Appendix: Combining Probabilities

Start with one flip of a coin. If you flip the coin once, you know the answer. Call the probability of getting heads P, so the probability of getting tails is $(1–P)$. The symbol ${}_kP_n$ represents the probability of getting k heads in n flips.

$${}_0P_1 = (1 - P)$$

$${}_1P_1 = P$$

Flip the coin a second time. Now there are several possibilities. You could have two tails, two heads, a head and a tail, or a tail and a head. I wrote down the probabilities for one flip. The probabilities for the second flip are the same, but I have to multiply the first flip by the second. The probability of getting two heads in a row is the probability of getting heads on the first toss times the probability of getting heads on the second toss:

$${}_0P_2 = {}_0P_1 \cdot {}_0P_1$$

$${}_1P_2 = {}_1P_1 \cdot {}_0P_1 + {}_0P_1 \cdot {}_1P_1$$

$${}_2P_2 = {}_1P_1 \cdot {}_1P_1$$

The most interesting example is the ${}_1P_2$. It says that the probability of getting one head in two tosses is the probability of getting a head then a tail, plus a tail then a head. Substituting in the values for ${}_0P_1$ and ${}_1P_1$ from the previous example gives:

$${}_0P_2 = (1 - P)^2$$

$${}_1P_2 = 2P(1 - P)$$

$${}_2P_2 = P^2$$

I want to flip the coin one more time before moving on to the general case:

$$
\begin{aligned}
{}_0P_3 &= {}_0P_1 \cdot {}_0P_1 \cdot {}_0P_1 \\
&= (1-P)^3 \\
{}_1P_3 &= {}_1P_1 \cdot {}_0P_1 \cdot {}_0P_1 + {}_0P_1 \cdot {}_1P_1 \cdot {}_0P_1 + {}_0P_1 \cdot {}_0P_1 \cdot {}_1P_1 \\
&= 3P(1-P)^2 \\
{}_2P_3 &= {}_1P_1 \cdot {}_1P_1 \cdot {}_0P_1 + {}_1P_1 \cdot {}_0P_1 \cdot {}_1P_1 + {}_0P_1 \cdot {}_1P_1 \cdot {}_1P_1 \\
&= 3P^2(1-P) \\
{}_3P_3 &= {}_1P_1 \cdot {}_1P_1 \cdot {}_1P_1 \\
&= P^3
\end{aligned}
$$

With the three-flip case, what is happening becomes more obvious. It finds all the ways that I can select k heads from n flips, and it multiplies by P^k to give the probability for this number of heads. Then I multiply again by $(1–P)^{n–k}$ to give the probability for this number of tails. Any statistics text will tell you that the number of ways of picking k from n is:

$$\frac{n!}{k! \cdot (n-k)!}$$

So:

$${}_kP_n = \frac{n! \cdot P^k(1-P)^{n-k}}{k! \cdot (n-k)!}$$

Checking this equation against the values already calculated for $n = 3$ in the previous example shows that it is correct:

$$
\begin{aligned}
{}_0P_3 &= \frac{3! \cdot P^0(1-P)^3}{0! \cdot 3!} \\
&= (1-P)^3
\end{aligned}
$$

Note that 0! and 1! are both equal to 1:

$$
\begin{aligned}
{}_1P_3 &= \frac{3! \cdot P^1(1-P)^2}{1! \cdot 2!} \\
&= 3P(1-P)^2
\end{aligned}
$$

$$
\begin{aligned}
{}_2P_3 &= \frac{3! \cdot P^2(1-P)^1}{2! \cdot 1!} \\
&= 3P^2(1-P) \\
{}_3P_3 &= \frac{3! \cdot P^3(1-P)^0}{3! \cdot 0!} \\
&= P^3
\end{aligned}
$$