^j \\ h_t^j &= (\Delta_h - \Theta_h \Pi^{-1}\Lambda) h_{t-1}^j + \Pi^{-1}\Theta_h s_t^j, \end{aligned} \tag{12.4.3}$$

h_{-1}^j given.

12.4.1. Consumption sharing rules

Our preference specification is an infinite-dimensional generalization of the one described in our digression on Gorman aggregation, a version in which goods are indexed by both dates and states of the world. The counterpart to the matrix V is determined by the probability distribution over states of the world conditioned on J_0 and on the parameters of the household technology. The counterpart to χ^j is determined by the preference shock process $\{b_t^j\}$ and the initial endowment of household capital h_{-1}^j. The allocation rule for consumption has the form:

$$c_t^j - \chi_t^j = (u^j/u^a)(c_t^a - \chi_t^a), \tag{12.4.4}$$

where the ratio (u^j/u^a) is time invariant and depends only on information available at time zero. We can express (12.4.4) as

$$\begin{aligned} c_t^j &= (u^j/u^a)c_t^a + \tilde{\chi}_t^j \\ \tilde{c}_t^j &= \tilde{\chi}_t^j, \end{aligned}$$

where $\tilde{\chi}_t^j \equiv \chi_t^j - (u^j/u^a)\chi_t^a$. Our goal is to show how to compute $\tilde{\chi}_t^j$ and (u^j/u^a). We shall show that the utility indexes can be set at the consumers' marginal utilities of wealth μ_0^{jw}, and that the 'deviation' baseline process for consumption $\{\tilde{\chi}_t^j\}$ can be computed by initializing the inverse canonical representation at a vector $\tilde{h}_{-1}^j$ and using a 'deviation' preference process $\{\tilde{b}_t^j\}$ as the 'driving' service process.

In terms of 'deviation' processes, the allocation rule for consumption services is

$$s_t^j - b_t^j = (\mu_0^{jw}/\mu_0^{aw})(s_t^a - b_t^a) \tag{12.4.5}$$

or

$$\tilde{s}_t^j = \tilde{b}_t^j,$$

where $\tilde{y}_t^j \equiv y_t^j - (\mu_0^{jw}/\mu_0^{aw})y_t^a$. The beauty of this representation is that it does not directly involve prices. The ˜ version of (12.4.3) is

$$\begin{aligned} \tilde{c}_t^j &= -\Pi^{-1}\Lambda\tilde{h}_{t-1}^j + \Pi^{-1}\tilde{s}_t^j \\ \tilde{h}_t^j &= (\Delta_h - \Theta_h\Pi^{-1}\Lambda)\tilde{h}_{t-1}^j + \Pi^{-1}\Theta_h\tilde{s}_t^j, \end{aligned} \tag{12.4.6}$$

$\tilde{h}_{-1}^j$ given. Associated with $\tilde{s}_t^j$ is a synthetic consumption process $\tilde{\chi}_t^j$ such that $\tilde{c}_t^j = \tilde{\chi}_j^t$ is the optimal sharing rule. To construct $\tilde{\chi}_t^j$ we simply substitute

$\tilde{s}_t^j = \tilde{b}_t^j$ into the inverse canonical representation:

$$\begin{aligned}\tilde{\chi}_t^j &= -\Pi^{-1}\Lambda\tilde{\eta}_{t-1}^j + \Pi^{-1}\tilde{b}_t^j \\ \tilde{\eta}_t^j &= (\Delta_h - \Theta_h\Pi^{-1}\Lambda)\tilde{\eta}_{t-1}^j + \Pi^{-1}\Theta_h\tilde{b}_t^j \\ \tilde{\eta}_{-1}^j &= \tilde{h}_{-1}^j.\end{aligned} \tag{12.4.7}$$

Since $\tilde{s}_t^j = \tilde{b}_t^j$ and $\tilde{\eta}_{-1}^j = \tilde{h}_{-1}^j$, it follows from (12.4.6) and (12.4.7) that $\tilde{c}_t^j = \tilde{\chi}_t^j$. Equivalently, allocation rule (12.4.4) holds with $\{\chi_t^j\}$ given by recursion (12.4.7), $\{\chi_t^a\}$ by its aggregate counterpart, and $(u^j/u^a) = (\mu_0^{jw}/\mu_0^{aw})$. Since the allocation rule for consumption can be expressed as

$$c_t^j = (\mu_0^{jw}/\mu_0^{aw})c_t^a + \tilde{\chi}_t^j, \tag{12.4.8}$$

we can append the recursion in (10.27) for c_t and χ_t from the aggregate, single-consumer economy to obtain a recursion for generating c_t^j.

12.5. Risk Sharing Implications

Because the *coefficient* (u^j/u^a) is invariant over time and across goods, allocation rule (12.4.4) implies a form of risk pooling in the deviation process $\{c_t^j - \chi_t^j\}$. Nonseparabilities (either over time or across goods) in the induced preference ordering for consumption goods appear only in the construction of the baseline process $\{\chi_t^j\}$ and in calculation of the risk-sharing coefficient (u^j/u^a) implied by the distribution of wealth. In the special case in which the preference shock processes $\{b_t^j\}$ are deterministic (in the sense that they reside in the information set J_0), individual consumption goods will be perfectly correlated with their aggregate counterparts (conditioned on J_0).[3]

[3] Need to add references to literature on risk sharing: Altug-Miller, MaCurdy, Mace, Cochrane, Townsend etc.

12.6. Implementing the Allocation Rule with Limited Markets

We have seen that one way to implement the allocation rule (12.4.4) is to introduce a complete set of markets in state and date contingent consumption. In some environments, a much smaller set of security markets suffices. An example occurs where a single consumption good is produced according to the linear technology:

$$\begin{aligned} c_t^a + i_t^a &= \gamma k_{t-1}^a + d_t^a \\ k_t^a &= \delta_k k_{t-1}^a + i_t^a, \quad \beta = 1/(\gamma + \delta_k). \end{aligned} \tag{12.6.1}$$

Each consumer has a common household technology with a heterogeneous preference shock process $\{b_t^j\}$ and a heterogeneous initial endowment of household capital h_{-1}^j. The preference shock process is constrained to be in J_0. Each consumer is endowed with a heterogeneous initial level of capital k_{-1}^j and an endowment process $\{d_t^j\}$ for consumption.

Instead of introducing a full array of contingent claims markets, there is a stock market for J securities that pay dividends $\{d_t^j\}$. In addition, one-period riskless claims to consumption are traded. To devise a way to implement allocation rule (12.4.4), note that

$$c_t^j - \chi_t^j + (u_j/u_a)\chi_t^a + (u_j/u_a)k_t^a = (u_j/u_a)[(\delta_k + \gamma)k_{t-1}^a + d_t^a]. \tag{12.6.2}$$

Let consumer j sell all its shares of stock j and purchase (u_j/u_a) shares of all securities traded in the stock market. Once purchased at date zero, let consumer j hold onto this portfolio for all time periods. The total dividends paid in period t will be $(u_j/u_a)d_t^a$. Suppose that the consumer purchases fraction (u_j/u_a) of the capital stock each period in the one period bond market. The time t payoff to the $t-1$ purchase will be $(u_j/u_a)(\delta_k + \gamma)k_{t-1}^a$ and the time t purchase will be $(u_j/u_a)k_t^a$. Taken together, these market transactions have a time t receipt of $(u_j/u_a)[(\delta_k + \gamma)k_{t-1}^a + d_t^a]$ and a time t payout of $(u_j/u_a)k_t^a$ for $t = 1, 2, \ldots$.

The difference between payouts and receipts in time t is not equal to c_t^j, but to $c_t^j - \chi_t^j + (u_j/u_a)\chi_t^a$. This deviation induces trading in the bond market. Note that $\chi_t^j - (u_j/u_a)\chi_t^a$ is in the time zero information set J_0 by assumption.

Let $\hat{k}_t^j$ denote additional purchases in the bond market by person j at time t. Construct $\hat{k}_t^j$ so that

$$\chi_t^j - (u_j/u_a)\chi_t^a + \hat{k}_t^j = (\delta_k + \gamma)\hat{k}_{t-1}^j, \quad t = 1, 2, \ldots \tag{12.6.3}$$

Solve this equation forward to determine an initial value $\hat{k}_0^j$:

$$\hat{k}_0^j = \sum_{t=1}^{\infty} \beta^t [\chi_t^j - (u_j/u_a)\chi_t^a]. \tag{12.6.4}$$

Notice that $\hat{k}_0^j$ is in J_0 so that it is feasible to construct the sequence $\{\hat{k}_t^j\}$. Modify the previous investment strategy so that the bond market purchases of person j at time t equals $(u_j/u_a)k_t^a + \hat{k}_t^j$ for $t = 1, 2, \ldots$. The time t receipts from the previous period purchases in the bond and stock markets equal $(\delta_k + \gamma)[(u_j/u_a)k_{t-1}^a + \hat{k}_{t-1}^j]$. In light of (12.6.2) and (12.6.3), the difference between time t payouts and receipts is c_t^j for $t = 1, 2, \ldots$. The coefficient (u_j/u_a) in the allocation rules is determined so that initial period consumption c_0^j can be purchased from the difference between the time zero receipts and payouts.

In this implementation, all consumers hold the same stock portfolio or mutual fund, but make a sequence of person-specific trades in the market for one-period bonds. We have allowed for nonseparabilities over time in the induced preference ordering for consumption goods, which have important effects on bond market transactions.

This construction displays a multiperiod counterpart to an aggregation result for security markets that was derived by Rubinstein (1974). In a two-period model, Rubinstein provided sufficient conditions on the preferences of consumers and asset market payoffs for the implementation of an Arrow-Debreu contingent claims economy in an environment with incomplete security markets. In Rubinstein's implementation, all consumers hold the same portfolio of risky assets. In our construction, consumers also hold the same portfolio of risky assets, and portfolio weights do not vary over time. All of the changes over time in portfolio composition take place through transactions in the bond market.

12.7. A Computer Example

The MATLAB program computes the allocation to individual i by executing the computations described above. The program `heter.m` requires that be run first, and that its output reside in memory. The program `heter.m` computes individual allocations in the form

$$c_t^i = S_c^i X_t,\ h_t^i = S_h^i X_t,$$

and so on. The matrices S_j^i are returned. The program also computes the matrices S_c^a, S_h^a, and so on, which determine the aggregate allocations $c_t,\ h_t,\ \ldots$ as functions of the augmented state variable X_t:

$$\begin{aligned} c_t &= S_c^a X_t \\ h_t &= S_h^a X_t, \end{aligned}$$

and so on. The MATLAB program can then be used to simulate the allocation to individual i and the aggregate allocation. The programs and must both be run for each individual i in a heterogeneous consumer economy.

We illustrate the workings of these programs with the following pure exchange economy. There are two households, each with identical preferences

$$-\left(\frac{1}{2}\right) E \sum_{t=0}^{\infty} \beta^t \left[(c_t^i - b_t^i)^2 + \ell_t^2\right] \mid J_0,\ i = 1,\, 2$$

We specify that $b_t^i = 15$ for $i = 1,\, 2$. The aggregate preference shock is $b_t = \sum_i b_t^i = 30$. We specify the following endowment processes. For consumer 1,

$$d_t^1 = 4 + .2\, w_t^1,$$

where w_t^1 is a Gaussian white noise with variance $(.2)^2$. For consumer 2, we specify

$$\begin{aligned} d_t^2 &= 3 + \tilde{d}_t^2 \\ \tilde{d}_t^2 &= 1.2\, \tilde{d}_{t-1}^2 - .22\, \tilde{d}_{t-2}^2 + .25\, w_t^2 \end{aligned}$$

where w_t^2 is a Gaussian white noise with variance $(.25)^2$. To capture the pure exchange setup, we specify $\Delta_k = 0,\ \Theta_k = 0,\ \Delta_h = 0,\ \Theta_h = 0,\ \Lambda = 0,\ \Pi = 1$. We set $\beta = 1/1.05$. We have used and to simulate a realization of this economy. Figure 12.7.1 reports the individual allocations to consumers 12.2.1 and 12.2.2. Notice how they appear perfectly correlated. Household one is wealthier than the other and so always consumes more (notice that the mean of the first household's endowment process is 4, while the mean of the second household's is 3). The perfect correlation between the two consumption services reflects the sharing present in Arrow-Debreu models with time separable preferences. Figure 12.7.2.a graphs $d_t^1 - c_t^1$ while figure 12.7.2.b graphs $d_t^2 - c_t^2$. These figures indicate the "balance of payments" between the two households.

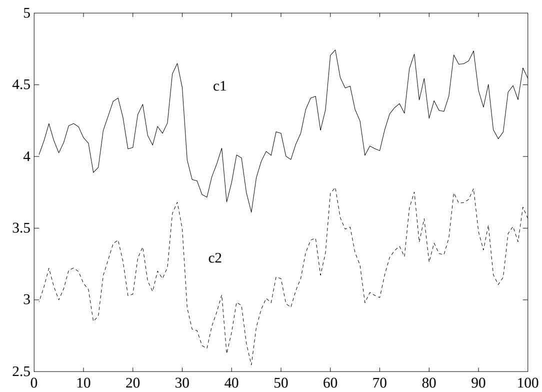

Figure 12.7.1: Consumption allocations of consumers one and two in pure endowment economy.

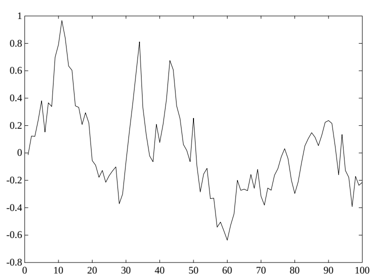

Fig. 12.7.2.a. Saving of consumer one.

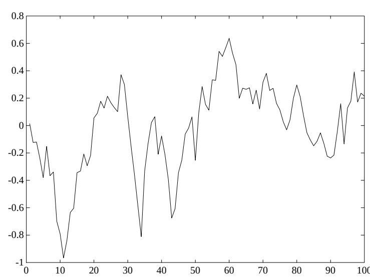

Fig. 12.7.2.ab5. Saving of consumer two.

12.8. Exercises

Exercise 1: The first part of this exercise is to be answered using "pencil and paper" as your tools. The second part is to be done on a computer using MATLAB.

12.8.1. Part one

An economy consists of identical numbers of two types of infinitely lived consumers. Type i consumers all have preferences that are ordered by

$$- .5\, E \sum_{t=0}^{\infty} \left[(c_{it} - b_i)^2\right] | J_0 \quad , \; i = 1, 2 \; , \quad 0 < \beta < 1, \tag{1}$$

where c_{it} is consumption of a single good by an consumer of type i, E is the mathematical expectation operator, and J_0 is information known at time 0. In (1), b_i is a parameter that determines the satiation level of consumption for consumers of type i.

This is a pure endowment (or pure exchange) economy. The only sources of the single consumption good are two types of trees, the first initially being owned by the first type of consumer, the second type of tree initially being owned by the second type of consumer. Initially, there is one tree owned by each consumer in the economy. The first type of tree yields a constant "dividend" of fruit at the rate

$$d_{1t} = d_1 \qquad \forall\; t \geq 0. \tag{2a}$$

The second type of tree yields dividends at time t of

$$\begin{aligned} d_{2,t+1} &= d_2 + g\, w_{t+1}, \qquad t \geq 0, \\ d_{20} &= d_2 \end{aligned} \tag{2b}$$

where d_2 is a constant and where $\{w_{t+1}\}_{t=0}^{\infty}$ is a martingale difference sequence with $E w_{t+1}^2 | J_t = 1$, where $J_t = w^t \equiv (w_t, w_{t-1}, \ldots, w_1)$. Notice that at time 0, $d_{2t} = d_2$, where d_2 is the stationary mean of $\{d_{2t}\}$.

The feasibility constraint is

$$\sum_{i=1}^{2} c_{it} = \sum_{i=1}^{2} d_{it}.$$

You are to consider the following decentralized version of this economy. At time 0, there are open a complete array of Arrow-Debreu contingent claims markets. We use the Harrison-Kreps commodity space and pricing system. At time 0, households of type i face the problem of maximizing (1) subject to

$$E \sum_{t=0}^{\infty} \beta^t \, p_t^0 \, c_{it} | J_0 = \sum_{t=0}^{\infty} \beta^t \, p_t^0 \, d_{it} | J_0. \tag{3}$$

A. Define a *competitive equilibrium* for this economy.

B. Construct an argument to show that this is a *representative consumer economy* in the sense that the equilibrium price system can be determined without simultaneously determining individual allocations.

C. Describe the preferences and constraints faced by the representative consumer in this economy.

D. Compute the equilibrium price system $\{p_t^0\}_{t=0}^{\infty}$.

E. At the equilibrium prices, compute the right hand side of (3), namely,

$$a_i^0 = E \sum_{t=0}^{\infty} \beta^t \, p_t^0 \, d_{it} | J_0, \tag{4}$$

for $i = 1, 2$. Go as far as you can in getting an analytic, closed form expression for a_i^0.

F. Use your answer to E to establish that the value of trees of type 2 is smaller the larger is the absolute value of g. Interpret this result.

G. Give an argument to establish that a_i^0 given by (4) would be the equilibrium price of type i trees if a market in trees (or equivalently, perpetual claims to the dividends from a tree) were opened at time 0.

H. Suppose that $d_1 = d_2 < b_1 = b_2$. Suppose that $g = 0$, so that the second tree has a perfectly sure yield. Compute the equilibrium consumption allocation.

I. Suppose that $d_1 = d_2 = 8$, that $b_1 = b_2 = 15$, and that $g = 1$. Compute the equilibrium consumption allocation (for *both* types of consumer).

  i. Verify that the equilibrium consumption allocations satisfy a "sharing rule".

ii. Which type of consumer consumes more in equilibrium? Why?

J. For this economy, compute the price at time t of a perfectly sure claim to one unit of consumption at time $t+1$.

12.8.2. Part two

Now you are to use the computer. Use MATLAB to do the computations.

HINT: We have written some programs that should be a big help in doing this problem. The main programs are and ; does the main calculations while reads in the parameter values for the economy. A program called does the asset pricing calculations. You can edit these files and run them to answer the question.

A. For the economy described in Exercise 1 of Part One, compute the equilibrium consumption allocation and simulate it. Write down the consumption allocations and the endowment realizations for $t=0,\dots,10$.

B. For the economy described in Exercise 1 of Part One, change the value of g from 1 to 0. Recompute the equilibrium consumption allocation, and simulate it. Write down the consumption allocations and endowment realizations for $t=0,\dots,10$.

C. How do the results of A and B conform to your answers in Part One?

D. Use the program to compute the value of trees at time 0 under the specification of parameter values given in Exercise 1 of Part One (in particular, set $g=1$). Do the results conform with your reasoning in Part One?

12.9. Economic integration

A. The world consists of two virtually identical but separated economies, $j = 1, 2$. The economies never trade with one another. Within each economy there exist Arrow-Debreu markets at time 0.

Economy j has the following structure:

12.9.1. Preferences:

$$-\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t[(c_{jt}-b_j)^2+\ell_t^2]$$

12.9.2. Technology

$$\begin{aligned} c_{jt}+i_{jt} &= \gamma k_{jt-1}+d_{jt}, \quad \gamma>0 \\ \phi i_{jt} &= g_{jt}, \quad \phi>0 \\ k_{jt} &= \delta_k k_{jt-1}+i_{jt}, \quad 0<\delta_k<1 \\ g_{jt}^2 &= \ell_{jt}^2 \\ k_{j,-1} &\text{ given} \end{aligned}$$

12.9.3. Information

$$\begin{aligned} z_{t+1} &= A_{22}z_t+C_2w_{t+1} \\ d_{jt} &= U_{dj}z_t \\ z_0 &\text{ given.} \end{aligned}$$

Here c_{jt} is consumption at t, i_{jt} is investment at t, k_{jt} is the capital stock at the end of period t, g_{jt}^2 is the square of labor absorbed in adjusting the capital stock, d_{jt} is an endowment shock, and b_j is a fixed (across time) preference parameter; all of these objects are scalars. The vector z_t is a set of information variables common to the two economies, and $\{w_{t+1}\}$ is a martingale difference sequence with $Ew_{t+1}w_{t+1}' = I$. The parameters $\gamma, \phi, \beta, \delta_k, A_{22}$, and C_2 are common across the two economies. The two economies differ in their values for b_j and U_{dj}.

a. Define a competitive equilibrium for economy j.

b. Describe how to compute a competitive equilibrium for economy j using dynamic programming.

c. Let p^0_{jt} be the (scaled) time zero Arrow-Debreu price for consumption in economy j. Show that there exists a representation for the equilibrium for economy j of the form

$$\begin{bmatrix} k_{jt} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} A^o_{11} & A^o_{12,j} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} k_{jt-1} \\ z_t \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} w_{t+1}$$

$$\begin{bmatrix} c_{jt} \\ p^0_{jt} \end{bmatrix} = \begin{bmatrix} S_{cj} \\ M_{cj} \end{bmatrix} \begin{bmatrix} k_{jt-1} \\ z_t \end{bmatrix}. \tag{1}$$

In particular, argue that A^o_{11} is the same across the two economies, but that $A^o_{12,j}, S_{cj}$, and M_{cj} depend on j.

HINT: Two approaches to this problem will bear fruit. First, one can obtain an Euler equation for capital and solve it, as in our treatment of Hall's model in chapter 3. Second, one can use the method of "adding speed by partitioning the state vector" described in chapter 9.

B. Consider a world consisting of two economies with preferences, technology, information, and initial capital stocks identical to the previous one. Now, however, the two economies are integrated, there being world-wide time zero Arrow-Debreu markets. Residents of country j own the initial capital stock $k_{j,-1}$ and the endowment process $\{d_{j,t}\}$.

a. Define a competitive equilibrium for the integrated economy.

b. Argue that the integrated economy is a *representative consumer economy*, being careful to define what you mean by a representative consumer economy.

c. Describe how to compute the equilibrium of the representative consumer economy.

d. Let $\hat{p}^0_t$ be the (scaled) time zero Arrow-Debreu price for consumption in the integrated economy, and let $\hat{k}_t = \hat{k}_{1t} + \hat{k}_{2t}$, and $\hat{c}_t = \hat{c}_{1t} + \hat{c}_{2t}$, where $(\hat{\cdot})$ denotes equilibrium objects for the integrated economy. Show that for the integrated economy, the equilibrium has a representation

$$\begin{bmatrix} \hat{k}_t \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} A^o_{11} & A^o_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \hat{k}_{t-1} \\ z_t \end{bmatrix} + \begin{bmatrix} 0 \\ C_2 \end{bmatrix} w_{t+1}$$

$$\begin{bmatrix} \hat{c}_t \\ \hat{p}_t^0 \end{bmatrix} = \begin{bmatrix} S_c \\ M_c \end{bmatrix} \begin{bmatrix} \hat{k}_{t-1} \\ z_t \end{bmatrix},$$

where A_{11}^o and A_{22} and the same objects that appear in (1).

e. Show that $\hat{k}_t = k_{1t} + k_{2t}$ and $\hat{c}_t = c_{1t} + c_{2t}$.

f. Find formulas for $\hat{c}_{1t}$ and $\hat{c}_{2t}$. Show that $\hat{c}_{1t} \neq c_{1t}$ and $\hat{c}_{2t} \neq c_{2t}$.

g. Show that $p_{1t}^0 + p_{2t}^0 = \alpha \, \hat{p}_t^0$ for some positive α.

Chapter 13
Permanent Income Models

This chapter describes a class of permanent income models of consumption, which stress a connection between consumption and income implied by present value budget balance, and which generate interesting predictions about the responses of various components of consumption to identifiable shocks to the information sets of economic agents. The models allow us to characterize how consumption of durables act as a form of savings and how habit persistence alters consumption-savings profiles.

13.1. Technology

To focus on dynamics induced by the household technology, it serves our purposes to adopt the following technology specification:

$$\begin{aligned} \phi_c \cdot c_t + i_t &= \gamma k_{t-1} + e_t \\ k_t &= k_{t-1} + i_t \end{aligned} \tag{13.1.1}$$

where ϕ_c is a vector of positive real numbers with n_c elements, e_t is a scalar exogenous endowment of consumption and k_{t-1} is a scalar capital stock. We set $\delta_k = 1$, thereby ignoring depreciation in capital so that i_t is net investment. Introducing depreciation in capital would add nothing to our analysis because we shall eliminate any additional input requirement for making new capital productive. With no intermediate inputs required for investment, even if there were depreciation in the capital stock, a version of the first equation of (13.1.1) would apply to net investment by suitably altering the marginal product of capital parameter γ.

The empirical counterpart to the scalar endowment process $\{e_t\}$ is typically labor income (e.g., see Flavin 1981, and Deaton 1992). Labor is supplied inelastically and produces e_t units of output independently of the level of capital. The absence of curvature in the technology has some troublesome implications for equilibrium prices that we will discuss later. Nevertheless, this technology provides a good laboratory for studying how the household technology alters consumption-savings profiles. Moreover, this specification has played a prominent role in the empirical literature on the permanent income theory of consumption.

To make the model behave well, we impose the restriction that $(1+\gamma)\beta = 1$. Relaxing this restriction to make capital more or less productive has unpleasant implications for the solution to the model. Thus, *reducing* the marginal product of capital will produce a solution in which the capital stock must eventually become negative.[4] *Increasing* the marginal product of capital typically produces a solution with asymptotic satiation in a deterministic version of the model; stochastic versions yield a marginal utility vector with mean zero in a stochastic steady state. This razor's edge linkage between the marginal product of capital and subjective discount factor is the price we pay for eliminating the role of intermediate goods in making new capital productive.

To put this technology within the general specification of Chapter 3, we include an additional equation

$$\phi_i i_t - g_t = 0, \tag{13.1.2}$$

where ϕ_i is a small positive number. Strictly speaking, this introduces a form of adjustment cost by requiring a household input be used to make capital productive. This small penalty makes capital satisfy the square summability constraint. When there are multiple consumption goods, to make the resulting matrix $[\Phi_c \ \Phi_g]$ nonsingular, we introduce $n_c - 1$ additional investment goods equal to the last $n_c - 1$ entries of c_t Thus, combining these constraints, when n_c is one, we form

$$\Phi_c = \begin{pmatrix} \phi_c \\ 0 \end{pmatrix} \ \Phi_i = \begin{pmatrix} 1 \\ \phi_i \end{pmatrix}, \Phi_g = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{13.1.3}$$

$$\Gamma = \begin{pmatrix} \gamma \\ 0 \end{pmatrix}, \Delta_k = 1, \Theta_k = 1;$$

and when n_c is greater than one,

$$\Phi_c = \begin{pmatrix} \phi_c' \\ 0 \\ 0 \ -I \end{pmatrix} \ \Phi_i = \begin{pmatrix} 1 & 0 \\ \phi_i & 0 \\ 0 & I \end{pmatrix}, \Phi_g = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \tag{13.1.4}$$

---

[4] A more interesting solution of the model imposes a period-by-period nonnegativity constraint on capital. Instead we limit the terminal behavior of the capital stock by imposing square summability.

$$\Gamma = \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}, \Delta_k = 1, \Theta_k = (1 \quad 0);$$

Thus, to embed this model into the general setup of Chapter 3, we have n_c investment goods (the original good and $n_c - 1$ additional ones introduced for technical purposes), and one intermediate good (used to enforce square summability in the capital stock).

13.2. Two Implications

We extract two sharp implications of this class of permanent income models of consumption. We obtain the first by substituting $k_t - k_{t-1}$ for i_t in (13.1.1) and solving the resulting difference equation for k_{t-1}:

$$k_{t-1} = \beta \sum_{j=0}^{\infty} \beta^j (\phi_c \cdot c_{t+j} - e_{t+j}). \tag{13.2.1}$$

From this formula, it follows that

$$k_{t-1} = \beta \sum_{j=0}^{\infty} \beta^j E(\phi_c \cdot c_{t+j} - e_{t+j}) | J_t. \tag{13.2.2}$$

Relation (13.2.2) will help us to find recursive solutions to the model.

Formula (13.2.1) also has important implications for how consumption and (endowment) income respond to the underlying shocks, as displayed by a set of dynamic multipliers - or impulse responses – $\{\chi_j\}$ and $\{\epsilon_j\}$ for $\{c_t\}$ and $\{e_t\}$, respectively, where $\chi_j w_t$ gives the response of c_{t+j} to w_t and $\epsilon_j w_t$ the response of e_{t+j} to w_t. Since the capital stock k_{t-1} cannot depend on w_t it follows from (13.2.1) that the shock must be present-value neutral. In other words, the impact of w_t on current and future values of e_t must be offset in a present-value sense by its impact on current and future values of c_t:

$$\sum_{j=0}^{\infty} \beta^j (\phi_c)' \chi_j = \sum_{j=0}^{\infty} \beta^j \epsilon_j. \tag{13.2.3}$$

Equality (13.2.3) is precisely the present-value relation studied by Flavin (1981), Hamilton and Flavin (1986), Sargent (1987), Hansen, Roberds and Sargent (1991), and Gali (1991).

The second implication pertains to the martingale behavior of the shadow price vector for consumption and capital. To begin, note that the forward evolution equation for the shadow price of capital is

$$\mathcal{M}_t^k = E(\beta \mathcal{M}_{t+1}^k | J_t) + E(\beta \gamma \mathcal{M}_{t+1}^e | J_t). \tag{13.2.4}$$

The first-order conditions for the first component of investment imply that

$$\mathcal{M}_t^e + \phi_i \mathcal{M}_t^g = \mathcal{M}_t^k, \tag{13.2.5}$$

where the left side captures the cost of an additional unit of investment and right side the benefit. The second term on the left side reflects the adjustment cost, and is zero in the limiting $\phi_i = 0$ case. By substituting (13.2.5) into (13.2.4) and using the fact that $\beta(1+\gamma) = 1$, we obtain the martingale implication for the shadow price of capital:

$$\mathcal{M}_t^k = E(\mathcal{M}_{t+1}^k | J_t), \tag{13.2.6}$$

and likewise for the multiplier process $\{\mathcal{M}_t^e\}$.

Finally, the shadow price of consumption is given by:

$$\mathcal{M}_t^c = (\Phi_c)' \mathcal{M}_t^d = \phi_c \mathcal{M}_t^e, \tag{13.2.7}$$

since the last $n_c - 1$ components of $\mathcal{M}_t^d$ are zero because these are the multipliers on a set of bookkeeping identities. Hence the shadow price process for consumption depends on a single scalar multiplier process $\{\mathcal{M}_t^e\}$, a martingale that we call the marginal utility process for income. We shall pursue the present-value budget balance and martingale implications further, and use them to find and represent the solution of the model.

13.3. Solution

To solve the model, we begin by deriving allocation rules for consumption and investment which can be represented in terms of the scalar martingale process $\{\mathcal{M}_t^e\}$. Then we use present-value relation (13.2.2) to compute $\mathcal{M}_t^e$. Our focus on the marginal utility of income imitates an aspect of the analysis in Bewley (1977).

To accomplish the first step, we use the notion of a canonical household technology. Recall that the household technology determines the sequence of consumption services associated with a given sequence of consumption goods and an initial condition for the household capital stock. When the household technology is canonical, we can construct an *inverse system* which maps a given sequence of consumption services and an initial condition on the household capital stock uniquely into a sequence of consumption goods required to support that service sequence. For a household technology to be canonical, there must be the same number of services as goods, the matrix Π must be nonsingular, and the absolute values of the eigenvalues of the matrix $(\Delta_h - \Theta_h \Pi^{-1}\Lambda)$ must be strictly less than $\beta^{-1/2}$. Under these restrictions, the inverted system can be represented recursively as:

$$c_t = \Lambda^* h_{t-1} + \Pi^* s_t \tag{13.3.1}$$

$$h_t = \Delta_h^* h_{t-1} + \Theta_h^* s_t$$

where

$$\Lambda^* \equiv -\Pi^{-1}\Lambda,\ \Pi^* \equiv \Pi^{-1};$$

and

$$\Delta_h^* \equiv (\Delta_h - \Theta_h \Pi^{-1}\Lambda),\ \Theta_h^* \equiv \Theta_h \Pi^{-1}.$$

In chapter **@preferences@**, we showed that there always exists a representation of the induced preferences for consumption goods in terms of a canonical technology.

Governing the multipliers is an analogous dual system:

$$\mathcal{M}_t^h = E[\beta(\Delta_h)'\mathcal{M}_{t+1}^h | J_t] + E[\beta\Lambda'\mathcal{M}_{t+1}^s | J_t] \tag{13.3.2}$$

$$\mathcal{M}_t^c = (\Theta_h)'\mathcal{M}_t^h + \Pi'\mathcal{M}_t^s;$$

and an associated inverse system:

$$\mathcal{M}_t^h = E[\beta(\Delta_h^*)'\mathcal{M}_{t+1}^h|J_t] - E[\beta(\Lambda^*)'\mathcal{M}_{t+1}^c|J_t] \tag{13.3.3}$$

$$\mathcal{M}_t^s = -(\Theta_h^*)'\mathcal{M}_t^h + (\Pi^*)'\mathcal{M}_t^c.$$

Since $\{\mathcal{M}_t^c\}$ is a martingale sequence, it follows from the inverse dual system that $\{\mathcal{M}_t^h\}$ and $\{\mathcal{M}_t^s\}$ are both martingales. In fact they are linear combinations of the scalar martingale sequence $\mathcal{M}_t^e$. For instance,

$$\mathcal{M}_t^s = M_s\mathcal{M}_t^e \tag{13.3.4}$$

where

$$M_s \equiv \{(\Pi^*)' + (\Theta_h^*)'[I - \beta(\Delta_h^*)']^{-1}\beta\Lambda^{*\prime}\}\phi_c \tag{13.3.5}$$

Consequently, we can solve for the service sequence in terms of the scalar martingale $\{\mathcal{M}_t^e\}$ from the simple link between the vector of services and the corresponding marginal utility vector:

$$s_t = -M_s\mathcal{M}_t^e + b_t. \tag{13.3.6}$$

From this relation and from the inverse household technology (13.3.1) it follows that

$$c_t = \Lambda^* h_{t-1} - \Pi^* M_s\mathcal{M}_t^e + \Pi^* b_t \tag{13.3.7}$$

$$h_t = \Delta_h^* h_{t-1} - \Theta_h^* M_s\mathcal{M}_t^e + \Theta_h^* b_t.$$

To characterize the decision rule for investment, we solve (13.1.1) for i_t and substitute the right-hand side of (13.3.7) for c_t:

$$i_t = \gamma k_{t-1} + e_t - \phi_c \cdot c_t \tag{13.3.8}$$

$$= \gamma k_{t-1} + e_t - (\phi_c)'\Lambda^* h_{t-1} - (\phi_c)'\Pi^*(b_t - M_s\mathcal{M}_t^e).$$

So far, we have derived a recursive representation for consumption, investment and household capital in terms of the scalar multiplier process $\{\mathcal{M}_t^e\}$. However, we have not shown how to initialize this sequence, and we do not yet have a formula relating the time t increment of this process to the underlying martingale difference sequence $\{w_t\}$. We now derive formulas for both of these objects.

To find an expression for the marginal utility of income process, we exploit the present-value budget balance restriction (13.2.2). In light of the inverse system (13.3.1) for the household technology, we are led to compute the expected discounted sum of services and household capital:

$$\sum_{j=0}^{\infty} \beta^j E(s_{t+j}|J_t) = -[1/(1-\beta)] M_s \mathcal{M}_t^e + \sum_{j=0}^{\infty} \beta^j E(b_{t+j}|J_t), \tag{13.3.9}$$

and

$$\beta \sum_{j=0}^{\infty} \beta^j E(h_{t+j}|J_t) = \beta \sum_{j=0}^{\infty} \beta^j E(\Delta_h^* h_{t+j-1} + \Theta_h^* s_{t+j}|J_t). \tag{13.3.10}$$

Rewriting (13.3.10), we see that

$$(I - \beta\Delta_h^*) \sum_{j=0}^{\infty} \beta^j E(h_{t+j-1}|J_t) = h_{t-1} + \beta\Theta_h^* \sum_{j=0}^{\infty} \beta^j E(s_{t+j}|J_t),$$

or

$$\begin{aligned} &\sum_{j=0}^{\infty} \beta^j E(h_{t+j-1}|J_t) = \\ &(I-\beta\Delta_h^*)^{-1} h_{t-1} + \beta(I-\beta\Delta_h^*)^{-1}\Theta_h^* \sum_{j=0}^{\infty} \beta^j E(s_{t+j}|J_t). \end{aligned} \tag{13.3.11}$$

Since $\phi_c \cdot c_t = (\phi_c)'(\Lambda^* h_{t-1} + \Pi^* s_t)$, it follows from equation (13.3.1) and (13.3.9) that

$$\begin{aligned} &(1+\gamma)k_{t-1} - (\phi_c)'\Lambda^*(I-\beta\Delta_h^*)^{-1} h_{t-1} = \\ &- M_s' M_s \mathcal{M}_t^e/(1-\beta) + \sum_{j=0}^{\infty} \beta^j E(M_s' b_{t+j} - e_{t+j}|J_t). \end{aligned} \tag{13.3.12}$$

Solving for $\{\mathcal{M}_t^e\}$ results in

$$\begin{aligned} \mathcal{M}_t^e = 1/(M_s' M_s)[(1-\beta) &\sum_{j=0}^{\infty} \beta^j E(M_s' b_{t+j} - e_{t+j}|J_t) \ - \\ &\gamma k_{t-1} + (1-\beta)(\phi_c)'\Lambda^*(I-\beta\Delta_h^*)^{-1} h_{t-1}]. \end{aligned} \tag{13.3.13}$$

To interpret this solution, it is useful to decompose the right-hand side of (13.3.13) into three components. First, we follow the permanent income literature by defining permanent income to be that amount of income to be spent on consumption that can be expected to persist in the future and still satisfy (13.2.2):

$$y_t^p \equiv \gamma k_{t-1} + (1-\beta)\sum_{j=0}^{\infty} E(\beta^j e_{t+j}|J_t). \tag{13.3.14}$$

Formally, this is obtained by letting $\{y_t^p\}$ be a hypothetical expenditure process for consumption, assuming it is a martingale, substituting y_{t+j}^p for $\phi_c \cdot c_{t+j}$ in equation (13.2.2), and solving for y_t^p.

Note that this measure of permanent income does not adjust for risk in the endowment sequence and hence even when divided by $(1-\beta)$ is distinct from equilibrium wealth. Nevertheless, it is an important component of the solution to the model. In fact, $c_t = y_t^p$ is the solution for consumption in the in the special case of single good, no preference shocks (b_t constant), and time separable preferences (Λ zero), which is Hall's (1978) permanent income model of consumption. In this simple case, the marginal utility process for endowment income is formed by translating the negative of permanent income: $\mathcal{M}_t^e = b - y_t^p$.

More generally, when the preference shock process is not expected to be constant, the term of interest is a 'permanent' measure of the preference shock sequence:

$$b_t^p \equiv (1-\beta)\sum_{j=0}^{\infty} E[\beta^j (M_s)' b_{t+j}|J_t]. \tag{13.3.15}$$

Finally, when preferences are not separable over time, the household capital stock also enters the solution for the marginal utility of endowment income. To interpret its coefficient, consider the sequence of consumption goods required to support zero consumption services from now into the future. To compute this sequence, simply feed a sequence of zeros into the inverse of the household technology. Discounting the resulting consumption sequence and premultiplying by $(\phi_c)'$ results in:

$$y_t^h \equiv (1-\beta)(\phi_c)'\Lambda^*(I-\beta\Delta_h^*)^{-1}h_{t-1}. \tag{13.3.16}$$

Hence y_t^h adjusts the permanent income measure to account for implicit consumption associated with a "baseline" zero service sequence.

To summarize, the marginal utility of endowment income can be represented as:

$$\mathcal{M}_t^e = (1-\beta)(1/M_s' M_s)(b_t^p - y_t^p + y_t^h). \tag{13.3.17}$$

Relation (13.3.17) gives a decomposition of marginal utility of income into three components: b_t^p, y_t^p and y_t^h. Increases in b_t^p result in an outward shift in preference for consumption, which increases the marginal utility of income; increases in e_t^p correspond to an outward shift of permanent income, which reduces the marginal utility of income; and alterations in y_t^h reflect movements in the initial household capital stock. The measurement of this third component is the discounted endowment-equivalent consumption sequence associated with a baseline (zero) sequence of services. Increasing it has an opposite impact on the marginal of income from increasing permanent income.

The final task of this section is to deduce a formula for the increment of $\mathcal{M}_t^e$ of the form μw_t for some μ. Note that y_t^h depends on time $t-1$ information. Hence only the b_t^p and y_t^p terms enter into consideration. Let $\{\psi_j\}$ denote the sequence of matrices of dynamic multipliers for the preference shock process $\{b_t\}$. It follows from (13.3.13) that

$$\mu = [(1-\beta)/(M_s' M_s)] \sum_{j=0}^{\infty} \beta^j [(M_s)'\psi_j - \epsilon_j]. \tag{13.3.18}$$

The dynamic multipliers $\{\chi_j\}$ for consumption can then be computed recursively from (13.3.7), and by construction satisfy the present-value budget balance restriction (13.2.3).

13.4. Deterministic Steady States

It is useful to study consumption in a deterministic steady state, partly to verify that there exist configurations of the model for which consumption of all goods is positive in this steady-state. Otherwise, the stochastic versions of the model would likely have some perverse implications. We consider cases in which $\{b_t\}$ and $\{e_t\}$ are constants set at the values b and e, respectively.

For a deterministic version of the model, the marginal utility of income is constant over time. Of course, we are only interested in initial conditions such that the initial marginal utility is positive and hence the entire sequence is positive. Thus from equation (13.3.17), we are lead to require that:

$$\mathcal{M}_0^e = (1/M_s' M_s)(M_s' b - e - \gamma k_{-1} + y_0^h) > 0. \tag{13.4.1}$$

Since $1/(M_s' M_s)$ is positive by construction, we only need to be concerned with requiring that $(M_s' b - e - \gamma k_{-1} + y_0^h)$ be positive. Any changes in (b, e, h_{-1}, k_{-1}) that alter the right side of (13.4.1) will clearly change the marginal utility of income (in all time periods).

Since the preference shifter sequence is fixed at a constant level, the constant marginal utility of income sequence implies a constant service sequence given by:

$$s = b - M_s \mathcal{M}_0^e. \tag{13.4.2}$$

The corresponding sequences of consumption goods and household capital need not be constant, and we now investigate the limiting behavior of these objects. Armed with the consumption service sequence, the consumption and household capital sequences can be computed from the inverse household technology.

The absolute values of the eigenvalues of Δ_h^* are less than $\beta^{-1/2}$ but can be greater than or equal to one. Without further restricting the eigenvalues of Δ_h^* to have absolute values that are strictly less than one, the consumption sequence may not converge to a steady state.[5] With the additional restriction that the absolute values of eigenvalues are strictly less than one, the consumption and household capital sequences will converge with limits:

$$\begin{aligned} h_\infty &= (I - \Delta_h^*)^{-1} \Theta_h^* s \\ c_\infty &= \Lambda^* h_\infty + \Pi^* s \end{aligned} \tag{13.4.3}$$

where variables with subscript ∞ denote limit points. By combining (13.4.2) and (13.4.3), it can be checked whether the limiting consumption vector is strictly positive.

As an illustration, consider a setting with a single consumption good, a single physical capital stock, and the following household technology:

$$\begin{aligned} h_t &= \delta_h h_{t-1} + (1 - \delta_h) c_t \\ s_t &= \lambda h_{t-1} + c_t, \end{aligned} \tag{13.4.4}$$

where δ is a depreciation factor between zero and one. Notice that the household capital stock is constructed to be a weighted average of current and past consumption. The inverse system is given by:

$$\begin{aligned} h_t &= [\delta_h - \lambda(1 - \delta_h)] h_{t-1} + (1 - \delta_h) s_t \\ c_t &= -\lambda h_{t-1} + s_t. \end{aligned} \tag{13.4.5}$$

[5] As emphasized by Becker and Murphy (1988), configurations with explosive eigenvalues can arise in models of 'rational addiction'.

In this simple case, the eigenvalue of Δ_h^* is simply the coefficient on the lagged capital stock in the evolution equation for household capital. This coefficient has an absolute value less than one if:

$$-1 < \lambda < (1+\delta)/(1-\delta). \tag{13.4.6}$$

When these inequalities are satisfied consumption and the household capital stock both converge to $s/(1+\lambda)$.

Negative values of λ that violate the inequalities in (13.4.6) display a form of 'rational addiction' as analyzed by Becker and Murphy (1988). For instance, when λ is -1, the coefficient on lagged capital is unity, and the consumption sequence required to support a constant service sequence must grow linearly over time. Lower values of λ (i.e., negative ones with larger absolute values) imply geometric growth in consumption.

Simply requiring the limiting value of consumption to be positive guarantees that consumption will be positive for initial levels of household capital close to the steady state, but it would be nice to obtain a stronger result. One strategy would require entries of the matrices of the inverse household technology all to be positive. Unfortunately, this restriction that would eliminate some important examples in which there is substitutability across goods or over time. For instance, in (13.4.4), when λ is positive as in the case of a durable good, the inverse household technology has a negative coefficient. Nevertheless, starting from an initial level of household capital below the steady state will result in a positive consumption sequence.

13.5. Cointegration

A key feature of our solution is that the marginal utility of income process is a martingale, which implies via (13.3.6) that the consumption service process is nonstationary. If in addition the preference shock process $\{b_t\}$ is asymptotically stationary, then the service process and consumption are each cointegrated.

Suppose that the preference shock process $\{b_t\}$ is asymptotically stationary, but unobservable to the econometrician. This implies that there are $n_s - 1$ linear combinations of consumption services that are asymptotically stationary. To show this, take any vector ψ that is orthogonal to M_s. It follows from

(13.3.6) that

$$\psi' s_t = \psi' b_t. \tag{13.5.1}$$

Evidently there are $n_s - 1$ linearly independent ψ's. Each such ψ is referred to as a *cointegrating vector*, in the terminology of Granger and Engle (19??).

An extensive literature treats the efficient estimation of cointegrating vectors. However, what interest us are not the cointegrating vectors but rather the vector M_s that is orthogonal to all of the cointegrating vectors for consumption services.

The cointegrating vectors for consumption services differ from the cointegrating vectors for consumption goods. To deduce the cointegrating vectors for the consumption flows, we shall build upon the deterministic steady-state calculations reported in (13.4.3). From (13.4.3), we know that for a deterministic steady state

$$c_\infty = [\Lambda^*(I - \Delta^*)^{-1}\Theta^* + \Pi^*]s_\infty. \tag{13.5.2}$$

The matrix on the right-hand side of (13.5.2) also gives the transformation mapping a date t shock to consumption services to the limiting response of consumption. Any vector ψ satisfying

$$\psi'[\Lambda^*(I - \Delta^*)^{-1}\Theta^* + \Pi^*]M_s = 0 \tag{13.5.3}$$

is a cointegrating vector for consumption. Let Ψ denote an $n_c - 1$ by n_c matrix with rows that are linearly independent cointegrating vectors. Notice that the implied model for Ψc_t and $c_{1,t} - c_{1,t-1}$ contains only stationary endogenous variables, so that it can be estimated using methods that require asymptotic stationarity, like the frequency-domain methods of chapter 9. An estimation strategy based on recursive formulations of the Gaussian conditional likelihood function would not require such a transformation to a stationary set of endogenous variables, but would require confronting the nonexistence of an asymptotically stationary distribution of the state vector from which to draw an initial estimator of the state. In chapter 9, we described a method based on ideas of Kohn and Ansley (1985) designed to construct an initial estimator of the state in such a circumstance.

If we were to assume that the preference shock process is itself nonstationary and that there does not exist any nontrivial cointegrating vector for this process, then it would follow that there exist no cointegrating vectors for either the service process or the consumption process. In this case, to utilize estimation methods requiring stationary, we would base parameter estimation on the implications for the differenced processes for consumption and household capital.

13.6. Constant Marginal Utility of Income

In the absence of uncertainty, the marginal utility of income process will be constant. In this section we introduce uncertainty in the endowment and preference shock processes, and ask: when will the marginal utility of income process remain constant through time? Constancy of the marginal utility of income is an extreme version of the prediction of permanent income theory that the ability to transfer consumption over time results in "smoothness of consumption" over time. In the absence of preference shocks, a constant marginal utility of income process implies that consumption services will also be constant through time.

We address this question from two angles. Initially, we investigate the limiting behavior as the subjective discount factor β approaches unity, and provide conditions on the stochastic structure sufficient to imply constant marginal utilities of income in the limit. In taking this limit, we will not concern ourselves with interpreting directly the limit economy. Instead we will study the limiting behavior of the solutions to the optimal resource allocation problems along with the associated marginal utility of income processes. The second attack on the question is to characterize the specifications of uncertainty that imply a constant marginal utility of income process for a given β that is strictly less than one.

The initial portion of our investigation will imitate and replicate features of Bewley's (1977) study of the permanent income model of consumption. Our analysis is mechanically simpler than Bewley's, but different. When we drive β to one, we maintain the link between the subjective discount factor and the marginal product of capital. Hence as β tends to one in our analysis, the marginal product of capital as measured by γ is simultaneously being driven to zero. In contrast, Bewley considered setups in which the counterpart to the marginal product of capital is always zero. Since capital is less productive, nonnegativity constraints are a central feature of his analysis of the economies with β strictly less than one.

Suppose that $\{z_t\}$ is a stationary stochastic process and that z_t has a finite second moment. Then it is known that the time series average $(1/N)\sum_{j=0}^{N-1} z_{t+j}$

converges.[6] Moreover, the limit vector is invariant to the starting date t of the average. Consistent with our setup in previous chapters, we assume that both b_t and e_t are linear functions of z_t. Recall that the portion of the solution for the marginal utility of income that is not predetermined (the portion that can respond to a current-period shock) is a linear combination, say ν, of a conditional expectation of the geometric average $(1-\beta)\sum_{j=0}^{\infty}\beta^j z_{t+j}$ where

$$\nu \equiv [1/(M_s' M_s)](M_s U_b - U_e) \tag{13.6.1}$$

$b_t = U_b z_t$ and $e_t = U_e z_t$ [see (13.3.13)]. While the simple time-series average and the geometric average will not typically agree, they can be made arbitrarily close by driving N to infinity and β to one. Under both limits, tail terms in the average become relatively more important as the limit point is approached. Formally, it follows from the theory of Cesaro and Abel summability that

$$\lim_{N\to\infty}(1/N)\sum_{j=0}^{N-1} z_{t+j} = \lim_{\beta\to 1}(1-\beta)\sum_{j=0}^{\infty}\beta^j z_{t+j} \tag{13.6.2}$$

(*e.g.* see Zygmond 1979, Theorem 1.33, page 80). Therefore, as the discount factor tends to one, the right side of (13.6.2) converges to a vector independent of t. Moreover, for the information structures we impose, the limit vector must be in the initial period information set.[7] The constancy of the marginal utility of income as β goes to unity follows immediately.

The argument just provided relies on stationarity but does not require linearity in the evolution equation for $\{z_t\}$. In fact, stationarity often can be replaced by a weaker notion of "asymptotic stationarity" as we now illustrate using the linear specification

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1} \tag{13.6.3}$$

imposed elsewhere in this book. This specification can be exploited to obtain an alternative demonstration of the constancy of the marginal utility of income.

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[6] The convergence is both with probability one and in mean square, where mean-square convergence is defined using the square root of the second moment as a norm. We use mean-square convergence in our subsequent analysis.

[7] This follows because the limiting random variable has a finite second moment. As long as the forecast error variance is independent of calendar time and information accumulates over time, the forecast error variance must be zero.

Recall that

$$(1-\beta)\sum_{j=0}^{\infty}\beta^j E(z_{t+j}|J_t) = (1-\beta)(I-\beta A_{22})^{-1} z_t. \tag{13.6.4}$$

To investigate the limit as β tends to one, it is convenient to uncouple the dynamics according to the eigenvalues. Let

$$A_{22} = TDT^{-1} \tag{13.6.5}$$

be the Jordan decomposition, and suppose that D can be partitioned as:

$$D = \begin{pmatrix} I & 0 \\ 0 & D_2 \end{pmatrix} \tag{13.6.6}$$

where the absolute values of the diagonal entries of D_2 are all strictly less than one. Using the Jordan decomposition, it follows that

$$(1-\beta)(I-\beta A_{22})^{-1} = T \begin{pmatrix} I & 0 \\ 0 & (1-\beta)(I-\beta D_2)^{-1} \end{pmatrix} T^{-1}. \tag{13.6.7}$$

Taking limits, we see that

$$\lim_{\beta\to 1}(1-\beta)(I-\beta A_{22})^{-1} = T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{13.6.8}$$

Therefore, $(1-\beta)\sum_{j=0}^{\infty}\beta^j E(z_{t+j}|J_t)$ depends only on

$$z_t^* \equiv (\, I \quad 0\,)\, T^{-1} z_t, \tag{13.6.9}$$

where z_t^* has law of motion

$$z_{t+1}^* = z_t^* + C^* w_{t+1}, \tag{13.6.10}$$

where

$$C^* \equiv (\, I \quad 0\,)\, T^{-1} C_2. \tag{13.6.11}$$

Sufficient conditions for the marginal utility of income to be constant are that the Jordan decomposition of A_{22} satisfies (13.6.6) and C^* be zero. When these restrictions are satisfied, the process $\{z_t\}$ will be asymptotically stationary because the process $\{z_t^*\}$ will be constant over time and because $\{(\, 0 \quad I\,)\, T^{-1} z_t\}$ will converge to a stationary process. This latter result follows since the diagonal entries of D_2 have absolute values that are strictly less than unity. Stationarity is only implied when the $\{z_t\}$ is initialized appropriately.

The arguments just given cannot be extended to $\{z_t\}$ processes with more fundamental forms of nonstationarity. For instance, time trends or unit roots in the endowment process would suffice to overturn constancy of the marginal utility of income in the limit. In the case of time trends, the averages may diverge as the limits are taken. For unit root processes (without drifts) the limits are well defined, but the uncertainty in the marginal utility of income process does not vanish.

We now change experiments and hold fixed the subjective discount factor and ask if it is still possible for the marginal utility of income to be constant. The answer to this question turns out to be yes. Assume the Jordan decomposition (13.6.5) and (13.6.6) along with restriction (13.6.11), except that D_2 can now have eigenvalues with absolute values that are equal or even greater than one (but less than $\beta^{-1/2}$). If

$$\nu(I - \beta A_{22})^{-1} = (\, \nu^* \quad 0 \,) \tag{13.6.12}$$

for some vector ν^*, then the marginal utility of income will be constant over time. While this clearly imposes a restriction on the matrix A_{22}, it is satisfied by some stationary and nonstationary specifications of the endowment and preference shock processes.

13.7. Consumption Externalities

One of the prime motivations for intertemporal complementarities put forth by Ryder and Heal (1973) is that individual consumers are concerned in part about their consumption relative to the past community average. In other words, there is an externality in consumption. This motivation is in contrast to that given by Becker and Murphy (1988) in which the complementarities are purely private. The solution described previously is applicable even if this consumption externality is present as a solution to an optimal resource allocation problem. However, the link between optimal resource allocation and competitive equilibrium may vanish when there is a consumption externality. We now investigate this issue.

To capture the externality, we endow the consumer with the household technology:

$$\begin{aligned} H_t &= \Delta_h H_{t-1} + \Theta_h C_t \\ s_t &= \Lambda H_{t-1} + \Pi c_t \end{aligned} \tag{13.7.1}$$

where H_t, H_{t-1} and C_t are interpreted as community-wide vectors that are beyond the control of the private consumers but are equal to their lower case counterparts in equilibrium.

The previous argument justifying the martingale properties of the marginal utilities of income and consumption relied only on the technology specification and still applies when the externality is present. In light of the externality interpretation, the marginal utility of services now satisfies:

$$\begin{aligned} \mathcal{M}_t^s &= (\Pi^*)' \mathcal{M}_t^c \\ &= (\Pi^*)' \phi_c \mathcal{M}_t^e. \end{aligned} \tag{13.7.2}$$

Recall that Π^* is equal to Π^{-1}. Although the link between the marginal utility of services and marginal utility of consumption goods is altered, the marginal utility of service process remains a martingale. The previous solution method can now be imitated by substituting the matrix $(\Pi^*)'\phi_c$ for M_s given by (13.3.4).

When there is a single consumption good and the household technology is canonical, the two solutions coincide. This alteration can be verified by taking the previous solution for the marginal utility of income and showing that all of the equilibrium conditions and first-order conditions remain satisfied. While the marginal utility of services is altered by a constant scale factor over time, this clearly has no impact on the implied marginal rates of substitution for consumption services and therefore the original quantity allocation remains intact with the externality interpretation. When there are multiple consumption goods, the quantity allocations can be altered. Also, when the original household technology is not 'canonical,' the quantity allocations can be altered even when there is a single consumption good. While there generally exists a canonical household technology that implies the same induced preferences for consumption goods, the externality version of the specification can give rise to a fundamentally different canonical technology, breaking the simple link between solution to the resource allocation problem and the decentralized economy.

To elaborate on this last point, suppose the original household technology is not canonical. In Chapter ? we showed how to find the corresponding canonical technology to be used in solving the optimal resource allocation problem. In the presence of consumption externalities, we can find the analog to a canonical household technology by first noting that

$$b_t - s_t = B_t - \Pi c_t, \tag{13.7.3}$$

where

$$B_t \equiv b_t - \Lambda H_{t-1}.$$

Consequently,

$$(b_t - s_t) \cdot (b_t - s_t) = B_t \cdot B_t - 2(B_t)'\Pi c_t + (c_t)'\Pi'\Pi c_t. \tag{13.7.4}$$

Suppose that $\Pi'\Pi$ is nonsingular, and obtain a factorization:

$$\hat{\Pi}'\hat{\Pi} = \Pi'\Pi \tag{13.7.5}$$

where $\hat{\Pi}$ is nonsingular. Also, define

$$\begin{aligned} \hat{\Lambda} &\equiv \hat{\Pi}'^{-1}\Pi'\Lambda \\ \hat{b}_t &\equiv \hat{\Pi}'^{-1}\Pi' b_t \\ \hat{s}_t &\equiv \hat{\Lambda} H_{t-1} + \hat{\Pi} c_t. \end{aligned} \tag{13.7.6}$$

Then $(b_t - s_t) \cdot (b_t - s_t)$ and $(\hat{b}_t - \hat{s}_t) \cdot (\hat{b}_t - \hat{s}_t)$ agree except for a term that is not controllable by the individual consumer. Consequently, technology (13.7.6) and the implied preferences for the original household technology are the same.

For this solution method to apply, we need this transformed version of the household technology to be canonical. Since the matrix $\hat{\Pi}$ is nonsingular by construction, it suffices for the matrix $\Delta_h - \Theta_h \hat{\Pi}^{-1}\hat{\Lambda}$ to have eigenvalues with absolute values that are strictly less than $\beta^{-1/2}$. If this restriction is not satisfied then there fails to exist a competitive equilibrium.

13.8. Tax Smoothing Models

By reinterpreting variables, our model can represent a class of linear models of optimal taxation, versions of which were studied by Barro (1979) and Judd (1990). Let τ_t be a vector of taxes collected from various sources (e.g., capital, labor, imports, etc.); G_t a scalar stochastic process of government expenditures; B_{t-1} the stock of risk-free one-period government debt bearing net one-period interest rate of $\gamma = \frac{1}{\beta} - 1$; and def_t the gross-of-interest 0 deficit. Match up variables as follows: $c_t \sim \tau_t, e_t \sim G_t, k_{t-1} \sim B_{t-1}, i_t \sim \text{def}_t$. Set ϕ_c to a vector of ones, so that $\phi_c' \tau_t$ measures total time t government tax revenues. With these associations, (13.1.1) become

$$\begin{aligned} \text{def}_t &= \gamma B_{t-1} + G_t - \phi_c' \tau_t \\ B_t &= B_{t-1} + \text{def}_t . \end{aligned}$$

The preference ordering is interpreted as minus the loss function associated with taxation, and measures the dynamics of tax distortions.

Consider three special cases of this model, each of which sets b_t to a vector of zeroes.

1. *Random walk taxes.* To capture Robert Barro's specification, set $\phi_c = 1$ (so there is only one kind of tax revenues), $\Lambda = 0, \Pi = 1, \Theta_h = \Delta_h = 0$. This version makes taxes follow a random walk. It is a relabelling of Hall's model of consumption.

2. *White noise taxes.* To capture a one-tax version of Judd's specification, again set $\phi_c = 1$, but now set $\Pi = \Delta_h = 1, \Theta_h = \Lambda = -1$. With these settings, the government's objective function is $-.5 E_0 \sum_{t=0}^{\infty} \beta^t (\sum_{j=0}^{t} \tau_{t-j})^2$. This specification is intended to capture the long-lived adverse effects of taxation on capital. The optimal policy makes taxes a white noise process, a feature that characterizes the asymptotic behavior of capital taxation in the model of Chari, Christiano, and Kehoe (1994). To deduce the white noise property for this model, use (13.3.7) and the relations defining $\Lambda^*, \Pi^*, \Delta_h^*, \Theta_h^*$ under (13.3.1). In particular, we obtain $\tau_t = -M_s(\mathcal{M}_t^e - \mathcal{M}_{t-1}^e)$.

3. *Two taxes.* Set $\phi_c = [1 \quad 1]$, and specify two taxes whose 'distortion technology' is obtained by stacking the two technologies described in examples 1 and 2. This is the kind of setup advocated by Judd (1990), and makes one tax a random walk, the other a white noise.

Chapter 14
Non-Gorman Heterogeneity Among Households

14.1. Introduction

The previous chapter studied a setting in which households have heterogeneous endowments and preference shock, but otherwise have identical preferences and household technologies, implying that all Engle curves are linear with the same slopes. The property of identically sloped linear Engle curves delivers a tidy and tractable theory of aggregation assuring the existence of a representative household. This theory applies when different households share the same household technology $(\Lambda, \Pi, \Delta_h, \Theta_h)$.

In this chapter, we maintain the linearity of households' Engle curves, but permit their slopes to vary across classes of households. In particular, we now allow households' technology parameters $(\Lambda_i, \Pi_i, \Delta_{hi}, \Theta_{hi})$ to differ across classes of households indexed by i. This alteration causes the existence of a representative household, in the sense of there being a preference ordering over stochastic processes for aggregate consumption *that is independent of the initial wealth distribution*, to vanish. Nevertheless, the structure still fits within a class that readily yields to linear quadratic dynamic programming algorithms. Competitive equilibria can be calculated using an algorithm based on Negishi's idea of finding a fixed point within a class of Pareto problems, where the fixed point is a list of Pareto weights that deliver budget balance at candidate equilibrium prices. In this chapter, we describe how the algorithms can be applied very efficiently within our class of economies. We also describe how a more limited form of aggregation than Gorman's can be carried out for this economy. In particular, implementation of the Negishi algorithm enables us to uncover a 'mongrel' preference ordering over aggregate consumption streams, where the preference ordering depends on the initial distribution of wealth, as do the parameters of any 'household technology' for representing those preferences.

14.2. Households' Preferences

There are equal numbers of two types of households, indexed by $i = 1, 2$. Households of type i have preferences ordered by

$$-\frac{1}{2} E \sum_{t=0}^{\infty} \beta^t \left[(s_{it} - b_{it}) \cdot (s_{it} - b_{it}) + \ell_{it}^2 \right] \mid J_0. \tag{14.2.1}$$

Here s_{it} is a consumption service vector for consumer i, b_{it} is a preference shock process, and ℓ_{it} is labor supplied by consumer i. Services s_{it} are produced via the technology

$$s_{it} = \Lambda_i h_{it-1} + \Pi_i c_{it} \tag{14.2.2}$$

$$h_{it} = \Delta_{h_i} h_{it-1} + \Theta_{h_i} c_{it} \ , \ i = 1, 2 \tag{14.2.3}$$

Here h_{it} is consumer i's stock of household durables at the end of period t, and c_{it} is consumer i's rate of consumption. The preference shock process b_{it} is governed by

$$b_{it} = U_{bi} z_t \tag{14.2.4}$$

where z_t continues to be governed by

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1}$$

Notice that this specification permits each class of households to have its own list of matrices $(\Lambda_i, \Pi_i, \Delta_{hi}, \Theta_{hi})$ that determine a technology for converting consumption goods into services.

14.2.1. Technology

Consumption goods (c_{1t}, c_{2t}) are produced via the technology

$$\Phi_c(c_{1t} + c_{2t}) + \Phi_g g_t + \Phi_i i_t = \Gamma k_{t-1} + d_{1t} + d_{2t} \tag{14.2.5}$$

$$k_t = \Delta_k k_{t-1} + \Theta_k i_t \tag{14.2.6}$$

$$g_t \cdot g_t = \ell_t^2, \qquad \ell_t = \ell_{1t} + \ell_{2t}. \tag{14.2.7}$$

As before, g_t denotes the quantity of labor-using intermediate production activities; d_{it} is the amount of the endowment vector of household i used in the production process. We assume that

$$d_{it} = U_{d_i} z_t \quad , \; i = 1, 2. \tag{14.2.8}$$

14.3. A Pareto Problem

The social welfare function is a weighted average of the utilities of the two households, with weight on household 1's utility being λ, $0 < \lambda < 1$. For fixed λ, we want to find an allocation that maximizes

$$-\frac{1}{2}\,\lambda E_0 \sum_{t=0}^{\infty} \beta^t [(s_{1t} - b_{1t}) \cdot (s_{1t} - b_{1t}) + \ell_{1t}^2]$$
$$-\frac{1}{2}\,(1-\lambda) E_0 \sum_{t=0}^{\infty} \beta^t [(s_{2t} - b_{2t}) \cdot (s_{2t} - b_{2t}) + \ell_{2t}^2]$$

subject to the constraints that describe the household and production technologies. By way of fitting it into an optimal linear regulator, it is convenient to note a property of the solution to the problem that permits us to avoid carrying along ℓ_{1t} and ℓ_{2t} as variables, and to replace them by functions of ℓ_t. The solution of the social planning problem implies a pair of simple 'sharing rules' for labor. We deduce these sharing rules before solving the full problem in order to economize the number of control variables.

Let $\mathscr{M}_t^\ell$ be the stochastic Lagrange multiplier associated with the constraint $\ell_{1t} + \ell_{2t} = \ell_t$. With respect to ℓ_{1t} and ℓ_{2t}, the first order conditions are $\mathscr{M}_t^\ell = \lambda \ell_{1t}$ and $\mathscr{M}_t^\ell = (1-\lambda)\ell_{2t}$. These conditions imply that $\ell_t = \ell_{1t} + \ell_{2t} = \mathscr{M}_t^\ell / (\lambda(1-\lambda))$, or $\mathscr{M}_t^\ell = \lambda(1-\lambda)\ell_t$. Substituting this last equality for $\mathscr{M}_t^\ell$ into the marginal conditions for ℓ_{1t} and ℓ_{2t} gives the 'sharing rules'

$$\ell_{1t} = (1-\lambda)\ell_t, \qquad \ell_{2t} = \lambda \ell_t.$$

Use these two equations to represent the terms in ℓ_{1t} and ℓ_{2t} in the social planning criterion as

$$\lambda \ell_{1t}^2 + (1-\lambda)\ell_{2t}^2 = \lambda(1-\lambda)\ell_t^2.$$

Substituting in the constraint $g_t \cdot g_t = \ell_t^2$, we can represent the social planning criterion as

$$\begin{aligned} -\frac{1}{2} E_0 \sum_{t=0}^{\infty} \beta^t [&\lambda(s_{1t} - b_{1t}) \cdot (s_{1t} - b_{1t}) + (1-\lambda)(s_{2t} - b_{2t}) \cdot (s_{2t} - b_{2t}) \\ &+ \lambda(1-\lambda) g_t \cdot g_t]. \end{aligned} \tag{14.3.1}$$

The objective function (14.3.1) is to be maximized subject to the following constraints:

$$s_{it} = \Lambda_i \, h_{it-1} + \Pi_i c_{it} \quad , \; i = 1,2 \tag{12.2}$$

$$h_{it} = \Delta_{h_i} \, h_{it-1} + \Theta_{h_i} c_{it} \quad , \; i = 1,2 \tag{12.3}$$

$$\Phi_c (c_{1t} + c_{2t}) + \Phi_g g_t + \Phi_i i_t = \Gamma k_{t-1} + d_{1t} + d_{2t} \tag{12.5}$$

$$k_t = \Delta_k \, k_{t-1} + \Theta_k i_t \tag{12.6}$$

$$\begin{aligned} d_{it} = U_{d_i} \, z_t \; , \quad b_{it} = U_{b_i} \, z_t \; , \quad i = 1,2 \\ z_{t+1} = A_{22} z_t + C_2 w_{t+1} \end{aligned} \tag{12.4}$$

This problem can be set up as an optimal linear regulator problem by following steps paralleling those for the single-household economy described in Chapter 4. Define the *state* and *controls* of the system as

$$x_t = \begin{pmatrix} h_{1t-1} \\ h_{2t-1} \\ k_{t-1} \\ z_t \end{pmatrix} \quad , \quad u_t = \begin{pmatrix} i_t \\ c_{1t} \end{pmatrix}.$$

Notice that from (14.2.5), (c_{2t}, g_t) can be expressed as functions of the state and controls at t:

$$\begin{bmatrix} c_{2t} \\ g_t \end{bmatrix} = [\Phi_c \ \Phi_g]^{-1} \Big\{ \Gamma\, k_{t-1} + (U_{d_1} + U_{d_2})\, z_t - \Phi_c \ c_{1t} - \Phi_i \, i_t \Big\}. \tag{14.3.2}$$

Substitution from the above equation into (14.2.3) for $i = 2$ shows that the law of motion for x_{t+1} can be represented

$$\begin{pmatrix} h_{1t} \\ h_{2t} \\ k_t \\ z_{t+1} \end{pmatrix} =$$

$$\begin{pmatrix} \Delta_{h1} & 0 & 0 & 0 \\ 0 & \Delta_{h2} & \Theta_{h2} U_c [\Phi_c \ \Phi_g]^{-1} \Gamma & \Theta_{h2} U_c [\Phi_c \ \Phi_g]^{-1} (U_{d1} + U_{d2}) \\ 0 & 0 & \Delta_k & 0 \\ 0 & 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} h_{1t-1} \\ h_{2t-1} \\ k_{t-1} \\ z_t \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & \Theta_{h1} \\ -\Theta_{h2} U_c [\Phi_c \ \Phi_g]^{-1} \Phi_i & -\Theta_{h2} \\ \Theta_k & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} i_t \\ c_{1t} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ C_2 \end{pmatrix} w_{t+1}$$

or

$$x_{t+1} = A x_t + B u_t + C\, w_{t+1}. \tag{14.3.3}$$

Here U_c is a matrix that selects the first n_c rows of the right hand side of (14.3.2), the rows corresponding to c_{2t}, and U_g is a matrix that selects the rows of (14.3.2) corresponding to g_{1t}. Here and below, we use the equalities

$U_c[\Phi_c\ \Phi_g]^{-1}\Phi_c = I, U_g[\Phi_c\ \Phi_g]^{-1}\Phi_g = I, U_c[\Phi_c\ \Phi_g]^{-1}\Phi_g = 0,\ U_g[\Phi_c\ \Phi_g]^{-1}\Phi_c = 0$. Now substitute from (14.3.2) into (14.2.2) for $i = 2$ to get

$$s_{2t} = \Lambda_2 h_{2t-1} + \Pi_2 U_c[\Phi_c\ \Phi_g]^{-1}\Big\{\Gamma k_{t-1} + (U_{d_1} + U_{d_2})z_t - \Phi_c c_{1t} - \Phi_i i_t\Big\}.$$

Use this equation and (14.2.2) for $i = 1$ to deduce

$$(s_{1t} - b_{1t}) = \begin{pmatrix} \Lambda_1 \\ 0 \\ 0 \\ -U_{b1} \\ 0 \\ \Pi_1 \end{pmatrix}' \begin{pmatrix} h_{1t-1} \\ h_{2t-1} \\ k_{t-1} \\ z_t \\ i_t \\ c_{1t} \end{pmatrix}$$

$$(s_{2t} - b_{2t}) = \begin{pmatrix} 0 \\ \Lambda_2 \\ \Pi_2 U_c[\Phi_c \Phi_g]^{-1}\Gamma \\ \Pi_2 U_c[\Phi_c\ \Phi_g]^{-1}(U_{d_1} + U_{d_2}) - U_{b_2} \\ -\Pi_2 U_c[\Phi_c\ \Phi_g]^{-1}\Phi_i \\ -\Pi_2 \end{pmatrix}' \begin{pmatrix} h_{1t-1} \\ h_{2t-1} \\ k_{t-1} \\ z_t \\ i_t \\ c_{1t} \end{pmatrix}$$

or

$$(s_{1t} - b_{1t}) = H_1 \begin{pmatrix} x_t \\ u_t \end{pmatrix} \tag{14.3.4}$$

$$(s_{2t} - b_{2t}) = H_2 \begin{pmatrix} x_t \\ u_t \end{pmatrix}. \tag{14.3.5}$$

Similarly, we have

$$g_t = U_g[\Phi_c\ \Phi_g]^{-1} \begin{pmatrix} 0 \\ 0 \\ \Gamma \\ (U_{d_1} + U_{d_2}) \\ -\Phi_i \\ 0 \end{pmatrix}' \begin{pmatrix} h_{1t-1} \\ h_{2t-1} \\ k_{t-1} \\ z_t \\ i_t \\ c_{1t} \end{pmatrix}$$

or

$$g_t = G \begin{pmatrix} x_t \\ u_t \end{pmatrix}. \tag{14.3.6}$$

Now notice that the current return function in (14.3.1) can be represented as

$$\begin{aligned} \lambda(s_{1t}-b_{1t})\cdot(s_{1t}-b_{1t}) \\ +(1-\lambda)(s_{2t}-b_{2t})\cdot(s_{2t}-b_{2t})+\lambda(1-\lambda)g_t\cdot g_t \\ =\begin{pmatrix} x_t \\ u_t \end{pmatrix}' S \begin{pmatrix} x_t \\ u_t \end{pmatrix} \end{aligned} \tag{14.3.7}$$

where

$$S=\lambda H_1'H_1+(1-\lambda)H_2'H_2+\lambda(1-\lambda)G'G. \tag{14.3.8}$$

The analysis on pages 79–80 now applies to the present system. In particular, let $x_t'Sx_t = x_t'Rx_t + u_t'Qu_t + 2x_t'Wu_t$, and write the law of motion in the form (14.3.3). This makes the Pareto problem with weight λ into a discounted optimal linear regulator problem. The solution of the Pareto problem is a law of motion

$$x_{t+1}=A^0(\lambda)x_t+CW_{t+1} \tag{14.3.9}$$

and a list of matrices $S_j(\lambda)$ such that optimal allocations are given by

$$\begin{aligned} c_{it} &= S_{c_i}(\lambda)x_t \quad , \ i=1,2 \\ i_t &= S_i(\lambda)x_t \\ h_{it} &= S_{h_i}(\lambda)x_t \quad , \ i=1,2 \\ s_{it} &= S_{s_i}(\lambda)x_t \quad , \ i=1,2 \end{aligned} \tag{14.3.10}$$

The value function for the Pareto problem has the form

$$V(x_t)=x_t'V_1(\lambda)x_t+V_2(\lambda). \tag{14.3.11}$$

Associated with the solution of the Pareto problem for a given λ are a set of Lagrange multiplier processes given by

$$
\begin{aligned}
\mathscr{M}_t^{h_1}(\lambda) &= 2\beta[I\ 0\ 0\ 0]\, V_1(\lambda)A^0(\lambda)x_t \\
\mathscr{M}_t^{h_2}(\lambda) &= 2\beta[0\ I\ 0\ 0]\, V_1(\lambda)A^0(\lambda)x_t \\
\mathscr{M}_t^{k}(\lambda) &= 2\beta[0\ 0\ I\ 0] V_1(\lambda)A^0(\lambda)x_t \quad \equiv M_k(\lambda)x_t \\
\mathscr{M}_t^{s_1}(\lambda) &= \lambda(S_{b1} - S_{s_1}(\lambda))x_t \\
\mathscr{M}_t^{s_2}(\lambda) &= (1-\lambda)(S_{b_2} - S_{s_2}(\lambda))x_t \\
\mathscr{M}_t^{c_1}(\lambda) &= \Theta_{h_1}' \mathscr{M}_t^{h_1}(\lambda) + \Pi_1' \mathscr{M}_t^{s_1}(\lambda) \\
\mathscr{M}_t^{c_2}(\lambda) &= \Theta_{h_2}' \mathscr{M}^{h_2}(\lambda) + \Pi_2' \mathscr{M}_t^{s_2}(\lambda) \\
\mathscr{M}_t^{i}(\lambda) &= M_i(\lambda)x_t\ , \quad M_i(\lambda) = \Theta_t' M_k(\lambda) \\
\mathscr{M}_t^{d}(\lambda) &= \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} \Theta_{h_i}' \mathscr{M}_t^{h_i}(\lambda) + \Pi_i' \mathscr{M}_t^{s_i}(\lambda) \\ -\lambda(1-\lambda)g_t \end{bmatrix},
\end{aligned}
\tag{14.3.12}
$$

From the structure of the Pareto problem and the fact that c_{1t}, c_{2t} appear additively in the technology (14.2.5), it follows that $\mathscr{M}_t^{c_1}(\lambda) = \mathscr{M}_t^{c_2}(\lambda)$.

14.4. Competitive Equilibrium

We will use the following

Definition: A *price system* is a list of stochastic processes $[\{p_{it}^0, w_t^0;\ q_t^0, r_t^0, \alpha_t^0\}_{t=0}^\infty, v_0]$, each element of which belongs to L_0^2.

Definition: An *allocation* is a list of stochastic processes $\{c_{it}, s_{it}, h_{it}, \ell_{it}, i = 1,2; k_t\}_{t=0}^\infty$ each element of which is in L_0^2.

14.4.1. Households

Households of type i face the problem of maximizing

$$-\frac{1}{2}E\sum_{t=0}^{\infty}\beta^t\big[(s_{it}-b_{it})\cdot(s_{it}-b_{it})+\ell_{it}^2\big]|J_0 \tag{12.1}$$

subject to the budget constraint

$$E\sum_{t=0}^{\infty}\beta^t p_t^0\cdot c_{it}\mid J_0=E\sum_{t=0}^{\infty}\beta^t[w_t^0\,\ell_{it}+\alpha_t^0\cdot d_{it}]\mid J_0+v_0\,k_{-1}^i, \tag{14.4.1}$$

the household technology

$$s_{it}=\Lambda_i\,h_{it-1}+\Pi_i\,c_{it} \tag{12.2}$$

$$h_{it}=\Delta_h\,h_{it-1}+\Theta_{h_i}\,c_{it}, \tag{12.3}$$

and the initial conditions $h_{i,-1}$, $k_{i,-1}$.

14.4.2. Firms of type I and II

Firms of type I and II face the problems described in chapter 6, with $c_t=c_{1t}+c_{2t}$ and $d_t=d_{1t}+d_{2t}$.

14.4.3. Definition of competitive equilibrium

We use the following standard definition:

DEFINITION: A *competitive equilibrium* is an allocation and a price system such that, given the price system, the allocation solves the optimum problem of households of each type and firms of each type.

14.5. Computation of Equilibrium

To compute an equilibrium, we use an iterative algorithm based on ideas of Negishi (1960). For each given value of the Pareto weight λ, we know that there exists a competitive equilibrium, though it will typically be associated with some distribution of wealth other than the one associated with the allocation of ownership of capital and endowment processes that we have assigned. An algorithm for computing an equilibrium with a pre-assigned distribution of ownership is to search for a $\lambda \in (0,1)$ that delivers budget balance for each household.

14.5.1. Candidate equilibrium prices

For a given λ, candidate equilibrium prices can be computed from the Lagrange multipliers associated with the solution of the Pareto problem for that value of λ. By pursuing arguments paralleling those of Chapter 6, we find

$$\begin{aligned} p_t^0 &= \mathscr{M}_t^{c_1}(\lambda)/\mu_0^w \\ r_t^0 &= \Gamma' \mathscr{M}_t^d(\lambda)/\mu_0^w \\ q_t^0 &= \Theta_k' \mathscr{M}_t^k(\lambda) \\ \alpha_t^0 &= \mathscr{M}_t^d(\lambda)/\mu_0^w \\ v_0 &= \Gamma' \mathscr{M}_0^d(\lambda)/\mu_0^w + \Delta_k' \mathscr{M}_0^k(\lambda)/\mu_0^w \\ w_t^0 &= \lambda(1-\lambda) \mid S_g(\lambda) x_t \mid /\mu_0^w \end{aligned} \tag{14.5.1}$$

These prices and the associated allocations are the ingredients used in the Negishi algorithm.

14.5.2. A Negishi algorithm

The algorithm consists of the following steps.

1. For a given $\lambda \in (0,1)$, solve the Pareto problem. Compute the Lagrange multipliers from (14.3.12) and use them to compute the candidate competitive equilibrium prices and quantities via (14.5.1).
2. At the candidate equilibrium prices and quantities, compute the left and right side of each household's budget constraint (14.4.1). In particular, compute

$$\mathscr{G}_i = E\sum_{t=0}^{\infty}\beta^t[w_t^0\ell_{it} + \alpha_t^0 \cdot d_{it}] \mid J_0 + v_0 \cdot k_{i,-1} - E\sum_{t=0}^{\infty}\beta^t p_t^0 \cdot c_{it} \mid J_0$$

   Use the method described in chapter 10 to compute this. For our two-household economy, $\mathscr{G}_1$ and $\mathscr{G}_2$ will either be of opposite signs, or both will equal zero.
3. If $\mathscr{G}_1 > 0$, increase λ and return to step 1. If $\mathscr{G}_1 = \mathscr{G}_2 = 0$, terminate the search and accept the allocation and price system associated with the current value of λ as equilibrium objects.[1]

In practice, one can improve on this algorithm by using any of a number of root finders to find the zero of the function $\mathcal{G}_i(\lambda)$ defined in step 2. We have found it efficient to use a 'secant method.'

[1] The Negishi algorithm is implemented in the MATLAB program `solvehet.m`. Some trial inputs are contained in the file `clex11h.m`, which inputs a two agent version of the economy in `clex11.m`, the one good stochastic growth model. As a benchmark, `clex11h.m` has the economy start out with identical endowments for the two households, and has them share identical household technologies. With these inputs, `solvehet.m` should find Pareto weight (which the program calls 'alpha') equal to .5, and should recover the same solution that `solvea.m` does with inputs `clex11.m`. Modify the inputs to get a non-Gorman aggregatable example. The program `simulhet.m` simulates the equilibrium computed by `solvehet.m`.

14.6. Mongrel Aggregation

Except in the special case that $\Lambda_1 = \Lambda_2$, $\Pi_1 = \Pi_2$, $\Delta_{h1} = \Delta_{h2}$, $\Theta_{h1} = \Theta_{h2}$, the specification of household technologies (14.2.2) – (14.2.3) violates the Gorman conditions for aggregation, so that there does not exist a representative household in the sense described in Chapter 7. However, for each Pareto weight λ, there does exist a representative household in the sense of a mongrel preference ordering over total consumption $(c_{1t} + c_{2t})$. This mongrel preference ordering depends on the distribution of wealth, i.e., the value of initial endowments and capital stocks evaluated at equilibrium prices.

14.6.1. Static demand

Mongrel aggregation of preferences is easiest to analyze in the special case that the demand curve is 'static' in the sense that time t demand is a function only of the current price p_t^0. Let the household technology be determined by a nonsingular square matrix Π, where each of $\Lambda, \Delta_h, \Theta_h$ are matrices of zeros of the appropriate dimensions. For this specification, the (canonical) demand curve is

$$c_t = \Pi^{-1} b_t - \mu_0 \Pi^{-1} \Pi^{-1\prime} p_t, \tag{14.6.1}$$

where μ_0 is the Lagrange multiplier on the household's budget constraint. The inverse demand curve is

$$p_t = \mu_0^{-1} \Pi' b_t - \mu_0^{-1} \Pi' \Pi c_t. \tag{14.6.2}$$

In equations (14.6.1) and (14.6.2), the price vector p_t can be interpreted as the marginal utility vector of the consumption vector p_t. Integrating the marginal utility vector shows that preferences can be taken to be

$$(-2\mu_0)^{-1} (\Pi c_t - b_t) \cdot (\Pi c_t - b_t). \tag{14.6.3}$$

From (14.6.2) or (14.6.3) it is evident that the preference ordering is determined only up to multiplication of Π, b_t by a common scalar. We are free to normalize preferences by setting $\mu_0 = 1$.

Now suppose that we have two consumers, i=1,2, with demand curves

$$c_{it} = \Pi_i^{-1} b_{it} - \mu_{0i} \Pi_i^{-1} \Pi_i^{-1\prime} p_t.$$

Adding these gives the total demand

$$c_{1t}+c_{2t}=(\Pi_1^{-1}b_{1t}+\Pi_2^{-1}b_{2t})-(\mu_{01}\Pi_1^{-1}\Pi_1^{-1\prime}+\mu_{02}\Pi_2\Pi_2^{-1\prime})p_t. \tag{14.6.4}$$

Setting $c_{1t}+c_{2t}=c_t$ and solving (14.6.4) for p_t gives

$$\begin{aligned}p_t &= (\mu_{01}\Pi_1^{-1}\Pi_1^{-1\prime}+\mu_{02}\Pi_2^{-1}\Pi_2^{-1\prime})^{-1}(\Pi_1^{-1}b_{1t}+\Pi_2^{-1}b_{2t})\\ &\quad -(\mu_{01}\Pi_1^{-1}\Pi_1^{-1\prime}+\mu_{02}\Pi_2^{-1}\Pi_2^{-1\prime})^{-1}c_t.\end{aligned} \tag{14.6.5}$$

We want to interpret (14.6.5) as an aggregate preference ordering associated with an aggregate demand curve of the form (14.6.2). To do this, we shall evidently have to choose the Π associated with the aggregate ordering to satisfy

$$\mu_0^{-1}\Pi'\Pi=(\mu_{01}\Pi_1^{-1}\Pi_2^{-1\prime}+\mu_{02}\Pi_2^{-1}\Pi_2^{-1\prime})^{-1}. \tag{14.6.6}$$

To find a matrix Π determining an aggregate preference ordering, we have to form and then factor the matrix on the right side of (14.6.6). This matrix looks like the inverse of a weighted sum of two moment matrices. Even after normalizing Π by setting $\mu_0=1$, a solution Π will in general depend on the ratio μ_{01}/μ_{02}, which functions like a Pareto weight on the two types of consumers.

There is a special case for which the aggregate or mongrel preference matrix Π *is* independent of μ_{01}/μ_{02}, namely:

$$\Pi_1=k\Pi_2 \quad \text{for scalar} \quad k>0 \tag{14.6.7}$$

Notice that when Π_1 and Π are scalars, condition (14.6.7) is automatically satisfied. So for the one consumption good case with this special specification (i.e., with Λ being zero), Gorman aggregation obtains.

In the more general case that demand curves are dynamic (quantities demanded at t depending on future prices), attaining a mongrel preference ordering becomes more difficult. In place of the problem of factoring a moment matrix as required in (14.6.6), we have to factor something that resembles a spectral density matrix, frequency by frequency. For studying mongrel preference orderings in the general case, it is convenient to work with a frequency domain representation of preferences.

14.6.2. Frequency domain representation of preferences

From chapter 9, recall the decomposition of services $s_t = s_{mt} + s_{it}$, where s_{mt} are services resulting from market purchases of consumption and s_{it} are services flowing from the initial household capital stock. Let $(\Delta_h, \Theta_h, \Lambda, \Pi)$ correspond to a canonical household service technology, and recall that

$$s_{mt} = \sigma(L) c_t$$

where

$$\sigma(L) = [\Pi + \Lambda L [I - \Delta_h L]^{-1} \Theta_h]$$
$$\sigma(L)^{-1} = \Pi^{-1} - \Pi^{-1} \Lambda [I - (\Delta_h - \Theta_h \Pi^{-1} \Lambda) L]^{-1} \Theta_h \Pi^{-1} L,$$

and

$$s_{it} = \Lambda \Delta_h^t h_{-1}.$$

We use the transform methods described in the appendix to chapter 9. For any matrix sequence $\{y_t\}$ satisfying $\sum_{t=0}^{\infty} \beta^t y_t y_t' < +\infty$, define $T(y)(\zeta) = \sum_{t=0}^{\infty} \beta^{t/2} y_t \zeta^t$. We define $S(\zeta) = \sigma(\beta^{.5}\zeta)$. Evidently, the transforms obey

$$T(s_m)(\zeta) = S(\zeta) T(c)(\zeta)$$
$$T(s_{it})(\zeta) = \Lambda [I - \beta^{1/2} \Delta_h \zeta]^{-1} h_{-1}.$$

As in chapter 9, express the one-period return as

$$\begin{aligned}(s_t - b_t) \cdot (s_t - b_t) = s_{mt} \cdot s_{mt} + 2 s_{mt} \cdot s_{it} - 2 s_{mt} \cdot b_t \\ + (b_t - s_{it}) \cdot (b_t - s_{it})\end{aligned} \tag{14.6.8}$$

The term $(b_t - s_{it}) \cdot (b_t - s_{it})$ is beyond control and therefore influences no decisions. So it can be ignored in describing a preference ordering. In terms of Fourier transforms, we have

$$\sum_{t=0}^{\infty} \beta^t s_{mt} \cdot s_{mt} = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(c)' S' S T(c) \, d\theta \tag{14.6.9}$$

$$\sum_{t=0}^{\infty} \beta^t s_{mt} \cdot s_{it} = \frac{1}{2\pi} \int_{\pi}^{\pi} T(c)' S' T(s_i) \, d\theta \tag{14.6.10}$$

$$\sum_{t=0}^{\infty} \beta^t s_{mt} \cdot b_t = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(c)' S' T(b) \, d\theta, \tag{14.6.11}$$

where it is understood that $S = S(\zeta), T(c) = T(c)(\zeta), T(b) = T(b)(\zeta)$, and $\zeta = e^{-i\theta}$. Here ($'$) denotes transposition *and* complex conjugation.

14.7. A Programming Problem for Mongrel Aggregation

To find a preference ordering over aggregate consumption, we can pose a nonstochastic optimization problem.[2] We rely on a certainty equivalence result to assert that the preference ordering over random consumption streams is given by the conditional expectation of the optimized value of this nonstochastic problem. Thus, our strategy in deducing the mongrel preference ordering over $c_t = c_{1t}+c_{2t}$ is to solve the programming problem: maximize over $\{c_{1t}, c_{2t}\}$ the criterion

$$\sum_{t=0}^{\infty} \beta^t [\lambda(s_{1t} - b_{1t}) \cdot (s_{1t} - b_{1t}) + (1-\lambda)(s_{2t} - b_{2t}) \cdot (s_{2t} - b_{2t})] \tag{14.7.1}$$

subject to

$$\begin{aligned} h_{jt} &= \Delta_{hj}\, h_{jt-1} + \Theta_{hj}\, c_{jt} \quad j = 1,2 \\ s_{jt} &= \Delta_j h_{jt-1} + \Pi_j c_{jt}\ , \quad j = 1,2 \\ c_{1t} + c_{2t} &= c_t \end{aligned}$$

subject to $(h_{1,-1},\, h_{2,-1})$ given, and $\{b_{1t}\}$, $\{b_{2t}\}$, $\{c_t\}$ being known and fixed sequences. Substituting the $\{c_{1t},\, c_{2t}\}$ sequences that solve this problem as functions of $\{b_{1t},\, b_{2t},\, c_t\}$ into the objective (14.7.1) will determine the mongrel preference ordering over $\{c_t\}$. In solving this problem, it is convenient to proceed by using Fourier transforms.

Using versions of (14.6.8), (14.6.9), (14.6.10), and (14.6.11) for households 1 and 2, in terms of transforms we can represent the Pareto-weighted average of discounted utility from consumption as

[2] These calculations are done by the MATLAB program .

$$
\begin{aligned}
-\sum_{t=0}^{\infty}&\beta^t[\lambda(s_{1t}-b_{1t})\cdot(s_{1t}-b_{1t})+(1-\lambda)(s_{2t}-b_{2t})\cdot(s_{2t}-b_{2t})]\\
&=-\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{[\lambda T(c_1)'S_1'S_1T(c_1)+(1-\lambda)T(c_2)'S_2'S_2T(c_2)]\\
&+2[\lambda T(c_1)'S_1'T(s_{1i})+(1-\lambda)T(c_2)'S_2'T(s_{2i})]\\
&-2[\lambda T(c_1)'S_1'T(b_1)+(1-\lambda)T(c_2)'S_2'T(b_2)]\Big\}\,d\theta\\
&+\text{ terms } \textit{not} \text{ involving } T(c_1) \text{ or } T(c_2).
\end{aligned}
\tag{14.7.2}
$$

where it is understood that each transform on the right side is to be evaluated at $\zeta = e^{-i\theta}$.

We want to maximize the right side of (14.7.2) over choice of $\{c_{1t}, c_{2t}\}_{t=0}^{\infty}$ or equivalently $T(c_1), T(c_2)$, subject to the constraint $c_{1t}+c_{2t}=c_t$, or equivalently the restriction

$$T(c_1)+T(c_2)=T(c).$$

To do this optimization, we form the Lagrangian

$$
\begin{aligned}
J=-\frac{1}{2\pi}\int_{-\pi}^{\pi}&\Big\{[\lambda T(c_1)'S_1'S_1T(c_1)+(1-\lambda)T(c_2)'S_2'S_2T(c_2)]\\
&+2[\lambda T(c_1)'S_1'T(s_{1i})+(1-\lambda)T(c_2)'S_2'T(s_{2i})]\\
&-2[\lambda T(c_1)'S_1'T(b_1)+(1-\lambda)T(c_2)'S_2'T(b_2)]\\
&+\mu[T(c)-T(c_1)-T(c_2)]\Big\}\,d\theta
\end{aligned}
\tag{14.7.3}
$$

where it is understood that there is a Lagrange multiplier $\mu = \mu(e^{-i\theta})$ for each frequency $\theta \in [-\pi, \pi]$. We can perform this maximization "frequency by frequency" (i.e., pointwise for each $\theta \in [-\pi, \pi]$). The first-order necessary conditions with respect to $T(c_1)$ and $T(c_2)$, respectively, are

$$
\begin{aligned}
\lambda S_1'S_1T(c_1)+\lambda S_1'T(s_{1i})-\lambda S_1'T(b_1)&=\mu/2\\
(1-\lambda)S_2'S_2T(c_2)+(1-\lambda)S_2'T(s_{2i})-(1-\lambda)S_2'T(b_2)&=\mu/2
\end{aligned}
$$

Using these two equations and the constraint $T(c_1)+T(c_2)=T(c)$ to solve for μ gives

$$
\begin{aligned}
\mu=2&\Big[\frac{1}{\lambda}\,(S_1'S_1)^{-1}+\frac{1}{1-\lambda}\,(S_2'S_2)^{-1}\Big]^{-1}\\
&\times\Big\{T(c)+S_1^{-1}\big(T(s_{1i})-T(b_1)\big)+S_2^{-1}\big(T(s_{2i})-T(b_2)\big)\Big\}.
\end{aligned}
\tag{14.7.4}
$$

Let

$$S'S = \left[\frac{1}{\lambda}(S_1'S_1)^{-1} + \frac{1}{1-\lambda}(S_2'S_2)^{-1}\right]^{-1} \tag{14.7.5}$$

where S is a matrix Fourier transform that satisfies the condition: $\det S(\zeta_o) = 0$ implies $|\zeta_o| > 1$. Equation (14.7.5) is the dynamic counterpart of (14.6.6), and collapses to (14.6.6) in the special case in which $S_j(L) = \Pi_j$. Recall that in the static case, constructing the aggregate preference ordering required factoring the inverse of a 'moment' matrix formed as a weighted sum of $\Pi_j^{-1}\Pi_j^{-1\prime}$.

The matrix $[\frac{1}{\lambda}(S_1'S_1)^{-1} + \frac{1}{1-\lambda}(S_2'S_2)^{-1}]^{-1}$ can be regarded as a spectral density matrix. Equation (14.7.5) expresses $S'S$ as a spectral factorization of $[\frac{1}{\lambda}(S_1'S_1)^{-1} + \frac{1}{1-\lambda}(S_2'S_2)^{-1}]^{-1}$. Later, we shall show how to achieve the factorization expressed in (14.7.5). For now, we just assume that we have an S that satisfies (14.7.5) and the condition that the zeros of $\det S(\zeta)$ all exceed unity in modulus.

Using (14.7.5) in (14.7.4) gives

$$\mu = 2S'S\Big[T(c) + S_1^{-1}\big(T(s_{1i}) - T(b_1)\big) + S_2^{-1}\big(T(s_{2i}) - T(b_2)\big)\Big].$$

The fact that the Lagrange multiplier is the derivative of the return function with respect to $T(c)$ implies that the mongrel return function has the representation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\begin{pmatrix} T(c) \\ T(s_{1i}) - T(b_1) \\ T(s_{2i}) - T(b_2)\end{pmatrix}' \begin{pmatrix} S'S & S'SS_1^{-1} & S'SS_2^{-1} \\ S_1^{-1\prime}S'S & - & - \\ S_2^{-1\prime}S'S & - & - \end{pmatrix} \begin{pmatrix} T(c) \\ T(s_{1i}) - T(b_1) \\ T(s_{2i} - T(b_2)\end{pmatrix} d\theta$$

where the blank terms do not involve $T(c)$, and do not affect the choice of $T(c)$.

Therefore, we can represent the mongrel preference ordering over $T(c)$ by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{T(c)S'ST(c) + (2T(c)'S')SS_1^{-1}(T(s_{1i}) - T(b_1)) + (2T(c)S')SS_2^{-1}(T(s_{2i}) - T(b_2))\Big\}\, d\theta. \tag{14.7.6}$$

Compare this with the single agent case of chapter 9, in which the preference ordering was shown to be

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big\{T(\bar c)'\bar S'\bar S T(\bar c)+2T(\bar c)'\bar S'(T(\bar s_i)-T(\bar b))\Big\}\,d\theta, \tag{14.7.7}$$

where we have put bars ($^-$) over the objects in (14.7.7) to represent the corresponding single agent-objects. Evidently, for the mongrel preference ordering (14.7.6) to match up with a single agent ordering (14.7.7), we can match objects up as

$$\begin{aligned} T(c) &\sim T(\bar c)\\ S &\sim \bar S\\ SS_1^{-1}(T(s_{1i})-T(b_1))+&\\ SS_2^{-1}(T(s_{2i})-T(b_2)) &\sim T(\bar s_i)-T(\bar b) \end{aligned} \tag{14.7.8}$$

where the object on the right of the right of the $\sim$ corresponds in each case to the single-agent 4.

We have two major tasks to complete our work. First, we have to show how to achieve the factorization (14.7.5). Second, we have to show how to interpret and to implement the correspondence given in (14.7.8).

14.7.1. Factoring $S'S$

To achieve the spectral factorization (14.7.5), we notice that $[\frac{1}{\lambda}(S_1'S_1)^{-1}+\frac{1}{1-\lambda}(S_2'S_2)^{-1}]$ can be regarded as the spectral density matrix of a stochastic process c_t^λ that is generated by the state space system

$$\begin{aligned} h_{1t} &= \beta^{.5}(\Delta_{h1}-\Theta_{h1}\Pi_1^{-1}\Lambda_1)h_{1t-1}+\Theta_{h1}\Pi_1^{-1}s_{1t}\\ h_{2t} &= \beta^{.5}(\Delta_{h2}-\Theta_{h2}\Pi_2^{-1}\Lambda_2)h_{2t-1}+\Theta_{h2}\Pi_2^{-1}s_{2t}\\ c_t^\lambda &= \beta^{.5}[-\frac{1}{\sqrt{\lambda}}\,\Pi_1^{-1}\Lambda_1,\quad -\frac{1}{\sqrt{1-\lambda}}\,\Pi_2^{-1}\Lambda_2]\begin{bmatrix}h_{1t-1}\\ h_{2t-1}\end{bmatrix}\\ &\quad +(\tfrac{1}{\sqrt{\lambda}}\,\Pi_1^{-1},\quad \tfrac{1}{\sqrt{1-\lambda}}\,\Pi_2^{-1})\begin{pmatrix}s_{1t}\\ s_{2t}\end{pmatrix} \end{aligned} \tag{14.7.9}$$

where $(\Delta_{hi},\Theta_{hi},\Lambda_i,\Pi_i)$ are each associated with a canonical representation, and where $s_t=\begin{pmatrix}s_{1t}\\ s_{2t}\end{pmatrix}$ is a white noise with covariance $Es_ts_t'=I$. Write this system compactly as

$$\begin{aligned} h_t &= \tilde{\Delta}_h \, h_{t-1} + H s_t \\ c_t^\lambda &= G_\lambda h_{t-1} + M_\lambda s_t . \end{aligned}$$

The spectral density of c_t^λ can be directly computed to be $[\frac{1}{\lambda}(S_1' S_1)^{-1} + \frac{1}{1-\lambda}(S_2' S_2)^{-1}]$.[3]

We can factor the *inverse* of

$$[\frac{1}{\lambda}(S_1' S_1)^{-1} + \frac{1}{1-\lambda}(S_2' S_2)^{-1}]$$

by obtaining an *innovations representation* for the system (14.7.9), then using it to form a 'whitener'. The innovations representation is

$$\begin{aligned} \hat{h}_t &= \tilde{\Delta}_h \hat{h}_{t-1} + K a_t \\ c_t^\lambda &= G_\lambda \hat{h}_{t-1} + a_t \end{aligned} \tag{14.7.10}$$

where $E a_t a_t' = \Omega = G_\lambda \Sigma G_\lambda' + M_\lambda M_\lambda'$, and $[K, \Sigma] =$ `kfilter` $(\tilde{\Delta}_h, G_\lambda, HH', M_\lambda M_\lambda', HM_\lambda')$, where `kfilter` is the matrix valued function defined in chapter 7 .

To get the inverse of $[\frac{1}{\lambda}(S_1' S_1)^{-1} + \frac{1}{1-\lambda}(S_2' S_2)^{-1}]$, let $r'r = \Omega$ be the Cholesky decomposition of Ω, and define $\hat{s}_t$ by $\hat{s}_t = r'^{\,-1} a_t$. Then use (14.7.10) to get the 'whitener'

$$\begin{aligned} \hat{h}_t &= (\tilde{\Delta}_h - K G_\lambda) \hat{h}_{t-1} + K c_t^\lambda \\ \hat{s}_t &= -r'^{-1} G_\lambda \hat{h}_{t-1} + r'^{-1} c_t^\lambda \end{aligned}$$

or

$$\begin{aligned} \hat{h}_t &= \hat{\Delta}_h \hat{h}_{t-1} + \hat{\Theta}_h c_t^\lambda \\ \hat{s}_t &= \hat{\Lambda} \hat{h}_{t-1} + \hat{\Pi} c_t^\lambda \end{aligned} \tag{14.7.11}$$

where

$$\begin{aligned} \hat{\Delta}_h &= (\tilde{\Delta}_h - K G_\lambda) \\ \hat{\Theta}_h &= K \\ \hat{\Lambda} &= -r'^{-1} G_\lambda \\ \hat{\Pi} &= r'^{-1} \end{aligned} \tag{14.7.12}$$

[3] To see this, we make use of the fact that

$$\sigma_j(\zeta)^{-1} = \Pi_j^{-1} - \Pi_j^{-1} \Lambda_j \, [I - (\Delta_{hj} - \Theta_{hj} \Pi_j^{-1} \Lambda_j) \zeta]^{-1} \, \Theta_{hj} \Pi_j^{-1} \zeta .$$

As a consequence of the factorization identity and associated matrix identities described in chapter 7,[4] we have that $S'S$ satisfies (14.7.5) where

$$\begin{aligned} S(\zeta) &= \left[\hat{\Pi} + \hat{\Lambda}\zeta[I - \hat{\Delta}_h\zeta]^{-1}\hat{\Theta}_h\right] \\ &= \left[\Pi + \Lambda\beta^{.5}\zeta[I - \Delta_h\beta^{.5}\zeta]^{-1}\Theta_h\right]. \end{aligned}$$

It follows that a (canonical) version of the mongrel household technology is

$$\begin{aligned} h_t &= \Delta_h h_{t-1} + \Theta_h(c_{1t} + c_{2t}) \\ s_t &= \Lambda h_{t-1} + \Pi(c_{1t} + c_{2t}), \end{aligned} \tag{14.7.13a}$$

where

$$\begin{aligned} \Delta_h = \beta^{-.5}\hat{\Delta}_h, &\quad \Theta_h = \hat{\Theta}_h \\ \Lambda = \beta^{-.5}\hat{\Lambda}, &\quad \Pi = \hat{\Pi}. \end{aligned} \tag{14.7.13b}$$

Collecting results, we have that the canonical household technology is determined by the matrices

$$\begin{aligned} \Delta_h &= \beta^{-.5}(\tilde{\Delta}_h - KG_\lambda) \\ \Theta_h &= K \\ \Lambda &= -{r'}^{-1}G_\lambda\beta^{-.5} \\ \Pi &= {r'}^{-1}. \end{aligned}$$

These equalities imply that

$$\Delta_h - \Theta_h\Pi^{-1}\Lambda = \begin{bmatrix} \Delta_{h1} - \Theta_{h1}\Pi_1^{-1}\Lambda_1 & 0 \\ 0 & \Delta_{h2} - \Theta_{h2}\Pi_2^{-1}\Lambda_2 \end{bmatrix}.$$

It follows that for the mongrel household technology, the counterpart to (BLANK.1) is

$$\begin{aligned} c_t &= [-\frac{1}{\sqrt{\lambda}}\Pi_1^{-1}\Lambda_1, \quad -\frac{1}{\sqrt{1-\lambda}}\Pi_2^{-1}\Lambda_2]h_{t-1} + r's_t \\ h_t &= \begin{bmatrix} \Delta_{h1} - \Theta_{h1}\Pi_1^{-1}\Lambda_1 & 0 \\ 0 & \Delta_{h2} - \Theta_{h2}\Pi_2^{-1}\Lambda_2 \end{bmatrix} h_{t-1} + Kr's_t. \end{aligned}$$

[4] In effect, we are using the factorization identity (7.33) and the matrix inversion identity (7.36). We are expressing $(S'S)^{-1}$ from (14.7.5) first in a form like (7.31a), then via the factorization identity in a form like (7.31b). Then we apply the inversion formula (7.36) to the two factors of (7.31b), replacing Ω with its Cholesky factorization, to construct a factored version of $S'S$.

Notice how the weight λ influences this representation: λ appears in the 'observer' matrix multiplying h_{t-1} in the first equation, and it appears indirectly through its influence on the matrices $[r', K]$. However, the state transition matrix $\Delta_h - \Theta_h \Pi^{-1} \Lambda$ is independent of λ.

14.8. Summary of Findings

We have found that the operator $\sigma_j(L)^{-1}$ is implemented by the state space system defined by the four matrices[5] $[\Delta_{hj} - \Theta_{hj} \Pi_j^{-1} \Lambda_j, \Theta_{hj} \Pi_j^{-1}, \Pi_j^{-1} \Lambda_j, \Pi_j^{-1}]$. The operator $\sigma(L)$ associated with the mongrel household technology is realized by the state space system $[\Delta_h, \Theta_h, \Lambda, \Pi]$ determined by (14.7.12) and (14.7.6). We can use these state space systems to derive a state space system for the mongrel preference shock.

14.9. The Mongrel Preference Shock Process

Our next goal is to construct a mongrel preference shock process that achieves the match up given in (14.7.8). Evidently, from (14.7.6), we have to operate on $T(s_{1i}) - T(b_1)$ with the "filter" SS_1^{-1}, operate on $T(s_{2i}) - T(b_2)$ with the "filter" SS_2^{-1}, then add the results to get a process that we can interpret as the mongrel $T(s_i) - T(b)$. The following cascading of state space systems evidently implements the required filtering and adding:

$$A \begin{cases} z_{t+1} = A_{22} z_t + C_2 w_{t+1} \\ b_{1t} = U_{b1} z_t \\ b_{2t} = U_{b2} z_t \\ h_{jt} = \Delta_{hj} h_{jt-1} \qquad j = 1, 2 \\ s_{jt} = \Lambda_j h_{jt-1} \qquad , \ j = 1, 2 \end{cases}$$

$$B \begin{cases} x_{jt} = (\Delta_{hj} - \Theta_{hj} \Pi_j^{-1} \Lambda_j) x_{jt-1} + \Theta_{hj} \Pi_j^{-1} (b_{jt} - s_{jt}) \\ y_{jt} = -\Pi_j^{-1} \Lambda_j x_{jt-1} + \Pi_j^{-1} (b_{jt} - s_{jt}) \qquad , j = 1, 2 \end{cases}$$

[5] Defined as usual as the matrices in the state equation followed by the matrices in the observation equation.

$$C \begin{cases} g_t = \Delta_h g_{t-1} + \Theta_h (y_{1t} + y_{2t}) \\ (b_t - \hat{s}_t) = \Lambda g_{t-1} + \Pi (y_{1t} + y_{2t}). \end{cases}$$

System A generates the "inputs" (b_{1t}, b_{2t}), (s^i_{1t}, s^i_{2t}). System B operates on $(b_{jt} - s^i_{jt})$ with σ_j^{-1}. System C operates on $\sum_j \sigma_j^{-1}(b_{jt} - s^i_{jt})$ with σ, as required by (14.7.8). In C, we are free to set $\hat{s}_t = 0$, and to regard the resulting b_t as our mongrel preference shock process. A recursive representation of $b_t - s_t$ is attained by linking the three systems in a series.[6]

It is evident how to use similar methods to break out the processes b_t and $\hat{s}_t$ separately.

14.9.1. Interpretation of $\hat{s}_t$ component

The term $SS_1^{-1}T(s_{1i})$ has the following interpretation. $T(s_{1i})$ is (the transform of) the contribution of services flowing to the household from the initial household capital stock h_{-1}. Then $S_1^{-1}T(s_{1i})$ is the (transform of the) equivalent amount of consumption that it would have taken to generate those services had they been acquired through new market purchases. The term $S_1^{-1}T(s_{1i})$ amounts to a consumption goods equivalent of the transient component of services flowing to the first household.

14.10. Choice of Initial Conditions

Our calculations do not tell us how to choose the correct initial condition at time 0 for the mongrel household capital stock vector h_{t-1}. Here is the reason. The first-order necessary conditions leading to (14.7.4)–(14.7.5) imply the following solution for the transform $T(c_1)$:

$$\begin{aligned} T(c_1) = {} & \frac{1}{\lambda}(S_1'S_1)^{-1}(S'S)T(c) + \frac{1}{\lambda}(S_1'S_1)^{-1}(S'S)S_2^{-1}(T(s_{2i} - T(b_2)) \\ & + (\frac{1}{\lambda}(S_1'S_1)^{-1}(S'S) - I)S_1^{-1}(T(b_1) - T(s_{1i})). \end{aligned} \tag{14.10.1}$$

[6] The MATLAB command `series` can be used to link the systems.

A similar expression holds for $T(c_2)$. Our calculations assure that $T(c_1) + T(c_2) = T(c)$. We assume that $T(c)$ is the transform of a sequence that is one-sided (i.e., $c_t = 0 \; \forall t < 0$), but this does not guarantee that $T(c_1)$ and $T(c_2)$ are *each* transforms of one-sided sequences, only that their *sum* is. When $S_j' S_j$ is not a constant times $S'S$ for $j = 1, 2$, as will generally be the case when the two household technologies are not identical, then $T(c_1)$ given by (14.10.1) will be the transform of a sequence that is nonzero for $t < 0$.[7] Thus, our frequency domain programming problem allows the 'mongrel planner' to reallocate *past* consumptions between the two types of households, subject to the restriction $c_{1t} + c_{2t} = 0$ for $t < 0$. These choices of c_{js} for $s < 0$ translate into choices of initial conditions for h_{-1}, the vector of mongrel household capital stocks at date $t = -1$.

We will not pursue calculations of the initial conditions here, because they are intricate and only effect the transient responses of services. Our main interest is not in the selection of the initial conditions but in the 'nontransient' part of the mapping from total consumption $c_t = c_{1t} + c_{2t}$ to the mongrel service vector s_t, which is given by (14.7.6).

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[7] The operator $(S_1' S_1)^{-1}(S'S)$ is two-sided except when it is proportional to the identity operator.

Part III

Extensions

Chapter 15

Equilibria with Distortions

15.1. Introduction

In earlier chapters, we often used the fact that the competitive equilibrium allocations solve a Pareto problem. This chapter describes classes of economies for which the connection between Pareto optimality and equilibrium breaks down, because distortions cause competitive equilibrium allocations not to be Pareto optimal. The distortions occur in the form of *externalities* in preferences and production technologies, and *distorting taxes*. These features can be accommodated using the invariant subspace methods described in chapter 8, which provide an approach to studying the existence and uniqueness of an equilibrium, and if one exists, to computing it.

This chapter describes ways of adapting our earlier methods to compute, represent, and manipulate equilibria.[1] We describe two classes of economies with distortions. The first class is designed partly as a warmup for the second, but is also interesting in its own right. In this first class of models, an equilibrium can be computed by one pass through the invariant subspace algorithm. This is possible possible because of two sorts of simplifying assumptions: first, that there is a representative agent (which permits Gorman-heterogeneity);and

[1] The invariant subspace computational approach follows the tradition of Blanchard and Khan (1980), Whiteman (1983), Anderson and Moore (1985), King, Plosser, and Rebelo (1989), and McGrattan (1991) in expressing the equilibrium conditions as a system of linear difference equations, then adapting the method of Vaughan to solve it. The principal caveat to keep in mind in applying these methods is that in distorted economies, there is no guarantee that the eigenvalues of the matrix 'characteristic equation' of the system of equilibrium conditions will 'split' half into those exceeding $1/\sqrt{\beta}$ in modulus, and half into those falling short of $1/\sqrt{\beta}$ in modulus. Failure of the eigenvalues to divide in that way signals either an existence or a uniqueness problem.

second, that the government has the ability to levy lump sum taxes or transfers.[2] Thus, the first setup is a representative agent setting with consumption and production externalities and distorting taxes.[3] .

In the second setup, equilibria must be computed with multiple passes through the invariant subspace algorithm. What necessitates this is that there are (non-Gorman) heterogeneous agents with consumption externalities and distorting taxes, but no lump sum taxes or transfers. In this setting, equilibrium computation means finding a set of taxes and households' marginal utilities of wealth that assure present value budget balance for households and the government. We find the equilibrium taxes and marginal utilities of wealth by repeatedly resorting to the invariant subspace algorithm, constructing candidate prices and quantities at each pass of the invariant subspace algorithm, then checking every agent's budget constraint at each pass.

It is convenient to describe these two classes of models sequentially, partly because the first one is complicated enough, and because it reveals about half of the difficulties involved in calculating models of the second class.

We shall use our machinery to compute equilibria of some simple models embodying preference and technology specifications that have occurred in the recent literature.

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[2] The MATLAB program `solvdist.m` computes equilibria for the first type of economy, while `disthet.m` computes equilibria for the second type of economy. The MATLAB program `compare.m` is useful for comparing the results of `disthet.m` with related undistorted economies whose equilibria have been computed with `solvea.m`.

[3] The first class of models are linear-quadratic relatives of the ones studied by Braun (1991) and McGrattan (1991b). In particular, both Braun and McGrattan use lump sum transfers to balance the government's budget.

15.2. A Representative Agent Economy with Distortions

We alter the model discussed in chapters 3 and 5 by adding two sorts of distortions: consumption and production externalities, and distorting taxes. We add these distortions in a way that is designed to preserve the applicability of the forms of equilibrium prices and quantities: we want the equilibrium law of motion to be linear in the state, quantities and (scaled Arrow-Debreu) prices to be linear functions of the state, and asset prices to be quadratic functions of the state. We add the following distortions to the Chapter 3–5 model.

15.2.1. a. Consumption externalities

The technology for producing household services is now taken to be

$$\begin{aligned} s_t &= \Lambda_h\, h_{t-1} + \Lambda_H\, H_{t-1} + \Pi c_t + \Pi_C C_t \\ h_t &= \Delta_h h_{t-1} + \Delta_H\, H_{t-1} + \Theta_h\, c_t + \Theta_H\, C_t, \end{aligned}$$

where H_{t-1} is the vector of aggregate household capital stocks, and C_t is the vector of aggregate consumption rates. In equilibrium, $c_t = C_t$ and $h_t = H_t$, but the representative household is assumed to take $\{H_t, C_t\}$ as given and beyond control when allocating its resources.

15.2.2. b. Production externalities

Firms produce subject to the linear technology

$$\Phi_c(c_t + G_t) + \Phi_i i_t + \Phi_g g_t = \Gamma_k k_{t-1} + \Gamma_K K_{t-1} + d_t$$

where K_{t-1} is the vector of aggregate capital stocks, and G_t is government purchases of consumption goods. In equilibrium, $k_{t-1} = K_{t-1}$, but the representative firm is assumed to regard $\{K_{t-1}\}$ as beyond its control when choosing its inputs and outputs. The presence of K_{t-1} on the right side of the technology constraint is designed to accommodate externalities of the sort analyzed by Paul Romer (1985). Below, we shall assign ownership of the 'technology' so that households sell the joint process $\Gamma_K K_{t-1} + d_t$ to firms. This assignment will confront the firm with a constant returns to scale technology. We assume that the government purchase vector process is exogenous and satisfies $G_t = U_G\, z_t$.

15.2.3. c. Taxes

We add to the model of chapters 3–5 a government that taxes consumption goods, investment goods, capital goods, and the intermediate labor activity. The government's budget constraint is

$$E\sum_{t=0}^{\infty}\beta^t\left\{p_t^{0\prime}\tau_c c_t + q_t^{0\prime}\tau_i i_t + r_t^{0\prime}\tau_k k_{t-1} + w_t^{0\prime}\tau_\ell g_t\right\}|J_0$$
$$= E\sum_{t=0}^{\infty}\beta^t p_t^0\cdot G_t|J_0 + T_0$$

where $G_t = U_G z_t$. Here τ_c is a diagonal matrix of tax rates on consumption goods, τ_i is a diagonal matrix of tax rates on investment goods, τ_k is a diagonal matrix of tax rates on rentals of capital, and τ_ℓ is a diagonal matrix of tax rates on the labor-using activity vector g_t; T_0 is lump sum transfers to the household. The government sets the tax rates τ_c, τ_i, τ_k, τ_ℓ and the expenditure process $U_G z_t = G_t$.

For this economy, a *price system* is a collection of stochastic processes $\{p_t^0,\, q_t^0, r_t^0,$
$w_t^0, \alpha_t^0\}_{t=0}^\infty$, each element of which belongs to L_0^2. A *tax system* is a list of diagonal matrices $[\tau_c,\, \tau_i,\, \tau_k,\, \tau_\ell]$. An *allocation* is a collection of stochastic processes $\{s_t,\, c_t,\, h_t,\, k_t,\, g_t,\, i_t\}_{t=0}^\infty$ each element of which belongs to L_0^2.

We now describe the choices faced by households and firms.

15.3. Households

Households own an 'endowment stream' $\{d_t + \Gamma_K K_{t-1}\}_{t=0}^\infty$ and the initial capital stocks, all of which they take as given. Households take the price system, the tax system, and $\{C_t,\, H_{t-1}, K_{t-1}\}_{t=0}^\infty$ as given, and choose contingency plans for $\{c_t,\, i_t,\, s_t,\, h_t,\, k_t,\, g_t\}_{t=0}^\infty$ to maximize

$$-\frac{1}{2}\; E\sum_{t=0}^{\infty}\beta^t[(s_t-b_t)\cdot(s_t-b_t) + g_t\cdot g_t] \mid J_0 \tag{15.3.1}$$

subject to the household technology,

$$s_t = \Lambda_h h_{t-1} + \Lambda_H H_{t-1} + \Pi c_t + \Pi_C C_t \tag{15.3.2}$$
$$h_t = \Delta_h h_{t-1} + \Delta_H H_{t-1} + \Theta_h c_t + \Theta_H C_t \tag{15.3.3}$$

the law of accumulation for physical capital,

$$k_t = \Delta_k k_{t-1} + \Delta_K K_{t-1} + \Theta_k i_t, \tag{15.3.4}$$

and the budget constraint

$$\begin{aligned} E \sum_{t=0}^{\infty} \beta^t [p_t^{0\prime}(I + \tau_c) c_t + q_t^{0\prime}(I + \tau_i) i_t] \mid J_0 \\ = E \sum_{t=0}^{\infty} \beta^t [w_t^{0\prime}(I - \tau_\ell) g_t + \alpha_t^0 \cdot (d_t + \Gamma_K K_{t-1}) \\ + r_t^{0\prime}(I - \tau_k) k_{t-1}] \mid J_0 + T_0 \end{aligned} \tag{15.3.5}$$

15.4. Firms

There is one kind of firm, a production firm that takes the price system and $\{K_{t-1}\}_{t=0}^{\infty}$ as fixed, and chooses an allocation and a process $\{\zeta_t\} \in L_0^2$ that maximize

$$E \sum_{t=0}^{\infty} \beta^t \{p_t^0 \cdot (c_t + G_t) + q_t^0 \cdot i_t - r_t^0 \cdot k_{t-1} - \alpha_t^0 \cdot \zeta_t - w_t^0 \cdot g_t | J_0\} \tag{15.4.1}$$

subject to the technology

$$\Phi_c (c_t + G_t) + \Phi_g g_t + \Phi_i i_t = \Gamma_k k_{t-1} + \zeta_t \cdot \zeta_t. \tag{15.4.2}$$

15.5. Information

We assume that

$$z_{t+1} = A_{22} z_t + C_2 w_{t+1},$$

where $\{w_t\}$ is a martingale difference sequence with identity covariance matrix. The stochastic process $\{z_t\}$ drives the exogenous processes

$$\begin{aligned} b_{it} &= U_{bi} z_t \\ d_{it} &= U_{di} z_t \\ G_t &= U_G z_t. \end{aligned}$$

15.6. Equilibrium

An *equilibrium* is defined as an allocation, a price system, and a tax system such that

(i) Given the price system, the tax system, and $\{H_{t-1}, C_{t-1}, K_{t-1}\}_{t=0}^{\infty}$, the allocation solves the household's problem;

(ii) Given the price system and $\{K_{t-1}\}_{t=0}^{\infty}$, the allocation solves the firm's problem with $\zeta_t \equiv \Gamma_K K_{t-1} + d_t$;

(iii) The representative household is representative, $H_{t-1} = h_{t-1}, C_t = c_t$; and the representative firm is representative, $K_{t-1} = k_{t-1}$ for all $t \geq 0$.

Given that conditions (i), (ii), (iii), are satisfied, the lump sum tax T_0, which is a present value, can be chosen to make the government budget constraint hold.

We have set things up so that the equilibrium law of motion takes the form

$$x_{t+1} = A^o x_t + C w_{t+1},$$

and equilibrium quantities and normalized Arrow-Debreu prices are linear functions of the state $x_t' = [h_{1t-1}' h_{2t-1}' k_{t-1}' z_t']$.

Appendix A describes how to compute the equilibrium of this model by manipulating agents' first order conditions and the other equilibrium conditions into a system susceptible to the application of Vaughan's algorithm. The MATLAB program `solvdist.m` computes an equilibrium. Equilibrium calculation is facilitated by the fact that there is a representative households (so that if there is heterogeneity among households, the features of Gorman aggregation isolate demand functions from effects of redistributions of wealth) and the permission that we give the government to levy lump sum taxes. In the next class of models, we give up both of these simplifying features.

15.7. Heterogeneous Households with Distortions

In this section, we describe how to extend the above setup to enable us to compute the equilibrium of a model with externalities, government expenditures and taxes, and two classes of agents with non-Gorman aggregable preferences. To produce the model, we combine elements of the preference specification treated in chapter 8 with the specification and methods described earlier in this chapter. We do not let the government raise lump sum taxes or dispense lump sum transfers, but require the government to balance its budget by levying only flat rate taxes. These changes vis a vis the first class of models have the consequence that equilibrium must now be computed by finding values of tax rates and marginal utilities of wealth for each household that make households' and the government's budgets balance.[4]

[4] Between these two classes of models, we have changed the way we price labor or 'intermediate goods' g_t. In the first class, the households own and sell to the firm the vector g_t, which is priced by the vector w_t^0. In the second model, households sell ℓ_{it} for w_t^0 to firms.

15.7.1. Households

There are two classes of households, indexed by $i = 1, 2$. Preferences of a household of type i are ordered by

$$-.5E \sum_{t=0}^{\infty} \beta^t [(s_{it} - b_{it}) \cdot (s_{it} - b_{it}) + \ell_{it}^2] | J_0, \tag{15.7.1}$$

where the household has access to the household technology

$$\begin{aligned} s_{it} &= \Lambda_{i1} h_{it-1} + \Lambda_{i2} H_{1t-1} + \Lambda_{i3} H_{2t-1} \\ &\quad + \Pi_{i1} c_{it-1} + \Pi_{i2} C_{1t} + \Pi_{i3} C_{2t} \\ h_{it} &= \Delta_{hi} h_{it-1} + \Delta_{Hi1} H_{it-1} + \Delta_{Hi2} H_{2t-1} \\ &\quad + \Theta_{hi} c_{it} + \Theta_{H1i} C_{1t} + \Theta_{H2i} C_{2t} \\ d_{it} &= U_{di} z_t \\ b_{it} &= U_{bi} z_t . \end{aligned} \tag{15.7.2}$$

Here C_{it}, H_{it} are the aggregate consumption and stock of durables, respectively, of household type i, and c_{it}, h_{it} are the individual consumption and durables, respectively, of a household of type i. There are firms of two types, to be described in detail below. A household of type i is assumed to own a a fraction f_i of the 'technology', which entitles it to sell $f_i(\Gamma_k + d_t)$ of an 'endowment' to a type I (production) firm, and to rent $f_i \Delta_K K_{t-1}$ of capital to a type II (capital renting) firm at time t. Thus, the intertemporal budget constraint of a household of type i is

$$\begin{aligned} E \sum_{t=0}^{\infty} \beta^t \{ & p_t^{0\prime} (I + \tau_c) c_{it} - f_i \alpha_t^0 \cdot (\Gamma_K K_{t-1} + d_t) \\ & - f_i r_t^0 \cdot \Delta_K K_{t-1} - w_t^0 (1 + \tau_\ell) \ell_{it} \} | J_0 - v_0 k_{-1,i} = 0 \end{aligned} \tag{15.7.3}$$

No lump sum transfers occur in (15.7.3), because we shall require the government to balance its budget using flat rate taxes only. Below, we will carry along a Lagrange multiplier μ_{0i} on (15.7.3) for each type of household.

15.7.2. Firms of type I

Firms of type I are production firms. They maximize

$$E\sum_{t=0}^{\infty}\beta^t\{p_t^0\cdot(c_t+G_t)+q_t^0\cdot i_t-r_t^0\cdot k_{t-1}-\alpha_t^0\cdot\zeta_t-w_t^0\ell_t\}|J_0 \tag{15.7.4}$$

subject to

$$\begin{aligned}\Phi_c(c_t+G_t)+\Phi_i i_t+\Phi_g g_t&=\Gamma_k k_{t-1}+\zeta_t,\\ g_t\cdot g_t&=\ell_t^2\end{aligned} \tag{15.7.5}$$

15.7.3. Firms of type II

Firms of type II are capital renting firms. They maximize

$$E_0\sum_{t=0}^{\infty}\beta^t\{r_t^{0\prime}(I-\tau_k)k_{t-1}-q_t^{0\prime}(I+\tau_i)i_t-\nu_t r_t\cdot\Delta_K K_{t-1}\}|J_0-v_0k_{-1} \tag{15.7.6}$$

subject to

$$k_t=\Delta_k k_{t-1}+\nu_t\Delta_K K_{t-1}+\Theta_k i_t.$$

Here ν_t is the amount of the 'endowment' $\Delta_K K_{t-1}$ purchased from households.

15.7.4. Government

The government makes a flow of expenditures G_t on consumption goods, governed by

$$G_t=U_G z_t. \tag{15.7.7}$$

The government's budget constraint is

$$\begin{aligned}E\sum_{t=0}^{\infty}\beta^t\{p_t^0\cdot G_t-p_t^{0\prime}\tau_c\cdot(c_{1t}+c_{2t})-q_t^{0\prime}\tau_i i_t-r_t^{0\prime}\tau_k k_{t-1}\\ -\tau_\ell w_t^0(\ell_{1t}+\ell_{2t})\}|J_0=0\end{aligned} \tag{15.7.8}$$

A *fiscal policy* is a collection of diagonal matrices $(\tau_c,\tau_i,\tau_k,\tau_\ell)$ determining taxes and a matrix U_G determining government expenditures.

15.7.5. Definition of equilibrium

An *equilibrium* is a price system $[\{p_t^0, q_t^0, r_t^0, \alpha_t^0, w_t^0\}_{t=0}^{\infty}, v_0]$, an individual allocation $\{c_{1t}, c_{2t}, h_{1t}, h_{2t}, k_t, i_t, g_t\}$, an aggregate allocation $\{C_{1t}, C_{2t}, H_{1t}, H_{2t}, K_t\}$, and a fiscal policy $(\tau_c, \tau_i, \tau_k, \tau_\ell, U_G)$ that satisfy the following conditions:

i. Given the aggregate allocation, the price system, and the fiscal policy, the individual allocation solves the optimum problems of households and the firms, with $\zeta_t \equiv 1$ and $\nu_t \equiv 1$.
ii. The allocations satisfy

$$\begin{aligned} c_{it} &= C_{it},\ i = 1, 2 \\ h_{it} &= H_{it},\ i = 1, 2 \\ k_t &= K_t \\ \zeta &= (\Gamma_K K_{t-1} + d_t). \end{aligned}$$

iii. The government budget constraint is satisfied.

As we have defined it, an equilibrium is typically not unique. In particular, there will usually be a set of taxes that serve to assure equilibrium. Below, we shall select one from among these equilibria partly by fixing enough of these taxes and 'solving' for others.[5]

15.7.6. Equilibrium computation

We compute an equilibrium by finding a fixed point in the parameters that index the tax system $\tau_c, \tau_i, \tau_k, \tau_\ell$ and a pair of marginal utilities of wealth (multipliers) μ_{01}, μ_{02} for households. For fixed values of the tax system and multiplier parameters, we can use the modified Vaughan algorithm described in the preceding section to compute a 'candidate' 0 allocation and price system. For that allocation and price system, we evaluate all elements of the budget constraints of the government and the two types of households. We use a nonlinear search algorithm We use a secant algorithm. to find a tax system and pair of multipliers μ_{01}, μ_{02} that assure that the budget constraints are all satisfied. Thus, we compute an equilibrium by solving a fixed point problem in a space of tax rates and marginal utilities of wealth for the different household types.

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[5] Even when we fix the 'right number' of the taxes there can still be multiple equilibria for reason of 'Laffer curves'.

Appendix B describes how the first order conditions for households and firms together with the other equilibrium conditions can be arranged to obtain a system of equations of the form (9.25), which with the invariant subspace methods of chapter 9 permits us to compute our candidate equilibrium. The equilibrium is computed by `disthet.m`.

15.8. Government Deficits and Debt

The government deficit at time t, measured in time t 'spot' prices, is

$$D_t^t = p_t^t \cdot (G_t - \tau_c(c_{1t} + c_{2t})) - q_t^t \cdot \tau_i i_t - r_t^t \cdot \tau_k k_{t-1} - w_t^t \tau_\ell(\ell_{1t} + \ell_{2t}).$$

Evidently, D_t^t can be represented as

$$D_t^t = \frac{x_t' Q_D x_t}{e_1 M_c x_t}, \tag{15.8.1}$$

where

$$\begin{aligned} Q_D = {} & M_c'[S_G - \tau_c(S_{c1} + S_{c2})] - M_i' \tau_i S_i \\ & - M_k' \tau_k S_{k_1} - \frac{\tau_\ell}{(1-\tau_\ell)(\mu_{01} + \mu_{02})} S_g' S_g. \end{aligned}$$

Let V_t denote the present value of the government deficit, which satisfies the difference equation

$$V_t = -D_t^t + \beta E_t \{p_{1,t+1}^t V_{t+1}\}, \tag{15.8.2}$$

where $p_{1,t+1}^t = e_1 M_c x_{t+1} / e_1 M_c x_t$ is the time t price of a state contingent claim to the first (numeraire) consumption good in time $t+1$. Notice that $\beta p_{1,t+1}^t$ acts as a stochastic discount factor for evaluating government indebtedness next period from the standpoint of this period. This equation is the counterpart of the following version of a one period government budget constraint, which occurs in various nonstochastic macroeconomic models: $-D_t + V_{t+1}/R_t = V_t$, where here V_t is interpreted as one-period debt falling due at time t.

Equations (15.8.1) and (15.8.2) imply that

$$V_t = \frac{x_t Q_V x_t + \sigma_V}{e_1 M_c x_t}, \tag{15.8.3}$$

where Q_V, σ_V are determined as follows. Define

$$\tilde{v} = \frac{\beta}{1-\beta} \texttt{doublej2}(\beta A^o, C, A^o, C).$$

Then $[Q_V] = \texttt{doublej2}(\beta A^{o\prime}, Q_D, A^{o\prime}, 1)$ and $\sigma_V = \text{trace}(Q_V \tilde{v})$. The matrix valued function `doublej2` was used repeatedly in our asset pricing calculations.

The forms of (15.8.2) and (15.8.3) mean that the econometric methods described in Hansen and Sargent (1993) for interpreting observations on asset prices can be used to model government budgets and bond holdings.[6]

15.9. Examples

15.9.1. A production externality

The following technology is designed to capture features of a specification of DeLong and Summers. There is one consumption good, but two capital goods ('machines' and 'structures') and two rates of investment. Machines generate a positive production externality, but not structures:

$$\begin{aligned} c_t + G_t + i_{1t} + i_{2t} &= \gamma_1 k_{1t-1} + \gamma_2 k_{2t-1} \\ &\quad + \Gamma_1 K_{1t-1} + \Gamma_2 K_{2t-1} + d_t \\ \phi_1 i_{1t} &= g_{1t} \\ \phi_2 i_{2t} &= g_{2t} \\ k_{1t} &= \delta_{k1} k_{1t-1} + i_{1t} \\ k_{2t} &= \delta_{k2} k_{2t-1} + i_{2t} \end{aligned}$$

To capture DeLong and Summers's idea, we set $\Gamma_1 = 0, \Gamma_2 > 0$, so that we interpret the first capital good as structures and the second as machines.

To complete this example, we incorporate a single-agent version of a simple quadratic utility specification. We suppress heterogeneity among consumers and instead make the two types of consumers be identical in their preferences and endowment sequences. Preferences are ordered by

$$-\frac{1}{2} \sum_{t=0}^{\infty} \beta^t [(c_{it} - b_{it})^2 + \ell_{it}^2].$$

[6] The restrictions embedded in (15.8.2) and (15.8.3) should be compared with those studied in the literature on linear models of 'present value budget balance.' See Hansen, Roberds, and Sargent (1991) for a summary of this literature.

The information process satisfies

$$A_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & .95 & 0 & 0 \\ 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & .1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{20\sqrt{2}} & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$d_{1t} = d_{2t} = [\,3.5 \quad 0 \quad .5 \quad 0\,]\, z_t,$$

$$G_t = [\,5 \quad 1 \quad 0 \quad 1\,]\, z_t.$$

$$b_{1t} = b_{2t} = 30.$$

These settings make endowments of the two types of households each the same first order autoregressive processes with mean 3.5 and serial correlation parameter $.8$, while government expenditures follows the process

$$G_t - 5 = \frac{(20\sqrt{2})^{-1}}{1 - .95L} w_{1t} + \frac{.5}{1 - .1L} w_{3t},$$

where L is the lag operator. This process approximates a mixture of a 'permanent shock' (the moving average in w_{1t}) and a 'transitory shock' (the moving average in w_{3t}. The mean of government expenditures is 5.

We set $\gamma_1 = \gamma_2 = .12, \Gamma_1 = 0, \Gamma_2 = .04, \phi_1 = \phi_2 = .5, \delta_{k1} = \delta_{k2} = .95, \beta = 1/1.05$. These parameter settings make the two types of capital symmetric with respect to adjustment costs and 'private productivity' (γ_i), but inject a positive production externality for the second capital good.

15.9.2. Consumption tax only

With these parameter settings, we computed an equilibrium where the only tax is the scalar consumption tax τ_c on the single consumption good. The equilibrium tax $\tau_c = .2497$. Figures 15.9.1.a and 15.9.1.b show a simulation of this equilibrium starting from the initial condition $k_{-1,1} = k_{-1,2} = 100$. Because we start from equal initial stocks of machines and structures, and because their private productivities are the same, the equilibrium must retain equality of structures and machines throughout time. Figures 15.9.1.a and 15.9.1.b embody this property, the rates of investment in machines and structures being identical. Figures 3 display realizations of government indebtedness V_t determined by (15.8.2). Figure 4 shows the government flow deficit.

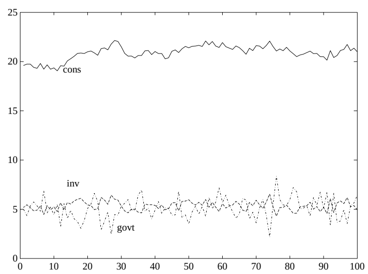

Fig. 15.9.1.a. Total consumption, investment, and government expenditures in Delong-Summers economy with $\tau_c =$ .2497 and no investment subsidy.

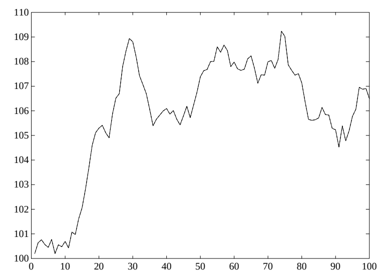

Fig. 15.9.1.b. Machines and structures in Delong-Summers economy with $\tau_c =$ .2497 and no investment subsidy.

15.9.3. Machinery investment subsidy

Figures 15.9.2 and 15.9.3 report the results of recomputing the equilibrium when we impose an investment subsidy $\tau_{i2} = -.04$. The only other tax that we permit the consumption tax, which must be set at $\tau_c = .2611$ to induce equilibrium. The simulations show how the investment rates for machines and structures now diverge in the direction that we would expect: the economy moves to a path with more machines and fewer structures than the first (no machine subsidy) equilibrium.[7]

[7] Realizations of the exogenous stochastic processes are held constant across these simulations.

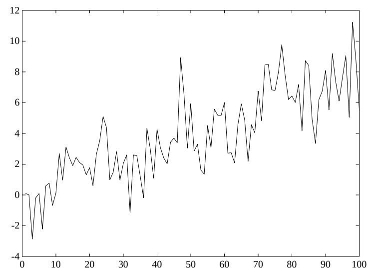

Fig. 15.9.2.a. Present value of government surplus in Delong-Summers economy with $\tau_c = .2497$ and no investment subsidy.

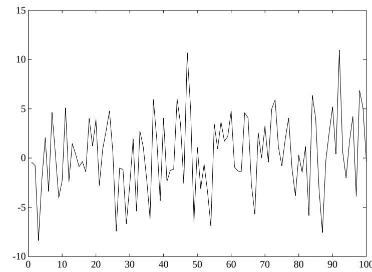

Fig. 15.9.2.b. Government deficit in Delong-Summers economy with $\tau_c = .2497$ and no investment subsidy.

15.9.4. 'Personal' habit persistence

We consider an economy with one consumption good that is allocated to two types of households. The first type of household has preferences ordered by

$$E_0 - .5 \sum_{t=0}^{\infty} \beta^t [(c_{1t} - 15)^2 + \ell_{1t}^2],$$

while the second has the habit-persistence indicated by the preferences

$$E_0 - .5 \sum_{t=0}^{\infty} \beta^t [(s_{2t} - 15)^2 + \ell_{2t}^2]$$

where

$$s_{2t} = -1 h_{2t-1} + 2 c_{2t}$$
$$h_{2t} = .8 h_{2t-1} + .2 c_{2t}$$

We set $\beta = 1/1.05$.

The production technology has

$$c_t + i_t = .11 k_{t-1} + .0001 K_{t-1} + d_t$$
$$k_t = .95 k_{t-1} + i_t$$
$$g_{1t} = .5 i_t$$

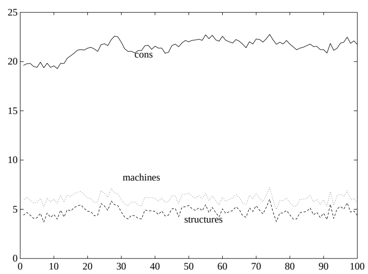

Fig. 15.9.3.a. Total consumption, investment, and government expenditures in Delong-Summers economy with $\tau_c = .2611$ and investment subsidy, $\tau_{i2} = -.04$.

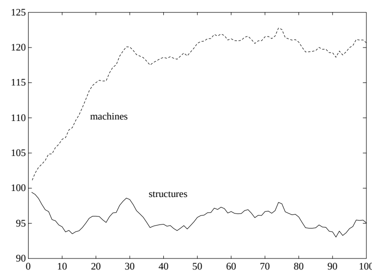

Fig. 15.9.3.b. Machines and structures in Delong-Summers economy with $\tau_c = .2611$ and investment subsidy, $\tau_{i2} = -.04$.

which is a version of our one-good "growth, adjustment cost" technology with a small production externality.

We set

$$A_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .8 & 0 \\ 0 & 0 & .5 \end{bmatrix}$$

$$U_G = [\,4 \quad .1 \quad 0\,]$$

$$U_{d2} = U_{d1} = \begin{bmatrix} 3.5 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k_{-1,1} = 125, \;\; k_{-1,2} = 25.$$

Notice that we endow the first household with more capital and equal claim to the endowment stream, so the first household is richer.

We do the following experiments with this economy. First, we set all taxes except τ_c equal to zero, and solve for the equilibrium values of (μ_{02}, τ_c). They are (2.0765,.2117). A simulation of this economy is in figure 7. Then we reset

$\tau_i = .08$, and solve again for $(\mu_{02}, \tau_c) = (2.0467, .27)$. A simulation of this economy is in figure 8. The simulations start from identical initial conditions.

In this economy, the habit persistence of the second type of consumer is something of an "engine of growth". The second type of household accumulates capital to support planned growth in its consumption. Taxing investment causes the household to cut back on these investments.

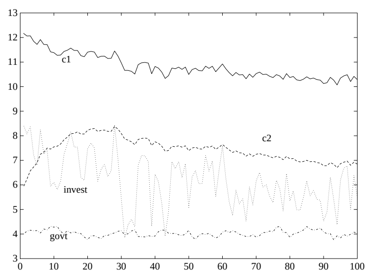

Fig. 15.9.4.a. Simulation of two-agent economy with $\tau_c = .2117, \tau_i = 0$. Four series are plotted: consumption of type 1 consumer, consumption of type 2 consumer, investment, and government purchases. Equilibrium marginal utilities of wealth are $\mu_{01} = 1, \mu_{02} = 2.0765$.

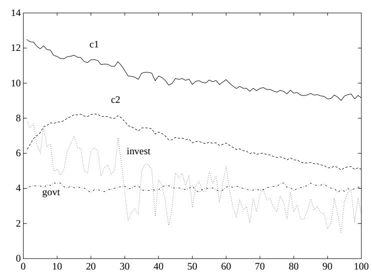

Fig. 15.9.4.b. Simulation of two-agent economy with $\tau_c = .1884, \tau_i = .08$. Equilibrium marginal utilities of wealth are $(1, 2.0467)$.

15.9.5. 'Social' habit persistence

Our next example is identical to the previous one, except that we alter the equation generating 'habits' of the second type of household to

$$h_{2t} = .8h_{2t-1} + .2C_{2t}.$$

We computed equilibria of this economy with the same government expenditure process and the same permissible tax instruments as for the previous economy. Figures 15.9.4 and 15.9.5 report simulations of this economy starting from the same initial conditions as for the previous economy. Notice that slightly lower consumption tax rates provide equilibria in this economy, for the reason that due to the 'social' rather than 'personal' nature of habit persistence, they have slightly lower adverse demand effects on the second type of household.

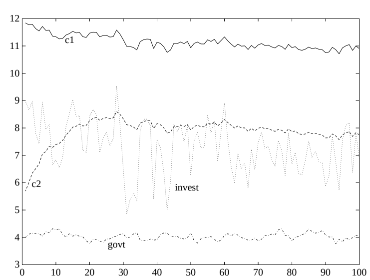

Fig. @Fg.feb1c31@.a. Simulation of two-agent economy with $\tau_c = .2101, \tau_i = 0$. The second type of agent has 'keeping up with the Jones' habit persistence with other agents of his type. Four series are plotted: consumption of type 1 consumer, consumption of type 2 consumer, investment, and government purchases.

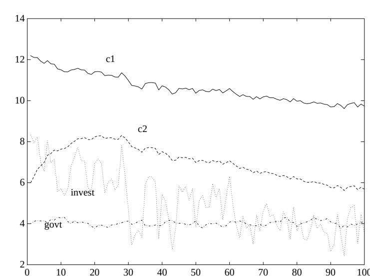

Fig. 15.9.5.b. Simulation of two-agent economy with $\tau_c = .1883, \tau_i = .08$.

15.10. Conclusions

The models that we have described are rigged to make it practical to extend the data matching exercises performed by Braun (1991) and McGrattan (1991b). Our calculations make possible various extensions of Braun and McGrattan. The ease of computing equilibrium quantities, prices, and present values means that we can use data on quantities, interest rates, and asset prices to do method of moment estimation along the lines described by Hansen and Sargent (1993, chapter 11). In particular, we can use data on government indebtedness and deficits as well as asset prices to help estimate parameters. By accepting our linear-quadratic specifications, we purchase the ability to get our hands on the Arrow-Debreu prices (which is harder with the approximations used by Braun, McGrattan and others in the real business cycle literature), which makes it feasible for us to do without lump sum taxes in our second class of models. It is possible for us to have numbers of capital stocks and enough heterogeneity among households to generate interesting tax incidence effects. Finally, the machinery in this chapter is a useful one within which to revisit issues in the literature on time series implications of 'present value budget balance' (e.g., Hansen, Roberds, and Sargent (1991)).

A. Invariant subspace equations for first specification

Our strategy is to use Lagrangian methods to obtain first order necessary conditions for households and firms. *After* obtaining those first order conditions, we substitute the equilibrium conditions ($H_{t-1} = h_{t-1}$, $K_{t-1} = k_{t-1}$, $C_t = c_t$, and so on) into them. Then we rearrange the system into the form of (9.25) so that it is susceptible to application of the modified Vaughan method. We can exploit the certainty equivalence principle and solve a nonstochastic version of the model first, and later adjust the solution to accommodate randomness.

15.A.1. Household's Lagrangian

A Lagrangian for a nonstochastic version of the household's problem is

$$
\begin{aligned}
L = \sum_{t=0}^{\infty} \beta^t \Big\{ & -\frac{1}{2}\ [(s_t - U_b z_t)\cdot(s_t - U_b z_t) + g_t \cdot g_t] \\
& + \mu_0 [w_t^{0\prime}(I-\tau_\ell) g_t + \alpha_t^0 (U_d z_t + \Gamma_K K_{t-1}) \\
& \quad + r_t^{0\prime}(I-\tau_k) k_{t-1} + T_t - p_t^{0\prime}(I+\tau_c) c_t - q_t^{0\prime}(I+\tau_i) i_t] \\
& + \mu_t^{s\prime}[\Lambda_h h_{t-1} + \Lambda_H H_{t-1} + \Pi c_t + \Pi_C C_t - s_t] \\
& + \mu_t^{h\prime}[\Delta_h h_{t-1} + \Delta_H H_{t-1} + \Theta_h c_t + \Theta_H C_t - h_t] \\
& + \mu_t^{k\prime}[\Delta_k k_{t-1} + \Delta_K K_{t-1} + \Theta_k i_t - k_t] \\
& + \mu_t^{z\prime}[A_{22} z_t - z_{t+1}] \Big\}
\end{aligned}
\tag{15.A.1}
$$

We can set the multiplier on the budget constraint $\mu_0 = 1$, which will amount to selecting a numeraire.

The first-order conditions for the household's problem are

$$\mu_t^s = (b_t - s_t) \tag{15.A.2}$$

$$(I+\tau_c) p_t^0 = \Pi' \mu_t^s + \Theta_h' \mu_t^h \tag{15.A.3}$$

$$\mu_t^h = \beta \Lambda_h' \mu_{t+1}^s + \beta \Delta_h' \mu_{t+1}^h \tag{15.A.4}$$

$$\mu_t^k = \beta \Delta_k' \mu_{t+1}^k + \beta (I-\tau_k) r_{t+1}^0 \tag{15.A.5}$$

$$g_t = (I-\tau_\ell) w_t^0 \tag{15.A.6}$$

$$(I+\tau_i) q_t^0 = \Theta_k' \mu_t^k \tag{15.A.7}$$

$$\mu_t^z = \beta [A_{22}' \mu_{t+1}^z + U_b'(s_{t+1} - U_b z_{t+1}) + U_d' \alpha_{t+1}^0] \tag{15.A.8}$$

The law of motion for the state variables chosen by the household and by nature is

$$\begin{pmatrix} h_t \\ k_t \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \Delta_h & 0 & 0 \\ 0 & \Delta_k & 0 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{pmatrix} + \begin{pmatrix} \Theta_h & 0 \\ 0 & \Theta_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_t \\ i_t \end{pmatrix} + \begin{pmatrix} \Delta_H & 0 & \Theta_H \\ 0 & \Delta_K & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} H_{t-1} \\ K_{t-1} \\ C_t \end{pmatrix} \tag{15.A.9}$$

15.A.2. Firm's first order conditions

The first-order conditions for the firm's problem are

$$p_t^0 = \Phi_c' \alpha_t^0 \tag{15.A.10}$$

$$q_t^0 = \Phi_i' \alpha_t^0 \tag{15.A.11}$$

$$w_t^0 = -\Phi_g' \alpha_t^0 \tag{15.A.12}$$

$$r_t^0 = \Gamma_k' \alpha_t^0 \tag{15.A.13}$$

The feasibility condition is

$$\Phi_c(c_t + G_t) + \Phi_i i_t + \Phi_g g_t = \Gamma_k k_{t-1} + \Gamma_K K_{t-1} + d_t \tag{15.A.14}$$

15.A.3. Representativeness conditions

Additional equilibrium conditions are

$$\begin{aligned} h_{t-1} &= H_{t-1} \\ k_{t-1} &= K_{t-1} \\ c_t &= C_t \end{aligned} \tag{15.A.15}$$

We describe in detail how to compute the equilibrium of the first type of model by arranging its equilibrium conditions into the form of equation (9.25).

Substituting the equilibrium conditions (15.A.15) into (15.A.9) gives

$$\begin{pmatrix} h_t \\ k_t \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \bar\Delta_h & 0 & 0 \\ 0 & \bar\Delta_k & 0 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{pmatrix} + \begin{pmatrix} \bar\Theta_h & 0 \\ 0 & \Theta_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_t \\ i_t \end{pmatrix} \tag{15.A.16}$$

where $\bar\Delta_h = \Delta_h + \Delta_H$, $\bar\Delta_k = \Delta_k + \Delta_K$, $\bar\Theta_h = \Theta_h + \Theta_H$. Equations (15.A.10) and (15.A.12) imply

$$\alpha_t^0 = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} p_t^0 \\ -w_t^0 \end{bmatrix}. \tag{15.A.17}$$

Substituting from (15.A.3) and (15.A.6) into (15.A.17) gives

$$\alpha_t^0 = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} (I+\tau_c)^{-1}[\Pi' \mu_t^s + \Theta_h' \mu_t^h] \\ -(I-\tau_\ell)^{-1} g_t \end{bmatrix} \tag{15.A.18}$$

Using $k_{t-1} = K_{t-1}$ in (15.A.14) and solving for $[c_t' \ g_t']'$

$$\begin{bmatrix} c_t \\ g_t \end{bmatrix} = [\Phi_c \ \Phi_g]^{-1} \ \{\bar\Gamma k_{t-1} + U_f z_t - \Phi_i i_t\} \tag{15.A.19}$$

where $U_f = U_d - \Phi_c U_G$ and $\bar\Gamma = \Gamma_k + \Gamma_K$.

Substituting $h_{t-1} = H_{t-1}$ and $c_t = C_t$ into equation (15.3.2) s_t gives

$$s_t = \Lambda h_{t-1} + \bar\Pi c_t$$

where $\Lambda = \Lambda_h + \Lambda_H$ and $\bar\Pi = \Pi + \Pi_C$.

Collecting our results to this point, we want to solve the following system of difference equations

$$\mu_t^s = (U_b z_t - s_t) \tag{15.A.20}$$

$$s_t = \Lambda h_{t-1} + \bar{\Pi} c_t \tag{15.A.21}$$

$$\mu_t^h = \beta \Lambda_h' \mu_{t+1}^s + \beta \Delta_h' \mu_{t+1}^h \tag{15.A.22}$$

$$\mu_t^k = \beta \Delta_k' \mu_{t+1}^k + \beta (I - \tau_k) \Gamma_k' \alpha_{t+1}^0 \tag{15.A.23}$$

$$\mu_t^z = \beta [A_{22}' \mu_{t+1}^z + U_b' (s_{t+1} - U_b z_{t+1}) + U_d' \alpha_{t+1}^0] \tag{15.A.24}$$

$$\begin{pmatrix} h_t \\ k_t \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} \bar{\Delta}_h & 0 & 0 \\ 0 & \bar{\Delta}_k & 0 \\ 0 & 0 & A_{22} \end{pmatrix} \begin{pmatrix} h_{t-1} \\ k_{t-1} \\ z_t \end{pmatrix} + \begin{pmatrix} \bar{\Theta}_h & 0 \\ 0 & \Theta_k \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_t \\ i_t \end{pmatrix} \tag{15.A.25}$$

$$\begin{bmatrix} c_t \\ g_t \end{bmatrix} = [\Phi_c \ \Phi_g]^{-1} \{\bar{\Gamma} k_{t-1} + U_f z_t - \Phi_i i_t\} \tag{15.A.26}$$

$$\alpha_t^0 = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} \begin{bmatrix} (I + \tau_c)^{-1} [\Pi' \mu_t^s + \Theta_h' \mu_t^h] \\ -(I - \tau_\ell)^{-1} g_t \end{bmatrix} \tag{15.A.27}$$

$$\Phi_i' \alpha_t = (I + \tau_i)^{-1} \Theta_k' \mu_t^k, \tag{15.A.28}$$

where the last equation comes from combining $(15.A.11)$ (among the firm's first-order necessary conditions) with $(15.A.7)$ (among the household's first-order necessary conditions).

Our goal is to manipulate this system into the form (9.25) that is susceptible to the application of the invariant subspace algorithm of chapter 9. Our strategy will be successively to eliminate i_t, c_t, g_t, α_t^0, s_t, and μ_t^s from the system, so that we are left with a difference equation in the "state" (h_{t-1}, k_{t-1}, z_t) and the "co-state" variables μ_t^h, μ_t^k, μ_t^z.

We begin by substituting $(15.A.27)$ and $(15.A.26)$ into $(15.A.28)$ to get

$$\begin{aligned} \Phi_i' \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} & \begin{bmatrix} (I + \tau_c)^{-1} [\Pi' \mu_t^s + \Theta_h' \mu_t^h] \\ -(I - \tau_\ell)^{-1} U_g [\Phi_c \ \Phi_g]^{-1} \{\bar{\Gamma} k_{t-1} + U_f z_t - \Phi_i i_t\} \end{bmatrix} \\ &= (I + \tau_i)^{-1} \Theta_k' \mu_t^k \end{aligned} \tag{15.A.29}$$

where as usual U_g is a selection matrix that picks off the components of the right side of $(15.A.26)$ corresponding to g_t. We adopt the partition

$$\tilde{\Phi} = \begin{bmatrix} \Phi_c' \\ \Phi_g' \end{bmatrix}^{-1} = [\tilde{\Phi}_1 \ \tilde{\Phi}_2] \tag{15.A.30}$$

where $\tilde{\Phi}_1$ is $(n_d \times n_c)$ and $\tilde{\Phi}_2$ is $(n_d \times n_g)$. Then equation $(15.A.29)$ can be written as

$$\begin{aligned} &\Phi_i' \tilde{\Phi}_1 (I + \tau_c)^{-1} [\Pi' \mu_t^s + \Theta_h' \mu_t^h] \\ &\quad - \Phi_i' \tilde{\Phi}_2 (I - \tau_\ell)^{-1} U_g [\Phi_c \ \Phi_g]^{-1} \ \{\bar{\Gamma} k_{t-1} + U_f z_t\} \\ &\quad - (I + \tau_i)^{-1} \Theta_k' \mu_t^k = -\Phi_i' \tilde{\Phi}_2 (I - \tau_\ell)^{-1} U_g \, [\Phi_c \ \Phi_g]^{-1} \, \Phi_i i_t \end{aligned}$$

Solving for i_t gives

$$i_t = L_1 \mu_t^s + L_2 \mu_t^h + L_3 \mu_t^k + L_4 k_{t-1} + L_5 z_t \tag{15.A.31}$$

where

$$\begin{aligned} L_1 &= G_1^{-1} \Phi_i' \tilde{\Phi}_1 (I + \tau_c)^{-1} \Pi' \\ L_2 &= G_1^{-1} \Phi_i' \tilde{\Phi}_1 (I + \tau_c)^{-1} \, \Theta_h' \\ L_3 &= -G_1^{-1} (I + \tau_i)^{-1} \Theta_k' \\ L_4 &= -G_1^{-1} \Phi_i' \tilde{\Phi}_2 (I - \tau_\ell)^{-1} U_g \ [\Phi_c \ \Phi_g]^{-1} \bar{\Gamma} \\ L_5 &= -G_1^{-1} \Phi_i' \tilde{\Phi}_2 (I - \tau_\ell)^{-1} U_g \ [\Phi_c \ \Phi_g]^{-1} \, U_f \\ G_1 &= -\Phi_i' \tilde{\Phi}_2 (I - \tau_\ell)^{-1} U_g \ [\Phi_c \ \Phi_g]^{-1} \, \Phi_i \end{aligned} \tag{15.A.32}$$

Substituting $(15.A.31)$ into $(15.A.26)$ and rearranging gives

$$c_t = L_6 k_{t-1} + L_7 z_t + L_8 \mu_t^s + L_9 \mu_t^h + L_{10} \mu_t^k \tag{15.A.33}$$

$$g_t = L_{11} k_{t-1} + L_{12} z_t + L_{13} \mu_t^s + L_{14} \mu_t^h + L_{15} \mu_t^k \tag{15.A.34}$$

where

$$\begin{aligned}
L_6 &= U_c[\Phi_c\ \Phi_g]^{-1}(\bar{\Gamma} - \Phi_i L_4)\\
L_7 &= U_c[\Phi_c\ \Phi_g]^{-1}(U_f - \Phi_i L_5)\\
L_8 &= -U_c[\Phi_c\ \Phi_g]^{-1}\Phi_i L_1\\
L_9 &= -U_c[\Phi_c\ \Phi_g]^{-1}\Phi_i L_2\\
L_{10} &= -U_c[\Phi_c\ \Phi_g]^{-1}\Phi_i L_3\\
L_{11} &= U_g[\Phi_c\ \Phi_g]^{-1}(\bar{\Gamma} - \Phi_i L_4)\\
L_{12} &= U_g[\Phi_c\ \Phi_g]^{-1}(U_f - \Phi_i L_5)\\
L_{13} &= -U_g[\Phi_c\ \Phi_g]^{-1}\Phi_i L_1\\
L_{14} &= -U_g[\Phi_c\ \Phi_g]^{-1}\Phi_i L_2\\
L_{15} &= -U_g[\Phi_c\ \Phi_g]^{-1}\Phi_i L_3
\end{aligned} \tag{15.A.35}$$

Substituting (15.A.34) into (15.A.27) and using our partition $[\tilde{\Phi}_1\ \tilde{\Phi}_2] = \begin{bmatrix}\Phi_c\\ \Phi_g\end{bmatrix}^{-1}$ gives

$$\alpha_t^0 = L_{16}\mu_t^s + L_{17}\mu_t^h + L_{18}\mu_t^k + L_{19}k_{t-1} + L_{20}z_t \tag{15.A.36}$$

where

$$\begin{aligned}
L_{16} &= \tilde{\Phi}_1(I+\tau_c)^{-1}\Pi' + \tilde{\Phi}_2(I-\tau_\ell)^{-1}U_g[\Phi_c\ \Phi_g]^{-1}\Phi_i L_1\\
L_{17} &= \tilde{\Phi}_1(I+\tau_c)^{-1}\Theta_h' + \tilde{\Phi}_2(I-\tau_\ell)^{-1}U_g[\Phi_c\ \Phi_g]^{-1}\Phi_i L_2\\
L_{18} &= \tilde{\Phi}_2(I-\tau_\ell)^{-1}U_g[\Phi_c\ \Phi_g]^{-1}\Phi_i L_3\\
L_{19} &= -\tilde{\Phi}_2(I-\tau_\ell)^{-1}U_g[\Phi_c\ \Phi_g]^{-1}\{\bar{\Gamma} - \Phi_i L_4\}\\
L_{20} &= -\tilde{\Phi}_2(I-\tau_\ell)^{-1}U_g[\Phi_c\ \Phi_g]^{-1}(U_f - \Phi_i L_5)
\end{aligned} \tag{15.A.37}$$

Substituting (15.A.33) for c_t into (15.A.20) and (15.A.21) gives

$$\mu_t^s = L_{21}h_{t-1} + L_{22}k_{t-1} + L_{23}z_t + L_{24}\mu_t^h + L_{25}\mu_t^k \tag{15.A.38}$$

where

$$\begin{aligned}
L_{21} &= -\mathscr{A}^{-1}\Lambda\\
L_{22} &= -\mathscr{A}^{-1}\bar{\Pi}L_6\\
L_{23} &= \mathscr{A}^{-1}(U_b - \bar{\Pi}L_7)\\
L_{24} &= -\mathscr{A}^{-1}\bar{\Pi}L_9\\
L_{25} &= -\mathscr{A}^{-1}\bar{\Pi}L_{10}\\
\mathscr{A} &= (I + \bar{\Pi}L_8)
\end{aligned} \tag{15.A.39}$$

Because $s_t = U_b z_t - \mu_t^s$, $(15.A.38)$ implies

$$s_t = L_{26} h_{t-1} + L_{27} k_{t-1} + L_{28} z_t + L_{29} \mu_t^h + L_{30} \mu_t^k \tag{15.A.40}$$

where

$$\begin{aligned} L_{26} &= -L_{21} \\ L_{27} &= -L_{22} \\ L_{28} &= U_b - L_{23} \\ L_{29} &= -L_{24} \\ L_{30} &= -L_{25} \end{aligned} \tag{15.A.41}$$

We can use $(15.A.38)$ to eliminate μ_t^s from the right sides of $(15.A.31)$, $(15.A.33)$, $(15.A.34)$, $(15.A.36)$ to obtain

$$i_t = N_1 h_{t-1} + N_2 k_{t-1} + N_3 z_t + N_4 \mu_t^h + N_5 \mu_t^k \tag{15.A.42}$$

$$c_t = N_6 h_{t-1} + N_7 k_{t-1} + N_8 z_t + N_9 \mu_t^h + N_{10} \mu_t^k \tag{15.A.43}$$

$$\begin{aligned} g_t = {} & N_{11} h_{t-1} + N_{12} k_{t-1} + N_{13} z_t \\ & + N_{14} \mu_t^h + N_{15} \mu_t^k \end{aligned} \tag{15.A.44}$$

$$\begin{aligned} \alpha_t = {} & N_{16} h_{t-1} + N_{17} k_{t-1} + N_{18} z_t \\ & + N_{19} \mu_t^h + N_{20} \mu_t^k \end{aligned} \tag{15.A.45}$$

where

$$\begin{cases} N_1 = L_1 L_{21} \\ N_2 = L_1 L_{22} + L_4 \\ N_3 = L_1 L_{23} + L_5 \\ N_4 = L_1 L_{24} + L_2 \\ N_5 = L_1 L_{25} + L_3 \end{cases} \tag{15.A.46}$$

$$\begin{cases} N_6 = L_8 L_{21} \\ N_7 = L_8 L_{22} + L_6 \\ N_8 = L_8 L_{23} + L_7 \\ N_9 = L_8 L_{24} + L_9 \\ N_{10} = L_8 L_{25} + L_{10} \end{cases} \tag{15.A.47}$$

$$\begin{cases} N_{11} = L_{13}L_{21} \\ N_{12} = L_{13}L_{22} + L_{11} \\ N_{13} = L_{13}L_{23} + L_{12} \\ N_{14} = L_{13}L_{24} + L_{14} \\ N_{15} = L_{13}L_{25} + L_{15} \end{cases} \tag{15.A.48}$$

$$\begin{cases} N_{16} = L_{16}L_{21} \\ N_{17} = L_{16}L_{22} + L_{19} \\ N_{18} = L_{16}L_{23} + L_{20} \\ N_{19} = L_{16}L_{24} + L_{17} \\ N_{20} = L_{16}L_{25} + L_{18} \end{cases} \tag{15.A.49}$$

We now substitute $(15.A.42)$, $(15.A.43)$, $(15.A.44)$, $(15.A.45)$ into $(15.A.22)$, $(15.A.23)$, $(15.A.24)$, $(15.A.25)$ to obtain the following system of difference equations:

$$\begin{aligned} \mu_t^h = {} & \beta\Lambda_h' L_{21} h_t + \beta\Lambda_h' L_{22} k_t + \beta\Lambda_h' L_{23} z_{t+1} \\ & + (\beta\Lambda_h' L_{24} + \beta\Delta_h')\mu_{t+1}^h + \beta\Lambda_h' L_{25}\mu_{t+1}^k \end{aligned} \tag{15.A.50}$$

$$\begin{aligned} \mu_t^k = {} & [\beta\Delta_k' + \beta(I-\tau_k)\Gamma_k' N_{20}]\mu_{t+1}^k \\ & + \beta(I-\tau_k)\Gamma_k' N_{16} h_t + \beta(I-\tau_k)\Gamma_k' N_{17} k_t \\ & + \beta(I-\tau_k)\Gamma_k' N_{18} z_{t+1} + \beta(I-\tau_k)\Gamma_k' N_{19}\mu_{t+1}^h \end{aligned} \tag{15.A.51}$$

$$\begin{aligned} \mu_t^z = {} & \beta A_{22}'\mu_{t+1}^z + \beta[U_b' L_{26} + U_d' N_{16}]h_t \\ & + \beta[U_b' L_{27} + U_d' N_{17}]k_t \\ & + \beta[U_b' L_{28} + U_d' N_{18} - U_b' U_b]\, z_{t+1} \\ & + \beta[U_b' L_{29} + U_d' N_{19}]\mu_{t+1}^h \\ & + \beta[U_b' L_{30} + U_d' N_{20}]\mu_{t+1}^k \end{aligned} \tag{15.A.52}$$

$$\begin{aligned} h_t = {} & (\bar\Delta_h + \bar\Theta_h N_6)h_{t-1} + \bar\Theta_h N_7 k_{t-1} \\ & + \bar\Theta_h N_8 z_t + \bar\Theta_h N_9 \mu_t^h + \bar\Theta_h N_{10}\mu_t^k \end{aligned} \tag{15.A.53}$$

$$\begin{aligned} k_t = {} & (\bar\Delta_k + \Theta_k N_2)k_{t-1} + \Theta_k N_1 h_{t-1} \\ & + \Theta_k N_3 z_t + \Theta_k N_4 \mu_t^h + \Theta_k N_5 \mu_t^k \end{aligned} \tag{15.A.54}$$

$$z_{t+1} = A_{22} z_t \tag{15.A.55}$$

We can arrange these equations in the form of (9.25) as follows:

$$m_2 \begin{bmatrix} x_t \\ \mu_t \end{bmatrix} = m_1 \begin{bmatrix} x_{t+1} \\ \mu_{t+1} \end{bmatrix},$$

where

$$m_2 = \begin{bmatrix} 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ (\bar{\Delta}_h + \bar{\Theta}_h N_6) & \bar{\Theta}_h N_7 & \bar{\Theta}_h N_8 & \bar{\Theta}_h N_9 & \bar{\Theta}_h N_{10} & 0 \\ \Theta_k N_1 & \bar{\Delta}_k + \Theta_k N_2 & \Theta_k N_3 & \Theta_k N_4 & \Theta_k N_5 & 0 \\ 0 & 0 & A_{22} & 0 & 0 & 0 \end{bmatrix}$$

and

$$m_1 = \begin{bmatrix} \beta \Lambda_h' L_{21} & \beta \Lambda_h' L_{22} & \beta \Lambda_h' L_{23} \\ \beta (I - \tau_k) \Gamma_k' N_{16} & \beta (I - \tau_k) \Gamma_k' N_{17} & \beta (I - \tau_k) \Gamma_k' N_{18} \\ \beta (U_b' L_{26} + U_d' N_{16}) & \beta (U_b' L_{27} + U_d' N_{17}) & \beta (U_b' L_{28} + U_d' N_{18} - U_b' U_b) \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\begin{matrix} \beta \Lambda_h' L_{24} + \beta \Delta_h' & \beta \Lambda_h' L_{25} & 0 \\ \beta (I - \tau_k) \Gamma_k' N_{19} & \beta \Delta_k' + \beta (I - \tau_k) \Gamma_k' N_{20} & 0 \\ \beta (U_b' L_{29} + U_d' N_{19}) & \beta (U_b' L_{30} + U_d' N_{20}) & \beta A_{22}' \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \Bigg]$$

where we set $x_t' = [h_{t-1}', \, k_{t-1}', \, z_t']$ and $\mu_t' = [\mu_t^{h\prime} \; \mu_t^{k\prime} \; \mu_t^{z\prime}]$. Then the equilibrium law of motion for $\{x_t\}$ is given by

$$x_{t+1} = A^o \, x_t \tag{15.A.56}$$

where

$$A^o = \mathscr{W}_{11} \Delta_1 (\mathscr{V}_{11} - \mathscr{V}_{12} \mathscr{V}_{22}^{-1} \mathscr{V}_{21}).$$

The shadow prices are determined via (12.5b), namely,

$$\mu_t = M x_t \tag{15.A.57}$$

where $M = \mathscr{W}_{21}\mathscr{W}_{11}^{-1}$. Using (15.$A$.42), (15.$A$.43), (15.$A$.44), and (15.$A$.45) with (15.$A$.56) we can write

$$i_t = \Big\{[N_1 \ N_2 \ N_3] + [N_4 \ N_5 \ 0]\, M\Big\}\, x_t \tag{15.A.58}$$

$$c_t = \Big\{[N_6 \ N_7 \ N_8] + [N_9 \ N_{10} \ 0]\, M\Big\}\, x_t \tag{15.A.59}$$

$$g_t = \Big\{[N_{11} \ N_{12} \ N_{13}] + [N_{14} \ N_{15} \ 0]\, M\Big\}\, x_t \tag{15.A.60}$$

$$\alpha_t^0 = \Big\{[N_{16} \ N_{17} \ N_{18}] + [N_{19} \ N_{20} \ 0]\, M\Big\}\, x_t \tag{15.A.61}$$

or

$$\begin{aligned} i_t &= S_i x_t \\ c_t &= S_c x_t \\ g_t &= S_g x_t \\ \alpha_t^0 &= S_\alpha x_t . \end{aligned} \tag{15.A.62}$$

Substituting (15.A.62) for α_t^0 into (15.A.10), (15.A.11), (15.A.12), (15.A.13) gives

$$\begin{aligned} p_t^0 &= S_p x_t \\ q_t^0 &= S_q x_t \\ w_t^0 &= S_w x_t \\ r_t^0 &= S_r x_t \end{aligned} \tag{15.A.63}$$

where

$$\begin{aligned} S_p &= \Phi_c' S_\alpha, \ S_q = \Phi_i' S_\alpha \\ S_w &= -\Phi_g' S_\alpha, \ S_r = \Gamma_k' S_\alpha \end{aligned} \tag{15.A.64}$$

We also have

$$h_t = S_h x_t \ , \ k_t = S_k x_t \tag{15.A.65}$$

where

$$S_h = [I \ 0 \ 0]\, A^o \quad , \quad S_k = [0 \ I \ 0]\, A^o . \tag{15.A.66}$$

These formulas express the law of motion for the state x_t and all quantities and prices in forms identical to those described in chapters 3–5. With these formulas in hand, all subsequent features of our analysis proceed identically as with that for an undistorted economy.

B. Invariant subspace equations for heterogeneous agent model

From the first-order conditions of the two types of households and firms the market clearing conditions we can deduce the following set of equations:

$$s_{1t} = \Lambda_1 h_{1t-1} + \Lambda_{13} h_{2t-1} + \Pi_1 c_{1t} + \Pi_{13} c_{2t} \tag{15.B.1}$$

$$s_{2t} = \Lambda_2 h_{2t-1} + \Lambda_{22} h_{1t-1} + \Pi_2 c_{2t} + \Pi_{22} c_{1t} \tag{15.B.2}$$

$$h_{1t} = \tilde{\Delta}_{h1} h_{1t-1} + \Delta_{H12} h_{2t-1} + \tilde{\Theta}_{h1} c_{1t} + \Theta_{H12} c_{2t} \tag{15.B.3}$$

$$h_{2t} = \tilde{\Delta}_{h2} h_{2t-1} + \Delta_{H21} h_{1t-1} + \tilde{\Theta}_{h2} c_{2t} + \Theta_{H21} c_{1t} \tag{15.B.4}$$

$$z_{t+1} = A_{22} z_t \tag{15.B.5}$$

$$(\Phi_c \Phi_g) \begin{pmatrix} c_t \\ g_t \end{pmatrix} = \Gamma k_{t-1} + d_t - \Phi_i i_t - \Phi_c U_G z_t \tag{15.B.6}$$

$$k_t = \tilde{\Delta}_k k_{t-1} + \Theta_k i_t \tag{15.B.7}$$

$$(I + \tau_c) \mu_{0i} p_t^0 = \Pi_{i1}' (U_{bi} z_t - s_{it}) + \Theta_{hi}' \mathcal{M}_t^{hi} \quad i = 1, 2 \tag{15.B.8}$$

$$\mathcal{M}_t^{hi} = \beta \Delta_{hi}' \mathcal{M}_{t+1}^{hi} + \beta \Lambda_{i1}' (U_{bi} z_{t+1} - s_{it+1}) \quad i = 1, 2 \tag{15.B.9}$$

$$\begin{aligned} \mathcal{M}_t^{zi} = \beta [& A_{22}' \mathcal{M}_{t+1}^{zi} + U_{bi}' (s_{it+1} - U_{bi} z_{t+1}) \\ & + U_{di}' \alpha_{t+1}^0 \mu_{0i}] \; i = 1, 2 \end{aligned} \tag{15.B.10}$$

$$\alpha_t^0 = \begin{pmatrix} \Phi_c' \\ \Phi_g' \end{pmatrix}^{-1} \begin{pmatrix} p_t^0 \\ -\frac{w_t^0}{l_t} g_t \end{pmatrix} \tag{15.B.11}$$

$$l_{it} = (1 - \tau_l) \mu_{0i} w_t^0 \qquad i = 1, 2 \tag{15.B.12}$$

$$\mathcal{M}_t^k = \beta \Delta_k' \mathcal{M}_{t+1}^k + \beta (I - \tau_k) \Gamma_k' \alpha_{t+1}^0 \tag{15.B.13}$$

$$(I + \tau_i) \Phi_i' \alpha_t^0 = \Theta_k' \mathcal{M}_t^k \tag{15.B.14}$$

where

$$
\begin{aligned}
\Lambda_1 &= \Lambda_{11} + \Lambda_{12} \\
\Pi_1 &= \Pi_{11} + \Pi_{12} \\
\Lambda_2 &= \Lambda_{21} + \Lambda_{23} \\
\Pi_2 &= \Pi_{21} + \Pi_{23} \\
\tilde{\Delta}_{h1} &= \Delta_{h1} + \Delta_{H11} \\
\tilde{\Delta}_{h2} &= \Delta_{h2} + \Delta_{H22} \\
\tilde{\Theta}_{h1} &= \Theta_{h1} + \Theta_{H11} \\
\tilde{\Theta}_{h2} &= \Theta_{h2} + \Theta_{H22} \\
\tilde{\Delta}_{k} &= \Delta_{k} + \Delta_{K} \\
\Gamma &= \Gamma_k + \Gamma_K
\end{aligned}
$$

Now, use $(15.B.8)$ twice for $i = 1, 2$ and $(15.B.1)$, $(15.B.2)$, we get

$$
\begin{aligned}
&\mu_{01}^{-1}(\Pi_{11}'(U_{b1}z_t - \Lambda_1 h_{1t-1} - \Lambda_{13}h_{2t-1} - \Pi_1 c_{1t} - \Pi_{13}c_{2t}) \\
&\qquad + \Theta_{h1}'\mathcal{M}_t^{h1}) = \\
&\mu_{02}^{-1}(\Pi_{21}'(U_{b2}z_t - \Lambda_2 h_{2t-1} - \Lambda_{22}h_{1t-1} - \Pi_2 c_{2t} - \Pi_{22}c_{1t}) \\
&\qquad + \Theta_{h2}'\mathcal{M}_t^{h2})
\end{aligned}
\qquad (15.B.15)
$$

From $(15.B.6)$ we get

$$
c_{1t} + c_{2t} = U_c(\Phi_c\ \Phi_g)^{-1}[\Gamma k_{t-1} - \Phi_i i_t + U_f z_t] \qquad (15.B.16)
$$

$$
g_t = U_g(\Phi_c\ \Phi_g)^{-1}[\Gamma k_{t-1} - \Phi_i i_t + U_f z_t] \qquad (15.B.17)
$$

Also, combining $(15.B.14)$, $(15.B.11)$, $(15.B.12)$ and $(15.B.8)$ gives us

$$
\begin{aligned}
&\mu_{01}^{-1}\Phi_i'\tilde{\Phi}_1(I + \tau_c)^{-1}[\Pi_{11}'(U_{b1}z_t - \Lambda_1 h_{1t-1} \\
&\qquad - \Lambda_{13}h_{2t-1} - \Pi_1 c_{1t} - \Pi_{13}c_{2t}) + \Theta_{h1}'\mathcal{M}_t^{h1}] \\
&\quad - \Phi_i'\tilde{\Phi}_2 g_t/(1 - \tau_l)(\mu_{01} + \mu_{02}) = (I + \tau_i)^{-1}\Theta_k'\mathcal{M}_t^k
\end{aligned}
\qquad (15.B.18)
$$

Here $\tilde{\Phi}_1$, $\tilde{\Phi}_2$, U_c and U_g are as defined in Chapter 10, $U_f = U_{d1} + U_{d2} - \Phi_c U_G$. Equations $(15.B.15)$ – $(15.B.18)$ can be used to solve 4 variables c_{1t}, c_{2t}, g_t and i_t in terms of the state variables z_t, h_{it-1}, $\mathcal{M}_t^{hi}$, k_{t-1} and $\mathcal{M}_t^k$, $i = 1, 2$, as:

$$\tilde{L}\begin{pmatrix} c_{1t} \\ c_{2t} \\ g_t \\ i_t \end{pmatrix} = \tilde{N}\begin{pmatrix} z_t \\ z_t \\ h_{1t-1} \\ h_{2t-1} \\ k_{t-1} \\ \mathcal{M}_t^{z1} \\ \mathcal{M}_t^{z2} \\ \mathcal{M}_t^{h1} \\ \mathcal{M}_t^{h2} \\ \mathcal{M}_t^{k} \end{pmatrix} \tag{15.B.19}$$

where

$$\tilde{L} = \begin{pmatrix} \mu_{01}\Pi'_{21}\Pi_{22} - \mu_{02}\Pi'_{11}\Pi_1 & \mu_{01}\Pi'_{21}\Pi_2 - \mu_{02}\Pi'_{11}\Pi_{13} & 0 & 0 \\ I_c & I_c & 0 & U_c(\Phi_c\ \Phi_g)^{-1}\Phi_i \\ 0 & 0 & I_g & U_g(\Phi_c\ \Phi_g)^{-1}\Phi_i \\ \mu_{01}^{-1}\Phi'_i\tilde{\Phi}_1(I+\tau_c)^{-1}\Pi'_{11}\Pi_1 & \mu_{01}^{-1}\Phi'_i\tilde{\Phi}_1(I+\tau_c)^{-1}\Pi'_{11}\Pi_{13} & (1-\tau_l)^{-1}(\mu_{01}+\mu_{02})^{-1}\Phi'_i\tilde{\Phi}_2 & 0 \end{pmatrix}$$

and

$$
\tilde{N} = \begin{pmatrix}
[-\mu_{02}\Pi'_{11}U_{b1}]' & [U_c(\Phi_c\ \Phi_g)^{-1}U_f]' & [U_g(\Phi_c\ \Phi_g)^{-1}U_f]' & [\mu_{01}^{-1}\Phi'_i\tilde{\Phi}_1(I+\tau_c)^{-1}\Pi'_{11}U_{b1}]' \\
[\mu_{01}\Pi'_{21}U_{b2}]' & 0 & 0 & 0 \\
[\mu_{02}\Pi'_{11}\Lambda_1 - \mu_{01}\Pi'_{21}\Lambda_{22}]' & 0 & 0 & [-\mu_{01}^{-1}\Phi'_i\tilde{\Phi}_1(I+\tau_c)^{-1}\Pi'_{11}\Lambda_1]' \\
[\mu_{02}\Pi'_{11}\Lambda_{13} - \mu_{01}\Pi'_{21}\Lambda_2]' & 0 & 0 & [-\mu_{01}^{-1}\Phi'_i\tilde{\Phi}_1(I+\tau_c)^{-1}\Pi'_{11}\Lambda_{13}]' \\
0 & [U_c(\Phi_c\ \Phi_g)^{-1}\Gamma]' & [U_g(\Phi_c\ \Phi_g)^{-1}\Gamma]' & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
[-\mu_{02}\Theta'_{h1}]' & 0 & 0 & [\mu_{01}^{-1}\Phi'_i\tilde{\Phi}_1(I+\tau_c)^{-1}\Theta'_{h1}]' \\
[\mu_{01}\Theta'_{h2}]' & 0 & 0 & 0 \\
0 & 0 & 0 & [-(I+\tau_i)^{-1}\Theta'_k]'
\end{pmatrix}'
$$

From equation $(15.B.19)$ c_{1t}, c_{2t}, g_t, i_t can be expressed as a linear combination of state vector x_t and costate vector $\mathcal{M}_t$, as:

$$
\begin{pmatrix} c_{1t} \\ c_{2t} \\ g_t \\ i_t \end{pmatrix} = N \begin{pmatrix} x_t \\ \mathcal{M}_t \end{pmatrix}
$$

where

$$
\begin{aligned}
N &= \tilde{L}^{-1}\tilde{N} \\
x_t &= \begin{pmatrix} z'_t & z'_t & h'_{1t-1} & h'_{2t-1} & k'_{t-1} \end{pmatrix}'
\end{aligned}
$$

and $\mathcal{M}_t$ is the corresponding multipliers of each component of x_t.

Divide N into blocks according to the dimensions of c_{it}, g_t, i_t, x_t and $\mathcal{M}_t$, such that:

$$\begin{aligned} c_{1t} &= N_{11}x_t + N_{12}\mathcal{M}_t = N_1\ (x_t'\ \mathcal{M}_t')' \\ c_{2t} &= N_{21}x_t + N_{22}\mathcal{M}_t = N_2\ (x_t'\ \mathcal{M}_t')' \\ g_t &= N_{31}x_t + N_{32}\mathcal{M}_t = N_3\ (x_t'\ \mathcal{M}_t')' \\ i_t &= N_{41}x_t + N_{42}\mathcal{M}_t = N_4\ (x_t'\ \mathcal{M}_t')' \end{aligned} \tag{15.B.20}$$

As an intermediate step, we express s_{1t}, s_{2t} and α_t^0 in terms of x_t and $\mathcal{M}_t$. This is done by using $(15.B.1)$, $(15.B.2)$, $(15.B.8)$, $(15.B.11)$ and $(15.B.12)$:

$$\begin{aligned} s_{1t} &= N_{s1}\ (x_t'\ \mathcal{M}_t')' \\ s_{2t} &= N_{s2}\ (x_t'\ \mathcal{M}_t')' \\ \alpha_t^0 &= N_\alpha\ (x_t'\ \mathcal{M}_t')' \end{aligned} \tag{15.B.21}$$

such that

$$\begin{aligned} N_{s1} &= \Pi_1 N_1 + \Pi_{13} N_2 + (0\ \ 0\ \ \Lambda_1\ \ \Lambda_{13}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0) \\ N_{s2} &= \Pi_{22} N_1 + \Pi_2 N_2 + (0\ \ 0\ \ \Lambda_{22}\ \ \Lambda_2\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0) \\ N_\alpha &= \mu_{01}^{-1}\tilde{\Phi}_1 (I+\tau_c)^{-1}[-\Pi_{11}' N_{s1} + (\Pi_{11}' U_{b1}\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ \Theta_{h1}'\ \ 0\ \ 0)] \\ &\quad - (1+\tau_l)^{-1}(\mu_{01}+\mu_{02})^{-1}\tilde{\Phi}_2 N_3 \end{aligned}$$

Now, it is possible to combine all equations $(15.B.1)$–$(15.B.14)$ to write down the Vaughan's linear equations:

$$\tilde{M}_1 \begin{pmatrix} x_{t+1} \\ \mathcal{M}_{t+1} \end{pmatrix} = \tilde{M}_2 \begin{pmatrix} x_t \\ \mathcal{M}_t \end{pmatrix} \tag{15.B.22}$$

where

$$\tilde{M}_1 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ -\beta U_{b1}' U_{b1} & 0 & 0 & 0 & 0 & \beta A_{22}' & 0 & 0 & 0 & 0 \\ 0 & -\beta U_{b2}' U_{b2} & 0 & 0 & 0 & 0 & \beta A_{22}' & 0 & 0 & 0 \\ \beta \Lambda_{11}' U_{b1} 0 & 0 & 0 & 0 & 0 & 0 & \beta \Delta_{h1}' & 0 & 0 & \\ 0 & \beta \Lambda_{21}' U_{b2} & 0 & 0 & 0 & 0 & 0 & 0 & \beta \Delta_{h2}' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta \Delta_k' \end{pmatrix} +$$

$$\beta \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ U_{b1}' N_{s1} + \mu_{01} U_{d1}' N_\alpha \\ U_{b2}' N_{s2} + \mu_{02} U_{d2}' N_\alpha \\ -\Lambda_{11}' N_{s1} \\ -\Lambda_{21}' N_{s2} \\ (I - \tau_k) \Gamma_k' N_\alpha \end{pmatrix}$$

and

$$\tilde{M}_2 = \begin{pmatrix} A_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{\Delta}_{h1} & \Delta_{H12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Delta_{H21} & \tilde{\Delta}_{h2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\Delta}_k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ \tilde{\Theta}_{h1} N_1 + \Theta_{H12} N_2 \\ \Theta_{H21} N_1 + \tilde{\Theta}_{h2} N_2 \\ \Theta_k N_4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Chapter 16
Recursive Risk Sensitive Control

16.1. Introduction

This chapter describes a class of preferences with a single additional parameter that permits us to represent altered attitudes toward risk *vis a vis* our earlier quadratic specification. We build on work by Jacobson (1973) and Whittle (1990), who proposed an exponential type of risk adjustment that, in linear–quadratic environments with Gaussian disturbances, preserves most of the computational conveniences of the standard *undiscounted* linear quadratic dynamic programming problem. But Whittle's (1981, 1990) way of introducing discounting into this problem had the unpleasant feature of introducing time dependence into the optimal decision rules, with the time dependence causing the effects of the risk parameter σ to wear off as the planning horizon is extended. By using a recursive specification of utility along the lines of Koopmans, Lucas and Stokey, Epstein and Zin, and Weil, Hansen and Sargent (1992) altered Whittle's specification to model discounting in a way that implies time invariance of optimal decision rules.

In this chapter we describe this preference specification and how it can be implemented with versions of our earlier formulas for computing equilibria and asset prices. The principal features of our specification are that: (a.) the equations for solving the optimal linear regulator problem are modified by replacing the operator associated with the usual matrix Riccati equation with the composition of that operator with another operator that is easily computed and interpreted; (b.) optimal decision rules remain linear in the state; (c.) 'certainty equivalence' no longer holds, i.e., the decision rules, although linear in the state, depend on the conditional covariance matrix of the innovations to the state; (d.) asset pricing formulas have an additional layer of 'risk adjustments,' because the covariance conditional covariance matrix of the innovations to the state now enter the quadratic form in the state in our asset pricing formulas.

16.2. A Control Problem

This section describes a ‘discounted linear quadratic exponential Gaussian control problem that we shall eventually use for the social planning problem of our standard economy. This is an infinite horizon control problem associated with iterations to convergence on the following equation in the value functions $V_j(x)$:

$$V_{j+1}(x) = \max_{u,x'} \{u'Qu + x'Rx + \frac{2\beta}{\sigma} \log E(\exp(\sigma V_j(x')/2)|J)\}, \tag{16.2.1}$$

where the maximization is subject to

$$x' = Ax + Bu + Cw,$$

where w is a Gaussian random vector with $Eww' = I$, and where $\beta \in (0,1)$ is a discount factor. Associated with the solution of this control problem are three operators:

$$\begin{aligned} D(V) &= V + \sigma VC(I - \sigma C'VC)^{-1}C'V \\ T(W) &= R + A'(\beta W - \beta^2 WB(Q + \beta B'WB)^{-1}B'W)A \\ d(k,V) &= \beta k - \left(\frac{\beta}{\sigma}\right) \log \det(I - \sigma C'VC). \end{aligned} \tag{16.2.2}$$

The operator T is the usual one associated with the matrix Riccati difference equation for the discounted optimal linear regulator problem. We shall give an interpretation of the new operator D shortly.

The value function associated with the solution of the infinite horizon control problem is

$$V_\infty(x) = x'U_1x + U_0 \tag{16.2.3}$$

where

$$U_1 = \lim_{j\to\infty} (T \circ D)^j(0)$$

$$U_0 = \lim_{j\to\infty} d^j(0,0)$$

The optimal decision rule is time invariant $u_t = -Fx_t$ where

$$F = \beta[Q + \beta B'D(U_1)B]^{-1}B'D(U_1)A. \tag{16.2.4}$$

These formulas can be derived by solving the maximization problem on the right side of (16.2.1) and using a Lemma stated by Jacobson (1973) on a property of the Gaussian distribution. For details, see Hansen and Sargent (1992).[1]

The optimal decision rule is linear in the state. When $\sigma = 0$, we get the standard linear–quadratic situation, because in that case $D = I$, so that the operator $T \circ D = T$. When $\sigma = 0$, the feedback rule F is independent of the conditional covariance parameters in C, so that 'certainty equivalence' holds when $\sigma = 0$. When $\sigma \neq 0$, the decision rule F depends on the parameters in C.

16.3. Pessimistic Interpretation

For some purposes, it is convenient to use an interpretation of the D operator due to Jacobson (1973) and Whittle (1990).

Evaluate the following 'aggregator function' associated with the functional equation (16.2.1):

$$\mathcal{A}(-Fx, x, y'Vy + k|J) = x'R^*x + \frac{2\beta}{\sigma} \log E\{\exp(\sigma(y'Vy + k)/2)|J\} \tag{16.3.1}$$

where $A^* = A - BF, R^* = F'QF + R$, and

$$y = A^*x + Cw.$$

We obtain

$$\mathcal{A}(-Fx, x, y'Vy + k|J) = x'[R^* + \beta A^{*\prime} D(V) A^*]x + d(k).$$

The piece $x'[R^* + \beta A^{*\prime} D(V) A^*]x$ has an interpretation in terms of 'pessimism,' to be seen as follows.

Consider the following *deterministic* minimum problem

$$\min_{w,y} \{-\frac{\beta}{\sigma} w'w + x'R^*x + \beta y'Vy\}$$

subject to

$$y = A^*x + Cw.$$

[1] This specification of discounting differs from the one used by Whittle (1990) in a way designed to assure time-invariant decision rules.

In this problem, w is treated as a control vector. As long as $(I - \sigma C'VC)$ is positive definite, this problem has a unique solution. The minimized value of the criterion is verified to be $x'[R^* + \beta A^{*\prime} D(V) A^*]x$, which is the quadratic in x part of $\mathcal{A}(-Fx, x, y'Vy + k|J)$. The 'solution' for w is

$$w = \sigma(I - \sigma C'VC)^{-1} C'VA^* x.$$

Thus, the optimum problem gives an interpretation of the operator D and a piece $\sigma(I - \sigma C'VC)^{-1} C'VA^* x$ in terms of a 'pessimistic' (when $\sigma < 0$) or 'optimistic' (when $\sigma > 0$) adjustment to the random term w_t in the law of motion for the state.

16.4. Recursive Preferences

The preceding generalization of linear–quadratic control theory can readily be applied in the context of representative agent versions of the class of economic models that we have been studying. We use the usual household technology

$$\begin{aligned} s_t &= \Lambda h_{t-1} + \Pi c_t \\ h_t &= \Delta_h h_{t-1} + \Theta_h c_t. \end{aligned}$$

We define the information process z_t as usual by

$$z_{t+1} = A_{22} z_t + C_2 z_t,$$

and let the law of motion for the complete state x_t be denoted

$$x_{t+1} = Ax_t + Cw_{t+1},$$

where now w_t is a Gaussian white noise. We define a utility index recursively by

$$U_t = -(s_t - b_t) \cdot (s_t - b_t)/2 + \beta \mathcal{R}_t(U_{t+1}) \tag{16.4.1}$$

where

$$\mathcal{R}_t(U_{t+1}) = \left(\frac{2}{\sigma}\right) \log E[\exp(\sigma U_{t+1}/2)|J_t].$$

Hansen, Sargent, and Tallarini (1993) have shown how to construct the prices induced by this preference specification. We describe these prices first in the context of an endowment economy.[2]

16.4.1. Endowment economy

Consider a pure endowment economy in which $c_t = U_c z_t$ is exogenous. Infinite recursions on (16.4.1) lead to the quadratic form

$$U_t^e = x_t' \Omega x_t + \rho,$$

where Ω, ρ are the limits as $j \to \infty$ of recursions on

$$\begin{aligned} \Omega_{j+1} &= -(S_s - S_b)'(S_s - S_b)/2 + \beta A' D(\Omega_j) A \\ \rho_{j+1} &= d(\rho_j, \Omega_j). \end{aligned}$$

For endowment economies, we use Ω as one of the ingredients in constructing a probability measure appropriate for the inner-product representation for asset pricing.

16.5. Asset Pricing

We want to represent asset prices in the usual way as a conditional mathematical expectation of an infinite discounted sum of an inner product of a 'scaled Arrow-Debreu price' vector and the payout of the asset. Hansen, Sargent, and Tallarini (1993) show that when $\sigma \neq 0$, the required conditional expectations operator for representing asset prices corresponds to a probability distribution that is distorted relative to the 'objective' one. In particular, they show that to attain the inner product representation of asset prices it is appropriate to construct a conditional expectations operator $\mathcal{F}_t$ defined by

$$\mathcal{F}_t U_{t+1} = E(V_{t+1} U_{t+1} | J_t) / E(V_{t+1} | J_t),$$

[2] The Matlab program `solvex.m` computes equilibrium quantities and prices in one of our economies with this preference specification; `solvex.m` calls `doublex.m`, which implements a doubling algorithm to solve the control problem.

where

$$V_{t+1} = \exp(\sigma U^e_{t+1}/2).$$

The operator $\mathcal{F}_t$ behaves like a conditional expectation operator in that it is linear, monotone, and maps bounded random variables that are measurable with respect to J_{t+1} into bounded random variables that are measurable with respect to J_t.[3] We can use $\beta\mathcal{F}_t$ to value a contingent claim to time $t+1$ utility, and $\mathcal{M}^s_t$ for the equilibrium valuation of services.

Recursively define the sequence of expectations operators

$$\mathcal{S}_{t,\tau} = \mathcal{F}_t \mathcal{F}_{t+1} \mathcal{F}_{t+2} \cdots \mathcal{F}_{t+\tau-1},$$

where $\mathcal{F}_{t,0} = I$. Then the valuation of a stream of consumption services $\{s_t\}$ is just

$$\sum_{t=0}^{\infty} \mathcal{S}_{t,\tau} \mathcal{M}^s_{t+\tau} \cdot s_{t+\tau}.$$

To compute equilibrium asset prices, we have to evaluate consumption rates c_t. We compute a multiplier $\mathcal{M}^c_t$ from $\mathcal{M}^s_t$ in the usual way, except that we substitute the conditional expectation operators $\mathcal{S}_{t,\tau}$ for the usual ones in equation (6.*) (refer to asset pricing chapter):

$$\mathcal{M}^c_t = \Pi' \mathcal{M}^s_t + \Theta'_h \sum_{\tau=1}^{\infty} \beta^\tau (\Delta_h)^{\tau\prime} \Lambda' \mathcal{S}_{t,\tau}(\mathcal{M}^s_{t+\tau}).$$

[3] $V_{t+1}/EV_{t+1}|J_t$ is used as a Radon-Nikodym derivative in this construction.

16.6. Characterizing the Pricing Expectations Operator

Hansen, Sargent, and Tallarini show how to evaluate the conditional expectations operators $\mathcal{S}_{t,\tau}$ by constructing a distorted probability measure for the state x_t.[4] In particular, consider the phony law of motion for x_t

$$x_{t+1} = \hat{A}x_t + \hat{C}w_{t+1}$$

where

$$\begin{aligned}\hat{A} &= [I + \sigma C(I - \sigma C'\Omega C)^{-1}C'\Omega]A \\ \hat{C}\hat{C}' &= C(I - \sigma C'\Omega C)^{-1}C'.\end{aligned}$$

Then we can compute $\mathcal{M}_t^c = M_c x_t$ where

$$M_c = [\Pi' + \Theta_h' \sum_{\tau=1}^{\infty} \beta^\tau \Delta_h^{\tau\prime} \Lambda' \hat{A}^\tau](S_b - S_s).$$

We can evaluate a claim on a stream $d_t = S_d x_t$ by

$$\begin{aligned}\sum_{\tau=0}^{\infty} \beta^\tau \mathcal{S}_{t,\tau} \mathcal{M}_{t+\tau}^c \cdot d_{t+\tau} &= \hat{E}_t \sum_{\tau=0}^{\infty} \beta^\tau \mathcal{M}_{t+\tau}^c \cdot d_{t+\tau} \\ &= x_t' \sum_{\tau=0}^{\infty} \beta^\tau (\hat{A}')^\tau S_d' M_c \hat{A}^\tau x_t \\ &\quad + \left(\frac{\beta}{1-\beta}\right) \text{trace}[S_d' M_c \sum_{\tau=0}^{\infty} \beta^\tau \hat{A}^\tau \hat{C}\hat{C}'(A')^\tau].\end{aligned}$$

[4] These calculations are implemented in the program `assetxq.m`.

16.7. Production Economies

The preceding formulas also apply to asset pricing in production economies, with the understanding that the matrix Ω now corresponds to the 'quadratic-form' matrix in the value function for the social planning problem.

16.8. Risk-Sensitive Investment under Uncertainty

For sake of illustration, we now consider a one consumption good, one capital good production economy that is a version of a Brock-Mirman stochastic growth model with adjustment costs. This delivers a linear-quadratic version of Lucas and Prescott's (1971) theory of investment under uncertainty. Consumption and investment satisfy:

$$c_t + i_t = \gamma k_{t-1} + d_t \tag{16.8.1}$$

where the capital stock k_t evolves according to:

$$k_t = \delta_k k_{t-1} + i_t \tag{16.8.2}$$

and $\{d_t\}$ is an exogenous endowment process. Labor input is required to adjust the capital stock, reflected in a quadratic adjustment cost in the preferences of the fictitious social planner. Also, there is an exogenous ideal consumption level process $\{b_{1t}\}$. The time t contribution to preferences is:

$$-(c_t - b_{1t})^2 - \phi^2 {i_t}^2 = -(\gamma k_{t-1} + d_t - i_t - b_{1t})^2 - \phi^2 {i_t}^2$$

which is quadratic in the control i_t, the endogenous state variable k_{t-1} and the exogenous states b_{1t} and d_t.

We begin with a parameter specification that implies no investment in a deterministic steady state, an unrealistic but useful starting point that we adopt to compare two alternative 'explanations' for investment. One is that altering the risk parameter σ induces a precautionary savings motive into the preference ordering. The other is that capital is more productive than in the benchmark economy, increasing the physical return to investment.

Benchmark Economy with No Steady-State Investment

We set $\beta = 1/1.05$, $\gamma = .1, \delta_k = .95, \phi = .5$. With this parameter configuration and constant values of d_t and b_{1t}, the model implies steady state values of zero for capital and investment. This follows from the fact that $\gamma+\delta_k=\beta^{-1}$, which equates the physical rate of return to investment net of adjustment costs to the subjective rate of time discount. Since adjustment costs are also present, reflected by a positive value of ϕ, physical investment in new capital becomes 'unattractive.' As a consequence, asymptotically there are no savings in the deterministic version of this model.

To see this, the Euler equation for capital is given by

$$\begin{aligned} E[(1-L^{-1})(1-\beta^{-1}L)k_t+\phi^2(1-\beta\delta_k L^{-1})(1-\delta_k L)k_t \mid J_t] = \\ E[(1-L)(b_{1t+1}-d_{t+1}) \mid J_t] \end{aligned} \tag{16.8.3}$$

where L is the lag operator. For future reference we have permitted there to be uncertainty in the forcing processes for $\{b_{1t}\}$ and $\{d_t\}$. The solution to this stochastic difference equation has representation:

$$k_t=\lambda k_{t-1}+\psi[(d_t-b_{1t})-(1-\beta\lambda)\sum_{j=0}^{\infty}(\beta\lambda)^j E(d_{t+j}-b_{1t+j} \mid J_t)]. \tag{16.8.4}$$

where $0<\lambda<1$, and $0<\psi$. In contrast to the familiar permanent income model in which λ and ψ are one, the presence of adjustment costs lowers λ. Notice that the term multiplying ψ is the difference between d_t-b_{1t} and a geometric average of current and future values of d_t-b_{1t}. This simple link to the forcing processes is a result of the $\gamma+\delta_k=\beta^{-1}$ restriction. For a model with b_{1t} and d_t constant, the term multiplying ψ will be zero and the capital stock sequence will converge to zero.

An Economy with Risk Sensitivity

As a precursor to illustrating investment induced by risk sensitivity, we now introduce a specific model of uncertainty in the endowment and preference shock processes. Let an exogenous state vector process $\{z_t\}$ follow a first-order vector autoregression:

$$z_{t+1}=A_{22}z_t+C_2w_{t+1},$$

and suppose that $b_{1t}=S_b z_t$ and $d_t=S_d z_t$, so that both the endowment and the preference shock are linear functions of exogenous state vector. Construct

$$A_{22}=\begin{bmatrix}1 & 0 & 0\\ 0 & .8 & 0\\ 0 & 0 & .5\end{bmatrix} \quad C_2=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & .1\end{bmatrix}, \tag{16.8.5}$$

and set $S_b = [\,30 \quad 0 \quad 1\,]\,, S_d = [5\ 1\ 0]$. We initialize the first component of z_t to be one, which replicates itself over time. Consequently, the preference shock process has mean 30 and the endowment shock process has mean 5.

For the $\sigma = 0$ version of this economy, ordinary certainty equivalence applies. Since $\{d_t\}$ and $\{b_{1t}\}$ are asymptotically stationary, so are the endogenous processes for consumption, capital, and investment. It again follows from (16.8.4) that investment and capital both have mean zero in the stochastic steady state, and consumption has a mean equal to the mean of the endowment shock d_t. Because of certainty equivalence, no precautionary savings occur in this model.

When σ is less than zero, ordinary certainty equivalence no longer holds, and mean investment is positive in a stochastic steady state. Table 2.1 reports the means and standard deviations computed with respect to both the objective probability distribution, and the appropriate distorted distribution for $\sigma = -.005$.

TABLE 2.1: Means and Standard Deviations for the Adjustment Cost Economy with $\sigma = -.005$

	True Process		*Pessimistic Process*	
	mean	standard deviation	mean	standard deviation
c_{1t}	9.83	1.37	2.07	1.57
i_t	4.83	1.22	0.00	1.32
b_{1t}	30.00	.12	30.01	.12
d_t	5.00	1.67	2.07	1.71

The mean of consumption is now notably higher than the mean endowment and the mean of investment is positive. In effect, there is a precautionary savings motive at work here.

A version of the pessimistic certainty equivalence principle derived by Jacobson (1973) and Whittle (1981) applies here. Thus, an alternative way to obtain the same solution is to endow the fictitious social planner with a pessimistic view of the world. The stochastic difference equation (16.8.3) and its solution (16.8.4) still apply except that the distorted expectation operator, call it $\hat{E}$, is used in place of E. As originally suggested by Jacobson (1973), we can imagine there being a second agent, say distinct from the social planner, that picks future values of the shocks in a pessimistic fashion. The degree of pessimism is governed by the value of σ, and the difference equations associated with these pessimistic forecasts for b_{1t+j} and d_{t+j} used for the time t decisions are given by:

$$(1-\beta^{-1}L)k_{t+j} - d_{t+j} + b_{1t+j} + (100/\sigma)(1-.5\beta L^{-1})(1-.5L)(b_{1t+j}-30)] = 0 \tag{16.8.6}$$

$$-(1-\beta^{-1}L)k_{t+j} + d_{t+j} - b_{1t+j} + (1/\sigma)(1-.8\beta L^{-1})(1-.8L)(d_{t+j}-5)] = 0.$$

for $j = 1, 2, \ldots$, where k_{t-1}, b_{1t} and d_t are given initial conditions. In effect, we can think of these as the Euler equations for the second agent. Notice that the capital stock enters these difference equations, and hence the pessimistic forecasts of future values of d_t and b_{1t} depend on k_{t-1}.

Certainty equivalence works here because the optimal choice for k_t can be obtained by (i) shifting (16.8.3) forward $j-1$ periods for $j = 1, 2, \ldots$ and eliminating the conditional expectation operator; (ii) combining the resulting difference equation with (16.8.6); and (iii) solving the composite system of difference equations for k_t, b_{1t+1}, d_{t+1} as a function of the state vector k_{t-1}, b_{1t} and d_t, imposing the appropriate terminal conditions. The composite solution gives the evolution equation for the pessimistic forecasts of b_{1t} and d_t, and the solution for k_t gives the optimal decision rule for k_t. This latter equation, when combined with the actual law of motion for b_{1t} and d_t, governs the evolution of the optimal capital stock process.

While λ is .9852 for this economy, once this forecasting dependence is incorporated, the feedback of k_t onto k_{t-1} drops slightly to .9817. A more dramatic implication is that there is now a positive constant term (1.766) in the decision rule for capital, whereas in the $\sigma = 0$ economy the constant term is zero. Table 2.1 also reports the pessimistic means and standard deviations for forcing processes and for consumption and investment. Notice that the perceived means for d_t and c_{1t} are considerably less than their true means. The mean of investment for the perceived process is again zero because of the applicability of (16.8.4) with distorted expectations. Hence the perceived long run average of the capital stock is not altered by changing expectations operators. Not surprisingly, the perceived standard deviations are larger than the true ones.

An Approximating Economy with $\sigma = 0$

Next we show that we can mimic the quantity implications of the preceding model by setting $\sigma = 0$ and modifying some of its parameters. The parameters that we alter are the mean of b_{1t}, the mean of d_t and the productivity parameter for capital γ. In light of our previous discussion, to make average investment positive it is necessary to relax the restriction that $\delta_k + \gamma = \beta^{-1}$ by making capital more productive.

We used the Kullback-Leibler (1951) information criterion as a device for picking the three parameters because of its well known link to maximum likelihood estimation (*e.g.*, see Akaike (1973), Ljung (1978), and White (1982)).[5] The information criterion was constructed in the same manner as in Hansen and Sargent (1993) using the consumption and investment processes implied by the original model and the approximating $\sigma = 0$ economy. The parameters that make the approximate ($\sigma = 0$) model as close as possible to the original model are .1032 for the productivity parameter γ and 47.0522 and 4.6812 for the means of b_{1t} and d_t. In addition to increasing γ to compensate for setting $\sigma = 0$, we were led to increase substantially the mean of b_{1t} and decrease slightly the mean of d_t.

It turns out that these three parameter adjustments are sufficient to make the quantity implications for the two models to be extremely close. This is shown in Figure 16.8.1 which displays the ratios of the spectral densities of consumption and investment, respectively, for the approximating economy to the corresponding spectral densities for the original economy. Departures from

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[5] Let each model be a member of our class of models, with parameters of the first model being denoted by a vector δ and those of the second model being denoted by α. For the first model, let the mean vector for the observables be $\nu(\delta)$, and the spectral density matrix be $\bar{S}_y(\omega, \delta)$. For the second model let the spectral density matrix be $S_y(\omega, \alpha)$, and the mean vector be $\mu(\alpha)$. The parameters α that make the second model as close as possible to the first are those that minimize the criterion

$$\begin{aligned} \Big\{ &- \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \det S_y(\omega, \alpha)\, d\omega \\ &- \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{trace}\Big[S_y(\omega, \alpha)^{-1}\, \bar{S}_y(\omega, \delta) \Big]\, d\omega \\ &- [\nu - \mu(\alpha)]\, S_y(0, \alpha)^{-1} [\nu - \mu(\alpha)]' \Big\} \,. \end{aligned}$$

unity are small, and confined to very low frequencies. Departures of this small magnitude and frequency-location would be virtually impossible to detect, via say a likelihood ratio test, without an extremely large time series sample. The adjustments in the means of b_{1t} and d_t are sufficient to make the means for consumption and investment the same for both economies. Hence, from the standpoint of data on consumption and investment, the precautionary savings version of the model is (almost) observationally equivalent to a specification in which savings are induced by making capital more productive.

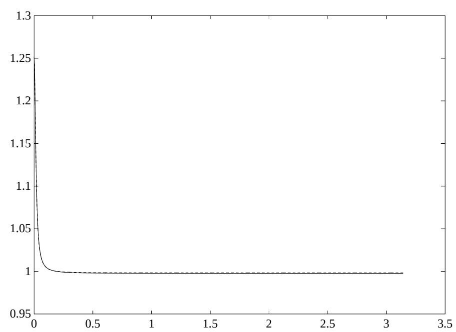

Figure 16.8.1: Ratio of spectral density of consumption and investment (dotted line) in the approximating economy to the spectral density of consumption and investment in the true economy.

16.9. Equilibrium Prices in the Adjustment Cost Economies

We now return to the two adjustment cost economies with their competing explanations for investment. While the quantities for the $\sigma = 0$ 'approximating' economy are, by design, close to those for the 'true' risk-sensitive economy, behaviors of asset prices and rates of return differ markedly. To illustrate this phenomenon, we first consider the implications for the two components of equilibrium wealth: the value of the endowment process from the current period forward (a Lucas tree), and the value of the existing capital stock. Figures 16.9.1.a and 16.9.1.b depict realizations of these two components to wealth for a common realization of the vector of Gaussian noises driving the composite preference and endowment shock process.

The value of the endowment process is always lower and the value of the capital stock higher for the original economy than for the approximating $\sigma = 0$ economy. Hence a bigger fraction of wealth is held in the form of capital in the original (risk-adjusted) economy. This is true even though the mean of the endowment process is higher for the original economy. The mean gross returns are (1.0444, 1.0786) for capital and the endowment process, respectively, in the *true* economy; and (1.0500, 1.0504) in the *approximating* economy.[6] These phenomena trace to the fact that the capital stock is a less risky investment than the endowment and that less risky investments are more highly valued in the original (risk-adjusted) economy than in the approximating economy.

Figure 16.9.2 shows probability densities for risk-free gross returns for both economies. Notice that the density for the approximating economy is centered around β^{-1}, and the modal value is higher than for the original economy with a risk adjustment. The density for the original (risk-adjusted) economy displays more dispersion than its counterpart for the approximating economy, reflecting in part the smaller mean of b_{1t}.

Consider next the market prices of risk for the two economies. The probability densities for these prices are reported in Figures 16.9.3.a and 16.9.3.b. The market price of risk is considerably higher in the risk-sensitive ($\sigma = -.005$) economy.

Finally, Figures 16.9.4.a and 16.9.4.b report spectral densities for the logarithmic one-period returns to holding a claim to the endowment process and

[6] These means were computed from simulations of length 100,000.

to holding capital, respectively. Both returns are more variable in the original (risk-adjusted) economy than in the approximating one. Moreover, the low frequency dip in the spectral density for the return to holding the endowment is substantially more pronounced.

Returning to the decomposition of equilibrium wealth described earlier, it is clear from Figures 16.9.4.a and 16.9.4.b that holding capital is much less risky than holding the endowment for both economies. It is also evident from Figure 16.9.2 that a riskless security is more valued (commands a lower equilibrium rate of return) in the original economy. This apparently underlies the fact that the capital stock is more highly valued in the original ($\sigma = -.005$) economy as depicted in Figure 16.9.1a. Moreover, as illustrated in Figures 16.9.3.a and 16.9.3.b, the market prices of risk tend to be higher in the original economy, so that the equilibrium risk adjustments are more pronounced. This helps to explain why equilibrium values for claims to the endowment process are lower for the original economy as depicted in Figure 16.9.1.a even though the mean of the endowment process is higher.

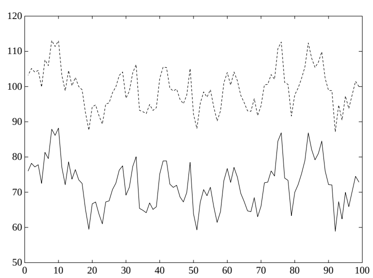

Fig. 16.9.1.a. Realizations of price of Lucas tree in true economy ($\sigma = -.005$) and approximating ($\sigma = 0$) economy (dashed line).

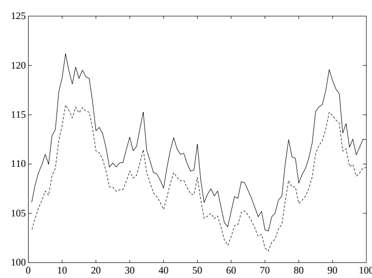

Fig. 16.9.1.b. Realizations of value of capital stock in true economy($\sigma = -.005$) and approximating ($\sigma = 0$) economy (dashed line).

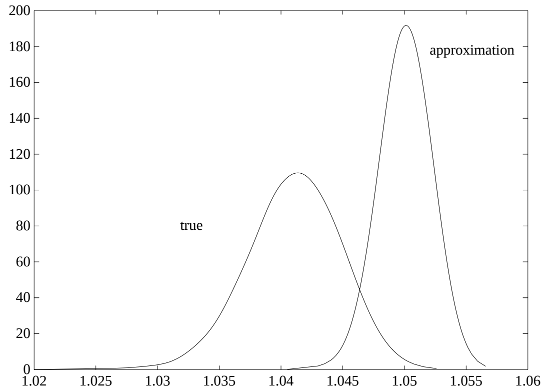

Figure 16.9.2: Figure 5.3 Densities of risk-free interest rate for true and approximating economies.

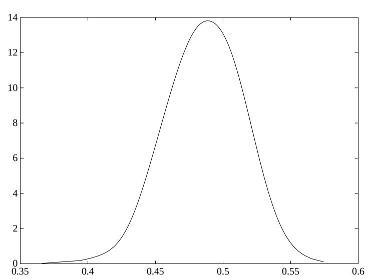

Fig. 16.9.3.a. Density for market price of risk for true adjustment cost true economy.

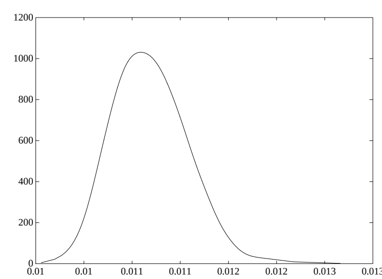

Fig. 16.9.3.b. Density for market price of risk for approximating adjustment cost economy.

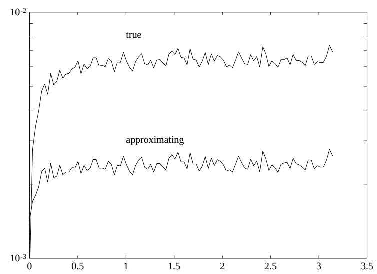

Fig. 16.9.4.a. Spectral densities of windsorized logarithmic returns to endowment streams for true and approximating adjustment costs economies.

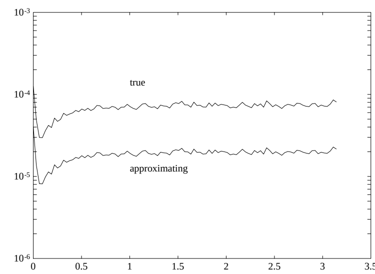

Fig. 16.9.4.b. Spectral densities of logarithmic returns on capital in true and approximating adjustment costs economies.

Chapter 17
Periodic Models of Seasonality

17.1. Introduction

Until now, each of the matrices defining the preferences, technology and information flows has been specified to be constant over time. In this chapter, we relax this assumption, and let the matrices be strictly periodic functions of time. Our interest is to apply and extend an idea of Denise Osborn (1988) and Richard Todd (1983, 1990) to arrive at a particular model of "seasonality".

Seasonality can be characterized in terms of the spectral density of a variable. A variable is said to "have a seasonal" if its spectral density displays peaks at or in the vicinity of the frequencies commonly associated with the seasons of the year, e.g., every twelve months for monthly data, every four quarters for quarterly data. Within one of our equilibria, it is possible to think of three ways of modelling seasonality. The first two ways can be represented within the time-invariant setup of our previous chapters, while the third way requires following Todd and departing from the assumption that the matrices that define our economies are time invariant.

The first model of seasonality is created by specifying the matrices $[A_{22}, C_{22}, U_b, U_d]$ that determine the information structure in the economy. We can exogenously inject a seasonal preference shock into the model by specifying $[A_{22}, U_b]$ in such a way that components of the shock process b_t have seasonals. Similarly, we can specify $[A_{22}, U_d]$ so that components of the endowment shock process d_t have seasonals. The seasonality of these exogenous processes will be transmitted to the prices and quantities determined in equilibrium. The way in which this seasonality is transmitted can be subtle, determined as it is by the restrictions across the parameters of the $\{b_t, d_t\}$ processes and the price and quantity processes that are determined by the equilibrium.[1]

[1] Sargent [1976, 1987,chap XI] described some of the ways in which the cross equation restrictions of linear rational expectations models determine the kind of seasonality in endogenous variables that is induced by imposing seasonality in the variables that agents within a model are implicitly forecasting.

The second model of seasonality is created by specifying the matrices $[\Delta_h,\, \Theta_h,\, \Lambda,\, \Pi]$ that determine preferences and the matrices $[\Phi_c,\, \Phi_i,\, \Phi_c,\, \Gamma,\, \Delta_k,\, \Theta_k]$ that determine the technology so that they make prices and quantities display seasonality even when the preference shocks b_t and the endowment shocks d_t do not display any seasonality. Seasonality can come either from the technology side or from the preference side. Notice that in the first kind of model the source of seasonality is imposed exogenously, while in this second kind of model the idea is that preferences and technology are such that the equilibrium of the economy creates a "propagation mechanism" that converts nonseasonal impulses into seasonal responses in prices and quantities.

This chapter is devoted to studying a third model of seasonality, by following Todd. We now specify an economy in terms of matrices that are periodic functions of time. This specification captures the idea, for example, that the technology is different in Winter than it is in Spring. You will get less corn if you plant in Minnesota in January than if you plant in May. As we shall see, this model of seasonality has properties that contrast in interesting ways to the other two models of seasonality.

17.2. A Periodic Economy

The social planner now faces the problem of maximizing

$$-.5 \sum_{t=0}^{\infty} \beta^t \left[(s_t - b_t) \cdot (s_t - b_t) + l_t^2 \right] \tag{17.2.1}$$

subject to

$$\begin{aligned}
\Phi_{c,s(t)}\; c_t + \Phi_{i,s(t)}\; i_t + \Phi_{g,s(t)}\; g_t &= \Gamma_{s(t)}\; k_{t-1} + d_t \\
k_t &= \Delta_{k,s(t)}\; k_{t-1} + \Theta_{k,s(t)}\; i_t \\
h_t &= \Delta_{h,s(t)}\; h_{t-1} + \Theta_{h,s(t)}\; c_t \\
s_t &= \Lambda_{s(t)}\; h_{t-1} + \Pi_{s(t)}\; c_t \\
z_{t+1} &= A_{22,s(t)}\; z_t + C_{22,s(t)}\; w_{t+1} \\
b_t &= U_b\, z_t \\
d_t &= U_d\, z_t
\end{aligned} \tag{17.2.2}$$

In (17.2.2), $s(t)$ is a periodic function that assigns integers to integers. In particular,

$$\begin{aligned} s &: (\ldots,\ -1,\ 0,\ 1,\ \ldots) \rightarrow [1,\ 2,\ \ldots,\ p] \\ s(t+p) &= s(t) \quad \forall\ t \\ s(t) &= t \ \text{ for } \ t = 1,\ 2,\ \ldots,\ p. \end{aligned} \tag{17.2.3}$$

A consequence of (17.2.3) is that the constraints in (17.2.2) can be represented in the form

$$\begin{aligned} \Phi_{c,j}\ c_{p\cdot t+j} + \Phi_{i,j}\ i_{p\cdot t+j} + \Phi_{g,j}\ g_{p\cdot t+j} &= \Gamma_j\ k_{p\cdot t+j+1} + d_{p\cdot t+j} \\ k_{p\cdot t+j} &= \Delta_{k,j}\ k_{p\cdot t+j+1} + \Theta_{k,j}\ i_{p\cdot t+j} \\ h_{p\cdot t+j} &= \Delta_{h,j}\ h_{p\cdot t+j+1} + \Theta_{h,j}\ c_{p\cdot t+j} \\ s_{p\cdot t+j} &= \Lambda_j\ h_{p\cdot t+j+1} + P\bar{e}_j\ c_{p\cdot t+j} \\ z_{p\cdot t+j+1} &= A_{22,j}\ z_{p\cdot t+j} + C_{22,j}\ w_{p\cdot t+j} \\ b_{p\cdot t+j} &= U_b\, z_{p\cdot t+j} \\ d_{p\cdot t+j} &= U_d\, z_{p\cdot t+j} \end{aligned} \tag{17.2.4}$$

where $t = 0,\ 1,\ 2,\ \ldots,$ and $j = 1,\ 2,\ \ldots,\ p$. Notice for $t = 0$, as j goes from 1 to p, that $p\cdot t + j$ goes from 1 to p; for $t = 1$, as j goes from 1 to p, that $p\cdot t+j$ goes from $p+1$ to $2p$, and so on.

Thus, (17.2.4) describes a setting in which the matrices that represent preferences and the technology are periodic with period p.

The social planning problem can be expressed in the form of a periodic optimal linear regulator problem. The social planner chooses a sequence of functions expressing u_t as functions of x_t, for all $t \geq 0$, to maximize

$$-E \sum_{t=0}^{\infty} \beta^t\ \{x_t'\ R_{s(t)}\, x_t + u_t'\ Q_{s(t)}\ u_t + 2u_t'\ W_{s(t)}\, x_t\} \tag{17.2.5}$$

subject to the constraints

$$x_{t+1} = A_{s(t)}\, x_t + B_{s(t)} + C_{s(t)}\, w_{t+1} \tag{17.2.6}$$

where $x_t' = [h_{t-1}',\ k_{t-1}',\ z_t]$. In (17.2.5), (17.2.6), the matrices $[R_{s(t)}, Q_{s(t)}, W_{s(t)},$ $A_{s(t)}, B_{s(t)}, C_{s(t)}]$ are the same functions of the matrices $[\Phi_{c,s(t)}, \Phi_{i,s(t)}, \Phi_{g,s(t)},$ $\Gamma_{s(t)}, \Delta_{k,s(t)}, \Theta_{k,s(t)}, \Delta_{h,s(t)}, \Theta_{h,s(t)}, \Lambda_{s(t)}, \Pi_{s(t)}, A_{22,s(t)}\,,\ C_{22,s(t)}, U_b, U_d]$ that the matrices $[R, Q, W, A, B, C]$ are of the matrices $[\Phi_c, \Phi_i, \Phi_c, \Gamma, \Delta_k, \Theta_k, \Delta_h, \Theta_h, \Lambda,$ $\Pi, A_{22}, C_{22}, U_b, U_d]$ in the constant coefficient case. These functions were described in chapter 3.

The Bellman equations for this problem are

$$\begin{aligned} V_t(x_t) = \max_{u_t} \{ & x_t' R_{s(t)} x_t + u_t' Q_{s(t)} u_t + 2 u_t' W_{s(t)} x_t \\ & + \beta E_t V_{t+1}(x_t) \} \end{aligned} \tag{17.2.7}$$

where the maximization is subject to

$$x_{t+1} = A_{s(t)} x_t + B_{s(t)} + C_{s(t)} w_{t+1}. \tag{13.6}$$

In (17.2.7), $V_t(x_t)$ is defined as the optimal value of the problem starting from state x_t at time t.

For the periodic optimal linear regulator problem, the optimal value function is quadratic but time varying:

$$V_t(x_t) = x_t' P_t x_t + \rho_t, \tag{17.2.8}$$

where the $n \times n$ matrix P_t satisfies the matrix Riccati difference equation

$$\begin{aligned} P_t = & R_{s(t)} + \beta A'_{s(t)} P_{t+1} A_{s(t)} - (\beta A'_{s(t)} P_{t+1} B_{s(t)} + W'_{s(t)}) \\ & \times (Q_{s(t)} + \beta B'_{s(t)} P_{t+1} B_{s(t)})^{-1} (\beta B'_{s(t)} P_{t+1} A_{s(t)} + W_{s(t)}), \end{aligned} \tag{17.2.9}$$

while the scalar ρ_t satisfies

$$\rho_t = \beta \rho_{t+1} + \beta \,\mathrm{trace}\, (P_{t+1} C_{s(t)} C'_{s(t)}). \tag{17.2.10}$$

Now think of solving Bellman's equation by iterating backwards on (17.2.9), (17.2.10), starting from some terminal values for P and ρ. Because the matrices $[R_{s(t)},\ Q_{s(t)},\ W_{s(t)},\ A_{s(t)},\ B_{s(t)}]$ are all functions of time when $p \geq 2$, it is too much to hope that $\{P_t, \rho t\}$ will converge in these iterations as $t \to -\infty$ to objects that are independent of time. What is reasonable to hope for, and what will indeed obtain under the assumptions made in our setup, is that iterations on (17.2.9) and (17.2.10) will each produce p convergent subsequences. In particular, backwards iterations on (17.2.9) and (17.2.10) will converge to a sequence that oscillates periodically among p value functions associated with the p seasons of the year. Thus, after many iterations, we will eventually have $V_t(x_t) = V_{s(t)}(x_t)$, where

$$V_{s(t)}(x_t) = x_t' P_{s(t)} x_t + \rho_{s(t)} \tag{17.2.11}$$

We can also represent these value functions as

$$V_j(x_{p\cdot t+j}) = x_{p\cdot t+j}' P_j x_{p\cdot t+j} + \rho_j, \tag{17.2.12}$$

where $t = 0, 1, 2, \ldots$ and $j = [1, 2, \ldots, p]$. Equation (17.2.12) summarizes the outcome that there are p value functions, one for each of the p seasons of the year.

The optimal decision rules can be represented as

$$u_t = -F_{s(t)} x_t \tag{17.2.13}$$

where

$$F_{s(t)} = -(Q_{s(t)} + \beta B_{s(t)}' P_{s(t+1)} B_{s(t)})^{-1} \beta B_{s(t)}' P_{s(t+1)} A_{s(t)}. \tag{17.2.14}$$

The optimal decision rules are thus periodic with period p. Substituting (17.2.13) into the law of motion (17.2.6) gives the following "closed loop" representation of the solution of the social planning problem:

$$x_{t+1} = \big(A_{s(t)} - B_{s(t)} F_{s(t)}\big) x_t + C_{s(t)} w_{t+1} \tag{17.2.15}$$

or

$$x_{t+1} = A_{s(t)}^o x_t + C_{s(t)} w_{t+1} \tag{17.2.16}$$

where $A_{s(t)}^o = A_{s(t)} - B_{s(t)} F_{s(t)}$. We can also represent (17.2.16) in the form

$$x_{p\cdot t+j+1} = A_j^o x_{p\cdot t+j} + C_j w_{p\cdot t+j+1} \tag{17.2.17}$$

for $t = 0, 1, 2, \ldots$ and $j = [1, 2, \ldots, p]$. Thus the laws of motion are periodic with a periodicity p that is inherited from that of the matrices specifying preferences, technology, and information flows.

The matrices $[A_j^o, P_j]$ for $j \in [1, 2, \ldots, p]$ can be used to construct the quantities and prices associated with the equilibrium of our model. Formulas for the matrices determining our equilibrium, namely the M and S matrices, are given by the very same formulas described in chapters 3 and 5, with the proviso that in the periodic case $s(t)$ or j subscripts appear on all objects in those formulas. Thus, we have that the quantities determined in our equilibrium are given by

$$\begin{aligned} h_t &= S_{h,s(t)} x_t & d_t &= S_{d,s(t)} x_t \\ k_t &= S_{k,s(t)} x_t & c_t &= S_{c,s(t)} x_t \\ k_{t-1} &= S_{k1,s(t)} x_t & g_t &= S_{g,s(t)} x_t \\ i_t &= S_{i,s(t)} x_t & s_t &= S_{s,s(t)} x_t \\ b_t &= S_{b,s(t)} x_t \end{aligned} \tag{17.2.18}$$

where

$$\begin{aligned} \begin{bmatrix} S_{h,s(t)} \\ S_{k,s(t)} \end{bmatrix} &= \begin{bmatrix} A^o_{11,s(t)} \\ A^o_{12,s(t)} \end{bmatrix} \\ S_{k1,s(t)} &= [0 \ I \ 0] \\ S_{i,s(t)} &= -F_{s(t)} \\ S_{d,s(t)} &= [0 \ 0 \ U_d] \\ S_{b,s(t)} &= [0 \ 0 \ U_b] \\ S_{c,s(t)} &= U_{c,s(t)} [\Phi_{c,s(t)} \ \ \Phi_{g,s(t)}]^{-1} \\ &\qquad [-\Phi_{i,s(t)} S_{i,s(t)} + \Gamma_{s(t)} S_{k1,s(t)} + S_{d,s(t)}] \\ S_{g,s(t)} &= U_{g,s(t)} [\Phi_{c,s(t)} \ \ \Phi_{g,s(t)}]^{-1} \\ &\qquad [-\Phi_{i,s(t)} S_{i,s(t)} + \Gamma_{s(t)} S_{k1,s(t)} + S_{d,s(t)}] \\ S_{s,s(t)} &= \Lambda_{s(t)} [I \ 0 \ 0] + \Pi_{s(t)} S_{c,s(t)} \end{aligned} \tag{17.2.19}$$

The Lagrange multipliers associated with the social planning problem are determined by the following counterparts of the formulas that we described in chapters 3 and 5:

$$\begin{aligned} M_{k,s(t)} &= 2\beta [0 \ I \ 0] \, P_{s(t)} A^o_{s(t)} \\ M_{h,s(t)} &= 2\beta [I \ 0 \ 0] \, P_{s(t)} A^o_{s(t)} \\ M_{s,s(t)} &= S_{b,s(t)} - S_{s,s(t)} \\ M_{d,s(t)} &= \begin{bmatrix} \Phi'_{c,s(t)} \\ \Phi'_{g,s(t)} \end{bmatrix}^{-1} \begin{bmatrix} \Theta'_{h,s(t)} M_{h,s(t)} + \Pi'_{s(t)} M'_{s,s(t)} \\ -S_{g,s(t)} \end{bmatrix} \\ M_{c,s(t)} &= \Theta'_{h,s(t)} M_{h,s(t)} + \Pi'_{s(t)} M_{s,s(t)} \\ M_{i,s(t)} &= \Theta_{k,s(t)} M_{k,s(t)} \end{aligned} \tag{17.2.20}$$

Formulas for the equilibrium price system can be stated in terms of the objects defined in (17.2.20):

$$
\begin{aligned}
p_{t'}^t &= M_{c,s(t')}x_{t'}/[\bar{e}_j M_{c,s(t)}x_t] \\
w_{t'}^t &= |S_{g,s(t')}x_{t'}|/[\bar{e}_j M_{c,s(t)}x_t] \\
r_{t'}^t &= \Gamma'_{s(t)}M_{d,s(t')}x_{t'}/[\bar{e}_j M_{c,s(t)}x_t] \\
q_{t'}^t &= M_{i,s(t')}x_{t'}/[\bar{e}_j M_{c,s(t)}x_t] \\
\alpha_{t'}^t &= M_{d,s(t')}x_{t'}/[\bar{e}_j M_{c,s(t)}x_t] \\
v_t &= [\Gamma_{s(t')}M_{d,s(t')} + \Delta'_{k,s(t')}]x_{t'}/[\bar{e}_j M_{c,s(t)}x_t]
\end{aligned}
\tag{17.2.21}
$$

These formulas give the time t price system for pricing goods to be delivered at all $t' \geq t$.

17.3. Asset Pricing

With the above formulas in hand, we can derive formulas for pricing assets. These formulas generalize those described in chapter 5 to the case in which the economy is strictly periodic. We begin by pricing an asset that entitles its owner to a stream of returns in the form of a vector of consumption goods described by $y_t = U_{a,s(t)}x_t$, where $U_{a,s(t)}$ is a periodic sequence of matrices. We let a_t denote the price of this asset at time t. By the same reasoning applied in chapter 5, a_t satisfies

$$a_t = E_t \sum_{h=0}^{\infty} \beta^h x'_{t+h} Z_{a,s(t+h)} x_{t+h}/[\bar{e}_j M_{c,s(t)}x_t], \tag{17.3.1}$$

where $Z_{aj} = U'_{a,j}M_{c,j}$. We shall show that (17.2.5) can be represented as

$$a_t = [x'_t \mu_{a,s(t)} x_t + \sigma_{a,s(t)}]/[\bar{e}_j M_{c,s(t)}x_t], \tag{17.3.2}$$

where $\mu_{a,s(t)}$ and $\sigma_{a,s(t)}$ satisfy

$$
\begin{aligned}
\mu_{a,1} &= Z_{a,1} + \beta A_1^{o\prime} Z_{a,2} A_1^o + \beta^2 A_1^{o\prime} A_2^{o\prime} Z_{a,3} A_2^o A_1^o + \cdots \\
&\quad + \beta^{p-1} A_1^{o\prime} A_2^{o\prime} \cdots A_{p-2}^{o\prime} A_{p-1}^{o\prime} Z_{a,p} A_{p-1}^o A_{p-2}^o \cdots A_2^o A_1^o \\
&\quad + \beta^p A_1^{o\prime} A_2^{o\prime} \cdots A_{p-1}^{o\prime} A_p^{o\prime} \mu_{a,1} A_p^o A_{p-1}^o \cdots A_2^o A_1^o \\
\mu_{a,p} &= Z_{a,p} + \beta A_p^{o\prime} \mu_{a,1} A_p^o \\
\mu_{a,p-1} &= Z_{a,p-1} + \beta A_{p-1}^{o\prime} \mu_{a,p} A_{p-1}^o \\
&\ \vdots \\
\mu_{a,2} &= Z_{a,2} + \beta A_2^{o\prime} \mu_{a,3} A_{p-2}^o
\end{aligned}
\tag{17.3.3}
$$

and

$$
\begin{aligned}
\sigma_{a,1} &= \beta \ \text{trace}(\mu_{a,2} C_1 C_1') + \beta \sigma_{a,2} \\
\sigma_{a,2} &= \beta \ \text{trace}(\mu_{a,3} C_2 C_2') + \beta \sigma_{a,3} \\
&\ \vdots \\
\sigma_{a,p} &= \beta \ \text{trace}(\mu_{a,1} C_p C_p') + \beta \sigma_{a,1}
\end{aligned}
\tag{17.3.4}
$$

The matrix $\mu_{a,1}$ can be computed from the first equation of (17.3.3) by using a doubling algorithm that is described in chapter 8. Then the remaining equations of (17.3.3) can be used to compute the remaining $\mu_{a,j}$'s. Given the $\mu_{a,j}$'s, (17.3.4) is a system of p equations that can be solved for the p $\sigma_{a,j}$'s.

To verify (17.3.3), (17.3.4), we can proceed as follows. Let the numerator of (17.3.1), (17.3.2) be denoted

$$
\begin{aligned}
\tilde{a}_t &= E_t \sum_{h=0}^{\infty} \beta^h x_{t+h}' Z_{a,s(t+h)} x_{t+h} \\
&= x_t' \mu_{a,s(t)} x_t + \sigma_{a,s(t)}
\end{aligned}
\tag{17.3.5}
$$

Recall the equilibrium transition laws (17.2.16):

$$
x_{t+1} = A_{s(t)}^o x_t + C_{s(t)} w_{t+1}. \tag{13.19}
$$

Evidently, (17.3.5) and (17.2.16) imply that

$$
\tilde{a}_t = x_t' Z_{a,s(t)} x_t + \beta \, E_t \tilde{a}_{t+1}
$$

or

$$
\begin{aligned}
x_t' \mu_{a,s(t)} x_t + \sigma_{a,s(t)} &= x_t' Z_{a,s(t)} x_t \\
+\beta E_t \left(A_{s(t)}^o x_t + C_{s(t)} w_{t+1}\right)' &\mu_{a,s(t+1)} \left(A_{s(t)}^o x_t + C_{s(t)} w_{t+1}\right) \\
&+ \beta \sigma_{a,s(t+1)}
\end{aligned}
$$

The above equation implies that

$$\mu_{a,s(t)} = Z_{a,s(t)} + \beta A^{o\prime}_{s(t)} \mu_{a,s(t+1)} A^{o}_{s(t)} \tag{17.3.6}$$

$$\sigma_{a,s(t)} = \beta\, \sigma_{a,s(t+1)} + \beta\, \text{trace}\,(\mu_{a,s(t+1)} C_{s(t)} C'_{s(t)}) \tag{17.3.7}$$

Equation (17.3.4) is equivalent with (17.3.7). The first equation of (17.3.3) is the result of recursions on (17.3.6) starting from $s(t)=1$, while the remaining equations of (17.3.3) are simply (17.3.6) for $s(t)=2,3,\ldots,p$.

This completes the verification of (17.3.3), (17.3.4).

We shall give a formula for the term structure of interest rates after we have described the prediction theory associated with (17.2.16).

17.4. Prediction Theory

For a model with period $p \geq 2$, there are two natural alternative ways of specifying the information sets upon which means, covariances, and linear least squares predictions are conditioned. First, we can calculate moments and forecasts by conditioning on the season. This amounts to computing different moments and different forecasting formulas for each of the p seasons. In the appendix to this chapter, we formally describe a sigma algebra, which we denote $\mathcal{I}^p$, that contains the information that corresponds to conditioning on the season. Second, we can calculate moments and forecasts by disregarding information about the season, which amounts to averaging data across seasons in a particular way. In the appendix, we formally describe a sigma algebra, denoted $\mathcal{I}$, which corresponds to not conditioning on the season.

In this section, we describe parts of the prediction theory for our periodic models that correspond to conditioning on the season. In this section,the notation $E_t(\,\cdot\,)$ denotes a mathematical expectation conditioned on x_t, under the assumption that we are also conditioning on the information in $\mathcal{I}^p$. We are assuming that the fictitious social planner uses this information to compute all relevant prices.

Recursions on (17.2.16) can be used to deduce the linear least squares predictions of the state vector x_t. There are p different sets of formulas for the j-step ahead predictions of x_{t+k} conditioned on x_t, one for each season of the year. Recursions on (17.2.16) lead directly to

$$\begin{aligned} x_{t+k} &= A^o_{s(t+k-1)} A^o_{s(t+k-2)} \cdots A^o_{s(t)} x_t \\ &\quad + A^o_{s(t+k-1)} A^o_{s(t+k-2)} \cdots A^o_{s(t+1)} C_{s(t)} w_{t+1} + \cdots \\ &\quad + A^o_{s(t+k-1)} C_{s(t+k-2)} w_{t+k-1} + C_{s(t+k-1)} w_{t+k} \end{aligned} \tag{17.4.1}$$

Equation (17.4.1) implies

$$E_t x_{t+k} = A^o_{s(t+k-1)} A^o_{s(t+k-2)} \cdots A^o_{s(t)} x_t \tag{17.4.2}$$

and

$$\begin{aligned} &E(x_{t+k} - E_t x_{t+k})(x_{t+k} - E_t x_{t+k})' \equiv \Sigma_{k,s(t)} \\ &\quad = A^o_{s(t+k-1)} A^o_{s(t+k-2)} \cdots A^o_{s(t+1)} C_{s(t)} C'_{s(t)} A^{o\prime}_{s(t+1)} \cdots \\ &\qquad A^{o\prime}_{s(t+k-2)} A^{o\prime}_{s(t+k-1)} \\ &\quad + \cdots + A^o_{s(t+k-1)} C_{s(t+k-2)} C'_{s(t+k-2)} A^{o\prime}_{s(t+k-1)} \\ &\qquad + C_{s(t+k-1)} C'_{s(t+k-1)} \end{aligned} \tag{17.4.3}$$

Recursive versions of (17.4.2) and (17.4.3) are available. Equation (17.4.2) implies

$$E_t x_{t+k} = A^o_{s(t+k-1)} E_t x_{t+k-1}.$$

Equation (17.4.3) implies

$$\Sigma_{k,s(t)} = A^o_{s(t+k-1)} \Sigma_{k-1,s(t)} A^{o\prime}_{s(t+k-1)} + C_{s(t+k-1)} C'_{s(t+k-1)}.$$

The prediction formulas (17.4.2), (17.4.3) are evidently predicated on the assumption that we know the matrices $[A^o_j, C_j]$ for $j = [1, \ldots, p]$. They also assume that x_t is in the information set of the forecaster.

Later in this chapter, we shall briefly describe how the Kalman filter can be used to compute the linear least squares forecast of y_t, conditioned only on the history of observed $y's$, and also on $\mathcal{I}^p$. We shall also describe a different theory of prediction, which assumes that we do not know the values of $[A^o_j, C_j]$, and that we cannot condition on the season, so that all that we possess is a time invariant representation for the $\{x_t, y_t\}$ process.

17.5. The Term Structure of Interest Rates

In light of formula (17.4.2), the same logic that led to formula (5.65) for the reciprocal of the risk-free interest rate on j-period loans, R_j^t, now leads to the following formula:

$$R_j^t = \beta^j \bar{e}_1 M_{c,s(t+j)} A^o_{s(t+j-1)} A^o_{s(t+j-2)} \cdots A^o_{s(t)} x_t / [\bar{e}_j M_{c,s(t)} x_t] \tag{17.5.1}$$

This formula gives the price at time t of a sure claim on the first consumption good j periods ahead.

17.6. Conditional Covariograms

In this section, we present formulas for the covariance function of x and y, conditioned on season, i.e., conditioned on $\mathcal{I}^p$. The conditional covariogram of $\{x_t\}$ can be expressed in terms of the conditional contemporaneous covariance function $c_{x,t}(0) = Ex_t x_t' | \mathcal{I}^p$ via the formulas

$$\begin{aligned} c_{x,t}(-k) &\equiv Ex_t x'_{t+k} | \mathcal{I}^p \\ &= Ex_t x_t' | \mathcal{I}^p A^{o\prime}_{s(t)} A^{o\prime}_{s(t+1)} \cdots A^{o\prime}_{s(t+k-2)} A^{o\prime}_{s(t+k-1)}, \ k \geq 1 \end{aligned}$$

or

$$\begin{aligned} c_{x,t}(-k) = c_{x,t}(0) A^{o\prime}_{s(t)} A^{o\prime}_{s(t+1)} \cdots \\ A^{o\prime}_{s(t+k-2)} A^{o\prime}_{s(t+k-1)}, \ k \geq 1. \end{aligned} \tag{17.6.1}$$

To solve for the matrices $c_{x,t}(0)$, we can solve the equations

$$Ex_{t+1} x'_{t+1} | \mathcal{I}^p = A^o_{s(t)} Ex_t x_t' | \mathcal{I}^p A^{o\prime}_{s(t)} + C_{s(t)} C'_{s(t)}$$

or

$$c_{x,t+1}(0) = A^o_{s(t)} c_{x,t}(0) A^{o\prime}_{s(t)} + C_{s(t)} C'_{s(t)}. \tag{17.6.2}$$

By solving the system formed by (17.6.2) for $t = 1, 2, \ldots, p$, we can determine the p contemporaneous covariance matrices $c_{x,1}(0), c_{x,2}(0), \ldots, c_{x,p}(0)$. Here is a fast way of solving this system. Iterating on (17.6.2) p times yields

$$\begin{aligned} &c_{x,t+p}(0) \\ &= A^o_{s(t+p-1)} A^o_{s(t+p-2)} \cdots A^o_{s(t)} \, c_{x,t}(0) \, A^{o\prime}_{s(t)} \cdots A^{o\prime}_{s(t+p-2)} A^{o\prime}_{s(t+p-1)} \\ &+ A^o_{s(t+p+1)} A^o_{s(t+p-2)} \cdots A^o_{s(t+1)} C_{s(t)} C'_{s(t)} A^{o\prime}_{s(t+1)} \cdots A^{o\prime}_{s(t+p-1)} \\ \cdots &+ A^o_{s(t+p-1)} C_{s(t+p-2)} C'_{s(t+p-2)} A^{o\prime}_{s(t+p-1)} + C_{s(t+p-1)} C'_{s(t+p-1)}. \end{aligned} \tag{17.6.3}$$

We compute $c_{x,t}(0)$ by setting $c_{x,t+p}(0)$ equal to $c_{x,t}(0)$ in (17.6.3). Equation (17.6.3) is a discrete Lyapunov equation that can be solved by a doubling algorithm that is described in chapter 8. Once (17.6.3) is solved for $t = 1$ to compute $c_{x,1}(0)$, (17.6.2) can be used to compute $c_{x,t}(0)$ for $t = 2, \ldots, p$. There is one covariance matrix $c_{x,t}(0)$ for each of the p seasons of the year.

Given $c_{x,t}(-k)$ for $k \geq 0$, we can compute $c_{y,t+k}(-k) = E y_t y'_{t+k} | \mathcal{I}^p$ by using (17.6.1). We obtain

$$E y_t y'_{t+k} \,|\, \mathcal{I}^p = G_{s(t)} c_{x,k}(-k) G'_{s(t+k)}, \;\; k \geq 0. \tag{17.6.4}$$

Although we are starting calendar time at $t = 0$, $c_{x,t}(k)$ and $c_{y,t}(k)$ are both defined for positive k so long as $t \geq k$. For any such t, $c_{x,t}(k) = c_{x,t-k}(-k)'$ and $c_{y,t}(k) = c_{y,t-k}(-k)'$, implying that $\{c_{y,t}(k)\}$ and $\{c_{y,t}(k)\}$ are both periodic starting from $t = k$. For notational convenience, we extend this construction for $0 \leq t \leq k$ by defining $c_{x,t}(k) = c_{x,t+\ell p}(k)$ and $c_{y,t}(k) = c_{y,t+\ell p}(k)$ for any ℓ such that $t + \ell p \geq k$. This guarantees that the conditional covariograms are periodic for all values of k.

17.7. The Stacked and Skip-Sampled System

The equilibrium has the system of periodic transition laws described in (17.2.16) or (17.2.17). The equilibrium stochastic process for x_t is time-varying, albeit in a highly structured way. We have seen that conditional on knowledge of the season, there are p covariograms, and p sets of formulas for linear least squares predictions that apply in the p seasons of the year. Using these formulas requires knowledge of the set of matrices $[A_j^o, C_j]$ for $j = [1, \ldots, p]$ that characterize the transition laws (17.2.16).

In this section, we describe a time invariant representation that also characterizes the system. We shall use this representation for several purposes. We shall use it to deduce two kinds of impulse response functions or moving average representations that can be defined for periodic models.[2] We shall also use it to compute a population version of a time-invariant vector autoregression for x_t.

[2] Each of these impulse response functions conditions on knowledge of the season. Later we shall describe yet another moving average representation that does not condition on season.

The p distinct covariograms as described by equations (17.6.1) and (17.6.4) are *conditional* covariograms, meaning that they are computed by conditioning on the season of the year. Sample counterparts of these conditional covariograms are computed by creating p distinct averages, averaging each over observations p periods apart. Sample covariograms can also be computed 'unconditionally', i.e., in a way that ignores the seasonal structure of the transition laws. This amounts to computing sample moments in the standard way, simply by averaging over adjacent observations, namely, as $T^{-1} \sum_{t=1}^{T} y_t \; y'_{t-j}$. For a periodic model, such averages will converge as $T \to \infty$, and they will converge to well defined functions of the parameters of the model. In particular, as $T \to \infty$, $T^{-1} \sum_{t=1}^{T} y_t y'_{t-k}$ would converge to an average of the p covariograms, namely, $p^{-1}[c_{y,1}(k) + c_{y,2}(k) + \ldots + c_{y,p}(k)]$. The convergence of these sample autocovariances assures the existence of a time invariant vector autoregressive representation for y_t.

We begin by defining for $t = 0, 1, \ldots$ the vector

$$X'_t = [x'_{p\cdot t-p+1}, x'_{p\cdot t-p+2}, \ldots, x'_{p\cdot t}]'. \tag{17.7.1}$$

Evidently, we have

$$X'_{t+1} = [x'_{p\cdot t+1}, x'_{p\cdot t+2}, \ldots, x'_{p\cdot t+p}]. \tag{17.7.2}$$

To verify this, substitute $(t+1)$ for t everywhere that t appears on the right side of (17.7.1). We also define

$$W'_{t+1} = [w'_{p\cdot t+1}, w'_{p\cdot t+2}, \ldots, w'_{p\cdot t+p}]. \tag{17.7.3}$$

It follows from (17.2.17) that

$$DX_{t+1} = FX_t + \mathcal{G}W_{t+1}, \tag{17.7.4}$$

where

$$D = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -A_1^o & I & 0 & \cdots & 0 & 0 \\ 0 & -A_2^o & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A_{p-1}^o & I \end{bmatrix} \tag{17.7.5}$$

$$F = \begin{bmatrix} 0 & A_p^o \\ 0 & 0 \end{bmatrix} \tag{17.7.6}$$

$$\mathcal{G} = \begin{bmatrix} C_p & 0 & 0 & \cdots & 0 \\ 0 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{p-1} \end{bmatrix} \tag{17.7.7}$$

Solving (17.7.4) for X_{t+1} gives

$$X_{t+1} = \hat{A}X_t + \hat{C}W_{t+1} \tag{17.7.8}$$

where $\hat{A} = D^{-1}F$ and $\hat{C} = D^{-1}\mathcal{G}$. We also define the vector $Y_t' = [y_{p\cdot t-p+1}', y_{p\cdot t-p+2}', \dots, y_{p\cdot t}']$. Then we have that

$$Y_t = HX_t \tag{17.7.9}$$

where $H = \begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_p \end{bmatrix}$. Thus we have that $\{Y_t\}$ is governed by the time invariant state space system

$$X_{t+1} = \hat{A}X_t + \hat{C}W_{t+1} \tag{17.7.10}$$

$$Y_t = HX_t \tag{17.7.11}$$

Notice that while $\{x_t, y_t\}$ is governed by a time varying linear state space system, the stacked and skip sampled process $\{X_t, Y_t\}$ is governed by a time invariant system.[3]

From representation (17.7.10) – (17.7.9), we can use standard formulas to deduce the moving average representation of Y_t in terms of W_t

$$Y_t = \sum_{j=0}^{\infty} \bar{C}_j W_{t-j}. \tag{17.7.12}$$

The moving average representation (17.7.12) implies the following representation for the components of Y_t in terms of the components of W_t:

$$y_{pt-p+k} = \sum_{j=0}^{\infty} \sum_{h=1}^{p} \bar{C}_j(k,h) w_{p(t-j)-p+h}, \; k = 1, \dots, p, \tag{17.7.13}$$

[3] In terms of the language introduced in the appendix, because S is of period p, S^p is of period one.

where $\bar{C}_j(k,h)$ denotes the $(k,h)^{\text{th}}$ $(m \times m)$ block of $\bar{C}_j$, where m is the dimension of y_t.

According to representation (17.7.13), there are two distinct concepts of a moving average representation, and p embodiments of each of these concepts. The first concept is a representation of y_{pt-p+k} in terms of current and lagged w_t's. The response of y_{pt-p+k} to lagged w's is evidently given by the sequence[4]

$$\begin{aligned}\{d_{k,v}\}_{v=0}^{\infty} = \{&\bar{C}_0(k,k), \bar{C}_0(k,k-1), \ldots \bar{C}_0(k,1)\bar{C}_1(k,p), \bar{C}_1(k,p-1), \ldots, \\ &\bar{C}_1(k,1), \bar{C}_2(k,p), \bar{C}_2(k,p-1), \ldots, \bar{C}_1(k,1), \ldots\}.\end{aligned} \tag{17.7.14}$$

In particular, we have from (17.7.13) that

$$y_{pt-p+k} = \sum_{v=0}^{\infty} d_{k,v}\ w_{pt-p+k-v}. \tag{17.7.15}$$

Notice that there is a different moving average of type (17.7.15) for each season $k = 1, \ldots, p$.

The second concept of a moving average is the response of the $\{y_t\}$ process to an innovation w_{pt-p+k} in a particular season k. The response of $\{y_t\}$ *to* w_{pt-p+k} is evidently given by the sequence

$$\begin{aligned}\{g_{k,v}\}_{v=0}^{\infty} = \{&\bar{C}_0(k,k), \bar{C}_0(k+1,k), \ldots, \bar{C}_0(p,k), \\ &\bar{C}_1(1,k), \bar{C}_1(2,k), \ldots, \bar{C}_1(p,k), \ldots\}.\end{aligned} \tag{17.7.16}$$

In the special case in which the true periodicity is one, it is straightforward to verify that for any $p > 1$, the impulse functions constructed from the stacked system (17.7.10) – (17.7.9) satisfy the restrictions:

$$\begin{aligned}\bar{C}_j(1,1) &= \bar{C}_j(2,2) = \ldots = \bar{C}_j(p-1,p-1) = \bar{C}_j(p,p) \\ \bar{C}_j(2,1) &= \bar{C}_j(3,2) = \ldots = \bar{C}_j(p,p-1) = \bar{C}_{j+1}(1,p) \\ \bar{C}_j(3,1) &= \bar{C}_j(4,2) = \ldots = \bar{C}_{j+1}(1,p-1) = \bar{C}_{j+1}(2,p) \\ &\vdots \\ \bar{C}_j(p-1,1) &= \bar{C}_j(p,2) = \ldots = \bar{C}_{j+1}(p-2,p-1) = \bar{C}_j(p-2,p) \\ \bar{C}_j(p,1) &= \bar{C}_{j+1}(1,2) = \ldots = \bar{C}_{j+1}(p-2,p-1) = \bar{C}_{j+1}(p-1,p)\end{aligned} \tag{4.14}$$

[4] Notice that by construction $\bar{C}_0(k,j) = 0$ for $k < j$.

Under these restrictions, it follows that

$$\begin{aligned} g_{k,v} &= g_{j,v} \quad \text{for all} \quad j,k, \quad \text{for all } v \\ d_{k,v} &= d_{j,v} \quad \text{for all} \quad j,k, \quad \text{for all } v \\ d_{k,v} &= g_{k,v} \quad \text{for all} \quad k, \quad \text{for all } v \end{aligned}$$

Thus, in the case in which the hidden periodicity is truly one, all of the impulse response functions defined in (17.7.15) and (17.7.16) are equal, and are equal to each other.

However, when the hidden periodicity is truly some $p > 1$, there are p distinct impulse response functions $\{d_{k,v}\}$ *of* y_{pt-p+k} to lagged w's, and p distinct impulse responses $\{g_{k,v}\}$ of $\{y_t\}$ *to* w_{pt-p_k}, for $k = 1, \ldots p$. In general the $\{d_{k,v}\}$ are different from one another and from the $\{g_{k,v}\}$'s for $k = 1, \ldots, p$. These differences provide a useful way of describing how the operating characteristics of a periodic model with $p \geq 2$ differ from a period one model.[5]

Later in this chapter, we compute the impulse response functions $\{d_{k,v}\}$ and $\{g_{k,v}\}$ for investment for a period 4 version of Hall's model described above. These impulse response functions are depicted in figures 17.10.1.a and 17.10.1.b. The impulse responses are with respect to the one shock in the model, which is a white noise endowment process. Figure 17.10.2 depicts the impulse responses $\{d_{k,v}\}$ for $k = 1, \ldots, 4$. Notice that they are smooth, but that they vary across quarters. Figure 17.10.3 shows the impulse response $\{g_{k,v}\}$ for $k = 1, \ldots, 4$. They vary across quarter k, and have shapes that are jagged, in contrast to the smooth $\{d_{k,v}\}$'s. Notice how the amplitude of the oscillations in $\{d_{k,v}\}$ grows as v increases from $v = 0$ to v about 30.

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[5] A MATLAB program `simpulse` performs these calculations.

17.8. Covariances of the Stacked, Skip Sampled Process

The stacked, skip-sampled process $\{Y_r\}$ is constructed to have periodicity one. We can compute for $k \geq 1$,

$$C_r(-k) \equiv E(Y_r Y_{r+k}' \mid \mathcal{I}^p) =$$
$$\begin{bmatrix} c_{y,pr}(-pk) & c_{y,pr}(-pk-1) & \cdots & c_{y,pr}(-pk-p+1) \\ c_{y,pr+1}(-pk+1) & c_{y,pr+1}(-pk) & \cdots & c_{y,pr+1}(-pk-p+2) \\ \vdots & \vdots & \vdots & \vdots \\ c_{y,pr+p-1}(-pk+p-1) & c_{y,pr+p-1}(-pk+p-2) & \cdots & c_{y,pr+p-1}(-pk) \end{bmatrix}. \tag{17.8.1}$$

An implication of the period 1 nature of $\{Y_r\}$ is that $C_r(-k)$ is independent of r. This follows immediately from (17.8.1). In particular, we have for $k \geq 1$,

$$\begin{aligned} C(k) &\equiv C_0(-k) \\ &= \begin{bmatrix} c_{y,0}(-pk) & c_{y,0}(-pk-1) & \dots & c_{y,0}(-pk-p+1) \\ c_{y,1}(-pk+1) & c_{y,1}(-pk) & \dots & c_{y,1}(-pk-p+2) \\ \vdots & \vdots & \ddots & \vdots \\ c_{y,p-1}(-pk+p-1) & c_{y,p-1}(-pk+p-2) & \dots & c_{y,p-1}(-pk) \end{bmatrix}. \end{aligned} \tag{17.8.2}$$

The $k = 0$ term must be treated separately. It is given by

$$\begin{aligned} C_r(0) &\equiv E(Y_r Y_r' \mid \mathcal{I}^p) \\ &= \begin{bmatrix} c_{y,pr}(0) & c_{y,pr}(-1) & \cdots & c_{y,pr}(-p+1) \\ c_{y,pr}(-1)' & c_{y,pr+1}(0) & \cdots & c_{y,pr+1}(-p+2) \\ \vdots & \vdots & \ddots & \vdots \\ c_{y,pr}(-p+1)' & c_{y,pr+1}(-p+2)' & \cdots & c_{y,pr+p-1}(0) \end{bmatrix}, \end{aligned} \tag{17.8.3}$$

which can also be shown to be independent of r.

The covariance generating function of the $\{Y_r\}$ process is given by

$$S(z) \equiv C(0) + \sum_{k=1}^{\infty} [C(-k)z^{-k} + C(-k)'z^k]. \tag{17.8.4}$$

It is useful to calculate the covariance generating function $S(z)$ of $\{Y_r\}$ by substituting (17.8.2) – (17.8.3) into (17.8.4). We obtain

$$S(z) = \begin{bmatrix} s_{11}^f(z) & s_{12}^f(z) & s_{13}^f(z) & \cdots & s_{1p}^f(z) \\ s_{21}^f(z) & s_{22}^f(z) & s_{23}^f(z) & \cdots & s_{2p}^f(z) \\ s_{31}^f(z) & s_{32}^f(z) & s_{33}^f(z) & \cdots & s_{3p}^f(z) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{p1}^f(z) & s_{p2}^f(z) & \cdots & \cdots & s_{pp}^f(z) \end{bmatrix} \tag{17.8.5}$$

where

$$s^f_{j,j+\ell}(z) = c_{j-1}(-\ell) + \sum_{k=1}^{\infty}[c_{y,j-1}(-pk-\ell)z^{-k} + c_{y,j+\ell-1}(-pk-\ell)'z^k], \tag{17.8.6}$$

and where the lower triangular terms of $S(z)$ are obtained from the upper by setting $S(z) = S(z^{-1})'$ for $z = e^{-i\omega}$.

The hypothesis that $\{y_t\}$ is of period one places restrictions on $S(z)$. Period one of $\{y_t\}$ implies that $c_{y,j}(k) = c_{y,1}(k)$ for all j. By using this equality in (17.8.6) it can be shown that

$$\begin{aligned} s^f_{11}(z) &= s^f_{22}(z) = \cdots = s^f_{pp}(z) \\ s^f_{12}(z) &= s^f_{23}(z) = \cdots s^f_{p-1,p}(z) = z^{-1}s^f_{p,1}(z) \\ s^f_{13}(z) &= s^f_{24}(z) = \cdots = s^f_{p-2,p}(z) = z^{-1}s^f_{p-1,1}(z) = z^{-1}s^f_{p,2}(z) \\ &\vdots \\ zs^f_{1,p}(z) &= s^f_{2,1}(z) = s^f_{3,2}(z) \cdots = s^f_{p,p-1}(z) \end{aligned} \tag{17.8.7}$$

The first line of equalities in (17.8.7) asserts that the block of matrices along the diagonal of $S(z)$ are equal to each other, and to a folded spectrum of the original unsampled $\{y_t\}$ process.

17.9. The Tiao-Grupe Formula

Define

$$r_{y,t}(-k) = E(y_t y'_{t+k} \mid \mathcal{I}).$$

It follows that $r_{y,t}(-k) = r_{y,1}(-k) \equiv r_y(-k)$ for all t. Furthermore, by the law of iterated expectations

$$r_y(-k) = r_{y,t}(-k) = E\big[c_{y,t}(-k) \mid \mathcal{I}\big]. \tag{17.9.1}$$

It follows from (17.9.1) that $r_y(-k)$ can be computed either by computing covariances without skip sampling, or by averaging across covariances that have been computed by skip sampling. That is,

$$r_y(-k) = p^{-1} \sum_{j=1}^{p} c_{y,j}(-k),$$

and by a law of large numbers

$$r_y(-k) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} y_t y'_{t+k}.$$

It is useful to derive Tiao and Grupe's (1980) formula for the covariance generating function of $\{y_t\}$, not conditioned on season, as a function of the covariance generating function conditioned on season. Tiao and Grupe's formula expresses the generating function for the covariances not conditioned on season in terms of the (conditional on season) covariance generating function of the stacked and skip sampled process Y_t.[6] We define the generating function for the covariances not conditioned on season to be:

$$\begin{aligned} s_y(z) &= \sum_{k=-\infty}^{\infty} r(k) z^k \\ \text{or} & \\ s_y(z) &= p^{-1} \sum_{k=-\infty}^{\infty} z^k \sum_{j=1}^{p} c_{y,j}(k). \end{aligned} \tag{17.9.2}$$

[6] Gladysev (1960) states a formula restricting the Cramer representations for Y_t and y_t that has the same content as the Tiao-Grupe formula.

To compute $s_y(z)$, define the operator

$$Q(z) = [I \ zI \ldots \ z^{p-1}I], \tag{17.9.3}$$

where each of the p identity matrices in (17.9.3) is $(n \times n)$. Note that for $k \geq 1$,

$$\begin{aligned} &Q(z)C(k)Q(z^{-1})' \\ &= [c_{y,0}(-pk) + c_{y,0}(-pk+1)z^{-1} + \cdots + c_{y,0}(-pk-p+1)z^{-p+1} \\ &+ c_{y,1}(-pk+1)z + c_{y,1}(-pk) + \cdots + c_{y,1}(-pk-p+2)z^{-p+2} \\ &\vdots \\ &+ c_{y,p-1}(-pk+p-1)z^{p-1} + c_{y,p-1}(-pk+p-2)z^{p-2} + \ldots + c_{y,p-1}(-pk)] \end{aligned} \tag{17.9.4a}$$

Notice also that for $k = 0$, we have

$$\begin{aligned} Q(z)C(0)Q(z^{-1})' &= [I \ z \ I \ \ldots \ z^{p-1}I]C(0) \begin{bmatrix} I \\ zI^{-1} \\ \vdots \\ z^{-p+1}I \end{bmatrix} \\ &= c_{y,0}(0) + c_{y,1}(0) + \ldots + c_{y,p-1}(0) \\ &+ z[c_{y,0}(-1)' + c_{y,1}(-1)' + \ldots + c_{y,p-2}(-1)'] \\ &+ z^{-1}[c_{y,0}(-1) + c_{y,1}(-1) + \ldots + c_{y,p-2}(-1)] + \\ &\vdots \\ &+ z^{p-1}c_{y,0}(-p+1)' + z^{-p+1}c_{y,0}(-p+1). \end{aligned} \tag{17.9.4b}$$

Applying (13.67) to (17.9.2) gives

$$\begin{aligned} s_y(z) &= p^{-1} \sum_{h=-\infty}^{\infty} z^{ph} Q(z)C(h)Q(z^{-1})' \\ s_y(z) &= p^{-1}Q(z)[\sum_{h=-\infty}^{\infty} z^{ph} C(h)]Q(z^{-1})' \\ s_y(z) &= p^{-1}Q(z)S(z^p)Q(z^{-1})', \end{aligned} \tag{17.9.5}$$

where $S(z)$ is the generating function for the $\{Y_r\}$ process, which is defined in (17.7.9) and (17.8.4). Equation (17.9.5) is the Tiao–Grupe formula.

Equation (17.9.5) shows how the generating function of the $\{y_t\}$ process can be obtained by transforming the generating function of the stacked, skip sampled process $\{Y_r\}$. Equation (17.9.5) is helpful in displaying the types of fluctuations that will occur in a periodic process $\{y_t\}$. Suppose that we were to take a realization $\{y_t\}_{t=1}^{T}$ of the $\{y_t\}$ process, compute the sample covariances as

$$\hat{r}(k) = \frac{1}{T} \sum_{t=k+1}^{T} y_t y_{t-k} \tag{17.9.6}$$

and the sample spectrum as

$$\hat{s}\left(e^{-i\omega_h}\right) = \sum_{k=-T+1}^{T} w(k)\hat{r}(k)e^{-i\omega_h j}, \omega_h = \frac{2\pi h}{T},\ h = 1, \ldots, T \tag{17.9.7}$$

where $w(k)$ is one of the popular windows. Notice that in computing (17.9.6) and (17.9.7) we are ignoring the hidden periodicity. In large samples, $\hat{r}(k)$ given by (17.9.6) will converge to $r(k)$ defined in (17.9.1), and $\hat{s}\left(e^{-i\omega_h}\right)$ will converge to $s_y\left(e^{-i\omega_h}\right)$.

17.9.1. A state space realization of the Tiao-Grupe formulation

We now return to representation (17.7.10) – (17.7.9). We will use this representation in conjunction with formula (17.9.3) to get a representation for the generating function $s_y(z)$ in terms of the parameters of our economic model. Then we shall describe how to use state space methods to factor this covariance generating function, thereby obtaining a Wold representation for y_t.

If the eigenvalues of $\hat{A}$ are bounded in modulus by unity,[7] then $\{X_t, Y_t\}$ will be asymptotically covariance stationary, with covariance generating matrices $S_X(z)$ and $S_Y(z)$ given by

$$\begin{gathered} S_X(z) = \sum_{k=-\infty}^{\infty} R_X(k) z^k \\ \text{or} \\ S_X(z) = (I - \hat{A}z)^{-1}\hat{C}\hat{C}'(I - \hat{A}z^{-1})^{-1\prime} \end{gathered} \tag{17.9.8}$$

[7] This is the condition alluded to in section 1.

and

$$\begin{aligned} S_Y(z) &= \sum_{k=-\infty}^{\infty} R_Y(k) z^k \\ S_Y(z) &= H S_X(z) H' \end{aligned} \tag{17.9.9}$$

where $R_X(k) = EX_t X'_{t-k}, R_Y(k) = EY_t Y'_{t-k}$.

By substituting $S_X(z)$ or $S_Y(z)$ for $S(z)$ in formula (17.9.2), we can compute the covariance generating function for the process $\{y_t\}$ by averaging across covariograms for different periods. For the y_t process under study here, we have[8]

$$\begin{aligned} &s_y(z) = Q(z) S_Y(z^p) Q(z^{-1})' \\ &\qquad \text{or} \\ &s_y(z) = Q(z) H (I - \hat{A} z^p)^{-1} \hat{C} \hat{C}' (I - \hat{A} z^{-p})^{-1\prime} H' Q(z^{-1})'. \end{aligned} \tag{17.9.10}$$

We now show how to use (17.9.10) to deduce a state-space representation for $\{y_t\}$. The first step involves recognizing that (17.9.10) is realized by the system

$$\begin{aligned} Z_{t+p} &= \hat{A} Z_t + \hat{C} V_{t+p} \\ y_t &= Q(L) H Z_t \end{aligned} \tag{17.9.11}$$

where L is the lag operator and $\{V_t\}$ is a vector white noise with identity contemporaneous covariance matrix.

It is convenient to stack (17.9.11) into the first order system

$$\begin{aligned} &\begin{bmatrix} Z_{t+p} \\ Z_{t+p-1} \\ Z_{t+p-2} \\ \vdots \\ Z_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 & \hat{A} \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \begin{bmatrix} Z_{t+p-1} \\ Z_{t+p-2} \\ Z_{t+p-3} \\ \vdots \\ Z_t \end{bmatrix} + \begin{bmatrix} \hat{C} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} V_{t+p} \\ &y_{t+p-1} = \tilde{H} \begin{bmatrix} Z_{t+p-1} \\ Z_{t+p-2} \\ Z_{t+p-3} \\ \vdots \\ Z_t \end{bmatrix} \end{aligned} \tag{17.9.12}$$

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[8] A MATLAB program `spectrs` implements (17.9.10) for the equilibrium of one of our periodic general equilibrium models.

where

$$\tilde{H} = p^{-.5}[G_1\ 0\ 0\ \ldots\ 0 \vdots 0\ G_2\ 0\ \ldots\ 0 \vdots \ldots \vdots 0\ 0\ 0\ \ldots\ G_p].$$

To see why $\tilde{H}$ takes this form, recall that the operator $Q(L)$ is defined as

$$\begin{aligned} p^{.5}Q(L) &= [I\ IL\ \ldots\ IL^{p-1}] \\ &= [I\ 0\ \ldots\ 0] \\ &+ [0\ I\ \ldots\ 0]L + \ldots \\ &+ [0\ 0\ \ldots\ I]L^{p-1}. \end{aligned}$$

This structure for $Q(L)$ and the form of H dictates that $\tilde{H}$ take the form that it does and that the state in (17.9.12) takes the form that it does in order to map (17.9.11) into a first-order system. Notice that the structure of $\tilde{H}$ implies that y_t is formed by averaging over linear combinations of the *first* n rows of Z_{t+p-r}, the *second* n rows of $Z_{t+p-2}, \ldots$, and the p^{th} n rows of Z_t. Furthermore, notice that according to (17.9.12), the $np \times np$ process Z_t consists of p completely uncoupled systems, each of which depends on its own past in exactly the same was as do the others. That is, (17.9.12) has the property that Z_t is independent of $Z_{t-1}, Z_{t-2}, \ldots, Z_{t-p+1}$ for all t; and that Z_t is correlated with Z_{t-p} in exactly the same way for all t. Thus, the "state equations" of (17.9.12) in effect describe p "parallel realizations" of the process Z_{t+p} defined in (17.9.11). Running p parallel processes is a way of realizing in the time domain the randomization over laws of motion that is involved in adopting a description of $\{y_t\}$ in terms of a stationary probability distribution. As noted above, y_t is formed by averaging across these p uncoupled realizations.

We can use Kalman filtering methods to derive a Wold representation for $\{y_t\}$. Modify and represent system (17.9.12) as

$$\begin{aligned} \tilde{Z}_{t+1} &= \tilde{A}\tilde{Z}_t + \tilde{C}\tilde{V}_{t+1} \\ y_t &= \tilde{H}Z_t + \tilde{\epsilon}_t \end{aligned} \tag{17.9.13}$$

where $\tilde{Z}_t' = [\tilde{Z}_{t+p-1}', \tilde{Z}_{t+p-2}', \ldots, \tilde{Z}_t']$ and

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \ldots & 0 & \hat{A} \\ I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \vdots & I & 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $\{\tilde{V}_t\}$ is a white noise, and where ϵ_t is a (potentially very small) measurement error which is a white noise process that is orthogonal to $\{\tilde{V}_t\}$ and satisfies $E\tilde{\epsilon}_t\tilde{\epsilon}_t' = R$.

To obtain the Wold representation for y_t which achieves the factorization of the spectral density matrix (17.9.10) for y_t, we use the Kalman filter to obtain an innovations representation associated with system (17.9.13). The innovations representation is

$$\begin{aligned} \hat{Z}_{t+1} &= \tilde{A}\hat{Z}_t + \tilde{K}\tilde{a}_t \\ y_t &= \tilde{H}\hat{Y}_t + \tilde{a}_t, \end{aligned} \tag{17.9.14}$$

where $\tilde{a}_t = y_t - E[y_t \mid y_{t-1}, y_{t-2}, \ldots], \hat{Z}_t = E[Z_t \mid y_{t-1}, y_{t-2}, \ldots]$,
$\Sigma = E(Z_t - Z_t)(Z_t - \hat{Z}_{kt})'$ and where Σ and $\tilde{K}$ are the state covariance matrix and the Kalman gain computed via the Kalman filter for system (17.9.13). The covariance matrix of the innovations is given by $E\tilde{a}_t\tilde{a}_t' = \tilde{H}\Sigma\tilde{H}' + R$.[9]

17.10. Some Calculations with a Periodic Hall Model

We use a periodic version of Hall's model as an example. The model is identical to the version of Hall's model described in chapters 3 and 5, except that the productivity parameter γ now varies periodically. The social planner chooses contingency plans $\{c_t, k_t, i_t\}_{t=0}^{\infty}$ to maximize the utility functional

$$\begin{gathered} -(\frac{1}{2})E\sum_{t=0}^{\infty}\beta^t[(c_t - b_t)^2 + \ell_t^2] \mid J_0 \\ 0 < \beta < 1 \end{gathered}$$

subject to the technology

$$\begin{aligned} c_t + i_t &= \gamma_{s(t)}k_{t-1} + d_t, \qquad \gamma_{s(t)} \geq 0 \\ k_t &= \delta_k k_{t-1} + i_t, \qquad 0 < \delta_k < 1 \\ \phi_1 i_t &= g_t, \qquad \phi_1 > 0, \qquad \phi_1 > 0 \\ g_t^2 &= \ell_t^2, \\ s(t+p) &= s(t), \quad \forall t, \quad s(t) = t \text{ for } t = 1, \ldots, p \end{aligned}$$

[9] These calculations are performed by the MATLAB program .

and subject to the (exogenous) laws of motion

$$b_t = 30$$
$$d_t = .8d_{t-1} + w_{1t} + 5 * (1 - .8)$$

We set $p = 4$, and $\gamma_1 = .13, \gamma_2 = .1, \gamma_2 = .1, \gamma_4 = .08$. We set $\phi_1 = .3, \delta_k = .95, \beta = 1/1.05$. The only source of disturbance in the model is the endowment shock, which is a first order autoregression. The variance of the innovation w_{1t} is unity.

The following MATLAB programs can be used to analyze the model.

`solves.m`:	computes the equilibrium of a periodic model,
`simuls.m`:	simulates a periodic equilibrium;
`steadsts.m`:	computes the means of variables from a periodic equilibrium, conditional on the season;
`assets.m`:	computes the objects in the formulas for equilibrium assets prices and the term structure of interest rates for a periodic model;
`assetss.m`:	simulates the asset prices in a periodic equilibrium;
`seasla.m`:	computes the time invariant state-space representation for the stacked, skip sampled version of a periodic model;
`simpulse.m`:	computes the two different concepts of period-dependent impulse response functions;
`spectrs.m`:	computes the spectral density of a periodic model, using the Tiao-Grupe formula;
`factors.m`:	factors a univariate spectral density computed via the Tiao-Grupe formula in order to obtain a univariate Wold representation for a single variable of a periodic equilibrium model.

We computed the equilibrium of the periodic version of Hall's model using `solves.m`. Figures 17.10.1.a and 17.10.1.b report the spectral density of consumption and investment, computed by using the Tiao-Grupe formula. Both consumption and investment display seasonality, it being more pronounced in investment than in consumption. This is a reflection of the consumption-smoothing property of the model. For the impulse response functions of investment with respect to the innovation in the endowment sequence, we used `simpulse.m` to compute the $\{d_{k,v}\}$ and $\{g_{k,v}\}$ sequences corresponding to the moving average representations defined in (17.7.15) and (17.7.16). Figure 17.10.3 reports $\{d_{k,v}\}$. The coefficients for each quarter are smooth functions of the lag, but they vary across quarters. Figure 17.10.3 reports $\{d_{k,v}\}$, which are each oscillatory functions of the lag. Recall that our periodic 1 of Hall's model is a one-shock model, with the only stochastic source of disturbances coming from the white noise endowment process. It follows that if we were to shut down the periodic time variation in the productivity of capital, *all* of the impulse response functions displayed in figure 17.10.3 and 17.10.4 would be equal to one another.[10] The discrepancies across these impulse response functions is a convenient "window" for examining the hidden periodic structure present in investment in this model.

Figure 17.10.4 reports the moving average coefficients associated with the univariate Wold representation for investment, which we have normalized by setting the innovation variance equal to unity (so that it is comparable in units with the impulse response functions in figures 17.10.2 and 17.10.3. The coefficient at zero lag in this moving average is .7528, while the coefficients at zero lag for the moving average kernels in figures 17.10.2 and 17.10.3 are (by quarters) .7075, .7069, .7062, .7002.[11] The squared values of each of these coefficients are the one-step ahead forecast error variances in investment, by quarter, when we condition on knowledge of the quarter. The squared value of the coefficient .7528 from the (time-invariant) Wold representation formed by not conditioning on quarter is larger, as we would expect.

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[10] For this statement to be true in general requires checking that the first of the "two difficulties" discussed by Hansen and Sargent [1990] is not present.

[11] The zero lag coefficients are equal for both the $\{d_{k,v}\}$ and the $\{g_{k,v}\}$ sequences.

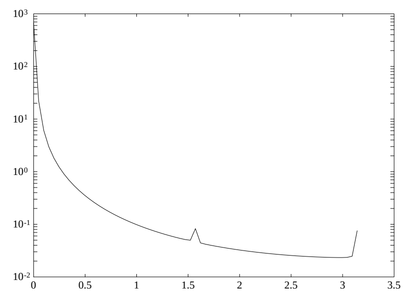

Fig. 17.10.1.a. Spectral density of consumption for a periodic version of Hall's model, calculated by applying the Tiao-Grupe formula.

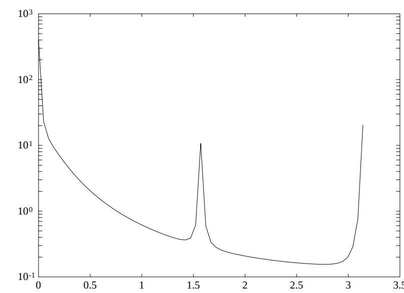

Fig. 17.10.1.b2. Spectral density of investment for a periodic version of Hall's model, calculated by applying the Tiao-Grupe formula.

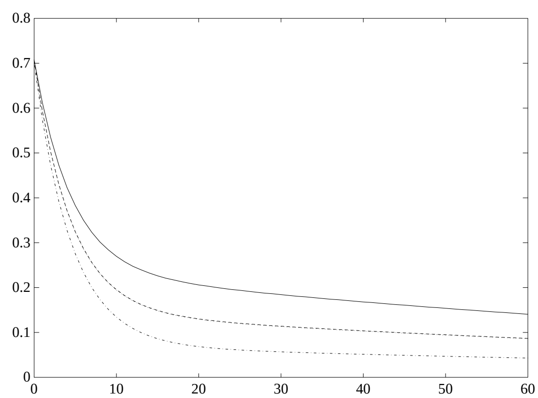

Figure 17.10.2: The response *of* the investment component of $y_{p \cdot t-p+k}$ to an innovation in the endowment shock in a periodic verions of Hall's model.

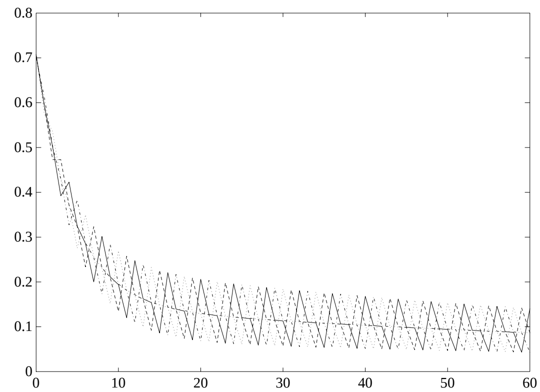

Figure 17.10.3: The response of investment *to* $w_{p\cdot t-p+k}$ in a periodic version of Hall's model.

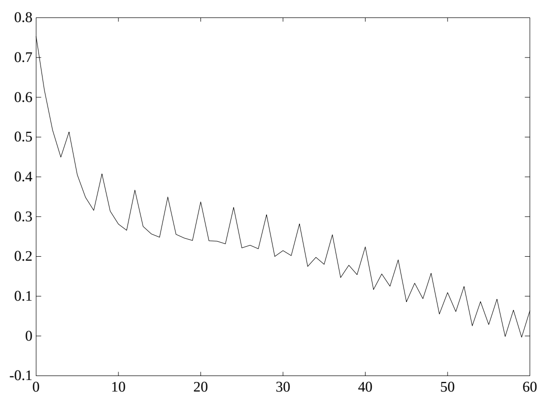

Figure 17.10.4: The moving average coefficients for a Wold moving average representation of investment, calculated by factoring the spectral density of investment given by the Tiao-Grupe formula.

17.11. Periodic Innovations Representations for the Periodic Model

An equilibrium can be represented as

$$x_{t+1} = A^o_{s(t)} x_t + C_{s(t)} w_{t+1} \tag{17.11.1}$$

$$y_t = G_{s(t)} x_t + \varepsilon_{yt} \tag{17.11.2}$$

where y_t is a vector of objects that are linear combinations of the state x_t, plus a white noise measurement error ε_{yt}. The matrix $G_{s(t)}$ is built up from components of the matrices $S_{\cdot,s(t)}$ and $M_{\cdot,s(t)}$ described above. We assume that the measurement error ε_{yt} is orthogonal to the w_{t+1} process, and that it is serially uncorrelated with contemporaneous covariance matrices

$$E\ \varepsilon_{yt}\ \varepsilon'_{yt} = \bar{R}_{s(t)}. \tag{17.11.3}$$

Associated with system (17.11.1) – (17.11.2) is a periodic innovations representation

$$\begin{aligned} \hat{x}_{t+1} &= A^o_{s(t)} \hat{x}_t + K_{s(t)} a_t \\ y_t &= G_{s(t)} \hat{x}_t + a_t \end{aligned} \tag{17.11.4}$$

where $\hat{x}^h_t = E[x_t \mid y_{t-1}, \ldots, y_1, \hat{x}_0], a_t = y_t - E[y_t \mid y_{t-1}, \ldots, y_1, \hat{x}_0]$, and $E a_t a'_t = \Sigma_{s(t)}$. In (17.11.4), $K_{s(t)}$ is the periodic Kalman gain. The matrices $\{\Sigma_{s(t)}, K_{s(t)}\}$ are the p limits of the p convergent subsequences of the Kalman filtering equations:

$$\begin{aligned} \Sigma_{t+1} &= A^o_{s(t)} \Sigma_t A^{o\prime}_{s(t)} + C_{s(t)} C'_{s(t)} \\ &\quad - A^o_{s(t)} \Sigma_t G'_{s(t)} (G_{s(t)} \Sigma_t G'_{s(t)} + \bar{R}_{s(t)})^{-1} G^o_{s(t)} \Sigma_t A'_{s(t)} \\ K_t &= A^o_{s(t)} \Sigma_t G'_{s(t)} (G_{s(t)} \Sigma_t G'_{s(t)} + \bar{R}_{s(t)})^{-1}. \end{aligned} \tag{17.11.5}$$

Because the matrices $[A^o_{s(t)}, C_{s(t)}, G_{s(t)}, \bar{R}_{s(t)}]$ are time-varying, system (17.11.5) will not converge. But because the matrices $[A^o_{s(t)}, C_{s(t)},$ $G_{s(t)}, \bar{R}_{s(t)}]$ periodic, there is a prospect that $\{\Sigma_t, K_t\}_{t=1}^{\infty}$ will consist of p convergent subsequences. This prospect is realized under regularity conditions that typically obtain for our problems.

The innovation covariance matrix associated with (17.11.4) is

$$\begin{aligned} Ea_t a_t' &= \Omega_{s(t)} \\ &= G_{s(t)} \Sigma_{s(t)} G_{s(t)}' + \bar{R}_{s(t)}. \end{aligned} \tag{17.11.6}$$

Given a sample of observations for $\{y_t\}_{t=1}^T$, the likelihood function conditioned in $\hat{x}_0$ can be expressed as

$$\begin{aligned} L^* = & - T\ ln\ 2\pi - .5 \sum_{t=1}^{T} ln \mid \Omega_{s(t)} \mid \\ & - .5 \sum_{t=1}^{t} a_t' \Omega_{s(t)}^{-1} a_t. \end{aligned} \tag{17.11.7}$$

A. A Model of Disguised Periodicity

This appendix characterizes a notion of *hidden periodicity* in a stationary time series, and describes a strategy for detecting its presence in a given vector time series.[12] The notion of hidden periodicity permits realizations of a stochastic process to be aperiodic, but requires that some particular functions of the tail of the stochastic process be periodic. As we shall see, these particular functions are time series averages of skip-sampled versions of the underlying process. It is averaging and skip sampling that causes the hidden periodicity to drop its 0.

Because the apparatus introduced in this appendix is abstract, we begin in section A1 with a heuristic account that is designed to indicate the motivation behind the formal apparatus introduced in section A2.

[12] Breiman [1968] is a useful background for the material presented in this section.

17.13. A1. Two Illustrations of Disguised Periodicity

Let $\{y_t\}$ be an n-dimensional stochastic process that is observed by an econometrician. We can use the Kolmogorov Extension Theorem to construct such a process on a sample space $\Omega = (\mathbf{R}^n)^\infty$, which is the infinite product space formed by taking copies of n-dimensional Euclidean space. A sample point in Ω can be expressed as an infinite-dimensional vector $(r_0, r_1, \ldots)$ where r_j is in $\mathbf{R}^n$ for each j. Probabilities are then defined over the product sigma algebra generated by taking products of the Borel sets of $\mathbf{R}^n$. Armed with this construction, for any $\omega = (r_0, r_1, \ldots)$, let

$$y_t(\omega) = r_t.$$

Thus, $y_t(\omega)$ is simply the t^{th} component of the sample point $\omega = (r_0, r_1, \ldots)$.

An alternative way to represent the process $\{y_t\}$ is in terms of a shift operator S. First, define a random variable $y : \Omega \to \mathbf{R}^n$ as

$$y(\omega) = r_0.$$

Define the *shift* transformation S via:

$$S[(r_0, r_1, r_2, \ldots)] = (r_1, r_2, r_3, \ldots).$$

Then because $y_t(\omega) = r_t$, an alternative representation of y_t is

$$y_t(\omega) = y[S^t(\omega)],$$

where S^t is interpreted as applying S t times in succession.

In thinking about hidden periodicity, the following example is of pedagogical interest.

EXAMPLE 1: Suppose that n is one and that all of the probability on Ω is concentrated onto two points, say a and b. Let a be a sequence of alternating ones and minus ones, beginning with a one. Let b be a similar sequence except that it begins with a minus one. Note that $S(a) = b$, and $S(b) = a$.

There are many probability structures that we can impose on Ω in Example 1. We can assign any probability between zero and one to a and the remaining probability to b. This assignment amounts to initializing the process. Unless we assign probability one half to each point, the resulting process will not be stationary.

In one sense, the initial assignment of the probability is quite irrelevant. The $\{y_t\}$ process is *deterministic* in the sense that given knowledge of the y_0, the entire future of process can be forecast perfectly. Since the initial condition tells the whole story, one might just as well condition on it.

However, from the vantage point of interpreting time averages of the process, the initial assignment of probability one half to each point is convenient. Independently of how we initialize the stochastic process, it obeys a *Law of Large Numbers*. Thus, take any Borel measurable function ϕ mapping Ω into $\mathbf{R}$ and form the sequence $\{z_t\}$ where

$$z_t = \phi(y_t, y_{t+1}, \ldots). \tag{17.13.1}$$

Then

$$\lim_{N\to\infty}(1/N)\sum_{t=0}^{N-1} z_t = (1/2)z(a) + (1/2)z(b) \tag{17.13.2}$$

where $z_0 \equiv z$. The equality holds when the left-side of equation (17.13.2) is evaluated at *either* a or b. When probability one half is assigned to each point, the right side of (17.13.2) can be expressed as Ez, so that we have the usual characterization of the limit points of sample averages as mathematical expectations. With this assignment of probabilities, the process $\{y_t\}$ is both stationary and ergodic.

For this particular example, realizations of both the original process $\{y_t\}$ and the constructed process $\{z_t\}$ are periodic sequences. While realizations of $\{y_t\}$ have period two, realizations of the constructed process $\{z_t\}$ can have period one for particular choices of z_t. For instance, let

$$z_t = y_t + y_{t+1}.$$

Then for either a or $b, \{z_t\}$ is a sequence of zeroes and hence has period one. More generally, for this example the periodicity of $\{z_t\}$ can never exceed two. This follows from the fact that $S^2(a) = a$ and $S^2(b) = b$ implying that $z_{t+2}(\omega) = z_t[S^2(\omega)] = z_t(\omega)$. Since the maximum periodicity of any constructed process $\{z_t\}$ is two, we will say that the periodicity of S is two.

There is something very special about Example 1. Since realizations of the original $\{y_t\}$ process are periodic, every constructed process $\{z_t\}$ turns out to be periodic. In this paper, we are interested in more general circumstances in which the periodicity is *disguised*. We do not wish to confine attention to processes $\{y_t\}$

whose realizations have an *exact* periodicity. The following example embodies what we mean by a hidden periodicity.

EXAMPLE 2: Let $\{w_t\}$ be an n_w-dimensional Gaussian white noise with covariance matrix I. Construct an n_x-dimensional stochastic process $\{x_t\}$ recursively via

$$x_{t+1} = A_t x_t + B_t w_{t+1}$$

where $\{(A_t, B_t)\}$ is a periodic sequence with period two, where A_t is an $(n_x \times n_x)$ matrix, and $\{B_t\}$ is an $(n_x \times n_w)$ matrix. Let $\{y_t\}$ be an n-dimensional process generated as a time-varying function of $\{x_t\}$

$$y_t = f_t(x_t)$$

where $\{f_t\}$ is a sequence of Borel measurable functions mapping n_x-dimensional Euclidean space into n-dimensional Euclidean space. Let f_t be a sequence of period two. Realizations of $\{y_t\}$ will not be periodic, but will inherit a sort of disguised periodicity from $\{(A_t, B_t, f_t)\}$.

There are two aspects of the process $\{y_t\}$ that we have left unspecified, namely x_0 and the periodic sequence $\{(A_t, B_t, f_t)\}$. As in Example 1, there is flexibility in the probabilistic specification of $\{(A_t, B_t, f_t)\}$. One possibility is, in effect, to condition on $\{(A_t, B_t, f_t)\}$, in which case the resulting process $\{y_t\}$ will not, in general, be stationary. Alternatively, we can view $\{(A_t, B_t, f_t)\}$ as emerging from a random draw from two possible sequences indexed by, say, a and b where $[A_t(b), B_t(b), f_t(b)] = [A_{t+1}(a), B_{t+1}(a), f_{t+1}(a)]$ for all t. As in Example 1, if we assign probability one half to each of these outcomes, under a restriction[13] on a matrix that is a function of $A_t(a)$ and $A_t(b)$, we can find an initial specification of x_0 under which $\{y_t\}$ is a stationary stochastic process. In this case, we can apply the Law of Large Numbers for stationary processes both to show that time series averages converge and to obtain a characterization of the limit points.

Suppose that it is possible to complete the specification in Example 2 so that $\{y_t\}$ is stationary. Consider how the hidden periodicity can be characterized and detected. Let ψ be a Borel measurable function mapping $\Omega \to \mathbf{R}$ and form a

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[13] The restriction is that the matrix $\hat{A}$ in equation (4.7) below have eigenvalues that are bounded in modulus by unity.

scalar stochastic process $\{z_t^*\}$ via

$$z_t^* \equiv \psi(y_t, y_{t+1}, \ldots) \quad \text{or} \quad z_t^* = z^*(S^t(\omega)) \tag{17.13.3}$$

We assume that

$$E \mid z^*(\omega) \mid < +\infty,$$

where $z^*(\omega) = z_0^*$.

In contrast to Example 1, when the periodicity is hidden, there is no necessity that $\{z_t^*\}$ form a periodic sequence. Hence we must have a weaker notion of periodicity, if the S implied by Example 2 is to be classified as periodic with period 2. A workable notion of hidden periodicity can be formulated in terms of a reduced class of constructed processes. Given an integer $j \geq 1$ and given ψ, define $\phi : \Omega \to \mathbf{R}$, via $\phi(y_t, y_{t+1} \ldots) = z_t$ where

$$z_t \equiv \lim_{N\to\infty} (1/N) \sum_{\tau=0}^{N-1} z_{t+\tau\cdot j}^*, \tag{17.13.4}$$

and where the right side of (17.13.4) is defined as an almost sure limit. Note that the process $\{z_t\}$ is constructed by taking time series averages of *skip samples* of the process $\{z_t^*\}$ with skip interval j. Notice that z_t depends only on the tail of the stochastic process $\{y_t\}$. It follows by construction that z_t is a periodic process with a period *not exceeding* j. The time series averages of skip samples will reveal the hidden periodicity. The idea is to compute (17.13.4) for $j = 2, 3, 4, \ldots$, and then to determine the period $\hat{p}$ of this sequence for each j. Thus, in example 2, it will turn out that for $j = 1, 3, 5, \ldots$ the number $\hat{p}$ is *one*. For $j = 2, 4, 6, 8, \ldots$, the number $\hat{p}$ will turn out to be 2. We shall define the hidden periodicity p as the maximum of these numbers $\hat{p}$ over $j = 1, 2, 3, \ldots$, where the maximum is also understood to be taken over a class of "test functions" ψ.

Thus, the notion of hidden periodicity in a stochastic process that we shall use is the periodicity to be found in time series averages of skip sampled versions of the data. In the next subsection, we develop these ideas formally, and define hidden periodicity precisely in terms of the properties of the shift operator S and its iterates.

17.14. A2. Mathematical Formulation of Disguised Periodicity

We now use the familiar formalism for stationary stochastic processes.[14] As in the previous subsection, let (Ω, F, Pr) denote the underlying probability space, and let S be a measurable, measure-preserving transformation mapping Ω into itself.

DEFINITION 1: A transformation S is measure preserving if $Pr(f) = Pr(S^{-1}f)$ for all $f \in F$.

Let $\mathcal{I}$ be the collection of invariant sets of the transformation S.

DEFINITION 2: $f \in F$ is an invariant set of S if $S^{-1}(f) = f$.

The collection $\mathcal{I}$ turns out to be a sigma algebra of events (see Breiman), so expectations conditioned on $\mathcal{I}$ are well defined. The invariant events of the transformation S given in example 1 are the null set and any set containing $\{a, b\}$.

DEFINITION 3: S is ergodic if all invariant events have probability zero or one.

Notice that S in example 1 is ergodic.

Let $\mathcal{L}$ be the space of random variables with finite absolute first moments, and let $\mathcal{M}$ be the subspace of $\mathcal{L}$ consisting of the random variables that are $\mathcal{I}$ measurable. The expectation operator $E(\cdot \mid \mathcal{I})$ maps $\mathcal{L}$ into $\mathcal{M}$. Throughout this section, we use the common convention that equality between random variables is interpreted formally as equality with probability one. Hence the equivalence class of random variables in $\mathcal{L}$ that are equal almost surely are treated as one element. Similarly, for a random variable to be in $\mathcal{M}$, it suffices for it to be in $\mathcal{L}$ and to be equal almost surely to a random variable that is measurable with respect to

$\mathcal{I}$. When S is ergodic, $\mathcal{M}$ contains only random variables that are constant almost surely.

A transformation S that is measure-preserving can be used to construct processes that are strictly stationary. Let z be a random variable in $\mathcal{L}$, and construct

$$z_t(\omega) \equiv z[S^t(\omega)]. \qquad (17.14.1)$$

[14] See Breiman [1968, chapter 6].

Then $\{z_t\}$ is strictly stationary and hence obeys a Law of Large Numbers. The limit point of the time series averages is given by $E(z|\mathcal{I})$. A $z \in \mathcal{L}$ has two interpretations. First, it indexes a stochastic process via (17.14.1); and second it denotes the time zero component of that stochastic process.

Our purpose is to define a notion of *periodicity* for the transformation S. Suppose there exists a random variable z such that the realizations of the resulting process $\{z_t\}$ are periodic. That is, for some j the resulting process satisfies:

$$z_{t+j} = z_t \quad \text{for all} \quad t \geq 0. \tag{17.14.2}$$

The fact that (17.14.2) holds for a particular stochastic process is informative about the periodicity of S but falls short of determining the periodicity of S. Notice that one can always find a random variable z such that (17.14.2) is satisfied for $j = 1$. In particular, let z be constant over states of the world. Since S is measure-preserving, $z_t = z$ for all t. Heuristically, we shall define the periodicity of S by forming a large set of periodic stochastic processes defined as in (17.14.1) and satisfying (17.14.2) for some j, and then calling the periodicity S the maximum j over these processes. Notice that all transformations S have periodicity of at least one.

To define formally the periodicity of S, we investigate the collection of invariant events of integer powers of the transformation S. Evidently, if S is measure-preserving, then S^j is measure-preserving for any positive integer j. We can think of S^j as corresponding to *skip-sampling* every j time periods. Let $\mathcal{I}^j$ denote the collection of invariant events of S^j, and let $\mathcal{M}^j$ denote the corresponding subspace of $\mathcal{L}$ of random variables that are $\mathcal{I}^j$ measurable. Any invariant event of S is also an invariant event of S^j. Consequently $\mathcal{M} \subset \mathcal{M}^j$. The converse is not true, however. Consider example 1. Note that $S^2(a) = a$ and $S^2(b) = b$. Consequently, $\{a\}$ and $\{b\}$ are invariant events of S^2 but not of S. In this case $\mathcal{I}^2 = F$. When $\mathcal{M}$ consists only of random variables that have the same values on a and $b, \mathcal{M}^2 = \mathcal{L}$. Processes that are generated (indexed) by elements of $\mathcal{M}$ are constant over time and hence have period one. On the other hand, processes generated by elements of $\mathcal{M}^2$ can oscillate with period two.

It of interest to obtain a characterization of $\mathcal{M}^j$ that applies more generally.

LEMMA 1: For any $z \in \mathcal{M}^j, z_{t+j} = z_t$ for all $t \geq 0$. Conversely, for any $z \in \mathcal{L}$ such that $z_t = z_{t+j}$ for all $t \geq 0, z_t \in \mathcal{M}^j$ for all $t \geq 0$.

Proof: Suppose that $z \in \mathcal{M}^j$. Then $S^{-j}(\{z \in b\}) = \{z \in b\}$ for any Borel set b of $\mathbf{R}$. Note that $S^{-j}(\{z \in b\}) = \{z_j \in b\}$. Consequently for any Borel set b, $\{z \in b\} = \{z_j \in b\}$. Equivalently, $z_j = z$. Repeating this same argument, it follows that $z = z_{\tau \cdot j}$ for any positive integer τ.

Recall that S is measure-preserving, as is S^t. Consequently, for any Borel set b,

$$Pr(\{z_t \in b\} \cap \{z_{t+\tau \cdot j} \in b\}) = Pr(\{z \in b\} \cap \{z_{\tau \cdot j} \in b\}),$$

$Pr(\{z_t \in b\}) = Pr(\{z \in b\})$, and $Pr(\{z_{t+\tau \cdot j} \in b\}) = Pr(\{z_{\tau \cdot j} \in b\})$. Since $z = z_{\tau \cdot j}$, it follows that $Pr\{z_t = z_{t+\tau \cdot j}\} = 1$ for any positive integer τ.

Next consider the converse. Suppose that $z \in \mathcal{L}$ such that $z_t = z_{t+j}$ for all $t \geq 0$. It remains to show that $z_t \in \mathcal{M}^j$. The sequence of time series averages

$$\{(1/N) \sum_{\tau=0}^{N-1} z_{t+\tau \cdot j}\}$$

converges almost surely to z_t as well as to $E(z_t \mid \mathcal{I}^j)$. Therefore, $Pr\{z_t = E(z_t \mid \mathcal{I})\} = 1$. ▮

In light of Lemma 1, processes generated by elements of $\mathcal{M}^j$ are periodic with a period that is no greater than j. We wish to use this insight to construct a formal definition of periodicity. Let $\mathcal{M}^{c\ell}$ be the closed linear space generated by $\{\mathcal{M}^j\}_{j=1}^{\infty}$ where closure is defined using the standard norm on $\mathcal{L}, E(|\cdot|)$.

Definition 4: The transformation S is said to have *periodicity* p if p is the smallest integer such that $\mathcal{M}^p = \mathcal{M}^{c\ell}$. Under this definition, random variables in $\mathcal{M}^{c\ell}$ generate periodic processes with maximum period p. Applying this definition to the transformation S given in example 1, we verify that S has period 2.

Next we describe an alternative way to deduce the periodicity of S. Mimicking the previous logic, we can show that for any positive integer j,

$$\mathcal{M}^j \subset \mathcal{M}^{\tau \cdot j} \text{ for } \tau = 1, 2, \ldots. \tag{17.14.3}$$

It turns out that if $\subset$ in (17.14.3) can be replaced by $=$, the period of S is no greater than j, and in fact j must be an integer multiple of the actual periodicity p. In other words, once skip-sampling reaches a point where further sampling fails to increase the collection of invariant events, this point is an integer multiple of the periodicity of S.

Lemma 2: Let j be any positive integer such that $\mathcal{M}^j = \mathcal{M}^{j\cdot\tau}$ for $\tau = 1, 2, \ldots$. Then $\mathcal{M}^{c\ell} = \mathcal{M}^j$ and S has periodicity p where $j = \ell \cdot p$ for some positive integer ℓ.

Proof: First we show that $\mathcal{M}^{c\ell} = \mathcal{M}^j$. Suppose to the contrary that there is some random variable in $\mathcal{M}^{c\ell}$ that is not in $\mathcal{M}^j$. Since $\mathcal{M}^j$ is closed and random variables in $\mathcal{M}^{c\ell}$ are limit points of sequences of random variables in $\bigcup \mathcal{M}^\tau$, there exists a positive integer τ and a random variable z such that z is in $\mathcal{M}^\tau$ but not in $\mathcal{M}^j$. However, $\mathcal{M}^{\tau\cdot j} = \mathcal{M}^j$ by assumption, which is a contradiction. Therefore $\mathcal{M}^{c\ell} = \mathcal{M}^j$ and $p \leq j$.

It remains to show that $j = p\cdot\ell$ for some integer ℓ. Note that $\mathcal{M}^p = \mathcal{M}^j = \mathcal{M}^{c\ell}$. In light of Lemma 1, random variables in $\mathcal{M}^j$ generate processes with period p and period j. Let ℓ be the smallest integer such that $\ell \cdot p \leq j$ and suppose that $\ell\cdot p < j$. Then $p > k > 0$ where $k \equiv j - \ell\cdot p$. For any $z \in \mathcal{M}^p$, with probability one $z = z_p = z_{\ell\cdot p} = z_{\ell\cdot p+k}$. Since S is measure-preserving, $\{z_t\}$ is periodic with period k. It follows from Lemma 1 that, $z \in \mathcal{M}^k$. Consequently, $\mathcal{M}^k = \mathcal{M}^p$ which is a contradiction. This in turn implies that the period of S is at least $j - \ell \cdot p$, which is a contradiction. Therefore $j = \ell \cdot p$. ∎

An implication of Lemma 2 is that processes generated by random variables in $\mathcal{M}^{c\ell}$ are periodic with a period equal to j, where $j = \ell \cdot p$ for some integer ℓ. Note that if S has periodicity p, then S^p has periodicity one.

Definition 4 of periodicity can be applied to any S transformation that is measure-preserving. Our interest is in the case in which S is the shift transformation described in section 1a. This transformation is measure-preserving by construction as long as the probability measure induced on Ω comes from a process $\{y_t\}$ that is strictly stationary. When the shift transformation is periodic with period p, we say that the process $\{y_t\}$ has *hidden periodicity* p.

Consider again constructions (17.13.3) and (17.13.4). The processes $\{z_t\}$ constructed via (17.13.4) are periodic by construction and hence it follows from Lemma 1 (or from the Law of Large Numbers for Stationary Processes) that the corresponding random variable z is in $\mathcal{M}^j$. The periodicity of $\{z_t\}$ can, in fact, be less than j. By choosing a sufficiently rich collection of test functions, we can span $\mathcal{M}^j$. Let $\hat{p}(j)$ be the maximum periodicity over such a class of functions. The hidden periodicity p of $\{y_t\}$ is then the supremum of the sequence $\{\hat{p}(j) : j = 1, 2, \ldots\}$. Lemma 2 describes a particular feature of subsequences of $\{\hat{p}(j) : j = 1, 2, \ldots\}$. For instance, for any $j = p \cdot \ell$ for some ℓ, the subsequence $\{\hat{p}(\tau \cdot j) : \tau = 1, 2, \ldots\}$ is constant. Turning this observation around, if one finds a constant subsequence of the form $\{p(\tau \cdot j) : \tau = 1, 2, \ldots\}$, then the hidden periodicity of $\{y_t\}$ must satisfy $j = p \cdot \ell$.

Part IV

Economies as Objects

Chapter 18
Introduction to Objects

This help manual is intended to help students use the MATLAB programs referenced in this book. To that end, it is divided into two main chapters, organized by increasing level of difficulty. This first chapter explains a little about object oriented programming (OOP), and why it's so useful in this context. The second chapter applies these ideas to the construction of an economy object, and offers a more in-depth coverage of all the features of an economy and all the nifty things one can do with it. This second chapter also provides the user with the tools, via examples, to invent new economies to experiment with.

Throughout, actual code and MATLAB file names will be in typewriter font, but references to an object will not be. For example, `economy.m` is in typewriter font, but when an economy is referred to, it is not. Also note that any actual MATLAB file has a short help section at the beginning, which can be accessed by typing `help file_name` at the MATLAB prompt.

18.1. Matlab Objects

For those users who have not encountered OOP before, this subsection goes over some definitions and some examples of what one can and cannot do with objects. Of course, all examples are in the context of MATLAB.

18.1.1. Definitions

We start with some definitions of basic concepts in OOP.

Class

The relevant analogy here is that of a type. A class is a new data type that you define. It includes not only the actual structure of the type, but also the functions that operate on it. So, for example, suppose we define a new class called a slde (short for stochastic linear difference equation) which is a collection of two matrices, and we need a function on it that displays it in a nice way. Then both the constructor `slde.m` and the display function `disp.m` are in the class.

Objects

An object is a run-time value which belongs to some class. If a class is a type, an object is a variable. A variable of a given class is called an instance of that class. For example, suppose we define a class that is a matrix. Then the identity matrix would be an object, or an instance of the matrix class.

Hierarchy

Classes in OOP are arranged in a tree-like hierarchy. A class' *superclass* is the class above it in the tree. The classes below it are *subclasses*. By convention, the root of the tree is called the "Object" class. The semantics of the hierarchy are that any class includes all the properties of its superclasses. In this way the hierarchy is general towards the root and specific towards its leaves. The hierarchy helps add logic to a collection of classes. It also enables similar classes to share properties through inheritance. Superclasses are often referred to as parent classes, and subclasses are often referred to as children.

Inheritance

A subclass *inherits* all of the data and functionality of its parent classes. In particular, a class inherits all of the methods. When an object receives a message, it checks for a corresponding method. If one is found, it is executed. Otherwise the search for a matching method travels up the tree to its parent and so on recursively. This means that a class automatically responds to all the messages of its superclasses. Most OOP languages include controls to limit how the data and methods are inherited. A subclass can also extend beyond the inherited functionality by adding data and defining new methods.

Overriding and Overloading

A class inherits all the methods of its superclasses, but a class can choose to respond to a message in a different way by re-defining a method. When an object receives a message, it checks its own methods before consulting it superclass. If the object's class and its superclass both contain a method for a message, the object's method is used. In other words, the first method found in the hierarchy takes precedence. When a subclass responds to a message in a different way than its superclass does, the subclass is said to have *overriden* its superclass's method—the class overrides and intercepts the message before it gets to the superclass. When a method has more than one definition depending on the context (i.e., it's defined for both a class and its parent), it is said to be overloaded. In MATLAB, most functions, including addition and subtraction, for example, can be overloaded to be class specific. For example, the function `disp.m` is overloaded for each class, so that it displays each class properly.

Fields

A field is a component of a class which is itself a class, or predefined type. For example, if we define a class called foo that consists of an integer and a matrix, then foo has two fields. If we defined a child class called foo_child that additionally has a vector, then foo_child also has two fields. The first is the foo field that is inherited, and the second is the vector field. There are not three fields, as the first two are subsumed into the foo field that foo_child inherits.

18.1.2. Matlab Specifics

There are two main ways in which MATLAB departs from the structure laid out above: accessing fields, and overloading certain functions. Additionally, there is a specified way in which MATLAB checks for methods, which we discuss here, as it may sometimes cause confusion.

Accessing Fields

Generally, the most direct way to access a field of an object is to call it via `foo.fieldname`. When defining new objects, this method works immediately within the functions of a class. However, to be able to do so at the command line, two extra functions are needed in each class directory: `subsref.m` and `subsasgn.m`. The first allows you to reference the value in the field, so if you type `foo_object.fieldname` on the command line, it returns the value contained in the `fieldname` field of object `foo_object`. The latter function, `subsasgn.m`, allows you to reassign a field of an object to be a different value. These two functions should be overloaded for every class. They work by taking an object both as an argument and as the output. This will turn out to be especially handy for reassigning values in an object in the economy class, as we will see in the next chapter.

Overloading Functions

One must be wary about overloading certain functions in MATLAB. For example, we defined our own method `sigma.m` to extract the sigma field from an economy object. However, sometimes it seems to work, and sometimes it seems to not work. We suspect that this is because there is already a sigma function defined by MATLAB, which it does not like to override. In general, it may be wise to try and name a function a different name, such as `sig.m`, rather than overloading. One can always check for taken names by typing `help function_name`; if MATLAB says it cannot find the function, it hasn't been taken.

Calling Functions

MATLAB has a specified protocol for where to seach when a new name is called. In the following order it looks for a: variable, subfunction, private function, or function on the search path. A subfunction is a function that resides in the same file as the calling function; we almost never use these. A private function is one that is in a private directory, and hence is only accessible to files in the directory immediately above it. An example of this is the `solve.m` function in the `@economy/private` directory. The search path is the predefined path along which MATLAB searches, to which we have added paths for `examples/econ` and `clex`. For more information on paths in MATLAB, type `help path`.

18.1.3. How to Define a Matlab Class

Suppose one wants to define a MATLAB class called foo. Then in a folder called `@foo` there must a constructor function called `foo.m`. If one defines a child class called foo_child, then there must be another folder `@foo_child` with a constructor function `foo_child.m`. Note that when constructing a foo object, one must be in a directory such that `@foo` is a subdirectory. The folders `@foo` and `@foo_child` may both be in the same directory.

The constructor function takes inputs and assigns them to fields. Once all fields are assigned, there is a declaration of the class, and the new object is returned. A short example for `@foo/foo.m` follows, where foo has two fields, an integer and a matrix.

Note that it is a good idea to include some comments about what a function does in the function definition file. In Matlab, the first comments section is displayed when `help function_name` is typed. The character for commenting out a line is the percent symbol: %. The help section should include definitions of all variables used, the first line of the function definition, and a short description of what the function does.

```
function f = foo(integer1, matrix2);
% function f = foo(integer1, matrix2);
% This function is the constructor called when a new
foo object % is defined.  It takes two arguments, an
integer and a matrix % and assigns them to the two fields
of a foo object.
foo.integerfield = integer1;
foo.matrixfield = matrix2;
f = class(f, ‘foo’);
```

To create an object called `foo1` using the integer 5 and the matrix `eye(4)`, one simply types at the command line: `foo1 = foo(5, eye(4))`.

Now suppose one wants to define a class called foo_child that is a child of foo. In the same directory as `@foo` create a folder called `@foo_child` that contains the constructor function `foo_child.m`. Suppose that in addition to the two fields in foo, one wants foo_child to have a vector field. The file `@foo_child/foo_child.m` is given for reference..

```
function fc = foo_child(foo1, vector2);
% function fc = foo_child(foo1, vector2);
% This function is the constructor called when a new
foo_child % object is defined.  It takes two arguments,
a foo object % and a vector, and assigns them to the
two fields of a foo_child % object.
foo_child.vectorfield = vector2;
fc = class(fc, ‘foo_child’, foo1);
```

Notice that, since `foo1` is an object of the foo class, foo_child automatically inherits from foo, and the inherited fields are filled with the values from `foo1`.

For a more thorough introduction to defining MATLAB objects, see Chapter 14 of the MATLAB manual "Using MATLAB".

18.2. Summary

In this chapter we have defined some of the basic concepts in object oriented programming: classes, objects, hierarchy, inheritance, overriding and overloading, and fields. We then went on to point out some pitfalls of MATLAB objects: field access and overloading. Finally, very briefly, we went over how to define a MATLAB class.

One can now see why object oriented programming would be a valuable tool in defining and using economies. An economy is defined to be a collection of matrices. These divide naturally into three categories, each of which define the information, technology and preferences structures. Thus one can create three classes representing these structures, and have an economy class inherit from all three. Also, most interesting operations are executed on an economy, and with an economy class, one can simply define functions that operate on an economy object. The exact definitions and functionality of these four classes are discussed in the next chapter.

Chapter 19
Economies as Matlab Objects

19.1. Introduction

We describe in the first few sections the structure of the economy class starting with the structure of the parent classes: information, technology and preferences. Notice that the field names correspond as much as possible with the names of matrices and vectors in the main book. We also describe the functions available to manipulate the various objects. To get more information on any of these functions, type `help function_name` in the MATLAB command window. Also, recall that the information, technology and preference divisions are introduced in Chapter 3 of Hansen and Sargent.

The last section deals with three different ways of working with objects of this class: using the built-in economies, mixing and matching the built-in parent objects (information, technology and preferences) and building a customized economy.

19.2. Parent Classes: Information

19.2.1. Structure

The first parent of the economy class is the information class. An information object contains matrices describing the laws of motion of taste and technology shocks.

$$\begin{aligned} z_{t+1} &= A_{22} z_t + C_2 w_{t+1} \\ b_t &= U_b z_t \\ d_t &= U_d z_t . \end{aligned}$$

An information object has four fields:

`a22:` The matrix A_{22}

`c2`: The matrix C_2

`ub`: The selector matrix U_b, which transforms the process z_t into taste shocks.

`ud`: The selector matrix U_d, which transforms the process z_t into technology shocks.

19.2.2. Functions

CONSTRUCTION: **`information`**, a constructor function that takes as argument the matrices `a22, c2, ub, ud`

19.3. Parent Classes: Technology

19.3.1. Structure

The technology class contains matrices that describe the technology of the economy :

$$\begin{aligned} \Phi_c c_t + \Phi_g g_t + \Phi_i i_t &= \Gamma k_{t-1} + d_t \\ k_t &= \Delta_k k_{t-1} + \Theta_k i_t. \end{aligned}$$

Recall that c_t is consumption, g_t a vector of intermediate goods, i_t investment, k_t capital, and d_t the production shock.

A technology object has six fields :

`phic`: The matrix Φ_c

`phig`: The matrix Φ_g

`phii`: The matrix Φ_i

`gamma`: The matrix Γ

`deltak`: The matrix Δ_k

`thetak`: The matrix Θ_k

19.3.2. Functions

CONSTRUCTION: `technology`, a constructor function which takes as arguments the matrices `phic, phig, phig, gamma, deltak, thetak`

19.4. Parent Classes: Preferences

19.4.1. Structure

The preferences object contains scalars β, σ and matrices $\Delta_h, \Theta_h, \Lambda, \Pi$ that describe the preferences of the representative agent. The household technology is:

$$h_t = \Delta_h h_{t-1} + \Theta_h c_t$$
$$s_t = \Lambda h_{t-1} + \Pi_h c_t$$

and preferences are ordered by

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[(s_t - b_t)^2 + \ell_t^2 \right].$$

Recall that h_t is a household stock of durables, c_t is a consumption, and s_t is services from the stock of durables. The vector b_t is used in the household objective function.

A preferences object has six fields :

`deltah:` The matrix Δ_h

`thetah:` The matrix Θ_h

`lambda:` The matrix Λ

`pihh:` The matrix Π_h

`beta:` The discount factor β

`sigma:` The risk sensitivity σ

19.4.2. Functions

CONSTRUCTION: `preferences`, a constructor function that takes as argument `deltah, thetah, lambda, phih, beta, sigma`

19.5. Child Class: Economy

19.5.1. Structure

The elements of an economy are contained in the parent fields: `information`, `technology`, and `preferences`. An economy as a child object inherits these three fields from its three parents (technology, preferences, and information). In addition, an economy object has several other fields that are not inherited, namely a set of matrices that characterize a competitive equilibrium. These are calculated automatically when an economy object is defined and become fields in the economy object. Furthermore, the function `subsasgn.m` for the economy class has been defined so that whenever one changes the value of a field of an object, the equilibrium is automatically recalculated. We shall illustrate this useful feature below.

Chapters 4 and 6 showed that an equilibrium has the representation

$$\begin{aligned} x_{t+1} &= A^0 x_t + C w_{t+1} \\ y_t &= G x_t. \end{aligned}$$

where $x_t' = \begin{bmatrix} h_{t-1} \\ k_{t-1} \\ z_{t-1} \end{bmatrix}'$, $A^o = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$.

The observables and the shadow prices are in the vector y_t. They are all linear combinations of the state variables in x_t. The coefficients of these linear combinations are in the matrix G, various rows of which were denoted M_j in Chapter 4 for a price of a quantity j.

The fields containing the solutions of the resource allocation problem are the following :

`ao` : The matrix A^0

`endo`: The eigenvalues of the block A_{11} of A^0

`exo`: The eigenvalues of the block A_{22} of A^0

`nnc`: The coordinate of the row of x containing a constant, if any

`c`: The matrix C

`sj, mj`: Quantities and shadow price are stored in sixteen different fields: `sb, sc, sd, sg, sh, si, sk, skl, ss, mc, md, mg, mh, mi, mk, ms`. These correspond with the matrices of analogous names.

For example, `sc` is the S_c of Chapter 4, which multiplies x_t to yield the optimal decision for c_t. And `mc` multiplies x_t to yield the Lagrange multiplier $\mathcal{M}_t^c$ (the shadow price of consumption).

`s_space`: An s_space (state space) object containing the state space representation of the economy (@**Eq**.**sspace**@). To have more information about what a state space object is, type `help s_space` at the MATLAB prompt. This field is useful for computations involving the MATLAB control toolbox.

19.5.2. Fields containing the history of the economy

An economy object contains also the history of the economy, that is an initial condition x_0. and a sequence of shocks $\{w_t\}_{t=0}^T$. (The sequence of shocks can be initialized at a null matrix, and has been in our sample economies. We include it as a potential field because it can be useful for generating simulations.) This information is stored in the following fields:

`hinitial`: The initial condition for the household capital goods

`kinitial`: The initial condition for the capital stock

`zinitial`: The initial condition for the information process

`shocks` : The sequence of shocks in the information process

19.5.3. Functions

19.5.4. Constructing the object and changing parameters

This part will be detailed in section 3 where we will explain what sequence of commands is needed to construct an economy object. For the time being, we simply list the functions :

CONSTRUCTION: `economy`, a constructor function that takes as arguments an information object, a technology object and a preferences object.

DISPLAY: A function `disp` which display the structure of the economy

19.5.5. Analyzing the economy

STEADY STATE: A function `steadst` to compute the steady state of the model. This uses the resource allocation solution fields.

ASSET PRICING:

A function `asset_price` to compute and simulate the price of an asset with payoffs that are linear combinations of the state variable of the economy

A function `riskprem` to compute the risk premium on an asset

A function `sure4j` to compute the prices of a j-period sure claim on consumption

SIMULATION: A function `simulate` to compute and graph time path for the observables and shadow prices.

REOPENING MARKETS: A function `reopening` to compute and graph the time path of prices in markets that reopen every period

IMPULSE RESPONSE: A function `impulse` to compute and graph the impulse response of the observables and shadow price to the shocks hitting the economy

ARMA REPRESENTATION: A function `arma_rep` to compute the arma representation of the

SPECTRAL DENSITY: A function `spect` to compute and graph the spectral densities of the observables and shadow prices.

19.6. Working with economies

19.6.1. The built-in economies

You will find in the directory `examples/econ` a series of script files which build standard economies – most of them described in the book. A typical script file is named *economy_name*`.m`. MATLAB will build the economy when you type *economy_name*`.m` at the command line. The program will create the following variable :

`e`*economy_name*: an economy object

19.6.2. Mixing and matching built-in parent objects

By using the built-in economies, your freedom is very restricted: you cannot set any structural parameters, the only thing you can modify is the history of the economy.

To give more freedom to your experiments, we have constructed some standard structures corresponding to the examples given in chapter 3. You can mix and match them and set *some* of their parameters.

TECHNOLOGY You will find in the directory `examples/tech` functions which create the technologies given as examples in chapter 3. A typical function is named `tech`*j*`.m`. By typing `help tech`*j* you will get a description of the parameters you are free to set. Note that all these parameters have default values. You build a technology by typing *name_of_tech* `= tech`*j*`(`*parameters*`)`

PREFERENCES You will find in the directory `examples/pref` functions which create the preferences given as examples in chapter 3. A typical function is named `pref`*j*`.m`. By typing `help pref`*j* you will get a description of the parameters you are free to set. Note that all these parameters have default values. You build a technology by typing *name_of_pref* `= pref`*j*`(`*parameters*`)`

INFORMATION You will find in the directory `examples/info` functions which create information processes, most of them following the examples given in chapter 2. A typical function is named `info`j`.m`. You will have to specify parameters of an underlying stochastic linear difference equation and extractor matrices `ub` and `ud`. This is a delicate step: you have to make sure that those matrices are comformable in columns with the process z_t and in rows with the consumption services vector (for `ub`) or with the technology shocks vector (for `ud`). Having a look at the script files in `examples/econ` can be useful, although the information constructor is designed to warn you when your matrices are not conformable.

ECONOMY After you have built the three parents objects `info, tech` and `pref` you are ready to create their child, the economy object. To do so you simply type *econ_name* `= economy(i,t,p)` and a new economy will be born.

You will probably immediately want to reset the initial conditions for the h_{-1}, k_{-1}, z_0, which the economy constructor sets at vectors of 1's as their default values. The initial conditions can be assessed from the economy object by typing `econ_name.hinitial`, `econ_name.kinitial`, and `econ_name.zinitial`, respectively. To reset z_0, for example, type `econ_name.zinitial = [ 5 2 0 ]'`. You can also directly reset other objects of any of the three parent objects (preferences, technology, or information), which will then be automatically inherited by the child economy object. We'll describe how to do this soon.

19.6.3. Building your own economy

In the directory `examples/econ` there is a script file `blank.m` which may be useful in building your first economies. Just fill in the blanks (the null matrices) with conformable matrices and run the script file; you'll be ready to experiment with your new economy.

```
% Creates an economy with null matrices everywhere.
% Required dimensions are given in comments.
%%% Technology %%%
deltak=[] ; % n_k by n_k
thetak=[]; % n_k by n_i
phic=[]; % m by n_c
phig=[] ; % m by n_g
phii=[]; % m by n_i
```

```
    gamma=[] ; % m by n_k
    %%% Preferences %%%
    deltah=[]; % n_h by n_h
    lambda=[]; % n_s by n_h
    thetah=[]; % n_h by n_c
    pih=[] ; % n_s by n_c
    beta=; % scalar
    sigma = 0; % scalar
    %%% Information %%%
    a22=[]; % n_z by n_z
    c2=[]; % n_z by n_w
    ud=[]; % n_d by n_z
    ub=[]; % n_b by n_z
    %%% Construction %%%
    iblank = information (a22, c2, ub, ud);
    tblank = technology (phic, phig, phii, gamma, deltak,
thetak);
    pblank = preferences (deltah, thetah, lambda, pih,
beta, sigma);
    eblank1 = economy (iblank, tblank, pblank);
    eblank1.hinitial = [];
    eblank1.kinitial = [];
    eblank1.zinitial = [];
    clear iblank1 tblank1 pblank1;
    clear phi gam sigma beta;
    clear a22 c2 u* phi*;
    clear gamma del* the* lambda pih;
```

19.7. Tutorial

Our object oriented programs are contained in a directory called `hansar1` that contains various subdirectories. Say that you keep the directory hansar1 in the location `c:\projects\hansar1`. After you start MATLAB type `addpath c:\projects\hansar1` then type `startup`. To read one of our existing economies, type for example `clex11`. The object `eclex11` is then created. Type `eclex11` to display it. Then to conserve notation rename the economy object simply `e` by typing `e=eclex11`. To access one of the fields of the economy e, a child object, type `e.j` where `j` is one of the economy fields described above, namely, ao, c, endo, exo, sv, mv, hinitial, kinitial, zinitial, where `v` denotes one of the variables `c, i, h, k`. To assess one of the parent objects, type either `e.information` or `e.technology` or `e.preferences`. To assess one of the fields of one of the parent objects, type either `e.information.j` where `j=a22, c` or `e.preferences.j` where `j= lambda, deltah, pi, thetah,beta, sigma` or `e.technology.j` where `j = deltak, thetak, gamma, phig, phii, phic`.

To reset an element of a parent object, type for example `e.information.a22(2,2) = .4`, a command that sets $A_{22}(2,2) = .4$, and that then recomputes all of the equilibrium objects in the child economy `e`.

Chapter 20
MATLAB Programs

This chapter consists of a manual of MATLAB programs that implement the calculations described in earlier chapters. Many of the programs use programs in MATLAB's Control Toolkit. You should load our programs into a subdirectory of MATLAB, and put this subdirectory on the matlabpath statement in your matlab.bat file.

There is a demonstration facility for some of our programs, which supplies a small course on how to use many of our programs. To use this program, just get into MATLAB, type `hsdemo`, and choose one of the options that the menu offers you.

20.1. Matlab programs

Our ordinary MATLAB programs are available via `ftp` at
<ftp://zia.stanford.edu/pub/~sargent/webdocs/matlab/hansar/hansarprograms.zip>.
Our object oriented programs are available at
<ftp://zia.stanford.edu/pub/~sargent/webdocs/matlab/hansarobjects.zip>

aarma

Purpose:
Creates arma representation for a recursive linear equilibrium model.

Synopsis:
[num,den,p,z]=aarma(ao,c,sy,i)

Description:
The equilibrium is computed by first running solvea. The equilibrium is

$$x_{t+1} = ao\ x_t + c\ w_{t+1}$$

A vector of observables is given by

$$y_t = sy\ x_t,$$

where sy is formed to pick off the described variables. For example, if we want $y_t = [c_t', i_t']$, we set sy=[sc; si]. aarma creates num and den, which pertain to the representation

$$den\ (F) y_t = num\ (F) w_{it}$$

where F is the *forward* shift operator defined by $F y_t = y_{t+1}$. This is an arma representation for the response of y_t to the i-th component of w_t. num(F) and den(F) are each stored with the coefficients being arranged in order of descending powers of F. The poles (zeros of den(F)) are returned in the vector p. The zeros of num(F) for each variable are returned in a column vector z, where each column corresponds to a variable.

References:
[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

aggreg

Purpose:
Computes state space representation of sampled (time aggregated) data.

Synopsis:
[A_r , C_r , aa, bb, cc, dd, V1] = aggreg (A,C,G,D,R).

Description:
The underlying model is

$$
\begin{aligned}
x_{t+1} &= Ax_t + Cw_{t+1} \\
y_t &= Gx_t
\end{aligned}
$$

where w_{t+1} is a martingale difference sequence. Error ridden observations on y are available only every r periods. The state space model for the data is then

$$
\begin{aligned}
x_{s+1} &= A_r x_s + C_r w_{rs+1} \\
y_s &= Gx_s + v_s \\
v_{t+1} &= Dv_s + u_{s+1}
\end{aligned}
$$

where $s = t{\cdot}r, Eu_t u_t' = R, A_r = A^r, C_r = I, Ew_{rt}w_{rt}' = V_r V_r = CC' + ACC'A' + \cdots A^{r-1}CC'A'^{\ r-1}$. The program uses innov to create an innovations representation for the sampled process $\{y_t, t = 0, r, 2r, 3r, \ldots\} = \{y_s, s = 0, 1, 2, \ldots\}$. varma2 can be used to compute an arma representation for the sampled data.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

aimpulse

Purpose:
Computes impulse response function for a recursive linear equilibrium model

Synopsis:
```
[z]=aimpulse(ao,c,sy,ii,ni)
```

Description:
The equilibrium is computed by first running `solvea`. The equilibrium is

$$x_{t+1} = ao\ x_t + c\ w_{t+1}.$$

A vector of observables is given by

$$y_t = sy\ x_t$$

where `sy` is formed to pick off the desired variables. For example, if we want $y_t = [c_t', i_t']'$, we set `sy=[sc;si]`. `aimpulse` computes the impulse response of y_t with respect to component `ii` of y_t for `ni` periods.

References:
[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

asimul

Purpose:

Simulate a recursive linear equilibrium model

Synopsis:

asimul, a script file. The outputs of solvea must be in memory, as must the matrix sy and the integer t_1.

Description:

The equilibrium is

$$x_{t+1} = A^o x_t + C w_{t+1}$$

A vector of observables y_t obeys

$$y_t = sy * x_t,$$

where sy is to be specified by the user. If we want $y_t = (c_t' \; i_t')'$, we would set sy = [sc; si]. asimul computes a simulation of y of length t_1 and stores the output in the matrix y.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

asseta

Purpose:
Computes and simulates asset prices for a recursive equilibrium model.

Synopsis:
asseta is a script file which requires that pay and nt, as well as the output of solvea, reside in memory.

Description:
Run solvea and asimul first. An asset pays out a stream of returns

$$y_t = pay * x_t$$

where pay is a vector and where x_t is governed by the equilibrium law of motion

$$x_{t+1} = A^o x_t + Cw_{t+1}$$

The asset is priced by

$$\text{asset price at } \; t = E_t \sum_{t=0}^{\infty} \beta^t p_{t+j}^t y_{t+j}.$$

The program computes the intertemporal marginal rate of substitution, the pay-off, the asset price, and the gross rate of return on the asset. A similation of these of length nt is stored in y. The program also calculates the prices of claims on sure j-period forward consumption for $j = 1, 2, 5$. A simulation of length nt of these for $j = 1, 2, 5$ are stored in $R1, R2$, and $R5$, respectively.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

assets

Purpose:
Creates matrices and scalars needed to price an asset in a strictly periodic equilibrium model of period p.

Synopsis:
assets is a script file. solves must be run first and its output must be in memory.

Description:
An asset with payoff $pay_t = U_a * x_t$ is to be priced, where x_t is the state vector for a dynamic linear equilibrium model that is periodic with period p. The asset price a_t is given by

$$x_t = [x_t' \mu_{a,s(t)} x_t + \sigma_{a,s(t)}]/[i_j \cdot M_{c,s(t)} x_t].$$

This program computes the matrices $\mu_{a,s(t)}$ and the scalars $\sigma_{a,s(t)}$ for $s(t) = 1, 2, \ldots, p$. These matrices and scalars are stored in memory. To simulate the asset price, use the program assetss.

See also:
simuls, assetss.

assetss

Purpose:

Simulates asset price and term structure of interest rates for a strictly periodic equilibrium model with period p.

Synopsis:

assetss is a script file. The programs solves, simuls, and assets must be run first and their outputs must reside in memory.

Description:

A simulation is constructed for the asset priced in assets. The term structure of interest rates is also computed.

The output of the simulation is returned in the vector y, which equals [mrs, pays, as, ret]. Here mrs is the marginal rate of substitution at time, pays is the payoff of the asset, as is the price of the asset and ret is the return on the asset. The prices of risk free claims on comsumption 1, 2, and 5 periods forward are returned in $R1, R2, R5$, respectively.

See also:

simuls, solves, assets

assetx

Purpose:
Computes and simulates asset prices for a recursive equilibrium model with Gaussian Exponential Quadratic specification.

Synopsis:
assetx is a script file which requires that pay and nt, as well as the output of solvex, reside in memory.

Description:
Run solvex and asimul first. An asset pays out a stream of returns

$$y_t = pay * x_t$$

where pay is a vector and where x_t is governed by the equilibrium law of motion

$$x_{t+1} = A^o x_t + C w_{t+1}$$

The asset is priced by

$$\text{asset price at } \; t = E_t \sum_{t=0}^{\infty} \beta^t p_{t+j}^t y_{t+j} \; .$$

The program computes the intertemporal marginal rate of substitution, the pay-off, the asset price, and the gross rate of return on the asset. A similation of these of length nt is stored in y. The program also calculates the prices of claims on sure j-period forward consumption for $j = 1, 2, 5$. A simulation of length nt of these for $j = 1, 2, 5$ are stored in $R1, R2,$ and $R5$, respectively.

See also:
asseta, solvex

References:
[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

avg

Purpose:

Prepares linear system for analysis of aggregation over time with "integrated" or "summed" data

Synopsis:

$[AA, CC] = avg(A, C, m)$

Description:

The state x_t evolves according to

$$x_{t+1} = Ax_t + Cw_{t+1}$$

Let $z_t = [x_t', x_{t-1}', \ldots, x_{t-m+1}']'$. Then z_t evolves according to

$$z_{t+1} = AA * z_t + CC * w_{t+1}$$

where

$$AA = \begin{bmatrix} A & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}, \ CC = \begin{bmatrix} C \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The program forms AA and CC.

See also:

aggreg

canonpr

Purpose:

Computes canonical representation of preferences.

Synopsis:

```
[lamh, pihh] = canonpr (beta, lamba, pih, deltah, thetah)
```

Description:

The program computes a canonical representation of preferences by solving the auxiliary consumer choice problem, maximize

$$-\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t s_t \cdot s_t$$

subject to

$$\begin{aligned} h_t &= \Delta_h h_{t-1} + \Theta_h c_t \\ s_t &= \Lambda h_{t-1} + \Pi c_t, \end{aligned}$$

h_1 given. The solution is a feedback rule $c_t = -F h_{t-1}$ where $F = (\Pi'\Pi + \beta\Theta + h'P\Theta_h)^{-1}(\beta\Theta_h' P\Delta_h + \Pi'\Lambda)$, and where P is the nonnegative definite P that solves the algebraic Riccati equation for the problem. A canonical $(\hat{\Lambda}, \hat{\Pi})$ is chosen for the equations

$$\begin{aligned} &\hat{\Pi}^{-1}\hat{\Lambda} = F \\ &\hat{\Pi}'\hat{\Pi} = (\Pi'\Pi' + \beta\Theta_h' P\Theta_h). \end{aligned}$$

clex 10, 11, 13, 14, 18, 35, 101c, 101f

Purpose:
Read in matrices defining an economy.

Synopsis:
clex*.m is always a script file.

Description:
Each clex*.m file creates a list of matrices Φ_c, Φ_g, Φ_i, Γ, Δ_k, Φ_k, Δ_h, Φ_h, Γ, Π, A_{22}, U_d, U_b, and U_d that define an economy. The economies are as follows:

clex 10	The Jones-Manuelli examples of chapter 3.
clex 11	Hall's model of chapter 3.
clex 13	Hall's model with higher adjustment costs.
clex 14	Lucas's model of chapter 3.
clex 18	The "seasonal preferences" model of chapter 7.
clex 35	The "heterogeneous agent" example of chapter 6.
clex 101c	The hog model of chapter 8.
clex 101f	The corn-hog model of chapter 8.

compn

Purpose:

compn creates companion matrix.

Synopsis:

$[B] = compn[a]$

Description:

The companion matrix B of the $1 \times n$ row vector a is defined as

$$B = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

disthet

Purpose:
Compute equilibrium of general equilibrium with two types of households, externalities, distorting taxes, and exogenous government expenditures.

Synopsis:
disthet is a script file. All matrices must be in memory.

Description:
disthet computes a competitive equilibrium of a distorted heterogeneous economy. Two types of agents live in an economy with a government. There are externalities. Type i agent's problem is to maximize:

$$E_0 - 0.5\{\sum_{t=0}^{\infty} \beta^t[(s_i(t) - b_i(t)) \cdot (s_i(t) - b_i(t)) + \ell_i(t)^2]\}$$

subject to:

$$\begin{aligned} s_i(t) &= \Lambda_{i1} h_i(t-1) + \Lambda_{i2} * H_1(t-1) + \Lambda_{i3} H_2(t-1) \\ &\quad + \Pi_{i1} c_i(t) + \Pi_{i2} C_1(t) + \Pi_{i3} C_2(t) \\ h_i(t) &= \Delta_{hi} * h_i(t-1) + \Delta_{Hi1} H_1(t-1) + \Delta_{Hi2} H_2(t-1) \\ &\quad + \Theta_{hi} c_i(t) + \Theta_{Hi1} C_1(t) + \Theta_{Hi2} C_2(t) \\ E\{\textstyle\sum_{t=0}^{\infty} &\beta^t[(I+\tau_c)p(t) \cdot c_i(t) - (1-\tau_\ell)w(t)l_i(t) - \alpha(t) \cdot d_i(t) \\ &\quad - f_i * (P_1(t) + P_2(t) - T_i(t)]\}|I_0 - v_0 * k_{0i} = 0 \end{aligned}$$

where $s_i, h_i, \ell_i, c_i, T_i, P_i$ are consumption services, household capital stock, labor, consumption, government transfer of type i agent and firms of type i's profit at t, i=1,2. Capital letters denote aggregate variables. τ_j is tax on j, j=c,l,k,i. Firms of type 1's problem is to maximize expected profit:

$$E \sum_{t=0}^{\infty} \{\beta^t[p(t)[c(t) + E(t)] + q(t) \cdot i(t) - r(t) \cdot k(t-1) - \alpha(t) \cdot d(t) - w(t)\ell(t)]\}$$

subject to:

$$\begin{gathered} \Phi_c(c(t) + E(t)) + \Phi_i i(t) + \Phi_g g(t) = \Gamma_k k(t-1) + \Gamma_K * K(t-1) + d(t) \\ g(t) \cdot g(t) = \ell(t)^2 \end{gathered}$$

where $c(t) = c_1(t) + c_2(t)$, and similarly for $d(t), \ell(t)$. $E(t), g(t)$ are government spending and intermediate goods, respectively. Firms of type 2's problem is to maximize expected profit:

$$E \sum_{t=0}^{\infty} \beta^t [(I - \tau_k) r(t) \cdot k(t-1) - (I + \tau_i) q(t) \cdot i(t)] - v_0 * k_0$$

subject to:

$$k(t) = \Delta_k k(t-1) + \Delta_K K(t-1) + \Theta_k i(t),$$

where $k_0 = k_{01} + k_{02}$. The state vector in this program is defined as $[z(t); z(t); h1(t-1); h2(t-1); k(t-1)]$.

dog

Purpose:
Compute 'mongrel' (i.e., non-Gorman) preference ordering over aggregate consumption for two households.

Synopsis:
function[Deltah,Thetah,Lambdah,Pih,Am3,Bm3,Cm3]=dog(alpha, beta,lambda1, pih1, deltah1,thetah1,lambda2,pih2,deltah2,thetah2, a22,c2,ub1,ub2)

Description:
Computes the canonical mongrel service technology for two households with parameter alpha (the Pareto weight on the first consumer). The mongrel household technology is

$$H(t) = \Delta_h H(t-1) + \Theta_h \, c(t)$$
$$s(t) = \Lambda H(t-1) + \Pi c(t)$$

The mongrel preference shock is given by the series connection of the three state space systems (A1,B1,C1,D1), (AA,BB,GG,HH), $(\Delta_h, \Theta_h, \Lambda, \Pi)$. We calculate a system representation (Am,Bm,Cm,Dm) for the mongrel shock. The mongrel shock is thus described by

$$Z(t+1) = Am2\, Z(t) + Bm3\, w(t+1)$$
$$bb(t) = Cm3\, Z(t).$$

In using this program, it is important to set the initial condition for the state appropriately. The given initial conditions for h01 and h02 are loaded into the SHOCK process, and the initial conditions for the MONGREL h01,h02 are set to zero.
Type [Amm,Bmm,Cmm,Dmm]=minreal(Am,Bm,Cm,Dm) to find minimal realization for preference shock.

doubleo

Purpose:
Computes time invariant Kalman filter or time invariant linear optimal control.

Synopsis:
$[K, S] =$ double (A, C, Q, R)

Description:
The program creates the Kalman filter for the following system:

$$x_{t+1} = Ax_t + e_{t+1}$$
$$y_t = Cx_t + v_t$$

where $Ee_{t+1}e'_{t+1} = Q, Ev_tv'_t = R$, and v_t is orthogonal to e_t for all t and s. Here A is $n \times n$, C is $k \times n$, Q is $n \times n$, and R is $k \times k$. The program creates the observer system

$$\hat{x}_{t+1} = A\hat{x}_t + Ka_t$$
$$y_t = C\hat{x}_t + a_t,$$

where K is the Kalman gain, and $S = E(x_t - \hat{x}_t)(x_t - \hat{x}_t)'$ where $\hat{x}_t = Ex_t \mid y_{t+1}y_{t-2}, \ldots$. Also, $a_t = y_t - Ey_t \mid y_{t-1}, y_{t-2}, \ldots$.

By using duality, the program can be used to solve optimal linear control problems. Let the control problem be to choose a feedback law $u_t = -Fx_t$ to maximize

$$-\sum_{t=0}^{\infty}\{x'_tQx_t + u'_tRu_t\}$$

subject to

$$x_{t+1} = A'x_t + B'u_t,$$

with x_0 given. The optimum control is then given by $F = K'$, where

$$[K, S] = \text{ double } (A, B, Q, R)$$

and where the optimal value function is x'_tSx_t.

The *doubling algorithm* is used to compute the solution.

See also:
mult and double3.

References:
[1] Anderson, B.D.O., and J. Moore, *Optimal Filtering*, 1979, p. 160.

doublex

Purpose:
Solves recursive undiscounted Gaussian Quadratic Exponential control problem.

Synopsis:
$[K, S, ST] =$ doublex (A, C, Q, R, c, sig)

Description:
This program uses the "doubling algorithm" to solve the Riccati matrix difference equations associated with the undiscounted quadratic-Gauusian linear optimal control problems. The control problem has the form

$$S(t) = \max_{u(t)} \{x(t)'Qx(t) + u(t)'Ru(t) + (2/\sigma) \log E_t \exp(\sigma/2) S(x(t+1))\},$$

subject to

$$x(t+1) = A'x(t) + C'u(t) + cw(t+1),$$

where $w(t+1)$ is a Gaussian martingale difference sequence with unit covariance matrix. The program returns the steady state value function in S. The optimal control law is $u(t) = -K' * x(t)$ The program also returns ST, which is the quadratic form in $E_t \exp(sig/2) S(x(t+1))$.

See also:
mult and double and solvex.

References:

[1] Anderson, B.D.O., and J. Moore, *Optimal Filtering*, 1979, p. 160.

[2] Jacobson, D.H. "Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games." *IEEE Transactions on Automatic Control*, **AC-18**, 124–31.

doublej

Purpose:
Computes infinite matrix sums of squares.

Synopsis:
$V =$ double (a_1, b_1)

Description:
The program computes the infinite sum V in

$$V = \sum_{j=0}^{\infty} a_1^j b_1 a^{j\prime},$$

where a_1 and b_1 are each $n \times n$ matrices. The program iterates to convergence on the following *doubling algorithm*, starting from $V_0 = 0$:

$$\begin{aligned} a_{1j} &= a_{1j-1} * a_{1j-1} \\ V_j &= V_{j-1} + a_{1j-1} * V_{j-1} * a_{1j-1}. \end{aligned}$$

The limiting value of V_j is returned in V.

References:

[1] Hansen, Lars P. and Thomas J. Sargent *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

doublej2

Purpose:
Computes infinite matrix sums of squares.

Synopsis:
$[V] =$ *doublej2* (a_1, b_1, a_2, b_2)

Description:
The program computes the infinite sum V in

$$V = \sum_{j=o}^{\infty} a_1^j (b_1 b_2) a_2^j$$

where a_1 and a_2 are each $n \times n$ matrices, b_1 is $n \times k$ and b_2 is $k \times n$. The program iterates to convergence on the following *doubling algorithm*, starting from $V_0 = 0$:

$$a_{1j} = a_{1j-1} * a_{1j-1}$$

$$a_{2j} = a_{2j-1} * a_{2j-1}$$

$$V_j = V_{j-1} + a_{1j-1} V_{j-1} a_{2j-1}.$$

The limit point is returned in V.

References:

[1] Hansen, Lars P. and Thomas J. Sargent *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

double3

Purpose:
Raw doubling algorithm for raising a symplectic matrix to higher and higher powers.

Synopsis:
$[aa, bb, gg] =$ double3 (a, b, g)

Description:
The algorithm iterates to convergence of g_j in the following recursions:

$$\begin{aligned} a_{j+1} &= a_j (I + b_j g_j)^{-1} g_j \\ g_{j+1} &= g_j + a'_j g_j (I + b_j g_j)^{-1} a_j \quad , \\ b_{j+1} &= b_j + a_j (I + b_j g_j)^{-1} b_j a'_j \end{aligned}$$

where a_j, b_j, g_j are each $n \times n$ matrices. If we let E_j, be the symplectic matrix

$$\begin{bmatrix} a_j^{-1} & a_j^{-1} b_j \\ g_j a_j^{-1} & a'_j + g_j a_j^{-1} b_j \end{bmatrix}$$

then $E_j = (E_0)^{2^j}$.

References:
[1] Anderson, B.D.O., and J. Moore, *Optimal Filtering*, 1979, p. 160.
[2] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

heter

Purpose:
Computes allocation to an individual who lives within a recursive linear equilibrium model.

Synopsis:
heter is a script file. The program solvea must be run first, and its inputs and outputs must be in memory. The matrices U_d^c and U_b^i, and the scalars $k0i, h0i$, and $tol > 0$ must all be in memory.

Description:
The consumer maximizes

$$-(\frac{1}{2})E\sum_{t=o}^{\infty}\beta^t[(s_t^i - b_t^i)\cdot(s_t^i - b_t^i) + \ell_t^2], \quad 0 < \beta < 1$$

subject to

$$\begin{aligned}
s_t^i &= \Lambda h_{t-1}^i + \Pi c_t^i \\
h_t^i &= \Delta_h h_{t-1}^i + \Theta_h c_t^i \\
&E\sum_{t=0}^{\infty}\beta^t p_t^0 c_t^i \mid I_o = E\sum_{t=0}^{\infty}\beta^t(w_t^0\ell_t^i + \alpha_t^0 d_t^i) \mid I_o \\
&+ v_0 k_{-1}^i \\
b_t^i &= U_b^i z_t \\
d_t^i &= U_d^i z_t
\end{aligned}$$

where $k_{-1}^i = k0i, h_{-1}^i = h0i$ are parameters to be fed in. The matrices $U_d^i = udi$ and $U_b^i = ubi$ must also be fed in. The

parameter $tol > 0$ must be fed in. The program computes the optimal solution for consumer i in the form $c_t^i = S_c^i x_t, h_t^i = S_h^i x_t, s_t^i = S_s^i x_t, b_t^i = S_b^i x_t, d_t^i = S_d^i x_t$, where x_t is the state variable of the economy augmented by the state variables k_{t-1}^i, h_{t-1}^i ideosyncratic to the individual. The program also computes the aggregate allocations $c_t = S_c^a x_t, h_t = S_h^a x_t$, and so on. The individual allocations are determined by the matrices $sci, shi, \ldots$, which are placed in memory. The aggregate allocation are placed in the matrices $sca, sha, \ldots$, which are placed in memory.

Algorithm:

See Hansen and Sargent, Chapter 6

See also:

simulh

innov

Purpose:
Compute the innovations representation for a recursive linear model whose observations are corrupted by first-order serially correlated measurement errors.

Synopsis:
$[aa, bb, cc, dd, V_1] =$ innov (ao, c, sy, D, R)

Description:
The model is assumed to have the state space representation

$$\begin{aligned} x_{t+1} &= a^o x_t + c w_{t+1} \\ y_t &= S_y x_t + e_{t+1} \end{aligned}$$

where w_t is a white noise with $E w_t {w_t}' = I$ and e_t is a measurement error process governed by

$$e_{t+1} = D e_t + \eta_{t+1}$$

where η_t is a white noise with contemporaneous covariance matrix R. The matrices R and D must each be $m \times m$ where $[m, n] = size(S_y)$. The program forms the innovations representation for y_t,

$$\begin{aligned} \hat{z}_{t+1} &= aa\hat{z}_t + bbu_t \\ y_t &= cc\hat{z}_t + ddu_t \end{aligned}$$

where $u_t = y_{t+1} - E[y_{t+1} \mid y_t, y_{t-1}, \ldots]$, and $E u_t u_t' = V_1$.

Algorithm:

$$aa = \begin{bmatrix} ao & 0 \\ GG & D \end{bmatrix}, bb = \begin{bmatrix} k1 \\ I \end{bmatrix}$$
$$cc = [0 \quad I], dd = [0],$$

where $k1$ is the Kalman gain associated with the Kalman filter for the original system.

References:
[1] Sargent, Thomas, "Two Models of Measurements and the Investment Accelerator," *Journal of Political Economy*, April 1989.

mult

Purpose:
Multiplies two symplectic matrices.

Synopsis:
$[a, b, g] =$ *mult* $(a_1, b_1, g_1, a_2, b_2, g_2)$

Description:
A symplectic matrix E_i is represented in the form

$$(*) \qquad E_i = \begin{bmatrix} a_i^{-1} & a_i^{-1} b_i \\ g_i a_i^{-1} & a_i' + g_i a_i^{-1} b_i \end{bmatrix}.$$

We desire to form $E = E_2 E_1$. We can compute

$$\begin{aligned} a &= a_2 (I + b_1 g_2)^{-1} g_1 \\ g &= g_1 + a_1' g_2 (I + b_1 g_2)^{-1} a_1 \\ b &= b_2 + a_2 (I + b_1 g_2)^{-1} b_1 a_2', \end{aligned}$$

and represent E as in representation $(*)$.

References:
[1] Anderson, B.D.O., and J. Moore, *Optimal Filtering*, 1979, p. 160.

seasla

Purpose:
Creates a time invariant representation for a strictly periodic, time varying linear equilibrium model

Synopsis:
seasla is a script file, which requires that the output of simuls reside in memory.

Description:
Let x_t be the state vector for a strictly periodic seasonal process of period p. Let $X_t' = [x_{pt-p+1}', x_{pt-p+2}', \ldots, x_{pt}']$. The law of motion for X_t is

$$X_{t+1} = \hat{A} X_t + \hat{C} W_{t+1}$$

where W_{t+1} is a vector white noise and $\hat{A}$ and $\hat{C}$ are defined as simuls. The spectral density matrix of the X_t process is given by $S(z) = (I - \hat{A}Z)^{-1} \hat{C}\hat{C}' (I - \hat{A}Z)^{-1\prime}$. Embedded in the spectral density matrix of the stacked process X_t are the spectral density matrices $s_1(z), s_2(z), \ldots, s_p(z)$ for the periodic process $\{x_t\}$. The process x_t whose spectral density is defined to be $s(z) = p^{-1} \sum_{k=1}^{p} s_k(z)$. It can be shown that

$$(*) \qquad s(z) = p^{-1} Q(z)(I - \hat{A}z^p)^{-1} \hat{C}\hat{C}'(I - \hat{A}z^{-p})^{-1\prime},$$

where $Q(z) = [I \; zI \cdots \; z^{p-1}I]$. A state space representation for a process x_t with spectral density matrix (*) is

$$(\dagger) \qquad \begin{aligned} Y_{t+p} &= \hat{A} Y_t + \hat{C} V_{t+p} \\ x_t &= p^{-.5} Q(L) Y_t \end{aligned}$$

where V_t is a vector white noise with identity covariance matrix.

The program seasla creates the spectral density for a univariate process that is a linear function of the state. Let the process be $c_t = sc_{s(t)} x_t$. We form the time invariant, averaged process, as $\tilde{c}_t$ which is determined by the system

$$(\ddagger) \qquad \begin{aligned} Y_{t+p} &= \hat{A} Y_t + \hat{C} V_{t+p} \\ c_t &= p^{-1} Q_c(L) Y_t \end{aligned}$$

where $Q_c(z) = [sc_1 \vdots sc_2 z \vdots \cdots \vdots sc_p z^{p-1}]$.

The program maps into a first-order system, then deduces the impulse response of c_t with respect to innovations V_t corresponding to representation (‡). This is stored in z_1. The program also uses the Kalman filter to obtain an innovations representation corresponding to (‡), and returns the impulse response of c_t with respect to the innovation in c_t in the vector z_2.

See also:

simuls, assets, assetss

seas1

Purpose:
To aid in creating the matrices that define a periodic recursive linear equilibrium model.

Synopsis:
seas1.m is a script file.

Description:
seas1 creates matrices $\Phi_{cs(t)}$, $\Phi_{gs(t)}$, $\Phi_{is(t)}$, $\Gamma_{s(t)}$, $\Delta_{ks(t)}$, $\Delta_{hs(t)}$, $A_{22s(t)}$, $C_{s(t)}$, $\Phi_{ks(t)}$, $\Phi_{hs(t)}$, $\Lambda_{s(t)}$, and $\Pi_{s(t)}$ that are needed to define a periodic linear recursive model. It creates *time invariant* versions of these matrices as follows. It first reads in $\Phi_c, \Phi_i, \Phi_y, \Gamma,$
$\Delta_k, \Delta_h, A_{22}, C, \Phi_k, \Phi_h, \Lambda_1$ and Π for a time invariant economy. One of our clex*.m files can be used to read in such matrices. Then seas1 simply sets the matrices $\Phi_{cs(t)} = \Phi_c$, and so on.

To create a periodic model, the user may find it useful to run seas1 first, and then to modify the resulting time invariant setup, rather than building up all of the matrices from scratch. In a typical periodic model, many of the matrices may in fact be time invariant.

See also:
solves.m

simpulse

Purpose:
Creates different impulse response functions for a periodic linear equilibrium model.

Synopsis:
simpulse is a script file. solves must be run first, and its outputs must be in memory.

Description:
A stacked version of a periodic model has state space form

$$(\dagger) \qquad \begin{aligned} X_{t+1} &= \hat{A} X_t + \hat{C} W_{t+1} \\ Y_t &= H X_t, \end{aligned}$$

where $X_t' = [x_{tp-p+1}',\, x_{tp-p+2}', \ldots, x_{tp}']$, $Y_t' = [y_{tp-p+1}',\, \ldots, y_{tp}']$, $W_t' = [w_{tp-p+1}',$ $\ldots, w_{tp}']$ and where $\hat{A}, \hat{C}$, are as defined in simuls.

This program first uses dimpulse to compute the impulse response function of the stacked system $(\dagger)$. From this impulse response function, it forms two impulse response functions for the periodic process y_t. First, it computes $\{d_{k,v}\}$ in the representation

$$y_{pt-p+k} = \sum_{v=0}^{\infty} d_{k,v} w_{pt-p+k-v}.$$

This is the response of y_{pt-p+k} (i.e., y_t in a particular season) to lagged w's. Second, the program computes the $\{h_{k,v}\}$ that give the response of $\{y_t\}$ *to* w_{pt-p+k} (i.e. an innovation in a particular season). The value of p must be in memory. The program prompts the user for the index of the innovation whose response functions are to be computed.

See also:
solves, simuls

simulh

Purpose:
Simulates allocation of individual i who lives within a recursive linear equilibrium model.

Synopsis:
simulh is a script file. heter must be run first and its output must be in memory.

Description:
The user is asked to specify which series he wants to simulate; e.g., to simulate the consumption allocation to agent i and the aggregate consumption allocation, respond $[sci; sca]$.

See also:
heter

simulhet

Purpose:
Simulates heterogeneous agent economy.

Synopsis:
simulhet is a script file, not a function.

Description:
Simulates the prices and quantities for a recursive linear equilibrium model with non-Gorman heterogeneity. solvehet must be run first and its output must be in memory. To simulate the individual consumption allocations, set sy=[sc1;sc2] when asked what series you want to simulate. To simulate the individual consumption service allocations, set sy= [ss1;ss2].

See also:
solvea and heter.

simuls

Purpose:
To simulate a strictly periodic recursive linear model.

Synopsis:
simuls is a script file, which requires that all of the outputs of solves be in memory. simuls prompts the user for the number of "years" to simulate.

Description:
simuls creates a simulation of the state vector x_t for a strictly periodic model of period p. The stacked state vector X_t is formed, where $X_t' = [x_{pt-p+1}', x_{pt-p+2}', \dots, x_{p_t}']$. The law of motion for X_t is

$$X_{t+1} = \hat{A} X_t + \hat{C} W_{t+1}$$

where $\hat{A} = D^{-1}F, \hat{C} = D^{-1}G$, where

$$D = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ -A_1^0 & I & 0 & \cdots & 0 & 0 \\ 0 & -A_2^0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -A_{p-1}^0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & A_p^0 \\ 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} C_p & 0 & 0 & \cdots & 0 \\ 0 & C_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_{p-1} \end{bmatrix},$$

and where $W_t' = [w_{pt+1}', w_{pt+2}', \dots, w_{pt+p}']'$.

The output of simuls is returned in the matrix X. The matrix X is arranged as follows:

$$X = \begin{bmatrix} x_1' & x_2' & \cdots & x_p' \\ x_{p+1}' & x_{p+2}' & \cdots & x_{2p}' \\ \vdots & \vdots & \ddots & \vdots \\ x_{Tp+1}' & x_{Tp+2}' & \cdots & x_{Tp+p}' \end{bmatrix}$$

where T is the number of "years" specified by the user.

To simulate c_t, i_t, etc., the user can write a program to put the relevant linear combinations off X. Alternatively, the user can edit the files simulc, simulk, simuli, simulg, simulb, or simuld.

solvea

Purpose:

Computes solution of recursive linear equilibrium model

Synopsis:

solvea is a script file. The matrices $A_{22}, C_2, U_d, U_b, \Phi_c, \Phi_g, \Phi_i, \Gamma, \Delta_k, \Theta_k, \Delta_h, \Theta_h, \Lambda$, and Π and the scalar β must be in memory.

Description:

The social planning problem is to maximize

$$-(\frac{1}{2})E\sum_{t=0}^{\infty}\beta^t[(s_t-b_t)\cdot(s_t-b_t)+\ell_t^2], \quad 0<\beta<1$$

subject to

$$\begin{aligned}
\Phi_c c_t + \Phi_g g_t + \Phi_i i_t &= \Gamma k_{t-1} + d_t \\
g_t \cdot g_t &= \ell_t^2 \\
k_t &= \Delta_k k_{t-1} + \Theta_k i_t \\
h_t &= \Delta_h h_{t-1} + \Theta_h c_t \\
s_t &= \Lambda h_{t-1} + \Pi c_t \\
z_{t+1} &= A_{22} z_t + C_2 w_{t+1} \\
b_t &= U_b z_t, \quad d_t = U_d z_t
\end{aligned}$$

Here s_t is consumption services, b_t a stochastic bliss process, ℓ_t is labor services, c_t is consumption rates, g_t is "intermediate goods", i_t is investment goods, d_t is an endowment shock process, k_t is physical capital, h_t is household capital, z_t is a vector of exogenous information variables, and w_{t+1} is a martingale difference sequence. Each of these is a vector, except for ℓ_t, which is scalar. Let $x_t = [h'_{t-1}, k'_{t-1}, z'_t]'$. The program computes the solution of the social planning problem in the form

$$\begin{aligned}
x_{t+1} &= A^o x_t + C w_{t+1} \\
k_t &= S_k x_t \ , g_t = S_q x_t \\
h_t &= S_h x_t \ , i_t = S_i x_t \\
c_t &= S_c x_t \ , b_t = S_b x_t \\
s_t &= S_s x_t \ , d_t = S_d x_t
\end{aligned}$$

The program also computes Lagrange multipliers $\mu_t^j = M_j x_t$ for variable $j = k, c, h, s, i$. The program computes and leaves in memory A^o, C, S_j (for $j = k, h, c, s, g, i, b$, and d) and M_j (for $j = k, c, h, s$, and i).

Algorithm:

The social planning problem is formulated and solved as an optimal linear regulator problem.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

solvdist

Purpose:

Computes equilibrium of representative agent economy with distorting taxes, exogenous govenment expenditures, and externalities.

Synopsis:

solvdist is a script file.

Description:

solvdist, a script file (not a function), finds a competitive equilibrium for a representative agent economy with distortions. Households maximize:

$$E_0 - .5 \sum \beta^t[(s(t)-b(t)) \cdot (s(t)-b(t)) + \ell(t)^2]$$

subject to

$$\begin{aligned} g(t)\cdot g(t) &= \ell(t)^2 \\ z(t+1) &= a22 * z(t) + c2 * w(t+1) \\ b(t) &= ub * z(t), \quad d(t) = ud * z(t), \quad E(t) = ue * z(t) \\ h(t) &= \Delta_h h(t-1) + \Delta_H H(t-1) + \Theta_h c(t) + \theta_H C(t) \\ s(t) &= \Lambda_h h(t-1) + \Lambda_H H(t-1) + \Pi_c c(t) + \Pi_C C(t) \\ k(t) &= \Delta_k k(t-1) + \Theta_k i(t) \end{aligned}$$

$$\sum_{t=0}^{\infty} \beta^t[(I+\tau_c)p(t)\cdot c(t) + (I+\tau_i)q(t)\cdot i(t) - (1-\tau_\ell)w(t)\cdot g(t)$$
$$-\alpha(t)\cdot(d(t)+\gamma_K K(t-1)) - (I-\tau_k)r(t)\cdot k(t-1) - T(t)] = 0$$

Firms maximize profits:

$$E_0 \sum_{t=0}^{\infty} \beta^t[p(t)\cdot(c(t)+E(t)) + q(t)\cdot i(t) - r(t)\cdot k(t-1) - \alpha(t)\cdot d(t) - w(t) * g(t)]$$

subject to

$$g(t)\cdot g(t) = \ell(t)^2$$
$$\Phi_c(c(t)+E(t)) + \Phi_g g(t) + \Phi_i i(t) = \Gamma_k k(t-1) + \Gamma_K K(t-1) + d(t)$$

Where $x(t) = [h(t-1)', k(t-1)', z(t)']'$, the solution of the problem is

$$x(t+1) = ao * x(t) + c * w(t+1)$$
$$j(t) = sj * x(t),$$

where $j = k, h, c, e, s, g, i, b, d, p, q, w, r, \alpha$. The program also computes the household's Lagrange multipliers $\mu_j = mj\ x(t)$ for $j = k, h, s, z$. (μ_0 is set to 1.)

solvehet

Purpose:

Solves Pareto problem for two-agent, non-Gorman preferences.

Description:

solvehet solves the pareto problem for two agents with heterogeneous household production functions, i.e. to maximize

$$E\sum_{t=0}^{\infty}\beta^t - .5\alpha([(s_1(t)-b_1(t))\cdot(s_1(t)-b_1(t))+\ell_1(t)^2]) \\ +(1-\alpha)([(s_2(t)-b_2(t)).(s_2(t)-b_2(t))+\ell_2(t)^2])$$

subject to

$$\begin{aligned}
\Phi_c c(t)+\Phi_g g(t)+\Phi_i i(t) &= \Gamma k(t-1)+d(t) \\
g_i(t)\cdot g_i(t) &= \ell_i(t)^2, \quad i=1,2 \\
g_1(t)+g_2(t) &= g(t) \\
k(t) &= \Delta_k k(t-1)+\Theta_k i(t) \\
h_i(t) &= \Delta_{hi} h_i(t-1)+\Theta_{hi} c_i(t) \\
s_i(t) &= \Lambda_i * h_i(t-1)+\Pi_{hi} c_i(t), \quad i=1,2 \\
c_1(t)+c_2(t) &= c(t) \\
z(t+1) &= a22*z(t)+c2*w(t+1) \\
i(t) &= ubi*z(t), \quad i=1,2; d(t)=ud*z(t)
\end{aligned}$$

The state vector is x(t) = [h1(t-1)',h2(t-1)',k(t-1)',z(t)]'. The control vector is u(t) = [c1(t)',i(t)']. The solution of the problem is given by:

$$\begin{aligned}
x(t+1) &= ao*x(t)+c*w(t+1) \\
j(t) &= sj\ x(t)
\end{aligned}$$

for j = k,c,g,i,d, and ci,bi,gi,hi,si, for i=1,2. The program also computes Lagrange multipliers.

solvex

Purpose:

Computes solution of recursive linear equilibrium model with Gaussian Quadratic Exponential preference specification.

Synopsis:

solvex is a script file. The matrices $A_{22}, C_2, U_d, U_b, \Phi_c, \Phi_g, \Phi_i, \Gamma, \Delta_k, \Theta_k,$ $\Delta_h, \Theta_h, \Lambda$, and Π and the scalars σ and β must be in memory.

Description:

Let $x_t = [h'_{t-1}, k'_{t-1}, z'_t]'$, and let the law of motion for x_t be $x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$. The social planning problem is to find a value function

$$\begin{aligned} V(x(t)) = &\max\{-.5[(s(t)-b(t)).(s(t)-b(t)) + l(t)^2] \\ &+\beta * (2/\sigma) * \log E_t \exp(\sigma/2 * (V(x(t+1))\} \end{aligned}$$

subject to

$$\begin{aligned} \Phi_c c_t + \Phi_g g_t + \Phi_i i_t &= \Gamma k_{t-1} + d_t \\ g_t \cdot g_t &= \ell_t^2 \\ k_t &= \Delta_k k_{t-1} + \Theta_k i_t \\ h_t &= \Delta_h h_{t-1} + \Theta_h c_t \\ s_t &= \Lambda h_{t-1} + \Pi c_t \\ z_{t+1} &= A_{22} z_t + C_2 w_{t+1} \\ b_t &= U_b z_t, \quad d_t = U_d z_t \end{aligned}$$

Here s_t is consumption services, b_t a stochastic bliss process, ℓ_t is labor services, c_t is consumption rates, g_t is "intermediate goods", i_t is investment goods, d_t is an endowment shock process, k_t is physical capital, h_t is household capital, z_t is a vector of exogenous information variables, and w_{t+1} is a martingale difference sequence. Each of these is a vector, except for ℓ_t, which is scalar. The program computes the solution of the social planning problem in the form

$$\begin{aligned} x_{t+1} &= A^o x_t + Cw_{t+1} \\ k_t &= S_k x_t \ , g_t = S_q x_t \\ h_t &= S_h x_t \ , i_t = S_i x_t \\ c_t &= S_c x_t \ , b_t = S_b x_t \\ s_t &= S_s x_t \ , d_t = S_d x_t \end{aligned}$$

The program also computes Lagrange multipliers $\mu_t^j = M_j x_t$ for variable $j = k, c, h, s, i$. The program computes and leaves in memory A^o, C, S_j (for $j = k, h, c, s, g, i, b,$ and d) and M_j (for $j = k, c, h, s,$ and i).

Algorithm:

The social planning problem is formulated and solved using `doublex`.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

solves

Purpose:

Computes the solution of recursive linear equilibrium model with periodic coefficients.

Synopsis:

solves is a script file. The matrices $A_{22s(t)},\ C_{2s(t)},\ U_d,\ U_b,\ \Phi_{cs(t)},\ \Phi_{gs(t)},\ \Phi_{is(t)},\ \Gamma_{s(t)},\ \Delta_{ks(t)},\ \Theta_{ks(t)},\ \Delta_{hs(t)},\ \Theta_{hs(t)},\ \Lambda_{s(t)}$, and $\Pi_{s(t)}$ and the scalar β must be in memory.

Description:

The social planning problem is to maximize

$$-(\frac{1}{2})E\sum_{t=0}^{\infty}\beta^t[(s_t-b_t)\cdot(s_t-b_t)+\ell_t^2],\ 0<\beta<1$$

subject to

$$\begin{aligned}
&\Phi_{cs(t)}c_t+\Phi_{gs(t)}g_t+\Phi_{is(t)}i_t=\Gamma_{s(t)}k_{t-1}+d_t\\
&g_t\cdot g_t=\ell_t^2\\
&k_t=\Delta_{ks(t)}k_{t-1}+\Phi_{ks(t)}i_t\\
&h_t=\Delta_{hs(t)}h_{t-1}+\Phi_{hs(t)}c_t\\
&s_t=\Lambda_{s(t)}h_{t-1}+\Pi_{s(t)}c_t\\
&z_{t+1}=A_{22s(t)}z_t+C_{2s(t)}w_{t+1}\\
&b_t=U_bz_t, d_t=z_t.
\end{aligned}$$

where $s(t{+}p)=s(t)$, where p is the period of the model. Here s_t is consumption services, b_t is a stochastic bliss process, ℓ_t is labor services, c_t is a vector of consumption rates, g_t is "intermediate goods", i_t is investment goods, d_t is an endowment shock process, k_t is physical capital, h_t is household capital, z_t is a vector of exogenous information variables, and w_{t+1} is a martingale difference sequence. Each of these is a vector, except for ℓ_t, which is a scalar. Let $x_t=[h'_{t-1},k'_{t-1},z'_t]'$. The program computes the solution of the social planning problem in the form

$$\begin{aligned} x_{t+1} &= A^o_{s(t)} x_t + C_{s(t)} w_{t+1} \\ k_t &= S_{ks(t)} x_t, g_t = S_{gs(t)} x_t \\ h_t &= S_{hs(t)} x_t, i_t = S_{is(t)} x_t \\ c_t &= S_{cs(t)} x_t, b_t = S_{bs(t)} x_t \\ s_t &= S_{ss(t)} x_t, d_t = S_{ds(t)} x_t \end{aligned}$$

The program also computes Lagrange multipliers $\mu^j_t = M_{js(t)} x_t$ for variables $j = k, c, h, s, i$. The program computes and leaves in memory $A^o_{s(t)}$, $C_{s(t)}$, $S_{js(t)}$ for $j = k, h, c, s, g, i, b$ and d, and M_j for $j = k, c, h, s$, and i.

The user is advised to use the Matlab program `seas1` an an aid in creating the matrices that must be fed into `solves`.

The user must edit `solves` to set the period p. Also, it will vastly accelerate computations if the user will load either the file `seas4.mat` (in the case $p = 4$) or the file `seas12.met` (in the case $p = 4$). The lines to edit occur immediately after the information provided by the help command, *i.e.* the first lines without %.

Algorithm:

The social planning problem is formulated as a periodic optimal linear regulator problem and solved using doubling algorithms described by Hansen and Sargent.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

spectr1

Purpose:
Computes spectral density of endogenous variables of a dynamic linear equilibrium model.

Synopsis:
spectr1 is a script file. The matrices ao, c, sy, R, D, and the scalar nnc must be in memory.

Description:
The equilibrium model is of the form

$$\begin{aligned} x_{t+1} &= ao\ x_t + cw_{t+1} \\ y_t &= sy\ x_t + v_t \\ v_{t+1} &= Dv_t + u_{t+1} \end{aligned}$$

where $Ew_t w_t' = I, E, u_t u_t' = R$. The constant corresponds to row number nnc of the state vector x_t. The eigenvalues of D and the eigenvalues of A (except for the unit eigenvalue associated with the constant term) must be less than unity in modulus. spectr1 computes the spectral densities variables in y_t.

Algorithm:
spectr1 deletes the nnc^{th} row and/or column of ao, c, and sy, which correspond to the constant term. Then spectral is used to compute the spectral density matrix of y_t.

See also:
spectral

spectral

Purpose:
Computes spectral density matrix for a linear system.

Synopsis:
spectral, a script file. The inputs A, C, G, D, R and T must reside in memory.

Description:
Let the system be

$$x_{t+1} = Ax_t + Ce_{t+1}$$
$$y_t = Gx_t + v_t$$
$$v_{t+1} = Dv_t + u_t$$

where $Ee_te'_t = I, Eu_tu'_t = R$, and where e_t and u_s are orthogonal for all t and s. The vector y_t is $rg \times 1$. The spectral density matrix for y is computed for ordinates $\omega j = 2\pi j/T, j = o, 1, \ldots, T-1$. The spectral density matrix for ordinate j is stored in $Syj, j = 0, 1, \ldots, T-1$. The spectral densities (diagonals of the spectral density matrices) are stored in the matrix S. The matrix S has rg rows and T columns, and $S(k,j) = Syj(k,k)$. The eigenvalues of A and D must all be less than unity in modulus.

Algorithm:
Let $Sy(\omega_j)$ be the spectral density matrix at frequency wj. Then

$$\begin{aligned} Sy(\omega_j) = {} & G(I - Ae^{-i\omega_{-j}})^{-1}CC'(I - Ae^{i\omega_j})^{-1}G' \\ & + (I - De^{-i\omega_j})^{-1}R(I - De^{+i\omega_j})^{-1} \end{aligned}.$$

spectrs

Purpose:
Computes spectral density matrix for set of variables determined by a periodic linear equilibrium model.

Synopsis:
spectrs is a script file. solves and simuls must be run first, and their outputs must reside in memory. The integer nnc (the index of the constant in the state vector) must be in memory.

Description:
The spectral density of a process y_t with hidden periodicity p is given by the Tiao-Grupe formula

$$S_y(z) = Q(z)H(I - \hat{A}z^p)^{-1}\hat{C}\hat{C}(I - \hat{A}z^{-p})^{-1\prime}H'Q(z^{-1})',$$

where $z = e^{-i\omega j}$; where $\hat{A}, \hat{C}$, and $\hat{H}$ are from the stacked state space system

$$\begin{aligned} X_{t+1} &= \hat{A}X_t + \hat{C}W_{t+1} \\ Y_t &= HX_t, \end{aligned}$$

and where $X_t' = [x'_{pt-p+1}, x'_{pt-p+2}, \dots, x'_{pt}]$, $Y_t' = [y'_{pt-p+1}, y'_{pt-p+2}, \dots, y'_{pt}]$, $W_t' = [w'_{pt-p+1}, w'_{pt-p+2}, \dots, w'_{pt}]$. The program returns the spectral density matrices for frequencies $\omega_j = 2\pi j/T$, for $j = 0, 1, \dots, T-1$ in the matrices $Sy0, Sy1, \dots, SyT-1$. The spectral densities of the individual series are returned in the matrix S.

The user can edit the file to specify T and the particular series whose spectrum is computed.

See also:
solves, simuls

steadst

Purpose:
steadst computes steady state values of observable variables determined by a recursive linear equilibrium model.

Synopsis:
steadst is a script file which requires that the scalar nnc and matrices $ao, sc, ss, si, sd, sb, sk, sh$ reside in memory.

Description:
The equilibrium model is represented as

$$\begin{aligned} x_{t+1} &= ao * x_t + c * w_{t+1} \\ y_t &= Gx_t \end{aligned}$$

where $G = [sc; ss; si; sd; sb; sk; sh]$. The integer nnc gives the row in the state vector x_t that corresponds to the constant term. steadst assumes that except for the eigenvalue associated with the constant term, all eigenvalues of ao are less than unity in modulus. The program calculates the steady state value of x_t, putting its value in zs. Then the program successively calculates the steady state values of $c, s, i, d, b, k,$ and h, which are the components of y.

Algorithm:
The steady state value of x is obtained as a basis vector for the null space of $(I - ao)$, normalized so that the component corresponding to the constant equals unity.

References:
[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

See also:
null

steadsts

Purpose:

Computes seasonal steady states and seasonal means for a periodic recursive linear equilibrium model.

Synopsis:

steadsts is a script file. solves and simuls must be run first, and their outputs must be in memory. So must nnc, the index of the constant term in the state vector.

Description:

The equilibrium for the stacked version of a periodic model can be represented as

$$X_{t+1} = \hat{A} X_t + \hat{C} W_{t+1}$$

where $X_t' = [x'_{tp-p+1,\ldots,} x'_{tp}], W_t' = [w_{pt-p+1,\ldots,} w_{pt}]$. The program computes the null space of $(I - \hat{A})$, which gives the steady for $X_t = \bar{X}$. Then seasonal means for individual variables are formed by pre-multiplying $\bar{X}$ by matrices formed from appropriate seasonal decision rules. The user must edit the file to compute seasonal means of the particular variables he is interested in.

See also:

steadst, solves, simuls.

vardec

Purpose:
Calculates variance of k-step ahead prediction errors in z_t for $k = 1, 2, \ldots, N$ for an "innovations system".

Synopsis:
[tab] = vardec (A, C, K, V, N)

Description:
Consider the innovations system

$$\begin{aligned} x_{t+1} &= Ax_t + Ku_t \\ z_t &= Cx_t + u_t \\ Eu_t u_t' &= V \end{aligned}$$

vardec prepares a table of diagonal elements of the covariance matrices of k-step ahead errors in predicting $z_t, k = 1, \ldots, N$. The output is returned in tab, which has N rows and max(size(V)) columns. The (k, h) element of *tab* gives the variance of the k-step ahead prediction errors for the h^{th} variable in z_t.

Algorithm:
Let the covariance matrix of k-step ahead prediction error in z be V_k. Then

$$\begin{aligned} V_1 &= V \\ V_2 &= CKVK'C' + V \\ V_k &= V_{k-1} + CA^{k-1}KVK'A^{k-1}C'. \end{aligned}$$

References:
[1] Sims, Christopher "Macroeconomics and Reality," *Econometrica*, 1980.
[2] Hansen, Lars Peter and Thomas Sargent, *Recursive Linear Models of Dynamic Economies*, (manuscript), Dec. 1988.

vardeci

Purpose:
Compute decomposition of k-step ahead prediction error variances for an "innovations system".

Synopsis:
$[tab]$ = vardeci (A, C, K, V, N, j)

Description:
Consider an innovations system

$$x_{t+1} = Ax_t + Ku_t$$
$$z_t = Cx_t + u_t$$

where $Eu_t u_t^{'} = V$. Let $r'r = V$ be a Cholesky decomposition of V. Form the innovations system with orthogonalized innovations

$$x_{t+1} = Ax_t + Bv_t$$
$$z_t = Cx_t + Dv_t$$

where $B = K \cdot r', D = r'$, and $Ev_t v_t' = I$. The program prepares a table of the part of the diagonal elements of the covariance matrix of the k-step ahead prediction errors, $k = 1, \ldots, N$, that is attributable to the j^{th} innovation. The table is returned in tab, which has dimension $N \times max(size(V))$. The (k, h) element of tab gives the variance in the k-step ahead variance in predicting the h^{th} component of z due to the j^{th} orthogonalized innovation in v_t.

Algorithm:
Let S_j be a selector matrix for j, equal to an $m \times m$ matrix of zeros except of a one in the (j, j) element. Let V_k be the covariance of the k-step ahead prediction error in z due the jth orthogonalized innovation. The V_k are calculated using the ecursions

$$\begin{aligned} V_1 &= DS_jS_j'D' \\ V_2 &= CBS_jS_j'B'C' + V_1 \\ V_k &= V_{k-1} + CA^{k-1}BS_jS_j'B'{A'}^{k-1}C' \end{aligned}$$

References:
[1] Sims, Christopher "Macroeconomics and Reality," *Econometrica*, 1980.
[2] Hansen, Lars Peter and Thomas Sargent, *Recursive Linear Models of Dynamic Economies.*

varma

Purpose:

varma computes an innovations representation for a recursive linear model whose observations are corrupted by first-order serially correlated measurement errors.

Synopsis:

varma is a script file, which requires that the matrices ao, c, sy, D, and R reside in memory.

Description:

The model is assumed to have the state space representation

$$x_{t+1} = ao * x_t + c * w_{t+1}$$
$$y_t = S_y * x_t + e_{t+1}$$

where w_t is a white noise with $Ew_t w_t{}' = I$, and e_t is a measurement error process governed by

$$e_{t+1} = De_t + \eta_{t+1},$$

where η_{t+1} is a vector white noise with contemporaneous covariance matrix R. The matrices R and D must each be $m \times m$, where $[m, n] = size(sy)$. The program uses the Kalman filter to form the innovations representation

$$\hat{x}_{t+1} = ao\hat{x}_t + k_1 * u_t$$
$$\tilde{y}_{t+1} = GGx_t + u_t$$

where $GG = [sya^o - DS_y], \tilde{y}_t = y_{t+1} - Dy_t$, and u_t is the innovation in y_{t+1}, $u_t = y_{t+1} - E[y_{t+1} \mid y_t, y_{t-1}, \ldots]$. The program uses evardec to compute a decomposition of variance for the innovations system.

References:

[1] Hansen, Lars Peter and T.J. Sargent, *Recursive Linear Models of Dynamic Economies*, manuscript, Dec. 1988.

[2] Sargent, Thomas, "Two Models of Measurements and the Investment Accelerator," *Journal of Political Economy*, April 1989.

varma2

Purpose:

varma2 creates impulse response functions associated with an innovations representation.

Synopsis:

varma2 is a script file which requires that the matrices aa, bb, cc, dd, $V1$ be in memory.

Description:

varma2 takes the output of innov and creates impulse response functions of y with respect to components of u. Impulse response functions with respect to the orthogonalized innovations $v_t = r'^{-1} u_t$ are also computed, where $r'r = V1$ is a Cholesky decomposition of V1.

Algorithm:

dimpulse is applied.

References:

[1] Sims, Christopher, "Macroeconomics and Reality," *Econometrica*, 1980.

varrep

Purpose:
Computes a vector autoregressive representation from a state space model with serially correlated measurement errors.

Synopsis:
function [AA,V1]=varrep(ao,c,sy,D,R,nj,nnc)

Description:
Computes (an infinite order) vector autoregressive representation for a recursive linear model whose observations are corrupted by first-order serially correlated measurement errors. The model occurs in the state space form

$$x(t+1) = ao * x(t) + c * w(t+1)$$
$$y(t) = sy * x(t) + e(t+1)$$

where e(t) is a measurement error process

$$e(t+1) = D * e(t) + ee(t+1)$$

and where ee(t+1) is a vector white noise with covariance matrix R. We assume that ee(t+1) and w(t+1) are orthogonal at all leads and lags. The program computes the autoregressive representation

$$y(t) = \sum_{j=1}^{\infty} A(j)y(t-j) + a(t)$$

where a(t)=y(t) - E[y(t)— y(t-1),y(t-2),...], and the A(j) are square matrices. The program creates the covariance matrix of a, which it stores in V1. The program returns nj of the matrices A(j), stacked into the ((m times nj) by m) matrix AA, where m is the number of rows of y. A(j) occurs in rows ((j-1)*m+1) to row j*m of AA. nnc is the location of the constant term in the state vector.

white1

Purpose:
Creates a state space system [AA,BB,CC,DD] that accepts the innovation to the (information) state vector w_{t+1} as in input and puts out the innovation u_t to y_t as an output.

Synopsis:
function[AA,BB,CC,DD]=white1(ao,c,sy,D,R)

Description:
The program couples the systems

$$\begin{aligned} x(t+1) &= ao * x(t) + c * w(t+1) \\ y(t) &= sy * x(t) + v(t) \\ v(t) &= D * v(t-1) + \eta(t) \\ E\eta(t)\eta(t)' &= R. \end{aligned}$$

and

$$\begin{aligned} xh(t+1) &= (ao - k1 * GG) * xh(t) + k1 * y(t) \\ u(t) &= -GG * xh(t) + u(t) \end{aligned}$$

where $w(t+1)$ is the innovation to agents' information sets and where $u(t)$ is the fundamental (Wold) representation innovation. A (minimum-realization) state space system [AA,BB,CC,DD] for the coupled system is returned. To compute the impulse response function, use dimpulse.

white2

Purpose:
Creates the state space system [AA,BB,CC,DD] that accepts the measurement error $v(t+1)$ as in input and puts out the innovation $u(t)$ to $y(t)$ as an output.

Synopsis:
function[AA,BB,CC,DD]=white2(ao,c,sy,D,R)

Description:
The program couples the systems

$$\begin{aligned} x(t+1) &= ao * x(t) + c * w(t+1) \\ y(t) &= sy * x(t) + v(t) \\ v(t) &= D * v(t-1) + \eta(t) \\ E\eta(t)\eta(t)' &= R. \end{aligned}$$

and

$$\begin{aligned} xh(t+1) &= (ao - k1 * GG) * xh(t) + k1 * y(t) \\ u(t) &= -GG * xh(t) + u(t) \end{aligned}$$

where $w(t+1)$ is the innovation to agents' information sets, $\eta(t)$ is the innovation to measurement error, and where $u(t)$ is the fundamental (Wold) representation innovation. The (minimum-realization) state space system [AA,BB,CC,DD] for the coupled system is returned. To compute the impulse response function, use dimpulse.

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Index

Author Index

Matlab Index