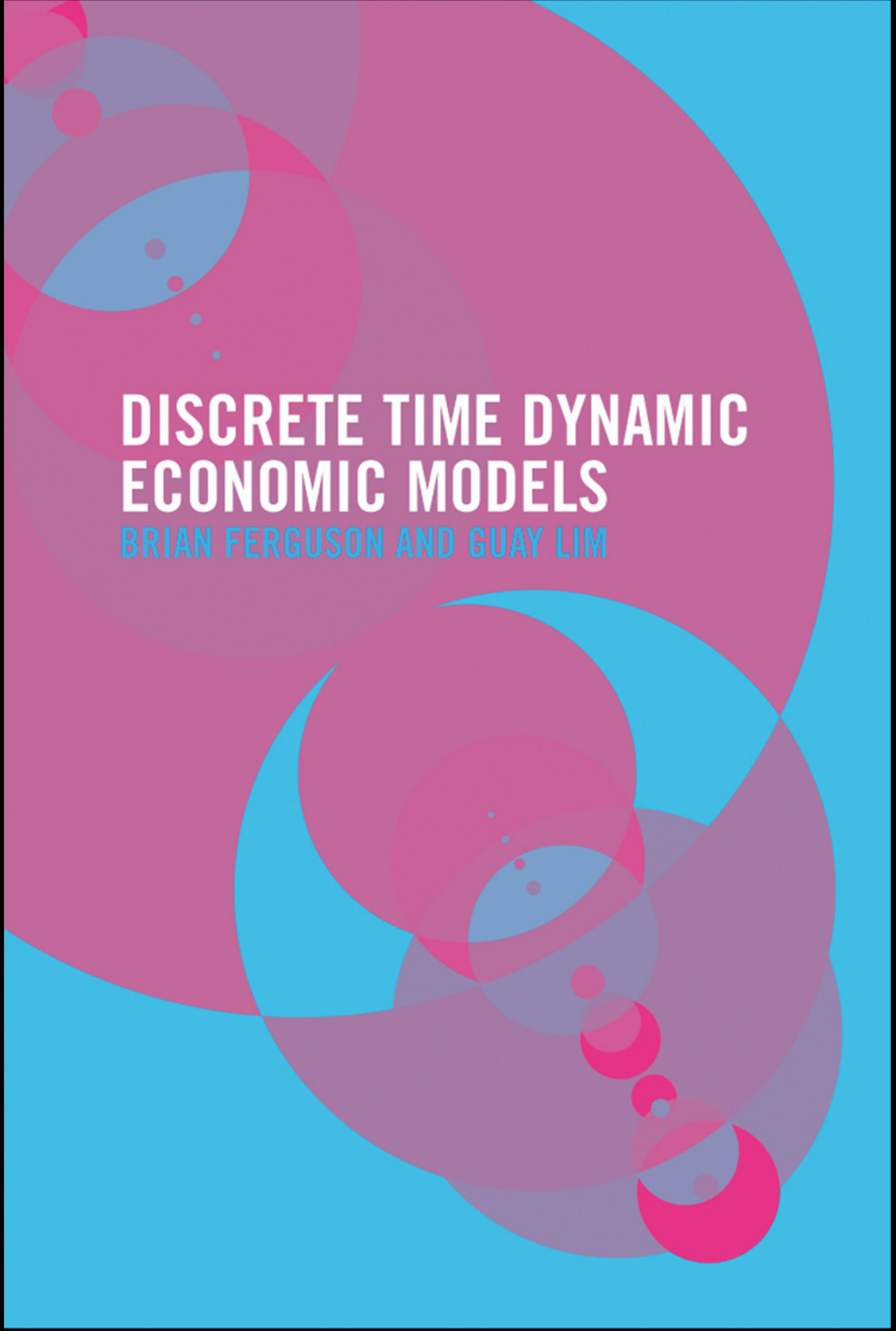

Dynamic Economic Models in Discrete Time

Economic behaviour is inherently dynamic. While things change continuously over time, much of economic analysis is based on discrete time, such as a month, a quarter, or a year, reflecting the periodic nature of data collecting and decision-making.

This book introduces and develops the techniques of discrete time modelling starting with first-order difference equation models and building up to systems of difference equations, covering the following topics along the way:

- nonlinear difference equation models
- random walks and chaotic processes
- optimization in discrete time models

This easy-to-follow book will primarily be of interest to upper-level students carrying out economic modelling. The nature of the book – bridging a gap between dynamic economic models and empirical analysis – will mean that it will also appeal to all academics with an interest in econometrics and mathematical economics.

Brian S. Ferguson is Associate Professor of Economics at the University of Guelph, Canada.

G. C. Lim is Associate Professor of Economics at Melbourne University, Australia.

Dynamic Economic Models in Discrete Time

Theory and empirical applications

Brian S. Ferguson and G.C. Lim

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Contents

1 Introduction

Economic behaviour is inherently dynamic at both the micro and the macro levels. Things change over time. Economic agents (be they consumers or firms) take time to respond to changes in prices, incomes and circumstances in general, so it takes time for the effects of those changes to work their way fully through the system. It can take a long time for the economy to reach its new equilibrium position after a shock and, indeed, given that the economy is subject to a continuous series of random shocks, it may never actually settle down into a new equilibrium. It is quite possible for every observation of an economic variable (be it of a micro variable like price, or a macro variable like the gross domestic product (GDP)) to represent transitional rather than equilibrium points.

However, economic theory focusses on equilibrium relations. This is logical, since equilibrium relations are generally determined by solving the optimization problem which drives the economic behaviour. The dynamic behaviour that we actually observe in markets ultimately derives from the efforts of individual economic agents to move towards a new optimum.

The need to consider transition dynamics is not ignored in courses which focus on theory, but time constraints often result in dynamic analysis being relegated to a minor place in the curriculum. It is not unusual for a bit of dynamics to be covered in a lecture or two, with the aim being to provide a brief cookbook on how to do the specific bits of analysis that will be relevant to that particular course. Of necessity, this provides a very incomplete introduction to dynamic economic analysis.

Our aim in this book is to fill some of that gap, at least with regard to discrete time analysis. This book is an introduction to discrete time economic dynamics, and so does not go into the mathematics or econometrics at a high technical level. Our aim is to give an introduction to the use of difference equation models in economic analysis.

A difference equation is a mathematical relation between the value of a variable y in period t, which we write as y_t and its value in one or more past periods, which we write as y_{t-i} where the value of i indicates how far into the past we are looking. When $i = 1$, we have y_{t-1}, and the relation $y_t = f(y_{t-1})$ is what is known as a first-order difference equation (FODE). If we were working with $y_t = f(y_{t-1}, y_{t-2})$, we would have a second-order difference equation (SODE).

Strictly speaking, $y_t = f(y_{t-1})$ is an autonomous FODE, since time appears only in the subscript on the y term. When we have something like $y_t = f(y_{t-1}, t)$ where time appears explicitly as an argument in its own right in the $f(\cdot)$ function, we refer to the difference equation as non-autonomous. Virtually all of the difference equations found in economic applications are autonomous.

In general, in dynamic economic models, we are working with equations of the form $y_t = f(y_{t-1}) + g(x_t, t)$, where x_t stands for any exogenous variable which might affect the value of y in period t. Written this way, we can regard the $f(y_{t-1})$ term as summarizing the intrinsic dynamics of y and the $g(x_t, t)$ term as summarizing the way the exogenous variables, including a deterministic trend, affect y.[1]

While we will go into all of this in more detail in later chapters, we should say an introductory word here about deterministic trends. We said above that economic variables change over time. They do so for a number of reasons.

One obvious reason for y to change is that x has changed. In comparative statics analysis, this is the only reason for y to change. In comparative statics analysis, then, we are basically looking at an expression of the form: $y_t = g(x_t)$. The y variable will change as the x variable changes, but we generally do not tell a story about why x changes over time – we leave that for a model in which x is the dependent variable.

Sometimes we extend the comparative statics model by adding a deterministic trend (we will explain the terminology 'deterministic' later, when we talk about other types of trends) giving us: $y_t = g(x_t, t)$. Here y can change even if x does not, because the presence of t as an explanatory variable (as distinct from its presence as a subscript) means that the simple passage of time is sufficient to cause the value of y to change. The most common explanation for the presence of this type of trend is that it represents something like technological change. For example, in a production function, technological change often enters as a variable which allows us to get more output out of unchanged levels of the inputs.

Despite the long-term popularity of models like this, it can be surprisingly difficult to think of a convincing reason for including a deterministic trend, either in theoretical or empirical dynamic models: strictly speaking, the presence of a deterministic trend often suggests that this process would continue to operate even if the world were to come to an end. The destruction of all capital and labour would disrupt the production process, of course, but there would still be an upward tendency in output.

Most often, a deterministic trend is taken as representing all of the factors which affect the value of y but which, usually for lack of data, we are unable to model. Sometimes it is the best we can do, but increasingly it is being seen as a last resort.

Modern dynamic economic analysis places greater weight than did the earlier literature on the fact that economic variables are not black box processes. They derive their values from other economic variables as a result of the decisions made by economic agents – consumers and producers. Dynamics arise from the same source – as we noted above, it takes time for people to react to changing circumstances.

This is where we get structures of the form: $y_t = f(y_{t-1}) + g(x_t, t)$ or even of the form: $y_t = f(y_{t-1}) + g(x_t, x_{t-1}, t)$. In this kind of equation, the $g(\cdot)$ term

summarizes how y responds to changes in the exogenous variables while the $f(\cdot)$ tells us about the time path y follows in the process of responding. Putting it oversimply, the $g(\cdot)$ term tells us where y is heading, and the $f(\cdot)$ term tells us about the path it follows as it heads there.

The range of types of time paths that a y variable can follow is remarkably wide. We will talk about this in more detail later, but the most obvious possibilities are a monotonic, or smooth approach path and a cyclical approach path (assuming it approaches at all – as we shall see, this is a testable hypothesis). As the $f(\cdot)$ term becomes more complicated, the range of types of paths that y could follow becomes much broader. It does not take a very complicated functional form for the $f(\cdot)$ term to yield chaotic behaviour.

The key message of dynamic economic modelling is that understanding the nature of the transition path is essential to understanding where y is going to wind up. If our true relation is $y_t = f(y_{t-1}) + g(x_t)$, and in our empirical analysis we assume that $y_t = g(x_t)$, there is a pretty good chance that we will wind up with a biased estimate of the form of the $g(x_t)$ function, giving us an erroneous picture of the true relation between y and x.

Difference equations arise quite naturally in economic models: the familiar multiplier derived from the Keynesian Cross macro model (which we shall consider in some detail later) is frequently described in terms of lagged responses to shocks. For example, a 'shock' increase in government spending today increases income today. However, while consumers will respond to the higher income, they do so with a lag – perhaps it takes a bit of time to decide what to spend the extra income on – so consumption does not increase until tomorrow. That increase in consumption results in a further increase in income which, after a lag, causes a further increase in consumption and so on until the multiplier effect of the initial increase in government spending has worked itself out.

While introductory macro textbooks tend to focus on comparative statics – differences between the pre- and post-shock equilibria – they generally make some mention of the length of time it might take for a real world multiplier process to work itself out. In more advanced macro courses, business cycles are often introduced in the form of lagged adjustments in IS–LM models. If both the goods and the money markets involve lagged responses to changes in exogenous variables, the economy can cycle around a new equilibrium point for quite some time before finally settling down. If we are doing empirical analysis of an IS–LM model, we have to take account of the possibility that a very large proportion of our observations might lie on neither the IS nor the LM curve, even though the equilibrium relations defined by those curves ultimately control the behaviour of the macro system over time.

Similarly, microeconomics analysis often includes dynamic adjustment processes. The Walrasian price adjustment process is generally described in terms of a time process of adjustment to market disequilibrium. When the market is in a state of excess demand, price rises until the market clears, and when the market is initially in a state of excess supply, price falls by whatever amount is needed to clear markets. In short, if we want to estimate the function which describes the optimal relation among the economic variables, we need to distinguish the adjustment dynamics from the underlying equilibrium relationship.

Much of the observed difference in behaviour between different markets can be explained in terms of the speed of their dynamic adjustment processes. Financial markets, for example, respond almost instantaneously to shocks. By definition, as soon as a shock hits (a bit of good news about a company's prospects, for example) the demand and supply curves for the relevant asset shift, causing the market equilibrium price to jump. Because financial markets adjust very quickly to changes in equilibrium values, the actual market price moves very rapidly towards the new equilibrium price. This rapid adjustment to shocks increases market volatility, at least by some measures, and creates the impression of market instability. In fact, rapid fluctuations in stock prices can be an evidence of rapid convergence to an equilibrium, albeit a moving one. But, even in this financial markets example, despite the rapid rate of market adjustments, if the market is subject to many, rapid shocks, a significant proportion of our observations are likely to reflect the dynamics of adjustments rather than the equilibrium relationship.

Labour markets, on the other hand, adjust slowly. The onset of a recession, shifting the labour demand curve back to the left, drives the equilibrium wage down. Because labour markets are slow to clear, the actual wage converges to the new equilibrium only after a long lag, meaning that the market spends a long time in a state of excess supply, and unemployment persists. In this case, a large proportion of our empirical observations are clearly points of disequilibrium.

Empirical economic dynamics has historically tended to focus on macro relations, for the simple reason that it was only for macro variables that we have had extensive time series data. However, at the micro level, the increased availability of panel data sets (which provide cross-sectional micro-information for economic agents over a period of years) have simulated more research into the dynamics of microeconomic behaviour.

This book develops the theory of discrete time dynamics, starting with the FODE then the SODE models, and then building up to higher order and systems of difference equations. It also includes a chapter on nonlinear difference equation models, and a brief introduction to the literature on the question of whether certain markets are better characterized as random walks or as chaotic processes.

This book also introduces techniques of optimization in discrete time models and discusses how they underpin dynamic testable equations of economic behaviour. It also includes an introduction to dynamic econometric modelling which includes a discussion of the more recent developments in time series econometrics; particularly, unit roots, cointegration and error correction forms. The focus here is on the interpretation, in terms of economic theory, of the results from empirical econometric modelling.

In summary, this book aims to introduce an Economics student to discrete time economic modelling – its theory and its empirical analysis – and to how dynamic optimizing acts as a bridge between the equilibrium modelling of economic behaviour and the applied econometric analysis of the adjustment process.

2 First-order difference equations

Introduction

The simplest type of difference equation is a linear, first-order equation, of the general form:

$$Y_t = \alpha Y_{t-1} + g \tag{2.1}$$

In an equation like this one, the time subscript, 't', should be thought of not as representing calendar time, but rather elapsed time – the amount of time which has passed since the dynamic process, which we are studying, began. As written, when the term g is non-zero, Equation (2.1) is called a non-homogeneous equation; when g is equal to zero, Equation (2.1) is called a homogeneous equation. Furthermore, since α is a constant, Equation (2.1) is also an example of a linear, constant coefficient first-order difference equation (FODE). Most economic applications of FODE involve constant coefficient models, although this is not a requirement.

According to this equation, the value which the variable Y takes on in period t is equal to a constant g plus a term which depends on the value which Y took on in the period $t - 1$. In economic applications, the g term represents all those variables which affect the current value of Y, other than Y's own lagged value.

Another way of looking at Equation (2.1) is to rewrite it in change form, that is, $\Delta Y_t = Y_t - Y_{t-1}$ form. This involves what is known as a linear reparametrization of Equation (2.1), which boils down to rearranging the terms in Equation (2.1) without altering its meaning. In the case of Equation (2.1), we simply subtract Y_{t-1} from both sides of Equation (2.1) and, for convenience of interpretation, rearrange it as:

$$\Delta Y_t = (\alpha - 1)Y_{t-1} + g \tag{2.2}$$

Equation (2.2) (which contains exactly the same information as in Equation (2.1), simply presented differently) tells us that the amount by which the value of Y changes from period $t - 1$ to period t depends on its value in period $t - 1$, and on the value of g.

For the difference equation structure to be useful in theoretical and econometric applications, we have to go beyond simply saying that it tells us how current and past values of Y are related. We have to extract precise information about the nature

of that relation. This information is referred to as the dynamic structure, or, more loosely, the dynamics of the relation.

To get an idea of how this is done, consider a simpler, homogeneous version of Equation (2.1):

$$Y_t = \alpha Y_{t-1} \tag{2.3}$$

Remember that we said that the relation between Y_t and Y_{t-1} had to be a genuine causal relation, meaning that there must be a continuing link between current and past values of Y over time. Given this, we can also write $Y_{t-1} = \alpha Y_{t-2}$ and $Y_{t-2} = \alpha Y_{t-3}$ and so on back in time. Since each of these expressions must hold, by definition, then by successive backward substitution we obtain:

$$Y_t = \alpha Y_{t-1} = \alpha^2 Y_{t-2} = \cdots = \alpha^t Y_{t-t} = \alpha^t Y_0 \tag{2.4}$$

where Y_{t-t} is obviously Y_0, which we refer to as the initial value of Y and which we assume is constant.

Since we derived Equation (2.4) from the sequence of equations beginning with Equation (2.3), it contains no new information, it just presents our prior information in a slightly different form. The reason this form is useful is because of the way the time element, t, appears in Equation (2.4). Instead of an equation relating Y_t to Y_{t-1}, we have an equation showing how the value of Y_t depends on the value of t itself. An expression like this, which gives the value of Y_t as a function of the value of t, not of Y_{t-1}, is generally referred to as a solution function for our FODE. The solution function throws light on the role of the α term in the evolution of Y_t.

- $0 < \alpha < 1$: suppose α is a constant, positive fraction. Then as time passes and t, the time index, gets steadily bigger, the term α^t gets steadily smaller, going to zero as t approaches infinity. Whatever our initial Y_0 value then, so long as Y_0 is not equal to zero, after enough time has elapsed, and t has become big enough, Equation (2.4) tells us that Y_t converges on zero.
- $\alpha > 1$: suppose that α is a constant, positive number larger than 1. In this case, as time passes and t gets steadily bigger, α^t also gets steadily bigger and, from Equation (2.4), no matter how small our initial value Y_0 is, again so long as it is not actually equal to zero,[1] as t goes to infinity, Y_t goes eventually to infinity.

Even more interesting behaviour comes out of the case where α is negative, but we shall set that case aside for a moment, and state a general result. Whenever the behaviour over time of a variable Y can be characterized by a first-order linear homogeneous difference equation such as Equation (2.3), we can, by substitution, find an expression like (2.4), in which the value of Y is shown as a function of the time index, t.

Solution functions

Homogeneous equations

For the majority of the examples we deal with, an equation like (2.3) has a solution function of the general form:

$$Y_t = A\lambda^t \tag{2.5}$$

where λ is referred to as the root of the difference equation, and A is a constant whose value is to be determined from the information given in the problem. Since the time subscript is arbitrary, this general expression will apply in each period, so that, for example $Y_{t-1} = A\lambda^{t-1}$. This means that we can use the solution function (2.5) to rewrite Equation (2.3) as:

$$A\lambda^t - \alpha A\lambda^{t-1} = A\lambda^{t-1}(\lambda - \alpha) = 0 \tag{2.6}$$

which yields:

$$(\lambda - \alpha) = 0 \tag{2.7}$$

known as the characteristic equation of our original FODE (2.3). From Equation (2.7) we see that $\lambda = \alpha$ and substituting this back into Equation (2.5) gives the solution of the FODE as:

$$Y_t = A\alpha^t \tag{2.8}$$

To see that (2.8) is indeed a solution, note that, since the form of the equation which determines the value of Y in each time period is assumed to be unchanged over time (all that changes is the particular value which t happens to take on) we can also write $Y_{t-1} = A\alpha^{t-1}$ giving $Y_t = \alpha Y_{t-1}$, which is just the homogeneous FODE that we started from, in Equation (2.3).

Next, we need to solve for the undetermined constant A. We do not actually have enough information in the problem as stated to do this: we need to bring in an additional bit of information. The additional information most commonly used is what is referred to as an initial condition, which is simply a statement that at time $t = 0$, Y_t takes on the specific, known value Y_0. The initial condition does not actually have to refer to $t = 0$, all we actually need is to know the value of Y at one specific value of t, but $t = 0$ is the most commonly used value.

Assume, then, that we know that at time $t = 0$, Y takes on the specific value Y_0. Since Equation (2.8) determines the value of Y_t for each value of t, we have, from Equation (2.8) $Y_0 = A$, since $\alpha^0 = 1$. Then since we know the specific numerical value represented by Y_0, we can use that knowledge to determine Y_t for $t \neq 0$. Plugging Y_0 into Equation (2.8) in place of A gives the complete expression for the solution to the difference Equation (2.3):

$$Y_t = Y_0\alpha^t \tag{2.9}$$

which is, happily, precisely the same as the solution we established in Equation (2.4), established through backwards substitution.

In establishing that a difference equation of the form $Y_t = \alpha Y_{t-1}$ has a solution of the form $Y_t = A\lambda^t$ where λ turns out to equal α and A to equal Y_0, we may appear to have been reinventing the wheel, since we had already established the solution $Y_t = Y_0\alpha^t$ as the function giving the value of Y at any time t. There is a purpose to the formal derivation, though, since the solutions to more complicated equations will be extensions of the basic form $Y_t = A\lambda^t$, and establishing that starting from this form yields the solution we had already found from direct substitution should give us confidence about the applicability of this approach to more complicated cases.

Dynamic properties

Before turning to those more complicated cases, however, let us return to the material discussed above and explore the dynamic path of Y_t. In the general homogeneous equation $Y_t = A\lambda^t$, where λ is what we refer to as the root (or characteristic root) of the difference equation, the behaviour of Y over time depends crucially on the value of λ. In what follows we will assume that A is not equal to zero, since if A is equal to zero, then Y_t will always equal zero, regardless of the value of t.

We turn now to the role of the characteristic root, λ:

- when λ is a positive fraction, and when the value of A is not itself equal to zero, then as time passes the value of Y_t will converge on zero; see Figure 2.1(a);
- when λ is a positive number greater than 1, as time passes, Y_t will tend to infinity, plus or minus as A is positive or negative; see Figure 2.1(b). Figure 2.1(a) and (b) shows that the sign of Y_t depends on the sign of A and we refer to the time path of Y as monotonic, since it converges on, or diverges from, zero in a smooth manner;
- when λ equals 1, then $Y_t = A$ for all values of t regardless of the value of A, see Figure 2.1(c). This is the case of a unit root, which used to be regarded as a rather pathological borderline case, but which has recently, as a result of developments in unit root econometrics, become very important in empirical applications of dynamic modelling. We shall discuss this case in detail in a later chapter;
- when λ is a negative fraction, for example, if $\lambda = -0.5$, then $\lambda^2 = 0.25$, $\lambda^3 = -0.125$, $\lambda^4 = 0.0625$, with the absolute value of λ^t tending to zero as time passes, but with the sign of λ^t alternating between positive and negative. In general, when λ is negative, we get behaviour that we refer to as being characterized by alternations. Figure 2.1(d) shows the case for a positive value of A; Y does converge on zero as time passes, but does not do so monotonically. Instead it jumps from above zero to below it to above it again and so on, at each jump getting closer to zero;
- when λ is a negative number less than -1, the path of Y alternates and diverges from zero. Consider the case where $\lambda = -2$. Then $\lambda^2 = 4$, $\lambda^3 = -8$, $\lambda^4 = 16$ and so on, with the absolute value of λ^t increasing steadily as time passes but with alternating signs. In the case shown in Figure 2.1(e) below, with A greater than zero, Y diverges from zero as time passes, again jumping from above to below and back, this time getting further from zero with each jump.

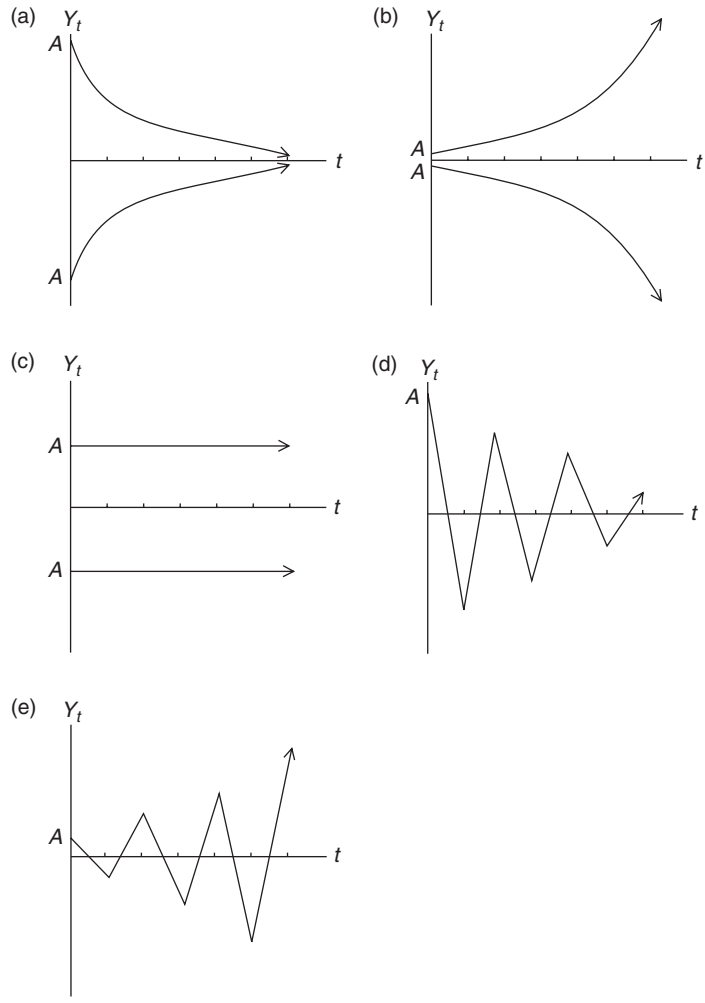

Figure 2.1 Time paths for different values of the characteristic root, λ.

It is tempting to think that alternating behaviour like that shown in Figure 2.1(d) and (e) gives us a mathematical representation of the business cycle, since alternations look very much like cyclical behaviour. While the business cycle does involve income, for example, rising above and then falling below its equilibrium value, this will generally be a gradual process, with income spending several periods above and then several periods below its equilibrium value. Alternating behaviour, of the sort generated when λ is negative, requires the value of Y to jump from above to below equilibrium from one period to the next, and then one period later to jump back above again, with an abruptness which is generally not a characteristic of

the business cycle. In fact, as it turns out, very few economic models legitimately yield negative roots and alternating behaviour,[2] and they tend to be very simplified models. In general, in economic applications, the most likely explanation for a negative root is that it is a mathematical artefact.

In most cases, especially in empirical applications of dynamic models, if we find a negative root our best strategy is to go back and reconsider the structure of the model. There are difference equation models which yield genuine cyclical behaviour, but those require difference equations of order higher than the first. We shall see examples of these equations later.

Non-homogeneous equations

Next, consider the non-homogeneous equation:

$$Y_t = \alpha Y_{t-1} + g \tag{2.10}$$

where g is first treated as a constant, and later generalized to cases where it is non-constant. Solving this equation is a two-step procedure.

The first thing we do is solve for what is known as the particular solution to Equation (2.10). Following that, we will find the solution to the homogeneous part of Equation (2.10), and then we will simply add the two parts together to give the general solution to Equation (2.10). In economic applications, the particular solution is simply what we refer to as the equilibrium of the (one equation) system (2.10).

In dynamic analysis an equilibrium of a difference equation is defined as having the property that, if the system is actually at that point there is no tendency for it to move away from it, regardless of the value of t. If Y_t is at its equilibrium value, it will stay at that value. Note that this says nothing about what happens to the value of Y if it is not equal to the equilibrium value, and in particular tells us nothing about whether Y will tend to converge on, or diverge from its equilibrium value as time passes. The behaviour of the actual value of Y over time, when Y is not initially at its equilibrium value, depends on the stability of the equilibrium. If the actual value of Y tends to converge on the equilibrium value as time passes, we say that the equilibrium is stable, while if the actual value of Y tends to diverge from the equilibrium value as time passes, we say the equilibrium is unstable.[3] In our discussion of the dynamic behaviour of homogeneous equations, zero was the equilibrium of all of the cases.

In general, the mathematical form of the equilibrium, or particular, solution to a difference equation will be determined by the mathematical form of the 'g' term on the right-hand side of Equation (2.10). When g is a constant, the particular solution will, in general, also be a constant. When g is a function of other, exogenous variables (meaning whose values are determined outside the system we are presently analysing) the particular solution will also be a function of those variables. We will see later a case in which g is itself a function of time, making the particular solution to Equation (2.10) a function of time.

g is a known constant

First, though, consider the case where g is a known constant. We noted above that, in dynamic economic applications, when we talk about an equilibrium value we mean a value which the system will tend to stay at, should it be reached.[4] If the system stays at that value as time passes, clearly the value of Y will not change over time, meaning that at equilibrium, $Y_t = Y_{t-1} = Y^*$ for all values of t, where Y^* denotes the equilibrium value of Y. From Equation (2.10) above we derive the equilibrium value Y^* as:

$$Y^* = \frac{g}{1-\alpha} \tag{2.11}$$

which turns out to be a constant whose value depends on, but is not equal to, the value of g. Note that when g equals 0, Y^* also equals 0, which supports our claim that, in the examples of homogeneous equations which was looked at above, zero was the equilibrium value of Y in each of the cases.

Digressing a little here, sometimes it will happen that this method fails because $(1-\alpha) = 0$ or $\alpha = 1$. In this case, the usual procedure[5] is to try as the form of Y^* a function which is of the same form as g but multiplied by t. In this case, that means trying a constant, say G, multiplied by t. Since the form we are trying for our particular solution, Gt, depends on t, we will denote the particular solution of Y by Y_t^*. Then, since we are trying as a solution form $Y_t^* = Gt$ (and hence $Y_{t-1}^* = G(t-1)$), we substitute this form into Equation (2.10), and rearrange as:

$$Gt(1-\alpha) + \alpha G = g \tag{2.12}$$

Because $\alpha = 1$, this becomes $G = g$, giving, as our solution form:

$$Y_t^* = gt \tag{2.13}$$

In general, in the theoretical sections that follow, we shall be dealing with cases where α is not equal to 1, but it is worth noting that the next step after finding that the natural first functional form which was being tried as a possible particular solution form has failed is generally to try the same, general form multiplied by t.

Returning to the case where α is not equal to 1, we have found as our particular, or equilibrium, solution, $Y^* = g/(1-\alpha)$. Note that we have omitted the time subscript from Y^* to emphasize that, in this case, where g is a constant and α is not equal to 1, the equilibrium value of Y does not change over time. The next thing we need to do is find the solution to the homogeneous part of Equation (2.10).

The homogeneous part of a difference equation like (2.10) is simply the homogeneous difference equation which is left when we drop the 'g' term, namely $Y_t = \alpha Y_{t-1}$. We have already solved a form identical to this, in our discussion of homogeneous difference equations, so we can write:

$$Y_t^h = A\lambda^t = A\alpha^t \tag{2.14}$$

where we have written the 'h' superscript on Y_t to indicate that it is the solution to the homogeneous part of a non-homogeneous difference equation. Note that

we have not replaced A by a specific value in Equation (2.14): when we are solving a non-homogeneous difference equation, that is the final step in the process.

Before reaching that step, we need to combine the particular solution with the solution to the homogeneous part to give us the form of the general solution to the difference equation:

$$Y_t = Y_t^h + Y_t^* \tag{2.15}$$

Note that we have added a 't' subscript to the equilibrium term, to allow for the possibility that the equilibrium value depends on time. Obviously a constant equilibrium value is a special case of a time-dependent one. In the case of Equation (2.10), combining solutions gives:

$$Y_t = Y_t^h + Y^* = A\alpha^t + \left(\frac{g}{1-\alpha}\right) \tag{2.16}$$

As our final step, we solve for the undetermined constant A, again using one initial condition, which tells us that at $t = 0, Y_t$ is equal to a precise, known numerical value Y_0. Substituting $t = 0$ into Equation (2.16), noting that $\alpha^0 = 1$, and rearranging gives:

$$A = Y_0 - \left(\frac{g}{1-\alpha}\right) \tag{2.17}$$

We get an insight into what this expression for A means if we note that we can also write it as $A = Y_0 - Y^*$. Since Y_0 is the actual initial value of Y, and Y^* is its (constant) equilibrium value, this tells us that A is just the initial deviation of the actual from the equilibrium value of Y, or the amount of the initial disequilibrium. This also tells us why we have to leave solving for A to one of the last steps in the process – we can not find the initial disequilibrium until we have both the expression for the equilibrium and the initial value of Y. Hence, substituting for A in Equation (2.16), yields the general solution of our difference equation as:

$$Y_t = (Y_0 - Y^*)\alpha^t + \left(\frac{g}{1-\alpha}\right) \tag{2.18}$$

Role of the adjustment coefficient

We can use the general solution to our difference equation to clarify the role of the α term. Rewriting Equation (2.18) as:

$$Y_t = (Y_0 - Y^*)\alpha^t + Y^* \tag{2.19}$$

consider the case where $t = 1$, that is, where one period has elapsed since our initial period. In that case, using Equation (2.19), we obtain an expression for α as:

$$\alpha = \frac{Y_1 - Y^*}{Y_0 - Y^*} \tag{2.20}$$

where $(Y_1 - Y^*)$ is the amount of the initial gap $(Y_0 - Y^*)$ which remains to be closed after one period, so that α is the proportion of the original gap which remains to be closed. Alternatively, we could rearrange Equation (2.20) as:

$$(1 - \alpha) = \frac{Y_0 - Y_1}{Y_0 - Y^*} \tag{2.21}$$

so that $(1-\alpha)$ shows the proportion of the original gap which has been closed after one period. Thus, if $\alpha = 0.6$, then after the first period of the dynamic adjustment process has passed, 40% of the original gap has been closed while 60% of the original gap remains to be closed.

In general, writing Equation (2.19) at time $t - 1$, and comparing it to Equation (2.19) at time t shows that α can also be written as:

$$\alpha = \frac{Y_t - Y^*}{Y_{t-1} - Y^*} \tag{2.22}$$

and hence α can also be interpreted as the ratio of the gaps remaining in periods t and $t-1$. From Equation (2.22), α tells us that the gap from the equilibrium which remains in period t is α times the gap which remained in period $t - 1$. If α is a fraction, so the equilibrium is stable, the gap remaining in period t is smaller than that which remained in $t - 1$, while if α is greater than 1, so that the equilibrium is unstable and Y is moving consistently away from its equilibrium value, the gap remaining in period t is bigger than that which existed in period 1.

Finally, we can show that:

$$(1 - \alpha) = \frac{Y_{t-1} - Y_t}{Y_{t-1} - Y^*} \tag{2.23}$$

so that $(1 - \alpha)$ shows the proportion of the remaining $t - 1$ gap which had been closed after period t. For example, for purposes of exposition, if we assume Y_0 was originally greater than Y^* and also assume that α is a positive fraction, so that Y^* is a stable equilibrium, then the value of Y will decrease over time towards Y^*, and Y_{t-1} will be greater than Y_t.

Case where g is a function of t

We have referred above to the case where g is a function of t as making the equilibrium value a function of time, Y_t^*. One simple example of this type of problem arises in growth models, when g is an exponential function of time, $g_t = G\delta^t$, giving as our difference equation:

$$Y_t = \alpha Y_{t-1} + G\delta^t \tag{2.24}$$

In this case, the process of solving the homogeneous part of the equation proceeds exactly as before because, even though time appears in the $G\delta^t$ term, the homogeneous part of the equation involves only terms in Y. The difference comes when we try to find a particular solution to Equation (2.24).

Again, we proceed by writing a trial solution expression which is of the same mathematical form as g. In this case, we write this as $Y_t^* = B\delta^t$, and we proceed by trying to find an expressions for B which will make this expression fit Equation (2.24). Note that δ appears in both the g function and in our trial expression for the particular solution. This means that we are assuming that the factor which determines the time path of g also determines that of Y_t^*. Replacing Y by Y^* in Equation (2.24), and then noting that $Y_{t-1}^* = B\delta^{t-1}$ and replacing the Y^* terms by their trial counterparts, we have:

$$B\delta^t - \alpha B\delta^{t-1} = G\delta^t \tag{2.25}$$

which, after dividing through by δ^{t-1} and rearranging gives:

$$B = \frac{G\delta}{\delta - \alpha} \tag{2.26}$$

and the time-varying equilibrium Y at time t as:

$$Y_t^* = \left(\frac{G\delta}{\delta - \alpha}\right)\delta^t \tag{2.27}$$

Note that the initial equilibrium value of Y, $Y_0^* = [G\delta/(\delta - \alpha)]$. Obviously this method will fail if $\delta = \alpha$, in which case we multiply our first trial solution form by t and try again.

Assuming then that $\delta \neq \alpha$, the general solution to Equation (2.24) is:

$$Y_t = A\alpha^t + \left(\frac{G\delta}{\delta - \alpha}\right)\delta^t \tag{2.28}$$

Again invoking the initial condition, $Y = Y_0$ when $t = 0$, we have:

$$Y_0 = A + \left(\frac{G\delta}{\delta - \alpha}\right) \tag{2.29}$$

Note that this gives $A = (Y_0 - Y_0^*)$ so the general solution to Equation (2.24) is:

$$Y_t = (Y_0 - Y_0^*)\alpha^t + \left(\frac{G\delta}{\delta - \alpha}\right)\delta^t \tag{2.30}$$

The stability of the equilibrium is, as in the case where g was a constant, dependent on the value of α: when α is a positive fraction, the equilibrium is stable and the approach trajectory monotonic. As in the earlier case, Equation (2.30) says that, in determining the time path of Y, the deviation of the actual value of Y from its equilibrium value equals the initial disequilibrium, multiplied by α^t. If α is a positive fraction, then as t goes to infinity, the effect of that initial disequilibrium term vanishes and Y converges on its equilibrium value. The fact that the value of the equilibrium is itself moving makes no difference to the dynamic adjustment

story – all that happens is that instead of converging on an unchanging equilibrium value, in this case Y converges on a moving target.

We can also manipulate Equation (2.30) to obtain another perspective on the dynamic process. First write Equation (2.30) in terms of the time-dependent equilibrium, Y_t^*:

$$Y_t = (Y_0 - Y_0^*)\alpha^t + Y_t^* \tag{2.31}$$

Then subtracting Equation (2.31) at time $t-1$ from Equation (2.31) at time t gives:

$$\Delta Y_t = (Y_0 - Y_0^*)\alpha^{t-1}(\alpha - 1) + \Delta Y_t^* \tag{2.32}$$

which can also be written as:

$$\Delta Y_t = (Y_{t-1} - Y_{t-1}^*)(\alpha - 1) + \Delta Y_t^* \tag{2.33}$$

Equation (2.33) shows us that the change in the value of Y between periods $t-1$ and t depends on the amount of the disequilibrium in period $t-1$ and also on the change in the location of the equilibrium between the two periods.

Phase diagrams

Diagrammatic representation of linear, FODE

When we were deriving the stability conditions for first-order linear difference equations, we illustrated our results with graphs which plotted the value of Y_t on the vertical axis and time on the horizontal. This type of diagram is very useful in terms of showing the trajectory that Y will follow, especially for cases when we have actual numerical values of coefficients and explicit expressions for the functions. Another useful graphical tool, at least in the case of FODE is a device known as a phase diagram, which plots Y_t against Y_{t-1}.[6]

Consider the simple linear FODE:

$$Y_t = \alpha Y_{t-1} + g, \quad 0 < \alpha < 1 \tag{2.34}$$

The equilibrium for this equation is $Y^* = g/(1-\alpha)$, and because α is a positive fraction, the equilibrium is stable. Our diagrammatic representation, then, should show Y converging on Y^* as time passes.

The phase diagram for Equation (2.34), shown in Figure 2.2(a), simply plots Y_t against Y_{t-1}, with the addition of one extra line. Equation (2.34) is plotted as a straight line with vertical intercept g and slope α. In addition, we have drawn a 45° line, defined as a line along which $Y_t = Y_{t-1}$ or, more formally, locus of points satisfying the expression $Y_t = Y_{t-1}$.

Equation (2.34) above defines the relation between Y_t and Y_{t-1} for all values of t. In terms of the phase diagram, this means that each observed (Y_t, Y_{t-1}) pair must lie on the graph of Equation (2.34). Points off the line representing Equation (2.34) are (Y_t, Y_{t-1}) pairs which by definition do not satisfy

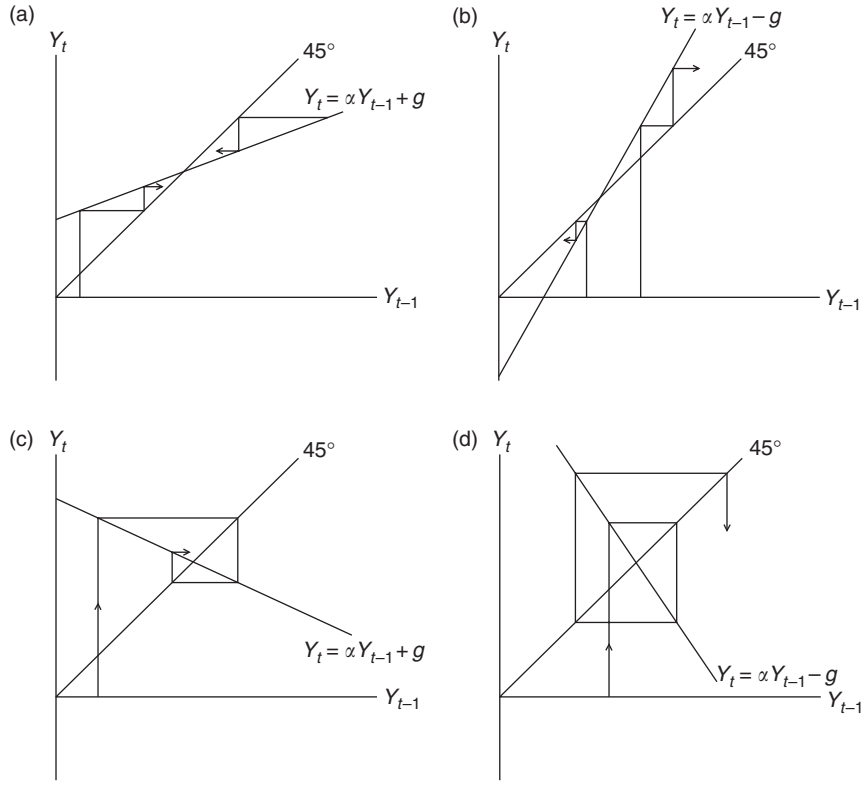

Figure 2.2 Phase diagrams for FODEs.

Equation (2.34), which means that the system cannot actually be at any of them. The graph of Equation (2.34) basically narrows down the set of points which we observe from all of the points in the diagram to just those satisfying Equation (2.34).

The 45° line is on the diagram because it enables us to find the equilibrium point for the system. So long as the g term in Equation (2.34) is a constant, the equilibrium value of Y will be a constant (since $\alpha \neq 1$), which means that, once we have reached the equilibrium value of Y, $Y_t = Y_{t-1} = Y^*$ for all future values of t. This means that, for the case where the equilibrium is a constant, the equilibrium value will lie somewhere on the 45° line.

The equilibrium value of the system, being the particular solution to Equation (2.34), must lie along the line showing which graphs Equation (2.34). Being a constant, it must also lie along the 45° line. This means that the equilibrium point for the system (2.34) must be a point of intersection between the line that graphs Equation (2.34) and the 45° line. In other words, it is the solution to the two-equation system:

$$Y_t = \alpha Y_{t-1} + g \tag{2.35}$$

$$Y_t = Y_{t-1} \tag{2.36}$$

Solving this pair of equations formally will yield $Y^* = g/(1-\alpha)$.

The phase diagram does more, however, than simply show us the equilibrium value. It also lets us plot the system's approach to its equilibrium (assuming, of course, that the equilibrium is stable). To see this, suppose that our initial value $Y_0 = 0$. Then in period 1, $t = 1$ and $t - 1 = 0$, so we can find Y_1 from Equation (2.34), that is, we get $Y_1 = g$. On the diagram, this lets us find our first (Y_t, Y_{t-1}) pair, $(Y_1, Y_0) = (g, 0)$. This point is just the vertical intercept of the graph of Equation (2.34). We shall refer to this graph in general terms as the graph of the $Y_t(Y_{t-1})$ function, since it shows the value of Y_t as a function of the value of Y_{t-1}.

To find our next point, note that after one more period has passed, Y_1, which was Y_t, has become Y_{t-1}. We make this point to emphasize the fact that the subscript 't' refers to the present time, whatever the value of t happens to be, and the subscript $t-1$ refers to one period ago, relative to t. Thus, once one more period has passed, $t = 2$ and the value Y took on in period 1 is now the value of Y_{t-1}. In order to find the value of Y_2, we need to find the point on the Y_{t-1} axis which equals the value of Y_1, then, using that as Y_{t-1} we can read the value of Y_2 as the vertical coordinate of the point on the $Y_t(Y_{t-1})$ function which lies directly above $Y_{t-1} = Y_1$.

There is, as it turns out, a simple way of doing this. What we want to find is a point on the horizontal axis whose value is equal to that of a point which we have already found on the vertical axis. Since the horizontal axis is the Y_{t-1} axis and the vertical axis is the Y_t axis, we are looking for a point where $Y_{t-1} = Y_t$, when we already know the value of Y_t.

It is sometimes easier to see this if we forget, for the moment, that t refers to time, which always moves from $t-1$ to t, and simply think of t and $t-1$ as subscripts identifying variables, in which case going from the value of Y_t to the value of Y_{t-1} presents no conceptual problems. Or simply note that since one period has passed, what was Y_t is now Y_{t-1}, so its value needs to be shown on the horizontal, rather than the vertical axis. Regardless of how we think of the process, though, we can find Y_2 given the value of Y_1 by mapping over from the value of Y_1 on the vertical axis to the 45° line, then, using that point to identify the new Y_{t-1}, go vertically up to the $Y_t(Y_{t-1})$ function to find Y_2.

On the phase diagram, to reduce clutter, this is typically shown as if the system were bouncing between the $Y_t(Y_{t-1})$ function and the 45° line, but in fact the system is always on the $Y_t(Y_{t-1})$ function – the mapping over to the 45° line is just there to help us find the next point on the $Y_t(Y_{t-1})$ function. In other words, we do not actually move along the steps shown on the diagram, but pass, between one period and the next, between the points where the corners of the steps touch the $Y_t(Y_{t-1})$ function.

Following this path along the $Y_t(Y_{t-1})$ function, we can see that the system is converging on the point of intersection between that function and the 45° line, which point we have already identified as the equilibrium of the system. We can also see that if instead of taking an initial value of Y equal to 0 we had picked some initial value above Y^* and followed the same procedure to find our next points – mapping horizontally across from the $Y_t(Y_{t-1})$ function to the 45° line – that we would wind up moving down along the $Y_t(Y_{t-1})$ function towards the equilibrium point. No matter where we picked our initial value of Y, over time we would move towards the equilibrium value. This is consistent with our having identified the intersection of the two lines on the diagram as a stable equilibrium point.

The phase diagram also illustrates that we cannot overshoot the equilibrium – if we started at an initial value of Y below the equilibrium value and tried to step to the right of Y^*, the mapping process we have just described would move us back to the equilibrium. Since this would apply to any step we took to the right of Y^*, regardless of how small a step it was, we conclude that we cannot pass from points to the left of Y^* to points to the right of that value. This means that we cannot overshoot the equilibrium, and since overshooting is the essence of cyclical behaviour, as we shall show when we discuss second order difference equations (SODE), it means that FODE cannot display cyclical behaviour. (They can, of course, display alternations, but we shall return to this point later.)

One further point before we move on to another case; the diagram suggests that each successive step which we take along the $Y_t(Y_{t-1})$ function will be smaller than the previous step, and this is in fact true. Strictly speaking, we converge on Y^* but never actually reach it in finite time, since in the expression:

$$Y_t = (Y_0 - Y^*)\lambda^t + Y^* \tag{2.37}$$

for Y_t to equal Y^*, we must have:

$$(Y_0 - Y^*)\lambda^t = 0 \tag{2.38}$$

Since Y_0 and Y^* are constants, and equal to each other only by accident $(Y_0 - Y^*)$ will in general not equal zero, so for Equation (2.38) to be satisfied it must be the case that:

$$\lambda^t = 0 \tag{2.39}$$

which, with λ a positive fraction but not equal to zero, requires $t = \infty$. So strictly speaking it takes an infinite amount of time for us to reach the equilibrium, but we can come so close to it in finite time that for practical purposes we are at the equilibrium point. We shall, therefore, continue to refer to the system as having reached the equilibrium when in fact it is arbitrarily close to it and still moving towards it, but closing a gap which is unobservably small by taking steps which

are themselves unobservably small. We shall now consider the phase diagrams for different values of α.

Positive adjustment coefficient

Figure 2.2(a) shows the case of a stable equilibrium, because we assumed that α was a positive fraction. In Figure 2.2(b) we show the case where $\alpha > 1$, and the equilibrium is unstable. We have also assumed, in Figure 2.2(b), that the vertical intercept of the $Y_t(Y_{t-1})$ function is negative, so that, using the convention of treating parameters like g as positive and making them negative by putting negative signs in front of them, our equation is:

$$Y_t = \alpha Y_{t-1} - g, \quad \alpha > 1 \tag{2.40}$$

This assumption will give us a positive equilibrium value for Y. We find the equilibrium of the system, as before, at the intersection of the $Y_t(Y_{t-1})$ function and the 45° line. Its value will be:

$$Y^* = -\left(\frac{g}{1-\alpha}\right) \tag{2.41}$$

which, because we have assumed that $\alpha > 1$, will be positive. The general solution to Equation (2.40) will be:

$$Y_t = (Y_0 - Y^*)\alpha^t - \left(\frac{g}{1-\alpha}\right) \tag{2.42}$$

and because α is greater than 1, the equilibrium will be unstable.

In terms of Figure 2.2(b), suppose that we pick an initial value of Y just below, but not equal to, the equilibrium value of Y. Finding this Y_0 value on the horizontal axis, we map vertically up to the $Y_t(Y_{t-1})$ function to find Y_1. Note that this time our initial point on the $Y_t(Y_{t-1})$ function lies below the 45° line; in Figure 2.2(a) the first point that we found on the $Y_t(Y_{t-1})$ function, starting from an initial value of Y below the equilibrium value, was above the 45° line. Next, as before, in order to find the next value of Y_{t-1} we map horizontally across to the 45° line. This requires us to move to the left from our initial point on the $Y_t(Y_{t-1})$ function, since if we were to move horizontally to the right from our initial point we would never intersect the 45° line.

This in fact gives us a general rule of thumb for analysing more complicated phase diagrams. Our initial point must always be on the $Y_t(Y_{t-1})$ function directly above our Y_0 value, even if the 45° line lies below the $Y_t(Y_{t-1})$ function at that point, and in mapping from the $Y_t(Y_{t-1})$ function to the 45° line we must always be moving horizontally. Following these rules of thumb in Figure 2.2(b) results in our moving steadily away from Y^*, in steps which get successively larger. This applies whether our initial value is above or below Y^*, and reflects the fact that the equilibrium is unstable.

Negative adjustment coefficient

Next, consider the cases shown in Figure 2.2(c) and (d). Here, our difference equation is:

$$Y_t = -\alpha Y_{t-1} + g \tag{2.43}$$

where the negative sign in front of the αY_{t-1} term means that the root of the system will be $-\alpha$, which is negative. This in turn means that in these phase diagrams the $Y_t(Y_{t-1})$ function will be negatively sloped. In Figure 2.2(c) we have assumed that $-\alpha$ is a negative fraction, so the equilibrium will be stable, and in Figure 2.2(d) $-\alpha$ is bigger than 1 in absolute value, so the equilibrium will be unstable.

Picking, in Figure 2.2(c), an initial value of Y equal to zero, our first point will be at g, the vertical intercept of the $Y_t(Y_{t-1})$ function, which in Equation (2.43) we have assumed is positive. Following the mapping rule set out above, we move horizontally across to the 45° line, which in this case takes us to a point to the right of the equilibrium, and then map vertically down to the $Y_t(Y_{t-1})$ function to find our next value of Y. Mapping horizontally across to the 45° line again and then vertically up to the $Y_t(Y_{t-1})$ function gives us our third Y value, which is again below the equilibrium value of Y. This illustrates the alternating behaviour which we showed earlier came out of the Equation (2.37) when the root λ, was negative. Again it is important to remember that despite the mapping procedure we are using, the system actually always lies on the $Y_t(Y_{t-1})$ function, so that when we are looking at actual data we observe only the successive points which we have found on that function.

In Figure 2.2(d) we have assumed that the root is negative and greater than 1 in absolute value – this means that the equilibrium is unstable and that the system's alternations cause it to jump to points which are further and further away from the equilibrium point. Finally, note that there exists a borderline case between those shown in Figure 2.2(c) and (d): when the slope of the $Y_t(Y_{t-1})$ function is exactly equal to -1, we still get alternating behaviour but now the successive jumps are exactly equal in magnitude, so we neither converge on, nor diverge from, the equilibrium. This is the negative root counterpart of a limit cycle, which is a concept we shall encounter when we discuss higher order difference equations. While it looks interesting, remember that negative roots are rare in economic models, and the chances of a root being exactly equal to -1 are even smaller, so in practice it is unlikely to be an important case. As we shall see in Chapter 7, though, the case of a root equal to 1, a unit root, does turn out to be of considerable importance in empirical applications of economic dynamics.

Examples of FODE models

The Keynesian multiplier

Probably the simplest of dynamic models which can be represented using FODE is the Keynesian multiplier, or Keynesian Cross, model. While this model is most

commonly written in static terms, the process of adjustment from old to new macroeconomic equilibrium after a shock is typically described in dynamic terms.

The basic closed economy Keynesian Cross model is as follows:

$$Y = C + I + G \tag{2.44}$$

$$C = C_0 + cY, \quad 0 < c < 1 \tag{2.45}$$

Here, Y is aggregate income, I is aggregate investment, G is government spending and C is aggregate consumption. Investment and government spending are autonomous in this basic version of the model. In the consumption function, Equation (2.45), C_0 is autonomous consumption and c is the marginal propensity to consume. Equation (2.44) is the national income accounting identity and also the Keynesian equilibrium condition.

The version of the model which we have written here is essentially static and therefore has no time subscripts. Equation (2.44), the equilibrium condition, says that the value of national income equals the value of aggregate expenditure, which is the sum of consumption, investment and government expenditure.

Substituting Equation (2.45) into (2.44) gives:

$$Y = C_0 + cY + I + G \tag{2.46}$$

from which we have:

$$Y^* = \left(\frac{1}{1-c}\right)(C_0 + I + G) \tag{2.47}$$

where $1/(1-c)$ is the simple Keynesian multiplier. This model tells us that a change in an autonomous component (e.g. a change, ΔG, in government spending) translates into a change in equilibrium income ΔY^*, with the magnitude of the change in equilibrium income dependent on the size of the multiplier: $\Delta Y^* = (1/(1-c))\Delta G$.

While the static model implies that actual income jumps instantaneously to the new equilibrium level (or, equivalently, that the macroeconomy is always in equilibrium) the analysis usually told in introductory economics texts to explain the adjustment process usually involves steps, with *equilibrium* income responding immediately to a change in government spending and *actual* income adjusting over several periods to the new equilibrium level. To represent this process mathematically it is necessary to put some adjustment lags into the model.

Lagged consumption

The simplest approach to putting a dynamic element into the Keynesian model has consumption adjusting to income with a one period lag, perhaps because it generally takes consumers some time to adjust their consumption patterns in response to changes in income.[7] Adding the appropriate time subscripts

gives us:

$$Y_t = C_t + I + G \tag{2.48}$$

$$C_t = C_0 + cY_{t-1} \tag{2.49}$$

where there is no time subscript on autonomous consumption, nor on I and G, because they do not change with time in this model.

Substituting Equation (2.49) into (2.48) as before, and rearranging the terms, gives:

$$Y_t = cY_{t-1} + I + G + C_0 \tag{2.50}$$

which is clearly a first-order, linear difference equation with the sum of the autonomous elements, $(I + G + C_0)$ as the 'g' term. Note that the g term is 'constant', in the sense that the values of its component elements, while they might be endogenous to some other models, are exogenous to this one. The term constant here does not mean never changing, but means rather that changes in the value of the term are driven by factors strictly external to this model. Later we will complicate the model by making investment endogenous.

Treating g as constant, then, we find the equilibrium solution to Equation (2.50) by setting $Y_t = Y_{t-1} = Y^*$, giving the equilibrium, or particular solution as:

$$Y^* = \left(\frac{1}{1-c}\right)(I + G + C_0) \tag{2.51}$$

which is exactly the same as the expression for equilibrium income in the static model.

Turning now to the dynamics of the model, the homogeneous part of Equation (2.50) is $Y_t - cY_{t-1} = 0$, and on the assumption that the solution to the homogeneous part will be of the form $Y_t = A\lambda^t$ this gives:

$$A\lambda^{t-1}(\lambda - c) = 0 \tag{2.52}$$

The characteristic equation of (2.50) is $(\lambda - c) = 0$, with characteristic root $\lambda = c$. This gives, as the solution to the homogeneous form of Equation (2.50):

$$Y_t^h = Ac^t \tag{2.53}$$

Note that we really did not have to go through the process of deriving the characteristic equation: its form is obvious from Equation (2.50). This will generally be the case with linear difference equations, as will become particularly clear when we begin to deal with higher order difference equations.

Combining the two parts into the general solution gives:

$$Y_t = Ac^t + Y^* \tag{2.54}$$

and invoking the usual initial condition for Y_0, gives $A = Y_0 - Y^*$, which then yields the dynamic form:

$$Y_t = (Y_0 - Y^*)c^t + \left(\frac{1}{1-c}\right)(I + G + C_0) \tag{2.55}$$

We noted above that the equilibrium for this version of the dynamic problem is the same as the equilibrium for the static problem. The stability of that equilibrium depends on the value of c, the root of the equation. On the usual Keynesian assumption, that c is a positive fraction, the equilibrium will be stable, and the system will approach equilibrium at a rate which depends on the value of c: as we noted earlier the root of a FODE shows the proportion of the initial disequilibrium which remains to be closed after one period has passed.

In the analysis of an equation like (2.55) we usually assume that the system was initially in equilibrium, so that our initial value, Y_0, was actually the original equilibrium value of income. The system is then shocked, perhaps by an increase in government spending, which causes the equilibrium to move to a new value as shown by Equation (2.51). The actual level of income then responds to the increase in equilibrium, rising towards it as time passes. Because this is a FODE model, it cannot display true cyclical behaviour, but shocks to G and I can still translate into fairly elaborate time paths of income.

Lagged income

An alternative approach to making the Keynesian model dynamic would be to make the adjustment of income itself explicitly dynamic. The Keynesian Cross model is demand-driven, with supply responding to changes in demand – one way of putting dynamics into the model is to make supply respond with a lag. In this model, of course, aggregate supply is the same as aggregate income, so we shall put the dynamics into the behaviour of Y.

In this version of the model, we replace the national income identity (2.44), with an expression for aggregate demand, D_t:

$$D_t = C_t + I + G \tag{2.56}$$

To keep the model very simple, we shall remove the dynamic element from consumption, making current consumption depend on current income:

$$C_t = C_0 + cY_t \tag{2.57}$$

but we shall introduce dynamics via the response of income to changes in demand:

$$Y_t - Y_{t-1} = \delta(D_t - Y_{t-1}), \quad 0 < \delta < 1 \tag{2.58}$$

where δ is a speed of adjustment term. This equation says that the change in income between period $t-1$ and period t is some fraction of the excess of demand

in period t over income in period $t-1$. Note that we have written the dynamic element (2.58), in difference form; we have done this because this is the way this sort of adjustment process is typically represented.

Substituting for D_t in Equation (2.58) and rearranging into the form of a FODE:

$$Y_t = \left(\frac{1-\delta}{1-\delta c}\right) Y_{t-1} + \left(\frac{\delta}{1-\delta c}\right)(I + G + C_0) \tag{2.59}$$

A little manipulation reveals that the equilibrium, or particular solution of the first order linear difference Equation (2.59) is:

$$Y^* = \left(\frac{1}{1-c}\right)(I + G + C_0) \tag{2.60}$$

exactly as was the case with the previous two versions of this model. Changing the adjustment dynamics of a model will in general not change its equilibrium, just the way it approaches that equilibrium.

The characteristic root of Equation (2.59) will obviously be:

$$\lambda = \left(\frac{1-\delta}{1-\delta c}\right) \tag{2.61}$$

so while the equilibrium will be the same in the two models, the adjustment speeds will differ unless it happens that (assuming c has the same value in both models) $\delta = (1-c)/(1-c^2)$.

Lagged consumption and income

There are, of course, other ways of making the Keynesian Cross model dynamic. We could introduce marginal propensity to invest, and assume that investment responds to income with a one period lag: $I_t = I_0 + \theta Y_{t-1}$. Alternatively, we could combine the two models we have just been looking at, so that we have:

$$D_t = C_t + I + G \tag{2.62}$$

$$C_t = C_0 + cY_{t-1} \tag{2.63}$$

$$Y_t - Y_{t-1} = \delta(D_t - Y_{t-1}), \quad 0 < \delta < 1 \tag{2.64}$$

In this case, with suitable substitutions our system can be reduced to:

$$Y_t = (1 - \delta(1-c))Y_{t-1} + \delta(C_0 + I + G) \tag{2.65}$$

which yields as its characteristic equation:

$$\lambda - (1 - \delta(1-c)) = 0 \tag{2.66}$$

so the root of the system is $\lambda = (1-\delta(1-c))$ which will be positive since $(1-c)$ is the marginal propensity to save, which is, on the usual assumptions of the model, a fraction, and δ is the income adjustment speed coefficient, also a fraction.

The equilibrium for this system is:

$$Y^* = \left(\frac{1}{1-c}\right)(I + G + C_0) \tag{2.67}$$

so adding the extra bit of dynamics once again does not affect the location of the equilibrium, just the process by which the system converges on equilibrium.

It is worth noting that, in all of the Keynesian Cross cases we have been considering, we ended up with a difference equation of the form:

$$Y_t = \alpha Y_{t-1} + \gamma(I + G + C_0) \tag{2.68}$$

where the α and γ terms have taken on different forms in different versions of the model. This raises a point which will prove to be important in empirical applications of economic dynamics – the fact that the behaviour over time of an economic series, such as gross domestic product (GDP), can be represented as a difference equation does not tell us which of the possible alternative dynamic models for that variable is actually driving it. Determining that requires careful economic and econometric analysis.

A simple Phillips stabilization model

Keynesian Cross models are, of course, associated with fiscal policy, and one obvious extension of the basic Keynesian Cross model involves incorporating a fiscal policy rule into the model. In this section we consider a version of a model developed by Phillips (1954) incorporating a proportional policy rule.

We begin with the model of Equations (2.48) and (2.49) above but now we modify the G_t term:

$$Y_t = C_t + I_t + G_t \tag{2.69}$$

$$C_t = C_0 + cY_{t-1} \tag{2.70}$$

$$G_t = G_0 + G_t^p \tag{2.71}$$

where G_t^p is the proportional fiscal policy component of government spending and G_0 is autonomous spending – spending which is essentially outside discretionary control, or at the very least spending which cannot be adjusted easily for fiscal policy purposes.

The reason this is referred to as a proportional policy model lies in the specification of G_t^p itself:

$$G_t^p = \gamma(Y^F - Y_{t-1}), \quad \gamma > 0 \tag{2.72}$$

where Y^F is full employment income and γ is an adjustment speed coefficient. Equation (2.72) says that this period's discretionary government spending is adjusted, for fiscal policy purposes, to be proportional to the gap between full employment income (or some target based on beliefs about where full employment

income is) and last period's actual income. If the economy was at full employment last period, discretionary spending will equal zero, and if the economy was in an inflationary gap situation last period, with actual income above the full employment level, discretionary spending will be negative, which in a more detailed model would probably mean a combination of spending cuts and tax increases.

Combining these equations gives the difference equation for the model as:

$$Y_t = (c - \gamma)Y_{t-1} + (C_0 + I + G_0 + \gamma Y^F) \tag{2.73}$$

The homogeneous part of this equation is $Y_t - (c - \gamma)Y_{t-1} = 0$, which gives the characteristic equation:

$$\lambda - (c - \gamma) = 0 \tag{2.74}$$

and root $\lambda = (c - \gamma)$. Here λ will probably be less than 1 in absolute value, but if γ is too large relative to the marginal propensity to consume, c, λ could be negative and the system could display alternations. On usual values of c and γ, though, this is highly unlikely. It is much more likely that the system will converge monotonically to equilibrium.

Note that, writing the solution to the homogeneous part as:

$$Y_t^h = A(c - \gamma)^t \tag{2.75}$$

and comparing this with the solution to the homogeneous part of a model without an explicit policy rule built in, $Y_t^h = Ac^t$, we note that the introduction of the policy rule has reduced the magnitude of the root from c to $(c - \gamma)$.

In this case, unlike others, the way we have introduced a new dynamic element has altered the location of the equilibrium. In the present case the equilibrium of the system is:

$$Y^* = \left(\frac{1}{1 - c + \gamma}\right)(C_0 + I + G_0 + \gamma Y^F) \tag{2.76}$$

so the introduction of the target aggregate income term, Y^F, increases the value of the equilibrium relative to the value it would have had there been no proportional fiscal policy rule and had government spending been equal to G_0.

Note also that the value of the Keynesian multiplier is reduced in this model from $1/(1 - c)$ to $1/(1 - c + \gamma)$, so that an exogenous increase in investment has a smaller impact in a model in which the government follows a proportional stabilization rule than it does in a model in which no such rule is followed. Of course, this also means that an exogenous reduction in investment will have a smaller downward effect on equilibrium income in the presence of an explicit stabilization policy rule.[8]

Note one more point about this model: the fact that we have introduced an explicit full employment target in government spending has not reduced our expression for equilibrium income to $Y^* = Y^F$ which would only occur if

$Y^F = (C_0 + I + G_0)/(1 - c)$; that is, if the equilibrium of the system would have been at Y^F anyway.

We can also rewrite Equation (2.76) as:

$$Y^* = \left(\frac{1-c}{1-c+\gamma}\right) Y^O + \left(\frac{\gamma}{1-c+\gamma}\right) Y^F \tag{2.77}$$

$$Y^O = \left(\frac{1}{1-c}\right)(C_0 + I + G_0) \tag{2.78}$$

where Y^O is the value of the no-policy equilibrium.[9] Since the terms multiplying Y^O and Y^F on the right-hand side are fractions which sum to one, the equilibrium in our policy model is a weighted average of the value that the equilibrium would be if the government applied no stabilizing fiscal policy at all, Y^O, and the full employment level, Y^F. This means that the introduction of the fiscal policy rule does pull equilibrium above its no-action level, but is not sufficient to pull it all the way up to full employment.

We will have to defer the question of whether Keynesian fiscal policy can ever actually move the economy to full employment until we have discussed higher order difference equations. Before leaving this very basic Phillips policy model, though, note that in the version which we have been discussing, the current values of both G and C depend on last period's income.

Suppose we modify the consumption function so that current consumption depends on current income, writing $C_t = C_0 + cY_t$, but leave the fiscal policy rule unchanged, dependent on last period's income. The rationale would be that consumers know what their current income is when they are making their spending decisions, but that it takes the government some time to calculate national income, so it can only be known (actually, estimated, but we do not introduce that complication here) after a lag.

Our model now becomes:

$$Y_t = C_t + I_t + G_t \tag{2.79}$$

$$C_t = C_0 + cY_t \tag{2.80}$$

$$G_t = G_0 + G_t^p \tag{2.81}$$

which gives a FODE of form:

$$(1 - c)Y_t = -\gamma Y_{t-1} + C_0 + I + G_0 + \gamma Y^F \tag{2.82}$$

from which we have, as a homogeneous part:

$$Y_t + \left(\frac{\gamma}{1-c}\right) Y_{t-1} = 0 \tag{2.83}$$

The characteristic equation for this homogeneous difference equation is:

$$\lambda + \left(\frac{\gamma}{1-c}\right) = 0 \tag{2.84}$$

giving, as the root for the system $\lambda = -\gamma/(1-c)$ and, as its equilibrium:

$$Y^* = \left(\frac{1}{1-c+\gamma}\right)(C_0 + I + G_0 + \gamma Y^F) \tag{2.85}$$

which is the same as the expression we had for equilibrium income in the previous model, in Equation (2.76). The root of the system, however, is not only different from the previous root, it is negative. We cannot even be as casually confident that, on reasonable values of γ and c, the equilibrium will be stable; stability requires $\gamma < (1-c)$, which would have to be checked empirically.

By changing the lag structure on consumption, then, while we have not changed the equilibrium value of the system, we have changed its dynamic behaviour from monotonic to displaying alternations. The explanation for the difference is fairly straightforward – in the first Phillips model which we considered, introducing the policy function did not add any new time structure to the problem, it just added a couple of terms. The induced, or endogenous elements of current aggregate expenditure, whether private consumption or government fiscal policy spending, all depended on last period's income.

In the second version of the model there is a change in dynamic structure, in that, while the induced part of this period's government spending still depends on last period's income, the induced part of this period's consumption expenditure depends on this period's income. The result is that, even in this simple model, the dynamics becomes rather more complicated. Our usual caveat about being wary of the validity of economic models with negative roots still applies, though, and quite frankly, the really interesting macroeconomic dynamics are associated with models which yield higher order difference equations, so at this point we shall turn from elementary macroeconomics to elementary microeconomics and consider the cobweb model of price behaviour.

The cobweb model

Having repeatedly warned against putting too much faith in models yielding negative roots, it seems appropriate that our first microeconomic example should be a model which really does, quite legitimately, yield negative roots.

The cobweb model, or hog cycle model, as it is also known (in honour of the commodity in whose market it was first analysed) is a simple demand and supply model with a one period response lag built in on the supply side.

On the demand side we have:

$$Q_t^D = \alpha_0 - \alpha_1 P_t + \alpha_2 Y_t \tag{2.86}$$

where Q_t^D is quantity demanded in period t, P_t is price in period t and Y_t is a demand shift factor, typically income. We have written the demand function with a negative sign in front of α_1, so α_1 itself is positive.

On the supply side we have:

$$Q_t^S = \beta_0 + \beta_1 P_{t-1} + \beta_3 Z_t \tag{2.87}$$

where Q_t^S is quantity supplied in period t, P_{t-1} is price in period $t-1$, Z_t is a supply shift factor, and β_1 is positive. Here quantity supplied today depends on yesterday's market price. While this can apply to any commodity with a significant lag between the beginning of the production process and the actual supplying of output to the market, the most common examples are from agriculture, where planting decisions are made on the basis of the price obtaining in the market at the time planting needs to be undertaken, but harvest, and the actual supplying of the output to market, occurs quite some time later. Note that we are assuming that the entire quantity which is produced is supplied to the market; that is, that there is no storage, and that none of the output is destroyed if the market price turns out to be very low at the time when the output is actually brought to market.

We complete the model with the standard equilibrium condition:

$$Q_t^D = Q_t^S \tag{2.88}$$

which means that, in each period, the current market price market adjusts to clear the market. The fact that we are assuming that the market is in equilibrium in each individual period does not mean that the equilibrium of the dynamic model is guaranteed to be stable.

Equation (2.88) is a short-run market clearing condition, giving us what is sometimes referred to as a series of short-run equilibrium prices. The equilibrium of the difference equation characterizing the system gives us what we refer to as the long-run equilibrium price, and the key question so far as the dynamics of the model are concerned is whether the short-run prices eventually converge to the long-run equilibrium price.

Substituting from Equations (2.86) and (2.87) into (2.88) gives:

$$\alpha_0 - \alpha_1 P_t + \alpha_2 Y_t = \beta_0 + \beta_1 P_{t-1} + \beta_3 Zt \tag{2.89}$$

from which we derive a first-order linear difference equation in price:

$$P_t = \left(\frac{\alpha_0 - \beta_0}{\alpha_1}\right) - \left(\frac{\beta_1}{\alpha_1}\right) P_{t-1} - \left(\frac{\beta_3}{\alpha_1}\right) Z_t + \left(\frac{\alpha_2}{\alpha_1}\right) Y_t \tag{2.90}$$

The homogeneous part of Equation (2.90) can be written as:

$$P_t + \left(\frac{\beta_1}{\alpha_1}\right) P_{t-1} = 0 \tag{2.91}$$

giving, as the characteristic equation for (2.90):

$$\lambda + \left(\frac{\beta_1}{\alpha_1}\right) = 0 \tag{2.92}$$

with characteristic root $\lambda = -(\beta_1/\alpha_1)$. Since both α_1 and β_1 are positive, the root is negative.

The equilibrium will be stable so long as $-(\beta_1/\alpha_1)$, which is less than zero, is greater than -1 (<1 in absolute value): a negative fraction. Thus, for stability we need:

$$1 > (\beta_1/\alpha_1) > 0 \tag{2.93}$$

which is satisfied when $\alpha_1 > \beta_1$ which says that for the equilibrium to be stable, the demand curve must be steeper than the supply curve. If we assume that all of the variables in our model are in log form, stability requires that the demand curve be more elastic than the supply curve.

Turning to the particular solution to Equation (2.90), writing the difference equation as:

$$P_t + \left(\frac{\beta_1}{\alpha_1}\right) P_{t-1} = \left(\frac{\alpha_0 - \beta_0}{\alpha_1}\right) - \left(\frac{\beta_3}{\alpha_1}\right) Z_t + \left(\frac{\alpha_2}{\alpha_1}\right) Y_t \tag{2.94}$$

we see that all of the terms on the right-hand side are constants, in the sense that their values are exogenously given, not determined within the model. This suggests that the equilibrium price, P^*, will also be a constant. Setting $P_t = P_{t-1} = P^*$ and solving, we find:

$$P^* = \left(\frac{\alpha_0 - \beta_0}{\alpha_1 + \beta_1}\right) - \left(\frac{\beta_3}{\alpha_1 + \beta_1}\right) Z_t + \left(\frac{\alpha_2}{\alpha_1 + \beta_1}\right) Y_t \tag{2.95}$$

Since $\alpha_0 + \alpha_2 Y_t$ is the value of quantity demanded when $P_t = 0$, which is the horizontal intercept of the demand curve in the usual demand and supply diagram, and $\beta_0 + \beta_3 Z_t$ is the value of quantity supplied when $P_t = 0$, which is the horizontal intercept of the supply curve, P^* will be positive. It is easily established that the expression for P^* in Equation (2.95) is the same as the equilibrium price from a static demand and supply model.

We have established, then, that so long as $\alpha_1 > \beta_1$, the price which the cobweb model eventually settles down to is the same as the static equilibrium price – that is, that once again the value of the equilibrium is not affected by the introduction of a dynamic element. The dynamic element does add something new to the model, though. Beyond the possibility that the equilibrium might be unstable, and the individual period market prices never converge on the long-run equilibrium value is the fact that this is one of the few economic models in which the root of the characteristic equation is negative. As we saw earlier, a negative root means that, even if the equilibrium is stable, the price will be alternately above and below its equilibrium value from period to period. If the long-run equilibrium is stable, then when the short-run equilibrium prices eventually converge on the long-run equilibrium price, price and quantity will settle down and stop jumping around.

The examples we have considered here are probably the most commonly used examples of FODE, primarily because they form the basis for a range of more complicated models. Since the really interesting action begins when we get into the analysis of theoretical models which involve higher order difference equations, we now turn to such models.

3 Second-order difference equations

Introduction

We now turn to the analysis of linear, second-order difference equations (SODEs). In economic applications, these second-order linear difference equation commonly appear in the general form:

$$Y_t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} = g \tag{3.1}$$

where the terms are basically as defined in the chapter on first-order equations.

The solution procedure is also, in general terms, the same as for first-order equations. We begin by looking at the particular, or equilibrium solution to Equation (3.1). Again, we find it by a process of trial and error, starting from the assumption that the functional form of the equilibrium value of Y will be the same as that of the right-hand-side term g, so, as before, if g is a constant we start by trying a constant as the particular solution, Y^*.

In the case of Equation (3.1), substituting $Y_t = Y_{t-1} = Y_{t-2} = Y^*$ and rearranging terms gives:

$$Y^* = \frac{g}{1 + \beta_1 + \beta_2} \tag{3.2}$$

As in the case of a first-order equation, if $(1 + \beta_1 + \beta_2) = 0$, Equation (3.2) will not work as a solution, and our next step is to see what happens if we try, as an equilibrium form, a constant term G multiplied by time t:

$$Y_t^* = Gt \tag{3.3}$$

Substituting into Equation (3.1) gives:

$$Gt + \beta_1 G(t-1) + \beta_2 G(t-2) = g \tag{3.4}$$

from which, since $(1 + \beta_1 + \beta_2) = 0$, yields:

$$G = \frac{-g}{\beta_1 + 2\beta_2} \tag{3.5}$$

This gives Y_t^* as:

$$Y_t^* = \frac{-gt}{\beta_1 + 2\beta_2} \tag{3.6}$$

To check this procedure for finding the particular solution, use Equation (3.6) to determine Y^*, with appropriate adjustments to the 't' terms, then substitute them into Equation (3.1) (for the case where g is a constant) to give:

$$\left(\frac{-gt}{\beta_1 + 2\beta_2}\right) + \beta_1\left(\frac{-g(t-1)}{\beta_1 + 2\beta_2}\right) + \beta_2\left(\frac{-g(t-2)}{\beta_1 + 2\beta_2}\right) = g \tag{3.7}$$

Rearranging terms, and remembering that we are dealing with the case when $(1 + \beta_1 + \beta_2) = 0$, shows that Equation (3.7) holds. In general, the procedure for finding the particular, or equilibrium solution for a SODE is the same as that for the first, so we shall spend no more time on it.

Characteristic roots

Real roots

The procedure for finding the solution to the homogeneous part is also basically the same. The homogeneous part of Equation (3.1) is:

$$Y_t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} = 0 \tag{3.8}$$

Again, as in the first-order case, we try a solution of the general form:

$$Y_t^h = A\lambda^t \tag{3.9}$$

Substituting Equation (3.9) into (3.8), again adjusting the time elements as necessary, gives us:

$$A\lambda^t + \beta_1 A\lambda^{t-1} + \beta_2 A\lambda^{t-2} = 0 \tag{3.10}$$

from which we find the characteristic equation:

$$\lambda^2 + \beta_1\lambda + \beta_2 = 0 \tag{3.11}$$

This is a quadratic equation, which has two solutions, found from the general expression:

$$\lambda_{1,2} = \frac{-\beta_1 \pm \sqrt{(\beta_1^2 - 4\beta_2)}}{2} \tag{3.12}$$

which gives us two possible roots for our characteristic equation[1] and two possible solutions to the homogeneous form:

$$\begin{aligned} Y_t^h &= A_1\lambda_1^t \\ Y_t^h &= A_2\lambda_2^t \end{aligned} \tag{3.13}$$

where we have put subscripts on the 'A' terms to help distinguish the cases. From the way these solutions were found, we know that either one of them will work as a solution to our homogeneous equation. Fortunately, it turns out that according to a result known as the superposition theorem, we can combine the two solutions into one:

$$Y_t^h = A_1\lambda_1^t + A_2\lambda_2^t \tag{3.14}$$

As in the case of first-order equations, the solution to the homogeneous part of the difference equation controls the dynamics of the system. In this case, there is obviously going to be at least one twist to take account of – in expression (3.12), the term $(\beta_1^2 - 4\beta_2)$, which is referred to as the discriminant of the roots, could perfectly well be negative, meaning that, when we take its square root, we will be dealing with complex numbers. Since this term appears in the expression for both roots, we know that if one of the roots is a complex number the other must also be a complex number – its complex conjugate. The case where the roots of our characteristic equation are complex numbers actually turns out to be very important in economic applications. We will deal with this issue shortly – for the moment, we shall assume that $(\beta_1^2 - 4\beta_2)$ is positive and that both of our roots are real.

Even when we restrict ourselves to the state where λ_1 and λ_2 are real, we still have four cases to deal with. Either of the two roots could be positive or negative, and either could be greater or less than 1 in absolute value. Note that we say either – if one of the roots is less than 1 in absolute value, nothing requires that the other root also be a fraction. In fact, in many economic applications, one root is a fraction while the other is bigger than one. These cases are often referred to as being inside and outside the unit circle, respectively – which is an alternative terminology to saying that a root is smaller or larger than 1 in absolute value.[2]

Both roots are positive fractions

First, consider the case where both roots are positive and inside the unit circle – that is, both are positive fractions. For the moment we have no information about the A terms, not even whether they are positive or negative, so we shall begin by assuming that both A_1 and A_2 are positive.

Under that assumption, it is fairly easy to characterize the dynamics of the Y_t^h term driven by Equation (3.14). Both roots are positive fractions, so as t goes to infinity each of the two elements which constitute Equation (3.14) go to zero. Which one gets there first (or strictly speaking, gets so close to zero that it might as well be zero, since neither term will actually reach zero until t actually reaches infinity) does not matter for present purposes, since in the long run, both are zero. In this case, the equilibrium of the system will be stable, since Y_t^h will converge on zero regardless of where it happened to start from. We refer to this as the case where both roots are stable – see example of this type of behaviour in Figure 3.1(a). Not surprisingly, it is indistinguishable from the graph we drew for the case of a FODE with a stable root.

Both roots are positive and greater than 1

The second case in which the dynamic behaviour of Y_t^h is easy to establish is the case where both roots are positive and outside the unit circle, that is, positive and greater than 1 in absolute value. In this case, both parts of Equation (3.14) get steadily bigger as time passes, converging on infinity as t goes to infinity, so Y_t^h is also driven to infinity, or, putting it another way, Y_t^h diverges. We refer to this as the case where both roots are unstable – for an example of this sort of trajectory see Figure 3.1(b).

When both our roots are real and stable, we refer to the equilibrium as a stable node, and when both roots are real and unstable, the equilibrium is called an unstable node.

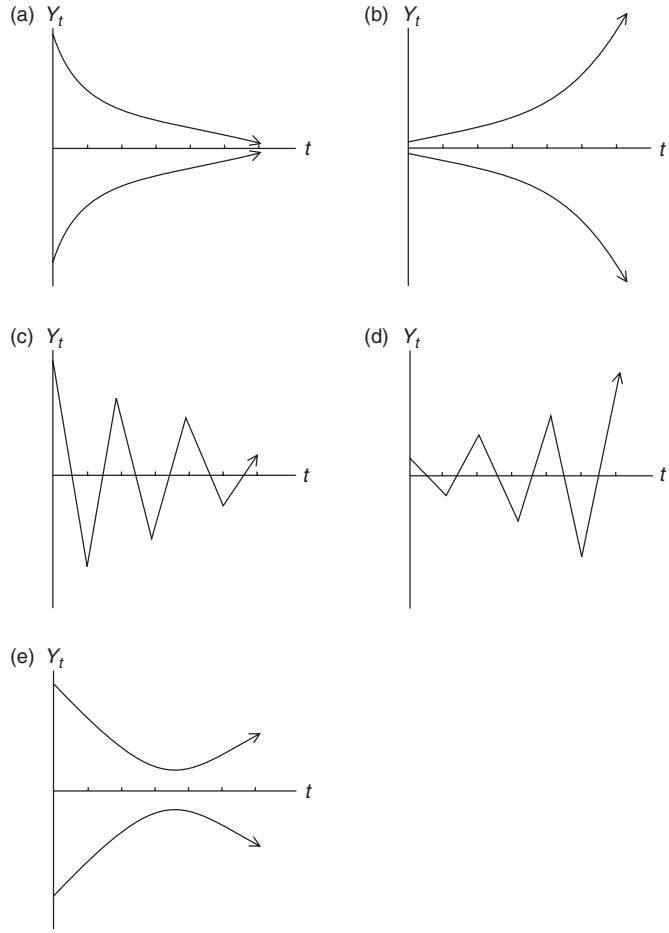

Figure 3.1 Dynamic behaviour of SODEs.

Both roots are negative

In this case, the equilibrium of the system will be stable if both roots are negative fractions and the equilibrium will be unstable if both are negative and outside the unit circle. As in the case of first-order equations, negative roots yield alternating behaviour, with the components of Equation (3.14) jumping from above to below and back above the equilibrium again, regardless of whether the roots are stable or unstable. Examples of time paths for the stable and unstable cases are illustrated in Figure 3.1(c) and (d). If we do happen to have two negative roots, there would seem to be a lot of scope for irregularities in the time path of the variable.

One stable and one unstable root

There is no reason, as it turns out, for both roots to be inside or both outside the unit circle, despite the fact that we started with those cases. We could perfectly well have one root inside and the other outside the unit circle – that is, we could perfectly well find that we had one stable and one unstable root, a case which turns out to be very important in problems in which the dynamic behaviour of some variable is a result of the intertemporal optimization decisions made by an economic agent.

Suppose that both roots are real and positive, and that λ_1 is inside the unit circle while λ_2 is outside, so that λ_1 is a stable root and λ_2 is an unstable root. In this case, regardless of the values of A_1 and A_2, the first element of Equation (3.14) will behave like a stable FODE, converging on zero, and the second part will behave like an unstable FODE, diverging, or heading off to infinity.[3]

The behaviour of Y_t^h will be the sum of the behaviours of its two component parts, and this means that ultimately the unstable part will dominate, since as time passes it will be getting steadily bigger while the stable part is getting steadily smaller, with the unstable part heading to infinity as the stable part heads to zero. Ultimately, in fact, the unstable part will be so much bigger than the stable part that the system will appear virtually to be moving along a trajectory driven by a single, unstable root of value λ_2.

We say 'ultimately' because it is quite possible for the system initially to appear to be following a stable path – that is, a path to a stable equilibrium. Suppose we have the case which we have just been discussing, suppose that both A_1 and A_2 are positive, and suppose that A_1 turns out to be very large and A_2 to be very small in absolute value. In that case, for values of t close to zero, the first term could easily dominate the second in the sum which yields the value of Y_t^h.

This situation could, depending on the relative magnitudes of the A terms, continue for some time before the unstable term actually came to dominate. Eventually, however, the unstable term must dominate, so that even if Y_t^h spends some considerable time apparently on a convergent path, which in the case of a homogeneous equation means heading towards zero, it must eventually turn around and head away from zero.

If we plot the value of Y_t^h against time, with time on the horizontal axis of our graph, this case would be as in Figure 3.1(e), with the time path of Y_t^h initially heading towards zero, then turning around and heading away. During the convergent stage of its time path the slope of the plot against time in Figure 3.1(e) is negative, during the divergent stage the slope is positive, and at the turnaround point the slope is zero, so we could use the derivative of Equation (3.14) with respect to time to solve for the value of t at which the slope becomes zero. If we do, we will find that it depends in part on the relative magnitudes of the A terms.

Clearly, there will exist a whole range of specific possible cases where one root is stable and the other unstable, but their time path diagrams will for the most part look like Figure 3.1(e). In some cases we will only observe divergent behaviour, but that will be a matter of the relative magnitudes of the A terms. If A_2 is sufficiently large, the unstable term will dominate from the beginning.

This type of time path, which arises when our SODE has one stable and one unstable root, is known as saddlepoint behaviour and the associated equilibrium is referred to as a saddlepoint.

Clearly, when we find that our equilibrium is a saddlepoint we should expect ultimately to observe the actual value of Y diverging from its equilibrium value, even if it seems initially to be converging on it. There is, however, one case in which a system with a saddlepoint equilibrium will actually converge on that equilibrium as time goes to infinity – that is, one case in which the unstable root will not eventually dominate. That is the case in which A_2, the term which determines the weight of the unstable part of Equation (3.14) gets in the overall behaviour of Y_t^h, is equal to zero. In that case, Y_t^h will behave as if it were the solution to the homogeneous part of a FODE, driven by a single, stable root.

When we discuss the determination of the values of A terms we will consider how this case can arise. It actually arises quite regularly when the values of the variables in the model are determined as the solution to an intertemporal optimization problem. When there is no optimization involved, and therefore no deliberate control of the variables, saddlepoints are much less likely to be observed, and when they are, the system is much less likely to be on the convergent path.

When we do have the case where A_2 is zero, so the system behaves as if it had only one root, and that root stable, we refer to the system as being on the stable branch to the equilibrium. The opposite case, where A_1 is zero and A_2 non-zero, so the system behaves as if it were a FODE with a single, unstable root, is referred to as being on the unstable branch. These two cases are extremely unlikely to occur by chance. In the vast majority of cases in which the problem being studied does not involve intertemporal optimization, the time path our variable actually follows will be a mixture – a weighted average – of the stable and unstable branches.

Complex roots

At this point we can no longer avoid dealing with the case where the roots of Equation (3.14) have imaginary parts – that is, are complex. This case arises when,

in Equation (3.12), the discriminant $(\beta_1^2 - 4\beta_2)$ is negative, so that in the process of solving for the roots we have to take the square root of a negative number.

Taking the square root of a negative number involves invoking 'i', defined as the square root of negative 1:

$$i = \sqrt{-1} \tag{3.15}$$

from which $i^2 = -1$. In the case of our expression (3.12), we can most easily see how i enters the problem by rewriting the expressions for the two roots as:

$$\lambda_{1,2} = \frac{-\beta_1 \pm \sqrt{-1(4\beta_2 - \beta_1^2)}}{2} = \frac{-\beta_1 \pm i\sqrt{(4\beta_2 - \beta_1^2)}}{2} \tag{3.16}$$

where the term $(4\beta_2 - \beta_1^2)$ is, by definition, positive in this example. In Equation (3.16) we are still taking the square root of a negative number, but now we have written the expression whose square root is being taken as the product of a negative term, -1, and a positive term, $(4\beta_2 - \beta_1^2)$.

The roots can also be written as:

$$\lambda_{1,2} = \left(\frac{-\beta_1}{2}\right) \pm \frac{i\sqrt{(4\beta_2 - \beta_1^2)}}{2} \tag{3.17}$$

which may be expressed as a complex conjugate pair[4] of values:

$$w \pm iz \qquad w = (-\beta_1/2) \quad z = \sqrt{(4\beta_2 - \beta_1^2)}/2 \tag{3.18}$$

Whenever we have an expression of the form $w \pm iz$, we can define terms which we denote by r and ω, where r, which is termed the modulus or absolute value of the complex conjugate pair, is defined as:

$$r = \sqrt{(w^2 + z^2)} \tag{3.19}$$

and ω is defined implicitly from the expressions:

$$r \cdot \cos(\omega) = w \tag{3.20}$$

$$r \cdot \sin(\omega) = z \tag{3.21}$$

Then any expression of the form $w \pm iz$ can also be written in the form $r(\cos(\omega) \pm i \sin(\omega))$. So we now have rather more complicated looking expressions for λ_1 and λ_2.

At this point it is worth going back and remembering that we started from expression (3.14), which, in view of the fact that λ_1 and λ_2 are a complex conjugate pair, we can also write as:

$$Y_t^h = A_1(w + iz)^t + A_2(w - iz)^t \tag{3.22}$$

and we should also note that $(w \pm iz)^t$ can be written:

$$(w \pm iz)^t = r^t(\cos(\omega) \pm i \cdot \sin(\omega))^t \tag{3.23}$$

This last step is important because, by a mathematical result known as De Moivre's Theorem, an expression of the form $(\cos(\omega) \pm i \cdot \sin(\omega))^t$ can also be written as $(\cos(t\omega) \pm i \cdot \sin(t\omega))$. Thus, our λ terms can also be written:

$$\lambda_1^t = r^t(\cos(t\omega) + i \cdot \sin(t\omega)) \tag{3.24}$$

$$\lambda_2^t = r^t(\cos(t\omega) - i \cdot \sin(t\omega)) \tag{3.25}$$

Next, we substitute these expressions into Equation (3.22), giving:

$$Y_t^h = A_1 r^t(\cos(t\omega) + i \cdot \sin(t\omega)) + A_2 r^t(\cos(t\omega) - i \cdot \sin(t\omega)) \tag{3.26}$$

While this does not look like much of an improvement over what we had before, especially since it still involves i, the square root of -1, it is a step in the process of eliminating i from the expression for the solution.

Our next step is to group terms, giving:

$$Y_t^h = r^t((A_1 + A_2)\cos(t\omega) + (A_1 - A_2)i \cdot \sin(t\omega)) \tag{3.27}$$

Now, write A_1 and A_2 as an arbitrary complex conjugate pair $v \pm i\xi$ where v and ξ are real numbers. Then $(A_1 + A_2) = 2v$ and $i(A_1 - A_2) = 2i^2\xi = -2\xi$. Denoting $(A_1 + A_2)$ by B_1 and $i(A_1 - A_2)$ by B_2, we now have:

$$Y_t^h = r^t(B_1 \cos(t\omega) + B_2 \sin(t\omega)) \tag{3.28}$$

and while all of that might seem a lot of work just to get an expression which still involves sines and cosines, we have succeeded in eliminating the 'i' term – everything in Equation (3.28) is real.

The reason this expression is important is what it tells us about the behaviour of a SODE with a complex conjugate pair of roots. Consider first the expression inside the brackets in Equation (3.28). Sines and cosines are cyclical variables, and the fact that we are taking the sin and cos of a term, $t\omega$, which involves t means that we are taking the sin and cos of an expression whose value changes as time passes. In other words $\cos(t\omega)$ and $\sin(t\omega)$ are not constants – their values change as the value of $t\omega$ changes with t. This in turn imparts a cyclical element to the behaviour of Y_t^h: as time passes its value will follow a path which depends in part on a pair of cyclical elements.

We say 'in part' because we have not yet discussed the role of the r^t term, where r is the modulus of the roots. From the expression for r, and the convention that, unless otherwise indicated when we take the square root of a number we take the positive square root, we see that r will be a positive real number. The value of r^t will necessarily change over time, unless $r = 0$ or $r = 1$. Excluding those cases, if $r > 1$, then as time passes, r^t will get steadily bigger. On the other hand, if $r < 1$, a positive fraction (since r is not negative), then as time passes, r^t will converge on zero.

Putting all of this together, as time passes the element inside the brackets in Equation (3.28) will follow a regular cyclical path of constant amplitude and frequency, with each of its elements cycling around zero but not tending to converge or diverge, and with specific values repeating at regular intervals forever. This puts the cyclical element into Y_t^h in Equation (3.28). Stability, in the sense of convergence or divergence, comes from the r^t term.

If $r < 1$, then as time passes, r^t will get steadily smaller, converging on zero, so the constant cyclical element inside the square brackets will be multiplied by a term which is getting steadily smaller, to the point that eventually the product of the two, which is the value of Y_t^h, will converge on zero. We refer to this as a stable cycle, and its behaviour is as shown in Figure 3.2(a).

On the other hand, if $r > 1$, the element multiplying the cyclical term will be growing steadily as time passes, causing the product of the two elements in Equation (3.28) to become bigger in absolute value. There will still be a regular cyclical pattern, but as time passes its amplitude will grow, yielding a time path as shown in Figure 3.2(b). We refer to this as an unstable cycle.

The modulus of the roots, then, determines the stability or instability of the equilibrium. The modulus, r, in the notation we have been using, (see Equation (3.19)) collapses to:

$$r = \sqrt{\beta_2} \tag{3.29}$$

that is r, which determines the stability of the equilibrium in the case of cyclical behaviour, is just equal to the square root of β_2 in the characteristic equation $(\lambda_2 + \beta_1\lambda + \beta_2) = 0$. It might be asked what happens if β_2 is negative: the answer is that in that case the discriminant $(\beta_1^2 - 4\beta_2)$ will be positive, and we will not have complex roots. This, in fact, gives us an easy check on whether cyclical behaviour is possible in any given model.

Before moving on, some pieces of terminology: when our difference equation has complex roots, we refer to the equilibrium of the system as a focus, either stable or unstable depending on the value of the modulus. In between the case of a stable focus and the case of an unstable focus is a case in which the equilibrium is referred to as a centre. This is the case where the modulus of the complex roots equals 1, meaning that the system cycles around its equilibrium point, neither converging nor diverging as time passes as shown in Figure 3.2(c). Clearly the coefficients of the system would have to take on a very precise set of values for the equilibrium to be a centre, and a small change in those coefficients would be all it took to change the equilibrium from a centre to a stable or unstable focus.

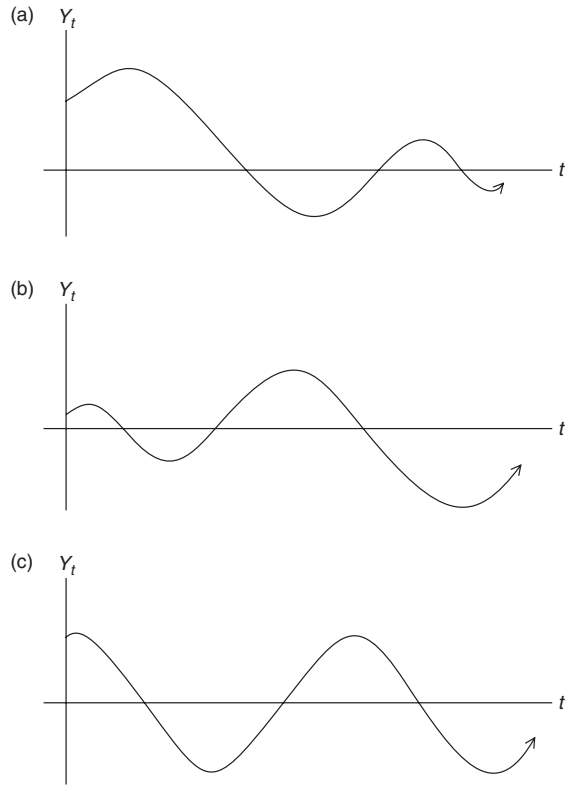

Figure 3.2 Cases of complex roots.

Properties of the characteristic equation

There are actually a number of bits of information that can be derived directly from the characteristic equation of a SODE, here written as:

$$\lambda^2 + \beta_1\lambda + \beta_2 = 0 \tag{3.30}$$

Gandolfo (1997) demonstrates a number of useful results.

Sign test

Consider first the case of a positive discriminant, so we know the roots are real. Then given the characteristic equation, we can invoke Descartes' Theorem, which says that, for our characteristic equation, the number of positive roots cannot exceed the number of changes in sign of the coefficients of the equation while the number of negative roots cannot exceed the number of continuations of sign.

To look for changes or continuations in sign, we read signs of the coefficient (the β's) in the characteristic equation from left to right, noting that since the coefficient on the first term is always 1 the sign of the first coefficient is always positive. Thus, when β_1 and β_2 are both positive, the sign pattern is $(+ + +)$, which displays two continuations and no changes in sign. This pattern means that the equation will have two negative roots. To guarantee that there will not be alternations we need two positive roots, which means, in a second-order equation, that we need to look for two changes in sign, or a sign pattern of $(+ - +)$.

When the term $\beta_1 = 0$, so our characteristic equation becomes $(\lambda^2 + \beta_2) = 0$, the discriminant of the roots becomes $-4\beta_2$. For the roots to be real we require β_2 to be negative, and the equation then factorizes into $(\lambda + \sqrt{\beta_2})(\lambda - \sqrt{\beta_2})$, giving us the sign pattern $(+\ 0\ -)$. In this case, as is clear from our discussion in this paragraph, the roots are real and of opposite sign, but equal in absolute value.

Stability test

We can also derive some information about the stability of the equilibrium directly from the characteristic equation. Gandolfo (1997) demonstrates that necessary and sufficient conditions for stability, for both the case of real roots and the case of complex roots, are:

$$1 + \beta_1 + \beta_2 > 0 \tag{3.31}$$

$$1 - \beta_2 > 0 \tag{3.32}$$

$$1 - \beta_1 + \beta_2 > 0 \tag{3.33}$$

In other words, if all three of these conditions are satisfied the roots of the characteristic equation will be stable regardless of whether they are real or complex (that is the sufficiency part) and if any of them are violated (necessity) the roots will not be stable.

To see where these conditions come from, consider the expressions for the roots λ_1 and λ_2:

$$\lambda_{1,2} = \frac{-\beta_1 \pm \sqrt{(\beta_1^2 - 4\beta_2)}}{2} \tag{3.34}$$

As we have written the roots, assuming for the moment that they are real, it can be shown that $\lambda_1 > \lambda_2$.

In this case, when we are evaluating the stability of the system, we are testing whether at least one of the roots is bigger than one in absolute value, which means greater than 1 or less than (i.e. a bigger negative number than) -1. In practice, we do not need to look for conditions that would place both roots outside the unit circle: if the larger of the two roots is greater than 1, the system is unstable regardless of the value of the smaller root, and if the smaller root is less than -1 (i.e. is a negative number greater than 1 in absolute value), the system is unstable regardless of the value of the larger root.

Similarly, if the larger root is less than 1 the smaller root must also be less than 1, and if the smaller root is greater than -1 (so that if it is negative, it is a negative fraction) the larger root must also be greater than -1. If we think of 1 and -1 as being the upper and lower bounds of the unit circle, so long as the value of the larger root is less then the upper bound and, at the same time, the value of the smaller root is greater than the lower bound, the system must be stable.

We can write the sufficiency condition which we have just derived as:

$$\lambda_1 = \frac{-\beta_1 + \sqrt{(\beta_1^2 - 4\beta_2)}}{2} < 1 \tag{3.35}$$

$$\lambda_2 = \frac{-\beta_1 - \sqrt{(\beta_1^2 - 4\beta_2)}}{2} > -1 \tag{3.36}$$

Manipulating Equation (3.35) yields $(1 + \beta_1 + \beta_2) > 0$, which is just Equation (3.31) above. Similarly, we can derive Equation (3.33) from (3.36). Thus, if the roots of the system are real Equation (3.31) and Equation (3.33) are satisfied, both roots must lie inside the unit circle, while Equation (3.32) is the stability condition for the case of complex roots. If all three of Equations (3.31), (3.32), (3.33) are satisfied, then, our system must be stable regardless of whether the roots are real or complex.

We also note here that from Equations (3.35) and (3.36), the sum of the roots, $(\lambda_1 + \lambda_2) = -\beta_1$ while the product of the roots, $\lambda_1\lambda_2 = \beta_2$. These relations can clearly be helpful to us in determining the stability of the system: if, for example, $\beta_2 > 1$, then at least one of the roots must be greater than 1 since if both were fractions their product would also be a fraction. Unfortunately, their product could be less than 1 even if one of the roots was greater than 1. This relation is the basis for condition (3.32), and explains why it applies to the case of real roots as well as to the case of complex roots.

If the roots are of opposite sign, β_2 will be negative and $1 - \beta_2$ will be positive regardless of the magnitude of the roots, but if both roots have the same sign, β_2 will be positive and violation of Equation (3.32) means that at least one of them must be greater than 1 in absolute value – that is, there must be an unstable root.

Similarly the sum of the roots could be greater than 1 even if both roots were less than 1, and, when one of the roots is negative, their sum could be less than 1 even if both were greater than 1 in absolute value. If the sum is greater than 2, though, one of them must be greater than 1. With difference equations it is easier to establish sufficient conditions for instability than it is to establish sufficient conditions for stability, at least for conditions which do not require actually calculating the values of the roots.

Unit roots

We can extend these results to add one which will be of importance when we discuss econometric applications of difference equations. Consider the case where

$(\beta_1 + \beta_2) = -1$, so the first of Gandolfo's conditions (3.31) is obviously violated. Then if we calculate the roots of our characteristic equation, letting $\beta_2 = (-\beta_1 - 1)$ we derive the roots of our characteristic equation as:

$$\lambda_1 = 1, \quad \lambda_2 = -(1 + \beta_1) \tag{3.37}$$

In other words, we have established a condition under which a SODE will have a root equal to 1; a unit root.

Repeated roots

Another special case arises when the discriminant $(\beta_1^2 - 4\beta_2) = 0$. In this case our roots are:

$$\lambda_1 = \lambda_2 = \frac{-\beta_1}{2} \tag{3.38}$$

that is, we have repeated roots. In this case, in order to find the second solution to our homogeneous equation, consider writing the homogenous equation as:

$$t\lambda^t + \beta_1(t-1)\lambda^{t-1} + \beta_2(t-2)\lambda^{t-2} = 0 \tag{3.39}$$

that is multiply each λ term by a value equal to the power on λ. Dividing Equation (3.39) through by λ^{t-2}, and substituting what we know: $\lambda = -\beta_1/2$, and $\beta_2 = \beta_1^2/4$ into Equation (3.39) shows that (3.39) indeed holds. All of which is to say that using an expression of the form $x\lambda^x$ as we did in Equation (3.39) also satisfies our original equation, which means that our solution form, which is generally written as $Y_t^h = A_1\lambda_1^t + A_2\lambda_2^t$ can, in the case of a single repeated root, be written:

$$Y_t^h = A_1\lambda^t + A_2 t\lambda^t \tag{3.40}$$

The reason this result is useful is because it will let us solve for expressions for the unknown constants A_1 and A_2 even when we have a repeated root. So we now turn to the question of solving for A_1 and A_2.

Completing the solution

As in the case of FODE, we leave solving for A_1 and A_2 to the end of the exercise. As in the first-order case, we begin by combining the particular solution with the solution to the homogeneous form, giving:

$$Y_t = A_1\lambda_1^t + A_2\lambda_2^t + Y^* \tag{3.41}$$

In the first-order case we had a single unknown constant to solve for, so we needed a single piece of outside information – a single initial condition. This time we have two constants to solve for so we need two initial conditions. As in the first-order case there are many possible initial conditions, but, as in the first-order case, the

most common pieces of information are indeed initial – we usually assume that we know the actual value of Y_t at $t = 0$ and $t = 1$. Substituting into Equation (3.41), we have:

$$Y_0 = A_1 + A_2 + Y^* \tag{3.42}$$

$$Y_1 = A_1\lambda_1 + A_2\lambda_2 + Y^* \tag{3.43}$$

Since Y_0 and Y_1 are assumed to be known, and we have already solved for λ_1 and λ_2, and for Y^* (which again need not be a constant), the only unknowns in Equations (3.42) and (3.43) are the terms A_1 and A_2. This means that, in Equations (3.42) and (3.43), we have a pair of linear equations in two unknowns, A_1 and A_2.

Solving for A_1 and A_2, we find:

$$A_1 = \frac{\lambda_2(Y_0 - Y^*) - (Y_1 - Y^*)}{(\lambda_2 - \lambda_1)} \tag{3.44}$$

$$A_2 = \frac{-\lambda_1(Y_0 - Y^*) + (Y_1 - Y^*)}{(\lambda_2 - \lambda_1)} \tag{3.45}$$

This time we do not have the neat interpretation that one of the A terms is the initial disequilibrium, although by definition (from Equation (3.42)), the two A terms sum to the initial disequilibrium.

Given expressions (3.44) and (3.45), we can now return to something we referred to in our discussion of the saddlepoint case, the case where we had one stable and one unstable root. In our discussion above, we assumed that λ_1 was the stable root and λ_2 the unstable root (although that was just for convenience – it could perfectly well be the other way around), and we said that the system would actually converge to its equilibrium in the case where A_2 was equal to zero. From Equation (3.45), this will happen when:

$$\lambda_1 = \frac{Y_1 - Y^*}{Y_0 - Y^*} \tag{3.46}$$

When Equation (3.46) is satisfied, then, our system is on the stable branch to the equilibrium. In practical terms, this means that, when we look at its evolution over time, the system will behave as if it were a first-order system with a stable root. From (3.44), if:

$$\lambda_2 = \frac{Y_1 - Y^*}{Y_0 - Y^*} \tag{3.47}$$

then $A_1 = 0$ and we will be on the unstable branch, which means that, observationally, the system will behave as if it were a first-order system with a single, unstable root.

g is a function of time

It should be clear by now that an economic system which can be represented by a SODE is capable of displaying a range of interesting behaviour. Suppose, for

example, that in our original Equation (3.1), the 'g' term is an exponential function of time:

$$g = G\delta^t \tag{3.48}$$

We handle this case exactly as we did its first-order counterpart – try, as our Y^* function, an expression of the form:

$$Y^* = B\delta^t \tag{3.49}$$

then, after a bit of manipulation, we find:

$$B = \frac{G\delta^2}{\delta^2 + \beta_1\delta + \beta_2} \tag{3.50}$$

where everything on the right-hand side of Equation (3.50) is a known value. This gives:

$$Y_t^* = \left(\frac{G\delta^2}{\delta^2 + \beta_1\delta + \beta_2}\right)\delta^t \tag{3.51}$$

so, as in the first-order case, the equilibrium value of Y moves as time passes, following an exponential time path of its own. The solution equation is now:

$$Y_t = A_1\lambda_1^t + A_2\lambda_2^t + \left(\frac{G\delta^2}{\delta^2 + \beta_1\delta + \beta_2}\right)\delta^t \tag{3.52}$$

Now suppose that the roots of our second-order equation are a complex conjugate pair, with a modulus which makes the system stable. In that case, our system will follow a cyclical path, converging on its equilibrium value, and that equilibrium value will itself be moving along an exponential path. The time path of Y can get interesting, and we are only up to second-order equations.

Not surprisingly, higher order systems can have even more interesting dynamics. Unfortunately, we cannot draw phase diagrams for difference equations of order greater than 1, or at least not easily, so it is very seldom done. If we want to look at time paths, we generally have to simulate the system. But, before turning to higher order system, we consider some examples of economic models which yield SODE.

Examples of SODE models

The multiplier-accelerator model

If the Keynesian Cross multiplier model is one of the most basic of all FODE models in economics, its extension to the multiplier-accelerator model is one of

the most basic of all SODE models. In this model, we add to the simpler model an investment equation, giving:

$$Y_t = C_t + I_t + G \tag{3.53}$$

$$C_t = C_0 + cY_t, \quad 0 < c < 1 \tag{3.54}$$

$$I_t = I_0 + v(C_{t-1} - C_{t-2}), \quad v > 0 \tag{3.55}$$

Equation (3.55) says that investment has two components – the first an autonomous element, which in the multiplier model was the whole of the investment term, and the second, a term which depends, with a lag, on the change in consumption spending. In this model the response of investment spending to consumer spending is generally explained as meaning that investment spending responds to expectations of profit, that profits increase with increased consumer spending, and that expectations about future profits are formed myopically – if consumer spending was up this period it is expected that next period will also be good. The lag in Equation (3.55), which says that investment spending today depends on yesterday's increase in consumer spending, reflects lags in converting investment plans into investment spending.

To derive a difference equation in Y from this model,[5] note that Equation (3.54) says that consumption in any period, t, $t-1$, $t-2$ etc., depends on income in that period according to a Keynesian consumption function. This lets us replace C_{t-1} and C_{t-2} with expressions in Y_{t-1} and Y_{t-2}, giving:

$$I_t = I_0 + vc(Y_{t-1} - Y_{t-2}) \tag{3.56}$$

Then, substituting Equations (3.56) and (3.54) into (3.53) and rearranging terms gives as the difference equation for the model:

$$Y_t(1-c) - vcY_{t-1} + vcY_{t-2} = C_0 + I_0 + G \tag{3.57}$$

Because the right-hand side of Equation (3.57) is composed of constants, and variables which are exogenous to the model, the equilibrium value of income will itself be a constant:

$$Y^* = \frac{C_0 + I_0 + G}{1-c} \tag{3.58}$$

so, if the value of I_0 in this model is the same as the value of investment in the multiplier model, (and the C and G terms also have the same values as in the simpler model) the value of equilibrium income in the multiplier-accelerator model will be the same as the value of equilibrium income in the multiplier model.

Turning to the dynamics of the system, write the homogeneous part of Equation (3.57) as:

$$Y_t - \left(\frac{vc}{1-c}\right) Y_{t-1} + \left(\frac{vc}{1-c}\right) Y_{t-2} = 0 \tag{3.59}$$

whose characteristic equation is:

$$\lambda^2 - \left(\frac{vc}{1-c}\right) \lambda + \left(\frac{vc}{1-c}\right) = 0 \tag{3.60}$$

The roots of this expression are:

$$\lambda_{1,2} = \frac{(vc/(1-c)) \pm \sqrt{(vc/(1-c))^2 - 4(vc/(1-c))}}{2} \tag{3.61}$$

At first glance, expression (3.61) is not terribly informative. We can, however, apply some of the results we cited earlier. First, the pattern of the signs of the elements of Equation (3.60) is $(+ - +)$ which, from Descartes' rule of signs, tells us that, if our roots are real, both are positive. We actually have an alternative approach to establishing this result: since the product of the roots, $vc/(1-c)$, is positive, the roots must have the same sign, either both positive or both negative. Also the sum of the roots, $vc/(1-vc)$, is positive. Since the roots, if real, must have the same sign, and they must sum to a positive value, both roots must be positive.

Applying conditions (3.31), (3.32), (3.33) to this problem, we see that Equations (3.31) and (3.33) are satisfied (assuming $(1-c)$ is a positive fraction). To satisfy condition (3.32) we require:

$$v < \frac{(1-c)}{c} \tag{3.62}$$

This clearly puts a tight limit on the values of v which are consistent with stability: if we take the common textbook value of $c = 0.8$, then stability requires that v be less than 0.25.

Furthermore, for the system's time path to be monotonic, we require the discriminant to be positive, which in turn requires that:

$$v > \frac{4(1-c)}{c} \tag{3.63}$$

To get a sense of what this implies, if we again set $c = 0.8$, monotonic behaviour requires that v be greater than 1.

Clearly Equations (3.62) and (3.63) cannot both be satisfied at the same time. If Equation (3.62) is satisfied, so that we have a stable equilibrium, then by Equation (3.63) the time path of Y must be cyclical. This particular version of the multiplier-accelerator model, then, imposes cyclical behaviour on the economy.

We say this version of the model because there are other versions of the same basic model. We have made investment depend on lagged changes of consumption and have made the level of current consumption a function of current income. One alternative version would put a lag into the consumption function as well, but that would yield a third-order difference equation, and we are not yet ready to deal with examples of that type of model. Another alternative version would put a one period lag into the consumption function and replace the investment function which we have used with:

$$I_t = I_0 + v(Y_{t-1} - Y_{t-2}), \quad v > 0 \tag{3.64}$$

In this version, investment depends directly on lagged changes in income, and in this case, even if we use, as our consumption function, $C_t = C_0 + cY_{t-1}$, we still wind up with a SODE:

$$Y_t - (c + v)Y_{t-1} + vY_t = C_0 + I_0 + G \tag{3.65}$$

We leave the analysis of this system as an exercise, noting only that in this example it is possible to have a time path which is both monotonic and stable.

Phillips stabilization policy model

For our second economic example of a SODE model, we again extend a model we considered in Chapter 2 on first-order models. In that chapter we introduced Phillips proportional stabilization model; here we add an extra element to the fiscal policy rule.

The basic model is as before:

$$Y_t = C_t + I + G_t \tag{3.66}$$

$$C_t = C_0 + cY_{t-1}, \quad 1 > c > 0 \tag{3.67}$$

$$G_t = G_0 + G_t^p + G_t^d \tag{3.68}$$

$$G_t^p = \gamma(Y^F - Y_{t-1}), \quad \gamma > 0 \tag{3.69}$$

$$G_t^d = -\delta(Y_{t-1} - Y_{t-2}), \quad \delta > 0 \tag{3.70}$$

Here investment is once again exogenous and we have added an extra government spending term G_t^d, which depends on the change in Y between periods $t-2$ and $t-1$. According to this term, if Y growth was positive an increase in that change results in a reduction in government spending. This policy term, known as a derivative policy term, is designed to prevent the economy from growing too quickly and, in a more complete macroeconomic model, letting inflationary pressures build up too quickly.[6]

Substituting Equations (3.67)–(3.70) into (3.66) and rearranging gives:

$$Y_t - (c - \gamma - \delta)Y_{t-1} - \delta Y_{t-2} = C_0 + I + G_0 + \gamma Y^F \tag{3.71}$$

which turns out to have the same expression for Y^* as in the simpler, proportional stabilization model:

$$Y^* = \frac{C_0 + I + G_0 + \gamma Y^F}{1 - c + \gamma} \tag{3.72}$$

Equation (3.72) tells us that, as in the simpler model, the introduction of the policy element does not automatically guarantee that the equilibrium will be at full employment. In fact, unlike the γ term, the δ term does not even enter the expression for the equilibrium. This is no surprise, since the δ term, the derivative stabilization coefficient, relates to the speed at which the system is moving, not to where it is heading.

The characteristic equation for (3.71) is:

$$\lambda^2 - (c - \gamma - \delta)\lambda - \delta = 0 \tag{3.73}$$

with roots:

$$\lambda_{1,2} = \frac{(c - \gamma - \delta) \pm \sqrt{(c - \gamma - \delta)^2 + 4\delta}}{2} \tag{3.74}$$

Stability requires:

$$1 - (c - \gamma - \delta) - \delta > 0 \tag{3.75}$$

$$1 + \delta > 0 \tag{3.76}$$

$$1 + (c - \gamma - \delta) - \delta > 0 \tag{3.77}$$

Looking at these conditions, the first is clearly satisfied under the usual assumptions about the magnitude of the marginal propensity to consume, and the second is satisfied since δ is positive. The third, however, depends on the relative magnitudes of the coefficients, and the best we can do is identify relative magnitudes which would guarantee stability.

Looking at Equation (3.74) we see that the discriminant of the roots is positive, so the roots are real and there will be no oscillations, but looking at Equation (3.73), we see that the sign pattern is either $(+--)$ or $(++-)$; in either case, there is one change and one continuation which, by Descartes' rule, means that we have one positive and one negative root. The presence of a negative root means that, while the system will not display oscillations, it will have an element of alternation to it.[7]

There are many macro models which can be reduced to second or higher order difference equations, and which have at least the potential to yield cyclical behaviour. Perhaps the broadest class of such models is the class of inventory adjustment models, beginning with Metzler (1941). We will return to macro models when we consider higher order systems: for our next example we turn back to microeconomics.

A cobweb model with firm entry

In this example we again return to a model which we saw in Chapter 2 on first-order systems: the cobweb model. This time, we add to the basic model an expression for firm entry. Doing this forces us to do some rather messy manipulation of the model, but we will be able, in Chapter 4, to use this example as the basis of a comparison between two approaches to dealing with models involving several difference equations.

The equations of our cobweb model are:

$$Q_t^D = \beta_0 - \beta_1 P_t + \beta_2 Y_t \tag{3.78}$$

$$Q_t^S = \alpha_0 + \alpha_1 P_{t-1} + \alpha_2 N_t \tag{3.79}$$

$$Q_t^D = Q_t^S \tag{3.80}$$

$$N_t = N_{t-1} + \gamma(P_{t-1} - P^c), \quad \gamma > 0 \tag{3.81}$$

where Q is quantity, P is price, Y is consumer income and N is the number of firms in the market.[8] Equation (3.81) says that the number of firms in the market in period t is equal to the number that were there in period $t-1$ plus an adjustment term which depends on the difference between the price level in $t-1$ and a critical value, P^c. When price in $t-1$ was above the critical value, new firms enter and $N_t > N_{t-1}$, when price in $t-1$ was below the critical value, existing firms leave and $N_t < N_{t-1}$, and when price in $t-1$ just equalled the critical value there was no tendency for firms to enter or leave the industry, so the number of firms remained unchanged between the two periods: $N_t = N_{t-1}$. In the case of a perfectly competitive market, we can think of the critical price level as being equal to the minimum point on the firms' (common) average cost curve. The term γ is an adjustment speed coefficient: the larger γ, the more firms enter or leave in response to a deviation of last period's price from the critical level.

Substituting Equations (3.78) and (3.79) into (3.80) gives us:

$$\beta_0 - \beta_1 P_t + \beta_2 Y_t = \alpha_0 + \alpha_1 P_{t-1} + \alpha_2 N_t \tag{3.82}$$

as in the simple cobweb. The problem is that we now have a difference equation for N, so in Equations (3.81) and (3.82) we have a system of two FODEs in two variables, N and P. Fortunately it turns out that there is a way of collapsing these two equations into a single difference equation.

First, note that because we are assuming that the market is always in short-run equilibrium, Equation (3.82) must always hold. That being the case, we can rearrange Equation (3.82) to obtain an expression for N_t:

$$N_t = \left(\frac{\beta_0 - \alpha_0}{\alpha_2}\right) - \left(\frac{\beta_1}{\alpha_2}\right) P_t - \left(\frac{\alpha_1}{\alpha_2}\right) P_{t-1} + \left(\frac{\beta_2}{\alpha_2}\right) Y_t \tag{3.83}$$

Lagging (3.83) by one period, then gives us an expression for N_{t-1}. Substituting these expressions into (3.81) and rearranging terms gives us a SODE:

$$P_t - \left(\frac{\beta_1 - \alpha_1 - \gamma\alpha_2}{\beta_1}\right) P_{t-1} - \left(\frac{\alpha_1}{\beta_1}\right) P_{t-2} = \left(\frac{\beta_2}{\beta_1}\right) Y_t - \left(\frac{\beta_2}{\beta_1}\right) Y_{t-1} + \left(\frac{\gamma\alpha_2}{\beta_1}\right) P^{\mathrm{c}} \tag{3.84}$$

Note that on the right-hand side of Equation (3.84) we have terms in Y_t and Y_{t-1}. This does not mean that we have a difference equation in Y. To have a difference equation in Y, we would have to have an equation reflecting the mechanism linking current to past values of Y. The presence of Y_t and Y_{t-1} reflects what is known as a lagged adjustment effect, something which we will be dealing with in Chapter 7. For the moment, we finesse the issue by assuming that consumer income is constant over time, so that $Y_t = Y_{t-1} = Y_0$. Conveniently, when we substitute this into Equation (3.84), the right-hand side Y terms disappear and we are left with:

$$P_t - \left(\frac{\beta_1 - \alpha_1 - \gamma\alpha_2}{\beta_1}\right) P_{t-1} - \left(\frac{\alpha_1}{\beta_1}\right) P_{t-2} = \left(\frac{\gamma\alpha_2}{\beta_1}\right) P^{\mathrm{c}} \tag{3.85}$$

Equation (3.85) is a SODE in P. Since we found Equation (3.85) by substituting the demand–supply equality condition directly into the firm entry equation, it combines the information from all of the equations in the system; the presence of the γ term indicates this. It is a bit unfortunate that we have lost sight of the N term, and in Chapter 4 we shall deal with this issue. For the moment, we have derived a SODE in price, which we can now analyse.

Since P^{c} is assumed to be constant (there is no technological change occurring, which might shift the firms' average cost curve), we assume the equilibrium price, P^*, is also constant. Making the usual substitutions in Equation (3.85) we find that:

$$P^* = P^{\mathrm{c}} \tag{3.86}$$

which says that the long-run equilibrium price for the model is the critical price, the price at which the number of firms remains unchanging over time. This is, of course, consistent with the definition of long-run market equilibrium in introductory microeconomic theory, and also bears out our claim that the information contained in Equation (3.81) was not lost to the system in the course of our manipulations. It tells us that if the current price is not equal to P^{c} the system cannot be in equilibrium, and given that P^{c} only appears in the firm entry equation, that must be because when the current price is not equal to P^{c}, new firms will enter or old ones will leave, shifting the supply curve and causing the equilibrium price to change.

Turning to the dynamics of the system, the characteristic equation is:

$$\lambda^2 - \left(\frac{\beta_1 - \alpha_1 - \gamma\alpha_2}{\beta_1}\right)\lambda - \left(\frac{\alpha_1}{\beta_1}\right) = 0 \tag{3.87}$$

The sign pattern of Equation (3.87) depends on the sign of $(\beta_1 - \alpha_1 - \gamma\alpha_2)/\beta_1$, and is either $(+, -, -)$ or $(+, +, -)$. In either case we have one change of sign

and one continuation, so we have one positive and one negative root. We can also tell this from the fact that $-(\alpha_1/\beta_1)$, which is the product of the roots, is negative. If the roots were complex, the final term on the right-hand side of Equation (3.87) would have to be positive, so the fact that it is negative means that the roots are real. Clearly for it to be negative the roots must be of opposite sign. The fact that one of the roots is negative means that the system will display alternations – this is clearly a consequence of the presence of the cobweb elements. Adding the firm entry equation has not changed that.

Checking the stability conditions, we need for stability:

$$1 - \left(\frac{\beta_1 - \alpha_1 - \gamma\alpha_2}{\beta_1}\right) - \left(\frac{\alpha_1}{\beta_1}\right) > 0 \tag{3.88}$$

$$1 + \left(\frac{\alpha_1}{\beta_1}\right) > 0 \tag{3.89}$$

$$1 + \left(\frac{\beta_1 - \alpha_1 - \gamma\alpha_2}{\beta_1}\right) - \left(\frac{\alpha_1}{\beta_1}\right) > 0 \tag{3.90}$$

Expression (3.88) condenses to $\gamma\alpha_2/\beta_1 > 0$ which is clearly satisfied. Condition (3.89) is also satisfied by construction. This leaves us with Equation (3.90) which can be collapsed to $\beta_1 > \alpha_1 + \gamma\alpha_2/2$ where β_1 is the (absolute value of the) slope of the demand curve, α_1 is the slope of the supply curve, γ is the firm entry speed parameter, and α_2 tells us how much the market supply curve shifts in response to the entry of new firms.

In the original cobweb model, stability required that the demand curve be steeper than (or, if the variables are in logs, more elastic than) the supply curve. In the present case that is not enough: the demand curve must be even steeper (or more price elastic) to compensate for the shift of the supply curve due to firm entry.

Basically, an increase in P in period $t-1$ has two effects in period t: it causes existing firms to increase their output by an amount which is determined by the slope of the supply curve, the α_1 term, and it also causes firms to enter. Thus, an increase in P in $t-1$ has a double effect on supply in period t, both effects tending to increase the quantity of output offered for sale on the market. Hence, the more stringent conditions placed on the slope of the demand curve.

Without actually evaluating the roots of Equation (3.87), then, we can say that the roots of the system will be real (so there will not be oscillations in price); that the system will have one positive and one negative root (so there will be alternations in price) and that the stability of the system depends on the slope of the demand curve relative to the two effects reflecting the response of supply in period t to changes in price in period $t-1$.

In developing the cobweb model with firm entry we had to do a fair bit of manipulation, and to collapse several equations into one. Quite a few higher order models can be derived from this kind of manipulation, but it also turns out that we can extract a lot of information out of systems of equations without actually having to collapse them. We shall consider models of this type in the next chapter.

4 Higher-order and systems of difference equations

Higher-order difference equations

It should be clear by now that we can keep adding lags of the Y variable, thereby raising the order of our difference equation. A third-order difference equation, for example, would have the general form:

$$Y_t + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \beta_3 Y_{t-3} = g \tag{4.1}$$

with characteristic equation:

$$\lambda^3 + \beta_1 \lambda^2 + \beta_2 \lambda + \beta_3 = 0 \tag{4.2}$$

The equilibrium value would be:

$$Y^* = g/(1 + \beta_1 + \beta_2 + \beta_3) \tag{4.3}$$

and the general solution is of form:

$$Y_t = A_1 \lambda_1^t + A_2 \lambda_2^t + A_3 \lambda_3^t + Y^* \tag{4.4}$$

where we would need three initial conditions to solve for the A terms.

Since Equation (4.2) has three roots, we now have the possibility of a wide range of time paths – we could now, for example, have one real and two complex roots.[1] Assuming the system was stable, Y would still converge on its equilibrium value over time, but the cyclical element could manifest as cycles around the convergent path generated by the monotonic (stable) root. Empirically, we could wind up with what looked like a very irregular, but still stable, cycle. We could also find ourselves dealing with more complicated saddlepoint behaviour, should we have two stable and one unstable root, for example, or two unstable and one stable.

In the first case, if λ_1 and λ_2 were stable and λ_3 unstable, the system would converge on the equilibrium if A_3 were zero, so that the system behaved as if it were a stable second-order system. In this case the stable branch would be a plane in two dimensions. If λ_1 and λ_2 were unstable and λ_3 stable, we would only converge if both A_1 and A_2 were zero, so that stable branch would now be a line,

but a line in three-dimensional space rather than (as in the second-order saddle point case) a line in two-dimensional space.

While higher order difference equations open a greater range of possible dynamic behaviour, the price of that greater flexibility is reduced analytical tractability. Unlike the second-order case there is no simple formula for the roots of Equation (4.4) (there is a formula, but it is not particularly revealing and involves generating several intermediate expressions).

In general terms, the best we can say is:

$$\lambda_1 + \lambda_2 + \lambda_3 = -\beta_1 \tag{4.5}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \beta_2 \tag{4.6}$$

$$\lambda_1\lambda_2\lambda_3 = -\beta_3 \tag{4.7}$$

which do not, in general, turn out to be terribly helpful unless we have actual, numerical values for the β terms. Expression (4.7) is a special case of the general result that, when a characteristic equation is written so that the coefficient on the highest power term is one, the product of the roots is equal to $(-1)^n$ times the constant term – hence in our second-order case the constant term was the product of the roots and here it is -1 times the product of the roots.

We can also write stability conditions in terms of the β coefficients: the necessary and sufficient stability conditions for Equation (4.1) are:

$$1 + \beta_1 + \beta_2 + \beta_3 > 0 \tag{4.8}$$

$$1 - \beta_1 + \beta_2 - \beta_3 > 0 \tag{4.9}$$

$$1 - \beta_2 + \beta_1\beta_3 - \beta_3^2 > 0 \tag{4.10}$$

A sufficient stability condition for the general case of a third-order difference equation is:

$$\sum |\beta_i| < 1 \tag{4.11}$$

while a necessary stability condition is:

$$-\sum \beta_i < 1 \tag{4.12}$$

Conditions (4.11) and (4.12) also apply to higher order difference equations. They are, however, in general only useful when we can place numerical values on the β terms.

Third and higher order difference equations are more commonly seen in econometric applications than in exercises in pure theoretical modelling, since in econometric applications we have estimates of the numerical values, and modern econometric software can estimate the values of, and perform hypothesis tests on the roots of a higher order difference equation.

Economic example

The simplest example of a model yielding a third-order difference equation is a variant on the multiplier-accelerator model, set out as follows:

$$Y_t = C_t + I_t + G \tag{4.13}$$

$$C_t = C_0 + cY_{t-1} \quad 0 < c < 1 \tag{4.14}$$

$$I_t = I_0 + v(C_{t-1} - C_{t-2}) \quad v > 0 \tag{4.15}$$

Here Equation (4.15) is the same investment function as we used in the second-order case, but we have returned to the consumption function which we used in a first-order example, with current consumption depending on lagged income. Substituting Equation (4.14) into (4.15) gives:

$$I_t = I_0 + vc(Y_{t-2} - Y_{t-3}) \tag{4.16}$$

and substituting Equations (4.14) and (4.16) into (4.13) and arranging gives:

$$Y_t - cY_{t-1} - vcY_{t-2} + vcY_{t-3} = C_0 + I_0 + G \tag{4.17}$$

Note that, from Equation (4.17) the equilibrium value of Y is:

$$Y^* = \frac{(C_0 + I_0 + G_t)}{(1 - c)} \tag{4.18}$$

so, once again, the introduction of a dynamic adjustment element – a lag – does not change the location of the equilibrium, although it can affect the trajectory the system follows as it approaches (in the stable case) that equilibrium.

The characteristic equation for (4.17) is:

$$\lambda^3 - c\lambda^2 - vc\lambda + vc = 0 \tag{4.19}$$

While the necessary stability condition (4.12) holds, the sufficient condition (4.11) does not necessarily hold, but depends on the relative magnitudes of c and v, so all we can say is that stability is not excluded (as it would be if the necessary condition failed to hold). Conditions (4.8)–(4.10) provide no immediate answers – while (4.8) holds, the other two hold only if certain conditions on the relative values of c and v are satisfied. This comes as no particular surprise, since v and c are Keynesian marginal propensity to spend terms, and if the marginal propensity to spend becomes too large, the system will be unstable.

The constant term, vc, which is equal to -1 times the product of the roots, is positive, meaning that the product of the roots is negative, so we have at least one negative root. Since the sign pattern in Equation (4.19) is $(+ - -+)$, the rule of signs also tells us that, if the roots are real, we have at most two positive (two changes of sign) and at most one negative (one continuation of sign) root. The other two roots, then, are either both positive,[2] or a complex conjugate pair.

This, then, is basically as far as we can go at the theoretical level. Given our earlier multiplier-accelerator examples, it is not surprising that we cannot preclude instability, nor is it particularly surprising that Y could follow an interesting time path.[3] The most we can say, though, is that to go any further we need values for the parameters – that is, we need to proceed to empirical implementation of the model.

Systems of difference equations

Rather than proceed further with higher order difference equations, since to do so would only be to add more indeterminacies, in this section we turn to systems of equations – that is, to cases in which we have several difference equations.

We have already dealt with examples of systems – such as in the cobweb example with firm entry, where we started with FODE in P and N, and reduced that system of two FODEs to a single SODE in P. We did that by judicious substitution of equations into each other.

In general, we can use substitution to reduce systems of several difference equations into a single equation, where that single equation will be of higher order than the component equations we started out with. In the cobweb example of a SODE, then, we reduced a system of two FODEs to a single second-order equation; this turns out to be a general result. We can, in general, reduce any system of two interrelated FODEs to a single second-order equation, and we can reduce a system of three interrelated FODEs to a single third-order equation.

To see this, consider the following system of two first-order equations:

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Z_{t-1} \tag{4.20}$$

$$Z_t = \beta_0 + \beta_1 Y_{t-1} + \beta_2 Z_{t-1} \tag{4.21}$$

At risk of belabouring the point, note that Equation (4.20) is a FODE in Y – the presence of the Z_{t-1} term does not alter that. In terms of the behaviour of Y as determined by this equation, Z is exogenous to the equation – part of the g_t term. Similarly, Equation (4.21) is a FODE in Z – it does not matter how many lagged Y terms might appear in Equation (4.21): the order of the equation is determined by the number of differences of the dependent variable.

This system, however, will ultimately reduce to a single SODE because of the feedback elements across the equations. *A* change in the value of Y in this period, period t, will, through Equation (4.21) lead to a change in the value of Z in one period's time, and that change in Z will affect Y a period later. Thus, a change in Y in period t will cause Z to change in $t + 1$, and that will feed back into a change in Y in $t + 2$.

To see how this works, we make the same kind of substitutions as we did in the cobweb example in Chapter 3 on SODEs. First, note that Equations (4.20) and (4.21) must always hold, which means that we can rearrange them without changing their nature. Thus, from Equation (4.20) obtain an expression for Z_{t-1} and then for Z_t (by moving all variables forward by one period) and substitute

these into Equation (4.21) to get:

$$\left(\frac{Y_{t+1}}{\alpha_2} - \frac{\alpha_0}{\alpha_2} - \frac{\alpha_1 Y_t}{\alpha_2}\right) = \beta_0 + \beta_1 Y_{t-1} + \beta_2 \left(\frac{Y_t}{\alpha_2} - \frac{\alpha_0}{\alpha_2} - \frac{\alpha_1 Y_{t-1}}{\alpha_2}\right) \quad (4.22)$$

Now lag Equation (4.22), simply to re-express things from the perspective of period t rather than $t + 1$, and collect terms to get the SODE:

$$Y_t - (\alpha_1 + \beta_2)Y_{t-1} + (\alpha_1\beta_2 - \beta_1\alpha_2)Y_{t-2} = (\alpha_0 + \alpha_2\beta_0 - \alpha_0\beta_2) \quad (4.23)$$

The characteristic equation is:

$$\lambda^2 - (\alpha_1 + \beta_2)\lambda + (\alpha_1\beta_2 - \beta_1\alpha_2) = 0 \quad (4.24)$$

from which we will find the appropriate terms to substitute into:

$$Y_t = A_1\lambda_1^t + A_2\lambda_2^t + Y^* \quad (4.25)$$

Thus, our system of two FODEs in Y and Z can be collapsed into a single SODE in Y. Note that since we never violated the equalities in the original system, this is a perfectly valid reduction of the system, and tells us that the time path of Y will in fact be determined by the two roots of Equation (4.24), even though (as we emphasized above) the difference equation with Y on the left-hand side in our original system was a FODE.

The key to this result is the presence of the spillover terms α_2 and β_1. In the absence of α_2, changes in Z would not affect Y, even after a lag, and Y would be driven by a FODE with a single root. In the absence of β_1, while changes in Z would continue to affect Y, changes in Y would not affect Z.

We have chosen to demonstrate the substitutions by reducing the system to a SODE in Y. Since that was an arbitrary choice (based ultimately on the fact that we happen to have written the Y equation first), we could equally well have reduced it to a SODE in Z, with the interesting result that, had we done so, the characteristic equation for that SODE would have been identical to Equation (4.24). In the equation for Z, then:

$$Z_t = B_1\lambda_1^t + B_2\lambda_2^t + Z^* \quad (4.26)$$

while Z^* would differ from Y^* (since we would not expect the two variables to have the same equilibrium values) and while A_1 and A_2 would not be the same as B_1 and B_2 (since the A terms will depend on $(Y_0 - Y^*)$ and $(Y_1 - Y^*)$ while the B terms will depend on $(Z_0 - Z^*)$ and $(Z_1 - Z^*)$), the roots, λ_1 and λ_2 will be the same in Equation (4.25) as in Equation (4.26).

This means that, while there will be differences in the details of the time paths (based on the relative values of the A and B terms), the same underlying dynamic forces drive both Y and Z. If one is stable, the other is stable, if one has complex roots the other obeys the same complex roots. They are, in short, part of a genuine

system, in the sense that there is a feedback mechanism linking the two variables together.

This result carries over to higher order equations – a system of three, interrelated FODEs can be collapsed into a single third-order difference equation. Which of the variables that third-order equation happens to be a difference equation in is to a degree arbitrary since, while their equilibria will differ, the dynamics of all three of the variables will be driven by the same three roots.

Our discussion has involved reducing systems of lower order difference equations to single, higher order difference equations. Of course, we can also do the reverse – for example, expand a single SODE into a pair of interrelated FODE. But this is a bit less straightforward and depends on the context of the problem.

Matrix techniques

Returning to Equations (4.20) and (4.21), we note that they can be written as a matrix system in the general form:

$$\begin{bmatrix} Y_t \\ Z_t \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} Y_{t-1} \\ Z_{t-1} \end{bmatrix} + \begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix} \tag{4.27}$$

For convenience of notation, we shall write Equation (4.27) as:

$$X_t = AX_{t-1} + A_0 \tag{4.28}$$

Now, assuming, as we have been, that α_0 and β_0 are constants, so that at equilibrium, $Y_t = Y_{t-1} = Y^*$ and $Z_t = Z_{t-1} = Z^*$, it is easy to see that in equilibrium, Equation (4.28) becomes:

$$X^* = AX^* + A_0 \tag{4.29}$$

where X^* is the vector whose elements are Y^* and Z^*. Equation (4.29) can be rewritten as:

$$(I - A)X^* = A_0 \tag{4.30}$$

where I is the identity matrix conformable with A and X, and Equation (4.30) can be solved as:

$$X^* = (I - A)^{-1}A_0 \tag{4.31}$$

Applying this set of operations to Equation (4.27) will yield the same expressions for Y^* and Z^* as would be found from Equations (4.20) and (4.21) directly. Obviously, we can only do this if the matrix $(I - A)^{-1}$ exists, which requires that the determinant of $(I - A)$ be non-zero.

Eigenvalues and eigenvectors

The fact that we can write our system of difference equations in matrix form means that we can make use of some other standard results from matrix algebra in analysing dynamic systems. The most important of these results from our point of view are those pertaining to the characteristic roots and vectors (eigenvalues and eigenvectors) of a matrix.

Define λ as a characteristic root, or eigenvalue, of A and define w as the characteristic vector, or eigenvector, associated with the root λ. Any $(n \times n)$ square matrix has associated with it n characteristic roots, $\lambda_1, \ldots, \lambda_n$, not necessarily distinct.[4] Each characteristic root has associated with it a characteristic vector $w_i, i = 1, \ldots, n$. When the characteristic roots of the matrix are distinct, their associated characteristic vectors are linearly independent – while the roots of a matrix are not necessarily distinct, in most economic applications they turn out to be distinct, so it is simplest to assume that they are distinct and only to make explicit mention of the cases in which they are not distinct (i.e. in which two or more roots take on the same value).

Each of the pairs of characteristic roots and vectors satisfy the expression:

$$Aw = \lambda w \tag{4.32}$$

which says that, if we post-multiply the square matrix A by the characteristic vector w, the result will be a vector equal to the characteristic vector multiplied by the corresponding (scalar) root λ. To solve Equation (4.32), we rewrite it as:

$$(A - \lambda I)w = 0 \tag{4.33}$$

where I is the appropriate conformable identity matrix.

One possible solution to Equation (4.33) is the trivial solution $w = 0$, but that solution is not very informative. We are really interested in non-trivial solutions – solutions in which the elements of w are not all equal to zero. For there to exist a non-trivial solution to Equation (4.33) it must be the case that the matrix $(A - \lambda I)$ cannot be inverted – that is, the value of λ must be chosen such that the matrix $(A - \lambda I)$ cannot be inverted. From matrix algebra, that will be the case if the determinant equals zero: $|A - \lambda I| = 0$.

Given our definition of A from Equation (4.27), the condition that the determinant $|A - \lambda I| = 0$ is:

$$|A - \lambda I| = (\alpha_1 - \lambda)(\beta_2 - \lambda) - \beta_1\alpha_2 = 0 \tag{4.34}$$

The two values of λ which solve this expression are the characteristic roots (or eigenvalues, or roots) of the matrix A. Given the values of the roots, we can substitute them into Equation (4.33) and solve for w (after normalization of the w vector). There will be one characteristic vector associated with each root. Note too that when we multiply expression (4.34) out, the result is identical to the characteristic equation we found in Equation (4.24). That means that the roots of Equation (4.34) are identical to the roots of Equation (4.24), which means that we

could have found the roots and the equilibrium of the system without doing all of that direct substitution.

However, finding the A_1, A_2, B_1 and B_2 terms for the final expressions for Y_t and Z_t still require that we establish specific initial conditions for each of Y and Z, but if all we are interested in is whether the system is stable, and whether its trajectory is monotonic or non-monotonic, we can find everything we need from the matrix version of the system.

Comparing Equation (4.34) (or more conveniently Equation (4.24)) to the matrix expression (4.27), we note that Equation (4.34) could also be written:

$$\lambda^2 - \text{Tr}(A)\lambda + \text{Det}(A) = 0 \tag{4.35}$$

where $\text{Tr}(A)$ is the trace of the coefficient matrix A in Equation (4.27) and $\text{Det}(A)$ is the determinant of that matrix. Recall the results from standard matrix theory which tell us that the trace of a (2×2) matrix is equal to the sum of its roots while the determinant of the matrix equals the product of its roots.

Return now to a system of two homogeneous FODEs which we shall write as $X_t = AX_{t-1}$, and let us explore further how we may use the information contained in eigenvalues and eigenvectors. Recall Equation (4.32): $Aw = \lambda w$, where, as before, λ is a characteristic root of A and w is the characteristic vector (eigenvector) associated with that root. Then, for a (2×2) system (for expository purposes we assume the roots are real and distinct) we have two roots, $\lambda_1 \lambda_2$ and their associated characteristic vectors:

$$w_1 = \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix}, \quad w_2 = \begin{bmatrix} w_{12} \\ w_{22} \end{bmatrix} \tag{4.36}$$

Note that in the w_{ij} notation in the vectors, the second subscript, j, tells us which characteristic vector we are dealing with, so the vectors themselves are labelled w_j, $j = 1, 2$.

Eigenvectors are unique only up to multiplicative scaling, which means that if w is an eigenvector of a matrix, so is ξw, where ξ is an arbitrary constant. This means that when we solve for the first eigenvector (for example), we will not find exact values of the w_{ij} terms, but rather will find an expression which tells us about the magnitude of w_{11} relative to w_{21}. To finish the solution we need to normalize the eigenvectors.

There are several common normalizations. Eigenvectors are frequently normalized by requiring their elements to sum to 1, or (in the case where the Euclidian norm is used) by requiring the sum of the squares of their elements to equal 1. For our purposes it is often the case that the most useful approach is to normalize one of the elements of an eigenvector to equal 1.

Now let W be the matrix whose columns are the characteristic vectors w_1 and w_2:

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \tag{4.37}$$

(Note again that the second subscript on each of the elements of W tells us which characteristic vector that particular w_{ij} term comes from.) Let Λ be the diagonal

matrix whose elements are the characteristic roots of A:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{4.38}$$

Expression (4.32): $Aw = \lambda w$ above shows the relation between one characteristic root and its corresponding characteristic vector; combining all of the roots and vectors into one expression gives us:

$$AW = W\Lambda \tag{4.39}$$

where, for matrix conformity, we post-multiply by the matrix Λ. (When we are dealing with a scalar root λ, order of multiplication does not matter.) From Equation (4.39), if W can be inverted, we can write:

$$A = W\Lambda W^{-1} \tag{4.40}$$

W can be inverted so long as its determinant is non-zero, which in turn requires that the rows and columns of the matrix be linearly independent. In difference equation modelling, the most likely violation of this requirement is the case where two of the columns of W are identical, which implies in turn that two of the roots are identical. For us to be able to write Equation (4.40), then, it must be the case that the roots of A are distinct, and while there are simple economic models which yield identical roots, this situation is not too likely to arise in practice.

Now, returning to our matrix system of homogeneous, FODEs, we can write:

$$X_t = AX_{t-1} = W\Lambda W^{-1}X_{t-1} \tag{4.41}$$

because we are dealing with homogeneous difference equations (for simplicity of exposition, not because it is crucial for our result) we can do the same type of substitution as we did in our discussion of a single, homogeneous FODE, and write:

$$X_t = A^2X_{t-2} = [W\Lambda W^{-1}][W\Lambda W^{-1}]X_{t-2} = W\Lambda^2 W^{-1}X_{t-2} \tag{4.42}$$

and, by continued substitution obtain:

$$X_t = W\Lambda^t W^{-1}X_0 \tag{4.43}$$

where X_0 is a (2×1) vector whose elements are Y_0 and Z_0. Now, note that since X_0 is a (2×1) vector and W^{-1} is a (2×2) matrix, the term $W^{-1}X_0$ is a (2×1)

vector, which we shall write as:

$$W^{-1}X_0 = C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{4.44}$$

Note that the elements of the C vector are constants, which lets us write Equation (4.43) as $X_t = W\Lambda^t C$, which set out in full is:

$$\begin{bmatrix} Y_t \\ Z_t \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{4.45}$$

and which can then be written in the general form for the solutions to a pair of homogeneous SODEs:

$$Y_t = c_1 w_{11}\lambda_1^t + c_2 w_{12}\lambda_2^t \tag{4.46}$$

$$Z_t = c_1 w_{21}\lambda_1^t + c_2 w_{22}\lambda_2^t \tag{4.47}$$

Our A_i and B_i terms, then, are actually functions of the initial values of the variables (which we already knew) and the elements of the characteristic vectors of the matrix of coefficients of the system. Note that the c_i elements each show up in more than one equation, whereas the w_{ij} elements each show up in only one equation. This helps clarify a result which we asserted earlier.

Suppose that the equilibrium of our system (which, in the case of a homogeneous system will be at the origin) is a saddlepoint, meaning that we have one stable and one unstable root. Let λ_1 be the stable root and λ_2 the unstable root. We said earlier that there exists a stable branch, by which we mean a trajectory which converges on the origin, and that our system will follow that branch if the weight on the unstable root in the solution is zero. Since we want both variables to converge on the origin, this requires that the weight on the unstable root equal zero in both Equations (4.46) and (4.47).

We can guarantee that result if c_2, the element which is common to both of the relevant terms, is equal to zero. Recall that $C = W^{-1}X_0$ and note that:

$$W^{-1} = \left[\frac{1}{w_{11}w_{22} - w_{21}w_{12}}\right] \begin{bmatrix} w_{22} & -w_{12} \\ -w_{21} & w_{11} \end{bmatrix} \tag{4.48}$$

Now let $\Delta = (w_{11}w_{22} - w_{21}w_{12})$, the determinant of the matrix formed from the characteristic vectors. From Equation (4.48) we can see that:

$$c_1 = (w_{22}Y_0 - w_{12}Z_0)/\Delta \tag{4.49}$$

$$c_2 = (w_{11}Z_0 - w_{21}Y_0)/\Delta \tag{4.50}$$

which show that the c_i terms are combinations of the initial values of Y and Z, using weights based on elements of the characteristic vectors of the system.

From Equation (4.50), c_2 will equal zero if:

$$\frac{Y_0}{Z_0} = \frac{w_{11}}{w_{21}} \tag{4.51}$$

where w_{11} and w_{21} are the elements of the first characteristic vector of the matrix of coefficients. Thus, if we are to converge to the saddlepoint equilibrium, our initial pair of values must lie on the vector generated by the characteristic vector of the stable root of the system.

While we do not make much use of characteristic vectors in economic analysis, it is worth making a note of the fact that they play a major role in determining the weights the individual roots get in the expressions that determine the time paths of the variables in the system.

The fact that we can write a system of difference equations in matrix form and that the roots of the matrix of coefficients of the system are the same as the roots we would find if we were to make the substitutions necessary to reduce our system of equations to a single, higher order equation does not, unfortunately, take us as far as we might have hoped. In particular, it does not give us simple rules for establishing whether the equilibrium of our dynamic system is stable.

Consider again matrix A shown in Equation (4.27). We know that the trace of A is the sum of the roots of the system and that the determinant of A is the product of the roots:

$$\alpha_1 + \beta_2 = \lambda_1 + \lambda_2 \tag{4.52}$$

$$\alpha_1\beta_2 - \beta_1\alpha_2 = \lambda_1\lambda_2 \tag{4.53}$$

but these relations are not sufficient in themselves to tell us whether the roots are inside the unit circle. Consider the determinant, for example. If it is negative, we know that we have one negative and one positive root, but if it is positive we could have two positive roots, two negative roots or a complex conjugate pair of roots. If in addition to having a positive determinant we happen to have a negative trace, we know that we have a pair of negative roots, but a positive trace along with a positive determinant would be consistent either with a pair of positive real roots or with the roots being a complex conjugate pair. Since we have no other way of determining whether we have a complex conjugate pair than to calculate the discriminant, we are not really a long way ahead: we can get the same information from Descartes' rule of signs.

The magnitudes of the trace and determinant are not as informative as we would like, either. Suppose we have a positive trace and a positive determinant. The rule of signs tells us that we have two positive roots. If the determinant is bigger than 1 we know that we have at least one unstable root, but if it is less than 1 we could either have two stable roots or one stable and one unstable root with the stable root being small enough to make the product of the two less than 1.

Even in the case where the determinant is greater than 1, we can not distinguish between the case where the equilibrium is a saddlepoint and the case where it is

monotonically unstable – that is, between the case of one stable and one unstable root and the case of two unstable roots.

If the trace is bigger than 2 we know that we must have at least one root bigger than 1, and possibly 2, but if it is bigger than 1 and less than 2 we might have two roots which are stable but just less than 1 each. If the trace is positive and less than 1 and if the determinant is positive so that in the case of real roots we know that the roots must be positive, we know that we have two stable roots, and if the determinant is less than 1 so that its square root is also less than 1, we know that even in the case of complex roots we have roots of modulus less than 1, but by the time we have gone through all of this we might as well have written out the expression for the characteristic equation of the system and checked the stability conditions which we discussed in Chapter 3, 'Second-order difference equations'. (Obviously, given their form, those conditions are derived from the relation between the roots of a matrix and its trace and determinant.)

Thus, while we get some information on stability from the matrix expression, it is certainly less than we would like, and it is frequently easiest to write out the expressions for the roots of the model we are analysing and establish conditions under which that particular model will have a stable equilibrium.

Examples of systems of difference equations

Cobweb model with firm entry

For our first example, we return to the case of the cobweb model with firm entry, which we discussed as an example of a SODE. In our earlier discussion we had to substitute and rearrange equations to get a second-order equation. The matrix approach, should, of course, lead to the same conclusions as the earlier approach.

Once again, the basic cobweb model is:

$$Q_t^D = \beta_0 - \beta_1 P_t + \beta_2 Y_t \tag{4.54}$$

$$Q_t^S = \alpha_0 + \alpha_1 P_{t-1} + \alpha_2 N_t \tag{4.55}$$

$$Q_t^D = Q_t^S \tag{4.56}$$

and the firm entry rule is:

$$N_t = N_{t-1} + \gamma(P_{t-1} - P^c), \quad \gamma > 0 \tag{4.57}$$

where the notation is familiar from our earlier discussion of this model. Making the same substitutions in Equations (4.54)–(4.56) as we did in our earlier discussions gives:

$$P_t = \frac{(\beta_0 - \alpha_0)}{\beta_1} + \frac{\beta_2}{\beta_1} Y_t - \frac{\alpha_1}{\beta_1} P_{t-1} - \frac{\alpha_2}{\beta_1} N_t \tag{4.58}$$

which we now put, together with Equation (4.57) in matrix form as:

$$\begin{bmatrix} 1 & \frac{\alpha_2}{\beta_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_t \\ N_t \end{bmatrix} = \begin{bmatrix} -\frac{\alpha_1}{\beta_1} & 0 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} P_{t-1} \\ N_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{(\beta_0 - \alpha_0)}{\beta_1} + \frac{\beta_2}{\beta_1} Y_t \\ -\gamma P^{\mathrm{c}} \end{bmatrix} \tag{4.59}$$

Multiplying through by the inverse of the matrix on the left-hand side of Equation (4.59) (which takes the place of doing many of the substitutions we did when we dealt with this model earlier) gives:

$$\begin{bmatrix} P_t \\ N_t \end{bmatrix} = \begin{bmatrix} \frac{-(\alpha_1 + \gamma\alpha_2)}{\beta_1} & \frac{-\alpha_2}{\beta_1} \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} P_{t-1} \\ N_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{(\beta_0 - \alpha_0)}{\beta_1} + \frac{\beta_2}{\beta_1} Y_t + \frac{\alpha_2}{\beta_1} \gamma P^{\mathrm{c}} \\ -\gamma P^{\mathrm{c}} \end{bmatrix} \tag{4.60}$$

The trace and the determinant of the matrix of coefficients in Equation (4.60) are respectively:

$$\mathrm{Tr}(A) = (\beta_1 - \alpha_1 - \gamma\alpha_2)/\beta_1 \tag{4.61}$$

$$\mathrm{Det}(A) = -\alpha_1/\beta_1 \tag{4.62}$$

giving, as the characteristic equation for the problem:

$$\lambda^2 - ((\beta_1 - \alpha_1 - \gamma\alpha_2)/\beta_1)\lambda - (\alpha_1/\beta_1) = 0 \tag{4.63}$$

which is the same expression as we found for this characteristic equation in our earlier discussion of this example, which is, of course, the desired result.

The determinant of the matrix of coefficients is negative, meaning that, as usual in a cobweb model, we have one negative and one positive root (and meaning that complex roots are excluded). The trace may be positive or negative depending on the sign of $(\beta_1 - \alpha_1 - \gamma\alpha 2)$. To test stability we need to evaluate the same three conditions as we looked at in our earlier discussion.

So far as the equilibrium values of P and N are concerned, if we assume, as before, that Y does not change over time, so that all of the elements in the final term on the right-hand side of Equation (4.60) are constant, we can assume that the equilibrium values of P and N are also constants. Setting $P_t = P_{t-1} = P^*$ and $N_t = N_{t-1} = N^*$ in Equation (4.60), we can bring all of the P and N terms over to the left-hand side of the expression, giving:

$$\begin{bmatrix} \frac{(\beta_1 + \alpha_1)}{\beta_1} & \frac{\alpha_2}{\beta_1} \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} P^* \\ N^* \end{bmatrix} = \begin{bmatrix} \frac{(\beta_0 - \alpha_0)}{\beta_1} + \frac{\beta_2}{\beta_1} Y_t + \frac{\alpha_2}{\beta_1} \gamma P^{\mathrm{c}} \\ -\gamma P^{\mathrm{c}} \end{bmatrix} \tag{4.64}$$

from which we can solve for P^* and N^*. P^* will, as before, equal P^{c}, which was the price level which determined whether firms were entering or leaving the market.

N^*, which we did not solve for before, will be a messy expression involving all of the terms on the right-hand side of Equation (4.64).

We cannot really claim that the matrix approach to this model is simpler or more transparent than the substitution approach, although it is, arguably, easier to keep track of what is going on in the matrix approach. The reason we did the cobweb example both ways was to support our claim that either approach will give the same result.

The cobweb is not, of course, the only form of market adjustment model. We can also add firm entry to model the standard Walrasian price adjustment model. Let our firm entry process be as in Equation (4.57) and let the demand and supply functions be as in Equations (4.54) and (4.55), respectively. However, instead of the $Q_t^D = Q_t^S$ equation, the version of the Walrasian price adjustment model we shall use here assumes that the change in price between periods $t-1$ and t is proportional to the amount of excess demand in period $t-1$:

$$P_t - P_{t-1} = \delta(D_{t-1} - S_{t-1}), \quad \delta > 0 \tag{4.65}$$

where, as before, we can think of the variables as being in log form so that the coefficients are elasticities.

This system is closed by the price adjustment Equation (4.65). From Equation (4.65) we see that when demand equals supply in period $t-1$, the price does not change between $t-1$ and t, when demand exceeds supply in $t-1$ the price is higher in t than it was in $t-1$, and when there is excess supply in period $t-1$ the price falls between periods $t-1$ and t.

Also note that we have not written an equation for Q. In the cobweb model, since the market cleared each period, Q was determined in each period at a short-run demand and supply intersection point. In a Walrasian model such as the present one, the most common assumption is what is known as a *Min condition*, which says that in any period, given the price level for that period, the quantity actually exchanged in the market is the lesser of quantity demanded and quantity supplied.

Now, substituting lagged Equation (4.54) and lagged Equation (4.55) into (4.65) and rearranging gives the FODE for P:

$$P_t = P_{t-1} - \delta(\alpha_1 + \beta_1)P_{t-1} + \delta(\beta_0 + \beta_2 Y_{t-1} - \alpha_0) - \delta\alpha_2 N_{t-1} \tag{4.66}$$

Equations (4.66) and (4.57) constitute our system. In matrix form, we have:

$$\begin{bmatrix} P_t \\ N_t \end{bmatrix} = \begin{bmatrix} 1 - \delta(\alpha_1 + \beta_1) & -\delta\alpha_2 \\ \gamma & 1 \end{bmatrix} \begin{bmatrix} P_{t-1} \\ N_{t-1} \end{bmatrix} + \begin{bmatrix} \delta(\beta_0 + \beta_2 Y_{t-1} - \alpha_0) \\ -\gamma P^{\mathrm{c}} \end{bmatrix} \tag{4.67}$$

Note that in Equation (4.67) we have left the $t-1$ subscript on Y: as usual, for simplicity, we shall assume that Y is constant over time, but it is important to remember that consumer income can in fact change, and that market dynamics will determine how prices respond to such a change. As in the case of the cobweb

model, the long-run equilibrium price for the system will be P^c, and again we leave the determination of the equilibrium value of N as an exercise.

The trace of the matrix of coefficients in Equation (4.67) is:

$$\mathrm{Tr}(A) = 2 - \delta(\alpha_1 + \beta_1) \tag{4.68}$$

which may be positive or negative, and the determinant is:

$$\mathrm{Det}(A) = 1 - \delta(\alpha_1 + \beta_1) + \gamma\delta\alpha_2 \tag{4.69}$$

which can also take on either sign. It is easy enough to establish stability conditions for this problem, so we leave that as an exercise.

The discriminant of the matrix of coefficients, $\Delta = \mathrm{Tr}(A)^2 - 4\mathrm{Det}(A)$ is:

$$\Delta = \delta^2(\alpha_1 + \beta_1)^2 - 4\gamma\delta\alpha_2 \tag{4.70}$$

which can also be either positive or negative. If the discriminant is negative we have complex roots and the trajectory followed over time by the market price, whether converging on the equilibrium (as is the most likely case) or diverging from it, will be cyclical. The discriminant will be negative if:

$$\frac{\delta(\alpha_1 + \beta_1)^2}{4\beta_2} < \gamma \tag{4.71}$$

Here, γ determines the speed with which new firms enter the market in response to an excess of P over P^c, and the speed with which they leave the market if P is below P^c. In terms of the supply equation (4.55) above, N is a supply shift factor, with firm entry shifting supply to the right (and driving the equilibrium price down) and firm exit shifting the supply curve to the left and driving the equilibrium price up.

The coefficient δ determines how rapidly the actual market price responds to changes in the equilibrium price, and that market price response feeds back into the firm entry equation, since it is the current market price P, and not the equilibrium price which enters that equation. The combination of the two speed of adjustment terms, δ and γ, determines whether we observe cyclical behaviour in the actual market price.

The Walrasian model serves to emphasize the need to take account of dynamics in empirical analysis of markets. It is quite likely that the majority of prices actually observed in the market are disequilibrium prices, even if the equilibrium is stable and the market price is converging on the equilibrium price.

When firm entry is easy, so that γ is large, we have an additional dynamic (supply shifting) element in play. If the roots of the matrix of coefficients are complex, as can easily happen when γ is large, we can observe periods in which both the market price and the number of firms in the market are increasing, or in which both are decreasing. Unless we estimate the demand and supply equations using an approach which allows for the possibility that the observed price–quantity points are Walrasian disequilibrium points, meaning that they are not at the intersection of demand and supply curves, we might well reach some rather strange conclusions about how markets work.

We now turn to an example which we have not considered before.

Cournot duopoly model

In this section we consider one of the simplest of duopoly models. We assume that there are two firms in the market, facing identical costs and producing an identical product for which there is a single market demand function. Because consumers cannot distinguish between the firms' products, market price depends on the total level of output. Letting $Q_{1,t}$ and $Q_{2,t}$ be the output levels of the two firms in period t, and assuming that the current market price depends on the total current output produced by the two firms, we can define the market inverse demand function as:

$$P_t = \alpha_0 - \alpha_1(Q_{1,t} + Q_{2,t}) \tag{4.72}$$

Each firm's goal is to maximize its profits in each period (we will consider intertemporal optimization problems later – for the moment we are dealing with extremely myopic economic agents) and each firm faces the same, constant, average cost c. Firm 1's profit in period t, then, is:

$$\Pi_{1,t} = (P_t - c)Q_{1,t} \tag{4.73}$$

with a similar expression for firm 2.

Each firm's objective, then, is to choose the current level of its output which will maximize its profit. There is, however, a complication. Each firm is assumed to have to make its production decisions for period t in the interval between periods t and $t-1$, which is just another way of saying that they have to make their production decisions for period t before they know what the price will be in period t. Each firm knows the market demand function (4.72), but while each knows (once it has done its planning) what its own output level will be, neither knows with certainty how much output the other firm is going to put on the market in period t. That means that neither firm knows with certainty what the price level will be in period t, and that each must make its production planning decisions on the basis of an expected price, P^{e}.

In arriving at their expectation of what the price will be in period t, each firm has to make an assumption about what level of output the other will decide to produce. In the most basic of Cournot models, both firms are, as we noted above, extremely myopic. Each, therefore, looks at the level of output the other firm produced in period $t-1$ and assumes that it will continue to produce that level of output. Thus, firm 1 assumes that, in period t, firm 2 will continue to produce output level $Q_{2,t-1}$ even though firm 1 is itself planning to change its output level.

Given this assumption about its competitor's production plans, firm 1 also forms an expectation about the market price in period t, which we denote $P^{\mathrm{e}}_{1,t}$:

$$P^{\mathrm{e}}_{1,t} = \alpha_0 - \alpha_1(Q_{1,t} + Q_{2,t-1}) \tag{4.74}$$

In expression (4.74), firm 1 takes $Q_{2,t-1}$ as given, and calculates the effect of his own output level, $Q_{1,t}$ on the market price level. It chooses $Q_{1,t}$ to maximize its

expected profits, which we denote $\Pi^{e}_{1,t}$ and which are given by:

$$\Pi^{e}_{1,t} = (P^{e}_{1,t} - c)Q_{1,t} \tag{4.75}$$

substituting for P^{e}_{t} in Equation (4.75), and rearranging gives firm 1 profit as:

$$\Pi^{e}_{1,t} = (\alpha_0 - c)Q_{1,t} - \alpha_1 Q^{2}_{1,t} - \alpha_1 Q_{2,t-1}Q_{1,t} \tag{4.76}$$

The first-order condition derived from maximizing Equation (4.76) with respect to $Q_{1,t}$ is:

$$(\alpha_0 - c) - 2\alpha_{1} Q_{1,t} - \alpha_1 Q_{2,t-1} = 0 \tag{4.77}$$

from which we find firm 1's decision rule for its output:

$$Q_{1,t} = \frac{(\alpha_0 - c)}{2\alpha_{1}} - \frac{Q_{2,t-1}}{2} \tag{4.78}$$

Assuming that firm 2 makes the same, myopic assumption about its competitor's output level, we also have:

$$Q_{2,t} = \frac{(\alpha_0 - c)}{2\alpha_1} - \frac{Q_{1,t-1}}{2} \tag{4.79}$$

We now have two equations in output, each of which involves a lagged output level, but neither of which is a difference equation in the sense in which we have defined that term since the lagged output in each is the other firm's output, not its own.

Nevertheless, it is easy to see that, if we lag Equation (4.79) one period, giving us an equation for $Q_{2,t-1}$ as a function of $Q_{1,t-2}$ and substitute that lagged equation into Equation (4.78) we will have a difference equation in $Q_{1,t}$ even though only $Q_{1,t-2}$ and not $Q_{1,t-1}$ will appear in it (or, equivalently, even though the coefficient on $Q_{1,t-1}$ in this SODE turns out to equal zero). Rather than do that, though, we shall set the system up in matrix form, as:

$$\begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} = \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} Q_{1,t-1} \\ Q_{2,t-1} \end{bmatrix} + \begin{bmatrix} \dfrac{(\alpha_0 - c)}{2\alpha_{1}} \\ \dfrac{(\alpha_0 - c)}{2\alpha_1} \end{bmatrix} \tag{4.80}$$

In expression (4.80), the trace of the matrix of coefficients is zero and the determinant is $-1/4$. Since the trace is the sum of the roots, a zero trace means either that both are zero (in which case their product, the determinant, would also have to be zero) or that they are of equal magnitude and opposite in sign. Since the determinant is the product of those roots, they must be equal to $1/2$ and $-1/2$. The equilibrium, therefore, is stable, but the presence of a negative root means that the path the system follows towards it displays alternations.

Turning to the equilibrium of the system, setting $Q_{1,t} = Q_{1,t-1} = Q_1^*$ and $Q_{2,t} = Q_{2,t-1} = Q_2^*$ we find that:

$$Q_1^* = \frac{(\alpha_0 - c)}{3\alpha_1}, \qquad Q_2^* = \frac{(\alpha_0 - c)}{3\alpha_1} \tag{4.81}$$

so that in equilibrium the two firms produce the same level of output. Since they both face the same price and they both have the same cost function this is hardly surprising. Note that in equilibrium, total industry output is $2(\alpha_0 - c)/3\alpha_1$; if the industry were a monopoly, profit maximizing output would be $(\alpha_0 - c)/2\alpha_1$. This means that under duopoly the total industry output is higher, and the price of that output therefore lower, than would be the case under monopoly.

Given that we have said several times that negative roots are unusual in economic models, it would seem appropriate to discuss why we have just found another one. There are two basic parts to the explanation. The first is that this model is, to a degree, a cobweb model, in the sense that production decisions have to be made in advance of pricing information becoming available, with the actual market price in any period being found at the intersection of the market demand curve and a vertical short-run supply curve. By itself, this would seem to weaken our caution against negative roots, since many production plans have to be put into motion on the basis of price expectations, before actual prices are known.

The second, and more important part of the answer is that we have assumed that there is no cost to adjusting the level of output beyond the marginal production cost of the new output, and that marginal cost is constant. It is more likely that there are additional costs which depend on the amount of the change in output from one period to the next – installation costs from adding new capital, for example, or perhaps financing costs associated with expanding productive capacity. It also seems unlikely that marginal production costs would be strictly constant – marginal costs are more likely to be increasing, at least to some degree, but we will not add this complication here.

As a simple illustration of the implications of some of these effects, then, we next add a nonlinear adjustment cost element to the two firms' cost functions, where we define adjustment costs to be costs arising from the fact of changing output from one period to the next. If output remains constant over time, no adjustment costs are incurred. Representing this by a quadratic cost term gives us, in place of Equation (4.73):

$$\Pi_{1,t}^{\mathrm{e}} = P_{1,t}^{\mathrm{e}} Q_{1,t} - cQ_{1,t} - v(Q_{1,t} - Q_{1,t-1})^2, \quad v > 0 \tag{4.82}$$

with a similar equation for firm 2. As before, neither firm knows what the actual market price will be in period t, so both have to work on the basis of their expectations about price, and as before both are extremely myopic and assume that their competitor will leave its output level unchanged from the period before even as they themselves change their own output, in other words, as described in Equation (4.74).

Substituting Equation (4.74) into (4.82) gives:

$$\Pi^{e}_{1,t} = [\alpha_0 - \alpha_1(Q_{1,t} + Q_{2,t-1})]Q_{1,t} - cQ_{1,t} - v(Q_{1,t} - Q_{1,t-1})^2 \qquad (4.83)$$

differentiating Equation (4.83) with respect to $Q_{1,t}$ and setting the result equal to zero gives the first-order condition for the problem:

$$(\alpha_0 - c) - \alpha_1(2Q_{1,t} + Q_{2,t-1}) - 2v(Q_{1,t} - Q_{1,t-1}) = 0 \qquad (4.84)$$

from which we can find $Q_{1,t}$, shown in Equation (4.85). Similarly, we can determine $Q_{2,t}$ also shown in matrix form (4.85):

$$\begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} = \begin{bmatrix} \dfrac{v}{(\alpha_1 + v)} & -\dfrac{\alpha_1}{2(\alpha_1 + v)} \\ -\dfrac{\alpha_1}{2(\alpha_1 + v)} & \dfrac{v}{(\alpha_1 + v)} \end{bmatrix} \begin{bmatrix} Q_{1,t-1} \\ Q_{2,t-1} \end{bmatrix} + \begin{bmatrix} \dfrac{(\alpha_0 - c)}{2(\alpha_1 + v)} \\ \dfrac{(\alpha_0 - c)}{2(\alpha_1 + v)} \end{bmatrix} \qquad (4.85)$$

Note that when $v = 0$, the matrix of coefficients in Equation (4.85) is the same as that in Equation (4.80), as we should expect.

The characteristic equation associated with system (4.85) is:

$$\lambda^2 - \frac{2v}{(\alpha_1 + v)}\lambda + \frac{(4v^2 - \alpha_1^2)}{4(\alpha_1 + v)^2} = 0 \qquad (4.86)$$

The discriminant of Equation (4.86) can be shown to equal $[\alpha_1^2/(\alpha_1 + v)^2]$, which is positive, so the roots of Equation (4.86) will be real. The determinant of the matrix of coefficients can be written as:

$$\text{Det}(A) = (2v - \alpha_1)(2v + \alpha_1)/4(\alpha_1 + v)^2 \qquad (4.87)$$

so the sign of the determinant depends on whether $(2v - \alpha_1)$ is positive or negative. So long as v is larger than $\alpha_1/2$, the two roots of Equation (4.86) will have the same sign. The trace of the matrix of coefficients in Equation (4.85) is positive, so in the case where the determinant is positive, the roots will be positive. This is also clear from the rule of signs, since if the determinant of the matrix of coefficients is positive, the sign pattern of Equation (4.86) will be $(+ - +)$, indicating the presence of two positive roots.

The roots of this example are easily calculated:

$$\lambda_1 = \frac{(2v + \alpha_1)}{2(\alpha_1 + v)}, \qquad \lambda_2 = \frac{(2v - \alpha_1)}{2(\alpha_1 + v)} \qquad (4.88)$$

since both are positive, so long as v is larger than $\alpha_1/2$, we can check stability by checking whether the larger of the two, λ_1, is less than 1: that is whether $(2v + \alpha_1) < (2\alpha_1 + 2v)$, or $\alpha_1 > 0$ which holds by assumption from

Equation (4.74). Thus, so long as v is sufficiently large, the system will approach its equilibrium monotonically.

Turning to the equilibrium, if we solve the system (4.85) for the equilibrium values of Q_1 and Q_2 we find that they are the same as in the case where v equals zero, so the introduction of the quadratic adjustment cost term does not alter the location of the equilibrium, just the path the system follows to it.

The introduction of the adjustment cost term changed the adjustment path by raising the cost of cobweb-type adjustment. In a cobweb model, the short-run supply curve jumps immediately to the new short-run profit maximizing position, even if the jump in output is a large one. By making large jumps particularly costly, the quadratic cost of adjustment term discourages them. If the cost of adjustment term, v, is smaller than its critical value, large jumps in output can still be profitable.

There are other, obvious extensions of the duopoly model which could be introduced. One particularly interesting result is found if we return to the case where $v = 0$ and introduce a third firm, operating on exactly the same rules as the two firms in the cases which we have been considering. In that case it can be shown that one of the roots of the system will equal -1.

The fact that this root is negative means that the three-firm model without adjustment costs will display alternations. This is not surprising, since we found alternations in the dynamics of the two-firm model with $v = 0$. What is more interesting is the fact that the root equals -1. In that case an expression like $A\lambda^t = A(-1^t)$, meaning that it alternates permanently between A and $-A$, introducing a dynamic element which neither diverges from nor converges on the equilibrium of the system. In the three-firm case with no adjustment costs, then, even given that the other two roots are stable, the system will never settle down.

Obviously we would expect that, after this had been going on for a few decades, one of the firms might figure out the pattern of the other firms' production decisions and build that into its own decision process. The myopic model applies only so long as firms are in fact myopic – only so long as they do not learn from experience. If we build a learning process in, the dynamics of the market will change, in a manner which depends on the precise learning process which we assume. Still, in the early stages of the development of a new market, when firms are still to a large degree guessing about what their rivals will do, we can expect to observe some interesting output dynamics.

A demography model

Our next example is not strictly speaking an economic example, although the dynamics generated can have significant economic implications. Consider a population which has arbitrarily been grouped into five age groups, P_1 through P_5. For simplicity in the theoretical model we assume that these age groups each span the same number of years, perhaps 15, and typically we define a time period, the interval between t and $t + 1$, which spans the same number of years as an age group. Assuming away immigration, populations evolve through time on the basis of birth rates and death or survival rates.

Let π_{ij} be the probability of an individual who is of age group i in period $t-1$ surviving to age j in period t. Since we are assuming that a time period covers the same number of years as an age group, the π are written $\pi_{i,i+1}$. The survival probabilities obviously reflect the probabilities of not surviving – of dying before reaching the next age group. In detailed demographic exercises the age groups would span one year and we would calculate one year survival probabilities.

We enter births into the system by assigning each age group between P_2 and P_4 a birth rate, $b_i, i = 2, \ldots, 4$. Whether there is a birth rate attached to P_1 and to P_5 depends on the width of the age groups we are using. Setting $b_1 = 0$ means that we are assuming that the first age group is too young to have children. Thus, for example, with 15-year age groups there would be a (generally very small) birth rate attached to P_1, while with 10-year age groups there generally would not.

These assumptions let us write a matrix expression for population growth:

$$\begin{bmatrix} P_{1,t} \\ P_{2,t} \\ P_{3,t} \\ P_{4,t} \\ P_{5,t} \end{bmatrix} = \begin{bmatrix} 0 & b_2 & b_3 & b_4 & 0 \\ \pi_{12} & 0 & 0 & 0 & 0 \\ 0 & \pi_{23} & 0 & 0 & 0 \\ 0 & 0 & \pi_{34} & 0 & 0 \\ 0 & 0 & 0 & \pi_{45} & 0 \end{bmatrix} \begin{bmatrix} P_{1,t-1} \\ P_{2,t-1} \\ P_{3,t-1} \\ P_{4,t-1} \\ P_{5,t-1} \end{bmatrix} \tag{4.89}$$

The matrix of coefficients in Equation (4.89) is referred to as a population projection matrix. From Equation (4.89) we see that the number of people in the first age group in period t is calculated as:

$$P_{1,t} = b_2 P_{2,t-1} + b_3 P_{3,t-1} + b_4 P_{4,t-1} \tag{4.90}$$

while the number in age group 2 in period t is:

$$P_{2,t} = \pi_{12} P_{1,t-1} \tag{4.91}$$

Immigration can be added in to Equation (4.89) as a vector whose elements reflect the number of people in each age group who immigrated during one period.

Since the fifth age group, P_5, contributes to population only by surviving, it is not uncommon to work with a reduced size population projection matrix, covering only the age groups up to the end of the reproductive years. In our case this would mean working with a 4×4 projection matrix:

$$\begin{bmatrix} P_{1,t} \\ P_{2,t} \\ P_{3,t} \\ P_{4,t} \end{bmatrix} = \begin{bmatrix} 0 & b_2 & b_3 & b_4 \\ \pi_{12} & 0 & 0 & 0 \\ 0 & \pi_{23} & 0 & 0 \\ 0 & 0 & \pi_{34} & 0 \end{bmatrix} \begin{bmatrix} P_{1,t-1} \\ P_{2,t-1} \\ P_{3,t-1} \\ P_{4,t-1} \end{bmatrix} \tag{4.92}$$

Expression (4.92) is just a matrix form of a difference equation system, and a much simpler one than many we derive from economic models. One thing is immediately obvious: the trace of the projection matrix is zero, which means that the sum of the roots is zero. In system (4.89) above the trace and determinant of the projection matrix – the matrix of coefficients of the difference equation

system – were both zero, meaning that at least one root of system (4.89) was zero (since the determinant is the product of the roots). That zero root reflects the limited contribution of age group P_5 to the population dynamics.

System (4.92), being a fourth-order system of difference equations, has four roots. Without going into the proof,[5] we note a few interesting features of population projection matrices as difference equation systems.

A population system virtually always has a single positive real root, larger in absolute value than the others. If the population being analysed is growing, the dominant root will be larger than 1. The other roots will virtually always be complex of modulus less than 1, although there may be small, stable negative roots.

Thus, when we write the solution form derived for P_1 from (4.92) as:

$$P_{1t} = A_{11}\lambda_1^t + A_{12}\lambda_2^t + A_{13}\lambda_3^t + A_{14}\lambda_4^t \tag{4.93}$$

the first root, λ_1, will be positive and larger than 1, the next two will probably be a complex conjugate pair of modulus less than 1 then the fourth root will (because it is a single root and therefore cannot be part of a complex conjugate pair) probably be negative and small in absolute value. In the long run, the dynamics of the population will be dominated by the first root. That root, being larger than 1, is what we have been referring to as an unstable root, but in the case of a population model there is nothing unexpected or undesirable about having an unstable root; if the largest root was less than 1 in absolute value the population would eventually vanish.[6] The presence of complex roots means that populations can exhibit cyclical behaviour, but the fact that those roots will be of modulus less than 1 means that in the long run the cycles will vanish.

In fact, if a population's birth and death (and therefore survival) rates have remained unchanged for a sufficiently long period, its population dynamics will be completely dominated by the first root. This dominance extends beyond the growth rate – if we calculate the eigenvector associated with the first root and normalize it so that it sums to 1, we will find that each element in the normalized eigenvector will be a positive fraction and that the eigenvector will represent the long-run age distribution of the population.

With λ_1 being greater than 1 the population will continue to grow over time, but if its birth and death rates have not changed over time the roots of the system will not have changed over time, and if those roots have remained unchanged for a period sufficiently long to allow the first root to come to dominate the dynamics of the population, then even though the population will be growing, its age distribution will remain stable and unchanging over time. In a population which is closed to immigration, a changing age distribution is a reflection of changes in birth and death rates which must have occurred in the relatively recent past.[7]

If birth and/or death rates have changed in the relatively recent past, as, for example, in the case of a sudden baby boom, the other roots in Equation (4.93) have a role to play. The complex roots will cause P_1 to follow a cyclical path, and since all of the age groups will obey equations analogous to Equation (4.93), with different A_{ij} weights but the same roots, all of the age groups will follow

cyclical paths. Depending on the modulus of the roots, those cycles can take a very long period to work themselves out. Populations do not settle into stable age distributions in the short run.

Population cycles can have significant economic effects. Different age groups have different preferences in consumption, based in part on factors like family formation. As the cycles resulting from a sudden baby boom work their way through the system, different sectors of the economy will advance or contract.

That there will be effects on the education and health sectors is quite clear. There will also be labour market effects: since different age groups of labour are less than perfect substitutes, we can conceive of them facing age-specific labour demand and supply curves. As the cyclical effects of a baby boom move through the labour force, the age-specific labour supply curves will shift in or out, depending on whether a cyclical bulge is moving in or out of that group.

If all age groups of labour were perfect substitutes there would have been a single outward shift of the labour supply curve as the baby boom group first entered the labour force and the equilibrium wage would have fallen (and, in a system with downward-sticky wages, unemployment would have risen) but the effect would have been spread out across the whole labour force. Because different age groups are in fact less than perfect substitutes, in reality the effect will fall most heavily on the age group whose numbers have just surged.

This effect will also come through in neoclassical growth models. When we set up the basic growth model later, we shall represent the population by a single variable growing at an exogenous rate. This is legitimate if the population age distribution is stable – in that case, even though different age groups of labour are less than perfect substitutes in production, because the proportion each age group accounts for in the total population will not change over time, we can model the population (and the labour force) as if it was a single entity.

When there are demographic cycles working their way through the labour force, we should really divide the overall labour force into age groups, each with its own labour productivity coefficient. Even if the total number of people in the labour force remains unchanged over time, the overall productivity of labour can change significantly as different age groups expand and then contract.

These demographic effects will not vanish once the baby boom[8] group leaves the working age years. If we think, for a moment, about the actual post-war baby boom, that cohort was large in absolute numbers, and even though its own reproductive behaviour seems to have been marked by a drop in age-specific birth rates (our π coefficients), the baby boom group itself will still produce a lot of children, and eventually grandchildren, in absolute terms. In calendar terms, a long time has to pass before the complex roots in Equation (4.93) and its counterparts for other age groups cease to play a significant part in determining the dynamics of a population.

As a concluding note, both here and in later discussion of the neoclassical growth model, we have treated demographic factors as exogenous. In fact, of course, reproductive behaviour (i.e. birth rates) and survival probabilities are sensitive to economic conditions. A full dynamic economic-demographic model should incorporate those effects, but that would move us well beyond our present scope.[9]

5 Intertemporal optimization

Introduction

It is important to remember that the systematic dynamic behaviour we observe in economic variables has to come from somewhere. The most important source of consistent relations between the past and future values of economic variables is intertemporal optimization on the part of economic agents.

Intertemporal optimization simply means recognizing that actions which are taken today have consequences for the future, and incorporating that recognition into decisions about what actions should be taken today. To take a simple example, the decision about how much of our current income to consume today and how much to save has implications not just for current consumption but also for future consumption. When we are deciding how much to consume today we have to take account of how that decision will affect our consumption tomorrow; or at least, we will take that into account if we are behaving in an intertemporally optimal manner.

To set things out a bit more formally, consider the two-period consumption–savings model found in most intermediate microeconomics texts. The individual has to decide on how much to consume in each of two periods, 1 and 2, where we label consumption in the two periods c_1 and c_2. He receives incomes y_1 and y_2 in the two periods, and knows that the prices of consumption goods will be p_1 and p_2. The interest rate he earns on savings or pays on borrowing is r, and his subjective discount factor is $\beta < 1$, where β is $1/(1+\delta)$, and δ is his subjective discount rate. His problem is to allocate consumption across the two periods in order to maximize the present (subjective) value of lifetime income, subject to the two-period budget constraint, which says that the present value (at the market interest rate) of his two-period consumption expenditure must equal the present value of his two-period income stream. We write his problem as:

$$\underset{c_1,\, c_2}{\text{Max:}}\, Ł = U(c_1) + \beta U(c_2) + \lambda \left[y_1 + \frac{y_2}{(1+r)} - p_1 c_1 - \frac{p_2 c_2}{(1+r)} \right] \tag{5.1}$$

The first-order conditions for this problem give:

$$\begin{aligned} U'(c_1) &= \lambda p_1 \\ \beta U'(c_2) &= \lambda \frac{p_2}{(1+r)} \end{aligned} \tag{5.2}$$

from which we have:

$$U'(c_1) = \frac{\beta U'(c_2)(1+r)p_1}{p_2} \tag{5.3}$$

Equation (5.3), which is an example of a relation known as an Euler equation, links consumption in period 2 with consumption in period 1, where the fact that the intertemporal relation is derived from the first-order conditions of an optimization problem means that the relation between the two periods' consumption levels is optimal.

We can interpret the first-order condition as a marginal benefit that equals marginal cost condition. In any intertemporal consumption problem like this one, the decision to increase consumption today by one unit results in a reduction in consumption in the future by an amount which depends on the relative prices of consumption in the present and the future and the rate of interest we could have earned had we saved the cost of that extra unit of consumption for one period. The β term on the right-hand side is there to discount future utility into present utility terms, so that we are comparing like with like.

The first-order condition tells us that if we increase consumption by one unit today, we receive a benefit equal to the marginal utility derived from consuming that unit. The cost of that benefit is the marginal utility we could have derived from the consumption we could have done had we saved the cost of that unit of consumption for one period at interest rate r.[1] At the optimum, marginal benefit equals marginal cost and we cannot increase the present value of our lifetime utility by saving (and shifting consumption into the future) or borrowing (and shifting consumption towards the present).

To take the example further, if we assume that utility in each period takes the log form:

$$U(c) = \ln(c) \tag{5.4}$$

and substitute the appropriate marginal utilities into Equation (5.3):

$$c_2 = \left(\frac{\beta p_1(1+r)}{p_2}\right) c_1 \tag{5.5}$$

which is really just a homogeneous FODE in c. If we replace the subscripts 1 and 2 by t and $(t+1)$ respectively, we have familiar dynamic notation – the 1, 2 notation is used in most intermediate micro texts to emphasize the similarity between this problem and the single-period problem of allocating one period's income over a number of consumption goods in that period.

Note that the relation between consumption in the two periods depends on prices in the two periods, and on the relation between the market and subjective discount

rates. To see this, replace β by $1/(1+\delta)$ to give:

$$c_2 = \left(\frac{p_1(1+r)}{p_2(1+\delta)}\right) c_1 \tag{5.6}$$

If, for simplicity, we assume that $p_2 = p_1$, then $c_2 = [(1+r)/(1+\delta)]c_1$ which says that whether consumption is greater or less in the second period than the first will depend on whether r, the market interest rate, is greater or less than δ, the subjective discount rate. If r is greater than δ, the interest the individual will earn on an extra dollar saved will more than compensate for the subjective cost of having to wait to consume that dollar, so he will, by saving, shift consumption from the present into the future. If r is less than δ, he will tend to shift consumption from the future into the present.

If $r = \delta$ we have $c_2 = c_1$, and he will, by judicious saving or borrowing, allocate his two periods incomes so that consumption is equal in each period. Finally, if we write $p_2 = (1+g)p_1$, where g is the inflation rate, we can write Equation (5.5) as:

$$c_2 = \left(\frac{\beta(1+r)}{(1+g)}\right) c_1 \tag{5.7}$$

which tells us that the way an individual allocates his income over time depends on his subjective discount rate, the market (nominal) interest rate and the inflation rate or, if we combine $(1+r)/(1+g)$ into a single term, on the subjective discount rate and the real interest rate. If $r = \delta$ in this case, then whether c_2 is greater or less than c_1 will depend on the inflation rate. If the individual knows (or, more realistically, expects) that prices are going to be higher in the second period than in the first, he will shift consumption into the present and c_1 will be greater than c_2.

Note that, while we have used the first-order conditions for the problem to derive a difference equation in consumption, consumption in each to the two periods actually depends on the exogenous variables in the problem: prices, the interest rate, the subjective discount rate and income in each of the periods. If, for example, inflation is zero and the interest and subjective discount rates are equal, the condition that consumption is equal in the two periods also means that consumption will equal half of the present value of lifetime wealth in each period. Changes in any of those variables will lead to changes in both c_1 and c_2, although the two periods' consumption levels will change in such a manner as to ensure that the first-order conditions continue to hold with the new values of the exogenous variables.

Dynamic programming

While the basic approach of the previous section applies to problems involving more than two periods, and many intertemporal optimization problems do involve more than two periods, writing a multi-period problem out in full as in Equation (5.1) above can quickly become cumbersome. The alternative approach most commonly used in discrete time problems is called Dynamic Programming,

a term which apparently was adopted because, at the time Richard Bellman developed the approach, linear programming was very popular in static analysis and the term dynamic sounded – dynamic.

In a sense, dynamic programming still divides the time horizon for the problem into two periods, the present and the future. The basic idea is that we make decisions about what to do now taking account of the fact that those decisions will have future repercussions, in a very specific manner.

We noted above that the first-order conditions for the intertemporal consumption problem amounted to weighing up the subjective benefit we derived from spending an extra dollar on consumption today against the utility we were foregoing by not saving that dollar for future consumption. The forgone marginal utility of the future consumption is the opportunity cost of increased consumption today.

The same principle applies to multi-period consumption–saving problems, but now when we spend an extra dollar today we are giving up a whole range of options. We might have saved it one period and spent it, plus interest, tomorrow, we might have saved it T periods into the future and used it, plus accumulated interest, for retirement consumption, or we might have spent part of it tomorrow and saved the rest for our retirement.

The opportunity cost of the extra spending today is the alternative use of that dollar which would have yielded the greatest increase in the present value of our lifetime utility. This is just an intertemporal version of the definition of opportunity cost – the opportunity cost of any action is the value (often subjective) of next best alternative use to which we could have put the resources used up in the action we are considering.

In a static consumption problem the opportunity cost of spending an amount of money in buying a unit of one commodity (and deriving the marginal utility associated with consuming one more unit of that commodity) is the largest extra utility we would have derived from spending that money on some other commodity. In a dynamic problem the opportunity cost of spending today is the largest extra lifetime utility we could have derived from saving the money and spending it at some point in the future.

In a dynamic programming problem, then, we are going to wind up with a marginal benefit equals marginal cost type of first-order condition in which the marginal benefit is the extra benefit we get from taking a certain action today and the marginal cost is the greatest possible future benefit we forego as a result of having taken that action today.

In the dynamic programming approach we select, optimally, the value of a particular choice variable today recognizing that things we do today have consequences for the future, and assuming that we will make all future choices optimally. This last assumption can jar a bit – it is easy enough to grasp the idea of making a choice today on the assumption that all relevant past decisions which we have made were made optimally. Here we are assuming that all future decisions will be made optimally, recognizing that the set of possible future choices open to us will be affected by what we do today. In other words, in an intertemporal consumption problem, we need to recognize that if we increase consumption

today, it also means that the amount we could consume in the future has been reduced.

To formalize the discussion, let x be our choice variable (also known as a control variable, since it is under our direct control) and let x_t be the value we choose for it in period t. Let s be what is termed a state variable, meaning a variable which is of interest in our problem but whose value is not under our direct control. It is, however, under our indirect influence in the sense that the behaviour of the state variable over time is determined by an equation of motion, a difference equation (usually first order) which we write in general form as: $s_t = Q(s_{t-1}, x_{t-1})$. It is the equation of motion for the state variable which ties the present and future together.[2]

In our consumption example above, consumption in each period was the control variable. As the state variable for an intertemporal consumption problem we usually select the consumer's stock of assets, a_t, where we define the two periods assets as:

$$\begin{aligned} a_1 &= y_1 \\ a_2 &= y_2 + (y_1 - p_1 c_1)(1 + r) \end{aligned} \tag{5.8}$$

which can be rewritten as an equation of motion:

$$a_2 = y_2 + (a_1 - p_1 c_1)(1 + r) \tag{5.9}$$

or rewritten in a more general, difference equation form as:

$$a_{t+1} = y_{t+1} + (a_t - p_t c_t)(1 + r) \tag{5.10}$$

Clearly Equation (5.10) would also be the equation of motion for assets in a problem with more than two periods.

The objective of a dynamic programming problem is to find a policy rule of the form $x_t = h_t(s_t)$ which tells us the optimal value of the control variable in period t, conditional on the value of the state variable in t, and taking into account the effect (through the equation of motion for the state variable) that our choice of x in period t will have for our options in the future.

Finite horizon problems

To get some idea of how dynamic programming works, consider a simplified version of the two-period consumption problem we considered earlier. We shall assume that $y_2 = 0$, so that the consumer is living off his first period assets y_1, which for consistency with the equation of motion for a_2, we shall label a_1. Assume that we have already found $h_2(a_2/p_2)$, the policy rule which gives us the optimal level of c_2 for any given value of a_2, where we have divided a_2 by p_2 because consumption in period 2 will depend on the real (in period 2 terms) value of assets in that period.[3]

The dynamic programming approach to analysing this problem involves:

$$\begin{aligned} &\underset{c_1}{\text{Max:}}\ \text{Ł} = U(c_1) + \beta U\left(h_2\left(\frac{a_2}{p_2}\right)\right) \\ &\text{s.t.: } a_2 = (a_1 - p_1 c_1)(1 + r) \end{aligned} \tag{5.11}$$

Clearly, a_2 is a function of c_1, so, when we differentiate Equation (5.11) with respect to c_1 in order to find the first-order condition we should differentiate a_2 with respect to c_1. Doing this, we find, as our first-order condition:

$$U'(c_1) = \beta U'(c_2) h_2'\left(\frac{a_2}{p_2}\right)\left(\frac{p_1(1+r)}{p_2}\right) \tag{5.12}$$

since $\partial a_2/\partial c_1 = -p_1(1+r)$. Now recall Equation (5.3), one of the versions of the Euler equation for the consumption problem: clearly Equations (5.12) and (5.3) will be the same expression if $h_2'(a_2/p_2) = 1$. In fact, that will prove to be the case, but we shall set demonstrating it aside for the moment.

The intertemporal consumption problem is not, of course, limited to two periods. Consider a simple extension to three periods: we can define c_3 and a_3 as above, and, also as above, define $h_3(a_3/p_3)$ as the policy function which gives us the utility maximizing value of c_3 for any value of period 3 assets. Period 3 assets are determined by the same equation of motion as defines period 2 assets, namely Equation (5.10), and we can write the consumer's problem as:

$$\underset{c_1}{\text{Max:}}\ \text{Ł} = U(c_1) + \beta U\left(h_2\left(\frac{a_2}{p_2}\right)\right) + \beta^2 U\left(h_3\left(\frac{a_3}{p_3}\right)\right) \tag{5.13}$$

In Equation (5.13), the term $h_3(a_3/p_3)$ is the optimal level of consumption in period 3 as a function of a_3, but the value of a_3 depends on a_2 and c_2. The term $h_2(a_2/p_2)$ is the solution value of c_2, which is also a function of a_2. We could chain back a period further by noting that a_2 depends on a_1 and c_1, but for the moment it suits our purpose to stop with a_2. That is because it lets us define the whole of the expression $\beta U(h_2(a_2/p_2)) + \beta^2 U(h_3(a_3/p_3))$ as being a function of a_2.

Expression (5.13) is in utility terms, discounted back so that the utility is measured in period 1 present value terms. However, we can also write it as:

$$\begin{aligned} &\underset{c_1}{\text{Max:}}\ \text{Ł} = U(c_1) + \beta J_2(a_2) \\ &J_2(a_2) = U(h_2(a_2/p_2)) + \beta U(h_3(a_3/p_3)) \end{aligned} \tag{5.14}$$

where the subscript on J indicates the number of periods included in its construction and where the term is now in period 2 present value terms.

$J_2(a_2)$ is an intertemporal maximum value function. This means that it shows the maximum present value of intertemporal utility, in period 2 terms, which the consumer can derive from asset level a_2, assuming he distributes his spending

across periods in a manner which satisfies the first-order conditions for utility maximization subject to an intertemporal budget constraint. If we then pre-multiply $J_2(a_2)$ by β, we have converted that utility level from period 2 terms to period 1 terms.

Our problem now becomes Equation (5.13) where, as always, the maximization is done subject to the intertemporal budget constraint as represented by the equation of motion (5.10). Looking at Equation (5.13), however, we can see that since a_2 and c_1 both depend on a_1 (and parameters like the prices and the interest rate), then if we solve for the policy rule $h_1(a_1/p_1)$, the maximized value of Equation (5.13) is in fact a function of a_1, letting us write:

$$J_3(a_1) = \underset{c_1}{\text{Max:}}\ (U(c_1) + \beta J_2(a_2)) \tag{5.15}$$

Note the subscript 3 on the left-hand side J term in Equation (5.15): as in the case of the subscript on J_2, this indicates the number of periods' utilities involved in the construction of the J term, or, alternatively, the number of periods remaining in the intertemporal optimization problem starting from the period in which we are doing the optimization. For consistency, we could write the term $u(h_3(a_3/p_3))$ as $J_1(a_3)$.

In terms of our intertemporal consumption problem, $J_3(a_1)$ indicates the present value of the maximum lifetime (using the term lifetime to indicate the length of the planning horizon) utility the consumer can derive if he starts with asset level a_1 and allocates his spending across the three periods according to the first-order conditions (which are, after all, necessary conditions for utility maximization). Equation (5.15) is Bellman's fundamental equation of optimality for our problem.

The fundamental equation of optimality is written in more general form as:

$$J_{T-t}(a_t) = \underset{c_t}{\text{Max:}}\ (U(c_t) + \beta J_{T-t-1}(a_{t+1})) \tag{5.16}$$

where the subscripts on a and c indicate the period in which the decision is being made, counting from the beginning of the planning horizon, and the subscripts on the J terms indicate the number of periods which will be affected by that decision, or equivalently the number of periods left until the end of the planning horizon, where T represents the end of the horizon. We place a subscript on the J terms because the functional form of J can change from period to period. Whatever the time period, though, J is, by definition, a maximum value function, so the J inside the '*Max*' operator on the right-hand side is unaffected by that operator because it is already maximized.

Similarly, when we find the optimal policy $h_t(s_t)$ we have to place a time subscript on h because the form of that function – of the policy rule which tells us what that period's choice of x should be given the value of the state variable at the beginning of that period – could change from period to period.

We could write Equation (5.16) in more general terms yet, with a_t appearing in the $U(\cdot)$ term: but what is ultimately important is the additive nature of the objective function $U(c_t)$. We are trying to maximize the intertemporally additive series:

$$U(c_1) + \beta U(c_2) + \beta^2 U(c_3) + \beta^3 U(c_4) + \cdots + \beta^{T-1} U(c_T)$$

subject to the equation of motion for a and any other constraints imposed on the problem. While we could, as we noted above, include the contemporaneous a term as an argument in each of the U elements, all of the arguments in the U function must have the same time subscript. This means that our utility function cannot take the form $U(c_1, c_2)$, for example, meaning that the utility we get from consumption today cannot depend on past or future consumption levels.

This additive structure means that the fundamental equation of optimality embodies Bellman's principle of optimality, which basically says that if we stop part-way through and consider the remainder of our plan, that remainder will still be optimal, given the level of the state variable at that point in the process. More formally, the principle of optimality says that if a policy is optimal, then whatever the initial value of the state variable, the rest of the plan will be optimal given that initial state. This principle, which says that whatever segment of an intertemporal plan we look at will be optimal given the level of the state variable at the beginning of the segment, only holds when we have the type of recursive intertemporal structure we set out above. It means that an optimal intertemporal plan is what is referred to as time consistent, which simply means that if we do stop the plan part way through and do a new optimization problem for the remainder of the planning horizon, then, given the value of the state variable at the point at which we do this, our new plan for the remainder of the horizon will be identical to our original plan – we would not regret our original plan and we would not change it.

In order to be able to use Bellman's equation in an optimization problem, we need to derive a couple of results. The first is simply the first-order condition for the maximization problem in Equation (5.16). Since c is our choice variable, the first-order condition is with respect to c in period t, giving:

$$U'(c_t) + \beta J'_{T-t-1}(a_{t+1}) \left(\frac{\partial Q(a_t, c_t)}{\partial c_t} \right) = 0 \tag{5.17}$$

where $a_{t+1} = Q(a_t, c_t)$ is the equation of motion for a, so the final element of the left-hand side of Equation (5.17) tells us how a_{t+1} changes in response to a change in c_t.

The second result which we need is what is sometimes known as the Benveniste–Scheinkman condition, or the envelope condition. To find it, we first substitute $h_t(a_t)$ in for c_t, indicating that we are working with an optimized function, and

then differentiate Equation (5.16) with respect to a_t, giving:

$$\frac{\partial J_{T-t}(a_t)}{\partial a_t} = \left[U'(h_t(a_t)) + \left(\beta \frac{\partial J_{T-t-1}(a_{t+1})}{\partial a_{t+1}}\right)\left(\frac{\partial Q(a_t, h_t(a_t))}{\partial h_t}\right)\right] \times \left[\frac{\partial h_t(a_t)}{\partial a_t}\right] + \left(\beta \frac{\partial J_{T-t-1}(a_{t+1})}{\partial a_{t+1}}\right)\left(\frac{\partial Q(a_t, h_t(a_t))}{\partial a_t}\right) \tag{5.18}$$

where we have made use of the Q function after substituting $h_t(a_t)$ into it. Now if we compare the first term on the right-hand side of Equation (5.18), with Equation (5.17) (noting again that we have replaced c_t by $h_t(a_t)$), we see that this whole, rather messy looking term is in fact equal to zero, leaving us, from Equation (5.18), with:

$$\frac{\partial J_{T-t}(a_t)}{\partial a_t} = \left(\beta \frac{\partial J_{T-t-1}(a_{t+1})}{\partial a_{t+1}}\right)\left(\frac{\partial Q(a_t, h_t(a_t))}{\partial a_t}\right) \tag{5.19}$$

In other words, even though, at the optimum, the value of c depends on the value of a, when we differentiate with respect to a on both sides of the optimized fundamental equation of optimality, we need only look at the partial derivatives of J with respect to a.

To get an idea of how dynamic programming works, consider an even simpler version of the intertemporal consumption problem; a special case known as the cake eating problem.

Cake eating problem

In the cake eating problem the consumer begins the planning horizon with a given stock of an asset, called cake, on which he must live for the remainder of the planning horizon. The cake, which is the state variable for the problem, and which we label 's', has no natural tendency to grow, meaning that there is no interest rate term in this problem, and the price of a unit of consumption is 1 in each period, so there are no prices. Letting 'c' be consumption, the equation of motion for the cake is:

$$s_{t+1} = s_t - c_t \tag{5.20}$$

which simply says that the amount of cake available at the beginning of the next period will be the amount the consumer did not eat in the present period. In this example we shall assume that $T = 5$, so the cake has to be consumed over five periods. The consumer discounts the future according to the discount factor β.

We could write out the consumer's problem in full, beginning with the objective function:

$$U(c_1) + \beta U(c_2) + \beta^2 U(c_3) + \beta^3 U(c_4) + \beta^4 U(c_5) \tag{5.21}$$

plus the equations of motion:

$$\begin{aligned} s_2 &= s_1 - c_1 \\ s_3 &= s_2 - c_2 \\ s_4 &= s_3 - c_3 \\ s_5 &= s_4 - c_4 \end{aligned} \tag{5.22}$$

and the intertemporal budget constraint:

$$c_1 + c_2 + c_3 + c_4 + c_5 \leq s_1 \tag{5.23}$$

which simply says that total lifetime consumption must be less than or equal to the initial stock of cake.

Rather than set up the Lagrangian for that problem, though, we shall use the dynamic programming, recursive approach to finding the optimal time path of consumption.

In a finite horizon (a qualification which will become important later) dynamic programming problem, we begin by solving the final period's problem, which in this case means that we must find a rule which tells us how much the consumer will consume in period 5 given whatever amount of cake happens to be available to him at the end of the fourth period (the beginning of the fifth period). The stock available to him is s_5, so we are looking for a policy rule $c_5 = h_5(s_5)$ which maximizes $U(c_5)$ subject to $c_5 \leq s_5$.

In solving this problem we are going to make use of what are known as terminal conditions, a special case of what are known as terminal transversality conditions. A terminal condition is a piece of extra information which tells us where the system is going to end up – it is the counterpart of the initial conditions of which we made use in our discussion of difference equations. Sometimes the terminal condition for a problem is given, as in the case where we are told that there must be a certain amount of cake left over, perhaps for the next generation, at the end of period 5. Other times it can be deduced from basic economic principles, which is the approach we shall follow in our problem.

As we wrote the intertemporal objective function in Equation (5.21), our planner derives no utility from anything that happens after the end of period 5. Sometimes an expression like (5.21) will have, as its final term, an expression like $B(s_6)$, which is called a bequest, or scrap value function. A scrap value function tells us how much benefit the planner derives from units of the state variable which are left over at the end of the planning horizon.[4] In a consumption problem like ours, $B(s_6)$ often reflects the utility the planner derives from knowing that he will be leaving a bequest to his children, who are the next generation of planners.

In Equation (5.21), there is no scrap value function, meaning that the planner derives no utility from any cake left uneaten at the end of the five-period planning horizon. We make the standard assumption that marginal utility is always positive, which means that he will never decide to stop consuming because he is satiated.

Given these assumptions, we can write his fifth-period problem in one-period Lagrangian form as:

$$\underset{c_5}{\text{Max:}}\ U(c_5) + \lambda(s_5 - c_5) \tag{5.24}$$

which has first-order condition:

$$U'(c_5) - \lambda = 0 \tag{5.25}$$

which, given the assumption that marginal utility is always positive, tells us that the Lagrange multiplier λ is also positive, which in turn tells us that the constraint in Equation (5.24) is binding. The fact that the constraint is binding tells us that $c_5 = s_5$ which means that our policy rule for the fifth period is:

$$c_5 = h_5(s_5) = s_5 \tag{5.26}$$

In other words, the optimal policy rule for consumption in the final period of the planning period is to eat all of the remaining cake, whatever quantity that may be.

This policy rule, then, tells us that the maximized value of utility, which, consistent with our earlier notation we will write as $J_1(s_5)$, is equal to $U(s_5)$. This, then, is the value function for the fifth and final period of our five-period problem. The next step in the recursive, dynamic programming approach is to find $J_2(s_4)$.

We know that we can write the fundamental equation of optimality – the Bellman equation – for the fourth (and next-to-last) period as:

$$J_2(s_4) = \underset{c_4}{\text{Max:}}\ U(c_4) + \beta J_1(s_5) \tag{5.27}$$

we also know, from the general equation of motion for cake, that $s_5 = s_4 - c_4$ which means that $\partial s_5/\partial c_4 = -1$ which gives the first-order condition for Equation (5.27) as:

$$U'(c_4) = \beta \frac{\partial J_1(s_5)}{\partial s_5} \tag{5.28}$$

To find $\partial J_1(s_5)/\partial s_5$, note that we have already established that $c_5 = s_5$ and that $J_1(s_5) = U(s_5)$, from which we can see that $\partial J_1(s_5)/\partial s_5 = U'(c_5)$. Substituting this into Equation (5.28) gives:

$$U'(c_4) = \beta U'(c_5) \tag{5.29}$$

which is just an Euler equation.

Our next step is to set up the problem for $t = 3$:

$$J_3(s_3) = \underset{c_3}{\text{Max:}}\ U(c_3) + \beta J_2(s_4) \tag{5.30}$$

The first-order condition for the choice of c_3 is:

$$U'(c_3) = \beta \left(\frac{\partial J_2(s_4)}{\partial s_4} \right) \tag{5.31}$$

since, from the equation of motion for cake, $\partial s_4/\partial c_3 = -1$. Our problem now is to find an expression for $\partial J_2(s_4)/\partial s_4$. We know that, by judicious substitution,

we can rewrite $J_2(s_4) = U(h_4(s_4)) + \beta J_1(s_4 - h_4(s_4))$, but this turns out not to be terribly helpful. It is at this point, though, that we can make use of the Benveniste–Scheinkman condition. That condition tells us that $[\partial J_2(s_4)/\partial s_4] = \beta[\partial J_1(s_5)/\partial s_5]$ and we know that, since $J_1(s_5) = U(c_5)$, remembering that $s_5 = c_5$ and $[\partial J_1(s_5)/\partial s_5 = U'(c_5)]$, this gives $[\partial J_2(s_4)/\partial s_4] = \beta U'(c_5)$ and, from Equation (5.31):

$$U'(c_3) = \beta(\beta U'(c_5)) = \beta^2 U'(c_5) \tag{5.32}$$

We can keep working backwards this way, with our next step being to evaluate:

$$J_4(s_2) = \underset{c_2}{\text{Max:}}\, U(c_2) + \beta J_3(s_3) \tag{5.33}$$

from which, by using the first-order condition and the Benveniste–Scheinkman condition we find $U'(c_2) = \beta U'(c_3) = \beta^3 U'(c_5)$ and, continuing back, we find $U'(c_1) = \beta U'(c_2) = \beta^4 U'(c_5)$ so we wind up with:

$$U'(c_1) = \beta U'(c_2) = \beta^2 U'(c_3) = \beta^3 U'(c_4) = \beta^4 U'(c_5) \tag{5.34}$$

Recall that, since β is a discount factor, it is less than 1. The utility function U, unlike the policy function h_t, has the same functional form in each period, so differences in the level of utility, and of marginal utility, from one period to the next must be due to differences in consumption levels. Equivalently, if consumption is the same in each period, the levels of utility and marginal utility will also be the same across periods.

From Equation (5.34), we see that marginal utility must be increasing over time, since $U'(c_1) = \beta U'(c_2)$ means that marginal utility in period 1 must be some fraction β of marginal utility in period 2 and so on up. That in turn means that consumption must be decreasing over time.

If we were to assume that $\beta = 1$, so that the consumer did not discount future utility relative to present utility, Equation (5.34) would tell us that the level of his marginal utility of consumption had to be constant over time, meaning that he consumed the same amount of cake in each period. Since he begins the problem with a fixed amount of cake, s_1, and the cake does not grow, he would, in the case where $\beta = 1$, consume one-fifth of his initial stock of cake in each period, regardless of the precise functional form of his utility function $U(c)$. When β is less than 1, we know that consumption decreases over time, but his actual time path of consumption depends on the form of $U(c)$.

To see this in more detail, assume that $U(c) = \ln(c)$, which means that $U'(c) = 1/c$. Then, from Equation (5.34), we have:

$$\begin{aligned} c_2 &= \beta c_1 \\ c_3 &= \beta c_2 = \beta^2 c_1 \\ c_4 &= \beta c_3 = \beta^3 c_1 \\ c_5 &= \beta c_4 = \beta^4 c_1 \end{aligned} \tag{5.35}$$

We know that optimality involves consuming the whole of the cake by the end of the fifth period, so that $(c_1 + c_2 + c_3 + c_4 + c_5) = s_1$, which, after substitution from Equation (5.35) gives:

$$\begin{aligned} c_1 &= \frac{s_1}{(1 + \beta + \beta^2 + \beta^3 + \beta^4)} \\ c_2 &= \frac{\beta s_1}{(1 + \beta + \beta^2 + \beta^3 + \beta^4)} \end{aligned} \tag{5.36}$$

and we can find similar expressions for c_3, c_4 and c_5.

Expressions like (5.36), though, while telling us how to find consumption levels in successive periods, are not the policy functions we are looking for. To find $h(s)$ we need to make use of the equations of motion for s. For example, using $s_2 = s_1 - c_1$ in Equation (5.36) gives us:

$$c_2 = \frac{\beta s_2}{(\beta + \beta^2 + \beta^3 + \beta^4)} = h_2(s_2) \tag{5.37}$$

If we continue this way and find the other $h(s)$ functions, we will find that they are all different. To take a simple example, if we assume $\beta = 1$, so that the planner consumes one-fifth of the original stock of cake in each period, we will find that $c_1 = s_1/5$, $c_2 = s_2/4$, $c_3 = s_3/3$ and so on. This result, that the $h(s)$ functions differ across periods, generalizes to the case where assets do grow and where the consumer has outside income.[5]

We can also use the results we have derived to this point to find the $J(s)$ functions. In our present example, we find that:

$$\begin{aligned} J_1(s_5) &= \ln(s_5) \\ J_2(s_4) &= \ln\left(\frac{1}{1+\beta}\right) + \beta \ln\left(\frac{\beta}{1+\beta}\right) + (1+\beta)\ln(s_4) \end{aligned} \tag{5.38}$$

Neither of these are particularly intuitive, nor are the other $J(s)$ functions which we could proceed to find. In fact, there is only a limited number of cases in which the $J(s)$ function can be solved for. We also note that the $J(s)$ functions differ across periods, a result which also applies to the other $J(s)$ functions which we have not derived here.

We can check these functional forms by testing that the Benveniste–Scheinkman condition holds, remembering that it only holds when the c values have been chosen optimally, which means that there must be a precise relationship between s_4 and s_5 if the condition is to hold. By applying the Benveniste–Scheinkman condition to Equation (5.38) above, we can find an expression for s_5 as a function of s_4. We can also derive a relation between s_4 and s_5 from our solved c functions and the equation of motion for the cake. If the expressions we have derived for the J functions in Equation (5.38) are correct, these two approaches to finding s_5 as a function of s_4 should give us the same expression.

To this point, our discussion has dealt with finite time problems. In the next section we shall consider the additional complications which arise when we turn to infinite horizon discrete time optimization problems.

Infinite horizon problems

The biggest difference between finite and infinite horizon problems is that, in an infinite horizon problem there is no last period. This may sound trivially obvious, but it turns out to be very important. The approach that we adopted in solving our finite horizon cake eating problem was the standard dynamic programming approach of starting in the final period, where we invoked a terminal condition to allow us to solve that period's problem as a one-period problem, then solving backwards, period by period, to the beginning of the planning horizon. If there is no last period, we cannot do this.

Fortunately, it is still possible to solve infinite horizon problems by dynamic programming. To see how, consider the Bellman equation:

$$J_{T-t}(s_t) = \underset{c_t}{\text{Max:}} \left(U(c_t) + \beta J_{T-t-1}(s_{t+1})\right) \tag{5.39}$$

where the time subscripts on the control variable c and the state variable s referred to the date as measured from the beginning of the problem, sometimes known as elapsed time, and the subscripts on the J terms referred to the number of periods left to go in the problem. In an infinite horizon problem, both $T-t$ and $T-t-1$ will equal infinity, meaning that, while the subscripts on c and s still make sense, the ones on the J terms really do not. No matter where we happen to be in the programme, the future looks as long as it ever did.

Since the subscripts on the J terms do not make much sense in this context, we can drop them, and rewrite Equation (5.39) as:

$$J(s_t) = \underset{c_t}{\text{Max:}} \left(U(c_t) + \beta J(s_{t+1})\right) \tag{5.40}$$

writing the Bellman equation this way gives us an idea of how the next step will work. In the finite time problem, the form of the J function changed as time passed and as we approach the end of the planning horizon. In an infinite horizon problem, no matter how much time has elapsed we are never any closer to the end of the horizon than we were before. This means that the reason for the J function to change over time has vanished. Dropping the T subscripts from the J term in Equation (5.40) indicates that, in an infinite horizon problem, while the value of the J function will change as s changes, the functional form of the J function does not change. Similarly, if we consider the policy rule – the $h(s)$ function – in this light, we can see that while the value of the function (and therefore of the optimal c) will change as s changes over time, the functional form of $h(s)$ will not change.

Cake eating problem

To see what this means for the process of solving an infinite horizon optimization problem, consider a variant on the cake eating problem, in which the cake is allowed to grow over time.

Specifically, consider the problem of maximizing utility from consumption, $U(c)$, over an infinite horizon, where future consumption is discounted according to the discount factor β and consumption is done out of accumulated assets, a. Our asset this time is a financial one which earns interest at the unchanging one-period rate r. The consumer has no income other than interest on his assets, so the equation of motion for a is:

$$a_{t+1} = (a_t - c_t)(1 + r) \tag{5.41}$$

Conceptually, Equation (5.41) says that the individual starts period t with a stock of assets equal to a_t. He spends, or commits to spending (perhaps by placing the appropriate amount in a non-interest-bearing account) an amount c_t in period t, leaving him to save $(a_t - c_t)$ out of the wealth he possessed at the beginning of period t. He earns interest at rate r on his savings through period t, giving him period $t + 1$ wealth as specified by Equation (5.41).

The Bellman equation for our problem is:

$$J(a_t) = \underset{c_t}{\text{Max:}} (U(c_t) + \beta J(a_{t+1})) \tag{5.42}$$

The first-order condition for the maximization problem in Equation (5.42), making use of the equation of motion for a when we differentiate through with respect to c_t, is:

$$U'(c_t) = \beta(1 + r)J'(a_{t+1}) \tag{5.43}$$

Since this condition has to hold for all values of t, we can also write:

$$U'(c_{t-1}) = \beta(1 + r)J'(a_t) \tag{5.44}$$

where, because the functional form of J does not change over time, neither does the functional form of J'.

Next, we write the Benveniste–Scheinkman condition for the infinite horizon problem as:

$$J'(a_t) = \beta(1 + r)J'(a_{t+1}) \tag{5.45}$$

Combining Equations (5.43), (5.44) and (5.45) gives the Euler equation for consumption for this problem:

$$U'(c_{t+1}) = \frac{U'(c_t)}{\beta(1 + r)} \tag{5.46}$$

which is just a nonlinear FODE in c.

Sometimes analysis of these problems stops at this point; other times authors will assume a functional form for $U(c)$, which is what we do here. As in our finite horizon examples, we assume that $U(c) = \ln(c)$. Then, from Equation (5.46), we have:

$$c_{t+1} = \beta(1 + r)c_t \tag{5.47}$$

When the objective of the analysis is to go beyond the difference equation which is the Euler equation for consumption for this problem and find the policy rule $c = h(a)$, the most common next step is to assume a functional form for $h(a)$ and try substituting it in the problem. In practice, there exists only a limited number of forms of utility function (and therefore $J(s)$ functions) for which this approach will work, which is why the papers using this approach all seem to draw on basically the same small set of utility functions.

In our case, assume:

$$c_t = Xa_t \tag{5.48}$$

where X is an unknown constant whose value is to be determined. We assume X is a constant because the $h(a)$ function is unchanging over time. Next, take the equation of motion for assets, $a_{t+1} = (a_t - c_t)(1 + r)$, and substitute Equation (5.48) for the a terms, with the appropriate time subscripts, giving, after some cancellation:

$$c_{t+1} = (1 - X)(1 + r)c_t \tag{5.49}$$

Combining Equation (5.49) with (5.47) above gives us a pair of equations relating c_{t+1} and c_t, both of which must be satisfied. The test of our assumed functional form in Equation (5.48) is whether we can find an expression for X which satisfies this requirement. Equating Equations (5.47) and (5.49) gives $\beta(1 + r)c_t = (1 - X)(1 + r)c_t$ which holds only if $X = (1 - \beta)$, making our policy rule:

$$c_t = (1 - \beta)a_t \tag{5.50}$$

Expression (5.50) satisfies our presumption that consumption in any period would be a constant fraction (since β is less than 1) of assets in that period. Substituting Equation (5.50) into the equation of motion for a gives:

$$a_{t+1} = \beta(1 + r)a_t \tag{5.51}$$

Note that according to Equation (5.47), the Euler equation for this problem, $c_{t+1} = \beta(1 + r)c_t$, which, combined with Equation (5.51) tells us that $(c_{t+1}/a_{t+1}) = (c_t/a_t)$ which says that the ratio of (optimal) current consumption to current assets remains unchanged over time. While this is a fairly simple example, it does demonstrate that the lack of a final period is not a fatal problem for dynamic programming in infinite horizon problems.

Uncertainty

Before we proceed to the next section, it is worth noting that the additive structure of the Bellman equation in dynamic programming is well suited to the study of problems involving an uncertain future. We have been assuming that there was no uncertainty in period t about what the world would look like in period $t+1$. In practice, of course, the future state of the world is uncertain, a fact which should be, and is, allowed for in intertemporal optimization problems. Much of finance theory, for example, can be characterized as trying to use asset allocation to solve intertemporal consumption problems when the return on each asset is uncertain. We know the expected return on an asset, but the actual return will be a random drawing from a probability distribution.

We will leave dealing with problems of intertemporal optimization under uncertainty until later in this chapter, but here, as a simple example, consider the case in which there are two possible states of the future world, w_1 and w_2 respectively, with state w_1 eventuating with probability π and state w_2 eventuating with probability $(1-\pi)$. We assume that the actual future value of the J function depends on which of these two possible states of the world actually materializes, so that we have $J(s_{t+1}; w_i), i = 1, 2$, where s_{t+1} is driven by an equation of motion but the value of w_i which arises is beyond the planner's control.

In this case, the maximum value function becomes an expected maximum value function, and the Bellman equation can be written as:

$$\begin{aligned} J(s_t) &= \text{Max: } E(U(x_t) + \beta J(s_{t+1})) = \text{Max: } U(x_t) + E\beta J(s_{t+1}) \\ &= \text{Max: } U(x_t) + \pi\beta J(s_{t+1}; w_1) + (1-\pi)\beta J(s_{t+1}; w_2) \end{aligned} \tag{5.52}$$

Consider the case where π is the probability of surviving the one period from t to $t+1$ and let $J(s_{t+1})$ be the maximum value function from the next period's intertemporal optimization problem, conditional on survival. Then $(1-\pi)$ is the probability of dying before the beginning of period $t+1$, and the conventional assumption is that in that case $J(s_{t+1}) = 0$.[6] In that case, Equation (5.52) becomes:

$$J(s_t) = \max(U(x_t) + \pi\beta J(s_{t+1})) \tag{5.53}$$

Written in this form, we see that the survival probability, π, enters the problem in the same way as does the discount factor, β. A lower probability of surviving into the next period has the same effect on the individual's optimal consumption plan as does an increase in the rate at which we discount the future. Since β is $1/(1+\delta)$, where δ is the subjective discount rate, an increase in δ translates into a reduction in β. If we assume that π can be written as $1/(1+\eta)$ where η reflects the appropriate mortality rate, we can write $\pi\beta$ as:

$$\pi\beta = \left(\frac{1}{1+\delta}\right)\left(\frac{1}{1+\eta}\right) = \left(\frac{1}{1+\delta+\eta+\delta\eta}\right) \tag{5.54}$$

If δ and η are both sufficiently small, this will be approximately equal to $1/(1+\delta+\eta)$ and we can treat observed discounting of the future as consisting of two

elements which enter in an identical fashion. One element reflects pure myopia and the other reflects expectation of mortality. Some authors have suggested that awareness of the probability of death is a more realistic explanation for the human propensity to discount the future than is simple myopia.

If this is an appropriate way to introduce uncertainty about length of life, it means that the results we have derived to this point do not have to be changed in any fundamental manner to accommodate uncertain life expectancies. All we have to do is increase the discount rate δ – reduce the discount factor β – and proceed as before. We will find that individuals with a lower probability of survival will discount the future more heavily and allocate their assets accordingly, tending to shift more consumption towards the present.

This conclusion has implications beyond the simple problems we have been dealing with here. Investment problems, whether investment in physical capital or investment in human capital, are simple extensions of the consumption–savings problems we have been discussing. Anything which reduces the individual's propensity to save (i.e. increases his propensity to shift consumption towards the present, away from the future) will also reduce his propensity to invest in physical or human capital. When we are studying the behaviour of individuals who live under circumstances beyond their control which reduce their probability of survival, we should (and in fact do) find that they will tend to save and invest less than do otherwise identical individuals who have a higher exogenous value of π. If higher community income translates into higher values of π, we should find a greater propensity to save and invest in richer communities than in poorer, which could tend to act against convergence between richer and poorer countries. On the other hand, if increases in π do indeed translate into a greater incentive to save and to invest in both physical and human capital, public health measures which reduce mortality rates might well prove to be important components of successful economic development policies; perhaps, in the case of countries with very low values of π, more important than policies focussing on investment in physical capital.

One final note – in practice, even within a single country, π is not constant. It changes as individuals age, and also changes over time as life expectancy in general, increases. Setting aside the general increase in life expectancy, the fact that π changes with age means that for the individual it changes over time, even if there is no general upward drift in life expectancy. This means, in general, that, as individuals age, they may well tend to discount the future more heavily. This in turn is a reason why empirical exercises starting from individual intertemporal optimizing behaviour but working with aggregate data might do well to take account of the age distribution of the population whose behaviour they are studying.

Lagrange Multiplier approach

While dynamic programming is the most common approach to solving discrete time intertemporal optimization problems, Chow (1997) has proposed an alternative, Lagrange Multiplier approach. Most authors have avoided the Lagrange

Multiplier approach, on the assumption that it required keeping track of too many summation signs to make it tractable, but Chow argues that the difficulties are overestimated and the benefits of this approach underestimated.

Following Chow's notation (but working in a world with no uncertainty), let $r(x_t, u_t)$ be the expression for the objective function for the problem at time t, where x_t is the value of the state variable for the system, x, at time t and u_t the value of the control variable, u, at time t. Labelling u a control variable means that it is a choice variable, that its value is chosen by the planner (sometimes subject to constraints). Labelling x a state variable means that its value describes the state of the system at any point in time. The value of x is not at the planner's discretion, but is determined by the equation of motion for x, which we shall write as:

$$x_{t+1} = f(x_t, u_t), \quad t = 0, \ldots, T \tag{5.55}$$

Here, Equation (5.55) tells us that, given the value of x in period t, and given the function $f(x, u)$ which describes how x evolves over time, when the planner has chosen the value of the control variable u in period t the value of x in $t + 1$ is also determined. In an intertemporal optimization problem, especially when x enters the objective function, $r(x, u)$, this means that in choosing the value of the control variable u in period t the planner has to take account not only of how that choice affects the value of the objective function at t but also of how it will affect the value of the objective function in $t + 1$.

In most intertemporal optimization problems, future values of the objective function are discounted, so that we are working in present value terms: let $\beta = 1/(1 + \delta)$ be the subjective discount factor, where δ is the planner's subjective discount rate. The Lagrangian for this intertemporal optimization problem is:

$$Ł = \sum_{t=0}^{T} \beta^t r(x_t, u_t) - \beta^{t+1} \lambda_{t+1}(x_{t+1} - f(x_t, u_t)) \tag{5.56}$$

where the summation is over the planning horizon, usually written 0 to T, where T could be ∞. Our problem is to choose u_t and x_t, $t = 0, \ldots, T$, to maximize Equation (5.56).

Note that in Equation (5.56) we have applied a discount factor to the Lagrange Multiplier λ_{t+1}. This is a convenience which puts all parts of the problem explicitly on present value terms, and means that the multiplier λ_{t+1} itself is in current value terms. Note also that the subscript on the multiplier, and the exponent on the discount factor applied to the multiplier, agree with the time subscript on the first x term in the multiplier part of the expression.

Next, find the first-order conditions for maximizing Equation (5.56) with respect to both u and x. We can treat x as a control variable because the equation of motion, which we have built into the Lagrangian, turns out always to be satisfied and so actually constrains the possible values x can take on. This is, in general, how a Lagrangian expression works, even in the static case – it converts a problem of optimization subject to constraint into an equivalent unconstrained problem.

To see this, try substituting in the intertemporal optimization problem for all of the values of x, using Equation (5.55) above (with appropriate time subscripts), to eliminate all of the x terms. The resultant expression will be extremely messy, and the first-order conditions for that problem will be identical (apart from a bit of equating and substituting) to those which we are about to derive.

The first-order conditions for Equation (5.56), $t = 0, \ldots, T$ are:

$$\begin{aligned} \frac{\partial Ł}{\partial u_t} &= \beta^t \frac{\partial r(x_t, u_t)}{\partial u_t} + \beta^{t+1}\lambda_{t+1}\frac{\partial f(x_t, u_t)}{\partial u_t} = 0 \\ \frac{\partial Ł}{\partial x_t} &= \beta^t \frac{\partial r(x_t, u_t)}{\partial x_t} - \beta^t \lambda_t + \beta^{t+1}\lambda_{t+1}\frac{\partial f(x_t, u_t)}{\partial x_t} = 0 \end{aligned} \tag{5.57}$$

There are a couple of things which should be noted about the first-order conditions. First, they hold for all values of t, so we do not really have two equations, we have as many equations as there are discrete time periods in our problem. Second, we will be able to cancel out quite a few β terms. And third, it is important to note the role of x_t as an overlap term. If we were to write Equation (5.56) out in full, expanding according to the summation operator, we would find that x_t appears twice in the constraint part, once as we have shown above and once as the 'leading' part in the period $t - 1$ version of Equation (5.56). This means that, when we differentiate with respect to x_t we will wind up with two elements, one from $-\beta^t \lambda_t x_t$ and one from $\beta^{t+1}\lambda_{t+1} f(x_t, u_t)$.

Finite horizon

To see how the Lagrange Multiplier approach works in an economic problem, consider the cake eating problem again. Our objective is to maximize utility over a horizon running from $t = 1$ to T where c_t stands for consumption in period t, the control variable, and utility in period t is given by $\ln(c_t)$. The amount of cake left in period $t + 1$, s_{t+1}, is the state variable, and is determined by the equation of motion:

$$s_{t+1} = s_t - c_t \tag{5.58}$$

Equation (5.58) says that the amount of cake left in $t + 1$ equals the difference between the amount of cake available at the beginning of period t and the amount that was consumed in t. Cake does not grow. For simplicity, we assume that $\beta = 1$, that is, that the individual does not discount the future. The Lagrangian expression for this problem is:

$$Ł = \sum_{t=1}^{T} \ln(c_t) - \lambda_{t+1}(s_{t+1} - s_t + c_t) \tag{5.59}$$

where we shall assume that the sum runs from $t = 1$ to T (i.e. there is no period $t = 0$ here).[7]

The first-order conditions for problem (5.59) $t = 1, \ldots, T$ are:

$$\begin{aligned} \frac{\partial Ł}{\partial c_t} &= \frac{1}{c_t} - \lambda_{t+1} = 0 \\ \frac{\partial Ł}{\partial s_t} &= -\lambda_t + \lambda_{t+1} = 0 \end{aligned} \tag{5.60}$$

where s_t is the overlap term. From the first-order conditions we note that $\lambda_t = \lambda_{t+1}$ which tells us that the value of the Lagrange Multiplier remains unchanged over time. We also note that the inverse of consumption in one period (which, given the natural log form we have chosen for the utility function is the marginal utility of consumption) is equal to the value of the multiplier in the next period. Since these conditions hold for all t, it is also the case that $(1/c_{t-1} = \lambda_t)$ so, combining the first-order conditions, we find $(1/c_{t-1} = 1/c_t)$ from which it is obvious that, for all t, $c_{t-1} = c_t$ which tells us that, given the absence of discounting, optimality requires the amount of cake consumed to be the same in each period.

Since this is a finite horizon problem (T is finite, because we are not trying to make the cake last forever) with no bequest element in the intertemporal utility function, there is no gain to be had from any cake left over after period T. This tells us that consumption in T, c_T, should equal the stock of cake remaining at the beginning of T, s_T, so that $s_{T+1} = 0$. But given that c_t is constant over time, if $c_T = s_T$, then so does c_{T-1}, and so also does c_{T-2} and every earlier value of c. Total consumption, then, is Ts_T. Since we want to consume the whole of the cake, and since we started out with a quantity of cake equal to s_1, by definition, all of this tells us that $Ts_T = Tc_T = s_1$ from which $c_T = s_1/T$ and, since c is constant over time, this gives us:

$$c_t = \frac{s_1}{T}, \quad t = 1, \ldots, T \tag{5.61}$$

If $T = 5$, we consume in each period a quantity of cake equal to one-fifth of the initial stock of cake, just as we concluded in the dynamic programming version of the cake eating problem.[8] This example certainly seems to provide support for Chow's argument that the Lagrange Multiplier method is actually much simpler to use than many people assume.

We should note, at this point, that the Lagrange Multiplier method only gave us the first-order conditions – the condition that said that the level of consumption had to be the same in each period. The final stage in the analysis, in which we determined what that level would be, required us to invoke an extra piece of information, a transversality condition. In this case, that information told us that there was no point in leaving any cake uneaten at the end of the planning horizon. Implicitly, we also assumed that we did not want to have consumed the whole of the cake before the end of the planning horizon. Cake was all we had to eat.

These two conditions, one made explicitly and the other implicitly (although in general it should be made explicit) tied down the solution to the problem. There are, after all, an infinite number of paths satisfying the condition that $c_{t-1} = c_t$,

but only one which also satisfies the condition that the stock of cake be completely exhausted precisely at the end of the planning horizon (and that path will vary depending on the size of the cake we start with and the number of periods in the planning horizon). In this, the Lagrange Multiplier approach is no different from the dynamic programming approach.

Infinite horizon

Next, let us consider an application of the Lagrange Multiplier approach to an extension of the cake eating problem. Again we have an intertemporal utility maximization problem, and again we shall assume that the utility function in period t is a natural log function, $\ln(c_t)$. In this problem, however, the planner is assumed to have inherited a stock of wealth, w_0, at the beginning of the planning horizon. His accumulated wealth earns interest at a constant rate r, and we shall define $R = (1 + r)$ as the interest factor. The equation of motion for his wealth is:

$$w_{t+1} = R(w_t - c_t), \quad t = 0, \ldots, T \tag{5.62}$$

so that, in period $t + 1$, his wealth consists of that portion of his period t wealth which he did not consume in t, plus interest earned on that wealth. We shall, in what follows, assume, at least initially, that $T = \infty$, so we have an infinite horizon problem, and we shall assume that he has a subjective discount factor β. The Lagrangian for this problem is:

$$Ł = \sum_{t=0}^{\infty} \beta^t \ln(c_t) - \beta^{t+1}\lambda_{t+1}(w_{t+1} - R(w_t - c_t)) \tag{5.63}$$

The first-order conditions for the problem are:

$$\begin{aligned} \frac{\partial Ł}{\partial c_t} &= \beta^t \left(\frac{1}{c_t}\right) - \beta^{t+1}\lambda_{t+1}R = 0 \\ \frac{\partial Ł}{\partial w_t} &= -\beta^t\lambda_t + \beta^{t+1}\lambda_{t+1}R = 0 \end{aligned} \tag{5.64}$$

These conditions can be simplified as $(1/c_t) = \beta\lambda_{t+1}R$ and $\lambda_t = \beta\lambda_{t+1}R$, and since the conditions must hold for all t, we also have $c_t = \beta R c_{t-1}$, a first-order difference equation in c. Whether the coefficient on c_{t-1} is greater or smaller than 1 depends on the relative sizes of the market interest rate, r, and the subjective discount rate, δ, since $\beta R = (1 + r)/(1 + \delta)$. If the planner's subjective discount rate is greater than the market rate, $\delta > r$ and $\beta R < 1$ so $c_t < c_{t-1}$, telling us that consumption declines over time. This is a standard result – if an individual discounts the future at a rate in excess of the market interest rate, they will tend to shift consumption away from the future, into the present.

Note that combining the first-order equations with the equation of motion for w gives us a system of two first-order, linear difference equations in c and w:

$$\begin{aligned} c_t &= \beta R c_{t-1} \\ w_t &= R(w_{t-1} - c_{t-1}) \end{aligned} \tag{5.65}$$

Before we analyse this system, note that while Equation (5.65) characterizes the behaviour of c and w over time, it is not necessarily in the form that most interests us for empirical purposes. In particular, Equation (5.65) tells us how consumption changes over time, but it may well be that what we are really interested in is an expression relating consumption in period t to wealth in period t – a type of consumption function (although based on wealth rather than current income).

The solution form that is usually given in the literature for that relation is:

$$c_t = (1 - \beta) w_t \tag{5.66}$$

which says that consumption in period t is a fraction $(1 - \beta)$ of wealth (since $\beta = 1/(1 + \delta)$, we could also write this as $(\delta/(1 + \delta))w_t$). Note that the absence of a time subscript on β means that this relation is assumed to hold unchanged in each period, so that consumption in any period is a constant fraction $(1 - \beta)$ of wealth in that period. Which simply says that the ratio of consumption to wealth is constant over time, even though the levels of c and w may be changing.

To see whether the form (5.66) fits with our results, assume $c_t = \gamma w_t$ where γ is some unknown constant. Next, substitute for the w terms in Equation (5.65), giving $(c_t/\gamma) = R(c_{t-1}/\gamma - c_{t-1})$ from which we have:

$$c_t = R(1 - \gamma) c_{t-1} \tag{5.67}$$

Expression (5.67) is consistent with (5.65) if $(1-\gamma) = \beta$ (from which $\gamma = (1-\beta)$), so Equation (5.66) is consistent with Equation (5.65).

Before considering this point further, we shall return to a point we made earlier, that Equation (5.65) is a system of two, linear difference equations. In matrix notation, it gives:

$$\begin{bmatrix} c_t \\ w_t \end{bmatrix} = \begin{bmatrix} \beta R & 0 \\ -R & R \end{bmatrix} \begin{bmatrix} c_{t-1} \\ w_{t-1} \end{bmatrix} \tag{5.68}$$

Note that the difference equations in this system are homogeneous. The roots of the matrix of coefficients in Equation (5.68) are R and βR. R is greater than 1, and we shall assume, for the sake of the illustration, that βR is less than 1, which means that the planner's subjective discount rate, δ, is greater than the interest rate, r. This means that the equilibrium of the system (which, since both equations are homogeneous, is at the origin) is a saddlepoint.

As our next step, consider the characteristic vectors associated with each of the roots. Taking the larger root, R, first, solving for its characteristic vector

(or eigenvector) requires solving:

$$\begin{bmatrix} (\beta-1)R & 0 \\ -R & 0 \end{bmatrix} \begin{bmatrix} w_{11} \\ w_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.69}$$

from which we see that w_{11} must be zero for the equation to be satisfied. The second term, w_{21}, can take on any value, so we normalize it to equal 1, giving $(w_{11}, w_{21})' = (0, 1)'$. Solving for the eigenvector associated with the second root, βR, which we shall write as $(w_{12}, w_{22})'$, we find, on normalizing w_{22} to equal 1, that $(w_{12}, w_{22})' = ((1-\beta), 1)'$.

Recalling that $X_t = AX_{t-1}$ can, when the eigenvectors of A are distinct, be written as $X_t = W\Lambda^t W^{-1} X_0$ where W is the matrix composed of the eigenvectors of A, and X_0 is the vector of initial values of, in this case, c and w, we have:

$$W = \begin{bmatrix} 0 & (1-\beta) \\ 1 & 1 \end{bmatrix}, \qquad W^{-1} = \begin{bmatrix} \dfrac{1}{\beta-1} & 1 \\ \dfrac{-1}{\beta-1} & 0 \end{bmatrix} \tag{5.70}$$

from all of which we find (with $\lambda_1 = R_1$ and $\lambda_2 = \beta R$):

$$\begin{bmatrix} c_t \\ w_t \end{bmatrix} = \begin{bmatrix} 0 & (1-\beta) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} \begin{bmatrix} \dfrac{1}{\beta-1} & 1 \\ \dfrac{-1}{\beta-1} & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ w_0 \end{bmatrix} \tag{5.71}$$

that is:

$$c_t = 0\lambda_1^t + c_0\lambda_2^t \tag{5.72}$$

$$w_t = \left(w_0 - \frac{c_0}{(1-\beta)}\right)\lambda_1^t + \left(\frac{c_0}{(1-\beta)}\right)\lambda_2^t \tag{5.73}$$

From Equations (5.72) and (5.73) we see that consumption is driven solely by the stable root of the system; hence the FODE in Equation (5.65). Recalling the point we noted above, that the solution to this problem is often given in the literature as $c_t = (1-\beta)w_t$ we see that, if we assume, as we should, this form applies to c_0 and w_0, the weight on the λ_1 term in Equation (5.73) also goes to zero, and w_t will also be driven only by the stable root of the system. Further, assuming $c_0 = (1-\beta)w_0$, the ratio c_t/w_t which we derive from Equations (5.72) and (5.73) is $(1-\beta)$ for all t, giving $c_t = (1-\beta)w_t$, as it should.

This manipulation, then, gives us an idea of why $c_t = (1-\beta)w_t$ works as a solution to the present problem, and while it will in general be faster to experiment with likely solution forms than to go through the whole of this derivation, having a bit more of an understanding of what is going on is generally useful.

These results also let us return to our discussion of transversality conditions. Because both of the difference equations in our system are homogeneous, the

equilibrium for the system is at the origin. Technically, while, under the assumptions we made above, our system does tend to converge to the equilibrium, it takes an infinite amount of time to do so. This suggests that the solution $c_t = (1 - \beta)w_t$ might only work in the infinite horizon case.

To consider this point, look at the case where T is finite. Assuming that there is no bequest motive, and therefore no reason to leave any wealth behind after T is reached, Equation (5.65) suggests that we should have $w_{T+1} = R(w_T - c_T) = 0$ from which we have:

$$c_T = w_T \tag{5.74}$$

which is not consistent with $c_t = (1 - \beta)w_t$ unless $\beta = 0$, which is not the case if we are assuming β to be constant, as we are. It is, however, consistent with Equation (5.65) $c_t = \beta R c_{t-1}$ since this requires only that $c_{T-1} = w_T/\beta R$, and so on back.[9] In other words, as we asserted earlier, the solution to the Lagrange Multiplier version of the optimization problem yields necessary conditions, which will always be satisfied along a solution path. The transversality condition, which depends on the planning horizon assumed, determines which candidate path will actually be followed.

While dynamic economic models can, as we have seen, display a wide range of dynamic behaviour, dynamic models derived from intertemporal optimization problems virtually always display saddlepoint dynamics. In fact, if an intertemporal optimization model does not display saddlepoint dynamics, you have probably made a mistake in calculations.

Saddlepoint dynamics in a 2×2 system, as we have seen, means that the system has one stable and one unstable root. We saw in the example preceding that saddlepoint dynamics does not have to mean explosive behaviour – if the weight on the unstable root is zero, only the stable root will operate and the system will converge on its long-run equilibrium point. This case is referred to as being on the stable branch; in general being on the stable branch will be the optimal trajectory for an infinite horizon problem. When the optimization problem has a finite horizon following the stable branch to the long-run equilibrium will, in general, not be optimal, and we will need some other transversality condition to pin down the trajectory the system will actually follow. In this case, both the stable and the unstable roots will be operative, in the sense that neither will have a zero weight. The system will still display saddlepoint dynamics: this type of behaviour is associated with optimization, not with a particular horizon.

While this could, in principle, involve negative roots, alternations, like cycles, are generally not associated with optimization. Basically, (thinking in infinite horizon terms) why spend time cycling or jumping around the long-run equilibrium of the system when you could head there directly? By far the most likely outcome of an intertemporal optimization problem involves having two positive roots, one greater than 1 and the other less than 1. There is actually a pair of conditions which can be used to test for this case (or at least to establish conditions under which it

will hold). We will have saddlepoint dynamics with positive roots if:

$$\begin{aligned} 1 - \mathrm{Tr}(A) + \mathrm{Det}(A) &< 0 \\ 1 + \mathrm{Tr}(A) + \mathrm{Det}(A) &> 0 \end{aligned} \tag{5.75}$$

Examples

Model of investment in health

We shall now consider an intertemporal optimization problem – a simple version the Grossman (1972) model of investment in health capital. In this model, the individual derives utility from consumption goods, C, and from his state of health, H. His utility function is written in general form as $U(C_t, H_t)$. He can buy consumption goods out of his current income, Y, but he cannot buy health directly. Instead, he buys goods which affect his health. We denote these goods by I_t and assume that he derives no direct utility from them.[10] His health in period t is determined by the equation of motion:

$$H_t = H_{t-1}(1 - \phi) + I_{t-1} \tag{5.76}$$

In Equation (5.76), ϕ represents the natural rate at which health declines per period if we take no measures to preserve it. For simplicity we assume ϕ to be constant. In practical applications it probably should be a function of age.

For convenience we assume that the individual spends the whole of his income, Y, in each period, giving:

$$Y_t = C_t + pI_t, \quad \forall t \tag{5.77}$$

Here the price of health investment goods is denoted by p and we have normalized the price of consumption goods to 1. We shall assume that his income is the same in each period, so that $Y_t = Y, \forall t$. Rearranging Equation (5.77), then, gives $C_t = Y - pI_t$ which we shall substitute into the utility function, giving us $U(Y - pI_t, H_t)$ and letting us treat I_t as the choice, or control, variable for the problem.

The fact that the individual's current state of health is determined by the difference equation (5.76) is the reason we refer to the individual as investing in health capital. Decisions which the planner makes about health-related behaviour today have implications for his health over many future periods. Since he derives no direct utility from I-type goods, but sacrifices utility from C-type goods when he decides to invest in his health, he has to decide the extent to which he is willing to sacrifice current utility in exchange for the utility he will derive in the future as a result of being in better health. The need to sacrifice today in order to benefit tomorrow is the mark of an investment decision.

The Lagrangian for this problem is:

$$Ł = \sum_{t=1}^{T} \beta^t U(Y - pI_t, H_t) - \beta^{t+1}\lambda_{t+1}(H_{t+1} - H_t(1-\phi) - I_t) \tag{5.78}$$

and the first-order conditions are:

$$\begin{aligned} -p\beta^t U_{C_t} + \beta^{t+1}\lambda_{t+1} &= 0 \\ \beta^t U_{H_t} - \beta^t \lambda_t + \beta^{t+1}\lambda_{t+1}(1-\phi) &= 0 \end{aligned} \tag{5.79}$$

where U_{C_t} is the marginal utility of consumption of C-type goods, evaluated at period t values, and U_{H_t} is the marginal utility of health capital, also evaluated at period t values. Combining the two equations yields:

$$pU_{C_{t-1}} = \beta U_{H_t} + pU_{C_t}(1-\phi)\beta \tag{5.80}$$

Equations (5.80) and (5.76) constitute a system of two FODE in I and H. Equation (5.80) is actually the implicit form of a nonlinear difference equation; to proceed further with this example we need to make some more simplifying assumptions. We shall assume that the marginal utilities are linear in H and C (we shall again use the budget constraint to substitute C out) and, particularly unrealistically, we shall assume that U_{CH}, the cross-partial of the utility function in C and H, is zero. Thus we have:

$$\begin{aligned} U_{C_t} &= f_C + f_{CC}C_t, \qquad f_C > 0, \quad f_{CC} < 0 \\ U_{H_t} &= f_H + f_{HH}H_t, \qquad f_H > 0, \quad f_{HH} < 0 \end{aligned} \tag{5.81}$$

To find the period $t-1$ marginal utilities we replace C_t and H_t by C_{t-1} and H_{t-1}. Making the substitutions in equation (5.80) gives, as our pair of difference equations:

$$\begin{aligned} \beta\Psi_1 + \beta p(1-\phi)\Psi_2 &= p\Psi_3 \\ H_t &= H_{t-1}(1-\phi) + I_{t-1} \end{aligned} \tag{5.82}$$

where $\Psi_1 = (f_H + f_{HH}H_t)$, $\Psi_2 = f_C + f_{CC}(Y - pI_t)$ and $\Psi_3 = f_C + f_{CC}(Y - pI_{t-1})$. The system (5.82) is actually in the form:

$$BX_t = AX_{t-1} + Z \tag{5.83}$$

Isolating the X_t vector on the left-hand side requires that we find $X_t = B^{-1}AX_{t-1} + B^{-1}Z$. The matrix of coefficients, $B^{-1}A$, which is key to analysing the dynamic behaviour of the system, is, in this case:

$$B^{-1}A = \begin{bmatrix} \dfrac{p^2 f_{CC} + \beta f_{HH}}{\beta p^2(1-\phi)f_{CC}} & \dfrac{f_{HH}}{p^2 f_{CC}} \\ 1 & (1-\phi) \end{bmatrix} \tag{5.84}$$

The trace and determinant of this matrix are both positive, so we have two positive roots. Checking the saddlepoint condition, we find that $(1 - \text{Trace} + \text{Determinant}) < 0$ and that $(1 + \text{Trace} + \text{Determinant}) > 0$, so one of our roots is greater than 1 and the other positive but less than 1, meaning that the equilibrium is a saddlepoint.

Finally, we have written the model as a pair of FODEs in I and H. In practice, while we may be able to observe I, we cannot observe H, at least not in cardinal units. If we want to do an empirical study of health investment behaviour at the individual level, we obviously need to be working with variables which we can measure. The best approach, therefore, might be to take advantage of the fact that a system of two FODEs can be collapsed into a single second-order equation. In the present case we would obviously want to work with the observable variable I, so we should be looking at estimating a SODE in I, with individual-level data.

Stochastic optimization

In this section we develop as an example of a stochastic optimization problem the model of Samuelson (1969). Consider the case of an investor facing a choice of two assets, a safe asset which pays a risk-free interest rate of r per period, and a risky asset which pays a stochastic return of z_t per period, with $E(z_t - r) > 0$. The risky asset's rate of return is characterized by a probability density function which we shall write as $f(z)$. The lack of a time subscript on r means that we are assuming the riskless interest rate to be constant over time.

The individual's problem is to choose a consumption–savings plan to maximize the expected present value of his lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{5.85}$$

where the summation is over all values of t, and for simplicity we shall deal with the infinite horizon case. The subscript on the expectation operator indicates that he is making his lifetime consumption–savings plan at the beginning of the planning horizon at time $t = 0$.

His wealth, W, evolves according to a difference equation which in turn depends on s_t, the proportion of his portfolio which he invests in the risky asset in each period. The proportion he invests in the riskless asset is, of course, $(1 - s_t)$ and $0 \leq s_t \leq 1$. For simplicity we shall assume an interior solution, meaning that s is a positive fraction. Given these assumptions, his wealth evolves according to:

$$W_{t+1} = (1 + r)(1 - s_t)(W_t - c_t) + (1 + z_t)s_t(W_t - c_t) \tag{5.86}$$

which we shall rearrange to give:

$$W_{t+1} = [1 + r + s_t(z_t - r)]\,(W_t - c_t) \tag{5.87}$$

The Lagrangian for our problem, then, is:

$$Ł = E_0\left[\sum_{t=0}^{\infty}\beta^t U(c_t) - \beta^{t+1}\lambda_{t+1}\left[W_{t+1} - (1 + r + s_t(z_t - r))(W_t - c_t)\right]\right] \tag{5.88}$$

Our choice variables are c_t, W_t and s_t, and the first-order conditions are respectively:

$$\begin{aligned} U_c(c_t) &= \beta E_t \lambda_{t+1}(1 + r + s_t(z_t - r)) \\ \lambda_t &= \beta E_t \lambda_{t+1}(1 + r + s_t(z_t - r)) \\ 0 &= E_t \lambda_{t+1}((z_t - r)(W_t - c_t)) \end{aligned} \tag{5.89}$$

Note that in each of these conditions we have written the expectations operator, E, with a 't' subscript. This is because at time t, all past values of all of the variables in the system are known, and we form expectations only about the unknown future values of the variables. Even c_t can be treated as known, since in period t it is a choice variable, meaning that we will wind up picking a single, non-stochastic, value of c. We treat z_t as stochastic since its actual value is not known until after consumption and savings decisions have been made for period t. Think of the return on an asset in period t as not being paid until the end of the period, but of the consumption and portfolio decisions for period t as having to be made at the beginning of the period.

From the first and second equations in (5.89), we have $\lambda_t = U_c(c_t)$ and since the choice of time subscript is arbitrary, we also have $\lambda_{t+1} = U_c(c_{t+1})$ which gives Equation (5.89) as:

$$U_c(c_t) = \beta E_t U_c(c_{t+1})(1 + r + s_t(z_t - r)) \tag{5.90}$$

which tells us about the optimal relation between the marginal utility of consumption over time, and therefore about the time pattern of consumption itself. Note, however, that since c_{t+1} and z_t are, from the perspective of period t, stochastic variables, the right-hand side of Equation (5.90) is the expectation of the product of two random variables, which means that we cannot simply divide through by $(1 + r + s_t(z_t - r))$.[11]

Our next step is to see if we can find a relation of the form $c_t = g(W_t)$ to characterize the relation between wealth and consumption in period t. We have found such expressions in non-stochastic problems, but there was always a nagging sense that they might have been artefacts. After all, in a strictly non-stochastic problem, time paths of c and W are determined at the beginning of the planning horizon and do not change so long as the conditions of the problem do not change. This creates the impression that any observed relation of the form $c_t = g(W_t)$ simply reflects the evolution of the two variables, rather than representing a causal relation between them. If the conditions of the non-stochastic problem do happen

to change, we have to solve the new problem treating the point at which the change occurred as the starting point for that new problem, but then things roll on smoothly again.

In a stochastic problem, W does change relative to its expected value as realizations of the random variable z arrive, and we are clearly interested in how c changes in response to random changes in W. If c does respond in a systematic manner to random changes in W, we have a genuine, causal relation between c and W which would be worth investigating econometrically.

In order to advance with this question, we need to assume a form for the utility function. As before, we assume that $U(c) = \ln(c)$, which gives, from Equation (5.90):

$$\frac{1}{c_t} = \beta E_t \left(\frac{1}{c_{t+1}} \right) (1 + r + s_t(z_t - r)) \tag{5.91}$$

Next, for no better reason than that it works in the non-stochastic counterpart of our problem, let us try, as a g-function, $c_t = \gamma W_t$. Since we are looking for a consumption–wealth relation which holds for all t in an infinite horizon problem, this implies that $c_{t+1} = \gamma W_{t+1}$. Substituting into Equation (5.91) gives the unpromising looking:

$$\frac{1}{\gamma W_t} = \beta E_t \left(\frac{1}{\gamma W_{t+1}} \right) (1 + r + s_t(z_t - r)) \tag{5.92}$$

Next, return to Equation (5.87), the equation of motion for W. Substituting for c_t on the right-hand side of Equation (5.87) gives $W_{t+1} = (1 + r + s_t(z_t - r))W_t (1 - \gamma)$. Then, note that at the beginning of period t, W_t is known, meaning that we can take it inside the expectations operator without seriously affecting anything, all of which is to say that:

$$1 = \beta E_t \frac{\gamma W_t (1 + r + s_t(z_t - r))}{\gamma W_t (1 + r + s_t(z_t - r))(1 - \gamma)} \tag{5.93}$$

We can simplify Equation (5.93) by doing some cancellation on the right-hand side, and with only non-stochastic terms on the right-hand side, we obtain $\gamma = 1 - \beta$, giving, as the relation we were looking for:

$$c_t = (1 - \beta) W_t \tag{5.94}$$

Note that Equation (5.94) says that c does indeed respond to the changes in W which result from the stochastic nature of z.

While we have found an expression for current consumption as a function of current wealth, we have not actually solved what may be the most interesting part of the problem, the choice of s_t. The reason for this is that we cannot solve for it, at least not without detailed information on the probability distribution of z. We can, however, get an idea of what is involved by returning to the third of the first-order condition in (5.89).

Maintaining the assumption that the utility function has the natural log form (with the type of risk aversion behaviour that this implies) and substituting Equation (5.94) in for c_t, we can write the third of the conditions in (5.89) as:

$$E_t\left[\frac{(z_t - r)}{1 + r + s_t(z_t - r)}\right] = 0 \tag{5.95}$$

where, since β is non-stochastic, we have taken $(1 - \beta)$ outside the expectations operator. We cannot actually go any further in solving for s. To see why, recall that, for a function $g(x)$ of a random variable x, $\mathrm{E}g(x) = \int g(x) f(x) dx$ where $f(x)$ is the probability density function for x. Applying this to Equation (5.95) gives:[12]

$$\int \left(\frac{(z - r)}{1 + r + s(z - r)}\right) f(z)\, \mathrm{d}z = 0 \tag{5.96}$$

Note that the integration is over the range of possible values of z, not over t. Note too that because the probability distribution of z does not change over time, we were able to drop the t subscript from z, which means that the same value of s will solve equation (5.96) in each period.

The optimal proportion of W to allocate to the risky asset in each period will be the value of s which guarantees that Equation (5.96) holds, but without knowing the exact functional form of $f(z)$ we cannot say anything about what this value will actually be. The convention is simply to say that this expression can be solved for s, and far be it from us to break with convention.

As a simple extension of the basic portfolio choice model, consider the case where, in addition to the riskless asset, we have two risky assets, with stochastic rates of return z_{1t} and z_{2t}, respectively. This means that we have two optimal portfolio share values to solve for, s_{1t} and s_{2t}. We shall assume that both s values are positive fractions, and that their sum is less than 1 (we could add these to the problem as constraints – for simplicity we are assuming an interior solution).

The basic analysis of the problem is as before, so we do not work it through. The difference arises in the equations which must be solved for s_{1t} and s_{2t}: in this case we have two integral equations:

$$\begin{aligned} E_t\left[\frac{(z_{1t} - r)}{1 + r + s_{1t}(z_{1t} - r) + s_{2t}(z_{2t} - r)}\right] &= 0 \\ E_t\left[\frac{(z_{2t} - r)}{1 + r + s_{1t}(z_{1t} - r) + s_{2t}(z_{2t} - r)}\right] &= 0 \end{aligned} \tag{5.97}$$

These are integral equations because they involve the expectations operator, E_t. They must be solved simultaneously for the optimal values of s_{1t} and s_{2t}. The integrals involve the density functions of the risky assets, and (as is clear from the fact that both assets appear in both equations) this clearly means looking at their joint density. Thus, in determining their optimal shares we must take account not only of the variances of the returns on the individual assets but also of the covariance between their returns. This is, of course, a standard result from portfolio theory.

6 Nonlinear difference equations

Introduction

The models we have been discussing to this point have basically been linear, and the analysis has been in terms of linear difference equations. Even in Chapter 5, 'Intertemporal optimization', when we ran into expressions like $U'(c_{t+1}) = U'(c_t)/\beta(1+r)$, which is a nonlinear first-order difference equation (FODE) in c, we assumed a form for the utility function which made things neatly linear: if $U(c) = \ln(c)$, we had $c_{t+1} = \beta(1+r)c_t$, a linear, homogeneous, FODE.

In practice, a great many economic models yield nonlinear dynamic relations. Probably the most familiar of such models are various models of economic growth, but consumption models of the sort we referred to above also yield nonlinear relations, especially when the utility function is not a member of a fairly restrictive class of functions.

In the broadest terms, the introduction of nonlinearity does not change the essence of a difference equation: we are still looking at an equation which describes the evolution of a variable over time. We just happen to be writing something like $x_{t+1} = f(x_t)$ instead of something like $x_{t+1} = a + bx_t$. The nonlinear form includes the linear form as a special case, and permits a much broader range of types of trajectories to develop. We can best see this by considering what nonlinearity of the $f(x)$ function means for the phase diagram for a FODE.

Phase diagrams

Consider the case where $x_{t+1} = f(x_t)$ with $f'(x) > 0$ and $f''(x) < 0$. Let $f(0) = 0$. Note that under these assumptions, while the slope of the $f(x)$ function flattens as x increases, it never becomes negative.

If we plot this curve on a graph which has x_{t+1} on the vertical axis and x_t on the horizontal, we get something which looks like Figure 6.1(a). We have also plotted a 45° line on Figure 6.1(a). As in the case of the phase diagram for a linear FODE, the intersection of the two curves marks an equilibrium point, a point where $x_{t+1} = x_t$.

Since both the $f(x)$ function and the 45° line go through zero, $x_{t+1} = x_t = 0$ is an equilibrium of the system. If we were dealing with a linear FODE, this would be the only equilibrium of the system.

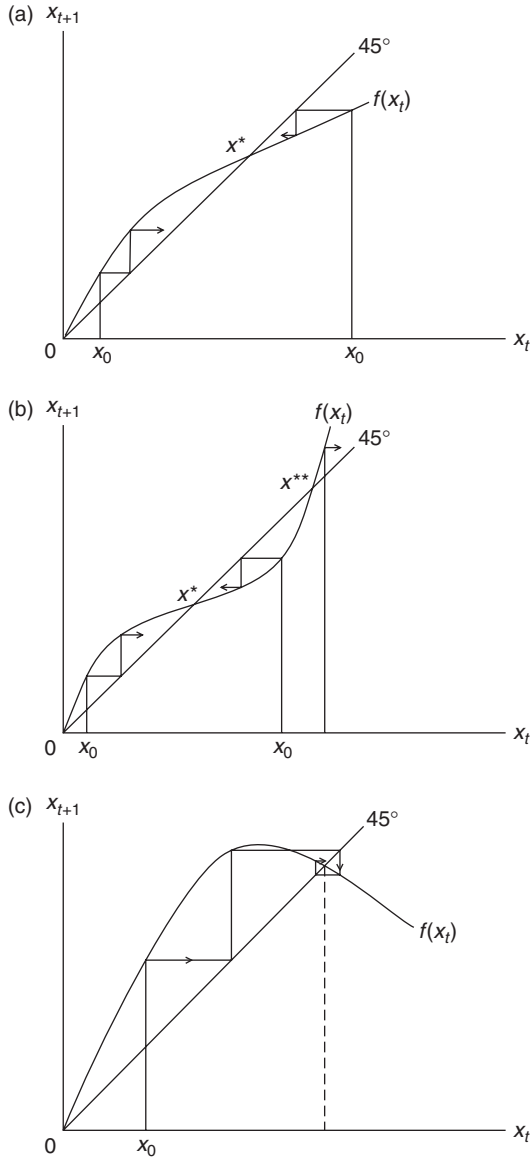

Figure 6.1 Phase diagrams for nonlinear difference equations.

In Figure 6.1(a), though, there is a second point at which the $f(x)$ curve cuts the 45° line, at the x value we have labelled x^*, and since this is also a point at which $x_{t+1} = x_t$, it is also an equilibrium point. A nonlinear difference equation can, then, have multiple equilibria, one for each time the $f(x)$ function crosses the 45° line.

In Figure 6.1(a), as we have drawn it, the second, upper, equilibrium point occurs at a point where the $f(x)$ line crosses the 45° line from above, with a slope which is positive and less than 1. In contrast, at the first, lower, equilibrium point, the one at the origin, the $f(x)$ curve has a positive slope greater than 1.

We know from our discussion of linear FODEs that, when the equilibrium of a linear equation is associated with a positive slope, x will approach it or diverge from it monotonically. That that result carries over to the nonlinear case is easily seen. We can also see by analogy with the linear case that when the slope of the $f(x)$ function at the equilibrium is less than 1 the equilibrium is stable and when the slope is greater than 1 the equilibrium is unstable. In terms of Figure 6.1(a) this means that the lower equilibrium, at $x = 0$, is unstable and that the upper one, at $x = x^*$, is stable.

Translating this into the behaviour of x, we can see that, if x_0, the initial value of x, is at either of the equilibria, the value of x will not change over time. If x_0 is either just above or just below 0, the system will diverge from zero, while if x_0 is just above or just below x^*, the system will converge on x^*.

In fact, as we have drawn Figure 6.1(a), if the initial value is anywhere above 0 the system will converge on x^*, either from above or from below, while if the initial value is anywhere below 0 (assuming negative values to be admissible, which they often are not in economic applications) the system will diverge from 0 below.

Strictly speaking, the stability of the equilibrium at x^* should be characterized as local stability, because the system will converge to x^* only if its initial value happens to fall in a local area around the equilibrium – in this case that local area happens to be all values strictly greater than zero. If the $f(x)$ function was linear, with a slope that was positive and less than 1 where it cuts the 45° line at x^*, the system would converge on x^* regardless of where its initial value happened to lie. In that case we would refer to x^* as being a globally stable equilibrium. Clearly whenever we have multiple equilibria, stability is going to be local rather than global.

In Figure 6.1(b) we have changed the form of $f(x)$ so that, after cutting the 45° line at x^*, as in Figure 6.1(a), it then curves back up and cuts again at x^{**}. Now x^{**} is also an equilibrium, and, from the fact that the slope of $f(x)$ is greater than 1 at that point, we can see that it is an unstable equilibrium.

If the system's initial value is above this new equilibrium, x will head off to infinity (assuming, of course, that there are no other equilibria above this one). The equilibrium at x^* is still locally stable, but now the neighbourhood within which the initial value of x must lie for the system to converge to x^* has shrunk: the system will converge to x^* if its initial value lies in the open interval between 0 and x^{**} (i.e. from anywhere just slightly above 0 to anywhere just slightly below x^{**} but not including either of those endpoint values – if it starts at 0 or at x^{**} it will stay there).

Convergence to an equilibrium like x^* does not have to be monotonic. In Figure 6.1(c) we have changed the $f(x)$ function so that its slope at x^* (which is still an equilibrium point) is negative but less than 1 in absolute value (i.e. a negative fraction). In this case the equilibrium is still stable, but the path along which the system converges to it displays alternations. In fact, as we have drawn

Figure 6.1(c), if the initial value of x is just above the lower equilibrium, the time path of x will initially be monotonic, with alternations only appearing as x approaches x^*.

Nonlinearity, then, can result in interesting mixtures of time-series properties in time series data sets. They can get even more interesting than we have suggested: suppose we draw a phase diagram with two equilibria, one at zero and one at x^*, as in Figure 6.1(a), but show the $f(x)$ curve cutting the 45° line at x^* with a slope which is negative and greater than 1 in absolute value so that both equilibria are unstable and the system displays alternations close to the upper equilibrium? Experimenting with the trajectories yielded by a diagram like that suggests that what we will see is pretty chaotic behaviour, but we will discuss chaos in a later section.

In our discussion to this point we have judged the stability of an equilibrium point by looking at how the $f(x)$ curve cuts it in a phase diagram. Since just looking at a diagram is never sufficient to prove anything, we need something a bit more formal. The obvious problem with testing stability by calculating the slope of the $f(x)$ curve is that the value of the slope changes as we move along the curve, which means that any statements we make about the slope only apply to the portion of the curve close to the point at which we calculate the slope.

This was implicit in our discussion of the diagrams where we talked about the slope of $f(x)$ close to the lower equilibrium as indicating that that equilibrium was unstable, and talked about the slope of $f(x)$ close to the upper equilibrium as indicating that that equilibrium was stable, but we drew no direct implications about stability from the slope of $f(x)$ at points between the two equilibria. But, given that we have drawn the $f(x)$ function as continuous and differentiable (i.e. with no corners), if its slope is greater than 1 at the lower equilibrium and less than 1 at the upper, there must be a point in between at which it is equal to 1, a fact which has not entered into our discussion of the stability of either equilibrium point.

Linearizing nonlinear difference equations

When we are investigating formally the stability of the equilibrium derived from a nonlinear difference equation, the best we can do is investigate stability in a relatively small area around the equilibrium. We do this by linearizing the nonlinear function at the equilibrium and testing the slope of that linear approximation. In essence, this is just a formalization of what we were doing when we looked at the slope of the $f(x)$ function on the phase diagram – we judged the stability of the equilibrium by the slope of $f(x)$ in the region of the equilibrium.

We linearize $f(x)$ by finding a first-order Taylor series expansion of this function with the equilibrium as the point of expansion. In general terms, a first-order Taylor series expansion produces a linear approximation to a nonlinear function. That approximation is only good for a limited range around what is known as the point of expansion, and the greater the degree of curvature of the original function the smaller that range will be.

Approximations do not have to stop with a first-order expansion – we can take the expansion to as high an order as we like, and the greater the curvature of the function the higher the order of expansion needed to approximate it closely. Those higher order terms, though, introduce nonlinear elements into the expansion, and, since the purpose of taking the approximation is to eliminate nonlinearities, we stop with a first order, or linear approximation.

To take a first-order Taylor series approximation to a general function $f(x)$, we first select the value of x which determines the point around which we are going to construct a linear approximation to the nonlinear function. For consistency with our other notation we shall denote this value of x by x^*, which means that the value of the function $f(x)$ at the point of approximation is $f(x^*)$. Then we can write, as the approximation to the function $f(x)$ at some arbitrary point x:

$$f(x) \approx f(x^*) + f_x(x^*)(x - x^*) \tag{6.1}$$

Note that the derivative on the right-hand side is also evaluated at x^*. The closer x is to x^*, the closer the value of the approximation (the expression on the right-hand side of Equation (6.1)) to the true value of the function (the expression on the left-hand side of Equation (6.1)).

Nonlinear FODE

To apply this to a nonlinear FODE, recall that $x_{t+1} = f(x_t)$ and that we have been using x^* to denote an equilibrium of the system. Approximating the function close to the equilibrium gives:

$$x_{t+1} = f(x_t) = f(x^*) + f_x(x^*)(x_t - x^*) \tag{6.2}$$

Next, note that since x^* is an equilibrium point (whether stable or unstable), $f(x^*) = x^*$. This lets us write Equation (6.2) as:

$$x_{t+1} = x^* + f_x(x^*)(x_t - x^*) \tag{6.3}$$

Now, define a new variable x^{d} which is defined as the deviation of the current value of x from its equilibrium value. Thus, $x_t^{\mathrm{d}} = x_t - x^*$ and $x_{t+1}^{\mathrm{d}} = x_{t+1} - x^*$ and we can rewrite Equation (6.3) as:

$$x_{t+1}^{\mathrm{d}} = f_x(x^*)x_t^{\mathrm{d}} \tag{6.4}$$

In interpreting Equation (6.4), remember that the first derivative, $f_x(x^*)$, is evaluated at a single point (here the equilibrium point), which means that it is a constant. Given this Equation (6.4) becomes a first-order homogeneous difference equation in x^{d}, with constant coefficients, which means that it is a linear first-order homogeneous difference equation.

The fact that Equation (6.4) is a homogeneous difference equation means that its equilibrium is at $x^{\mathrm{d}} = 0$, but since x^{d} is the deviation of the original, untransformed x from its equilibrium, when $x^{\mathrm{d}} = 0$, it must be the case that $x = x^*$. So if

Equation (6.4) is stable in the sense that x^{d} converges on its equilibrium, it must also be the case that x converges on its equilibrium.

The equilibrium of Equation (6.4) will be stable under the same conditions as the equilibrium of any other linear difference equation; when the slope term is less than 1 in absolute value. The trick here is that the slope must be evaluated at x^*.

We can extend this process to the case of a system of two nonlinear FODEs. Let:

$$\begin{aligned} y_{t+1} &= f(y_t, x_t) \\ x_{t+1} &= g(y_t, x_t) \end{aligned} \tag{6.5}$$

where $f(\cdot)$ and $g(\cdot)$ are nonlinear functions. Let the equilibrium point whose stability properties we are trying to establish (and again it may be one of several equilibria) be denoted (x^*, y^*) and let y^{d} and x^{d} once again represent variables defined as deviations of the original x and y variables from their equilibrium values. The expressions for first-order (i.e. linear) approximations to Equations (6.5) are:

$$\begin{aligned} y_{t+1} = f(y_t, x_t) &\approx f(y^*, x^*) + f_x(y^*, x^*)(x_t - x^*) + f_y(y^*, x^*)(y_t - y^*) \\ x_{t+1} = g(y_t, x_t) &\approx g(y^*, x^*) + g_x(y^*, x^*)(x_t - x^*) + g_y(y^*, x^*)(y_t - y^*) \end{aligned} \tag{6.6}$$

Now since $f(y^*, x^*) = y^*$ and $g(y^*, x^*) = x^*$, Equation (6.6) can be written in deviation form as:

$$\begin{aligned} y^{\mathrm{d}}_{t+1} &\approx f_x(y^*, x^*)x^{\mathrm{d}}_t + f_y(y^*, x^*)y^{\mathrm{d}}_t \\ x^{\mathrm{d}}_{t+1} &\approx g_x(y^*, x^*)x^{\mathrm{d}}_t + g_y(y^*, x^*)y^{\mathrm{d}}_t \end{aligned} \tag{6.7}$$

The system (6.7) contains two homogeneous linear FODEs with a coefficient matrix whose elements are the first partial derivatives of the f and g functions, all evaluated at the equilibrium point. Within the local region around the equilibrium, we can work with system (6.7), instead of the original nonlinear system, so long as the expansion yields a good approximation. In particular, we can solve for the roots of system (6.7) and evaluate the stability of the equilibrium point (x^*, y^*).

Clearly, in a system with multiple equilibria, it is not sufficient for us to evaluate system (6.7) at only one of the equilibria. Even if we establish that the equilibrium point under consideration is stable, the roots of Equation (6.7) do not tell us whether it is locally or globally stable nor, if it is locally stable, how large or small its relevant locality is. All of which means that a thorough evaluation of a system like (6.5) requires that we identify all of its equilibria and then evaluate the stability properties of each of those equilibria. George and Oxley (1999) are justifiably critical of researchers who, in effect, linearize around their preferred equilibrium and treat local stability properties as if they are global ones.

Even doing all of that leaves us with an incomplete picture of the dynamics of the system represented by Equation (6.5). We have already seen that, with a single nonlinear FODE, the time path of the variable it represents can involve a mix

of trajectories – monotonic and alternating. Furthermore, as with linear systems, when we start dealing with cases with more than one variable and more than one root we can very quickly get into interesting dynamics. Adding nonlinearities just increases the range of types of transitional dynamics which we might encounter. This would not matter so much if we were sure our system was always close to equilibrium, perhaps because of extremely fast speeds of adjustment, but if we believe that most observations are disequilibrium rather than equilibrium points, it can become very important for empirical purposes.

To see how linearization works in a simple first-order example, suppose our nonlinear difference equation is:

$$x_t = Ax_{t-1}(1 - x_{t-1}), \quad A > 1 \tag{6.8}$$

This quadratic expression shows up a lot in expositions on nonlinearity in economics, since it is one of the simplest forms of nonlinear difference equation and yet, with suitable choice of value for the scaling term A, is capable of generating quite complex time paths.

Since Equation (6.8) is a first-order nonlinear difference equation we can draw a phase diagram for it, as shown in Figure 6.2. The diagram shows that the $f(x_{t-1})$ function described by Equation (6.8) has horizontal intercepts at $x_{t-1} = 0$ and at $x_{t-1} = 1$, and that the function has an inverted-U shape in between its horizontal intercepts, reaching a maximum at $x_{t-1} = 1/2$, at which point $x_t = A/4$. The equilibria for this difference equation are found at the points of intersection between the $f(x_{t-1})$ function itself and the 45° line: in the case of Equation (6.8) the equilibria are at $x = 0$ and $x = 1 - 1/A$.

From Figure 6.2 it is clear that the lower equilibrium is unstable, but whether the upper one is stable or not depends on the value of A. If $A = 2$ the equilibrium value of x coincides with the value at which $f(x_{t-1})$ reaches its maximum, see Figure 6.2(a).

If $A > 2$, the value of x_{t-1} which maximizes $f(x_{t-1})$ is to the left of the equilibrium value of x, see Figure 6.2(b). While if $A < 2$, the equilibrium value of x, $1 - 1/A$, is less than $1/2$ and the $f(x_{t-1})$ function cuts the 45° line to the left of its maximum, see Figure 6.2(c).

In Figure 6.2(c), at the equilibrium, the slope of $f(x_{t-1})$ is positive and less than 1, and hence the upper equilibrium is stable and the approach to it is monotonic. In contrast, in Figure 6.2(b), the slope of the $f(x_{t-1})$ function is negative at the upper equilibrium, which means that the approach to equilibrium will display alternations. Whether the alternations will be stable or not will depend on the precise value of the slope at the equilibrium.

Differentiating Equation (6.8) gives us the general expression for the slope:

$$\frac{\partial x_t}{\partial x_{t-1}} = A(1 - 2x_{t-1}) \tag{6.9}$$

Evaluating this at the lower equilibrium, $x = 0$, gives $\partial x_t / \partial x_{t-1} = A$ and we have already assumed that $A > 1$. At the upper equilibrium, since

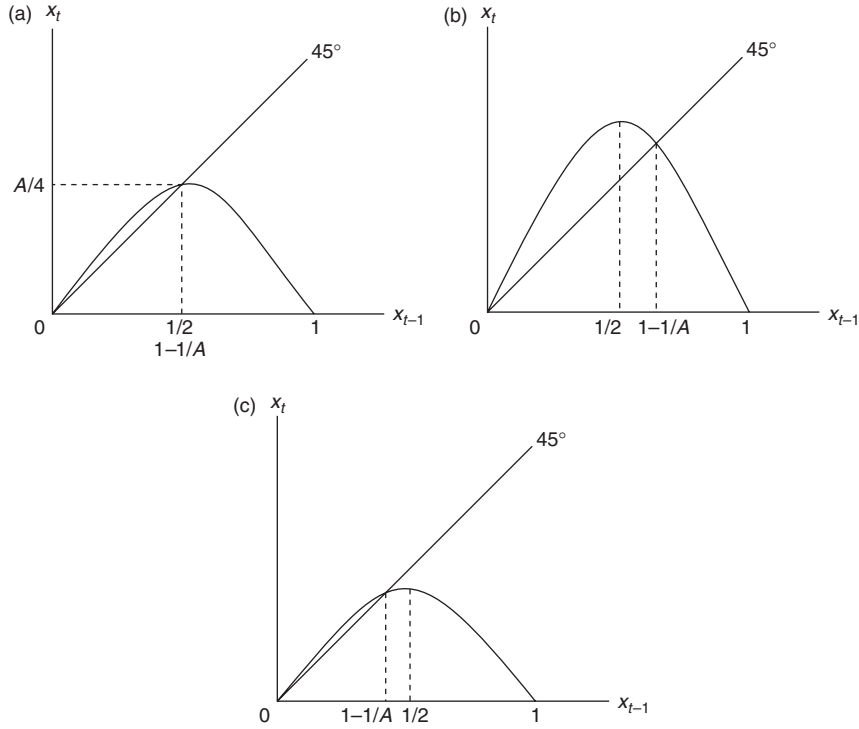

Figure 6.2 Linearizing nonlinear difference equations.

$x_t = x_{t-1} = (1 - 1/A)$, the expression for the slope of $f(x_{t-1})$ becomes:

$$\frac{\partial x_t}{\partial x_{t-1}} = A(1 - 2(1 - 1/A)) = A(2/A - 1) = 2 - A \tag{6.10}$$

which is positive (or negative) as A is less than (greater than) 2. If Equation (6.10) is negative, the upper equilibrium will still be stable so long as Equation (6.10) lies between -1 and 0. This requires A to lie between 2 and 3. If A is bigger than 3, the upper equilibrium is unstable. Note, incidentally, that the value of A does not affect the value of x_{t-1} at which Equation (6.9) is equal to zero – the maximum of this particular $f(x_{t-1})$ function will always be at $x_{t-1} = 1/2$, although the value of x_t at that point, $A/4$, does depend on the value of A. Since this particular $f(x_{t-1})$ function will always cut the horizontal axis at 0 and 1, and will always reach its peak at $x_{t-1} = 1/2$, the role of A is clearly to stretch (or compress) the function vertically.

If A is greater than 3, we are in the interesting situation of having two adjacent unstable equilibria. Normally, in economic analysis, we assume that equilibria will alternate, stable and then unstable, but nonlinearity requires us to reconsider

that assumption. When a model has adjacent unstable equilibria, and the initial value of x lies between them, the best we can hope for is that the system will be Lyapunov stable – meaning that it stays within a well-defined region, but never converges to a single point. Since, in this case, the upper equilibrium is associated with a negative slope of $f(x_{t-1})$, the system will clearly display alternations. For some values of A it will settle down into a regular, repeating pattern of alternations around the upper equilibrium point, while for other values of A the system never settles down in the sense of repeating one, possibly complicated, trajectory over and over. In this last case the trajectory is alternating but aperiodic, and it is in this case that the behaviour of the system is referred to as chaotic. We will return to the question of chaos below. Before doing that, though, we consider an economic model which involves a nonlinear difference equation.

A basic neoclassical growth model

The economic example which we consider here is the basic neoclassical growth model. This model contains difference equations for two variables, but by a trick common to growth models we are able to reduce it to a single difference equation model.

We begin with an aggregate production function:

$$Y_t = F(K_t, L_t) \tag{6.11}$$

where Y is aggregate output, K is aggregate capital and L is aggregate labour. The time subscripts on each variable indicate that there are no lags in the production process. Recall that we referred to this case in our discussion of population dynamics. We are treating population as a single, homogeneous unit, at least as far as the production function is concerned. We can get away with this, even when different age groups of labour actually have different marginal productivities, so long as the age distribution of our population is unchanging over time. In a more detailed model we would enter the different age groups of labour as separate inputs in the production function, and add the population dynamics matrix to our system.

For simplicity here, then, labour (which is here assumed to be identical to population; that is, the labour force participation rate is 100 per cent) is assumed to grow at an exogenous proportional rate η, according to the difference equation:

$$L_t = (1 + \eta)L_{t-1} \tag{6.12}$$

We note that Equation (6.12) can be rewritten as $(L_t - L_{t-1})/L_{t-1} = \eta$, hence our referring to η as a proportional growth rate.

Capital grows as a result of net investment, which is defined as gross investment minus an allowance for depreciation, and gross investment is equal to savings – this is a neoclassical model, so all savings are invested in productive physical capital:

$$K_t = sF(K_{t-1}, L_{t-1}) + (1 - \delta)K_{t-1} \tag{6.13}$$

Here δ is the depreciation rate and s is the (exogenous) savings rate.[1] Note that there is a one period lag between when saving is done and when capital appears.

This equation tells us that this period's capital is equal to the undepreciated part remaining from last period's, plus any savings/investment done out of last period's income (output), which has turned into new capital equipment in this period.

At this point, we introduce a simplifying assumption. Specifically, we assume that the aggregate production function, $F(K_t, L_t)$, displays constant returns to scale. The good thing about a constant returns to scale production function is that (it can be shown that) we can write:

$$F(K_t, L_t) = L_t F\left(\frac{K_t}{L_t}, 1\right) \tag{6.14}$$

where K_t/L_t is the current capital–labour ratio, and $F(K_t/L_t, 1)$ is the amount of output a single worker could produce if he had available to him an amount of capital equal to the current aggregate capital–labour ratio. Under constant returns to scale, aggregate output is just that single worker's output level, multiplied by the total labour force. We usually write Equation (6.14) as:

$$F(K_t, L_t) = L_t f(k_t) \tag{6.15}$$

where k_t is the current capital–labour ratio and $f(k_t)$ is just a more convenient piece of notation for $F(K_t/L_t, 1)$, the amount of output a single worker could produce. Rearranging Equation (6.15) gives:

$$f(k_t) = \frac{F(K_t, L_t)}{L_t} \tag{6.16}$$

which says that if we calculate current per worker output by taking total output and dividing it by the total labour force (i.e. calculate the average product of labour), the value we get will be identical to the output level a single worker could produce under the conditions described above. We usually denote this output per worker as y_t. This scalability property of a constant returns to scale production function[2] means that we can analyse the model in per capita terms, which turns out to be a way of getting around the problem of having too many difference equations.

Consider our expression for the current period's aggregate capital stock, as set out in Equation (6.13). Dividing through on both sides of Equation (6.13) by L_t gives:

$$\frac{K_t}{L_t} = s\frac{F(K_{t-1}, L_{t-1})}{L_t} + \frac{(1-\delta)K_{t-1}}{L_t} \tag{6.17}$$

which really does not look terribly helpful, since, while the left-hand side is the current capital–labour ratio, k_t, the time subscripts on the right-hand side do not match up neatly. However, if we multiply and divide all terms on the right-hand side by L_{t-1}, which amounts to multiplying by 1 and which therefore makes no

difference to the expression, we have:

$$\frac{K_t}{L_t} = \left(\frac{sF(K_{t-1}, L_{t-1})}{L_{t-1}}\right)\left(\frac{L_{t-1}}{L_t}\right) + \left(\frac{(1-\delta)K_{t-1}}{L_{t-1}}\right)\left(\frac{L_{t-1}}{L_t}\right) \tag{6.18}$$

Here, the term $F(K_{t-1}, L_{t-1})/L_{t-1}$ is obviously output per worker in period $t-1$, and the term K_{t-1}/L_{t-1} is the capital–labour ratio in period $t-1$. The term (L_{t-1}/L_t) is easily shown, from Equation (6.13) above, to be $1/(1+\eta)$ so, using the notation we developed above, we can rewrite Equation (6.18) as:

$$k_t = \frac{sf(k_{t-1})}{(1+\eta)} + \left(\frac{1-\delta}{1+\eta}\right)k_{t-1} \tag{6.19}$$

which, since η and δ are exogenous, is a nonlinear FODE in k.

Because we have not specified a precise functional form for $f(k_t)$, we are limited to qualitative, phase diagram analysis of Equation (6.19), but phase diagrams can be very revealing things. In this case, we note, without proving, that the per capita production function $sf(k_t)$ has all of the usual marginal productivity properties, even though it shows output per worker as a function of capital per worker. Most importantly, the marginal product of k is positive and diminishing:[3] $f'(k) > 0$, $f''(k) < 0$.

Using these assumptions, we can draw the phase diagram for Equation (6.19) with k_t on the vertical and k_{t-1} on the horizontal, see Figure 6.3. The curved line is the $k_t(k_{t-1})$ function.

Note that it starts from the origin, on the (fairly standard) argument that when k_{t-1} equals zero, $f(k_{t-1})$ equals zero.[4] The slope of the $k_t(k_{t-1})$ function is found

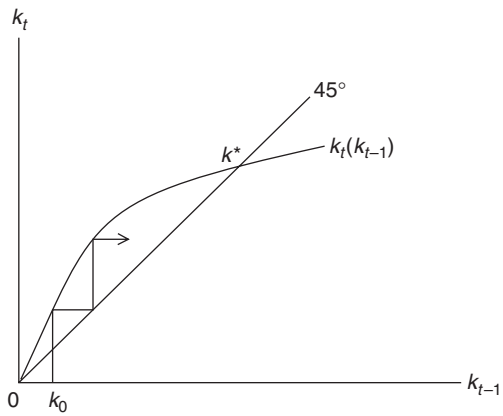

Figure 6.3 Phase diagram for a neoclassical growth model.

by differentiating Equation (6.19) with respect to k_{t-1}:

$$\frac{\partial k_t}{\partial k_{t-1}} = \left(\frac{s}{1+\eta}\right) f'(k_{t-1}) + \left(\frac{1-\delta}{1+\eta}\right) \tag{6.20}$$

with second derivative:

$$\frac{\partial^2 k_t}{\partial k_{t-1}^2} = \left(\frac{s}{1+\eta}\right) f''(k_{t-1}) < 0 \tag{6.21}$$

From Equations (6.20) and (6.21) the $k_t(k_{t-1})$ function is initially positively sloped with slope decreasing, and reaching zero where, from Equation (6.20):

$$f'(k_{t-1}) = \frac{-(1-\delta)}{s} < 0 \tag{6.22}$$

it is certainly possible for the marginal product of capital per worker to become negative – there might be so much capital around that workers start tripping over it – but at the macro level this is probably not too serious a possibility. In the phase diagram for this problem, we draw the $k_t(k_{t-1})$ function with a slope which, while diminishing, is always positive.

From Equation (6.19), we see that the equilibrium value of k, k^*, for which we cannot actually find an expression without having a precise mathematical expression for $f(k)$, must satisfy the condition:

$$\frac{f(k^*)}{k^*} = \frac{(\eta+\delta)}{s} \tag{6.23}$$

where it can be shown that $f(k)/k = F(K, L)/K$, the average product of capital. In equilibrium, then, the capital–labour ratio must be such that the average product of capital equals $(\eta+\delta)/s$, where δ represents the amount that has to be set aside per unit of capital to replace worn out capital, and η represents the amount of capital which has to be put aside just to ensure that, in the current period, each newly arrived worker is equipped with the same amount of capital as that available to existing workers.[5]

One of the simplest, and most commonly used, functional forms for the production function $f(k)$ is:

$$f(k) = k^{\beta}, \quad 0 < \beta < 1 \tag{6.24}$$

If we substitute Equation (6.24) into our model, we find, for the equilibrium level of k:

$$k^* = \left(\frac{s}{\eta+\delta}\right)^{1/(1-\beta)}$$

Substituting this expression into Equation (6.20), we find that the slope of the $k_t(k_{t-1})$ function is indeed positive and less than 1, so the upper equilibrium point

is stable. Evaluating the slope at the lower equilibrium confirms that $k = 0$ is an unstable equilibrium.

Note that the equilibrium for this problem is in terms of k, the capital–labour ratio. Our population and labour force do not stop growing when we reach equilibrium, nor does the total capital stock. Since population is growing at a rate η per period, our net (i.e. after subtracting some capital to replace depreciated stock) capital must grow at the same rate, otherwise k will not remain constant. Thus, in this model, equilibrium is defined in terms of capital per worker and therefore in terms of output per worker; and if we define consumption per capita as $(1-s)f(k)$, in terms of per capita consumption.

It is the per capita values which remain constant once we reach equilibrium. The aggregate variables continue to increase, but at a constant proportional rate so their size relative to that of the population remains unchanged. Because population continues to grow, all of our other key aggregate variables must grow at the same rate in order that their per capita values can remain unchanged. This type of growth, however, does not raise consumption per capita and therefore does not raise the standard of living. An underdeveloped country which had reached equilibrium at a low, perhaps subsistence, value k^* might then grow very rapidly in aggregate terms because its population was growing very rapidly while the standard of living of its population does not improve at all.[6]

Chaos in economics

We referred earlier in this chapter to the case of a nonlinear difference equation with two equilibria, both of which were unstable. For example, consider a nonlinear difference equation of the form:

$$x_t = Ax_{t-1}(1 - x_{t-1}), \quad 2 < A \leq 4 \tag{6.25}$$

where the lower bound on A is necessary to ensure that the system displays oscillations at the upper equilibrium and the upper bound is simply a matter of custom in expositions using this form. The phase diagram would look like those in Figure 6.2 where the $x_t(x_{t-1})$ function is an inverted-U, cutting the horizontal axis at $x_{t-1} = 0$ and again at $x_{t-1} = 1$. Looking at the points of intersection between this function and the 45° line, we see that it has two equilibria, the lower equilibrium at zero and the upper one at some positive value x^*. Also, as we noted earlier, the maximum of Equation (6.25) is found at $x_{t-1} = 1/2$, and that at that value of x_{t-1}, $x_t = A/4$. By restricting A not to be greater than 4, we keep both x_{t-1} and x_t from going outside the range [0, 1].

When our system yields this type of phase diagram, there are thresholds in its behaviour such that small changes in the value of A can result in dramatic changes in the type of trajectory x follows. For values of A between 2 and 3, we have straightforward convergent alternations, but as A rises above 3, the trajectories become increasingly complicated. For some values of A between 3 and 4, x_t settles into periodic alternations – a limit cycle, in which it takes on the same series of values over and over.

At $A = 3.2$, for example, if allowed to run long enough, the system will settle into what is known as a period 2 cycle, jumping back and forth between (approximately) 0.513045 and 0.799455. In this case we have the alternations version of a limit cycle. Limit cycles proper require the system to be capable of producing oscillations, which means we have to be dealing with a higher order nonlinear system before they can appear. The alternations version gives us the essence of limit cycles, though.

If we start the system just above the lower equilibrium, it will diverge from that equilibrium, rising towards the upper one but, because that equilibrium is also unstable, never actually reaching it. Instead, it will settle into the pattern of repeated alternations between the two values we gave above. Similarly, if we start the system from some point near, but not equal to, the upper equilibrium, its trajectory will take it out, away from the upper equilibrium, and again it will settle into repeated alternations between 0.513045 and 0.799455. In the long run, then, the cycle itself is a stable attractor for the system.

There can also be unstable limit cycles, which have the same basic properties as unstable equilibria. If we start at a value in the limit cycle we will stay in that same cyclical path forever, neither converging nor diverging, but if we start on either side of it we will diverge from it. Here the interesting case arises when we start from a value just inside the trajectory of the limit cycle – in this case there will generally be an equilibrium which is a stable focus, towards which we will converge as time passes. In this case, the limit cycle defines the locality within which that focus is a locally stable equilibrium.

If we increase A slightly, to 3.4, the pattern changes, alternating between 0.451963 and 0.842154, but if we take A up to 3.5 the system settles into a period 4 cycle, going from 0.38282 to 0.826941 to 0.500884 to 0.874997. At A equal to 3.84 we are back to a period 3 cycle, with x (eventually) settling into jumping between 0.149407, 0.488004 and 0.959447.

The transition between periodicity of alternations is not smooth, though, and this is where chaos enters the picture. When we set A equal to 3.58, the system alternates around the upper equilibrium (which in this case is at $x_t = 0.72067$) but never repeats itself and never settles into a pattern which repeats over and over again – it becomes aperiodic, or chaotic.

Chaos is of interest to economists as more than a mathematical curiosity. Most importantly, it is extremely difficult to distinguish a chaotic series from a random series. To the naked eye, and to most standard statistical tests for randomness, a chaotic series has the same properties as a sequence of random numbers. In fact, the sequences of numbers produced by the random number generators used in calculators and computer programs typically actually produce chaotic series, not random series.

While chaotic and random series are practically indistinguishable, especially in small samples, they are actually fundamentally different. A random series can never be forecast exactly. If we know the probability density function of the random variable we can assign probabilities to ranges of values within which the next observed value might fall, but we can never predict the next value with certainty.

In contrast, if we know the function generating a chaotic series, and if we know the current value of the series exactly, we can predict the next value in the series with absolute certainty.

The information requirements for predicting a chaotic series are enormous – we have to know the exact values of the parameters of the difference equation which is generating the series, and we have to know what its current value is exactly. One of the defining features of a chaotic series is extreme sensitivity to initial values.

Suppose we take a chaos function, pick an initial value of x, and let the function run. We will wind up with a long series of numbers, showing the values x takes on in each period. Suppose we then go back and run the same function, with exactly the same parameters, but with a very slightly different initial value. The two series will track reasonably well for a little while, but before long the new series will diverge dramatically from the old one. If a difference equation exhibits chaotic behaviour, small changes to the initial value will, in the long run, produce dramatically different histories.

The fact that chaotic behaviour can be practically indistinguishable from random behaviour at first appeared to be no more than a mathematical curiosity. After a while, though, examples of apparently chaotic behaviour began to appear in other fields, including physics and medicine. This prompted a number of economists to begin to wonder if the irregularities which were observed in most economic time series, and which were generally assumed to be the product of introducing random shocks to systems of well-behaved difference equations, might actually be chaotic behaviour.[7] Day (1982, 1983), for example, showed that familiar models in economic dynamics could be modified to produce chaotic behaviour.

One of the cases he considered was the neoclassical model of economic growth, which we considered above. Day proposed modifying the production function (written in per worker terms) from the standard Cobb–Douglas form $f(k) = Bk^{\beta}$ to:

$$f(k) = Bk^{\beta}(m-k)^{\gamma}, \quad m > k, \quad \gamma > 0 \tag{6.26}$$

where the new term is a congestion effect, modifying the Cobb–Douglas production function to admit a case which we excluded in our earlier discussion – the case where the marginal productivity of k becomes negative. Day notes (Day 1982: 409) that so long as γ is small, the congestion term would not have much effect until k gets close to m, but when k is close to m, congestion can have a powerful effect on output. Day establishes that there are parameter values which, when inserted in Equation (6.26), will yield chaotic patterns in k and therefore in output per capital.

Day's early work prompted others to look for ways to persuade familiar models to produce chaotic behaviour. As Frank and Stengos (1988) note, in many cases this amounted to no more than taking a model which included a difference equation and replacing the most common functional form for that difference equation with some variant on Equation (6.25). In other cases, though, researchers were more careful about how they introduced chaotic dynamics.

Day and Pavlov (2002) took one of the earliest of the modern Keynesian business cycle models, a cobweb model developed in Goodwin (1967), a standard

Keynesian macro model in which dynamics entered through the lagged response of investment spending to the interest rate and showed how, with reasonable assumptions about the nonlinear equation driving investment,[8] goods and money market interactions could induce a range of dynamic behaviour in aggregate income.

Day's demonstrations that simple economic models could display chaos did not, of course, prove that the irregular time series observed in so many economic series were actually chaotic. After all, the thing which made chaos so intriguing was the fact that chaotic behaviour was so difficult to distinguish from random behaviour, so the fact that a model could be modified to produce deterministic patterns which looked random did not prove that actual, random-looking economic time series were in fact chaotic.

Furthermore, while most of the theoretical literature restricted itself to simple FODEs using simple nonlinear functional forms, the equations driving actual economic variables are not restricted to easily manipulable forms. This raised obvious problems for econometric investigations of chaos, since it meant that there did not exist a single, well-defined chaos equation which could be tested empirically. Instead, chaos had to be sought in the time series behaviour of economic variables.

The econometric techniques used are well beyond the scope of this book, so we simply refer the reader to such references as Frank and Stengos (op. cit.), Hsieh (1991), Liu *et al*. (1992) and Abhyankar *et al*. (1997). Cunningham (1993) does an interesting job of bringing together two hot topics in econometric dynamics, chaos and unit root dynamics, in the process raising doubts about the techniques used to test for each.

While some of the econometric literature has dealt with empirical testing for chaos in business cycles, most of it has focussed on looking for evidence of chaos in financial markets. Outside the economics literature, authors seem to take the presence of chaos for granted, inspired in part by metaphors about sensitivity to initial conditions – a butterfly flapping its wings in China will produce a tornado in Kansas. A good part of the interest in financial market chaos among non-economists is undoubtedly due to the term itself – simply looking at the recent behaviour of financial markets makes it obvious to many people that those markets are chaotic. It is much easier to wrap one's mind around a term like 'chaotic' than around 'stochastic'. For most people, indeed, chaotic and random mean basically the same thing. The fact that they have very different mathematical meanings and tell very different stories about what drives markets is generally not understood.

Some observers of financial markets do, of course, understand the subtleties, and indeed for some the fundamental difference between chaotic and random behaviour is a powerful attractor. After all, if financial markets are chaotic rather than random they are, at least in principle, perfectly predictable. Frank and Stengos (1988) give a simple example of a simulated price series, generated by a known chaos function, which passed all of the standard tests for efficient markets, and note that 'You can make a lot of money in an "efficient market" like this one'. (p. 104). Undoubtedly for some people the hope of predictability and of beating the market is a large part of the lure of chaos.

At one time economic chaos, particularly as applied to financial markets, looked like an immensely promising research programme. In 1988, Frank and Stengos argued that, while empirical work to that point had not produced much evidence of chaos, the prospect was good in the case of markets for which large, high-quality data sets were becoming available. They noted financial markets and foreign exchange markets in particular, but held out less hope for the discovery of chaos in aggregate time series data (even assuming it was in fact present). Writing just a few years later, Granger (1991) concluded that there was no evidence of chaos in economic data. An admittedly cursory review of the literature to date suggests that Granger's conclusion still holds, although there are dissenters. George and Oxley (1999) find evidence of chaos in the S&P 500.

This does not mean that chaos has been a blind alley as an economic research programme. Improvements in testing procedures may well yield evidence of chaos, although it has to be recognized that, if an economic variable which is driven by a chaos function is also subject to random shocks, the deterministic chaotic component of its observed time series is likely to be swamped by the stochastic element. Given the sensitivity of chaos functions to initial conditions, the stochastic component would have to be extracted very thoroughly before the deterministic element could come through.

On a more positive note, the chaos research programme drew attention to the importance of nonlinearities in structural economic relations. Linearity in economic structure has always been regarded as a first-order approximation to the real world, but the limitations of that approximation have tended to be overlooked. In the empirical literature on economic chaos it is very common for authors to conclude that their investigation has not produced evidence of chaos, but has produced evidence of structural nonlinearity.[9] In the long run, focussing attention on nonchaotic nonlinearities and on approaches to modelling them, may prove to be the most important contribution of the chaos research programme.

7 Empirical analysis of economic dynamics

Introduction

The purpose of this chapter is to introduce readers to some empirical applications of dynamic economic models. The literature is vast, and we make no effort to be exhaustive or representative, aiming simply to give the flavour of the area. Nor do we attempt to deal with the econometrics of dynamic modelling in any detail. Readers interested in the detail of econometric issues are referred to books such as Hendry (1995) and Enders (1995) and the references therein.

Our neglect of econometric detail does not mean that it is unimportant; rather that it is impossible to do the issues justice in a single chapter. The econometric issues and problems which arise when dynamic models are being used are crucial to correct interpretation of the results and neglect of them may lead to incorrect conclusions.

Basic empirical dynamic models

Probably the most familiar, and historically the most often used dynamic model in economics is the partial adjustment (PA) model. This model assumes that there is an underlying long-run relation towards which the system is tending, but that because of adjustment costs and lagged responses it cannot jump immediately to the long-run relation, but must adjust towards it over time. Hence, the aim of much empirical analysis of economic dynamics is to identify the underlying long-run relationship using actual observations which reflect short-run adjustments.

The terminology 'long run' and 'short run' merit a bit of clarification at this point. The term 'long run' derives from the mathematics of difference equations. Consider the difference equation:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 x_t, \quad \alpha_1 < 1 \tag{7.1}$$

where x_t is an exogenous variable, meaning that its behaviour is not affected by that of y, although the fact that it has a 't' subscript means that it can change over time. This equation has, as the solution to its homogeneous form:

$$y_t = A\alpha_1^t \tag{7.2}$$

and as its equilibrium solution:

$$y_t^* = \frac{\alpha_0}{(1-\alpha_1)} + \frac{\alpha_2}{(1-\alpha_1)} x_t \tag{7.3}$$

where the t subscript indicates that we may have a moving equilibrium, if the value of x_t changes over time. Combining these into the general solution gives:

$$y_t = A\alpha_1^t + \frac{\alpha_0}{(1-\alpha_1)} + \frac{\alpha_2}{(1-\alpha_1)} x_t \tag{7.4}$$

where $A = y_0 - y_0^*$ as usual.

In Equation (7.4), if x_t remains unchanged over time, as t goes to infinity (i.e. in the long run) the value of y will converge to an unchanging equilibrium solution, (7.3).[1] This is the reason the equilibrium solution (7.3) is often referred to, in empirical work, as the long-run solution. The actual time path of y_t will follow an adjustment path towards this long-run solution.

If x_t changes over time, the actual time path of y_t will consist in part of adjustment dynamics and in part of responses to changes in x – hence the actual time path of y will involve an adjustment towards a moving target. And, in this case, while y is being pulled by the equilibrium, or long-run relation in Equation (7.3), it is quite possible that y will never actually reach a particular equilibrium value defined by the relation (7.3).[2]

Thus, one aim of applied dynamic modelling is to separate out the influences of changes in the equilibrium solution from those of the adjustment dynamics. This involves identifying Equation (7.3) – the fundamental relation between y and x – while paying due attention to the dynamic adjustment process.

Whatever the philosophical or mathematical interpretation of 'long run' and 'short run', there is an additional consideration which arises in empirical analysis. Whether the data contains long- and short-run elements depends on the calendar period of the data we are working with relative to the speed of adjustment of the underlying economic processes. Suppose we are looking at a relation like (7.1), and a single change in the value of x occurs. If the adjustment process is fast enough, y_t could be arbitrarily close to its new equilibrium value within a year.[3] If we are working with annual data, we would probably not be able to observe much of a long-run–short-run distinction in the data. In quarterly data, and more so in monthly data, however, the adjustment dynamics might be quite noticeable.[4] In general, econometric modelling should include some testing for dynamic structure, even if ultimately it proves that the behaviour being analysed can be modelled perfectly satisfactorily as a static process.

The partial adjustment model

As noted above, historically the most frequently used dynamic model is probably the PA model. The basic structure of the model is as follows. Define:

$$y_t^* = \alpha_0 + \alpha_1 x_t + \alpha_2 z_t \tag{7.5}$$

as the underlying behavioural relation between the dependent variable, y and the explanatory variables, x and z. It might be, for example, that Equation (7.5) is the demand function for a commodity, where y is consumer demand for the product, x is the product's price and z is consumer income. This expression gives us the desired, or optimal, level of y.[5]

When price, x, changes, quantity demanded changes in the opposite direction. Consumers will react to price changes immediately, but it may be that they are constrained in how strong their immediate response can be. Basic microeconomic theory says that price elasticity of demand is greater in the long run than in the short run, because in the long run consumers have more time to adjust the way they allocate their budgets across the various commodities they buy – more time for complementarity and substitutability among products to come into play.

To allow for this, let y_t be the actual value of the dependent variable, as opposed to the optimal level y_t^*. The PA model assumes:

$$y_t - y_{t-1} = \delta(y_t^* - y_{t-1}), \quad 0 \leq \delta \leq 1 \tag{7.6}$$

where δ is the adjustment coefficient. Equation (7.6) says that the actual change in y from period $t-1$ to period t is a fraction, δ of the gap between period $t-1$'s actual level and period t's desired level.

Equation (7.6) could also be written as:

$$y_t = \delta y_t^* + (1-\delta) y_{t-1} \tag{7.7}$$

Equation (7.6) says that the actual value of y in period t is found by starting from the actual value of y in $t-1$ and adding to it a fraction, δ, of the gap between y_{t-1} and y_t^*. Equation (7.7) restates this by defining y_t as a weighted average of y_{t-1}, with weight $(1-\delta)$, and y_t^*, which has weight δ. Either way, we are saying that y moves part of the way from where it was in the last period to where it would optimally be this period.

If $\delta = 1$, the y_{t-1} term on the right-hand side of Equation (7.7) vanishes and we have $y_t = y_t^*$. In this case there is no dynamic adjustment process, so the model is essentially a static one. This introduces another slight oddity in the language of empirical dynamics: a static model is actually one whose dynamic adjustment occurs instantaneously – essentially, 'static' means 'infinitely fast'.

If $\delta = 0$, the y_t^* term disappears from the right-hand side of Equation (7.7), leaving $y_t = y_{t-1}$. In this case there is no adjustment at all; a change in y^* has no effect on the actual value of y. Clearly in this case the long-run Equation (7.5) has no impact on y at all, and we have probably got the wrong model.

If $0 < \delta < 1$, y_t adjusts part of the way from where it was, y_{t-1}, to where it would like to be, y_t^* in a single period. This is the case which gives the PA model its name. The closer δ is to 1, the larger the proportion of the gap between the previous actual and current optimal values which is closed in a single period, so the faster the adjustment process, while the closer δ is to 0, the slower the adjustment process.

Clearly it is desirable to be able to put values on both δ and the coefficients of Equation (7.5). One problem with doing this is that while theory can define the y_t^* equation – as in the case of the demand function – we cannot in general observe y_t^*.

To get around this problem, and implement the model econometrically, substitute for y_t^* from Equation (7.5) in (7.7) giving:

$$y_t = \delta\alpha_0 + \delta\alpha_1 x_t + \delta\alpha_2 z_t + (1 - \delta)y_{t-1} \tag{7.8}$$

The coefficient on y_{t-1} is $(1 - \delta)$ while the coefficients on the intercept and the x and z variables are products of δ and the respective α coefficients. Equation (7.8) is the version of the PA model which is actually estimated.[6] The unobservable y_t^* has been substituted out of Equation (7.7) using (7.5), leaving us with no variables which are not in principle observable.

If Equation (7.8) is estimated using standard linear regression techniques giving:

$$y_t = \beta_0 + \beta_1 x_t + \beta_3 z_t + \beta_3 y_{t-1} \tag{7.9}$$

we will not have separate estimates of the δ and α components, but these can be derived since $\beta_3 = (1 - \delta)$, $\beta_0 = \delta\alpha_0$, $\beta_1 = \delta\alpha_1$ and $\beta_2 = \delta\alpha_2$. Also because the estimated α values are nonlinear combinations of the estimated β coefficients, standard errors for the α values can be generated and hypothesis tests performed.

When the β_3 coefficient is significantly different from zero, we have partial adjustment to the optimal value of y (optimal conditional on the values of x and z). If β_3, the coefficient on y_{t-1} is not significantly different from zero, it must be the case that δ is not significantly different from 1 and we have a static (or instantaneously adjusting) relation. And in the special case when we cannot reject the hypothesis that δ is equal to 1, we can take the estimated coefficient values in Equation (7.9) as estimates of the parameters of the optimal relation (7.5) since in this case $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$.

In Equation (7.9), the estimated coefficient β_1 represents the effect of x_t on y_t with y_{t-1} held constant. This means that β_1 shows the immediate response of y to changes in x. A similar interpretation can be given to β_2, the coefficient on z_t. By definition, y_{t-1} cannot change until one period after the change in x and/or z, by which time the present has become the past. The β coefficients in Equation (7.9) are interpreted as short-run coefficients, giving the initial, immediate response of y to changes in the explanatory variables.

Equation (7.9) is a FODE in y. Its particular, or equilibrium, solution is:

$$y_t^* = \frac{\beta_0}{(1 - \beta_3)} + \frac{\beta_1}{(1 - \beta_3)} x_t + \frac{\beta_2}{(1 - \beta_3)} z_t \tag{7.10}$$

which, given the interpretation which Equation (7.8) lets us put on the β coefficients, is the same as the optimal relation set out in Equation (7.5). The coefficients in Equation (7.10) are referred to as the long-run coefficients of the relation between y and the explanatory variables.

Returning now to Equation (7.9) and looking at it as a FODE in y, we see that the solution to its homogeneous part is:

$$y_t = A\beta_3^t \tag{7.11}$$

which, given the definition of β_3, can be written as:

$$y_t = A(1-\delta)^t \tag{7.12}$$

This gives the general solution to the difference equation as:

$$y_t = A(1-\delta)^t + y_t^* \tag{7.13}$$

When $\delta = 1$, $y_t = y_t^*$ and we have instantaneous adjustment, while when $\delta = 0$, the A term, the initial disequilibrium term, never vanishes while the y_t^*, is undefined. Thus, the properties of the PA model are derived from the properties of the estimated difference equation which it generates.

Although this easy interpretation of the PA model in terms of the dynamic forms we have been considering elsewhere in this book is certainly appealing, the PA model has one major drawback.

To find the long-run coefficients, we divide each of the short run, β coefficients, by $(1-\beta_3)$. If β_1 and β_2 are the short-run effects on y of x and z respectively, and $(\beta_1/(1-\beta_3))$ and $(\beta_2/(1-\beta_3))$ are the long-run effects, then the relation between the short-run effects of the two variables can be written as β_1/β_2 while the relation between their long-run effects can be written as $(\beta_1/(1-\beta_3))/(\beta_2/(1-\beta_3)) = \beta_1/\beta_2$, so if the effect of x on y is twice as strong as that of z on y in the short run $(\beta_1/\beta_2 = 2)$, then the long-run effect of x must also be twice as great as that of z.

Clearly this is a very restrictive condition to impose on a model. Empirically it is quite plausible that different explanatory variables could take different lengths of time to have their effect on the dependent variable. For institutional reasons, one might have a strong immediate effect but a relatively weak long-run effect while another could have a weak short-run effect (because it takes a while for its effect to manifest itself) but a much stronger long-run effect.

The PA model, while appealing in its simplicity and in its obvious relationship to the theoretical models we have been considering, and despite its widespread use, is really too restrictive for empirical work. Fortunately the problems with the PA model can be corrected quite easily, by going to the Error Correction Model (ECM). This is the form which we will consider next.

The error correction model

The ECM is rapidly becoming the standard applied model in dynamic econometrics. It is best known in the context of unit root models, which we shall consider later, but it is applicable more generally. The ECM is basically a reparametrization of the general dynamic form (GDF):

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 z_t + \beta_3 y_{t-1} + \beta_4 x_{t-1} + \beta_5 z_{t-1} \tag{7.14}$$

where z and x are exogenous variables.

This is a first-order difference equation (FODE) in three variables, although the only feedback is from x and z to y, with no feedback between x and z and no feedback from y to either x or z. Because of this lack of feedback effects, the dynamics of the equation are derived from the y-terms alone, meaning that this equation has only one root. The lags on the x and z terms are there to allow greater flexibility than the PA model permits. The PA model can be seen to be nested in the GDF model in the sense that setting both β_4 and β_5 equal to zero reduces Equation (7.14) to the short-run version of the PA model (7.9).

The dynamic analysis of the GDF model can also be seen as an extension of that of the PA model. As in the PA model, the coefficients on x_t and z_t, β_1 and β_2 respectively, represent the short-run effects of those variables on y, since, when we look at one of those coefficients we are looking at the partial derivative of y with respect to the variable in question with all other variables, including the lagged value of that variable, held constant.

Again we define the long run as a period in which the exogenous variables have settled down to unchanging values and the dependent variable has a chance to converge on its new equilibrium without further changes in explanatory variables. Thus, the long run is defined as having $x_t = x_{t-1}$, $z_t = z_{t-1}$ and $y_t = y_{t-1}$. The long-run effect of an exogenous variable in the GDF equation is defined as the effect of that variable calculated with these conditions imposed.

For x, the long-run coefficient is defined as $(\beta_1 + \beta_4)/(1 - \beta_3)$, while for z the long-run coefficient is defined as $(\beta_2 + \beta_5)/(1 - \beta_3)$. If we set $\beta_4 = \beta_5 = 0$, we have the same relation between long- and short-run coefficients as in the PA model. Allowing β_4 and β_5 to differ from zero gives us greater flexibility. In the GDF the ratio of short-run effects is β_1/β_2 while the ratio of long-run effects is:

$$\left[\frac{(\beta_1 + \beta_4)}{(1 - \beta_3)}\right] \Big/ \left[\frac{(\beta_2 + \beta_5)}{(1 - \beta_3)}\right] = \left[\frac{(\beta_1 + \beta_4)}{(\beta_2 + \beta_5)}\right] \neq \frac{\beta_1}{\beta_2} \tag{7.15}$$

so the identity between the relative long- and short-run effects in the PA model is not present in the GDF model.

Since the general dynamic form in Equation (7.14) is a FODE in y, it can be solved like any other FODE. Its homogeneous form has solution: $y_t = A\beta_3^t$ while to find its particular, or equilibrium solution, we must incorporate the assumption that $x_t = x_{t-1}$ and $z_t = z_{t-1}$. Based on this assumption, the long run or equilibrium solution to Equation (7.14) is:

$$y_t^* = \frac{\beta_0}{(1 - \beta_3)} + \frac{(\beta_1 + \beta_4)}{(1 - \beta_3)} x_t + \frac{(\beta_2 + \beta_5)}{(1 - \beta_3)} z_t \tag{7.16}$$

These two terms can then be combined into the general solution to Equation (7.14) and, assuming that β_3 is a fraction, the equilibrium will be stable.

Equation (7.16) can be utilized to reparametrize the GDF into what is known as the ECM:

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2 \Delta z_t + (\beta_3 - 1)\,(y_{t-1} - \delta_0 - \delta_1 x_{t-1} - \delta_2 z_{t-1}) \tag{7.17}$$

where $\delta_0 = \beta_0/(1-\beta_3)$; $\delta_1 = (\beta_1+\beta_4)/(1-\beta_3)$; and $\delta_2 = (\beta_2+\beta_5)/(1-\beta_3)$. It is simply a rearrangement of the GDF Equation (7.14), so none of the interpretation of coefficients and combinations of coefficients is changed, but the ECM form is in many cases easier to interpret.

We can see the rationale for the rearrangement once we recognize δ_1 and δ_2 as the long-run coefficients for x and z, respectively. Short-run effects of x and z are now captured in the coefficients on the Δx and Δz terms, while long-run effects are captured in the coefficients on x_{t-1} and z_{t-1}.

Note too that defining the 'long-run' value of y at $t-1$ as:

$$y^*_{t-1} = \delta_0 + \delta_1 x_{t-1} + \delta_2 z_{t-1} \tag{7.18}$$

allows us to rewrite Equation (7.17) as:

$$\Delta y_t = \beta_1 \Delta x_t + \beta_2 \Delta z_t + \delta_3 (y_{t-1} - y^*_{t-1}) \tag{7.19}$$

where $\delta_3 = (\beta_3 - 1)$. Equation (7.19) shows why this version of the GDF is called an error correction form. Changes in y in any period are shown to depend on changes in the values of the exogenous variables in the current period and on the previous period's disequilibrium $(y_{t-1} - y^*_{t-1})$. This lagged disequilibrium is the 'error' which is being corrected this period. Since the coefficients of the underlying economic relations are assumed to be constant over time, the form of the equilibrium solution does not change over time.

Equation (7.17) could be estimated as:

$$\Delta y_t = \gamma_0 + \gamma_1 \Delta x_t + \gamma_2 \Delta z_t + \gamma_3 y_{t-1} + \gamma_4 x_{t-1} + \gamma_5 z_{t-1} \tag{7.20}$$

but comparison of Equations (7.20) and (7.17) shows that there is no particular difficulty translating the coefficients of Equation (7.20) into those of (7.17). In particular, $\gamma_3 = (\beta_3 - 1) = \delta_3$, so γ_3 in this form is the same as the coefficient on the 'error' to be corrected in Equation (7.19).

Thus, when we estimate an equation of the form (7.20) we have an equation from which we can derive both long- and short-run effects of the exogenous variables. If the model being estimated is correctly specified, γ_3, the coefficient on the 'error' to be corrected, should be negative, indicating that when y_{t-1} is above its long-run value (determined conditional on the value of the explanatory variables in period $t-1$), so that $(y_{t-1} - y^*_{t-1})$ is positive, the effect on Δy_t should be negative, indicating that y is falling towards equilibrium; while if y was below its equilibrium value last period, making the error component negative, it should be rising towards it in the present period, making Δy_t positive. The γ_3 term is equal to $(\beta_3 - 1)$,

where β_3 is the coefficient on the lagged dependent variable in the original general dynamic form, so γ_3 being statistically significantly negative is also the stability condition for that equation.

Although it is seldom done in a single equation context, we can work with higher order equations in ECM form. Consider the equation:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \beta_0 x_t + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 x_{t-3} \tag{7.21}$$

The error correction form of this equation is:

$$\begin{aligned} \Delta y_t = {} & \alpha_0 + (\alpha_1 - 1)\Delta y_{t-1} + (\alpha_2 + \alpha_1 - 1)\Delta y_{t-2} \\ & + (\alpha_3 + \alpha_2 + \alpha_1 - 1) y_{t-3} + \beta_0 \Delta x_t + (\beta_1 + \beta_0)\Delta x_{t-1} \\ & + (\beta_2 + \beta_1 + \beta_0)\Delta x_{t-2} + (\beta_3 + \beta_2 + \beta_1 + \beta_0) x_{t-3} \end{aligned} \tag{7.22}$$

In this equation, we can still find the long-run effect of x on y by dividing the coefficient on the x_{t-3} term by minus the coefficient on the y_{t-3} term, and a negative coefficient on y_{t-3} is still one part of the stability conditions, but there are other conditions which also have to be satisfied for stability (see discussion on higher order difference equations) so we can no longer treat a negative coefficient on the largest lagged level of y as sufficient for stability – all of the stability conditions should be checked, either in terms of the coefficients of the estimated equations, or by direct calculation of the roots themselves. Since the stability conditions can be checked in terms of the estimated coefficients, they can be tested for statistical significance as part of the estimation procedure in most econometrics packages, so that is probably the preferred approach.

Time series analysis

Unit roots

The ECM is encountered most often in econometric studies involving unit root, non-stationary variables, in which the objective of the research is to establish whether what is known as a cointegrating vector exists. It is sometimes thought that the ECM form can only be used in this context, but as we saw in the previous section the ECM is simply a reparametrization of the GDF, and establishing the interpretation of the form does not require us to make any assumptions about the roots of the variables involved.

For example, Anderson and Blundell (1982, 1983) used a system ECM form in estimating singular systems of equations. Their purpose was to distinguish between the long- and short-run properties of demand systems, testing the hypothesis that the homogeneity and symmetry properties which consumer theory predicts for demand systems hold only in the long run and not the short run. Their results indicated that dynamic modelling of demand systems was preferable to static

models. Anderson and Blundell started from the argument that adjustment costs and lags made it inappropriate to model consumer behaviour in a strictly static form, rather than taking as their starting point the time series properties of their individual data series, as is done in the recent unit root literature.[7]

A cointegrating vector is essentially the long-run or equilibrium vector of the previous section, but the term cointegrating vector applies only when the individual variables involved display unit root properties. We will discuss the nature of cointegrating vectors below. First, we briefly consider what is meant by a unit root process.

A variable y_t has a unit root when its difference equation representation has a root $\lambda = 1$. For a FODE this means that:

$$y_t = y_{t-1} + \varepsilon_t \tag{7.23}$$

where we have departed from our usual notation, which would have used g_t in place of ε_t, for the sake of conformity with the econometrics literature. For the moment, ε_t is simply the term that makes Equation (7.23) a non-homogeneous difference equation.

In Equation (7.23) the coefficient on y_{t-1} is equal to 1, making the solution to the homogeneous form:

$$A\lambda^{t-1}(\lambda - 1) = 0 \tag{7.24}$$

giving $\lambda = 1$ as noted above. We cannot solve for the particular solution in the normal way, because $1/(1-\alpha)$ in this case $= 1/0$. We can see the implications of this in Figure 7.1, where we have drawn the phase diagram for Equation (7.23).

The $y_t(y_{t-1})$ function is a straight line, with vertical intercept ε_t (which for the moment we will treat as a constant) and slope 1, so that it is parallel to the 45° line. Since the $y_t(y_{t-1})$ function never crosses the 45° line, there is no equilibrium

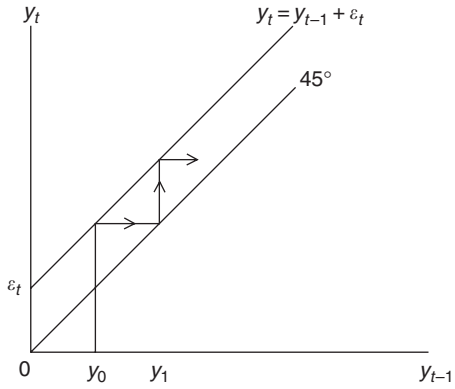

Figure 7.1 Phase diagram for a unit root process.

to be found in the diagram. If we do the usual exercise of picking an initial value of y_{t-1}, in this case 0, then finding the corresponding y_t, in this case ε_t, and then reflecting off the 45° line to find successive values of y_{t-1} and y_t, we see that y_t simply keeps growing over time.

We can see this, in fact, if we rewrite Equation (7.23) as:

$$\Delta y_t = y_t - y_{t-1} = \varepsilon_t \tag{7.25}$$

which shows that y will grow by an amount equal to ε_t from one period to the next. If y is in log form this is a rate of growth, if y is in levels it is the one period amount of growth.

As in the case of any difference equation, the non-homogeneity term, ε_t, plays the role of determining where the $y_t(y_{t-1})$ function lies on the phase diagram. An increase in ε_t shifts the whole function up by the amount of the increase. When the difference equation is stable, this shifts the equilibrium up, and the system converges on the new equilibrium. In the case of a unit root, the value of y_t shifts up by the amount of the increase in ε_t, and thereafter y continues to grow, starting from that higher level, without ever converging on an equilibrium.

In econometric applications, ε_t is not a constant, but rather a random variable, generally normally distributed, with mean zero, constant variance and no auto-correlation. Rather than changing occasionally, ε_t changes every period, by an unpredictable amount. From Equation (7.23), each change in ε_t has a permanent effect on the value of y.

We can see more clearly what this means if we consider a difference equation of the form:

$$y_t = \alpha_1 y_{t-1} + \varepsilon_t, \quad 0 < \alpha_1 < 1 \tag{7.26}$$

where the assumption about α_1 means that the difference equation is stable – its root, $\lambda = \alpha_1$, lies within the unit circle.

From Equation (7.25) we see that:

$$y_{t-1} = \alpha_1 y_{t-2} + \varepsilon_{t-1} \tag{7.27}$$

and so on back. Substituting Equation (7.27) into (7.26) gives:

$$\begin{aligned} y_t &= \alpha_1^2 y_{t-2} + \alpha_1 \varepsilon_{t-1} + \varepsilon_t \\ &= \alpha_1^3 y_{t-3} + \alpha_1^2 \varepsilon_{t-2} + \alpha_1 \varepsilon_{t-1} + \varepsilon_t \end{aligned} \tag{7.28}$$

and so on as far back as we choose to go. After each of these substitutions we will wind up with a single y-term, $\alpha_1^n y_{t-n}$, and a series of ε terms, each weighted by powers of α_1 (the ε_t term is weighted by α_1^0).

So long as α_1 is a positive fraction, successive powers of α_1 get successively smaller. This means that eventually we will reach a lag n where α_1^n is effectively equal to zero and the impact of all previous lag terms can be disregarded. While past ε terms show up in the expression for y_t, they do so with geometrically diminishing

weights, meaning that as time passes the impact of any given random shock to y fades away.

When $\alpha = 1$, however, the effect of a past shock never fades away. Instead, all past shocks accumulate into the present and future values of y. This makes y_t an extreme case of what is known as a long memory variable – when $\alpha_1 = 1$, y_t never forgets a past shock. The cumulation of later shocks will eventually overwhelm the impact of a past shock, no matter how large that past shock may be, but its effect never vanishes. It is because of this permanent accumulation of past events into the value of y now that y_t is known as an integrated variable – we are effectively integrating over its past behaviour to determine its current value.

Unit roots may also arise in the case of variables characterized by higher order difference equations:

$$y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} = 0 \tag{7.29}$$

In this case y has a unit root if $\alpha_1 + \alpha_2 = -1$. The second root in this case will be $-(1 + \alpha_1)$. Recall from our discussion of theoretical difference equation models that for stability in a single-order difference equation (SODE) we required three conditions to be satisfied, one of which was $\alpha_1 + \alpha_2 > 1$. Clearly a unit root model is not stable in the sense in which we have been using the term. It is not explosive, however, since in this case $A\lambda^t = A$ does not explode. A unit root is a borderline unstable case.

When the difference equation for a unit root process contains a non-zero intercept, as in:

$$y_t = \alpha_0 + y_{t-1} + \varepsilon_t \tag{7.30}$$

we find that:

$$\Delta y_t = \alpha_0 + \varepsilon_t \tag{7.31}$$

The phase diagram for Equation (7.30) would have a $y_t(y_{t-1})$ function with vertical intercept α_0 and slope 1, while the ε_t term would cause the $y_t(y_{t-1})$ function in any period to be distributed randomly around $y_t = \alpha_0 + y_{t-1}$. From Equation (7.31) we see that the one period growth of y will equal α_0 plus a random term. Because that random term has mean zero, over the long run $\Delta y_t = \alpha_0$, although in any single period, growth in y will be expected to deviate from α_0. The α_0 term is known as the drift in y_t.

A variable which follows a difference equation like (7.23) or (7.30) (or a higher order counterpart of Equation (7.30) so long as none of the other roots are explosive) is said to have a stochastic trend, also referred to as being difference stationary. This is in contrast to a variable with a deterministic trend, which is referred to as trend stationary.

Difference and trend stationarity

Interest in unit root processes in applied economic dynamics really derives from the work of Nelson and Plosser (1982). Until that time it was customary to model

trended variables as:

$$y_t = \alpha_0 + \alpha_1 t + \varepsilon_t \tag{7.32}$$

where t stands for time, and is a deterministic trend, which means that y is trend stationary. Nelson and Plosser presented evidence that most trended macroeconomic variables are better characterized by Equation (7.30) than by Equation (7.32).[8]

The difference between Equation (7.30) and (7.32) is significant for two reasons, one theoretical and one econometric.

On the theoretical side, Equations (7.30) and (7.32) imply very different forms of behaviour for key macroeconomic variables. Suppose, for example, that y_t is real gross domestic product (GDP). According to Equation (7.32) the expected value of real GDP in period $t + 1$, when the expectation is being formed in period t, is:

$$E_t y_{t+1} = \alpha_0 + \alpha_1 (t + 1) \tag{7.33}$$

Since Equation (7.33) does not contain y_t explicitly, the expectation we form in period t about y_{t+1} is completely independent of the actual value of y_t. Even if period t is a period of deep recession, the recession will be regarded as resulting from an unusually bad value of ε, and the expectation (since there is assumed to be no autocorrelation in ε) is that in period $t + 1$ GDP will be back on the trend line characterized by $y_t = \alpha_0 + \alpha_1 t$.

Even if we modify Equation (7.32) to give:

$$y_t = \alpha_0 + \alpha_1 t + \alpha_2 y_{t-1} + \varepsilon_t, \quad 0 < \alpha_2 < 1 \tag{7.34}$$

so that there may be a lag in returning to the trend line, the behaviour of y is still governed by a stable difference equation with a moving equilibrium. The solution to the non-stochastic version of Equation (7.34) is:

$$y_t = A\alpha_2^t + \frac{\alpha_0(1 - \alpha_2) - \alpha_1\alpha_2}{(1 - \alpha_2)^2} + \frac{\alpha_1}{(1 - \alpha_2)} t \tag{7.35}$$

so that regardless of what shocks may strike y_t, in the long run they will wash out and y_t will follow a trend line with a constant slope.

On the other hand, if GDP obeys Equation (7.30), there is no tendency to return to any particular level after a shock. If we take the expectation of y_{t+1} based on the form of the equation in (7.30), we have:

$$E_t y_{t+1} = \alpha_0 + y_t \tag{7.36}$$

In other words, instead of returning to a value on a deterministic trend line, y starts growing at rate α_0 from its current position, generating a stochastic trend.

This has significant economic implications. If GDP is trend stationary, then after it goes into a recession, it will tend to rise towards the trend line and also move

along the trend line simply because of the passage of time. Departures from the trend line, on either side, are therefore transitory.

If GDP is difference stationary, it will grow from wherever it happens to be today. If the economy is in a recession today, GDP will be expected to grow by α_0 between this period and the next, but it will not be expected to put on a spurt of growth to pull it back up to some trend line. In practical terms this means that if real GDP is trend stationary, any output lost in a recession will be recovered in the following recovery, while if real GDP is difference stationary, the output that was lost during a recession is lost forever. A difference stationary world would require very different macroeconomic policies than would a trend stationary world.

Implications for empirical analysis

The econometric problems associated with unit root variables simply add another layer of difficulty to empirical dynamics. Unit root variables display what is known as non-stationarity. A variable y_t is stationary if:

$$\begin{aligned} \mathrm{E}(y_t) &= \text{constant for all } t \\ \mathrm{Var}(y_t) &= \text{constant for all } t \\ \mathrm{Cov}(y_t, y_{t+j}) &= \text{constant for all } t \end{aligned} \tag{7.37}$$

For a unit root variable, the variance of y_t increases as t increases, so the stationarity conditions are violated.[9]

Enders (1995) states that if a variable y_t is driven by a linear stochastic difference equation (i.e. a difference equation with a random ε_t term in it), the stability condition for the difference equation is a necessary condition for the time series of values of y generated by that difference equation to be stationary. Thus, difference equations with unit roots or unstable roots are associated with non-stationary variables.

The problem with non-stationary variables is that the underlying distribution theory used in econometric testing assumes the variables are stationary. This means that the t-tables found in virtually all econometrics texts cannot be counted on to give accurate critical values for hypothesis testing in the case of non-stationary values.

This problem extends right down to the problem of determining whether a variable has a unit root. We noted above that stability was necessary for stationarity. This means that, in the case of the Equation (7.30) where the root $\lambda = \alpha_1$, y will be a stationary variable if α_1 is significantly less than 1 in absolute value.[10]

Unfortunately we cannot simply estimate Equation (7.30) and compare the calculated t-statistic from our regression output with the critical values in the standard t-tables, since the calculated statistic, which is (as usual) the ratio of the estimated value of α_1 divided by its estimated standard error, no longer has the symmetric, bell-shaped distribution of the tabulated t-distribution when y_t has a unit root.

Fortunately, the distribution of unit root test statistics has been generated by simulation methods, the most complete set to date being those generated by

MacKinnon (1995). While these tabulated values must be treated with some caution, since they are based on computer simulations and are therefore sensitive to the particular assumptions underlying the simulations which generated them, they are important additions to the tools of dynamic econometrics.

ARIMA models

Unit root processes and the non-stationarity they introduce are not new to econometrics. Box and Jenkins (1976) ARIMA analysis[11] recognizes non-stationarity and attempts to correct for it. The particular non-stationarity introduced by unit root variables is in one sense easily corrected by differencing the variables.

Recall from Equation (7.30) how a unit root variable y_t could be rewritten in its first difference form, Equation (7.31) where ε_t has all of the usual, desirable (from the econometric point of view) properties. The dependent variable Δy_t is clearly a stationary variable.

Because a unit root variable can be transformed into a stationary variable by differencing it once, it is known as an $I(1)$ variable. By extension, if we tested the dynamic properties of Δy_t and found that it was unit root, so that $\Delta^2 y_t$, rather than Δy_t, was stationary, then y_t would be an $I(2)$ variable, the '2' referring to the fact that we have to difference y_t twice to transform it into a stationary variable.

The counterpart Equation to (7.31) for an $I(2)$ variable is found as:

$$\begin{aligned}\Delta^2 y_t &= \Delta\Delta y_t = \Delta(y_t - y_{t-1}) \\ &= \Delta y_t - \Delta y_{t-1} = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\ &= y_t - 2y_{t-1} + y_{t-2}\end{aligned} \tag{7.38}$$

which is a SODE with characteristic equation:

$$\lambda^2 - 2\lambda + 1 = 0 \tag{7.39}$$

whose two roots are both equal to 1.

We should note here that an $I(1)$ variable is not restricted to having only one root, nor is an $I(2)$ variable restricted to having exactly 2 roots. An $I(1)$ variable has exactly one unit root, and as many stable roots as it likes, and similarly an $I(2)$ can have many stable roots in addition to its two unit roots.

We should note too that unit root behaviour is not the only form of non-stationarity which economic variables can display, but it is in many ways the easiest to deal with. Variables with unstable roots are non-stationary, but their particular non-stationarity cannot be removed by differencing. As it happens, though, macroeconomic variables are virtually never dynamically unstable, while there is a great deal of evidence in favour of unit root behaviour. At the micro level, there are cases where an unstable root can manifest itself, but there is very little literature on econometric issues arising from this case.

In Box–Jenkins analysis, the $I(1)$ nature of most macroeconomic variables is corrected for by first differencing and working with the $I(0)$, or stationary,

counterparts. Thus, instead of dealing with an equation in their levels form, such as:

$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t \tag{7.40}$$

or, when explanatory variables are present:

$$y_t = \beta_0 + \beta_1 x_t + \beta_2 z_t + \beta_3 y_{t-1} + \beta_4 x_{t-1} + \beta_5 z_{t-1} + \varepsilon_t \tag{7.41}$$

ARIMA analysis tends to first difference all of the variables and regress first difference on first difference with no levels of variables present:

$$\Delta y_t = \beta_0 + \beta_1 \Delta x_t + \beta_2 \Delta z_t + \varepsilon_t \tag{7.42}$$

Equation (7.42) can be seen to be a restricted version of the ECM equation (7.17). The particular restrictions which reduce Equation (7.17) to (7.42) involve setting the coefficients on the lagged levels to zero: in the notation of (7.17) this involves:

$$\beta_1 + \beta_4 = 0 \tag{7.43}$$

$$\beta_2 + \beta_5 = 0 \tag{7.44}$$

$$\beta_3 - 1 = 0 \tag{7.45}$$

all of these are linear, easily testable restrictions, so, degrees of freedom permitting, the ECM form should always be estimated and restrictions (7.45) tested before the ARIMA form is used, particularly because, by imposing restrictions (7.45), we are eliminating the part of the ECM form which gives us information about the long-run relation underlying the variables. ARIMA can only tell us about short-run relations, while the ECM form can tell us about both long- and short-run relations.

While the realization that many macroeconomic variables seem to display unit root behaviour has caused problems for econometric analysis of economic dynamics, it has also brought some benefits to the analysis of economic systems.

We have already seen that when variables are related in a system of equations, they all have the same roots. This means that variables with different roots to their characteristic equations cannot be part of the same system of variables. Even in the case of Equation (7.14) where there is no feedback from y to x and z, the general principle is that if variables are to be causally related, their underlying dynamic behaviour should be marked by the same order of integration. In particular, $I(1)$ variables should be related to, and driven by, other $I(1)$ variables. It is not reasonable for an $I(1)$ variable to be driven by an $I(0)$ variable.

This means that the first step in applied economic dynamics should be, and increasingly is, to test the order of integration of the variables which are hypothesized to be related.[12] If the variables prove all to be $I(0)$, we can proceed with standard econometric analysis, if some are $I(1)$ and some $I(0)$, this should be taken as evidence against our theory, and if they are all $I(1)$ we should augment standard econometric procedures with cointegration analysis.

Cointegration analysis

Cointegration analysis essentially adds testing of dynamic properties to the standard toolkit of econometric evaluation. Consider a two-equation system in y_t and x_t:

$$\begin{aligned} y_t &= \alpha_1 y_{t-1} + \alpha_2 x_t + \alpha_3 x_{t-1} + \varepsilon_t \\ x_t &= \gamma_1 x_{t-1} + \nu_t \end{aligned} \tag{7.46}$$

where the random terms ε_t and ν_t are assumed to have all the econometrically desirable properties. This two-equation system has one endogenous variable, y_t (expressed in a general dynamic form) and a variable, x_t which is exogenous because there is no feedback from y to x.[13] For simplicity, we have assumed that there are no intercept terms.

If we set the equations up in system form we will find that y_t has two roots, α_1 and γ_1. It may appear that there is a contradiction between our assumption that there is no feedback from y to x and the result that the characteristic equation for the system (7.46) has two roots, one from each variable, since we have shown elsewhere that all of the equations in the system are driven by linear combinations of the same roots.

Recall, however, that while the roots driving y and x are the same, they enter the solution equations for the homogeneous parts of y and x with different weights:

$$\begin{aligned} y_t &= A_{11}\lambda_1^t + A_{12}\lambda_2^t \\ x_t &= A_{21}\lambda_1^t + A_{22}\lambda_2^t \end{aligned} \tag{7.47}$$

where the A_{ij} terms are the undetermined constants, which cannot be solved until we have added the initial conditions to the problem, and that in our example $\lambda_1 = \alpha_1$ and $\lambda_2 = \gamma_1$. It can be shown, however that in the present case, $A_{21} = 0$ so that while x is technically driven by the same roots as is y, the root derived from the y equation has a weight of zero in the equation explaining the dynamics of x.

Stationary variables

Assuming for the moment that both roots are inside the unit circle (i.e. strictly stable: we will introduce unit root complications later), we can then derive particular solutions for y_t and x_t in the usual way. We can set $y_t = y_{t-1} = y_t^*$ and $x_t = x_{t-1} = x_t^*$, and derive:

$$\begin{aligned} (1-\alpha_1)y_t^* - (\alpha_2 + \alpha_3)x_t^* &= \varepsilon_t \\ (1-\gamma_1)x_t^* &= \nu_t \end{aligned} \tag{7.48}$$

Normally, we would use Equations (7.48) to solve for y^* and x^* in terms of the right-hand side terms. We have stopped at an intermediate step, however, for purposes of relating the theoretical solution structure to the forms used in dynamic econometrics.

By rewriting Equations (7.48) we derive:

$$y_t^* = \delta_1 x_t^* + \eta_t \tag{7.49}$$

where $\eta_t = \varepsilon_t/(1-\alpha_1)$ is a random disturbance term with all of the econometrically desirable properties. If we drop the '*' superscript, we have an econometrically estimable long-run equation relating y and x, using the actual observed values of y and x. This is what we refer to as the 'long-run' relation between y and x.

While we could proceed to estimate Equation (7.49) directly, if y and x are both stationary variables this would not be appropriate. This is because actual, observed values of y and x include dynamic effects; hence running Equation (7.49) without the lagged terms may result in a significant omitted variable bias problem.

Non-stationary variables

In the case of unit root variables, the situation is slightly different. To make a unit root system, assume that γ_1 in Equation (7.46) is equal to 1, while the other root of the system, α_1 is a positive fraction. The first thing to note about this assumption is that we can no longer proceed from Equations (7.48) to finding a particular solution for the system in terms of the right-hand side terms of those equations. If we set Equation (7.48) up in matrix form, $Az = g$, where A is the matrix of coefficients on y and x, and z is the column vector whose elements are y and x, then with $\gamma_1 = 1$, A is not invertible.

Note, however, that in deriving Equation (7.48), we did not need to use the assumption that both roots were stable, so even with γ_1 set as the unit root of the system, we can still find Equation (7.49). Since both y and x are $I(1)$ variables in this case, both are integrated, and we refer to Equation (7.49) as the cointegrating equation for y and x.

In deriving Equation (7.49), we assumed that y and x are causally related. This, of course, is the kind of theoretical hypothesis that must be tested in applied economic dynamics. At this point, the non-stationary nature of y and x introduces another problem. When two (or more) economic variables display unit root behaviour with drift, as appears to be the case for most macroeconomic variables, they are prone to generating spurious regression results. Even if x and y are completely unrelated variables, if both are $I(1)$ with drift, a regression of y on x will tend to yield an R^2 value of at least 0.4, and the calculated t-statistics will be extremely large, even, again, in the case of unrelated variables.

To test whether Equation (7.49) represents a spurious or a cointegrated relationship, the econometric techniques of cointegration are applied. One method exploits the fact that if two $I(1)$ variables, y and x, are cointegrated – that is, if a stable long-run relation like Equation (7.49) exists between them – then when y is regressed on x the regression residual should be $I(0)$. The Engle–Granger (1987) approach to testing for the existence of a cointegrating relation involves estimating Equation (7.49) by ordinary least squares and using critical values such as those developed by MacKinnon to test the hypothesis that the regression residual displays unit root behaviour. If y and x are cointegrated, the residuals should

be $I(0)$, if y and x are not cointegrated, the residuals will be $I(1)$. Thus, even if the R^2 value from regressing Equation (7.49) is high, if we cannot reject the null hypothesis that the residuals are $I(1)$, the regression is spurious.

Engle and Granger also show that if a set of economic variables are cointegrated, there exists an error correction representation of the relation between them. Hendry (1995) advocates always using the ECM form for the sake of the greater efficiency of estimation resulting from allowing for both the long- and short-run relation between y and x in the same estimating equation. Note that the t-statistics generated as part of the OLS estimation of Equation (7.49) in the Engle–Granger approach cannot be compared with the critical values found in tables of Student's t values, since the calculated t's will not have the Student's t distribution. Other estimation techniques must be used if we want to do hypothesis tests on the individual coefficients.[14]

In a case like Equation (7.49), when we have only two variables, y and x, there can be at most one cointegrating relation between them. When we have several variables, however, $\{y_{1t}, y_{2t}, y_{3t}, y_{4t}\}$, there can be several cointegrating relations among them – the maximum potential number of cointegrating relations is equal to the number of y variables less 1, in fact. We consider this case in the next section.

Cointegration in the case of several variables

When we have several variables, $y_{1t}, y_{2t}, y_{3t}, y_{4t}$, it is customary to begin by stacking the variables in matrix form, so that $y_t = (y_{1t}\, y_{2t}\, y_{3t}\, y_{4t})'$. Then, assuming for the moment that there is only one lag, so we are dealing with FODEs in each of the variables, and letting ε_t be a (4×1) vector of disturbance terms with all of the usual econometrically desirable properties, we have:

$$y_t = A_1 y_{t-1} + \varepsilon_t \tag{7.50}$$

where A_1 is the coefficient matrix on the lagged dependent variables (for simplicity we have assumed that there are no intercept terms). It is not necessary for our theoretical model to start with Equation (7.50). We can start with a structural form like:

$$B_0 y_t = B_1 y_{t-1} + v_t \tag{7.51}$$

and derive Equation (7.50) as a dynamic reduced form: $A_1 = B_0^{-1} B_1$. Both Equations (7.51) and (7.50) are familiar forms in time series analysis (Enders 1995): both are cases of Vector Autoregressions: VARs. Equation (7.51) is known as a structural-form VAR, while Equation (7.50) describes a standard reduced-form VAR.

Equation (7.50) is nothing more than the matrix representation of a system of FODEs with a random disturbance vector added. If all of the roots of A_1 are stable, there is no particular problem with econometric analysis of Equation (7.50). If some of the roots of A_1 are equal to 1, however, we run into generalized versions of the same sort of problems that arise in the single equation case.

One possible approach to determining whether the system (7.50) has unit roots would be to estimate the coefficients in the matrix A_1, calculate the roots using

those estimated coefficients, and use the technique of Theil and Boot (1962) or Oberhofer and Kmenta (1973) to estimate the standard errors of the roots.[15]

Rather than estimating and testing the roots of A_1 directly, most researchers, if they decide to use a systems approach rather than a single equation approach, now use Johansen's (1988, 1991) approach to estimation and testing. Rather than estimating standard errors for the estimated coefficients, Johansen's approach tests the number of roots of A_1 which are significantly less than unity.

In Johansen's approach, having first established that all of our y_{it} variables are $I(1)$, we rewrite system (7.50) as:

$$\Delta y_t = \Pi y_{t-1} + \varepsilon_t \tag{7.52}$$

where $\Pi = (A_1 - I)$. Normally, for two square, conformable matrices A and B, the roots of $A + B$ are not equal to the roots of A plus the roots of B. In this case, however, it can be shown that each root of $(I - A_1)$ is equal to 1 minus the corresponding root of A_1. Thus, each unit root of A_1 will be associated with a zero root of $(I - A_1)$, while each stable root of A_1 will be associated with a positive, fractional root of $(I - A_1)$. If A_1 has complex roots it is the modulus of those roots which matters, while negative roots in A_1, though rare in economics, lead to roots of $(I - A_1)$ which are greater than 1. It also happens that the eigenvectors of the matrix $(I - A_1)$ are the same as the eigenvectors of A_1.

Johansen's maximum likelihood technique involves calculating the roots of $(I - A_1)$ and testing the null that they are significantly different from zero. His test statistics are based on log transformations of the roots of $(I - A_1)$, so an unstable root in A_1, which would yield a negative root in $(I - A_1)$, would cause the method to fail, but as we have noted before, macroeconomic systems are generally not unstable, regardless of how unexpected their behaviour may occasionally be.

Johansen's technique estimates the number of unit roots in A_1, yielding at the same time the number of stable roots. Because all of the y variables are driven by the same roots, the unit roots introduce what are known as common trends in the behaviour of the y variables while the stable roots yield cointegrating vectors.

To see where this terminology derives from, consider the two variable case:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} \tag{7.53}$$

which is just $y_t = A_1 y_{t-1}$, a non-stochastic version of Equation (7.50). We saw this case in our discussion of systems of difference equations. Letting c_1 and c_2 be the characteristic vectors of (7.53), corresponding to roots λ_1 and λ_2 respectively, we can define a matrix:

$$C = [c_1 c_2] = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{7.54}$$

in which each column of C is the corresponding characteristic vector c_1 or c_2. Further, let Λ be the matrix:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{7.55}$$

Given these definitions, A_1 can be written $A_1 = C\Lambda C^{-1}$ and $A_1^t = C\Lambda^t C^{-1}$ so we have $y_t = C\Lambda C^{-1} y_{t-1} = C\Lambda^t C^{-1} y_0$ where y_0 is the vector of known initial values of y_1 and y_2. Written out in full:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} \begin{bmatrix} c^{11} & c^{12} \\ c^{21} & c^{22} \end{bmatrix} \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} \tag{7.56}$$

where the c^{ij} terms are the elements of C^{-1}. It is customary to perform the multiplication $C^{-1} y_0$ and define:

$$\begin{aligned} z_1 &= c^{11} y_{10} + c^{12} y_{20} \\ z_2 &= c^{21} y_{10} + c^{22} y_{20} \end{aligned} \tag{7.57}$$

Substituting the z terms into Equation (7.56) we wind up with a pair of equations:

$$\begin{aligned} y_{1t} &= z_1 c_{11} \lambda_1^t + z_2 c_{12} \lambda_2^t \\ y_{2t} &= z_1 c_{21} \lambda_1^t + z_2 c_{22} \lambda_2^t \end{aligned} \tag{7.58}$$

Now suppose we want to plot y_{1t} against y_{2t}. As time passes, the two variables must obey time paths defined by Equations (7.58). Differentiating Equation (7.58) with respect to t (and assuming for convenience that the λ_i have positive, real values) we have:

$$\begin{aligned} \frac{\partial y_{1t}}{\partial t} &= z_1 c_{11} \lambda_1^t \ln(\lambda_1) + z_2 c_{12} \lambda_2^t \ln(\lambda_2) \\ \frac{\partial y_{2t}}{\partial t} &= z_1 c_{21} \lambda_1^t \ln(\lambda_1) + z_2 c_{22} \lambda_2^t \ln(\lambda_2) \end{aligned} \tag{7.59}$$

and dividing we have:

$$\frac{\partial y_{1t}}{\partial y_{2t}} = \frac{z_1 c_{11} \lambda_1^t \ln(\lambda_1) + z_2 c_{12} \lambda_2^t \ln(\lambda_2)}{z_1 c_{21} \lambda_1^t \ln(\lambda_1) + z_2 c_{22} \lambda_2^t \ln(\lambda_2)} \tag{7.60}$$

In most cases, this is not particularly revealing. Suppose, however, that $\lambda_1 = 1$ and that λ_2 is a positive fraction. Then $\ln(\lambda_1) = \ln(1) = 0$ and we have:

$$\frac{\partial y_{1t}}{\partial y_{2t}} = \frac{c_{12}}{c_{22}} \tag{7.61}$$

where c_{12} and c_{22} are the elements of the characteristic vector associated with λ_2, the stable root of the system.

Now return to Equation (7.50). Normally we would proceed to write $(I - A_1) y_t^* = \varepsilon_t$ and invert $(I - A_1)$ to find the particular solution for the y vector. If all of the roots of A_1 are stable, we can still do this, giving:

$$y_t^* = (I - A_1)^{-1} \varepsilon_t \tag{7.62}$$

In this case, since there is no unit root present (by assumption), all of the y variables are stationary. Conversely, if $(I - A_1)$ can be inverted, all of the y variables must be stationary. In that case, none are integrated of order greater than 0, so the question of cointegrating relations does not arise.

Cointegration comes into play when some of the roots of A_1 are unitary, meaning that some of the roots of $(I - A_1)$ are zero, meaning in turn that $(I - A_1)$ is not invertible. At this point, we invoke a bit of matrix algebra. The rank of a matrix is equal to the number of non-zero characteristic roots of that matrix. If $(I - A_1)$ has zero roots, it is of less than full rank, meaning that at least one row of $(I - A_1)$ can be found as a linear combination of the others. The rank of $(I - A_1)$, meaning the number of linearly independent rows of $(I - A_1)$ and therefore the number of linearly independent combinations of the y's generated by the system, equals the number of stable roots of A_1.

Consider now the first difference form of Equation (7.50):

$$\Delta y_t = (A_1 - I)y_{t-1} + \varepsilon_t \tag{7.63}$$

Since all of the y variables are $I(1)$ (otherwise we would not be looking at any of this material), all of the elements on the left-hand side of Equation (7.63) are $I(0)$, that is, stationary. By definition, ε_t is stationary. Thus, $(A_1 - I)y_{t-1}$ must be a set of stationary combinations of the y's, and a stationary combination of the y's is what we have termed a cointegrating vector.

If $(A_1 - I)$ is of full rank, it implies that all of the y variables can, individually, be written as linear combinations of the ε_t variables and, since each of the elements in the ε_t vector is $I(0)$ and since no linear combination of $I(0)$ variables can ever be $I(1)$, each of the elements of the y_t vector must be $I(0)$. So for us to have a non-zero number of cointegrating vectors, $(A_1 - I)$ must be of less than full rank, meaning that there must be fewer cointegrating vectors than there are elements in the vector y_t.

In the simple case of two y variables, then, there can be at most one cointegrating relation. When y contains three variables there can be at most two cointegrating relations among them. Note that when A_1 has a unit root, $|A_1 - \lambda I| = |A_1 - I| = 0$.

The matrix $[A_1 - I]$ is $(n \times n)$ where n is the number of elements in the y_t vector. Any $(n \times n)$ matrix of rank $0 < r < n$ can be written as the product of an $(n \times r)$ matrix and an $(r \times n)$ matrix, which in the cointegration literature are usually identified as α and β', respectively. (Note that this means that β is actually $(n \times r)$.) The matrix β is made up of the eigenvectors associated with the non-zero roots of $(A_1 - I)$.

In the Johansen procedure, then, r is the number of cointegrating combinations of the y variables. We can rewrite Equation (7.63), using this notation, as:

$$\Delta y_t = \alpha\beta' y_{t-1} + \varepsilon_t \tag{7.64}$$

Assume for the moment that y_t is a (2×1) vector. Then we can write Equation (7.64) as:

$$\begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{7.65}$$

In expression (7.65), β_1 and β_2 are defined so that $\beta_1 y_{1t} + \beta_2 y_{2t}$ is a cointegrating vector. In more familiar notation, we would write the cointegrating relationship between y_{1t} and y_{2t} as:

$$y_{1t} = (-\beta_2/\beta_1) y_{2t} + \nu_t \tag{7.66}$$

with the coefficient on y_{1t} normalized to equal 1. The term ν_t is the deviation between the actual value of y_{1t} and the value predicted by the cointegrating vector $(-\beta_2/\beta_1) y_{2t}$. In other words, it is the error correction associated with the normalized cointegrating vector. Thus, in Equation (7.65), the error correction term associated with the cointegrating vector is without normalization. Writing this as ECM_{t-1} we can rewrite Equation (7.65) as:

$$\begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \begin{bmatrix} \text{ECM}_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{7.67}$$

Written this way, Equation (7.67) shows that the α vector shows the extent to which the ECM term enters the determination of each element of the y_t vector.

Now we return to the system of Equation (7.46) where, in what follows, we shall set $\gamma_1 = 1$. We have two possible approaches to estimating this system, single equation and system. Within the single equation approach, we also have two possibilities. One is to adopt the single equation ECM structure and estimate:

$$\Delta y_t = \alpha_2 \Delta x_t + (\alpha_1 - 1) y_{t-1} + (\alpha_3 + \alpha_2) x_{t-1} + \varepsilon_t \tag{7.68}$$

Within this approach, two alternative methods have been widely adopted in the literature. One is to adopt Hendry's approach and estimate Equation (7.68) directly by OLS. We then test for the existence of a cointegrating relation between y and x by testing whether the coefficient on y_{t-1} is significantly different from zero on the basis of Hendry's critical values. If it is, we have a cointegrating relation, if it is not, we have a spurious relation.

The other is to adopt the Engle–Granger two-step approach: in this method we first estimate the candidate cointegrating relation:

$$y_t = \kappa_1 x_t + \nu_t \tag{7.69}$$

and test the regression residuals, which are the estimates of the ν_t values, for unit root properties. If ν_t is stationary, we have a cointegrating relation and ν_t represents the error to be corrected – the difference $(y_t - y_t^*)$. If Equation (7.69) is a cointegrating relation, we then enter υ_{t-1} in Equation (7.68) and estimate:

$$\Delta y_t = \alpha_2 \Delta x_t + (\alpha_1 - 1) \upsilon_{t-1} + \varepsilon_{1t} \tag{7.70}$$

Since all of the terms in Equation (7.70) are $I(0)$, we can use the Student's t distribution as the basis for hypothesis testing both about the significance of $(\alpha_1 - 1)$, which should be negative and significant, and about α_2.

The alternative to the single equation approach is the system approach. In this approach we begin with the matrix system:

$$\begin{bmatrix} 1 & -\alpha_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_3 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \tag{7.71}$$

From this we get the dynamic reduced form:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_3 + \alpha_2\gamma_1 \\ 0 & \gamma_1 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix} \tag{7.72}$$

which is the form from which the Johansen procedure begins. From here on we shall set $\gamma_1 = 1$, since this gives us the unit root which justifies what we are doing.

Converting Equation (7.72) to first difference form, with the unit root assumption imposed, by subtracting the identity matrix from the matrix of coefficients on the lagged variables on the right-hand side gives:

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} \alpha_1 - 1 & \alpha_3 + \alpha_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix} \tag{7.73}$$

where the $(A_1 - I)$ matrix, which contains a row of zeroes, is clearly of less than full rank. Johansen's procedure would indicate the presence of one cointegrating vector, associated with the stable root, and one common trend, associated with the unit root.

There are a number of ways we can rewrite Equation (7.73). One would give us

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} \alpha_1 - 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{\alpha_3 + \alpha_2}{\alpha_1 - 1} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix} \tag{7.74}$$

which would, when written out, give us a version of the error correction term from the single equation form. We might also, though, rewrite Equation (7.73) as:

$$\begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [\alpha_1 - 1 \;\; \alpha_3 + \alpha_2] \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix} \tag{7.75}$$

Here the product of the (1×2) vector $(\alpha_1 - 1, \alpha_3 + \alpha_2)$ and the (2×1) vector $(y_{t-1}, x_{t-1})'$ is $(\alpha_1 - 1)y_{t-1} + (\alpha_3 + \alpha_2)x_{t-1}$ which is just another version of the error correction term from the single equation form without the coefficient on the y term having been normalized to 1.

The absence of the normalization in Equation (7.75) brings out one point about the Johansen procedure – because the precise expressions for the eigenvectors from which they are derived depend on an arbitrary normalization, the cointegrating vectors Johansen's procedure yields are unique only up to a scalar multiple, so we must use judgement and theory in deciding which variable should be given a coefficient of 1, since setting its coefficient to 1 makes that variable the dependent variable, in the usual notation, and means that we are interpreting that cointegrating vector as explaining that variable.

In the present example we have assumed that y is the dependent variable and x the explanatory variable, but in cases where we have several y variables the assignment may not be so clear. In the empirical Johansen literature, in cases where the tests indicate the presence of several cointegrating vectors, we often find authors basically guessing about the structural interpretation of cointegrating vectors, and so guessing about the normalization to apply.

In Equation (7.75), the (2×1) column vector $(1 \quad 0)'$ on the right-hand side indicates that the ECM term enters the Δy_t equation with a weight of 1 and the Δx_t equation with a weight of zero, meaning that adjustments of y towards its equilibrium value have no impact on x. This is a consequence, and also a test, of the assumption that x is an exogenous variable.

Note that there is a difference between the Δy_t equation coming out of Equation (7.75) and that coming out of Equation (7.68) in that the Δx_t term does not appear to be present in Equation (7.75). In fact it is present, in v_{1t}, the disturbance term of the y_t equation. The v_{1t} term is equal to $(\varepsilon_{1t} + \alpha_2 \varepsilon_{2t})$, while the $\alpha_2 \Delta x_t$ term in Equation (7.68), making use of the fact that $\Delta x_t = \varepsilon_{2t}$, is the second part of the v_{1t} term. Thus, we have not lost any terms, we have simply arranged them differently.

Johansen's systems approach can also be used when there are several y variables and several lags. In the case of two lags, we have:

$$y_t = A_1 y_{t-1} + A_2 y_{t-2} + \varepsilon_t \tag{7.76}$$

where y_t is a vector of variables, ε_t is a vector of disturbance terms and A_1 and A_2 are coefficient matrices. This system has already been translated from structural form to dynamic reduced form. In first difference form Equation (7.76) becomes $\Delta y_t = (A_1 - I)\Delta y_{t-1} + (A_2 + A_1 - I)y_{t-2} + \varepsilon_t$ which in the econometrics literature is typically written:

$$\Delta y_t = (A_1 - I)\Delta y_{t-1} - (I - A_1 - A_2)y_{t-2} + \varepsilon_t \tag{7.77}$$

Assuming that equilibrium involves $y_t = y_{t-1} = y_{t-2}$, the matrix $(I - A_1 - A_2)$ gives the long-run relations among the variables. Each cointegrating relation among the y's, and there may be several, so long as the number of cointegrating relations is less than the number of elements in the y_t vector, yields an ECM term in Equation (7.77). Note that some of the elements in the y_t vector may be exogenous, since this simply involves the coefficients governing feedback from the endogenous variables to those exogenous variables being equal to zero. Thus, we have not excluded the possibility of there being exogenous variables in this system, we have simply not given them separate notation. Any element in Δy_t which is not affected by any of the ECM terms is exogenous as far as the system is concerned, and we can save on computation costs by taking advantage of that fact.

Johansen's procedure, applied to Equation (7.77), calculates the roots of $(I - A_1 - A_2)$. In the case of a single lag, the roots of $(I - A_1)$ are simply equal to 1 minus the roots of A_1. In the present case, this does not hold – the roots of $(I - A_1 - A_2)$ cannot be found by subtracting the roots of A_1 and A_2 from 1. The

matrix $(I - A_1 - A_2)$ will have as many roots as there are elements in the vector y_t, while the dynamic system (7.76) will have that number of roots multiplied by the number of lags. Thus, if there are four elements in y_t, $(I - A_1 - A_2)$ will have 4 roots while the system (7.76) will have eight. The characteristic Equation for (7.76) can be written in matrix form as:

$$\left|\lambda^2 I - \lambda A_1 - A_2\right| = 0 \tag{7.78}$$

There is, however, one point of contact between the roots of $(I - A_1 - A_2)$ and those of the system (7.76). For any unit root in Equation (7.76), the characteristic Equation (7.78) becomes:

$$|I - A_1 - A_2| = 0 \tag{7.79}$$

so unit roots of system (7.76) will correspond to zero roots of $(I - A_1 - A_2)$. The non-zero roots of $(I - A_1 - A_2)$ do not have immediate counterparts in the system, but still determine the number of cointegrating vectors. While the link between the Johansen cointegration methodology and the structure of the theoretical dynamic model is not as close in the case of several variables and several lags, it is necessarily still present.

The cointegration methodology makes use of the time series properties of the variables in a system to isolate, econometrically, the long-run relation among them, and uses that information to increase the efficiency with which all of the coefficients, long and short run, are estimated. While the cointegration – Error Correction Mechanism approach is sometimes derived directly from theoretical modelling, assuming the decision makers are minimizing a quadratic loss function, it is often adopted without direct theoretical justification, as a general form of dynamic relation. It is not the only form of adjustment mechanism, however, and there may be occasions on which theory suggests a particular dynamic process – the Walrasian adjustment mechanism in a competitive market model, for example. If that is the case, econometric testing of the dynamic adjustment process should be regarded as part of the process of evaluating the theoretical model itself.

Examples of empirical analysis

The literature on applications of dynamic econometric models is far too large to survey here. Single equation applications are particularly numerous, with the older literature generally using PA models and the more recent literature using the ECM form. In general it is preferable to work with the ECM form and test the reduction to the PA form. Further, while the ECM form can be derived as part of the solution to certain explicit optimization models, it is also a reasonable general form to adopt when dynamic behaviour is being added *ad hoc* to a static theoretical model. In that case, in particular, it makes sense to include several lags and reduce the dynamics to the smallest lag structure consistent with the statistical properties of the data. For references on applying ECM forms, the reader is referred to Hendry

(1995) and to just about any recent issue of journals dealing with applied economic analysis.

There is also a massive literature on multi-equation models, although unlike the single equation literature, which has continued relatively uninterrupted since its initial stages, the systems literature has had a more cyclical history. In its latest incarnation it involves applications of the Johansen approach to cointegration analysis, working from dynamic reduced form equations, and attempting to put economic interpretations on the estimated cointegrating relations. This approach is summarized by Hendry and Doornik (1994). The earlier strand of systems modelling was associated with work such as that by Theil and Boot (1962) analysing macroeconometric models.[16]

Large scale macro modelling

Theil and Boot (1962) begin with Klein's Model I of the US economy, a six-equation Keynesian structural macro model. In Klein's (1950) model, dynamics entered the structural equations directly, with investment, for example, assumed to depend on lagged profits and the lagged capital stock, rather than through a specific adjustment process like the PA model.

Theil and Boot convert Klein's structural model to a dynamic reduced form in lagged dependent variables and current and lagged values of the exogenous variables. They then analyse the time path of the endogenous variables in response to specific shocks to specific exogenous variables, using simulation analysis in a manner similar to that used by Taylor(1993). In addition to analytical simulations, Theil and Boot calculate the characteristic roots of the six-equation system and, in a step which is not often followed in more recent work, estimate the standard error for the largest root. Because of reductions the model turns out to have three roots, one positive real root and a pair of complex conjugate roots, the latter meaning that the system displays oscillations. Because the modulus of the complex roots is larger than the real root, the complex roots are the dominant roots. On calculating the standard error of the largest root, Theil and Boot find that the root, with modulus 0.838 and estimated (asymptotic) standard error 0.22, is not significantly different from unity (based on what we would now term conventional t-values).[17] Theil and Boot conclude that the model characterizes the US economy as being close to instability, and as, in fact, leaving the question of the stability of the system still open. This interpretation of their result indicates the impact of the unit root-cointegration literature on econometric dynamics since Theil and Boot wrote. What Theil and Boot regard as a worrisome feature of Klein's Model I, current researchers would take for granted and, indeed, would be surprised if it were not to appear.

Authors like Klein and Theil and Boot put more emphasis on structural modelling as the first stage of empirical dynamic macroeconomic modelling than would many current researchers, perhaps because of the influence of Sims' (1980) criticisms of macroeconometric modelling. The more recent Johansen approach to cointegration systems estimation has a lot in common with Sims' proposal that macroeconomists should work with unrestricted reduced form systems, treating all

variables as endogenous. This should not, however, be taken to mean that modern macroeconomic dynamics is simply VAR modelling devoid of theoretical structure (see Hendry 1995). Systems macroeconometrics fell somewhat out of favour in the 1970s, partly because estimation of even moderate scale dynamic macro systems (as opposed to Klein-type models, which came to be regarded as small macro models) quickly became computationally very expensive, and partly because of the failure of existing macro models to explain the behaviour of inflation and unemployment during the 1970s. It should not, however, be thought that dynamic macro modelling disappeared until the appearance of the Johansen cointegration methodology – see Fair (1994), for example.

Microeconomic examples

There are numerous empirical papers dealing with economic dynamics in microeconomic markets. To conclude this book, we consider a pair of papers summarizing research projects in which we gained first-hand experience with the tricks of econometric dynamics.

The market for physician services

Ferguson and Crawford (1989) set up a disequilibrium model of the market for physicians' services, which they estimate on Canadian data from the period before the introduction of national health insurance. Their basic model is of a competitive market with Walrasian adjustment to disequilibrium.

In the first stage of their paper, Ferguson and Crawford specified demand and supply functions for physicians services as:

$$D_t = \alpha_1 P_t + \alpha_2 X_t + u_{1t} \quad \alpha_1 < 0 \tag{7.80}$$

$$S_t = \beta_1 P_t + \beta_2 Z_t + u_{2t} \quad \beta_1 > 0 \tag{7.81}$$

$$Q_t = \text{Min}(D_t, S_t) \tag{7.82}$$

$$\Delta P_t = \delta(D_t - S_t), \quad \delta > 0 \tag{7.83}$$

where X and Z are demand and supply-side vectors of exogenous variables and u_1 and u_2 are random disturbance terms. Equation (7.82) is the Min condition, that specifies the actual quantity traded as the lesser of supply and demand. The econometrics of models of this type, termed the quantitative method of disequilibrium analysis, is discussed by Maddala (1983).

To implement the model, note that:

$$Q_t = D_t - (D_t - S_t) \tag{7.84}$$

in the case of excess demand (when $Q_t = S_t$) and, for the case of excess supply:

$$Q_t = S_t + (D_t - S_t) \tag{7.85}$$

From Equation (7.83), $(D_t - S_t) = \Delta P_t/\delta$, which can be generalized to $(D_t - S_t) = \Delta P_t^+/\delta_{\text{d}}$ where $\Delta P_t^+ = \Delta P_t$ when P is rising, that is, for the case of excess

demand, and zero otherwise, and for the case of excess supply, to $(D_t - S_t) = \Delta P_t^- / \delta_s$ where $\Delta P_t^- = \Delta P_t$ when P is falling and zero otherwise. Thus, the sign of ΔP_t identifies the nature of the disequilibrium; positive for excess demand, negative for excess supply and zero for equilibrium.

The use of different adjustment coefficients, δ_d and δ_s, allows for a different speed of adjustment in response to excess demand than excess supply. This is, of course, a testable hypothesis. Thus, we have:

$$\begin{aligned} Q_t &= D_t - \Delta P_t^+ / \delta_d \\ Q_t &= S_t + \Delta P_t^- / \delta_s \end{aligned} \tag{7.86}$$

for the excess demand and excess supply cases, respectively. In equilibrium, $\Delta P_t = 0$ and $Q_t = D_t = S_t$. From Equation (7.86) it is possible to estimate the coefficients of the demand and supply functions: in the case of excess demand, $\Delta P_t^- = 0$ and Equation (7.86) becomes $Q_t = S_t$ while in the case of excess supply, $\Delta P_t^+ = 0$ and $Q_t = D_t$.

In the standard Walrasian model, δ is non-negative, but has no upper bound. It can be transformed, however, into a coefficient which is bounded between 0 and 1 if we define:

$$\mu = 1/(1 + \delta(\beta_1 - \alpha_1)) \tag{7.87}$$

With Equation (7.87) and the expression for the equilibrium value of price, P_t^*, which can be found by equating the demand and supply functions, we can rewrite Equation (7.83) as:

$$P_t = \mu P_{t-1} + (1 - \mu) P_t^* \tag{7.88}$$

where δ_d or δ_s can be used in Equation (7.87) as appropriate. The term μ is bounded between 0 and 1; if $\mu = 0$ there is instantaneous adjustment to equilibrium, while if $\mu = 1$ there is no adjustment. The form of Equation (7.88) is the same as that of Equation (7.7) in the PA model, but is derived from a specific dynamic adjustment mechanism.

Ferguson and Crawford estimated the disequilibrium model on pooled data from eight Canadian provinces for the period 1963–68, using iterative three stage least squares, and found that the speed of adjustment of price to excess demand differed from the speed of price response to excess supply. Specifically they found that μ_d was not significantly different from zero, indicating very rapid upward adjustment of the price of physicians' services in the face of excess demand, while μ_s was not significantly different from 1.0, indicating extremely slow downward adjustment of price in response to excess supply (the point estimate was actually 1.24, which technically yields an unstable difference equation for price, but the estimated value was not significantly different from 1 or from values slightly below 1). According to these results, Canadian physicians were faster to raise fees than to lower them.

Adjustments in interest rates

Lim (2001) applies a number of the techniques we have discussed in this volume. Her objective is to study the dynamic behaviour of the interest rates paid and charged by Australian banks. Her focus is on three rates: a representative rate charged by banks on their loans (r_L), a representative rate paid on deposits (r_D) and a broader money market rate (r_B). Her basic model is a Cournot multi-product oligopoly model, which is simply an extension to a market with several firms of the Cournot single product duopoly model which we considered earlier.

The loan and deposit rates are assumed to be endogenously determined and, like prices in our Cournot duopoly example, to depend on total market quantities. The aggregate demand curve for loans, $L(r_L)$, is a decreasing function of the loan interest rate and the aggregate supply of funds brought to banks for deposits, $D(r_D)$, is an increasing function of the deposit interest rate, and as in the case of our Cournot example, Lim works with the inverse functions, $r_L(L)$ and $r_D(D)$. The money market rate is assumed to be exogenous, a hypothesis which is supported in the later econometric results. Banks are assumed to be profit maximizers, facing operating cost curves which have constant marginal costs of loans and deposits.

Since in this model banks are assumed to be Cournot oligopolists, each bank is assumed to select its profit-maximizing level of loans and deposits on the assumption that the other banks in the market hold their levels of loans and deposits constant. As in our duopoly example, this assumption can be used to find the long-run equilibrium position for the system. In our duopoly example we found the long-run equilibrium in terms of output quantities, but since market price was determined by aggregate market output, we could also find a long-run equilibrium price. Since Lim's interest is the behaviour of interest rates, which are the prices in her model, she solves for equilibrium expressions for the loan and deposit rates (r_L^* and r_D^*, respectively). She shows that, in the long run, bank loan and deposit rates are determined by the number of banks in the market (N), the marginal costs of administering loans and deposits (γ_L and γ_D), the money market rate and the functional forms of the loan demand and deposit supply curves. In other words, economic theory provided the two fundamental equations (i.e. the long-run cointegrating relationships) sought in the empirical analysis:

$$r_L^* = \gamma_{11} + \beta_{11} r_B \tag{7.89}$$

$$r_D^* = \gamma_{21} + \beta_{21} r_B \tag{7.90}$$

where $\gamma_{11} = \gamma_L - r_L'(L^*)(L^*/N)$, $\gamma_{21} = -\gamma_D - r_D'(D^*)(D^*/N)$. In the simplest case, both slope coefficients are expected to be unity, ($\beta_{11} = \beta_{21} = 1$) while the intercept terms γ_{11} and γ_{21} may be treated as constant deposit and loan intermediation margins. These long-run equations yield an explicit, testable prediction: in the long run, and increase in the money market rate should be passed through one-for-one to both the bank loan and deposit rates.

In the empirical analysis, the interest rates were first shown to have unit root behaviour, which means that the cointegration approach is the appropriate

framework for empirical investigation. Since it is possible that there are several cointegrating relations among the interest rates, Johansen's approach was applied. The data shows the presence of two cointegrating relations, one which can be normalized to express the loan rate as a function of the money market rate and the other which can be normalized to express the deposit rate as a function of the money market rate; both as suggested by Equations (7.89) and (7.90).

The paper is particularly concerned with whether loan and deposit rates adjust differently, depending on whether the change in the money market rate was positive or negative. Hence, Lim uses an extension of the basic Johansen cointegration and error correction framework which allows for the possibility of asymmetries in both long- and short-run behaviour of the system. In particular, she allows for the possibility that banks respond differently to increases in the money market rate than they do to decreases, and further that the response of deposit rates to changes in the money market rate is different from the response of loan rates. In essence she allows the error correction terms and other parameters in the model to change with the stance of monetary policy. A generalized version of the model estimated is set out below, where the superscript 's' indicates that the parameters are affected by the stance of monetary policy:

$$\Delta r_{L,t} = \alpha_{11}^{s}[r_{L,t-1} - \gamma_{11}^{s} - \beta_{11}^{s} r_{B,t-1}] + \alpha_{12}^{s}[r_{D,t-1} - \gamma_{21}^{s} + \beta_{21}^{s} r_{B,t-1}] + \varepsilon_t$$

$$\Delta r_{D,t} = \alpha_{21}^{s}[r_{L,t-1} - \gamma_{11}^{s} - \beta_{11}^{s} r_{B,t-1}] + \alpha_{22}^{s}[r_{D,t-1} - \gamma_{21}^{s} + \beta_{21}^{s} r_{B,t-1}] + \varepsilon_t$$

The model is applied to quarterly Australian data. The results show that in the long run, an increase in the money market rate has the same effect on both the loan and deposit rates, as theory predicts, although the magnitude is slightly less than predicted. The Cournot model predicts that, in the long run, a 1 percentage point increase in the money market rate will lead to increases of 1 percentage point in both the loan and deposit rates, whereas the estimated cointegrating relations indicate that in the long run a 1 percentage point increase in the money market rate increases both the loan and deposit rates by just over 0.8 percentage points. The results show no asymmetry in the long-run responses, meaning that in the long run whether the change in the money market rate is an increase or a decrease has no effect on how much of that change is passed through to loan and deposit rates.

The adjustment path to the long run, however, differs depending on whether the monetary policy change is an increase or a decrease. The results suggest that banks adjust their loan and deposit rates, in response to a change in the money market rate, at a faster rate during periods of monetary easings (negative changes) than during periods of monetary tightenings (increases in interest rates).

Conclusion

In this chapter we have made no claims to comprehensiveness. Our objective was simply to give some idea of the relation between the tools of theoretical dynamic

modelling and those of econometric dynamics. These two topics are often treated as completely separate, to the point of using different, and apparently contradictory language to refer to the same thing. For example, the theoretical literature will refer to stable roots as lying inside the unit circle (as we have above, meaning lying strictly between -1 and $+1$) whereas the econometric literature will often say that stability requires the roots to lie outside the unit circle. The explanation is that because of the mathematical notation used (in particular the use of the lag operator) the roots which econometricians calculate are the inverse of the roots calculated by theorists. Same thing, different perspective.

Whichever approach is used, the key point is that neglect of dynamic relations can lead to very misleading conclusions about economic relationships. Even if it turns out that there is no significant dynamic structure present, the possibility should always be considered and tested.

Notes

1 Introduction

1 The presence of the t term in the $g(x_t, t)$ function does not make the difference equation non-autonomous: that would only happen if the mathematical form of the $f(\cdot)$ function itself depended critically on the value of t.

2 First-order difference equations

1 Obviously, if Y_0 equals 0, the right-hand side of Equation (2.4), and therefore Y_t, will always equal zero regardless of how big t gets.
2 We shall consider one model which does yield a negative root.
3 Strictly speaking we should refer to an equilibrium as globally stable if the actual value of Y converges on the equilibrium value regardless of what that actual value might be. We refer to an equilibrium as locally stable if Y converges on its equilibrium value only if the initial value of Y lies within some local neighbourhood around Y^e. The distinction between local and global stability will become important when we get to more complicated forms of difference equation.
4 If the equilibrium is unstable, of course, it will never actually be reached, but that does not change the nature of the equilibrium point itself.
5 The obvious question here is, why do we try this? The answer is: because it generally works.
6 Phase diagrams can also be drawn for higher order difference equations, but at higher orders we lose the diagrammatic simplicity of two axes.
7 Empirically, whether this argument makes sense depends on the length of the time period involved. With monthly or quarterly data the story is quite plausible. With annual data, it is rather less so. In theoretical dynamics we can simply refer to a 'period', without specifying a calendar interval.
8 Drawing the phase diagram for the two models, one with and one without the proportional fiscal policy rule, is a useful exercise in sorting out the importance of the differences between two very similar models, in terms of effects on both the slope and the intercept of the $Y_t(Y_{t-1})$ function.
9 And note the assumption that this really is a no-policy equilibrium – this is, the equilibrium the system would reach if all government spending were non-discretionary and did not respond in any way to deflationary or inflationary gaps, so the Phillips policy rule is, in this example, best seen as introducing an extra element of government spending which otherwise would not have been present. Note also that we have not discussed how this spending is to be financed.

3 Second-order difference equations

1 In general the number of roots equals the degree of the difference equation – one root for a FODE, two for a SODE and so on up. They will not always have distinct values, however.
2 There is a drawback to it. As we shall see when we deal with empirical applications of difference equation models, econometricians use the terms in precisely the opposite sense to that in which theorists use them. An econometrician refers to a root as being outside the unit circle, while a theorist would refer to it as being inside the unit circle. In both cases, they mean what we shall characterize in a moment as a stable root. The difference in terminology comes out of a difference in the way the expressions for the roots are found, which we shall discuss in detail in Chapter 7.
3 Plus or minus, depending on whether A_2 is positive or negative. Our neglect of the A terms to this point does not mean that they are unimportant to the ultimate behaviour of the system, it just means that they are not crucial for the general behaviour which we are discussing here. We shall see an illustration of the type of role they play in just a moment.
4 Note that complex numbers always come in conjugate pairs, so if one root is complex we must have a second complex root.
5 We do this primarily for comparability with the earlier multiplier model – we could easily derive a difference equation in consumption.
6 It may seem more natural to think in terms of percentage changes in Y rather than changes in the level of Y – if so, think of all of the Y, C, I and G terms in the model as being logs.
7 Gandolfo (1997) notes that in the case of sign pattern $(+\ -\ -)$, the positive root will be larger in absolute value than the negative root, with the reverse being true when the sign pattern is $(+\ +\ -)$.
8 If entering the number of firms linearly seems implausible, we can think of all of these terms as being log transformations of the original variables.

4 Higher-order and systems of difference equations

1 In fact, since complex roots come in conjugate pairs, if we have a pair of complex roots the third root must be real. Similarly, if we have two real roots the third must also be real.
2 This is because if only one of the other two were negative the product of the roots would be positive, and we know that it must be negative, and while three negative roots would also give a positive product, that would violate the rule of signs.
3 Note that our discussion assumes that the g_t term in this example is a constant. If there is an exogenous (to this model) growth element present, those interesting intrinsic dynamics could be chasing a moving equilibrium, which could make the time path more interesting still.
4 When the matrix A is square and all of its elements are real, any complex roots of A must occur in conjugate pairs. This is the basis for our earlier assertion that (in economic models, at least) complex roots come in conjugate pairs.
5 A good source on the dynamics of population growth is Keyfitz (1968).
6 A number of developed countries are actually in that state now – if it were not for immigration their populations would be tending to decline.
7 Again note that we can add immigration into our model, we do not do so solely to keep the exposition simple.
8 Just to complicate matters, in practice the post-war baby boom observed in most English speaking countries was followed what is sometimes known as the baby bust, a dramatic drop in births. The baby bust can be modelled as a further change in birth rates, following the change which produced the baby boom, and both the baby boom and the baby bust have sent their own (cyclical) shock waves through the population.

9 The best known early presentation of such a model was, of course, the Rev. Thomas Robert Malthus's *Essay on the Principle of Population*. On early models of economic growth in general, see Eltis (2000) and for general economic-demographic modelling, see Denton and Spencer (1975).

5 Intertemporal optimization

1 Alternatively, we could think in terms of spending an extra dollar on consumption today and giving up the marginal utility we could have derived from saving that dollar until tomorrow, at market interest rate r, and increasing our future consumption by the future purchasing power of that dollar plus accumulated interest.
2 There may be other constraints on our choice of x; we set those aside so as not to complicate the problem too much at this point.
3 This is just a reminder that utility depends on real consumption, not on nominal consumption expenditure. Effectively, in this problem, the policy rule is a consumption function at the level of the individual consumer, where consumption is a function of accumulated assets rather than current income alone.
4 The term 'scrap value' comes from the fact that much of the early work on intertemporal optimization problems dealt with investment decisions, where a piece of capital equipment would be used in production for a number of years, then sold for its scrap value, that value depending on how hard it had been run in the previous periods.
5 One empirical implication of this result is that the relation between aggregate consumption and aggregate assets (or their per capita counterparts) will depend on how many individuals in the population are in each period of the planning horizon. Basically, this means that the form of the aggregate consumption function will depend on the age distribution of the population. This suggests that, if we are estimating aggregate consumption relations derived, at least in principle, from the optimization procedure we have discussed here, we should include demographic explanatory variables among our explanatory variables.
6 Whether zero is an appropriate valuation to put on death we leave to philosophers and health economists.
7 This assumption is strictly a matter of convenience, as we shall see later. The choice of whether the first period should be labelled period 0 or period 1 really depends on the conditions of the problem. Labelling it period 0 means that the discount term (when there is one) for the first period is β^0=1, which is consistent with the convention that the first period is not discounted, although if planning is done at the beginning of the period and consumption not done until the end, discounting that first period might seem natural. One of the catches of discrete time modelling is the need to decide when, during a period, things happen. They can be assumed to happen at the beginning, or at the end, or it can be determined that, for a particular problem, it does not really matter when they happen. What is important from the analytical point of view is to decide at the beginning of the analysis which possibility applies, and to remain consistent in that assumption throughout the analysis.
8 That is why we started at $t = 1$ instead of $t = 0$: starting at $t = 1$ means that T tells us the total number of periods over which the cake had to last. If we had started at $t = 0$, we would have had $T + 1$ periods. It was an assumption made strictly for purposes of avoiding what might have been a bit of notational inconvenience.
9 Note that if $\beta R = 1$, meaning that $\delta = r$, we are back in the cake eating problem, consuming a constant amount per period.
10 This is, of course, a simplifying assumption. Some goods which are good for you yield utility, as do some goods which are harmful to your health. Both types could be added into the model.

11 We could use Equation (5.89) to simplify Equation (5.90) and eliminate some of the stochastic elements from the right-hand side, but we shall leave this aside for the moment.
12 See Samuelson (1969) for the development of a similar expression.

6 Nonlinear difference equations

1 In optimal growth models, savings becomes endogenous.
2 Basically, what we are doing is noting that if we draw the isoquant diagram for our production function in capital–labour space, and expand output along a ray from the origin, meaning that we expand the levels of output, labour and capital without changing the ratio of capital to labour, each successive isoquant is just a radial expansion of the previous one, with constant returns meaning that, along that ray from the origin, the isoquant for two units of output is twice as far from the origin as the isoquant for one unit of output and so on.
3 This does not violate the constant returns assumption, since k is capital per worker and the marginal productivity of k shows what happens to output per worker when capital per worker is increased. This is equivalent to asking, when looking at $F(K, L)$, about the effects of increasing K while holding L constant or, since L is growing at a constant proportional rate η, about the effects of increasing K faster than L. When we talk about marginal productivity of k in $f(k)$, we are just converting those effects to per-worker terms.
4 This is the assumption that all inputs are essential, so that regardless of how much labour might be present, if the workers have no capital to work with, output will be zero. This property is present, for example, in the constant returns Cobb–Douglas production function $Y = AL^{\alpha}K^{1-\alpha}$.
5 Another way of looking at this expression would be to write it as $sf(k^{\mathrm{e}})/k^{\mathrm{e}} = (\eta + \delta)$ which has a clear interpretation in terms of gross and net investment, the sorting out of which we leave as an exercise.
6 Which obviously raises the question of whether an economy is really large, in any meaningful sense, if it has a large gross domestic product (GDP) because it has a very large population along with a very low per capita income.
7 For references, see Frank and Stengos (1988).
8 That is, the investment function took the general nonlinear form $I_{t+1} = I(r_t)$.
9 For a discussion of some cases of empirical research in which nonlinearity proved important, see Zellner (2002).

7 Empirical analysis of economic dynamics

1 Paradoxically, if x_t is unchanging over time, we will never be able to estimate the value α_2 econometrically, even though it is pulling the value of y, since we will not have sufficient information on how changes in x cause y to change. All we will observe is y_t following a FODE, and while we might be able to identify the point y^* towards which that difference equation is tending we will not be able to isolate the various factors involved in determining the location of the equilibrium.
2 Which is why data on economic variables are often viewed as observations from disequilibrium, not equilibrium states, or in other words reflecting short-run adjustment, rather than long-run behaviour.
3 Recall that technically we never actually reach the new equilibrium in finite time, but we can get so close to it that the gap is effectively zero.
4 Even in annual data, some dynamic effect may be noticeable, since shocks generally are not so obliging as to occur precisely at the beginning of the calendar year. A shock

occurring part way through the year and observable in the data on the exogenous variables in the year in which it occurs, will have part of its effect on the dependent variable in the year of the shock and part in the next year.

5 While we have not formally derived the optimal equation, it can be done for certain types of optimization problems – see Pagan (1985), Nickell (1985) and Domowitz and Hakkio (1990).

6 A random disturbance term ε_t is typically simply added to this point. There are econometric modelling considerations associated with the way ε_t is defined, but we will not be considering those here.

7 In contrast, Ng (1995) also investigated demand systems, starting explicitly from time series considerations but reaching the same general conclusion as Anderson and Blundell, to the effect that one reason that theoretical restrictions are often rejected when consumer demand systems are investigated is the neglect of the long-run/short-run distinction.

8 The question of the correct representation is by no means closed – see Perron (1989).

9 Note that this is the unconditional variance of y_t. The conditional variance of y_t given the value of y_{t-1} is equal to $\text{var}(\varepsilon_t)$ which is constant by assumption. When we speak of the unconditional variance of y_t we are essentially standing at period 0 and looking into the future with no notion of what particular values will eventuate – the farther ahead we look, the wider the range of values which y_t could take on with any given probability, depending on the set of values it could take on between period 0 and period t. When we look at period t from one period before, knowing the value of y_{t-1}, the variance of y_t will always equal $\text{var}(\varepsilon_t)$. The non-stationarity arises from the fact that y_t will incorporate, with no attenuation, the values which all of the random shocks between period 0 and period t could possibly take on.

10 In practice, since economic variables seldom display alternations, α_1 will be positive, so we need only test whether it is significantly less than 1 in magnitude.

11 The 'I' in ARIMA stands for Integrated, the AR stands for Autoregressive and the MA stands for Moving Average. If the variables are not Integrated, and therefore do not have to be differenced to achieve stationarity, we have ARMA analysis. See Enders (1995) for details.

12 Order of integration is not a universal constant. Dixit and Pindyck (1994), show that whether the constant dollar prices of oil and copper are $I(1)$ or $I(0)$ depends critically on the sample period being used. Nevertheless, the principle still holds – for two variables to be related, they should display similar dynamic behaviour over the data period being investigated.

13 Strictly speaking we should model the exogenous factors driving x and set up a block recursive system, but all modelling exercises must stop somewhere.

14 For a discussion of regression-based estimation techniques see Lim and Martin (1995).

15 This approach never really found wide acceptance, however, probably because of the heavy computing required. Further, the use of macroeconometric models fell into some disfavour in the 1970s due to their apparent inability to explain or predict the behaviour of economies during that period. By the time the cost of computing power had fallen enough to make these techniques widely accessible, they had fallen out of the mainstream. Even the applications of system ECM forms by Anderson and Blundell in the early 1980s, to which we have already referred, were regarded as computationally very expensive, and these did not involve testing for unit roots. The expansion in computing capacity and falling cost of computing power in recent years has made dynamic systems modelling much more widespread.

16 See also, Adelman and Adelman (1959).

17 For recent work on related issues, see Bierens (2001).

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Index